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The Asymmetric Leximin Solution

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Abstract

In this article we define and characterize a class of *asymmetric leximin solutions*, that contains both the symmetric leximin solution of Imai [5] and the two-person asymmetric Kalai-Smorodinsky solution of Dubra [3] as special cases. Solutions in this class combine three attractive features: they are defined on the entire domain of convex n -person bargaining problems, they generally yield Pareto efficient solution outcomes, and asymmetries among bargainers are captured by a single parameter vector.

The characterization is based on a strengthening of Dubra's [3] property *Restricted Independence of Irrelevant Alternatives* (RIIA). RIIA imposes Nash's [9] IIA, under the added condition that the contraction of the feasible set preserves the mutual proportions of players' utopia values. Our axiom, entitled *RIIA for Independent Players* (RIP), says RIIA holds for a *group* of players, given that the contraction of the feasible set does not affect players outside that group.

JEL-Classification: C78

Keywords: Bargaining, asymmetric bargaining solution, leximin solution.

1 Introduction

In the literature on axiomatic bargaining, few axioms are as prolific as the one of Anonymity. Anonymity says that the utility a bargaining solution allocates to an individual bargainer, does not depend on that bargainer's identity. While desirable in many situations, such an assumption might not always be entirely

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appropriate, as it does not leave for the possibility to account for individual levels of skill, effort, or commitment. The Nash bargaining solution, introduced by Nash [9], is one example in which Anonymity has successfully been relaxed. Harsanyi and Selten [4] and Kalai [6] defined and characterized a family of asymmetric Nash solutions, that capture the asymmetries among bargainers in a single parameter vector.

In this article, we present a similar generalization for the lexicographic maxmin – or *leximin* – solution, which is a solution for multilateral bargaining problems introduced by Imai [5]. It satisfies a highly attractive monotonicity property which says that no individual bargainer can gain from destroying his own utility. In normalized bargaining problems, it is found through the following optimization procedure. First maximize the payoff of the worst-off agent; among the maximizers thus obtained, maximize the payoff of the next to worst-off agent, and so on, until no individual agent’s utility can be further increased within the feasible set. For general bargaining problems, the solution is found by applying the described procedure to a normalized version, and subsequently scaling back.

We introduce an asymmetric version of this solution concept, named the *asymmetric leximin solution*. It is obtained by the optimization procedure outlined above, with the added feature that players’ payoffs are given strictly positive weights. The interpretation is that the societal value of an individual bargainer’s wealth, depends on that bargainer’s identity. This idea is also fundamental to Thomson’s [14] asymmetric version of the Kalai-Smorodinsky solution (Raiffa [10], Kalai & Smorodinsky [7]). The Kalai-Smorodinsky solution is defined as the unique point in which the boundary of the feasible set is intersected by the line segment that connects the problem’s disagreement and utopia point; Thomson’s weighted version is similarly obtained, but with a *weighted* utopia point. It is wellknown that the leximin solution is a lexicographic extension of the Kalai-Smorodinsky solution. In much the same way, the asymmetric leximin solution constitutes a lexicographic extension of Thomson’s asymmetric variation.

An alternative asymmetric version of the Kalai-Smorodinsky solution was proposed by Dubra [3], for the specific case of two-person bargaining problems. Dubra’s solution is defined as the unique Pareto efficient outcome that (weakly) Pareto dominates the outcome found by Thomson’s solution concept. He also provides an axiomatic characterization for this solution class, that obtains by removing Anonymity from Kalai and Smorodinsky’s [7] axiom set, and replacing

it with a weaker version of Nash's [9] independence axiom IIA, named *Restricted* IIA (RIIA); RIIA imposes that a collective destruction of joint utility that preserves the mutual proportions of the bargainers' utopia values, does not affect the solution outcome, whenever that outcome is still available.

On the domain of bilateral bargaining problems, the asymmetric leximin solution coincides with Dubra's solution concept, meaning that our results can be seen as a multilateral generalization of those obtained by Dubra. Our characterization of the asymmetric leximin solution is based on Imai's characterization of the symmetric leximin solution. However, where Imai's characterization result depends on Anonymity, ours depends on an axiom named *RIIA for Independent Players* (RIP).¹ This is a strengthening of RIIA that applies to *groups* of bargainers. In particular, whenever a *group* of bargainers collectively destroys its joint utility as stipulated in Dubra's axiom, and this does not interfere with the solution payoffs accorded to the nonmembers of that group, then the solution outcome should not change at all.

The rest of this article is organized as follows. Section 2 collects relevant assumptions, notations and definitions. In Section 3 we obtain an axiomatic characterization result for the family of asymmetric leximin solutions. Section 4 concludes.

2 Preliminaries

A *bargaining problem* is defined by the *player set* $N := \{1, \dots, n\}$, with $n \geq 2$ finite, and a set $S \subset \mathbb{R}^N$, where

- S is nonempty, closed and convex,
- contains the zero vector $\mathbf{0}$,
- there exist $x \in S$ with $x > \mathbf{0}$,²
- the set $S \cap \mathbb{R}_+^N$ is bounded.

The set of all bargaining problems is denoted \mathcal{B} .

¹In addition, we include an axiom named *Strong Individual Rationality*, which also appeared in Harsanyi and Selten's [4] and Kalai's [6] axiomatization of the asymmetric Nash solution. It ensures that weights vectors are strictly positive, and thus, that the solution is well-defined.

²For $x, y \in \mathbb{R}^N$, $x > y$ denotes $x_i > y_i$ for all $i \in N$, $x \geq y$ means $x_i \geq y_i$ for all $i \in N$. The relations ' $<$ ' and ' \leq ' are similarly defined.

A vector $x \in \mathbb{R}^N$ is called an *outcome* or *point*; it represents a payoff profile for the players of the game in the sense that each x_i , $i \in N$, specifies the utility realized by player i . The set S is called the *feasible set*; it contains all the outcomes available to the players. The outcome $\mathbf{0}$, the *disagreement point*, is the outcome that obtains if the bargainers fail to reach agreement on a single outcome. The condition that the feasible set holds outcomes that are strictly larger than the disagreement point, provides an incentive for the players to reach agreement. The agreement bargainers reach is represented by a map $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ that assigns a single outcome $\varphi(S)$ to any problem S in \mathcal{B} . Such a map is referred to as a *bargaining solution*.

Bargaining solutions are typically characterized by a number of properties, or *axioms*. Given a problem S , the set $P(S)$ denotes the *Pareto set*. It is defined as $P(S) := \{z \in S \mid x \geq z \text{ implies } x = z\}$. For $x, y \in \mathbb{R}^N$ and $S \subset \mathbb{R}^N$, let $xy := (x_1y_1, \dots, x_ny_n)$ and $xS := \{xz \mid z \in S\}$. For a permutation $\pi : N \rightarrow N$ of N , $z \in \mathbb{R}^N$, and $S \subset \mathbb{R}^N$, define $\pi(z) := (z_{\pi(1)}, \dots, z_{\pi(n)})$ and $\pi(S) := \{\pi(x) \mid x \in S\}$. The following is a list of classic axioms.

- (SIR) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Strong Individual Rationality* if $\varphi(S) > \mathbf{0}$ for all $S \in \mathcal{B}$.
- (PO) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Pareto Optimality* if $\varphi(S) \in P(S)$ for all $S \in \mathcal{B}$.
- (SI) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Scale Invariance* if for all $S \in \mathcal{B}$ and $b \in \mathbb{R}_{++}^N$, $\varphi(bS) = b\varphi(S)$.
- (AN) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Anonymity* if for all $S \in \mathcal{B}$, any permutation $\pi : N \rightarrow N$, and any $i \in N$, we have $\varphi_{\pi(i)}(\pi(S)) = \varphi_i(S)$.
- (IIA) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Independence of Irrelevant Alternatives* if for all $S, T \in \mathcal{B}$ with $\varphi(T) \in S \subseteq T$, we have $\varphi(S) = \varphi(T)$.

For a discussion of these properties, see Nash [9] and Roth [11].

Kalai and Smorodinsky [7] proposed a monotonicity axiom, which imposes that an expansion of the feasible set in favor of a single player, makes that player (weakly) better off. We present Imai's [5] version of this property. The *utopia point* of a problem S , is defined as a vector $u(S) := (u_i(S))_{i \in N}$ where $u_i(S) := \max\{x_i \mid x \in S \cap \mathbb{R}_+^N\}$ for each $i \in N$. For $x \in \mathbb{R}^N$ and nonempty $Q \subset N$, x is equivalently written as (x_Q, x_{-Q}) where x_Q denotes the vector $(x_i)_{i \in Q}$, and x_{-Q} the vector $(x_i)_{i \in N \setminus Q}$. For $Q = N$, $x_Q \equiv x$. Let \mathbb{R}^\emptyset be a

singleton containing the empty vector; then for $Q = \emptyset$, we say $x_Q \in \mathbb{R}^Q$. For $S \in \mathcal{B}$, let S_{-i} denote the closure of the set $\{x_{-i} \mid x \in S \text{ and } x \leq u(S)\}$.

(IM) $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Individual Monotonicity* if for all $S, T \in \mathcal{B}$ with $T \subseteq S$ and $S_{-i} = T_{-i}$ for some $i \in N$, we have $\varphi_i(S) \geq \varphi_i(T)$.

In response to the criticism on IIA that it renders a bargaining solution too irresponsible to alterations in the feasible set (cf. Luce and Raiffa [8], Kalai and Smorodinsky [7]), Roth [12] proposed a weaker version that imposes invariance under the conditions of IIA, if in addition the problem's utopia point does not change.

(IAIP) Let $S, T \in \mathcal{B}$ with $T \subseteq S$ and $u(T) = u(S)$. $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *IIA other than Ideal Point* if $\varphi(S) \in T$ implies $\varphi(S) = \varphi(T)$.

Dubra [3] introduced a stronger version of Roth's property, which rather than equality, requires that the utopia points of the considered problems are *proportional*.

(RIIA) Let $S, T \in \mathcal{B}$ with $T \subseteq S$ and $u(T) = \beta u(S)$ for some $\beta > 0$. $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *Restricted IIA* if $\varphi(S) \in T$ implies $\varphi(T) = \varphi(S)$.

Dubra argues that a contraction of the feasible set as described in the axiom definition, retains the mutual bargaining powers among the players. Then, since the original solution outcome remains available, players should convene on the same outcome. The following is a strengthening of RIIA, that imposes the same reasoning on a *group* of players.

(RIP) Let $S, T \in \mathcal{B}$ with $T = S \cap V$, where V is such that $u_Q(T) = \beta u_Q(S)$ for some nonempty $Q \subseteq N$ and $\beta > 0$, and $x \in V$ implies $(x_Q, y) \in V$ for all $y \in \mathbb{R}^{N \setminus Q}$.

$\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies *RIIA for Independent Players* if $\varphi(S) \in T$ and $\varphi_{-Q}(T) = \varphi_{-Q}(S)$ implies $\varphi(T) = \varphi(S)$.

The axiom describes a contraction of the feasible set along the axes of the players in Q , that leaves the solution outcomes of players not in Q unaffected. If this contraction preserves the mutual bargaining powers among the players in Q , and the original solution values of these players are still available, then they should convene on those same values.

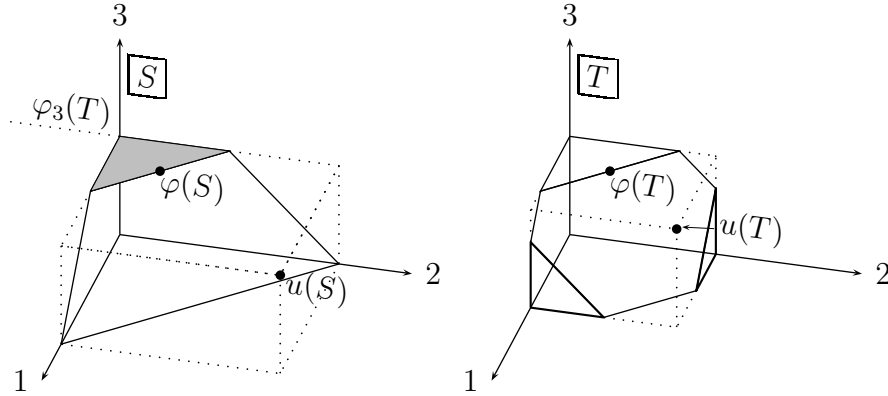


Figure 1: A visual illustration of RIP with $N = \{1, 2, 3\}$ and $Q = \{1, 2\}$.

Next we define the asymmetric leximin solution. The set of weights vectors is given by

$$\Delta := \left\{ w \in \mathbb{R}_{++}^N \mid \sum_{i \in N} w_i = 1 \right\},$$

and for $x > \mathbf{0}$, define $x^{-1} := (1/x_1, \dots, 1/x_n)$.

Let μ be a vector-specific permutation for N , that puts the vector elements in ascending order. For $x, y \in \mathbb{R}^N$, $x \succ^l y$ whenever $x_i > y_i$ for some $i \in N$ and $x_j = y_j$ for all $j < i$. The *lexicographic maxmin ordering*, denoted \succ , is defined as follows: for $x, y \in \mathbb{R}^N$, $x \succ y$ if and only if $\mu(x) \succ^l \mu(y)$. For $S \in \mathcal{B}$ denote the unique maximum in S with respect to ‘ \succ ’ by $\xi(S)$.³ For $w \in \Delta$, the *asymmetric leximin solution* $L^w : \mathcal{B} \rightarrow \mathbb{R}^N$ is defined as

$$L^w(S) := \xi(S), \tag{1}$$

whenever $u(S) = w^{-1}$. For $S \in \mathcal{B}$ with $u(S) \neq w^{-1}$, $L^w(S) := b\xi(b^{-1}S)$ where $b := wu(S)$. The set $\mathcal{L} := \{L^w \mid w \in \Delta\}$ denotes the class of all asymmetric leximin solutions.

The *leximin solution* $L : \mathcal{B} \rightarrow \mathbb{R}^N$, defined in [5], is the unique symmetric solution in \mathcal{L} . More specifically, let $\mathbf{1}$ denote the n -dimensional vector with all entries equal to 1; then $L \equiv L^{\frac{1}{n}\mathbf{1}}$.

³For the existence and uniqueness proof, see Lemma 4 in [5].

Theorem 2.1 (Imai [5]). *A bargaining solution $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies PO, SI, AN, IM and IAIP, if and only if $\varphi = L$.*

Dubra [3] defined the *asymmetric Kalai-Smorodinsky solution* K^w , $w \in \Delta$, for the specific case of two-player problems. L^w defined on a two-player problem coincides with K^w for each $w \in \Delta$.⁴

Theorem 2.2 (Dubra [3]). *For $N = \{1, 2\}$, a bargaining solution $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies SIR, PO, SI, IM and RIIA if and only if $\varphi = K^w$ for some $w \in \Delta$.*

Let \mathcal{B}^* be the class of problems S with $u(S) = \mathbf{1}$, and take $S \in \mathcal{B}^*$. Then $L^w(S)$, $w \in \Delta$, is obtained by the following procedure. Starting from the disagreement point $x^0 \equiv \mathbf{0}$, increase the utilities of all players simultaneously in the direction w , until the boundary of the set is reached, say in the point x^1 . There is a number of players for whom a further improvement would result in an infeasible alternative. Fix the payoffs of these players at their x^1 -levels, and continue increasing the utilities of the remaining players in the direction w . This leads again to a point – call it x^2 – from which further increase of utilities means stepping out of S . Then once again, fix the payoffs of the problematic players at their x^2 -levels, and continue the process for the players that remain. Since the total number of players is finite, and since at each iteration a finite number of players is excluded from further improvement, this procedure leads in a finite number of steps to an outcome in $P(S)$. This outcome corresponds with $L^w(S)$.

3 The Characterization Result

The aim of this section is to obtain a characterization result for the class of asymmetric leximin solutions.

Theorem 3.1. *A bargaining solution $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ satisfies SIR, PO, SI, IM and RIP if and only if $\varphi \in \mathcal{L}$.*

The following is the first of two propositions that establish Theorem 3.1.

Proposition 3.2. *Any solution $\varphi \in \mathcal{L}$ satisfies SIR, PO, SI, IM and RIP.*

⁴Dubra's solution class also includes the two corner solutions, where one player's weight is zero. Consequently, his characterization does not include SIR. However, this extension is trivially obtained from Dubra's result.

Before moving to the proof of this proposition, we introduce two helpful lemmas. A set S is said to be *comprehensive* whenever $x \in S$ and $y \geq x$ together imply $y \in S$. In the context of bargaining, this may be interpreted as free disposal of utility. For $S \in \mathcal{B}$, $\text{com } S$ denotes the smallest comprehensive set in \mathbb{R}^N that contains the set S . Given nonempty $Q \subseteq N$, a problem $S \in \mathcal{B}$, and a point $x \in \text{com } S \cap \mathbb{R}_+^N$, the set $m_Q^x(S)$ denotes the intersection of S with the hyperplane through x , parallel to \mathbb{R}^Q , seen as a subset of \mathbb{R}^Q . Specifically,

$$m_Q^x(S) := \{x' \in \mathbb{R}^Q \mid (x', x_{-Q}) \in S\}.$$

Thomson and Lensberg [15, pp. 132–133] show that ξ satisfies a property named *Multilateral Stability*. This means that given a problem, the outcome selected by ξ is not only optimal for that problem, but also for the subproblem faced by any subgroup of bargainers, that results when all others are paid out their solution outcome, but do not further participate in the bargaining process. For nonempty $Q \subseteq N$, let \mathcal{B}^Q be the class of bargaining problems for Q .

Lemma 3.3 (Thomson & Lensberg [15]). *For $S \in \mathcal{B}$ and $T \in \mathcal{B}^Q$ where $T = m_Q^{\xi(S)}(S)$ for some nonempty $Q \subseteq N$, we have $\xi(T) = \xi_Q(S)$.*

Driesen [2, Lemma 3.4] showed that the map ξ is first-degree homogeneous.

Lemma 3.4 (Driesen [2]). *For all $S \in \mathcal{B}^Q$ (with $Q \neq \emptyset$) and $\gamma > 0$, we have $\xi(\gamma S) = \gamma \xi(S)$.*

Proof of Proposition 3.2. Consider a solution $L^w \in \mathcal{L}$. It follows directly from the definition that L^w satisfies SI. Let $S \in \mathcal{B}$, and without loss of generality, assume $u(S) = w^{-1}$, such that $L^w(S) = \xi(S)$. It is easily seen that $\xi(S) > \mathbf{0}$ and $\xi(S) \in P(S)$. Hence, L^w satisfies SIR and PO. The proof that L^w satisfies IM is obtained by applying Imai's [5] arguments to the problem S (i.e. a problem with w^{-1} as utopia point), rather than some problem $S' \in \mathcal{B}^*$.

To establish RIP, consider a problem $T = S \cap V$, where V is such that $u_Q(T) = \beta u_Q(S)$ for some nonempty $Q \subseteq N$ and $\beta > 0$, and $x \in V$ implies $(x_Q, y) \in V$ for all $y \in \mathbb{R}^{N \setminus Q}$. Assume $L^w(S) \in T$ and $L_{-Q}^w(T) = L_{-Q}^w(S)$, and let b be a vector such that $u(T) = bu(S)$. Then b_i is equal to β if $i \in Q$, and equal to 1 otherwise. We claim the following.

$$\xi(T) = b\xi(b^{-1}T). \tag{2}$$

In the generic case where $Q = N$, this follows directly from Lemma 3.4. Thus, assume $Q \neq N$. Eq. (2) is obtained in two steps:

1. $\xi_{-Q}(T) = b_{-Q}\xi_{-Q}(b^{-1}T)$;
2. $\xi_Q(T) = b_Q\xi_Q(b^{-1}T)$.

We first establish point 1.

1. By definition of L^w we have $L^w(S) = \xi(S)$ and $L^w(T) = b\xi(b^{-1}T)$, implying $L^w_{-Q}(S) = \xi_{-Q}(S)$ and $L^w_{-Q}(T) = b_{-Q}\xi_{-Q}(b^{-1}T)$. By assumption, we have $L^w_{-Q}(S) = L^w_{-Q}(T)$. Hence,

$$\xi_{-Q}(S) = b_{-Q}\xi_{-Q}(b^{-1}T). \quad (3)$$

Since $T \subseteq S$, $L^w(S) = \xi(S)$, and $L^w(S) \in T$, we have $\xi(T) = \xi(S)$. Thus,

$$\xi_{-Q}(S) = \xi_{-Q}(T). \quad (4)$$

Point 1 follows from (3) and (4).

Using this, we can also establish the second point.

2. By Lemmas 3.3 and 3.4, we have

$$\xi_Q(T) = \xi\left(m_Q^{\xi(T)}(T)\right) = \beta\xi\left(\frac{1}{\beta}m_Q^{\xi(T)}(T)\right).$$

For $V \in \mathcal{B}$, and vectors $x, y \in \text{com } V$ with $x_{-Q} = y_{-Q}$, we have $m_Q^x(V) = m_Q^y(V)$. By point 1 and the definition of b , we have $\xi_{-Q}(T) = \xi_{-Q}(b^{-1}T)$. Moreover, $\xi(T) \in \text{com } b^{-1}T$. Thus,

$$\frac{1}{\beta}m_Q^{\xi(T)}(T) = m_Q^{\xi(T)}(b^{-1}T) = m_Q^{\xi(b^{-1}T)}(b^{-1}T).$$

By Lemma 3.3 and the above, it follows that $\xi_Q(T) = \beta\xi_Q(b^{-1}T)$. This proves point 2.

By the definition of L^w and Eq. (2), $L^w(T) = \xi(T)$. Since $\xi(S) \in T \subseteq S$, we have $\xi(T) = \xi(S)$. Hence, $L^w(T) = L^w(S)$. This establishes the proof. ■

The property RIP imposes RIIA on a group of players under the condition that the contraction of the feasible set does not affect the solution outcomes of

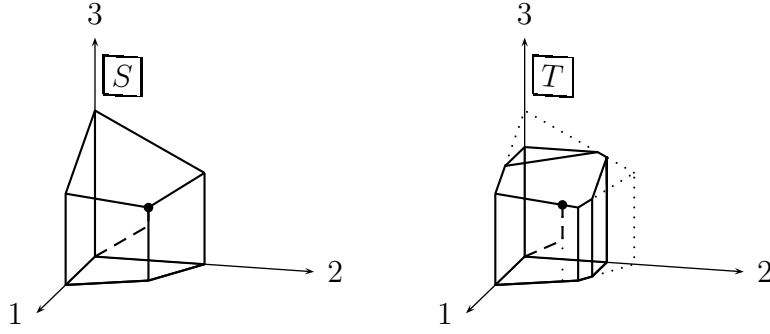


Figure 2: T is contraction of S along the axes of players 2 and 3, with $u_2(T)/u_3(T) = u_2(S)/u_3(S)$. Since player 1's payoff is not preserved by this contraction, RIP does not apply.

the nonmembers of that group. Figure 2 illustrates visually that this condition cannot be dispensed with.

Next, it is shown that for any solution φ satisfying the axioms of Theorem 3.1, there exists a weights vector w such that $\varphi = L^w$.

Proposition 3.5. *If a solution φ satisfies SIR, PO, SI, IM and RIP, then $\varphi \in \mathcal{L}$.*

The proof requires some extra notation. Let e^i be the vector for which the i -th entry is 1, and all others 0. For nonempty $Q \subseteq N$, we write $\sum_{i \in Q} e^i$ as $e(Q)$. Note that $e(N) = \mathbf{1}$. For $S \subset \mathbb{R}^N$ and $y \in S$, let $Q(S, y) := \{i \in N \mid y + \varepsilon e^i \in S \text{ for some } \varepsilon > 0\}$. For $y \in S$ with $Q(S, y) \neq \emptyset$, define

$$z(S, y) := y + a(S, y) e(Q(S, y))$$

where

$$a(S, y) := \max \{a \in \mathbb{R} \mid y + a e(Q(S, y)) \in S\}.$$

For $y \in S$ with $Q(S, y) = \emptyset$, $a(S, y) := 0$ and $z(S, y) := 0$, by convention.

For $S \in \mathcal{B}$, let $z^0 := \mathbf{0}$ and $z^j := z(\text{com } S, z^{j-1})$ for $j \geq 1$. Let k be the smallest integer such that $z^j = z^{j+1}$. Then for $j = 1, \dots, k$, define

$$Q^j := Q(\text{com } S, z^{j-1}), \quad \text{and} \quad a^j := a(\text{com } S, z^{j-1}).$$

The sequence $\{z^j\}_{j=0}^k$ is referred to as the *defining sequence* of $\xi(S)$.

For $x, y \in \mathbb{R}^N$, let $x \cdot y$ denote the inner product $\sum_{i \in N} x_i y_i$. For $p \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$, let

$$H(p, \beta) := \{x \in \mathbb{R}^N \mid p \cdot x \leq \beta\}.$$

The following lemma collects two useful results.

Lemma 3.6 (Imai [5]). *Let $S \in \mathcal{B}$. Then*

- (i) $\xi(S)$ is given by z^k ;
- (ii) for $T = S \cap H(p, \beta)$ with $p > \mathbf{0}$ and $\beta \in \mathbb{R}$ such that $u(S) = u(T)$, we have $S_{-i} = T_{-i}$ for all $i \in N$.

The proof of Proposition 3.5 is constructed as follows. First the weights vector w is obtained as the solution outcome of a generic problem. We fix a normalized problem S , and obtain the defining sequence $\{z^j\}_{j=0}^k$ of $\xi(w^{-1}S)$. From this, a sequence $\{x^j\}_{j=0}^k$ results, with $x^j = wz^j$ for all j . Then auxiliary bargaining problems are constructed for each j . By means of an induction argument, it is established that x^j is the common solution outcome of all stage- j auxiliary problems. This implies that x^k is the common solution outcome of the final-stage auxiliary problems. The observation that x^k is efficient in the problem S is then sufficient to conclude that $\varphi(S) = x^k$. The desired characterization result follows by point (i) of Lemma 3.6.

The recursive structure of the proof is similar to the one found in Imai's characterization proof of L . However, Imai's argument makes use of the axiom AN to make the induction step. We achieve the same objective by redefining his auxiliary bargaining problems, and then exploiting the axioms SI and RIP. Roughly said, we first transform a stage $j - 1$ auxiliary problem, with solution x^{j-1} , into a problem with solution x^j ; next, using RIP, the obtained problem is reduced to a stage j auxiliary problem without affecting the solution outcome.

Proof of proposition 3.5. Let $\varphi : \mathcal{B} \rightarrow \mathbb{R}^N$ be a bargaining solution satisfying SIR, PO, SI, IM and RIP. Consider the problem $H := H(\mathbf{1}, 1)$, and define $w := \varphi(H)$. Since φ satisfies SIR and PO, we have $w \in \Delta$.

Fix a problem $S \in \mathcal{B}^*$ with $S = \text{com } S \cap (\mathbf{1} - \mathbb{R}_+^N)$. By SI and RIP, this choice is without loss of generality. We show $\varphi(S) = L^w(S)$. Let $\{z^j\}_{j=0}^k$, $\{Q^j\}_{j=1}^k$ and $\{a^j\}_{j=1}^k$ be as in the defining sequence of $\xi(w^{-1}S)$, and for each j define $x^j := wz^j$. In view of point (i) of Lemma 3.6, it is sufficient to show $\varphi(S) = x^k$.

First we introduce a number of auxiliary bargaining problems. Define $p^1 := \mathbf{0}$ and $q^1 := (1/a^1)\mathbf{1}$. For $j = 2, \dots, k$, let $p^j := e(N \setminus Q^j)$ and $q^j := (q_i^j)_{i \in N}$,

where for each $i \in N$,

$$q_i^j := \begin{cases} 1/a^j & \text{if } i \in Q^j \\ q_i^{j-1} & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} \overline{S}^j &:= H(q^j, 1) \cap \left(\bigcap_{j'=1}^j H(p^{j'}, p^{j'} \cdot x^{j'}) \right) \cap (\mathbf{1} - \mathbb{R}_+^N) & j = 1, \dots, k; \\ \underline{S}^j &:= \overline{S}^j \cap H(p^{j+1}, p^{j+1} \cdot x^{j+1}) & j = 1, \dots, k-1; \\ S^j &:= H(q^j, 1) \cap S & j = 1, \dots, k; \\ S'^j &:= \overline{S}^j \cap S & j = 1, \dots, k. \end{aligned}$$

Each halfspace $H(q^j, 1)$ is a rescaling of H , specifically constructed such that its corner points are given by $a^j e^i$ for all $i \in Q^j$, $a^{j-1} e^i$ for all $i \in Q^{j-1} \setminus Q^j$, $a^{j-2} e^i$ for all $i \in Q^{j-2} \setminus Q^{j-1}$, etc. Consequently, SI implies $\varphi(H(q^j, 1)) = x^j$ for each j . It is shown next that all auxiliary problems are normalized.

Claim 3.7. $\overline{S}^j, \underline{S}^j, S^j, S'^j \in \mathcal{B}^*$ for each j .

Proof. It is sufficient to show that e^i is in $\overline{S}^j, \underline{S}^j, S^j$, and S'^j for each j and i . The claim follows from four observations.

- (a) $e^i \in H(p^j, p^j \cdot x^j)$ for each i and j ;
- (b) $e^i \in H(q^j, 1)$ for each i and j ;
- (c) $e^i \in S$ for all i ;
- (d) $e^i \in (\mathbf{1} - \mathbb{R}_+^N)$ for all i .

Note that (a), (b) and (d) together imply $e^i \in \overline{S}^j$ for each i and j . Then by (c), we have $e^i \in S'^j$ for each i and j ; for $j < k$, it is implied by (a) that each e^i is in \underline{S}^j . Finally, (b) and (c) together imply $e^i \in S^j$ for each i and j .

Observation (d) is trivial. Observation (c) follows from comprehensiveness of S and $S \in \mathcal{B}^*$. We now show (a) and (b). Denote $N \setminus Q^2$ by Q , Q^2 by Q' , and for $i \in Q$, denote $Q' \cup \{i\}$ by Q_i . Let $\bar{w} := (\bar{w}_1, \dots, \bar{w}_n)$ where $\bar{w}_i := w_i / \sum_{i' \in Q} w_{i'}$ for all $i \in N$. Note that $\sum_{i \in Q} \bar{w}_i = 1$.

By the supporting hyperplane theorem and the definition of x^1 , it follows that there is a $p \in \mathbb{R}_+^N$ with $p_i = 0$ for all $i \in Q'$, such that $p \cdot z \leq p \cdot x^1$ for all

$z \in S$. By observation (c) it follows that $p \cdot e^i \leq p \cdot x^1$ for all $i \in Q$. Since $p_i = 0$ for all $i \in Q'$, this implies $p \cdot e(Q_i) = p \cdot e^i \leq p \cdot x^1$ for all $i \in Q$. It follows that $p \cdot \sum_{i \in Q} \bar{w}_i e(Q_i) \leq p \cdot x^1$. Note that $\sum_{i \in Q} \bar{w}_i e(Q_i) = \bar{w} + (e(Q') - \bar{w}e(Q'))$. Since $p_i = 0$ for all $i \in Q'$, we obtain $p \cdot \bar{w} \leq p \cdot x^1$. Note that $x^1 = a^1 w = [a^1 \sum_{i \in Q} w_i] \bar{w}$. Then $p \cdot \bar{w} \leq p \cdot x^1$ is equivalent to $a^1 \sum_{i \in Q} w_i \geq 1$. Hence, $x^1 \geq \bar{w}$.

Since $p^1 \cdot e^i = 0$ for all $i \in N$, each e^i is trivially in $H(p^1, p^1 \cdot x^1)$. Consider some $j \in \{2, \dots, k\}$. Note that $p_i^j = 0$ for all $i \in Q^j$. Hence,

$$p^j \cdot \bar{w} = \sum_{i \in N \setminus Q^j} \bar{w}_i = \frac{\sum_{i \in N \setminus Q^j} w_i}{\sum_{i \in N \setminus Q^2} w_i} \geq \frac{\sum_{i \in N \setminus Q^2} w_i}{\sum_{i \in N \setminus Q^2} w_i} = 1.$$

The inequality follows from the observation that $N \setminus Q^2$ is a subset of $N \setminus Q^j$. By the above, we have $x^j \geq x^1 \geq \bar{w}$. Hence, $p^j \cdot x^j \geq p^j \cdot x^1 \geq p^j \cdot \bar{w} \geq 1$. Since $p^j \cdot e^i \leq 1$ for all $i \in N$, we obtain $p^j \cdot x^j \geq p^j \cdot e^i$ for all $i \in N$. This establishes observation (a).

Since $x^1 \geq \bar{w}$ and $\sum_{i \in N} \bar{w}_i \geq \sum_{i \in Q} \bar{w}_i = 1$, we have $\mathbf{1} \cdot e^i = 1 \leq \mathbf{1} \cdot \bar{w} \leq \mathbf{1} \cdot x^1 = a^1$ for all $i \in N$. Recall that $q^1 = (1/a^1)\mathbf{1}$. Then $q^1 \cdot e^i \leq 1$, implying $e^i \in H(q^1, 1)$ for all $i \in N$. Since $H(q^j, 1) \subset H(q^{j+1}, 1)$ for each $j < k$, observation (b) follows. ■

The proof proceeds with an induction argument. In particular, we want to show that

$$\varphi(\bar{S}^j) = \varphi(\underline{S}^j) = \varphi(S^j) = \varphi(S'^j) = x^j \quad (5)$$

for each j . First, this is established for $j = 1$.

Claim 3.8. $\varphi(\bar{S}^1) = \varphi(\underline{S}^1) = \varphi(S^1) = \varphi(S'^1) = x^1$.

Proof. First observe that $H(q^1, 1) \cap H(p^1, p^1 \cdot x^1) = H(q^1, 1) = H(\mathbf{1}, a^1) = a^1 H$. Then by SI, we have $\varphi(H(q^1, 1) \cap H(p^1, p^1 \cdot x^1)) = a^1 w = x^1$. Since $x^1 \leq u(S) = \mathbf{1}$ we have $x^1 \in \bar{S}^1$. By RIP this implies $\varphi(\bar{S}^1) = x^1$. For all $i \in N \setminus Q^2$, we have $x_i^1 = x_i^2$, implying $p^2 \cdot x^1 = p^2 \cdot x^2$. Hence, $x^1 \in H(p^2, p^2 \cdot x^2)$. Since $x^1 \in \bar{S}^1$ as well, we have $x^1 \in \underline{S}^1 \subseteq \bar{S}^1$. Moreover, $u(\underline{S}^1) = u(\bar{S}^1) = \mathbf{1}$. Then by RIP, it follows that $\varphi(\underline{S}^1) = x^1$. Since $x^1 \in \bar{S}^1$ and $x^1 \in S$, we have $x^1 \in S'^1$. Since $S'^1 \subseteq \bar{S}^1$ and $u(S'^1) = u(\bar{S}^1)$, RIP implies $\varphi(S'^1) = \varphi(\bar{S}^1) = x^1$. Observe that $S'^1 = S^1 \cap (\mathbf{1} - \mathbb{R}_+^N)$. Since $\varphi(S^1) \leq u(S^1) = \mathbf{1}$, we have $\varphi(S^1) \in (\mathbf{1} - \mathbb{R}_+^N)$. Hence, $\varphi(S^1) \in S'^1$. Since $S'^1 \subseteq S^1$ and $u(S'^1) = u(S^1) = \mathbf{1}$, RIP implies $\varphi(S^1) = \varphi(S'^1)$. Hence, $\varphi(S^1) = x^1$. ■

Fix some $j \in \{2, \dots, k\}$, and assume Eq. (5) holds for $j - 1$. As part of the induction argument, we need to show that $[\varphi(\underline{S}^{j-1}) = x^{j-1}]$ implies $[\varphi(\overline{S}^j) = x^j]$. This is established in the next three claims.

Claim 3.9. $\varphi_i(\overline{S}^j) = x_i^j$ for all $i \in N \setminus Q^j$.

Proof. Observe that $H(q^{j-1}, 1) \subset H(q^j, 1)$, and thus

$$\underline{S}^{j-1} = \overline{S}^j \cap H(q^{j-1}, 1).$$

By point (ii) of Lemma 3.6 this implies $\underline{S}_{-i}^{j-1} = \overline{S}_{-i}^j$ for all $i \in N$. By n -fold application of IM we obtain $\varphi(\overline{S}^j) \geq x^{j-1}$. By the definition, if for some x we have $x^{j-1} \leq x \in \overline{S}^j$, then $x_i = x_i^{j-1}$ for all $i \in N \setminus Q^j$. Since $x_i^{j-1} = x_i^j$ for all $i \in N \setminus Q^j$, the claim follows. ■

Let t be a vector that gives one to all players who are not in Q^j , and (a^j/a^{j-1}) to all players who are. Observe that $tx^{j-1} = x^j$.

Claim 3.10. $\overline{S}^j = t\underline{S}^{j-1} \cap \{x \in \mathbb{R}^N \mid x_i \leq 1 \text{ for all } i \in Q^j\}$.

Proof. Since $tH(p^{j'}, p^{j'} \cdot x^{j'}) = H(p^{j'}, p^{j'} \cdot x^{j'})$ for all $j' \in \{1, \dots, j\}$ and $tH(q^{j-1}, 1) = H(q^j, 1)$, we have

$$\begin{aligned} t\underline{S}^{j-1} &= t\overline{S}^{j-1} \cap tH(p^j, p^j \cdot x^j) \\ &= tH(q^{j-1}, 1) \cap \left(\bigcap_{j'=1}^j tH(p^{j'}, p^{j'} \cdot x^{j'}) \right) \cap t(\mathbf{1} - \mathbb{R}_+^N) \\ &= H(q^j, 1) \cap \left(\bigcap_{j'=1}^j H(p^{j'}, p^{j'} \cdot x^{j'}) \right) \cap t(\mathbf{1} - \mathbb{R}_+^N). \end{aligned}$$

Intersecting both sides of the equation with $\{x \in \mathbb{R}^N \mid x_i \leq 1 \text{ for all } i \in Q^j\}$ establishes the claim. ■

Claim 3.11. $\varphi(\overline{S}^j) = x^j$.

Proof. By Claim 3.10 we have $t\underline{S}^{j-1} = \overline{S}^j \cap V$ where $V := \{x \in \mathbb{R}^N \mid x_i \leq 1 \text{ for all } i \in Q^j\}$. Then $u_Q(\overline{S}^j) = (a^{j-1}/a^j)u_Q(t\underline{S}^{j-1})$. Moreover, $x \in V$ implies $(x_{Q^j}, y) \in V$ for all $y \in \mathbb{R}^{N \setminus Q^j}$. The following two observations show that $x^j \in \overline{S}^j$.

- By construction, $x^j \in t\underline{S}^{j-1}$.
- By definition, $x^j \in S$, implying $x_i^j \leq 1$ for all $i \in Q^j$. Thus, $x^j \in V$.

Since $\varphi(\underline{S}^{j-1}) = x^{j-1}$, it follows from SI that $\varphi(t\underline{S}^{j-1}) = x^j$. Hence, $\varphi(t\underline{S}^{j-1}) \in \overline{S}^j$. By Claim 3.9, $\varphi_{-Q^j}(\overline{S}^j) = \varphi_{-Q^j}(t\underline{S}^{j-1})$. The claim then follows by RIP. ■

This is sufficient to make the induction step, and to complete the proof.

Claim 3.12. $\varphi(\overline{S}^j) = \varphi(\underline{S}^j) = \varphi(S^j) = \varphi(S'^j) = x^j$ for each $j = 1, \dots, k$ (or $j = 1, \dots, k-1$ for $\varphi(\underline{S}^j)$).

Proof. By Claim 3.11 we have $\varphi(\overline{S}^j) = x^j$. Using the same arguments as in Claim 3.8, this implies $\varphi(S'^j) = x^j$. We now show $\varphi(S^j) = x^j$. The first step is to show that $\varphi(S^j) \in \overline{S}^j$.

- $\varphi(S^j) \in H(q^j, 1)$.
- Using the same arguments as in Claim 3.9, we have $\varphi(S^j) \geq \varphi(S^{j-1}) = x^{j-1}$. Like before, $x \in S^j$ and $x \geq x^{j-1}$ implies $x_i = x_i^{j-1}$ for all $i \in N \setminus Q^j$.
Then

$$p^{j'} \cdot x = p^{j'} \cdot x^j = p^{j'} \cdot x^{j'} \text{ for all } j' \in \{1, \dots, j\}.$$

$$\text{Hence, } \varphi(S^j) \in \bigcap_{j'=1}^j H(p^{j'}, p^{j'} \cdot x^{j'})$$

- $\varphi(S^j) \leq \mathbf{1}$, so $\varphi(S^j) \in (\mathbf{1} - \mathbb{R}_+^N)$.

Since $\varphi(S^j) \in \overline{S}^j$ and $\varphi(S^j) \in S$ (the latter follows from $S^j \subseteq S$), we have $\varphi(S^j) \in S'^j$. Since $\overline{S}^j \subseteq H(q^j, 1)$, we have $S'^j \subseteq S^j$. Then $\varphi(S^j) \in S'^j \subseteq S^j$. By RIP we obtain $\varphi(S^j) = \varphi(S'^j) = x^j$.

Finally, if $j < k$, then $\varphi(\underline{S}^j) = x^j$ by the same arguments as in Claim 3.8. ■

Claim 3.13. $\varphi(S) = x^k$.

Proof. Since $S^k = S \cap H(q^k, 1)$, point (ii) of Lemma 3.6 implies $S_{-i} = S_{-i}^k$ for all $i \in N$. Hence, by n -fold application of IM it follows that $\varphi(S) \geq \varphi(S^k)$, and thus by Claim 3.12, $\varphi(S) \geq x^k$. The claim follows from the observation that $x^k \in P(S)$. This concludes the proof. ■

Theorem 3.1 is established by Propositions 3.2 and 3.5. Next we argue that the axioms of the theorem are independent.

SIR: For $n = 2$, the solution $F(S) := \{x \in P(S) \mid x \geq (u_1(S), 0)\}$ satisfies PO, SI, IM and RIP, but violates SIR.

PO: The *Kalai-Smorodinsky solution* (Kalai & Smorodinsky [7]) is defined as $K(S) := \alpha^* u(S)$, where $\alpha^* := \max\{\alpha \mid \alpha u(S) \in S\}$. This solution satisfies SIR, SI, IM and RIP, but violates PO.

SI: The lexicographic egalitarian solution ξ satisfies SIR, PO, IM, and RIP, but violates SI.

IM: The Nash bargaining solution (Nash [9]) is defined as the unique point x^* that maximizes the function $\prod_{i \in N} x_i$ on $S \cap \mathbb{R}_+^N$. It satisfies SIR, PO, SI and RIP, but violates IM.

RIP: To show that RIP is independent from the other axioms, we consider a solution $D : \mathcal{B} \rightarrow \mathbb{R}^N$ that improves all player's payoffs simultaneously in a certain direction, until the boundary is reached. Subsequently, the players whose payoffs can be further increased within the feasible set, are divided in two distinct groups. For one of the groups, optimization continues as usual, until no further improvement is possible. From the point thus obtained, the payoffs of players in the second group are improved simultaneously until no further improvement is possible. The resulting outcome corresponds with D . Such a solution satisfies SIR, PO, SI and IM, but violates RIP. This is illustrated by the following example.

Example. Consider the 3-person problem S defined as the convex hull of the points $(0, 0, 0)$, $(4, 0, 0)$, $(0, 4, 0)$, $(0, 0, 1)$, $(4, 0, 1)$ and $(0, 4, 1)$.⁵ The solution D improves players' payoffs in the direction $(1/6, 1/6, 2/3)u(S)$, until the boundary is reached in the point $(1, 1, 1)$. From there on, improvement of player 1's payoff takes precedence over player 2's. Then $D(S) = (3, 1, 1)$. Next, consider the problem $T := S \cap \{x \in \mathbb{R}^N \mid x_i \leq 3.6 \text{ for } i = 1, 2\}$. When optimizing in the direction $(1/6, 1/6, 2/3)u(T)$, the boundary is first reached in the point $(0.9, 0.9, 1)$. Further improvement for player 1 leads to the solution $D(T) = (3.1, 0.9, 1)$.

4 Concluding Remarks

The method for characterizing a weighted version of Imai's lexicographic Kalai-Smorodinsky solution presented in this article, crucially depends on the property SI. Therefore, it does not extend to weighted versions of the lexicographic

⁵This is a rescaling of the example of Roth [13].

egalitarian solution (Thomson & Lensberg [15, pp. 139]), or the lexicographic equal-loss solution (Chun & Peters [1]).

The property RIIA in itself is too weak to characterize the asymmetric leximin solution. This is illustrated by the observation that the solution D described in the example above – in addition to SIR, PO, SI and IM – also satisfies RIIA. The reason it is sufficient in the case of two-player problems is that for these, RIP is implied by the combination of PO, IM and RIIA.

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