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This is dedicated to my family.

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Abstract. In [**Ban10**] a spatial version of intersection homology is defined. A key step is fiberwise homology truncation of the link bundle of a pseudomanifold. This is implemented in [**Ban10**] for trivial link bundles. The difficulty of extending said results to more general link bundles is informed by two factors: firstly, the type of fiber (which is also the link of the pseudomanifold), and secondly, the base space of the bundle (which is the singular set of the pseudomanifold). We extend the methods introduced in [**Ban10**] to link bundles of two types: (1) Fibers CW-complexes with (amongst other conditions) evenly graded homology and base space a sphere. (2) Using a fiber admitting truncation only in selected degrees and base space such that the bundle is glued from two trivial bundles.

Different methods are required in each setting. In the first setting, truncation of the fiberwise gluing homeomorphisms yields only homotopy equivalences. Hence homotopy theory is necessary to build a truncated bundle with the right properties. In the second case, this difficulty is not encountered, and no homotopy theory is necessary. Here, we use sheaf theory. In both cases we require the link bundle to be glued from trivial bundles by means of cellular homeomorphisms. Generalized Poincaré duality is shown for pseudomanifolds with each type of link bundle.

Zusammenfassung. In [**Ban10**] wird eine räumliche Version der Schnitthomologie definiert. Ein wichtiger Baustein ist das Faserweise Abschneiden der Homologie eines Linkbündels einer Pseudomannigfaltigkeit. In [**Ban10**] wird dies für triviale Linkbündel ausgeführt. Zwei Faktoren bestimmen den Schwierigkeitsgrad einer Verallgemeinerung dieser Technik: zum einen die Form der Faser (die der Link der Pseudomannigfaltigkeit ist) und zum anderen die Art der Basis des Bündels (welche gleichzeitig die singuläre Menge der Pseudomannigfaltigkeit ist). Die Methoden aus [**Ban10**] werden in dieser Arbeit für zwei Typen von Linkbündeln erweitert: (1) Bündel mit Fasern CW-Komplexe mit u. a. verschwindender Homologie in ungeraden Graden und Basis eine Sphäre. (2) Für Fasern, die das Abschneiden der Homologie nur in bestimmten Graden zulassen, und einem Bündel, welches aus zwei trivialen Bündeln verklebt wird.

Verschiedene Methoden sind in beiden Fällen angebracht. Im ersten Fall ergibt eine Anwendung des Abschneidefunktors auf die Verklebungshomöomorphismen nur Homotopieäquivalenzen. Daher ist es notwendig, Homotopietheorie einzusetzen um ein abgeschnittenes Bündel mit den gewünschten Eigenschaften zu konstruieren. Im zweiten Fall treten diese Probleme nicht auf, und es wird keine Homotopietheorie benötigt. Stattdessen wird Garbentheorie verwendet. In beiden Fällen ist es notwendig, zu fordern dass die Linkbündel mittels zellulärer Homöomorphismen aus trivialen Bündeln verklebt werden. Verallgemeinerte Poincaré Dualität wird für Pseudomannigfaltigkeiten mit beider Art Linkbündel gezeigt.

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Contents

Preface

Banagl defined in [**Ban10**] a spatial homology truncation functor which can be used to define a spatial "sibling" of intersection homology theory. Roughly speaking, this intersection space homology of a pseudomanifold X is calculated in two steps: Given a perversity $\overline{p}$, one firstly assigns to X an intersection space $I^{\overline{p}}X$. This space $I^{\overline{p}}X$ arises from X by making certain geometric changes to X in a way dictated by the perversity $\overline{p}$. Secondly, the (ordinary) homology $H_*(I^{\overline{p}}X)$ is calculated, and called the perversity $\overline{p}$ intersection space homology of X. One may ask whether intersection space homology satisfies generalized Poincaré duality. The construction of the intersection space, along with the proof of generalized Poincaré duality, has been carried out in [**Ban10**] for several classes of spaces X, among them pseudomanifolds with isolated singularities and two-strata pseudomanifolds with trivial link bundles.

We extend these results to two-strata pseudomanifolds with two types of link bundles, some of which include nontrivial fiber bundles, and obtain generalized Poincaré duality in each case. The process of showing this is broadly similar in both cases. We therefore outline the first case as well as the general idea.

The first type of setting is a two-strata pseudomanifold with link bundle with cellular structure group (i.e. consisting only of cellular self-homeomorphisms with cellular inverses), interleaf links and spheres as singular sets. An interleaf link is a link which is an object in the interleaf category **ICW**, defined in [**Ban10**, Definition 1.62] as comprising simply connected CW-complexes with finitely generated even-dimensional homology and vanishing odd-dimensional homology for any coefficient group as well as cellular maps. The focus on links from the interleaf category **ICW** is due to the availability of a certain lift: namely, for a fixed $N \in \operatorname{Ob}\mathbf{ICW}$ and for some maps $f \in \operatorname{Hom}_{\mathbf{ICW}}(N,N)$, the truncation $t_{<k}(f)$ can be lifted from the homotopy category **HoCW** of CW-complexes and homotopy classes of cellular maps to the category **CW** of CW-complexes and cellular maps. This setting is addressed in Chapter 2.

To outline the method, consider a pseudomanifold

$$X = X_n \supset X_{n-c},$$

composed of two strata, and with interleaf link and trivial link bundle. The last condition requires an explanation; there is a neighborhood of the singular set $\Sigma = X_{n-c}$ which admits the structure of a fiber bundle. In the present setting, this is a trivial bundle,

$$\Sigma \times \overset{\circ}{\operatorname{cone}}(L),$$

with base space Σ and fiber $\overset{\circ}{\operatorname{cone}}(L)$, the open cone of the link L of X. Excision of an open neighborhood of the total space $\Sigma \times \overset{\circ}{\operatorname{cone}}(L)$ yields M. The latter space is a compact manifold, and its boundary

$$\partial M = \Sigma \times L$$

is called the link bundle of X. By assumption, this is a trivial bundle. Also by assumption, L is an interleaf CW-complex, and hence admits spatial homology truncation in arbitrary degrees, as explained in [**Ban10**]. To be precise, there is a map

$$t_{<k}(L) \to L.$$

Applying truncation to each fiber of ∂M yields a map

$$\Sigma \times t_{<k}(L) \to \Sigma \times L = \partial M \hookrightarrow M,$$

which we denote i_k. The parameter k depends on a perversity $\overline{p}$. Ultimately, it is the defining characteristic of the space

$$I^{\overline{p}}X = M \cup_{i_k} \operatorname{cone}(\Sigma \times t_{<k}(L)).$$

which is accordingly referred to as the perversity $\overline{p}$ intersection space $I^{\overline{p}}X$ of X.

Assume now that the link bundle is a non-trivial fiber bundle. Any such bundle may be thought of as consisting of two segments of data:

(1) a covering $\{U_\alpha\}$ and a corresponding collection of trivial bundles $\{U_\alpha \times L\}$, together with
(2) a collection of gluing functions $\{g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \operatorname{Homeo}(L)\}$ satisfying the cocycle condition.

We can apply the spatial homology truncation functor in a fiberwise fashion to the trivial bundles $U_\alpha \times L$, yielding a collection of trivial bundles $U_\alpha \times t_{<k}(L)$. Often, but not always, these can then be glued back together to a space $\operatorname{ft}_{<k}(\partial M)$ by means of the truncated gluing functions $\tilde{t}_{<k} \circ g_{\alpha\beta}$, using functoriality of spatial homology truncation. This process is called fiberwise homology truncation, because it is a fiberwise application of the spatial homology truncation functor $t_{<k}$. Then a map

$$\mu \colon \operatorname{ft}_{<k}(\partial M) \to \partial M$$

is defined, in order to attach the new link bundle to M. The composition of μ with the standard inclusion $\partial M \hookrightarrow M$ plays a role analogous to that of i_k, above. This technique was introduced at the level of differential forms in [**Ban11**].

Comparing this situation to that of a trivial link bundle, two new obstacles to carrying out the construction arise:

(1) Fiber bundles are glued by means of gluing functions which take value in the group of homeomorphisms of the fiber. Yet the output of our fiberwise truncation process are only homotopy equivalences.
(2) In order to glue the truncated bundle to M we need a continuous map μ. The construction of the truncated bundle must enable the existence of this map. Additionally, we would like μ to be a fiberwise map, so that we can exploit naturality of the Leray-Serre spectral sequence for the proof of generalized Poincaré duality.

The first problem is addressed by a modified gluing process. We do not glue two trivial bundles $U_\alpha \times t_{<k}(L), U_\beta \times t_{<k}(L)$ directly, but instead attach them to opposite ends of a mapping cylinder. The resulting bundle is a Dold fibration due to a result from [**Pup74**]. Of course, using a modified gluing process creates additional difficulties for the construction of the map μ. This leads to the second problem, which we have approached by adapting a process from [**Bau89**] for constructing maps between mapping cylinders.

Exploiting naturality of the Leray-Serre spectral sequence, we are then able to show that interleaf fiber bundles over spheres of dimension greater than one with cellular structure groups have cohomology and homology of product bundles. This result holds for Dold fibrations with the same properties as well. Thus we can calculate the cohomology and homology of both, the link bundle of X and its truncated version. This enables a proof of generalized Poincaré duality which follows the proof for the case of trivial link bundles in [**Ban10**]. The isomorphism we obtain is

$$\tilde{H}^{n-i}(I^{\overline{p}}X) \cong \tilde{H}_i(I^{\overline{q}}X),$$

for complementary perversities.

In summary, one may state that in Chapter 2 the focus is placed on links which admit truncation in arbitrary degrees. Fitting with this theme, generalized Poincaré duality is shown to hold for arbitrary complementary perversities. In the second setting, detailed in Chapter 3, we change this viewpoint. To be more specific, one may ask whether generalized Poincaré duality is retained if we demand that the link need only admit truncation in certain degrees. A partial answer is given. If a given pair of complementary perversities governs the degrees in which a link admits truncation, then we can show that a pseudomanifold with this link satisfies a statement of generalized Poincaré duality governed by these same two perversities. Similarly to the first setting, an intersection space is defined by excising a neighborhood of the singular set, truncating the link bundle, defining a map μ and then showing that μ^* has a right inverse. In this setting, we are able to construct a right inverse to

$$\mu^* = \mathrm{incl}^* \colon H^*(\partial M) \to H^*(\mathrm{ft}_{<k}\,(\partial M))$$

by means of sheaf theory. A proof of generalized Poincaré duality will then proceed in a manner similar to Chapter 2.

If the structures of the proofs in the two chapters are broadly similar, the main difference lies in how the map μ^* is shown to have a right inverse. In Chapter 2, the proof is accomplished by using homotopy theory, while in Chapter 3, the backbone of the same proof is sheaf theory. In both cases, the corresponding cohomological Leray-Serre spectral sequences collapse. But only in Chapter 2 can this aspect be used to show the total space to have the homology of a product bundle. In Chapter 3, the bundle under consideration need not be equipped with a simply connected base space. Therefore the local cohomology of the bundle can be twisted.

To complete the outline of this text, in Chapter 1 some preliminary results are established and recalled. Amongst these are results on bundles in general and weak fibrations in particular. Finally, Appendix A recalls some results on cofibrations and the cohomology of mapping cylinders for the convenience of the reader.

Perhaps it is of interest to note that in [**IS08**] for a given functor $\mathbf{Top} \to \mathbf{Top}$ with certain properties a fiberwise application is introduced, which maps certain quasifibrations to quasifibrations. Comparing this with the fiberwise application of spatial homology truncation, we see that the two settings are different. In the setting discussed in the present text, the input is a functor $\mathbf{ICW} \to \mathbf{HoCW}$, which is then applied to each fiber of a fiber bundle. The output is then a Dold fibration.

Explanation of Limitations. It may be interesting to briefly explain the reasons behind some of the conditions of the main results in this text. Both of the main chapters address exclusively links F which are finite CW-complexes. This is due to the fact that we need the group of cellular self-homeomorphisms with cellular inverses, $\mathrm{Homeo}_{CW}(F)$, to be a topological group in order to enable the fiberwise truncation process. The link bundles under consideration in both chapters are fiber bundles glued from trivial bundles by maps from $\mathrm{Homeo}_{CW}(F)$. Of course, not all cellular self homeomorphisms possess a cellular inverse. But in Section 2.7, we show that at least for finite CW-complexes, a cellular self-homeomorphism always has a cellular inverse. Moreover, this restriction to maps with cellular inverses is frequently used in the theory of transformation groups. See e.g. the definition of an isomorphism of relative G-CW-complexes in [**Lüc89**, p. 7].

Some pseudomanifolds can be treated with both the methods of Chapter 2 and 3. It should be noted that the output of the two procedures is slightly different geometrically, even if the statement of generalized Poincaré duality is not. By geometrically we mean that the intersection spaces are not homeomorphic. The underlying cause for this is the presence of homotopy theory in Chapter 2, where we do not glue two trivial bundles outright but rather attach them to opposite ends of a mapping cylinder. Since this gluing procedure is not replicated in Chapter 3, where no homotopy theory is required and direct attachment without an intermediary mapping cylinder is possible, the resulting intersection spaces differ by design.

Prerequisites. Terminology and results from [**Ban10**] are used throughout this text. Chapter 3 employs sheaf theory and draws on results, and in particular notation, from [**Bre97**].

Results on weak bundles such as Serre and Dold fibrations are employed. There is unfortunately no comprehensive reference dedicated exclusively to weak fibrations, but [**Rud08**, Chapter IV, Section 1] contains an overview of the topic.

Quoted results are marked by the quotation of the source in the header of their environments.

Notation. Following the notation in [**Ban10**], we define the mapping cylinder M_f of a map $f\colon X \to Y$ to be

$$M_f = \frac{Y \sqcup X \times I}{(x,1) \sim f(x) \forall x \in X},$$

i.e. the gluing takes place at the 1-end of $X \times I$. Likewise, the mapping cone (or homotopy cofiber) of f is

$$\operatorname{cone}(f) = Y \cup_f \operatorname{cone}(X) = M_f / X \times \{0\}.$$

We sometimes use the more compact notation $C_f = \operatorname{cone}(f)$ as well as $CX = \operatorname{cone}(X)$.

We use the American English spelling of the word "fiber". For reasons of uniformity, quotations containing the British English spelling "fibre" have been changed to the American English version. When not explicitly stated otherwise, the ring of coefficients for both, cohomology and homology, is $\mathbb{Q}$. Definitions and notation for spatial homology truncation and intersection space homology are used as introduced in [**Ban10**]. The word fibration will refer to a Hurewicz fibration unless indicated otherwise. The symbol i_k will always denote – for a given space Y – a map

$$Y \to Y \times I, \ y \mapsto (y,k).$$

At times, the range of i_k may be a mapping cone, mapping cylinder, or some other quotient space involving $Y \times I$. The symbol j_k will be used analogously.

For spaces, for which spatial homology truncation is defined, we abbreviate the notation for the spatial homology truncation and cotruncation functors of [**Ban10**] by writing

$$L_{<k} = t_{<k}(L),$$
$$L_{\geq k} = t_{\geq k}(L).$$

When discussing the Leray-Serre spectral sequence, we use the notation of [**McC01**]. In this book, much of the necessary background on this spectral sequence may be found.

For maps, the standard exponential notation

$$Y^X = \{f\colon X \to Y \mid f \text{ continuous}\}$$

is used. Notice that we explicitly demand that any element $f \in Y^X$ be continuous.

When discussing pairs, we use the shorthand

$$(X \times (Y,Z)) = (X \times Y, X \times Z),$$

as well as (for a subset $K \subset M$)

$$(M \mid K) = (M, M - K).$$

These may be combined when writing

$$((M \mid K) \times X) = (M \times X, (M-K) \times X).$$

When discussing sheaf theory, we follow the notation of [**Ban07**]. In particular, $H_n(X)$ will denote both ordinary cohomology and sheaf cohomology. The distinction will be clear from the context.

Depending on the situation, we use both notations for the interior of a subspace,

$$\operatorname{int} U = \mathring{U}.$$

We say that a spectral sequence *collapses* if all differentials in the E_2-term and in all higher terms vanish.

Suppose there is a map

$$F\colon X \times Y \to X \times Y$$

of product spaces such that for

$$\mathrm{proj}_1\colon X \times Y \to X,\ (x, y) \mapsto x$$

it holds that

$$\mathrm{proj}_1(F(x, y)) = \mathrm{proj}_1(x, y).$$

Then it holds that

$$F(x, y) = (x, \mathrm{proj}_2 \circ F(x, y)). \tag{1}$$

If we define for $x \in X$ the map

$$f(x)\colon Y \to Y,\ y \mapsto \mathrm{proj}_2 \circ F(x, y),$$

then we can write equation (1) as

$$F(x, y) = (x, f(x)(y)).$$

In such circumstances, we define

$$(x, F(x)y) = (x, f(x)(y)).$$

CHAPTER 1

Preliminaries

1.1. Bundles

1.1.1. Generalities on Bundles. This section defines some terms relating to bundles and collects some of the classical results pertaining to the topic at hand.

DEFINITION 1.1.1. A triple $\xi = (X, p, A)$ with X and A being topological spaces and $p\colon X \to A$ being a continuous map is called a *bundle* over A. The space X is referred to as the *total space*, A is called the *base space* and p is the *projection* of the bundle. For any point $a \in A$, the space $p^{-1}(a)$ is called the *fiber* over a of the bundle ξ.

We use the terms *fiber bundle*, *Hurewicz fibration* and *Serre fibration* in the usual sense. For some bundles, all fibers are equivalent in some sense. E. g. in a Hurewicz fibration $\xi = (X, p, A)$ with connected base, the fibers over any two points are homotopy equivalent. Thus there is some justification for referring to this homotopy class as the "fiber of the bundle". By abuse of notation, we sometimes include the fiber of a fibration by writing the bundle as a quadruple

$$\xi = (F, X, p, A).$$

The abuse is of course that we should include only the homotopy class, but in fact we include the actual fiber F over some distinguished point. We assume that it is understood that the fiber may change up to homotopy equivalence when moving along the base space, and we proceed analogously for fiber bundles.

We alternatively refer to the bundle $\xi = (X, p, A)$ as ξ, X or p, depending on the situation. For a bundle ξ, let $F(\xi)$ denote the fiber of ξ and let $B(\xi)$ denote the base space of ξ. Recall that we can factor any bundle $\xi = (X, p, A)$ as

$$X \xrightarrow[\simeq]{p_{\mathrm{he}}} W(p) \xrightarrow{p_{\mathrm{Hf}}} A$$

such that p_{he} is a homotopy equivalence, p_{Hf} is a Hurewicz fibration, and the diagram

$$\begin{tikzcd} X \arrow[rr, "p_{\mathrm{he}}", "\simeq"'] \arrow[dr, "p"'] & & W(p) \arrow[dl, "p_{\mathrm{Hf}}"] \\ & A & \end{tikzcd} \tag{2}$$

commutes. The bundle $\xi_{\mathrm{aHf}} = (W(p), p_{\mathrm{Hf}}, A)$ is called the *associated Hurewicz fibration* of ξ.

Given bundles $\xi = (X, p, A)$ and $\eta = (Y, q, B)$, a *fiberwise map*

$$(\phi, \Phi)\colon \xi \to \eta$$

is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "\Phi"] & B \end{tikzcd}$$

If base spaces are equal, $A = B$, a fiberwise map (ϕ, id) may be written simply as

$$\phi\colon X \to Y.$$

Definition 1.1.2 (Definition 6.16 on p. 123 in [**DK01**]). Given two bundles over A, $\xi = (X, p, A)$ and $\eta = (Y, q, A)$, and two fiberwise maps $\phi_0, \phi_1\colon X \to Y$, we say that $H\colon X \times I \to Y$ is a *fiber homotopy* over A from ϕ_0 to ϕ_1 if the diagram

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "p\times \mathrm{id}_I"'] & Y \arrow[d, "q"] \\ A \times I \arrow[r, "\mathrm{proj}_1"] & A \end{tikzcd} \tag{3}$$

commutes, and furthermore we have $H(\cdot, 0) = \phi_0$ and $H(\cdot, 1) = \phi_1$. In this case, we say that ϕ_0 and ϕ_1 are *fiber homotopic*. Likewise, a *fiber homotopy equivalence* is a fiberwise map $\phi\colon X \to Y$ such that there exists a fiberwise map $\psi\colon Y \to X$ with $\phi\psi\colon Y \to Y$ and $\psi\phi\colon X \to X$ being fiber homotopic to the respective identities.

In the following, the words *fiberwise homotopy* will mean "fiber homotopy". Commutativity of diagram (3) means for $(x, t) \in X \times I$ that

$$\begin{aligned} q(H(x,t)) &= \mathrm{proj}_1(p \times \mathrm{id}_I(x,t)) \\ &= \mathrm{proj}_1(p(x), t) \\ &= p(x), \end{aligned}$$

which implies $H(x, t) \in q^{-1}(p(x))$. So a fiber homotopy is a homotopy between total spaces which moves points only within a fiber. A fiberwise map between Hurewicz fibrations that is a homotopy equivalence of total spaces is a fiber homotopy equivalence by [**Hat01**, Exercise 4H.3]. Conversely, a fiber homotopy equivalence induces homotopy equivalences if restricted to fibers. Furthermore, under some mild restrictions, a fiberwise map which restricts to a homotopy equivalence on each fiber can be shown to be a fiber homotopy equivalence:

Theorem 1.1.3 (Satz 1 and Bemerkung 3 on p. 120ff. in [**Dol55**]). *Let $\xi = (X, p, A)$ and $\eta = (Y, q, A)$ be fiber bundles over a finite CW-complex A. Let $\phi\colon X \to Y$ be a fiberwise map. Then ϕ is a fiber homotopy equivalence if and only if for all $a \in A$, the restriction*

$$\phi|\colon p^{-1}(a) \to q^{-1}(a)$$

is a homotopy equivalence.

The requirement that A be a finite CW-complex is indeed needed. See [**Dol55**, Bemerkung 3] for an example of a fiber bundle $\xi = (X, p, A)$ with A not a finite CW-complex and a fiberwise map $(\phi, \mathrm{id})\colon \xi \to \xi$ with ϕ restricting to a homeomorphism on each fiber, such that ϕ is not a fiber homotopy equivalence.

Definition 1.1.4 (Dold fibration, Definition 5.1 and Proposition 5.13 on p. 238ff. in [**Dol63**]). A bundle $\xi = (X, p, A)$ such that p has the *weak covering homotopy property* (WCHP) is called a *Dold fibration*. This means that for every homotopy $H\colon Z \times I \to A$, and every initial position $h\colon Z \to X$ such that $ph(z) = H(z, 0)$, there exists a homotopy $\tilde{H}\colon Z \times [-1, 1] \to X$ such that $\tilde{H}(z, -1) = h(z)$ and

$$p\tilde{H}(z, t) = H(z, t)\, \forall t \in [0, 1]$$

while

$$p\tilde{H}(z, t) = H(z, 0)\, \forall t \in [-1, 0].$$

In other words, H is covered by a homotopy $\tilde{H}|_{Z\times[0,1]}$ whose initial position $\tilde{H}(\cdot, 0)$ is vertically homotopic to h. Dold fibrations over connected base spaces share a property with fibrations: Both posses fibers which are defined up to homotopy.

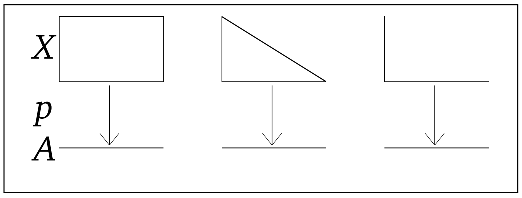


FIGURE 1. Examples of (from left to right): a fiber bundle, a Hurewicz fibration and a Dold fibration. These examples are due to V. Puppe.

Contrast this with the situation in a fiber bundle; here we have a fiber defined up to homeomorphism.

Another key aspect is that a Dold fibration remains a Dold fibration under fiber homotopy equivalence.

LEMMA 1.1.5 (Korollar 6.7 on p. 111 in [**tDKP70**]). *Let $\xi = (X, p, A)$ and $\eta = (Y, q, A)$ be bundles such that there is a fiber homotopy equivalence $\phi\colon \xi \to \eta$. Then ξ is a Dold fibration if η is.*

There is a useful local criterion for Dold fibrations involving numerable coverings. Notice that a numerable covering need not be open.

THEOREM 1.1.6 (Theorem 1.25 on p. 195 in [**Rud08**]). *Let $\{U_\alpha\}$ be a numerable covering of a space A, and let ξ be a bundle over A. If $\xi|_{U_\alpha}$ is a Dold fibration for every α, then so is ξ.*

PROPOSITION 1.1.7 (Recollection 1.24 on p. 195 in [**Rud08**]). *Every locally finite covering of a paracompact space is numerable.*

1.1.2. Dold Fibrations and Mapping Cylinders. Given two (Hurewicz or Dold) fibrations $\xi = (X, p, A)$ and $\eta = (Y, q, A)$ as well as a fiberwise map $\phi\colon X \to Y$, the mapping cylinder M_ϕ, viewed as a bundle over $A \times I$, is in general no longer a fibration.

EXAMPLE 1.1.8 (p. 263 in [**Wir74**]). Let $\xi = \eta = (A \times \{1, 2\}, \mathrm{proj}_1, A)$ and
$$\phi\colon A \times \{1, 2\} \to A \times \{1, 2\}, (a, k) \mapsto (a, 1).$$
Then M_ϕ does not have path lifting properties and thus is neither a Hurewicz nor Dold fibration.

So Dold fibrations are not preserved under the operation of taking mapping cylinders of arbitrary fiberwise maps. One may suspect that mapping cylinders of fiber homotopy equivalences of Dold fibrations are again Dold fibrations. For some fiber homotopy equivalences, this is confirmed by a result of Puppe in [**Pup74**], which we will now describe.

We first need to introduce some notation. Fix a topological space F. Then $\mathbf{Top}^F$ is the category with objects bundles $p\colon X \to A$ such that the associated Hurewicz fibration $p_{\mathrm{Hf}}\colon W(p) \to A$ has only fibers homotopy equivalent to F. Morphisms in $\mathbf{Top}^F$ are fiberwise maps

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "\Phi"] & B \end{tikzcd}$$

such that the canonical map $\psi\colon X \to \Phi^*(W(q))$ is a homotopy equivalence. We consider this canonical map now. We can factorize q as

$$Y \xrightarrow[\simeq]{q_{\mathrm{he}}} W(q) \xrightarrow{q_{\mathrm{Hf}}} B,$$

with q_{Hf} being the associated Hurewicz fibration. This fits into a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & Y \arrow[r, "q_{\mathrm{he}}"] \arrow[d, "q"'] & W(q) \arrow[dl, "q_{\mathrm{Hf}}"] \\ A \arrow[r, "\Phi"] & B & \end{tikzcd} \tag{4}$$

We can pull back $W(q)$ along Φ to obtain:

$$\begin{tikzcd} \Phi^*(W(q)) \arrow[dr, "\mathrm{proj}_1"'] & X \arrow[l, "\psi"'] \arrow[d, "p"] \\ & A \end{tikzcd}$$

Thus,

$$\Phi^*(W(q)) = \{(a, w) \in A \times W(q) \mid q_{\mathrm{Hf}}(w) = \Phi(a)\},$$

and the image of $x \in p^{-1}(a)$ under the canonical map is

$$\psi\colon x \mapsto (a, q_{\mathrm{he}} \circ \phi(x)).$$

This is well-defined if and only if

$$q_{\mathrm{Hf}}(q_{\mathrm{he}} \circ \phi(x)) = \Phi(a);$$

which does in fact hold:

$$\begin{aligned} \Phi(a) &= q \circ \phi(x) && \text{(square in (4), and recall that } x \in p^{-1}(a)) \\ &= q_{\mathrm{Hf}} \circ q_{\mathrm{he}} \circ \phi(x) && \text{(triangle in (4)).} \end{aligned}$$

Lastly, we note that the restriction of the canonical map ψ to a fiber $p^{-1}(a)$ of X is

$$\psi|_{p^{-1}(a)} = q_{\mathrm{he}}|_{q^{-1}(\Phi(a))} \circ \phi|_{p^{-1}(a)}. \tag{5}$$

Remark 1.1.9.
(1) Let $\eta = (Y, q, A)$ be a Hurewicz fibration. Then $q_{\mathrm{he}} = \mathrm{id}$ and $q_{\mathrm{Hf}} = q$. Therefore η is an object of $\mathbf{Top}^F$ if and only if the fiber of η over each point $a \in A$ is homotopy equivalent to F.
(2) Every Dold fibration is fiber homotopy equivalent to its associated Hurewicz fibration by [**Dol66**, Satz on p. 6.6]. I.e. for a given Dold fibration $\xi = (X, p, A)$ we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "p_{\mathrm{he}}", "\simeq"'] \arrow[dr, "p"'] & & W(p) \arrow[dl, "p_{\mathrm{Hf}}"] \\ & A & \end{tikzcd}$$

in which p_{he} is even a fiber homotopy equivalence. In particular, every fiber of p is homotopy equivalent to a fiber of p_{Hf}. Therefore, if ξ is an object of $\mathbf{Top}^F$, the fiber of ξ over each point $a \in A$ is homotopy equivalent to F. Conversely, if the fiber of ξ over each point $a \in A$ is homotopy equivalent to F, then the same holds for the fibers of p_{Hf}, making ξ an object of $\mathbf{Top}^F$.
(3) The first two items in this remark can be condensed to [**Pup74**, Remark on p. 2]: If $\xi = (X, p, A)$ is a fibration (Hurewicz or Dold), then ξ is an object of $\mathbf{Top}^F$ if and only if the fiber of ξ over each point $a \in A$ is homotopy equivalent to F.

The last remark facilitates checking whether a given Dold fibration is an object of $\mathbf{Top}^F$. We want to establish a similar criterion for fiberwise maps between Dold fibrations. We need some preliminary considerations:

DEFINITION 1.1.10 (p. 364 in [**Pup71**]). A topological space X is called *numerably contractible* if it admits a numerable cover $\{U_\alpha\}$ such that each inclusion $U_\alpha \hookrightarrow X$ is nullhomotopic.

THEOREM 1.1.11 (Theorem 6.3 on p. 243 in [**Dol63**]). *Let A be numerably contractible. Let $\xi = (X, p, A)$ and $\eta = (Y, q, A)$ be Dold fibrations. Then a fiberwise map $(\phi, \mathrm{id})\colon \xi \to \eta$ is a fiber homotopy equivalence if and only if for each $a \in A$, the restriction*

$$\phi|_{p^{-1}(a)} \colon p^{-1}(a) \to q^{-1}(a)$$

is a homotopy equivalence.

REMARK 1.1.12. Note that [**Dol63**, Theorem 6.3] also mentions that this same result holds if we exchange "numerably contractible space" with "locally contractible paracompact space". Furthermore, [**Dol63**, Proposition 6.7] notes that every connected CW-complex is numerably contractible. If a space A is numerably contractible with a numerable cover of contractible spaces $\{U_\alpha\}$, then the same holds for subspaces formed by taking unions of sets U_α, provided that these unions are connected.

PROPOSITION 1.1.13. *Let*

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "\Phi"] & B \end{tikzcd}$$

be a fiberwise map between Dold fibrations $\xi = (X, p, A)$ and $\eta = (Y, q, B)$. Let ξ and η have only fibers homotopy equivalent to F, let A be either numerably contractible or locally contractible paracompact, and let the restriction of ϕ to each fiber be a homotopy equivalence. Then (ϕ, Φ) is a morphism in $\mathbf{Top}^F$.

PROOF. We have to show that the induced map

$$\psi\colon X \to \Phi^*(W(q))$$

is a homotopy equivalence. The map

$$q_{\mathrm{Hf}} \colon W(q) \to B$$

is a Hurewicz fibration, hence so is $\Phi^*(W(q)) \to A$ by [**Rud08**, Proposition 1.14]. From equation (5) we know the restriction of ψ to a fiber to be

$$\psi|_{p^{-1}(a)} = q_{\mathrm{he}}|_{q^{-1}(\Phi(a))} \circ \phi|_{p^{-1}(a)}.$$

By assumption, the map $\phi|$ is a homotopy equivalence and so is $q_{\mathrm{he}}|$. Hence the composition $\psi|$ is a homotopy equivalence. We can invoke Theorem 1.1.11 and Remark 1.1.12 to show that ψ is a fiber homotopy equivalence because A is numerably contractible. In particular, ψ is a homotopy equivalence. □

REMARK 1.1.14. (1) If $\mathbf{DTop}^F_{\mathrm{CW}}$ is the full subcategory of $\mathbf{Top}^F$ with objects Dold fibrations $\xi = (X, p, A)$ such that ξ has fiber F and A is a connected CW-complex, we can give a sufficient condition for a fiberwise map to be a morphism: A fiberwise map is a morphism in $\mathbf{DTop}^F_{\mathrm{CW}}$ if it restricts to a homotopy equivalence on each fiber. This is due to the fact that A, being a connected CW-complex, is numerably contractible by [**Dol63**, Proposition 6.7], which enables an application of Proposition 1.1.13.

(2) Let $\mathbf{HTop}^F_{\mathrm{CW}}$ be the full subcategory of $\mathbf{DTop}^F_{\mathrm{CW}}$ with objects Hurewicz fibrations. Then $(\phi, \mathrm{id})\colon \xi \to \eta$ is a morphism in $\mathbf{HTop}^F_{\mathrm{CW}}$ if and only if ϕ restricts to a homotopy equivalence on each fiber. To see this, that one side of this equivalency was already shown, above. For the other side, let (ϕ, id) is a morphism in $\mathbf{HTop}^F_{\mathrm{CW}}$. Now $\eta = (Y, q, A)$ being a Hurewicz fibration implies $W(q) = Y$. Then

$$\psi\colon X \to \mathrm{id}^*(W(q)) = \mathrm{id}^*(Y)$$

is a homotopy equivalence. It is also, by definition, a fiberwise map. As both spaces involved are Hurewicz fibrations (X by definition and $\mathrm{id}^*(Y)$ as the pull-back of a Hurewicz fibration), we can invoke [**Hat01**, Exercise 4H.3] to show that ψ is a fiber homotopy equivalence. In particular, the restriction to a fiber $\psi| = \phi|$ is a homotopy equivalency.

LEMMA 1.1.15. *Let*

$$\begin{tikzcd} X_1 \arrow[d, "p_1"'] & X_0 \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] \arrow[d, "p_0"'] & X_2 \arrow[d, "p_2"] \\ A_1 & A_0 \arrow[l, "\Phi_1"'] \arrow[r, "\Phi_2"] & A_2 \end{tikzcd}$$

be a commutative diagram such that the vertical maps are Dold fibrations and objects of $\mathbf{Top}^F$. *Let furthermore both squares be morphisms in* $\mathbf{Top}^F$. *Then the induced map* $p\colon X \to A$ *between the row-wise double mapping cylinders is a Dold fibration.*

PROOF. This follows from the proof of [**Pup74**, Lemma 2, p. 4]. □

1.2. Leray-Serre Spectral Sequences

1.2.1. Naturality of Leray-Serre Spectral Sequences. In the present text, we make extensive use of the Leray-Serre spectral sequence. In particular, we employ its naturality properties. In fact, there are two sorts of such naturality properties: On the one hand, the Leray-Serre spectral sequence is natural with respect to fiberwise maps. We review this in Subsection 1.2.1. On the other hand, in a certain sense the Leray-Serre spectral sequence behaves in a non-natural fashion: Namely, in order to recover a (co)homology group from a convergent Leray-Serre spectral sequence, we need to choose splittings of certain short exact sequence. In general, it may not be possible to choose natural splittings. However, in the cases in which we are interested, natural splittings exist. We deal with this aspect in Subsection 1.2.2. Both naturality properties will be exploited in later sections. For the remainder of this chapter, all results and expressions concerning spectral sequences are quoted from [**McC01**].

Recall the cohomological version of the Leray-Serre spectral sequence:

THEOREM 1.2.1 (Theorem 5.2 on p. 135 in [**McC01**]). *Let* $\xi = (F, X, p, A)$ *be a fibration with connected fiber and path connected base space. Then there is a first quadrant spectral sequence of algebras,* $\{E_r^{*,*}, d_r\}$*, converging to* $H^*(X)$ *as an algebra, with an isomorphism*

$$(\phi_2^*)^{-1}\colon E_2^{p,q} \xrightarrow{\cong} H^p(A; \mathcal{H}^q(F; \mathbb{Q})).$$

The range of $(\phi_2^*)^{-1}$ *is the cohomology of the space* A *with local coefficients in the cohomology of the fiber of* ξ. *This sequence is natural with respect to fiberwise maps.*

If A is simply connected then $\pi_1(A)$ acts trivially on $H^*(F; \mathbb{Q})$, and the E_2-term can be identified as

$$E_2^{p,q} \cong H^p(A; H^q(F; \mathbb{Q})).$$

What, precisely, does the last sentence in the theorem mean? There are two statements contained here: firstly the fact that the spectral sequence is functorial in a certain sense, and secondly the fact that the identification of the E_2-term is through a natural isomorphism. We consider both parts in detail.

Concerning the functoriality, we need some terminology. Following [**McC01**, p. 65], we denote by **SpecSeq** the category of spectral sequences and morphisms thereof. We let Hurewicz fibrations with simply connected base and connected fiber form the objects of a category **Fib**. Morphisms in this category are fiberwise maps. Then we can state that the assignment

$$\mathrm{LS}^*\colon \xi \mapsto \{E_r^{*,*}, d_r\},$$

which assigns to a fibration ξ its cohomological Leray-Serre spectral sequence is actually a contravariant functor from **Fib** to **SpecSeq**. Thus a fiberwise map

$$\Xi = (\psi, \Psi)\colon \xi \to \eta$$

of fibrations in **Fib** induces a morphism

$$\{\Xi\} = \mathrm{LS}^*(\Xi)\colon \{E_r^{p,q}(\eta), d_r(\eta)\} \to \{E_r^{p,q}(\xi), d_r(\xi)\}$$

of the associated spectral sequences:

$$\begin{tikzcd}
\eta & \{E_r^{p,q}(\eta), d_r(\eta)\} \arrow[d, "\{\Xi\}"] \\
\xi \arrow[u, "\Xi"] \arrow[ur, phantom, "\overset{\mathrm{LS}^*}{\rightsquigarrow}"] & \{E_r^{p,q}(\xi), d_r(\xi)\}
\end{tikzcd}$$

This functorial formulation can be found for example in [**Spa66**, Chapter 9].

Concerning the second part of the naturality statement, we have for any fiberwise map $\Xi = (\psi, \Psi) \in \mathrm{Hom}_{\mathbf{Fib}}(\xi, \eta)$ a commutative diagram:

$$\begin{tikzcd}
E_2^{p,q}(\eta) \arrow[r, "(\phi_2^*)^{-1}", "\cong"'] \arrow[d, "\{\Xi\}_2^{p,q}"'] & H^p(B(\eta); H^q(F(\eta))) \arrow[d, "\overline{(\psi,\Psi)}"] \\
E_2^{p,q}(\xi) \arrow[r, "(\phi_2^*)^{-1}", "\cong"'] & H^p(B(\xi); H^q(F(\xi)))
\end{tikzcd}$$

Here $\overline{(\psi, \Psi)}$ is the map induced on cohomology by $\psi|\colon F(\xi) \to F(\eta)$ and $\Psi\colon B(\xi) \to B(\eta)$. Thus we can combine the two diagrams to obtain the following diagram with commutative squares:

$$\begin{tikzcd}
\eta & \{E_r^{p,q}(\eta), d_r(\eta)\} \arrow[d, "\{\Xi\}"] & E_2^{p,q}(\eta) \arrow[l, hook'] \arrow[r, "(\phi_2^*)^{-1}"] \arrow[d, "\{\Xi\}_2^{p,q}"'] & H^p(B(\eta); H^q(F(\eta))) \arrow[d, "\overline{(\psi,\Psi)}"] \\
\xi \arrow[u, "\Xi"] \arrow[ur, phantom, "\overset{\mathrm{LS}^*}{\rightsquigarrow}"] & \{E_r^{p,q}(\xi), d_r(\xi)\} & E_2^{p,q}(\xi) \arrow[l, hook'] \arrow[r, "(\phi_2^*)^{-1}"] & H^p(B(\xi); H^q(F(\xi)))
\end{tikzcd}$$

This is naturality of the cohomological Leray-Serre spectral sequence for fiberwise maps.

Analogously, naturality of the homological Leray-Serre spectral sequence can be summarized as commutativity of the squares in the following diagram:

$$\begin{tikzcd}
\xi \arrow[d, "{(\psi,\Psi)}"'] \arrow[r, phantom, "\overset{\mathrm{LS}_*}{\rightsquigarrow}"] & \{E^r_{p,q}(\xi), d^r(\xi)\} \arrow[d, "\{\Xi\}"] & E^2_{p,q}(\xi) \arrow[l, hook'] \arrow[r, "\phi^2_*"] \arrow[d, "\{\Xi\}^2_{p,q}"'] & H_p(B(\xi); H_q(F(\xi))) \arrow[d, "\overline{(\psi,\Psi)}"] \\
\eta & \{E^r_{p,q}(\eta), d^r(\eta)\} & E^2_{p,q}(\eta) \arrow[l, hook'] \arrow[r, "\phi^2_*"] & H_p(B(\eta); H_q(F(\eta)))
\end{tikzcd}$$

1.2.2. Leray-Serre Spectral Sequences and the Recovery Problem. We want to look at what may be called the *recovery problem*:

$$\boxed{\text{Given a convergent spectral sequence, recover its limit.}}$$

On first glance, this seems an odd question to ask; the intuition is that if one has some objects converging to another object, then the limit of the former should be exactly the latter. The reason that this picture does not hold up for convergence of spectral sequences is its very definition. Firstly, convergence of a spectral sequence is not necessarily unique. Secondly, even when convergence is unique, we recover the desired group only up to isomorphism. The task of finding such an isomorphism is known as "solving the extension problems". The reason for this terminology will become apparent later.

The requirement that the underlying filtration be bounded ensures unique convergence. In this case the recovery problem reduces to the *extension problems*. As we are only concerned with spectral sequences arising from bounded filtrations, unique convergence does not concern us. Hence we will now describe the extension problems. To do so, we must first examine how the spectral sequences with which we are concerned arise.

A filtered differential graded module (A, d, F^*) induces a spectral sequence by [**McC01**, Theorem 2.6]. This spectral sequence converges to $H^*(A, d)$. More specifically, if we assume that the spectral sequence at hand is cohomological, then convergence is by definition the fact that the map d_∞ (see [**McC01**]) is an isomorphism

$$E^{p,q}_\infty \xrightarrow[\cong]{d_\infty} \tfrac{F^p H^{p+q}(A,d)}{F^{p+1} H^{p+q}(A,d)}.$$

Concerning the terms on the RHS, note that the filtration F^* of A induces a filtration of $H^*(A, d)$ by

$$F^l H^k(A, d) = \operatorname{im}[H^k(F^l A) \to H^k(A)].$$

When the differential graded module (A, d) in question is understood we often omit it from the notation by writing $F^l H^k$ for $F^l H^k(A, d)$. It should be noted that one may also consider the Wang sequence of a given bundle. This is done in the proof of Proposition 2.5.3.

Proposition 1.2.2. *Let $\xi = (X, p, B)$ be a fibration such that $B \cong S^{n-c}$. Then the cohomological Leray-Serre spectral sequence converges to $H^*(X)$, and there is a short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & F^{n-c}H^{n-r} \arrow[r, "i"] & H^{n-r}(X) \arrow[r] & E^{0,n-r}_\infty \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

which allows us to recover the group $H^{n-r}(X)$ by choosing a splitting s. Any splitting induces an isomorphism

$$a^* \colon F^{n-c}H^{n-r} \oplus E^{0,n-r}_\infty \xrightarrow{\cong} H^{n-r}(X),\ (x_{n-c}, x_0) \mapsto i(x_{n-c}) + s(x_0).$$

PROOF. As said previously, the cohomological Leray-Serre spectral sequence is constructed according to [**McC01**, Theorem 2.6]. The input for this theorem is a filtered differential graded module (A, d, F^*). In our case, $A = C^*(X)$ and $d = \partial^*$ is the usual coboundary operator. The filtration F^* of $A = C^*(X)$ which induces the cohomological Leray-Serre spectral sequence is given as

$$F^s C^*(X) = \ker[C^*(X) \to C^*(J^{s-1})],$$

with

$$J^s = p^{-1}(B^s).$$

We use a cell structure

$$B \cong S^{n-c} = e^0 \cup e^{n-c}$$

on the base. So

$$F^0 C^*(X) = \ker[C^*(X) \to C^*(\underbrace{J^{-1}}_{=\varnothing})] = C^*(X).$$

We also know that

$$\begin{aligned} &F^0 H^{n-r} \\ =&\mathrm{im}\,[H^{n-r}(F^0 C^*(X)) \to H^{n-r}(C^*(X))] \\ =&\mathrm{im}\,[H^{n-r}(C^*(X)) \to H^{n-r}(C^*(X))] \\ =&\mathrm{im}\,[H^{n-r}(X) \to H^{n-r}(X)] \\ =&H^{n-r}(X), \end{aligned} \tag{6}$$

and furthermore that

$$\begin{aligned} F^1 C^*(X) &= \ker[C^*(X) \to C^*(J^0)] \\ &= \ker[C^*(X) \to C^*(p^{-1}(e^0))] \\ &= F^2 C^*(X) = \cdots = F^{n-c} C^*(X). \end{aligned}$$

These are the cochains in X which vanish on $p^{-1}(e^0)$. Thus, $F^1 C^*(X)$ is equal to $C^*(X, p^{-1}(e^0))$. Also,

$$F^{n-c+1} C^*(X) = \ker[C^*(X) \to C^*(J^{n-c+1-1})] = \ker[C^*(X) \to C^*(X)] = \{0\}.$$

Then, by (6),

$$\begin{aligned} E_\infty^{0,n-r} \xrightarrow[\cong]{d_\infty} &\frac{F^0 H^{n-r}}{F^1 H^{n-r}} \\ =& \frac{H^{n-r}(X)}{\mathrm{im}\,[H^{n-r}(C^*(X, p^{-1}(e^0))) \to H^{n-r}(X)]} \\ =& \frac{H^{n-r}(X)}{\mathrm{im}\,[H^{n-r}(X, p^{-1}(e^0)) \to H^{n-r}(X)]} \\ =& \frac{H^{n-r}(X)}{\ker[H^{n-r}(X) \to H^{n-r}(p^{-1}(e^0))]}, \end{aligned}$$

with the last equality being due to the long exact sequence of the pair. We note

$$F^1 H^{n-r} = F^2 H^{n-r} = \cdots = F^{n-c} H^{n-r}$$

and

$$\begin{aligned} F^{n-c+1} H^{n-r} &= \mathrm{im}\,[H^{n-r}(\{0\}) \to H^{n-r}(X)] \\ &= \{0\}. \end{aligned}$$

This constitutes a decreasing filtration of H^{n-r} such that

$$\{0\} = F^{n-c+1} H^{n-r} \subset F^{n-c} H^{n-r} = \cdots = F^1 H^{n-r} \subset F^0 H^{n-r} = H^{n-r}.$$

Concerning the associated graded module

$$\operatorname{Gr}(H^{n-r},F^*)^p = \frac{F^pH^{n-r}}{F^{p+1}H^{n-r}} \xleftarrow[\cong]{d_\infty} E_\infty^{p,n-r-p}$$

of H^{n-r}, we obtain for any $p \in \mathbb{N}$ a short exact sequence

$$0 \longrightarrow F^{p+1}H^{n-r} \xrightarrow{\text{incl}} F^pH^{n-r} \xrightarrow{\text{quot}} \operatorname{Gr}(H^{n-r},F^*)^p \longrightarrow 0.$$

Setting $\operatorname{Gr}^p = \operatorname{Gr}(H^{n-r},F^*)^p$, we see that this set of short exact sequence is as follows:

$$\begin{tikzcd}
 & 0 \arrow[r, "="] & F^{n-c+1}H^{n-r} \arrow[r, "="] & \operatorname{Gr}^{n-c+1} \arrow[r] & 0 \\
0 \arrow[r, "="] & F^{n-c+1}H^{n-r} \arrow[r] & F^{n-c}H^{n-r} \arrow[r, "="] & \operatorname{Gr}^{n-c} \arrow[r] & 0 \\
0 \arrow[r] \arrow[d, dotted, no head] & F^{n-c}H^{n-r} \arrow[r, "="] \arrow[d, dotted, no head] & F^{n-c-1}H^{n-r} \arrow[r] \arrow[d, dotted, no head] & \operatorname{Gr}^{n-c-1} \arrow[r, "="] \arrow[d, dotted, no head] & 0 \arrow[d, dotted, no head] \\
0 \arrow[r] & F^2H^{n-r} \arrow[r, "="] & F^1H^{n-r} \arrow[r] & \operatorname{Gr}^1 \arrow[r, "="] & 0 \\
0 \arrow[r] & F^1H^{n-r} \arrow[r] \arrow[d, equal] & F^0H^{n-r} \arrow[r] \arrow[d, equal] & \operatorname{Gr}^0 \arrow[r] & 0 \\
 & F^{n-c}H^{n-r} \arrow[d, equal] & H^{n-r} & E_\infty^{0,n-r} \arrow[u, "\cong"', "d_\infty"] & \\
 & \operatorname{Gr}^{n-c} & & &
\end{tikzcd}$$

Splitting these sequences means solving the extension problems. In the present case, the splittings exist because all groups involved are in fact rational vector spaces. In fact, to obtain $H^{n-r} = F^0H^{n-r}$ from

$$\{\operatorname{Gr}(H^{n-r},F^*)^k \cong E_\infty^{k,n-r-k} \mid k \in \mathbb{Z}\}$$

we only need to regard the bottom split exact sequence, namely

$$\begin{tikzcd}
0 \arrow[r] & F^{n-c}H^{n-r} \arrow[r, "\text{incl}"] & H^{n-r}(X) \arrow[r, "(d_\infty)^{-1}\circ\text{quot}"] & E_\infty^{0,n-r} \arrow[l, bend left, "s"] \arrow[r] & 0.
\end{tikzcd}$$

Any splitting s of this sequence induces an isomorphism

$$a^*\colon F^{n-c}H^{n-r} \oplus E_\infty^{0,n-r} \xrightarrow{\cong} H^{n-r}(X),\ (x_{n-c},x_0) \mapsto i(x_{n-c}) + s(x_0).$$

$\square$

To complete our picture, we need to consider the homology side.

Proposition 1.2.3. *Let $\xi = (X,p,B)$ be a fibration such that $B \cong S^{n-c}$. Then the homological Leray-Serre spectral sequence converges to $H_*(X)$, and there is a short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & F_{n-c-1}H_{r-1} \arrow[r] & H_{r-1}(X) \arrow[r] & E^\infty_{n-c,r-1-n+c} \arrow[l, bend left, "s"] \arrow[r] & 0
\end{tikzcd}$$

which allows us to recover the group $H_{r-1}(X)$ by choosing a splitting s. This splitting induces an isomorphism

$$a_*\colon F_{n-c-1}H_{r-1} \oplus E^\infty_{n-c,r-1-n+c} \xrightarrow{\cong} H_{r-1}(X).$$

Proof. The proof is analogous to the cohomological version. The filtration F_* of $C_*(X)$ is given as

$$F_sC_*(X) = \operatorname{im}[C_*(J^s) \to C_*(X)].$$

Thus,

$$\begin{aligned}
F_{-1}C_*(X) &= \operatorname{im}[C_*(\underbrace{J^{-1}}_{=\varnothing}) \to C_*(X)] = \{0\} \\
F_0C_*(X) &= \operatorname{im}[C_*(\underbrace{J^0}_{\cong p^{-1}(e^0)}) \to C_*(X)] = C_*(p^{-1}(e^0)) \\
&= F_1C_*(X) = \cdots = F_{n-c-1}C_*(X) \\
F_{n-c}C_*(X) &= \operatorname{im}[C_*(\underbrace{J^{n-c}}_{=X}) \to C_*(X)] = C_*(X).
\end{aligned}$$

Hence

$$\begin{aligned}
F_{-1}H_{r-1} &= \operatorname{im}[H_{r-1}(F_{-1}C_*(X)) \to H_{r-1}(C_*(X))] \\
&= \operatorname{im}[H_{r-1}(\{0\}) \to H_{r-1}(X)] \\
&= \{0\} \\
F_0H_{r-1} &= \operatorname{im}[H_{r-1}(F_0C_*(X)) \to H_{r-1}(C_*(X))] \\
&= \operatorname{im}[H_{r-1}(C_*(p^{-1}(e^0))) \to H_{r-1}(X)] \\
&= \operatorname{im}[H_{r-1}(p^{-1}(e^0)) \to H_{r-1}(X)] \\
&= F_1H_{r-1} = \cdots = F_{n-c-1}H_{r-1} \\
F_{n-c}H_{r-1} &= H_{r-1} = H_{r-1}(X).
\end{aligned}$$

To explain the splitting in the homological case, there are two relevant exact sequences. The first is induced by the composition

$$F_0H_{r-1} \xrightarrow{\text{quot}} \frac{F_0H_{r-1}}{F_{-1}H_{r-1}} \xrightarrow[\cong]{(d^\infty)^{-1}} E^\infty_{0,r-1}$$

In fact, this sequence is

$$0 \longrightarrow F_{-1}H_{r-1} \xrightarrow{\text{incl}} F_0H_{r-1} \xrightarrow{(d^\infty)^{-1}\circ\text{quot}} E^\infty_{0,r-1} \longrightarrow 0. \tag{7}$$

Analogously, there is an isomorphism

$$d^\infty\colon E^\infty_{n-c,r-1-n+c} \xrightarrow{\cong} \frac{F_{n-c}H_{r-1}}{F_{n-c-1}H_{r-1}},$$

which induces the bottom exact sequence:

$$\begin{tikzcd}
0 \arrow[d] \\
F_{n-c-1}H_{r-1} \arrow[d, "\text{incl}"'] \\
F_{n-c}H_{r-1} \arrow[d, "(d^\infty)^{-1}\circ\text{quot}"'] \\
E^\infty_{n-c,r-1-n+c} \arrow[d] \\
0
\end{tikzcd} \tag{8}$$

□

CHAPTER 2

Spheres as Base Spaces

2.1. Setting

2.1.1. Introduction. We begin with some remarks concerning notation, before introducing the setting in which we will work. By default, we will consider S^m to be covered by two closed hemispheres D_0 and D_1 defined as

$$D_0 = S^m \cap (\mathbb{R}^m \times \mathbb{R}_{\geq 0}),$$
$$D_1 = S^m \cap (\mathbb{R}^m \times \mathbb{R}_{\leq 0}).$$

These intersect in an equatorial $(m-1)$-sphere, as shown in the following sketch:

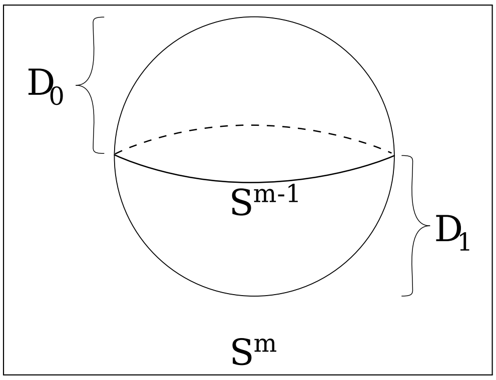


FIGURE 1. The m-sphere covered by two closed sets intersecting at the equator.

Before we can move on to define the link bundles, we quote definition of the desired links.

DEFINITION 2.1.1 (Definition 1.62 on p. 71 in [**Ban10**]). Let **ICW** be the full subcategory of **CW** whose objects are simply connected CW-complexes K with finitely generated even-dimensional homology and vanishing odd-dimensional homology for any coefficient group. We call **ICW** the *interleaf category*.

The following definition specifies the type of link bundle in which we are interested.

DEFINITION 2.1.2. Let $\xi = (N, Y, p, S^m)$ be a fiber bundle with

(1) $N \in \operatorname{Ob}\mathbf{ICW}$ consisting of finitely many cells,
(2) $m \geq 2$, and
(3) structure group a subgroup of $\operatorname{Homeo}_{CW}(N)$, the topological group of cellular homeomorphisms with cellular inverses of N. (We call this a *cellular structure group*).

A bundle fitting this description will be called an *interleaf fiber bundle over S^m*.

Using the local coordinate description of a fiber bundle (cf. [**Hus94**]), we are going to construct a non-trivial fiber bundle η with fiber S^2 and base space S^2, which is an interleaf fiber bundle over S^2. Cover S^2 by two closed hemispheres E_1 and E_2 such that $E_1 \cap E_2 = S^1$. If we denote a rotation of the plane by

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

we see that

$$SO(2) = \{A_\theta \mid \theta \in [0, 2\pi)\}$$

acts on S^2 by rotation around the x_3-axis,

$$SO(2) \times S^2 \to S^2,\ (A_\theta, (x_1, x_2, x_3)^T) \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

This action makes S^2 a left $SO(2)$-space. We give a system of transition functions $\{g_{11}, g_{12}, g_{21}, g_{22}\}$ associated with the closed cover $\{E_1, E_2\}$ by setting

$$\begin{aligned}
E_1 \cap E_1 = E_1 &\xrightarrow{g_{11}} SO(2),\ x \xmapsto{\qquad g_{11} \qquad} \mathrm{id}_2 \\
E_1 \cap E_2 = S_1 &\xrightarrow{g_{12}} SO(2),\ x =\!=\!= e^{i\theta} \xmapsto{g_{12}} A_\theta \\
E_2 \cap E_1 = S_1 &\xrightarrow{g_{21}} SO(2),\ x =\!=\!= e^{i\theta} \xmapsto{g_{21}} A_{-\theta} \\
E_2 \cap E_2 = E_2 &\xrightarrow{g_{22}} SO(2),\ x \xmapsto{\qquad g_{22} \qquad} \mathrm{id}_2
\end{aligned}$$

THEOREM 2.1.3 (Theorem 3.2 on p. 64 in [**Hus94**]). *Let $\{V_i\}_{i\in I}$ be an open covering of a space B, let G be a topological group, let Y be a left G-space, and let $\{g_{i,j}\}_{i,j\in I}$ be a system of transition functions associated with the open sets $\{V_i\}_{i\in I}$. Then there exists a fiber bundle $\eta = \xi[Y]$ and an atlas $\{(h_i, V_i)\}_{i\in I}$ for η such that the set of transition functions of this atlas is $\{g_{i,j}\}_{i,j\in I}$.*

Then Theorem 2.1.3 shows that above data suffice to construct a fiber bundle $\eta = \xi[S^2]$. Carrying out the calculations suggested in the proof of Theorem 2.1.3, one concludes that η is the fiber bundle

$$\eta = (S^2, \frac{(E_1 \times S^2) \sqcup (E_2 \times S^2)}{(e^{i\theta}, y) \sim (e^{i\theta}, A_\theta y) \forall e^{i\theta} \in E_1 \cap E_2 = S^1 \forall y \in S^2}, \mathrm{proj}_1, S^2)$$

i.e. we attach a fiber S^2 to each point $x \in E_1$ and to each point $x \in E_2$, and we glue the resulting trivial bundles over the equator, but with a rotation: $\{e^{i\theta}\} \times S^2$ is identified with $\{e^{i\theta}\} \times A_\theta S^2$.

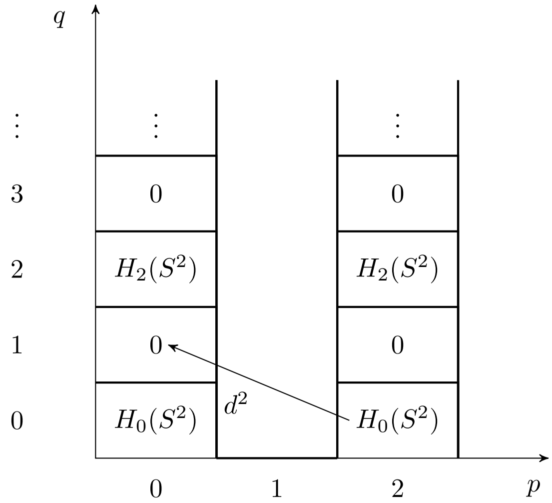
The fiber bundle η is non-trivial. This is a consequence of a result from [**Ste51**] concerning bundles over S^m in normal form. A bundle over S^m is considered to be in *normal form*, if it is given in coordinate description with a closed cover consisting of two hemispheres $E_1 \cap E_2 = S^{m-1}$, and if a reference point $x_0 \in S^{m-1}$ is mapped by g_{12} to $e \in G$. Steenrod calls the map g_{12} the *characteristic map* of the bundle.

THEOREM 2.1.4 (Equivalence Theorem, Theorem 18.3 on p. 97 in [**Ste51**]). *Let η and η' be bundles over S^m in normal form and having the same fiber and path-connected structure group G. Let g_{12} and g'_{12} be their characteristic maps. Then η and η' are equivalent if and only if $g_{12} \simeq g'_{12}$.*

PROPOSITION 2.1.5. *The fiber bundle η is non-trivial.*

PROOF. The characteristic map $g_{12}\colon S^1 \to SO(2)$ can be viewed as a map $g\colon S^1 \to S^1$, $e^{i\theta} \mapsto e^{i\theta}$, or, in other words, as id_{S^1}. It is clearly not homotopic to $g'_{12}\colon S^1 \to SO(2)$, $e^{i\theta} \mapsto \mathrm{id}_2 = e \in SO(2)$, the characteristic map of the trivial bundle $(S^2, S^2 \times S^2, \mathrm{proj}_1, S^2)$. Therefore η is non-trivial. □

The gluing of the total space of η involves rotations around the x_3-axis only. We can assume the cell structure of the fiber S^2 to be $S^2 = e^0 \cup e^2$ with e^0 being placed on the x_3-axis, making the rotation maps cellular homeomorphisms with cellular inverses. Thus η is a non-trivial interleaf fiber bundle over S^2. Yet its homology is that of a trivial bundle, for we have the following E^2-term in the homology Leray-Serre spectral sequence:



We see that all differentials in the E^2-term vanish, and furthermore that this is also true of higher differentials. Therefore the interleaf fiber bundle η exhibits the same homology as a trivial bundle. This is in accordance with Proposition 2.5.11. An analogous result concerning the cohomology of η can be established in the same way, confirming Corollary 2.5.8 in this case. This ends our discussion of the example.

In the present chapter, the focus is on a compact stratified topological pseudomanifold X which is composed of two strata,

$$X = X_n \supset X_{n-c}.$$

Accordingly, X is of dimension n while the second stratum $\Sigma = X_{n-c}$ is the singular set. The latter $(n-c)$-dimensional closed manifold. An important aspect of pseudomanifolds in general is the link, and in the present case this is a $(c-1)$-dimensional closed manifold L. Moreover, we assume that a neighborhood U_Σ of the singular set Σ exists, which is characterized by the fact that it can be equipped with the structure of a locally trivial fiber bundle $\eta = (\mathring{\text{cone}}\,(L), U_\Sigma, q, \Sigma)$. Given these data, we aim to define for a given perversity $\overline{p}$ an intersection space $I^{\overline{p}}X$, and ultimately to show that the cohomology of this space satisfies generalized Poincaré duality. In [**Ban10**], the case of a trivial link bundle was treated, with just the assumption that the fiber be a simply connected space. We assume for a moment that η is a trivial bundle, $U_\Sigma = \Sigma \times \mathring{\text{cone}}\,(L)$, and recall the definition of [**Ban10**]. In detail, one subtracts the total space $\Sigma \times \mathring{\text{cone}}\,(L)$ from X. This yields a manifold M^n, and the boundary of M exhibits the structure of a fiber bundle

$$\partial M = \Sigma \times L.$$

This is called the *link bundle*, due to the fact that its fiber is the link of X. Next, spatial homology truncation enables the existence of a fiberwise map

$$\Sigma \times t_{<k}(L) \to \Sigma \times L = \partial M$$

This map is then composed with the inclusion $\partial M \hookrightarrow M$, and this composition is called g. The procedure is completed by defining the intersection space as

$$I^{\overline{p}}X = \text{cone}\,(g) = M \cup_g \text{cone}\,(\Sigma \times t_{<k}(L))$$

which Banagl does on [**Ban10**, p. 177]. Having quoted this construction, we wish to modify it to accommodate nontrivial link bundles.

Assume therefore that η is possibly nontrivial. As in Banagl's construction, we remove the total space of η from X and obtain M. By assumption, the link bundle

$$\xi = (L, \partial M, p, \Sigma),$$

of X may be nontrivial. Accordingly, we may not assume that ∂M is a product and cannot rely on this to construct the fiberwise truncation of ξ, and a different approach is necessary.

In [**Ban10**, Section 2.9], generalized Poincaré duality was shown to hold for the homology of the intersection space of pseudomanifolds X such that the link L is a simply-connected space while the link bundle is trivial, $\partial M = \Sigma \times L$. We address a similar statement of duality for a different class of pseudomanifolds: Roughly speaking, we restrict to spheres as singular sets but allow for a controlled amount of twist in the link bundle.

To be more precise, additionally to X being a compact, n-dimensional two-strata pseudomanifold, our restrictions are threefold: Firstly, let the singular set be a sphere $\Sigma = S^{n-c}$ with $n - c \geq 2$. Secondly, let the link L be oriented[1] and an object in the interleaf category **ICW** consisting of finitely[2] many cells. Since L is in **ICW**, this excludes odd-dimensional links. We can therefore assume c to be odd. Thirdly, let the link bundle be an interleaf fiber bundle over S^{n-c}.

To sum up, we demand that X^n be a stratified pseudomanifold which

(1) is compact and composed of two strata, has a
(2) interleaf fiber bundle over S^{n-c}, $n - c \geq 2$ as link bundle and
(3) an oriented link L.

To complete the overview of our setting, we introduce the intersection space of X. For a perversity $\overline{p}$, we set $k = c - 1 - \overline{p}(c)$. Then, following the exposition in Subsection 2.1.2, we will obtain for ξ a fiberwisely truncated bundle $\mathrm{ft}_{<k}(\xi) = (L_{<k}, \mathrm{ft}_{<k}(\partial M), \mathrm{ft}_{<k}(p), B)$. We also obtain a commutative diagram:

$$\begin{tikzcd} L \arrow[r, hook] & \partial M \arrow[r, "p"] & S^{n-c} \\ L_{<k} \arrow[u, "\mu|"] \arrow[r, hook] & \mathrm{ft}_{<k}(\partial M) \arrow[u, "\mu"] \arrow[r, "\mathrm{ft}_{<k}(p)"] & B \arrow[u, "\simeq"', "\mathrm{M}"] \end{tikzcd} \tag{9}$$

Given the existence of the composition g, given as

$$\mathrm{ft}_{<k}(\partial M) \xrightarrow{\mu} \partial M \overset{j}{\hookrightarrow} M,$$

one can define the intersection space of X.

Definition 2.1.6 (Definition 2.41 on p. 177 in [**Ban10**]). Given the restrictions (1), (2) and (3), above, the perversity $\overline{p}$ intersection space $I^{\overline{p}}X$ of X is defined to be $I^{\overline{p}}X = \mathrm{cone}(g) = M \cup_g \mathrm{cone}(\mathrm{ft}_{<k}(\partial M))$.

Notice that g – and thus in turn $I^{\overline{p}}X$ – depends on μ.

2.1.2. Fiberwise Homology Truncation. We want to be able to apply the process of fiberwise homology truncation to arbitrary interleaf fiber bundles over S^m. Accordingly, let $\xi = (N, Y, p, S^m)$ be an interleaf fiber bundle over S^m. By [**Ste51**, p. 96], any fiber bundle over the m-sphere can be assumed to be given as the row-wise pushout of the diagram

$$\begin{tikzcd} D_0 \times N \arrow[d] & S^{m-1} \times N \arrow[l, "\mathrm{incl} \circ \alpha"'] \arrow[r, hook] \arrow[d] & D_1 \times N \arrow[d] \\ D_0 & S^{m-1} \arrow[l, hook'] \arrow[r, hook] & D_1 \end{tikzcd}$$

which consists of two squares representing fiberwise maps. The vertical maps are projections to the first component. From a topological standpoint, the complexity

[1]It is actually not necessary to demand this, as L is a simply connected manifold.
[2]The finite cell structure is actually implied by the fact that L is compact.

of the bundle is determined entirely by

$$\alpha\colon S^{m-1}\times N \to S^{m-1}\times N,\ (b,y)\mapsto (b,g_{01}(b)(y)).$$

This map, in turn, is specified by the choice of a *clutching function*

$$g_{01}\colon S^{m-1}\to \mathrm{Homeo}_{CW}(N).$$

Given $k\in\mathbb{N}$, the fiberwise homology truncation of ξ is

$$\mathrm{ft}_{<k}(\xi) = (N_{<k}, \mathrm{ft}_{<k}(Y), \mathrm{ft}_{<k}(p), B),$$

with entries defined as follows:

- $N_{<k} = t_{<k}(N)$ is the spatial homology truncation in degree k of N.
- $\mathrm{ft}_{<k}(Y)$ is defined as the double mapping cylinder of the upper row of the diagram

$$\begin{tikzcd} D_0\times N_{<k} \arrow[d] & S^{m-1}\times N_{<k} \arrow[l, "\mathrm{incl}\circ\alpha_{<k}"'] \arrow[r, hook] \arrow[d] & D_1\times N_{<k} \arrow[d] \\ D_0 & S^{m-1} \arrow[l, hook'] \arrow[r, hook] & D_1 \end{tikzcd} \tag{10}$$

  while the base space B is defined to be the double mapping cylinder of the lower row in the same diagram. This implies $B\cong S^m$ by the topological Poincaré conjecture.
- $\alpha_{<k}\colon S^{m-1}\times N_{<k}\to S^{m-1}\times N_{<k}, (b,y)\mapsto (b,\tilde{t}_{<k}(g_{01}(b))(y))$, wherein $\tilde{t}_{<k}$ denotes the lift discussed in [**Ban10**, Section 1.10].

Following [**Ban10**, Section 1.10], for a given space W, we denote by $G(W)$ the group of homotopy self-equivalences $W\xrightarrow{\simeq} W$.

PROPOSITION 2.1.7. *The clutching function of the truncated bundle,*

$$\tilde{t}_{<k}\circ g_{01}\colon S^{m-1}\to G(N_{<k})$$

is continuous, and so is $\alpha_{<k}$.

PROOF. Note that the composition

$$\tilde{t}_{<k}\circ g_{01}\colon S^{m-1}\to G(N_{<k})$$

is continuous by [**Ban10**, Theorem 1.78]. Furthermore, $G(N_{<k})$ acts on $N_{<k}$ by an action $\beta\colon G(N_{<k})\times N_{<k}\to N_{<k}, (h,y)\mapsto h(y)$. This action is continuous, see [**Fuc71**, Section 7]. Thus the composition

$$\begin{array}{rcccl} A\colon S^{m-1}\times N_{<k} & \xrightarrow{(\tilde{t}_{<k}\circ g_{01})\times \mathrm{id}_N} & G(N_{<k})\times N_{<k} & \xrightarrow{\beta} & N_{<k} \\ (b,y) & \longmapsto & (\tilde{t}_{<k}(g_{01}(b)),y) & \longmapsto & (\tilde{t}_{<k}(g_{01}(b))(y)) \end{array}$$

is continuous. Accordingly, if Δ is the diagonal map, the map

$$\alpha_{<k} = (\mathrm{id}\times A)\circ(\Delta\times\mathrm{id})\colon S^{m-1}\times N_{<k}\to S^{m-1}\times N_{<k},\ (b,y)\mapsto (b,\tilde{t}_{<n}(g_{01}(b))(y))$$

is continuous because it is just a product with the composition A above. $\square$

PROPOSITION 2.1.8. *The truncated bundle* $\mathrm{ft}_{<k}(\xi)$ *is a Dold fibration.*

PROOF. We want to apply Lemma 1.1.15 to diagram (10). Since all vertical maps in this diagram are trivial Dold fibrations with fiber $N_{<k}$, it remains to show that both squares are morphisms in $\mathbf{Top}^{N_{<k}}$. The right square is trivial, and therefore a morphism in $\mathbf{Top}^{N_{<k}}$. Concerning the left square: Firstly, S^{m-1} is a connected CW-complex and thus numerably contractible by [**Dol63**, Proposition 6.7]. Secondly, $\alpha_{<k}$ is continuous, thus so is

$$\mathrm{incl}\circ\alpha_{<k}\colon S^{m-1}\times N_{<k}\to D_0\times N_{<k}$$

and the left square is a fiberwise map. Lastly, the restriction of $\mathrm{incl} \circ \alpha_{<k}$ to an individual fiber is

$$\alpha_{<k}|_{N_{<k}} = \tilde{t}_{<k}(g_{01}(b)) \colon N_{<k} \to N_{<k},$$

which is an element of $G(N_{<k})$ and hence a homotopy equivalence. Therefore Proposition 1.1.13 shows that the left square is a morphism in $\mathbf{Top}^{N_{<k}}$. Thus $\mathrm{ft}_{<k}(\xi)$ is a Dold fibration by Lemma 1.1.15. □

2.2. Maps on Mapping Cylinders

We continue to work with an arbitrary interleaf fiber bundle over S^m, say $\xi = (N, Y, p, S^m)$. The present section is concerned with the construction of a "middle map" (cf. diagram (9) regarding the name)

$$\mu \colon \mathrm{ft}_{<k}(Y) \to Y.$$

To construct μ, we adapt a technique from [**Bau89**, p. 260f] which enables the construction of maps between mapping cones within a more general categorical framework. We summarize it in the version for **Top**, then we generalize it to allow for mapping cylinders in **Top**. The result will then be used to define μ. All material in this section is quoted from [**Bau89**] unless noted otherwise.

Recall that for a topological space A, the *cone* CA on A is defined as

$$CA = A \times I / A \times \{0\},$$

while for a map $g \colon A \to B$, the *mapping cone* C_g is defined as

$$C_g = \frac{CA \sqcup (B \times \{1\})}{(a,1) \sim (g(a),1)\ \forall a \in A}.$$

Of course, the mapping cone could just as well have been defined as a quotient space of $CA \sqcup B$, but the use of the product enables a clean way of referring to equivalence classes within C_g in the present subsection.

Assume now that we are given a square which commutes up to homotopy,

$$\begin{tikzcd} A \arrow[r, "g"] & B \times \{1\} \\ X \arrow[u, "v"] \arrow[r, "f"] \arrow[ur, phantom, "\boxed{\overset{H}{\simeq}}"] & Y \times \{1\} \arrow[u, "w"'] \end{tikzcd} \tag{11}$$

with a homotopy $H \colon g \circ v \simeq w \circ f$. We can define a homotopy

$$G \colon X \times I \to CA$$

from the constant map

$$c_{[a,0]} \colon X \to CA,\ x \mapsto [a,0]_{CA}$$

to $i \circ v$, with i defined as

$$i \colon A \hookrightarrow CA,\ a \mapsto [a,1]_{CA},$$

by setting $G(x,t) = [v(x),t]_{CA}$. This yields a diagram

(12)
$$\begin{tikzcd}
CA & & A \arrow[ll, hook', "i"'] \\
 & \arrow[r, phantom, "\boxed{\overset{G}{\simeq}}"] & \\
 & & X \arrow[uu, "v"'] \arrow[uull, bend left, "c_{[a,0]}"]
\end{tikzcd}$$

which commutes up to homotopy.

Suppose we are given these data, then our task is to construct a map $F\colon C_f \to C_g$. Baues uses a diagram to illustrate the situation:

(13)
$$\begin{tikzcd}
CA \arrow[rrr, "\pi_g"] & & \arrow[phantom, "\boxed{\text{push}}"] & C_g \\
\arrow[r, phantom, "\boxed{\overset{G}{\simeq}}"] & A \arrow[ul, hook', "i"'] \arrow[r, "g"] & B \times \{1\} \arrow[ur, hook, "i_g"'] & \\
 & X \arrow[u, "v"] \arrow[r, "f"] \arrow[uul, bend left, "\mathrm{const}"] \arrow[ur, phantom, "\boxed{\overset{H}{\simeq}}"] \arrow[dl, hook', "j"'] & Y \times \{1\} \arrow[u, "w"'] \arrow[dr, hook, "i_f"] & \\
CX \arrow[rrr, "\pi_f"] & & \arrow[phantom, "\boxed{\text{push}}"] & C_f \arrow[uuu, dashed, "F"']
\end{tikzcd}$$

The map $\pi_g\colon CA \to C_g$ is defined as

$$\pi_g([a,t]_{CA}) = [a,t]_{C_g}$$

with π_f being defined in an analogous fashion. The map $i_f\colon Y \times \{1\} \hookrightarrow C_f$ is defined as $i_f(y,1) = [y,1]_{C_f}$, and i_g is defined in the same way. Analogously to i, the map $j\colon X \hookrightarrow CX$ is the inclusion of X at the 1-end of CX.

Note that we get commutativity of the top and bottom squares for free – they are pushout squares. Baues shows that we can get the map $F\colon C_f \to C_g$ by setting

(14)
$$\begin{cases} F \circ i_f = i_g \circ w \\ F \circ \pi_f = (\pi_g \circ G) + (i_g \circ H) \end{cases}$$

The "+" in equation (14) represents *track addition* (or *addition of homotopies*). Track addition, first defined in [**Bar55**], is just concatenation in the case of homotopies $H\colon X \times I \to Y$. It is more complicated for higher homotopies, but these do not concern us here.

In the present case, the track addition may be easily written explicitly. On $\pi_f(CX) \subset C_f$, the function F is defined as

(15)
$$F\colon [x,t]_{C_f} \mapsto \begin{cases} \pi_g(G(x,2t)), & t \in [0,1/2] \\ i_g(H(x,2t-1)), & t \in [1/2,1] \end{cases}$$

Clearly, there is a choice involved here: F assigns to the equivalence class $[x,t]_{C_f}$ a value that depends on the choice of a representative (x,t) of that class. We need to show that F is constant on each equivalence class, or, in other words, that F is well-defined. Considering C_f, there are three cases of equivalence classes: $[x,t]_{C_f}$ with $t=0$, $t \in (0,1)$, and $t=1$.

(1) "$t=0$": For any choice of a representative $(x,0)$ of $[x,0]_{C_f}$ the value of F is

$$F([x,0]_{C_f}) = \pi_g(G(x,0)) = \pi_g(c_{[a,0]}(x)) = \pi_g([a,0]_{CA}) = [a,0]_{C_g}$$

and this clearly does not depend on the choice of the representative $(x,0)$ of $[x,0]_{C_f}$.

(2) "$t \in (0,1)$": For $t \in (0,1)$, any equivalence class $[x,t]_{C_f}$ contains just one element, namely (x,t). Hence the value of F does not depend on a choice in this case.

(3) "$t=1$": We need to show that the two definitions of F contained in equations (14) and (15) agree. To this end, for an equivalence class $[x,1]_{C_f}$ choose two representatives, namely $(x,1)$ and $(f(x),1)$. By equation (14), we have

$$\begin{aligned} F([f(x),1]_{C_f}) &= i_g \circ w(f(x)) \\ &= i_g((w \circ f)(x)) \\ &= [w(f(x)),1]_{C_g} \end{aligned}$$

and by equation (15), we have

$$\begin{aligned} F([x,1]_{C_f}) &= i_g(H(x,1)) \\ &= i_g((w \circ f)(x)) \qquad\qquad \text{(because } H(\cdot,1) = w \circ f) \\ &= [w(f(x)),1]_{C_g}. \end{aligned}$$

And so the two definitions of F agree. Next, consider a pair of representatives $(x,1),(x',1) \in [x,1]_{C_f}$ such that $x \neq x'$. This implies $f(x) = f(x')$. The two equations

$$F([x,1]_{C_f}) = [w(f(x)),1]_{C_g} \text{ and } F([x',1]_{C_f}) = [w(f(x')),1]_{C_g}$$

hold, and $f(x) = f(x')$ implies

$$F([x,1]_{C_f}) = F([x',1]_{C_f}).$$

Thus the value of F does not depend on the choice of a representative. We consider the question whether F is continuous. Recall the following result concerning maps defined on quotient spaces.

Theorem 2.2.1 (Theorem 22.2 on p. 142 in [**Mun00**]). *Let $p\colon X \to Y$ be a quotient map. Let Z be a space and let $g\colon X \to Z$ be a map that is constant on each set $p^{-1}(y)$, for $y \in Y$. Then g induces a map $f\colon Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous.*

View C_f as the quotient space of $CX \sqcup (Y \times \{1\}) \xrightarrow{q_f} C_f$. The quotient map q_f may be explained as

$$\begin{cases} q_f|_{CX} = \pi_f \\ q_f|_{Y\times\{1\}} = i_f. \end{cases} \tag{16}$$

According to above result, we can show that F is continuous, if it is induced by a continuous map $\tilde{F}\colon CX \sqcup (Y \times \{1\}) \to C_g$ which is constant on the preimage (under

q_f) of any equivalence class in C_f. The situation presents itself as a diagram.

$$\begin{tikzcd} CX \sqcup (Y \times \{1\}) \arrow[d, "q_f"'] \arrow[dr, dashed, "\tilde{F}"] & \\ C_f \arrow[r, "F"] & C_g \end{tikzcd}$$

Set $\tilde{F} = F \circ q_f$. Firstly, we have to check that $\tilde{F}|_{CX}$ and $\tilde{F}|_{Y\times\{1\}}$ are continuous. For the former map, we obtain

$$\begin{aligned} \tilde{F}|_{CX}([x,t]_{CX}) &= (F \circ q_f)|_{CX}([x,t]_{CX}) \\ &= (F \circ \pi_f)([x,t]_{CX}) \\ &\overset{(14)}{=\!=} ((\pi_g \circ G) + (i_g \circ H))\,[x,t]_{C_f} \end{aligned}$$

Both maps in the track addition in (15) are continuous, and they agree on the overlap of their respective domains: For $[x, 1/2]_{C_f}$, we have

$$\pi_g\left(G\left(x, 2\frac{1}{2}\right)\right) = \pi_g(G(x,1)) \overset{(12)}{=\!=} \pi_g((i \circ v)(x)) = (\pi_g \circ i \circ v)(x)$$

and

$$i_g\left(H\left(x, 2\frac{1}{2} - 1\right)\right) = i_g(H(x,0)) \overset{(11)}{=\!=} i_g((g \circ v)(x)) = (i_g \circ g \circ v)(x).$$

So we have to show

$$(\pi_g \circ i \circ v)(x) = (i_g \circ g \circ v)(x)\ \forall x \in X$$

but this equation holds by virtue of the upper pushout square in diagram (13). Furthermore, $(x,0) \mapsto (\pi_g \circ G) + (i_g \circ H)(x,0)$ is constant and thus Theorem 2.2.1 shows that the map $(\pi_g \circ G) + (i_g \circ H) = \tilde{F}|_{CX} \colon CX \to C_g$ is continuous.

For $\tilde{F}$ restricted to $Y \times \{1\}$, we have

$$\begin{aligned} \tilde{F}|_{Y\times\{1\}}(y,1) &= (F \circ q_f)(y,1) \\ &= F(q_f(y,1)) \\ &\overset{(16)}{=\!=} F(i_f(y,1)) \\ &\overset{(14)}{=\!=} (i_g \circ w)(y,1) \end{aligned}$$

which is clearly continuous, since both w and i_g are continuous.

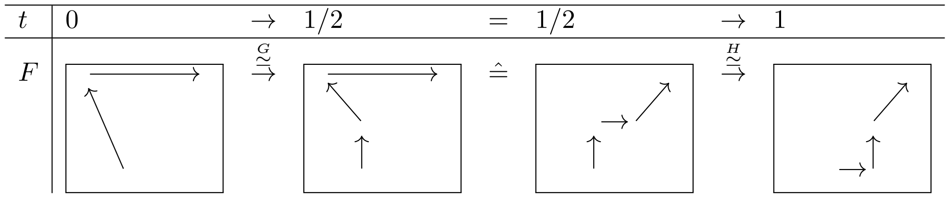
Since the continuous map $\tilde{F}$ induces F in the sense of Theorem 2.2.1, we need to show that $\tilde{F}$ is constant on the preimage (under q_f) of each equivalence class in C_f. As noted previously, there are three types of equivalence classes $[x,t]_{C_f}$ in C_f, corresponding to $t = 0$, $t \in (0,1)$ and $t = 1$.

(1) "$t = 0$": That F is well-defined implies that $\tilde{F}$ is constant.
(2) "$t \in (0,1)$": Each equivalence class contains only one element, thus there is nothing to show.
(3) "$t = 1$": The map F is well-defined and hence $\tilde{F}$ is constant on the preimage of each equivalence class.

The next task is to glean a better understanding of the behavior of the map F. Recall the boundary relations of the homotopies G and H:

$$\begin{aligned} H\colon g \circ v \simeq w \circ f, \quad & H(\cdot,0) = g \circ v \\ & H(\cdot,1) = w \circ f \\ G\colon c_{[a,0]} \simeq i \circ v, \quad & G(\cdot,0) = c_{[a,0]} \\ & G(\cdot,1) = i \circ v \end{aligned} \tag{17}$$

Clearly, $Y \hookrightarrow C_f$ is mapped to $w(Y) \hookrightarrow C_g$. The other part of F is not so straightforward. Here, the time parameter of the cylindrical part of the mapping cone is used as an argument of the homotopies. It is perhaps better to try to explain this visually. The path which F takes in diagram (13) is illustrated in the following schematic:



The next sketch shows where F takes which value on the mapping cone.

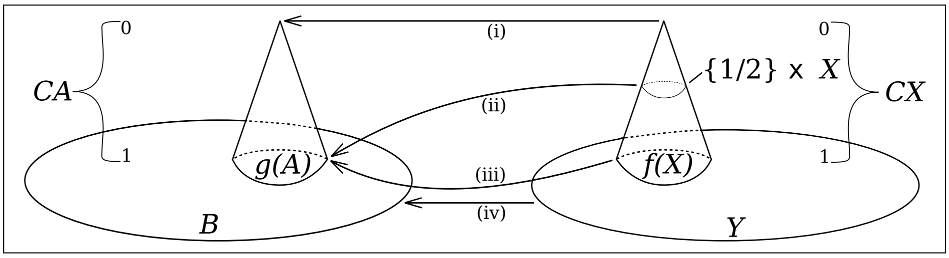


FIGURE 2. The four arrows represent the value of F at the respective base points in CX: (i) $\pi_g(G(\cdot,0)) = \pi_g \circ \mathrm{const}_{\mathrm{pt}}$, (ii) $\pi_g(G(\cdot,1)) = \pi_g \circ i \circ v = i_g \circ g \circ v = i_g(H(\cdot,0))$, (iii) $i_g(H(\cdot,1)) = i_g \circ w \circ f$, (iv) $i_g \circ w$.

The case of a map between mapping cylinders is similar to that of a map between mapping cones. Essentially, we need a different homotopy G, and to adapt F accordingly. Consider the following diagram, in which the spaces A, B, X, Y and the maps f, g, v, w as well as the homotopy H and the commutativity up to homotopy of the central square induced by H are assumed to be given:

$$\begin{tikzcd}
A \times I \arrow[rrr, "\pi_g"] \arrow[drrr, phantom, "\boxed{\text{push}}"] & & & M_g \\
\boxed{\overset{G}{\simeq}} & A \arrow[ul, hook', "i_1"'] \arrow[r, "g"] \arrow[dr, phantom, "\boxed{\overset{H}{\simeq}}"] & B \times \{1\} \arrow[ur, hook, "i_g"'] & \\
& X \arrow[u, "v"] \arrow[r, "f"] \arrow[uul, bend left, "i_0 \circ v"] \arrow[dl, hook', "j"'] & Y \times \{1\} \arrow[u, "w"'] \arrow[dr, hook, "i_f"] & \\
X \times I \arrow[rrr, "\pi_f"] \arrow[urrr, phantom, "\boxed{\text{push}}"] & & & M_f \arrow[uuu, dashed, "F"']
\end{tikzcd} \tag{18}$$

Note that the maps $i_{(l)}$ are defined as

$$i_{(l)} \colon A \hookrightarrow A \times I,\ a \mapsto (a, l)$$

for $l = 0, 1$. These are just the inclusions of A at the 0-end and at the 1-end of $A \times I$, respectively. There is an obvious homotopy $G\colon X \times I \to A \times I$, namely

$$G(x,t) = (v(x), t) \tag{19}$$

Then

$$G\colon i_{(0)} \circ v \simeq i_{(1)} \circ v,\ G(\cdot, 0) = i_{(0)} \circ v,\ G(\cdot, 1) = i_{(1)} \circ v.$$

After above discussion of the case of mapping cones, it should be clear that setting

$$\begin{cases} F \circ i_f = i_g \circ w, \\ F \circ \pi_f = (\pi_g \circ G) + (i_g \circ H) \end{cases} \tag{20}$$

achieves the desired result of a continuous map. Well-definition and continuity follow as in the case of mapping cones. For convenience, we write down the track addition explicitly. On $\pi_f(X \times I) \subset M_f$, the function F is defined as

$$F\colon [x,t]_{M_f} \mapsto \begin{cases} \pi_g(G(x, 2t)), & t \in [0, 1/2] \\ i_g(H(x, 2t-1)), & t \in [1/2, 1]. \end{cases} \tag{21}$$

We summarize this:

Proposition 2.2.2. *Given a square*

$$\begin{tikzcd} A \arrow[r, "g"] & B \times \{1\} \\ X \arrow[u, "v"] \arrow[r, "f"'] \arrow[ur, phantom, "\boxed{\overset{H}{\simeq}}"] & Y \times \{1\} \arrow[u, "w"'] \end{tikzcd}$$

which commutes up to homotopy, there is a continuous map $F\colon M_f \to M_g$ with restrictions

$$F|_{X \times \{0\}} = v,$$
$$F|_{Y \times \{1\}} = w.$$

Proof. A function F may be defined as in equation (20). We have already established that this yields a well-defined and continuous map. □

2.3. Construction of μ

Let $\xi = (N, Y, p, S^m)$ be an arbitrary interleaf fiber bundle over S^m, and recall from Subsection 2.1.2 the clutching function g_{01}: For a fixed $b \in S^{m-1}$, there is a cellular homeomorphism $g_{01}(b)\colon N \to N$ and a natural transformation of functors

$$\mathrm{emb}_k\,(N)\colon t_{<k} \to t_{<\infty}.$$

This natural transformation as well as the functor $t_{<\infty}$ are introduced in [**Ban10**, p. 78].

Hence there is a diagram

$$\begin{tikzcd} N_{<k} \arrow[r, "\mathrm{emb}_k\,(N)"] \arrow[d, "t_{<k}(g_{01}(b))"'] & N \arrow[d, "{[g_{01}(b)]}"] \\ N_{<k} \arrow[r, "\mathrm{emb}_k\,(N)"] & N \end{tikzcd}$$

which commutes in **HoCW**. Commutativity in **HoCW** is equivalent to commutativity up to homotopy of the corresponding diagram in **CW**, for an arbitrary choice of representatives. We choose the canonical representatives

$$\begin{aligned}
&[g_{01}(b)],\\
t_{<k}(g_{01}(b)) &= [\tilde{t}_{<k}(g_{01}(b))] = [(h_N \circ g_{01}(b) \circ h'_N)^{k-1}],\\
\mathrm{emb}_k\,(N) &= [h'_N \circ \mathrm{incl}].
\end{aligned}$$

Only the last choice needs an explanation, for which we direct the reader to [**Ban10**, p. 78f]. The definition of the homotopy equivalence $h_N\colon N \to E(N)$ as well as the definition of the CW-complex $E(N)$ can also be found in [**Ban10**, Section 1.9]. We obtain a homotopy $H_b\colon N_{<k} \times I \to N$ such that

$$\begin{tikzcd}
N_{<k} \arrow[r, "{[h'_N \circ \mathrm{incl}]}"] \arrow[d, "t_{<k}(g_{01}(b))"'] & N \arrow[d, "{[g_{01}(b)]}"] \\
N_{<k} \arrow[r, "{[h'_N \circ \mathrm{incl}]}"] & N
\end{tikzcd}$$

commutes outright in **HoCW** and, for the choice of representatives given, up to homotopy H_b in **CW**. We want to show that the assignment $b \rightsquigarrow H_b$ is continuous in the following sense: We can use the assignment to construct a fiber homotopy $H\colon S^{m-1} \times N \times I \to S^{m-1} \times N_{<k}$. To this end, we recall from [**Ban10**, p. 78f] the definition of the homotopy H_b. In the diagram

$$\begin{tikzcd}
E(N)^{k-1} \arrow[r, "{[\mathrm{incl}]}"] \arrow[d, "{[(h_N \circ g_{01}(b) \circ h'_N)^{k-1}]}"'] & E(N) \arrow[r, "{[h'_N]}"] \arrow[d, "{[h_N \circ g_{01}(b) \circ h'_N]}"'] & N \arrow[d, "{[g_{01}(b)]}"] \\
E(N)^{k-1} \arrow[r, "{[\mathrm{incl}]}"] & E(N) \arrow[r, "{[h'_N]}"] & N
\end{tikzcd} \tag{22}$$

the left square commutes in **HoCW** – it commutes in **CW** for the choice of representatives indicated since $g_{01}(b)$ is cellular. For the right square, we notice

$$\begin{aligned}
[h'_N] \circ [h_N \circ g_{01}(b) \circ h'_N] &= \underbrace{[h'_N \circ h_N]}_{=[\mathrm{id}_N]} \circ [g_{01}(b) \circ h'_N]\\
&= [g_{01}(b) \circ h'_N],
\end{aligned}$$

for h_N is a homotopy equivalence with homotopy inverse h'_N. We obtain a homotopy

$$\tilde{G}\colon N \times I \to N,\ \tilde{G}(\cdot, 0) = h'_N \circ h_N,\ \tilde{G}(\cdot, 1) = \mathrm{id}_N.$$

Importantly, $\tilde{G}$ does not depend on b. Thus, composing

$$E(N) \times I \xrightarrow{h'_N \times \mathrm{id}} N \times I \xrightarrow{g_{01}(b) \times \mathrm{id}} N \times I \xrightarrow{\tilde{G}} N$$

yields a homotopy $G_b\colon E(N) \times I \to N$ such that

$$G_b(\cdot, 0) = h'_N \circ h_N \circ g_{01}(b) \circ h'_N, \quad G_b(\cdot, 1) = g_{01}(b) \circ h'_N,$$

and the square

$$\begin{tikzcd} E(N) \arrow[r, "h'_N"] \arrow[d, "h_N \circ g_{01}(b) \circ h'_N"'] \arrow[dr, phantom, "\boxed{\begin{smallmatrix} G_b \\ \simeq \end{smallmatrix}}"] & N \arrow[d, "g_{01}(b)"] \\ E(N) \arrow[r, "h'_N"] & N \end{tikzcd}$$

commutes up to homotopy G_b. We can therefore compose

$$H_b\colon E(N)^{k-1} \times I \xrightarrow{\mathrm{incl}\times\mathrm{id}} E(N) \times I \xrightarrow{G_b} N, \tag{23}$$

and obtain

$$\begin{aligned} H_b(\cdot, 0) &= h'_N \circ h_N \circ g_{01}(b) \circ h'_N \circ \mathrm{incl} \\ &= h'_N \circ \mathrm{incl} \circ (h_N \circ g_{01}(b) \circ h'_N)^{k-1}. \end{aligned}$$

The last equality is due to commutativity (in **CW**) of the left square in (22) for the choice of representatives indicated. Furthermore,

$$H_b(\cdot, 1) = g_{01}(b) \circ h'_N \circ \mathrm{incl}.$$

This is H_b as it was defined in [**Ban10**].

We introduce the notation e_k^N for the canonical representative $h'_N \circ \mathrm{incl}$ of $\mathrm{emb}_k(N)$. We cross the diagram

$$\begin{tikzcd} N_{<k} \arrow[r, "e_k^N"] \arrow[d, "\tilde{t}_{<k}(g_{01}(b))"'] \arrow[dr, phantom, "\boxed{\begin{smallmatrix} H_b \\ \simeq \end{smallmatrix}}"] & N \arrow[d, "g_{01}(b)"] \\ N_{<k} \arrow[r, "e_k^N"] & N \end{tikzcd}$$

with S^{m-1} to obtain

$$\begin{tikzcd} S^{m-1} \times N \arrow[r, "\alpha"] & S^{m-1} \times N \\ S^{m-1} \times N_{<k} \arrow[r, "\alpha_{<k}"] \arrow[u, "\mathrm{id}\times e_k^N"] \arrow[ur, phantom, "\boxed{\begin{smallmatrix} H \\ \simeq \end{smallmatrix}}"] & S^{m-1} \times N_{<k} \arrow[u, "\mathrm{id}\times e_k^N"'] \end{tikzcd} \tag{24}$$

which commutes up to fiber homotopy H: On each fiber, there is a homotopy H_b, and regarding equation (23) we see that these patch together continuously since G_b is continuous in b (since g_{01} is). We use diagram (24) as input for Proposition 2.2.2 and obtain a map

$$\phi\colon M_{\alpha_{<k}} \to M_\alpha. \tag{25}$$

For convenience, we note which variable mentioned in Proposition 2.2.2 becomes which variable in this case:

f	$\alpha_{<k}$
g	α
v	$\mathrm{id} \times e_k^N$
w	$\mathrm{id} \times e_k^N$

LEMMA 2.3.1. *Let $\xi = (N, Y, p, S^m)$ be an interleaf fiber bundle over S^m, with clutching function $g_{01}\colon S^{m-1} \to \mathrm{Homeo}_{CW}(N)$ determining a map*

$$\alpha\colon S^{m-1} \times N \to S^{m-1} \times N,\ (b, y) \mapsto (b, g_{01}(b)(y)).$$

Let $\alpha_{<k}$ be defined as

$$\alpha_{<k}\colon S^{m-1} \times N_{<k} \to S^{m-1} \times N_{<k},\ (b, y) \mapsto (b, \tilde{t}_{<k}(g_{01}(b))(y)).$$

Then there is a fiberwise map

$$(\phi, \Phi)\colon M_{\alpha_{<k}} \to M_\alpha,$$

which up to homotopy restricts to e_k^N on each fiber.

PROOF. We would like to show that there is a map $\Phi\colon S^{m-1} \times I \to S^{m-1} \times I$, making

$$\begin{tikzcd} M_{\alpha_{<k}} \arrow[r, "\phi"] \arrow[d, "q_{<k}"'] & M_\alpha \arrow[d, "q"] \\ S^{m-1} \times I \arrow[r, "\Phi"] & S^{m-1} \times I \end{tikzcd} \tag{26}$$

a fiberwise map. (Here, $q_{<k}$ and q are the respective canonical projections.) We define

$$\Phi\colon S^{m-1} \times I \to S^{m-1} \times I,\ (b, t) \mapsto \begin{cases} (b, 2t), & t \in [0, 1/2], \\ (b, 1), & t \in [1/2, 1]. \end{cases}$$

This is clearly continuous. It remains to show the commutativity of diagram (26). It suffices to show this for a fixed $b \in S^{m-1}$. We consider several cases:

(1) "$t = 0$":

$$\begin{aligned} \phi([b, y, 0]_{M_{\alpha_{<k}}}) &\overset{(20),(21)}{=\!=} \pi_\alpha(G(b, y, 0)) \\ &\overset{(19)}{=\!=} \pi_\alpha(i_0 \circ (\mathrm{id} \times e_k^N)(b, y)) \\ &= \pi_\alpha(i_0(b, e_k^N(y))) \\ &= \pi_\alpha(b, e_k^N(y), 0) \\ &= [b, e_k^N(y), 0]_{M_\alpha} \in q^{-1}(b, 0) \end{aligned} \tag{27}$$

(2) "$t \in (0, 1/2]$":

$$\begin{aligned} \phi([b, y, t]_{M_{\alpha_{<k}}}) &\overset{(20),(21)}{=\!=} \pi_\alpha(G(b, y, 2t)) \\ &\overset{(19)}{=\!=} \pi_\alpha((\mathrm{id} \times e_k^N)(b, y), 2t) \\ &= \pi_\alpha(b, e_k^N(y), 2t) \\ &= [b, e_k^N(y), 2t]_{M_\alpha} \in q^{-1}(b, 2t) \end{aligned}$$

(3) "$t \in [1/2, 1]$":

$$\begin{aligned} \phi([b, y, t]_{M_{\alpha_{<k}}}) &\overset{(20),(21)}{=\!=} i_\alpha(H(b, y, 2t - 1)) \in q^{-1}(b, 1) \\ &= i_\alpha(b, \underbrace{H_b(y, 2t - 1)}_{\in N}) \\ &= [b, H_b(y, 2t - 1), 1] \in q^{-1}(b, 1). \end{aligned}$$

We see that diagram (26) commutes in all three cases, and we have shown the lemma. $\square$

REMARK 2.3.2. The boundary relations of the fiberwise map (ϕ, Φ) of Lemma 2.3.1 can be determined as follows. Firstly, we note that

$$\Phi(b,l) = (b,l)\ \forall b \in S^{m-1} \forall l \in \{0,1\}.$$

So fibers at both ends are kept in place at their base points under (ϕ, Φ). Secondly, for the 1-end we have from equation (20):

$$\begin{aligned}
\phi([b,y,1]_{M_{\alpha_{<k}}}) &= \phi \circ i_{\alpha_{<k}}(b,y) && \text{(definition of } \phi) \\
&= i_\alpha \circ (\mathrm{id} \times e_k^N)(b,y) && \text{(RHS square in (18))} \\
&= i_\alpha(b, e_k^N(y)) \\
&= [b, e_k^N(y), 1]_{M_\alpha}.
\end{aligned}$$

Lastly, for the 0-end,

$$\phi([b,y,0]_{M_{\alpha_{<k}}}) = [b, e_k^N(y), 0]_{M_\alpha}$$

was already shown in equation (27).

What remains is to use $(\phi, \Phi)\colon M_{\alpha_{<k}} \to M_\alpha$ to create a fiberwise map

$$(\mu, \mathrm{M})\colon \mathrm{ft}_{<k}(\xi) \to \xi.$$

We will define (μ, M) to be the composition

$$\mathrm{ft}_{<k}(\xi) \xrightarrow{(\mu_{<k}, \mathrm{M}_{<k})} \xi_{\mathrm{exp}} \xrightarrow{(\mu_{\mathrm{exp}}, \mathrm{M}_{\mathrm{exp}})} \xi.$$

The bundle

$$\xi_{\mathrm{exp}} = (N, X_{\mathrm{exp}}, p_{\mathrm{exp}}, B)$$

is ξ "exploded": We use a total space

$$X_{\mathrm{exp}} = (D_0 \times N) \cup M_\alpha \cup (D_1 \times N)/\sim_{\mathrm{exp}}$$

with obvious gluing "$\sim_{\mathrm{exp}}$" and obvious projection p_{exp}. Then the Gluing Lemma together with the boundary relations of (ϕ, Φ) tell us that we can glue the fiberwise maps

$$\begin{aligned}
D_0 \times N_{<k} &\xrightarrow{(\mathrm{id} \times e_k^N, \mathrm{id})} D_0 \times N, \\
M_{\alpha_{<k}} &\xrightarrow{(\phi, \Phi)} M_\alpha, \text{ and} \\
D_1 \times N_{<k} &\xrightarrow{(\mathrm{id} \times e_k^N, \mathrm{id})} D_1 \times N,
\end{aligned}$$

to form the fiberwise map $(\mu_{<k}, \mathrm{M}_{<k})\colon \mathrm{ft}_{<k}(\xi) \to \xi_{\mathrm{exp}}$. The maps

$$\begin{aligned}
S^{m-1} \times N \times I \to S^{m-1} \times N \times \{1\},\ (b,y,t) &\mapsto (b, g_{01}(b)(y), 1), \\
S^{m-1} \times N \times \{1\} \to S^{m-1} \times N \times \{1\},\ (b,y,1) &\mapsto (b,y,1)
\end{aligned}$$

define a function

$$\iota\colon (S^{m-1} \times N \times I) \sqcup (S^{m-1} \times N \times \{1\}) \to S^{m-1} \times N \times \{1\}.$$

We use the notation "$\times\{1\}$" for the base of the mapping cone to make the following clearer. If $\zeta\colon (S^{m-1} \times N \times I) \sqcup (S^{m-1} \times N \times \{1\}) \to M_\alpha$ is the appropriate quotient map, then a preimage under ζ is either

$$\zeta^{-1}([b,y,1]_{M_\alpha}) = \{(b, (g_{01}(b))^{-1}(y), 1) \in S^{m-1} \times N \times I, (b,y,1) \in S^{m-1} \times N \times \{1\}\},$$

or, for $t \in [0,1)$,

$$\zeta^{-1}([b,y,t]_{M_\alpha}) = \{(b,y,t) \in S^{m-1} \times N \times I\}.$$

In both cases, ι is clearly constant on the preimage. Hence, by Theorem 2.2.1, ι induces a continuous "collapsing" map

$$\kappa\colon M_\alpha \to S^{m-1} \times N,$$

which can be used to constitute a fiberwise map:

$$\begin{tikzcd} M_\alpha \arrow[r, "\kappa"] \arrow[d, "q"'] & S^{m-1} \times N \arrow[d, "\mathrm{proj}_1"] \\ S^{m-1} \times I \arrow[r, "\mathrm{K}=\mathrm{proj}_1"] & S^{m-1} \end{tikzcd}$$

Taking into account the identifications in X, the Gluing Lemma allows us to form a continuous fiberwise map $(\mu_{\exp}, \mathrm{M}_{\exp})\colon \xi_{\exp} \to \xi$ by gluing the three fiberwise maps

$$D_0 \times N \xrightarrow{(\mathrm{id}\times\mathrm{id},\mathrm{id})} D_0 \times N,$$
$$M_\alpha \xrightarrow{(\kappa,\mathrm{K})} S^{m-1} \times N, \text{ and}$$
$$D_1 \times N \xrightarrow{(\mathrm{id}\times\mathrm{id},\mathrm{id})} D_1 \times N.$$

Thus we obtain

$$(\mu, \mathrm{M}) = (\mu_{\exp}, \mathrm{M}_{\exp}) \circ (\mu_{<k}, \mathrm{M}_{<k})\colon \mathrm{ft}_{<k}(\xi) \to \xi.$$

which is a composition of fiberwise maps and therefore fiberwise. We sum up:

LEMMA 2.3.3. *Let $\xi = (N, Y, p, S^m)$ be an interleaf fiber bundle over S^m. Then there is a fiberwise map $(\mu, \mathrm{M})\colon \mathrm{ft}_{<k}(\xi) \to \xi$.*

2.4. Natural Transformations of Truncation Functors

Before we can examine the homology of interleaf bundles in greater detail, we need to consider the natural transformations of functors

$$\mathrm{emb}_n\colon t_{<n} \to t_{<\infty},$$
$$\mathrm{pro}_n\colon t_{<\infty} \to t_{\geq n},$$

of [**Ban10**] in greater detail. For each $N \in \mathrm{Ob}\,\mathbf{ICW}$, there are canonical representatives

$$e_n^N\colon E(N)^{n-1} \xhookrightarrow{\mathrm{incl}} E(N) \xrightarrow{h'_N} N,$$
$$p_n^N\colon N \xrightarrow{h_N} E(N) \xrightarrow{\mathrm{proj}} E(N)/E(N)^{n-1}$$

in **CW**, as stated in [**Ban10**].

PROPOSITION 2.4.1. *If $N \in \mathrm{Ob}\,\mathbf{ICW}$, then (for any combination of n and r) it holds that*

(1) $p_{n*}^N\colon H_r(N) \to H_r(N_{\geq n})$ *is surjective,*
(2) $e_{n*}^N\colon H_r(N_{<n}) \to H_r(N)$ *is injective, and*
(3) $e_n^{N*}\colon H^r(N) \to H^r(N_{<n})$ *is surjective.*

PROOF. Concerning (1), for $r \geq n$ and $0 < r < n$, this follows from [**Ban10**, Proposition 1.75]. For $r = 0$, notice that N and $E(N)$ are path connected (for N is simply connected). As $E(N)$ is path connected, so is $N_{\geq n} = E(N)/E(N)^{n-1}$ for any n. Thus,

$$H_0(N) \cong \mathbb{Q} \cong H_0(N_{\geq n}).$$

But both of these groups are generated by any point in the respective space, and thus p_{n*}^N is an isomorphism.

The proofs for cases (2) and (3) are analogous to [**Ban10**, Proposition 1.75]. □

Corollary 2.4.2. *For $r < n$,*

$$e_n^{N*}\colon H^r(N) \to H^r(N_{<n})$$

is an isomorphism. For other values of r, it is the zero map.

Proof. For $r < n$, the induced map e_n^{N*} is a linear map of finite-dimensional vector spaces of the same dimension,

$$H^r(N) \xrightarrow{e_n^{N*}} H^r(N_{<n}) \cong H^r(N).$$

It is surjective by Proposition 2.4.1. Hence it is an isomorphism. For $n \leq r$, it is a map

$$H^r(N) \xrightarrow{e_n^{N*}} H^r(N_{<n}) = 0,$$

and hence the zero map. □

In the same way, one shows:

Corollary 2.4.3. *For $r < n$,*

$$e_{n*}^{N}\colon H_r(N_{<n}) \to H_r(N)$$

is an isomorphism. For other values of r, it is the zero map.

Corollary 2.4.4. *For $r \geq n$,*

$$p_{n*}^{N}\colon H_r(N) \to H_r(N_{\geq n})$$

is an isomorphism. For other values of r, it is the zero map.

2.5. Homology of Interleaf Bundles

In this subsection, we show that interleaf fiber bundles over S^m have the (co)homology of product bundles. We notice that the E_2-term of the cohomological Leray-Serre spectral sequence of an interleaf bundle is the same as that of a product bundle, and proceed to show that the former spectral sequence collapses at the E_2-term. This allows us to exploit convergence to recover the cohomology of the total space from the E_2-term. Here, collapse at the E_2-term is understood to mean the vanishing of all higher differentials. We will employ the theorem of Leray-Hirsch to show collapse for some cases, and then proceed to show collapse directly in the remaining cases.

Definition 2.5.1 (p. 148 in [**McC01**]). Let $\xi = (F, Y, p, A)$ be a Hurewicz fibration with inclusion $i\colon F \hookrightarrow Y$. We say that F is *totally nonhomologous to zero* in Y with respect to the ring R if the inclusion $i\colon F \hookrightarrow Y$ induces a surjective homomorphism $i^*\colon H^*(Y;R) \twoheadrightarrow H^*(F;R)$.

Note that this means a surjective homomorphism is induced in *every* degree. The name "totally nonhomologous to zero" stems from [**McC01**, p. 148], while the very same property is referred to as "totally non*co*homologous to zero" (or *TNCZ*) in other works, including [**Mar90**] and [**Yam05**]. Actually, TNCZ seems to be the more commonly used term.

Theorem 2.5.2 (Leray-Hirsch, Theorem 5.10 on p. 148 in [**McC01**]). *Let $\xi = (F, Y, p, A)$ be a Hurewicz fibration, with F connected, A of finite type, path connected, and for which the system of local coefficients on A is simple.*

1. *Then F is totally non-homologous to zero in Y with respect to a field k if and only if the Leray-Serre spectral sequence for cohomology of ξ collapses at the E_2-term.*
2. *If F is totally non-homologous to zero in Y with respect to a field k, then we have*

$$H^*(Y;k) \cong H^*(A;k) \otimes_k H^*(F;k)$$

as vector spaces.

We consider only simply connected base spaces in this chapter. By [**McC01**, Proposition 5.20], any bundle of groups over a simply-connected space is simple. Accordingly, so is a system of local coefficients. The space A being of *finite type* means that $H_k(A)$ is finitely generated for all k.

As stated before, we want to use Leray-Hirsch to show that some higher differentials vanish. To this end, we show that the fiber in an interleaf fiber bundle over S^m is totally nonhomologous to zero in the total space in some cases. In fact, we show a stronger result: Namely, we do not need to pose any requirements on the structure group. By Theorem 2.5.2(1), the fact that the fiber is totally nonhomologous to zero in the total space then implies the vanishing of the differentials.

In the following, d_m^k is the differential $d_m^k\colon E_m^{0,k} \to E_m^{m,k-m+1}$.

Proposition 2.5.3. *Given a fibration $\xi = (N, Y, p, S^m)$ with $N \in \mathrm{Ob}\,\mathbf{ICW}$, we have*

(1) *if m is even, then $H^k(Y) \cong H^k(N) \oplus H^{k-m}(N)$, and*
(2) *if m is odd, then*

$$H^k(Y) \cong \begin{cases} H^k(N)/\mathrm{im}\,(d_m^k), & k \text{ even}, \\ H^{k-m}(N), & k \text{ odd}. \end{cases}$$

Proof. We consider the Wang sequence:

k	m	$H^{k-1}(N) \xrightarrow{d_m^{k-1}}$	$H^{k-1-m+1}(N) \xrightarrow{j}$	$H^k(Y) \xrightarrow{i}$	$H^k(N) \xrightarrow{d_m^k}$	$H^{k-m+1}(N)$
odd	odd	$H^{k-1}(N)$	$H^{k-1-m+1}(N)$	$H^k(Y)$	0	0
odd	even	$H^{k-1}(N)$	0	$H^k(Y)$	0	$H^{k-m+1}(N)$
even	odd	0	0	$H^k(Y)$	$H^k(N)$	$H^{k-m+1}(N)$
even	even	0	$H^{k-1-m+1}(N)$	$H^k(Y)$	$H^k(N)$	0

In the first, second and fourth rows, the map $i\colon H^k(Y) \to H^k(N)$ is surjective. Accordingly, in these cases N is totally nonhomologous to zero in Y with respect to $\mathbb{Q}$, and we can invoke Leray-Hirsch (Theorem 2.5.2) to see that $H^*(Y) \cong H^*(S^m) \otimes_{\mathbb{Q}} H^*(N)$.

The third row treats the case of k being even while m is odd. We obtain a long exact sequence

$$0 \to H^k(Y) \overset{i}{\hookrightarrow} H^k(N) \xrightarrow{d_m^k} H^{k-m+1}(N) \overset{j}{\twoheadrightarrow} H^{k+1}(Y) \to 0 \tag{28}$$

from the first and third rows, above. From the long exact sequence (28) we get the short exact sequence of vector spaces

$$0 \to H^k(Y) \overset{i}{\hookrightarrow} H^k(N) \xrightarrow{d_m^k} \mathrm{im}\,(d_m^k) \to 0.$$

This splits, yielding $H^k(N) \cong H^k(Y) \oplus \mathrm{im}\,(d_m^k)$, which implies

$$H^k(Y) \cong H^k(N)/\mathrm{im}\,(d_m^k) \cong H^k(N)/\ker(j).$$

Hence $i\colon H^k(Y) \to H^k(N)$ is surjective if and only if d_m^k is the zero map. □

Given an $\xi = (N, Y, p, S^m)$, interleaf fiber bundle over S^m, the last proof shows that all differentials vanish if m is even. This can also be seen by examining the spectral sequence directly (by using a method which will be introduced below), but it is quicker to just look at the Wang sequence. We want to establish that the differentials vanish if m is odd as well. To this end, we want to exploit naturality of the spectral sequence with respect to certain fiberwise maps. We consider these maps now: As noted in equation (2), for the Dold fibration $\mathrm{ft}_{<k}(\xi)$ there is the

associated Hurewicz fibration $\mathrm{ft}_{<k}(\xi)_{\mathrm{aHf}}$ with

$$\begin{tikzcd} \mathrm{ft}_{<k}(Y) \arrow[rr, "\mathrm{ft}_{<k}(p)_{\mathrm{he}}", "\simeq"'] \arrow[dr, "\mathrm{ft}_{<k}(p)"'] & & W(\mathrm{ft}_{<k}(p)) \arrow[dl, "\mathrm{ft}_{<k}(p)_{\mathrm{Hf}}"] \\ & B & \end{tikzcd}$$

while we get from Lemma 2.3.3 a fiberwise map

$$\begin{tikzcd} Y \arrow[r, "p"] & S^m \\ \mathrm{ft}_{<k}(Y) \arrow[u, "\mu"] \arrow[r, "\mathrm{ft}_{<k}(p)"] & B \arrow[u, "\simeq"', "\mathrm{M}"] \end{tikzcd}$$

of fibrations with path connected fiber and simply connected base. Concatenating these diagrams, we obtain the following commutative diagram:

$$\begin{tikzcd} N \arrow[r, hook] & Y \arrow[r, "p"] & S^m \\ N_{<k} \arrow[u, "\mu|"] \arrow[r, hook] & \mathrm{ft}_{<k}(Y) \arrow[u, "\mu"] \arrow[r, "\mathrm{ft}_{<k}(p)"] & B \arrow[u, "\simeq"', "\mathrm{M}"] \\ N_{<k} \arrow[u, "\mathrm{ft}_{<k}(p)_{\mathrm{he}}|"', "\simeq"] \arrow[r, hook] & W(\mathrm{ft}_{<k}(p)) \arrow[u, "\mathrm{ft}_{<k}(p)_{\mathrm{he}}"', "\simeq"] \arrow[r, "\mathrm{ft}_{<k}(p)_{\mathrm{Hf}}"] & B \arrow[u, "\mathrm{id}_B"'] \end{tikzcd} \tag{29}$$

LEMMA 2.5.4. *For the fiberwise restriction $\mu|$ of μ, we have*

$$(\mu|)^* = (e_k^N)^*.$$

PROOF. Considering the definition of μ, namely

$$\mu = \mu_{\exp} \circ \mu_{<k},$$

we obtain $\mu| = \mu_{\exp}| \circ \mu_{<k}|$, and we have to distinguish several cases:

(1) The upper left square of (29) factors as:

$$\begin{tikzcd} N \arrow[r, hook] & D_0 \times N \arrow[r, hook] & Y \\ N_{<k} \arrow[u, "\mu|"] \arrow[r, hook] & D_0 \times N_{<k} \arrow[r, hook] & \mathrm{ft}_{<k}(Y) \arrow[u, "\mu"'] \end{tikzcd}$$

In this case,

$$\mu| = \mu_{\exp}| \circ \mu_{<k}| = \mathrm{id} \circ e_k^N = e_k^N.$$

(2) Case (1) with "D_1" substituted for "D_1".

(3) The upper left square of (29) factors as:

$$\begin{tikzcd} N \arrow[r, hook] & M_\alpha \arrow[r] & Y \\ N_{<k} \arrow[u, "\mu|"] \arrow[r, hook] & M_{\alpha_{<k}} \arrow[r, hook] & \mathrm{ft}_{<k}(Y) \arrow[u, "\mu"'] \end{tikzcd}$$

Consult the proof of Lemma 2.3.1 for the equations for ϕ: If $t \in \left[0, \frac{1}{2}\right]$, then it holds that

$$\begin{aligned}\mu([b,f,t]_{M_{\alpha_{<k}}}) &= \kappa \circ \phi([b,f,t]_{M_{\alpha_{<k}}}) \\ &= \kappa([b, e_k^N(f), 2t]_{M_\alpha}) \\ &= [b, g(b) \circ e_k^N(f)]_Y \quad \text{(by the definition of } \kappa) \\ &= [b, e_k^N(f)]_Y.\end{aligned}$$

Thus, $\mu| = e_k^N$. If, on the other hand, $t \in \left[\frac{1}{2}, 1\right]$, then

$$\begin{aligned}\mu([b,f,t]_{M_{\alpha_{<k}}}) &= \kappa \circ \phi([b,f,t]_{M_{\alpha_{<k}}}) \\ &= \kappa([b, H_b(f, 2t-1), 1]_{M_\alpha}) \\ &= [b, H_b(f, 2t-1)]_Y \quad \text{(by the definition of } \kappa)\end{aligned}$$

Now, H_b is a homotopy with

$$H_b(\cdot, 1) = g(b) \circ e_k^N.$$

We can define a homotopy $\mu| \simeq e_k^N$ by setting

$$H' \colon \underbrace{(q_{<k})^{-1}(b)}_{=N_{<k}} \times I \to N \subset Y,\ H'([b,f,t]_{M_{\alpha_{<k}}}, s) = [b, H_b(f, (2t-1)(1-s)+s)]_Y.$$

$\square$

Lemma 2.5.5. *Let R be a ring, and let*

$$\Phi = \mathrm{Hom}(- \otimes_R -, -) \text{ and } \Gamma = \mathrm{Hom}(-, \mathrm{Hom}(-,-)),$$

be functors

$$R\text{-}\mathbf{Mod}^{\mathrm{op}} \times R\text{-}\mathbf{Mod}^{\mathrm{op}} \times R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$$

with arguments in the same order. Then there is a natural equivalence $\eta \colon \Phi \to \Gamma$.

Proof. We have to show that for any morphisms

$$X \xleftarrow{f} X'$$
$$Y \xleftarrow{g} Y'$$
$$Z \xrightarrow{h} Z'$$

we can complete the diagram

$$\begin{tikzcd} \mathrm{Hom}(X \otimes_R Y, Z) \arrow[r, "\mathrm{Hom}(f \otimes_R g, h)"] \arrow[d, dashed, "\eta", "\cong"'] & \mathrm{Hom}(X' \otimes_R Y', Z') \arrow[d, dashed, "\cong", "\eta'"'] \\ \mathrm{Hom}(X, \mathrm{Hom}(Y,Z)) \arrow[r, "\mathrm{Hom}(f, \mathrm{Hom}(g,h))"] & \mathrm{Hom}(X', \mathrm{Hom}(Y', Z')) \end{tikzcd}$$

commutatively. Let $i \in \mathrm{Hom}(X \otimes_R Y, Z)$ be R-linear. We set

$$\eta(i)(x) = i(x, \cdot) \colon Y \to Z.$$

Concerning injectivity, let $i, i' \in \mathrm{Hom}(X \otimes_R Y, Z)$, and $i \neq i'$ but $\eta(i) = \eta(i')$. The latter equation implies

$$\forall x \in X \colon i(x, \cdot) = i'(x, \cdot),$$

and therefore $i = i'$, which is a contradiction. Therefore η is injective. To show surjectivity, let $j \in \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$. We have to find $\eta^{-1}(j) \in \mathrm{Hom}(X \otimes_R Y, Z)$ such that $\eta(\eta^{-1}(j)) = j$. Setting

$$\eta^{-1}(j) \colon X \otimes_R Y \to Z,\ (x,y) \mapsto j(x)(y)$$

proves surjectivity. Lastly, to show commutativity, let $i \in \mathrm{Hom}(X \otimes_R Y, Z)$, $x' \in X'$ and $y' \in Y'$. Then

$$\begin{aligned}
[\mathrm{Hom}\,(f, \mathrm{Hom}\,(g,h)) \circ \eta(\,i)](x')(y') &= [\mathrm{Hom}(g,h) \circ \eta(i) \circ f](x')(y') \\
&= [(\mathrm{Hom}(g,h) \circ \eta(i))(f(x'))](y') \\
&= [\mathrm{Hom}(g,h)(\eta(i)(f(x')))](y') \\
&= [\mathrm{Hom}(g,h)(i(f(x'), \cdot))](y') \\
&= [h \circ i(f(x'), \cdot) \circ g](y') \\
&= h(i(f(x'), g(y'))),
\end{aligned}$$

and

$$\begin{aligned}
[\eta' \circ \mathrm{Hom}(f \otimes_R g, h)(i)](x')(y') &= [\eta' \circ (h \circ i \circ (f \otimes_R g))](x')(y') \\
&= [\eta' \circ (h \circ i \circ (f(x') \otimes_R g(\cdot)))](y') \\
&= [\eta' \circ (h \circ i(f(x'), g(\cdot)))](y') \\
&= [\eta'(h(i(f(x'), g(\cdot))))](y') \\
&= h(i(f(x'), g(y'))).
\end{aligned}$$

□

LEMMA 2.5.6. *Let $m \geq 2$ be a natural number, and let $\xi = (F, Y, p, A)$ be a fibration in* **Fib**. *For the associated cohomological Leray-Serre spectral sequence $\{E_r^{p,q}, d_r\}$, and for any coordinates (p, q), there is a natural (for morphisms in* **Fib***) isomorphism*

$$S_2^*\colon E_2^{p,q} \xrightarrow{\cong} H^p(A) \otimes H^q(F).$$

PROOF. We have already seen that the isomorphism

$$\phi_2^*\colon E_2^{p,q} \xrightarrow{\cong} H^P(A; H^q(F))$$

of Theorem 1.2.1 is natural in the required sense. Furthermore, there are natural isomorphisms (with "natural" referring to an ordinary natural transformation)

$$\begin{aligned}
&H^p(A; H^q(F)) && \\
\cong&\mathrm{Hom}(H_p(A), H^q(F)) && \text{(natural, [\textbf{May99}, p. 134], UCT)} \\
=&\mathrm{Hom}(H_p(A), \mathrm{Hom}(H_q(F), \mathbb{Q})) && \text{(obviously natural)} \\
\cong&\mathrm{Hom}(H_p(A) \otimes H_q(F), \mathbb{Q}) && \text{(natural by Lemma 2.5.5)} \\
\cong&\mathrm{Hom}(H_p(A) \otimes H_q(F), \underbrace{\mathbb{Q} \otimes \mathbb{Q}}_{\cong \mathbb{Q}}) && \text{(obviously natural)} \\
\cong&H^p(A) \otimes H^q(F) && \text{(natural, [\textbf{May99}, p. 136]).}
\end{aligned}$$

Hence the composition of these natural isomorphisms with ϕ_2^* yields a natural (for morphisms in **Fib**) isomorphism S_2^*. □

To make this naturality statement precise, consider a fiberwise map $(\psi, \Psi) \in \mathrm{Hom}_{\mathbf{Fib}}(\xi, \eta)$. Then naturality implies that the diagram

$$\begin{tikzcd}
E_2^{p,q}(\eta) \arrow[r, "(\phi_2^*)^{-1}"] \arrow[d, "\{\Xi(\psi,\Psi)\}"'] & H^p(B(\eta); H^q(F(\eta))) \arrow[r, "\cong"'] \arrow[d, "\overline{(\psi,\Psi)}"'] & H^p(B(\eta)) \otimes H^q(F(\eta)) \arrow[d, "\Psi^* \times (\psi|)^*"] \\
E_2^{p,q}(\xi) \arrow[r, "(\phi_2^*)^{-1}"] & H^p(B(\xi); H^q(F(\xi))) \arrow[r, "\cong"'] & H^p(B(\xi)) \otimes H^q(F(\xi))
\end{tikzcd}$$

commutes.

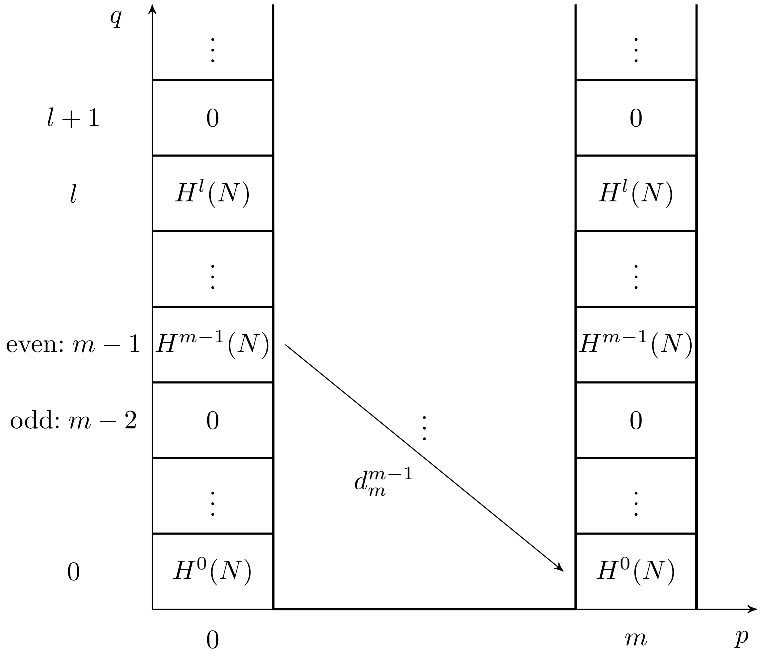
PROPOSITION 2.5.7. *Let $\xi = (N^l, Y, p, S^m)$ be an interleaf fiber bundle over S^m. Then the cohomological Leray-Serre spectral sequence collapses at the E_2-term.*

Proof. The fiber bundle ξ has a paracompact Hausdorff base space S^m and is thus a fibration by [**Spa66**, Cor. 14 on p. 96]. Hence the proof of Proposition 2.5.3 shows that all differentials vanish if m is even.

If m is odd, we have to consider the differentials

$$\{d_m^k \colon E_m^{0,k} \to E_m^{m,k-m+1} \mid k \in \mathbb{Z}\}.$$

The E_m-term is as follows:



The only such differentials which do not trivially vanish are d_m^k for $0 \leq k \leq l$. However, all differentials d_m^k for $0 \leq k < m-1$ do vanish, because

$$k < m-1 \Rightarrow k-m+1 < 0$$

implies that d_m^k is a map

$$E_m^{0,k} \to E_m^{m,k-m+1} = 0.$$

It remains to consider the differentials d_m^k for k such that $m-1 \leq k \leq l$. For the remainder of this proof, we fix such a k. Recall diagram (29). Due to the naturality of the Leray-Serre spectral sequence, the composition of fiberwise maps

$$\Xi = (\mathrm{ft}_{<k}(p)_{\mathrm{he}}, \mathrm{id}) \circ (\mu, \mathrm{M})$$

induces a map of spectral sequences

$$\{\Xi\} \colon \{E_r^{p,q}, d_r\} \to \{\tilde{E}_r^{p,q}, \tilde{d}_r\}$$

between the cohomological Leray-Serre spectral sequences $\{E_r^{p,q}, d_r\}$, $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$ related to the cohomology of the Hurewicz fibrations ξ and

$$\mathrm{ft}_{<k}(\xi)_{\mathrm{aHf}} = (W(\mathrm{ft}_{<k}(p)), \mathrm{ft}_{<k}(p)_{\mathrm{Hf}}, S^m),$$

respectively. Invoking Lemma 2.5.6 for both spectral sequences, we obtain natural isomorphisms

$$\begin{aligned} E_2^{p,q} &\cong H^p(S^m) \otimes H^q(N), \\ \tilde{E}_2^{p,q} \cong H^p(B) \otimes H^q(N_{<k}) &\cong H^p(S^m) \otimes H^q(N_{<k}). \end{aligned} \tag{30}$$

But it holds that

$$H^p(S^m) \cong \begin{cases} \mathbb{Q}, & p = 0, m, \\ 0, & \text{else.} \end{cases}$$

Hence the first equation (30) reduces to

$$E_2^{p,q} \cong H^p(S^m) \otimes H^q(N) \cong \begin{cases} \mathbb{Q} \otimes H^q(N) \cong H^q(N), & p = 0, m, \\ 0 \otimes H^q(N) = 0, & \text{else.} \end{cases}$$

So we get the following E_2-term:

$$\begin{array}{r|c|ccc|c|l}
q & \vdots & & & & \vdots & \\ \cline{2-2} \cline{6-6}
l+1 & 0 & & & & 0 & \\ \cline{2-2} \cline{6-6}
l & H^l(N) & & & & H^l(N) & \\ \cline{2-2} \cline{6-6}
 & \vdots & & & & \vdots & \\ \cline{2-2} \cline{6-6}
0 & H^0(N) & & & & H^0(N) & \\ \hline
 & 0 & & & & m & p
\end{array}$$

Analogously, using

$$H^q(N_{<k}) \cong \begin{cases} H^q(N), & q < k, \\ 0, & \text{else,} \end{cases}$$

we obtain the $\tilde{E}_2$-term:

$$\begin{array}{r|c|ccc|c|l}
q & \vdots & & & & \vdots & \\ \cline{2-2} \cline{6-6}
k & 0 & & & & 0 & \\ \cline{2-2} \cline{6-6}
k-1 & H^{k-1}(N) & & & & H^{k-1}(N) & \\ \cline{2-2} \cline{6-6}
 & \vdots & & & & \vdots & \\ \cline{2-2} \cline{6-6}
0 & H^0(N) & & & & H^0(N) & \\ \hline
 & 0 & & & & m & p
\end{array}$$

For q arbitrary and $p = 0$ or $p = m$, naturality of the isomorphisms (30) implies that the map of spectral sequences $\{\Xi\}\colon \{E_r^{p,q}, d_r\} \to \{\tilde{E}_r^{p,q}, \tilde{d}_r\}$ induces commutative squares:

(31)
$$\begin{tikzcd}
E_2^{p,q} \arrow[r, "\{\Xi\}_2^{p,q}"] \arrow[d, "\cong"'] & \tilde{E}_2^{p,q} \arrow[d, "\cong"] \\
H^q(N) \arrow[r, "(\mu| \circ \mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^*"] & H^q(N_{<k})
\end{tikzcd}$$

We examine the map

$$(\mu| \circ \mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|)^* = (\mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|)^* \circ (\mu|)^* \colon H^q(N) \to H^q(N_{<k})$$

more closely. Firstly, $(\mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|)^*$ is an isomorphism, because $\mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|$ is a homotopy equivalence. Secondly, due to Lemma 2.5.4, $(\mu|)^* = (e_k^N)^*$. Corollary 2.4.2 shows that the map $(e_k^N)^*$ is an isomorphism for degrees $q < k$ and the zero map else.

The map of spectral sequences $\{\Xi\}$ induces for a fixed k such that $m-1 \leq k \leq l$ the following commutative diagram:

$$\begin{tikzcd}
E_m^{0,k} \arrow[rr, "\{\Xi\}_m^{0,k}"] \arrow[dr, "d_m^k"'] & & \tilde{E}_m^{0,k} \arrow[dr, "\tilde{d}_m^k"] & \\
& E_m^{m,k-m+1} \arrow[rr, "\{\Xi\}_m^{m,k-m+1}"] & & \tilde{E}_m^{m,k-m+1}
\end{tikzcd}$$

Then $\tilde{E}_m^{0,k} = H^k(N_{<k}) = 0$, which implies $\tilde{d}_m^k = 0$. Moreover, $m \geq 2 \Rightarrow -m+1 \leq -1 \Rightarrow k-m+1 \leq k-1 = k-1$ which implies $\tilde{E}_m^{m,k-m+1} \cong H^{k-m+1}(N) \cong E_m^{m,k-m+1}$. Therefore, considering diagram (31) and using naturality, we have

$$\begin{tikzcd}
H^k(N) \arrow[rr, "0=(\mu|\circ\mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|)^*"] \arrow[dr, "d_m^k"'] & & 0 \arrow[dr, "0"] & \\
& H^{k-m+1}(N) \arrow[rr, "(\mu|\circ\mathrm{ft}_{<k}\,(p)_{\mathrm{he}}|)^*", "\cong"'] & & H^{k-m+1}(N)
\end{tikzcd} \tag{32}$$

which implies, by commutativity and by considering that the bottom horizontal map is an isomorphism that d_m^k vanishes. Since we can do this for any k such that $m-1 \leq k \leq l$, we have shown that all differentials d_m^k vanish. □

Hence we have established that all differentials in the cohomological Leray-Serre spectral sequence of an interleaf fiber bundle over S^m do in fact vanish. This will allow us to identify the cohomology of the total space in question as the cohomology of a product bundle.

Corollary 2.5.8. *Given an interleaf fiber bundle $\xi = (N, Y, p, S^m)$ over S^m, we have an isomorphism*

$$S^*\colon H^q(Y) \xrightarrow{\cong} H^q(N) \oplus H^{q-m}(N).$$

Proof. Proposition 2.5.7 implies $E_2^{p,q} = E_\infty^{p,q}$ for arbitrary coordinates p, q. Furthermore we get from Lemma 2.5.6 an isomorphism

$$S_2^*\colon E_2^{p,q} \xrightarrow{\cong} H^p(S^m) \otimes H^q(N).$$

We form the following composition:

$$\begin{tikzcd}
H^q(N) \oplus H^{q-m}(N) \arrow[d, "\cong"] \\
[\mathbb{Q} \otimes H^q(N)] \oplus [\mathbb{Q} \otimes H^{q-m}(N)] \arrow[d, "\cong"] \\
[H^0(S^m) \otimes H^q(N)] \oplus [H^m(S^m) \otimes H^{q-m}(N)] \arrow[d, "(S_2^*)^{-1}\oplus(S_2^*)^{-1}"', "\cong"] \\
E_2^{0,q} \oplus E_2^{m,q-m} \arrow[d, equal] \\
E_\infty^{0,q} \oplus E_\infty^{q,q-m} \arrow[d, "d_\infty\oplus d_\infty"', "\cong"] \\
\frac{F^0 H^q(C^*(Y))}{F^1 H^q(C^*(Y))} \oplus \frac{F^q H^q(C^*(Y))}{F^{q+1} H^q(C^*(Y))} \arrow[d, "a^*"', "\cong"] \\
H^q(Y)
\end{tikzcd}$$

Note that

$$d_\infty\colon E_\infty^{p,q} \xrightarrow{\cong} \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

comes from the proof of [**McC01**, Theorem 2.6] while the isomorphism a^* is induced from solving the extension problems, as described in Subsection 1.2.2. At this point, the choice of the splitting inducing a^* is not important for our argument, but this will change later on. □

Note that this is precisely the cohomology of a product bundle $S^m \times N \to S^m$.

PROPOSITION 2.5.9. *Given an interleaf fiber bundle $\xi = (N^l, Y, p, S^m)$ over S^m, we have for any integer $k \geq 1$ an isomorphism*

$$H^q(\mathrm{ft}_{<k}(Y)) \cong H^q(t_{<k}(N)) \oplus H^{q-m}(t_{<k}(N)).$$

PROOF. By convention, if $k > l$, the bundle is not modified, and in this case Corollary 2.5.8 confirms the desired result. On the other hand, if $k \leq l$, we recall - using the same terminology as in the proof of Proposition 2.5.7 - that there is an induced map of spectral sequences

$$\{\Xi\}\colon \{E_r^{p,q}, d_r\} \to \{\tilde{E}_r^{p,q}, \tilde{d}_r\}$$

between the cohomological Leray-Serre spectral sequences $\{E_r^{p,q}, d_r\}$, $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$ related to the cohomology of the Hurewicz fibrations ξ and $\mathrm{ft}_{<k}(\xi)_{\mathrm{aHf}}$. For a parameter $m-1 \leq j \leq k-1$, we again consider the following diagram:

$$\begin{tikzcd}
E_m^{0,j} \arrow[rr, "\{\Xi\}_m^{0,j}"] \arrow[dr, "d_m^j"'] & & \tilde{E}_m^{0,j} \arrow[dr, "\tilde{d}_m^j"] & \\
& E_m^{m,j-m+1} \arrow[rr, "\{\Xi\}_m^{m,j-m+1}"] & & \tilde{E}_m^{m,j-m+1}
\end{tikzcd} \tag{33}$$

Which is just:

$$\begin{tikzcd}
H^j(N) \arrow[rr, "(\mu|\circ \mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^*", "\cong"'] \arrow[dr, "d_m^j"'] & & H^j(N) \arrow[dr, "\tilde{d}_m^j"] & \\
& H^{j-m+1}(N) \arrow[rr, "(\mu|\circ \mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^*", "\cong"'] & & H^{j-m+1}(N)
\end{tikzcd}$$

We know $d_m^j = 0$ from Proposition 2.5.7, therefore $\tilde{d}_m^j$ vanishes as well. Theorem 2.5.2 then shows that $W(\mathrm{ft}_{<k}(p))$ has the desired cohomology, which means that the same holds for $\mathrm{ft}_{<k}(Y) \simeq W(\mathrm{ft}_{<k}(p))$. □

PROPOSITION 2.5.10. *Given a fibration $\xi = (F, Y, p, A)$ in* **Fib***, there is a natural (for morphisms in* **Fib***) isomorphism*

$$S_*^2\colon E_{p,q}^2 \xrightarrow{\cong} H_p(S^m) \otimes H_q(F).$$

PROOF. By [**McC01**, Theorem 5.1], the isomorphism $E_{p,q}^2 \cong H_p(A; H_q(F))$ is natural for morphisms in **Fib**. Furthermore, there is an isomorphism

$$H_p(A; H_q(F)) \cong H_p(A) \otimes H_q(F),$$

which is natural by [**May99**, p. 130] and the UCT. Thus the composition $E_{p,q}^2 \cong H_p(A) \otimes H_q(F)$ is natural in the required sense as well. □

Completely analogously to the cohomological result above, one shows:

PROPOSITION 2.5.11. *Let $\xi = (N^l, Y, p, S^m)$ be an interleaf fiber bundle over S^m. Then all differentials in the homological Leray-Serre spectral sequence vanish, and the homology of the total space is trivial:*

$$H_q(Y) \cong H_q(N) \oplus H_{q-m}(N).$$

Furthermore, for $\mathrm{ft}_{<k}(\xi)$, there is an isomorphism

$$H_q(\mathrm{ft}_{<k}(Y)) \cong H_q(N_{<k}) \oplus H_{q-m}(N_{<k}).$$

2.5.1. Canonical Splittings. We now return to the setting of the pseudomanifold X and with link bundle $\xi = (L, \partial M, p, S^m)$, $m = n - c$, described in Section 2.1. By definition, ξ is an interleaf fiber bundle over S^m. Thus we obtain from Lemma 2.3.3 a fiberwise map (μ, M), which fits into the following commutative diagram:

$$\begin{tikzcd}
L \arrow[r, hook] & \partial M \arrow[r, "p"] & S^m \\
L_{<c-k} \arrow[r, hook] \arrow[u, "\mu|"] & \mathrm{ft}_{<c-k}\,(\partial M) \arrow[r, "\mathrm{ft}_{<c-k}\,(p)"] \arrow[u, "\mu"] & B \arrow[u, "\simeq", "\mathrm{M}"'] \\
L_{<c-k} \arrow[r, hook] \arrow[u, "\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|", "\simeq"'] & W(\mathrm{ft}_{<c-k}\,(p)) \arrow[r, "\mathrm{ft}_{<c-k}\,(p)_{\mathrm{Hf}}"] \arrow[u, "\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}", "\simeq"'] & B \arrow[u, "\mathrm{id}_B"']
\end{tikzcd}$$

By naturality of the homological Leray-Serre spectral sequence, the fiberwise map

$$\Xi = (\mu, \mathrm{M}) \circ (\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}, \mathrm{id}_B)$$

induces a morphism of spectral sequences

$$\{\Xi\}\colon \{\tilde{E}^r_{p,q}, \tilde{d}^r\} \to \{E^r_{p,q}, d^r\}$$

between the homological Leray-Serre spectral sequences $\{E^r_{p,q}, d^r\}$ (associated with ξ) and $\{\tilde{E}^r_{p,q}, \tilde{d}^r\}$ (associated with $\mathrm{ft}_{<c-k}\,(\xi)_{\mathrm{aHf}}$):

$$\begin{tikzcd}
\mathrm{ft}_{<c-k}\,(\xi)_{\mathrm{aHf}} \arrow[d, "\Xi"'] \arrow[r, phantom, ""] & \{\tilde{E}^r_{p,q}, \tilde{d}^r\} \arrow[d, "\{\Xi\}"] & \tilde{E}^2_{p,q} \arrow[l, hook'] \arrow[r, "\phi^2_*", "\cong"'] \arrow[d, "\{\Xi\}^2_{p,q}"'] & H_p(B; H_q(L_{<c-k})) \arrow[d, "\Xi"] \\
\xi & \{E^r_{p,q}, d^r\} & E^2_{p,q} \arrow[l, hook'] \arrow[r, "\phi^2_*", "\cong"'] & H_p(S^{n-c}; H_q(L))
\end{tikzcd}$$

$$\Xi \overset{\mathrm{LS}_*}{\rightsquigarrow} \{\Xi\}$$

Both spectral sequences collapse, thus it is permissible to substitute 2 for ∞ in the rightmost square to obtain the following diagram:

$$(34) \qquad \begin{tikzcd}
\tilde{E}^\infty_{p,q} \arrow[r, "\cong"] \arrow[d, "\{\Xi\}^\infty_{p,q}"'] & H_p(B; H_q(L_{<c-k})) \arrow[d, "\Xi"] \\
E^\infty_{p,q} \arrow[r, "\cong"] & H_p(S^{n-c}; H_q(L))
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
H_p(B; H_q(L_{<c-k})) \arrow[d, "\Xi"'] & H_p(B) \otimes H_q(L_{<c-k}) \arrow[l, "\cong"'] \arrow[d, "\cong"', "(\mathrm{M}\circ\mathrm{id}_B)_* \times (\mu|\circ\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|)_*"] \\
H_p(S^{n-c}; H_q(L)) & H_p(S^{n-c}) \otimes H_q(L) \arrow[l, "\cong"']
\end{tikzcd}$$

commutes due to naturality. We concatenate it with (34) and write Ω_* for $(\mathrm{M}\circ \mathrm{id}_B)_* \times (\mu| \circ \mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|)_*$ to obtain the following diagram:

$$\begin{tikzcd}
\tilde{E}^\infty_{p,q} \arrow[r, "\cong"] \arrow[d, "\{\Xi\}^\infty_{p,q}"'] & H_p(B; H_q(L_{<c-k})) \arrow[d, "\Xi"] & H_p(B)\otimes H_q(L_{<c-k}) \arrow[l, "\cong"'] \arrow[d, "\cong"', "\Omega_*"] \\
E^\infty_{p,q} \arrow[r, "\cong"] & H_p(S^{n-c}; H_q(L)) & H_p(S^{n-c})\otimes H_q(L) \arrow[l, "\cong"']
\end{tikzcd}$$

The RHS square commutes by Proposition 2.5.10 while the LHS square commutes in an obvious fashion. Take two copies of the last diagram, substitute

$$(p,q) = (0, r-1)$$

in the first copy and

$$(p,q) = (n-c, r-1-n+c)$$

in the second. Then form the direct sum of these two diagrams. The result is a diagram

$$\begin{tikzcd}
H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[r, "\cong"] \arrow[d, "Z_*^{r-1}\oplus Z_*^{r-1-n+c}"'] & \tilde{E}^\infty_{0,r-1} \oplus \tilde{E}^\infty_{n-c,r-1-n+c} \arrow[d, "\{\Xi\}^\infty_{0,r-1}\oplus\{\Xi\}^\infty_{n-c,r-1-n+c}"] \\
H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[r, "\cong"] & E^\infty_{0,r-1} \oplus E^\infty_{n-c,r-1-n+c}
\end{tikzcd}$$

with

$$Z_*^{r-1} \oplus Z_*^{r-1-n+c} = [(e^L_{c-k})_* \circ (\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|)_*] \oplus [(e^L_{c-k})_* \circ (\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|)_*]. \tag{35}$$

Incidentally, the last diagram forms the top square of the diagram

$$\begin{tikzcd}
H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[d, "\cong"] & H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[l, "Z_*\oplus Z_*"'] \arrow[d, "\cong"'] \\
E^\infty_{0,r-1} \oplus E^\infty_{n-c,r-1-n+c} \arrow[d, "d^\infty\oplus d^\infty"', "\cong"] & \tilde{E}^\infty_{0,r-1} \oplus \tilde{E}^\infty_{n-c,r-1-n+c} \arrow[l, "\{\Xi\}^\infty\oplus\{\Xi\}^\infty"'] \arrow[d, "\cong"', "\tilde{d}^\infty\oplus\tilde{d}^\infty"] \\
\frac{F_0H_{r-1}}{F_{-1}H_{r-1}} \oplus \frac{F_{n-c}H_{r-1}}{F_{n-c-1}H_{r-1}} \arrow[d, "a_*"', "\cong"] & \frac{\tilde{F}_0H_{r-1}}{\tilde{F}_{-1}H_{r-1}} \oplus \frac{\tilde{F}_{n-c}H_{r-1}}{\tilde{F}_{n-c-1}H_{r-1}} \arrow[d, "\cong"', "\tilde{a}_*"] \\
H_{r-1}(\partial M) & H_{r-1}(W(\mathrm{ft}_{<c-k}\,(p))) \arrow[l, "(\mu\circ\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}})_*"']
\end{tikzcd} \tag{36}$$

in which the maps a_*, $\tilde{a}_*$ are induced by splittings of two short exact sequences (as explained in Subsection 1.2.2) while the notation is abbreviated by setting

$$F_pH_q = F_pH_q(C_*(\partial M)),$$
$$\tilde{F}_pH_q = \tilde{F}_pH_q(C_*(W(\mathrm{ft}_{<c-k}\,(p)))).$$

Finally, we can state the aim of the present section. We want to show that is is always possible to choose splittings such that the bottom rectangle of the last diagram commutes.

Lemma 2.5.12. *Given a commutative diagram of rational vector spaces,*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[u, "\alpha", "\cong"'] & B' \arrow[r, "g'"] \arrow[u, "\beta", "\cong"'] & C' \arrow[r] \arrow[u, "\cong", "\gamma"'] & 0 \end{tikzcd}$$

such that the rows are exact, there is for any given splitting $h\colon C \to B$ of the upper row a canonical choice of a splitting $h'\colon C' \to B'$ of the lower row. For the isomorphisms

$$\phi\colon A \oplus C \xrightarrow{\cong} B,$$
$$\phi'\colon A' \oplus C' \xrightarrow{\cong} B',$$

induced by these splittings, the diagram

$$\begin{tikzcd} A \oplus C \arrow[d, "\phi"'] & A' \oplus C' \arrow[l, "\alpha\oplus\gamma"'] \arrow[d, "\phi'"] \\ B & B' \arrow[l, "\beta"] \end{tikzcd} \tag{37}$$

commutes.

Proof. Given a splitting $h\colon C \to B$, we induce a splitting of the lower row by setting
$$h' = \beta^{-1} \circ h \circ \gamma\colon C' \to B'.$$
We have
$$\begin{aligned} g' \circ h' &= g' \circ \beta^{-1} \circ h \circ \gamma \\ &= \gamma^{-1} \circ g \circ \beta \circ \beta^{-1} \circ h \circ \gamma \\ &= \gamma^{-1} \circ g \circ h \circ \gamma \\ &= \gamma^{-1} \circ \gamma \\ &= \mathrm{id}. \end{aligned}$$
So h' is indeed a splitting for the lower row. The splittings induce the aforementioned isomorphisms if we set
$$\phi(x, y) = f(x) + h(y),\ \phi'(x', y') = f'(x') + h'(y').$$
Thus we can complete the diagram to the following:

$$\begin{tikzcd} & & A \oplus C \arrow[d, "\phi"', "\cong"] \arrow[dr, "\mathrm{proj}_2"] & & \\ 0 \arrow[r] & A \arrow[r, hook, "f"] \arrow[ur, "x\mapsto(x,0)"] & B \arrow[r, "g"] & C \arrow[r] \arrow[l, bend left, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, hook, "f'"] \arrow[u, "\alpha", "\cong"'] \arrow[dr, "x'\mapsto(x',0)"'] & B' \arrow[r, "g'"'] \arrow[u, "\beta", "\cong"'] & C' \arrow[r] \arrow[u, "\cong", "\gamma"'] \arrow[l, bend right, "h'"'] & 0 \\ & & A' \oplus C' \arrow[u, "\phi'", "\cong"'] \arrow[ur, "\mathrm{proj}_2"'] & & \end{tikzcd}$$

Now, let $(x', y') \in A' \oplus C'$. Then,

$$\begin{aligned}\phi \circ (\alpha \oplus \gamma)(x', y') &= \phi(\alpha(x'), \gamma(y'))\\ &= f(\alpha(x')) + h(\gamma(y'))\end{aligned}$$

and also

$$\begin{aligned}\beta \circ \phi'(x', y') &= \beta(f'(x') + h'(y'))\\ &= \beta(f'(x') + \beta^{-1} \circ h \circ \gamma(y'))\\ &= \beta(f'(x')) + \beta(\beta^{-1} \circ h \circ \gamma(y'))\\ &= f(\alpha(x')) + h(\gamma(y')).\end{aligned}$$

Thus diagram (37) commutes as required. □

LEMMA 2.5.13. *Given a commutative diagram of rational vector spaces,*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, equal, "f=\mathrm{id}_A"] & B \arrow[r, "g=0"] & 0 \arrow[r] & 0 \\ 0 \arrow[r] & A' \arrow[u, two heads, "\alpha"] \arrow[r, "f'"] & B' \arrow[u, "\beta"] \arrow[r, "g'"] & C' \arrow[u, "\gamma=0"'] \arrow[r] & 0 \end{tikzcd}$$

such that the rows are exact, there is only one possible splitting of the top row: $0 = h\colon 0 \to B$. It is always possible to choose a splitting $h'\colon C' \to B'$ of the lower row such that for the isomorphisms

$$\phi\colon A \oplus 0 \xrightarrow{\cong} B,$$
$$\phi'\colon A' \oplus C' \xrightarrow{\cong} B',$$

induced by these splittings, the diagram

$$\begin{tikzcd} A \oplus 0 \arrow[d, "\phi"'] & A' \oplus C' \arrow[l, "\alpha\oplus\gamma"'] \arrow[d, "\phi'"] \\ B & B' \arrow[l, "\beta"] \end{tikzcd} \tag{38}$$

commutes. Furthermore, β is surjective.

PROOF. For $x \in A = B$, there is a $x' \in A'$ such that $\alpha(x') = x$. But commutativity of the left square implies

$$x = \alpha(x') = f(\alpha(x')) = \beta(f'(x')).$$

Therefore, β is surjective.

By the snake lemma, there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0,$$

which reduces to the short exact sequence

$$0 \to \ker\alpha \xrightarrow{f'} \ker\beta \xrightarrow{g'} C' \to 0.$$

We choose an arbitrary splitting $h'\colon C' \to \ker\beta$ of this sequence. Then h' is also a splitting of

$$0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0.$$

We want to show:

$$\forall (x', y') \in A' \oplus C'\colon \phi \circ \alpha \oplus 0(x', y') = \beta \circ \phi'(x', y').$$

So,

$$\phi' \circ \alpha \oplus 0(x', y') = \phi(\alpha(x'), 0) = f \circ \alpha(x'),$$

while

$$\begin{aligned}\beta\circ\phi'(x',y') &= \beta(f'(x')+h'(y'))\\ &= \beta\circ f'(x')+\beta\circ h'(y')\\ &= f\circ\alpha(x')+\beta\circ h'(y').\end{aligned}$$

Thus we need to show that $\beta\circ h'(y')$ vanishes. This is true because our chosen splitting satisfies

$$\operatorname{im} h' \subset \ker\beta.$$

Therefore, diagram (38) commutes. □

LEMMA 2.5.14. *Given a commutative diagram of rational vector spaces and linear maps*

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ C \arrow[ur, "g"] \arrow[rr, "h", "\cong"'] & & B \end{tikzcd}$$

the identity $f\circ g\circ h^{-1}=\mathrm{id}_B$ holds.

PROOF. For $c\in C$, we have $f\circ g(c) = h(c)$. For any $b\in B$, there is a $c=h^{-1}(b)\in C$. Thus,

$$\forall b\in B\colon\ f\circ g\circ h^{-1}(b)=b=\mathrm{id}_B(b).$$

□

REMARK 2.5.15. Consider a fiberwise map of Serre fibrations, $(\phi,\mathrm{id})\colon\eta\to\xi$. Denote by $\{E_r^{p,q},d_r\}$ and $\{\tilde{E}_r^{p,q},\tilde{d}_r\}$ the cohomological Leray-Serre spectral sequence of ξ and η, respectively. Then the following diagram commutes (see [**McC01**, p. 65f]):

$$\begin{tikzcd} 0 \arrow[r] & F^pH^n \arrow[r, "\mathrm{incl}"] \arrow[d, "H(\phi)"'] & F^{p-1}H^n \arrow[r, "(d_\infty)^{-1}\circ\mathrm{quot}"] \arrow[d, "H(\phi)"'] & E_\infty^{p-1,n-p+1} \arrow[r] \arrow[d, "\phi_\infty"] & 0 \\ 0 \arrow[r] & \tilde{F}^p\tilde{H}^n \arrow[r, "\mathrm{incl}"] & \tilde{F}^{p-1}\tilde{H}^n \arrow[r, "(\tilde{d}_\infty)^{-1}\circ\mathrm{quot}"] & \tilde{E}_\infty^{p-1,n-p+1} \arrow[r] & 0 \end{tikzcd}$$

Notice furthermore that ϕ_∞ is the map induced by ϕ on the E_∞-term. There is an analogous commutative diagram for homology. We will see in the next proof that the splittings of the exact rows in this diagram represent the solution of the extension problems.

PROPOSITION 2.5.16. *It is possible to choose the splittings inducing the maps a_*, $\tilde{a}_*$ in diagram* (36) *such that the lower rectangle of diagram* (36) *commutes. The same holds for the cohomological analog of this diagram.*

PROOF. By Corollary 2.4.3, we know that $(e^L_{c-k})_*$ is either an isomorphism or the zero map, depending on the degree in which homology is considered. Due to the fact that $\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|$ is a homotopy equivalence, the composition

$$(e^L_{c-k})_*\circ(\mathrm{ft}_{<c-k}\,(p)_{\mathrm{he}}|)_*$$

behaves in the same way as $(e^L_{c-k})_*$. Thus, there are three possible cases:

(1) "$Z_*^{r-1}\oplus Z_*^{r-1-n+c}=0$",
(2) "$Z_*^{r-1}=0$ while $Z_*^{r-1-n+c}$ is an isomorphism", or
(3) "both Z_*^{r-1} and $Z_*^{r-1-n+c}$ are isomorphisms".

It is not possible for Z_*^{r-1} to be an isomorphism while $Z_*^{r-1-n+c}$ vanishes. This is due to the fact that $Z_*^{r-1-n+c} = 0$ implies

$$\begin{aligned} & r-1-n+c \geq c-k \\ \Rightarrow & r-1 \geq c-k \\ \Rightarrow & Z_*^{r-1} = 0. \end{aligned}$$

In case (1), we have to show commutativity of the following diagram:

$$\begin{tikzcd}
H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[d, "\cong"] & 0 \arrow[l, "0"'] \arrow[d, "\cong"'] \\
E^\infty_{0,r-1} \oplus E^\infty_{n-c,r-1-n+c} \arrow[d, "\cong", "d^\infty \oplus d^\infty"'] & \tilde{E}^\infty_{0,r-1} \oplus \tilde{E}^\infty_{n-c,r-1-n+c} \arrow[l] \arrow[d, "\cong"', "\tilde{d}^\infty \oplus \tilde{d}^\infty"] \\
\frac{F_0 H_{r-1}}{F_{-1} H_{r-1}} \oplus \frac{F_{n-c} H_{r-1}}{F_{n-c-1} H_{r-1}} \arrow[d, "\cong", "a_*"'] & \frac{\tilde{F}_0 H_{r-1}}{\tilde{F}_{-1} H_{r-1}} \oplus \frac{\tilde{F}_{n-c} H_{r-1}}{\tilde{F}_{n-c-1} H_{r-1}} \arrow[d, "\cong"', "\tilde{a}_*"] \\
H_{r-1}(\partial M) & H_{r-1}(W(\mathrm{ft}_{<c-k}(p))) \arrow[l, "(\mu \circ \mathrm{ft}_{<c-k}(p)_{\mathrm{he}})_*"']
\end{tikzcd}$$

Commutativity is obvious. In case (2), the relevant diagram is the following:

(39)
$$\begin{tikzcd}
H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[d, "\cong"] & 0 \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[l, "0 \oplus Z_*^{r-1-n+c}"'] \arrow[d, "\cong"'] \\
E^\infty_{0,r-1} \oplus E^\infty_{n-c,r-1-n+c} \arrow[d, "\cong", "d^\infty \oplus d^\infty"'] & 0 \oplus \tilde{E}^\infty_{n-c,r-1-n+c} \arrow[l, "0 \oplus \{\Xi\}^2_{n-c,r-1-n+c}"'] \arrow[d, "\cong"', "0 \oplus \tilde{d}^\infty"] \\
F_0 H_{r-1} \oplus \frac{F_{n-c} H_{r-1}}{F_{n-c-1} H_{r-1}} \arrow[d, "\cong", "a_*"'] & 0 \oplus \tilde{F}_{n-c} H_{r-1} \arrow[d, "\cong"', "\tilde{a}_*"] \\
H_{r-1}(\partial M) & H_{r-1}(W(\mathrm{ft}_{<c-k}(p))) \arrow[l, "(\mu \circ \mathrm{ft}_{<c-k}(p)_{\mathrm{he}})_*"']
\end{tikzcd}$$

To show commutativity, we need to consider the splittings. The diagram

(40)
$$\begin{tikzcd}
 & F_0 H_{r-1} \arrow[d, equal] & & & \\
0 \arrow[r] & F_{n-c-1} H_{r-1} \arrow[r, hook, "i"] & F_{n-c} H_{r-1} \arrow[r, "\mathrm{quot}"] & \frac{F_{n-c}}{F_{n-c-1}} \arrow[r] \arrow[l, bend right, "s"'] & 0 \\
 & & & E^2_{n-c,r-1-n+c} \arrow[u, "\cong", "d^\infty"'] & \\
 & & & \tilde{E}^2_{n-c,r-1-n+c} \arrow[u, "\cong", "\{\Xi\}^2_{n-c,r-1-n+c} = H(\Xi)"'] & \\
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[uuu, "H(\Xi)"] & \tilde{F}_{n-c} H_{r-1} \arrow[r, "\mathrm{id}"] \arrow[uuu, "H(\Xi)"] & \tilde{F}_{n-c} H_{r-1} \arrow[r] \arrow[u, "\cong", "(\tilde{d}^\infty)^{-1}"'] \arrow[l, bend left, "\tilde{s} = \mathrm{id}"] & 0
\end{tikzcd}$$

commutes and has exact rows. It is a homological version of Remark 2.5.15. Thus $\tilde{s} = \mathrm{id}$ is the only possible choice for the splitting of the lower row. Set

$$s = H(\Xi) \circ \tilde{d}^\infty \circ H(\Xi)^{-1} \circ (d^\infty)^{-1}.$$

Then,

$$\mathrm{quot} \circ s = \mathrm{quot} \circ H(\Xi) \circ \tilde{d}^\infty \circ H(\Xi)^{-1} \circ (d^\infty)^{-1} = \mathrm{id}$$

by Lemma 2.5.14. Considering commutativity of the lower rectangle in diagram (39) as well as diagram (40), we have for $(0,x) \in 0 \oplus \tilde{E}^{\infty}_{n-c,r-1-n+c}$,

$$\begin{aligned}
[0+s] \circ [0 \oplus d^{\infty}] \circ [0 \oplus H(\Xi)](0,x) &\stackrel{!}{=} H(\Xi) \circ \tilde{a}_* \circ [0 \oplus \tilde{d}^{\infty}](0,x) \\
\Leftrightarrow [0+s] \circ [0 \oplus d^{\infty}](0, H(\Xi)(x)) &\stackrel{!}{=} H(\Xi) \circ \tilde{a}_*(0, \tilde{d}^{\infty}(x)) \\
\Leftrightarrow [0+s](0, d^{\infty}(H(\Xi)(x))) &\stackrel{!}{=} H(\Xi)(0 + \tilde{s}(\tilde{d}^{\infty}(x))) \\
\Leftrightarrow s(d^{\infty}(H(\Xi)(x))) &\stackrel{!}{=} H(\Xi)(\tilde{d}^{\infty}(x)) \\
\Leftrightarrow H(\Xi) \circ \tilde{d}^{\infty} \circ H(\Xi)^{-1} \circ (d^{\infty})^{-1} \circ d^{\infty} \circ H(\Xi)(x) &\stackrel{!}{=} H(\Xi)(\tilde{d}^{\infty}(x))
\end{aligned}$$

and the latter equation clearly holds. Thus the lower rectangle commutes.

Dealing with case (3), we recall two of the underlying exact sequences. Firstly, there is the sequence (7). This fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & & & \\
0 \arrow[r] & F_{-1}H_{r-1} \arrow[r, hook, "i"] & F_0 H_{r-1} \arrow[r, "(d^{\infty})^{-1}\circ\mathrm{quot}"] & E^2_{0,r-1} \arrow[r] & 0 \\
0 \arrow[r] & \tilde{F}_{-1}H_{r-1} \arrow[r, hook, "\tilde{i}"] \arrow[u, "H(\Xi)", "\cong"'] \arrow[d, equal] & \tilde{F}_0 H_{r-1} \arrow[r, "(\tilde{d}^{\infty})^{-1}\circ\mathrm{quot}"] \arrow[u, "H(\Xi)"] & \tilde{E}^2_{0,r-1} \arrow[r] \arrow[u, "\cong", "\{\Xi\}^2_{0,r-1}"'] & 0 \\
 & 0 & & &
\end{tikzcd}$$

The 5-lemma shows the middle map to be an isomorphism. Secondly, the sequence (8) fits into the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & F_0 H_{r-1} \arrow[d, equal] & & & \\
0 \arrow[r] & F_{n-c-1}H_{r-1} \arrow[r, hook, "i"] & F_{n-c}H_{r-1} \arrow[r, "\mathrm{quot}"] & \frac{F_{n-c}}{F_{n-c-1}} \arrow[r] & 0 \\
 & & & E^2_{n-c,r-1-n+c} \arrow[u, "\cong", "d^{\infty}"'] & \\
 & & & \tilde{E}^2_{n-c,r-1-n+c} \arrow[u, "\cong", "\{\Xi\}^2_{n-c,r-1-n+c}"'] & \\
0 \arrow[r] & \tilde{F}_{n-c-1}H_{r-1} \arrow[r, hook, "\tilde{i}"] \arrow[uuu, "H(\Xi)", "\cong"'] \arrow[d, equal] & \tilde{F}_{n-c}H_{r-1} \arrow[r, "\mathrm{quot}"] \arrow[uuu, "H(\Xi)"] & \frac{\tilde{F}_{n-c}}{\tilde{F}_{n-c-1}} \arrow[r] \arrow[u, "\cong", "(\tilde{d}^{\infty})^{-1}"'] & 0 \\
 & \tilde{F}_0 H_{r-1} & & &
\end{tikzcd} \tag{41}$$

Again, the 5-lemma shows the middle map to be an isomorphism. Applying Lemma 2.5.12 to (41), we see that a choice of splittings is possible such that the lower rectangle of diagram (36) commutes in case (3). The cohomological version is shown in an analogous fashion. □

Remark 2.5.17. Given the data contained in the upper rectangle of diagram (36), Proposition 2.5.16 tells us that we may always choose splittings such that the lower rectangle of the same diagram commutes. From now on, we will assume that given a choice, we have chosen splittings as described in Proposition 2.5.16. For such a choice of splittings, we denote the composition of the vertical isomorphisms on the left side of diagram (36) by

$$(S_*)^{-1}\colon H_{r-1}(L) \oplus H_{r-1-n+c}(L) \xrightarrow{\cong} H_{r-1}(\partial M).$$

Accordingly, the inverse isomorphism S_* is defined as well. Analogously, we denote the composition of vertical isomorphisms on the right side by

$$(S_{W*})^{-1}\colon H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \xrightarrow{\cong} H_{r-1}(W(\mathrm{ft}_{<c-k}(p)).$$

Furthermore, by the same procedure, we obtain isomorphisms on cohomology:

$$S^*\colon H^{n-r}(\partial M) \xrightarrow{\cong} H^{n-r}(L) \oplus H^{c-r}(L)$$

$$S^*_W\colon H^{n-r}(W(\mathrm{ft}_{<k}(p))) \xrightarrow{\cong} H^{n-r}(L_{<k}) \oplus H^{c-r}(L_{<k}).$$

The full version of diagram (41) used in this proof is the following:

$$\begin{tikzcd}
 & & F_{n-c-1} \oplus \frac{F_{n-c}}{F_{n-c-1}} \arrow[d, "a_*"', "\cong"] \arrow[dr, "\mathrm{proj}_2"] & & \\
0 \arrow[r] & F_{n-c-1}H_{r-1} \arrow[ur, "x\mapsto(x,0)"] \arrow[r, hook, "i"] & F_{n-c}H_{r-1} \arrow[r, "\mathrm{quot}"] & \frac{F_{n-c}}{F_{n-c-1}} \arrow[r] & 0 \\
 & & & E^2_{n-c,r-1-n+c} \arrow[u, "\cong"', "d^\infty"] & \\
 & & & \tilde{E}^2_{n-c,r-1-n+c} \arrow[u, "\cong"', "\{\Xi\}^2_{n-c,r-1-n+c}"] & \\
0 \arrow[r] & \tilde{F}_{n-c-1}H_{r-1} \arrow[uuu, "H(\Xi)"', "\cong"] \arrow[dr, "\tilde{x}\mapsto(\tilde{x},0)"'] \arrow[r, hook, "\tilde{i}"] & \tilde{F}_{n-c}H_{r-1} \arrow[uuu, "H(\Xi)"', "\cong"] \arrow[r, "\mathrm{quot}"] & \frac{\tilde{F}_{n-c}}{\tilde{F}_{n-c-1}} \arrow[u, "\cong"', "(\tilde{d}^\infty)^{-1}"] \arrow[r] & 0 \\
 & & \tilde{F}_{n-c-1} \oplus \frac{\tilde{F}_{n-c}}{\tilde{F}_{n-c-1}} \arrow[u, "\tilde{a}_*"', "\cong"] \arrow[ur, "\mathrm{proj}_2"'] & &
\end{tikzcd}$$

2.5.2. Crossing and E_2-terms. It is the aim of the present subsection to show the following result:

Lemma 2.5.18. *The isomorphism*

$$S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1}$$

can be written as a direct sum of isomorphisms:

$$\begin{tikzcd} H^{n-r}(L) \arrow[d, "C^{n-r}"', "\cong"] & \bigoplus & H^{c-r}(L) \arrow[d, "\cong"', "C^{c-r}"] \\ H_{r-1-n+c}(L) & \bigoplus & H_{r-1}(L) \end{tikzcd}$$

We refer to this behavior as *crossing*. Before we can give a proof, we need several preliminary results. More specifically, there are two types of results needed. Firstly, we show that in some special cases, the maps d_∞ and d^∞ are just the identity. Secondly, we make an analysis of some E_2- and E^2-terms. Both types of results are then used in the proof of Lemma 2.5.18, which consists of showing commutativity of a diagram involving the aforementioned objects. If not mentioned otherwise, in this entire subsection $\xi = (L, Y, p, A)$ will be a Hurewicz fibration with B simply connected and F connected.

As a geometric intuition, consider the situation of a trivial bundle over S^m. Here, crossing can be shown directly by considering cells and Poincaré duality.

2.5.2.1. *The Maps d_∞ and d^∞ and the Second Terms.* The desired result follows from a homological version of [**McC01**, Theorem 5.9]. Thus we need to provide the latter result first. This is a general result on first-quadrant spectral sequences. Consider such a spectral sequence, the ascending filtration on $H_q = H_q(Y)$,

$$\{0\} = F_{-1}H_q \subset F_0 H_q \subset \cdots \subset F_{q-1}H_q \subset F_q H_q = H_q,$$

and its associated graded module $E^\infty_{n,q-n} \cong F_n H_q / F_{n-1} H_q$. Recall that $F_q H_q = H_q$ follows from the fact that we are dealing with a first quadrant spectral sequence. Then

$$E^\infty_{0,q} = F_0 H_q / F_{-1} H_q = F_0 H_q \subset H_q(Y).$$

The modules along the left column of the E^2-term of a first quadrant (homological) spectral sequence are determined by quotients $E^2_{0,q}$ by the image of the incoming differentials. Thus we have a series of quotients

$$E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{q-1}_{0,q} \twoheadrightarrow E^q_{0,q} = E^\infty_{0,q}.$$

Proposition 2.5.19. *Given a fibration $\xi = (L, Y, p, A)$ with A simply connected, and L connected; the composite*

$$H_q(L) \hookrightarrow E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{q-1}_{0,q} \twoheadrightarrow E^q_{0,q} = E^\infty_{0,q} \hookrightarrow H_q(Y)$$

is the homomorphism $\mathrm{incl}_*\colon H_q(L) \to H_q(Y)$.

Proof. Consider the fibrations ξ and $(L, L, \mathrm{const}_{\mathrm{pt}}, \mathrm{pt})$. Let $g\colon \{\mathrm{pt}\} \to A$ be the map $g(\mathrm{pt}) = a$ for some fixed $a \in A$. We obtain a fiberwise map

$$(L, L, \mathrm{const}_{\mathrm{pt}}, \mathrm{pt}) \xrightarrow{(\mathrm{incl}, g)} \xi,$$

which we can visualize as a commutative diagram:

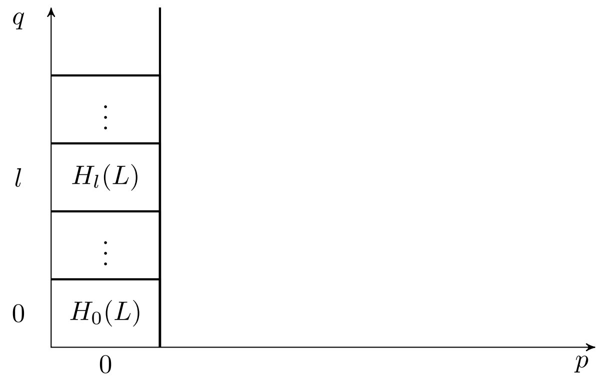
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "g"'] & L \arrow[l] \arrow[d, "\mathrm{incl}"'] & L \arrow[l, "="'] \arrow[d, "="] \\ A & Y \arrow[l, "p"'] & L \arrow[l, "\mathrm{incl}"'] \end{tikzcd}$$

If we denote by $\{E^r_{*,*}(\eta)\}$ the homological Leray-Serre spectral sequence of the fibration η, then we get from naturality an induced mapping

$$\{E^r_{*,*}(L, L, \mathrm{const}_{\mathrm{pt}}, \mathrm{pt})\} \xrightarrow{\{(\mathrm{incl}, g)\}} \{E^r_{*,*}(\xi)\}$$

which converges to $\mathrm{incl}^\infty = \mathrm{incl}_*$ at E^∞. We consider the E^2-terms. For the fibration $(L, L, \mathrm{const}_{\mathrm{pt}}, \mathrm{pt})$, it is characterized as

$$E^2_{p,q} \cong H_p(\mathrm{pt}; H_q(L)) \cong H_p(\mathrm{pt}) \otimes H_q(L) \cong \begin{cases} H_q(L), & p = 0, \\ 0, & \text{else.} \end{cases}$$



Entries which are not noted, vanish. Thus the term consists of a single column, and the spectral sequence collapses at this second term. The induced mapping injects $H_*(L)$ into $H_0(A; H_*(L))$. Now consider how a map of spectral sequences behaves with respect to the limit term. In our case,

$$\mathrm{incl}_* = \mathrm{incl}^\infty = H(\mathrm{incl}^{q-1}) = H(H(\mathrm{incl}^{q-2})) = \cdots = H(\ldots(H(\mathrm{incl})))$$

is just the composition

$$H_q(L) \hookrightarrow E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{q-1}_{0,q} \twoheadrightarrow E^q_{0,q} = E^\infty_{0,q} \hookrightarrow H_q(Y).$$

□

Remark 2.5.20. Given a fibration $\xi = (L, Y, p, A)$ with A simply connected, L connected and furthermore such that the associated homological Leray-Serre spectral sequence collapses. Then $\mathrm{incl}_* \colon H_*(L) \to H_q(Y)$ is injective. This can be seen by considering the composition

$$H_q(L) \hookrightarrow E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{q-1}_{0,q} \twoheadrightarrow E^q_{0,q} = E^\infty_{0,q} \hookrightarrow H_q(Y).$$

If the spectral sequence collapses, then all surjections in this composition are equalities. Hence incl_* is injective.

This remark is the desired general result on spectral sequences. We need to consider the E_2- and E^2-terms, before we can finish our analysis of the maps d_∞ and d^∞. Here, we specialize to an interleaf fiber bundle $\xi = (L, \partial M, p, S^{n-c})$ arising from a link bundle as described in Section 2.1. We can describe explicitly some of the modules in the E_2-term of the associated Leray-Serre spectral sequence. Recall from [**McC01**] the definition of the entries in the E_r-term. These are

$$E^{p,q}_r = \frac{Z^{p,q}_r}{Z^{p+1,q-1}_{r-1} + B^{p,q}_{r-1}},$$

with

$$Z^{p,q}_r = F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1}),$$
$$B^{p,q}_r = F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1}).$$

In our case, the underlying differential graded algebra is $(A, d) = (C^*(\partial M), \partial^*)$, and the filtration is

$$F^p(C^*(\partial M))^q = \ker[C^*(\partial M) \to C^*(J^{p-1})] \cap C^q(\partial M) = \ker[C^q(\partial M) \to C^q(J^{p-1})].$$

Let $k \in \mathbb{N}$ such that $k \geq 1$, then $J^{n-c+k-1} = \partial M$, and hence

$$F^{n-c+k}C^q(\partial M) = \ker[C^q(\partial M) \to C^q(J^{n-c+k-1})] = \{0\}.$$

Since all sequences involved collapse, we are only interested in the E_2-terms. Thus, let $r = 2,\ p = n - c,\ q = c - r$. Then we have the following modules:

$$\begin{aligned}
Z_2^{n-c,c-r} &= F^{n-c}C^{n-r}(\partial M) \cap (\partial^*)^{-1}(\underbrace{F^{n-c+2}C^{n-r+1}(\partial M)}_{=0}) \\
&= \ker[C^{n-r}(\partial M) \to C^{n-r}(L)] \cap \ker \partial^* \\
&= C^{n-r}(\partial M, L) \cap \ker \partial^*, \\
Z_1^{n-c+1,c-r-1} &= \underbrace{F^{n-c+1}C^{n-r}(\partial M)}_{=0} \cap (\partial^*)^{-1}(F^{n-c+2}C^{n-r+1}(\partial M)) \\
&= \{0\}, \\
B_1^{n-c,c-r} &= F^{n-c}C^{n-r}(\partial M) \cap \partial^*(F^{n-c-1}C^{n-r-1}(\partial M)) \\
&= C^{n-r}(\partial M, L) \cap \partial^*(F^{n-c-1}C^{n-r-1}(\partial M)) \\
&= C^{n-r}(\partial M, L) \cap \partial^*(\ker[C^{n-r-1}(\partial M) \to C^{n-r-1}(L)]) \\
&= C^{n-r}(\partial M, L) \cap \partial^*(C^{n-r-1}(\partial M, L)).
\end{aligned}$$

Thus,

$$\begin{aligned}
E_2^{n-c,c-r} &= \frac{Z_2^{n-c,c-r}}{Z_1^{n-c+1,c-r-1} + B_1^{n-c,c-r}} \\
&= \frac{C^{n-r}(\partial M, L) \cap \ker \partial^*}{C^{n-r}(\partial M, L) \cap \partial^*(C^{n-r-1}(\partial M, L))} \qquad (42) \\
&= H^{n-r}(\partial M, L)
\end{aligned}$$

We know that L is TNCZ in ∂M. Thus the map

$$H^{n-r}(\partial M) \xrightarrow{\text{incl}^*} H^{n-r}(L)$$

is surjective for any r. Therefore, the long exact sequence for cohomology of the pair $(\partial M, L)$ looks like this:

$$\cdots \xrightarrow{0} H^{n-r}(\partial M, L) \xhookrightarrow{j^*} H^{n-r}(\partial M) \xrightarrow{\text{incl}^*} H^{n-r}(L) \xrightarrow{0} \cdots$$

Thus we have a short exact sequence

$$0 \longrightarrow H^{n-r}(\partial M, L) \xhookrightarrow{j^*} H^{n-r}(\partial M) \xrightarrow{\text{incl}^*} H^{n-r}(L) \longrightarrow 0.$$

We would like to show:

Proposition 2.5.21.

$$H^{n-r}(\partial M, L) = \operatorname{im}[H^{n-r}(\partial M, L) \to H^{n-r}(\partial M)].$$

Proof. Let ξ represent an element of $j(H^{n-r}(\partial M, L))$. Then

$$\xi \in \ker[C^{n-r}(\partial M, L) \to C^{n-r+1}(\partial M, L)].$$

Furthermore,

$$\xi \in \operatorname{im}[C^{n-r-1}(\partial M) \to C^{n-r}(\partial M)] \supset \operatorname{im}[C^{n-r-1}(\partial M, L) \to C^{n-r}(\partial M, L)].$$

Thus, ξ also represents an element of $H^{n-r}(\partial M, L)$. On the other hand, let ξ be a non-zero cocycle representing an element of $H^{n-r}(\partial M, L)$. Let ξ be given as a cochain

$$\xi = \sum_\alpha m_\alpha \xi_\alpha,\ m_\alpha \in \mathbb{Q}.$$

If the sets $\{\xi_\alpha \mid \alpha\}$ and

$$\operatorname{im}[C^{n-r-1}(\partial M) \to C^{n-r}(\partial M)] \setminus \operatorname{im}[C^{n-r-1}(\partial M, L) \to C^{n-r}(\partial M, L)]$$

are disjoint, then $[\xi] = j([\xi])$. If there is some α such that

$$\xi_\alpha \in \operatorname{im}[C^{n-r-1}(\partial M) \to C^{n-r}(\partial M)] \setminus \operatorname{im}[C^{n-r-1}(\partial M, L) \to C^{n-r}(\partial M, L)],$$

then there exists a simplex $\sigma \in C_{n-r}(L)$ such that $\xi_\alpha(\sigma) \neq 0$, which is a contradiction. Hence the two sets are always disjoint, and $[\xi] = j([\xi])$. $\square$

Analogously, on the homology side, we have

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}},$$

with

$$\begin{aligned} Z^r_{p,q} &= F_p A_{p+q} \cap d^{-1}(F_{p-r} A_{p+q-1}), \\ B^r_{p,q} &= F_p A_{p+q} \cap d(F_{p+r} A_{p+q+1}). \end{aligned}$$

In our case, $(A, d) = (C_*(\partial M), \partial_*)$ with a filtration

$$F_p(C_*(\partial M))_q = \operatorname{im}[C_*(J^p) \to C_*(\partial M)] \cap C_q(\partial M) = \operatorname{im}[C_*(J^p) \to C_*(\partial M)].$$

We are interested in the case $r = 2,\ p = 0,\ q = r - 1$:

$$\begin{aligned} Z^2_{0,r-1} &= F_0 C_{r-1}(\partial M) \cap (\partial_*)^{-1}(\underbrace{F_{-2} C_{r-2}(\partial M)}_{=0}) \\ &= C_{r-1}(L) \cap \ker \partial_*, \\ Z^1_{-1,r} &= \underbrace{F_{-1} C_{r-1}(\partial M)}_{=0} \cap (\partial_*)^{-1}(F_{-2} C_{r-2}(\partial M)) \\ &= \{0\}, \\ B^1_{0,r-1} &= F_0 C_{r-1}(\partial M) \cap \partial_*(F_1 C_r(\partial M)) \\ &= C_{r-1}(L) \cap \partial_*(C_r(L)). \end{aligned}$$

Thus,

$$\begin{aligned} E^2_{0,r-1} &= \frac{Z^2_{0,r-1}}{Z^1_{-1,r} + B^1_{0,r-1}} \\ &= \frac{C_{r-1}(L) \cap \ker \partial_*}{C_{r-1}(L) \cap \partial_*(C_r(L))} \\ &= H_{r-1}(L). \end{aligned}$$

By [**McC01**, p. 36f], d_∞ is defined as being induced from

$$\ker \partial^* \to H(C^*(\partial M), \partial^*),\ \xi \mapsto [\xi].$$

We know from equation (42) that $E_2^{n-r,c-r} = H^{n-r}(\partial M, L)$. Thus, in our case, d_∞ is the following map:

$$\begin{tikzcd} H^{n-r}(\partial M, L) \arrow[d, equal] \\ E_2^{n-r,c-r} \arrow[d, "d_\infty"'] \\ F^{n-c} H^{n-r} \arrow[d, equal] \\ \operatorname{im}[H^{n-r}(\partial M, L) \to H^{n-r}(\partial M)] \end{tikzcd}$$

But we have seen that the last term is equal to $H^{n-r}(\partial M, L)$ as well. Thus, d_∞ is just the identity. Using Remark 2.5.20, one can show that d^∞ is the identity as well.

2.5.2.2. *The Crossing Lemma.*

LEMMA 2.5.22. *The isomorphism*

$$S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1}$$

can be written as a direct sum of isomorphisms:

$$\begin{tikzcd} H^{n-r}(L) \arrow[d, "C^{n-r}"', "\cong"] & \bigoplus & H^{c-r}(L) \arrow[d, "\cong"', "C^{c-r}"] \\ H_{r-1-n+c}(L) & \bigoplus & H_{r-1}(L) \end{tikzcd}$$

We refer to this behavior as *crossing*.

PROOF. We first show that the isomorphism

$$S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1}$$

has the following restriction:

$$C^{c-r} = (S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1})|_{H^{c-r}(L)} \colon H^{c-r}(L) \xrightarrow{\cong} H_{r-1}(L).$$

Let $\{\sigma_p\}$ be a basis for $H^p(S^{n-c})$. (So $\sigma_p \neq 0$ if and only if $p = 0,\ n-c$.) Let $\{\nu_q^{(j)}\}_j$ be a basis for $H^q(L)$. Then $\{\sigma_p \otimes \nu_q^{(j)}\}_j$ is a basis for $H^p(S^{n-c}) \otimes H^q(L)$. We examine what happens to the elements of this basis under the maps in question. Consider the diagram on the next page.

$$
\begin{tikzcd}
\sigma_{n-c} \otimes \nu^{(j)}_{c-r} \arrow[rr, mapsto, "(-\cap[S^{n-c}])\otimes(-\cap[L])", "\cong"'] \arrow[dd, mapsto] & & \sigma_{n-c} \cap [S^{n-c}] \otimes \nu^{(j)}_{c-r} \cap [L] \arrow[rrr, equal, "!"] & & & \phi^2_*(\phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M]) \\
& H^{n-c}(S^{n-c}) \otimes H^{c-r}(L) \arrow[rrr, "(-\cap[S^{n-c}])\otimes(-\cap[L])", "\cong"'] \arrow[d, "\phi_2^*"', "\cong"] & & & H_0(S^{n-c}) \otimes H_{r-1}(L) & \\
\phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \arrow[d, equal] & E_2^{n-c,c-r} \arrow[r, equal] \arrow[d, equal] & H^{n-r}(\partial M, L) & H_{r-1}(L) \arrow[r, equal] & E^2_{0,r-1} \arrow[u, "\cong"', "\phi^2_*"] \arrow[d, equal] & \phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M] \arrow[uu, mapsto] \arrow[d, equal] \\
\phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \arrow[d, equal] & E_\infty^{n-c,c-r} \arrow[d, "d_\infty = \mathrm{id}"'] & & \mathrm{im}\,[H_{r-1}(L) \to H_{r-1}(\partial M)] \arrow[dr, equal] & E^\infty_{0,r-1} \arrow[d, "d^\infty = \mathrm{id}"] & \phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M] \arrow[d, equal] \\
\phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \arrow[dd, equal] & \frac{F^{n-c}H^{n-r}}{F^{n-c+1}H^{n-r}} \arrow[r, equal] \arrow[d, "i^*"'] & \mathrm{im}\,[H^{n-r}(\partial M, L) \to H^{n-r}(\partial M)] & & \frac{F_0 H_{r-1}}{F_{-1} H_{r-1}} \arrow[d, dashed, "i_*"] & \phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M] \arrow[dd, equal] \\
& H^{n-r}(\partial M) \arrow[rrr, "-\cap[\partial M]", "\cong"'] & & & H_{r-1}(\partial M) & \\
\phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \arrow[rrrrr, mapsto] & & & & & \phi_2^*(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M]
\end{tikzcd}
$$

In the upper right corner, the equality marked with an exclamation mark can be seen as follows:

$$\begin{aligned}\sigma_{n-c} \cap [S^{n-c}] \otimes \nu^{(j)}_{c-r} \cap [L] &= (\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap ([S^{n-c}] \otimes [L]) \\ &= (\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap \phi^2_*([\partial M]) \\ &\overset{!}{=} \phi^2_*(\phi^*_2(\sigma_{n-c} \otimes \nu^{(j)}_{c-r}) \cap [\partial M])\end{aligned}$$

The second line uses that ϕ^2_* is an isomorphism. The third line is the cup-cap formula, which usually requires the morphism ϕ^2_* to be induced from a continuous map ϕ^2. In fact, all that is required is a morphism of chain complexes which induces ϕ^2_*. This can be seen from any proof of said formula – see e. g. [**Hat01**, p. 241]. Hence it holds for ϕ^2_* because that morphism is induced from the morphism of chain complexes

$$\phi\colon F_pC_{p+q}(\partial M) \to C_p(S^{n-c}; C_q(L))$$

as stated in [**McC01**, Lemma 5.24]. Therefore, the diagram on p. 51 commutes. The map

$$H^{n-c}(S^{n-c}) \otimes H^{c-r}(L) \xrightarrow{(-\cap[S^{n-c}])\otimes(-\cap[L])} H_0(S^{n-c}) \otimes H_{r-1}(L)$$

is an isomorphism. Thus, due to commutativity, so is C^{c-r}. Then, the required crossing behavior occurs. ☐

2.6. Duality

Lemma 2.6.1. *For any homotopy equivalence*

$$s\colon L_{<c-k} \overset{\simeq}{\rightarrow} L_{<c-k},$$

the sequence

$$\begin{array}{c} \vdots \\ \downarrow{\scriptstyle 0} \\ H_{r-1}(L_{<c-k}) \\ \downarrow{\scriptstyle (e^L_{c-k})_*\circ s_*} \\ H_{r-1}(L) \\ \downarrow{\scriptstyle (p^L_{c-k})_*} \\ H_{r-1}(L_{\geq c-k}) \\ \downarrow{\scriptstyle 0} \\ H_{r-2}(L_{<c-k}) \\ \downarrow{\scriptstyle (e^L_{c-k})_*\circ s_*} \\ H_{r-2}(L) \\ \downarrow{\scriptstyle (p^L_{c-k})_*} \\ \vdots \end{array}$$

is exact.

Proof. By Proposition 2.4.1, the induced map $(p^L_{c-k})_*$ is surjective in all dimensions, and the induced map $(e^L_{c-k})_*$ is injective in all dimensions. Thus it remains to show exactness at $H_{r-1}(L)$. We have to show

$$\operatorname{im}((e^L_{c-k})_* \circ s_*) = \ker(p^L_{c-k})_*.$$

Consider the involved maps

$$\begin{aligned}(e^L_{c-k})_* \circ s_* &= h'_{L*} \circ \mathrm{incl}_* \circ s_* \\ (p^L_{c-k})_* &= \mathrm{proj}_* \circ h_{L*}\end{aligned}$$

Let $y \in \operatorname{im}((e^L_{c-k})_* \circ s_*)$, say $y = h'_{L*}(s_*(x))$ for some x in $H_{r-1}(L_{<c-k}) = H_{r-1}(E(L)^{(c-k)-1})$. Then

$$(p^L_{c-k})_*(y) = \mathrm{proj}_* \circ h_{L*} \circ h'_{L*} \circ s_*(x) = \mathrm{proj}_*(s_*(x)) = 0,$$

because h_L is a homotopy equivalence. Conversely, let $y \in \ker(p^L_{c-k})_*$. This implies

$$(p^L_{c-k})_*(y) = \mathrm{proj}_* \circ h_{L*}(y) = 0,$$

and in turn, that $h_{L*}(y) \in H_{r-1}(E(L)^{(c-k)-1})$. Thus, with $x = (s_*)^{-1}(h_{L*}(y))$, we have

$$\begin{aligned}(e^L_{c-k})_* \circ s_*(x) &= h'_{L*} \circ \mathrm{incl}_* \circ s_*(x) \\ &= h'_{L*} \circ \mathrm{incl}_* \circ s_* \circ (s_*)^{-1} \circ h_{L*}(y) \\ &= h'_{L*} \circ \mathrm{incl}_* \circ h_{L*}(y) \\ &= h'_{L*} \circ h_{L*}(y) \\ &= y \in \operatorname{im}((e^L_{c-k})_* \circ s_*).\end{aligned}$$

□

LEMMA 2.6.2. *Let $(A_i, f_i)_{i\in\mathbb{Z}}$ and $(B_i, g_i)_{i\in\mathbb{Z}}$ be long exact sequences, and let $k \in \mathbb{Z}$. (The integer k will be used as an offset.) Then the sequence*

$$\cdots \to A_{i+1} \oplus B_{i+1+k} \xrightarrow{f_{i+1}\oplus g_{i+1+k}} A_i \oplus B_{i+k} \xrightarrow{f_i \oplus g_{i+k}} A_{i-1} \oplus B_{i-1+k} \to \cdots$$

is exact.

PROOF. We have to show $\operatorname{im}(f_{i+1} \oplus g_{i+1+k}) = \ker(f_i \oplus g_{i+k})$. Let

$$(a, b) = (f_{i+1}(\overline{a}), g_{i+1+k}(\overline{b})).$$

Then $a \in \operatorname{im} f_{i+1} = \ker f_i$ and $b \in \operatorname{im} g_{i+1+k} = \ker g_{i+k}$. Thus,

$$f_i \oplus g_{i+k}(a, b) = (0, 0),$$

and $\operatorname{im}(f_{i+1} \oplus g_{i+1+k}) \subset \ker(f_i \oplus g_{i+k})$. The converse statement,

$$\operatorname{im}(f_{i+1} \oplus g_{i+1+k}) \supset \ker(f_i \oplus g_{i+k}),$$

is shown in an analogous fashion. Hence the sequence is exact. □

LEMMA 2.6.3. *For $n - c \geq 2$, there is an isomorphism*

$$S_{\mathrm{rel}*} \colon H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \xrightarrow{\cong} H_{r-1}(L_{\geq c-k}) \oplus H_{r-1-n+c}(L_{\geq c-k}).$$

PROOF. We want to use [**Ban10**, Lemma 2.46] to show the existence of $S_{\mathrm{rel}*}$. Therefore, we need to show the existence of a commutative diagram of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \tag{43}$$

such that the rows are exact. The existence of the vertical map between the vector spaces in the middle is then guaranteed by said lemma. The upper row will come from an application of Lemma A.5 to μ, while the lower row will be provided by a direct sum of two copies (for different degrees) of the long exact sequence of Lemma

2.6.1. Naturality of the homological Leray-Serre spectral sequence will provide the vertical maps.

Turning to the specific nature of the upper row, the map $\mu\colon \mathrm{ft}_{<c-k}(\partial M) \to \partial M$ induces a long exact sequence

$$\begin{tikzcd}
H_{r-1}(\mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\mu_*"'] \\
H_{r-1}(\partial M) \arrow[d, "q_*\circ j_{1*}"'] \\
H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\partial_*"'] \\
H_{r-2}(\mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\mu_*"'] \\
H_{r-2}(\partial M)
\end{tikzcd}$$

as was shown in Lemma A.5.

For the lower row, we form a direct sum of two copies of the long exact sequence of Lemma 2.6.1, with an offset in degrees of $-n+c$ and with $s = \mathrm{ft}_{<c-k}(p)_{\mathrm{he}}|$. The result is the following long exact (by Lemma 2.6.2) sequence:

$$\begin{tikzcd}
H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[d, "(e^L_{c-k})_*\circ s_* \oplus (e^L_{c-k})_*\circ s_*"'] \\
H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[d, "(p^L_{c-k})_* \oplus (p^L_{c-k})_*"'] \\
H_{r-1}(L_{\geq c-k}) \oplus H_{r-1-n+c}(L_{\geq c-k}) \arrow[d, "0\oplus 0"'] \\
H_{r-2}(L_{<c-k}) \oplus H_{r-2-n+c}(L_{<c-k}) \arrow[d, "(e^L_{c-k})_*\circ s_* \oplus (e^L_{c-k})_*\circ s_*"'] \\
H_{r-2}(L) \oplus H_{r-2-n+c}(L)
\end{tikzcd}$$

Notice that the maps do actually depend on the degree. Thus the same symbol may stand for different maps on the two summands, respectively.

In order to complete the proof we need to explain the vertical maps of diagram (43). Recall from Remark 2.5.17 the isomorphisms S_*, S_{W*}, and observe that the definition of these isomorphisms entails a choice of splittings such that diagram (36) commutes. Letting $Z_* = Z_*^{r-1} \oplus Z_*^{r-1-n+c}$, we see that the square

$$\begin{tikzcd}
H_{r-1}(\mathrm{ft}_{<c-k}(\partial M)) \arrow[rr, "\mu_*"] & & H_{r-1}(\partial M) \arrow[dd, "\cong"', "S_*"] \\
H_{r-1}(W(\mathrm{ft}_{<c-k}(p))) \arrow[u, "(\mathrm{ft}_{<c-k}(p)_{\mathrm{he}})_*"', "\cong"] \arrow[d, "S_{W*}"', "\cong"] & & \\
H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[rr, "Z_*"] & & H_{r-1}(L) \oplus H_{r-1-n+c}(L)
\end{tikzcd}$$

commutes as well. We set $S_{<*} = S_{W*} \circ (\mathrm{ft}_{<c-k}(p)_{\mathrm{he}})_*^{-1}$. Thus for any $r \in \mathbb{Z}$ we obtain a commutative diagram consisting of rational vector spaces and exact

columns:

$$
\begin{tikzcd}
H_{r-1}(\mathrm{ft}_{<c-k}(\partial M)) \arrow[r, "S_{<*}", "\cong"'] \arrow[d, "\mu_*"'] & H_{r-1}(L_{<c-k}) \oplus H_{r-1-n+c}(L_{<c-k}) \arrow[d, "Z_*"] \\
H_{r-1}(\partial M) \arrow[r, "S_*", "\cong"'] \arrow[d, "q_* \circ j_{1*}"'] & H_{r-1}(L) \oplus H_{r-1-n+c}(L) \arrow[d, "(p^L_{c-k})_* \oplus (p^L_{c-k})_*"] \\
H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\partial_*"'] & H_{r-1}(L_{\geq c-k}) \oplus H_{r-1-n+c}(L_{\geq c-k}) \arrow[d, "0 \oplus 0"] \\
H_{r-2}(\mathrm{ft}_{<c-k}(\partial M)) \arrow[r, "S_{<*}", "\cong"'] \arrow[d, "\mu_*"'] & H_{r-2}(L_{<c-k}) \oplus H_{r-2-n+c}(L_{<c-k}) \arrow[d, "Z_*"] \\
H_{r-2}(\partial M) \arrow[r, "S_*", "\cong"'] & H_{r-2}(L) \oplus H_{r-2-n+c}(L)
\end{tikzcd}
\tag{44}
$$

The middle rectangle commutes by virtue of the fact that the columns are exact. The middle map

$$S_{\mathrm{rel}*} \colon H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \to H_{r-1}(L_{\geq c-k}) \oplus H_{r-1-n+c}(L_{\geq c-k})$$

is obtained by applying [**Ban10**, Lemma 2.46] to this diagram. The 5-lemma then shows $S_{\mathrm{rel}*}$ to be an isomorphism. □

Remark 2.6.4. A less concise statement of Lemma 2.6.3 is that the group

$$H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M))$$

depends on c, k, n and r, as indicated in the following table:

$n-c$	r	Relation to k	$H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \cong$
odd	even	$c-k > r-2$	0
		$r-2 \geq c-k > r-1-n+c$	0
		$r-1-n+c \geq c-k$	$H_{r-1-n+c}(L)$
	odd	$c-k > r-1$	0
		$r-1 \geq c-k > r-2-n+c$	$H_{r-1}(L)$
		$r-2-n+c \geq c-k$	$H_{r-1}(L)$
even	even	$c-k > r-2$	0
		$r-2 \geq c-k > r-2-n+c$	0
		$r-2-n+c \geq c-k$	0
	odd	$c-k > r-1$	0
		$r-1 \geq c-k > r-1-n+c$	$H_{r-1}(L)$
		$r-1-n+c \geq c-k$	$H_{r-1}(L) \oplus H_{r-1-n+c}(L)$

By Lemma A.7, the pair $(M, \partial M)$ and the map $\mu\colon \mathrm{ft}_{<c-k}(\partial M) \to \partial M$ induce the following exact sequence:

$$
\begin{tikzcd}
H_n(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "i_*"'] \\
H_n(M_g, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "d_*"'] \\
H_n(M, \partial M) \arrow[d, "q_* \circ j_{1*} \circ \partial_*"'] \\
H_{n-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "i_*"'] \\
H_{n-1}(M_g, \mathrm{ft}_{<c-k}(\partial M))
\end{tikzcd}
\tag{45}
$$

Notice that ∂_* is the connecting homomorphism of the long exact sequence of the pair $(M, \partial M)$. Towards proving generalized Poincaré duality, it is helpful to show the following lemma:

Lemma 2.6.5 (Lemma 2.45 on p. 181 in [**Ban10**]). *There is an isomorphism $D_<$ which completes the diagram*

$$
\begin{tikzcd}
H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[d, dashed, "\cong"', "D_<"] \\
H_r(M, \partial M) \arrow[r, "q_* \circ j_{1*} \circ \partial_*"] & H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M))
\end{tikzcd}
\tag{46}
$$

to a commutative square. Here, ∂_* is the connecting homomorphism of the long exact sequence of the pair $(M, \partial M)$. (There is no sign here.)

Proof. By [**Ban10**, Lemma 2.45], the square

$$
\begin{tikzcd}
H^{n-r}(M) \arrow[r, "j^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\partial M) \arrow[d, "\cong"', "-\cap[\partial M]"] \\
H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(\partial M)
\end{tikzcd}
\tag{47}
$$

commutes. We construct the isomorphism $D_<$ such that

$$
\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "\mu^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[d, dashed, "\cong"', "D_<"] \\
H_{r-1}(\partial M) \arrow[r, "q_* \circ j_{1*}"] & H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M))
\end{tikzcd}
\tag{48}
$$

commutes. To this, end, consider again the diagram

$$
\begin{tikzcd}
L \arrow[r, hook] & X \arrow[r, "p"] & S^m \\
L_{<k} \arrow[r, hook] \arrow[u, "\mu|"] & \mathrm{ft}_{<k}(X) \arrow[r, "\mathrm{ft}_{<k}(p)"] \arrow[u, "\mu"] & B \arrow[u, "\simeq"', "\mathrm{M}"] \\
L_{<k} \arrow[r, hook] \arrow[u, "\mathrm{ft}_{<k}(p)_{\mathrm{he}}|", "\simeq"'] & W(\mathrm{ft}_{<k}(p)) \arrow[r, "\mathrm{ft}_{<k}(p)_{\mathrm{Hf}}"] \arrow[u, "\mathrm{ft}_{<k}(p)_{\mathrm{he}}", "\simeq"'] & B \arrow[u, "\mathrm{id}_B"']
\end{tikzcd}
$$

this time as input for the cohomological version of Proposition 2.5.16 and Remark 2.5.17. We obtain a commutative square

$$
\begin{tikzcd}
H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[d, "(\mathrm{ft}_{<k}(p)_{\mathrm{he}})^*"', "\cong"] & & H^{n-r}(\partial M) \arrow[ll, "\mu^*"'] \arrow[dd, "\cong"', "S^*"] \\
H^{n-r}(W(\mathrm{ft}_{<k}(p))) \arrow[d, "S_W^*"', "\cong"] & & \\
H^{n-r}(L_{<k}) \oplus H^{c-r}(L_{<k}) & & H^{n-r}(L) \oplus H^{c-r}(L) \arrow[ll, "Z^*"']
\end{tikzcd}
\tag{49}
$$

with

$$Z^* = [(\mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^* \circ (e_k^L)^*] \oplus [(\mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^* \circ (e_k^L)^*].$$

In analogy to the proof of Lemma 2.6.3, we set

$$S_<^* = S_W^* \circ (\mathrm{ft}_{<k}(p)_{\mathrm{he}})^*.$$

We want to complete the diagram

$$
\begin{tikzcd}
H^{n-r}(L) \oplus H^{c-r}(L) \arrow[r, "Z^*"] & H^{n-r}(L_{<k}) \oplus H^{c-r}(L_{<k}) \\
H^{n-r}(\partial M) \arrow[u, "S^*", "\cong"'] \arrow[r, "\mu^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[u, "\cong", "S_<^*"'] \arrow[d, dashed, "\cong"', "D_<"] \\
H_{r-1}(\partial M) \arrow[r, "q_* \circ j_{1*}"] \arrow[d, "S_*"', "\cong"] & H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\cong"', "S_{\mathrm{rel}*}"] \\
H_{r-1}(L) \oplus H_{r-1-m}(L) \arrow[r, "(p_{c-k}^L)_* \oplus (p_{c-k}^L)_*"] & H_{r-1}(L_{\geq c-k}) \oplus H_{r-1-m}(L_{\geq c-k})
\end{tikzcd}
\tag{50}
$$

commutatively, and show the ensuing map $D_<$ to be an isomorphism. The top square is just diagram (49). It commutes. The bottom square is the upper middle rectangle in diagram (44) and commutes as well. The idea is to use the fact that Z^* admits a right inverse map $(Z^*)^{-1}$ to define

$$D_< = (S_{\mathrm{rel}*})^{-1} \circ [(p_{c-k}^L)_* \oplus (p_{c-k}^L)_*] \circ S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \circ (Z^*)^{-1} \circ S_<^*.$$

Commutativity of the bottom and top squares then yields the desired result.

Hence we examine the map Z^*. Consider the first component. We see from Corollary 2.4.2 that $(e_k^L)^*$ is an isomorphism for $n-r<k$ and the zero map else. The same holds for the composition with the isomorphism $(\mathrm{ft}_{<k}(p)_{\mathrm{he}}|)^*$. Hence we can find to any nonzero element $\alpha \in H^{n-r}(L_{<k})$ a unique nonzero preimage $\zeta^{n-r}(\alpha) \in H^{n-r}(L)$ under the first component of Z^*. This defines an injective map

$$\zeta^{n-r} \colon H^{n-r}(L_{<k}) \to H^{n-r}(L).$$

Similar considerations hold for the other component, yielding an injective map

$$\zeta^{c-r} \colon H^{c-r}(L_{<k}) \to H^{c-r}(L).$$

The direct sum of these maps forms an injective inverse map $(Z^*)^{-1}$, and we can thus define $D_<$ as above.

We want to show $D_<$ to be an isomorphism. Concerning injectivity, we see that $S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \circ (Z^*)^{-1} \circ (S_<^*)^{-1}$ is injective. Thus we have to show that $(p_{c-k}^L)_* \oplus (p_{c-k}^L)_*$, restricted to

$$
\begin{aligned}
&S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \circ (Z^*)^{-1}(H^{n-r}(L_{<k}) \oplus H^{c-r}(L_{<k})) \\
=&S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \circ (\zeta^{n-r}(H^{n-r}(L_{<k})) \oplus \zeta^{c-r}(H^{c-r}(L_{<k})))
\end{aligned}
$$

is injective. In order to do this, we notice that Lemma 2.5.18 shows that the isomorphism

$$S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \tag{51}$$

can be written as a direct sum of two isomorphisms:

$$\begin{tikzcd}
H^{n-r}(L) \arrow[d, "C^{n-r}"', "\cong"] & \bigoplus & H^{c-r}(L) \arrow[d, "\cong"', "C^{c-r}"] \\
H_{r-1-n+c}(L) & \bigoplus & H_{r-1}(L)
\end{tikzcd}$$

Notice that this means that the components of the direct sums – as written in diagram (50) – are interchanged by the isomorphism. This "crossing" behavior is crucial. Heuristically, our interest in it can be justified by the fact that the maps do in fact cross if we consider a trivial link bundle.

We still have to show injectivity of $D_<$. To do this, we need to consider many cases. This becomes possible only when we introduce a more efficient notation. As an abstract representation of the diagram

$$\begin{tikzcd}
 & & & H^{n-r}(L_{<k}) & \bigoplus \arrow[dlll, "\zeta^{n-r}\oplus\zeta^{c-r}"'] \arrow[ddd, "S_{\mathrm{rel}*}\circ D_< \circ (S^*_<)^{-1}"] & H^{c-r}(L_{<k}) \\
H^{n-r}(L) \arrow[drr, "C^{n-r}"] & \bigoplus & H^{c-r}(L) \arrow[dll, "C^{c-r}"'] & & & \\
H_{r-1}(L) & \bigoplus \arrow[drrr, "(p^L_{c-k})_*\oplus(p^L_{c-k})_*"'] & H_{r-1-n+c}(L) & & & \\
 & & & H_{r-1}(L_{\geq c-k}) & \bigoplus & H_{r-1-n+c}(L_{\geq c-k})
\end{tikzcd} \tag{52}$$

we can use a symbol consisting of boxes:

$$\begin{tikzcd}
\square \arrow[drr] & \oplus & \square \arrow[dll] & \square & \oplus \arrow[d] \arrow[lll] & \square \\
\square & \oplus & \square \arrow[r] & \square & \oplus & \square
\end{tikzcd}$$

The boxes are drawn black (■) or white (□). A white box is drawn if and only if a necessary condition for the vanishing of the corresponding vector space in diagram (52) arises from the values of (and relations between) c, k, n and r. A black box is drawn in all other cases. Thus, a vector space represented by a black box could still vanish, the black box merely signifies that we can not infer vanishing from the given data alone.

For example, let n be even while r is odd, and let $c - k > r - 1$. Then

$$r - 1 - n + c < r - 1 < c - k,$$

which implies

$$\begin{aligned} r-1-n<-k &\Leftrightarrow \\ n-r+1>k &\Leftrightarrow \\ n-r>k-1 &\Leftrightarrow \\ n-r\geq k & \end{aligned}$$

Also, $c-k>r-1$ implies $c-r\geq k$. Thus we get the following schematic:

$$\begin{tikzcd} \square \arrow[drr] & \oplus & \blacksquare \arrow[dll] & \square \arrow[l] & \oplus \arrow[d] & \blacksquare \\ \blacksquare & \oplus & \square \arrow[r] & \blacksquare & \oplus & \square \end{tikzcd}$$

which we compress to:

$$\begin{matrix} \square\blacksquare\square\blacksquare \\ \blacksquare\square\blacksquare\square \end{matrix}$$

There is an interesting dynamic exhibited here. Roughly speaking, the four boxes on the LHS act as a filter for the four boxes on the RHS. This fact allows us to give the following table representing all possible cases:

n	r	$\Rightarrow$ LHS	$\sim$ to k	$r-1 \,\square\, c-k$	$r-1-n+c \,\square\, c-k$	$n-r \,\square\, k$	$c-r \,\square\, k$	$\Rightarrow$ RHS	total
odd	odd	$\begin{smallmatrix}\blacksquare&\blacksquare\\ \blacksquare&\blacksquare\end{smallmatrix}$	$c-k>r-1$	$<$	$<$	$\geq$	$\geq$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\square&\square&\square\\ \square&\square&\square&\square\end{smallmatrix}$
			$c-k=r-1$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\blacksquare&\square&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&\square\end{smallmatrix}$
			$r-1-n+c \leq r-1 < c-k$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\blacksquare&\square&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&\square\end{smallmatrix}$
			$r-1-n+c=c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\blacksquare&\blacksquare\\ \blacksquare&\blacksquare\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\blacksquare&\blacksquare&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&\blacksquare\end{smallmatrix}$
			$r-1-n+c>c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\blacksquare&\blacksquare\\ \blacksquare&\blacksquare\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\blacksquare&\blacksquare&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&\blacksquare\end{smallmatrix}$
	even	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\square&\square&\square\\ \square&\square&\square&\square\end{smallmatrix}$
even	odd	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$c-k>r-1$	$<$	$<$	$\geq$	$\geq$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\blacksquare&\square&\square\\ \blacksquare&\square&\square&\square\end{smallmatrix}$
			$c-k=r-1$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\blacksquare&\square&\blacksquare\\ \blacksquare&\square&\blacksquare&\square\end{smallmatrix}$
			$r-1-n+c \leq r-1 < c-k$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\blacksquare&\square&\blacksquare\\ \blacksquare&\square&\blacksquare&\square\end{smallmatrix}$
			$r-1-n+c=c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\blacksquare&\square&\blacksquare\\ \blacksquare&\square&\blacksquare&\square\end{smallmatrix}$
			$r-1-n+c>c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\square&\blacksquare\\ \blacksquare&\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\blacksquare&\square&\blacksquare\\ \blacksquare&\square&\blacksquare&\square\end{smallmatrix}$
	even	$\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}$	$c-k>r-1$	$<$	$<$	$\geq$	$\geq$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\square&\square&\square\\ \square&\blacksquare&\square&\square\end{smallmatrix}$
			$c-k=r-1$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\square&\square&\square\\ \square&\blacksquare&\square&\square\end{smallmatrix}$
			$r-1-n+c \leq r-1 < c-k$	$\geq$	$<$	$\geq$	$<$	$\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\square&\square&\square\\ \square&\blacksquare&\square&\square\end{smallmatrix}$
			$r-1-n+c=c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\square&\blacksquare&\square\\ \square&\blacksquare&\square&\blacksquare\end{smallmatrix}$
			$r-1-n+c>c-k$	$\geq$	$\geq$	$<$	$<$	$\begin{smallmatrix}\blacksquare&\square\\ \square&\blacksquare\end{smallmatrix}$	$\begin{smallmatrix}\blacksquare&\square&\blacksquare&\square\\ \square&\blacksquare&\square&\blacksquare\end{smallmatrix}$

Drawing on the data of this table, we will see that $D_<$ is injective in each case. By Corollary 2.4.4, we know that $(p^L_{c-k})_*$ is an isomorphism if $c-k$ is greater or equal to the degree of the homology groups in question, and zero else. It can be seen from the table that in all cases were $(p^L_{c-k})_*$ vanishes, the corresponding input data

$$C^{p-r} \circ \zeta^{p-r}(H^{p-r}(L_{<k})),\ p = c, n,$$

vanishes as well. Hence the compositions

$$(p^L_{c-k})_* \circ C^{n-r} \circ \zeta^{n-r},$$
$$(p^L_{c-k})_* \circ C^{c-r} \circ \zeta^{c-r}$$

are injective, because the first two factors in each composition are injective. Finally,

$$D_< = (S_{\mathrm{rel}*})^{-1} \circ [(p^L_{c-k})_* \oplus (p^L_{c-k})_*] \circ S_* \circ (- \cap [\partial M]) \circ (S^*)^{-1} \circ (Z^*)^{-1} \circ S^*_<.$$

is an injective linear map between vector spaces of the same dimension, and hence an isomorphism.

Now $g^* = \mu^* \circ j^*$ while we also have a factorization

$$H_r(M, \partial M) \xrightarrow{\partial_*} H_{r-1}(\partial M) \xrightarrow{q_* \circ j_{1*}} H_{r-1}(M_\mu, \mathrm{ft}_{<c-k}(\partial M)).$$

Thus we can compose squares (47) and (48) to obtain a commutative square (46).

□

We now come to the main result on duality.

Theorem 2.6.6 (Theorem 2.47 on p. 183 in [**Ban10**]). *Let X be an n-dimensional compact, oriented, stratified pseudomanifold with one singular stratum $\Sigma = S^{n-c}$ of dimension $n - c \geq 2$. The link L is assumed to be an object of the interleaf category* **ICW**, *and we assume the link bundle to have a cellular structure group. We assume X, Σ and L to be oriented compatibly. Let $I^{\bar{p}}X$ and $I^{\bar{q}}X$ be $\bar{p}$- and $\bar{q}$-intersection spaces of X with $\bar{p}$ and $\bar{q}$ complementary perversities. Then there exists a generalized Poincaré duality isomorphism*

$$D\colon \tilde{H}^{n-r}(I^{\bar{p}}X) \xrightarrow{\cong} \tilde{H}_r(I^{\bar{q}}X).$$

Proof. The proof is virtually the same as the proof of [**Ban10**, Theorem 2.47]. We obtain a diagram

$$\begin{array}{ccc}
H^{n-r-1}(M) & \xrightarrow[\cong]{-\cap[M,\partial M]} & H_{r+1}(M, \partial M) \\
\downarrow{\scriptstyle g^*_{\bar{p}}} & & \downarrow{\scriptstyle \lambda} \\
H^{n-r-1}(\mathrm{ft}_{<k}(\partial M)) & \xrightarrow[\cong]{D_<} & H_r(M_{\mu_{\bar{q}}}, \mathrm{ft}_{<c-k}(\partial M)) \\
\downarrow & & \downarrow \\
H^{n-r}(M_{g_{\bar{p}}}, \mathrm{ft}_{<k}(\partial M)) & & H_r(M_{g_{\bar{q}}}, \mathrm{ft}_{<c-k}(\partial M)) \\
\downarrow & & \downarrow \\
H^{n-r}(M) & \xrightarrow[\cong]{-\cap[M,\partial M]} & H_r(M, \partial M) \\
\downarrow{\scriptstyle g^*_{\bar{p}}} & & \downarrow{\scriptstyle \lambda} \\
H^{n-r}(\mathrm{ft}_{<k}(\partial M)) & \xrightarrow[\cong]{D_<} & H_{r-1}(M_{\mu_{\bar{q}}}, \mathrm{ft}_{<c-k}(\partial M))
\end{array}$$

in which the outer squares commute by Lemma 2.6.5. Using this diagram as input for [**Ban10**, Lemma 2.46], we obtain a map

$$D'\colon H^{n-r}(M_{g_{\bar{p}}}, \mathrm{ft}_{<k}(\partial M)) \to H_r(M_{g_{\bar{q}}}, \mathrm{ft}_{<c-k}(\partial M)),$$

which is an isomorphism by the 5-lemma. Composition yields

$$D\colon \tilde{H}^{n-r}(I^{\bar{p}}X) \xrightarrow{\cong} \tilde{H}_r(I^{\bar{q}}X).$$

□

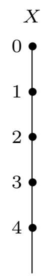
2.7. Cellular Self-Homeomorphisms of CW-complexes

One demand on an interleaf fiber bundle was that the structure group should consist of cellular homeomorphisms with cellular inverses. In this section, we show that for certain CW-complexes, any cellular homeomorphism satisfies the demand that its inverse be cellular.

Let X^n be a finite CW-complex, and let n be the dimension of a cell of maximal dimension. Furthermore let $f\colon X \to X$ be a cellular self-homeomorphism. This means that f is bijective, continuous, open and that

$$f(X^k) \subset X^k \ \forall k \in N.$$

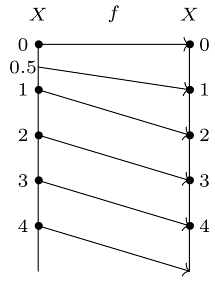
In this setting, is f^{-1} also cellular? If X consists of an infinite number of cells, f^{-1} is in general not cellular. Consider the following example: Take X to be the CW-complex composed of 0-cells in correspondence to the natural numbers $\mathbb{N}$, and 1-cells connecting the 0-cells corresponding to k and $k+1$, for all $k \in \mathbb{N}$.



Define a map $f\colon X \to X$ by setting

$$f(0) = 0, f(0.5) = 1, f(1) = 2, f(2) = 3,$$

and so forth, with f continuous on the intervals between these points. This yields:



The map f is clearly continuous, open and bijective. It is also cellular, since $f(X^1) \subset X^1$ and $f(X^0) \subset X^0$. But the inverse f^{-1} maps $1 \in X^0$ to $0.5 \notin X^0$, and is thus not cellular. So the inverses of cellular self-homeomorphisms of *infinite* CW-complexes are not always cellular. However, it can be shown that the inverse f^{-1} is always cellular for X a finite CW-complex:

Proposition 2.7.1. *Let X^n be a CW-complex with only finitely many cells, and let $f\colon X \to X$ be a cellular self-homeomorphism. Then f^{-1} is also cellular.*

Proof. The proof is by induction over skeletons. Since there are only finitely many 0-cells and since $f(X^0) \subset X^0$, the restriction of f to X^0 is a permutation. Hence f^{-1} is clearly cellular on X^0. Assume now that f^{-1} has been shown to be cellular on X^{k-1}, for some $k \geq 1$. We want to show that f^{-1} is also cellular on X^k.

Claim 0: If f^{-1} is cellular on X^{k-1}, then $f|_{X^{k-1}}\colon X^{k-1} \to X^{k-1}$ is bijective. To see this, note that $f|_{X^{k-1}}$ is the restriction of an injective map, and hence injective.

It is also surjective: consider $x \in X^{k-1}$. We know that f^{-1} is cellular on X^{k-1},

$$f^{-1}(X^{k-1}) \subset X^{k-1}.$$

Hence the preimage of x under f must also lie in X^{k-1}. This makes $f|_{X^{k-1}}$ surjective. Accordingly, Claim 0 holds.

Claim 1: If f^{-1} is cellular on X^{k-1}, then the image of any open k-cell under f is contained in exactly one open k-cell. Consider an open k-cell $e = e^k$. Note that $f(e) \subset X^k$, because f is cellular. Claim 0 shows that

$$f(e) \subset X^k - X^{k-1},$$

because $f|_{X^{k-1}}$ is bijective - there is “no space left” for $f(e)$ in the $(k-1)$-skeleton. But $X^k - X^{k-1}$ is a disjoint union of open k-cells. Accordingly, $f(e)$ is contained in exactly one open k-cell, for e is connected and so, in turn, is $f(e)$ (connectedness is a topological property). This shows Claim 1 to be true.

Claim 2: If $e, e' \in X^k$ are open k-cells such that $f(e) \subset e'$, then $f|_e \colon e \to e'$ is surjective. In order to prove this, we will show that $f(e)$ is both open and closed in e', which implies $f(e) = e'$ since e' is connected. To see this, consider the characteristic maps

$$\phi, \phi' \colon (D^k, S^{k-1}) \to (X^k, X^{k-1})$$

of e and e', respectively. The restrictions of ϕ and ϕ' to $\mathring{D}^k$ are homeomorphisms.

Furthermore, notice that $X^k \subset X$ and $e, e' \subset X^k$ are equipped with the subspace topology, respectively. Thus $f|_{X^k} \colon X^k \to X$ is continuous, and by further restriction to the subspace e of X^k, we see that $f|_e \colon e \to X$ is continuous as well. By assumption it holds that $f(e) \subset e'$, and hence the map $\tilde{f} \colon e \to e'$, obtained by restricting the range of $f|_e \colon e \to X$ to e' is continuous.

This enables us to form a composition

$$F \colon \mathring{D}^k \xrightarrow{\phi} e \xrightarrow{\tilde{f}} e' \xrightarrow{(\phi')^{-1}} \mathring{D}^k.$$

Now $\tilde{f}$ agrees with f on e, and hence is the restriction of an injective map. Thus it is still injective. Since $\phi, {\phi'}^{-1}$ restricted to $\mathring{D}^k$ are both homeomorphisms, and in particular, injective, this makes F injective. It is also continuous: Both ϕ and ${\phi'}^{-1}$ are continuous, and $\tilde{f}$ was shown to be continuous above.

Given that $F \colon \mathring{D}^k \to \mathring{D}^k \subset \mathbb{R}^k$ is both continuous and injective, an application of the theorem on invariance of domain yields that $F(\mathring{D}^k)$ is open in $\mathbb{R}^k$ and hence in $\mathring{D}^k$, and furthermore that F is a homeomorphism. Note that

$$\begin{aligned} f(e) &= \tilde{f} \circ \phi(\mathring{D}^k) && \text{(because } f \text{ and } \tilde{f} \text{ agree on } e\text{)} \\ &= \phi' \circ {\phi'}^{-1} \circ \tilde{f} \circ \phi(\mathring{D}^k) && \text{(because } \phi' \text{ is a homeomorphism)} \\ &= \phi' \circ F(\mathring{D}^k). \end{aligned}$$

The image of the open (in $\mathring{D}^k$) set $F(\mathring{D}^k)$ under the homeomorphism ϕ' is open in e'. Thus we have shown that $f(e)$ is open in e'.

On the other hand, $\bar{e} = e \cup \partial e$ is closed in X. Hence $f(\bar{e}) = f(e) \cup f(\partial e)$ is closed in X. The fact that X is endowed with the weak topology implies that $f(\bar{e}) = f(\bar{e}) \cap X^k$ is closed. Hence $f(\bar{e}) \cap e'$ is closed in e', for e' is a subspace of X^k. But

$$f(\bar{e}) \cap e' = f(e) \cap e' = f(e),$$

since $f(\partial e) \subset X^{k-1}$.

Thus $f(e)$ is both closed and open in the connected space e'. This implies that $f(e)$ is equal to e' (for $f(e) = \varnothing$ is impossible), and Claim 2 is proven.

In particular, Claim 2 shows that no two open k-cells are mapped into the same open k-cell. We claim that even more is true, namely that f acts by permutation on the set of k-cells. In order to see this, note that there are only finitely many open k-cells by assumption. Claim 1 shows that each open k-cell is mapped into exactly one other open k-cell, and the finite number of open k-cells ensures that each open k-cell is the image of another open k-cell under f. Hence f is a permutation on the set of open k-cells, and, by Claim 2, f restricts to a surjection on each open k-cell. In other words, $f(X^k - X^{k-1}) = X^k - X^{k-1}$, which implies $f^{-1}(X^k - X^{k-1}) = X^k - X^{k-1}$. Combining this with the assumption that $f^{-1}(X^{k-1}) \subset X^{k-1}$, we know that $f^{-1}(X^k) \subset X^k$. This is the definition of f^{-1} being cellular on X^k, and the induction step is finished. $\square$

Definition 2.7.2 (p. 56 in [**FP90**]). Let $f\colon X \to Y$ be a map between CW-complexes. If f is cellular and takes every open cell of X onto an open cell of Y, it is called regular.

Remark 2.7.3. Let X^n be a CW-complex with only finitely many cells, and let $f\colon X \to X$ be a cellular self-homeomorphism. Then f is regular.

Accordingly, we can modify one of the conditions in Theorem 2.6.6. If the link $L \in \mathrm{Ob}\,\mathbf{ICW}$ consists of finitely many cells, we need not demand that the link bundle have a structure group consisting exclusively of cellular homeomorphisms with cellular inverses. It suffices to demand that the structure group consist exclusively of cellular homeomorphisms.

CHAPTER 3

Non-simple Systems of Local Coefficients

3.1. Introduction

The previous chapters focused on links which could be truncated in arbitrary degrees. Accordingly, we obtained generalized Poincaré duality for arbitrary complementary perversities. We now change this viewpoint. For a given pair of complementary perversities, we want to show that the homology of the intersection space of a pseudomanifold satisfies Poincaré duality. In order to do this, we need to be able to truncate the corresponding link in only two degrees. For the given pair of perversities, we can thus relax our demands on links – they need only admit truncation in the two degrees mentioned. In this setting, we are able to construct a right inverse to

$$\mathrm{incl}^*\colon H^*(\partial M) \to H^*(\mathrm{ft}_{<k}\,(\partial M))$$

by means of sheaf theory. A proof of generalized Poincaré duality will then proceed in a manner similar to the last chapter.

The existence of a right inverse to the morphism incl^* implies the surjectivity of incl^*. It may come as a surprise that this holds in the absence of any further demands on the fiber bundles involved. But the proof of surjectivity of incl^* is a byproduct of the sheaf theoretic Ansatz pursued below.

We now proceed to introduce the links in question.

Definition 3.1.1. Let $k \in \mathbb{N}$, and let N be a CW-complex. If in the homological cellular chain complex $(H_*(N^*, N^{*-1}), d_*)$ the boundary map

$$d_k\colon H_k(N^k, N^{k-1}) \to H_{k-1}(N^{k-1}, N^{k-2})$$

vanishes, then we refer to N as *k-admissible*.

For a given k-admissible space N, we have

$$H_{k-1}(N^{k-1}) = \frac{\ker d_{k-1}}{\mathrm{im}\, d_k} = \ker d_{k-1} = H_{k-1}(N)$$

while $H_i(N^{k-1}) = H_i(N)$ for $i < k-1$, and $H_i(N^{k-1})$ vanishes for $i \geq k$. Hence we call

$$t_{<k}(N) = N^{k-1}$$

the *spatial homology truncation* in degree k of N. The usual shorthand $N_{<k} = t_{<k}(N)$ is employed. The spatial homology truncation is a subcomplex and as such, a closed subspace of N.

Given two k-admissible spaces N and O, as well as a cellular map $f\colon N \to O$, we can define

$$t_{<k}(f) = f^{k-1}\colon N_{<k} \to O_{<k}.$$

If $g \in \mathrm{Homeo}_{CW}(N)$, then $t_{<k}(g)$ is in $\mathrm{Homeo}_{CW}(N_{<k})$. To see this, proceed as follows:

Remark 3.1.2. (1) Truncation of k-admissible spaces preserves some homeomorphisms. Given $f \in \mathrm{Homeo}_{CW}(N)$, i.e. a cellular homeomorphism

$$f\colon N \to N$$

with cellular inverse, we note that

$$t_{<k}(f)\colon t_{<k}(N) \to t_{<k}(N)$$

is injective and continuous. It must also be surjective, as cellularity of f^{-1} implies $f^{-1}(t_{<k}(N)) \subset t_{<k}(N)$. Furthermore, $(t_{<k}(f))^{-1} = f^{-1}|_{t_{<k}(N)}$ is continuous. Thus, $t_{<k}(f)$ is again a (cellular) homeomorphism $N_{<k} \to N_{<k}$ with cellular inverse.

(2) Assuming that N is a finite CW-complex, $\mathrm{Homeo}_{CW}(N)$, endowed with the compact-open topology, is a topological group, see [**Ban10**, p. 86]. The same holds for any skeleton N^k, and the morphism,

$$t_{<k}\colon \mathrm{Homeo}_{CW}(N) \to \mathrm{Homeo}_{CW}(N_{<k})$$

is a homomorphism of topological groups. In particular, it is a continuous map.

Definition 3.1.3. Let $(\overline{p}, \overline{q})$ be a pair of complementary perversities, and let N^n be a CW-complex. If N is both $(n - \overline{p}(n+1))$- and $(1 + \overline{p}(n+1))$-admissible, then we refer to N as $(\overline{p}, \overline{q})$-*admissible*.

Thus if N is $(\overline{p}, \overline{q})$-admissible these are the two degrees in which we can truncate the homology of N.

Definition 3.1.4. A $(\overline{p}, \overline{q})$-*admissible bundle* is a fiber bundle $\xi = (N, Y, p, A)$ such that

(1) N is oriented, $(\overline{p}, \overline{q})$-admissible and its CW-structure is finite,
(2) A is a closed topological manifold, and
(3) the structure group of ξ takes values in $\mathrm{Homeo}_{CW}(N)$.

3.2. Cohomology with Compact Support and Cap Products

We need the Künneth theorem for cohomology with compact support.

Theorem 3.2.1. *Let M^m and N^n be oriented manifolds. Then*

$$H^i_c(M \times N; \mathbb{Q}) \cong \bigoplus_{i=p+q} H^p_c(M; \mathbb{Q}) \otimes H^q_c(N; \mathbb{Q}).$$

Proof. The proof follows from [**BT82**, Exercise I.5.12]. □

Proposition 3.2.2. *Let (X, Y) be a pair of compact spaces with X being a Hausdorff space, and let $f\colon Y \hookrightarrow X$ be an inclusion. Then f is proper.*

Proof. Let $K \subset X$ be a compact subset. A compact subset of a Hausdorff space is closed. Then $f^{-1}(K)$ is closed in Y because f is continuous. The preimage $f^{-1}(K)$ of K under f is thus a closed set in a compact space, and hence compact itself. Thus f is proper. □

3.3. Cap Products

We need some results on homological truncations. Most of these results are versions of results in [**Ban10**, Section 2.9], adapted to the setting of cohomology with compact support and $(\overline{p}, \overline{q})$-admissible spaces.

Proposition 3.3.1 (Lemma 2.42 on p. 177 in [**Ban10**]). *Let N be $(\overline{p}, \overline{q})$-admissible and set $k = n - \overline{p}(n+1)$. Then the map*

$$\pi_*\colon H_r(N) \to H_r(N, N_{<k})$$

induced on homology by the inclusion is an isomorphism when $r \geq k$, while

$$H_r(N, N_{<k}) = 0$$

when $r < k$.

We need the following result, which we obtain by replacing every occurrence of $H^*(N_{<k})$ with $H_c^*(N_{<k})$ (since these groups are equal for the space is compact) in [**Ban10**, Proposition 2.43]. From now on, for a given k-admissible space N, we will let z denote the inclusion $N_{<k} \hookrightarrow N$.

Proposition 3.3.2 (Proposition 2.43 on p. 178 in [**Ban10**]). *Let N^n be an oriented, closed, connected manifold and a $(\overline{p}, \overline{q})$-admissible space. Let $k = n - \overline{p}(n+1)$.*

(1) *There exists a cap product*

$$H_c^{n-r}(N_{<k}) \otimes H_n(N) \xrightarrow{\cap} H_r(N, N_{<n-k+1})$$

such that

$$\begin{tikzcd} H_c^{n-r}(N_{<k}) \otimes H_n(N) \arrow[r, "\cap"] & H_r(N, N_{<n-k+1}) \\ H_c^{n-r}(N) \otimes H_n(N) \arrow[u, "z^* \otimes \mathrm{id}"] \arrow[r, "\cap"] & H_r(N) \arrow[u, "\pi_*"'] \end{tikzcd}$$

commutes.

(2) *Capping with the fundamental class $[N] \in H_n(N)$ is an isomorphism*

$$- \cap [N] \colon H_c^{n-r}(N_{<k}) \xrightarrow{\cong} H_r(N, N_{<n-k+1}).$$

The cap product of Proposition 3.3.2 is defined as

$$\begin{gathered} \cap \colon H_c^{n-r}(N_{<k}) \otimes H_n(N) \to H_r(N, N_{<n-k+1}), \\ \xi \cap x \mapsto \begin{cases} \pi_*((z^*)^{-1}(\xi) \cap x), & n - r < k, \\ 0, & n - r \geq k. \end{cases} \end{gathered} \tag{53}$$

Proposition 3.3.3 (Proposition 2.44 on p. 179 in [**Ban10**]). *Let N^n be an oriented, closed, connected manifold and a $(\overline{p}, \overline{q})$-admissible space. Let U^s be an oriented manifold. Let $k = n - \overline{p}(n+1)$.*

(1) *There exists a cap product*

$$H_c^{s+n-r}(U \times N_{<k}) \otimes H_{s+n}(U \times N) \xrightarrow{\cap} H_r(U \times (N, N_{<n-k+1}))$$

such that

$$\begin{tikzcd} H_c^{s+n-r}(U \times N_{<k}) \otimes H_{s+n}(U \times N) \arrow[r, "\cap"] & H_r(U \times (N, N_{<n-k+1})) \\ H_c^{s+n-r}(U \times N) \otimes H_{s+n}(U \times N) \arrow[u, "(\mathrm{id}_U \times z)^* \otimes \mathrm{id}"] \arrow[r, "\cap"] & H_r(U \times N) \arrow[u, "\mathrm{incl}_*"'] \end{tikzcd} \tag{54}$$

commutes.

(2) *Capping with the fundamental class $[U \times N] \in H_{s+n}(U \times N)$ is an isomorphism*

$$- \cap [U \times N] \colon H_c^{s+n-r}(U \times N_{<k}) \xrightarrow{\cong} H_r(U \times (N, N_{<n-k+1}))$$

The proof is the proof of Proposition 2.44 in [**Ban10**], with cohomology with compact support substituted for singular cohomology.

3.4. Setting

We are now able to state the intention of this chapter in greater detail. Firstly, we restrict the class of pseudomanifolds under consideration in several ways. Assume that X is an n-dimensional, compact, stratified topological pseudomanifold with two strata

$$X = X_n \supset X_{n-c}.$$

Let

$$\eta = (\mathring{\text{cone}}\,(L), U_\Sigma, q, \Sigma)$$

be a fiber bundle with total space an open neighborhood U_Σ of the singular set Σ of X. Remove the total space of η from X to obtain

$$M = X - U_\Sigma. \tag{55}$$

As before, M is a compact manifold-with-boundary. Assume that the link bundle

$$\xi = (L, \partial M, p, \Sigma) \tag{56}$$

of X is (p, q)-admissible, and furthermore that there are open subsets $U, V \subset \Sigma$ such that ξ restricts to a trivial bundle over each of U and V.

To sum up, we demand that X^n be a stratified pseudomanifold which

(1) is compact and composed of two strata, has a
(2) $(\overline{p}, \overline{q})$-admissible link bundle, which
(3) trivializes over U, trivializes over V and $U \cup V = \Sigma$.

It follows that the link L^{c-1} is $(\overline{p}, \overline{q})$-admissible. By definition, it may be truncated in the degrees

$$\begin{aligned} c - 1 - \overline{p}(c - 1 + 1) &= c - 1 - \overline{p}(c), \\ 1 + \overline{p}(c - 1 + 1) &= 1 + \overline{p}(c). \end{aligned}$$

Set $k = 1 + \overline{p}(c)$. Thus, k and $c - k = c - 1 - \overline{p}(c)$ are the two degrees in which the link admits spatial homology truncation.

Definition 3.4.1. Given ξ as in equation (56), the fiberwise homology truncation of ξ in degree s for $s = k$ or $s = c - k$ is the bundle

$$\operatorname{ft}_{<s}(\xi) = (L_{<s}, \operatorname{ft}_{<s}(\partial M), \operatorname{ft}_{<s}(p), \Sigma),$$

with entries defined as follows:

- $L_{<s}$ is the spatial homology truncation in degree s of L.
- Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Delta}$ be a transition system (in the sense of [**Hus94**, Definition 5.2.4]) for ξ. Then $\{t_{<s} \circ g_{\alpha\beta}\}_{\alpha,\beta\in\Delta}$ is a transition system as well. We use this as input for [**Hus94**, Theorem 3.2] and obtain a bundle $\operatorname{ft}_{<s}(\xi)$ with total space
$$\operatorname{ft}_{<s}(\partial M) = \bigsqcup_{\alpha,\beta\in\Delta} U_\alpha \times L_{<s}/t_{<s} \circ g_{\alpha\beta}.$$
- The map $\operatorname{ft}_{<s}(p)$ is the restriction $p|_{\operatorname{ft}_{<s}(\partial M)}$.

Proposition 3.4.2. *For both, $s = k$ and $s = c-k$, the truncated bundle $\operatorname{ft}_{<s}(\xi)$ is a fiber bundle.*

Proof. It is a fiber bundle due to [**Hus94**, Theorem 3.2]. □

For $s = k$, we obtain the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, hook] & \partial M \arrow[r, "p"] & \Sigma \arrow[d, equal] \\ L_{<k} \arrow[u, hook, "\mu| = z = \mathrm{incl}"] \arrow[r, hook] & \operatorname{ft}_{<k}(\partial M) \arrow[u, hook, "\mu = \mathrm{incl}"] \arrow[r, "\operatorname{ft}_{<k}(p)"] & \Sigma \end{tikzcd}$$

So we have a composition of trivial inclusions

$$\operatorname{ft}_{<k}(\partial M) \overset{\mu}{\hookrightarrow} \partial M \overset{j}{\hookrightarrow} M$$

which we denote

$$g_{\overline{p}} = j \circ \mu \colon \operatorname{ft}_{<k}(\partial M) \hookrightarrow M.$$

The same considerations apply to $\operatorname{ft}_{<c-k}(\xi)$. Analogously to [**Ban10**, Section 2.9], we can now introduce the homotopy cofiber of g as intersection space.

DEFINITION 3.4.3 (Definition 2.41 on p. 177 in [**Ban10**]). Let X, $(\overline{p}, \overline{q})$, M and L be defined as described above. Then the perversity $\overline{p}$ intersection space $I^{\overline{p}}X$ of X is defined to be

$$I^{\overline{p}}X = \operatorname{cone}(g_{\overline{p}}) = M \cup_{g_{\overline{p}}} \operatorname{cone}(\operatorname{ft}_{<k}(\partial M)).$$

Analogously, we set $I^{\overline{q}}X = \operatorname{cone}(g_{\overline{q}}) = M \cup_{g_{\overline{q}}} \operatorname{cone}(\operatorname{ft}_{<c-k}(\partial M))$.

3.5. Metric Spaces

DEFINITION 3.5.1 (p. 261 in [**Mun00**]). A space W is *locally metrizable* if every point x of W has a neighborhood U that is metrizable in the subspace topology.

THEOREM 3.5.2 (Smirnov, e.g. Theorem 42.1 on p. 261 in [**Mun00**]). *A space W is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.*

The topological manifold Σ is locally metrizable. It is also compact and hence paracompact. We have shown that it is Hausdorff. Hence it is metric by Smirnov.

PROPOSITION 3.5.3. *The total space* $\operatorname{ft}_{<k}(\partial M)$ *is locally contractible.*

PROOF. In [**Hur55**, p. 957], there is the definition of a *regular* Hurewicz fibration. While the details need not concern us here, the important fact is this: By [**Hur55**, Theorem 3] any fibration with a metric base space is regular. Hence $\operatorname{ft}_{<k}(\xi)$ is regular. By [**AF62**, Theorem 4.13], a regular fibration such that each fiber as well as the base is an *absolute neighborhood retract* (ANR) has a locally contractible total space. Hence the result follows if we can show that Σ and the fibers are ANRs.

By [**Hav73**, Theorem on p. 281], the metric, locally contractible n-manifold Σ is an ANR. At each point $b \in \Sigma$, the fiber $\operatorname{ft}_{<k}(p)^{-1}(b) = L_{<k}$ is a CW-complex and hence locally contractible. The same argument from [**Hav73**] implies that it is an ANR. $\square$

3.6. Sheaf Theory

This section will collect some results and definitions from sheaf theory. Morphisms of sheaves over the same base space induce morphisms on sheaf cohomology. If the sheaves do not share the same base space, it is still possible to induce maps on sheaf cohomology, using the following definition:

DEFINITION 3.6.1 (Definition I.4.2 on p. 14 in [**Bre97**]). Given sheaves $\mathbf{A}$ and $\mathbf{B}$ on X and Y respectively, as well as a continuous map $f \colon X \to Y$, an *f-cohomomorphism* $g \colon \mathbf{B} \rightsquigarrow \mathbf{A}$ is a collection of homomorphisms

$$g = \{g_x \colon \mathbf{B}_{f(x)} \to \mathbf{A}_x\}_{x \in X}$$

such that for any section $s \in \mathbf{B}(U)$ the function $x \mapsto g_x(s(f(x)))$ is a section of $\mathbf{A}$ over $f^{-1}(U)$ (i.e. this function is continuous).

Note that this definition does not preclude an f-cohomomorphism from assigning multiple values to the same argument. If f is injective, this does not happen. Furthermore the f-cohomomorphism is only defined on all of $\mathbf{B}$ if f is surjective. Accordingly, we see that an f-cohomomorphism $\mathbf{B} \rightsquigarrow \mathbf{A}$ is not in general a function $\mathbf{B} \to \mathbf{A}$.

REMARK 3.6.2. (1) Given a map $f \colon X \to Y$ and a sheaf $\mathbf{B}$ on Y, there is a canonical f-cohomomorphism

$$f^* = \{f^*_x\} \colon \mathbf{B} \rightsquigarrow f^*\mathbf{B},$$

given as

$$f^*_x \colon \mathbf{B}_{f(x)} \to (f^*\mathbf{B})_x,\ b \mapsto [x, b]_{f^*\mathbf{B}},$$

see [**Bre97**, p. 12-14]. Analogously, given a sheaf $\mathbf{A}$ on X, there is a canonical f-cohomomorphism $f_*\colon f_*\mathbf{A} \rightsquigarrow \mathbf{A}$.

(2) Given a pair (X, Y) of topological spaces such that Y is closed in X as well as an abelian group G, the canonical inclusion
$$i = \mathrm{incl}\colon Y \hookrightarrow X$$
satisfies $i^*(\underline{G}_X) = \underline{G}_Y$ by [**Har08**, Section 4.4.3].

(3) Given (X, Y) as in (2) and using (1), we see in [**Bre97**] that i induces a canonical i-cohomomorphism
$$i^* = \{i^*_y\}\colon \underline{G}_X \rightsquigarrow \underline{G}_Y = i^*(\underline{G}_X),$$
with
$$i^*_y\colon (\underline{G}_X)_{i(y)} \to (\underline{G}_Y)_y$$
the morphism induced from id_G. This may still fail to be a function, for i may not be surjective.

(4) An f-cohomomorphism g induces a morphism on sheaf cohomology, which is denoted g^*. Following [**Bre97**], we commit an abuse of notation: An inclusion i as in (2) induces an i-cohomomorphism i^*, which induces a morphism on sheaf cohomology – which is again denoted i^*, and should not be confused with the morphism i^* induced by i on singular cohomology. The context will serve to clarify what the symbol i^* denotes in each instance.

As we often quote results from [**Bre97**], we need to introduce supports.

Definition 3.6.3 (Definition I.6.1 on p. 21 in [**Bre97**]). A *family of supports* on a space X is a family $\mathbf{\Phi}$ of closed subsets of X such that:

(1) a closed subset of a member of $\mathbf{\Phi}$ is a member of $\mathbf{\Phi}$;
(2) $\mathbf{\Phi}$ is closed under finite unions.

A family of supports $\mathbf{\Phi}$ is said to be *paracompactifying* if

(3) each element of $\mathbf{\Phi}$ is paracompact;
(4) each element of $\mathbf{\Phi}$ has a (closed) neighborhood which is in $\mathbf{\Phi}$.

A trivial example is the family of all closed subsets of a space.

Remark 3.6.4. (1) In the notation of [**Bre97**], the family of supports $\mathbf{\Phi}$ used in the definition of a cohomology theory H^* is indicated as a subscript, $H^*_{\mathbf{\Phi}}$. For example, given a sheaf $\mathbf{A}$ over a space X and an injective resolution $\mathbf{A} \to \mathbf{I}^\bullet$, the i-th sheaf cohomology group is defined in [**Bre97**, Definition II.2.2] as
$$H^i_{\mathbf{\Phi}}(X; \mathbf{A}) = H^i(\Gamma_{\mathbf{\Phi}}(\mathbf{I}^\bullet)),$$
with
$$\Gamma_{\mathbf{\Phi}}(\mathbf{I}^j) = \{s \in \mathbf{I}^j(X) \mid \operatorname{supp} s \in \mathbf{\Phi}\}.$$
We mostly work with the family of all closed subsets of X, which is denoted *cld* in [**Bre97**], and usually dropped from notation. Thus, if we quote results from [**Bre97**], and write H^*, the original statement in [**Bre97**] will likely be stated in terms of $H^*_{\mathbf{\Phi}}$, but will be valid for $\mathbf{\Phi} = cld$.

(2) For a locally compact space, the family of all compact subsets c is paracompactifying, see [**Bre97**].
(3) For a compact Hausdorff space, $c = cld$. Again, see [**Bre97**].

Some of the results quoted from [**Bre97**] require the family of supports to be paracompactifying. Hence we need to check this.

Proposition 3.6.5. *For $\partial M, \mathrm{ft}_{<k}(\partial M)$ and Σ, the family $c = cld$ is paracompactifying.*

Proof. The compact manifold ∂M is Hausdorff, as is its compact subspace $\mathrm{ft}_{<k}(\partial M)$. The singular set Σ is a closed manifold, and hence also compact and Hausdorff. Any compact Hausdorff space is locally compact. Thus each of the three mentioned spaces is compact, locally compact and Hausdorff. Hence in each case, $c = cld$ is paracompactifying. $\square$

Definition 3.6.6 (p. 35 in [**Bre97**]). A space X is said to be HLC^n_L (*homologically locally connected*) if for each $x \in X$ and neighborhood U of x, there is a neighborhood $V \subset U$ of x, depending on p, such that the homomorphism ${}_\Delta\tilde{H}_p(V;L) \to_\Delta \tilde{H}_p(U;L)$ is trivial for $p \leq n$.

When the coefficients L are clear from context, they are dropped from notation. If X is HLC^n for all n, we say that is X is *HLC*.

Remark 3.6.7 (p. 35 in [**Bre97**]). Any locally contractible space is HLC. As is any manifold or CW-complex.

Theorem 3.6.8 (Theorem I.1.1 on p. 184 in [**Bre97**]). *Let* $\mathbf{A}$ *be a locally constant sheaf. Then there exist natural transformations of functors (of* X *as well as of* $\mathbf{A}$*)*

$$H^*_{\mathbf{\Phi}}(X;\mathbf{A}) \xrightarrow{\theta} {}_S H^*_{\mathbf{\Phi}}(X;\mathbf{A}) \xleftarrow{\psi^*} {}_\Delta H^*_{\mathbf{\Phi}}(X;\mathbf{A}).$$

When $\mathbf{\Phi}$ *is paracompactifying, the groups* ${}_\Delta H^*_{\mathbf{\Phi}}(X;\mathbf{A})$ *are the classical singular cohomology groups. If in addition to* $\mathbf{\Phi}$ *being paracompactifying,* X *is HLC, then both* θ *and* ψ^* *are isomorphisms.*

3.7. Duality

It is our intention to construct for a given $r \in \mathbb{N}$ a cap product

$$H^{n-1-r}(\mathrm{ft}_{<k}(\partial M)) \otimes H_{n-1}(\partial M) \overset{\cap}{\dashrightarrow} H_r(\partial M, \mathrm{ft}_{<c-k}(\partial M)). \tag{57}$$

In order to do so, we consider the situation described in the following diagram:

$$\begin{tikzcd}
\underline{\mathbb{Q}}_Y \arrow[r, equal] \arrow[d, "\pi_Y"'] & i^*\underline{\mathbb{Q}}_X \arrow[dl, hook'] & \underline{\mathbb{Q}}_X \arrow[l, squiggly, "i^*"'] \arrow[d, "\pi_X"] \\
Y \arrow[rr, hook, "i"] \arrow[d, equal] & & X \arrow[d, equal] \\
\mathrm{ft}_{<k}(\partial M) \arrow[rr, hook, "i"] \arrow[d, "q=\mathrm{ft}_{<k}(p)"'] & & \partial M \arrow[d, "p"] \\
\Sigma \arrow[rr, equal] & & \Sigma
\end{tikzcd} \tag{58}$$

The topmost equality is due to [**Har08**, Section 4.4.3]. In this diagram, commutativity holds where it is defined. The topmost part involves the canonical i-cohomomorphism $\underline{\mathbb{Q}}_X \rightsquigarrow i^*\underline{\mathbb{Q}}_X$, and accordingly commutativity of this part is not defined.

We consider the canonical injective resolutions:

$$\begin{aligned} \underline{\mathbb{Q}}_X &\to \mathbf{I}^\bullet, \\ \underline{\mathbb{Q}}_Y = i^*\underline{\mathbb{Q}}_X &\to \mathbf{J}^\bullet, \\ i_*i^*\underline{\mathbb{Q}}_X &\to \mathbf{K}^\bullet. \end{aligned} \tag{59}$$

Since we can regard any sheaf as a complex concentrated in degree zero, the resolutions $\mathbf{I}^\bullet, \mathbf{J}^\bullet$ and $\mathbf{K}^\bullet$ double as injective resolutions of the complexes $\underline{\mathbb{Q}}^\bullet_X, i^*\underline{\mathbb{Q}}^\bullet_X$ and $i_*i^*\underline{\mathbb{Q}}^\bullet_X$, respectively. There is an induced i-cohomomorphism of resolutions

$$i^* \colon \mathbf{I}^\bullet \rightsquigarrow \mathbf{J}^\bullet.$$

I.e. this morphism commutes with the differentials of the resolutions. Hence it induces a morphism $\mathcal{H}^i(X;\underline{\mathbb{Q}}^\bullet_X) \xrightarrow{i^*} \mathcal{H}^i(Y;i^*\underline{\mathbb{Q}}^\bullet_X)$. This morphism factors. To see this, let $\beta_X\colon \underline{\mathbb{Q}}_X \to i_*i^*\underline{\mathbb{Q}}_X$ be the canonical adjunction morphism. Following [**Bre97**, p. 63] we denote as $i^\dagger$ the morphism induced on sheaf cohomology by the canonical i-cohomomorphism $i_*\colon i_*i^*\underline{\mathbb{Q}}_X \rightsquigarrow i^*\underline{\mathbb{Q}}_X$. Then the factorization of i^* is as follows, see [**Bre97**, p. 63]:

$$\begin{tikzcd}
\mathcal{H}^i(X;\underline{\mathbb{Q}}^\bullet_X) \arrow[r, "\beta_X^*"] \arrow[d, "i^*"'] & \mathcal{H}^i(X;i_*i^*\underline{\mathbb{Q}}^\bullet_X) \arrow[dl, "i^\dagger"] \\
\mathcal{H}^i(Y;i^*\underline{\mathbb{Q}}^\bullet_X) &
\end{tikzcd} \tag{60}$$

One may adopt a more general definition as follows:

Definition 3.7.1 (p. 63 in [**Bre97**]). Let $f\colon X \to Y$ be a continuous map, let $\mathbf{A}$ and $\mathbf{B}$ be sheaves on X and Y, respectively. There is a canonical f-cohomomorphism $f^*\colon \mathbf{B} \rightsquigarrow \mathbf{A}$, which induces a morphism on cohomology. The latter morphism is denoted $f^\dagger$.

Theorem 3.7.2 (Corollary 4.4.19 on p. 73 in [**Har08**]). *Assume that X is paracompact, that Y is locally compact and Hausdorff, and that $f\colon X \to Y$ is a proper map. Furthermore, let $\mathbf{F}$ be a sheaf on X. Then $f^\dagger$ is an isomorphism if*

$$H^q(f^{-1}(y);j_y^*(\mathbf{F})) = 0 \;\forall q \geq 1 \forall y \in Y.$$

Here, $j_y\colon f^{-1}(y) \hookrightarrow X$ is the inclusion.

Proposition 3.7.3. *Given the factorization* (60), *$i^\dagger$ is an isomorphism.*

Proof. For $i\colon Y \hookrightarrow X$, we let

$$j_x\colon i^{-1}(x) \hookrightarrow Y$$

be the inclusion. The inclusion i is a proper map. Thus, Theorem 3.7.2 implies the result if

$$H^q(i^{-1}(x);j_x^*(i^*\underline{\mathbb{Q}}_X)) = 0 \;\forall q \geq 1 \forall x \in X. \tag{61}$$

In the present case, the preimage of $x \in X$ under i is either the empty set or it contains just one element, $x \in Y$. In the first case, equation (61) is satisfied. In the latter case, j_x is the inclusion

$$\{x\} \hookrightarrow Y.$$

Therefore, there are isomorphisms

$$H^q(i^{-1}(x);j_x^*(i^*\underline{\mathbb{Q}}_X)) \cong H^q(\{x\};\mathbb{Q}).$$

and the latter term vanishes for $q \geq 1$. Hence $i^\dagger$ is an isomorphism. □

We define

$$\mathbf{S}^\bullet = Rp_*\underline{\mathbb{Q}}^\bullet_X,$$

and

$$\mathbf{T}^\bullet = Rq_*\underline{\mathbb{Q}}^\bullet_Y.$$

Then

$$\begin{aligned}
\mathbf{T}^\bullet &= Rq_*(i^*\underline{\mathbb{Q}}^\bullet_X) && \text{(diagram (58))}\\
&= R(p\circ i)_*(i^*\underline{\mathbb{Q}}^\bullet_X) && \text{(diagram (58))}\\
&= Rp_*\circ Ri_*(i^*\underline{\mathbb{Q}}^\bullet_X) && \text{([\textbf{Dim08}, Proposition 2.3.3])}\\
&= Rp_*(i_*i^*\underline{\mathbb{Q}}^\bullet_X) && \text{(Y is closed in X).}
\end{aligned}$$

We will introduce isomorphisms

$$\mathcal{H}^i(\Sigma;\mathbf{S}^\bullet) \xrightarrow{a_X} H^i(X;\mathbb{Q}),$$
$$\mathcal{H}^i(\Sigma;\mathbf{T}^\bullet) \xleftarrow{b_Y} H^i(Y;\mathbb{Q}),$$

on p. 74. Given these isomorphisms, we would like to construct a morphism $s\colon \mathbf{T}^\bullet \to \mathbf{S}^\bullet$ such that for the diagram

$$\begin{tikzcd}
\mathcal{H}^i(\Sigma;\mathbf{S}^\bullet) \arrow[r, "a_X", "\cong"'] & H^i(X;\mathbb{Q}) \arrow[d, "i^*"] \\
\mathcal{H}^i(\Sigma;\mathbf{T}^\bullet) \arrow[u, dashed, "s^*"] & H^i(Y;\mathbb{Q}) \arrow[l, "b_Y"', "\cong"]
\end{tikzcd}$$

it holds that

$$i^* \circ a_X \circ s^* \circ b_Y = \mathrm{id}. \tag{62}$$

The construction will proceed as indicated in the diagram on p. 74.

REMARK 3.7.4. Recall that c denotes the collection of all compact subsets of a space. Often, the space in question may be clear from context. When this is not so, we indicate the name of the space (say, X) by writing the family of supports of compact subsets of X as c_X. The only deviations from this rule are the abbreviations

$$c_U = c_{U\times L},$$
$$c_U^{<k} = c_{U\times L_{<k}}.$$

$$
\begin{tikzcd}
\mathcal{H}^i(\Sigma;\mathbf{S}^\bullet) \arrow[r, equal] \arrow[d, "(Rp_*(\beta_X))^*"'] & \mathcal{H}^i(\Sigma;Rp_*(\underline{\mathbb{Q}}^\bullet_X)) \arrow[r, "k_X", "\cong"'] \arrow[d, "(Rp_*(\beta_X))^*"'] & \mathcal{H}^i(X;\underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] \arrow[d, "\beta_X^*"'] & \mathcal{H}^i(X;\underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] \arrow[d, "i^*"'] & H^i_{c_X}(X;\underline{\mathbb{Q}}_X) \arrow[d, "i^*"'] & H^i_{c_X}(X;\mathbb{Q}) \arrow[l, "j_X"', "\cong"] \arrow[d, "i^*"] \\
\mathcal{H}^i(\Sigma;\mathbf{T}^\bullet) \arrow[r, equal] \arrow[u, dashed, bend left=60, "s^*"] & \mathcal{H}^i(\Sigma;Rp_*(i_*i^*\underline{\mathbb{Q}}^\bullet_X)) \arrow[r, "k_Y", "\cong"'] & \mathcal{H}^i(X;i_*i^*\underline{\mathbb{Q}}^\bullet_X) \arrow[r, "i^\dagger", "\cong"'] & \mathcal{H}^i(Y;i^*\underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] & H^i_{c_Y}(Y;\underline{\mathbb{Q}}_Y) & H^i_{c_Y}(Y;\mathbb{Q}) \arrow[l, "j_Y"', "\cong"]
\end{tikzcd}
\tag{63}
$$

E **D** **C** **B** **A**

Commutativity of the squares is due to:

A	Theorem 3.6.8 – X, Y are compact and Hausdorff, X is a manifold and hence locally contractible while Y is locally contractible due to Remark 3.5.3
B	Equality
C	Commutativity of diagram (60)
D	[**Dim08**, Corollary 2.3.4]
E	Equality

Define $a_X = (j_X)^{-1} \circ k_X$ and $b_Y = (k_Y)^{-1} \circ (i^\dagger)^{-1} \circ j_Y$.

REMARK 3.7.5. Let $f\colon X \to Y$ be continuous, let $\mathbf{F}$ be a sheaf on X, let $\mathbf{F} \to \mathbf{I}^\bullet$ be an injective resolution and let $q \in \mathbb{N}$. Let furthermore f be proper, let X be paracompact and let Y be locally compact and Hausdorff. Following [**Har08**, Equation (4.29)], the *higher direct image sheaves* are

$$R^q f_*(\mathbf{F}) = \frac{\ker(f_*(\mathbf{I}^q) \to f_*(\mathbf{I}^{q+1}))}{\operatorname{im}(f_*(\mathbf{I}^{q-1}) \to f_*(\mathbf{I}^q))}.$$

It holds that

$$R^q f_*(\mathbf{F}) = \mathbf{H}^q(f_*\mathbf{I}^\bullet),$$

if the RHS is defined as in [**Ban10**, Definition 1.3.1]. Let $j_y\colon f^{-1}(y) \hookrightarrow X$, and let $\gamma\colon \mathbf{F} \to \mathbf{G}$ be a morphism of sheaves on X.

$$\begin{tikzcd}
(f_*\mathbf{F})_y \arrow[r, equal] \arrow[d, "(f_*(\gamma))_y"'] & \varinjlim_{U\colon f^{-1}(y)\subset U} \mathbf{F}(U) \arrow[r, equal] & j_y^*\mathbf{F}(f^{-1}(y)) \arrow[r, equal] & \Gamma(f^{-1}(y); j_y^*\mathbf{F}) \arrow[d, "s\mapsto\gamma\circ s"] \\
(f_*\mathbf{G})_y \arrow[r, equal] & \varinjlim_{U\colon f^{-1}(y)\subset U} \mathbf{G}(U) \arrow[r, equal] & j_y^*\mathbf{G}(f^{-1}(y)) \arrow[r, equal] & \Gamma(f^{-1}(y); j_y^*\mathbf{G})
\end{tikzcd} \tag{64}$$

Going from left to right, the first and second equalities are explained in the proof of [**Har08**, Theorem 4.4.17]. The third equality holds by definition. The diagram commutes. This induces a map of chain complexes (cf. proof of [**Har08**, Theorem 4.4.17]), which in turn induces the following commutative square:

$$\begin{tikzcd}
(R^q f_*(\mathbf{F}))_y \arrow[r, equal] \arrow[d, "(R^q f_*(\gamma))_y"'] & H^q(f^{-1}(y); j_y^*\mathbf{F}) \arrow[d, "(j_y^*(\gamma))^*"] \\
(R^q f_*(\mathbf{G}))_y \arrow[r, equal] & H^q(f^{-1}(y); j_y^*\mathbf{G})
\end{tikzcd}$$

Fix $b \in \Sigma$, let

$$u\colon \underbrace{p^{-1}(b)}_{=L} \hookrightarrow X,$$

recall the inclusions

$$i\colon Y \hookrightarrow X,$$
$$z\colon L_{<k} \hookrightarrow L,$$

and denote the canonical adjunction morphism by

$$\beta_L\colon \underline{\mathbb{Q}}_L \to z_* z^* \underline{\mathbb{Q}}_L.$$

LEMMA 3.7.6. *There are isomorphisms of sheaves λ and $\kappa \circ \eta$ such that the diagram*

$$\begin{tikzcd}
u^*(\underline{\mathbb{Q}}_X) \arrow[r, "\lambda", "\cong"'] \arrow[d, "u^*(\beta_X)"'] & \underline{\mathbb{Q}}_L \arrow[d, "\beta_L"] \\
u^*(i_* i^* \underline{\mathbb{Q}}_X) \arrow[r, "\kappa\circ\eta", "\cong"'] & z_* z^* \underline{\mathbb{Q}}_L
\end{tikzcd}$$

commutes.

PROOF. We let

$$\lambda\colon u^*(\underline{\mathbb{Q}}_X) \xrightarrow{\cong} \underline{\mathbb{Q}}_L$$

be the isomorphism of sheaves described in [**Har08**, Section 4.4.3]. For a closed inclusion

$$\mathrm{incl}_Z\colon Z \hookrightarrow X$$

there is a functor

$$(-)_Z\colon \mathbf{A} \mapsto \mathbf{A}_Z = (\mathrm{incl}_Z)_*(\mathrm{incl}_Z)^*(\mathbf{A}),$$

see [**KS90**, p. 93]. By [**KS90**, Remark 2.3.11], there is a natural transformation

$$\eta\colon u^*((-)_Y) \simeq (u^*(-))_{u^{-1}(Y)}.$$

Note that $u^{-1}(Y) = L_{<k}$. We obtain a diagram

$$\begin{tikzcd}
u^*(\underline{\mathbb{Q}}_X) \arrow[rr, "\lambda", "\cong"'] \arrow[d, "u^*(\beta_X)"'] & & \underline{\mathbb{Q}}_L \arrow[dd, "\beta_L"] \\
u^*((\underline{\mathbb{Q}}_X)_Y) \arrow[d, "\eta"', "\cong"] & & \\
(u^*(\underline{\mathbb{Q}}_X))_{u^{-1}(Y)} \arrow[rr, "\kappa=(\lambda)_{u^{-1}(Y)}", "\cong"'] & & (\underline{\mathbb{Q}}_L)_{u^{-1}(Y)}
\end{tikzcd}$$

for which we want to show commutativity. By [**Ive86**, Lemma 2.2], it suffices to check this on stalks. Letting $x \in L_{<k} \subset L$, the diagram becomes

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, equal] & u^*(\underline{\mathbb{Q}}_X)_x \arrow[rr, "\lambda_x=\mathrm{id}"] \arrow[d, "(u^*(\beta_X))_x=\mathrm{id}"'] & & (\underline{\mathbb{Q}}_L)_x \arrow[r, equal] \arrow[dd, "(\beta_L)_x=\mathrm{id}"] & \mathbb{Q} \\
\mathbb{Q} \arrow[r, equal] & u^*((\underline{\mathbb{Q}}_X)_Y)_x \arrow[d, "\eta_x=\mathrm{id}"'] & & & \\
\mathbb{Q} \arrow[r, equal] & ((u^*(\underline{\mathbb{Q}}_X))_{u^{-1}(Y)})_x \arrow[rr, "\kappa_x=\mathrm{id}"] & & ((\underline{\mathbb{Q}}_L)_{u^{-1}(Y)})_x \arrow[r, equal] & \mathbb{Q}
\end{tikzcd}$$

and commutativity holds. For $x \in L - L_{<k}$, the same diagram is

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, equal] & u^*(\underline{\mathbb{Q}}_X)_x \arrow[rr, "\lambda_x=\mathrm{id}"] \arrow[d, "(u^*(\beta_X))_x=0"'] & & (\underline{\mathbb{Q}}_L)_x \arrow[r, equal] \arrow[dd, "(\beta_L)_x=0"] & \mathbb{Q} \\
0 \arrow[r, equal] & u^*((\underline{\mathbb{Q}}_X)_Y)_x \arrow[d, "\eta_x=0"'] & & & \\
0 \arrow[r, equal] & ((u^*(\underline{\mathbb{Q}}_X))_{u^{-1}(Y)})_x \arrow[rr, "\kappa_x=0"] & & ((\underline{\mathbb{Q}}_L)_{u^{-1}(Y)})_x \arrow[r, equal] & 0
\end{tikzcd}$$

and commutativity again holds. ☐

LEMMA 3.7.7. *The morphism*

$$\tau_{<k}(Rp_*(\beta_X))\colon \tau_{<k}(\mathbf{S}^\bullet) \to \tau_{<k}(\mathbf{T}^\bullet)$$

is a quasi-isomorphism.

PROOF. The diagram

$$\begin{tikzcd}
\mathbf{H}^m(\mathbf{T}^\bullet)_b \arrow[d, equal] & \mathbf{H}^m(\mathbf{S}^\bullet)_b \arrow[l, "\mathbf{H}^m(Rp_*(\beta_X))_b"'] \arrow[d, equal] \\
\mathbf{H}^m(Rp_*(i_*i^*\underline{\mathbb{Q}}_X))_b \arrow[d, equal] & \mathbf{H}^m(Rp_*(\underline{\mathbb{Q}}_X))_b \arrow[d, equal] \\
R^m p_*(i_*i^*\underline{\mathbb{Q}}_X)_b \arrow[d, equal] & R^m p_*(\underline{\mathbb{Q}}_X)_b \arrow[l, "(R^m p_*(\beta_X))_b"'] \arrow[d, equal] \\
H^m(p^{-1}(b); u^*(i_*i^*\underline{\mathbb{Q}}_X)) \arrow[d, equal] & H^m(p^{-1}(b); u^*(\underline{\mathbb{Q}}_X)) \arrow[l, "(u^*(\beta_X))^*"'] \arrow[d, equal] \\
H^m(L; u^*(i_*i^*\underline{\mathbb{Q}}_X)) & H^m(L; u^*(\underline{\mathbb{Q}}_X))
\end{tikzcd}$$

commutes: the upper rectangle by definition, and the lower square by Remark 3.7.5 (note that p is proper). Lemma 3.7.6 shows that the diagram

$$\begin{tikzcd}
u^*(\underline{\mathbb{Q}}_X) \arrow[r, "\lambda", "\cong"'] \arrow[d, "u^*(\beta_X)"'] & \underline{\mathbb{Q}}_L \arrow[d, "\beta_L"] \\
u^*(i_*i^*\underline{\mathbb{Q}}_X) \arrow[r, "\kappa\circ\eta", "\cong"'] & z_*z^*\underline{\mathbb{Q}}_L
\end{tikzcd}$$

commutes. As $H^m(L; -)$ is a functor, the diagram

$$\begin{tikzcd}
H^m(L; u^*(\underline{\mathbb{Q}}_X)) \arrow[r, "\cong"] \arrow[d, "(u^*(\beta_X))^*"'] & H^m(L; \underline{\mathbb{Q}}_L) \arrow[d, "(\beta_L)^*"] \\
H^m(L; u^*(i_*i^*\underline{\mathbb{Q}}_X)) \arrow[r, "\cong"] & H^m(L; z_*z^*\underline{\mathbb{Q}}_L)
\end{tikzcd}$$

commutes. From [**Bre97**, p. 93], we know that $(\beta_L)^*$ factorizes as follows:

$$\begin{tikzcd}
H^m(L; \underline{\mathbb{Q}}_L) \arrow[dd, "(\beta_L)^*"'] \arrow[dr, "z^*"] & & \\
& H^m(z^{-1}(L); z^*\underline{\mathbb{Q}}_L) \arrow[r, equal] & H^m(L_{<k}; \underline{\mathbb{Q}}_{L_{<k}}) \\
H^m(L; z_*z^*\underline{\mathbb{Q}}_L) \arrow[ur, "z^\dagger"'] & &
\end{tikzcd}$$

The inclusion z is proper, hence $z^\dagger$ is an isomorphism. Furthermore,

$$z^*\colon H^m(L; \underline{\mathbb{Q}}_L) \to H^m(L_{<k}; \underline{\mathbb{Q}}_{L_{<k}})$$

is an isomorphism for $m < k$, because

$$\begin{tikzcd}
H^m(L; \underline{\mathbb{Q}}_L) \arrow[r, "\cong"] \arrow[d, "z^*"'] & H^m(L; \mathbb{Q}) \arrow[d, "z^*"] \\
H^m(L_{<k}; \underline{\mathbb{Q}}_{L_{<k}}) \arrow[r, "\cong"] & H^m(L_{<k}; \mathbb{Q})
\end{tikzcd}$$

commutes while

$$z^*\colon H^m(L;\mathbb{Q}) \xrightarrow{\cong} H^m(L_{<k};\mathbb{Q}),\ \forall m < k.$$

Therefore $(\beta_L)^*$ is an isomorphism for $m < k$. Using commutativity of the various diagrams, we can see that this implies that $\mathbf{H}^m(Rp_*(\beta_X))$ is an isomorphism for $m < k$. Thus,

$$\tau_{<k}(Rp_*(\beta_X))\colon \tau_{<k}(\mathbf{S}^\bullet) \to \tau_{<k}(\mathbf{T}^\bullet)$$

is a quasi-isomorphism. □

PROPOSITION 3.7.8. *The canonical inclusion*

$$\zeta\colon \tau_{<k}(\mathbf{T}^\bullet) \to \mathbf{T}^\bullet$$

is a quasi-isomorphism.

PROOF. This is easily seen from

$$\mathbf{H}^m(\mathbf{T}^\bullet)_b \cong H^m(L_{<k};\mathbb{Q}).$$

□

Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{S}^\bullet \arrow[r, "Rp_*(\beta_X)"] & \mathbf{T}^\bullet \\
\tau_{<k}(\mathbf{S}^\bullet) \arrow[u, "\sigma"] \arrow[r, "\tau_{<k}(Rp_*(\beta_X))"] & \tau_{<k}(\mathbf{T}^\bullet) \arrow[u, "\zeta"']
\end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^m(\Sigma;\mathbf{S}^\bullet) \arrow[r, "(Rp_*(\beta_X))^*"] & \mathcal{H}^m(\Sigma;\mathbf{T}^\bullet) \\
\mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{S}^\bullet)) \arrow[u, "\sigma^*"] \arrow[r, "(\tau_{<k}(Rp_*(\beta_X)))^*", "\cong"'] & \mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{T}^\bullet)) \arrow[u, "\cong", "\zeta^*"']
\end{tikzcd}$$

in which the isomorphisms are due to [**Ban07**, p. 23]. Set $\chi = (\tau_{<k}(Rp_*(\beta_X)))^*$, $s^* = \sigma^* \circ \chi^{-1} \circ (\zeta^*)^{-1}$ and

$$t^* = a_X \circ s^* \circ b_Y. \tag{65}$$

Then equation (62) holds due to Lemma 3.7.7 and Proposition 3.7.8. In other words, the equation $i^* \circ t^* = \mathrm{id}$ holds. Thus t^* is a right inverse for i^*.

REMARK 3.7.9. To understand the morphism σ, consider the following diagram:

$$\begin{tikzcd}
\mathbf{S}^\bullet \arrow[r, equal] & \cdots \arrow[r] & \mathbf{S}^{k-3} \arrow[r] & \mathbf{S}^{k-2} \arrow[r] & \mathbf{S}^{k-1} \arrow[r] & \mathbf{S}^k \arrow[r] & \cdots \\
\tau_{<k}(\mathbf{S}^\bullet) \arrow[u, "\sigma"] \arrow[r, equal] & \cdots \arrow[r] & \mathbf{S}^{k-3} \arrow[u, "\cong"'] \arrow[r] & \mathbf{S}^{k-2} \arrow[u, "\cong"'] \arrow[r] & \mathbf{ker} d^{k-1} \arrow[u, hook] \arrow[r] & 0 \arrow[u, "0"'] \arrow[r] & \cdots
\end{tikzcd}$$

Recalling the inclusions

$$\begin{tikzcd} U \times L \arrow[d, hook, "j"'] & U \times L_{<k} \arrow[l, hook', "i_U"'] \arrow[d, hook, "j_{<k}"] \\ X & Y \arrow[l, hook', "i"'] \end{tikzcd} \tag{66}$$

we would like to define right inverses t_U^* and t_V^* of i_U^* and i_V^* respectively. Fitting these morphisms into the square

$$\begin{tikzcd} H_c^{n-r}(U \times L) \oplus H_c^{n-r}(V \times L) \arrow[r, "i_U^* \oplus i_V^*"] \arrow[d, "\beta^*"'] & H_c^{n-r}(U \times L_{<k}) \oplus H_c^{n-r}(V \times L_{<k}) \arrow[d, "\beta_{<k}^*"] \arrow[l, bend right, dashed, "t_U^* \oplus t_V^*"'] \\ H_c^{n-r}(\partial M) \arrow[r, "i^*"] & H_c^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[l, bend left, dashed, "t^*"] \end{tikzcd}$$

in which the vertical morphisms stem from Mayer-Vietoris sequences, we need the constructed right inverses to satisfy

$$\beta^* \circ (t_U^* \oplus t_V^*) = t^* \circ \beta_{<k}^*. \tag{67}$$

It suffices to construct t_U^* as the construction of t_V^* is similar. Note that

$$\beta^* \colon H_c^{n-r}(U \times L) \oplus H_c^{n-r}(V \times L) \to H_c^{n-r}(\partial M)$$

is by definition the direct sum of morphisms

$$\begin{aligned} \epsilon(j) \colon H_c^i(U \times L; \mathbb{Q}) &\to H_c^i(X; \mathbb{Q}), \\ -\epsilon(\mathrm{incl}) \colon H_c^i(V \times L; \mathbb{Q}) &\to H_c^i(X; \mathbb{Q}) \end{aligned}$$

induced by the inclusions j of diagram (66) and

$$\mathrm{incl} \colon V \times L \hookrightarrow X.$$

Hence we focus on the square

$$\begin{tikzcd} H_c^{n-r}(U \times L) \arrow[r, "i_U^*"] \arrow[d, "\epsilon(j)"'] & H_c^{n-r}(U \times L_{<k}) \arrow[d, "\epsilon(j_{<k})"] \arrow[l, bend right, dashed, "t_U^*"'] \\ H_c^{n-r}(\partial M) \arrow[r, "i^*"] & H_c^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[l, bend left, dashed, "t^*"] \end{tikzcd} \tag{68}$$

and recall diagram (63), which was used to analyze the morphism i^* in diagram (68) and to define t^*. The same procedure will now be carried out for the other morphisms in diagram (68), using diagrams akin to (63). This will ultimately enable us to define t_U^* in terms of sheaf cohomology and to show equation (67). This sheaf-theoretic approach has an advantage over a construction of t_U^* via the Künneth theorem: equation (67) is a byproduct of the sheaf-theoretic Ansatz.

The first step will be the diagram on p. 82, addressing the morphism i_U. Some preliminary considerations are necessary. The map i_U is proper, because it is the inclusion of a closed subspace. (Cf. e.g. [**May99**, p. 161].)

REMARK 3.7.10. (1) We have for $c_X \mid U = \{K \mid K \subset U \wedge K \in c_X\}$ that
$$K \in c_X \mid U \underset{K \text{ compact}}{\Rightarrow} K \in c_U.$$
On the other hand, $K \in c_U$ is certainly compact and a subset of U. Thus, $c_X \mid U = c_U$.
(2) The spaces U, L and $L_{<k}$ are locally compact. Hence $c_U = c_X \mid U \times L$ and $c_U^{<k}$ are paracompactifying by [**Bre97**, p. 22].
(3) Likewise, U, L and $L_{<k}$ are locally contractible (the first is a manifold while the latter two are CW-complexes), and hence $U \times L$ and $U \times L_{<k}$ are locally contractible.

Set
$$\begin{aligned} l = i_U &: U \times L_{<k} \hookrightarrow U \times L, \\ \mathbf{V}^\bullet &= Rp_*(j_! \underline{\mathbb{Q}}^\bullet_{U \times L}), \\ \mathbf{W}^\bullet &= Rp_*(j_! l_* l^* \underline{\mathbb{Q}}^\bullet_{U \times L}). \end{aligned}$$

REMARK 3.7.11. The closure of the open subspace $U \times L$ is $\overline{U} \times L$. Thus
$$U \times L = U \times L \cap \overline{U} \times L$$
is the intersection of an open subspace with a closed subspace. Hence, by [**Bou66**, Proposition I.3.3.5], $U \times L$ is locally closed in X. The space X is compact and Hausdorff, thus by [**Lee00**, Lemma 4.29] the open subspace $U \times L$ is locally compact Hausdorff. By Remark 3.6.4(2), the family c_U is paracompactifying, as is c_X. Hence [**Bre97**, Corollary II.10.2] applies to show that there are natural isomorphisms
$$\begin{aligned} \zeta &: H^i_{c_X|U\times L}(U \times L; \underline{\mathbb{Q}}_X \mid U \times L) \xrightarrow{\cong} H^i_{c_X}(X; (\underline{\mathbb{Q}}_X)_{U \times L}) \\ \zeta_{<k} &: H^i_{c_Y|U\times L_{<k}}(U \times L_{<k}; \underline{\mathbb{Q}}_Y \mid U \times L_{<k}) \xrightarrow{\cong} H^i_{c_Y}(Y; (\underline{\mathbb{Q}}_Y)_{U \times L_{<k}}) \\ \bar{\zeta} &: H^i_{c_U}(U \times L; l_* l^* \underline{\mathbb{Q}}_{U\times L}) \xrightarrow{\cong} H^i_{c_X}(X; j_! l_* l^* \underline{\mathbb{Q}}_{U \times L}). \end{aligned}$$
of sheaves on $U \times L$.

REMARK 3.7.12. As $U \times L$ is open in X, $(\underline{\mathbb{Q}}_X)_{U\times L}$ is a subsheaf of $\underline{\mathbb{Q}}_X$. Let
$$\eta : (\underline{\mathbb{Q}}_X)_{U \times L} \hookrightarrow \underline{\mathbb{Q}}_X$$
be the corresponding inclusion. Likewise, there is the inclusion
$$\eta_{<k} : (\underline{\mathbb{Q}}_Y)_{U \times L_{<k}} \hookrightarrow \underline{\mathbb{Q}}_Y.$$

REMARK 3.7.13. From Theorem 3.6.8 there are natural isomorphisms
$$\begin{aligned} j_X &: H^i_{c_X}(X; \mathbb{Q}) \xrightarrow{\cong} H^i_{c_X}(X; \underline{\mathbb{Q}}_X), \\ j_U &: H^i_{c_U}(U \times L; \mathbb{Q}) \xrightarrow{\cong} H^i_{c_U}(U \times L; \underline{\mathbb{Q}}_{U\times L}) = H^i_{c_X|U\times L}(U \times L; \underline{\mathbb{Q}}_X \mid U \times L). \end{aligned}$$

The amalgam of the previous three remarks can be used to express commutativity of the diagram,

$$\begin{tikzcd} H^i_{c_X|U\times L}(U \times L; \underline{\mathbb{Q}}_X \mid U \times L) \arrow[d, "\zeta"', "\cong"] & H^i_{c_{U\times L}}(U \times L; \mathbb{Q}) \arrow[l, "j_U"', "\cong"] \arrow[dd, "\epsilon(j)"] \\ H^i_{c_X}(X; (\underline{\mathbb{Q}}_X)_{U\times L}) \arrow[d, "\eta^*"'] & \\ H^i_{c_X}(X; \underline{\mathbb{Q}}_X) & H^i_{c_X}(X; \mathbb{Q}) \arrow[l, "j_X"', "\cong"] \end{tikzcd} \tag{69}$$

wherein $\epsilon(j)$ is the morphism induced by j on cohomology with compact supports. The coefficient sheaves satisfy

$$\begin{aligned}(\underline{\mathbb{Q}}_X)_{U\times L} &= (\underline{\mathbb{Q}}_X \mid U \times L)^X && \text{(by definition in [\textbf{Bre97}, p. 11])}\\ &= (\underline{\mathbb{Q}}_{U\times L})^X && \text{(due to } \underline{\mathbb{Q}}_X \mid U \times L = \underline{\mathbb{Q}}_{U\times L})\\ &= j_!\underline{\mathbb{Q}}_{U\times L},\end{aligned}$$

and

$$(\underline{\mathbb{Q}}_Y)_{U\times L_{<k}} = (j_{<k})_!\underline{\mathbb{Q}}_{U\times L_{<k}}.$$

Hence the diagram

$$\begin{tikzcd} H^i_{cX}(X; j_!\underline{\mathbb{Q}}_{U\times L}) \arrow[d, "\eta^*"'] & H^i_{cX|U\times L}(U \times L; \underline{\mathbb{Q}}_{U\times L}) \arrow[l, "\zeta"', "\cong"] \arrow[d, "\eta^*\circ\zeta"] \\ H^i_{cX}(X; \underline{\mathbb{Q}}_X) \arrow[r, equal] & H^i_{cX}(X; \underline{\mathbb{Q}}_X) \end{tikzcd} \tag{70}$$

is commutative. We are now ready to introduce the next diagram overleaf.

$$
\begin{tikzcd}
 & \mathcal{H}^i(\Sigma;\mathbf{V}^\bullet) & & & H^i_{c_X|U\times L}(U\times L;\underline{\mathbb{Q}}_{U\times L}) & \\
\mathcal{H}^i(\Sigma;Rp_*(j_!\underline{\mathbb{Q}}^\bullet_{U\times L})) \arrow[ur, equal] \arrow[r, "k_U", "\cong"'] \arrow[d, "(Rp_*(j_!(\beta_U^*)))^*"'] & \mathcal{H}^i(X;j_!\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[d, "j_!(\beta_U^*)"'] & \mathcal{H}^i_{c_U}(U\times L;\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[l, "\zeta"', "\cong"] \arrow[r, equal] \arrow[d, "\beta_U^*"'] & \mathcal{H}^i_{c_X|U\times L}(U\times L;\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[r, equal] \arrow[d, "l^*"'] & H^i_{c_X|U\times L}(U\times L;\underline{\mathbb{Q}}_X\mid U\times L) \arrow[u, equal] \arrow[d, "l^*"'] & H^i_{c_U}(U\times L;\mathbb{Q}) \arrow[ul, "j_U"', "\cong"] \arrow[d, "l^*"] \\
\mathcal{H}^i(\Sigma;Rp_*(j_!l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L})) \arrow[r, "k_U^{<k}", "\cong"'] \arrow[dr, equal] & \mathcal{H}^i(X;j_!l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) & \mathcal{H}^i_{c_U}(U\times L;l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[l, "\bar{\zeta}"', "\cong"] \arrow[r, "l^\dagger", "\cong"'] & \mathcal{H}^i_{c_Y|U\times L_{<k}}(U\times L_{<k};l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[r, equal] & H^i_{c_Y|U\times L_{<k}}(U\times L_{<k};\underline{\mathbb{Q}}_Y\mid U\times L_{<k}) \arrow[d, equal] & H^i_{c_U^{<k}}(U\times L_{<k};\mathbb{Q}) \arrow[dl, "j_U^{<k}", "\cong"'] \\
 & \mathcal{H}^i(\Sigma;\mathbf{W}^\bullet) & & & H^i_{c_Y|U\times L_{<k}}(U\times L_{<k};\underline{\mathbb{Q}}_{U\times L_{<k}}) &
\end{tikzcd}
$$

E **D** **C** **B** **A**

Commutativity of the squares is due to:

A	Theorem 3.6.8 and Remark 3.7.10
B	Equality
C	Commutativity of diagram (60)
D	[**Bre97**, II.10.1], noting that we have omitted the supports c_X on the LHS since X is compact, and noting that $c_X \mid U = c_U$
E	[**Dim08**, Corollary 2.3.4]

Define $a_U = (j_U)^{-1} \circ (\zeta)^{-1} \circ k_U$ and $b_U^{<k} = (k_U^{<k})^{-1} \circ \bar{\zeta} \circ (l^\dagger)^{-1} \circ j_U^{<k}$.

We need more diagrams. But first, some preparation is required. By [**Dim08**, Corollary 2.3.4] the diagram

$$
\begin{tikzcd}
\mathcal{H}^i(\Sigma; Rp_*(j_!\underline{\mathbb{Q}}^\bullet_{U\times L})) \arrow[r, "k_U", "\cong"'] \arrow[d, "(Rp_*(\eta))^*"'] & \mathcal{H}^i(X; j_!\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[r, equal] \arrow[d, "\eta"] & H^i_{c_X}(X; j_!\underline{\mathbb{Q}}_{U\times L}) \\
\mathcal{H}^i(\Sigma; Rp_*(\underline{\mathbb{Q}}^\bullet_X)) \arrow[r, "k_X", "\cong"'] & \mathcal{H}^i(X; \underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] & H^i_{c_X}(X; \underline{\mathbb{Q}}_X)
\end{tikzcd}
\tag{71}
$$

commutes. Note that herein we have discontinued writing the supports c_X since X is compact. Now we concatenate diagrams (69), (70) and (71) to obtain the top diagram on p. 85. It commutes.

Proceeding analogously for the morphism $\epsilon(j_{<k})$, we obtain the diagram

$$
\begin{tikzcd}
H^i_{c_Y}(Y; \underline{\mathbb{Q}}_Y) & & H^i_{c_Y}(Y; \mathbb{Q}) \arrow[ll, "j_Y", "\cong"'] \\
H^i_{c_Y}(Y; (\underline{\mathbb{Q}}_Y)_{U\times L_{<k}}) \arrow[u, "\eta^*_{<k}"] & & \\
H^i_{c_Y|U\times L_{<k}}(U\times L_{<k}; \underline{\mathbb{Q}}_Y \mid U\times L_{<k}) \arrow[u, "\zeta_{<k}"] & & H^i_{c_U^{<k}}(U\times L_{<k}; \mathbb{Q}) \arrow[ll, "j_U^{<k}", "\cong"'] \arrow[uu, "\epsilon(j_{<k})"']
\end{tikzcd}
$$

and the diagram

$$
\begin{tikzcd}
H^i_{c_Y}(Y; i^*\underline{\mathbb{Q}}_X) \arrow[r, equal] & H^i_{c_Y}(Y; \underline{\mathbb{Q}}_Y) \\
H^i_{c_U}(U\times L_{<k}; l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[u, "\eta^*_{<k}\circ\zeta_{<k}"] \arrow[r, equal] & H^i_{c_Y|U\times L_{<k}}(U\times L_{<k}; \underline{\mathbb{Q}}_Y \mid U\times L_{<k}) \arrow[u, "\eta^*_{<k}\circ\zeta_{<k}"']
\end{tikzcd}
$$

both of which commute. Recalling the inclusions

$$
\begin{tikzcd}
U\times L \arrow[d, hook, "j"'] & U\times L_{<k} \arrow[l, hook', "l=i_U"'] \arrow[d, hook, "j_{<k}"] \\
X & Y \arrow[l, hook', "i"']
\end{tikzcd}
$$

we see that the diagram

$$
\begin{tikzcd}
H^i_{c_X}(X; i_*(j_{<k})_!l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[r, "i^\dagger", "\cong"'] \arrow[d, equal] & H^i_{c_Y}(Y; (j_{<k})_!l^*\underline{\mathbb{Q}}_{U\times L}) \\
H^i_{c_X}(X; (i\circ j_{<k})_!l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[d, equal] & \\
H^i_{c_X}(X; (j\circ l)_!l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[d, equal] & \\
H^i_{c_X}(X; j_!l_*l^*\underline{\mathbb{Q}}_{U\times L}) & \\
H^i_{c_U}(U\times L; l_*l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[u, "\bar{\zeta}", "\cong"'] \arrow[r, "l^\dagger", "\cong"'] & H^i_{c_U^{<k}}(U\times L_{<k}; l^*\underline{\mathbb{Q}}_{U\times L}) \arrow[uuuu, "\cong", "\zeta_{<k}"']
\end{tikzcd}
\tag{72}
$$

commutes. Additionally, the diagram

$$\begin{tikzcd}
H^i_{c_X}(X; i_* i^* \underline{\mathbb{Q}}_X) \arrow[r, "i^\dagger", "\cong"'] & H^i_{c_Y}(Y; i^* \underline{\mathbb{Q}}_X) \\
H^i_{c_X}(X; j_! l_* l^* \underline{\mathbb{Q}}_{U\times L}) \arrow[r, "i^\dagger", "\cong"'] \arrow[u, "\xi"] & H^i_{c_Y}(Y; (l^* \underline{\mathbb{Q}}_{U\times L})^Y) \arrow[u, "\eta_{<k}"']
\end{tikzcd} \tag{73}$$

commutes. Recall

$$\eta_{<k} \colon (\underline{\mathbb{Q}}_Y)_{U\times L_{<k}} \hookrightarrow \underline{\mathbb{Q}}_Y,$$

and the inclusion of sheaves

$$\begin{aligned}
j_! l_* l^* \underline{\mathbb{Q}}_{U\times L} &= (l_* l^* \underline{\mathbb{Q}}_{U\times L})^X \\
&= (l_* \underline{\mathbb{Q}}_{U\times L_{<k}})^X \\
&= ((\underline{\mathbb{Q}}_{U\times L_{<k}})^{U\times L})^X \\
&= (\underline{\mathbb{Q}}_{U\times L_{<k}})^X \overset{\xi}{\hookrightarrow} (\underline{\mathbb{Q}}_Y)^X \\
&= i_* \underline{\mathbb{Q}}_Y \\
&= i_* i^* \underline{\mathbb{Q}}_X,
\end{aligned}$$

which is used in the last diagram.

$$
\begin{tikzcd}
\mathcal{H}^i(\Sigma; \mathbf{V}^\bullet) \arrow[r, equal] & \mathcal{H}^i(\Sigma; Rp_*(j_!\underline{\mathbb{Q}}^\bullet_{U\times L})) \arrow[r, "k_U", "\cong"'] \arrow[d, "(Rp_*(\eta))^*"'] & \mathcal{H}^i(X; j_!\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[r, equal] \arrow[d, "\eta^*"'] & H^i_{c_X}(X; j_!\underline{\mathbb{Q}}_{U\times L}) \arrow[d, "\eta^*"'] & H^i_{c_X|U\times L}(U\times L; \underline{\mathbb{Q}}_{U\times L}) \arrow[l, "\zeta"', "\cong"] \arrow[d, "\eta^*\circ\zeta"'] & H^i_{c_U}(U\times L; \mathbb{Q}) \arrow[l, "j_U"', "\cong"] \arrow[d, "\epsilon(j)"] \\
\mathcal{H}^i(\Sigma; \mathbf{S}^\bullet) \arrow[r, equal] & \mathcal{H}^i(\Sigma; Rp_*(\underline{\mathbb{Q}}^\bullet_X)) \arrow[r, "k_X", "\cong"'] & \mathcal{H}^i(X; \underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] & H^i_{c_X}(X; \underline{\mathbb{Q}}_X) \arrow[r, equal] & H^i_{c_X}(X; \underline{\mathbb{Q}}_X) & H^i_{c_X}(X; \mathbb{Q}) \arrow[l, "j_X"', "\cong"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{H}^i(\Sigma; \mathbf{T}^\bullet) \arrow[r, equal] & \mathcal{H}^i(\Sigma; Rp_*(i_*i^*\underline{\mathbb{Q}}^\bullet_X)) \arrow[r, "k_Y", "\cong"'] & \mathcal{H}^i(X; i_*i^*\underline{\mathbb{Q}}^\bullet_X) \arrow[r, "i^\dagger", "\cong"'] & \mathcal{H}^i(Y; i^*\underline{\mathbb{Q}}^\bullet_X) \arrow[r, equal] & H^i_{c_Y}(Y; \underline{\mathbb{Q}}_Y) & H^i_{c_Y}(Y; \mathbb{Q}) \arrow[l, "j_Y"', "\cong"] \\
\mathcal{H}^i(\Sigma; \mathbf{W}^\bullet) \arrow[r, equal] & \mathcal{H}^i(\Sigma; Rp_*(j_!l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L})) \arrow[r, "k_U^{<k}", "\cong"'] \arrow[u, "(Rp_*(\xi))^*"] & \mathcal{H}^i(X; j_!l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[u, "\xi"] & \mathcal{H}^i_{c_Y|U\times L_{<k}}(U\times L_{<k}; l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[r, equal] \arrow[u, "\eta^*_{<k}\circ\zeta_{<k}"] & H^i_{c_Y|U\times L_{<k}}(U\times L_{<k}; \underline{\mathbb{Q}}_Y \mid U\times L_{<k}) \arrow[u, "\eta^*_{<k}\circ\zeta_{<k}"] & H^i_{c_U^{<k}}(U\times L_{<k}; \mathbb{Q}) \arrow[l, "j_U^{<k}"', "\cong"] \arrow[u, "\epsilon(j_{<k})"'] \\
& & & \mathcal{H}^i_{c_U}(U\times L; l_*l^*\underline{\mathbb{Q}}^\bullet_{U\times L}) \arrow[ul, "\bar{\zeta}", "\cong"'] \arrow[u, "l^\dagger", "\cong"'] & &
\end{tikzcd}
$$

Commutativity of the top diagram was discussed previously. Concerning the squares in the bottom diagram, commutativity is due to, from left to right: [**Dim08**, Corollary 2.3.4], commutativity of the concatenation of the commutative diagrams (72) and (73), as well as Theorem 3.6.8.

Joining the diagram (63) with the diagrams of pages 82 and 85, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}^m(\Sigma;\mathbf{V}^\bullet) \arrow[rrr, "(Rp_*(j_!(\beta_U^*)))^*"] \arrow[dr, "a_U", "\cong"'] \arrow[ddd, "(Rp_*(\eta))^*"'] & & & \mathcal{H}^m(\Sigma;\mathbf{W}^\bullet) \arrow[ddd, "(Rp_*(\xi))^*"] \\
& H_c^{n-r}(U\times L) \arrow[r, "l^*"] \arrow[d, "\epsilon(j)"'] & H_c^{n-r}(U\times L_{<k}) \arrow[ur, "b_U^{<k}"', "\cong"] \arrow[d, "\epsilon(j_{<k})"'] & \\
& H_c^{n-r}(X) \arrow[r, "i^*"] & H_c^{n-r}(Y) \arrow[dr, "b_Y", "\cong"'] & \\
\mathcal{H}^m(\Sigma;\mathbf{S}^\bullet) \arrow[ur, "a_X", "\cong"'] \arrow[rrr, "(Rp_*(\beta_X))^*"] & & & \mathcal{H}^m(\Sigma;\mathbf{T}^\bullet)
\end{tikzcd}$$

The outer square of the latter diagram is the inner square of the induced diagram

(74)
$$\begin{tikzcd}
\mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{V}^\bullet)) \arrow[r, "(\tau_{<k}(Rp_*(j_!(\beta_U^*))))^*", "\cong"'] \arrow[d, "\theta^*"'] \arrow[ddd, bend right=60, "(\tau_{<k}(Rp_*(\eta)))^*"'] & \mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{W}^\bullet)) \arrow[d, "\rho^*"'] \arrow[ddd, bend left=60, "(\tau_{<k}(Rp_*(\xi)))^*"] \\
\mathcal{H}^m(\Sigma;\mathbf{V}^\bullet) \arrow[r, "(Rp_*(j_!(\beta_U^*)))^*"] \arrow[d, "(Rp_*(\eta))^*"'] & \mathcal{H}^m(\Sigma;\mathbf{W}^\bullet) \arrow[d, "(Rp_*(\xi))^*"'] \\
\mathcal{H}^m(\Sigma;\mathbf{S}^\bullet) \arrow[r, "(Rp_*(\beta_X))^*"] & \mathcal{H}^m(\Sigma;\mathbf{T}^\bullet) \\
\mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{S}^\bullet)) \arrow[r, "(\tau_{<k}(Rp_*(\beta_X)))^*", "\cong"'] \arrow[u, "\sigma^*"] & \mathcal{H}^m(\Sigma;\tau_{<k}(\mathbf{T}^\bullet)) \arrow[u, "\zeta^*"']
\end{tikzcd}$$

in which

$$\rho\colon \tau_{<k}(\mathbf{W}^\bullet) \to \mathbf{W}^\bullet$$
$$\theta\colon \tau_{<k}(\mathbf{V}^\bullet) \to \mathbf{V}^\bullet$$

are the canonical inclusions. Finally, we define

(75) $$t_U^* = a_U \circ \theta^* \circ ((\tau_{<k}(Rp_*(j_!(\beta_U^*))))^*)^{-1} \circ (\rho^*)^{-1} \circ b_U^{<k}.$$

This is a right inverse for much the same reasons that t^* is. It remains to check that diagram (68) commutes in the desired fashion, i.e. that the equation

(76) $$\epsilon(j) \circ t_U^* = t^* \circ \epsilon(j_{<k})$$

holds. Commutativity of diagram (74) is all that is required to show this:

$$\begin{aligned}
\epsilon(j) \circ t_U^* &= \epsilon(j) \circ a_U \circ \theta^* \circ ((\tau_{<k}(Rp_*(j_!(\beta_U^*))))^*)^{-1} \circ (\rho^*)^{-1} \circ b_U^{<k} \\
&= a_X \circ (Rp_*(\eta))^* \circ \theta^* \circ ((\tau_{<k}(Rp_*(j_!(\beta_U^*))))^*)^{-1} \circ (\rho^*)^{-1} \circ b_U^{<k} \\
&= a_X \circ \sigma^* \circ ((\tau_{<k}(Rp_*(\eta)))^*)^{-1} \circ ((\tau_{<k}(Rp_*(j_!(\beta_U^*))))^*)^{-1} \circ (\rho^*)^{-1} \circ b_U^{<k} \\
&= a_X \circ \sigma^* \circ ((\tau_{<k}(Rp_*(\beta_X)))^*)^{-1} \circ (\tau_{<k}(Rp_*(\xi)))^* \circ (\rho^*)^{-1} \circ b_U^{<k} \\
&= a_X \circ \sigma^* \circ ((\tau_{<k}(Rp_*(\beta_X)))^*)^{-1} \circ (\zeta^*)^{-1} \circ (Rp_*(\xi))^* \circ b_U^{<k} \\
&= a_X \circ \sigma^* \circ ((\tau_{<k}(Rp_*(\beta_X)))^*)^{-1} \circ (\zeta^*)^{-1} \circ b_Y \circ \epsilon(j_{<k}) \\
&= a_X \circ \sigma^* \circ \chi^{-1} \circ (\zeta^*)^{-1} \circ b_Y \circ \epsilon(j_{<k}) \\
&= a_X \circ s^* \circ b_Y \circ \epsilon(j_{<k}) \\
&= t^* \circ \epsilon(j_{<k})
\end{aligned}$$

Recall that we fixed a complementary pair of perversities $(\overline{p}, \overline{q})$, and that we introduced the variable k. The desired cap product may now be defined.

Proposition 3.7.14 (Proposition 2.44 on p. 179 in [**Ban10**]). *Let X be an n-dimensional compact, oriented, stratified pseudomanifold with one singular stratum X_{n-c} and $(\overline{p}, \overline{q})$-admissible link bundle. Assume that M^n is defined as in equation* (55)*. There exists a cap product*

$$H^{n-1-r}(\mathrm{ft}_{<k}(\partial M)) \otimes H_{n-1}(\partial M) \xrightarrow{\cap} H_r(\partial M, \mathrm{ft}_{<c-k}(\partial M))$$

such that the diagram

$$\begin{tikzcd}
H^{n-1-r}(\mathrm{ft}_{<k}(\partial M)) \otimes H_{n-1}(\partial M) \arrow[r, "\cap"] & H_r(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \\
H^{n-1-r}(\partial M) \otimes H_{n-1}(\partial M) \arrow[u, "i^* \otimes \mathrm{id}"] \arrow[r, "\cap"] & H_r(\partial M) \arrow[u, "\pi_*"']
\end{tikzcd} \tag{77}$$

commutes.

Proof. From the definition of t^* in equation (65) it follows that t^* is a right inverse for i^*. Accordingly, i^* is surjective, and t^* yields a canonical choice of preimage under i^*. Furthermore, t^* is injective. Given $\xi \in H^{n-1-r}(\mathrm{ft}_{<k}(\partial M))$ and $x \in H_r(\partial M)$, we set

$$\xi \cap x = \pi_*(t^*(\xi) \cap x).$$

This yields commutativity of diagram (77). ☐

Recall our assumption that the link bundle be trivial over U and V, respectively. On p. 89, we consider a diagram, aspects of which include:

- The horizontal isomorphisms are due to Proposition 3.3.3.
- The columns form Mayer-Vietoris sequences. Hence they are exact. To be specific:
  - **I** Cohomology with compact support of $\partial M = (U \times L) \cup (V \times L)$.
  - **II** Cohomology with compact support of
  $$\mathrm{ft}_{<k}(\partial M) = (U \times L_{<k}) \cup (V \times L_{<k}).$$
  - **III** Relative homology of
  $$(\partial M, \mathrm{ft}_{<c-k}(\partial M)) = ((U \times L) \cup (V \times L), (U \times L_{<c-k}) \cup (V \times L_{<c-k})).$$
  - **IV** Homology of $\partial M = (U \times L) \cup (V \times L)$.
- If the squares $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ commute, then for the cap product of Proposition 3.7.14 it holds that the cap product with the fundamental class $[\partial M]$ is an isomorphism. We provide an outline of the proofs of commutativity.

A: One can glean commutativity from any proof of (ordinary) Poincaré duality.

B: We show this by using the induced nature of the cap product.

C: Commutativity is shown in much the same way as for square **B**.

D: Commutativity follows from commutativity of square **A**.

$$
\begin{tikzcd}
H_c^{n-r}(U \cap V \times L) \arrow[r, "i^*_{U\cap V}"] \arrow[d, "\alpha^*"'] & H_c^{n-r}(U \cap V \times L_{<k}) \arrow[r, "-\cap[U\cap V\times L]", "\cong"'] \arrow[d, "\alpha^*_{<k}"'] \arrow[dr, phantom, "\mathbf{A}"] & H_{r-1}(U \cap V \times (L, L_{<c-k})) \arrow[d, "\alpha_*^{\mathrm{rel}}"'] & H_{r-1}(U \cap V \times L) \arrow[l, "\pi_{U\cap V*}"'] \arrow[d, "\alpha_*"] \\
H_c^{n-r}(U \times L) \oplus H_c^{n-r}(V \times L) \arrow[r, "i^*_U \oplus i^*_V"] \arrow[d, "\beta^*"'] & H_c^{n-r}(U \times L_{<k}) \oplus H_c^{n-r}(V \times L_{<k}) \arrow[r, "(-\cap[U\times L])\oplus(-\cap[V\times L])", "\cong"'] \arrow[d, "\beta^*_{<k}"'] \arrow[dr, phantom, "\mathbf{B}"] & H_{r-1}(U \times (L, L_{<c-k})) \oplus H_{r-1}(V \times (L, L_{<c-k})) \arrow[d, "\beta_*^{\mathrm{rel}}"'] & H_{r-1}(U \times L) \oplus H_{r-1}(V \times L) \arrow[l, "\pi_{U*}\oplus\pi_{V*}"'] \arrow[d, "\beta_*"] \\
H_c^{n-r}(\partial M) \arrow[r, "i^*"] \arrow[d, "\gamma^*"'] & H_c^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[r, "-\cap[\partial M]"] \arrow[d, "\gamma^*_{<k}"'] \arrow[dr, phantom, "\mathbf{C}"] & H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \arrow[d, "\gamma_*^{\mathrm{rel}}"'] & H_{r-1}(\partial M) \arrow[l, "\pi_*"'] \arrow[d, "\gamma_*"] \\
H_c^{n-r+1}(U \cap V \times L) \arrow[r, "i^*_{U\cap V}"] \arrow[d, "\delta^*"'] & H_c^{n-r+1}(U \cap V \times L_{<k}) \arrow[r, "-\cap[U\cap V\times L]", "\cong"'] \arrow[d, "\delta^*_{<k}"'] \arrow[dr, phantom, "\mathbf{D}"] & H_{r-2}(U \cap V \times (L, L_{<c-k})) \arrow[d, "\delta_*^{\mathrm{rel}}"'] & H_{r-2}(U \cap V \times L) \arrow[l, "\pi_{U\cap V*}"'] \arrow[d, "\delta_*"] \\
H_c^{n-r+1}(U \times L) \oplus H_c^{n-r+1}(V \times L) \arrow[r, "i^*_U \oplus i^*_V"] & H_c^{n-r+1}(U \times L_{<k}) \oplus H_c^{n-r+1}(V \times L_{<k}) \arrow[r, "(-\cap[U\times L])\oplus(-\cap[V\times L])", "\cong"'] & H_{r-2}(U \times (L, L_{<c-k})) \oplus H_{r-2}(V \times (L, L_{<c-k})) & H_{r-2}(U \times L) \oplus H_{r-2}(V \times L) \arrow[l, "\pi_{U*}\oplus\pi_{V*}"'] \\
\mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV}
\end{tikzcd}
$$

We summarize our knowledge concerning the right inverses in the following statement.

Lemma 3.7.15. *For each $m \in \mathbb{N}$, there exist morphisms*

$$\begin{aligned} t^*_{U\cap V}&\colon H^m_c(U\cap V\times L_{<k}) \to H^m_c(U\cap V\times L)\\ t^*_{U,V}&\colon H^m_c(U\times L_{<k})\oplus H^m_c(V\times L_{<k}) \to H^m_c(U\times L)\oplus H^m_c(V\times L)\\ t^*&\colon H^m_c(\mathrm{ft}_{<k}(\partial M)) \to H^m_c(\partial M)\end{aligned}$$

such that

$$\begin{aligned} i^*_{U\cap V}\circ t^*_{U\cap V} &= \mathrm{id},\\ (i^*_U\oplus i^*_V)\circ t^*_{U,V} &= \mathrm{id},\\ i^*\circ t^* &= \mathrm{id}.\end{aligned}$$

Proof. For i^*, the right inverse t^* was defined in (65). The morphism $t^*_{U,V}$ is the direct sum of the morphism t^*_U, defined in equation (75), with the morphism t^*_V. The latter morphism is defined analogously to t^*_U. □

Proposition 3.7.16. *In the diagram on p. 89, square* **B** *commutes.*

Proof. Let $\xi \in H^{n-r}_c(U\times L_{<k})$. Then

$$\begin{aligned} \epsilon(j_{<k})(\xi)\cap[\partial M] &= \pi_*(t^*(\epsilon(j_{<k})(\xi))\cap[\partial M]) && \text{(by definition)}\\ &= \pi_*(\epsilon(j)(t^*_U(\xi))\cap[\partial M]) && \text{(by equation (76))}\\ &= j_*(\pi_*(t^*_U(\xi)\cap j^*([\partial M]))) && \text{(naturality, see [\textbf{May99}, p. 161])}\\ &= j_*(\pi_*(t^*_U(\xi)\cap[U\times L]))\\ &= j_*(\xi\cap[U\times L])\end{aligned}$$

and one can analogously show the corresponding result regarding $\eta \in H^{n-r}_c(V\times L_{<k})$. Therefore, square **B** commutes. □

Proposition 3.7.17. *In the diagram on p. 89, square* **C** *commutes.*

Proof. The proof is very similar to the proof of Proposition 3.7.16. □

Proposition 3.7.18. *For the cap product defined in Proposition 3.7.14, capping with the fundamental class $[\partial M]$ is an isomorphism*

$$-\cap[\partial M]\colon H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \xrightarrow{\cong} H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)).$$

Proof. This follows from commutativity of the diagram on p. 89 in conjunction with the 5-lemma. □

The remainder of the argument supporting generalized Poincaré duality now progresses as it did in the previous two chapters.

Lemma 3.7.19 (Lemma 2.45 on p. 181 in [**Ban10**]). *There is an isomorphism $D_<$ which completes the diagram*

$$\begin{tikzcd} H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[d, dashed, "D_<"] \\ H_r(M,\partial M) \arrow[r, "\partial_*"] & H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \end{tikzcd} \tag{78}$$

to a commutative square. Here, ∂_* is the connecting homomorphism for the long exact sequence of the triple $(M, \partial M, \mathrm{ft}_{<c-k}(\partial M))$. (There is no sign here.)

PROOF. By [**Ban10**, Lemma 2.45], the square

$$\begin{tikzcd} H^{n-r}(M) \arrow[r, "j^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\partial M) \arrow[d, "\cong"', "-\cap[\partial M]"] \\ H_r(M,\partial M) \arrow[r, "\partial_*"] & H_{r-1}(\partial M) \end{tikzcd} \tag{79}$$

commutes. Thus we need to show that an isomorphism $D_<$ exists such that the diagram

$$\begin{tikzcd} H^{n-r}(\partial M) \arrow[r, "\mu^*=\mathrm{incl}^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}(\mathrm{ft}_{<k}(\partial M)) \arrow[d, dashed, "\cong"', "D_<"] \\ H_{r-1}(\partial M) \arrow[r, "\mathrm{incl}^*"] & H_{r-1}(\partial M, \mathrm{ft}_{<c-k}(\partial M)) \end{tikzcd} \tag{80}$$

commutes. But this was done in Proposition 3.7.14. $\square$

The main theorem now follows precisely as it did in the previous chapter.

THEOREM 3.7.20 (Theorem 2.47 on p. 183 in [**Ban10**]). *Let $(\overline{p},\overline{q})$ be a pair of complementary perversities. Let X be an n-dimensional compact, oriented, stratified pseudomanifold with one singular stratum $\Sigma = X_{n-c}$ of dimension $n-c \geq 2$. The link bundle is assumed to be $(\overline{p},\overline{q})$-admissible. Assume that there are open subsets $U, V \subset \Sigma$ such that $U \cup V = \Sigma$ while the link bundle restricts to a trivial bundle over U and V, respectively. We assume X, Σ and L to be oriented compatibly. Let $I^{\overline{p}}X$ and $I^{\overline{q}}X$ be $\overline{p}$- and $\overline{q}$-intersection spaces of X. Then there exists a generalized Poincaré duality isomorphism*

$$D\colon \tilde{H}^{n-r}(I^{\overline{p}}X) \xrightarrow{\cong} \tilde{H}_r(I^{\overline{q}}X).$$

APPENDIX A

Background

1.1. Cofibrations

A map $i\colon A \to X$ has the *homotopy extension property* (HEP) if we can fill in the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[rr, "i_0"] \arrow[dd, "i"'] & & Y \times I \arrow[dl, "h"'] \arrow[dd, "i\times \mathrm{id}"] \\
& Z & \\
X \arrow[ur, "f"] \arrow[rr, "j_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

So we are given a homotopy h and an initial position f of a possible extension of h.

Definition A.1 (p. 43 in [**May99**]). A map $i\colon Y \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP). This means that if $h \circ i_0 = f \circ i$ in the diagram then there exists a map $\tilde{h}$ that makes the diagram commute.

So if (Y, X) is a pair, a given homotopy $h\colon Y \times I \to Z$ with given initial position f may be extended to a homotopy $\tilde{h}\colon X \times I \to Z$. For X, Y Hausdorff spaces a cofibration $i\colon X \to Y$ is an inclusion with closed image.

An inclusion may be "replaced up to homotopy" by a cofibration: Given an inclusion $i\colon Y \hookrightarrow X$, note that there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, hook, "j_0"] \arrow[dr, hook, "i"'] & & M_i \arrow[dl, "r"] \\
& X &
\end{tikzcd}$$

with maps $j_0\colon Y \to M_i, y \mapsto [y, 0]$ and

$$r\colon M_i \to X, \begin{cases} [y,t] \mapsto i(y), \\ [x,1] \mapsto x. \end{cases}$$

Then j_0 is a cofibration by [**Spa66**, Theorem 12, Section 4, Chapter 1]. Thus we may replace the range of i by a homotopy equivalent space $M_i \simeq X$ and obtain a replacement of i by a cofibration j_0. Regarding computations of homology groups, this is a convenient fact, due to the next theorem.

Theorem A.2 (p. 108 in [**May99**]). *For any cofibration $i\colon Y \to X$, the quotient map $q\colon (X, Y) \to (X/Y, \mathrm{pt})$ induces an isomorphism*

$$H_*(X, Y) \cong H_*(X/Y, \mathrm{pt}) = \tilde{H}_*(X/Y).$$

Proposition A.3. *Let $i\colon Y \hookrightarrow X$ be an inclusion. Then $H_*(X, Y) \cong \tilde{H}_*(\mathrm{cone}\,(i))$.*

Proof. Replace i by the canonical cofibration $Y \to M_i$, as above, and obtain

$$H_*(M_i, Y) \cong \tilde{H}_*(M_i/Y)$$

But $M_i/Y \simeq \mathrm{cone}\,(i)$ (recall that the canonical replacement cofibration j_0 includes Y at the top or 0-end of M_i) and furthermore $M_i \simeq X$. Thus we obtain

$$H_*(X, Y) \cong H_*(M_i, Y) \cong \tilde{H}_*(M_i/Y) \cong \tilde{H}_*(\mathrm{cone}\,(i)) \tag{A.1}$$

which proves the claim. □

Suppose we are given a map $f\colon Y \to X$ and its homotopy cofiber (or mapping cone) $\mathrm{cone}\,(f) = X \cup_f \mathrm{cone}\,(Y)$. If f is an inclusion, then Proposition A.3 gives information concerning the reduced homology of $\mathrm{cone}\,(f)$ in terms of the homology of the pair (X, Y). What about the case of f not being an inclusion? We define

$i_0 \colon Y \to M_f$ by $y \mapsto [y, 0]$ and let j_1 be the trivial inclusion. We obtain a non-commutative diagram

(A.2)
$$\begin{tikzcd} Y \arrow[rr, hook, "f"] \arrow[dr, hook, "i_0"'] & & X \arrow[dl, hook', "j_1"] \\ & M_f & \end{tikzcd}$$

which commutes up to homotopy. A homotopy $H \colon i_0 \simeq j_1 \circ f$ may be defined as

$$H \colon Y \times I \to M_f,\ (y, t) \mapsto [y, t]$$

This is continuous and $H(\cdot, 0) = i_0$, $H(\cdot, 1) = j_1 \circ f$. A basic result on mapping cones (see e.g. [**Coh70**]) shows $i_0 \simeq j_1 \circ f \Rightarrow \mathrm{cone}\,(i_0) \simeq \mathrm{cone}\,(j_1 \circ f)$. But

$$\begin{aligned} \mathrm{cone}\,(j_1 \circ f) &= M_f \cup_{j_1 \circ f} \mathrm{cone}\,(Y) \\ &\simeq X \cup_f \mathrm{cone}\,(Y) && (M_f \simeq X) \\ &= \mathrm{cone}\,(f). \end{aligned}$$

To the inclusion i_0 we may apply the earlier considerations regarding homology, yielding

$$\begin{aligned} \tilde{H}_*(\mathrm{cone}\,(f)) &\cong \tilde{H}_*(\mathrm{cone}\,(j_1 \circ f)) && (\mathrm{cone}\,(f) \simeq \mathrm{cone}\,(j_1 \circ f)) \\ &\cong \tilde{H}_*(\mathrm{cone}\,(i_0)) && (i_0 \simeq j_1 \circ f) \\ &\cong H_*(M_f, Y). \end{aligned}$$

Thus we obtain a relationship analogous to the one in Proposition (A.3).

LEMMA A.4. *Let*

$$\cdots \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} A_{i-2} \xrightarrow{d_{i-2}} \cdots$$

be an exact sequence of rational vector spaces. Let there be an integer $i \in \mathbb{Z}$, a rational vector space B, as well as linear maps f and g such that the triangle

$$\begin{tikzcd} A_i \arrow[rr, "d_i"] \arrow[dr, "f"'] & & A_{i-1} \\ & B \arrow[ur, "g"', "\cong"] & \end{tikzcd}$$

commutes. Then

$$\cdots \xrightarrow{d_{i+1}} A_i \xrightarrow{f} B \xrightarrow{d_{i-1} \circ g} A_{i-2} \xrightarrow{d_{i-2}} \cdots$$

is an exact sequence

PROOF. Exactness at A_i: We have to show that $\mathrm{im}\, d_{i+1} = \ker f$. Now, $\mathrm{im}\, d_{i+1} = \ker d_i$, so it is equivalent to show $\ker d_i = \ker f$. Let $x \in \ker d_i$. Then commutativity implies $x \in \ker g \circ f$. If $x \neq 0$, then g being an isomorphism implies $x \in \ker f$. On the other hand, $x \in \ker f$ is also an element of $\ker g \circ f = \ker d_i$.

Exactness at B: We have to show that $\mathrm{im}\, f = \ker d_{i-1} \circ g$. Assume $y = f(x) \in \mathrm{im}\, f$. Then $g(y) \in \mathrm{im}\, g \circ f = \mathrm{im}\, d_i = \ker d_{i-1}$. On the other hand, let $y \in \ker d_{i-1} \circ g$. Then $d_{i-1} \circ g(y) = 0$ implies $g(y) \in \ker d_{i-1} = \mathrm{im}\, d_i = \mathrm{im}\, g \circ f$ and thus, $y \in \mathrm{im}\, f$.

Exactness at A_{i-2}: We have to show that $\mathrm{im}\, d_{i-1} \circ g = \ker d_{i-2}$. Let $x \in \ker d_{i-2} = \mathrm{im}\, d_{i-1}$. Then $x \in \mathrm{im}\, d_{i-1} \circ g$, because g is an isomorphism. On the other hand, $x \in \mathrm{im}\, d_{i-1} \circ g$ implies $x \in \mathrm{im}\, d_{i-1} = \ker d_{i-2}$. $\square$

Consider the homology long exact sequence of the pair (M_f, Y). Diagram (A.2) fits into this sequence:

$$\begin{tikzcd} \dots \arrow[r] & H_q(Y) \arrow[r, "i_{0*}"] \arrow[dr, "f_*"'] & H_q(M_f) \arrow[r, "q_*"] & H_q(M_f, Y) \arrow[r, "\partial_*"] & H_{q-1}(Y) \arrow[r] & \dots \\ & & H_q(X) \arrow[u, "\cong"', "j_{1*}"] & & & \end{tikzcd}$$

An application of Lemma A.4 to this sequence yields:

Lemma A.5. *Given a continuous map $f\colon Y \to X$, the sequence*

$$\dots \longrightarrow H_q(Y) \xrightarrow{f_*} H_q(X) \xrightarrow{q_* \circ j_{1*}} H_q(M_f, Y) \xrightarrow{\partial_*} H_{q-1}(Y) \longrightarrow \dots$$

is exact. The analogous result in cohomology holds as well.

Given a pair (A, B) and a map $f\colon C \to B$, we can form the map

$$\mathrm{incl} \circ f\colon C \to A.$$

We consider the homology long exact sequence of the triple $(M_{\mathrm{incl}\circ f}, M_f, C)$:

$$\cdots \xrightarrow{j_*} H_{n+1}(M_{\mathrm{incl}\circ f}, M_f) \xrightarrow{\overline{\partial}_*} H_n(M_f, C) \xrightarrow{i_*} H_n(M_{\mathrm{incl}\circ f}, C) \xrightarrow{j_*} H_n(M_{\mathrm{incl}\circ f}, M_f)$$

The connecting homomorphism $\overline{\partial}_*$ is induced from the connecting homomorphism δ_* of the long exact sequence of the pair $(M_{\mathrm{incl}\circ f}, M_f)$. Here, the inclusions

$$i\colon (M_f, C) \hookrightarrow (M_{\mathrm{incl}\circ f}, C)$$
$$j\colon (M_{\mathrm{incl}\circ f}, C) \hookrightarrow (M_{\mathrm{incl}\circ f}, M_f)$$

induce the maps i_* and j_*. Notice that i includes M_f in $M_{\mathrm{incl}\circ f}$ in the only way possible, while j includes C in the 0-end of M_f. In a similar vein, the inclusion at the 1-end,

$$k\colon (A, B) \hookrightarrow (M_{\mathrm{incl}\circ f}, M_f)$$

is a homotopy equivalence of pairs (i.e. the corresponding homotopy respects the pairs). We define a map

$$d\colon (M_{\mathrm{incl}\circ f}, C) \to (A, B),\ d([x,t]_{M_{\mathrm{incl}\circ f}}) = \begin{cases} \mathrm{incl}\circ f(x), & x \in C, \\ x, & x \in A. \end{cases}$$

Then the triangle

$$\begin{tikzcd} H_n(M_{\mathrm{incl}\circ f}, C) \arrow[rr, "j_*"] \arrow[dr, "d_*"'] & & H_n(M_{\mathrm{incl}\circ f}, M_f) \\ & H_n(A, B) \arrow[ur, "\cong", "k_*"'] & \end{tikzcd}$$

commutes because the corresponding triangle

$$\begin{tikzcd} (M_{\mathrm{incl}\circ f}, C) \arrow[rr, "j"] \arrow[dr, "d"'] & & (M_{\mathrm{incl}\circ f}, M_f) \\ & (A, B) \arrow[ur, "k"'] & \end{tikzcd}$$

commutes up to homotopy. Thus an application of Lemma A.4 yields:

Lemma A.6. *Given a pair (A, B) and continuous map $f\colon C \to B$, the sequence*

$$\cdots \xrightarrow{\overline{\partial}_* \circ k_*} H_n(M_f, C) \xrightarrow{i_*} H_n(M_{\mathrm{incl}\circ f}, C) \xrightarrow{d_*} H_n(A, B) \xrightarrow{\overline{\partial}_* \circ k_*} H_{n-1}(M_f, C) \xrightarrow{i_*} \cdots$$

is exact.

We require a restatement of the last result.

LEMMA A.7. *Given a pair (A, B) and continuous map $f\colon C \to B$, the sequence*

$$\begin{tikzcd} \cdots \arrow[r, "q_*\circ j_{1*}\circ\partial_*"] & H_n(M_f, C) \arrow[r, "i_*"] & H_n(M_{\mathrm{incl}\circ f}, C) \arrow[r, "d_*"] & H_n(A, B) \arrow[dl, "q_*\circ j_{1*}\circ\partial_*"'] \\ & & H_{n-1}(M_f, C) \arrow[r, "i_*"] & \cdots \end{tikzcd}$$

is exact. Notice that ∂_* is the connecting homomorphism of the long exact sequence of the pair (A, B).

PROOF. From Lemma A.6, we get a sequence

$$\cdots \xrightarrow{\overline{\partial}_*\circ k_*} H_n(M_f, C) \xrightarrow{i_*} H_n(M_{\mathrm{incl}\circ f}, C) \xrightarrow{d_*} H_n(A, B) \xrightarrow{\overline{\partial}_*\circ k_*} H_{n-1}(M_f, C) \xrightarrow{i_*} \cdots$$

which is exact. For $n \in \mathbb{Z}$, consider the following diagram:

$$\begin{tikzcd} H_{n+1}(M_{\mathrm{incl}\circ f}, C) \arrow[r, "j_*"] \arrow[dr, "d_*"'] & H_{n+1}(M_{\mathrm{incl}\circ f}, M_f) \arrow[rr, "\overline{\partial}_*"] \arrow[dr, "\delta_*"] & & H_n(M_f, C) \\ & H_{n+1}(A, B) \arrow[u, "k_*", "\cong"'] \arrow[dr, "\partial_*"] & H_n(M_f) \arrow[ur, "\mathrm{incl}_*"] & \\ & & H_n(B) \arrow[u, "k_*"] & \end{tikzcd}$$

This diagram commutes. Commutativity of the left triangle was shown in the proof of Lemma A.6. The right triangle commutes by definition. The remaining part of the diagram also commutes. $\square$

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