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Thema

Iwasawa Theory for One-Parameter Families of Motives

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Abstract. In this thesis we build on the work of Fukaya and Kato [**FK06**] in which they presented equivariant Tamagawa Number conjectures that implied a very general (noncommutative) Iwasawa main conjecture for rather general motives. We apply their methods to the case of one-parameter families of motives to derive a main conjecture for such families (theorem 4.31). On our way there we get some unconditional results on the variation of the (algebraic) λ- and the μ-invariant in many cases (theorem 3.33 and corollary 3.34). We focus on the results dealing with Selmer complexes instead of the more classical notion of Selmer groups. However, where possible we give the connection to the classical notions. The final chapter deals with the deformation theory of the representations occurring in our theory and the existence of one-parameter families. In particular we recover and generalize some results on the variation of Iwasawa invariants in Hida families.

Zusammenfassung. Diese Arbeit baut auf dem Artikel [**FK06**] von Fukaya und Kato auf. In diesem Artikel werden äquivariante Tamagawazahl Vermutungen formuliert, von denen eine sehr allgemeine (nichtkommutative) Iwasawa Hauptvermutung für eine breite Klasse von Motiven abgeleitet wird. Wir wenden die dort verwendeten Methoden auf den Fall von Ein-Parameter Familien von Motiven an, um eine Hauptvermutung für diese Familien (theorem 4.31) abzuleiten. Auf dem Weg dorthin erhalten wir einige Resultate über die Variation der (algebraischen) λ- und μ- Invarianten (theorem 3.33 und corollary 3.34), die nicht die Vermutungen von Fukaya und Kato voraussetzen. Unser Hauptaugenmerk liegt dabei auf Resultaten, die Selmer Komplexe an Stelle der klassischeren Selmer Gruppen verwenden. Wo immer es möglich ist, werden wir aber den Zusammenhang mit der klassischen Situation herstellen. Das letzte Kapitel beschäftigt sich mit der Deformationstheorie der Darstellungen, die in der Theorie auftauchen, und der Existenz von Ein-Parameter Familien. Insbesondere verallgemeinern wir einige bekannte Resultate über die Variation von Iwasawa Invarianten in Hida Familien.

Contents

CHAPTER 0

Introduction

This thesis studies the Iwasawa theory of families of motives. The idea to study a whole family at once instead of just a single motive was introduced by Hida in [**Hid86**] in which he studied what are now called Hida families of modular forms, i.e., families that consist of all ordinary modular cusp forms of a given level and nebentype but arbitrary weight that are congruent modulo p. Among the early successes of this technique is its use by Mazur and Wiles in [**MW86**] and [**Wil88**] in the proof of the Iwasawa main conjecture for the Tate motive along the cyclotomic $\mathbb{Z}_p$-extension.

There are basically two ways how one can utilize the fact that the motives are members of a family: the easier one, by far, is to study how certain algebraic invariants of the Selmer groups of different members of the families are related so that the invariants for many motives can be computed by just knowing one of them. This approach has been carried out for the modular forms along the cyclotomic extension by Emerton, Pollack, and Weston in [**EPW06**] and for modular forms along the false Tate extension by Aribam in his PhD thesis [**Sha09**]. In this thesis, we will give a version for very general motives, which subsumes many of the results of [**EPW06**] and [**Sha09**]. More importantly, we give a uniform treatment for all the motives and for many p-adic Lie groups.

The second way to use families is to construct a two-variable p-adic L-function for the family that interpolates all the L-functions of the members of the family. This approach was initiated by Greenberg and followed by Ochiai in classical (commutative) settings (see, for instance, [**Och06**]). It is worth mentioning that Ochiai has studied the case of families of Hilbert modular forms using this approach quite successfully. However, it seems that this method can not be generalized in non-commutative settings. As a remedy we will use the results of Fukaya and Kato in [**FK06**] to formulate a main conjecture for the family of such a kind that it is compatible with equivariant Tamagawa number conjectures and gives a two-variable algebraic p-adic ζ-function.

Before delving deeper into the details, it might be helpful to recall the setup in non-commutative Iwasawa theory for motives:

Let p be an odd prime and let F be a number field. Moreover, let M be an F-motive and F_∞/F be a Galois extension, such that $G := Gal(F_\infty/F)$ is a p-adic Lie group. In this setting, (non-commutative) Iwasawa theory investigates the following Pontryagin dual of of the Selmer group:

$$\mathcal{X}(M, F_\infty) := Sel(M, F_\infty)^\vee$$

Assuming, for simplicity, that M has coefficients in $\mathbb{Q}$, then $\mathcal{X}(M, F_\infty)$ has a structure as a module under the Iwasawa algebra

$$\Lambda := \mathbb{Z}_p[\![G]\!] := \varprojlim_{U \lhd G} \mathbb{Z}[G/U],$$

where U runs over the open normal subgroups. One of the main goals of Iwasawa theory is to describe this structure. With this aim in mind, we follow two approaches:

- The first one is to study the λ- and the μ-invariant as defined by Coates and Howson in [**CH97**], [**CH01**], and most notably in [**How02**] as Euler characteristics, provided that G does not have any p-torsion and F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension of F.
- The second strategy is to formulate an Iwasawa main conjecture. Such a conjecture predicts the existence of a p-adic ζ-function $\zeta(M, F_\infty/F)$ that is related to the module structure of $\mathcal{X}(M, F_\infty)$, at the same time interpolating values of the complex L-function of twists of M by Artin characters at zero.

In this thesis, we work with the formulation of the theory given by Fukaya and Kato in [**FK06**]. The last article continues a line in noncommutative Iwasawa theory started by the habilitation thesis of Venjakob [**Ven05**] and generalized in the article [**CFKSV**]. Fukaya and Kato (loc.cit.) succeed in formulating an Iwasawa main conjecture for motives in a setup generalizing the case of ordinary reduction at p. Furthermore, they were able to show that this main conjecture can be derived assuming Tamagawa number conjectures.

The approach of Fukaya and Kato can be applied to motives satisfying the Dabrowski-Panchishkin condition (condition 2.9), a vast generalization of ordinarity. Their method uses two crucial steps. Firstly, the p-adic realization of the motive together with the subrepresentations given by the condition induces a pair $(\mathbb{T}, \mathbb{T}^0)$ of free representations over Λ. Here, $\mathbb{T}$ is a representation of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ and $\mathbb{T}^0$ is a free direct summand that is stable under the local Galois group $G_{\mathbb{Q}_p}$. Secondly, Fukaya and Kato assume equivariant Tamagawa number conjectures, which we will denote by (FK) in the following. Assuming that these conjectures hold, they associate a ζ-function to any pair of representations $(\mathbb{T}, \mathbb{T}^0)$ over an adic ring Λ (see definition 1.11 for "adic ring"), provided $\mathbb{T}^0$ is isomorphic as a Λ-module to the 1-eigenspace $\mathbb{T}^+$ of the complex conjugation on $\mathbb{T}$. However, in this generality it is not possible to work with Selmer groups. Remarkably, if the Λ-module $\mathcal{X}(M, F_\infty/F)$ is replaced by a complex $SC(M, F_\infty/F) := SC(\mathbb{T}, \mathbb{T}^0)$ of Λ-modules that is closely related to it, then a main conjecture can be formulated linking ζ to this complex.

The ζ-function $\zeta(M, F_\infty/F)$ is related to the Selmer complex $SC(M, F_\infty/F)$ via the long exact sequence of K-theory. Assuming that G has no p-torsion and $SC(M, F_\infty/F)$ has S^*-torsion cohomology groups for the denominator set $S^* \subset \Lambda$ described in the quoted articles, the complex $SC(M, F_\infty/F)$ then describes a class in $K_0(S^*\text{-tor})$, the Grothendieck group of the finitely generated S^*-torsion Λ-modules. In that situation the p-adic ζ-function $\zeta(M, F_\infty/F)$ should be an element in $K_1(\Lambda_{S^*})$, the K_1-group of the localization of Λ at S^*. The $\zeta(M, F_\infty/F)$ and the class of $SC(M, F_\infty/F)$ are connected through the fact that the former maps to the class of the latter under the connection morphism ∂ in the exact sequence:

$$K_1(\Lambda) \to K_1(\Lambda_{S^*}) \xrightarrow{\partial} K_0(S^* - tor) \to 0$$

As $\zeta(M, F_\infty/F) \in K_1(\Lambda_{S^*})$, it can be evaluated at Artin characters $\rho : G \to Gl_n(\mathbb{Q}_p)$. This evaluation is defined via an extension of the maps $K_1(\Lambda) \to K_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$, using the functoriality of K-groups.

The precise main conjecture derived by Fukaya and Kato will be reproduced in this thesis as theorem 4.24. For the purpose of the introduction and to give a first impression of this kind of results, we state the following simplified version of the theorem:

THEOREM 0.1 (Iwasawa main conjecture for motives). *We assume that the conjectures (FK) hold. Let M be a critical F-motive satisfying the Dabrowski-Panchishkin condition and let F_∞/F be a p-adic Lie extension with Galois group G without p-torsion. Moreover, let $\beta : \mathbb{T}^0 \to \mathbb{T}^+$ be an isomorphism of Λ-modules. If $SC(M, F_\infty/F)$ has S^*-torsion cohomology groups, then there is an element $\zeta_\beta(M, F_\infty/F)$ in $K_1(\Lambda_{S^*})$ that can be uniquely described and has the following properties: Firstly, under the connection morphism ∂, the element $\zeta_\beta(M, F_\infty/F)$ maps to the class of $SC(M, F_\infty/F)$. Secondly, the values of $\zeta_\beta(M, F_\infty/F)$ at Artin characters ρ can be computed using the value of the complex L-function of $M(\rho^*)$ at 0.*

The classical main conjecture for elliptic curves or modular forms can be presented in this form.

Finally, we note that the case of the Iwaswa main conjecture, where the motive M is the Tate motive, F is a totally real field, and some μ-invariants vanish, was most recently proven by Kakde in [**Kak10**]. Another proof of a similar main conjecture was provided even slightly earlier by Ritter and Weiss in [**RW10**]. However, very little is known about motives other than twists of the Tate motive and motives associated with elliptic curves. Furthermore, the vanishing of the μ-invariant remains an open problem in many cases.

Let us now turn to the setup for families. Recall that we fixed a number field F and a Galois extension F_∞/F with Galois group G. We assume that G does not contain any p-torsion and that the cyclotomic $\mathbb{Z}_p$-extension of F is contained in F_∞. A (height one) specialization is a continuous $\mathbb{Z}_p$-algebra morphism $\phi : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p$. We fix a non-empty set Σ of specializations. A family of motives is a free representation $\rho : G_F \to GL(T)$ of finite rank over $\mathbb{Z}_p[[t]]$ of the absolute Galois group G_F of F together with a collection of motives M_ϕ over F such that the p-adic realization of $(M_\phi)_{\phi\in\Sigma}$ is just $M_{\phi,p} = T \otimes_{\mathbb{Z}_p,\phi} \mathbb{Q}_p$.

As mentioned earlier, studying the non-commutative main conjecture for a family of motives by following the methods of Ochiai does not seem to yield results. Alternatively, we use the basic idea that the construction of the pair $(\mathbb{T}, \mathbb{T}^0)$ can be applied to the representation ρ of the family instead of the p-adic realization of a motive to give a pair of Galois representations over the adic ring $\Lambda[[t]]$. For the representation $\mathbb{T}$, this does not provide any problems. However, to apply the machinery of Fukaya and Kato, we need a free $\Lambda[[t]]$-direct summand $\mathbb{T}^0$ of $\mathbb{T}$, which is stable under $G_{\mathbb{Q}_p}$ and isomorphic to the invariant-module of the complex conjugation on $\mathbb{T}$. Concerning this problem, we get the crucial result (lemma 2.14):

LEMMA 0.2. *Assume that for every place v of F dividing p we can choose a submodule $T^0(v) \subset T$ which is invariant under the decomposition group of v which is a direct summand as a $\mathbb{Z}_p[[t]]$-module. We assume in addition:*

(1) *For every $\phi \in \Sigma$, the canonical map*

$$D_{dR}(F_v, T^0(v) \otimes_{\mathbb{Z}_p[[t]],\phi} \mathbb{Q}_p) \to D_{dR}(F_v, T \otimes_{\mathbb{Z}_p[[t]],\phi} \mathbb{Q}_p)/D^0_{dR}(F_v, T \otimes_{\mathbb{Z}_p[[t]],\phi} \mathbb{Q}_p)$$

is an isomorphism.

(2) *There is one $\phi \in \Sigma$, such that we can construct the pair $(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ for the motive M_ϕ.*

Then it is possible to construct the pair of big representations for all motives M_ϕ with $\phi \in \Sigma$ and to construct $\mathbb{T}$ and $\mathbb{T}^0$ for the family such that the constructions are compatible with specialization.

The conditions on the individual ϕ are just technical and cannot be removed in the described setting. A major restriction is the existence of the $T^0(v)$. In fact, the existence of submodules of this kind is a deformation problem with one particular nearly ordinary condition. In the last chapter, we will study this problem more closely and give some examples of when such a representation exists or when it can never exist.

Much of the work relies on rather explicit computations with power series rings. However, it seems likely that the last result also holds if we replace $\mathbb{Z}_p[[t]]$ with a finite flat extension. This would be a worthy generalization, as it would increase the number of known examples considerably.

Assuming from now on that the conditions of the above lemma are satisfied, we can study the relations between the Iwasawa invariants of Selmer complexes of the specializations and the invariant of the complex of the family. In the general case where the coefficients of the motives are bigger than $\mathbb{Q}$, there are some technical problems that need to be solved. The general statements can be found as theorem 3.35 and corollary 3.36. But the case described above (with rational coefficients) is already contained in corollary 3.34:

THEOREM 0.3. *We assume that the extension F_∞/F contains the cyclotomic $\mathbb{Z}_p$-extension of F. Let $(T, \Sigma, (M_\phi))$ be a family and let $\phi \in \Sigma$ be a specialization map. We denote the kernel of $\phi : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p$ by (f). Furthermore, we assume that $(\mathbb{T}, \mathbb{T}^0)$ is the pair of representations over $\Lambda[[t]]$ associated to the family and $(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ the pair of representations over Λ associated to M_ϕ. If $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ has S^*-torsion cohomology groups and condition 2.15 is satisfied, then the following holds:*

(1) *There is an n depending only on the pair $(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ such that*

$$\mu_{\Lambda/f}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)) = \mu_{\Lambda/g}(SC(\mathbb{T}_\psi, \mathbb{T}^0_\psi))$$

for any $\psi \in \Sigma$ with kernel (g) such that $p^n | f - g$.

(2) *Assuming that the cohomology groups of $SC(\mathbb{T}, \mathbb{T}^0)$ are S-torsion, we have*

$$\lambda_\Lambda(SC(\mathbb{T}, \mathbb{T}^0)) = \lambda_{\Lambda_\phi}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)).$$

However, we have to be careful as now the cohomology groups of $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are automatically S^*-torsion, but need not be S-torsion.

Note that condition 2.15 required in the theorem states that the extension F_∞/F has to be infinitely ramified at places where the representation of the family is infinitely ramified. Apart from this requirement, the extension F_∞/F can be chosen rather arbitrarily. This theorem is a vast generalization of the analogous results by Emerton, Pollack, and Weston [**EPW06**] as well as those of Aribam [**Sha09**], as mentioned earlier.

In cases where G does not have any p-torsion, we can compare the Iwasawa invariants of the Selmer groups with those of the Selmer complexes. Thus, in this situation we get similar results for invariants of the Selmer groups (corollary 3.40):

THEOREM 0.4. *Let $(\mathbb{T},\mathbb{T}^0)$ be a pair of big Galois representations associated with a family of motives and assume that the according p-adic Lie group $G = Gal(F_\infty/F)$ does not have any p-torsion and that condition 2.12 on freeness is met. Moreover, let ϕ and ψ be two specializations of the family. Then the following holds:*

(1) *If ϕ has kernel (f) and $\mathcal{X}(\mathbb{T},\mathbb{T}^0)$ is S^*-torsion, then there is an n depending only on $(\mathbb{T}_\phi,\mathbb{T}^0_\phi)$ such that*
$$\mu_{\Lambda_\phi}(\mathcal{X}(\mathbb{T}_\phi,\mathbb{T}^0_\phi)) = \mu_{\Lambda_\psi}(\mathcal{X}(\mathbb{T}_\psi,\mathbb{T}^0_\psi)),$$
whenever there is a $g \in \mathbb{Z}_p[[t]]$ generating the kernel of ψ such that $\pi^n|f-g$.

(2) *If the cohomology groups of $SC(U,\mathbb{T}_\phi,\mathbb{T}^0_\phi)$ are S-torsion and all the groups H_v for every place v of F dividing p as well as the group H admit infinite pro-p quotients without p-torsion, then:*
$$\lambda_\Lambda(\mathcal{X}(\mathbb{T}_\phi,\mathbb{T}^0_\phi)) = \lambda_{\Lambda_\phi}(\mathcal{X}(\mathbb{T}_\psi,\mathbb{T}^0_\psi))$$

We also prove a few slightly different versions of these theorems. The main ingredients in the proof of these theorems are twofold. On the one hand, we prove generalizations of many of the results in [**FK06**] on modules over adic rings. On the other hand, we make use of the fact that the Iwasawa invariants of Coates and Howson as well as the Selmer complexes are both defined in terms of homological algebra and therefore relate well to each other.

As mentioned above, the ζ-function of a motive is closely related to the ζ-function of the corresponding pair of representations. Basically, the only difference are some Euler factors at bad primes. It is possible to mimic this construction for families in many cases. Assuming the conjectures (FK), we can thus apply the theorems of Fukaya and Kato to derive the existence of a ζ-function for families. This result is summed up in theorem 4.31:

THEOREM 0.5 (Iwasawa main conjecture for families). *Assume that the conjectures (FK) hold. Let M_t be a family of motives satisfying the condition 2.12, let F_∞/F be a Lie extension as in section 2.2 inducing a pair of Galois representations $(\mathbb{T},\mathbb{T}^0)$, and assume that β is an isomorphism as above. We assume furthermore that the condition 2.15 is satisfied and that the cohomology groups of $SC(\mathbb{T},\mathbb{T}^0)$ are S^*-torsion. Then there is a ζ-element $\zeta_\beta(M_t,F_\infty/F) \in K_1(\Lambda[[t]]_{S^*}) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda})$ with the following properties:*

(1) *Under the boundary map of the long exact sequence of K-theory, the element $\zeta_\beta(M_t,F_\infty/F)$ maps to the class of $SC(\mathbb{T},\mathbb{T}^0)$ in $K_0(S^*\text{-}tor)$.*
(2) *Under specialization maps ϕ, the isomorphism $\zeta_\beta(M_t,F_\infty/F)$ is mapped to $\zeta_{\beta_\phi}(M_\phi,F_\infty/F)$ in $K_1(\Lambda_{S^*}) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$.*
(3) *Assume that ϕ is a specialization, ρ is an Artin character of G, and j is an integer such that $M_\phi(\rho)(j)$ is critical as in theorem 4.24, and let $\rho' : \mathbb{Z}_p \times G \to K'$ be ρ on G and ϕ on $\mathbb{Z}_p$. Then, the value of $\zeta_\beta(M_t,F_\infty/F)$ at $\rho'\kappa^{-j}$ can be described using the value of the complex L-function of $M_\phi(\rho^*,j)$ at 0.*

There are cases where a similar result holds for the Selmer groups in place of the Selmer complexes. However, in general, the classes of the two in $K_0(\Sigma)$ do not coincide, so we have to introduce a correction factor, which weakens the result.

The thesis is organized as follows: In the first chapter, some technical lemmata are proven and we reproduce some theorems from different fields. Please note that the theorems from homological algebra, while well-known, are carefully stated to produce canonical morphisms in later chapters. Chapter two reproduces those parts of the theory of Selmer complexes that are needed later and proves the relations to the Selmer groups. The third chapter presents our results on the variation of Iwasawa invariants. In the fourth chapter, we recall the theory of ζ-isomorphism and derive our versions of the Iwasawa main conjecture for families. Finally, in the last chapter we discuss some deformation theory specialized to the nearly ordinary case and present the classical examples of families.

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CHAPTER 1

Preliminaries

Before we actually start working with the Selmer complexes and ζ-isomorphisms, it is essential to name some basic facts about noncommutative rings and homological algebra and to fix the notation. For this purpose, we fix once and for all a rational prime p. Please note that some of our definitions depend on this specific prime even though it is not always explicitly mentioned.

1.1. Some facts on modules and representations

In this section, we collect some facts from representation theory we will need later. For this whole section, let Λ be a not necessarily commutative, left and right Noetherian ring without zero divisors. Moreover, let G be an abstract group and T a finitely generated (left) Λ-module equipped with a Λ-linear (left) G-action.

Let us first note:

PROPOSITION 1.1. *Let $f \in \Lambda$ be an element of the center. Assume that T does not have any f-torsion. Then, the canonical map $T^G/f \to (T/f)^G$ is injective. If, in addition, G is finite, then the cokernel is annihilated by the order of G.*

PROOF. As T does not have any f-torsion, we have an exact sequence:

$$0 \to T \xrightarrow{f\cdot} T \to T/f \to 0$$

The long exact cohomological sequence implies the short exact sequence:

$$0 \to T^G/f \to (T/f)^G \to H^1(G,T)[f] \to 0$$

But any cohomology group of G is annihilated by the order of G. Thus the claim follows.

□

>From that, we deduce the following application:

COROLLARY 1.2. *Let $\mathcal{O} \subset \mathcal{O}'$ be a finite extension of commutative principal ideal domains such that $\mathcal{O}'$ is free as an $\mathcal{O}$-module. Let $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ be an $\mathcal{O}$-algebra homomorphism. Then the kernel of ϕ is a principal ideal. Let T be an $\mathcal{O}[[t]]$-module that does not have any torsion by a generator of the kernel of ϕ. Let G operate $\mathcal{O}[[t]]$-linearly on T. Then, the natural map $\mathcal{O}' \otimes_{\mathcal{O}[[t]]} T^G \to (\mathcal{O}' \otimes_{\mathcal{O}[[t]]} T)^G$ is injective, and if G is finite, the cokernel is annihilated by the order of G.*

PROOF. As $\mathcal{O}[[t]]$ is factorial and the kernel of ϕ is a prime ideal of height 1, the kernel is principal.

Look at the induced map $\mathcal{O}'[[t]] \to \mathcal{O}'$. This is a surjective map and the kernel is still principal, say generated by f. Clearly, $\mathcal{O}'[[t]] \otimes_{\mathcal{O}[[t]]} T$ does not have any f torsion. We can therefore apply the last proposition to the $\mathcal{O}'[[t]] \otimes_{\mathcal{O}[[t]]} \Lambda$-module

$\mathcal{O}'[[t]] \otimes_{\mathcal{O}[[t]]} T$ to show that the map $\mathcal{O}' \otimes_{\mathcal{O}[[t]]} (\mathcal{O}'[[t]] \otimes_{\mathcal{O}'[[t]]} T)^G \to (\mathcal{O}' \otimes_{\mathcal{O}[[t]]} \mathcal{O}'[[t]] \otimes_{\mathcal{O}'[[t]]} T)^G$ has the desired properties. But as $\mathcal{O}'[[t]]$ is free over $\mathcal{O}[[t]]$, we conclude that $\mathcal{O}'[[t]] \otimes_{\mathcal{O}[[t]]} T^G = (\mathcal{O}'[[t]] \otimes_{\mathcal{O}[[t]]} T)^G$, proving our assertion. □

Before we develop this theory further, let us remark that the situation with coinvariants in place of invariants is much better:

LEMMA 1.3. *Let R and S be (not necessarily commutative) rings. And let T be an S-module with a linear action of a group G. Moreover let Y be an R-S-bimodule. We define a G action on $Y \otimes_S T$ by taking the trivial operation on Y. If, furthermore, $()_G$ denotes the G coinvariants, then we have a canonical isomorphism of R-modules:*

$$(Y \otimes_S T)_G \cong Y \otimes (T_G)$$

PROOF. We can view T as an $S[G]$-module. Let $Y[G] := R[G] \otimes_R Y = Y \otimes_S S[G]$ as an $R[G]$-$S[G]$-bimodule. Then, we conclude:

$$Y \otimes_S T = Y[G] \otimes_{S[G]} T$$

as $R[G]$-modules where the G-action on the left hand side is the one described in the assertion. And further

$$\begin{aligned}(Y[G] \otimes_{S[G]} T)_G &= R \otimes_{R[G]} Y[G] \otimes_{S[G]} T = Y \otimes_{S[G]} T \\ &= Y \otimes_S S \otimes_{S[G]} T = Y \otimes_S (T_G)\end{aligned}$$

as required. □

We will spend the rest of this section giving criteria when Λ-modules, in particular the module T^G, are free.

Let us first develop some conditions under which modules are free. Firstly:

LEMMA 1.4. *For a ring Λ and a finitely generated projective Λ-module T, we assume that $I \subset \Lambda$ is contained in the radical and T/IT is free as a Λ/I module; then, T is free as a Λ-module.*

PROOF. We choose lifts $t_1, ..., t_n$ in T of a basis of T/IT. By Nakayama's lemma they generate T, so that we get an exact sequence

$$0 \to K \to \Lambda^n \to T \to 0$$

where the right map is given by sending the standard basis of Λ^n to the t_i and where K is defined to be the kernel. As T is projective, this sequence splits, so K is isomorphic to a quotient of Λ^n. Consequently it is finitely generated. Moreover, again making us of the fact that T is projective, we conclude that the sequence

$$0 \to K/I \to (\Lambda/I)^n \to T/I \to 0$$

is still exact. The map $(\Lambda/I)^n \to T/I$ is given by the chosen basis, thus it is an isomorphism. We can then deduce that $K/I = 0$ and, using the Nakayama lemma again, we arrive at $K = 0$, proving our assertion. □

Secondly, we have some compatibility for free ring extensions:

LEMMA 1.5. *Let Λ be a ring with radical J, such that Λ/J is a finite dimensional algebra over a skew field (compare the definition of adic rings in the next section). Let Λ' be a finite extension of Λ, which is free as a Λ-module and possesses a basis consisting of central elements. Then, for any finitely generated*

Λ-module T we have: $\Lambda' \otimes_\Lambda T$ is free as a Λ'-module if and only if T is free as a Λ-module.

PROOF. The "only if" part is trivial.

To prove the "if" part, assume that $T' := \Lambda' \otimes_\Lambda T$ is free as a Λ' module, then it is free as a Λ-module, too, so T is projective as a Λ-module. We can thus utilize the last lemma to see that it is enough to show that T/J is free as a Λ/J-module. If $r_1, ..., r_s$ is a central basis of Λ' over Λ, then the (both-sided) ideal J' of Λ' generated by J is $\Lambda' J = J\Lambda' = \bigoplus_i r_i J$. Therefore, Λ'/J' is a free Λ/J-module. We may thus replace Λ by Λ/J and prove the lemma in the case that $J = 0$.

Wedderburn's theorem tells us that in this case Λ is a finite product of matrix algebras over skew fields. As the extension Λ'/Λ is generated by central elements, the central idempotents corresponding to the product decomposition of Λ are also central in Λ'. It follows that Λ' has a corresponding product decomposition. As being a free module of finite rank n over a product of rings is the same as being a free module of rank n over all the factors, we have reduced the lemma to the case where Λ is simple, if we show in addition that the rank of T only depends on the Λ' rank of $\Lambda' \otimes_\Lambda T$ and the Λ-rank of Λ'.

Finally, in the case that $\Lambda = M_n(k)$ with some skew field k, the (explicit) Morita equivalence tells us that the category of modules over $M_n(k)$ is canonically equivalent to the category of vector spaces over k. Thus, the isomorphism classes of finitely generated $M_n(k)$-modules T are classified by the dimension of the corresponding vector space denoted by $r(T)$. Moreover, it is not hard to see that T is free if and only if $n|r(T)$, and in this case the rank of T is $\frac{r(T)}{n}$. In the situation of our lemma, we have that $r(T) = dim_k(T) \cdot \frac{1}{n} = dim_k(T') \cdot \frac{1}{sn} = dim_k(\Lambda') \cdot \frac{t}{sn}$ where t is the Λ'-rank of the free module T'. The assertion now follows from the fact that $dim_k(\Lambda') = n^2 s$. □

The last lemma is quite well known, but we state it for completeness:

LEMMA 1.6. *Let Λ be an integral domain with the field of fractions F. A Λ-module T is generated by $n := dim_F(F \otimes T)$ elements, if and only if it is free.*

PROOF. The module T is free if and only if it is free of rank n, and if that is the case, then it is clearly generated by n elements. If T is generated by n elements, however, we have an exact sequence:

$$0 \to R \to \Lambda^n \to T \to 0$$

Tensoring with F, we conclude that $R \otimes F$ is 0. In other words: R is a torsion module. But since R is a submodule of a free module, it is torsion-free. Consequently it follows that $R = 0$ and that $\Lambda^n \to T$ is an isomorphism. □

COROLLARY 1.7. *Assume that Λ is a Noetherian integral domain, $f \in \Lambda$ is contained in the Jacobson radical and Λ/f is a principal ideal domain. Then, for any finitely generated free Λ-module T with a group G acting Λ-linearly on T, the module of G-invariant elements T^G is a free Λ-module.*

PROOF. By proposition 1.1, T^G/f maps injectively into the free module T/f. It follows that the Λ/f-module T^G/f is torsion-free hence it is free as Λ/f is a principal ideal domain. Therefore, by the last lemma it is generated by $n := dim_{Quot(\Lambda/f)}(Quot(\Lambda/f) \otimes (T^G)/f))$ elements. By Nakayama's lemma, T^G is also generated by n elements. Localizing at (f) we observe that $\Lambda_{(f)}$ is a discrete valuation ring and so the torsion-free module $\Lambda_{(f)} \otimes T^G \subset \Lambda_{(f)} \otimes T$ is free. Thus we conclude that

$$dim_{Quot(\Lambda)}(Quot(\Lambda) \otimes T^G)) = dim_{Quot(\Lambda/f)}(Quot(\Lambda/f) \otimes (T^G)/f)) = n.$$

Together with the last lemma the claim follows. □

1.2. Homological algebra

Cohomology theories with compact support and the Selmer complexes are defined as mapping cones. In our applications it turns out, however, that the numbering of the mapping cone should be shifted by one. The shifted cone is called the mapping fiber. Adopting this notion, we establish the following definition:

DEFINITION 1.8. *Let $f : B^\bullet \to C^\bullet$ be a morphism of complexes. The mapping fiber $A^\bullet = cone(f)[-1]$ of f is the mapping cone of f shifted by one, i.e., the complex with the modules $A^i = B^i \oplus C^{i-1}$ and differential $d_A : (b, c) \mapsto (d_B(b), -d_C(c) - f(b))$. This complex makes the following a distinguished triangle in the derived category:*

$$A^\bullet \to B^\bullet \xrightarrow{f} C^\bullet \xrightarrow{+}$$

To get maps between mapping fibers and to compute the differences between them, we will adopt the following proposition:

PROPOSITION 1.9. *Assume we are given the following diagram of complexes:*

$$\begin{array}{ccc} B & \xrightarrow{g} & B'' \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi''} \\ C & \xrightarrow{h} & C'' \end{array}$$

We denote the mapping fibers of g and h by B' and C' and the mapping fibers of ϕ and ϕ'' by A and A''. Then the following holds:

1. *There is a natural morphism $\phi' : B' \to C'$ making $(\phi', \phi, \phi'') : (B', B, B'') \to (C', C, C'')$ a morphism of triangles. The same argument gives a natural morphism $f : A \to A''$.*
2. *The mapping fibers of ϕ' and f are naturally isomorphic as complexes; denoting them by A', we get the following natural diagram in the derived category, where all the rows and columns are distinguished triangles in the derived category and all the squares commute except for the lower right one, which commutes up to sign:*

$$\begin{array}{ccccccc} A' & \longrightarrow & A & \xrightarrow{f} & A'' & \longrightarrow & A'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & B & \xrightarrow{g} & B'' & \longrightarrow & B'[1] \\ \downarrow{\scriptstyle\phi'} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi''} & & \downarrow \\ C' & \longrightarrow & C & \xrightarrow{h} & C'' & \longrightarrow & C'[1] \\ \downarrow & & \downarrow & & \downarrow & - & \downarrow \\ A'[1] & \longrightarrow & A[1] & \longrightarrow & A''[1] & \longrightarrow & A'[2] \end{array}$$

Moreover, all the maps between complexes without a shift are actually maps of complexes.

This is a well-known fact, and easily computed using the explicit construction of the mapping fibers (or cones). A version of it can be found in chapter 10 of [**Wei94**]. Let us just point out for the second assertion that viewing the i-th module of the mapping fibers of ϕ' and f as $B^i \oplus C^{i-1} \oplus (B'')^{i-1} \oplus (C'')^{i-2}$ the isomorphism is the identity on the first three summands and minus the identity on $(C'')^{i-2}$. The rest of the proof consists of comparing sings.

This proposition has a generalization in general triangulated categories. In this setting one can still complete the diagram in the form of the second part, however there are no natural choices for some of the morphisms leading to ambiguities we would like to avoid.

For later reference and to avoid confusions with the direction of shifting we note the following special case:

COROLLARY 1.10. *In the situation of the last proposition and keeping the notations we assume in addition that $\phi : B \to C$ is a quasi-isomorphism. In this situation, we have a canonical distinguished triangle in the derived category:*

$$B' \to C' \to A'' \to B'[1].$$

Equivalently, the mapping cone of the map of the mapping fibers $B' \to C'$ is canonically quasi-isomorphic to the mapping fiber of $B'' \to C''$.

PROOF. The last proposition gave us a canonical triangle of complexes:

$$B' \to C' \to A'[1] \to B'[1].$$

As ϕ is a quasi-isomorphism, A is acyclic and thus the canonical map $A'' \to A'[1]$ is a quasi-isomorphism. This allows us to replace $A'[1]$ in the above triangle by A'' in a canonical way. □

Turning back to more specific situations we want to note some facts on adic rings. For reasons of completeness, we give the definition:

DEFINITION 1.11. *A ring Λ is called an adic ring if there is a two-sided ideal $I \subset \Lambda$ such that for all $n \geq 1$, Λ/I^n is finite of p-power order and $\Lambda = \varprojlim_n \Lambda/I^n$.*

REMARK 1.12. *Recall that the prime occurring in the definition is our fixed prime p. So by definition, "adic ring" implies "pro-p ring", even if p is not explicitly mentioned.*

If Λ is an adic ring with respect to some ideal I, then I is contained in the radical J of Λ and $\Lambda = \varprojlim \Lambda/J^n$. Moreover, we have that $J^n \subset I$ for some n, so that the topology induced on Λ is independent of the choice of I. In the following Λ will thus be viewed as a topological ring. Furthermore, Λ is semi-local and Λ/J is a finite product of full matrix algebras over finite fields.

This is shown in paragraph 1.4, of [**FK06**] in particular in lemma 1.4.4. From the same paragraph, we take the next lemma, which is our main source of examples of adic rings.

LEMMA 1.13. *If G is a profinite group which contains an open finitely generated pro-p subgroup and if $\mathcal{O}$ is the ring of integers of a p-adic field then the Iwasawa algebra $\mathcal{O}[\![G]\!]$ is an adic ring. In particular, p-adic Lie groups fulfill this condition.*

PROOF. For the proof, see part 1.4.2 of the mentioned paragraph. □

Modules over adic rings have the advantage that they behave well with projective limits, and thus many computations can be reduced to finite cases. As a first example of this fact, we note the following behavior of K_1 groups:

PROPOSITION 1.14. *Let Λ be an adic ring and J its radical. Then we have:*

$$K_1(\Lambda) \cong \varprojlim_n K_1(\Lambda/J^n)$$

This is proposition 1.5.1 of [**FK06**].

1.3. Galois cohomology

Since this thesis will deal with a variety of cohomolog theories we will first lay down the main definitions and theorems to fix the notation. To deal with Galois cohomology, one needs to have Galois groups, so from now on we assume every given field to be equipped with the choice of an algebraic closure.

Firstly, for a profinite group G, a topological Ring R, and a topological $R[G]$ module M, we define $C(G, M)$ to be the complex of inhomogeneous continuous cochains, $R\Gamma(G, M)$ to be the complex viewed as an object in the derived category of abstract R-modules, and $H^m(G, M)$ to be its m-th cohomology group. Please note that, in this generality, this is not a derived functor. As a first fact we note:

REMARK 1.15. *Let $\phi : R' \to R$ be a homomorphism of topological rings and let $()_{R'}$ be the functor "view as R'-module." Then $C(G, M_{R'}) = (C(G, M))_{R'}$ and therefore $R\Gamma(G, M_{R'}) = (R\Gamma(G, M))_{R'}$ and $H^m(G, M_{R'}) = (H^m(G, M))_{R'}$.*

While this is a well-known and obvious fact, it is rarely stated in this generality.

Now let us note the compatibility of the cohomology groups with projective limits. The following is the first part of proposition 1.6.5 in [**FK06**]:

PROPOSITION 1.16. *Assume that Λ is an adic ring and $H^m(G, M)$ is finite for all finite abelian groups M of p-power order endowed with a continuous G action. Then for all finitely generated (topological) Λ-modules T with a continuous, Λ-linear G-action, we have a canonical isomorphism:*

$$H^m(G, T) \overset{\cong}{\to} \varprojlim_n H^m(G, T/J^nT)$$

for any m with J being the radical of Λ.

Next we give the main notations for Galois cohomology groups: If F is a field, then G_F denotes its absolute Galois group and for a continuous G_F-module M we denote by $H^i(F, M)$, $C^i(F, M)$, and $R\Gamma(F, M)$) the continuous cohomology, the cochains, and the derived complex for G_F, respectively. Moreover, for $F = \mathbb{R}$ and $F = \mathbb{C}$ we define the Tate complexes $\widehat{C}(F, T)$ to be the 2-periodic complexes computing the cohomology for the cyclic groups $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$.

We observe the following well-known facts:

PROPOSITION 1.17. *If M is finite of p power order, then in the following cases the groups $H^i(F, M)$ are finite:*

- *If F is an l-adic field or $\mathbb{R}$ or a finite field (l is an arbitrary prime).*
- *If F is a finite extension of the maximal unramified extension $\mathbb{Q}_l^{ur}$ and $l \neq p$.*

Moreover, the following is known about the cohomological p-dimensions:

- *l-adic fields have cohomological p-dimension 2.*
- *Finite fields have cohomological p-dimension 1.*
- *$cd_p(\mathbb{R}) = 0$ unless $p = 2$.*
- *$\widehat{C}(F, T)$ is acyclic unless $F = \mathbb{R}$ and $p = 2$.*
- *Finite extensions of $\mathbb{Q}_l^{ur}$ have cohomological p dimension 1 if $p \neq l$.*

PROOF. All of this is well known and most of it is easily proven (if one knows the Galois groups). The only aspect requiring some work is the case of the l-adic fields. The respective proof can be found, for instance, in [**NSW08**] Theorem 7.1.8. □

As usual in arithmetic, we need the "finite part" of the local Galois cohomology. This describes a subcomplex of the local cohomology, which computes the unramified cohomology:

DEFINITION 1.18. *For an l-adic field F with $l \neq p$ and a Galois module M, let $C_f(F, M) \subset C(F, M)$ be the following subcomplex: $C_f^i(F, M) = 0$ unless $i \in \{0, 1\}$, in degree 0 it is the full module $C^0(F, M)$ and in degree 1 it is the kernel of the map $(C^1(F, M))_{d=0} \to H^1(F^{ur}, M)$. Here, the index $d = 0$ should be read as taking the cocycles and F^{ur} is the maximal unramified extension of F. Moreover, we write the cohomology of this complex as $H_f^i(F, M)$.*

LEMMA 1.19. *In the derived category, we have canonical isomorphisms:*

$$C_f(F, M) \cong C(F^{ur}/F, M^I) \cong [1 - \phi : M^I \to M^I],$$

where $I \subset G_F$ is the inertia subgroup, ϕ denotes the geometric Frobenius, and the last complex lives in degrees 0 and 1.

PROOF. We will not give a complete proof of this fact, but we state what the canonical morphisms are: The second one is simple: it is given by the map which is just the identity for the degree 0 parts: $C^0(F^{ur}/F, M^I) = M^I$ and in degree 1 sends a map $c \in Maps(Gal(F^{ur}/F), M^I)) = C^1(F^{ur}/F, M^I)$ to $-c(\phi)$.

The other quasi-isomorphism is a bit harder to describe: First note that as the higher cohomology groups of $C(F^{ur}/F, M^I)$ vanish, the inclusion of the subcomplex $[C^0(F^{ur}/F, M^I) \to C^1(F^{ur}/F, M^I)_{d=0}]$ is a quasi-isomorphism. The image of this complex in $C(F, M)$ under the inflation map is then contained in $C_f(F, M)$ and the quasi-isomorphism we are looking for is the induced map

$$[C^0(F^{ur}/F, M^I) \to C^1(F^{ur}/F, M^I)_{d=0}] \to C_f(F, M).$$

□

Turning to global cohomology, let us now fix a number field F and an open subset $U \subset spec(\mathcal{O}_F)$ not containing the primes dividing p. Let G_U denote the Galois group of the maximal extension of F which is unramified in U. For any G_U-module T, we then denote the chain-complex, derived complex, and the cohomology groups by $C(U, T)$, $R\Gamma(U, T)$, and $H^m(U, T)$, respectively.

We fix embeddings of the algebraic closures $\overline{F} \to \overline{F_v}$ for all places v of F to define cohomology with compact support, of which two versions will be used:

DEFINITION 1.20. *Let F be a number field and $U \subset \mathcal{O}_F$ be an open subset. We then define for every G_U-module T the following complexes:*

(1) *The complex $C_c(U,T)$ is the mapping fiber of the map:*

$$C(U,T) \to \bigoplus_{v \notin U} C(F_v, T)$$

Here the sum runs over all (finite or infinite) places not in U. The cohomology of this complex is denoted by $H^i_c(U,T)$.

(2) *The complex $C_{(c)}(U,T)$ is the mapping fiber of the map:*

$$C(U,T) \to \bigoplus_{v|\infty} \widehat{C}(F_v, T) \oplus \bigoplus_{l \notin U} C(F_l, T)$$

Here l runs over all finite places not in U. The cohomology of this complex is denoted by $H^i_{(c)}(U,T)$.

These definitions follow the ones made in [**FK06**].

We observe the following facts about these cohomology groups:

PROPOSITION 1.21. *If T is finite of order a power of p, then all the groups $H^i(U,T)$, $H^i_c(U,T)$, and $H^i_{(c)}(U,T)$ are finite for any m. Moreover, $H^i_c(U,T) = H^i_{(c)}(U,T) = 0$ if $i > 3$, and if $i > 2$ and $p \neq 2$, then $H^i(U,T) = 0$.*

PROOF. The cases where $p \neq 2$ follow directly from proposition 8.3.18 and theorem 8.3.20 in [**NSW08**]. Indeed, this theorem is the assertion on $H^i(U,T) = 0$, and the other two cases follow directly from the cohomological dimensions of the local fields and the long exact sequences coming for the distinguished triangles defining the cohomology with compact support. Making use of the same exact sequences, it remains to be shown that $H^i(U,T) \to \bigoplus_{v|\infty} H^i(F_v,T)$ is an isomorphism for all $i \geq 3$. This last statement is part of the Poitou-Tate theorem and can be found, for instance, as the second part of theorem 8.6.10 in [**NSW08**]. □

The perfectness and the base change properties of the Selmer complex depend on the similar facts for the cohomology theories. These are parts 2 and 3 of proposition 1.6.5. in [**FK06**]:

PROPOSITION 1.22. *Assume that the groups $H^i(G,M)$ are finite whenever M is a finite module of p-power order. Furthermore, the cohomological p-dimension of G is assumed to be finite. If T is a finitely generated projective Λ-module for an adic ring Λ endowed with a continuous G-action, then the following holds:*

(1) *The complex $R\Gamma(G,T)$ is perfect.*

(2) *If Y is a finitely generated projective Λ'-module for some other adic ring Λ' endowed with a compatible right Λ-action, then there is a canonical isomorphism:*

$$Y \otimes^L_\Lambda R\Gamma(G,T) \overset{\cong}{\to} R\Gamma(G, Y \otimes_\Lambda T)$$

This applies in particular to $Y = \Lambda'$ if Λ' is a Λ algebra.

Similarly, if F number field and U is an open subset of spec$(\mathcal{O}_F)$, then for any G_U-module T, that is finitely generated and projective as a Λ-module, $R\Gamma_c(U,T)$ is perfect, and in the situation of the second part, there is again an isomorphism:

$$Y \otimes^L_\Lambda R\Gamma_c(U,T) \overset{\cong}{\to} R\Gamma_c(U, Y \otimes_\Lambda T)$$

Finally, we state two duality theorems that we will use. The standard references for these theorems include [**Mil06**] and [**NSW08**]. A treatment of the exact theorems can be found in [**Lim11**]

For some Λ-module T, by $T^{\vee} := Hom_{cont}(T, \mathbb{Q}_p/\mathbb{Z}_p)$ we denote the Pontryagin dual and by $T^{\vee}(1) := Hom_{cont}(T, \mu_{p^{\infty}})$ the Kummer dual. For a complex $C^{\bullet}$, let us denote $(C^{\bullet})^{\vee} := RHom_{cont}(C^{\bullet}, \mathbb{Q}_p/\mathbb{Z}_p)$.

Firstly, there is the local duality:

PROPOSITION 1.23. *Let T be a finitely generated projective module over an adic ring Λ. We assume that T is endowed with a Λ-linear G_F action for some l-adic field F. Then we have a canonical isomorphism:*

$$\Phi(F,T) : R\Gamma(F,T) \cong R\Gamma(F, T^{\vee}(1))^{\vee}[-2].$$

Under this perfect pairing, the orthogonal complement of $H^1(F^{ur}, T) \hookrightarrow H^1(F,T)$ is $H^1(F^{ur}, T^{\vee}(1)) \hookrightarrow H^1(F, T^{\vee}(1))$ and vice versa.

For a proof, see for instance [**NSW08**] theorems (7.2.6) and (7.2.15).

Secondly, we have the global duality:

PROPOSITION 1.24. *Let T be a finitely generated projective module over an adic ring Λ. We assume that T is endowed with a Λ-linear G_F action for some number field F. We fix an open subset $U \subset spec(\mathcal{O}_F)$ not containing any places of F dividing p. Then we have the canonical isomorphisms*

$$\Phi_{(c)}(U,T) : R\Gamma_{(c)}(U,T) \cong R\Gamma(U, T^{\vee}(1))^{\vee}[-3]$$

and

$$\Phi(U,T) : R\Gamma(U,T) \cong R\Gamma_{(c)}(U, T^{\vee}(1))^{\vee}[-3].$$

For a proof, see [**Mil06**] corollary 3.3. Also the main theorem of [**Lim11**] treats an even bigger class of modules.

CHAPTER 2

Selmer complexes

Inherent to the theory of zeta isomorphisms as it is developed in [**FK06**] is the use of Selmer complexes instead of Selmer groups. Basically, for every set of local conditions leading to a Selmer group, one gets a Selmer complex. As Nekovar has already remarked in [**Nek06**], the only sets of local conditions that are accessible with elementary means are the ones suggested by Greenberg in [**Gre89**]. In this chapter, we will develop the theory of Selmer complexes as far as we need it. Finally, we will show how they are connected to classical Selmer groups.

2.1. Definitions and basic facts

Let us first discuss how to associate a Selmer complex to a representation of the absolute Galois group of $\mathbb{Q}$ over some adic ring Λ. The next section will focus on how such a representation is associated to a motive or a family of motives.

Let us fix embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_v}$ for all places v of $\mathbb{Q}$ for the rest of the thesis. These embeddings induce restriction maps of the corresponding Galois groups and therefore the following is well defined:

DEFINITION 2.1. *Let T be a finitely generated module over an adic ring Λ and let $T^0 \subset T$ be a $G_{\mathbb{Q}_p}$-subrepresentation for the chosen embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. We choose an open subset U of $Spec(\mathbb{Z})$ not containing p, so that the representation T is unramified in U and such that the complement of U contains at least one prime different from p. Then we define the imprimitive Selmer complex $SC(U,T,T^0)$ to be the mapping fiber of the map*

$$C(U,T) \to C(\mathbb{Q}_p, T/T^0) \oplus \bigoplus_{l \notin U \cup \{p\}} C(\mathbb{Q}_l, T).$$

Similarly, we define the primitive Selmer complex $SC_U(T,T^0)$ to be the mapping fiber of

$$C(U,T) \to C(\mathbb{Q}_p, T/T^0) \oplus \bigoplus_{l \notin U \cup \{p\}} C(\mathbb{Q}_l, T)/C_f(\mathbb{Q}_l, T).$$

The U in the index of the primitive Selmer complex can and will be omitted by the following Lemma:

LEMMA 2.2. *Let $U' \subset U$ be two open subsets of $spec(\mathbb{Z})$, both satisfying the conditions on ramification. There is then a natural map of complexes $SC_U(T,T^0) \to SC_{U'}(T,T^0)$ which is a quasi-isomorphism.*

PROOF. This is a standard argument, which is given here for the sake of completeness. Firstly, we look at the following commutative diagram of complexes:

$$\begin{array}{ccc} C(U,T) & \longrightarrow & C(\mathbb{Q}_p, T/T^0) \oplus \bigoplus_{l \notin U \cup \{p\}} C(\mathbb{Q}_l, T)/C_f(\mathbb{Q}_l, T) \\ \downarrow & & \downarrow \\ C(U',T) & \longrightarrow & C(\mathbb{Q}_p, T/T^0) \oplus \bigoplus_{l \notin U' \cup \{p\}} C(\mathbb{Q}_l, T)/C_f(\mathbb{Q}_l, T) \end{array}$$

Here, the left vertical map is the inflation and the right vertical map is the canonical inclusion. The first part of proposition 1.9 applies to our situation and produces the natural morphism

$$\phi : SC_U(T, T^0) \to SC_{U'}(T, T^0).$$

Of course, if $U' \subset V \subset U$, this map factors naturally via $SC_V(T, T^0)$. Thus by induction it is enough to prove the lemma in the case $U' = U \cup \{l\}$. The second part of proposition 1.9 tells us that it suffices to show that the induced map on the mapping fibers of the vertical maps is a quasi-isomorphism. The mapping fiber of the right vertical map is canonically quasi-isomorphic to the mapping fiber of $0 \to C(\mathbb{Q}_l, T)/C_f(\mathbb{Q}_l, T)$, which in turn is quasi-isomorphic to the mapping fiber of $C(\mathbb{Q}_l^{nr}/\mathbb{Q}_l, T) \to C(\mathbb{Q}_l, T)$. Combining those findings we have a commutative diagram

$$\begin{array}{ccc} C(U,T) & \longrightarrow & C(\mathbb{Q}_l^{nr}/\mathbb{Q}_l, T) \\ \downarrow & & \downarrow \\ C(U',T) & \longrightarrow & C(\mathbb{Q}_l, T) \end{array}$$

where the horizontal arrows are restriction maps and the vertical arrows are inflation maps. It remains to be shown that the induced map on the mapping fibers is a quasi-isomorphism. Moreover, using proposition 1.16, we may assume that T is finite of p power order. As p is invertible on U and U', the group cohomology coincides in this case with the étale cohomology of the associated sheaf (see for instance [**Mil06**] chapter 2 proposition 2.9).

In the language of étale cohomology, both of the mapping fibers actually compute the cohomology relative to l on U and $spec(\mathbb{Z}_l)$ and those are isomorphic by the excision lemma. That the isomorphism from the excision lemma is the one induced by our map on the mapping fibers follows from the fact that any acyclic resolution can be used to compute derived functors. □

By the definition of Nekovar in [**Nek06**], the cohomology groups of a Selmer complex should vanish outside the degrees 1, 2, and 3. This is noted in the following lemma:

LEMMA 2.3. *Assume that* $p \neq 2$ *or invert* p*, then we have* $H^i(SC(U, T, T^0)) = H^i(SC(T, T^0)) = 0$ *unless* $i = 1, 2, 3$.

PROOF. We have already seen, in the section about Galois cohomology, that the complexes of which the Selmer complexes are mapping fibers have trivial cohomology groups outside the degrees 0, 1, and 2. So the Selmer complexes can only have nontrivial cohomology groups in the degrees 0, 1, 2, and 3. The degree 0 part of $SC(U, T, T^0)$ vanishes as a single localization map in degree 0 is already injective (being the inclusion of G_U-invariants into G_l invariants) and the complement of $U \cup \{p\}$ is nonempty by definition. For the complex $SC(T, T^0)$ we note that the degree 0 cohomology group is contained in the one of the imprimitive Selmer complex (compare lemma 2.5) and must therefore vanish. □

Note the similarity in the definitions of the Selmer complexes and the definitions of the cohomology with compact support (including the degrees in which cohomology can occur). This is not by coincidence: The Selmer complex with Greenberg's local conditions should be thought of as the "right" cohomology theory for arithmetics in the sense that it is a compactly supported cohomology theory ($H^0 = 0$), which admits duality results with itself (i.e. not switching between compactly supported and not compactly supported theories like global duality 1.24). This point of view is explained by Nekovar in [**Nek06**], in particular in the paragraphs 0.9 and 0.12, where he presents his results.

To be more precise, proposition 1.9 gives us a morphism of exact triangles:

$$\begin{array}{ccccccc} C_{(c)}(U,T) & \longrightarrow & C(U,T) & \longrightarrow & \widehat{C}(\mathbb{R},T) \oplus \bigoplus_{l \notin U} C(\mathbb{Q}_l,T) & \xrightarrow{+} & \\ \downarrow & & \downarrow & & \downarrow & & \\ SC(U,T,T^0) & \longrightarrow & C(U,T) & \longrightarrow & C(\mathbb{Q}_p,T/T^0) \oplus \bigoplus_{l \notin U \cup \{p\}} C(\mathbb{Q}_l,T) & \xrightarrow{+} & \end{array}$$

Now remarking that $\widehat{C}(\mathbb{R},T/T^0)$ is acyclic and using the distinguished triangle related to the exact sequence

$$0 \to T^0 \to T \to T/T^0 \to 0,$$

the corollary 1.10 following this proposition gives the first part of the next lemma. The second part follows completely analogously and is stated as equation 4.4 in [**FK06**]:

LEMMA 2.4. *If $p \neq 2$, we have two canonical distinguished triangles:*

(1) $C_{(c)}(U,T) \to SC(U,T,T^0) \to C(\mathbb{Q}_p,T^0) \xrightarrow{+}$ *and*

(2) $C_{(c)}(U,T) \to SC(T,T^0) \to C(\mathbb{Q}_p,T^0) \oplus \bigoplus_{l \notin U \cup \{p\}} C_f(\mathbb{Q}_l,T) \xrightarrow{+}$

The same results hold if $p = 2$ and we invert p in every module.

We give one result towards the special situation of the next section: Let us assume that we fix an embedding $F \to \overline{\mathbb{Q}}$ and thus identify $\overline{F}$ and $\overline{\mathbb{Q}}$ via this embedding. The primitive and the imprimitive Selmer complex are related by the following fact:

LEMMA 2.5. *Assume $T = Ind_{G_F}^{G_\mathbb{Q}} T'$ is induced from some representation T' of the absolute Galois group of a number field F which is unramified at all primes v of F which lie over U. In this case, we have a canonical, distinguished triangle*

$$SC(T,T^0) \to SC(U,T,T^0) \to \bigoplus_v C_f(F_v,T') \xrightarrow{+},$$

where the sum is taken over all finite places v of F not lying above U or p.

PROOF. First observe that the case $F = \mathbb{Q}$, i.e., the not induced situation, is an easy application of the corollary 1.10 after the "3×3-lemma." Thus, for the general case it remains to be shown that there is a canonical quasi-isomorphism $C_f(\mathbb{Q}_l,T) \to \bigoplus_{v|l} C_f(F_v,T')$ whenever $l \notin U$.

By applying Mackey decomposition, we have the following equality of $G_{\mathbb{Q}_l}$-modules:

$$Ind_{G_F}^{G_{\mathbb{Q}}} T' = \bigoplus_{G_{\mathbb{Q}_l} g G_F} Ind_{G_{\mathbb{Q}_l} \cap (G_F)^g}^{G_{\mathbb{Q}_l}} T'^g = \bigoplus_{v|l} Ind_{G_{F_v}}^{G_{\mathbb{Q}_l}} T'$$

Here in the middle term the sum is taken over a representing system of the double cosets $G_{\mathbb{Q}_l} g G_F$ and T'^g denotes the module T' with the conjugated group operation. Applying Shapiro's lemma to the cohomology on the right-hand side gives the desired result. □

REMARK 2.6. *The choice of the system of representatives of double cosets corresponds to the choice of embeddings* $\overline{F} = \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_l} = \overline{F_v}$ *for the place* v *corresponding to the double coset* $G_{\mathbb{Q}_l} g G_F$ *via the chosen embedding* $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_l}$. *Also, the influence these choices have is marginal, we will stick to them from now on.*

We conclude this section with two remarks on the imprimitive Selmer complex:

PROPOSITION 2.7. *Assume that* $p \neq 2$, *then the Selmer complex* $SC(U, T, T^0)$ *is perfect. If, in addition, the invariants under the complex conjugation* T^+ *and* T^0 *map to the same class in* $K_0(\Lambda)$, *then* $[SC(U, T, T^0)] = 0$ *in* $K_0(\Lambda)$.

PROOF. The first part follows from the fact that $R\Gamma(U, T)$ and the local summands are perfect (see propositions 1.17 and 1.22). The second part can be found in [**FK06**] 4.1.2.. □

2.2. The representations associated to motives and families

This section will explain, what kind of representations we will apply the theory of Selmer complexes to:

As before, p is a fixed odd prime and F is a number field. The completion of F at p is denoted by $F_p = \prod_{v|p} F_v$. We fix a Galois extension F_∞/F with Galois group G such that F_∞/F is unramified outside a finite set of places and G is a p-adic Lie-group.

DEFINITION 2.1.

(1) In the rest of this chapter, a family of Galois representations of rank $n < \infty$ over a topological ring R will always be a continuous group morphism $G_F \to GL(T)$ of the absolute Galois group G_F of F, where T is a free module of rank $n < \infty$ over the power series ring $R[[t]]$.
(2) Moreover, if R is an integral domain with field of fractions K, representations V over K are said to be continuous if there is an R-lattice (i.e., a finitely generated R- submodule which generates V as a K-module) that is G_F stable, such that $G_F \to GL(T)$ is continuous. All representations are assumed to be continuous if not stated otherwise.
(3) In the above situation, (continuous) representations over $K[[t]]$ are defined in an analogous fashion.
(4) A (height one) specialization is a continuous R-algebra morphism $R[[t]] \to R'$, where R' is a finite R algebra. The specialization of the representation is the induced representation on $T \otimes R'$.

The above notion of continuity is necessary since we do not have natural topologies on all the fields of fractions and there is no choice of a topology making the ring an open subset. Nevertheless the definition is well-behaved:

LEMMA 2.8. *Let $\rho : G \to GL(V)$ be a not necessarily continuous representation over $K = \mathcal{Q}(R)$, where R is a compact local Noetherian integral domain endowed with its $\mathfrak{m}$-adic topology. Then the following holds:*

(1) *If R is a complete discrete valuation ring, then K has a natural topology, and a representation over K is continuous by the above definition if and only if it is continuous in the natural topology.*
(2) *The definition is independent of the lattice: If $T, T' \subset V$ are ρ-stable lattices, then $\rho : G \to GL(T)$ is continuous if and only if $\rho : G \to GL(T')$ is.*
(3) *The standard way to get ρ-stable lattices works in this case as well: If ρ is continuous and $T \subset V$ is a (not necessarily ρ-stable) lattice, then $\bigcup_{g\in G} gT$ is a ρ-stable lattice.*

PROOF. The first assertion is a simple compactness argument.

The second one is a straightforward application of the Artin-Rees lemma once we note that there are r and r' in R, such that $rT \subset T'$ and $r'T' \subset T$.

For the last assertion, we have only to show that $T' := \bigcup_{g\in G} gT$ is finitely generated as an R-module. But if $L \subset V$ is a ρ-stable lattice, then there is a non-zero element $r \in R$, such that $rT \subset L$. As L is ρ-stable we conclude that $T' \cong rT' \subset L$ and is therefore finitely generated. □

For reasons of convenience, we give a list of notations used in the remainder of this chapter:

(1) Let K be a number field and λ a place of K dividing p. We will then look at specializations $\phi : K_\lambda[[t]] \to \overline{K_\lambda}$ and denote the image of ϕ with K_ϕ.
(2) Modules over $K_\lambda[[t]]$ and K_ϕ will be denoted by V or V_ϕ, respectively.
(3) $\mathcal{O} := \mathcal{O}_\lambda \subset K_\lambda$ and $\mathcal{O}_\phi \subset K_\phi$ are the rings of integers.
(4) Modules over $\mathcal{O}$ and $\mathcal{O}_\phi$ will be denoted by T and T_ϕ.
(5) The Iwasawa algebras are $\Lambda := \Lambda(G) := \mathcal{O}[\![G]\!][[t]]$ and $\Lambda_\phi := \Lambda_\phi(G) := \mathcal{O}_\phi[\![G]\!]$.
(6) Finally, the modules over Λ and Λ_ϕ will be denoted by $\mathbb{T}$ and $\mathbb{T}_\phi$.

We are interested in families of Galois representations with the additional property that certain specializations are the p-adic realization of motives which satisfy some extra conditions.

To put it more concretely: Let V be a free linear representation of G_F over $K_\lambda[[t]]$ of rank n. We will always assume this representation to be unramified outside a finite set of primes of F. Let $\Sigma \subset Hom(K_\lambda[[t]], \overline{K_\lambda})$ be a finite or infinite set of height one specializations, such that for every $\phi \in \Sigma$ there is a motive M_ϕ over K', a finite extension of K, such that for a place λ_ϕ of K' dividing λ the completion of K' at λ_ϕ is K_ϕ and the module $V_\phi := V \otimes_{K_\lambda[[t]],\phi} K_\phi$ is the λ_ϕ-adic realization of M_ϕ.

For all $\phi \in \Sigma$, the λ-adic realizations M_λ of the motives M_ϕ are assumed to satisfy the Dabrowski-Panchishkin condition:

CONDITION 2.9. *For every place v of F dividing p, there is a G_{F_v} submodule $M^0_\lambda(v)$ of M_λ such that*

$$D_{dR}(F_v, M^0_\lambda(v)) \overset{\cong}{\to} D_{dR}(F_v, M_\lambda, M_\lambda)/D^0_{dR}(F_v, M_\lambda, M_\lambda).$$

Let us recall how to define the big Galois representation $\mathbb{T}$ over $\Lambda := \Lambda(G)$ associated to motives as treated in [**FK06**] and, in the process, on the way define the analog for families.

Recall that the Λ-adic representation T for a single motive M only depends on the λ-adic realization of M. The first step in the process for single members of the family is to choose an $\mathcal{O}_\lambda$-lattice in M_p. It will be crucial in the following that this lattice is free. While such a lattice evidently exists, as $\mathcal{O}_\lambda$ is a PID, in the case of families the analog properties need a careful study of modules over Iwasawa-algebras:

PROPOSITION 2.10. *Let R be a commutative, Noetherian and integrally closed domain and M be a torsion-free finitely generated R-Module. By $M^\circ := Hom_R(M, R)$, we demote the linear dual. Then $M^{\circ\circ} = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$ is reflexive. The natural morphism $M \to M^{\circ\circ}$ is a pseudo isomorphism. Moreover, if R is regular of Krull dimension ≤ 2, then M is free if and only if M is reflexive.*

PROOF. See [**NSW08**] lemma 5.1.2 for the first part; the second statement is obvious, and the last assertion is proposition 5.1.9 (loc.cit.). □

COROLLARY 2.11. *Let $\mathbb{K} := \mathcal{Q}(\mathcal{O}[[t]])$ be the field of fractions and G be a profinite group. For every finite dimensional continuous representation $\rho : G \to Aut_{\mathbb{K}}(W)$, there is a free $\mathcal{O}[[t]]$ sublattice of W stable under ρ.*

Moreover, let $R \subset K$ be any ring containing $\mathcal{O}[[t]]$ such that p is invertible in R and t is not. Then starting with a continuous (free) representation V over R, the lattice T can be taken in V. Any such lattice has the property that for every specialization map $\phi : K_\lambda[[t]] \to K_\phi$ the submodule $T \otimes_{\mathcal{O}[[t]]} \mathcal{O}_\phi$ is a ρ-stable $\mathcal{O}_\phi$-lattice in the K_ϕ-vector space $V \otimes_{\mathcal{O}[[t]]} \mathcal{O}_\phi = V \otimes_{K_\lambda[[t]]} K_\phi$.

PROOF. By continuity, we have a ρ-stable finitely generated $\mathcal{O}[[t]]$-submodule T' in W such that T' generates W as a $\mathbb{K}$-vector space. We set $T := \bigcap_{ht(\mathfrak{p})=1} T'_{\mathfrak{p}}$. This is again finitely generated and still ρ-stable. As $T \supset T'$, it contains a $\mathbb{K}$-basis of W. The last proposition shows that T is reflexive and, hence, free.

In the second situation we use the first part, to choose a free ρ-stable $\mathcal{O}[[t]]$-lattice T' in $W = \mathbb{K} \otimes V$. Then, as T' is finitely generated, there is a nonzero r in $\mathcal{O}[[t]]$, such that $rT' \subset V$. We can take $T := rT'$.

As t is not invertible in R the specialization map induces $\phi : R \to K_\phi$. The The module $T \otimes_{\mathcal{O}[[t]]} \mathcal{O}_\phi$ is clearly ρ-stable and compact. Therefore is is a $\mathcal{O}$-lattice and hence free, because $\mathcal{O}$ is a principal ideal domain. □

We apply this corollary to choose a G_F-invariant free $\mathcal{O}_\lambda[[t]]$-lattice T in a given family of Galois representations V.

The next step is to find $\mathcal{O}[[t]]$-direct summands $T^0(v)$ of T corresponding under the specializations $\phi \in \Sigma$ to the tangent space, as in the Dabrowski-Panchishkin condition. Of course this is not possible in general; it is a special nearly ordinary condition (see chapter 5 for the corresponding deformation theory).

CONDITION 2.12. *For every place v of F dividing p, there is a free $G_v := Gal(\overline{F_v}/F_v)$ stable $\mathcal{O}[[t]]$-direct summand $T^0(v)$ of T such that its image in V_ϕ generates the K_λ-subspace $V^0_\phi(v)$ from the Dabrowski-Panchishkin condition.*

Now everything is ready to impose the constructions of Fukaya and Kato in [**FK06**] to get the *big* Galois representations of $G_{\mathbb{Q}}$ over $\Lambda := \Lambda(G)$: First fix a system $(F_n)_{n\in\mathbb{N}}$ of finite extensions F_n of F in F_∞ such that $\bigcup F_n = F_\infty$. Our big representation is defined to be $\mathbb{T} := \varprojlim g_*(f_n)_*(f_n)^*(T)$ as a pro-étale sheaf, where $g : Spec(F) \to Spec(\mathbb{Q})$ and $f_n : Spec(F_n) \to Spec(F)$ are the natural maps. (That is to say: The module is defined to be $\mathbb{T} := \mathbb{Z}[G_\mathbb{Q}] \otimes_{\mathbb{Z}[G_F]} (\Lambda \otimes_{\mathcal{O}_\lambda} T)$ as a $G_\mathbb{Q}$-module, and the structure as a Λ-module is the one on $\Lambda \otimes T$, which carries over to $\mathbb{T}$, as $\mathbb{Z}[G_\mathbb{Q}]$ is free over $\mathbb{Z}[G_k]$.)

We repeat this process for the module $\mathbb{T}^0$ with the corresponding local fields: Define T^0 to be the pro-étale sheaf on $Spec(F \otimes \mathbb{Q}_p)$ that is given on the points $Spec(F_v)$ of the scheme by $T^0(v)$. Let g_p and $f_{n,p}$ denote the induced maps of $\mathbb{Q}_p$ and $F_{n,p} = F_n \otimes \mathbb{Q}_p$. Completely analogously, we define $\mathbb{T}^0 := \mathbb{T}^0_\Lambda := \varprojlim g_p^*(f_{n,p})_*(f_{n,p})^*(T^0)$ as a pro-étale sheaf.

Remark 2.13. *In terms of Galois modules, this is induction after restriction. Thus, even if the constructions of* $\mathbb{T}$ *and* $\mathbb{T}^0$ *are done by maps of different spaces, all those maps are projections of different (finite) Galois coverings, where the Galois groups coincide. That implies that they are compatible, and we can view* $\mathbb{T}^0$ *as a sub-*$G_{\mathbb{Q}_p}$*-representation of the* $G_\mathbb{Q}$*-representation* $\mathbb{T}$.

This construction sends free $\mathcal{O}[[t]]$*-modules to free* Λ*-modules. Moreover it is an additive functor and thus sends direct sums to direct sums. In particular as* T^0 *is a* Λ*-direct summand of the pullback of* T *as a pro-étale sheaf of* $\mathcal{O}[[t]]$*-modules on* $Spec(F \otimes \mathbb{Q}_p)$ *the construction shows that* $\mathbb{T}^0$ *is* Λ*-direct summand of* $\mathbb{T}$.

In order to be able to apply the general machinery for big representations, there is only one thing missing: an isomorphism between $\mathbb{T}^0$ and $\mathbb{T}^+$. We are just stating the compatibility with specializations here:

Lemma 2.14. *Assume that* $p \neq 2$. *We have* $(\mathbb{T}_\phi)^+ = \Lambda_\phi \otimes_\Lambda (\mathbb{T}^+)$ *and the space* $(\mathbb{T}_phi)^0$ *induced from the* T^0_ϕ *is canonically isomorphic to* $\Lambda_\phi \otimes_\Lambda \mathbb{T}^0$ *we denote these spaces by* $\mathbb{T}^0_\phi$. *In particular, if* $\beta : \mathbb{T}^+ \to \mathbb{T}^0$ *is an isomorphism of* Λ*-modules, then* β *induces isomorphisms of the* Λ_ϕ *modules* $(\mathbb{T}_\phi)^+$ *and* $(\mathbb{T}_\phi)^0$ *for every* $\phi \in \Sigma$. *Conversely, assume that there is one* $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ *in* Σ *such that* $(\mathbb{T}_\phi)^+$ *is isomorphic to* $(\mathbb{T}_\phi)^0$ *as a* Λ_ϕ*-module. Then* $\mathbb{T}^0$ *and* $\mathbb{T}^+$ *are isomorphic. This will be the case if* M_ϕ *is critical; compare property C1 in paragraph 4.2 in* [**FK06**].

Proof. The first assertion follows from the fact that the 1 eigenspace of the complex conjugation (or any involution) acting linear on a module over a ring where 2 is invertible is automatically a direct summand and thus $(\mathbb{T}_\phi)^+ = \Lambda_\phi \otimes_\Lambda (\mathbb{T}^+)$. The analogous assertion on $(\mathbb{T}_\phi)^0$ and $\Lambda_\phi \otimes_\Lambda \mathbb{T}^0$ follows directly from the construction. The first statement on β is then obvious.

For the converse part, we first demonstrate the assertion for the case that ϕ is surjective. The kernel of the surjective map $\Lambda \to \Lambda_\phi$ is contained in the radical. Thus we may apply lemma 1.4 to conclude the freeness of $\mathbb{T}^+$ from the freeness of $\mathbb{T}^+_\phi$. The assertion is therefore reduced to counting the ranks, which can be done in any specialization. For a general ϕ by an application of lemma 1.5 it suffices to show that $\mathcal{O}' \otimes_\mathcal{O} \mathbb{T}$ is free. We can extend ϕ naturally to specialization map $\phi : \mathcal{O}'[[t]] \to \mathcal{O}'$, which is clearly surjective, thus this is covered by the last case.

The last part is lemma 4.2.8. of [**FK06**] (and can be reduced to dimension counting, too). □

So, under the condition 2.12 and assuming again that $p \neq 2$, we find that $\mathbb{T}^+$ and $\mathbb{T}^0$ are isomorphic and can choose an isomorphism.

2.3. The Selmer complexes of motives and families

The Selmer complexes associated to motives and families of motives are simply the ones associated to the representations introduced in the last section. However, in this induced situation we can make some more precise statements. In particular, we will prove some good properties of the Selmer complex, that are only true in the induced situation.

First we will need another condition making $SC(\mathbb{T}, \mathbb{T}^0)$ behave well under specializations: Let R be either $\mathcal{O}$, the ring of integers of a p-adic field, or $\mathcal{O}[[t]]$, the ring of power series over $\mathcal{O}$. Moreover, let T be a free finitely generated R-module equipped with a continuous R-linear G_F action for a number field F. Assume as always that this action is unramified outside a finite set. Finally, let $G = Gal(F_\infty/F)$ be a p-adic Lie group and $\Lambda = R[\![G]\!]$. Intending to study the $G_{\mathbb{Q}}$-module $\Lambda \otimes_R T$, we look at the following condition:

CONDITION 2.15. *For every finite place v of F not lying over p, if the ramification index of T is divisible by p^∞, then the ramification index of G at v is also infinite.*

REMARK 2.16.

- *If the operation of G_F on T factors through a finite extension of G, then the condition is trivially satisfied.*
- *If the condition is not satisfied, it is possible to make G bigger, so that the condition is fulfilled: Let S be the finite set where T is ramified, then the false Tate extension F_{ft}/F, where we take all p^n-th roots of one element of F contained in all members of S but not in any of their squares, is a p-adic Lie extension of dimension 2. Accordingly, the composition $F_\infty F_{ft}/F$ is a p-adic Lie extension with G as a quotient of relative dimension at most 2. In most cases, we will assume that the cyclotomic $\mathbb{Z}_p$-extension is contained in F_∞/F, so that our new group has relative dimension at most 1.*

Let us denote by $()_p$ the localization at the multiplicative set consisting of the powers of p.

PROPOSITION 2.17. *Let v be a finite place of F not lying over p and $I_v \subset G_F$ be the inertia group. We assume that $R = \mathcal{O}$ or $R = \mathcal{O}[[t]]$ and the condition 2.15 is satisfied. Then, we have:*

(1) *The module $(\Lambda \otimes_R T)^{I_v}$ is trivial, if the ramification index of G in v is infinite.*
(2) *The Λ-module $((\Lambda \otimes_R T)^{I_v})_p = ((\Lambda \otimes_R T)_p)^{I_v}$ is finitely generated and projective. If we assume additionally that G does not have any p-torsion, then $(\Lambda \otimes_R T)^{I_v}$ is projective.*
(3) *Now let $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ be a specialization morphism and denote $\Lambda = \mathcal{O}[\![G]\!][[t]]$ and $\Lambda' = \mathcal{O}'[\![G]\!]$. Then, we have an isomorphism*
$$\Lambda' \otimes^L_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}_p \xrightarrow{\cong} (\Lambda' \otimes_{\mathcal{O}[[t]]} T)^{I_v}_p.$$
Moreover, if G does not have any p-torsion or $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ is surjective, then $\Lambda' \otimes^L_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v} = \Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}$ in the derived category and the above morphism is induced by the natural inclusion
$$\Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v} \hookrightarrow (\Lambda' \otimes_{\mathcal{O}[[t]]} T)^{I_v}.$$

PROOF. The first assertion with $R = \mathcal{O}$ is the contend of proposition 4.2.14(3) in [**FK06**]. For the proof, no particular properties of $\mathcal{O}$ were used, so it carries over literally to the case of $R = \mathcal{O}[[t]]$, we reproduce it here for completeness: By the orthogonality statement of the local duality 1.23, the module $(\Lambda \otimes_R T)^{I_v}$ is the Pontryagin dual of $H^1(F^{ur}, (\Lambda \otimes_R T)^\vee)$. As an étale cohomology group this is isomorphic to $H^1(F_\infty \otimes_F F_v^{ur}, T^\vee(1))$ by Shapiro's lemma. The last cohomology group is a direct sum over groups isomorphic to $H^1(F_\infty F_v^{ur}, T^\vee(1))$, so that it is enough to show that the p-cohomological dimension of the composition $F_\infty F_v^{ur}$ is zero or, equivalently, that the p-Sylow group vanishes. But that is obvious, since the p-Sylow group of $G_{F_\infty F_v^{ur}}$ is a subgroup of $\mathbb{Z}_p$, the Sylow group of $G_{F_v^{ur}}$, such that the factor group is finite unless the p-Sylow subgroup is trivial.

To prove the second statement, we can restrict ourselves to the case where G is of finite ramification index at v, as the case of infinite ramification is covered by the first part. Let $G = \varprojlim G_i$ with finite groups G_i and set $\Lambda_i := R[G_i]$. Then, $\Lambda = \varprojlim \Lambda_i$, and, as T is finitely generated over R, we get $\Lambda \otimes_R T = \varprojlim \Lambda_i \otimes T$. If we define J_v to be the kernel of $I_v \to G$, then we find that $(\Lambda_i \otimes_R T)^{I_v} = (\Lambda_i \otimes_R T^{J_v})^{I_v/J_v}$, as the Λ_i are finitely generated free R modules. From this, we conclude that $(\Lambda \otimes_R T)^{I_v} = (\Lambda \otimes_R T^{J_v})^{I_v/J_v}$. It suffices now to show that T^{J_v} is free, because then $\Lambda \otimes_R T^{J_v}$ is free and, as the order of the finite group I_v/J_v is invertible, in both situation it follows that $(\Lambda \otimes_R T^{J_v})^{I_v/J_v}$ is projective as a direct summand of a free module.

Now we observe that T^{J_v} is torsion-free and it is therefore free if $R = \mathcal{O}$. If $R = \mathcal{O}[[t]]$, then, using our condition, J_v operates through a finite quotient and therefore we can apply corollary 1.7 to see that it is free.

For the third assertion, we remark that if G does not have any p-torsion (respectively we invert p) then by the second part $(\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}$ (respectively $(\Lambda \otimes_{\mathcal{O}[[t]]} T)_p^{I_v}$) is projective, so that $\Lambda' \otimes_\Lambda^L (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v} \cong \Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}$. If ϕ is surjective, the higher Tor groups vanish, too, as $(\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}$ is still torsion-free and ϕ means nothing else than dividing out its kernel, which is a principal ideal, so that we get the same isomorphism.

Applying corollary 1.2 to the I_v/J_v-module $(\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v}$, we get an injective morphism $\Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{I_v} \hookrightarrow (\Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}[[t]]} T)^{J_v})^{I_v/J_v}$, the cokernel of which is annihilated by some power of p. This is clearly an isomorphism if we invert p.

Finally, we apply the same corollary 1.2 to the J_v module $\Lambda \otimes_{\mathcal{O}[[t]]} T$, to arrive at the assertion.

□

REMARK 2.18. *The third part of the proposition is not as strong as one might have hoped: It would simplify matters considerably if the inclusion* $\Lambda' \otimes_\Lambda (\Lambda \otimes_{\mathcal{O}_\lambda[[t]]} T)^{I_v} \hookrightarrow (\Lambda' \otimes_{\mathcal{O}_\lambda[[t]]} T)^{I_v}$ *was actually an isomorphism. However, it follows directly from the proof that this is the case if we strengthen our condition on the ramification 2.15. We have to assume the following for all finite places* v *of* F *not dividing* p*: If the ramification index of* v *in* G *is finite, then the ramification index of* v *in the representation* T *is not divisible by* p*.*

Of course, this stronger condition can be fulfilled, for example, by requiring the Galois extension corresponding to G *to contain the false Tate extension, which is infinitely ramified at all (finitely many) ramified places of the Galois representation* T*.*

The proposition leads immediately to the following:

COROLLARY 2.19. *Let $\mathbb{T}$ and $\mathbb{T}^0$ be defined as in the last section 2.2 (for a family or a motive), and assume that $p \neq 2$ and that G does not have any p-torsion. If the condition 2.15 is satisfied, then $SC(\mathbb{T}, \mathbb{T}^0)$ is perfect. If, moreover, $[\mathbb{T}^0] = [\mathbb{T}^+]$ in $K_0(\Lambda)$, then $[SC(\mathbb{T}, \mathbb{T}^0)] = 0$ in $K_0(\Lambda)$.*

PROOF. According to lemma 2.5, it is sufficient to prove the claimed properties for $SC(U, \mathbb{T}, \mathbb{T}^0)$ and the $C_f(F_v, \Lambda \otimes_R T)$ instead of $SC(\mathbb{T}, \mathbb{T}^0)$. The assertions for $SC(U, \mathbb{T}, \mathbb{T}^0)$ are shown in proposition 2.7. The properties of $C_f(F_v, \Lambda \otimes_R T)$ follow from the last proposition together with the quasi-isomorphism $C_f(F_v, \Lambda \otimes_R T) \cong [1 - \phi : (\Lambda \otimes_R T)^{I_v} \to (\Lambda \otimes_R T)^{I_v}]$ from lemma 1.19. □

2.4. The relation to Selmer groups

While Selmer complexes come more naturally to our approaches, all classical theorems and conjectures use Selmer groups instead. Selmer groups and complexes are connected by an exact sequence, as we will see below. This sequence for the primitive objects is described in [**FK06**].

As mentioned above, we are using Greenberg's conditions (see [**Gre89**]):

DEFINITION 2.20. *Let F be a number field and $(T, (T^0(v))_{v|p})$ be a pair arising form the λ-adic realization of a motive satisfying the Dabrowski-Panchishkin condition. Moreover, let F_∞/F be a p-adic Lie extension. Then, we define the primitive Selmer group $Sel(T^\vee(1), F_\infty)$ to be the kernel of*

$$H^1(\mathbb{Q}, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1)) \oplus \bigoplus_{l \neq p} H^1(\mathbb{Q}_l^{ur}, \mathbb{T}^\vee(1)) \oplus H^1(\mathbb{R}, \mathbb{T}^\vee(1)),$$

where l ranges over all prime numbers apart from p.

For some open set $U \subset spec(\mathbb{Z})$ not containing p, the imprimitive Selmer group $Sel_U(T^\vee(1), F_\infty)$ is defined to be the kernel of

$$H^1(\mathbb{Q}, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1)) \oplus \bigoplus_{l \in U} H^1(\mathbb{Q}_l^{ur}, \mathbb{T}^\vee(1)) \oplus H^1(\mathbb{R}, \mathbb{T}^\vee(1)).$$

REMARK 2.21. *Classically, one would define the Selmer groups over F_∞ to be the direct limits over Selmer groups over finite subextensions $F_\infty \supset K \supset F$. The primitive Selmer group over K, for instance, would be the kernel of*

$$H^1(K, T^\vee(1)) \to \bigoplus_{v \nmid p\infty} H^1(K_v^{ur}, T^\vee(1)) \oplus \bigoplus_{v|p} H^1(K_v, (T^0(v))^\vee(1)) \oplus \bigoplus_{v|\infty} H^1(K_v, T).$$

The two definitions describe two naturally isomorphic modules by some standard application of Shapiro's lemma.

It is very common to define the Selmer group, as we did, namely as a subgroup of the Galois cohomology of the absolute Galois group of $\mathbb{Q}$ respectively F_∞. But the Selmer complexes are more naturally related to the cohomology of the Galois group with restricted ramification. This is yet again a different angle from which to look at the Selmer groups, and we note the following standard fact:

Lemma 2.22. *Let $U \subset Spec(\mathbb{Z})$ be an open subset, in which the representation $\mathbb{T}$ is unramified. Then, the primitive Selmer group $Sel(T^\vee(1), F_\infty)$ is the kernel of*

$$H^1(U, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1)) \oplus \bigoplus_{l \notin U \cup \{p\}} H^1(\mathbb{Q}_l^{ur}, \mathbb{T}^\vee(1)) \oplus H^1(\mathbb{R}, \mathbb{T}^\vee(1)).$$

Likewise, the imprimitive Selmer $Sel_U(T^\vee(1), F_\infty)$ group is the kernel of

$$H^1(U, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1)) \oplus H^1(\mathbb{R}, \mathbb{T}^\vee(1)).$$

Proof. The inflation map $H^1(U, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}, \mathbb{T}^\vee(1))$ is injective, and, as a subset, $H^1(U, \mathbb{T}^\vee(1))$ is given as the kernel of the restriction map $H^1(\mathbb{Q}, \mathbb{T}^\vee(1)) \to H^1(\mathbb{Q}_U, \mathbb{T}^\vee(1))$, where $\mathbb{Q}_U$ is the maximal in U unramified extension of $\mathbb{Q}$. Now let $H \subset G_\mathbb{Q}$ be the absolute Galois group of $\mathbb{Q}_U$. For $l \in U$, the restriction map to $H^1(\mathbb{Q}_l^{ur}, \mathbb{T}^\vee(1))$ clearly factors over $H^1(H, \mathbb{T}^\vee(1))$. As in these cases the group operation is always trivial, the H^1-groups actually are groups of homomorphism. Therefore, all we have to show is that the local inertia groups I_l with $l \in U$ generate H as a normal subgroup of $G_\mathbb{Q}$. In arithmetic terms this translates as: $\mathbb{Q}_U$ does not have any nontrivial extension which is unramified for all $l \in U$. This is the definition of $\mathbb{Q}_U$. □

By $\mathcal{X}(\mathbb{T}, \mathbb{T}^0)$ and $\mathcal{X}(U, \mathbb{T}, \mathbb{T}^0)$ we denote the Pontryagin dual of the primitive respectively the imprimitive Selmer group. Then Selmer group and complex are connected by the following:

Proposition 2.23. *Recall that $G = Gal(F_\infty/F)$ is our chosen Lie group. Let $\mathcal{G}$ be the kernel of $G_F \to G$. Fixing an embedding $\overline{F} \to \overline{F_v}$ for every place v of F, let $\mathcal{G}(v)$ be the kernel of $G_{F_v} \to G$ and G_v be its image. Then, we have two exact sequences of Λ-modules:*

$$0 \to \mathcal{X}(\mathbb{T}, \mathbb{T}^0) \to H^2(SC(\mathbb{T}, \mathbb{T}^0)) \to \bigoplus_{v|p} \Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1))_{\mathcal{G}(v)}$$
$$\to T(-1)_\mathcal{G} \to H^3(SC(\mathbb{T}, \mathbb{T}^0)) \to 0$$

and

$$0 \to \mathcal{X}(U, \mathbb{T}, \mathbb{T}^0) \to H^2(SC(U, \mathbb{T}, \mathbb{T}^0)) \to \bigoplus_{v|p} \Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1))_{\mathcal{G}(v)}$$
$$\to T(-1)_\mathcal{G} \to H^3(SC(U, \mathbb{T}, \mathbb{T}^0)) \to 0$$

Proof. The first sequence is the main part of proposition 4.2.35 of [**FK06**]. With the additional work already done, the proof of the second sequence follows analogously. We give it here for reasons of completeness: We take the long exact cohomology sequences that comes from the distinguished triangles in lemma 2.4. Take the first one, for instance:

$$H^1(\mathbb{Q}_p, \mathbb{T}^0) \to H^2_{(c)}(U, \mathbb{T}) \to H^2(SC(U, \mathbb{T}, \mathbb{T}^0)) \to H^2(\mathbb{Q}_p, \mathbb{T}^0) \to H^3_{(c)}(U, \mathbb{T})$$
$$\to H^3(SC(U, \mathbb{T}, \mathbb{T}^0)) \to 0$$

This sequence will prove the second part of our proposition once we have shown that:

- $\mathcal{X}(U, \mathbb{T}, \mathbb{T}^0) = coker(H^1(\mathbb{Q}_p, \mathbb{T}^0) \to H^2_{(c)}(U, \mathbb{T}))$
- $\bigoplus_{v|p} \Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1))_{\mathcal{G}(v)} = H^2(\mathbb{Q}_p, \mathbb{T}^0)$ and

- $T(-1)_{\mathcal{G}} = H^3_{(c)}(U, \mathbb{T})$

The other long exact sequence from lemma 2.4 yields the first assertion once we have shown in addition that:

$$\mathcal{X}(\mathbb{T}, \mathbb{T}^0) = coker(H^1(\mathbb{Q}_p, \mathbb{T}^0) \oplus \bigoplus_{l \notin U \cup \{p\}} H^1_f(\mathbb{Q}_l, \mathbb{T}) \to H^2_{(c)}(U, \mathbb{T}))$$

For the first part, we note that $H^1(\mathbb{Q}_p, \mathbb{T}^0)^\vee = H^1(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1))$ by local duality (see proposition 1.23) and $H^2_{(c)}(U, \mathbb{T})^\vee = H^1(U, \mathbb{T}^\vee(1))$ by global duality (proposition 1.24). Therefore according to lemma 2.22 the first assertion holds. The similar assertion in the primitive case needs as an extra input that the orthogonal complement of $H^1_f(\mathbb{Q}_l, \mathbb{T})$ under the local duality is $H^1_f(\mathbb{Q}_l, \mathbb{T}^\vee(1))$. Thus,

$$H^1_f(\mathbb{Q}_l, \mathbb{T})^\vee = H^1(\mathbb{Q}_l, \mathbb{T}^\vee(1))/H^1_f(\mathbb{Q}_l, \mathbb{T}^\vee(1)) = H^1(\mathbb{Q}_l^{ur}, \mathbb{T}),$$

reducing again to the characterization of the Selmer group from lemma 2.22.

To show the second equality, we compute

$$H^2(\mathbb{Q}_p, \mathbb{T}^0) \cong H^0(\mathbb{Q}_p, (\mathbb{T}^0)^\vee(1))^\vee \cong \bigoplus_{v|p} H^0(F_v \otimes F_\infty, (T^0(v))^\vee(1))^\vee,$$

where the first equality is local duality and the second one is Shapiro's lemma and Mackey decomposition. The summands in the rightmost term can easily be shown to equal $\Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1))_{\mathcal{G}(v)}$ as required.

The third statement follows by the observation

$$H^2_{(c)}(U, \mathbb{T}) \cong H^0(U, \mathbb{T}^\vee(1))^\vee \cong H^0(\mathcal{G}, T(-1))^\vee,$$

where the first equality is global duality and the second one is Shapiro's lemma. Lastly, the Pontryagin dual of the invariants are the coinvariants of the Pontryagin dual as required.

□

CHAPTER 3

Variation of Selmer complexes

This chapter will discuss how properties of the families and the specializations are related. In a few cases, we will also relate properties of different specializations directly. In particular, we will investigate how the Iwasawa invariants behave in families. Many special instances of this behavior have been treated directly. Compare, for instance, the article of Emerton, Pollack, and Weston [**EPW06**] on the cyclotomic case or Aribam's PhD thesis [**Sha09**], on the case of a false Tate extension.

3.1. Specialization of the Selmer complex

In the following we would like to analyze the relation between the objects of the big representation and the specializations.

The case of the imprimitive complex is very easy:

PROPOSITION 3.1. *Let Λ and Λ' be adic rings, let $U \subset Spec(\mathbb{Z})$ and T, T^0 be a Galois representation over Λ, unramified in U. Furthermore, let Y be a finitely generated projective (left) Λ'-module endowed with a compatible right Λ-action. If we then set $T' := Y \otimes_\Lambda T$ and $(T')^0 := Y \otimes_\Lambda T^0$, we have a canonical isomorphism:*

$$Y \otimes_\Lambda^L SC(U, T, T^0) \xrightarrow{\cong} SC(U, T', (T')^0)$$

In particular, if $\mathcal{O} \subset \Lambda$ is a ring of integers of a p-adic field and if $\mathcal{O}'$ is a finite extension of $\mathcal{O}$, then we have a canonical isomorphism:

$$\mathcal{O}' \otimes_{\mathcal{O}} SC(U, T, T^0) \xrightarrow{\cong} SC(U, \mathcal{O}' \otimes_{\mathcal{O}} T, \mathcal{O}' \otimes_{\mathcal{O}} T^0)$$

PROOF. The main assertion is remark 4.1.4 in [**FK06**] (with the small typing error of using $\otimes$ instead of $\otimes^L$). One can easily deduce the base change property from the analog statement for the cohomology groups stated in proposition 1.22. The finiteness assumption of this proposition is satisfied in the relevant cases, by other assertions in the same section.

In the special case of a scalar extension, we are allowed to drop the L from the tensor product, as $\mathcal{O}'$ is projective as a right $\mathcal{O}$-module, too. □

Obviously, we are more interested in the analog statement for the primitive Selmer complex. Unfortunately, the result is not quite as strong, but it will be sufficient for what follows.

THEOREM 3.2. *We assume that $\Lambda = \mathcal{O}[\![G]\!][[t]]$ and $\Lambda_\phi = \mathcal{O}_\phi[\![G]\!]$ and that $\phi : \mathcal{O} \to \mathcal{O}_\phi$ is a specialization map as in the last chapter. Let $(\mathbb{T}, \mathbb{T}^0)$ be a pair of representations associated to a family such that the condition 2.15 on the ramification is satisfied. We then have an exact triangle*

$$\Lambda_\phi \otimes_\Lambda^L SC(\mathbb{T}, \mathbb{T}^0) \to SC(\mathbb{T}_\phi, \mathbb{T}_\phi^0) \to C \xrightarrow{+},$$

with C being the mapping fiber of a map of complexes $C' \to C'[-1]$, where C' is a perfect complex, the cohomology groups of which are annihilated by some power of p.

Furthermore, if $\mathcal{O}'$ is a finite extension of $\mathcal{O}$, which is a maximal order, and $\mathcal{O}'_\phi$ is a finite extension of $\mathcal{O}_\phi$, we then have two canonical isomorphisms:

$$\mathcal{O}' \otimes_{\mathcal{O}} SC(\mathbb{T}, \mathbb{T}^0) \xrightarrow{\cong} SC(\mathcal{O}' \otimes_{\mathcal{O}} \mathbb{T}, \mathcal{O}' \otimes_{\mathcal{O}} \mathbb{T}^0)$$

and

$$\mathcal{O}'_\phi \otimes_{\mathcal{O}_\phi} SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi) \xrightarrow{\cong} SC(\mathcal{O}'_\phi \otimes_{\mathcal{O}_\phi} \mathbb{T}_\phi, \mathcal{O}'_\phi \otimes_{\mathcal{O}_\phi} \mathbb{T}^0_\phi)$$

PROOF. Using the exact triangle from lemma 2.5, connecting the primitive and the imprimitive Selmer complex we can reduce the assertions to the last proposition. However, to do this, we have to show the related base change properties of $C_f(F_v, \Lambda \otimes T)$. Since the complex $C_f(F_v, \Lambda \otimes T)$ is quasi-isomorphic to the complex $(\Lambda \otimes T)^{I_v} \to (\Lambda \otimes T)^{I_v}$ in degrees 0 and 1, as we have already seen, the first claim follows from the third part of proposition 2.17. The analog of the second claim is obviously true for C_f. □

REMARK 3.3. *One could replace the ramification condition 2.15 by the stronger assumption that the Galois extension F_∞/F is infinitely ramified at all those finite places at which the ramification index of T is divisible by p. Under this condition, the morphism $\Lambda_\phi \otimes^L_\Lambda SC(\mathbb{T}, \mathbb{T}^0) \to SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ is actually an isomorphism in the derived category. This follows directly from the remark after proposition 2.17.*

Let us note the special case of surjective specializations:

COROLLARY 3.4. *We keep the assumption of the last propositions and assume in addition that $\phi \in \Sigma$ is a surjective specialization map. The kernel of ϕ is a principal ideal denoted by (f). We then have a canonical distinguished triangle:*

$$SC(U, \mathbb{T}, \mathbb{T}^0) \xrightarrow{f\cdot} SC(U, \mathbb{T}, \mathbb{T}^0) \to SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi) \xrightarrow{+}$$

If, moreover, the condition 2.15 is satisfied, we also have a canonical triangle:

$$SC(\mathbb{T}, \mathbb{T}^0)_p \xrightarrow{f\cdot} SC(\mathbb{T}, \mathbb{T}^0)_p \to SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)_p \xrightarrow{+}$$

In this sequence $()_p$ again denotes the localization with respect to the multiplicative set $\{1, p, p^2, ...\}$. In particular, there is an exact sequence of Λ_ϕ-modules for every integer i:

$$0 \to H^i(SC(U, \mathbb{T}, \mathbb{T}^0))/f \to H^i(SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi) \to H^{i+1}(SC(U, \mathbb{T}, \mathbb{T}^0))[f] \to 0$$

Likewise, under the conditions for the second triangle we have exact sequences:

$$0 \to H^i(SC(\mathbb{T}, \mathbb{T}^0)_p)/f \to H^i(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)_p) \to H^{i+1}(SC(\mathbb{T}, \mathbb{T}^0))_p)[f] \to 0$$

PROOF. According to proposition 2.7 and corollary 2.19 all Selmer complexes in question are perfect. It follows that we have a distinguished triangle

$$SC(U, \mathbb{T}, \mathbb{T}^0) \xrightarrow{f\cdot} SC(U, \mathbb{T}, \mathbb{T}^0) \to \Lambda/f \otimes_\Lambda SC(U, \mathbb{T}, \mathbb{T}^0) \xrightarrow{+},$$

and after inverting p the same is true without U. On the other hand by proposition 3.1 there is a canonical isomorphism

$$\Lambda/f \otimes_\Lambda SC(U, \mathbb{T}, \mathbb{T}^0) \cong SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$$

and again we get the same isomorphism without U after inverting p.

□

3.2. The canonical Ore sets S and S^*

Let us first recall the notion of a non-commutative Ore set:

A (left and right) Ore set in a ring Λ is a multiplicatively closed set S, such that, for all elements $r \in \Lambda$ and $s \in S$, there are $s', s'' \in S$ and $r', r'' \in \Lambda$ satisfying $rs' = r's$ and $s''r = r''s$; i.e., one can write left fractions as right fractions and vice versa. If we have a left and right Ore set, then left and right localizations exist and coincide. We intend to compare Ore sets and thus make the following definition:

Definition 3.1. Let S and S' be two multiplicative sets in Λ, then we say that S' is divisible by S if, for every $s \in S$, there are s' and s'' in S' and r' and r'' in Λ such that $sr' = s'$ and $r''s = s''$. If S is divisible by S' and S' is divisible by S, then we call them codivisible.

Remark 3.5.

- *Codivisibility is an equivalence relation.*
- *For codivisible Ore sets S and S', a Λ-module is S-torsion if and only if it is S'-torsion.*
- *The localizations on codivisible Ore sets coincide.*

We keep the general assumptions of the last sections and assume in addition that F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F and that G is a p-adic Lie group. As usual, we denote $H := Gal(F_\infty/F_{cyc}) \subset G$ and $\Gamma := Gal(F_{cyc}/F) = G/H \cong \mathbb{Z}_p$. The first part of this section is purely group-theoretic, so the assumption of G and H actually being Galois groups is not really needed here.

Under these assumptions, all characterizations of the set S introduced in [**CFKSV**] are equivalent and define a left and right Ore set. The most important ones are:

Definition 3.2. As before, let K_λ be a p-adic field with the ring of integers $\mathcal{O}$ and the residue field k. Let Λ be either the Iwasawa-algebra $\mathcal{O}[\![G]\!]$ or the ring of power series $\mathcal{O}[\![G]\!][[t]]$. In the former case, let $G' = G$ and $H' = H$, in the latter $G' = G \times \mathbb{Z}_p$ and $H' = H \times \mathbb{Z}_p$ such that $\Lambda = \mathcal{O}[\![G']\!]$. Then, an element $s \in \Lambda$ is in the subset S if and only if it satisfies the following equivalent conditions:

(1) There is an open pro-p subgroup U of H' which is normal in G' such that the image of s in $k[\![G/U]\!]$ is not a left zero divisor.
(2) For every open pro-p subgroup U of H' which is normal in G', the image of s in $k[\![G/U]\!]$ is not a left zero divisor.
(3) The Λ module $\Lambda/\Lambda s$ is finitely generated as a module over $\mathcal{O}[\![H']\!]$ for some H' as above.
(4) The Λ module $\Lambda/\Lambda s$ is finitely generated as a module over $\mathcal{O}[\![H']\!]$ for all H' as above.
(5) The right Λ module $s\Lambda\backslash\Lambda$ is finitely generated as a module over $\mathcal{O}[\![H']\!]$ for some H' as above.
(6) The right Λ module $s\Lambda\backslash\Lambda$ is finitely generated as a module over $\mathcal{O}[\![H']\!]$ for all H' as above.

Let $\pi \in \mathcal{O}$ be a uniformizer and $g \in \mathcal{O}[[t]]$ be an arbitrary nonzero element. We define

$$S^* := \bigcup_{n\geq 0} \pi^n S \qquad \text{and}$$
$$\mathbb{S}_g^* := \bigcup_{n,k\geq 0} \pi^n g^k S.$$

REMARK 3.6.

- *The definitions of S^* and $\mathbb{S}_g^*$ do not depend on the choice of π as two such choices differ by some factor in $\mathcal{O}^\times$ and units are elements of S.*
- *The paragraph 2 and parts of the paragraphs 3 and 4 of* [**CFKSV**] *contain a careful study of these denominator sets. This applies to the S and S^* of the families because we view them as the Ore sets of the group $G \times \mathbb{Z}_p$.*
- *The set S^* defined here is somewhat bigger than $\bigcup_{n\geq 0} p^n S$, the one used in* [**CFKSV**]. *However, it is clear that they are codivisible.*
- *A Λ-module M is S^*-torsion if and only if $M/M(p)$ is S-torsion.*
- *It may happen that some module is S^*-torsion after specialization (i.e., after tensoring with Λ_ϕ), but the module itself is not S^*-torsion. That is why we defined the extended set $\mathbb{S}_g^*$. This does not solve the problem completely, but the author is not aware of an Ore set extending the class of torsion modules substantially while still specializing to a set codivisible with S^*.*

Of course we would not study these sets if we did not have the following theorem:

THEOREM 3.7. *The sets S, S^*, and $\mathbb{S}_g^*$ of the last definition are (left and right) Ore sets in Λ.*

PROOF. The statement for S is theorem 2.4 in [**CFKSV**]. The generalizations from S to S^* and $\mathbb{S}_g^*$ are obvious as we extend the multiplicative set by central elements. □

Let us first remark that these sets behave well under scalar extensions:

LEMMA 3.8. *Let G and H be groups as in the definition of S. Let $\mathcal{O}'/\mathcal{O}$ be an extension of rings of integers of p-adic fields. By S and S' we denote the denominator sets in $\Lambda := \mathcal{O}[\![G]\!]$ and $\Lambda' := \mathcal{O}'[\![G]\!]$. Then S and S' are codivisible as subsets of Λ'. The same statement holds for the accoding sets $S*$ and $(S*)'$. And, if $g \in \mathcal{O}[[t]]$ is a non-zero element and $\mathbb{S}_g^*$ and $(\mathbb{S}_g^*)'$ are the multiplicative sets in $\mathcal{O}[\![G]\!][[t]]$ and $\mathcal{O}'[\![G]\!][[t]]$, respectively, then they are codivisible.*

PROOF. Let us first proof the assertion on S. According to the characterizations of S, it is the preimage of the corresponding set in $\mathcal{O}[\![G/U]\!]$ when U is a normal pro-p subgroup of G contained in H. We may thus assume H to be finite. Moreover, G contains an open subgroup isomorphic to $\mathbb{Z}_p$. Therefore, Λ is a finite extension of a ring isomorphic to $\mathcal{O}[\![\mathbb{Z}_p]\!] = \mathcal{O}[[t]]$ and any element of this subring not divisible by a uniformizer π of $\mathcal{O}$ is an element of S by Weierstrass preparation theorem. As S' is clearly divisible by S, it is enough to show that every element of S' divides some power series in this subring, the coefficients of which are not divisible by π. For any $s' \in S'$, the kernel I of the canonical map $\mathcal{O}[[t]] \to \Lambda'/\Lambda' s'$

should contain such a power series to give the right divisibility. But using the fourth of the equivalent characterizations of S', we see that $\Lambda'/\Lambda's'$ is a finitely generated $\mathcal{O}'[\![H]\!]$-module; thus, it is finitely generated as an $\mathcal{O}$-module. The quotient $\mathcal{O}[[t]]/I$ is therefore generated by finitely many monomials as an $\mathcal{O}$-module. It follows that it is possible to write one (bigger) monomial as a finite sum of smaller ones in the quotient giving the desired prower series (in fact: polynomial) in I. The left divisor property follows completely analogously.

The assertions on the other sets follow directly from the one on S. □

Now we will study how these sets behave under specialization. The sets are all compatible with each other with the small differences coming purely from non-trivial scalar extensions.

Proposition 3.9. *Let $\mathcal{O}'/\mathcal{O}$ be an extension of rings of integers of p-adic fields. Let $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ be a continuous morphism and G a compact p-adic Lie group with a closed normal subgroup H, as in the definition of the denominator sets. Moreover, let g be a distinguished polynomial in $\mathcal{O}[[t]]$ prime to the kernel (f) of ϕ. If we set $\Lambda := \mathcal{O}[\![G]\!][[t]]$ and $\Lambda_\phi := \mathcal{O}'[\![G]\!]$, then ϕ induces a map: $\Lambda \to \Lambda_\phi$. By S and $\mathbb{S}_g^*$ we denote the denominator sets of Λ and by S_ϕ and S_ϕ^* the ones of Λ_ϕ. Then the following holds:*

1. *The denominator sets are related by $S = \phi^{-1}(S_\phi)$. Moreover, $\phi(S)$ is an Ore set and S_ϕ and $\phi(S)$ are codivisible. In particular, if the extension of the residue fields of $\mathcal{O}'$ and $\mathcal{O}$ is trivial, then $\phi(S)$ and S_ϕ coincide.*
2. *We have $\phi(S^*) \subset \phi(\mathbb{S}_g^*) \subset S_\phi^*$, and all three sets are codivisible Ore sets. Moreover, if $\mathcal{O} = \mathcal{O}'$, then they coincide.*

For the proof we make use of the following elementary fact:

Lemma 3.10. *Let Γ be a profinite group and k'/k an extension of finite fields of characteristic p. An element $s \in k[\![\Gamma]\!]$ is a left (respectively, right) zero divisor in $k'[\![\Gamma]\!]$ if and only if it is one in $k[\![\Gamma]\!]$.*

Proof. The ring extension $k'[\![\Gamma]\!]/k[\![\Gamma]\!]$ is free, hence faithfully flat both as a left or a right module. Therefore, the property of the right (respectively, left) multiplication by s to be injective is preserved. □

Proof (of the proposition). For the first statement, let H' be any open pro-p subgroup of H which is normal in G. We have a commutative square of topological rings

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad\phi\quad} & \Lambda_\phi \\ \downarrow & & \downarrow \\ k[\![(G \times \mathbb{Z}_p)/(H' \times \mathbb{Z}_p)]\!] & \longrightarrow & k'[\![G/H']\!] \end{array},$$

where the lower horizontal arrow is induced by the identification $(G \times \mathbb{Z}_p)/(H' \times \mathbb{Z}_p) = G/H'$ and the inclusion $k \hookrightarrow k'$. All maps are continuous ring morphisms and the commutativity of the diagram, restricted to the subring $\mathcal{O}[\![G]\!]$ of $\Lambda = \mathcal{O}[\![G]\!][[t]]$ is obvious. Thus, all we have to check is the commutativity for t. But as ϕ is continuous, it maps t to an element of the maximal ideal of $\mathcal{O}'$, it follows that the image projects to 0 in k'. On the other hand, because we divide out the full extra $\mathbb{Z}_p$ factor from the group in the left vertical projection, t maps to zero.

Now, as the denominator sets S and S_ϕ are the preimages of the sets of nonzero divisors, we get $S = \phi^{-1}(S_\phi)$ from the lemma. As Λ_ϕ is an extension of $\phi(\Lambda)$ by central elements, it is easy to check that the images of Ore sets are still Ore sets. If now S'_ϕ is the analogously defined denominator set in the subring $\mathcal{O}[\![G]\!]$ of Λ_ϕ, then $S'_\phi \subset \phi(S) \subset S_\phi$. By lemma 3.8 we know that S'_ϕ and S_ϕ are codivisible, thus the same holds for S_ϕ and S proving our claims.

The second part follows from the first one once we handle the images of g and the chosen uniformizer in $\mathcal{O}$. But $S*_\phi$ contains all nonzero elements of $\mathcal{O}'$, so the inclusions $\phi(S^*) \subset \phi(\mathbb{S}^*_g) \subset S^*_\phi$ is proven. The codivisibility is obvious. □

We will need the following well-known proposition (compare for instance [**CFKSV**] proposition 2.3):

PROPOSITION 3.11. *Let M be a finitely generated $\mathcal{O}[\![G]\!]$-module, then M is S-torsion if and only if it is finitely generated over $\mathcal{O}[\![H]\!]$.*

From this proposition we deduce immediately:

THEOREM 3.12. *Let $C^\bullet$ be a perfect complex over $\Lambda := \mathcal{O}[\![G]\!][[t]]$ and $\phi : \mathcal{O}[[t]] \to \mathcal{O}_\phi$ be a specialization map. If the cohomology groups of $C^\bullet$ are S-torsion (respectively, $\mathbb{S}^*_g$-torsion for some g prime to $(f) = ker(\phi)$), then the cohomology groups of $\Lambda_\phi \otimes^L_\Lambda C^\bullet$ are S_ϕ-torsion (respectively, S^*_ϕ-torsion). Conversely, if the cohomology groups of $\Lambda_\phi \otimes^L_\Lambda C^\bullet$ are S-torsion the same is true for those of $C^\bullet$.*

PROOF. We replace $C^\bullet$ with a quasi-isomorphic bounded complex of projective modules and may thus replace $\otimes^L$ by $\otimes$. As $\mathcal{O}_\phi[\![G]\!][[t]]$ is free over $\mathcal{O}[\![G]\!][[t]]$, we have

$$H^i(\mathcal{O}_\phi[\![G]\!][[t]] \otimes_\Lambda C^\bullet) = \mathcal{O}_\phi[\![G]\!][[t]] \otimes_\Lambda H^i(C^\bullet).$$

Therefore, using that the corresponding denominator sets are codivisible (lemma 3.8), we can replace $C^\bullet$ by $\mathcal{O}_\phi[\![G]\!][[t]] \otimes_\Lambda C^\bullet$ and Λ by $\mathcal{O}_\phi[\![G]\!][[t]]$. It is thus enough to prove the theorem for the case, where ϕ is surjective.

The kernel of $\phi : \mathcal{O}[[t]] \to \mathcal{O}$ is generated by one element $f \neq 0$. As f is not a zero divisor and central in Λ, the multiplication with f is a injective morphism on projective Λ-modules. Thus, we have an exact sequence of complexes:

$$0 \to C^\bullet \xrightarrow{f\cdot} C^\bullet \to \Lambda_\phi \otimes_\Lambda C^\bullet \to 0$$

We take the long exact cohomology sequence and obtain a short exact sequence for every integer i:

$$0 \to H^i(C^\bullet)/f \to H^i(\Lambda_\phi \otimes_\Lambda C^\bullet) \to H^{i+1}(C^\bullet)[f] \to 0$$

Using proposition 3.9 we conclude that if all $H^i(C^\bullet)$ are S-torsion (resp. $\mathbb{S}^*_g$-torsion) then the $H^i(\Lambda_\phi \otimes^L_\Lambda C^\bullet)$ are S_ϕ-torsion (resp., S^*_ϕ-torsion).

In the other direction, if $H^i(\Lambda_\phi \otimes C^\bullet)$ is S_ϕ-torsion for some i, then $H^i(C^\bullet)/f$ is S-torsion. Using the characterization from proposition 3.11, the topological Nakayama lemma (see for instance lemma 5.2.18 in [**NSW08**]) shows that $H^i(C^\bullet)$ is S-torsion, thus proving the last assertion. □

We are most interested in the application of these compatibilities to Selmer complexes:

COROLLARY 3.13. *Let $SC(U,\mathbb{T},\mathbb{T}^0)$ be the Selmer complex over $\Lambda := \mathcal{O}[\![G]\!][[t]]$ of a family and $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ be a specialization map. If the cohomology groups of $SC(U,\mathbb{T},\mathbb{T}^0)$ are S-torsion (respectively, $\mathbb{S}_g^*$-torsion for some g prime to $(f) = ker(\phi)$), then the cohomology groups of $SC(U,\mathbb{T}_\phi,\mathbb{T}_\phi^0)$ are S_ϕ-torsion (respectively, S_ϕ^*-torsion). Conversely, if the cohomology groups of $SC(U,\mathbb{T}_\phi,\mathbb{T}_\phi^0)$ are S-torsion, then the same is true for those of $SC(U,\mathbb{T},\mathbb{T}^0)$.*

PROOF. We need only note that the Selmer complex is perfect, and by proposition 3.1 the tensor product can be computed as

$$\Lambda_\phi \otimes^L_\Lambda SC(U,\mathbb{T},\mathbb{T}^0) = SC(U,\mathbb{T}_\phi,\mathbb{T}_\phi^0).$$

□

As the base change property for $SC(\mathbb{T},\mathbb{T}^0)$ is not as strong as the one of $SC(U,\mathbb{T},\mathbb{T}^0)$, one should not expect the analog torsion properties for $SC(\mathbb{T},\mathbb{T}^0)$. We get the following:

COROLLARY 3.14. *In the situation of the last corollary, if the cohomology groups of $SC(\mathbb{T},\mathbb{T}^0)$ are $\mathbb{S}_g^*$-torsion for some g prime to $(f) = ker(\phi)$, then the cohomology groups of $SC(\mathbb{T},\mathbb{T}_\phi^0)$ are S_ϕ^*-torsion. Conversely, if the cohomology groups of $SC(\mathbb{T}_\phi,\mathbb{T}_\phi^0)$ are S-torsion, then the cohomology groups of $SC(\mathbb{T},\mathbb{T}^0)$ are S^*-torsion.*

PROOF. This follows easily from the theorem together with lemma 2.5. □

It is not clear how to ensure that the $H^i(SC(U,\mathbb{T},\mathbb{T}^0))$ are $\mathbb{S}_g^*$-torsion, as the S^*-torsion property for the specializations is not enough. The problem is illustrated by the following:

EXAMPLE 1. *In the situation where $G = \mathbb{Z}_p$ we know that $\Lambda := \mathbb{Z}_p[\![G]\!][[t]] = \mathbb{Z}_p[[x,t]]$ is the power series ring in two variables. Now let $f_1,...,f_k$ be any prime elements of the factorial ring $\mathbb{Z}_p[[t]]$ which are prime to p. Then $M := \Lambda/(p^n - f_1 \cdot ... \cdot f_k tx)\Lambda$ is a torsion Λ-module without $\mathbb{Z}_p[[t]]$- or S-torsion, but M/f_iM is annihilated by p^n for all $i = 1...k$.*

3.3. Basics on the algebraic Iwasawa invariants

The previous section has ended on a negative example for the S^*-torsion property of the family. We will study the variation of the $\mu-$ and $\lambda-$invariants in different specializations and show how to partially work around these problems in later sections. First, however, let us make some general remarks, and recall basic properties of these invariants:

LEMMA 3.15. *Let $\Lambda = \mathcal{O}[\![G]\!][[t]]$ or $\Lambda = \mathcal{O}[\![G]\!]$ and let π be a uniformizer of $\mathcal{O}$. For a finitely generated Λ-module M and a non-negative number n, the following properties are equivalent:*

1. *For every $m \in M$, there is an $s \in S$ such that $\pi^n sm = 0 \in M$.*
2. *There is a system of generators $m_1,...,m_k$ of M such that for every $i = 1,...,k$, there is an $s_i \in S$ with $\pi^n s_i m_i = 0$.*
3. *$M/M[\pi^n]$ is S-torsion.*

We call a module M with these properties $\pi^n S$-torsion.

PROOF. This is an easy computation using the the Ore set property of S and the fact that π is central. □

REMARK 3.16. *A finitely generated Λ-module is S^*-torsion if and only if it is $\pi^n S$-torsion for all sufficiently large n.*

Even though we do not necessarily get an S^*-torsion property for the family, it is still possible to get some variational results.

PROPOSITION 3.17. *Let $\Lambda = \mathcal{O}[\![G]\!][[t]]$ and let M be a finitely generated Λ-module. We denote the maximal ideal of $\mathcal{O}[[t]]$ by $\mathfrak{m}$ and a uniformizer of $\mathcal{O}$ by π. Let $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$ be a surjective specialization map with the kernel generated by a prime element $f \in \mathcal{O}[[t]]$. We assume in addition that $\mathcal{O}' \otimes_{\mathcal{O}[[t]]} M$ is $\pi^n S_\phi$-torsion. Then, for every specialization map $\psi : \mathcal{O}[[t]] \to \mathcal{O}''$ the kernel of which is generated by a prime element $g \in \mathcal{O}[[t]]$ with $f - g \in \pi^n \mathfrak{m}$, the module $\mathcal{O}'' \otimes_{\mathcal{O}[[t]]} M$ is $\pi^n S_\psi$-torsion.*

PROOF. First, we reduce the assertion to an analog for quotients of M: As the image of S in Λ_ϕ and S_ϕ are codivisible (proposition 3.9) and the analog statement holds for ψ, we can view all modules as Λ-modules and show that if $\mathcal{O}' \otimes_{\mathcal{O}[[t]]} M$ is $\pi^n S$-torsion, then $\mathcal{O}'' \otimes_{\mathcal{O}[[t]]} M$ is $\pi^n S$-torsion. Moreover, if M/gM is $\pi^n S$-torsion, then so is $\mathcal{O}'' \otimes M$. Thus, it suffices to show that if $M/fM = \mathcal{O}' \otimes_{\mathcal{O}[[t]]} M$ is $\pi^n S$-torsion, then so is M/gM.

Let f and g be as in the proposition and denote $f - g = \pi^n r$ with $r \in \mathfrak{m}$. Furthermore we set $\tilde{M} := (M/g)/(M/g)[\pi^n]$. It is enough to show that $\tilde{M}$ is S-torsion. But by proposition 3.11, a finitely generated module M is S-torsion if and only if it is finitely generated over $\mathcal{O}[\![H]\!][[t]]$. Therefore, using the topological Nakayama lemma (and the fact, that $\mathfrak{m}$ is contained in the radical of Λ), we conclude that $\tilde{M}$ is S-torsion if and only if $\tilde{M}/r$ is.

Now for an arbitrary element $m \in M$ there is an $s \in S$ and an $m' \in M$ such that $\pi^n sm = fm' = gm' + \pi^n rm'$; thus, we get $\pi^n(sm - rm') = gm'$. This in turn implies $sm = rm'$ in $\tilde{M}$ and so $sm = 0$ in $\tilde{M}/r$. Therefore, we have shown that $\tilde{M}/r$ is S-torsion, and by the former observations this implies the first assertion. □

With these preparations in place, we can finally turn to the invariants. From now on, we assume that the p-adic Lie group G does not have any p-torsion. For the assertions on the λ-invariant, we will always consider the following situation:

(*) The group G contains a closed normal subgroup H, such that $\Gamma = G/H$ is isomorphic to $\mathbb{Z}_p$. Moreover, the denominator sets S, S^* and $\mathbb{S}_g^*$ are defined with respect to this subgroup.

For reasons of simplicity, we set $\Lambda(G) := \mathcal{O}[\![G]\!]$ and $\Omega(G) := \mathbb{F}_q[\![G]\!]$, with $\mathbb{F}_q = \mathbb{F}_{p^k}$ being the residue field of $\mathcal{O}$. Both of these rings have finite global dimension, and for a finitely generated Λ-module M and a finitely generated $\Omega(G)$-module N we define:

$$rank_{\Lambda(G)}(M) := \sum_i (-1)^i rank_{\mathcal{O}}(H_i(G,M))$$
$$rank_{\Omega(G)}(N) := \sum_i (-1)^i dim_{\mathbb{F}_q}(H_i(G,N))$$
$$\mu_{\Lambda(G)}(M) := \sum_i (-1)^i log_q(ord(H_i(G,M(\pi)))) \quad = \sum_{i=1}^{\infty} rank_{\Omega(G)} M[\pi^i]/M[\pi^{i-1}]$$
$$\lambda_{\Lambda(G)}(M) := \sum_i (-1)^i rank_{\mathcal{O}}(H_i(H,M))$$

Here, for the λ-invariant we assume that we are in the situation (*) and that M is S-torsion (equivalently, M is finitely generated over $\Lambda(H)$). The right-hand sides are then always finite sums over finite numbers, so that the ranks are well defined.

Remark 3.18.

- *It is one of the main results of Susan Howson's article* [**How02**] *(namely theorem 1.1) that if the $\Lambda(G)$ does not contain any nontrivial zero divisors (equivalently, G has no torsion), then the rank as defined above coincides with the naive rank, namely the dimension of the localized module over the skew field of fraction which exists in this situation.*
- *These ranks are called homological ranks and are written $hmrank_{\Lambda(G)}$ etc. in Howson's article (loc.cit.). But as the two ranks are shown to be equal when both are defined, we do not need to distinguish between them.*
- *The λ-invariant depends on the choice of the subgroup $H \subset G$. If G is a Galois group, H should be thought of as the subgroup related to the cyclotomic $\mathbb{Z}_p$-extension.*

Before we prove first properties of these ranks, let us recall the following fact from homological algebra:

Proposition 3.19. *Assume that there is a commutative square of (not necessarily commutative) Noetherian rings of finite global dimension*

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

and M is a finitely generated R-module. Then, taking $[\bullet]$ to denote the class in $K_0(S')$, all objects in the following formula are well defined, the sums are finite, and equality holds:

$$\sum_{i,j} (-1)^{i+j} [Tor_i^S(S', Tor_j^R(S,M))] = \sum_{i,j} (-1)^{i+j} [Tor_i^{R'}(S', Tor_j^R(R',M))]$$

In particular, for $\Lambda := \mathcal{O}[\![G]\!][[t]]$ and $\phi : \mathcal{O}[[t]] \to \mathcal{O}_\phi$, a specialization map, we set again $\Lambda_\phi := \mathcal{O}_\phi[\![G]\!]$. If M is a finitely generated Λ-module, then for any invariant I on finitely generated $\mathcal{O}_\phi$ modules, which is additive on short exact sequences, we *have:*

$$\sum_{i,j}(-1)^{i+j}I(H_i(G,Tor_j^{\Lambda}(\Lambda_\phi,M))) = \sum_{i,j}(-1)^{i+j}I(Tor_i^{\mathcal{O}[[t]]}(\mathcal{O}_\phi,H_j(G,M)))$$

PROOF. The Tor groups are clearly finitely generated by the Noetherian hypothesis, and thus admit perfect resolutions by the assumption on the global dimension. It remains to be shown that the equality holds. This can be accomplished by comparing each side of the equality to $\sum_i(-1)^i[Tor_{i+j}^R(S',M)]$, using the two base-change spectral sequences for Tor (see for instance [**Wei94**] theorem 5.6.6):

$$Tor_i^S(S',Tor_j^R(S,M)) \Rightarrow Tor_{i+j}^R(S',M)$$

and

$$Tor_i^{R'}(S',Tor_j^R(R',M)) \Rightarrow Tor_{i+j}^R(S',M)$$

The second assertion follows by taking $R=\mathcal{O}[\![G]\!][[t]]$, $S=\Lambda_\phi$, $R'=\mathcal{O}[\![G]\!][[t]]/I_G=\mathcal{O}[[t]]$, and $S'=\mathcal{O}_\phi[\![G]\!]/I_G=\mathcal{O}_\phi$, as well as the maps to be the canonical ones. $\square$

From that we conclude immediately:

COROLLARY 3.20. *Let $\mathcal{O}'/\mathcal{O}$ be an extension of rings of integers of p-adic fields of ramification index e. We set $\Lambda := \mathcal{O}[\![G]\!]$ and $\Lambda' := \mathcal{O}'[\![G]\!]$. Then, for any Λ-module M we have $rank_\Lambda(M) = rank_{\Lambda'}(\Lambda'\otimes_\Lambda M)$ and $e\cdot\mu_\Lambda(M)=\mu_{\Lambda'}(\Lambda'\otimes_\Lambda M)$. Moreover, if in situation (*) the module M is S-torsion, we have $\lambda_\Lambda(M) = \lambda_{\Lambda'}(\Lambda'\otimes_\Lambda M)$.*

PROOF. This follows directly from the last proposition together with the fact that $\mathcal{O}'/\mathcal{O}$ and Λ'/Λ are free, hence flat, ring extensions. $\square$

We can now summarize some basic facts about the homological ranks:

PROPOSITION 3.21. *Assume there is an exact sequence of finitely generated $\Lambda(G)$-modules:*

$$0\to M'\to M\to M''\to 0$$

Let π be a uniformizer of $\mathcal{O}$. We set $\mathbb{F}_q := \mathcal{O}/\pi$ and $\Omega(G) := \mathbb{F}_q[\![G]\!]$. Then we have:

1. $rank_{\Lambda(G)}(M) = rank_{\Lambda(G)}(M') + rank_{\Lambda(G)}(M'')$.
2. $rank_{\Lambda(G)}(M) = rank_{\Omega(G)}(M/\pi M) - rank_{\Omega(G)}(M[\pi])$.
3. *If $U \triangleleft G$ is a normal subgroup such that G/U does not have any torsion and is infinite and M is finitely generated as $\Lambda(U)$-module, then $rank_{\Lambda(G)}(M)=0$.*
4. *In situation (*), if M is S-torsion, then $\mu_{\Lambda(G)}(M)=0$.*
5. *In situation (*), if M is S^*-torsion, then*
$$\mu_{\Lambda(G)}(M) = \mu_{\Lambda(G)}(M') + \mu_{\Lambda(G)}(M'').$$
6. *If M is finitely generated over $\Lambda(H)$, then*
$$\lambda_{\Lambda(G)}(M) = \lambda_{\Lambda(G)}(M') + \lambda_{\Lambda(G)}(M'').$$
7. *If again M is finitely generated over $\Lambda(H)$, then*
$$\lambda_{\Lambda(G)}(M) = rank_{\Omega(H)}(M/\pi) - rank_{\Omega(H)}(M[\pi]).$$

Remark 3.22. *We stress that one should not expect to many properties of an naive rank to hold for the homological rank. In particular, the rank can be negative and there are submodules of nonzero rank of modules of rank zero. See for instance* [**How02**] *the second remark after corollary 2.4 for an example of a module of negative rank. This example is due to Venjakob and also provides an other negative result: It is finitely generated over* $\mathbb{Z}_p$ *and the group* G *has dimension 1, but it still has nonzero rank, showing that the assertion in part 3 of our proposition that* U *has to be normal is crucial.*

Proof. The first assertion is lemma 2.1 from [**How02**]. For the naive rank, this is a very general fact (if a skew field exists).

The second assertion is corollary 1.10 from the same article if G is assumed to be pro-p in addition. To prove it in the general case we compute with the last proposition:

$$\begin{aligned} rank_{\Lambda(G)}(M) &= \sum_i (-1)^i rank_{\mathcal{O}}(H_i(G,M)) \\ &= \sum_i (-1)^i \Big(dim_{\mathbb{F}_q}(H_i(G,M)/\pi) - dim_{\mathbb{F}_q}(H_i(G,M)[\pi]) \Big) \\ &= \left(\sum_i (-1)^i dim_{\mathbb{F}_q}(H_i(G,M/\pi)) \right) - \left(\sum_i (-1)^i dim_{\mathbb{F}_q}(H_i(G,M[\pi])) \right) \\ &= rank_{\Omega(G)}(M/\pi) - rank_{\Omega(G)}(M[\pi]) \end{aligned}$$

Here, we used the assertion in the case that $G = 1$ in the second line and the last lemma in the third one.

The third claim is immediately reduced to the case that $U = 1$ and that G is a p-adic Lie group without any torsion. Then M is a torsion $\Lambda(G)$-module and its rank is zero by comparison with the classical definition.

The fourth assertion is analogous to the third one, only replacing the $\Lambda(G)$-rank with the $\Omega(G)$-rank and U with H.

For the fifth statement, first assume that M is p-primary. Then, like the first assertion, this one is just the additivity of homological ranks. In the general case, set $C := coker(M(\pi) \to M''(\pi))$, so that we have an exact sequence of p-primary modules:

$$0 \to M'(\pi) \to M(\pi) \to M''(\pi) \to C \to 0$$

It is thus enough to show that $\mu(C) = 0$. We know that C is the epimorphic image of a subset of $M/M(\pi)$, therefore it is S-torsion. It follows from the previous statement of the lemma that $\mu(C) = 0$, as required.

The last two assertions are simply the first two applied to H instead of G. □

If M is a finitely generated $\Lambda(G)$-module which is annihilated by π, then we have $M/\pi = M = M[\pi]$. We conclude by induction that $rank_{\Omega(H)}(M/\pi) = rank_{\Omega(H)}(M[\pi])$ for any p-primary module which is in addition finitely over $\Lambda(H)$. It follows from the last part of the proposition that $\lambda(M) = 0$ for such a module M. Therefore, we can compute the λ-invariant for any finitely generated S-torsion module M as $\lambda_{\Lambda(G)}(M) = \lambda_{\Lambda(G)}(M/M(\pi))$. For this reason, the following definition extends the previous ones:

DEFINITION 3.3. In situation (*), if M is an S^*-torsion module, then we define

$$\lambda_{\Lambda(G)}(M) := \lambda^{old}_{\Lambda(G)}(M/M(\pi)),$$

where λ^{old} is the λ-invariant as defined before.

This generalized λ-invariant does not behave well. In particular, it is not additive on exact sequences. Where possible we will try to avoid these cases. At one point, however, we will need the following fact:

LEMMA 3.23. *Let $F : M \to M'$ be a morphism of finitely generated S^*-torsion $\Lambda(G)$-modules such that the kernel and cokernel are annihilated by some power of p. Then, $\lambda(M) = \lambda(M')$.*

PROOF. We look at the induced morphism $M/M(\pi) \to M'/M'(\pi)$. It is injective, and its cokernel is the surjective image of the cokernel of f, thus it is annihilated by the same power of p. We may therefore apply part 6 of the proposition to the exact sequence

$$0 \to M/M(\pi) \to M'/M'(\pi) \to N \to 0$$

to conclude the assertion. □

An immediate consequence of the additivity of the ranks and the definition as an Euler characteristic is the following one:

PROPOSITION 3.24. *Let us assume again that M is a finitely generated $\Lambda(G)$-module and N is a finitely generated $\Omega(G)$-module. Moreover, we assume that $G' \subset G$ is a normal (closed) subgroup such that G/G' does not have any p-torsion. We then have:*

$$rank_{\Lambda(G)}(M) = \sum_i (-1)^i rank_{\Lambda(G/G')}(H_i(G', M))$$

$$rank_{\Omega(G)}(N) = \sum_i (-1)^i rank_{\Omega(G/G')}(H_i(G', N))$$

$$\mu_{\Lambda(G)}(M) = \sum_i (-1)^i \mu_{\Lambda(G/G')}(H_i(G', M(\pi)))$$

$$\lambda_{\Lambda(G)}(M) = \sum_i (-1)^i \lambda_{\Lambda(G/G')}(H_i(H', M))$$

For the λ-invariant we here assume that we are in situation (*) and that M is S-torsion, and set $H' = G' \cap H$. The invariant $\lambda_{\Lambda(G/G')}$ is defined with respect to the subgroup H/H'.

PROOF. Recall the Hochschild-Serre spectral sequence for group homology:

$$H_i(G/G', H_j(G', M)) \Rightarrow H_{i+j}(G, M)$$

Then, by general nonsense, for any invariant I that is additive on short exact sequences, the alternating sum can be computed as

$$\sum_i (-1)^i I(H_i(G, M)) = \sum_{i,j} (-1)^{i+j} I(H_i(G/G', H_j(G', M))).$$

Writing the right hand side as a double sum yields the desired results. For the assertion on the λ-invariant, we have to replace G by H and G' by H'. □

Remark 3.25. *Note that for the statement on the μ-invariant we had to take the π-primary part first to use the spectral sequence, making this result considerably weaker. In particular, taking the π-primary part will not be compatible with specialization maps.*

We will use this last proposition to prove variation results. The last remark displays the source of the problems with the μ-invariant that we run into. These problems can only be avoided, if one restricts the statements to certain subsets of the specializations.

3.4. The variation of the algebraic Iwasawa invariants

Now that we have fixed our notation and stated the basic facts, we can start to prove statements on the variational behavior of the invariants. From now on, we will always assume that we are in the situation (*), i.e., we fix a subgroup H of G such that $\Gamma = G/H$ is isomorphic to $\mathbb{Z}_p$.

For the μ-invariant we have to do some explicit calculations:

Lemma 3.26. *If M is a finitely generated $\Lambda(G)$-module, which is $\pi^n S$-torsion for some $n \geq 0$, then $\mu(M) = \mu(M[\pi^n]) = \mu(M/\pi^n M)$.*

Proof. The module M is $\pi^n S$ torsion if and only if either of the following two equivalent statements holds

(1) The submodule $\pi^n M \subset M$ is S-torsion, or
(2) the quotient $M/M[\pi^n]$ is S-torsion.

Putting the inclusion and the projection, in their respective tautological short exact sequences and using the fact that S-torsion modules have trivial μ-invariant (part 4 of proposition 3.21) together with the additivity of the μ-invariant (part 5 of the same proposition) yields the assertion. □

For the variation of the μ-invariant, this gives the following result:

Corollary 3.27. *Let $\Lambda := \mathcal{O}[\![G]\!][[t]]$, let M be a finitely generated Λ-module, and let $f, g \in \mathcal{O}[[t]]$ be prime elements with $\pi^n | f - g$ such that $\mathcal{O}[[t]]/f$ and $\mathcal{O}[[t]]/g$ are maximal orders. If M/f and M/g are both $\pi^n S$-torsion (compare proposition 3.17), then $\mu_{\Lambda/f}(M/f) = \mu_{\Lambda/g}(M/g)$ and $\mu_{\Lambda/f}(M[f]) = \mu_{\Lambda/g}(M[g])$.*

Proof. From the last lemma, we get $\mu_{\Lambda/f}(M/f) = \mu_{\Lambda/f}(M/(\pi^n, f))$ and likewise $\mu_{\Lambda/g}(M/g) = \mu_{\Lambda/g}(M/(\pi^n, g))$. By the assumption we know that $(\pi^n, f) = (\pi^n, g)$ as ideals in Λ, so the modules coincide as $\Lambda/(\pi^n, f)$-modules. This smaller quotient, however, is still enough to compute the μ-invariants. For the second part, observe that by the other assertion of the last lemma $\mu_{\Lambda/f}(M[f]) = \mu_{\Lambda/f}((M[f])[\pi^n]) = \mu_{\Lambda/f}(M[(f, \pi^n)])$. Here, we denote $M[(f, \pi^n)] := \{m \in M | rm = 0 \; \forall r \in (f, \pi^n)\}$. The last form again only depends on the ideal, which is the same for f and g. □

Remark 3.28. *The proof shows that we could have replaced the condition "$\pi^n | f - g$" by "$(\pi^n, f) = (\pi^n, g)$ as ideals in $\mathcal{O}[[t]]$." Clearly, the second assumption on f and g is weaker. But we are less interested in f and g themselves than in the ideals (f) and (g) they generate. The p-adic Weierstraß preparation theorem tells us that each ideal is generated by a unique distinguished polynomial and the difference of those two is divisible by π^n, provided the ideals are the same.*

For the λ-invariant, the result would not be as good if we looked at $\lambda(M/f)$ and $\lambda(M[f])$ separately. We can, however, use homological methods to get even stronger statements for their difference:

Firstly, we define any of the above invariants for complexes with finitely many nonzero cohomology groups by setting for instance $\mu(C^\bullet) := \sum_i (-1)^i \mu(H^i(C^\bullet))$ or $\lambda(C^\bullet) := \sum_i (-1)^i \lambda(H^i(C^\bullet))$. Compare for the theory of generalize Iwasawa invariant also the third section of **[BV11]** where in Burns and Venjakob develop a theory for the μ-invariant.

If the cohomology groups of the complexes in question are S-torsion, then the λ-invariant is compatible with specializations:

THEOREM 3.29. *Let $\phi : \mathcal{O}[[t]] \to \mathcal{O}_\phi$ be a specialization map and denote $\Lambda(G) := \mathcal{O}[\![G]\!][[t]]$ and $\Lambda_\phi(G) := \mathcal{O}_\phi[\![G]\!]$. Moreover, let $C^\bullet$ be a perfect complex of Λ-modules. Then, we have*

$$rank_{\Lambda(G)}(C^\bullet) = rank_{\Lambda_\phi(G)}(\Lambda_\phi(G) \otimes^L_{\Lambda(G)} C^\bullet).$$

Similarly, if the cohomology groups of $C^\bullet$ are finitely generated as $\Lambda(H)$-modules, then we have

$$\lambda_{\Lambda(G)}(C^\bullet) = \lambda_{\Lambda_\phi(G)}(\Lambda_\phi(G) \otimes^L_{\Lambda(G)} C^\bullet),$$

where the λ-invariant over $\Lambda(G) = \mathcal{O}[\![G \times \mathbb{Z}_p]\!]$ is computed with respect to $H \times \mathbb{Z}_p$ and $\lambda_{\Lambda_\phi(G)}$ with respect to H.

REMARK 3.30. *Recall that we have shown in theorem 3.12 that if $C^\bullet$ has S-torsion cohomology groups, then so has $\Lambda_\phi(G) \otimes^L_{\Lambda(G)} C^\bullet$. Accordingly, both sides of the assertion on the λ-invariants are well-defined.*

We will need two lemmata for the proof:

LEMMA 3.31. *Keeping the notations of the theorem, let M be a finitely generated $\Lambda(G)$-module, which is S-torsion. Then for all $i \geq 0$ there is a canonical isomorphism of $\Lambda_\phi(H)$-modules:*

$$Tor_i^{\Lambda(G)}(\Lambda_\phi(G), M) \cong Tor_i^{\Lambda(H)}(\Lambda_\phi(H), M)$$

PROOF. The assertion follows directly from the fact that $\Lambda_\phi(G) = \Lambda(G) \otimes_{\Lambda(H)} \Lambda_\phi(H)$ and that $\Lambda(G)$ is flat over $\Lambda(H)$ applied to the flat base change theorem of Tor. □

Secondly, we will need the special case of the theorem, where $C^\bullet$ is replaced by a single module:

LEMMA 3.32. *Still keeping the notation of the theorem, we assume that M is a finitely generated $\Lambda(G)$-module. We then have*

$$rank_{\Lambda(G)}(M) = \sum_i (-1)^i rank_{\Lambda_\phi(G)}(Tor_i^{\Lambda(G)}(\Lambda_\phi(G), M)).$$

If, moreover, M is finitely generated over $\Lambda(H)$, then

$$\lambda_{\Lambda(G)}(M) = \sum_i (-1)^i \lambda_{\Lambda_\phi(G)}(Tor_i^{\Lambda(G)}(\Lambda_\phi(G), M)).$$

PROOF. Using the last lemma and replacing the λ-invariant with the $\Lambda(H)$-rank, we can rewrite the assertion on λ-invariants as:

$$rank_{\Lambda(H)}(M) = \sum_i (-1)^i rank_{\Lambda_\phi(H)}(Tor_i^{\Lambda(H)}(\Lambda_\phi(H), M))$$

Thus, we have reduced the second statement to the first one. (As long as we do not use the fact that G has the special subgroup H in the proof of the first assertion.)

Using proposition 3.24 to rewrite the ranks, we translate the first assertion to:

$$\sum_i (-1)^i rank_{\mathcal{O}[\![\mathbb{Z}_p]\!]}(H_i(G, M)) = \sum_{i,j} (-1)^{i+j} rank_{\mathcal{O}_\phi}(H_j(G, Tor_i^{\Lambda(G)}(\Lambda_\phi(G), M)))$$

The special case of proposition 3.19 shows that the right-hand side of the last equation is equal to:

$$\sum_{i,j} (-1)^{i+j} rank_{\mathcal{O}_\phi}(Tor_j^{\mathcal{O}[[t]]}(\mathcal{O}_\phi, H_i(G, M)))$$

Taking only the i-th summand and replacing $H_i(G, M)$ by an arbitrary finitely generated $\mathcal{O}[\![\mathbb{Z}_p]\!]$-module N we have reduced the problem to the identity:

$$rank_{\mathcal{O}[\![\mathbb{Z}_p]\!]}(N) = \sum_j (-1)^j rank_{\mathcal{O}_\phi}(Tor_j^{\mathcal{O}[\![\mathbb{Z}_p]\!]}(\mathcal{O}_\phi, N))$$

I.e., it suffices to prove the assertion in the case $G = 1$.

For this last claim, choose a finite free resolution $P_\bullet$ of N as $\mathcal{O}[\![\mathbb{Z}_p]\!]$-module with $P_i = \mathcal{O}[\![\mathbb{Z}_p]\!]^{n_i}$. Such a resolution exists as $\mathcal{O}[\![\mathbb{Z}_p]\!]$ has finite homological rank and is local. It follows that $rank_{\mathcal{O}[\![\mathbb{Z}_p]\!]}(N) = \sum_i (-1)^i n_i$. On the other hand, we can compute the Tor-groups with this resolution and get

$$\begin{aligned}
\sum_i (-1)^i rank_{\mathcal{O}_\phi}(Tor_i^{\mathcal{O}[\![\mathbb{Z}_p]\!]}(\mathcal{O}_\phi, M)) &= \sum_i (-1)^i rank_{\mathcal{O}_\phi}(H_i(\mathcal{O}_\phi \otimes_{\mathcal{O}[\![\mathbb{Z}_p]\!]} P_\bullet)) \\
&= \sum_i (-1)^i rank_{\mathcal{O}_\phi}(\mathcal{O}_\phi \otimes_{\mathcal{O}[\![\mathbb{Z}_p]\!]} P_i) \\
&= \sum_i (-1)^i rank_{\mathcal{O}_\phi}(\mathcal{O}_\phi^{n_i}) \\
&= \sum_i (-1)^i n_i
\end{aligned}$$

as required. □

PROOF (OF THE THEOREM). Let us only focus on the assertion on the λ-invariant, the one on the rank is proven in a similar manner. First, recall that we already know from theorem 3.12 that the cohomology groups of the derived tensor product are S-torsion. Recall the spectral sequence for the cohomology groups of the derived total tensor product (see [**Wei94**] application 5.7.8):

$$Tor_i^{\Lambda(G)}(\Lambda_\phi(G), H^j(A^\bullet)) \Rightarrow H^{j-i}(\Lambda_\phi(G) \otimes^L_{\Lambda(G)} A^\bullet)$$

As λ is an invariant which is additive on short exact sequences of S-torsion modules, it follows that

$$\begin{aligned}\lambda_{\Lambda_\phi(G)}(\Lambda_\phi(G)\otimes^L_{\Lambda(G)} C^\bullet) &= \sum_j (-1)^j \lambda_{\Lambda_\phi(G)}(H^j(\Lambda_\phi(G)\otimes^L_{\Lambda(G)} C^\bullet))\\ &= \sum_{i,j}(-1)^{j-i}\lambda_{\Lambda_\phi(G)}(Tor_i^{\Lambda(G)}(\Lambda_\phi(G), H^j(C^\bullet)))\\ &= \sum_j (-1)^j \sum_i (-1)^i \lambda_{\Lambda_\phi(G)}(Tor_i^{\Lambda(G)}(\Lambda_\phi(G), H^j(C^\bullet)))\\ &= \sum_j (-1)^j \lambda_{\Lambda(G)}(H^j(C^\bullet)) = \lambda_{\Lambda(G)}(C^\bullet),\end{aligned}$$

where, in the final line, we used the last lemma. Thus, the assertion is proven. □

3.5. The Iwasawa invariants of the Selmer complexes

Using the theorems of the last section, we intend to show that the Iwasawa invariants of the Selmer complexes have good behavior under specialization:

THEOREM 3.33. *Let G, H, $\mathbb{T}$, $\mathbb{T}^0$, and U be as in the previous sections (see section 2.2). We assume that condition 2.12 on the freeness of the subrepresentations is satisfied. We set $\Lambda := \mathcal{O}[\![G]\!][[t]]$ and, for a specialization map $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$, we set $\Lambda_\phi = \mathcal{O}'[\![G]\!]$. Assuming that the cohomology groups of the Selmer complex $SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are S^*_ϕ-torsion, the following holds:*

1. *If ϕ is surjective with kernel (f), then there is an n depending only on the pair $(\mathbb{T}/f, \mathbb{T}^0/f)$ such that*
$$\mu_{\Lambda/f}(SC(U, \mathbb{T}/f, \mathbb{T}^0/f)) = \mu_{\Lambda/g}(SC(U, \mathbb{T}/g, \mathbb{T}^0/g))$$
for all prime elements $g \in \mathcal{O}[[t]]$ with $\pi^n | f - g$ such that $\mathcal{O}[[t]]/g$ is a maximal order.
2. *If the cohomology groups of the Selmer complex $SC(U, \mathbb{T}, \mathbb{T}^0)$ (equivalently, of $SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$) are S-torsion, then we have:*
$$\lambda_\Lambda(SC(U, \mathbb{T}, \mathbb{T}^0)) = \lambda_{\Lambda_\phi}(SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi))$$

PROOF. The first assertion follows from the short exact sequences

$$0 \to H^i(SC(U, \mathbb{T}, \mathbb{T}^0))/f \to H^i(SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)) \to H^i(SC(U, \mathbb{T}, \mathbb{T}^0))[f] \to 0$$

from corollary 3.4. The S^*_ϕ-torsion condition tells us that all three modules in these sequences are $\pi^n S$- torsion for some n. Then, we can apply proposition 3.17 and corollary 3.27 to the left and right modules to conclude the assertion.

The second assertion is just the combination of the last theorem with the base change property of the Selmer complex (proposition 3.1). □

For the primitive Selmer complex, we have a weaker specialization property described in theorem 3.2. Therefore, we have to use the extended definition of the λ-invariant:

COROLLARY 3.34. *Keeping the notation of ϕ,f, Λ, and Λ_ϕ from the last theorem, we still assume that $(\mathbb{T}, \mathbb{T}^0)$ is a pair of Λ-representations as in section 2.2 fulfilling condition 2.12, such that $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ has S^*_ϕ-torsion cohomology groups. Moreover, we assume that condition 2.15 holds. Then we conclude:*

(1) *If ϕ is surjective with kernel (f), then there is an n depending only on the pair $(\mathbb{T}/f, \mathbb{T}^0/f)$ such that*
$$\mu_{\Lambda/f}(SC(\mathbb{T}/f, \mathbb{T}^0/f)) = \mu_{\Lambda/g}(SC(\mathbb{T}/g, \mathbb{T}^0/g))$$
for all prime elements $g \in \mathcal{O}[[t]]$ with $\pi^n | f - g$ such that $\mathcal{O}[[t]]/g$ is a maximal order.

(2) *Assuming that the cohomology groups of $SC(\mathbb{T}, \mathbb{T}^0)$ are S-torsion, we have*
$$\lambda_\Lambda(SC(\mathbb{T}, \mathbb{T}^0)) = \lambda_{\Lambda_\phi}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)).$$
However, we have to be careful as now the cohomology groups of $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are automatically S^*-torsion, but need not be S-torsion.

PROOF. Recall the exact triangle in theorem 3.2
$$\Lambda_\phi \otimes^L_\Lambda SC(\mathbb{T}, \mathbb{T}^0) \to SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi) \to C \xrightarrow{+},$$
where C was the mapping fiber of a map $C' \to C'[-1]$, with C' a perfect complex with p-primary cohomology groups. As the cohomology groups of $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are S^*_ϕ-torsion and the ones of $C^\bullet$ even p-primary; it follows that the ones of $\Lambda_\phi \otimes^L_\Lambda SC(\mathbb{T}, \mathbb{T}^0)$ are S^*_ϕ-torsion, too. Thus, the μ-invariant is additive on the long exact sequence. Moreover, $\mu(C) = \mu(C') - \mu(C') = 0$, and we conclude $\mu_{\Lambda_\phi}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)) = \mu_{\Lambda_\phi}(\Lambda_\phi \otimes^L_\Lambda SC(\mathbb{T}, \mathbb{T}^0))$. The assertion on the μ-invariant is therefore reduced to the one of the imprimitive Selmer complex.

For the claim on the λ-invariant, we use the long exact sequence coming out of the same triangle and then apply lemma 3.23 to show that $\lambda_{\Lambda_\phi}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)) = \lambda_{\Lambda_\phi}(\Lambda_\phi \otimes^L SC(\mathbb{T}, \mathbb{T}^0))$. Replacing the term in the assertion, reduces it to the second part of theorem 3.29. □

The condition that we are only allowed to take surjective specializations if we want to have results on the variation of the μ-invariant is mainly due to the fact that this variation property has been proven with very elementary means, and we wanted to keep our notation slim. However, we will next establish the invariance under scalar extension, which allows us to drop this condition:

THEOREM 3.35. *Let $\Lambda = \mathcal{O}[\![G]\!][[t]]$ or $\Lambda = \mathcal{O}[\![G]\!]$ and let $(\mathbb{T}, \mathbb{T}^0)$ be a pair of Λ-representations. Moreover, let $\mathcal{O}'$ be a finite extension of $\mathcal{O}$, which is a maximal order, too. We denote the ramification index of $\mathcal{O}'/\mathcal{O}$ by e and set $\Lambda' = \mathcal{O}' \otimes_\mathcal{O} \Lambda$, $\mathbb{T}' = \mathcal{O}' \otimes \mathbb{T}$, and $\mathbb{T}'^0 = \mathcal{O}' \otimes \mathbb{T}^0$. Then, we have:*
$$e \cdot \mu_\Lambda(SC(U, \mathbb{T}, \mathbb{T}^0)) = \mu_{\Lambda'}(SC(U, \mathbb{T}', \mathbb{T}'^0))$$
$$e \cdot \mu_\Lambda(SC(\mathbb{T}, \mathbb{T}^0)) = \mu_{\Lambda'}(SC(\mathbb{T}', \mathbb{T}'^0))$$
Moreover, if the cohomology groups of $SC(U, \mathbb{T}, \mathbb{T}^0)$ are S-torsion, we have:
$$\lambda_\Lambda(SC(U, \mathbb{T}, \mathbb{T}^0)) = \lambda_{\Lambda'}(SC(U, \mathbb{T}', \mathbb{T}'^0))$$
$$\lambda_\Lambda(SC(\mathbb{T}, \mathbb{T}^0)) = \lambda_{\Lambda'}(SC(\mathbb{T}', \mathbb{T}'^0))$$

PROOF. We have already seen in proposition 3.1 and theorem 3.2 that taking the Selmer complex commutes with scalar extension for every case. Then the assertions follow from corollary 3.20 and the fact that for any complex $C^\bullet$ of $\mathcal{O}$-modules we have $H^i(\mathcal{O}' \otimes_\mathcal{O} C^\bullet) = \mathcal{O}' \otimes H^i(C^\bullet)$ as $\mathcal{O}'/\mathcal{O}$ is free. □

Finally, we can deduce the general variation property for the μ-invariant:

COROLLARY 3.36. *Let G, H, $\mathbb{T}$, $\mathbb{T}^0$ and U be as above. We set $\Lambda := \mathcal{O}[\![G]\!][[t]]$ and, for a specialization map $\phi : \mathcal{O}[[t]] \to \mathcal{O}_\phi$ with kernel (f), we set $\Lambda_\phi = \mathcal{O}'[\![G]\!]$. Moreover, let e denote the ramification index of $\mathcal{O}_\phi/\mathcal{O}$. Assuming that the cohomology groups of the Selmer complex $SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are S^*_ϕ-torsion, there is a positive integer n depending only on $(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$ and the degree $\mathcal{O}_\phi/\mathcal{O}$, so that for all specializations $\psi : \mathcal{O}[[t]] \to \mathcal{O}_\psi$ whose kernel (g) satisfies $\pi^n | f - g$ we have:*

$$e' \cdot \mu_{\Lambda_\phi}(SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)) = e \cdot \mu_{\Lambda_\psi}(SC(U, \mathbb{T}_\psi, \mathbb{T}^0_\psi))$$

Here, e' denotes the ramification degree of $\mathcal{O}_\psi/\mathcal{O}$. Moreover, if in addition condition 2.15 is satisfied, we also get:

$$e' \cdot \mu_{\Lambda_\phi}(SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)) = e \cdot \mu_{\Lambda_\psi}(SC(\mathbb{T}_\psi, \mathbb{T}^0_\psi))$$

PROOF. Take $\mathcal{O}'$ to be an extension of $\mathcal{O}$ which contains $\mathcal{O}_\phi$ and $\mathcal{O}_\psi$. We intend to apply the first two theorems of this section to the induced specialization maps $\phi' : \mathcal{O}'[[t]] \to \mathcal{O}'$ and $\psi' : \mathcal{O}'[[t]] \to \mathcal{O}'$. Possibly changing f and g each by a unit factor, we may assume that both are distinguished polynomials (p-adic Weierstraß preparation). Then, the degrees of f and g coincide (say they are equal to d) and the kernels of ϕ' and ψ' are generated by linear factors f' of f, respectively, g' of g. As f and g were irreducible, we conclude, if $\pi^n | f - g$ it follows that $\pi^{n'} | f' - g'$ with $n' = [n/d]$. Therefore, taking n large enough such that n' is as in the last theorems, we have proven the assertions. □

REMARK 3.37. *It can be seen from the proof that the integer n needed in the corollary is actually larger than the one needed for surjective specializations. Thus, although it is not explicitly noted, the first two theorems of this section have advantages.*

3.6. The Iwasawa invariants of the Selmer groups

We want to apply the results of the last section to get some results for the Iwasawa invariants of the Selmer groups. The first observation is that there are cases where $[SC(\mathbb{T}, \mathbb{T}^0)] = [\mathcal{X}(\mathbb{T}, \mathbb{T}^0)]$ in $K_0(\mathfrak{M}_H(G))$ (i.e., K_0 of the category of complexes with S^*-torsion cohomology groups). As the μ-invariant factors over the projection into this group, the μ-invariants of the Selmer group and the Selmer complex will coincide. This is the case, for instance, if $p \geq 5$ and the motive comes from an elliptic curve over $\mathbb{Q}$ with ordinary reduction at p, where we take G to be the image of the absolute Galois group in the automorphisms of the Tate module (see **[FK06]** 4.5.3). But we can say more about the relation of the Iwasawa invariants of Selmer complexes and the ones of the related Selmer groups. In general, it is hard to compare the classes in K_0; it is much easier, however, to show that the Iwasawa invariants coincide for many examples.

THEOREM 3.38. *Assume that $(\mathbb{T}, \mathbb{T}^0)$ is a pair of big Galois representations associated to a single motive (not a family) and the p-adic Lie group $G = Gal(F_\infty/F)$ as above. We assume that G does not have any p-torsion. If $\mathcal{X}(\mathbb{T}, \mathbb{T}^0)$ has S^*-torsion cohomology groups, we have*

$$\mu_{\Lambda(G)}(\mathcal{X}(U, \mathbb{T}, \mathbb{T}^0)) = \mu_{\Lambda(G)}(SC(U, \mathbb{T}, \mathbb{T}^0)) \text{ and}$$
$$\mu_{\Lambda(G)}(\mathcal{X}(\mathbb{T}, \mathbb{T}^0)) = \mu_{\Lambda(G)}(SC(\mathbb{T}, \mathbb{T}^0)).$$

For any place v of F dividing p, take $H_v = G_v \cap H$ to be the intersection of the decomposition group of v and H. If, in addition, all the groups H_v and H have an infinite pro-p quotient without p-torsion and the cohomology groups of the Selmer complexes are S-torsion, then so is the Selmer group, and

$$\lambda_{\Lambda(G)}(\mathcal{X}(U,\mathbb{T},\mathbb{T}^0)) = \lambda_{\Lambda(G)}(SC(U,\mathbb{T},\mathbb{T}^0)) \text{ and}$$
$$\lambda_{\Lambda(G)}(\mathcal{X}(\mathbb{T},\mathbb{T}^0)) = \lambda_{\Lambda(G)}(SC(\mathbb{T},\mathbb{T}^0)).$$

PROOF. The proof has two steps: Firstly, we show that the invariants of the Selmer groups coincide with those of the second cohomology groups of the Selmer complex, secondly, we show that the invariants of the other cohomology groups of the Selmer complex vanish.

For the first step, we take the exact sequences relating the Selmer complex and the Selmer groups (see proposition 2.23). We have to show that the μ-invariant of any submodule of $\bigoplus_{v|p} \Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1))_{\mathcal{G}(v)}$ vanishes, and, in the case described in the theorem, the same holds for the λ-invariant. For $v|p$ the map $G_v \to \Gamma = G/H$ is surjective, so that we have $\mathcal{O}[\![G]\!] = \mathcal{O}[\![H]\!]\hat{\otimes}_{\mathcal{O}[\![H_v]\!]}\mathcal{O}[\![G_v]\!]$ as $\mathcal{O}[\![H]\!]$-$\mathcal{O}[\![G_v]\!]$-bimodules. The representation $T^0(v)(-1)$ is finitely generated as an $\mathcal{O}$-module; consequently, it is finitely generated as an $\mathcal{O}[\![G_v]\!]$- and as an $\mathcal{O}[\![H_v]\!]$-module, too. It follows that $\mathcal{O}[\![G]\!] \otimes_{\mathcal{O}[\![G_v]\!]} T^0(v)(-1) = \mathcal{O}[\![H]\!] \otimes_{\mathcal{O}[\![H_v]\!]} T^0(v)(-1)$ is finitely generated as an $\mathcal{O}[\![H]\!]$-module, showing that the module is S-torsion, so that the μ-invariant of any submodule vanishes by part 4 of proposition 3.21.

To demonstrate the vanishing of the λ-invariant, we remark that, taking U to be the kernel of the projection of H_v onto an infinite pro-p group, we can apply part 3 of proposition 3.21 to show that any submodule of $T^0(v)(-1)$ has zero $\mathcal{O}[\![H_v]\!]$ rank. This implies that $\mathcal{O}[\![H]\!] \otimes_{\mathcal{O}[\![H_v]\!]} T^0(v)(-1)$ has zero $\mathcal{O}[\![H]\!]$ rank. (For this part, compare proposition 4.3.16 in [**FK06**].)

For the second step, we observe that again due to the exact sequences in proposition 2.23 the module $H^3(SC(U,\mathbb{T},\mathbb{T}^0)) = H^3(SC(\mathbb{T},\mathbb{T}^0))$ is the epimorphic image of $T(-1)_{\mathcal{G}}$ and therefore of $T(-1)$. But T - hence also every quotient of it - is a finitely generated $\mathbb{Z}_p$-module and has therefore μ-invariant 0. If H admits an infinite pro-p quotient without torsion, then the $\mathcal{O}[\![H]\!]$-rank is zero, too. It follows that the invariants of the third cohomology group vanish.

By lemma 2.3, apart from H^2 and H^3 there is only one cohomology group that might not vanish: the first one. By proposition 4.3.13 in [**FK06**], the group $H^1(SC(\mathbb{T},\mathbb{T}^0))$ is zero in the case of $dim(G) > 1$ such that we only need to focus on the case of the μ-invariant. In the case of $dim(G) = 1$, however, the same proposition computes $H^1(SC(\mathbb{T},\mathbb{T}^0))$ as a subset of $T^{\mathcal{G}}$, consequently it is finitely generated over $\mathbb{Z}_p$ and thus has trivial μ-invariant. To prove the same statement for $H^1(SC(U,\mathbb{T},\mathbb{T}^0))$, we use the distinguished triangle form lemma 2.5, relating the primitive and the imprimitive Selmer complex, to get an exact sequence

$$\bigoplus_v H^0_f(F_v,\Lambda\otimes T) \to H^1(SC(U,\mathbb{T},\mathbb{T}^0)) \to H^1(SC(\mathbb{T},\mathbb{T}^0)) \to \dots\ .$$

As the invariants of all subsets of $H^1(SC(\mathbb{T},\mathbb{T}^0))$ vanish in the appropriate situations, it is enough to show that the $H^0_f(F_v,\mathbb{T})$ vanish. By assumption, however, $H^1(SC(U,\mathbb{T},\mathbb{T}^0))$ is S^*-torsion and $H^0_f(F_v,\Lambda\otimes T) \subset \Lambda \otimes T$ is torsion-free as a subset of a free module. Thus, it is the 0 module.

□

REMARK 3.39. *The conditions on the group H for the λ-invariant are a bit technical. But it is certainly enough that if F_∞/F_{cyc} contains an infinite (normal) pro-p subextension, such that all primes dividing p are at most finitely decomposed. This is for instance the case if F_∞ contains $K = \cup_i F_{cyc}(\sqrt[p^i]{p})$*

Finally, let us note what this implies for the behavior of the invariants in families:

COROLLARY 3.40. *Let $(\mathbb{T}, \mathbb{T}^0)$ be a big Galois representation associated to a family of motives and assume that the according p-adic Lie group $G = Gal(F_\infty/F)$ does not have any p-torsion and that condition 2.12 on freeness is met. Moreover, let ϕ and ψ be two specializations of the family. Then the following holds:*

(1) *If ϕ has kernel (f) and $\mathcal{X}(\mathbb{T}, \mathbb{T}^0)$ is S^*-torsion, then there is an n such that:*
$$e_\psi \cdot \mu_{\Lambda_\phi}(\mathcal{X}(\mathbb{T}_\phi, \mathbb{T}^0_\phi)) = e_\phi \cdot \mu_{\Lambda_\psi}(\mathcal{X}(\mathbb{T}_\psi, \mathbb{T}^0_\psi))$$
for all specializations ψ with kernel (g) such that $\pi^n | f - g$. Here, e_ϕ and e_ψ are the ramification indices of $\mathcal{O}_\phi/\mathcal{O}$, respectively, $\mathcal{O}_\psi/\mathcal{O}$.

(2) *If the cohomology groups of $SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are S-torsion and for every place v of F dividing p the groups H_v and H admit infinite pro-p quotients without p-torsion, then:*
$$\lambda_\Lambda(\mathcal{X}(\mathbb{T}_\phi, \mathbb{T}_\phi)) = \lambda_{\Lambda_\phi}(\mathcal{X}(\mathbb{T}_\psi, \mathbb{T}^0_\psi))$$

PROOF. This is the combination of the last theorem with corollaries 3.36 and 3.34. □

REMARK 3.41. *There is also an obvious imprimitive version of this corollary, which is obtained by replacing corollary 3.34 with theorem 3.33.*

CHAPTER 4

p-adic zeta isomorphisms

In this chapter, we would like to give a brief summary about the p-adic ζ-isomorphisms conjecturally constructed by Fukaya and Kato in our main reference [**FK06**]. While, for reasons of brevity, it is not possible to give all the details of the construction, we have to review some of their tools, just to state the results. Assuming the conjectures of Fukaya and Kato, we will then be able to show an Iwasawa main conjecture for families of the kind described above.

4.1. Determinant categories

We will not give a full account of determinant categories over noncommutative rings here; mainly we reproduce the review in subsection 1.2 of [**FK06**]. For more details please consult Venjakob's explanations in the first section of [**Ven07**]. A more conceptual account can be found in [**BF03**]. However, the categories are obtained in a different way there and the explicit construction of Fukaya and Kato are used to describe the ζ-isomorphism.

Let us start by recalling the definition of the the determinant category $\mathcal{C}_\Lambda$ over a (possibly noncommutative) ring Λ: The objects of $\mathcal{C}_\Lambda$ are pairs (P,Q) of finitely generated projective Λ-modules with the morphisms defined as follows. The set $Hom_{\mathcal{C}_\Lambda}((P,Q),(P',Q'))$ is not empty, if and only if $[P]-[Q]=[P']-[Q']$ in $K_0(\Lambda)$. In that case, there is a finitely generated projective Λ-module R such that $P\oplus Q'\oplus R$ is isomorphic to $P'\oplus Q\oplus R$. The set of isomorphisms is a torsor for the groups of automorphisms of both sides, and we define the Set of homomorphisms $Hom_{Det(\Lambda)}((P,Q),(P',Q'))$ to be

$$K_1(\Lambda)\times^{Aut_\Lambda(P\oplus Q'\oplus R)} Isom_\Lambda(P\oplus Q'\oplus R, P'\oplus Q\oplus R)$$

This construction can be shown to be independent of R. There is a (functorial) multiplicative structure on this category induced by the direct sum. Via this structure, the neutral object is $0=[(0,0)]$ and the inverse of the object $[(P,Q)]$ is $[(Q,P)]$. Moreover, there is a functor from the category of finitely generated projective Λ-modules with Λ-isomorphisms as morphisms into this category. This functor is called Det_Λ. In particular, this functor will map Λ-automorphisms to automorphisms of determinant objects, which are canonically isomorphic to $K_1(\Lambda)$.

If $C^\bullet$ is a bounded complex of finitely generated projective Λ-module, then we define $Det_\Lambda(C^\bullet)$ to be (C_{even}, C_{odd}), where C_{even} is the direct sum of the even degree modules and C_{odd} is the direct sum of the odd degree ones.

We conclude our introducing remarks by noting that for any second ring Λ' and any Λ'-Λ-bimodule Y which is projective and finitely generated as a Λ'-module, the tensor product $Y\otimes_\Lambda$ induces a "change of rings" functor also denoted as a tensor product between the determinant categories.

We will need a few canonical isomorphisms. The first one is:

FACT 4.1. *If $A^\bullet : [A^0 \to A^1]$ is a complex and $\phi : A^1 \to A^0$ is an isomorphism, then ϕ induces a natural isomorphism of determinants:*

$$Det_\Lambda(0) \to Det_\Lambda(A^\bullet)$$

In most cases, A^1 and A^0 will be naturally identified.

Next, we get a canonical isomorphism for short exact sequences:

FACT 4.2. *If*

$$0 \to P' \overset{\iota}{\to} P \to P'' \to 0$$

is an exact sequence of finitely generated projective Λ-modules (or of complexes of such modules), then there is a natural isomorphism:

$$Det_\Lambda(P') \cdot Det_\Lambda(P'') \to Det_\Lambda(P)$$

Moreover, using the multiplicative structure of $\mathcal{C}_\Lambda$ we get an canonical isomorphism:

$$Det_\Lambda(0) \to Det_\Lambda(P) \cdot Det_\Lambda(P')^{-1} \cdot Det_\Lambda(P'')^{-1}$$

Indeed, the isomorphism is constructed by choosing a splitting $s : P'' \to P$ and taking the canonical morphism to be the image of $P' \oplus P'' \overset{\iota \oplus s}{\to} P$ under the Det_Λ-functor. This construction is independent of the choice of s.

More generally, one obtains:

FACT 4.3. *If C is an acyclic, bounded complex of finitely generated projective Λ-modules, then there is a canonical isomorphism*

$$Det(0) \to Det(C).$$

The morphism in question is given as follows: As a bounded exact sequence of projective modules C splits. Thus after choosing splittings s we get that $s + d : C_{odd} \cong C_{even}$ is an isomorphism. Thus inducing an isomorphism $Det(0) \to Det(C)$ as required.

Finally, we note that the construction can be extended to the derived category:

FACT 4.4. *Quasi-isomorphisms of complexes induce isomorphisms in the determinant category. In particular, the functor Det factors over the derived category with quasi-isomorphisms.*

The mapping cones of quasi-isomorphisms are acyclic. The last fact applied to the mapping cone gives the desired morphism when we view the modules of the cone as direct sums of the modules of the two complexes.

With these preparations in place, one can state some compatibilities for the "change of ring" functor:

LEMMA 4.5. *Assume that Λ and Λ' are two rings and Y is an Λ'-Λ-bimodule that is projective as a Λ'-module. Then we have:*

(1) *The functor "$Y \otimes_\Lambda$" commutes with the det functor. I.e., for any bounded complex of finitely generated Λ-modules, we have $(Y \otimes_\Lambda) \circ (Det_\Lambda) \cong Det_{\Lambda'} \circ (Y \otimes_\Lambda)$.*

(2) *If $f : C \to C'$ is a quasi-isomorphism of complexes of finitely generated projective Λ-modules and $Y \otimes_\Lambda C$ (hence also $Y \otimes_\lambda C'$) is acyclic, then $Y \otimes_\Lambda Det_\Lambda(f) = Det_{\Lambda'}(id_Y \otimes f) = 1 \in Aut(Det_{\Lambda'}(0)) = K_1(\Lambda')$, where we identify the determinants of acyclic complexes with 0 via the canonical isomorphism from fact 4.3.*

Proof. As the first part is an easy calculation, we only demonstrate the second part:

The first equality follows directly from the first part of the lemma. For the second equality, we look at the commutative diagram:

$$\begin{array}{ccc} Det_{\Lambda'}(Y \otimes_\Lambda C) & \xrightarrow{id_y \otimes f} & Det_{\Lambda'}(Y \otimes_\Lambda C') \\ {\scriptstyle Det(0)} \uparrow & & \uparrow {\scriptstyle Det(0)} \\ Det_{\Lambda'}(0) & \xrightarrow{\quad id_0 \quad} & Det_{\Lambda'}(0) \end{array}$$

There, the $Det(0)$ on the vertical arrows denotes the determinant of the zero map, which is a quasi-isomorphism by assumption. It follows that $Det(id_Y \otimes f) = Det(id_0) = 1$. □

4.2. The localized K_1

The noncommutative Iwasawa main conjecture as introduced in special cases in [**Ven05**] and generalized in [**CFKSV**] predicts a p-adic ζ-function as an element of $K_1(\Lambda_{S^*})$. Thus, it can only be a characteristic element of the Selmer group if the $\mathfrak{M}_H(G)$ conjecture is fulfilled, i.e., if the dual of the Selmer group is S^*-torsion. It seems unreasonable, however, to believe there is a similar statement for very general families, and even for the classical settings this conjecture is known to hold only in a few special cases. Fortunately, there is a way to work around this obstruction: Fukaya and Kato introduced to the theory some localized K_1 group relative to some subcategory Σ. This group is equal to $K_1(\Lambda_{S^*})$ in the case that Σ is the subcategory of S^*-torsion modules and satisfies standard functorialities. For the moment we will therefore adopt this notion. Only in the end of this chapter we will note, how to deduce results involving the classical notions assuming some kinds of $\mathfrak{M}_H(G)$ conjectures are satisfied.

Let us first recall the construction: By $\Sigma \subset \mathcal{P}(\Lambda)$ we denote a full subcategory of the category $\mathcal{P}(\Lambda)$ of bounded complexes of finitely generated Λ-modules satisfying the following conditions:

Condition 4.6.

(1) *If C is in Σ, then so are all complexes quasi-isomorphic to it.*
(2) *Σ contains all acyclic complexes.*
(3) *For any $C \in \Sigma$, all the translations $C[r]$ belong to Σ, too.*
(4) *For any short exact sequence in $\mathcal{P}(\Lambda)$,*

$$0 \to C' \to C \to C'' \to 0.$$

If C' and C'' belong to Σ, then so does C.

By a slight abuse of the term, we will call such a Σ a triangulated subcategory of $\mathcal{P}(\Lambda)$, meaning that it is the preimage of a triangulated subcategory of the derived category.

REMARK 4.7. *If we have any collection of $\mathcal{C}_i$ of triangulated subcategories, then the full subcategory of objects that are contained in all $\mathcal{C}_i$ is a triangulated subcategory, too. Thus, there is always a smallest triangulated subcategory containing some objects.*

With this description, we define:

DEFINITION 4.8. *Let Σ be a triangulated subcategory of $\mathcal{P}(\Lambda)$. Then the group $K_1(\Lambda, \Sigma)$ is the (multiplicatively written) abelian group defined by the generators $[C, a]$, where C is an object of Σ and a is an isomorphism $Det_\Lambda(0) \to Det_\Lambda(C)$, subject to the following relations:*

1. *If C is acyclic, then $[C, can] = 1$. Here, can is the isomorphism from fact 4.3.*
2. *For exact sequences $0 \to C' \to C \to C'' \to 0$ of objects of Σ, set $B = C \oplus C'[1] \oplus C''[1]$; then $[B, can] = 1$ where this time can is the second morphism from 4.2.*
3. *The multiplicative structure of Det_Λ is preserved: For two generators $[C, a]$ and $[C', a']$, we have*
$$[C, a] \cdot [C', a'] = [C \oplus C', aa'].$$

REMARK 4.9. *In general, our first canonical isomorphism (fact 4.1) will in general not be identified with 1.*

Before we state more properties of the localized K_1 we note two cases where we get classical K-groups (proposition 1.3.7 in [**FK06**]).

PROPOSITION 4.10.

1. *The full subcategory* acycl *of acyclic complexes in $\mathcal{P}(\Lambda)$ is a triangulated subcategory. The localized K_1 group with respect to this subcategory is $K_1(\Lambda, acyc) = K_1(\Lambda)$.*
2. *If S is an Ore set in Λ, then the full subcategory* S-tor *of complexes with S-torsion cohomology groups in $\mathcal{P}(\Lambda)$ is a triangulated subcategory and $K_1(\Lambda, S - tor) = K_1(\Lambda_S)$.*

The second part of this proposition is recounts proposition 1.3.7. in [**FK06**]. The first part can be deduced from the second part by taking $S = 1$. Please note that the identification is sending a pair $[C, a] \in K_1(\Lambda, acyc) \to K_1(\Lambda)$ to the composition $can^{-1} \circ a \in Aut(Det(0)) = K_1(\Lambda)$, where can^{-1} is the inverse map of the one described in fact 4.3.

Let us note the following fact: If Λ and Λ' are two rings and if Σ and Σ' are triangulated subcategories of $\mathcal{P}(\Lambda)$, respectively, $\mathcal{P}(\Lambda')$, then any Λ'-Λ-bimodule Y which is projective and has the property that for every element C of Σ we have $Y \otimes_\Lambda C \in \Sigma'$ induces a morphism $Y \otimes : K_1(\Lambda, \Sigma) \to K_1(\Lambda', \Sigma')$. Using this map, the identification from the second part can be written as

$$K_1(\Lambda, S - tor) \overset{\Lambda_S \otimes}{\to} K_1(\Lambda_S, acyc) \to K_1(\Lambda_S).$$

Next we state the generalization of the usual localization sequence of K-theory (this is theorem 1.3.15 of [**FK06**]):

THEOREM 4.11. *For any triangulated subcategory Σ of $\mathcal{P}(\Lambda)$, the sequence*

$$K_1(\Lambda) \overset{f}{\to} K_1(\Lambda, \Sigma) \overset{g}{\to} K_0(\Sigma) \overset{h}{\to} K_0(\Lambda)$$

is exact. The maps can be defined in the following way: f sends the class of an automorphism $\phi : M \to M$ of projective Λ-modules to $[[M \stackrel{\phi}{\to} M], id]$, where id denotes the isomorphism $Det(0) \to Det(M) \cdot Det(M)^{-1}$ given by the fact 4.1 via the identity map on M. The morphism g sends a pair $[C, a]$ to the class of C in $K_0(\Sigma)$. Finally, h sends the class of a complex in $K_0(\Sigma)$ to the class of the same complex in $K_0(\Lambda)$.

These maps are compatible with the change of ring functors induced by tensoring with bimodules Y.

In some classical cases (e.g. when Σ is the subcategory of complexes with S^*-torsion modules as cohomology groups), one often knows that the map h in the above sequence is the zero map (see for instance [**CFKSV**]). This generalizes directly to families if we view our Iwasawa algebra $\mathcal{O}[\![G]\!][[t]]$ as $\mathcal{O}[\![G \times \mathbb{Z}_p]\!]$. But again, it is not even clear what the $\mathfrak{M}_H(G)$ conjecture for families should state exactly.

For many results, $\mathfrak{M}_H(G)$ can be replaced by one of the following subcategories:

Definition 4.12. *For any pair of Galois representations $(\mathbb{T}, \mathbb{T}^0)$, an open set U of spec($\mathbb{Z}$) such that we can define the Selmer complexes $SC(U, \mathbb{T}, \mathbb{T}^0)$ and $SC(\mathbb{T}, \mathbb{T}^0)$, we take $\Sigma(U, \mathbb{T}, \mathbb{T}^0)$ to be the smallest triangulated subcategory of $\mathcal{P}(\Lambda)$ containing all complexes quasi-isomorphic to $SC(U, \mathbb{T}, \mathbb{T}^0)$. Similarly, $\Sigma(\mathbb{T}, \mathbb{T}^0)$ is the smallest triangulated subcategory containing all complexes quasi-isomorphic to $SC(\mathbb{T}, \mathbb{T}^0)$.*

Of course, this definition only makes sense if the Selmer complexes are perfect. With some additional conditions we can say more:

Corollary 4.13. *Assume that $p \neq 2$, $\Lambda = \mathcal{O}[\![G]\!]$, or $\Lambda = \mathcal{O}[\![G]\!][[t]]$ for a p-adic Lie group G and that we have a pair of representations $(\mathbb{T}, \mathbb{T}^0)$ obtained from a motive, respectively, a family of motives as in Section 2.2, such that $\mathbb{T}^0$ and $\mathbb{T}^+$ are isomorphic as Λ-modules. Suppose we are in one of the two situations:*

1. *The subcategory is $\Sigma = \Sigma(U, \mathbb{T}, \mathbb{T}^0)$, where $U \subset$ spec($\mathbb{Z}$) is a subset inside which both the extension F_∞/F and the representation $\mathbb{T}$ are unramified.*
2. *The group G does not have any p-torsion, the ramification condition 2.15 is satisfied, and the subcategory is $\Sigma = \Sigma(\mathbb{T}, \mathbb{T}^0)$.*

Then there is an exact sequence:

$$K_1(\Lambda) \stackrel{f}{\to} K_1(\Lambda, \Sigma) \stackrel{g}{\to} K_0(\Sigma) \stackrel{h}{\to} 0$$

Proof. In fact, by lemma 2.7 and corollary 2.19 the classes of the Selmer complexes in $K_0(\Lambda)$ vanish in these situations. Thus, their classes in $K_0(\Sigma)$ map to zero under h. But the full subcategory of Σ consisting of all objects whose classes map to zero clearly is a triangulated subcategory. So, as Σ is by definition the smallest triangulated subcategory containing the Selmer complex, h is the zero map. □

We conclude this section by defining the evaluation map for elements of $K_1(\Lambda, \Sigma)$. Assume that L is a p-adic field and $\rho : \Lambda \to M_n(L)$ is a ring homomorphism such that all complexes C of Σ become acyclic after tensoring with $M_n(L)$. Then the value $\xi(\rho) \in L^\times$ of an element $\xi \in K_1(\Lambda, \Sigma)$ at ρ is just its image under the following natural map:

$$K_1(\Lambda,\Sigma) \overset{M_n(L)\otimes_\Lambda}{\to} K_1(M_n(L), acycl) = K_1(M_n(L)) = K_1(L) = L^\times$$

REMARK 4.14.

(1) *As a right $M_n(\Lambda)$-module, $M_n(\Lambda)$ is the sum of n copies of L^n interpreted as row vectors acted on by matrix multiplication from the right. Thus, becoming acyclic after tensoring with $M_n(\Lambda)$ is the same as being acyclic after tensoring with the $L-\Lambda$-bimodule L^n. Moreover, when we applied Morita invariance to see that $K_1(M_n(L)) = K_1(L)$, we effectively tensorized with $L^n \otimes_{M_n(L)}$. Accordingly, we could have written the map as:*

$$K_1(\Lambda,\Sigma) \overset{L^n\otimes_\Lambda}{\to} K_1(L) = L^\times$$

(2) *If Σ is the minimal subcategory containing some set of complexes M and satisfying condition 4.6, then it is enough to ask that all complexes of M become acyclic after tensoring with $M_n(L)$, respectively, L^n. Indeed, in this case all complexes of Σ become acyclic, as the ones that do form a triangulated subcategory and Σ is minimal.*

We intend to evaluate elements in slightly different sets, thus we introduce:

DEFINITION 4.15. *For an adic ring Λ, we set*

$$\tilde{\Lambda} := \varprojlim (W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \Lambda/J^n),$$

where $W(\overline{\mathbb{F}_p})$ are the Witt vectors and J is the radical of Λ. Similarly, for a p-adic field L with ring of integers $\mathcal{O}_L$, we write $\tilde{L} := L \otimes_{\mathcal{O}_L} \tilde{\mathcal{O}_L}$.

Then we extend our evaluation map to elements in $K_1(\Lambda,\Sigma) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$
For a map $\rho : \Lambda \to M_n(L)$ as before we look at the canonical map:

$$K_1(\Lambda,\Sigma) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}) \to K_1(L) \times^{K_1(L)} K_1(\tilde{L}) \to K_1(\widehat{L^{ur}}),$$

where the first map is the one induced by the map above and the second one is induced by the natural map $\tilde{L} \to \widehat{L^{ur}}$.

4.3. ζ-isomorphisms for Galois representations

The main result of Fukaya and Kato in [**FK06**] is to give a conjectural construction of the p-adic zeta element of a pair of Galois representations $(\mathbb{T}, \mathbb{T}^0)$ together with some open unramified set U.

The results depends on some conjectures, we denote them by (FK) in the following. This notation includes:

- The Beilinson-Deligne conjecture (given in [**FK06**] 2.2.8): The description of the classical L-function of a motive at the critical spot $s = 0$, in particular the association of a ζ-isomorphism describing the value.
- An equivariant global Tamagawa number conjecture (given in 2.3.2 loc.cit.): A compatible way to associate isomorphisms

$$\zeta : Det_\Lambda(0) \to Det_\Lambda(R\Gamma_c(U,T))^{-1}$$

for representations of the absolute Galois group of $\mathbb{Q}$ over adic rings Λ extending the ζ-isomorphisms from the Beilinson-Deligne conjecture.

- A local Tamagawa number conjecture (given in 3.4.3 and 3.5.2 loc.cit.): A compatible way to associate epsilon factors as isomorphisms in determinant categories to representations of the absolute Galois groups of $\mathbb{Q}_p$ and $\mathbb{Q}_l$ over adic rings.
- Some functional equation relating the zeta isomorphisms of a representation with the one of its Kummer dual (given in 3.5.5 loc.cit.).

Without going into the details, we will simply assume that these conjectures are fulfilled when needed and will work with the results deduced from them by Fukaya and Kato, which are state below.

Firstly, there is a natural choice of a characteristic element (See **[FK06]** 4.1.3):

THEOREM 4.16. *Assuming conjectures (FK), let* (T, T^0) *is a pair of representations of* $G_{\mathbb{Q}}$ *and* $G_{\mathbb{Q}_p}$ *which are finitely generated projective modules over an adic ring* Λ *such that* $[T^0] = [T^+]$ *in* $K_0(\Lambda)$*. Furthermore, let* U *be an open subset of* $spec(\mathbb{Z})$ *such that the* $G_{\mathbb{Q}}$*-representation* T *is unramified in* U*. If we choose an isomorphism* $\beta : \tilde{\Lambda} \otimes Det_{\Lambda}(T^+) \to \tilde{\Lambda} \otimes Det_{\Lambda}(T^0)$*, then the construction* **[FK06]** *produces an element* $\zeta_\beta(U, T, T^0) \in K_1(\Lambda, \Sigma(U, T, T^0)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$*. This element can be written as* $([C, a], k)$*, with* $[C, a] \in K_1(\Lambda, \Sigma(U, T, T^0)$ *in the above notation and* $C = SC(U, T, T^0)$*. In particular, under the canonical map from theorem 4.11, the element* $\zeta_\beta(U, T, T^0)$ *maps to the class of the Selmer complex* $SC(U, T, T^0)$ *in* $K_0(\Sigma(U, T, T^0))$*.*

This ζ-element behaves well under base change(see **[FK06]** 4.1.4):

THEOREM 4.17. *In the situation of the last theorem, assume* Λ' *is another adic ring and* Y *is a* Λ'*-*Λ*-bimodule, projective over* Λ'*. We set* $T' := Y \otimes_\Lambda T$ *and* $(T')^0 := Y \otimes_\Lambda T^0$*. Then the isomorphism* β *induces* $\beta' : \tilde{\Lambda}' \otimes Det_{\Lambda'}((T')^+) \to \tilde{\Lambda}' \otimes Det_{\Lambda'}((T')^0)$*, and under the canonical map*

$$K_1(\Lambda, \Sigma(U, T, T^0)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}) \to K_1(\Lambda', \Sigma(U, T', (T')^0)) \times^{K_1(\Lambda')} K_1(\tilde{\Lambda}'),$$

the isomorphism $\zeta_\beta(U, T, T^0)$ *maps to* $\zeta_{\beta'}(U, T', (T')^0)$*.*

Please note that this natural map exists, as the Selmer complex over Λ is mapped onto the one over Λ' (shown in proposition 3.1) and as the preimage of $\Sigma(U, \mathbb{T}', (\mathbb{T}')^0)$ is a triangulated subcategory, it contains $\Sigma(U, \mathbb{T}, \mathbb{T}^0)$.

The following corollary is the case of this theorem, that is most important for what follows:

COROLLARY 4.18. *Assume that the pair of representations comes from a family of motives as in section 2.2 and* $\Lambda := \mathcal{O}[\![G]\!][[t]]$*. Let* $\beta : \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^+) \to \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^0)$ *be an isomorphism and let* $\beta_\phi : \tilde{\Lambda}_\phi \otimes Det_{\Lambda_\phi}(\mathbb{T}^+_\phi) \to \tilde{\Lambda}_\phi \otimes Det_{\Lambda_\phi}(\mathbb{T}^0_\phi)$ *be the induced isomorphism. Moreover, let* U *be a set outside of which* $\mathbb{T}$ *is unramified. Then, for any specialization map* ϕ*, we have that* $\zeta_\beta(U, \mathbb{T}, \mathbb{T}^0)$ *maps to* $\zeta_{\beta_\phi}(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$ *under the canonical map:*

$$K_1(\Lambda, \Sigma(U, \mathbb{T}, \mathbb{T}^0)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}) \to K_1(\Lambda_\phi, \Sigma(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)) \times^{K_1(\Lambda_\phi)} K_1(\tilde{\Lambda}_\phi)$$

Lastly, these ζ-isomorphisms have some interpolation properties, but to describe them we have to introduce some notations first:

Let V be a finite dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ over a p-adic number field L. Then the Euler polynomial at $l \neq p$ is

$$P_{L,l}(V, u) := det_L(1 - \phi_l u; V^{I_l})$$

where ϕ_l is the geometric Frobenius. If $l = p$, the polynomial at p is

$$P_{L,p}(V,u) := det_L(1 - \phi_p u; D_{crys}(V|Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)).$$

Now, if M is a K-motive over $\mathbb{Q}$ and M_λ is the λ-adic realization for some place λ of K dividing p, then the L-function of M should be given by the Euler product:

$$L_K(M,u) = \prod_l P_{K_\lambda,l}(M_\lambda,u)^{-1}$$

Assuming in addition that L is again some p-adic field with an embedding $K_\lambda \to L$ such that there is some $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ invariant subspace V^0 in $V = L \otimes_{K_\lambda} M_\lambda$ with $D_dR(V^0) \overset{\cong}{\to} D_dR(V)/D_dR^0(V)$, let us fix some isomorphism $\beta : V^+ \to V^0$. Moreover, all the cohomology groups $H^0(\mathbb{Q}, M_p)$, $H^1_f(\mathbb{Q}, M_p)$, $H^0(\mathbb{Q}, (M_p)^*(1))$, and $H^1_f(\mathbb{Q}, (M_p)^*(1))$ are zero. If the conjectures (FK) hold, then we can define the complex and p-adic periods as follows (see [**FK06**] 4.1.11 for the details):

As usual, one has to choose a K-basis γ of M_B^+, the part of the Betti realization of M fixed under the complex conjugation and δ of $t_M = M_{dR}/M_{dR}^0$. Then, the complex period $\Omega_\infty(M) \in \mathbb{C}^\times$ is just the Deligne period given as the determinant of the period map $\mathbb{C} \otimes_K M_B^+ \to \mathbb{C} \otimes_K t_M$ with respect to the chosen basis. The p-adic period $\Omega_{p,\beta}(M) \in (\hat{L^{ur}})^\times$ essentially captures the information on the epsilon factors from the local Tamagawa number conjecture and depends on our chosen β. Moreover, both Ω_∞ and $\Omega_{p,\beta}$ depend on the chosen basis γ and δ, but an other choice will change them both by the same factor in $K^\times$.

With these notations, we can write down the conjectured values of our zeta-element (which is theorem 4.1.12.(2) in [**FK06**]):

THEOREM 4.19. *Assuming the conjectures (FK) as before, let (T, T^0) be a pair of Galois representations over an adic ring Λ and let U be an open set as above. Moreover, assume that there is a morphism $\rho : \Lambda \to M_n(L)$ and a K-motive M over $\mathbb{Q}$, satisfying the Dabrowski-Panchishkin condition 2.9 such that (V_ρ, V_ρ^0) are the p-adic realization and the local subrepresentation of the Dabrowski-Panchishkin condition. Furthermore, the following supposed to hold:*

1. *$H^0(\mathbb{Q}, V_\rho)$, $H^1_f(\mathbb{Q}, V_\rho)$, $H^0(\mathbb{Q}, (V_\rho)^*(1))$, and $H^1_f(\mathbb{Q}, (V_\rho)^*(1))$ are zero.*
2. *For any prime $l \neq p$ not contained in U, we have $P_{L,p}(V_\rho, 0) \neq 0$.*
3. *The polynomials $P_{L,p}(V_\rho, u)P_{L,p}(V_\rho^0, u)^{-1}$ and $P_{L,p}((V_\rho^0)^*(1), u)$ do not have a zero at $u = 1$.*

Then, the value of $\zeta_\beta(U, T, T^0)$ at ρ is:

$$L_K(M,0)\Omega_\infty(M)^{-1}\Omega_{p,\beta}(M) \prod_{r \geq 1} \Gamma(r)^{h(-r)}$$
$$[P_{L,p}(V_\rho,u)P_{L,p}(V_\rho^0,u)^{-1}]_{u=1} P_{L,p}((V_\rho^0)^*(1),1) \prod_{l \notin U \cup \{p\}} P_{L,l}(V_\rho,1)$$

A motive satisfying the conditions of this theorem is called critical. Following the definitions of Fukaya and Kato in section 4.2. we set:

DEFINITION 4.20. *Let M be an F motive with coefficients in K. We assume that M satisfies the Dabrowski-Panchishkin condition 2.9. Moreover, we choose an extension F_∞/F with Galois group G and $U \subset spec(\mathbb{Z})$ as in section 2.2. Then, for the induced representations $(\mathbb{T}, \mathbb{T}^0)$, the invariants under the complex* *conjugation are isomorphic to $\mathbb{T}^0$ and we choose a Λ-isomorphism $\beta : \mathbb{T}^+ \to \mathbb{T}^0$. In this situation the imprimitive ζ-function is denoted as:*

$$\zeta_\beta(U, M, F_\infty/F) := \zeta_\beta(U, \mathbb{T}, \mathbb{T}^0)$$

Moreover, if we replace the motive M in the above considerations by a family M_t and and assume in addition that it satisfies condition 2.12 on freeness, we can again choose some $\beta : \mathbb{T}^+ \to \mathbb{T}^0$ and take again $\zeta_\beta(U, \mathbb{T}, \mathbb{T}^0)$ as the imprimitive ζ-function $\zeta_\beta(U, M_t, F_\infty/F)$ for M_t.

Please note that these are the imprimitive ζ-elements; they are lacking some Euler factors outside U. To get a ζ-isomorphism for $SC(\mathbb{T}, \mathbb{T}^0)$ we have to define Euler factors as appropriate isomorphisms in the determinant categories:

Definition 4.21. *We look at the setting of a pair of big Galois representations $(\mathbb{T}, \mathbb{T}^0)$ coming from a motive or a family. In the case of a family, we assume in addition that the condition on the ramification 2.15 holds. For a prime v of F not dividing p, we set $\Sigma(v)$ to be the smallest subcategory of $\mathcal{P}(\Lambda)$ that satisfies condition 4.6 and contains all complexes quasi-isomorphic to $C_f(F_v, \Lambda \otimes T) \cong [(\Lambda \otimes T)^{I_v} \to (\Lambda \otimes T)^{I_v}]$. Then, $\zeta(v, \mathbb{T}) \in K_1(\Lambda_p, \Sigma(v)_p)$ is defined to be the canonical isomorphism $Det_{\Lambda_p}(0) \to Det_{\Lambda_p}(C_f(F_v, \Lambda \otimes T))^{-1}$ from fact 4.1. The lower p indicates that we invert the multiplicative set $\{1, p, p^2, ...\}$ and thus we can apply fact 4.1 by proposition 2.17.*

If p is not 2 and G does not have any p-torsion then every finitely generated Λ-module (including $(\Lambda \otimes T)^{I_v}$) admits a perfect resolution, therefore we may drop the index p in this situation.

Assuming the conjectures (FK) again, we can define a primitive ζ-isomorphism:

Definition 4.22. *Fix a pair of Galois representations $(\mathbb{T}, \mathbb{T}^0)$ coming from a motive M or a family of motives M_t and a set U as above. Recall that this fixes a Galois extension F_∞/F with Galois group G. Then, for any isomorphism $\beta : \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^+) \to \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^0)$, we set:*

$$\zeta_\beta(M_{(t)}, F_\infty/F) := \zeta_\beta(U, \mathbb{T}, \mathbb{T}^0) \cdot \prod_{v \notin U \cup \{p\}} \zeta(v, \mathbb{T}).$$

Here, the product is to be understood in the following way: We set $\zeta_\beta(M_{(t)}, F_\infty/F) := ([C, a], k)$, where C is a complex quasi-isomorphic to $SC(\mathbb{T}, \mathbb{T}^0)$, $k \in \tilde{\Lambda}_p$ the corresponding element of $\zeta_\beta(U, \mathbb{T}, \mathbb{T}^0)$ and, finally, where $a : Det_\Lambda(0) \to Det_\Lambda(SC(\mathbb{T}, \mathbb{T}^0))$ is the product of the determinant morphisms of $\zeta_\beta(U, \mathbb{T}, \mathbb{T}^0)$ and $\zeta(v, \mathbb{T})$ with respect to the product structure on the determinant category. The ζ-isomorphism is therefore an element in $K_1(\Lambda_p, \Sigma(\mathbb{T}, \mathbb{T}^0)_p) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda}_p)$.

If p is not 2 and if G does not have any p-torsion, we can drop the index p in this definition.

Remark 4.23. *The construction from the above definition is possible using the description of $\zeta(U, T, T^0)$ from theorem 4.16 and the distinguished triangle*

$$SC(\mathbb{T}, \mathbb{T}^0) \to SC(U, \mathbb{T}, \mathbb{T}^0) \to \bigoplus_v C_f(F_v, \mathbb{T}) \xrightarrow{+}$$

from lemma 2.5 to identify $Det_\Lambda(SC(\mathbb{T}, \mathbb{T}^0))$ canonically with $Det_\Lambda(SC(U, \mathbb{T}, \mathbb{T}^0) \cdot \prod_v Det_\Lambda(C_f(F_v, \mathbb{T}))$.

4.4. The main conjecture for families

Assuming the conjectures (FK), Fukaya and Kato derived the following version of the Iwasawa main conjecture (theorem 4.2.22 in [**FK06**]):

THEOREM 4.24. *Assume the conjectures (FK) and let* $(\mathbb{T}, \mathbb{T}^0)$ *be a pair of Galois representations associated to a motive* M. *Let* $\beta : \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^+) \to \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^0)$ *be an isomorphism as above.*

(1) *If* Σ *is* $\Sigma(\mathbb{T}, \mathbb{T}^0)_p$ *(resp.,* $\Sigma(U, \mathbb{T}, \mathbb{T}^0)_p$*) is the subcategory associated to the Selmer complex, then the canonical map* $K_1(\Lambda_p, \Sigma) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda}_p) \to K_0(\Sigma)_p$ *sends* $\zeta_\beta(M, F_\infty/F)$ *(resp.,* $\zeta_\beta(U, M, F_\infty/F)$ *) to the class of the Selmer complex* $SC(\mathbb{T}, \mathbb{T}^0)$ *(resp.,* $SC(U, \mathbb{T}, \mathbb{T}^0)$*). We can drop the index* p *if* G *does not have any* p*-torsion and* $p \neq 2$.

(2) *For some extension* K' *of* K, *let* $\rho : G \to GL_n(K')$ *be a homomorphism factoring over a finite quotient of* G *and* $M(\rho^*) := [\rho^*] \otimes_K M$ *be the twisted motive. For some integer* j, *we assume that* $M(\rho^*)(j)$ *is critical (i.e., the conditions from theorem 4.19 are satisfied) and let* $(V, V^0(v))$ *be the* λ*-adic representation associated to* $M(\rho^*)(j)$. *Then the value of* $\zeta_\beta(M, F_\infty/F)$ *(resp.* $\zeta_\beta(U, M, F_\infty/F)$ *) at* $\rho\kappa^{-j}$ *is*

$$L'_K(M(\rho^*), j)\Omega_\infty(\tau(M(\rho^*))(j))^{-1}\Omega_p(\tau(M(\rho^*))(j)) \prod_{r\geq 1} \Gamma(r)^{h(j-r)}$$

$$\prod_{v|p}[P_{L,v}(V, u)P_{L,v}(V^0(v), u)^{-1}]_{u=1} P_{L,v}((V^0(v))^*(1), 1) \prod_{v\in A} P_{L,v}(V, 1) \ ,$$

where A *is the set of all places of* F *not dividing* p *and where* F_∞/F *is infinitely ramified (resp., the set of all places not lying over* U *or* p*).*

The specialization properties of the imprimitive Selmer complexes which we noted at the beginning of the last chapter make it easy to generalize the main conjecture for motives that Fukaya and Kato derived from their conjectures (FK) to families in the imprimitive setup: We have already seen that characteristic elements of the big Selmer complex specialize to characteristic elements of the Selmer complexes of the specializations. In the last section, we also discussed that the ζ-isomorphism of the family maps onto the one of the specialization. Thus, we get as a corollary:

COROLLARY 4.25. *Let* $(\mathbb{T}, \mathbb{T}^0)$ *be a pair of Galois representations associated to a family of motives* M_t *and a* p*-adic Lie extension* F_∞/F. *Assuming the conjecture (FK), the* ζ*-isomorphism of a family* $\zeta(U, M_t, F_\infty/F)$ *maps to the class of the Selmer complex* $SC(U, \mathbb{T}, \mathbb{T}^0)$ *under the boundary map and maps to the* ζ*-isomorphisms of the specializations* $\zeta(U, M_\phi, F_\infty/F)$ *under the specialization map* ϕ. *Moreover, it interpolates the critical values of the* L*-functions of the specializations in the sense of the last theorem.*

PROOF. Theorem 4.16 tells us that $\zeta(U, \mathbb{T}, \mathbb{T}^0)$ maps to the class of the Selmer complex, corollary 4.18 tells us that the specializations are the correct ones, and theorem 4.24 gives us the values in terms of values of the L-functions. □

The analog statement in the primitive setup is derived from the imprimitive one: Firstly, we note that characteristic elements of $SC(\mathbb{T}, \mathbb{T}^0)$ map to characteristic elements of $SC(\mathbb{T}_\phi, \mathbb{T}_\phi)$ under our standard assumptions:

PROPOSITION 4.26. *Again, let* $(\mathbb{T}, \mathbb{T}^0)$ *be a pair of Galois representations induced from a family of motives* M_t *and a Lie extension* F_∞/F. *Assuming that the condition on the ramification* 2.15 *is fulfilled, there is a characteristic element* $c(\mathbb{T}, \mathbb{T}^0)$ *for the Selmer complex* $SC(\mathbb{T}, \mathbb{T}^0)$. *Moreover, any such element has the following property: Take* Σ *to be the smallest subcategory of* $\mathcal{P}(\Lambda)$ *containing* $\Sigma(\mathbb{T}, \mathbb{T}^0)$ *and all complexes with p primary cohomology groups, and define* Σ_ϕ *accordingly. Then, for any specialization* $\phi : \mathcal{O}[[t]] \to \mathcal{O}'$, *we have that under the map*

$$K_1(\Lambda, \Sigma) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}) \to K_1(\Lambda_\phi, \Sigma_\phi) \times^{K_1(\Lambda_\phi)} K_1(\tilde{\Lambda}_\phi)$$

the element $c(\mathbb{T}, \mathbb{T}^0)$ *maps to a characteristic element of* $SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$.

PROOF. The only extra input we need here is that in the triangle

$$\Lambda_\phi \otimes^L_\Lambda SC(\mathbb{T}, \mathbb{T}^0) \to SC(\mathbb{T}_\phi, \mathbb{T}^0_\phi) \to C \xrightarrow{+}$$

from theorem 3.2, the class of the complex C in $K_1(\Sigma)$ is zero. But C can be obtained as the mapping fiber of $C' \to C'[-1]$ with C' in Σ. Thus, the class of C is $[C] = [C'] - [C'] = 0$.

□

REMARK 4.27. *If we take the stronger ramification condition from the remark after proposition 2.17, then we could replace the category* Σ *in the assertion by the smaller one* $\Sigma(\mathbb{T}, \mathbb{T}^0)$.

Next, we remark that, assuming the conjectures (FK) hold, the ζ-isomorphism is still a characteristic element for the Selmer complex in the primitive setup:

FACT 4.28. *The canonical map* $K_1(\Lambda, \Sigma(v)) \to K_0(\Sigma(v))$ *sends* $\zeta(v, \mathbb{T})$ *to the class* $-[C_f(F_v, \Lambda \otimes T)]$. *Moreover, if the conjectures (FK) are satisfied and* $(\mathbb{T}, \mathbb{T}^0)$ *is a pair of Galois representations associated to a family of motives* M_t *satisfying the condition 2.15, then* $\zeta_\beta(M_t, F_\infty/F)$ *is a characteristic element of* $SC(\mathbb{T}, \mathbb{T}^0)$.

PROOF. This is obvious from the definition of the ζ elements and the boundary map sending a pair $[C, a] \in K_1(\Lambda, \Sigma)$ to the class C in $K_0(\sigma)$. □

Finally, we show that the class of the ζ-element of a family maps onto the one of a specialization.

PROPOSITION 4.29. *We assume the conjectures (FK) hold. Let* $(\mathbb{T}, \mathbb{T}^0)$ *be a pair of Galois representations associated to a family* M_t *and* F_∞/F *be a p-adic Lie extension satisfying the condition 2.15. Moreover, let* $\beta : \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^+) \to \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^0)$ *be an isomorphism as above and let* $\phi : \mathcal{O}[[t]] \to \mathcal{O}_\phi$ *be a specialization of the family. If we write* $\beta' : \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^+_\phi) \to \tilde{\Lambda} \otimes Det_\Lambda(\mathbb{T}^0_\phi)$ *for the induced isomorphism, then the canonical morphism*

$$K_1(\Lambda_p, \Sigma(\mathbb{T}, \mathbb{T}^0)_p) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda}_p) \to K_1(\Lambda_{\phi,p}, \Sigma(\mathbb{T}_\phi, \mathbb{T}^0_\phi)_p) \times^{K_1(\Lambda_{\phi,p})} K_1(\tilde{\Lambda}_{\phi,p})$$

sends $\zeta(M_t, F_\infty/F)$ *to* $\zeta(M_\phi, F_\infty/F)$. *In particular, if* $\rho : G \times \mathbb{Z}_p \to K'$ *is a character with values in a p-adic field* K' *containing* $\mathcal{O}_\phi$ *such that* $\rho|_{\mathbb{Z}_p} = \phi|_{\mathbb{Z}_p \subset \mathcal{O}[[\mathbb{Z}_p]] = \mathcal{O}[[t]]}$, *then the value of* $\zeta_\beta(M_t, F_\infty/F)$ *at* ρ *coincides with the value of* $\zeta_{\beta'}(M_\phi, F_\infty/F)$ *at* $\rho|_G$.

PROOF. The second assertion follows obviously from the first one.

If we replaced $\zeta_\beta(M_t, F_\infty/F)$ with $\zeta_\beta(U, M_t, F_\infty/F)$ and the specialized elements accordingly, then the claim would be contained in corollary 4.25. The only things we have to take care of are thus the Euler factors. To put it differently: We have to show that the difference of the Euler factor of the specialization and the image of the Euler factor of the family is 1 if we invert p. However, that is an easy application of lemma 4.5 together with the last part of proposition 2.17. □

REMARK 4.30. *The reason why we had to invert p here is that, in general, the Euler factors of the family will not map to those of the specialization, when we do not invert p. This is due to the fact that the last part of proposition 2.17 is not as good as one would hope. As remarked directly after this proposition, this issue can be fixed by replacing the condition 2.15 on the ramification by a stronger one. It is easy to see that in that case we are able to prove the last proposition without inverting p. The same holds for the next theorem.*

Combining the results, we have proven:

THEOREM 4.31 (Iwasawa main conjecture for families). *Assume that the conjectures (FK) hold. Let M_t be a family of motives satisfying the condition 2.12 and let F_∞/F be a Lie extension as in section 2.2 inducing a pair of Galois representations $(\mathbb{T}, \mathbb{T}^0)$, and let β be an isomorphism as above. If we assume furthermore that the condition 2.15 is satisfied, then there is a ζ-element $\zeta_\beta(M_t, F_\infty/F) \in K_1(\Lambda_p, \Sigma(\mathbb{T}, \mathbb{T}^0)_p) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda}_p)$ with the following properties:*

(1) *Under the boundary map of the long exact sequence of K-theory, the element $\zeta_\beta(M_t, F_\infty/F)$ maps to the class of $SC(\mathbb{T}, \mathbb{T}^0)$ in $K_0(\Sigma(\mathbb{T}, \mathbb{T}^0)_p)$.*
(2) *Under specialization maps ϕ, the isomorphism $\zeta_\beta(M_t, F_\infty/F)$ is mapped to $\zeta_{\beta_\phi}(M_\phi, F_\infty/F)$ in $K_1(\Lambda_{\phi,p}, \Sigma(\mathbb{T}_\phi, \mathbb{T}^0_\phi)_p) \times^{K_1(\Lambda_{\phi,p})} K_1(\tilde{\Lambda}_{\phi,p})$.*
(3) *Assume that ϕ is a specialization, ρ is an Artin character of G, and j is an integer such that $M_\phi(\rho^*)(j)$ is critical as in theorem 4.24, and let $\rho' : \mathbb{Z}_p \times G \to K'$ be ρ on G and ϕ on $\mathbb{Z}_p$. Then, the value of $\zeta_\beta(M_t, F_\infty/F)$ at $\rho'\kappa^{-j}$ is given by*

$$L_{K'}(M_\phi(\rho^*), j)\Omega_\infty(\tau(M_\phi(\rho^*))(j))^{-1}\Omega_p(\tau(M_\phi(\rho^*))(j)) \prod_{r\geq 1} \Gamma(r)^{h(j-r)}$$

$$\prod_{v|p} [P_{L,v}(V, u) P_{L,v}(V^0(v), u)^{-1}]_{u=1} P_{L,v}((V^0(v))^*(1), 1) \prod_{v\in A} P_{L,v}(V, 1) \ ,$$

where again $(V, V^0(v))$ is the representation associated to $M_\phi(\rho^*)(j)$ and A is as above.

It would be desirable to have a similar result with the dual Selmer groups in place of the Selmer complexes. It seems likely that we can take the same ζ-element for the Selmer group if we take $\mathfrak{M}_H(G)$ as the subcategory defining our localized K_1 and assume certain conditions on G and our motive. Fukaya and Kato gave an example when this is the case (see corollary 4.3.18 in [**FK06**]). In more general settings, we get a correction factor which also occurs in interpolation properties. But at least, we get one factor for the family, which specializes to factors for the motives:

COROLLARY 4.32. *Let M_t, $G = Gal(F_\infty/F)$ and $(\mathbb{T}, \mathbb{T}^0)$ be as in the last theorem. Again, we require condition 2.15 to be fulfilled and assume in addition* *that $dim(G) \geq 2$ as a p-adic Lie group. Let $S-tor$ denote the triangulated subcategory of $P(\Lambda)$ of complexes with S-torsion cohomology groups. Moreover, let $\mathcal{G} := ker(G_F \to G)$ and set $\mathcal{G}(v) := ker(G_{F_v} \to G)$ and $G_v := Im(G_{F_v} \to G)$ for places v of F dividing p. Then, there is an element $\xi \in K_1(\Lambda, S-tor)$ mapping to the class $[(T(-1)_{\mathcal{G}})] - \sum_{v|p}[\Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1)_{\mathcal{G}(v)})]$ in $K_0(S-tor)$. Furthermore, if we take Σ to be the smallest triangulated subcategory containing $S-tor$ and $\Sigma(\mathbb{T}, \mathbb{T}^0)$, then Σ contains a complex quasi-isomorphic to $\mathcal{X}(\mathbb{T}, \mathbb{T}^0)$, and $\zeta'_\beta(M_t, F_\infty/F) := \xi \cdot \zeta_\beta(M_t, F_\infty/F) \in K_1(\Lambda, \Sigma) \times^{K_1(\Lambda_p)} K_1(\tilde{\Lambda}_p)$ has the following properties:*

1. *Under the boundary map of the long exact sequence of K-theory, the element $\zeta'_\beta(M_t, F_\infty/F)$ maps to the class of $\mathcal{X}(\mathbb{T}, \mathbb{T}^0)$ in $K_0(\Sigma_p)$.*
2. *For a specialization map ϕ we define ξ_ϕ to be the image of ξ in $K_1(\Lambda_\phi, \Sigma_\phi)$. Then, the isomorphism $\zeta'_\beta(M_t, F_\infty/F)$ is mapped to $\zeta'_{\beta_\phi}(M_\phi, F_\infty/F) := \xi_\phi \cdot \zeta'_{\beta_\phi}(M_\phi, F_\infty/F)$ in $K_1(\Lambda_{\phi,p}, \Sigma_p) \times^{K_1(\Lambda_{\phi,p})} K_1(\tilde{\Lambda}_{\phi,p})$, and $\zeta'_{\beta_\phi}(M_\phi, F_\infty/F)$ is a characteristic element for $\mathcal{X}(\mathbb{T}_\phi, \mathbb{T}^0_\phi)$.*
3. *Assume that ϕ is a specialization, ρ is an Artin character of G, and j is an integer such that $M_\phi(\rho^*)(j)$ is critical as in theorem 4.24, and let $\rho' : \mathbb{Z}_p \times G \to K'$ be ρ on G and ϕ on $\mathbb{Z}_p$. Then, the value of $\zeta'_\beta(M_t, F_\infty/F)$ at $\rho'\kappa^{-j}$ is given by*

$$L_{K'}(M_\phi(\rho^*), j)\Omega_\infty(\tau(M_\phi(\rho^*))(j))^{-1}\Omega_p(\tau(M_\phi(\rho^*))(j)) \prod_{r \geq 1} \Gamma(r)^{h(j-r)}$$
$$\prod_{v|p}[P_{L,v}(V, u)P_{L,v}(V^0(v), u)^{-1}]_{u=1} P_{L,v}((V^0(v))^*(1), 1)$$
$$\prod_{v \in A} P_{L,v}(V, 1) \cdot \xi(\rho) \ ,$$

   where again $(V, V^0(v))$ is the representation associated to $M_\phi(\rho^*)(j)$ and A is as above.

PROOF. As remarked before, it is well known that $K_1(\Lambda, S-tor) = K_1(\Lambda_S) \to K_0(S-tor)$ is surjective. Moreover, $T(-1)_{\mathcal{G}}$ is finitely generated over $\mathcal{O}$; therefore it is S-torsion and as $G_v \to G/H$ is surjective, all the modules $\Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0(v)(-1)_{\mathcal{G}(v)})$ are S-torsion, too. Thus, ξ exits.

It follows from the long exact sequence in proposition 2.23 and the vanishing of $H^1(SC(\mathbb{T}, \mathbb{T}^0))$ that $\zeta'_\beta(M_t, F_\infty/F)$ is a characteristic element for the family. For the second assertion all we have to show is that ξ_ϕ maps to the class $[(T_\phi(-1)_{\mathcal{G}})] - \sum_{v|p}[\Lambda \otimes_{\mathcal{O}[\![G_v]\!]} (T^0_\phi(v)(-1)_{\mathcal{G}(v)})]$, but that is obvious as tensoring commutes with taking coinvariants (as seen in lemma 1.3).

Finally, the third part follows directly from the third part of the theorem. □

CHAPTER 5

Complements and Examples

In this chapter, we will discuss what kind of families satisfy our conditions and will start by summing up the deformation theory for our situation. Then we will go on to show that for any finite number of given motives the one parameter setting we look at is as good as the n-parameter setting, and we will generalize our results of the variational behavior of the Iwasawa invariants. Finally, we would like to give examples of some families of motives allowing infinitely many specializations.

5.1. Basics on deformation theory

In this section, we would like to give the basic definitions and results of the deformation theory of the kind of Galois representations that we are interested in.

Let $\mathcal{O}$ be the ring of integers of a p-adic field with residue field $\mathbb{F}$. Very generally speaking, by a deformation problem we understand a functor D from $\mathcal{C}_{\mathcal{O}}$, the category of local complete commutative Noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$, to the category *Set* of sets. More precisely, for our discussion it is sufficient to take $\mathcal{O}$ to be the ring of Witt vectors of a finite field $\mathbb{F}$ and for some profinite group G. Recall that a subfunctor F of a set valued functor G is a functor such that for any object M the set $F(M)$ is naturally a subset of $G(M)$ and for any morphism f the map $F(f)$ is the restriction of $G(f)$. The functors we are interested in are subfunctors of one of the following two functors:

(1) Fixing a finite dimensional G-representation $V_{\mathbb{F}}$ over $\mathbb{F}$, there is the universal deformation functor D_G sending a local $\mathcal{O}$-algebra $(A, \mathfrak{m})$ to the set of equivalence classes (V_A, ϕ) of (free) G-representations V_A over A, together with an $\mathbb{F}$-isomorphism $\phi : A/\mathfrak{m} \otimes_a V_A \to V_{\mathbb{F}}$.

(2) Keeping in mind the notation of (1), we assume in addition that we are given an $\mathbb{F}$-basis β_β of $V_{\mathbb{F}}$. We then define the deformation functor of framed representations $D_G^{\square}$ to send A to the set of equivalence classes of pairs (V_A, β_A) of a free G-representation V_A over A with a basis β_A, such that the identification of $\beta_{\mathbb{F}}$ and the image of β_A in $A/\mathfrak{m} \otimes V_A$ induces an isomorphism of G-representations.

In general, the first functor is not representable. However, we have the following theorem due to Mazur:

THEOREM 5.1. *Let us assume that the maximal pro-p quotient of any open subgroup of G is finitely generated. Then, if $End_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, the functor D_G is representable. If we drop the condition on the endomorphism, then there is still a versal hull, i.e., there is a ring $R^v \in \mathcal{C}_{\mathcal{O}}$ and a natural transformation $Hom_{\mathcal{C}_{\mathcal{O}}}(R^v, \bullet) \to D_G$, which is surjective, and an isomorphism for $Hom_{\mathcal{C}_{\mathcal{O}}}(R^v, \mathbb{F}[\varepsilon]/\varepsilon^2) \to D_G(\mathbb{F}[\varepsilon]/\varepsilon^2)$. Finally, still assuming the condition on G, the functor $D_G^{\square}$ is representable.*

A proof of the statements on D_G can be found in [**Maz89**]. The assertion on $D_G^\square$ is well known. A short proof of it can be found in [**Böc10**] (proposition 5.1 in lecture 1).

In particular the condition on G is satisfied for many Galois groups: We are interested in Galois representations of G_F, the absolute Galois group of a number field F. In section 2.2 we mentioned that our representation should be unramified outside a finite set, so let S be a finite set of places of F containing the set S_p of all places above the fixed prime p and all infinite places. We are interested in representations of $G_S := G_{S,F} := Gal(F_S/F)$, where F_S is the maximal extension of F, which is unramified outside S. Moreover, for a place v of F we denote by G_v the absolute Galois group of F_v. Then we note that G_S and G_v both satisfy the finiteness condition. We abbreviate the deformation functors as follows: $D_S := D_{G_S}$, $D_S^\square := D_{G_S}^\square$ and $D_v^\square := D_{G_v}^\square$.

Now let us turn to condition 2.12. Please recall that it states there are subrepresentations $T(v)$ for all places v dividing p that interpolate the ones given by the Dabrowski-Panchishkin condition for that specializations. This translates to the representation being nearly ordinary:

DEFINITION 5.2. *We will call a free representation V of G_v over a ring A nearly ordinary of rank n_0 if there is a free A-direct G_v-stable summand V^0 of V of rank n_0. A representation of G_S is called nearly ordinary of rank n_0 at p if for all places v of F dividing p the restriction to G_v is nearly ordinary of rank n_0.*

We fix some n_0 for the rest of this and the following sections. All nearly ordinary representations will b tacitely assumed to be nearly ordinary of rank n_0.

REMARK 5.3. *This is a very narrow class of nearly ordinary representations. There is a wide array of more general notions. A much broader class containing the case described above was studied by Tilouine in his book* [**Til96**].

Moreover, we assume that $V_{\mathbb{F}}$ is nearly ordinary and fix subrepresentations $(V_{\mathbb{F}})_v^0$ as in the definition. Then, the ordinary deformation functor $D_v^{n.o.}$ is defined to send a ring A in $\mathcal{C}_{\mathcal{O}}$ to equivalence classes of triples (V_A, V_A^0, ϕ) of a G_v-representation V_A and a subrepresentation V_A^0 of rank n_0 and an isomorphism $\phi : A/\mathfrak{m} \otimes V_A \to V_{\mathbb{F}}$ that identifies the image of V_A^0 with $V_{\mathbb{F}}^0$. Similarly, $D_S^{n.o.}(A)$ is defined to be a triple $(V_A, (V_{A,v}^0)_{v|p}, \phi)$ with the obvious compatibilities.

To define the functor $D_v^{\square,n.o.}$, we agree to the convention that the basis is chosen in such a way that the subspace V_0 is given as the span of the first n_0 basis vectors. Finally $D_S^{\square,n.o.}$ is defined with a little twist: Instead of just one basis on $V_{\mathbb{F}}$ resp. a deformation, we fix a family $(\beta_v)_{v|p}$ of bases such that the submodule for G_v is given by the first n_0 basis vectors of β_v.

The question when (a much more general class than) the ordinary deformation functors are representable has been studied by Tilouine in [**Til96**] with some corrections in Mauger's thesis [**Mau04**]. Many results where reproduced and generalized by Böckle in [**Böc99**] and [**Böc07**]. In particular propositions 3.3 and 3.4 in [**Böc07**] can be applied to our situation as follows:

PROPOSITION 5.4. *The functors $D_v^\square$ and $D_v^{\square,n.o.}$ are representable. If the functors $D_v^{n.o.}$ are subfunctors of D_v, then $D_v^{n.o.}$ and $D_S^{n.o.}$ admit versal hulls. If furthermore D_S is representable, then so is $D_S^{n.o.}$, and if D_v is representable, then so is $D_v^{n.o.}$.*

PROOF. This is almost exactly the statement of the quoted theorems. These theorems need two conditions to be fulfilled: Firstly that the functors D_v admit a versal hull. But that is the theorem of Mazur quoted above as theorem 5.1. Secondly, we have to observe that the natural transformation $D_v^{n.o.} \to D_v$ is relatively representable in the sense of definition 3.1 (loc.cit.). But this condition can be stated as follows: Assume $A_1 \to A_0$ and $A_2 \to A_0$ are surjective morphisms of Artinian rings in $\mathcal{C}_{\mathcal{O}}$ and set $A := A_1 \times_{A_0} A_2$. Furthermore, assume that V_A is a representation of G_v and that there are subrepresentations $V_{A_i}^0 \subset V_{A_i} := A_i \otimes V_A$ for $i = 1, 2$ making V_{A_i} nearly ordinary such that $A_0 \otimes_{A_1} V_{A_1}^0 = A_0 \otimes_{A_2} V_{A_2}^0$ in $V_{A_0} := A_0 \otimes_A V_A$. There is then a unique subrepresentation V_A^0 in V_A, making V_A ordinary and mapping to V_{A_i} for $i = 1, 2$. As we have $V_A = V_{A_1} \times_{V_{A_0}} V_{A_2}$, we can thus take $V_A^0 := V_{A_1}^0 \times_{V_{A_0}} V_{A_1}^0$ and the claim follows. □

REMARK 5.5. *That $D_v^{n.o.}$ is a subfunctor of D_v is the case if there is only one choice for lifts of $V_{\mathbb{F}}^0$. This is the case, for instance, if there is a subgroup I of G_v such that $V_{\mathbb{F}}^0 = V_{\mathbb{F}}^I$ and $(V_{\mathbb{F}}/V_{\mathbb{F}}^0)^I = 0$. Or if $V_{\mathbb{F}}/V_{\mathbb{F}}^0$ are the I coinvariants of $V_{\mathbb{F}}$ and the coinvariants of $V_{\mathbb{F}}^0$ vanish.*

These are the two likely instances of a more general criterion used by Tilouine in [**Til96**]*: Let ad denote the module of endomorphisms of $V_{\mathbb{F}}$ where the G_v-action is the conjugation. Denote by $ad' \subset ad$ the submodule of endomorphisms that send $V_{\mathbb{F}}^0$ to itself. If $H^0(G_v, ad/ad')$ vanishes, then $D_v^{n.o.}$ is a relative representable subfunctor of D_v.*

5.2. Obstructions for ordinary deformations

We have thus found that our functors are usually representable or at least admit a versal hull. Thereby, we have translated the problem of finding families containing some specified representations over $\mathcal{O}$ to finding $\mathcal{O}$-algebra morphisms of the universal or versal ring into the $\mathcal{O}[[t]]$ over which the morphisms given by the specified $\mathcal{O}$ representations factorize. We will see below that this is possible for a finite number of given representations lifting the same residual representation, provided that the (uni-)versal ring is regular. This is the case, if the deformation problem is unobstructed. More precisely, in general the problem of lifting a given representation to an infinitesimally larger ring produces an obstruction class in some H^2-group. The problem is called "unobstructed" if this the H^2 vanishes. We will see that in the case of nearly ordinary deformation problems we are dealing with the second cohomology group of a Selmer complex (for a finite representation).

Let us first recall how deformation rings are constructed in general: Firstly, the tangent space of a deformation functor D is defined to be $\mathfrak{t}_D := D(\mathbb{F}[\varepsilon]/\varepsilon^2)$. If the functor is representable by a ring R or - more generally - admits a versal hull, then the tangent space can be recovered as the mod p Zariski tangent space of R: $(\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}_{\mathcal{O}}))^*$, where the asterisk denotes the $\mathbb{F}$ linear dual and $\mathfrak{m}_{\mathcal{O}}$ is (the ideal in R generate by) the maximal ideal of $\mathcal{O}$. Thus, if the $\mathbb{F}$-dimension of $\mathfrak{t}_D$ is d, then R has a presentation:

$$0 \to J \to \mathcal{O}[[t_1, ..., t_d]] \to R \to 0$$

The minimal number of variables t_i needed in such a presentation is d. Therefore this is a minimal presentation and R is regular (equivalently smooth over $\mathcal{O}$) if and only if $J = 0$. As the ring $\mathcal{O}[[t_1, ..., t_d]]$ is local Noetherian, it is equivalent to show that $J = 0$ and that $J' := J/\mathfrak{m}J = 0$. One can typically show that the ($\mathbb{F}$-linear) dual of J' maps injectively into some second order cohomology group. In particular, the following cases are known:

PROPOSITION 5.6. *We denote tangent spaces, (uni-)versal deformation rings, and the ideals J' with the same indices as the deformation functors they are associated to. Moreover, let* ad *denote the $\mathbb{F}$-endomorphisms of $V_{\mathbb{F}}$ made into a G_S module by conjugation. Then the following holds:*

(1) *The tangent space $\mathfrak{t}_v$ of D_v is naturally isomorphic to $H^1(G_v, ad)$ and $(J'_v)^*$ maps injectively into $H^2(G_v, ad)$.*
(2) *Assuming that $V_{\mathbb{F}}$ is ordinary as a G_v-representation, with $V^0_{\mathbb{F}}$ being the distinguished subspace we set ad' to be the submodule of endomorphisms that send $V^0_{\mathbb{F}}$ to itself. We assume that $H^0(G_v, V_{\mathbb{F}}) = 0$. Then the tangent space of $D_v^{n.o.}$ is isomorphic to $H^1(G_v, ad'_v)$ the functor admits a versal hull and $(J_v^{n.o.\prime})^*$ maps injectively into $H^2(G_v, ad'_v)$.*
(3) *The global tangent space $\mathfrak{t}_S$ is canonically isomorphic to $H^1(G_S, ad)$. Moreover, the dual $(J'_S)^*$ maps injectively into $H^2(G_S, ad)$.*

The first and the last assertion are special cases of a deformation functor without extra conditions. They are due to Mazur and can be found in [**Maz89**]. The middle assertion is proven in the same manner. A very conceptual view of it can be found in [**Böc07**] as proposition 6.3.

As mentioned above, these are not the rings we are most interested in: The case of the global nearly ordinary deformation rings, or rather, the case of just one place v dividing p, can be understood with a certain Selmer complex for the representations ad and ad'_v. In the case that $H^0(G_v, ad/ad'_v) = 0$ this result can be found spread out through chapter 6 of the book of Tilouine [**Til96**]. However, as we the deformation functors defined above are slightly different from the ones of Tilouine if the above condition is not satisfied, in general our statement has a different appearance and seems to be a bit more uniform.

THEOREM 5.7. *Assume that v is the only place of F dividing p. We define $SC(ad, ad'_v)$ to be the mapping fiber of the following map induced by the restriction morphism:*

$$C^\bullet(G_S, ad) \to C^\bullet(G_v, ad/ad'_v)$$

With this definition we have: The tangent space $\mathfrak{t}_S^{n.o.}$ is canonically isomorphic to $H^1(SC(ad, ad'_v))$. Furthermore, if $D_S^{n.o.}$ has a versal hull, then we define $(J_S^{n.o.})'$ as above and its $\mathbb{F}$-linear dual maps injectively into the group $H^2(SC(ad, ad'_v))$.

PROOF. Firstly, let us observe that the Selmer complex can be described as the subcomplex of those cochains in $C^\bullet(G_S, ad)$, whose restriction to G_v has images contained in ad'_v.

To define the isomorphism with the tangent space $\mathfrak{t}_{D_S^{n.o.}} = D_S^{n.o.}(\mathbb{F}[\varepsilon]/\varepsilon^2)$, recall that an element of the tangent space is given by a triple $(V_{\mathbb{F}[\varepsilon]/\varepsilon^2}, V^0_{\mathbb{F}[\varepsilon]/\varepsilon^2}, \phi)$. The morphism ϕ identifies $V_{\mathbb{F}[\varepsilon]/\varepsilon^2}/\varepsilon$ canonically with $V_{\mathbb{F}}$ sending $V^0_{\mathbb{F}[\varepsilon]/\varepsilon^2}$ to $V^0_{\mathbb{F}}$. Moreover the multiplication with ε induces an isomorphism $V_{\mathbb{F}[\varepsilon]/\varepsilon^2}/\varepsilon \overset{\cong}{\to} V_{\mathbb{F}[\varepsilon]/\varepsilon^2}[\varepsilon]$. Using this decomposition we conclude, that every triple as above is isomorphic to one of the form $(V_{\mathbb{F}} \oplus V_{\mathbb{F}}, V^0_{\mathbb{F}} \oplus V^0_{\mathbb{F}}, \pi_1)$ where the multiplication by ε sends a pair (v, w) to $(0, v)$ and π_1 is the projection to the first component. The G_S-operation

ρ on $V_{\mathbb{F}} \oplus V_{\mathbb{F}}$ is of the form $\rho(g)(v,w) = (\bar{\rho}(g)v, \bar{\rho}(g)w + \psi(g)(w))$, where ψ is a map from G_S to ad. For the G_S-operation to stabilize $V^0_{\mathbb{F}} \oplus V^0_{\mathbb{F}}$ we need that for all $g \in G_v$ the morphism $\psi(g)$ is contained in ad'_v.

A small computation shows that $c := \psi \cdot \bar{\rho}^{-1}$ is actually a 1-cocycle in $SC(ad, ad'_v)$ and every such cocycle corresponds to a ψ induced by an operation ρ as above. Finally, straightforward computations also show that two morphisms ψ and ψ' coming from isomorphic triples of the described form, differ by a boundary in $SC(ad, ad'_v)$. We have thus proven the assertion on the tangent space.

Now, to map $(J_S^{n.o.\prime})^*$ into $H^2(SC(ad, ad'_v))$, we will follow the usual strategy of lifting the versal representation to a bigger ring, as applied, for instance, for the deformation ring R_S in [**Böc99**] theorem 2.4:

Firstly, let $f \in (J_S^{n.o.\prime})^*$ be a nonzero element, i.e., f is a surjective $\mathbb{F}$-linear morphism $J_S^{n.o.\prime} \to \mathbb{F}$. Composing f with the natural projection $J_S^{n.o.} \to J_S^{n.o.\prime}$, the projection $\pi\mathcal{O}[[t_1,...,t_d]] \to R_S^{n.o.}$ factors as $\mathcal{O}[[t_1,...,t_d]] \xrightarrow{f} R' \xrightarrow{\pi'} R_S^{n.o.}$, where the kernel of π' is identified with $\mathbb{F}$ and the restriction of f to $J_S^{n.o.}$ is our chosen f.

The versal deformation ring $R_S^{n.o.}$ comes together with an ordinary versal representation $\rho : G_S \to V_{R_S^{n.o.}}$, with a G_v-invariant submodule $V^0_{R_S^{n.o.}}$. We extend these $R_S^{n.o.}$ modules to R' modules, such that $V^0_{R'}$ is still a free R'-direct summand of the free R'-module $V_{R'}$. Then, we can choose a continuous (but not necessarily homomorphic) lift ρ' of ρ:

$$\begin{array}{ccc} G_S & \xrightarrow{\rho'} & Aut_{R'}(V_{R'}) \\ & \searrow^{\rho} & \downarrow \\ & & Aut_R(V_R) \end{array}$$

Moreover, we can choose ρ' in such a way that the images of elements of G_v stabilize $V^0_{R'}$. Setting $c_f(g,h) := \rho'(gh)\rho'(h)^{-1}\rho'(g)^{-1}$, we observe that ρ' is a homomorphism if and only if $c_f = 0$ as a map. As ρ is a homomorphism, it follows that the image of c_f is contained in the kernel of $Aut'_R(V_{R'}) \to Aut_{R_S^{n.o.}}(V_{R_S^{n.o.}})$, which is canonically isomorphic to ad and elements of G_V are mapped to ad'_v. One can easily verify that $c_f(gh,k)c_f(g,h) = \rho'(ghk)\rho'(k)^{-1}\rho'(h)^{-1}\rho'(g)^{-1} = c_f(g,hk)c_f(h,k)^g$; therefore, c_f is a 2-cycle in $SC(ad, ad'_v)$. Moreover, choosing a different lift $\rho'' = z \cdot \rho'$ with $z : G_S \to ad$ corresponds to changing c_f by minus the boundary of z. Therefore we have shown: The choice of f gives a well-defined class in $H^2(SC(ad, ad'_v))$ which is zero if and only if it is possible to lift ρ to a homomorphism ρ', satisfying the usual ordinary condition.

It remains to show that nonzero f never map to zero classes. So assume that there is a homomorphic lift ρ' of ρ. The versal property of $R_S^{n.o.}$ tells us that there is a map $R_S^{n.o.} \to R'$. By construction, this morphism is an isomorphism on the tangent spaces, and $R_S^{n.o.}$ is complete; thus, the morphism is surjective. However, $R' \to R_S^{n.o.}$ is surjective by definition. It is well known that surjective ring endomorphisms of Noetherian local complete rings are isomorphisms, thus the projection $R' \to R_S^{n.o.}$ is injective, which is a contradiction to the assumption that f was surjective. □

5.3. Finite families

In the previous chapters, we were looking at one-parameter families of Galois representations. When one is looking for families admitting multiple specializations, allowing only one parameter seems to be a rather strong restriction. In the last section we studied deformation problems and discussed how common it is for them to be unobstructed. In the case of a finite number of Galois representations corresponding to points of such an unobstructed deformation functor, we will show that they are indeed (almost) members of such a family.

In this section, all rings are assumed to be commutative and Noetherian.

Before we turn to the general statement, let us discuss the case of two representations: We assume that there are two representations over some p-adic rings of integers $\mathcal{O}_1$ and $\mathcal{O}_2$ that correspond to two ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in a deformation ring R, i.e., we assume that $\mathcal{O}_i = R/\mathfrak{p}_i$. In general, there is just some morphism $\mathcal{O}[[t]] \to \mathcal{O}_i$, but we assume it to be surjective. We aim to find an ideal $\mathfrak{q} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ such that $R/\mathfrak{q}$ is regular. We note that for the results concerning the variation of Iwasawa invariants this is already everything we need. However, for the predictions of the main conjecture a family interpolating more than two representations carries additional information.

We will deduce the following result:

Theorem 5.8. *Let R be a local (commutative Noetherian) ring and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be ideals such that $R/\mathfrak{p}_i$ is regular of dimension one for $i = 1, 2$, Then, there is an ideal $\mathfrak{q} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ such that $R/\mathfrak{q}$ is regular of dimension at most 2 if and only if there is an ideal $\mathfrak{q}'$ contained in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ such that $R/\mathfrak{q}'$ is regular.*

Some preliminary work has to be done before proving this theorem. Recall that a regular parameter sequence in a local ring R is a sequence $x_1, ..., x_n$ of elements of the maximal ideal $\mathfrak{m}$ such that their images in the $R/\mathfrak{m}$ vector space $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. With this definition, one can characterize the ideals in regular rings such that the quotient rings are regular:

Proposition 5.9. *Let R be a regular local ring and $\mathfrak{p} \subset R$ be an ideal. Then $R/\mathfrak{p}$ is regular if and only if $\mathfrak{p}$ is generated by a regular parameter sequence.*

A proof of this fact can be found in [**Mat89**] theorem 14.2.

Thus, with $\mathfrak{p}_1$ and $\mathfrak{p}_2$ as in the theorem it is equivalent to ask for an ideal $\mathfrak{q}$ contained in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ either that $R/\mathfrak{q}$ is regular or that the images of both the ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $R/\mathfrak{q}$ are generated by regular sequences.

We will use the following well-known fact:

Fact 5.10. *Let R be a commutative ring and $\mathfrak{a}_1$ and $\mathfrak{a}_2$ two ideals, then*

$$R/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \cong R/\mathfrak{a}_1 \times_{R/(\mathfrak{a}_1+\mathfrak{a}_2)} R/\mathfrak{a}_2.$$

To put it in geometrical terms: The set theoretic union of two reduced closed subschemes is isomorphic to the scheme one gets by gluing them along their intersection.

Next, we need a special way of expressing elements of R with respect to ideals, which are almost maximal:

Lemma 5.11. *Let $(R, \mathfrak{m})$ be a local ring and let $\mathfrak{b}$ be an ideal of R and $x \in \mathfrak{m}$ such that $xR + \mathfrak{b} = \mathfrak{m}$. We assume in addition that there is a natural number*

k such that $\mathfrak{m}^k \subset \mathfrak{b}$. Then, every element $r \in \mathfrak{m}^2$ can be written in the form $r = bm + x^s u$, with $b \in \mathfrak{b}$, $m \in \mathfrak{m}$, $u \in R^\times$, and $s \geq 2$ an integer.

PROOF. Firstly, as $\mathfrak{b} + xR = \mathfrak{m}$, we have $\mathfrak{m}^2 = \mathfrak{b}\mathfrak{m} + x\mathfrak{m}$. Inserting the first equation again for the last $\mathfrak{m}$, we get $\mathfrak{m}^2 = \mathfrak{b}\mathfrak{m} + x(\mathfrak{b} + xR) = \mathfrak{b}\mathfrak{m} + x^2R$. Now $R = R^\times \cup (\mathfrak{b} + xR)$ and by induction we get $R = R^\times \cup xR^\times \cup ... \cup x^{k-1}R^\times \cup \mathfrak{b}$, where in the last step we used that by assumption $x^k \in \mathfrak{m}^k \subset \mathfrak{b}$. Inserting the right-hand side for R in the equation $\mathfrak{m}^2 = \mathfrak{b}\mathfrak{m} + x^2R$ gives the assertion. □

Now we are prepared to prove the theorem. The "only if" part is obvious and we rephrase the "if" part in the next lemma to streamline our notation:

LEMMA 5.12. *Let R be a regular local ring of dimension n and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two ideals of R such that $R/\mathfrak{p}_i$ are local regular rings of dimension 1, then there is an ideal $\mathfrak{q} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ such that $R/\mathfrak{q}$ is a local regular ring of dimension at most 2.*

PROOF. We first note that we may assume that the dimension of R is more than 2, because otherwise choosing $\mathfrak{q}$ to be the 0 ideal will trivially meet the requirements.

If the dimension of R is at least 3 we will find $\mathfrak{q}$ as in the lemma such that $dim(R/\mathfrak{q}) = 2$. Using proposition 5.9, it is enough to find a subset of a regular parameter sequence of length $n-2$, which is contained in $\mathfrak{p}_1 \cap \mathfrak{p}_2$. Now, geometrically speaking, the ideals correspond to two curves and we have to distinguish two cases: case 1, these curves intersect transversally in the sole closed point, and case 2, they do not. More precisely:

CASE 1 (transversal intersection): $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{m}$ in R.

In this case, $(\mathfrak{p}_1 + \mathfrak{m}^2)/\mathfrak{m}^2$ and $(\mathfrak{p}_2 + \mathfrak{m}^2)/\mathfrak{m}^2$ generate the n-dimensional $R/\mathfrak{m}$ vector space $\mathfrak{m}/\mathfrak{m}^2$. As they are both $n-1$ dimensional, the intersection has the dimension $n-2$. We want to lift a basis of it to $\mathfrak{p}_1 \cap \mathfrak{p}_2$, thus we have to prove that:

$$((\mathfrak{p}_1 \cap \mathfrak{p}_2) + \mathfrak{m}^2)/\mathfrak{m}^2 \cong ((\mathfrak{p}_1 + \mathfrak{m}^2)/\mathfrak{m}^2) \cap ((\mathfrak{p}_2 + \mathfrak{m}^2)/\mathfrak{m}^2)$$

The canonical map is clearly injective, and it remains to be shown that it is surjective. We prove that by showing dually that the map

$$(R/(\mathfrak{p}_1 \cap \mathfrak{p}_2))/\mathfrak{m}^2 \to (R/\mathfrak{m}^2)/((\mathfrak{p}_1 + \mathfrak{m}^2)/\mathfrak{m}^2 \cap (\mathfrak{p}_2 + \mathfrak{m}^2)/\mathfrak{m}^2)$$

is injective. Using the above fact, this map can be written as

$$(R/\mathfrak{p}_1 \times_{R/\mathfrak{m}} R/\mathfrak{p}_2)/\mathfrak{m}^2 \to R/(\mathfrak{m}^2 + \mathfrak{p}_1) \times_{R/\mathfrak{m}} R/(\mathfrak{m}^2 + \mathfrak{p}_2)\ .$$

Therefore, all we have to show is that $\mathfrak{m}^2(R/\mathfrak{p}_1 \times_{R/\mathfrak{m}} R/\mathfrak{p}_2) = \mathfrak{m}^2 R/\mathfrak{p}_1 \times \mathfrak{m}^2 R/\mathfrak{p}_2$ as ideals in $R/\mathfrak{p}_1 \times_{R/\mathfrak{m}} R/\mathfrak{p}_2$. As the first term is simply the image of $\mathfrak{m}^2 \subset R$ in this fiber product, it is enough to find preimages of all elements of the right-hand side in R. This last assertion follows from our assumption $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{m}$.

CASE 2 (non transversal intersection): $\mathfrak{p}_1 + \mathfrak{p}_2 \neq \mathfrak{m}$.

An application of Nakayama's lemma shows that the images of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $\mathfrak{m}/\mathfrak{m}^2$ cannot generate the whole $R/\mathfrak{m}$-vector space. But the images are $n-1$ dimensional, so that we conclude that the images coincide. Now choose elements $x_1, ..., x_{n-1}$ in $\mathfrak{p}_1$ which map to a basis of $(\mathfrak{p}_1 + \mathfrak{m}^2)/\mathfrak{m}^2 \subset \mathfrak{m}/\mathfrak{m}^2$ and $x'_1, ..., x'_{n-1}$ in $\mathfrak{p}_2$ which map to the same basis. Let $x_n \in \mathfrak{m}$ be an element such that $x_1, .., x_n$ map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. By definition, each difference $x_i - x'_i$ $(i = 1, ..., n-1)$ is contained in $\mathfrak{m}^2$. Using the last lemma we can write each difference as $x_i - x'_i = q_i^{(1)} - q_i^{(2)} + x_n^{k_i} u_i$, where $q_i^{(s)} \in \mathfrak{m}\mathfrak{p}_i$ and $u_i \in R^\times$. Let us assume without loss of generality that $k_{n-1} \leq k_i$ for all i. Then, we define $y_i := x_i - x_n^{k_i - k_{n-1}} u_i u_{n-1}^{-1}(x_{n-1} + q_{n-1}^{(1)}) - q_i^{(1)}$ and $y'_i := x'_i - x_n'^{k_i - k_{n-1}} u_i u_{n-1}^{-1}(x'_{n-1} + q_{n-1}^{(2)}) - q_i^{(2)}$ for $i = 1, ..., n-2$. It follows that $y_1, ..., y_{n-2}$ is a regular parameter sequence, because, modulo $\mathfrak{m}^2$, we changed the x_i's only by a multiple of the linear independent element x_{n-1}. Moreover, $y_i \in \mathfrak{p}_1$, $y'_i \in \mathfrak{p}_2$ and $y_i = y'_i$. Thus, we have found a regular parameter sequence of cardinality $n-2$, which is contained in the intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ as required. We have thus finished the proof of the first theorem, too. □

Before we to the case of a finite number of given representations, let us state the consequences for the variation of the Iwasawa invariants: Basically, one can replace the one-parameter family by an n-parameter family, as far as the invariants are concerned. More precisely following theorem 3.33, we get for the μ-invariants:

COROLLARY 5.13. *Let G, H, and U be as in the previous sections (see particularly section 2.2), moreover, assume that we are given a complete regular local ring R with a finite residue field of characteristic p and a free R-representation T of the absolute Galois group G_K over K together with sub-G_v-representations $T^0(v)$ for any place v of F satisfying the condition 2.12. We associate the $R[\![G]\!]$-representations $(\mathbb{T}, \mathbb{T}^0)$ as in section 2.2. Assume that $\phi : R \to \mathcal{O}_\phi$ is a surjective homomorphism and we define Λ_ϕ, $\mathbb{T}_\phi$, and $\mathbb{T}_\phi^0$ as in the previous sections. Assuming that the cohomology groups of the Selmer complex $SC(U, \mathbb{T}_\phi, \mathbb{T}_\phi^0)$ are S_ϕ^*-torsion, the following holds: There is an n depending only on the pair $(\mathbb{T}_\phi, \mathbb{T}_\phi^0)$ such that*

$$\mu_{\Lambda_\phi}(SC(U, \mathbb{T}_\phi, \mathbb{T}_\phi^0)) = \mu_{\Lambda_\psi}(SC(U, \mathbb{T}_\psi, \mathbb{T}_\psi^0))$$

for all morphisms $\psi : R \to \mathcal{O}'$ such that $ker(\phi) + (p^n) = ker(\psi) + (p^n)$.

PROOF. Assume we are given maps ϕ and ψ as in the corollary, where the n is chosen to be the one of theorem 3.33. By theorem 5.8, there is a $\mathfrak{q} \subset R$ such that ϕ and ψ factor over the regular quotient $R/\mathfrak{q}$ and the dimension of $R/\mathfrak{q}$ is at most 2. If the dimension is one, then $\psi = \phi$ and the assertion is obvious. If the dimension is 2, then $R/\mathfrak{q}$ is isomorphic to $\mathcal{O}[[t]]$, where $\mathcal{O}$ is the ring of integers of a p-adic number field.

To be able to apply theorem 3.34 to this situation, it remains to be shown that the images of $ker(\phi)$ and $ker(\psi)$ in $R/\mathfrak{q}$ are generated by elements f and g such that $p^n | f - g$. But by our assumption $(f) + (p^n) = (g) + (p^n)$. Therefore, if we take f and g to be the unique distinguished polynomials generating the ideals, the assertion follows.

□

REMARK 5.14.

- *The corresponding properties for the invariant of the primitive Selmer complex (compare corollary 3.34) and the Selmer group (compare theorem 3.38) carry over accordingly.*
- *It is again possible to get rid of the condition on the surjectivity of ϕ and ψ, using the known properties of the μ-invariant under scalar extensions (theorem 3.35).*

- *In the language of rigid geometry, $RigSpec(R)$ is the unit ball and the maps ϕ and ψ correspond to points. Then the condition $ker(\phi) + (p^n) = ker(\psi) + (p^n)$ translates to "ψ is in a ball with radius $|p^n|$ around ϕ."*

For the λ-invariant, the following assertion is a direct consequence:

COROLLARY 5.15. *Let G, H, U, R, and $\mathbb{T}$ be as in the last corollary. We assume that $\phi : R \to \mathcal{O}_\phi$ is a homomorphism and the cohomology groups of $SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)$ are S-torsion. Then, for any other homomorphism $\psi : R \to \mathcal{O}_\psi$ the cohomology groups of $SC(U, \mathbb{T}_\psi, \mathbb{T}^0_\psi)$ are also S-torsion and we have:*

$$\lambda_{\Lambda_\phi}(SC(U, \mathbb{T}_\phi, \mathbb{T}^0_\phi)) = \lambda_{\Lambda_\psi}(SC(U, \mathbb{T}_\psi, \mathbb{T}^0_\psi))$$

PROOF. Utilizing the invariance of the λ-invariant under scalar extension (theorem 3.35), we may assume that ϕ and ψ are surjective. Therefore, we can apply theorem 5.8 to our situation and find again that ϕ and ψ are members of a one-parameter family. Then the assertion follows from theorem 3.33. □

Let us now discuss the case when n representations can be viewed as members of the same one-parameter family. The exact analogue of the case of two given representations cannot hold, as we typically do not find a 2-dimensional subspace of the Zariski tangent space on R containing all the directions given by the n ideals $\mathfrak{p}_i$. However, if we allow finite integral ring extensions, we get a similar result, as will be stated detailed in the following theorem:

THEOREM 5.16. *Let $\mathcal{O}$ be the ring of integers of a p-adic field. We set $R := \mathcal{O}[[t_1, ..., t_n]]$ and assume we are given $\mathcal{O}$-algebra maps $\phi_i : R \to \mathcal{O}_i$ for $i = 1, ..., k$, where the $\mathcal{O}_i$ are finite extensions of $\mathcal{O}$. Then there is a finite extension $\mathcal{O}'$ of $\mathcal{O}$ and rings of p-adic integers $\mathcal{O}'_i$ containing $\mathcal{O}'$ and the corresponding $\mathcal{O}_i$ together with an $\mathcal{O}$-algebra map $\psi : R \to \mathcal{O}'[[t]]$ and $\mathcal{O}'$-algebra morphisms $\psi_i : \mathcal{O}'[[t]] \to \mathcal{O}_i$ such that the following diagram is commutative for all $i = 1, ..., k$:*

$$\begin{array}{ccc} R & \xrightarrow{\phi_i} & \mathcal{O}_i \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathcal{O}'[[t]] & \xrightarrow{\psi_i} & \mathcal{O}'_i \end{array}$$

Moreover, $\mathcal{O}'$ and $\mathcal{O}'_i$ can be chosen in a way such that $\mathcal{O}'[[t]]$ is finite over the image of ψ and, for every i, the ring $\mathcal{O}'_i$ is the normalization of the image of ψ_i.

As in the first part of this section, we need some technical preparations before we can prove this theorem:

LEMMA 5.17. *Let $\mathcal{O}$ be the ring of integers of a p-adic number field and π be a uniformizer. We fix some integer $k \geq 1$ and denote $\mathcal{O}' = \mathcal{O}[\sqrt[k^2]{\pi}]$. In $\mathcal{O}'$, the element $\pi' := \sqrt[k^2]{\pi}$ is a uniformizer and the following holds:*

1. *There are polynomials $P_i \in \mathcal{O}'[t]$ for $i = 1, ..., k$ such that $P_i(\pi'^j) = \delta_{ij} \cdot \pi$ for $i, j \in \{1, ..., k\}$ and all P_i are contained in the maximal ideal of $\mathcal{O}'[[t]]$.*
2. *For every choice of elements $x_1, ..., x_k$ in $\pi\mathcal{O}$, there is a polynomial $P_{\underline{x}} \in \mathcal{O}_s[t]$ such that $P_{\underline{x}}(\pi'^i) = x_i$ for $i = 1, ..., k$ and P is contained in the maximal ideal of $\mathcal{O}'[[t]]$.*

PROOF. Clearly, the second part follows from the first one. For the first part, let us denote by v a valuation on $\mathcal{O}'$. We define $Q_i \in \mathcal{O}'[t]$ by $Q_i(t) := \prod_{j \neq i}(t - \pi'^j)$ where the product ranges over all $j = 1, ..., k$ different from i. Then $Q_i(\pi'^j) = 0$ for $i \neq j$ and we have to show that $Q_i(\pi'^i)$ divides π. The valuation of $v(\pi'^i - \pi'^j)$ is $\min(i, j) \cdot v(\pi')$; thus, it is bounded by $k \cdot v(\pi') = \frac{1}{k} \cdot v(\pi)$. It follows that $v(Q_i(\pi'^i)) \leq v(\pi)$. Denoting $c_i := \pi / Q_i(\pi'^i)$, we set $P_i := c_i \cdot Q_i$ and the claim follows. □

The next lemma is a special instance of the theorem:

LEMMA 5.18. *With $\mathcal{O}$ again a p-adic ring of integers, we set $R := \mathcal{O}[[x, y]]$ and assume that we are given $\mathcal{O}$-algebra morphisms $\phi_i : R \to \mathcal{O}$ for $i = 1, ..., k$. Then there is an extension $\mathcal{O}'/\mathcal{O}$ and morphisms $\psi : R \to \mathcal{O}'[[t]]$ and $\psi_i : \mathcal{O}'[[t]] \to \mathcal{O}'$ such that the following diagram commutes for every i:*

$$\begin{array}{ccc} R & \xrightarrow{\phi_i} & \mathcal{O} \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathcal{O}'[[t]] & \xrightarrow{\psi_i} & \mathcal{O}' \end{array}$$

Moreover, it is possible to chose $\mathcal{O}'$, ψ, and ψ_i in such a way that $\mathcal{O}[[t]]$ is finite over the image of ψ.

PROOF. Let π be again some uniformizer of $\mathcal{O}$ and $\mathcal{O}'$ and π' be as in the last lemma. We set $x_i := \phi_i(x) \in \pi\mathcal{O}$ and $y_i := \phi_i(y) \in \pi\mathcal{O}$. Then, using the notations of the last lemma, we have two polynomials $P_{\underline{x}}$ and $P_{\underline{y}}$ in $\mathcal{O}'[t]$. As these polynomials are contained in the maximal ideal of $\mathcal{O}[[t]]$, we can define the algebra morphism ψ by $\psi(x) = P_{\underline{x}}$ and $\psi(y) = P_{\underline{y}}$. Finally, defining the morphisms ψ_i by $\psi_i(t) = \pi'^i$, we see that the diagrams commute.

In general the described construction does not have the property that $\mathcal{O}'[[t]]$ is finite over the image of ψ. However, we may change the construction as follows: Let v be the valuation on $\mathcal{O}'$ normalized such that $v(\pi') = 1$ for a uniformizer π' of $\mathcal{O}'$. Denoting $P_{\underline{x}}(t) =: \sum_{i=0}^k a_i t^i$ and $P_{\underline{y}}(t) = \sum_{i=0}^k b_i t^i$ we define n as follows:

$$n := min\{\frac{v(a_1)}{1}, \frac{v(a_2)}{2}, ... \frac{v(a_k)}{k}, \frac{v(b_1)}{1}, \frac{v(b_2)}{2}, ... \frac{v(b_k)}{k}\}$$

Replacing $\mathcal{O}'$ with a ramified extension and adjusting π', v, and n accordingly, we may assume that n is an integer. It follows that $P_{\underline{x}}$ and $P_{\underline{y}}$ are elements in the subring $\mathcal{O}'[[\pi'^n t]]$. Thus, the whole image of ψ is contained in $\mathcal{O}'[[\pi'^n t]]$.

Assuming without loss of generality that $n = \frac{v(a_i)}{i}$ then $\mathcal{O}'[[\pi'^n t]]/P_{\underline{x}}$ is generated by the finite set $\{1, t, ..., t^{i-1}\}$ over $\mathcal{O}'[[P_{\underline{x}}]]$. An application of the topological Nakayama lemma tells us that $\mathcal{O}'[[\pi'^n t]]$ is finite over $\mathcal{O}'[[P_{\underline{x}}]]$. As $\mathcal{O}'$ is finite over $\mathcal{O}$ we conclude that $\mathcal{O}'[[P_{\underline{x}}]]$ is finite over the image. Replacing $\pi'^n t$ with t we have shown the assertion. □

With these preparations we are ready to prove the theorem.

PROOF (OF THEOREM). In the situation of the theorem, we replace $\mathcal{O}$ and $\mathcal{O}_i$ by a ring of integers $\tilde{\mathcal{O}}$ of a p-adic number field containing all $\mathcal{O}_i$ and extend the maps ϕ_i accordingly. Then we can apply the last lemma to the restriction of the morphisms to $\tilde{\mathcal{O}}[[t_{n-1}, t_n]]$. Extending ψ and ψ_i as the identity on $t_1, ..., t_{n-2}$, we get a commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{O}}[[t_1, ..., t_n]] & \xrightarrow{\phi_i} & \tilde{\mathcal{O}} \\ \big\downarrow \psi & & \big\downarrow \\ \mathcal{O}'[[t_1, ..., t_{n-2}, t]] & \xrightarrow{\psi_i} & \mathcal{O}' \end{array}$$

The map ψ in this diagram is chosen such that it is finite. Using induction and composing the maps with the inclusions $\mathcal{O} \hookrightarrow \tilde{\mathcal{O}}$ and $\mathcal{O}_i \hookrightarrow \tilde{\mathcal{O}}$ yields the commutative square:

$$\begin{array}{ccc} \mathcal{O}[[t_1, ..., t_n]] & \xrightarrow{\phi_i} & \mathcal{O}_i \\ \big\downarrow \psi & & \big\downarrow \\ \mathcal{O}'[[t]] & \xrightarrow{\psi_i} & \mathcal{O}' \end{array}$$

Here we possibly replaced the $\mathcal{O}'$ from the last diagram with a bigger one. As compositions of finite maps are finite the ψ in the new diagram is still finite. If we replace the $\mathcal{O}'$ in the lower right corner by $\mathcal{O}'_i$, the integral closures of the images of ψ_i, we get the diagram from the assertion in the theorem. $\square$

5.4. Example 1: Twists by characters

Firstly, a theory of families should be able to deal with the easiest case. So, although not much knowledge can be gained thereby, we treat the case of a trivial character first:

We assume that we are given a single F-motive M with a K operation. We denote the λ-adic realization by M_λ; it is unramified outside a finite set S. Moreover, we assume that this motive satisfies the Dabrowski-Panchishkin condition 2.9. The trivial extension of the λ-adic realization given by $Aut_{\mathcal{O}}(M_\lambda) \subset Aut_{\mathcal{O}[[t]]}(\mathcal{O}[[t]] \otimes_{\mathcal{O}} M_\lambda)$ satisfies then all our conditions on families in section 2.2, in particular 2.12, the one on freeness. For this family, we can allow every continuous morphism $\mathcal{O}[[t]] \to \mathcal{O}$ as a specialization.

If the main conjecture 4.24 for the motive M holds, then it is easy to describe what happens in this case: The ζ-function of the family is just the image of the one of the motive, under the map induced on the K-groups by the natural inclusion $\mathcal{O} \to \mathcal{O}[[t]]$. In fact, this inclusion is a section for any specialization map $\phi : \mathcal{O}[[t]] \to \mathcal{O}$, as those are $\mathcal{O}$-algebra morphisms; therefore, the ζ-function of the family does indeed specialize to the one of M under any specialization map, as conjectured.

Now let us turn to the case of non-trivial characters. We still assume that we fixed the finite set S. Let $\mathcal{O}$ be the ring of integers of a p-adic field and $\chi : G_S \to \mathcal{O}^\times$ be a (continuous) character. Firstly, let us note:

FACT 5.19. *There is an integer k (depending only on $\mathcal{O}$) such that for any two integers n_1 and n_2 with $k | n_2 - n_1$ we have $\chi^{n_1} \equiv \chi^{n_2}$ mod p^s, where s is the p-adic valuation of $(n_2 - n_1)/k$.*

PROOF. Let v be the valuation normalized to $v(p) = 1$ and let t be the smallest natural number such that for a uniformizer π of $\mathcal{O}$ we have $v(\pi^t) \geq \frac{1}{p-1}$. Then it is known that $v(x^p - 1) = v(x-1) + 1$ for all $x \equiv 1 \bmod \pi^t$. Thus, we can take k to be the smallest natural number such that $\chi^k \equiv 1 \bmod \pi^t$. □

With this fact in place we note:

LEMMA 5.20. *Assume that k is as in the last fact and $s < k$, then there is a character $\tilde{\chi} : G_S \to \mathcal{O}[[t]]$ such that for every $z \in \mathbb{Z}$ we have*

$$\tilde{\chi}/(t-z) = \chi^{zk+s} \ .$$

PROOF. We prove this fact using a well-known strategy in the theory of pseudo-representations:

Let $I_i = \prod_{z=0}^{i}(t-z)$ as an ideal of $\mathcal{O}[[t]]$. Then we construct $\chi_i : G_S \to \mathcal{O}[[t]]/I_i$ inductively: The character χ_0 is set to be χ^s. For $i > 0$, we note that $\mathcal{O}[[t]]/I_i \cong \mathcal{O}[[t]]/I_{i-1} \times_{\mathcal{O}[[t]]/(I_{i-1})+(t-i)} \mathcal{O}[[t]]/(t-i)$ by fact 5.10. We set χ_i to be χ_{i-1} on the left factor and χ^{ik+s} on the right factor. This is possible as the definitions agree on $\mathcal{O}[[t]]/(I_{i-1}) + (t-i)$ by the last fact. Now, taking $\tilde{\chi}$ to be the limit of the χ_i on $\mathcal{O}[[t]] = \varprojlim \mathcal{O}[[t]]/I_i$, we fist note that the assertion is fulfilled for all $i \geq 0$ by the definition of $\tilde{\chi}$. For $i < 0$, take some subsequence $i_0, i_1, ...$ of the non-negative integers that converges p-adically to i. Then, utilizing the last fact again, we see that the χ^{ki_r+s} converge to χ^{ki+s} and the reductions $\tilde{\chi}/(t-i_r)$ converge to $\tilde{\chi}/(t-i)$. So the assertion follows for all $i < 0$, too. □

If now χ is the p-adic realization of a motive $\mathcal{X}$ of rank one and $\rho : G_S \to Aut(M_\lambda)$ is the p-adic realization of our given motive M, then $\tilde{\chi}\rho$ is a family of motives specializing to the twists $\mathcal{X}^{ki+r} \otimes M$. In particular, all finite characters can be achieved as the p-adic realization of a Dirichlet motive.

As the twists might change the weights, in general one cannot expect all the specializations of this family to satisfy the Dabrowski-Panchishkin condition with respect to the given subrepresentation $\mathbb{T}^0_\phi$. However, the only problem that might occur is that after twisting with χ^n some previously non-negative weights become positive or vice versa. And for high enough (resp., low enough) powers of χ, all weights that will eventually become positive or non-negative will have done so. Thus, there are two subfamilies with infinitely many specializations satisfying the condition each with respect to one given subrepresentation.

THEOREM 5.21. *Let χ be a character as above. Let $\rho : G_S \to Aut(M_\lambda)$ be the p-adic realization of the motive M. Then there is a family of representations $\tilde{\chi}\rho \to \mathcal{O}[[t]]$ such that for k and s as above we can specialize modulo $t - z$ for any big enough integer z to get the p-adic realization of $M(\chi^{zk+s})$. This family with the set of specializations being $t \mapsto z$ for sufficiently large integers z, satisfies all our conditions, including 2.12, the one on freeness.*

5.5. A counterexample: No families of elliptic curves

This short section is aiming to show that there are no examples of infinite families of elliptic curves. This is probably known to the experts, but in the opinion of the author this knowledge has not spread wide enough in the community yet.Moreover, we will hint at a generalization for abelian varieties.

The naive idea would be to look at situations (i.e., moduli problems), where the (rigid) moduli space X of some geometric objects exists (as a scheme) and has infinitely many K-rational points for a number field K. Then finding a curve in X that passes through infinitely many of the rational points should give us a family. However, there are two problems with that approach: Firstly, the p-adic representations of two rational points which are p-adically close should be congruent modulo some high power of p which is not obvious at all. Secondly, it turns out, that the requirement of the representations to be unramified outside a finite set of primes results in only finitely many points (i.e., objects) in classical situations.

Let us now turn to the case that is probably best understood: elliptic curves. As is well known, the moduli space of elliptic curves over a number field with some moduli structure is again a curve. Thus, to have infinitely many rational points, we have to be in one of the cases where the genus of the modular curve is zero or one. That at first glance this seems to be achievable: In fact, it is remarked by Rubin and Silverberg in [**RS95**] that there are indeed infinitely many elliptic curves with full level 3, respectively level 5, structure and thus with trivial mod 3 (resp. mod 5) representation. The result of the cited paper is even better: By twisting the curves of this infinite set with an arbitrary elliptic curve, we get infinitely many elliptic curves with any given mod 3 (resp. mod 5) representation that actually occurs for one curve. So this setting seems to be rather promising.

However, we need much more: For any given n, we want infinitely many elliptic curves over the same number field K, which induce the same representation modulo p^n. But that turns out to be impossible for any $n > 1$: In fact, still fixing the number field K and denoting the kernel of a given mod p^n Galois representation by U, all elliptic curves over K with this mod p^n representation have a full level p^n structure after base change to $\overline{K}^U$. As the genus of (a connected component of) the modular curve with full level N structure is independent of the number field and bigger than 1 as soon as $N > 6$, there are always at most finitely many elliptic curves with a given structure. As the Galois group with restricted ramification G_S for a finite set S is finitely generated (compare: 1.21), there are only finitely many mod p^n representations for any given S. Thus, the above reasoning shows that even though we find infinitely many elliptic curves with the same mod p representations of the absolute Galois group, we only find finitely many of them, such that the full Tate module is unramified outside any given finite set S, so this attempt failed.

Therefore one might be inclined to believe that this is due to the fact that, in any case, there were only very few moduli spaces with infinitely many K-rational points to start with. The question arises, can this defect be fixed by looking at situations where the moduli space is higher dimensional? It is not entirely clear to the author, if there can be such families of abelian varieties. However, the above argument can be reduced to the fact, that the complex points of the modular curve have the upper half-plane as an analytic cover which is unramified, provided that we fix a big enough level structure. It was remarked that algebraic curves in Shimura varieties classifying polarized abelian varieties should have the same property in most cases.

5.6. Example 2: Hida families of modular forms

So far, we have not seen many natural occurring families. However, there is one general principle that provided the motivating examples for Iwasawa theory of families: Hida families of modular forms. The general idea is that the universal ordinary deformation ring of the residual representation associated to an ordinary Hecke eigenform should be naturally isomorphic to a Hecke algebra, which is in turn a local complete intersection. As the deformation ring is denoted by R and this Hecke algebra is often denoted by T, these theorems are referred to as "R=T" theorems in deformation theory.

Firstly, the case of elliptic modular forms is a famous theorem of Taylor and Wiles [**TW95**]. It can be found in chapter 7 of Hida's book [**Hid00**] as theorem 5.29:

THEOREM 5.22. *Assume that S is a finite set of places of $\mathbb{Q}$ consisting of all infinite places, and those places dividing pN for an integer N prime to p. Let $G_S = Gal(\mathbb{Q}_S/\mathbb{Q})$ and let $\overline{\rho} : G_S \to V_{\mathbb{F}_p}$ be the residual representation associated to an elliptic Hecke eigen cusp form of tame conductor N and fixed nebentype χ. Suppose that $\overline{\rho}$ restricted to $\mathbb{Q}_S/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ is absolutely irreducible and that one of the two characters of $\overline{\rho}$ is in fact ramified (not just the character with coefficients in $\mathbb{Z}_p$). If we suppose in addition that $p > 3$, then the universal ordinary deformation ring $R^{n.o.}$ can be identified with the Hecke algebra $T^{n.o.}$ operating on the ordinary cusp forms of nebentype χ over $\mathbb{Q}$ with coefficients in $\Lambda := \mathcal{O}[[t]]$.*

This theorem is particularly interesting together with the next one (theorem 5.30 in [**Hid00**]):

THEOREM 5.23. *In the situation of the last theorem $T^{n.o.}$ is a finite flat complete intersection over Λ.*

While this is not enough to imply that $R^{n.o.}$ is regular, we have $\Lambda = T^{n.o.}$ in "most" of the cases. More precisely, it is noted by Hida (remark b after corollary 5.50 in [**Hid00**]) that if $\overline{\rho}$ is the residual representation attached to a Hecke eigen cusp form f of weight k, then $\Lambda = T^{n.o.}$ holds if and only if there is no other Hecke eigenform of the same weight that is congruent to f modulo p.

In this situation, our universal ordinary deformation ring is therefore already of the form we are looking for. Therefore, we get variation results for Iwasawa invariants of Hida families of elliptic modular cuspforms as a corollary of the theorems above and the results in sections 3.5 and 3.6:

COROLLARY 5.24. *Let $\mathbb{Q}_\infty/\mathbb{Q}$ be a p-adic Lie extension with Galois group G without p-torsion. Assume that f is an elliptic modular Hecke eigen cusp form, which is p-ordinary, of nebentype χ satisfying the conditions of the last two theorems. We assume that f has coefficients in a p-adic integer ring $\mathcal{O}$ and set $\Lambda := \mathcal{O}[[G]]$. Moreover, we assume that there is no other form of the same weight that is congruent to f modulo p. Then f arises as the specialization of a one parameter Hida family as above. Moreover, every modular eigen cusp form f' of the same nebentype which is congruent to f modulo p arises as a specialization of this family. Let $(\mathbb{T}_{f'}, \mathbb{T}^0_{f'})$ be a pair of big Galois representations associated to f' as in section 2.2 and assume for some admissible $U \subset spec(\mathbb{Z})$ that the cohomology groups of $SC(U, \mathbb{T}_f, \mathbb{T}_f)$ are S^*- torsion. Then, the invariant $\mu_\Lambda(SC(U, \mathbb{T}_f, \mathbb{T}^0_f))$ vanishes if and only if $\mu_\Lambda(SC(U, \mathbb{T}_{f'}, \mathbb{T}^0_{f'}))$ vanishes. Moreover, in general, there is an n depending only on f such that*

$$\mu_\Lambda(SC(U, \mathbb{T}_f, \mathbb{T}^0_f)) = \mu_\Lambda(SC(U, \mathbb{T}_{f'}, \mathbb{T}^0_{f'})) = \mu_\Lambda(\mathcal{X}(U, \mathbb{T}_{f'}, \mathbb{T}^0_{f'})) \text{ and}$$
$$\mu_\Lambda(SC(\mathbb{T}_f, \mathbb{T}^0_f)) = \mu_\Lambda(SC(\mathbb{T}_{f'}, \mathbb{T}^0_{f'})) = \mu_\Lambda(\mathcal{X}(\mathbb{T}_{f'}, \mathbb{T}^0_{f'}))$$

for all f' in the family congruent to f modulo p^n. Furthermore, if $SC(U, \mathbb{T}_f, \mathbb{T}_f^0)$ has S-torsion cohomology groups, then the analogous statement for the λ-invariant holds:

$$\lambda_\Lambda(SC(U, \mathbb{T}_f, \mathbb{T}_f^0)) = \lambda_\Lambda(SC(U, \mathbb{T}_{f'}, \mathbb{T}_{f'}^0)) = \lambda_\Lambda(\mathcal{X}(U, \mathbb{T}_{f'}, \mathbb{T}_{f'}^0)) \text{ and}$$
$$\lambda_\Lambda(SC(\mathbb{T}_f, \mathbb{T}_f^0)) = \lambda_\Lambda(SC(\mathbb{T}_{f'}, \mathbb{T}_{f'}^0)) = \lambda_\Lambda(\mathcal{X}(\mathbb{T}_{f'}, \mathbb{T}_{f'}^0))$$

REMARK 5.25. *The special case where $\mathbb{Q}_\infty/\mathbb{Q}$ is a false Tate extension was treated by Aribam in* [**Sha09**] *with a more explicit approach and without the congruence assumption.*

In the case that $\mathbb{Q}_\infty/\mathbb{Q}$ is the cyclotomic $\mathbb{Z}_p$-extension, Ochiai formulated a different two-variable Iwasawa main conjecture (see [**Och06**]*) for Hida families. It would be interesting to compare the two approaches; however, Ochiai uses different period elements and there is no obvious way to compare them to the motivic periods.*

There are other cases where the ordinary deformation ring is known to equal some Hecke algebra and where in many cases this Hecke algebra can be recovered as an Iwasawa algebra. Most notable is the case of Hilbert modular forms. Much of this theory is the work of Fujiwara. Firstly, the $R = T$ theorem is the main result in [**Fuj**] (theorem 0.2 resp. 11.1). Again, the Hecke algebra is presented over an Iwasawa algebra (corollaries 4.21-4.24 of [**Hid06**]) and is known to equal this Iwasawa algebra in many cases (Hida's remark after question (q9) in [**Hid06**]).

Please note that even in the case that the universal ring is regular, we are not exactly in the situation of the main conjectures stated in this thesis: Instead of a treatment of the one parameter case, one would need a version for n-parameter families. However, in section 5.3 on finite families, we saw that this does not pose a problem for assertions on the Iwasawa invariants. Moreover, an iteration of the methods presented in the previous chapters should also predict an n-variable version of the ζ-element.

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