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STEADY-STATE GROWTH AND THE ELASTICITY OF SUBSTITUTION

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Abstract: In a neoclassical economy with endogenous capital- and labor-augmenting technical change the steady-state growth rate of output per worker is shown to increase in the elasticity of substitution between capital and labor. This confirms the assessment of Klump and de La Grandville (2000) that the elasticity of substitution is a powerful engine of economic growth. However, unlike their findings my result applies to the steady-state growth rate. Moreover, it does not hinge on particular assumptions on how aggregate savings come about. It holds for any household sector allowing savings to grow at the same rate as aggregate output.

Keywords: Capital Accumulation, Elasticity of Substitution, Direction of Technical Change, Neoclassical Growth Model.

JEL-Classification: E22, O11, O33, O41.

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1 Introduction

Is the measured degree of factor substitution an indicator for an economy's growth potential? The debate surrounding this question began with the contributions by de La Grandville (1989) and Klump and de La Grandville (2000). These authors study the link between the elasticity of substitution, being treated as a parameter of an aggregate CES production function, and economic growth in the neoclassical economy of Solow (1956). They conclude that the degree of factor substitution is a powerful engine of economic growth in the sense that a higher elasticity of substitution between capital and labor leads to a higher growth rate along the transition and a higher steady-state level of output per worker. This assessment has been challenged by Miyagiwa and Papageorgiou (2003) and Irmen (2003). These authors emphasize the role of the underlying savings hypothesis of Solow (1956) and find cases in a model of overlapping generations where a higher elasticity of substitution is an impediment to growth.[1]

Such conflicting results arise in a neoclassical setting since the elasticity of substitution affects the two main pillars on which aggregate one-sector models of economic growth are based. First, there is a direct impact on aggregate production since differing degrees of factor substitution affect the shape of the aggregate production function. Second, there is an indirect effect on aggregate savings and capital accumulation since the degree of factor substitution affects the functional income distribution.

This paper takes a new look at the link between the elasticity of substitution and economic growth from the perspective of the so-called Endogenous Growth Theory. From this point of view the central question is whether and how the elasticity of substitution affects the steady-state growth rate of an economy. I address this question in a *neoclassical economy with endogenous capital- and labor-augmenting technical change*. In this framework, I show that the elasticity of substitution is a determinant of the long-run growth rate of an economy. More importantly, I establish that a greater elasticity of substitution means faster steady-state growth of per-worker variables. Although this finding confirms the spirit of the claim of Klump and de La Granville, it is not subject to the above mentioned criticism. In

[1] Irmen and Klump (2009) reconcile these findings by pointing out that the positive growth effects of a high elasticity of substitution materialize as long as the propensity to save out of capital income exceeds the propensity to save out of wage income. See Xue and Yip (2009) for a comprehensive discussion surrounding the growth effects of the elasticity of substitution in one-sector models of economic growth.

fact, my steady-state result holds for any household sector that allows for a constant aggregate consumption growth rate equal to the growth rate of the economy. Hence, the effect of the elasticity of substitution through savings and capital accumulation on the steady state, i. e., the second pillar of aggregate one-sector growth models, is mute. To the best of my knowledge, the present paper is the first that derives such a result in an endogenous growth model.

I consider an economy in an infinite sequence of periods described by an endogenous growth model with capital- and labor-augmenting technical change. My analytical framework is *neoclassical* since it maintains the assumptions of perfect competition, of an aggregate production function with constant returns to scale and positive and diminishing marginal products, and of capital accumulation. It has *endogenous growth* since economic growth results from innovation investments undertaken by profit-maximizing firms. To allow for innovation investments in *capital- and labor-augmenting technical change*, I introduce two intermediate-good sectors, one producing a capital-intensive intermediate, the other a labor-intensive intermediate.[2] Innovation investments increase the productivity of capital and labor at the level of these intermediate-good firms. Moreover, they feed into aggregate productivity indicators that evolve cumulatively, i. e., in a way often referred to as 'standing on the shoulders of giants'.

Competitive final-good firms use both intermediates and produce according to the normalized CES production function of de La Grandville (1989) and Klump and de La Grandville (2000). In equilibrium, the quantity of either intermediate-good input is equal to the amount of capital and labor in efficiency units. Therefore, the elasticity of substitution of the CES of the final-good sector coincides with the (partial) elasticity of substitution between capital and labor, i. e., the one that applies if the level of capital- and labor-augmenting technical change is kept constant.

The positive effect of a greater elasticity of substitution on the steady-state growth rate appears as Theorem 1 in Section 3. It is derived in a series of steps. First, I lay out the details of the competitive production sector and define its equilibrium in Section 2. Here, a central finding is that the equilibrium incentives to engage in capital- and labor-augmenting technical change depend on the intensity with which

[2] The production sector extends and complements the one devised in Irmen (2005) by allowing for capital-augmenting technical change. In turn, the latter builds on Hellwig and Irmen (2001) and Bester and Petrakis (2003). See Acemoglu (2003) for an alternative model of endogenous capital- and labor-augmenting technical change where innovation investments are financed through rents that accrue in an environment with monopolistic competition.

both intermediates are used in the production of the final good, and on the elasticity of substitution between these intermediates. These dependencies arise since the price that intermediate-good firms charge in equilibrium is equal to the marginal product of the respective intermediate good they produce. As a consequence, this intensity serves as the state variable of the whole production sector in each period.

In Section 3, I deal with the analysis of the steady-state. Along such a trajectory all variables must grow at a constant rate. It follows that the efficient capital intensity must be constant, too. Yet, the production sector alone does not pin down the steady-state capital intensity. Therefore, I embed the production sector in the broader concept of a neoclassical economy with endogenous capital- and labor-augmenting technical change by adding a difference equation for capital accumulation and a resource constraint. For this setting, I establish that in a steady state i) the growth rate of capital-augmenting technical change is zero, and ii) the growth rate of output per worker is the growth rate of labor-augmenting technical change. Both properties follow from the so-called Steady-State Growth Theorem of Uzawa (1961), which applies here since aggregate innovation investments in capital- and labor-augmenting technical change are proportionate to efficient capital and efficient labor, respectively.

To establish the steady-state growth rate effect of the elasticity of substitution one needs to account for both properties. Theorem 1 reveals that the driving force behind it is the *efficiency effect* of the elasticity of substitution established by Klump and de La Grandville (2000): ceteris paribus, an increase in this elasticity increases output. In the context of endogenous capital- and labor-augmenting technical change, this efficiency effect implies faster steady-state growth.

2 The Competitive Production Sector

The production sector has a final-good sector and an intermediate-good sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The *manufactured final good* can be consumed or invested. If invested it may either become future capital or serve as an input in current innovation activity undertaken by intermediate-good firms. Intermediate-good firms produce one of two types of intermediates and sell it to the final-good sector. The production of the *labor-intensive intermediate good* uses labor as the sole input, the only input in the production of the *capital-intensive intermediate good* is capital. Labor- and capital-augmenting technical change is the result of innovation investments undertaken by intermediate-good firms. *Labor* and *capital* are supplied to the intermediate-good sector. The final good serves as numéraire.

2.1 The Final-Good Sector

The final-good sector produces with the following CES production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$Y_t = F\left(Y_{K,t}, Y_{L,t}\right) = \Gamma\,\left[\gamma\, Y_{K,t}^{\psi} + (1-\gamma)\, Y_{L,t}^{\psi}\right]^{1/\psi}, \tag{1}$$

where Y_t is aggregate output in t, $Y_{K,t}$ is the aggregate amount of the capital-intensive intermediate input, and $Y_{L,t}$ denotes the aggregate amount of the labor-intensive intermediate input.[3] The parameters satisfy $\Gamma > 0$, $1 > \gamma > 0$, and $1 > \psi > -\infty$. Moreover, $\sigma = 1/(1-\psi)$ is the elasticity of substitution between $Y_{K,t}$ and $Y_{L,t}$. I show below that σ is also the (partial) elasticity of substitution between capital and labor. In terms of the labor-intensive intermediate-good input, let $y_t = F(\kappa_t, 1) \equiv f(\kappa_t)$, where $\kappa_t \equiv Y_{K,t}/Y_{L,t}$. Then

$$y_t = f(\kappa_t) = \Gamma\,\left[\gamma\,\kappa_t^{\psi} + (1-\gamma)\right]^{1/\psi}. \tag{2}$$

To identify the effect of the elasticity of substitution on otherwise identical economies, I follow de La Grandville (1989) and Klump and de La Grandville (2000) and normalize (2) by choosing some baseline values for the following variables: $\bar{\kappa}$, $\bar{y} = f(\bar{\kappa})$, and the marginal rate of substitution $\bar{m} = \left[f(\bar{\kappa}) - \bar{\kappa}\, f'(\bar{\kappa})/f'(\bar{\kappa})\right] > 0$. The normalized CES production function that satisfies these criteria is then equal to (see, Klump and de La Grandville (2000), eq. 5)

$$f_\sigma(\kappa) = \Gamma(\sigma)\,\left[\gamma(\sigma)\,\kappa^{\psi} + (1-\gamma(\sigma))\right]^{1/\psi} \tag{3}$$

with

$$\Gamma(\sigma) \equiv \bar{y}\left(\frac{\bar{\kappa}^{1-\psi} + \bar{m}}{\bar{\kappa} + \bar{m}}\right)^{1/\psi} \quad \text{and} \quad \gamma(\sigma) = \frac{\bar{\kappa}^{1-\psi}}{\bar{\kappa}^{1-\psi} + \bar{m}}. \tag{4}$$

Also, I follow Klump and de La Grandville (2000) and denote partial derivatives of f with respect to κ by a prime so that $f_\sigma' \equiv \partial f_\sigma/\partial\kappa$ and $\partial f_\sigma'/\partial\sigma \equiv \partial^2 f_\sigma/\partial\kappa\partial\sigma$.

[3] I shall stick as close as possible to the notation of Klump and de La Grandville (2000). For reasons that become obvious below, I have to replace their constants A and a by Γ and γ, respectively. See Chapter 3 in de La Grandville (2009) for a careful derivation of the normalized CES production function. Klump and Saam (2008) discuss how to calibrate the normalized CES in dynamic macroeconomic models.

In units of the final good of period t the profit in t of the final-good sector is

$$Y_t - p_{K,t}Y_{K,t} - p_{L,t}Y_{L,t}, \tag{5}$$

where $p_{j,t},\ j = K, L$, is the price of the respective intermediate factor. The final-good sector takes the sequence $\{p_{K,t}, p_{L,t}\}_{t=1}^{\infty}$ of factor prices as given and maximizes the sum of the present discounted values of profits in all periods. Since it simply buys both intermediates in each period, its maximization problem is equivalent to a series of one-period maximization problems. Focussing on configurations where both intermediates are used, the profit-maximizing first-order conditions for $t = 1, 2, ...$ are

$$Y_{K,t} \quad : \quad p_{K,t} = f_{\sigma}'(\kappa_t), \tag{6}$$

$$Y_{L,t} \quad : \quad p_{L,t} = f_{\sigma}(\kappa_t) - \kappa_t f_{\sigma}'(\kappa_t). \tag{7}$$

2.2 The Intermediate-Good Sector

There are two different sets of intermediate-good firms, each represented by the set $\mathbb{R}_+$ of nonnegative real numbers with Lebesgue measure. Intermediate-good firms may either belong to the sector that produces the labor- or the capital-intensive intermediate. In other words, they are part of the labor- or the capital-intensive intermediate-good sector.

At any date t, all firms of a sector have access to the same sector-specific technology with production function

$$y_{l,t} = \min\{1, a_t l_t\} \quad \text{or} \quad y_{k,t} = \min\{1, b_t k_t\}, \tag{8}$$

where $y_{l,t}$ and $y_{k,t}$ is output, 1 a capacity limit,[4] a_t and b_t denote the firms' labor and capital productivity in period t, l_t and k_t is the labor and the capital input. The firms' respective labor and capital productivity is equal to

$$a_t = A_{t-1}(1 - \delta + q_t^A) \quad \text{or} \quad b_t = B_{t-1}(1 - \delta + q_t^B); \tag{9}$$

here $A_{t-1} > 0$ and $B_{t-1} > 0$ denote aggregate indicators of the level of technological knowledge to which innovating firms in period t have access for free. Naturally,

[4] The assumption of a capacity constraint is by no means restrictive for my results. The capacity choice may be endogenized along the lines of Hellwig and Irmen (2001).

$\delta \in (0,1)$ is the rate of depreciation of technological knowledge in both sectors, and q_t^A and q_t^B are indicators of productivity growth gross of depreciation.

To achieve a productivity growth rate $q_t^j > 0$, $j = A, B$, a firm must invest $i(q_t^j)$ units of the final good in period t. The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for both sectors, time invariant, $\mathcal{C}^2$, and strictly convex. Moreover, with the notation $i'(q^j) \equiv di(q^j)/dq^j$ for $j = A, B$, it satisfies

$$i(0) = \lim_{q^j \to 0} i'\left(q^j\right) = 0, \quad i'\left(q^j\right) > 0 \text{ for all } q^j > 0, \quad \lim_{q^j \to \infty} i\left(q^j\right) = \infty. \tag{10}$$

Hence, higher rates of productivity growth require ever-growing investments.

If a firm innovates, the assumption is that an innovation in period t is proprietary knowledge of the firm only in t, i. e., in the period when the innovation materializes. Subsequently, the innovation becomes embodied in the sector specific productivity indicators (A_t, B_t), (A_{t+1}, B_{t+1}), ..., with no further scope for proprietary exploitation. The evolution of these indicators will be specified below.[5]

Per-period profits in units of the current final good are

$$\pi_{L,t} \;=\; p_{L,t}\, y_{l,t} - w_t\, l_t - i(q_t^A), \quad \pi_{K,t} = p_{K,t}\, y_{k,t} - R_t\, k_t - i(q_t^B), \tag{11}$$

where $p_{L,t}\, y_{l,t}$, $p_{K,t}\, y_{k,t}$ is the respective firm's revenue from output sales, $w_t l_t$, $R_t k_t$ its wage bill at the real wage rate w_t and its capital cost at the real rental rate of capital R_t, and $i(q_t^j)$, $j = A, B$, its investment outlays.

Firms choose a production plan $(y_{l,t}, l_t, q_t^A)$ or $(y_{k,t}, k_t, q_t^B)$ taking the sequence $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$ of real prices and the sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of aggregate productivity indicators as given. They choose a production plan that maximizes the sum of the present discounted values of profits in all periods. Because production choices for different periods are independent of each other, for each period t, they choose the plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ that maximizes the profit $\pi_{L,t}$ and $\pi_{K,t}$, respectively.

If a firm innovates, it incurs an investment cost $i(q_t^j) > 0$ that is associated with a given innovation rate $q_t^j > 0$ and is independent of the output $y_{l,t}$ or $y_{k,t}$. An innovation investment is only profit-maximizing if the firm's margin is strictly positive, i. e., if $p_{L,t} > w_t/a_t$ or $p_{K,t} > R_t/b_t$. Then, there is a positive scale effect, namely if

[5] As will become clear below, all firms innovate in equilibrium since $i(0) = \lim_{q^j \to 0} i'\left(q^j\right) = 0$. To save space, I shall disregard throughout the discussion of what would happen if firms did not innovate. Details on this are available from the author upon request.

the firm innovates, it wants to apply the innovation to as large an output as possible and produces at the capacity limit, i. e., $y_{l,t} = 1$ or $y_{k,t} = 1$. The choice of (l_t, q_t^A) and (k_t, q_t^B) must then minimize the costs of producing the capacity output. Assuming $w_t > 0$ and $R_t > 0$ these input combinations satisfy

$$l_t = \frac{1}{A_{t-1}(1-\delta+q_t^A)}, \quad k_t = \frac{1}{B_{t-1}(1-\delta+q_t^B)}, \tag{12}$$

and

$$\begin{aligned} \hat{q}_t^A &\in \arg\min_{q^A \geq 0} \left[\frac{w_t}{A_{t-1}(1-\delta+q^A)} + i\left(q^A\right) \right], \\ \hat{q}_t^B &\in \arg\min_{q^B \geq 0} \left[\frac{R_t}{B_{t-1}(1-\delta+q^B)} + i\left(q^B\right) \right]. \end{aligned} \tag{13}$$

Given the convexity of the innovation cost function and the fact that $\lim_{q^j \to 0} i'(q^j) = 0$, the conditions (13) determine a unique level $\hat{q}_t^A > 0$ and $\hat{q}_t^B > 0$ as the solution to the first-order conditions

$$\frac{w_t}{A_{t-1}(1-\delta+\hat{q}_t^A)^2} = i'\left(\hat{q}_t^A\right) \quad \text{and} \quad \frac{R_t}{B_{t-1}(1-\delta+\hat{q}_t^B)^2} = i'\left(\hat{q}_t^B\right). \tag{14}$$

The latter relate the marginal reduction of a firm's wage bill/capital cost to the marginal increase in its investment costs.

Recall that the set of each intermediate-good sector is $\mathbb{R}_+$ with Lebesgue measure. Then, the maximum profit of any intermediate-good firm producing the labor- or the capital-intensive intermediate must be zero at any t. Indeed, since the supply of labor and capital is bounded in each period, the set of intermediate-good firms employing more than some $\varepsilon > 0$ units of labor or capital must have bounded measure and hence must be smaller than the set of all intermediate-good firms. Given that inactive intermediate-good firms must be maximizing profits just like the active ones, we need that maximum profits of all active intermediate-good firms at equilibrium prices are equal to zero.

Using (11), (12), and (14), we find that for profit-maximizing intermediate-good firms earning zero profits in equilibrium, it holds that

$$p_{L,t} = (1-\delta+\hat{q}_t^A)i'(\hat{q}_t^A) + i(\hat{q}_t^A), \quad p_{K,t} = (1-\delta+\hat{q}_t^B)i'(\hat{q}_t^B) + i(\hat{q}_t^B), \tag{15}$$

i. e., the price is equal to variable costs plus fixed costs when w_t/a_t and R_t/b_t are consistent with profit-maximization as required by (14). Upon combining the equilibrium conditions of the final-good sector and both intermediate-good sectors we find the following proposition.

Proposition 1 *If (6), (7), and (15) hold, then there are maps,* $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $g^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$*, such that* $\hat{q}^A_t = g^A(\kappa_t, \sigma) > 0$ *and* $\hat{q}^B_t = g^B(\kappa_t, \sigma) > 0$ *for any* $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$.

Proof Upon substitution of (6) and (7) in the respective zero-profit condition of (15) gives

$$f_\sigma(\kappa_t) - \kappa_t f'_\sigma(\kappa_t) = (1 - \delta + \hat{q}^A_t) i'(\hat{q}^A_t) + i(\hat{q}^A_t), \tag{16}$$

$$f'_\sigma(\kappa_t) = (1 - \delta + \hat{q}^B_t) i'(\hat{q}^B_t) + i(\hat{q}^B_t). \tag{17}$$

Denote the right-hand side of both conditions by $RHS(\hat{q}^j)$, $j = A, B$. Due to the properties of the function i defined in (10), the range of $RHS(\hat{q}^j)$ is $\mathbb{R}_+$. Moreover, $RHS'(\hat{q}^j) > 0$ on $\mathbb{R}_+$.

Similarly, denote by $LHS^j(\kappa_t, \sigma)$, $j = A, B$, the left-hand sides of these conditions. Due to the properties of the CES function, $LHS^j(\kappa_t, \sigma)$ is continuous and strictly positive on $\mathbb{R}_{++}$. Hence, for any pair $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$ there is a unique $\hat{q}^A_t = g^A(\kappa_t, \sigma) > 0$ that satisfies (16) and a unique $\hat{q}^B_t = g^B(\kappa_t, \sigma) > 0$ that satisfies (17). ■

Proposition 1 emphasizes two important properties of the production sector. First, the equilibrium incentives to engage in labor- and capital-augmenting technical change depend on the factor intensity of the final-good sector and on the elasticity of substitution. Second, for all $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$, we have $\hat{q}^A_t > 0$ and $\hat{q}^B_t > 0$.

The first property is due to the fact that $p_{K,t}$ and $p_{L,t}$ depend on κ_t and σ according to (6) and (7). This dependency feeds back onto $\hat{q}^A_t$ and $\hat{q}^B_t$ through the zero-profit condition (15). The second property hinges on the characteristics of the input requirement function given in (10). If $i(0) > 0$, then maximum profits of innovating firms could be strictly negative even at low levels of $\hat{q}^j_t > 0$. Then, these firms would not enter and the intermediate-good production of the respective sector would collapse. If $i'(0) > 0$, then the first marginal unit of q^j would no longer be costless. Consequently, the cost-minimization problem (13) would not necessarily admit an interior solution. Indeed, firms would choose $\hat{q}^j_t = 0$ if the marginal reduction of their wage bill/capital cost at $q^j = 0$ was smaller than $i'(0)$.

2.3 Evolution of Technological Knowledge

The evolution of the economy's level of technological knowledge is given by the evolution of the aggregate indicators $(A_t, B_t) \in \mathbb{R}^2_{++}$. Labor- and capital-augmenting productivity growth occurs at the level of those intermediate-good firms that produce at t. Denoting the measure of these firms by n_t and m_t, respectively, their contribution to A_t and B_t is equal to the highest level of labor and capital productivity attained by one of them, i. e.,

$$\begin{aligned} A_t &= \max\{a_t(n) = A_{t-1}\left(1-\delta+q_t^A(n)\right) \mid n \in [0, n_t]\} \\ B_t &= \max\{b_t(m) = B_{t-1}\left(1-\delta+q_t^B(m)\right) \mid m \in [0, m_t]\}. \end{aligned} \tag{18}$$

Since in equilibrium $q_t^A(n) = q_t^A$ and $q_t^B(m) = q_t^B$, we have $a_t = A_{t-1}\left(1-\delta+q_t^A\right)$ and $b_t = B_{t-1}\left(1-\delta+q_t^B\right)$. Hence, for all $t = 1, 2, ...$

$$A_t = A_{t-1}\left(1-\delta+q_t^A\right) \quad \text{and} \quad B_t = B_{t-1}\left(1-\delta+q_t^B\right) \tag{19}$$

with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

2.4 Dynamic Competitive Equilibrium of the Production Sector

For given sequences of capital $\{K_t\}_{t=1}^{\infty}$, $K_t \in \mathbb{R}_{++}$, and labor $L_t = L_1 (1+g_L)^{t-1}$, $g_L > (-1)$, $L_1 > 0$, the dynamic competitive equilibrium of the production sector determines a sequence of prices $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$, a sequence of allocations $\{Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, y_{l,t}, y_{k,t}, q_t^A, q_t^B, a_t, b_t, l_t, k_t\}_{t=1}^{\infty}$, and a sequence of indicators of the level of technological knowledge $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$.

Definition 1 *In a dynamic competitive equilibrium of the production sector, the above mentioned sequences satisfy the following conditions:*

(E1) At all t, all firms maximize profits and earn zero-profits.

(E2) At all t, the market for both intermediates clears, i. e.,

$$Y_{L,t} = n_t \quad \textit{and} \quad Y_{K,t} = m_t. \tag{20}$$

(E3) At all t, there is full employment of labor and capital, i. e.,

$$n_t l_t = L_t \quad \textit{and} \quad m_t k_t = K_t. \tag{21}$$

(E4) The productivity indicators A_t and B_t evolve according to (19) with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

Condition (E1) is satisfied if Proposition 1 holds. (E2) and (E3) require market clearing of the market for both intermediates and both factors. To avoid more complicated notation, the market-clearing conditions (20) and (21) use the fact that all entering intermediate-good firms at t produce the capacity output and hire the same amount of workers and capital, respectively.

By (12) the equilibrium amount of labor and capital employed by some intermediate-good firm is $l_t = 1/A_{t-1}\left(1-\delta+q_t^A\right)$ and $k_t = 1/B_{t-1}\left(1-\delta+q_t^B\right)$. Then, from (E3), $n_t = A_{t-1}\left(1-\delta+q_t^A\right)L_t$ and $m_t = B_{t-1}\left(1-\delta+q_t^B\right)K_t$. With (E2) and (E4) we have

$$\begin{aligned} Y_{L,t} &= A_{t-1}\left(1-\delta+q_t^A\right)L_t = A_t L_t, \\ Y_{K,t} &= B_{t-1}\left(1-\delta+q_t^B\right)K_t = B_t K_t. \end{aligned} \tag{22}$$

Hence, at a semantic level, technical change is capital- and labor-saving at the level of the individual firm and capital- and labor-augmenting at the level of economic aggregates. If, ceteris paribus, A_{t-1} and B_{t-1} increase, the capacity output is produced with less labor and less capital. At the aggregate level, these gains in productivity translate into more entry through the requirement of full employment of labor and capital. Accordingly, aggregate output of each intermediate-good is equal to the respective input in efficiency units, and a higher A_t or B_t is equivalent to having more labor or capital, respectively.

Moreover, from (22) we have in equilibrium

$$Y_t = F\left(B_t K_t, A_t L_t\right) \quad \text{and} \quad \kappa_t = \frac{B_t K_t}{A_t L_t}. \tag{23}$$

We can use the latter to derive the relative marginal product of capital, MPK_t/MPL_t as

$$\frac{MPK_t}{MPL_t} = \frac{\gamma(\sigma)}{1-\gamma(\sigma)}\left(\frac{B_t}{A_t}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{K_t}{L_t}\right)^{\frac{-1}{\sigma}}. \tag{24}$$

The relative marginal product of capital decreases in its relative abundance, K_t/L_t. This is the substitution effect, and σ is the *partial* elasticity of substitution between capital and labor. 'Partial' refers to the fact that B_t/A_t remains constant. However, changes in the relative abundance of capital induce technical change. To see how, write κ_t of (23) as

$$\kappa_t = \frac{B_{t-1}\left(1-\delta+g^B\left(\kappa_t\right)\right)}{A_{t-1}\left(1-\delta+g^A\left(\kappa_t\right)\right)}\frac{K_t}{L_t}. \tag{25}$$

The latter implicitly defines a functional relationship between κ_t and K_t/L_t characterized by the elasticity

$$\frac{\partial \ln \kappa_t}{\partial \ln (K_t/L_t)} = \frac{1}{1+\varepsilon_\kappa^A - \varepsilon_\kappa^B} \in (0,1), \tag{26}$$

where

$$\varepsilon_\kappa^A \equiv \frac{g_\kappa^A(\kappa_t,\sigma)\kappa_t}{1-\delta+g^A(\kappa_t,\sigma)} > 0 \quad \text{and} \quad \varepsilon_\kappa^B \equiv \frac{g_\kappa^B(\kappa_t,\sigma)\kappa_t}{1-\delta+g^B(\kappa_t,\sigma)} < 0, \tag{27}$$

are the elasticities of the equilibrium growth factors with respect to κ_t. With these relationships at hand, it is straightforward to derive the *total elasticity of substitution*, $\hat{\sigma}$, as a measure of the relative change in the relative marginal product of capital to the relative change in the relative abundance of capital taking induced innovation into account, i. e.,

$$\hat{\sigma} \equiv -\left[\frac{\partial \ln (MPK_t/MPL_t)}{\partial \ln (K_t/L_t)}\right]^{-1} = \sigma\left[\frac{1+\varepsilon_\kappa^A-\varepsilon_\kappa^B}{1+\sigma\left(\varepsilon_\kappa^A-\varepsilon_\kappa^B\right)}\right]. \tag{28}$$

Hence, $\hat{\sigma} = \sigma$ either if $\sigma = 1$ or if $\varepsilon_\kappa^A = \varepsilon_\kappa^B = 0$. In the former case, final-good production is Cobb-Douglas, and the term B_t/A_t vanishes in (24). In the latter case, firms do not respond to changes in κ_t, and technical change would be either absent or exogenous. It is worth noting that (28) implies $\hat{\sigma} \gtreqless 1 \Leftrightarrow \sigma \gtreqless 1$. Hence, intermediates are gross complements (substitutes) whenever capital and labor are gross complements.

To prepare for the steady-state analysis, we state and prove the following proposition.

Proposition 2 *Let κ_t be the state variable that determines the behavior of the production sector in t. Given a sequence $\{\kappa_t\}_{t=1}^\infty$, $\kappa_t \in \mathbb{R}_{++}$, there is a unique dynamic equilibrium that satisfies Definition 1.*

Proof In total there are 19 variables and 19 equations. By (6) and (7) the prices $p_{L,t}$ and $p_{K,t}$ depend on κ_t. Proposition 1 shows that $\hat{q}_t^A$ and $\hat{q}_t^B$ depend on κ_t. Then, an application of the equilibrium conditions of Definition 1 reveals that l_t, k_t, w_t, R_t, a_t, b_t, A_t, B_t, n_t, m_t, $Y_{K,t}$, $Y_{L,t}$, Y_t depend on κ_t through either $\hat{q}_t^A$ or $\hat{q}_t^B$. Recall that $y_{l,t} = y_{k,t} = 1$ for all $t \geq 1$ in accordance with Proposition 1. ∎

3 Steady-State Analysis

I define a steady state, or equivalently a balanced growth path, as a path along which all variables mentioned in Definition 1 grow at constant exponential rates (possibly zero) for all $t \geq \tau \geq 1$.

It is immediate from Proposition 1 and 2 that in a steady state $\kappa_t = \kappa^*$ and both rates, $\hat{q}_t^A$ and $\hat{q}_t^B$, are constant. Yet, the dynamic competitive equilibrium of the production sector on its own does not pin down κ^*. Therefore, I embed the production sector into a richer macroeconomic environment that accounts for capital investment and a resource constraint. I refer to this environment as the *neoclassical economy with endogenous capital- and labor-augmenting technical change*. This environment delivers a single steady-state condition for κ^*. With this condition at hand, I study the role of the elasticity of substitution for steady-state growth.

Definition 2 *The neoclassical economy with endogenous capital- and labor-augmenting technical change is defined by the following environment:*

1. *The (normalized) CES production function (1)*

$$Y_t = F_\sigma\left(B_t K_t, A_t L_t\right) = \Gamma\left(\sigma\right) \left[\gamma\left(\sigma\right) \left(B_t K_t\right)^\psi + \left(1-\gamma\left(\sigma\right)\right)\left(A_t L_t\right)^\psi\right]^{1/\psi}. \quad (29)$$

2. *Capital accumulation according to*

$$K_{t+1} = I_t^K + (1-\delta^K)K_t, \quad K_1 > 0, \quad (30)$$

where $I_t^K > 0$ is gross investment of current output in the capital stock, $\delta^K \in [0,1]$ is the depreciation rate of capital, and $K_1 > 0$ the initial condition.

3. *Two indicators of technological knowledge, A_t and B_t, that evolve according to (19).*

4. *Innovation investments of current output, $I_t^A > 0$ and $I_t^B > 0$, are necessary and sufficient for $q_t^A > 0$ and $q_t^B > 0$. Moreover,*

$$I_t^A = A_t L_t\, i\left(q_t^A\right) \quad \text{and} \quad I_t^B = B_t K_t\, i\left(q_t^B\right). \quad (31)$$

5. *A resource constraint according to which consumption, $C_t > 0$, gross investment in the capital stock, $I_t^K > 0$, and innovation investments, $I_t^A > 0$ and $I_t^B > 0$, add up to aggregate output, i. e.,*

$$C_t + I_t^K + I_t^A + I_t^B = Y_t. \quad (32)$$

6. *The labor force grows at a constant rate $g_L > (-1)$, i. e., $L_t = L_1(1+g_L)^{t-1}$ with $L_1 > 0$ as initial condition.*

Definition 2 adds capital accumulation according to (30) and the resource constraint (32) to the production sector of Section 2. Moreover, it uses equilibrium conditions. In accordance with (22), we replace $Y_{L,t}$ and $Y_{K,t}$ by $A_t L_t$ and $B_t K_t$ in (29) and use (E2) to conclude that $I_t^A = n_t\, i\left(q_t^A\right) = A_t L_t\, i\left(q_t^A\right)$ and $I_t^B = m_t\, i\left(q_t^B\right) = B_t K_t\, i\left(q_t^B\right)$ in (31). In addition to consumption, the three ways to invest current output show up on the left-hand side of the resource constraint (32).

The following proposition establishes the key properties of a steady state in a neoclassical economy with endogenous capital- and labor-augmenting technical change.

Proposition 3 *Suppose the neoclassical economy with endogenous capital- and labor-augmenting technical change exhibits a steady state starting at period τ with $I_t^K > 0$, $I_t^A > 0$, $I_t^B > 0$ for $t \geq \tau$. Then, for all $t \geq \tau$*

$$Y_t = F_\sigma\left(B_\tau\, K_t, A_t\, L_t\right) = \Gamma\left(\sigma\right)\,\left[\gamma\left(\sigma\right)\,\left(B_\tau K_t\right)^\psi + \left(1-\gamma\left(\sigma\right)\right)\left(A_t L_t\right)^\psi\right]^{1/\psi}, \qquad (33)$$

and output per worker grows at rate

$$g^* \equiv q^A - \delta. \qquad (34)$$

Proof For a steady state, the evolution of A_t and B_t as given by (19) requires $q_t^A = q^A$ and $q_t^B = q^B$ for all $t \geq \tau$. Since $I_t^j > 0$ we have $q^j > 0$, $j = A, B$. With this in mind, the neoclassical economy with endogenous capital- and labor-augmenting technical change is isomorphic to the environment to which the Steady-State Theorem of Uzawa (1961) applies.

To see this, consider the resource constraint (32), which may be written as

$$C_t + I_t^K + A_t L_t i\left(q^A\right) + B_t K_t i\left(q^B\right) = Y_t. \qquad (35)$$

Define net output as

$$\tilde{Y}_t = \tilde{F}_\sigma\left(B_t K_t, A_t L_t\right) \equiv F_\sigma\left(B_t K_t, A_t L_t\right) - A_t L_t i\left(q^A\right) - B_t K_t i\left(q^B\right). \qquad (36)$$

One readily verifies that the net production function $\tilde{F}$ has constant returns to scale in K_t and L_t, and, using (16), (17) and and the properties of f_σ and i, positive and diminishing marginal products of K_t and L_t.

Hence, the environment described by (i) $\tilde{Y}_t = \tilde{F}_\sigma\left(B_t K_t, A_t L_t\right)$, (ii) the resource constraint $C_t + I_t^K = \tilde{Y}_t$, (iii) capital accumulation according to (30), and (iv) growth of the labor force at a constant rate is the one to which the Steady-State Growth Theorem of Uzawa (1961) applies (see, Schlicht (2006) and Jones and Scrimgeour (2008)). Hence, in a steady-state it must be that $q^B = \delta$ and $g^* = q^A - \delta$. ■

Proposition 3 states that in a steady state the growth rate of output per worker coincides with the net growth rate of labor-saving technical change g^*; there is no capital-saving technical progress, i. e., $B_t = B_\tau$ for all $t \geq \tau$.[6] These findings mimic the predictions of the so-called Steady-State Growth Theorem of Uzawa (1961). In fact, the proof of Proposition 3 shows that in a steady state, the neoclassical economy of Definition 2 is isomorphic to the environment to which Uzawa's theorem applies. This is due to the fact that aggregate innovation investments in capital- and labor-augmenting technical change are proportionate to efficient capital and efficient labor, respectively. If such an economy is equipped with the production sector of Section 2, Proposition 3 means that

$$\delta = q^B = g^B\left(\kappa^*, \sigma\right) \quad \text{and} \quad g^* = q^A - \delta = g^A\left(\kappa^*, \sigma\right) - \delta. \tag{37}$$

The first of these conditions pins down the steady-sate capital intensity κ^*, the second gives the steady-sate growth rate of output per worker. This is quite remarkable: the steady-state intensity of efficient capital per unit of efficient labor and the steady-state growth rate of the economy depend only on parameters that characterize the economy's production sector. The following theorem exploits this fact.

Theorem 1 *Consider two neoclassical economies with endogenous capital- and labor-augmenting technical change equipped with a production sector set out in Section 2. Let these economies differ only with respect to σ. Then, the economy with the greater σ experiences faster steady-state growth of output per worker.*

Proof From (37) a change in σ affects κ^* since such a change must leave g^B unaffected. Denote this relationship by $\kappa^* = \kappa^*(\sigma)$. An application of the implicit function theorem to (17) reveals that $\kappa^*(\sigma)$ satisfies

$$\frac{d\kappa^*}{d\sigma} = -\frac{\partial f'_\sigma}{\partial \sigma}\frac{1}{f''_\sigma}, \tag{38}$$

[6]Observe that $B_t = B_\tau$ means that innovation investments in capital-augmenting technical occur in the steady state. In fact, they are equal to $I_t^B = B_\tau K_t\, i\left(\delta\right)$ and offset the effect of depreciation on B_τ.

where the argument of f_σ is κ^*.

To study the effect of σ on the growth rate of output per worker, write $g^* = g^A(\kappa^*(\sigma), \sigma) - \delta$ such that

$$\frac{dg^*}{d\sigma} = \frac{\partial g^A(\kappa^*, \sigma)}{\partial \kappa^*} \frac{d\kappa^*}{d\sigma} + \frac{\partial g^A(\kappa^*, \sigma)}{\partial \sigma}. \tag{39}$$

From (16), we derive

$$\frac{\partial g^A}{\partial \kappa} = -\frac{\kappa^* f''_\sigma}{(1-\delta+g^A)\, i''(g^A) + 2i'(g^A)}, \tag{40}$$

$$\frac{\partial g^A}{\partial \sigma} = \frac{\frac{\partial f_\sigma}{\partial \sigma} - \kappa^* \frac{\partial f'_\sigma}{\partial \sigma}}{(1-\delta+g^A)\, i''(g^A) + 2i'(g^A)}, \tag{41}$$

where the argument of g^A is (κ^*, σ) and the argument of f_σ is κ^*. Upon substitution of (38), (40), and (41) in (39) gives

$$\frac{dg^*}{d\sigma} = \frac{\frac{\partial f_\sigma}{\partial \sigma}}{(1-\delta+g^A)\, i''(g^A) + 2\, i'(g^A)} > 0. \tag{42}$$

The sign of $dg^*/d\sigma$ follows since $sign[\partial f_\sigma(\kappa^*)/\partial\sigma] > 0$ for $\kappa^* \neq \bar{\kappa}$ in accordance with the proof of Theorem 1 of Klump and de La Grandville (2000). ■

To grasp the intuition of Theorem 1, consider the left-hand side of (16) at the steady state, i. e.,

$$f_\sigma(\kappa^*(\sigma)) - \kappa^*(\sigma)\, f'_\sigma(\kappa^*(\sigma)). \tag{43}$$

For the Theorem to hold this expression must increase in σ. Since the steady state requires $\kappa^*(\sigma)$ to adjust such that $\delta = g^B(\kappa^*(\sigma), \sigma)$, changing σ must leave $g^B(\kappa^*(\sigma), \sigma)$ constant. Hence, from (17), $f'_\sigma(\kappa^*(\sigma))$ must remain unchanged. Then, changing σ has two effects on (43). First, there are two (indirect) effects through a change of $\kappa^*(\sigma)$ which cancel out. Second, there is the (direct) efficiency effect identified by Klump and de La Grandville (2000), i. e., $\partial f_\sigma(\kappa^*(\sigma))/\partial\sigma > 0$ for $\bar{\kappa} \neq \kappa^*$. Hence, it is due to the efficiency effect that the economy with the greater elasticity of substitution has faster steady-state growth of output per worker.

It is worth noting that Theorem 1 does not depend on the assumption that the requirement functions, i, is the same in both sectors. If, for some reason, the innovation process in one sector is more difficult than in the other such that $i^A(q^A) = \alpha\, i(q^A) \neq i^B(q^B) = \beta\, i(q^B)$ with $\alpha \neq \beta$, then α becomes a parameter of g^A and β one of g^B. Accordingly, the steady-state efficient capital intensity and the steady-state growth rate of output per worker depend on these parameters in accordance with (37). However, the qualitative effect of the elasticity of substitution on steady-state growth remains unaffected.

Theorem 1 is also robust with respect to modifications in the way the indicators A_t and B_t evolve. For instance, I assume in (18) that their evolution depends only on the innovation activity of the respective sector. To relax this assumption, one may allow for spillovers. As long as these are not too strong, e. g., such that the evolution of A_t depends only on the innovation investments in capital-augmenting technical change and vice versa, the steady-sate growth effects of the elasticity of substitution remain valid.[7]

Finally, the growth effect of Theorem 1 may also be related to the total elasticity of substitution between capital and labor.

Corollary 1 *If the production technology of two economies is characterized by $\sigma_2 > \sigma_1$ and both partial elasticities of substitution are close to unity, then $\hat{\sigma}_2 > \hat{\sigma}_1$ and the economy with the greater total elasticity of substitution between capital and labor has the higher steady-state growth rate.*

Proof From (28) it follows that

$$\frac{d\hat{\sigma}}{d\sigma} = \frac{1}{1+\sigma\left(\varepsilon_\kappa^A - \varepsilon_\kappa^B\right)}\left[1 + \varepsilon_\kappa^A - \varepsilon_\kappa^B + \sigma(1-\sigma)\left(\frac{\partial \varepsilon_\kappa^A}{\partial \sigma} - \frac{\partial \varepsilon_\kappa^B}{\partial \sigma}\right)\right]. \tag{44}$$

Evaluated at $\sigma = 1$, we have $d\hat{\sigma}/d\sigma = 1$. In view of Theorem 1, Corollary 1 follows immediately. ■

Hence, if we believe to measure the total rather than the partial elasticity of substitution, the prediction of faster steady-state growth under a greater elasticity of substitution remains valid. Though, since little is known about the derivatives $\partial \varepsilon_\kappa^j/\partial\sigma$, $j = A, B$, this result may only be locally valid.

[7]For instance, one may replace (18) by $A_t = A_{t-1}\left[\left(1-\delta+q_t^A\right)\right]^\phi \left[\left(1-\delta+q_t^B\right)\right]^\lambda$ and $B_t = B_{t-1}\left[\left(1-\delta+q_t^B\right)\right]^\phi \left[\left(1-\delta+q_t^A\right)\right]^\lambda$, with $\phi > 0$ and $\lambda \in [0, \phi]$. Here, λ measures the strength of the spillover from current research in one sector on the productivity indicator of the other sector. Details are available upon request.

4 Concluding Remarks

This paper suggests that there are new effects linking the predicted growth performance of an economy to the elasticity of substitution once we leave the setting of one-sector growth models of de La Grandville (1989) and Klump and de La Grandville (2000). In the three-sector economy under scrutiny here the direction of technical change, i. e., the economy's choice between capital- and labor-augmenting technical change, is endogenous. I find that an economy with a greater elasticity of substitution between capital and labor has a greater steady-state growth rate of output per worker. This is due to the efficiency effect of the elasticity of substitution of Klump and de La Grandville (2000), i. e., for a given efficient capital intensity, output per efficient labor increases in the elasticity of substitution. In the present context, innovation investments that raise the productivity of labor become more profitable due to the efficiency effect. Therefore, the steady-state growth rate is higher.

Unlike other channels linking the elasticity of substitution to a country's growth performance, the central result of this paper does not depend on particular assumptions on the household side of the economy. All relevant steady-state conditions on growth rates follow from Uzawa's Steady-State Growth Theorem. Theorem 1 derives a great deal of its generality from this fact. The price, however, is that it does not include predictions concerning transitional dynamics. These will depend on the way the household sector is modeled. Finding out whether and how the elasticity of substitution affects these dynamics is not a trivial task. I leave it for future research.[8]

[8] Adding Solow's savings hypothesis to the production sector of Section 2 leads to a two-dimensional dynamical system of first-order, autonomous, and non-linear difference equations. Calibration exercises for reasonable parameter values indicate that the steady state may be locally stable for values of σ close to unity. Details are available from the author upon request.

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