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Equivariant ϵ-conjecture for unramified twists of $\mathbb{Z}_p(1)$

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Abstract

Let $\mathbb{Q}_p \subseteq K \subseteq L$ be a tower of finite Galois extensions and V be a p-adic de Rham representation of the absolute Galois group G_K of K. In the first part of my work I extend the conjecture of D. Benois and L. Berger in [BB] to the case of a not necessary abelian extension L/K of p-adic fields, which relates the equivariant local epsilon constant attached to V (and to the extension L/K) to a natural algebraic invariant coming from the Galois cohomology groups of V. I show the functorial properties of this conjecture and prove its validity for unramified extensions L/K with $G = Gal(L/K)$ of order prime to p and $V = \mathbb{Q}_p(1)(\chi^{ur})$.

In the second part of the work we study Galois descent of K_1-groups of group algebras with coefficients in certain subrings of the ring of integers of $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. These results were important for the above mentioned reformulation of the conjecture.

Zusammenfassung

Seien $\mathbb{Q}_p \subseteq K \subseteq L$ endliche galoissche Körpererweiterungen und V eine p-adische de Rham Darstellung der absoluten Galoisgruppe G_K von K. Im ersten Teil meiner Arbeit setze ich die Vermutung von D. Benois und L. Berger auf den Fall einer nicht notwendigerweise abelschen Erweiterung L/K von p-adischen Körpern fort. Diese Vermutung stellt einen Zusammenhang zwischen den equivarianten lokalen ϵ-Faktoren assoziiert zu V (und zu der Körpererweiterung L/K) und der natürlichen algebraischen Invariante kommend von den Galoiskohomologiegruppen von V her. Ich zeige die funktoriellen Eigenschaften der Vermutung und beweise ihre Gültigkeit im Falle einer unverzweigten Körpererweiterung L/K mit der Galoisgruppe $G = Gal(L/K)$, deren Ordnung prim zu p ist, und $V = \mathbb{Q}_p(1)(\chi^{ur})$.

In dem zweiten Teil dieser Arbeit gewinnen wir Galoisabstiegsresultate für die K_1-Gruppen von den Gruppenalgebren mit Koeffizienten in einem Unterring des Ganzheitsrings von $\mathbb{C}_p$, der Komplettierung des algebraischen Abschlusses von $\mathbb{Q}_p$. Diese Resultate waren für die oben genannte Umformulierung der Vermutung wichtig.

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Contents

1 Introduction

The central aim of Iwasawa theory consists of describing the relation between special values of L-functions and Galois cohomology of objects M occurring in arithmetic algebraic geometry, i.e. sets of solutions of Diophantine equations or more generally motives. This is a very actual and active research area in modern number theory. The most prominent example besides the analytic class number formula is certainly the Birch-Swinnerton-Dyer conjecture, which we would like to recall here briefly: Let E be an elliptic curve defined over $\mathbb{Q}$; then attached to E we have the complex Hasse-Weil L-function $L(E,s)$ and the Birch-Swinnerton-Dyer conjecture predicts, firstly, that the order of vanishing at $s=1$ of this complex analytic function (known to be holomorphic on the full complex plane $\mathbb{C}$ by Taylor, Wiles et al.) equals the rank of the Mordell-Weil group $E(\mathbb{Q})$ of E. Secondly, the conjecture states that the leading term (of the Taylor expansion of $L(E,s)$ at $s=1$) can be expressed in terms of the most important arithmetic invariants of the curve E - up to the period and regulator all these invariants can be rephrased using certain Galois cohomology groups.

To arbitrary motives this conjecture has been extended by Fontaine, Perrin-Riou and Kato and is called the Tamagawa Number Conjecture (TNC). Again we have the complex L-function $L(M,s)$ attached to a motive M, which is believed to satisfy the following functional equation relating $L(M,s)$ to the L-function $L(M^*(1),s)$ of the (Kummer) dual motive $M^*(1)$ of M:

$$L(M,s) = \epsilon(M,s)\frac{L_\infty(M^*(1),-s)}{L_\infty(M,s)}L(M^*(1),-s).$$

Here L_∞ are so called Euler-factors at infinity (attached to M and $M^*(1)$, respectively), which is built up by certain Γ-factors and certain powers of 2 and π (depending on the Hodge structure of M and $M^*(1)$, respectively), while the so called ϵ-factor decomposes into local factors

$$\epsilon(M,s) = \prod_{\nu\in S}\epsilon_\nu(M,s),$$

where $\epsilon_\infty(M,s)$ is a constant equal to a power of i[1].

In the following we assume the validity of the functional equation. Then, taking leading coefficients L^* of the Taylor expansion of $L(M,s)$ at $s=0$ induces

$$L^*(M) = (-1)^\eta\epsilon(M,0)\frac{L^*_\infty(M^*(1))}{L^*_\infty(M)}L(M^*(1)),$$

where η denotes the order of vanishing at $s=0$ of the completed L-function $L_\infty(M^*(1),s)L(M^*(1),s)$.

Example 1.1 *For the motive $M = h^1(A)(1)$ of an abelian variety we have*

$$L_\infty(M^*(1),s) = L_\infty(M,s) = 2(2\pi)^{-(s+1)}\Gamma(s+1), \qquad \eta = 0,$$

$$L^*_\infty(M^*(1)) = L^*_\infty(M) = \pi^{-1}, \quad \epsilon_\infty(M,0) = -1.$$

[1] We fix once and for all the complex period $2\pi i$, i.e. a square root of -1.

It is by no means evident that the validity of the TNC for M is equivalent to the validity of the TNC for $M^*(1)$ under this functional equation on the complex analytic side and under Artin/Verdier- or Poitou/Tate-duality on the p-adic Galois cohomology side. To the contrary, they are only compatible if and only if the (local) ϵ-factors are in a specific way related to certain (local) cohomology groups, see the explanation in [V]. It is the content of the (absolute) ϵ-conjecture that these required properties of the ϵ-factors (attached to p-adic representations) hold; sometimes this conjecture is also called Local Tamagawa Number Conjecture. It is known for certain semi-stable representations by the work [Be1].

The key idea of Iwasawa theory is now to study the above conjectural properties of L-functions and p-adic Galois cohomology for whole (p-adicly varying) families of motives simultaneously. In this spirit Fukaya and Kato [FK] formulated recently a vast generalization of the TNC, which they call the ζ-isomorphism conjecture. Their sort of 'meta'-conjecture builds now the framework of actual research in this area. In particular, its validity would imply the validity of the Equivariant Tamagawa Number Conjecture (ETNC) as formulated by Burns and Flach [BF]. The compatibility with the functional equation relies now on the behavior of the ϵ-constants in families. This is the content of the (equivariant) ϵ-isomorphism conjecture again due to Fukaya and Kato, which we would like to refer to simply as the equivariant ϵ-conjecture. Besides the above stated compatibility property the significance of this conjecture is the following: Coates, Fukaya, Kato, Sujatha and Venjakob [CFKSV] have formulated and investigated a (non-commutative) Iwasawa main conjecture for elliptic curves without complex multiplication with respect to the GL_2-tower of number fields, which arises by adjoining the p-power torsion points of E to $\mathbb{Q}$. This conjecture, in particular, claims the existence of (non-commutative) p-adic L-functions satisfying natural interpolation properties. Fukaya and Kato have proved that the validity of both their ζ-isomorphism conjecture and their equivariant ϵ-conjecture implies the existence of this totally new type of p-adic L-function for E. On the other hand, by a result of Burns and Venjakob [BuV] the validity of the GL_2-main conjecture for an elliptic curve $E/\mathbb{Q}$ implies under certain conditions the validity of the ETNC for E with respect to any finite quotient G of the GL_2-tower, if the corresponding ϵ-isomorphism for subquotients of T_pE with respect to G exists.

But it is not only this rather philosophical relation between these conjectures which draws the attention to the ϵ-conjecture: one hopes that a proof of this conjecture also would give hints for a construction of these new p-adic L-functions for GL_2. Indeed, one of Iwasawa's constructions of the p-adic L-function for the cyclotomic $\mathbb{Z}_p$-extension is based on the construction of the so called Coleman-homomorphism [Co]. On the other hand, Perrin-Riou [P-R] and Kato (unpublished) have constructed a map associated with crystalline p-adic representations V, which interpolates the exponential map of Bloch-Kato for V in the cyclotomic $\mathbb{Z}_p$-extension of a local field and which is roughly speaking the inverse of the Coleman-map in case of the Tate-module $V = \mathbb{Q}_p(1)$. Using these techniques cases of the Local Tamagawa Number Conjecture for the Tate-motive could be proved by Burns and Flach [BF1]. Other cases ($l \neq p$) have been proved by S. Yasuda [Y].

2 Known formulations of the ϵ-conjecture

In this section we shortly recall the known formulations of the ϵ-conjecture starting with the absolute one and finishing with the most general equivariant statement of [FK]. Through this section let p be a prime number and let K denote a finite extension of $\mathbb{Q}_p$. Also we fix once and for all a generator ξ of $\mathbb{Z}_p(1)$), where $\mathbb{Z}_p(1)$ is the twist of the trivial representation $\mathbb{Z}_p$ of $G_{\mathbb{Q}_p}$ by the cyclotomic character of $G_{\mathbb{Q}_p}$.

2.1 $C_{ep,K}(V)$ by B. Perrin-Riou

We consider a potentially semi-stable (pst) p-adic representation V of the absolute Galois group G_K. The formulation of the conjecture $C_{ep,K}(V)$ needs the following steps to be done (for details see [P-R]):

1. Define $\tilde{\Delta}_{ep,K}(V) := \bigotimes_{0\le i\le 2} (\det_{\mathbb{Q}_p} \mathrm{H}^i(K,V))^{(-1)^i} \otimes \det_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p} V)$, where $\mathrm{Ind}_{K/\mathbb{Q}_p} V$ is the induced representation of V from G_K to $G_{\mathbb{Q}_p}$. Here the determinants over $\mathbb{Q}_p$ are just the highest exterior products of the corresponding $\mathbb{Q}_p$-vector spaces.

2. Define $\Delta_K(V) := (\det_{\mathbb{Q}_p}(\mathrm{D}^K_{dR}(V)))^{-1} \otimes \det_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p} V)$, where B_{dR} is a Fontaine's period ring for de Rham representations and $\mathrm{D}^K_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

3. Set $t_V(K) := \mathrm{D}^K_{dR}(V)/Fil^0\mathrm{D}^K_{dR}(V)$ and $\mathrm{D}_{cris}(V) := (B_{cris} \otimes_{\mathbb{Q}_p} V)^{G_K}$, where B_{cris} is another Fontaine's period ring for crystalline representations. Using the exponential map of Bloch-Kato for V and its Kummer dual representation $V^*(1)$ deduce an exact sequence (of $\mathbb{Q}_p$-spaces) connecting the Galois cohomology groups $\mathrm{H}^i(K,V)$ with $\mathrm{D}^K_{dR}(V)$
$$\begin{aligned} 0 \to \mathrm{H}^0(K,V) \to \mathrm{D}_{cris}(V) \to \mathrm{D}_{cris}(V) \oplus t_V(K) \to \mathrm{H}^1(K,V) \\ \to \mathrm{D}_{cris}(V^*(1))^* \oplus t^*_{V^*(1)}(K) \to \mathrm{D}_{cris}(V^*(1))^* \to \mathrm{H}^2(K,V) \to 0. \end{aligned} \tag{2.1}$$
From this exact sequence construct $\tilde{e}_V : \Delta_K(V) \xrightarrow{\cong} \tilde{\Delta}_{ep,K}(V)$ identifying $t^*_{V^*(1)}(K) \cong Fil^0\mathrm{D}^K_{dR}(V)$ such that
$$0 \to t^*_{V^*(1)}(K) \to \mathrm{D}^K_{dR}(V) \to t_V(K) \to 0 \tag{2.2}$$
is exact.

4. Choose a G_K-stable $\mathbb{Z}_p$-lattice T of V and define
$$\tilde{\Delta}_{ep,K}(T) := \bigotimes_{0\le i\le 2} (\det_{\mathbb{Z}_p}\mathrm{H}^i(K,T))^{(-1)^i} \otimes \det_{\mathbb{Z}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p} T).$$
The general theory of (not necessary commutative) determinants can be found in Appendix B. Note that $\tilde{\Delta}_{ep,K}(T)$ is a $\mathbb{Z}_p$-submodule of $\tilde{\Delta}_{ep,K}(V)$, which does not depend on the choice of T.

5. Set $\Delta_{ep,K,\mathbb{Z}_p}(V) := \tilde{e}_V^{-1}(\bigotimes_{0\le i\le 2}(\det_{\mathbb{Z}_p}\mathrm{H}^i(K,T))^{(-1)^i}\otimes\det_{\mathbb{Z}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}T))$ – a $\mathbb{Z}_p$-submodule of $\Delta_K(V)$.

6. For $j\in\mathbb{Z}$ set $h_j = \dim_K(\mathrm{Fil}^j\mathrm{D}^K_{dR}(V)/\mathrm{Fil}^{j+1}\mathrm{D}^K_{dR}(V))$ and

$$\Gamma^*(j) = \begin{cases} \Gamma(j) = (j-1)! & \text{if } j>0 \\ \lim_{s\to j}(s-j)\Gamma(s) = (-1)^j((-j)!)^{-1} & \text{if } j\le 0. \end{cases}$$

Conjecture 2.1 **($C_{ep,K}(V)$)** *Let V be a pst p-adic representation of G_K. Then for every $\omega\in\Delta_K(V)$ we have*

$$\Delta_{ep,K,\mathbb{Z}_p}(V) = \mathbb{Z}_p\det(-\phi\mid \mathrm{D}_{cris}(V^*(1)))\prod_j\Gamma^*(-j)^{-h_j(V)[K:\mathbb{Q}_p]}\eta_V(\omega)\omega,$$

where ϕ denote the Frobenius map acting on $\mathrm{D}_{cris}(V^*(1))$ and $\eta_V:\Delta_K(V)\to\mathbb{Q}_p^{ur}$ is described below.

Remark 2.2 *$C_{ep,K}(V)$ has the following functorial properties:*

1. *The conjectures $C_{ep,K}(V)$ and $C_{ep,K}(V^*(1))$ are equivalent.*

2. *Let $0\to V'\to V\to V''\to 0$ be an exact sequence of pst p-adic representations of G_K. If $C_{ep,K}$ holds for two of the representations V,V',V'', then it also holds for the third one.*

3. *If L/K is a finite extension and if V is a pst representation of G_L, the induced representation $\mathrm{Ind}_{L/K}(V)$ from G_L to G_K is also pst, and we check that the conjectures $C_{ep,L}(V)$ and $C_{ep,K}(\mathrm{Ind}_{L/K}(V))$ are equivalent.*

Proof. See [P-R, C.2.9]. □

Theorem 2.3 *Let K be unramified over $\mathbb{Q}_p$. The conjecture $C_{ep,K}(V)$ holds for every ordinary p-adic representation V.*

Proof. See [P-R, C.2.10]. □

The comparison isomorphism

$$B_{dR}\otimes_K\mathrm{D}^K_{dR}(V)\xrightarrow{\cong}B_{dR}\otimes_{\mathbb{Q}_p}V$$

gives a canonical $\mathbb{Q}_p$-linear injection

$$\xi_V:\Delta_K(V)\to\overline{K}t^{t_H(V)}\subset B_{dR}$$

for a suitable integer $t_H(V)$ and an additive generator t of $\mathbb{Z}_p(1)$ in B_{cris}. We set $\tilde{\xi}_V := t^{-t_H(V)}\xi_V$. Let d_K be the discriminant of K over $\mathbb{Q}_p$ and $\epsilon(V)$ be the ϵ-constant associated to V (see [Ta1]).

Lemma 2.4 *If $\omega \in \Delta_K(V)$, then*

$$\frac{\tilde{\xi}_V}{|d_K|^{\dim(V)/2}\,\epsilon(V)} \in \mathbb{Q}_p^{ur}.$$

In particular, its absolute value $\eta_V(\omega)$ belongs to $p^{\mathbb{Z}}$.

Proof. See [P-R, C.2.8]. □

Remark 2.5 *If we set*

$$\delta_{V,K/K} := \det(-\phi \mid \mathrm{D}_{cris}(V^*(1))) \prod_j \Gamma^*(-j)^{-h_j(V)[K:\mathbb{Q}_p]} \eta_V \circ \tilde{e}_V^{-1}$$

then $C_{ep,K}(V)$ says that the image of $\tilde{\Delta}_{ep,K}(T)$ under $\delta_{V,K/K}$ is $\mathbb{Z}_p^{ur}$. In other words, the determinant of the restriction of $\delta_{V,K/K}$ to $\tilde{\Delta}_{ep,K}(T)$, which a priori is an element of $(\mathbb{Q}_p^{ur})^\times$, belongs to $(\mathbb{Z}_p^{ur})^\times = K_1(\mathbb{Z}_p^{ur})$.

Here the notation for the map $\delta_{V,K/K}$ was taken from the conjecture 2.6 below, in order to show that all conjectures $C_{ep,K}(V)$, 2.6 and later 2.18 can be formulated in one way. Note that the integrality of the determinant of $\delta_{V,K/K}$ gives the required relation (mentioned in the introduction) between the ϵ-constant and the Galois cohomology groups associated to V, thus is the content of the absolute ϵ-conjecture.

2.2 $C_{ep}(L/K, V)$ by D. Benois and L. Berger

Let L be a finite abelian extension of K and let $G = Gal(L/K)$ denote the Galois group of L/K. We consider a pst p-adic representation V of G_K. The formulation of the conjecture $C_{ep}(L/K, V)$ (an equivariant version of $C_{ep,K}(V)$) needs the following steps to be done (for details see [BB]):

1. Define

$$\begin{aligned}\Delta_{ep}(L/K, V) &:= \det_{\mathbb{Q}_p[G]} R\Gamma(L, V) \otimes \det_{\mathbb{Q}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \cong \\ &\cong \bigotimes_{0\le i\le 2} (\det_{\mathbb{Q}_p[G]} \mathrm{H}^i(L, V))^{(-1)^i} \otimes \det_{\mathbb{Q}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} V).\end{aligned}$$

2. Construct $\Delta_{ep}(L/K, V) \xrightarrow{\cong} (\det_{\mathbb{Q}_p[G]}(\mathrm{D}_{dR}^L(V)))^{-1} \otimes \det_{\mathbb{Q}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)$ using exact sequence of $\mathbb{Q}_p[G]$-modules similar to (2.1) and (2.2) connecting the Galois cohomology groups $\mathrm{H}^i(L, V)$ with $\mathrm{D}_{dR}^L(V)$.

3. Let $\hat{\sigma}$ be an element of $Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$, which acts trivially on the p^nth roots of unity for all n. Let σ denote the restriction of $\hat{\sigma}$ to $\mathbb{Q}_p^{ur}$ and let $a_{V,L/K} := \det_{\mathbb{Q}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)(\hat{\sigma}) \in \mathbb{Z}_p[G]^\times = K_1(\mathbb{Z}_p[G])$.
   Define
   $$\mathbb{Z}_p[G]_{V,L/K} = \left\{ x \in \widehat{\mathbb{Z}_p^{ur}}[G] \mid \sigma(x) = a_{V,L/K}\, x \right\}$$
   and $\mathbb{Q}_p[G]_{V,L/K} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G]_{V,L/K}$.

4. Normalize the comparison isomorphism

$$B_{dR} \otimes_L \mathrm{D}^L_{dR}(V) \xrightarrow{\cong} B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V$$

by the equivariant ϵ-constant $\epsilon(L/K, V)$ (cf. [BB, Lem. 2.4.1]), a power of ξ, $\Gamma(j)$-factors (defined in the similar way as in the subsection 2.1) and some other constants depending on L/K and $\dim_{\mathbb{Q}_p} V$ to construct an isomorphism

$$\beta_{V,L/K} : (\det_{\mathbb{Q}_p[G]}(\mathrm{D}^L_{dR}(V)))^{-1} \otimes \det_{\mathbb{Q}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \xrightarrow{\cong} \mathbb{Q}_p[G]_{V,L/K}.$$

5. Combining (2) and (4) obtain $\delta_{V,L/K} : \Delta_{ep}(L/K, V) \xrightarrow{\cong} \mathbb{Q}_p[G]_{V,L/K}$.

6. Choose a G_K-stable $\mathbb{Z}_p$-lattice T of V and define

$$\Delta_{ep}(L/K, T) := \det_{\mathbb{Z}_p[G]} R\Gamma(L, T) \otimes \det_{\mathbb{Z}_p[G]}(\mathrm{Ind}_{L/\mathbb{Q}_p} T).$$

Note that $\Delta_{ep}(L/K, T)$ is a $\mathbb{Z}_p[G]$-submodule of $\Delta_{ep}(L/K, V)$, which does not depend on the choice of T.

Conjecture 2.6 **($C_{ep}(L/K, V)$)** *Let $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ be a pst p-adic representation of G_K and L/K be a finite abelian extension, then*

$$\delta_{V,L/K} : \Delta_{ep}(L/K, T) \xrightarrow{\cong} \mathbb{Z}_p[G]_{V,L/K}.$$

Remark 2.7 *$C_{ep}(L/K, V)$ has the following functorial properties:*

1. *The conjectures $C_{ep}(L/K, V)$ and $C_{ep}(L/K, V^*(1))$ are equivalent.*

2. *Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of pst p-adic representations of G_K. If $C_{ep}(L/K)$ holds for two of the representations V, V', V'', then it also holds for the third one.*

3. *If $C_{ep}(L/K, V)$ holds, and M/K is an extension of K contained in L, then the conjectures $C_{ep}(L/M, V)$ and $C_{ep}(M/K, V)$ hold, too.*

4. *If $C_{ep}(L/K, V)$ holds, and $\eta : G \to \overline{\mathbb{Q}_p}^{\times}$ is a character of G, then the conjecture $C_{ep,K}(V(\eta))$ holds.*

Theorem 2.8 *If K is an unramified extension of $\mathbb{Q}_p$ and V is a crystalline representation of G_K, then:*

1. *the conjecture $C_{ep}(L/K, V)$ holds for all extensions L/K contained in the cyclotomic extension $K_\infty = \cup_{n=1}^{\infty} K(\zeta_{p^n})$.*

2. *the conjecture $C_{ep}(L, V) = C_{ep,L}(V)$ holds for all L contained in the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$.*

3. *the conjectures $C_{ep}(K, V(\eta))$ hold for all Dirichlet characters η of the Galois group $Gal(K_\infty/K)$ of the cyclotomic extension.*

Proof. See [BB, Thm. 4.1.3 and Cor. 4.4.5]. □

Remark 2.9 *$C_{ep}(L/K,V)$ is equivalent to the following statement: the determinant of $\delta_{V,L/K}$ restricted to $\Delta_{ep}(L/K,T)$ is an element of $\mathbb{Z}_p[G]^{\times}_{V,L/K} = \left\{x \in \widehat{\mathbb{Z}_p^{ur}}[G]^{\times} = K_1(\widehat{\mathbb{Z}_p^{ur}}[G]) \mid \sigma(x) = a_{V,L/K}x\right\}$, which gives a relation between the equivariant ϵ-constant and the Galois cohomology groups associated to V, hence is the content of the equivariant ϵ-conjecture.*

Remark 2.10 *The property (4) of Remark 2.7 relates the equivariant ϵ-conjecture for V to the absolute ϵ-conjectures for the twisted representations $V(\eta)$. Later we will formulate a (possibly non-abelian) equivariant ϵ-conjecture using similar relations.*

Remark 2.11 *The $C_{ep,K}(V)$ is a special case of $C_{ep}(L/K,V)$ (cf. Remarks 2.5 and 2.9). In this case the group algebra $\mathbb{Z}_p[G]$ degenerates to $\mathbb{Z}_p$.*

2.3 LTNC by T. Fukaya and K. Kato

2.3.1 ϵ-isomorphisms of de Rham representations

[FK] gives a way to construct a generalization of the isomorphism $\delta_{V,K/K}$ of Remark 2.5. We give a short description of this ϵ-isomorphism (for details see (loc. cit.)).

Let F be a finite extension of $\mathbb{Q}_p$, we set

$$\tilde{F} := \widehat{\mathbb{Q}_p^{ur}} \otimes_{\mathbb{Q}_p} F = W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} F,$$

where $W(\overline{\mathbb{F}_p})$ is the Wittring of $\overline{\mathbb{F}_p}$.

Let V be a finite dimensional vector space over F endowed with the continuous action of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ which is de Rham, hence pst by [Be, Cor. 5.22]. Then analogously to the subsection 2.1 we can define an isomorphism

$$\epsilon_{F,\xi}(V) = \epsilon_{F,\xi}(\mathbb{Q}_p, V) : \mathbf{d}_{\tilde{F}}(0) \xrightarrow{\cong} \tilde{F} \otimes_F \tilde{\Delta}_{ep,\mathbb{Q}_p}(F,V),$$

$$\tilde{\Delta}_{ep,\mathbb{Q}_p}(F,V) := \mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_F(V).$$

Here $\mathbf{d}_F$ is the not necessary commutative determinant, which can be replaced in this case by the commutative one. For the definition of not necessary commutative determinants, K_1-groups and the connection between them see Appendix B.

The ϵ-isomorphisms above have the following properties:

(1) Let $\Psi(\mathbb{Q}_p, V))$ denote the canonical isomorphism

$$\Psi(\mathbb{Q}_p, V)) : R\Gamma(\mathbb{Q}_p, V) \xrightarrow{\cong} R\mathrm{Hom}_F(R\Gamma(\mathbb{Q}_p, V^*(1)), F)[-2]$$

coming from the local Tate's duality theory for the Galois cohomology and let $\overline{\mathbf{d}_F(\Psi(\mathbb{Q}_p, V))}$ denote the map, which is inverse to $\mathbf{d}_F(\Psi(\mathbb{Q}_p, V))$ with respect to the composition (cf. Appendix B). Then

$$\epsilon_{F,\xi}(V) \cdot \epsilon_{F,-\xi}(V^*(1))^* = \overline{\mathbf{d}_F(\Psi(\mathbb{Q}_p, V))} \cdot \mathbf{d}_F(\xi : V(-1) \to V)$$

(see the remark below).

(2) For an exact sequence $0 \to V' \to V \to V'' \to 0$ of de Rham representations we have

$$\epsilon_{F,\xi}(V) = \epsilon_{F,\xi}(V') \cdot \epsilon_{F,\xi}(V'').$$

(3) Let $\varphi_p : \tilde{F} \to \tilde{F}$ be the ring homomorphism induced by $x \mapsto x^p$ of $\overline{\mathbb{F}_p}$ and the identity map of F. Let $\tau_p \in Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ be an element such that $\kappa(\tau_p) = 1$, which induces $x \mapsto x^p$ on $\overline{\mathbb{F}_p}$. Then, as an element of

$$\mathrm{Isom}(\mathbf{d}_F(0), \tilde{\Delta}_{ep,\mathbb{Q}_p}(F,V)) \times^{K_1(F)} K_1(\tilde{F}),$$

$\epsilon_{F,\xi}(V)$ belongs to

$$\mathrm{Isom}(\mathbf{d}_F(0), \tilde{\Delta}_{ep,\mathbb{Q}_p}(F,V)) \times^{K_1(F)} \left\{ x \in K_1(\tilde{F}) \mid \varphi_p(x) = \det(\tau_p \mid V)^{-1} \cdot x \right\}.$$

(4) Let $c \in \mathbb{Z}_p^\times$ and let σ_c be the unique element of the inertia subgroup of $Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ such that $\sigma_c(\zeta_{p^n}) = \zeta_{p^n}^c$ for any $n \geq 1$. Then

$$\epsilon_{F,\xi^c}(V) = [V, \sigma_c] \cdot \epsilon_{F,\xi}(V),$$

where $[V, \sigma_c]$ is an element of $K_1(F)$.

Remark 2.12 *Here we give a little modified version of the property (1) of the ϵ-isomorphisms of de Rham representations (cf. [FK, Prop. 3.3.8]). Note also, that both sides of the equality of the property (1) are invariant when we replace V by $V^*(1)$ and then take the F-dual.*

Proof. First we write down the definitions of the maps, which appear in the property (1):

- $\epsilon_{F,\xi}(V) : \mathbf{d}_{\tilde{F}}(0) \xrightarrow{\cong} \tilde{F} \otimes_F \{\mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_F(V)\}$,
- $\epsilon_{F,-\xi}(V^*(1))^* : \tilde{F} \otimes_F \left\{ \left(\mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V^*(1))) \cdot \mathbf{d}_F(V^*(1)) \right)^* \right\} \xrightarrow{\cong} \mathbf{d}_{\tilde{F}}(0)^*$,
- $\overline{\mathbf{d}_F(\Psi(\mathbb{Q}_p, V))} : \mathbf{d}_F\left(R\Gamma(\mathbb{Q}_p, V^*(1))^* \right) \xrightarrow{\cong} \mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V))$,
- $\mathbf{d}_F(\xi : V(-1) \to V) : \mathbf{d}_F(V(-1)) \xrightarrow{\cong} \mathbf{d}_F(V)$.

Using the fact $\mathbf{d}_F(-)^* = \mathbf{d}_F((-)^*)$ we get:

$$\mathbf{d}_F(V^*(1))^* = \mathbf{d}_F(V(-1)), \quad \mathbf{d}_{\tilde{F}}(0)^* = \mathbf{d}_{\tilde{F}}(0).$$

If we drop the tensor product with $\tilde{F}$ on both sides, then we have the following maps in the equality of the property (1):

(left hand side) : $\mathbf{d}_{\tilde{F}}(0) \cdot \mathbf{d}_F(V(-1)) \cdot \mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V^*(1))^*) \xrightarrow{\cong}$

$$\xrightarrow{\cong} \mathbf{d}_F(V) \cdot \mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_{\tilde{F}}(0);$$

(right hand side) : $\mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V^*(1))^*) \cdot \mathbf{d}_F(V(-1)) \xrightarrow{\cong} \mathbf{d}_F(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_F(V)$.

Thus the both sides are isomorphisms of the same objects. The rest of the proof is same as the proof of Proposition 3.3.8 in [FK]. □

Remark 2.13 *Let $F = \mathbb{Q}_p$. If we consider the map, which is inverse to $\epsilon_{F,\xi}(V)$ with respect to the composition, then we get the map $\delta_{V,K/K}$ of $C_{ep}(L/K,V)$ also mentioned in Remark 2.5. Of course the same is true for an F if we extend the construction of $\delta_{V,K/K}$ to this case.*

For $F \neq \mathbb{Q}_p$ we can formulate a generalization of $C_{ep,K}(V)$. Let $\mathcal{O}_F$ be the ring of integers of F and $\widetilde{\mathcal{O}_F} := W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$. We choose a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_F$-lattice T in V and define an $\mathcal{O}_F$-submodule of $\tilde{\Delta}_{ep,\mathbb{Q}_p}(F,V)$ by

$$\tilde{\Delta}_{ep,\mathbb{Q}_p}(\mathcal{O}_F,T) := \mathbf{d}_{\mathcal{O}_F}(R\Gamma(\mathbb{Q}_p,T)) \cdot \mathbf{d}_{\mathcal{O}_F}(T),$$

which is independent of the choice of T. Then we get the following conjecture:

Conjecture 2.14 ($C_{ep,K}(V,F)$) *The restriction of $\epsilon_{F,\xi}(V)$ to $\tilde{\Delta}_{ep,\mathbb{Q}_p}(\mathcal{O}_F,T)$ is an element of*

$$\mathrm{Isom}(\mathbf{d}_{\mathcal{O}_F}(0), \tilde{\Delta}_{ep,\mathbb{Q}_p}(\mathcal{O}_F,T)) \times^{K_1(\mathcal{O}_F)} \left\{ x \in K_1(\widetilde{\mathcal{O}_F}) \mid \varphi_p(x) = \det(\tau_p \mid T)^{-1} \cdot x \right\}.$$

In other words, there exists a unique element $\epsilon_{\mathcal{O}_F,\xi}(T)$ of the set above such that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \epsilon_{\mathcal{O}_F,\xi}(T) = \epsilon_{F,\xi}(V)$. Here we use the fact that the maps $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \to K_1(F) = F^\times$ and $K_1(\widetilde{\mathcal{O}_F}) \to K_1(\tilde{F})$ are injective, for both $\mathcal{O}_F$ and $\widetilde{\mathcal{O}_F}$ are commutative semilocal rings (see [CR 2, Prop. 45.12]).

2.3.2 ϵ-isomorphism. The general case

We formulate the conjecture in the original setting of [FK].

Definition 2.15 *A (not necessary commutative) ring Λ is called an **adic ring**, if it satisfies the following condition:*

There exists a two sided ideal I of Λ, such that Λ/I^n is finite of order a power of p for any $n \geq 1$ and such that

$$\Lambda \xrightarrow{\cong} \varprojlim_n \Lambda/I^n.$$

Definition 2.16 *For an adic ring Λ we define $\tilde{\Lambda} := \varprojlim_n (W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \Lambda/J^n)$, where J denotes the Jacobson radical of Λ.*

Remark 2.17 *Let $\mathbb{Q}_p^{ab}$ be the maximal abelian extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$. Then the canonical map $\sigma \mapsto [\mathbb{T},\sigma] \in K_1(\Lambda)$ factors through the quotient $Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ by the closure of the commutator subgroup, this is because the target group $K_1(\Lambda)$ is abelian (hence the commutator subgroup is killed). Thus we have the induced homomorphism:*

$$[\mathbb{T}, ?] : Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \to K_1(\Lambda).$$

Let $\mathbb{T}$ be a finitely generated (f.g.) projective Λ-module endowed with a continuous Λ-linear action of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. We set

$$\tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda,\mathbb{T}) := \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}))\cdot\mathbf{d}_\Lambda(\mathbb{T}).$$

Conjecture 2.18 (*LTNC*) *There exists a unique way to associate an isomorphism*

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) = \epsilon_{\Lambda,\xi}(\mathbb{Q}_p,\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \xrightarrow{\cong} \tilde{\Lambda}\otimes_\Lambda \tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda,\mathbb{T}),$$

i.e.

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) \in \mathrm{Isom}(\mathbf{d}_\Lambda(0),\tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda,\mathbb{T}))\times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$$

to each triple $(\Lambda,\mathbb{T},\xi)$ satisfying the following (1)-(6).

1. *Let $\Psi(\mathbb{Q}_p,\mathbb{T}))$ denote the canonical isomorphism*

$$\Psi(\mathbb{Q}_p,\mathbb{T})) : R\Gamma(\mathbb{Q}_p,\mathbb{T}) \xrightarrow{\cong} R\mathrm{Hom}_{\Lambda^{\mathrm{o}}}(R\Gamma(\mathbb{Q}_p,\mathbb{T}^*(1)),\Lambda^{\mathrm{o}})[-2]$$

   coming from the local Tate's duality theory for the Galois cohomology and Λ^{o} denote the opposite ring of Λ. Then

$$\epsilon_{\Lambda,\xi}(\mathbb{T})\cdot\epsilon_{\Lambda,-\xi}(\mathbb{T}^*(1))^* = \overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p,\mathbb{T}))}\cdot\mathbf{d}_\Lambda(\xi:\mathbb{T}(-1)\to\mathbb{T}).$$

2. *For triples $(\Lambda,\mathbb{T},\xi)$, $(\Lambda,\mathbb{T}',\xi)$, $(\Lambda,\mathbb{T}'',\xi)$ with common Λ and ξ and with an exact sequence*

$$0 \longrightarrow \mathbb{T}' \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}'' \longrightarrow 0,$$

   the canonical isomorphisms

$$\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T})) \cong \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}'))\cdot\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}''))$$

   and

$$\mathbf{d}_\Lambda(\mathbb{T}) \cong \mathbf{d}_\Lambda(\mathbb{T}')\cdot\mathbf{d}_\Lambda(\mathbb{T}'')$$

   send $\epsilon_{\Lambda,\xi}(\mathbb{T})$ to $\epsilon_{\Lambda,\xi}(\mathbb{T}')\cdot\epsilon_{\Lambda,\xi}(\mathbb{T}'')$.

3. *Let $\varphi_p : \tilde{\Lambda}\to\tilde{\Lambda}$ be the ring homomorphism induced by $x\mapsto x^p$ of $\overline{\mathbb{F}_p}$ and the identity map of Λ. Let $\tau_p \in Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ be the unique element such that $\kappa(\tau_p)=1$ which induces $x\mapsto x^p$ on $\overline{\mathbb{F}_p}$. Then $\epsilon_{\Lambda,\xi}(\mathbb{T})$ belongs to*

$$\mathrm{Isom}(\mathbf{d}_\Lambda(0),\tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda,\mathbb{T}))\times^{K_1(\Lambda)}\left\{x\in K_1(\tilde{\Lambda}) \mid \varphi_p(x) = [\mathbb{T},\tau_p]^{-1}\cdot x\right\}.$$

4. *Let $(\Lambda,\mathbb{T},\xi)$, $(\Lambda',\mathbb{T}',\xi)$ be triples with common ξ, and let Y be a finitely generated projective Λ'-module endowed with a continuous right action of Λ, which is compatible with the action of Λ'. Assume $\mathbb{T}' = Y\otimes_\Lambda\mathbb{T}$. Then $Y\otimes_\Lambda$ sends $\epsilon_{\Lambda,\xi}(\mathbb{T})$ to $\epsilon_{\Lambda',\xi}(\mathbb{T}')$.*

5. *Let $c \in \mathbb{Z}_p^\times$ and let σ_c be the unique element of the inertia subgroup of $Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ such that $\sigma_c(\zeta_{p^n}) = \zeta_{p^n}^c$ for any $n \geq 1$. Then*

$$\epsilon_{\Lambda,\xi^c}(\mathbb{T}) = [\mathbb{T}, \sigma_c] \cdot \epsilon_{\Lambda,\xi}(\mathbb{T}).$$

6. *Let L be a finite extension of $\mathbb{Q}_p$, let V be a finite dimensional L-vector space endowed with a continuous action of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ which is de Rham, and let T be an O_L-sublattice of V which is stable under the action of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Then $\epsilon_{O_L,\xi}(T)$ induces $\epsilon_{L,\xi}(V)$ of the subsubsection 2.3.1.*

Remark 2.19 *The ϵ-isomorphisms $\epsilon_{\Lambda,\xi}(\mathbb{T})$ for (not necessary commutative) Λ are related to ϵ-isomorphisms of de Rham representations in the following way. Let L be a finite extension of $\mathbb{Q}_p$, let $n \geq 1$, and let $\rho : \Lambda \to M_n(L)$ be a continuous ring homomorphism. Assume $V = L^n \otimes_\Lambda \mathbb{T}$ is a de Rham representation of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over L. Then the above conditions (4) and (6) tell that $L^n \otimes_\Lambda$ sends $\epsilon_{\Lambda,\xi}(\mathbb{T})$ to $\epsilon_{L,\xi}(V)$, hence $LTNC$ is the content of the (not necessary abelian) equivariant ϵ-conjecture in the sense of the introduction (cf. [V, Conj. 5.9]).*

In the next section we formulate an analogue of $C_{ep}(L/K,V)$ for not necessary abelian extensions L/K (see Conjecture 3.6) and compare the conjectures $C_{ep}(L/K,V)$, $LTNC$ and 3.6 (see Remarks 3.12 and 3.13). For a different formulation of the local not necessary commutative Tamagawa number conjecture see [BF, §5].

3 Non-abelian analogue of $C_{ep}(L/K,V)$

3.1 Formulation

The goal of this subsection is to formulate an analogue $C_{ep}^{na}(L/K,V)$ of the $C_{ep}(L/K,V)$ for an arbitrary (not necessary abelian) finite extension L of K, which will be a generalization of conjectures 2.1 and 2.6 and will give a right candidate for the ϵ-isomorphism associated to the triple $(\mathbb{Z}_p[G], \mathrm{Ind}_{L/\mathbb{Q}_p}T, \xi)$ in $LTNC$.

Let p be a prime, we introduce a tower of finite Galois field extensions $\mathbb{Q}_p \subseteq K \subseteq L$ and denote by $G = Gal(L/K)$ the Galois group of L/K. Further, let V be a p-adic de Rham representation of G_K and T be a G_K-stable $\mathbb{Z}_p$-lattice in V. We denote $\mathrm{Ind}_{L/\mathbb{Q}_p}T$ by $\mathbb{T}$, $\mathbb{Z}_p[G]$ by Λ and $\mathbb{Q}_p[G]$ by Ω. Then Λ is an adic ring in the sense of subsubsection 2.3.2 and $\mathbb{T}$ is a f.g. projective Λ-module endowed with a continuous Λ-linear action of $G_{\mathbb{Q}_p}$ (see Appendix A).

To formulate the conjecture we have to work with not necessary commutative determinants. For the definition and the properties of them see Appendix B. To apply the determinants we have to check, that every considered module (resp. complex of modules) admits a finite projective resolution (resp. is perfect) over R, where R is either Λ or Ω. For this see Appendix A and use the following theorem:

Theorem 3.1 *Let R be an adic ring and X be a f.g. projective R-module endowed with a continuous R-linear action of $G_{\mathbb{Q}_p}$. Then*

1. $\mathrm{H}^m(G_{\mathbb{Q}_p}, X) \xrightarrow{\cong} \varprojlim_n \mathrm{H}^m(G_{\mathbb{Q}_p}, X/J^nX)$, *where J is the Jacobson radical of R.*

2. $R\Gamma(\mathbb{Q}_p, X)$ *is a perfect complex over R.*

3. *Let R' be another adic ring. Let Y be a f.g. projective R'-module endowed with a continuous right action of R which commutes with the action of R'. Then*
$$Y \otimes_R^L R\Gamma(\mathbb{Q}_p, X) \xrightarrow{\cong} R\Gamma(\mathbb{Q}_p, Y \otimes_R X).$$
Here an element σ of $G_{\mathbb{Q}_p}$ acts on $Y \otimes_R X$ by $1 \otimes \sigma$. In particular, for a ring homomorphism $R \to R'$, we have
$$R' \otimes_R^L R\Gamma(\mathbb{Q}_p, X) \xrightarrow{\cong} R\Gamma(\mathbb{Q}_p, R' \otimes_R X).$$

Proof. See [FK, Prop. 1.6.5]. □

Remark 3.2 *For the theorem above we used the following fact: $G_{\mathbb{Q}_p}$ is a profinite group satisfying the following conditions*

- $\mathrm{H}^m(G_{\mathbb{Q}_p}, M)$ *is finite for any finite discrete abelian group M of order a power of p endowed with a continuous action of $G_{\mathbb{Q}_p}$ and for any m.*

- *There exists an integer $d > 0$ such that $\mathrm{H}^m(G_{\mathbb{Q}_p}, M) = 0$ for any finite discrete abelian group M of order a power of p endowed with a continuous action of $G_{\mathbb{Q}_p}$ and for any $m > d$.*

Proof. See [FK, Rem. 1.6.2]. $\square$

As next we prove

Proposition 3.3 *Let R be an adic ring, then*

1. *$\tilde{R} \cong \widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} R$.*
2. *Let R' be another adic rings. Let Y be a f.g. projective R'-module endowed with a continuous right action of R which commutes with the action of R' such that Y is also f.g. as a right R-module, then $\tilde{R}' \otimes_{R'} Y \cong Y \otimes_R \tilde{R}$ as $R' - R$-bimodules.*

Proof. (1) By Definition 2.16 $\tilde{R} = \varprojlim_n (W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} R/J^n)$, where J denotes the Jacobson radical of R.

Using $W(\overline{\mathbb{F}_p}) = \widehat{\mathbb{Z}_p^{ur}}$, we have

$$\tilde{R} = \varprojlim_n (\widehat{\mathbb{Z}_p^{ur}} \otimes_{\mathbb{Z}_p} R/J^n).$$

Since R/J^n are finite $\mathbb{Z}_p$-modules by [Br, Lem. 2.1(ii)]

$$\varprojlim_n (\widehat{\mathbb{Z}_p^{ur}} \otimes_{\mathbb{Z}_p} R/J^n) \cong \varprojlim_n (\widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} R/J^n),$$

and by [Br, Lem. A.4]

$$\varprojlim_n (\widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} R/J^n) \cong \widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} \varprojlim_n R/J^n \cong \widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} R.$$

(2) Since Y is f.g. as a R'-module (resp. a right R-module), we have again by [Br, Lem. 2.1(ii)]

$$\tilde{R}' \otimes_{R'} Y \cong \tilde{R}' \widehat{\otimes}_{R'} Y \cong \widehat{\mathbb{Z}_p^{ur}} \widehat{\otimes}_{\mathbb{Z}_p} Y \cong Y \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{ur}} \cong Y \otimes_R \tilde{R}.$$

$\square$

We introduce $\Delta_{ep}(L/K, T) := \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})) \cdot \mathbf{d}_\Lambda(\mathbb{T})$. The following is a basic support for the conjecture.

Theorem 3.4 *Let R be an adic ring and let Y be a f.g. projective R-module endowed with a continuous $G_{\mathbb{Q}_p}$-action, then $[R\Gamma(\mathbb{Q}_p, Y)] + [Y] = 0$ in $K_0(R)$. That is, the set $\mathrm{Isom}\Big(\mathbf{d}_R(0), \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, Y)) \cdot \mathbf{d}_R(Y)\Big)$ is not empty.*

Proof. See [FK, Prop. 3.1.3]. $\square$

The conjecture $C^{na}_{ep}(L/K,V)$ will say, that there exists a special isomorphism

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \to \tilde{\Lambda} \otimes_{\Lambda} \Delta_{ep}(L/K,T),$$

i.e.

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) \in \mathrm{Isom}\Big(\mathbf{d}_{\tilde{\Lambda}}(0), \tilde{\Lambda} \otimes_{\Lambda} \Delta_{ep}(L/K,T)\Big).$$

Remark 3.5 *Consider $\epsilon \in \mathrm{Isom}(\mathbf{d}_{\tilde{\Lambda}}(0), \tilde{\Lambda} \otimes_{\Lambda} \Delta_{ep}(L/K,T)) =: \tilde{I}$. By the definition the set $\tilde{I}$ is a $K_1(\tilde{\Lambda})$-torsor, such that if we fix some element $\widetilde{can}$ of $\tilde{I}$, then $\epsilon = \widetilde{can} \cdot \tilde{k}$ for some $\tilde{k} \in K_1(\tilde{\Lambda})$.*

As next we define the set $I := \mathrm{Isom}(\mathbf{d}_{\Lambda}(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$ as the quotient of $\mathrm{Isom}(\mathbf{d}_{\Lambda}(0), \Delta_{ep}(L/K,T)) \times K_1(\tilde{\Lambda})$ by the action of $K_1(\Lambda)$ given by

$$(x,y) \mapsto (x \cdot k, \tilde{k} \cdot y), \quad \forall x \in \mathrm{Isom}(\mathbf{d}_{\Lambda}(0), \Delta_{ep}(L/K,T)),\ y \in K_1(\tilde{\Lambda}),\ k \in K_1(\Lambda).$$

Note, that $\mathrm{Isom}(\mathbf{d}_{\Lambda}(0), \Delta_{ep}(L/K,T))$ is a $K_1(\Lambda)$-torsor by the definition and $\tilde{k}$ denotes the image of $k \in K_1(\Lambda)$ under the map $K_1(\Lambda) \xrightarrow{\tilde{\Lambda}\otimes_{\Lambda}} K_1(\tilde{\Lambda})$. We want to show

$$I \cong \tilde{I} \quad \textit{(not canonically)}.$$

For this we fix an element $can \in \mathrm{Isom}(\mathbf{d}_{\Lambda}(0), \Delta_{ep}(L/K,T))$ and denote by $\widetilde{can}$ its image in $\tilde{I}$, i.e. $\widetilde{can} = \tilde{\Lambda} \otimes_{\Lambda} can$. We define the maps:

$$\begin{aligned} &\varphi : \tilde{I} \to I, \quad \tilde{i} = \widetilde{can} \cdot \tilde{k} \mapsto (can, \tilde{k}); \\ &\psi : I \to \tilde{I}, \quad (x,y) = (can \cdot s_x, y) \mapsto \tilde{x} \cdot y = \widetilde{can} \cdot \widetilde{s_x} \cdot y,\ s_x \in K_1(\Lambda), \end{aligned}$$

where $\tilde{x} = \tilde{\Lambda} \otimes_{\Lambda} x$. The maps are inverse to each other, thus are bijections.

Finally, we want to define the actions of $K_1(\Lambda)$ on the sets I and $\tilde{I}$. Let ϵ be an element of $\tilde{I}$, then the product of ϵ with some element $s \in K_1(\Lambda)$ is defined as multiplication of ϵ with the image $\tilde{s}$ of s in $K_1(\tilde{\Lambda})$. Further, for each pair $(x,y) \in I$, we define

$$s \cdot (x,y) := (x \cdot s, y) \overset{!}{=} (x \cdot s \cdot s^{-1}, \tilde{s} \cdot y) = (x, \tilde{s} \cdot y),$$

where the equality $\overset{!}{=}$ follows from the definition of the set I. Then the maps φ and ψ commute with the $K_1(\Lambda)$-action, thus are bijections of $K_1(\Lambda)$-sets. It follows from Theorem 6.52, that the actions of $K_1(\Lambda)$ factorize over the quotient $\mathrm{Det}(K_1(\Lambda)) \cong K_1(\Lambda)/SK_1(\Lambda)$, as $SK_1(\Lambda)$ is the kernel of the map $K_1(\Lambda) \xrightarrow{\tilde{\Lambda}\otimes_{\Lambda}} K_1(\tilde{\Lambda})$.

At last we need the following fact:

Let $\rho : G \to GL_n(F)$, $n \geq 1$ be a (irreducible) representation of G over some finite extension F of $\mathbb{Q}_p$. Then ρ induces a (continuous) ring homomorphism

$\rho : \Lambda \to M_n(\mathcal{O}_F)$. From the properties of the determinants and Proposition 3.3(2) we know that the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under the functor $\mathcal{O}_F^n \otimes_\Lambda -$ is a map

$$\mathbf{d}_{\widetilde{\mathcal{O}_F}}(0) \to \widetilde{\mathcal{O}_F} \otimes_{\mathcal{O}_F} \{\mathbf{d}_{\mathcal{O}_F}(R\Gamma(\mathbb{Q}_p, \mathcal{O}_F^n \otimes_\Lambda \mathbb{T})) \cdot \mathbf{d}_{\mathcal{O}_F}(\mathcal{O}_F^n \otimes_\Lambda \mathbb{T})\},$$

which after $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ becomes

$$\mathbf{d}_{\tilde{F}}(0) \to \tilde{F} \otimes_F \{\mathbf{d}_F(R\Gamma(\mathbb{Q}_p, F^n \otimes_\Lambda \mathbb{T})) \cdot \mathbf{d}_F(F^n \otimes_\Lambda \mathbb{T})\}.$$

In the following we abuse the notation and speak about (irreducible) representations $\rho : \Lambda \to M_n(F)$ and the images under the functor $F^n \otimes_\Lambda -$. Such an image always means the map induced by the image under the functor $\mathcal{O}_F^n \otimes_\Lambda -$.

Setting $W_{\mathbb{T},\rho} := F^n \otimes_\Lambda \mathbb{T}$ and using the following commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & \mathrm{Ind}_{L/\mathbb{Q}_p} T \arrow[r] & F^n \otimes_\Lambda \mathrm{Ind}_{L/\mathbb{Q}_p} T \arrow[d, "\cong"] \\ V \arrow[r] & \rho^* \otimes_{\mathbb{Q}_p} V \arrow[r] & \mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Q}_p} V), \end{tikzcd} \tag{3.1}$$

where ρ^* denotes the contragredient (=dual) representation of ρ and the right hand side vertical map is an isomorphism of representations of $G_{\mathbb{Q}_p}$ over F, we see that $W_{\mathbb{T},\rho}$ is a de Rham representation of $G_{\mathbb{Q}_p}$ over F for all ρ.

Now we are ready to formulate the conjecture keeping the above notation.

Conjecture 3.6 ($C_{ep}^{na}(L/K, V)$) *For any choice of T and ξ, there exists a unique (see Proposition 3.9 below) isomorphism*

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \to \tilde{\Lambda} \otimes_\Lambda \Delta_{ep}(L/K, T)$$

(i.e. $\epsilon_{\Lambda,\xi}(\mathbb{T}) \in \mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$ by Remark 3.5) satisfying the following condition:

($\star$) Let $\rho : \Lambda \to GL_n(F)$, $n \geq 1$ be a (irreducible) representation. Then the image $\epsilon_{F,\xi}(W_{\mathbb{T},\rho})$ of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F^n \otimes_\Lambda -$ is the ϵ-isomorphism of the de Rham representation $W_{\mathbb{T},\rho}$ described in subsubsection 2.3.1.

The following propositions are comments on the conjecture.

Proposition 3.7 *$C_{ep}^{na}(L/K, V)$ is independent of the choice of ξ, i.e. if $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists for one choice of ξ satisfying ($\star$), then also for all other choices.*

Proof. Let $\xi' = \xi^c$ for some $c \in \mathbb{Z}_p^\times$ be another generator of $\mathbb{Z}_p(1)$. We assume, that the conjecture $C_{ep}^{na}(L/K, V)$ holds, i.e. there exists a unique isomorphism $\epsilon_{\Lambda,\xi}(\mathbb{T})$. Let σ_c be the unique element of the inertia subgroup of $Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$, such that $\sigma_c(\zeta_{p^n}) = \zeta_{p^n}^c$ for any $n \geq 1$, then we define an ϵ-isomorphism for ξ' as

$$\epsilon_{\Lambda,\xi'}(\mathbb{T}) := [\mathbb{T}, \sigma_c] \cdot \epsilon_{\Lambda,\xi}(\mathbb{T}).$$

We have to prove, that $\epsilon_{\Lambda,\xi'}(\mathbb{T})$ is a well defined map satisfying condition ($\star$).

$\epsilon_{\Lambda,\xi}(\mathbb{T})$ is an element of a $K_1(\Lambda)$-set $\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$, thus $\epsilon_{\Lambda,\xi'}(\mathbb{T})$ is also an element of this set, as $[\mathbb{T}, \sigma_c] \in K_1(\Lambda)$, hence $\epsilon_{\Lambda,\xi'}(\mathbb{T})$ is well defined.

Further, let $\rho : \Lambda \to M_n(F)$ be a (irreducible) representation. Since $\epsilon_{\Lambda,\xi}(\mathbb{T})$ satisfies the condition $(\star)$, the image $\epsilon_{F,\xi}(W_{\mathbb{T},\rho})$ of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F^n \otimes_\Lambda -$ is the ϵ-isomorphism of the de Rham representation $W_{\mathbb{T},\rho}$ described in subsubsection 2.3.1.

On the other hand, the image of $\epsilon_{\Lambda,\xi'}(\mathbb{T})$ under $F^n \otimes_\Lambda -$ is $[W_{\mathbb{T},\rho}, \sigma_c] \cdot \epsilon_{F,\xi}(W_{\mathbb{T},\rho})$. The condition $(\star)$ for $\epsilon_{\Lambda,\xi'}(\mathbb{T})$ follows now from the property (4) of the ϵ-isomorphisms of subsubsection 2.3.1. □

Proposition 3.8 *$C^{na}_{ep}(L/K,V)$ is independent of the choice of T, i.e. if $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists for one choice of T satisfying $(\star)$, then also for all other choices.*

Proof. Let T' be an another $\mathbb{Z}_p$-sublattice of V. By replacing T by $p^nT \subseteq T \cap T'$ we can assume that $T \subseteq T'$. Then there is an exact sequence of $\mathbb{Z}_p$-representations of G_K:

$$0 \longrightarrow \mathrm{T} \longrightarrow \mathrm{T}' \longrightarrow \mathrm{T}'/\mathrm{T} \longrightarrow 0.$$

Using the facts, that $R\Gamma(\mathbb{Q}_p, \mathrm{Ind}_{L/\mathbb{Q}_p}(-)) \cong R\Gamma(L,-)$ and that $\mathrm{Ind}_{L/\mathbb{Q}_p}-$ is an exact functor, we get from the exact sequence above an exact triangle

$$R\Gamma(L,T) \longrightarrow R\Gamma(L,T') \longrightarrow R\Gamma(L,T'/T) \longrightarrow R\Gamma(L,T)\,[1]$$

of perfect complexes of Λ-modules and an exact sequence of Λ-modules:

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}' \longrightarrow \mathrm{Ind}_{L/\mathbb{Q}_p}(T'/T) \longrightarrow 0.$$

Note, that all Λ-modules in the exact sequence admit finite projective resolutions (see Appendix A). Thus we have the following canonical isomorphisms

$$\mathbf{d}_\Lambda(\mathbb{T}') \cong \mathbf{d}_\Lambda(\mathbb{T}) \cdot \mathbf{d}_\Lambda(\mathrm{Ind}_{L/\mathbb{Q}_p}(T'/T)),$$

$$\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}')) \cong \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T})) \cdot \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathrm{Ind}_{L/\mathbb{Q}_p}(T'/T))).$$

Now let $C^{na}_{ep}(L/K,V)$ holds for T, i.e $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists. From Theorem 3.4 we know, that $\mathbf{d}_\Lambda(0)$ is (not canonically) isomorphic to $\Delta_{ep}(L/K,T'/T)$, so that we choose such an isomorphism $\Phi : \mathbf{d}_\Lambda(0) \to \Delta_{ep}(L/K,T'/T)$ and define $\epsilon_{\Lambda,\xi}(\mathbb{T}') = \epsilon_{\Lambda,\xi}(\mathbb{T}) \cdot \Phi$. We see that $\epsilon_{\Lambda,\xi}(\mathbb{T}')$ is an isomorphism from $\mathbf{d}_{\tilde{\Lambda}}(0)$ to $\tilde{\Lambda} \otimes_\Lambda \Delta_{ep}(L/K,T)$, so it only remains to prove, that it satisfies condition $(\star)$. Since T'/T is a finite $\mathbb{Z}_p$-module, hence torsion, we have

$$W_{\mathbb{T},\rho} = \mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Q}_p} V) = W_{\mathbb{T}',\rho}$$

for all (irreducible) representations $\rho : \Lambda \to M_n(F)$ and Φ becomes a canonical isomorphism after $\mathbb{Q}_p \otimes \mathbb{Z}_p -$. It follows that the images of $\epsilon_{\Lambda,\xi}(\mathbb{T}')$ and $\epsilon_{\Lambda,\xi}(\mathbb{T}) \cdot \Phi$ under $F^n \otimes_\Lambda -$ are equal for all ρ, thus $\epsilon_{\Lambda,\xi}(\mathbb{T}')$ satisfies condition $(\star)$, for $\epsilon_{\Lambda,\xi}(\mathbb{T})$ does and $F^n \otimes_\Lambda \Phi$ is trivial. □

Proposition 3.9 (uniqueness) *If $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists, it is uniquely determined by condition $(\star)$.*

Proof. Let $\epsilon_{\Lambda,\xi}(\mathbb{T})$ and $\epsilon'_{\Lambda,\xi}(\mathbb{T})$ be two ϵ-isomorphisms, then

$$\overline{\epsilon'_{\Lambda,\xi}(\mathbb{T})} \circ \epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \xrightarrow{\cong} \mathbf{d}_{\tilde{\Lambda}}(0)$$

can be viewed as an element k of $K_1(\tilde{\Lambda})$ (see Appendix B) and we have to prove that $k = 1$. For this we introduce the following commutative diagram of homomorphisms of K_1-groups (see section 6)

$$\begin{tikzcd}
 & & 1 \arrow[d] & \\
 & & SK_1(\Lambda) \arrow[d] & \\
1 \arrow[r] & SK_1(\Lambda) \arrow[r] & K_1(\Lambda) \arrow[r] \arrow[d] & K_1(\tilde{\Lambda}) \arrow[d, hook] \\
 & & K_1(\Omega) \arrow[r, hook] & K_1(\tilde{\Omega}).
\end{tikzcd} \tag{3.2}$$

By definition and Remark 3.5 the element k lies in the image of $K_1(\Lambda)$ in $K_1(\tilde{\Lambda})$ and can be mapped further to some element in $K_1(\tilde{\Omega})$. Using Wedderburn decomposition of $K_1(\Omega)$ and the commutativity of the square in the diagram (3.2) we can also view k as an image in $K_1(\tilde{\Omega})$ of some element of $K_1(\Omega)$, such that $k = (k_\chi) \in \prod_{\chi \in E} (\widehat{\mathbb{Q}_p^{ur}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\chi))^\times$, where the product extends over the set E of representatives of the $\mathbb{Q}_p$-equivalence classes of the irreducible characters of G over $\overline{\mathbb{Q}_p}$ and $\mathbb{Q}_p(\chi)$ is obtained by adjoining to $\mathbb{Q}_p$ the values of χ.

On the other hand, let $\rho_\chi : \Lambda \to M_{n_\chi}(\mathbb{Q}_p(\chi))$ be an irreducible (continuous) representation, such that χ is the character of ρ_χ, then we get the component k_χ as the image of k under the map induced by $\mathbb{Q}_p(\chi)^{n_\chi} \otimes_\Lambda$. Since $\epsilon_{\Lambda,\xi}(\mathbb{T})$ and $\epsilon'_{\Lambda,\xi}(\mathbb{T})$ satisfy condition $(\star)$, every component $k_\chi = 1$, thus $k = 1$, as the map $K_1(\tilde{\Lambda}) \to K_1(\tilde{\Omega})$ is injective. □

Proposition 3.10 *Let R be an adic ring. Let $\varphi_p : \tilde{R} \to \tilde{R}$ be the ring homomorphism induced by $x \mapsto x^p$ of $\overline{\mathbb{F}_p}$ and the identity map of R. Let κ denote the cyclotomic character from $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ to $\mathbb{Z}_p^\times$ characterized by $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\kappa(\sigma)}$ $(\forall n \geq 1)$, and let $\tau_p \in Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ be the unique element such that $\kappa(\tau_p) = 1$ which induces $x \mapsto x^p$ on $\overline{\mathbb{F}_p}$. For any $a \in K_1(\tilde{R})$, the set*

$$K_1(\tilde{R})_a := \left\{ x \in K_1(\tilde{R}) \mid \varphi_p(x) = a \cdot x \right\}$$

is not empty.

Proof. See [FK, Prop. 3.4.5]. □

Note, that by Theorem 6.52 the set $K_1(\tilde{\Lambda})_{[\mathbb{T},\tau_p]^{-1}}$ is a $\mathrm{Det}(K_1(\Lambda))$-torsor.

Proposition 3.11 *If $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists, then it belongs to*

$$\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})_{[\mathbb{T},\tau_p]^{-1}}$$

(cf. the remark above and the comment at the end of Remark 3.5).

Proof. Let

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) = (\epsilon_1, \epsilon_2) \in \mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}).$$

We have to prove that ϵ_2 is in $K_1(\tilde{\Lambda})_{[\mathbb{T},\tau_p]^{-1}}$. There exists a $k \in K_1(\tilde{\Lambda})$ such that $\varphi_p(\epsilon_2) = k \cdot \epsilon_2$. Similar to the proof of Proposition 3.9 we view $\epsilon_2 = ((\epsilon_2)_\chi)_{\chi\in E}$ as an element of $K_1(\tilde{\Omega})$. The map $K_1(\tilde{\Lambda}) \to K_1(\tilde{\Omega})$ is a group homomorphism commuting with the action of φ_p, hence $\varphi_p(\epsilon_2) = \big(k_\chi \cdot (\epsilon_2)_\chi\big)_{\chi\in E}$.

It follows from the condition $(\star)$ for $\epsilon_{\Lambda,\xi}(\mathbb{T})$ and the property (3) of ϵ-isomorphisms of de Rham representations described in subsubsection 2.3.1, that $k_\chi = \big[\mathbb{Q}_p^{n_\chi}(\chi) \otimes_\Lambda \mathbb{T}, \tau_p\big]^{-1}$. But the element $[\mathbb{T},\tau_p]^{-1} \in K_1(\tilde{\Lambda})$ has the same image in $K_1(\tilde{\Omega})$ as k, thus again from the injectivity of the map $K_1(\tilde{\Lambda}) \to K_1(\tilde{\Omega})$ we deduce that $k = [\mathbb{T},\tau_p]^{-1}$ in $K_1(\tilde{\Lambda})$. □

Remark 3.12 *For L/K abelian, $C_{ep}(L/K,V)$ and $C_{ep}^{na}(L/K,V)$ are equivalent.*

Proof. We prove the equivalence of conjectures $C_{ep}(L/K,V)$ and $C_{ep}^{na}(L/K,V)$ for L/K abelian. We have $\delta_{V,L/K} : \Delta_{ep}(L/K,T) \to \mathbb{Z}_p[G]_{V,L/K}$ and $\epsilon_{\Lambda,\xi}(\mathbb{T})$ should be a map $\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbb{Z}_p[G]_{V,L/K} \xrightarrow{\cong} \Delta_{ep}(L/K,T)$ satisfying condition $(\star)$.

Now if $C_{ep}(L/K,V)$ holds, i.e. $\delta_{V,L/K}$ is an isomorphism, then we set $\epsilon_{\Lambda,\xi}(\mathbb{T}) := \overline{\delta_{V,L/K}} : \mathbb{Z}_p[G]_{V,L/K} \xrightarrow{\cong} \Delta_{ep}(L/K,T)$. From the commutative diagram 3.1 and the construction of $\delta_{V,L/K}$, which for $L = K$ is the same as the construction of (an inverse of) ϵ-isomorphisms of de Rham representations described in subsubsection 2.3.1, we deduce that $\overline{\delta_{V,L/K}}$ (and thus $\epsilon_{\Lambda,\xi}(\mathbb{T})$) satisfies condition $(\star)$, thus $C_{ep}^{na}(L/K,V)$ holds.

Conversely, if $C_{ep}^{na}(L/K,V)$ holds, i.e. $\epsilon_{\Lambda,\xi}(\mathbb{T})$ exists, then we view $\delta_{V,L/K}$ as a map $\Delta_{ep}(L/K,V) \xrightarrow{\cong} \mathbb{Q}_p[G]_{V,L/K}$ and show that the restriction of $\delta_{V,L/K}$ to $\Delta_{ep}(L/K,T)$ is an isomorphism. For this we consider

$$\delta_{V,L/K} \circ \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbb{Q}_p[G]_{V,L/K} \xrightarrow{\cong} \mathbb{Q}_p[G]_{V,L/K}$$

as an element $k \in K_1(\tilde{\Omega})$. From the construction of $\delta_{V,L/K}$ and the commutative diagram 3.2 we know, that $k = (k_\chi)_{\chi\in E}$ lies in the image of $K_1(\Omega)$ in $K_1(\tilde{\Omega})$, where χ are one-dimensional characters, as G is abelian. By previous considerations $\overline{\delta_{V,L/K}}$ satisfies condition $(\star)$, where we extend each character χ to a homomorphism $\chi : \Omega \to M_n(F)$. As $\epsilon_{\Lambda,\xi}(\mathbb{T})$ also satisfies condition $(\star)$ we deduce that $k_\chi = 1$ for all χ, hence $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \epsilon_{\Lambda,\xi}(\mathbb{T}) = \overline{\delta_{V,L/K}}$. Since $\epsilon_{\Lambda,\xi}(\mathbb{T})$ is the restriction of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \epsilon_{\Lambda,\xi}(\mathbb{T})$ to the integral structure induced by T, its inverse gives the restriction of $\delta_{V,L/K}$ to $\Delta_{ep}(L/K,T)$ and is an isomorphism. This implies $C_{ep}(L/K,V)$. □

Remark 3.13 *The LTNC implies $C^{na}_{ep}(L/K,V)$. Conversely, $C^{na}_{ep}(L/K,V)$ implies the existence and uniqueness of the ϵ-isomorphism associated to the triple $(\mathbb{Z}_p[G], \mathrm{Ind}_{L/\mathbb{Q}_p}T, \xi)$ in LTNC.*

Proof. Indeed, $\Lambda = \mathbb{Z}_p[G]$ is an adic ring and $\mathbb{T} = \mathrm{Ind}_{L/\mathbb{Q}_p}T$ is a f.g. projective (left) Λ-module endowed with a continuous action of $G_{\mathbb{Q}_p}$, so that if $LTNC$ holds, then the conjectures $C^{na}_{ep}(L/K,V)$ hold for all extensions L/K and all de Rham representations V simultaneously (for condition $(\star)$ see Remark 2.19). Conversely, if $C^{na}_{ep}(L/K,V)$ holds, then the map $\epsilon_{\Lambda,\xi}(\mathbb{T})$ gives a unique (cf. Proposition 3.9) candidate for the ϵ-isomorphism associated to the triple $(\mathbb{Z}_p[G], \mathrm{Ind}_{L/\mathbb{Q}_p}T, \xi)$ in $LTNC$. □

For the better illustration we put all conjectures stated in previous sections into the following diagram

$$\begin{tikzcd}
C_{ep,K}(V) \arrow[r, Leftrightarrow, "L=K"] \arrow[d, Leftrightarrow, "F=\mathbb{Q}_p"] & C_{ep}(L/K,V) \arrow[r, Leftrightarrow, "L/K \text{ abelian}"] & C^{na}_{ep}(L/K,V) \arrow[d, dotted, bend right] \\
C_{ep,K}(V,F) \arrow[rr, dotted, bend right] & & LTNC, \arrow[ll, Rightarrow, "(\mathcal{O}_F,T,\xi)"'] \arrow[u, Rightarrow, "(\mathbb{Z}_p[G],\mathrm{Ind}_{L/\mathbb{Q}_p}T,\xi)"']
\end{tikzcd}$$

where dotted arrows mean that the corresponding conjecture gives a candidate for the ϵ-isomorphism in $LTNC$.

3.2 Functorial properties

Proposition 3.14 *$C^{na}_{ep}(L/K,V)$ has the following functorial properties:*

1. *The conjectures $C^{na}_{ep}(L/K,V)$ and $C^{na}_{ep}(L/K,V^*(1))$ are equivalent.*
2. *Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of p-adic de Rham representations of G_K. If $C^{na}_{ep}(L/K)$ holds for two of the representations V, V', V'', then it also holds for the third one.*
3. *Let M/K be a Galois extension of K contained in L. If $C^{na}_{ep}(L/K,V)$ holds, then the conjectures $C^{na}_{ep}(L/M,V)$ and $C^{na}_{ep}(M/K,V)$ hold, too.*

4. *Let $\rho : \Lambda \to M_n(\mathcal{O}_F)$ be a (irreducible) representation of G over F. If $C^{na}_{ep}(L/K, V)$ holds, then $C_{ep,K}(V, F)$ holds for $V = \rho^* \otimes_{\mathbb{Q}_p} V$.*

The proof of the properties $(1) - (3)$ is pretty long and is given in the following subsubsections for each statement of the proposition separately. The property (4) follows directly from condition $(\star)$, the fact before $C^{na}_{ep}(L/K, V)$ (the definition of $F^n \otimes_\Lambda -$) and the formulation of $C_{ep,K}(V, F)$.

Proof of (1)

We give the proof in the situation, where the dual conjecture $C^{na}_{ep}(L/K, V^*(1))$ holds, the other direction being analogous. We choose a G_K-stable $\mathbb{Z}_p$-lattice T in V and set $T^*(1)$ to be the dual $\mathbb{Z}_p$-lattice in $V^*(1)$. First we notice

Proposition 3.15 *Let V be a p-adic representation of $G_{\mathbb{Q}_p}$, then V is de Rham if and only if $V^*(1)$ is de Rham.*

Proof. See [Fo, Prop. 1.5.2]. □

The canonical isomorphism:

$$\Psi(\mathbb{Q}_p, \mathbb{T}) : R\Gamma(\mathbb{Q}_p, \mathbb{T}) \xrightarrow{\cong} R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^\circ)\,[-2]$$

induces

$$\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T})) : \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})) \xrightarrow{\cong} \mathbf{d}_\Lambda(R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^\circ)),$$

where

$$\mathbf{d}_\Lambda(R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^\circ)) = (\mathbf{d}_{\Lambda^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1))))^*$$

by the definition (see Appendix B).

Next we show the comparison between two duality theories:

Proposition 3.16 *The modules $\mathbb{T}^*(1)$ and $\mathrm{Ind}_{L/\mathbb{Q}_p}(T^*(1))$ are isomorphic as right Λ-modules endowed with a continuous $G_{\mathbb{Q}_p}$-action.*

Proof. We view left Λ°-modules (for example $\mathbb{T}^*$) as right Λ-modules, then from Proposition 7.4 we have

$$\mathrm{Ind}_{L/\mathbb{Q}_p}(T^*(1)) \cong \mathrm{Ind}_{K/\mathbb{Q}_p}\Big((\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \Lambda\Big)$$

and

$$(\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \Lambda \cong \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Lambda(1)$$

as right Λ-modules with a continuous $G_{\mathbb{Q}_p}$-action, where $\Lambda(1)$ is the twist of the trivial G_K-module Λ by the cyclotomic character of G_K.

On the other hand,

$$\mathbb{T}^*(1) \cong \mathrm{Ind}_{K/\mathbb{Q}_p}(\mathrm{Hom}_\Lambda(\Lambda \otimes_{\mathbb{Z}_p} T, \Lambda)) \otimes_\Lambda \Lambda(1)$$

again as right Λ-modules with a continuous $G_{\mathbb{Q}_p}$-action, where $\Lambda(1)$ is now the twist of the trivial $G_{\mathbb{Q}_p}$-module Λ by the cyclotomic character of $G_{\mathbb{Q}_p}$.

Since $(\mathrm{Ind}_{K/\mathbb{Q}_p} M)(1) \cong \mathrm{Ind}_{K/\mathbb{Q}_p}(M(1))$ for an arbitrary G_K-module M (see [S, p. 29]), it is enough to prove that $\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Lambda \cong \mathrm{Hom}_\Lambda(\Lambda \otimes_{\mathbb{Z}_p} T, \Lambda)$, as right Λ-modules with a continuous G_K-action. But this is obvious, for all modules are projective Λ-modules (resp. $\mathbb{Z}_p$-modules) and the G_K-action is Λ-linear (resp. $\mathbb{Z}_p$-linear). □

Now we define

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) := \mathbf{d}_\Lambda(\xi : \mathbb{T}(-1) \to \mathbb{T}) \cdot (\epsilon_{\Lambda,\xi^{-1}}(\mathbb{T}^*(1))^*)^{-1} \cdot \overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T}))},$$

then

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \to \tilde{\Lambda} \otimes_\Lambda \tilde{\Delta}_{ep}(L/K, T).$$

To see it we recall all maps used in the definition of $\epsilon_{\Lambda,\xi}(\mathbb{T})$:

- $\mathbf{d}_\Lambda(\xi : \mathbb{T}(-1) \to \mathbb{T}) : \mathbf{d}_\Lambda(\mathbb{T}(-1)) \to \mathbf{d}_\Lambda(\mathbb{T})$,
- $\overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T}))} : \mathbf{d}_\Lambda(R\mathrm{Hom}_{\Lambda^o}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^o)) \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))$,
- $\theta = \epsilon_{\Lambda,\xi^{-1}}(\mathbb{T}^*(1))^* : \tilde{\Lambda} \otimes_\Lambda \tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda^o, \mathbb{T}^*(1))^* \to \mathbf{d}_{\tilde{\Lambda}}(\mathrm{Hom}_{\Lambda^o}(0, \Lambda^o)) = \mathbf{d}_{\tilde{\Lambda}}(0)$,
- $\theta^{-1} : \tilde{\Lambda} \otimes_\Lambda \left(\tilde{\Delta}_{ep,\mathbb{Q}_p}(\Lambda^o, \mathbb{T}^*(1))^*\right)^{-1} \to (\mathbf{d}_{\tilde{\Lambda}}(0))^{-1} = \mathbf{d}_{\tilde{\Lambda}}(0)$.

Since $\mathrm{Hom}_{\Lambda^o}(\mathbb{T}^*(1), \Lambda^o) \cong \mathbb{T}(-1)$, it follows that

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) = \mathbf{d}_\Lambda(\xi : \mathbb{T}(-1) \to \mathbb{T}) \cdot (\epsilon_{\Lambda,\xi^{-1}}(\mathbb{T}^*(1))^*)^{-1} \cdot \overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T}))}$$

is an isomorphism between

$$\mathbf{d}_{\tilde{\Lambda}}(0) = \tilde{\Lambda} \otimes_\Lambda \{\mathbf{d}_\Lambda(\mathbb{T}(-1)) \cdot (\mathbf{d}_\Lambda(R\mathrm{Hom}_{\Lambda^o}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^o)))^{-1} \cdot$$
$$\cdot (\mathbf{d}_\Lambda(\mathrm{Hom}_{\Lambda^o}(\mathbb{T}^*(1), \Lambda^o)))^{-1} \cdot \mathbf{d}_\Lambda(R\mathrm{Hom}_{\Lambda^o}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), \Lambda^o))\}$$

and

$$\tilde{\Lambda} \otimes_\Lambda \{\mathbf{d}_\Lambda(\mathbb{T}) \cdot \mathbf{d}_\Lambda(0) \cdot \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))\} = \tilde{\Lambda} \otimes_\Lambda \{\mathbf{d}_\Lambda(\mathbb{T}) \cdot \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))\}.$$

Next we verify condition $(\star)$. Let $\rho : \Lambda \to M_n(F)$ be a (irreducible) representation of G, then the contragredient $\rho^* : \Lambda^o \to M_n(F)$ of ρ induces a functor $(F^n)^t \otimes_{\Lambda^o} -$, where $(F^n)^t$ denotes the transpose of F^n.

Proposition 3.17 *$(F^n)^t \otimes_{\Lambda^o} (\mathbb{T}^*(1))$ is isomorphic to $(F^n \otimes_\Lambda \mathbb{T})^*(1)$ as a representation of $G_{\mathbb{Q}_p}$ over F, i.e.*

$$W_{\mathbb{T}^*(1),\rho^*} := (F^n)^t \otimes_{\Lambda^o} (\mathbb{T}^*(1)) \cong (F^n \otimes_\Lambda \mathbb{T})^*(1) =: (W_{\mathbb{T},\rho})^*(1).$$

Proof. The proof is similar to the proof of Proposition 3.16 and uses the facts, that all modules are projective Λ-modules (resp. F-modules) and the $G_{\mathbb{Q}_p}$-action is Λ-linear (resp. F-linear). □

The tensor product $F^n \otimes_\Lambda -$ defined via ρ is an additive exact functor on the category of projective left Λ-modules (resp. perfect complexes of left Λ-modules), as a consequence, the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F^n \otimes_\Lambda$ is equal to the product

$$F^n \otimes_\Lambda (\mathbf{d}_\Lambda(\xi : \mathbb{T}(-1) \to \mathbb{T})) \cdot F^n \otimes_\Lambda ((\epsilon_{\Lambda,\xi^{-1}}(\mathbb{T}^*(1))^*)^{-1}) \cdot F^n \otimes_\Lambda (\overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T}))}),$$

where

$$\begin{aligned} F^n \otimes_\Lambda (\mathbf{d}_\Lambda(\xi : \mathbb{T}(-1) \to \mathbb{T})) &= \mathbf{d}_F(\xi : W_{\mathbb{T},\rho}(-1) \to W_{\mathbb{T},\rho}), \\ F^n \otimes_\Lambda (\overline{\mathbf{d}_\Lambda(\Psi(\mathbb{Q}_p, \mathbb{T}))}) &= \overline{\mathbf{d}_F(\Psi(\mathbb{Q}_p, W_{\mathbb{T},\rho}))} \end{aligned}$$

and since $\epsilon_{\Lambda^\circ,\xi^{-1}}(\mathbb{T}^*(1))$ satisfies condition $(\star)$

$$\begin{aligned} F^n \otimes_\Lambda ((\epsilon_{\Lambda,\xi^{-1}}(\mathbb{T}^*(1))^*)^{-1}) &= \left(\left((F^n)^t \otimes_{\Lambda^\circ} (\epsilon_{\Lambda^\circ,\xi^{-1}}(\mathbb{T}^*(1))) \right)^* \right)^{-1} \\ &= (\epsilon_{F,\xi^{-1}}(W_{\mathbb{T}^*(1),\rho^*})^*)^{-1}, \end{aligned}$$

$\epsilon_{F,\xi^{-1}}(W_{\mathbb{T}^*(1)})$ being the ϵ-isomorphism of the de Rham representation $W_{\mathbb{T}^*(1)}$ described in subsubsection 2.3.1. Next by the property (1) of ϵ-isomorphisms of subsubsection 2.3.1 $\epsilon_{\Lambda,\xi}(\mathbb{T})$ satisfies condition $(\star)$.

Proof of (2)

We prove the case, where the conjectures $C^{na}_{ep}(L/K, V')$ and $C^{na}_{ep}(L/K, V'')$ hold (the other cases being analogous to this one). Choosing appropriate G_K-stable $\mathbb{Z}_p$-lattices T, T', T'' in V, V', V'', respectively, we obtain an exact sequence

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

which give rise to an exact sequence of Λ-modules

$$0 \longrightarrow \mathbb{T}' \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}'' \longrightarrow 0, \tag{3.3}$$

and an exact triangle of perfect complexes of Λ-modules

$$R\Gamma(\mathbb{Q}_p, \mathbb{T}') \longrightarrow R\Gamma(\mathbb{Q}_p, \mathbb{T}) \longrightarrow R\Gamma(\mathbb{Q}_p, \mathbb{T}'') \longrightarrow R\Gamma(\mathbb{Q}_p, \mathbb{T}')\,[1]\,,$$

such that

$$\mathbf{d}_\Lambda(\mathbb{T}) = \mathbf{d}_\Lambda(\mathbb{T}') \cdot \mathbf{d}_\Lambda(\mathbb{T}''),$$
$$\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})) = \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}')) \cdot \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}'')).$$

We define $\epsilon_{\Lambda,\xi}(\mathbb{T}) := \epsilon_{\Lambda,\xi}(\mathbb{T}') \cdot \epsilon_{\Lambda,\xi}(\mathbb{T}'')$, then

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \xrightarrow{\cong} \tilde{\Lambda} \otimes_\Lambda \{\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})) \cdot \mathbf{d}_\Lambda(\mathbb{T})\}$$

according to the identification above.

Further, let $\rho : \Lambda \to M_n(F)$ be a (irreducible) representation. Using condition $(\star)$ for $\epsilon_{\Lambda,\xi}(\mathbb{T}')$ and $\epsilon_{\Lambda,\xi}(\mathbb{T}'')$ we get:

$$F^n \otimes_\Lambda : \epsilon_{\Lambda,\xi}(\mathbb{T}') \to \epsilon_{F,\xi}(W_{\mathbb{T}'}), \quad F^n \otimes_\Lambda : \epsilon_{\Lambda,\xi}(\mathbb{T}'') \to \epsilon_{F,\xi}(W_{\mathbb{T}''}),$$

where $\epsilon_{F,\xi}(-)$ are ϵ-isomorphisms of de Rham representations described in subsubsection 2.3.1.

The tensor product $F^n \otimes_\Lambda -$ is an additive exact functor on the category of projective left Λ-modules (resp. perfect complexes of left Λ-modules), as a consequence, the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F^n \otimes_\Lambda$ is equal to the product $\epsilon_{F,\xi}(W_{\mathbb{T}'}) \cdot \epsilon_{F,\xi}(W_{\mathbb{T}''})$, which by the property (2) of ϵ-isomorphisms of subsubsection 2.3.1 is the ϵ-isomorphism of the de Rham representation $W_{\mathbb{T}}$ sitting in the following exact sequence of F-spaces

$$0 \longrightarrow W_{\mathbb{T}'} \longrightarrow W_{\mathbb{T}} \longrightarrow W_{\mathbb{T}''} \longrightarrow 0$$

obtained from (3.3), hence $\epsilon_{\Lambda,\xi}(\mathbb{T})$ satisfies condition $(\star)$.

Proof of (3)

Set $G_1 := Gal(L/M)$, $G_2 := Gal(M/K)$ and $\Lambda_1 := \mathbb{Z}_p[G_1]$, $\Lambda_2 := \mathbb{Z}_p[G_2]$. We first give the proof for $G_2 = Gal(M/K) = G/G_1$. The projection of the group rings $f : \Lambda \to \Lambda_2$ induces the base change functor $\Lambda_2 \otimes_\Lambda -$ and homomorphisms on the K-groups:

$$\begin{array}{lll} f^* & : & K_0(\tilde{\Lambda}) \to K_0(\tilde{\Lambda}_2), \qquad Y \mapsto \Lambda_2 \otimes_\Lambda Y, \\ f^* & : & K_1(\tilde{\Lambda}) \to K_1(\tilde{\Lambda}_2), \qquad [Y, \alpha] \mapsto [\Lambda_2 \otimes_\Lambda Y, \mathrm{id}_{\Lambda_2} \otimes \alpha]. \end{array}$$

Remark 3.18 *Since $G_{\mathbb{Q}_p}$ acts Λ- (resp. Λ_2-) linear, we have*

$$\Lambda_2 \otimes_\Lambda \mathrm{Ind}_{L/\mathbb{Q}_p} T \cong \mathrm{Ind}_{M/\mathbb{Q}_p} T$$

and by Theorem 3.1

$$\Lambda_2 \otimes^L_\Lambda R\Gamma(\mathbb{Q}_p, \mathrm{Ind}_{L/\mathbb{Q}_p} T) \cong R\Gamma(\mathbb{Q}_p, \mathrm{Ind}_{M/\mathbb{Q}_p} T).$$

We define $\epsilon_{\Lambda_2,\xi}(\mathrm{Ind}_{M/\mathbb{Q}_p} T)$ as the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $\Lambda_2 \otimes_\Lambda -$. Then $\epsilon_{\Lambda_2,\xi}(\mathrm{Ind}_{M/\mathbb{Q}_p} T)$ is by Proposition 3.3 an isomorphism

$$\mathbf{d}_{\tilde{\Lambda}_2}(0) \xrightarrow{\cong} \tilde{\Lambda}_2 \otimes_{\Lambda_2} \left\{ \mathbf{d}_{\Lambda_2}(R\Gamma(\mathbb{Q}_p, \mathrm{Ind}_{M/\mathbb{Q}_p} T)) \cdot \mathbf{d}_{\Lambda_2}(\mathrm{Ind}_{M/\mathbb{Q}_p} T) \right\}.$$

Now let $\bar{\rho} : \Lambda_2 \to M_n(F)$ be a (irreducible) representation of G_2. We also consider $\overline{\rho}$ as a representation of G, which is trivial on G_1. Together with the projection $\Lambda \to \Lambda_2$ it induces two isomorphic functors

$$F^n \otimes_\Lambda - \cong F^n \otimes_{\Lambda_2} (\Lambda_2 \otimes_\Lambda -).$$

As a consequence, the images of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ and $\epsilon_{\Lambda_2,\xi}(\mathrm{Ind}_{M/\mathbb{Q}_p} T)$ under $F^n \otimes_\Lambda -$ and $F^n \otimes_{\Lambda_2} -$, respectively, are equal, thus $\epsilon_{\Lambda_2,\xi}(\mathrm{Ind}_{M/\mathbb{Q}_p} T)$ satisfies condition $(\star)$, as $\epsilon_{\Lambda,\xi}(\mathbb{T})$ does.

Next we proof the second part of the statement. Λ is a f.g. free Λ_1-module via the inclusion of the group rings $f : \Lambda_1 \to \Lambda$. There is a forgetful functor from the category of projective Λ-modules to the category of projective Λ_1-modules. The functor is to regard a Λ-module (resp. a perfect complex of Λ-modules) as a Λ_1-module (resp. a perfect complex of Λ_1-modules) and is represented by Λ, where Λ is considered as a Λ_1-Λ bimodule. The induced homomorphism f_* on the K-groups is called the transfer map (norm homomorphism).

We define $\epsilon_{\Lambda_1,\xi}(\Lambda \otimes_\Lambda \mathbb{T})$ as the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under the forgetful functor, then by Proposition 3.3 and Theorem 3.1

$$\epsilon_{\Lambda_1,\xi}(\Lambda \otimes_\Lambda \mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}_1}(0) \xrightarrow{\cong} \tilde{\Lambda}_1 \otimes_{\Lambda_1} \{\mathbf{d}_{\Lambda_1}(R\Gamma(\mathbb{Q}_p, \Lambda \otimes_\Lambda \mathbb{T})) \cdot \mathbf{d}_{\Lambda_1}(\Lambda \otimes_\Lambda \mathbb{T})\}.$$

Let $\rho_1 : \Lambda_1 \to M_n(F)$ be a (irreducible) representation of G_1. We set

$$\rho := \mathrm{Ind}_{M/K}(\rho_1) : \Lambda \to M_n(F)$$

and obtain two isomorphic functors defined via ρ_1 and ρ, respectively:

$$F^n \otimes_{\Lambda_1} (\Lambda \otimes_\Lambda -) \cong F^n \otimes_\Lambda -$$

The images of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ and $\epsilon_{\Lambda_1,\xi}(\Lambda \otimes_\Lambda \mathbb{T})$ under $F^n \otimes_\Lambda -$ (resp. $F^n \otimes_{\Lambda_1} -$) are the same, hence $\epsilon_{\Lambda_2,\xi}(\mathrm{Ind}_{M/\mathbb{Q}_p} T)$ satisfies condition $(\star)$, for $\epsilon_{\Lambda,\xi}(\mathbb{T})$ satisfies it.

3.3 Generalization to Iwasawa algebras

Let L/K be a compact p-adic Lie extension of a local filed K with the Galois group $\mathcal{G}$. We write $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$ for the complete group algebra of $\mathcal{G}$ with coefficients in $\mathbb{Z}_p$ (cf. [NSW, §2]). The ring Λ is a semilocal adic ring, so that $LTNC$ also applies to Λ and some f.g. projective Λ-module $\mathbb{T}$ with a continuous $G_{\mathbb{Q}_p}$-action. Next we generalize $C^{na}_{ep}(L/K, V)$ (as a special case of $LTNC$) to the case of complete group algebras. Let V, T and ξ be as in the previous subsection. Consider

$$\mathbb{T} := \mathrm{Ind}_{L/\mathbb{Q}_p} T = \varprojlim_U \mathbb{T}_U,$$

where U runs through the open normal subgroups of $\mathcal{G}$ and $\mathbb{T}_U := \mathrm{Ind}_{L^U/\mathbb{Q}_p} T$ are f.g. projective $\mathbb{Z}_p[\mathcal{G}/U]$-modules endowed with a continuous action of $G_{\mathbb{Q}_p}$, then by [NSW, Cor. 5.2.12] $\mathbb{T}$ is a f.g. (compact) projective Λ-module with a continuous $G_{\mathbb{Q}_p}$-action. Using Theorems 3.1, 3.4 and the fact before $C^{na}_{ep}(L/K, V)$ we formulate

Conjecture 3.19 ($C^{na}_{ep}(L/K, V)$) *Let L/K be a compact p-adic Lie extension and let V be a de Rham representation of G_K, then for any choice of T and ξ, there exists an isomorphism*

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \to \tilde{\Lambda} \otimes_\Lambda \Delta_{ep}(L/K, T)$$

(i.e. $\epsilon_{\Lambda,\xi}(\mathbb{T}) \in \mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$*)*
satisfying the following condition:

($\star\star$) *Let* $\rho : G \to GL_n(F)$, $n \geq 1$ *be an (irreducible) Artin representation of* $\mathcal{G}$. *Then the image* $\epsilon_{F,\xi}(W_{\mathbb{T},\rho})$ *of* $\epsilon_{\Lambda,\xi}(\mathbb{T})$ *under* $F^n \otimes_\Lambda -$ *is the* ϵ*-isomorphism of the de Rham representation* $W_{\mathbb{T},\rho}$ *described in subsubsection 2.3.1.*

The proofs of Propositions 3.7 and 3.8 generalize immediately to the case of complete group algebras, inverse limits being exact on the category of compact Λ-modules, thus Conjecture 3.19 is independent of the choices of T and ξ.

Remark 3.20 *We believe that* $SK_1(\tilde{\Lambda}) = 1$, *so that*

$$K_1(\tilde{\Lambda}) = \mathrm{Det}(\tilde{\Lambda}) = \varprojlim_U \mathrm{Det}(\widehat{\mathbb{Z}_p^{ur}}[\mathcal{G}/U])$$

by Theorem 6.29. If this is true, then $\epsilon_{\Lambda,\xi}(\mathbb{T}) = \varprojlim_U \epsilon_{\mathbb{Z}_p[\mathcal{G}/U],\xi}(\mathbb{T}_U)$ *(projections are compatible by Proposition 3.14(3)) is unique and belongs to*

$$\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})_{[\mathbb{T},\tau_p]^{-1}}.$$

The reason for that is the vanishing of $SK_1(\widehat{\mathbb{Z}_p^{ur}}[G])$ *for every finite group* G *(see Corollary 6.51). In particular, for abelian extensions* L/K *it is always true, as* $\tilde{\Lambda}$ *is a commutative semilocal ring in this case.*

Remark 3.21 *Let* $\mathcal{H}$ *be an open normal subgroup of* $\mathcal{G}$ *and set* $\Lambda' := \mathbb{Z}_p[\mathcal{G}/\mathcal{H}]$. *It follows from condition* ($\star\star$) *of Conjecture 3.19 that the image of* $\epsilon_{\Lambda,\xi}(\mathbb{T})$ *under* $\Lambda' \otimes_\Lambda$ *is* $\epsilon_{\Lambda',\xi}(\mathrm{Ind}_{L^{\mathcal{H}}/\mathbb{Q}_p} T)$ *– the* ϵ*-isomorphism of* $C_{ep}^{na}(L/K,V)$. *In particular, if* L/K *is finite, then Conjectures 3.19 and* $C_{ep}^{na}(L/K,V)$ *coincide.*

Proposition 3.22 *There are the following functorial properties of Conjecture 3.19*

1. *Conjectures* $C_{ep}^{na}(L/K,V)$ *and* $C_{ep}^{na}(L/K,V^*(1),\xi)$ *are equivalent.*
2. *Let* $0 \to V' \to V \to V'' \to 0$ *be an exact sequence of* p*-adic de Rham representations of* G_K. *If* $C_{ep}^{na}(L/K,\xi)$ *holds for two of the representations* V, V', V'', *then it also holds for the third one.*
3. *Let* $\mathcal{H}$ *be an open normal subgroup of* $\mathcal{G}$. *If* $C_{ep}^{na}(L/K,V)$ *holds, then the conjectures* $C_{ep}^{na}(L/L^{\mathcal{H}},V)$ *and* $C_{ep}^{na}(L^{\mathcal{H}}/K,V)$ *hold, too.*
4. *Let* $\rho : \Lambda \to M_n(\mathcal{O}_F)$ *be an (irreducible) Artin representation of* $\mathcal{G}$ *over* F. *If* $C_{ep}^{na}(L/K,V)$ *holds, then* $C_{ep,K}(V,F)$ *holds for* $V = \rho^* \otimes_{\mathbb{Q}_p} V$.

Proof. The proof is similar to the proof of Proposition 3.14. It only uses the fact that inverse limits are exact on the category of compact Λ-modules. □

Remark 3.23 *Conjecture 3.19 is proved in [BB, Thm. 4.4.4] for* K *being a finite unramified extension of* $\mathbb{Q}_p$, $L = \cup_{n=1}^\infty K(\zeta_{p^n})$ *– the cyclotomic extension of* K *– and* V *being a crystalline representation of* G_K.

4 The conjecture 3.6. Ideas of the proof

4.1 Reduction step

Let $\mathbb{Q}_p \subseteq K \subseteq L$ be a tower of finite Galois field extensions. Let L/K be a Galois field extension with the Galois group G, such that $(|G|, p) = 1$. Let V be a p-adic de Rham representation of G_K. Then we have the following

Proposition 4.1 *The conjecture $C_{ep}^{na}(L/K, V)$ holds if and only if $C_{ep,K}(W, F)$ hold for all de Rham representations $W_{\mathbb{T},\rho}$ over F_ρ, where $\rho : G \to GL_{n_\rho}(F_\rho)$ runs through the set of all (irreducible) representations of G.*

Proof. The "only" part is given by Proposition 3.14 (4).

For the other direction we choose an element

$$\lambda \in \left\{ x \in K_1(\tilde{\Lambda}) = \mathrm{Det}(\tilde{\Lambda}) \mid \varphi_p(x) = [\mathbb{T}, \tau_p] \cdot x \right\}$$

and multiply each $\epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ by $\lambda_\rho := F_\rho^{n_\rho} \otimes_\Lambda \lambda$, respectively. Then φ_p acts trivially on each $\lambda_\rho \cdot \epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$, so that it belongs to

$$\mathrm{Isom}(\mathbf{d}_{\mathcal{O}_{F_\rho}}(0), \tilde{\Delta}_{ep,\mathbb{Q}_p}(\mathcal{O}_{F_\rho}, \mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))) \times^{\mathcal{O}_{F_\rho}^\times} \mathcal{O}_{F_\rho}^\times.$$

The order n of the group G is a unit in $\mathbb{Z}_p$, thus Λ is the maximal order in Ω and we have the Wedderburn decompositions:

$$\Omega = \prod_{\rho \in E} \mathrm{A}_\rho,\ Cent(\mathrm{A}_\rho) = F_\rho \quad \text{and} \quad \Lambda = \prod_{\rho \in E} \mathcal{A}_\rho,$$

where E denotes the set of representatives of the $\mathbb{Q}_p$-equivalence classes of the irreducible representations (or corresponding characters χ_ρ) of G. These decompositions induce the following commutative diagram

$$\begin{tikzcd} \mathrm{Det}(K_1(\Lambda)) \arrow[r] \arrow[d, "f_1"] & K_1(\Omega) \arrow[d, "f_2", "\cong"'] \\ \prod_{\rho \in E} \mathcal{O}_{F_\rho}^\times \arrow[r] & \prod_{\rho \in E} F_\rho^\times, \end{tikzcd} \tag{4.1}$$

where the map f_1 is given as $\prod_{\rho \in E} \mathcal{O}_{F_\rho}^{n_\rho} \otimes_\Lambda -$.

Next we find an element ϵ' of

$$\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K, T)) \times^{K_1(\Lambda)} \mathrm{Det}(K_1(\Lambda)),$$

such that its image under $\mathcal{O}_{F_\rho}^{n_\rho} \otimes_\Lambda -$ is $\lambda_\rho \cdot \epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ for each $\rho \in E$. Such an element always exists, because the set $\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K, T))$ is not empty (see Theorem 3.4) and the map f_1 realizing $\mathcal{O}_{F_\rho}^{n_\rho} \otimes_\Lambda -$ on K_1-groups is surjective by [Fr, Lem. 1.5].

Now we define the ϵ-isomorphism $\epsilon_{\Lambda,\xi}(\mathbb{T})$ of the conjecture $C_{ep}^{na}(L/K,V)$ to be $\epsilon' \cdot \lambda^{-1}$, where we abuse the notation and write ϵ' also for the image of ϵ' in the set

$$\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda})$$

via the inclusion $\mathrm{Det}(K_1(\Lambda)) \hookrightarrow K_1(\tilde{\Lambda})$.

We only have to check the condition $(\star)$. But from the definition of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ we see that for each $\rho \in E$ the image of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F_\rho^{n_\rho} \otimes_\Lambda -$ is precisely the ϵ-isomorphism $\epsilon_{F_\rho,\xi}(W_{\mathbb{T},\rho})$ induced by $\epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$. Moreover, for ρ and ρ' lying in the same $\mathbb{Q}_p$-equivalence class the images of $\epsilon_{\Lambda,\xi}(\mathbb{T})$ under $F_\rho^{n_\rho} \otimes_\Lambda -$ and $F_{\rho'}^{n_{\rho'}} \otimes_\Lambda -$ defined via ρ and ρ', respectively, are the same. □

Remark 4.2 *Let $W_{\mathbb{T},\rho}$ be a representation of $G_{\mathbb{Q}_p}$ over F_ρ. We also can view $W_{\mathbb{T},\rho}$ as a representation over $\mathbb{Q}_p$. Then the ϵ-isomorphism $\epsilon_{\mathbb{Z}_p,\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ (the inverse of $\delta_{V,K/K}$ in $C_{ep,K}(V)$) is the image under the transfer map (norm map) $N_{F_\rho/\mathbb{Q}_p}$ of $\epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ in $C_{ep,K}(V,F)$. Further, the norm of an element $x \in F_\rho$ belongs to $\mathbb{Z}_p$ if and only if $x \in \mathcal{O}_{F_\rho}$, thus $\epsilon_{\mathbb{Z}_p,\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ exists if and only if $\epsilon_{\mathcal{O}_{F_\rho},\xi}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho^* \otimes_{\mathbb{Z}_p} T))$ exists, i.e. $C_{ep,K}(V)$ and $C_{ep,K}(V,F)$ are equivalent.*

Remark 4.3 *Let $\mathbb{Q}_p \subseteq K \subseteq L$ be a tower of finite Galois field extensions, such that the Galois group G of L/K is a product $H \times H'$ with $(|H'|,p) = 1$. Let V be a p-adic de Rham representation of G_K. Then using the same arguments as in Proposition 4.1 and Remark 4.2 one can show that the conjecture $C_{ep}^{na}(L/K,V)$ holds if and only if $C_{ep}^{na}(L^{H'}/K,W)$ hold for all de Rham representations $W_{\mathbb{T},\rho}$ over F_ρ, where $\rho : H' \to GL_{n_\rho}(F_\rho)$ runs through the set of all (irreducible) representations of H'.*

4.2 Applications

We keep the notation of the previous subsection. Let $\mathbb{Q}_p \subseteq K \subseteq L \subseteq \mathbb{Q}_p^{ab}$. By local theorem of Kronecker-Weber every irreducible representation ρ of G over F_ρ is a tensor product of an unramified character χ^{ur} and a character χ^{cyc} of the the cyclotomic extension $K_\infty = \cup_{n=1}^\infty K(\zeta_{p^n})$. In this case we have the following examples

Let V be a crystalline representation of G_K.

Example 4.4 *Let L/K be an unramified extension of degree prime to p. Every twist $V(\chi)$ of V by an (irreducible) unramified character χ of $G = Gal(L/K)$ is again a crystalline representation, such that by Theorem 2.8 (2) the conjectures $C_{ep}(K,V(\chi))$ hold for all χ and thus by Proposition 4.1 the conjecture $C_{ep}(L/K,V)$ holds.*

Now let K be an unramified extension of $\mathbb{Q}_p$.

Example 4.5 *Let $L/K \subset \mathbb{Q}_p^{ab}$ be a tamely ramified extension with $(|G|,p) = 1$. Every twist $V(\chi)$ of V by a character χ of $G = Gal(L/K)$ is a crystalline representation $V(\chi^{ur})$ twisted by a character χ^{cyc} of the cyclotomic extension. By*

Theorem 2.8 (3) the conjectures $C_{ep}(K, V(\chi))$ hold for all χ, thus the conjecture $C_{ep}(L/K, V)$ is true.

Remark 4.6 *Since the conjecture $C_{ep}(L/K, V)$ is stable under exact sequences of p-adic representations, we deduce it also for those semi-stable representations of G_K, whose irreducible subquotients are crystalline representations and for $L/K \subset \mathbb{Q}_p^{ab}$ being as in the two examples above.*

Let V be an ordinary p-adic representation of G_K and K be still unramified over $\mathbb{Q}_p$.

Example 4.7 *Let L/K be an unramified extension of degree prime to p. Every twist $V(\chi)$ of V by an (irreducible) unramified character χ of $G = Gal(L/K)$ is an ordinary representation. By Theorem 2.3 the conjectures $C_{ep,K}(V(\chi))$ hold for all χ, hence the conjecture $C_{ep}^{na}(L/K, V) = C_{ep}(L/K, V)$ holds.*

5 One-dimensional Lubin-Tate groups

We keep the following notation till the end of this section. Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$ and $\chi^{ur} : G_K \to \mathbb{Z}_p^\times$ be a continuous unramified character. We consider a continuous representation $T = \mathbb{Z}_p(\chi^{ur})(1)$ of G_K and set $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. Let L be a finite tamely ramified extension of K and let $G = Gal(L/K)$. We aim to compute the terms appearing in $C_{ep}^{na}(L/K, V)$ and prove the conjecture for L/K being an unramified extension of degree prime to p. For this we extend the ideas of [Breu] from the case of the multiplicative group $\mathbb{G}_m$ to arbitrary one-dimensional Lubin-Tate groups. We omit the case, in which χ^{ur} factors over L, as been proved by (loc. cit.) and assume from now on that $\chi^{ur}(G_L) \neq 1$.

Warning: in contrast to the previous subsection in this section we introduce arbitrary tamely ramified extensions (not only with $(|G|, p) = 1$).

5.1 One-dimensional Lubin-Tate groups and Galois representations

In this subsection we collect some useful facts about one-dimensional Lubin-Tate groups. The main references for this subsection are [N] and [St].

Definition 5.1 *A one-dimensional commutative formal group over a ring R is a formal power series $\mathcal{F}(X, Y) \in R[[X, Y]]$ with the following properties:*

1. $\mathcal{F}(X, Y) \equiv X + Y \bmod \deg 2$,
2. $\mathcal{F}(X, Y) = \mathcal{F}(Y, X)$,
3. $\mathcal{F}(X, \mathcal{F}(Y, Z)) = \mathcal{F}(\mathcal{F}(X, Y), Z)$.

By evaluating in a domain $\mathfrak{p}$, where the power series converge, we get an ordinary group from a formal group. For example let R be a complete valuation ring and $\mathfrak{p}$ its maximal ideal, then the operation

$$x +_{\mathcal{F}} y := \mathcal{F}(x, y), \quad x, y \in \mathfrak{p}$$

defines a new abelian group structure on $\mathfrak{p}$.

Example 5.2 *$\hat{\mathbb{G}}_a = X + Y$ the formal additive group.*

Example 5.3 *$\hat{\mathbb{G}}_m = X + Y + XY$ the formal multiplicative group. Indeed,*

$$X + Y + XY = (1 + X) \cdot (1 + Y) - 1,$$

such that the new operation $+_{\hat{\mathbb{G}}_m}$ is obtained from multiplication $\cdot$ via the translation $x \mapsto x + 1$.

Definition 5.4 *A homomorphism $f : \mathcal{F} \to \mathcal{G}$ between two formal groups is a power series $f(X) = \sum_{i=1}^{\infty} a_i X^i \in R[[X]]$, such that*

$$f(\mathcal{F}(X, Y)) = \mathcal{G}(f(X), f(Y)).$$

A homomorphism is an isomorphism, if a_1 is a unit, i.e. there is a power series

$$f^{-1}(X) = a_1^{-1} X + \ldots \in R[[X]], \quad f^{-1} : \mathcal{G} \to \mathcal{F},$$

such that $f(f^{-1}(X)) = f^{-1}(f(X)) = X$.

If the coefficients of a homomorphism $f : \mathcal{F} \to \mathcal{G}$ belongs to an extension ring R', then we call this homomorphism defined over R'.

The homomorphisms $f : \mathcal{F} \to \mathcal{F}$ of a formal group $\mathcal{F}$ over R form a ring $\mathrm{End}_R(\mathcal{F})$ with the operations

$$(f +_{\mathcal{F}} g)(X) = \mathcal{F}(f(X), g(X)), \quad (f \circ g)(X) = f(g(X)).$$

Definition 5.5 *A formal R-module is a formal group $\mathcal{F}$ over R together with a ring homomorphism*

$$R \to \mathrm{End}_R(\mathcal{F}), \quad a \longmapsto [a]_{\mathcal{F}}(X),$$

such that $[a]_{\mathcal{F}}(X) \equiv aX \bmod \deg 2$.

A homomorphism (over $R' \supseteq R$) between formal R-modules $\mathcal{F}$, $\mathcal{G}$ is a homomorphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups (over R') with the property

$$f([a]_{\mathcal{F}}(X)) = [a]_{\mathcal{G}}(f(X)) \quad \text{for all } a \in R.$$

Now let $R = \mathcal{O}_K$ be the valuation ring of a local field K with the residue class field $\mathbb{F}_q = \mathcal{O}_K/\mathfrak{p}_K$ of cardinality q.

Definition 5.6 *A (one-dimensional commutative) Lubin-Tate group over $\mathcal{O}_K$ for the prime element π is a formal $\mathcal{O}_K$-module $\mathcal{F}$ such that*

$$[\pi]_{\mathcal{F}}(X) \equiv X^q \bmod \pi.$$

The simplest candidate for $[\pi]_{\mathcal{F}}(X)$ is $[\pi]_{\mathcal{F}}(X) = \pi X + X^q$. The next theorem shows

Theorem 5.7 *There is a unique (one-dimensional commutative) Lubin-Tate group $\mathcal{F}_\pi$ over $\mathcal{O}_K$ for the prime element π such that $[\pi]_{\mathcal{F}_\pi}(X) = \pi X + X^q \in \mathrm{End}_{\mathcal{O}_K}(\mathcal{F}_\pi)$.*

Proof. See [St, Thm. 74]. □

Proposition 5.8 *The Lubin-Tate formal groups $\mathcal{F}$, $\mathcal{F}'$ for the primes π, π', respectively, are isomorphic over $\mathcal{O}_K$ if and only if $\pi = \pi'$. There is an isomorphism defined over $\mathcal{O}_{\widehat{K^{ur}}}$*

$$\mathcal{F}_\pi \to \mathcal{F}_{\pi'}, \quad X \longmapsto uX + \dots,$$

where $\pi' = u\pi$ and $u \in \mathcal{O}_K^\times$.

Proof. See [St, Prop. 77] and [N, Thm. 4.6]. □

From the last proposition we see that it is enough to consider the unique Lubin-Tate group $\mathcal{F}_\pi$ for each prime π.

Proposition 5.9 *The Lubin-Tate formal group $\mathcal{F}_\pi$ over $\mathcal{O}_K$ is a p-divisible commutative formal Lie group of height $[K : \mathbb{Q}_p]$. In particular, we have a notion of a tangent space.*

Proof. See [St, Prop. 75]. □

Proposition 5.10 *There is an isomorphism $\log_{\mathcal{F}_\pi} : \mathcal{F}_\pi \to \hat{\mathbb{G}}_a$ of formal $\mathcal{O}_K$-modules defined over K.*

Proof. See [St, Prop. 76]. □

Let $\bar{\mathfrak{p}}$ denote the maximal ideal of the valuation ring of $\overline{\mathbb{Q}_p}$. We evaluate $\mathcal{F}_\pi$ in $\bar{\mathfrak{p}}$ to get an abelian group $\mathcal{F}_\pi(\bar{\mathfrak{p}})$. As next we study the structure of this group.

Proposition 5.11 *$\mathcal{F}_\pi(\bar{\mathfrak{p}})$ is p-divisible.*

Proof. The statement follows from 5.9 and [St, Cor. 84 and Thm. 70]. □

The group $\mathcal{F}_\pi$ is a formal $\mathcal{O}_K$-module, thus for any algebraic extension L of K the group $\mathcal{F}_\pi(\mathfrak{p}_L)$ admits an $\mathcal{O}_K$-module structure.

Proposition 5.12 *Let $K = \mathbb{Q}_p$ and L be a finite extension of K. Then $\mathcal{F}_\pi(\mathfrak{p}_L)$ is a $\mathbb{Z}_p$-module of rank $[L : \mathbb{Q}_p]$.*

Proof. The statement follows from 5.9 and [St, Cor. 87(2)]. □

The endomorphism $[\pi] : \mathcal{F}_\pi \to \mathcal{F}_\pi$ is a finite flat map of order q, hence the kernel $\mathcal{F}_\pi[\pi]$ of $[\pi]$ is a finite flat group over $\mathcal{O}_K$ of order q. The π-adic Tate-module of $\mathcal{F}_\pi$ is

$$T_\pi \mathcal{F}_\pi = \varprojlim_n \mathcal{F}_\pi[\pi^n](\bar{\mathfrak{p}})$$

with transfer maps given by $[\pi]$. The group $T_\pi = T_\pi \mathcal{F}_\pi$ has an $\mathcal{O}_K$-module structure by functoriality and carries an $\mathcal{O}_K$-linear G_K-action.

Lemma 5.13 *T_π is a free $\mathcal{O}_K$-module of rank 1.*

Proof. See the proof before [St, Lem. 78]. □

From the lemma above we deduce that the Galois action on T_π is given by a character

$$\chi_\pi : G_K \to GL_1(T_\pi) = \mathcal{O}_K^\times.$$

Lemma 5.14 *The character χ_π is surjective.*

Proof. See [St, Lem. 78]. □

Lemma 5.15 *For $\pi' = u\pi$ the characters $\chi_{\pi'}$ and χ_π differ by an unramified character $\chi_{\pi'}/\chi_\pi$ sending the Frobenius of $\mathcal{O}_{\widehat{K^{ur}}}$ to u.*

Proof. See [St, Prop. 80]. □

Remark 5.16 *For a different prime π' the π'-adic Tate-module*

$$T_{\pi'} \mathcal{F}_\pi = \varprojlim_n \mathcal{F}_\pi[\pi'^n](\bar{\mathfrak{p}})$$

is isomorphic to T_π as an $\mathcal{O}_K[G_K]$-module. Indeed, in K we can factor $\pi' = u\pi$ with a unit u. The map $[u]$ is an automorphism of $\mathcal{F}_\pi$ and commutes with the action of G_K.

Later we will need a description of points of a formal group given by the lemma on p. 237 of [LR]. For the convenience of the reader we recall the statement and the proof of this lemma here.

Definition 5.17 *A (d-dimensional) formal group $\mathcal{F}$ over $\mathcal{O}_K$ is called **toroidal** if $\mathcal{F}$ is isomorphic to $\hat{\mathbb{G}}_m^d$ over $\mathcal{O}_{\widehat{K^{ur}}}$.*

If $f : \mathcal{F} \to \hat{\mathbb{G}}_m^d$ is an isomorphism over $\mathcal{O}_{\widehat{K^{ur}}}$, there is another, f^σ, obtained by applying σ – the Frobenius of $\mathcal{O}_{\widehat{K^{ur}}}$ – to the coefficients of the power series describing f. Then $f^\sigma \circ f^{-1}$ is an automorphism of $\hat{\mathbb{G}}_m^d$ and thus corresponds in a natural way to a non-singular $d \times d$ matrix w over $\mathbb{Z}_p$ (since $\mathrm{End}_{\mathcal{O}_{\widehat{K^{ur}}}}(\hat{\mathbb{G}}_m) \cong$

$\mathrm{End}_{\mathbb{Z}_p}(\hat{\mathbb{G}}_m) \cong \mathbb{Z}_p$). This matrix is called a **twist matrix** of $\mathcal{F}$. Any two such will be similar.

Now suppose that L/K is a totally ramified Galois extension and that $\mathcal{F}$ is toroidal. Extend σ to $L\widehat{K^{ur}}$ by defining it to be the identity on L. Finally, let w be a twist matrix of $\mathcal{F}$. We set

$$V(L) = V_w(L) := \left\{ \beta \in U^1(L\widehat{K^{ur}})^d \mid \beta^\sigma = \alpha^w \right\},$$

where $U^1(L\widehat{K^{ur}})$ denote the principal units of $\mathcal{O}_{L\widehat{K^{ur}}}$, σ acts diagonally on β and u acts in the obvious way. The elements of $G(L/K)$ commute with the action of σ and thus $V(L)$ is a $G(L/K)$-module.

Lemma 5.18 *$\mathcal{F}(\mathfrak{p}_L) \cong V(L)$ as $G(L/K)$-modules.*

Proof. Let $f : \mathcal{F} \to \hat{\mathbb{G}}_m^d$ be an isomorphism over $\mathcal{O}_{\widehat{K^{ur}}}$ such that $f^\sigma \circ f^{-1} = w$. Then f induces a group isomorphism from $\mathcal{F}(\mathfrak{p}_{L\widehat{K^{ur}}})$ to $\hat{\mathbb{G}}_m^d(\mathfrak{p}_{L\widehat{K^{ur}}}) = U^1(L\widehat{K^{ur}})^d$. Let $\beta \in \mathcal{F}(\mathfrak{p}_{L\widehat{K^{ur}}})$. Then $\beta \in \mathcal{F}(\mathfrak{p}_L)$ if and only if $\beta^\sigma = \beta$. Now if $\beta^\sigma = \beta$ then

$$f(\beta)^\sigma = f^\sigma(\beta^\sigma) = f^\sigma(\beta) = f(\beta)^w.$$

Thus $\beta \in \mathcal{F}(\mathfrak{p}_L)$ implies $f(\beta) \in V(L)$. Similarly if $\beta \in V(L)$ then $\beta \in \mathcal{F}(\mathfrak{p}_L)$. □

From now on we set $K = \mathbb{Q}_p$.

Example 5.19 *The formal multiplicative group $\hat{\mathbb{G}}_m$ is a Lubin-Tate group over $\mathbb{Z}_p$ for the prime element p with respect to the mapping*

$$\mathbb{Z}_p \to \mathrm{End}_{\mathbb{Z}_p}(\hat{\mathbb{G}}_m), \quad a \longmapsto [a]_{\hat{\mathbb{G}}_m}(X) = (1+X)^a - 1 = \sum_{i=1}^{\infty} \tbinom{a}{i} X^i.$$

In particular, $[p^n]_{\hat{\mathbb{G}}_m}(X) = (1+X)^{p^n} - 1$, so that $\hat{\mathbb{G}}_m[p^n]$ consists of the elements $\zeta - 1$, where ζ varies over the p^n-th roots of unity. Thus the p-adic Tate module $T_p\hat{\mathbb{G}}_m$ is isomorphic to $\mathbb{Z}_p(1)$ (the isomorphism depending on a choice of ξ). By Proposition 5.8 $\hat{\mathbb{G}}_m$ is isomorphic to $\mathcal{F}_p$ over $\mathbb{Z}_p$, whence $T_p\hat{\mathbb{G}}_m \cong T_p\mathcal{F}_p$ as $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-modules.

Example 5.20 *We consider $\mathcal{F}_\pi$ for $\pi = up$ with $u \neq 1 \in \mathbb{Z}_p^\times$. From the previous example, Lemma 5.15 and Remark 5.16 we deduce $T_p\mathcal{F}_\pi \cong \mathbb{Z}_p(1)(\chi)$, where $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is an unramified character sending the Frobenius of $\mathbb{Q}_p^{ur}$ to u.*

Further, the group $\mathcal{F}_\pi$ is toroidal of dimension 1. Next we compute the twist matrix $w \in \mathbb{Z}_p^\times$ of $\mathcal{F}_\pi$. The isomorphism $f : \mathcal{F}_\pi \to \hat{\mathbb{G}}_m$ defined over $\widehat{\mathbb{Z}_p^{ur}}$ is the inverse of the unique isomorphism

$$g : \hat{\mathbb{G}}_m \hat{\otimes} \mathbb{Q}_p^{ur} \xrightarrow{\log_{\hat{\mathbb{G}}_m}} \hat{\mathbb{G}}_a \xrightarrow{\log_{\mathcal{F}_\pi}^{-1}} \mathcal{F}_\pi \hat{\otimes} \mathbb{Q}_p^{ur}$$

also defined over $\widehat{\mathbb{Z}_p^{ur}}$ *(see step 3 in the proof of Proposition 77 in [St]). From the construction of* g *it follows, that*

$$g^\sigma = g \circ [u]_{\hat{\mathbb{G}}_m},$$

whence

$$w = f^\sigma \circ f^{-1} = (g^{-1})^\sigma \circ g = [u^{-1}]_{\hat{\mathbb{G}}_m} \circ g^{-1} \circ g = [u^{-1}]_{\hat{\mathbb{G}}_m}.$$

From Lemma 5.18 we have for a totally ramified Galois extension $L/\mathbb{Q}_p$

$$\mathcal{F}_\pi(\mathfrak{p}_L) \cong \left\{\beta \in U^1(L\widehat{\mathbb{Q}_p^{ur}}) \mid \beta^\sigma = [u^{-1}]_{\hat{\mathbb{G}}_m}(\beta)\right\}$$

as $G(L/\mathbb{Q}_p)$*-modules. Furthermore, since* $G(L^{ur}/L) \cong G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$ *is topologically generated by* σ *and* $\chi(\sigma) = u$*, there is an isomorphism of* $G(L/\mathbb{Q}_p)$*-modules*

$$\mathcal{F}_\pi(\mathfrak{p}_L) \cong U^1(\widehat{L^{ur}})(\chi)^{G(L^{ur}/L)},$$

where the $\mathbb{Z}_p$*-module structure of* $U^1(\widehat{L^{ur}})$ *is given by the mapping in the previous example. Similar results can be obtained also for the base change of* $\mathcal{F}_\pi$ *to a finite extension* $K/\mathbb{Q}_p$*.*

5.2 Galois cohomology

First we compute the Galois cohomology groups $\mathrm{H}^i(L,T)$ as Λ-modules. We point out, that by this we also determine the Galois cohomology groups of $T^*(1)$ in view of the local duality theorems in the Galois cohomology theory.

Proposition 5.21 *With the notation as above we have:*

- $\mathrm{H}^i(L,T) = 0$ *for* $i \neq 1$.
- $H^1(L,T) \cong \left(\widehat{(L^{ur})^\times}^p(\chi^{ur})\right)^{G(L^{ur}/L)}$*, where* $\widehat{-}^p$ *denotes the p-completion of a group.*
- *There is an isomorphism of* Λ*-modules*

$$\mathcal{F}(\mathfrak{p}_L) \xrightarrow{\cong} \mathrm{H}^1(L,T).$$

Proof. $\mathrm{H}^i(L,T) = 0$ for $i \neq 0,1,2$ because the cohomological dimension of G_K is 2. Further, $\mathrm{H}^0(L,T) = T^{G_L} = 0$, as the character $\chi^{ur} \otimes \chi^{cyc} : G_L \to \mathbb{Z}_p^\times$ is not trivial. Using the local duality theorem we get $\mathrm{H}^2(L,T) = \mathrm{H}^0(L,T^*(1))^*$. The last group is zero because $(\chi^{ur})^{-1}(G_L) \neq 1$.

To compute the group $\mathrm{H}^1(L,T)$ we use the Hochschild-Serre spectral sequence for the closed subgroup $G_{L^{ur}}$ of G_L, which exists first only for finite discrete modules T/p^n, but with [NSW, Thm. 2.7.5] also for the compact module T. Note that the character χ^{ur} factors over L^{ur}, such that $G_{L^{ur}}$ acts via the cyclotomic character on T. The five-term exact sequence takes the form

$$0 \to \mathrm{H}^1(Gal(L^{ur}/L), T^{G_{L^{ur}}}) \to \mathrm{H}^1(L,T) \to$$

$$\mathrm{H}^1(G_{L^{ur}}, T)^{G(L^{ur}/L)} \to \mathrm{H}^2(Gal(L^{ur}/L), T^{G_{L^{ur}}}) \to \mathrm{H}^2(L,T).$$

The module of invariants $T^{G_{L^{ur}}}$ is a zero-module, since the cyclotomic character is not trivial, thus the first and the fourth terms in the exact sequence above vanish and we get a canonical isomorphism

$$\mathrm{H}^1(L,T) \cong \mathrm{H}^1(G_{L^{ur}}, T)^{G(L^{ur}/L)} = \mathrm{H}^1(G_{L^{ur}}, \mathbb{Z}_p(\chi^{ur})(1))^{G(L^{ur}/L)}.$$

From the Kummer theory and the isomorphism

$$\mathrm{H}^1(G_{L^{ur}}, \mathbb{Z}_p(\chi^{ur})(1))^{G(L^{ur}/L)} = \Big(\mathrm{H}^1(G_{L^{ur}}, \mathbb{Z}_p(1))(\chi^{ur})\Big)^{G(L^{ur}/L)}$$

we obtain $H^1(L,T) \cong \Big(\widehat{(L^{ur})^{\times}}^{p}(\chi^{ur})\Big)^{G(L^{ur}/L)}$.

By taking G_L-invariants of the exact sequence

$$0 \longrightarrow \mathcal{F}[p^n](\bar{\mathfrak{p}}) \longrightarrow \mathcal{F}(\bar{\mathfrak{p}}) \xrightarrow{[p^n]} \mathcal{F}(\bar{\mathfrak{p}}) \longrightarrow 0 \tag{5.1}$$

we get the following exact sequence of Λ-modules

$$0 \longrightarrow {}^{\mathcal{F}(\mathfrak{p}_L)}/_{[p^n](\mathcal{F}(\mathfrak{p}_L))} \longrightarrow \mathrm{H}^1(L, \mathcal{F}[p^n](\bar{\mathfrak{p}})) \longrightarrow \mathrm{H}^1(L, \mathcal{F}(\bar{\mathfrak{p}}))[p^n] \longrightarrow 0$$

for each $n \geq 1$.

The inverse limit over n of the exact sequences above results in the exact sequence of Λ-modules

$$0 \longrightarrow \mathcal{F}(\mathfrak{p}_L) \longrightarrow \mathrm{H}^1(L,T) \longrightarrow \mathrm{H}^1(L,T)/\mathcal{F}(\mathfrak{p}_L) \longrightarrow 0$$

$\mathcal{F}(\mathfrak{p}_L)$ being a f.g. $\mathbb{Z}_p$-module (cf. [Breu, 4.5.1]). From Example 5.20 we deduce that the quotient $\mathrm{H}^1(L,T)/\mathcal{F}(\mathfrak{p}_L)$ is isomorphic to

$$\Big(\frac{\widehat{(L^{ur})^{\times}}^{p}}{U^1(\widehat{L^{ur}})}(\chi^{ur})\Big)^{G(L^{ur}/L)}. \tag{5.2}$$

For the extension $L^{ur}/\mathbb{Q}_p$ we have (both algebraically and topologically)

$$(L^{ur})^{\times} = (\pi_L) \times \mathcal{O}_{L^{ur}}^{\times} \cong \mathbb{Z} \oplus \mathcal{O}_{L^{ur}}^{\times},$$

where π_L is a prime element of $\mathcal{O}_L$. Let κ_L denote the residue class field of L, then we have a split exact sequence

$$1 \longrightarrow U^1(L^{ur}) \longrightarrow \mathcal{O}_{L^{ur}}^{\times} \longrightarrow \overline{\kappa_L}^{\times} \longrightarrow 1.$$

The group $\overline{\kappa_L}^\times$ is p-divisible, thus

$$\widehat{\overline{\kappa_L}^\times}^p := \varprojlim_n \overline{\kappa_L}^\times/(\overline{\kappa_L}^\times)^{p^n} = 1$$

and

$$\widehat{(L^{ur})^\times}^p = \widehat{(\pi_L)}^p \times \widehat{U^1(L^{ur})}^p = \widehat{(\pi_L)}^p \times U^1(\widehat{L^{ur}}),$$

whence the quotient in (5.2) is isomorphic to

$$\left(\widehat{(\pi_L)}^p(\chi^{ur})\right)^{G(L^{ur}/L)}.$$

The last is trivial, since the group $G(L^{ur}/L)$ acts trivially on (π_L) and χ^{ur} is a non-trivial character . □

Next we compute the finite part $\mathrm{H}^1_f(L,T) \subseteq \mathrm{H}^1(L,T)$ defined as a preimage of $\mathrm{H}^1_f(L,V)$ under the map $i : \mathrm{H}^1(L,T) \to \mathrm{H}^1(L,V)$.

Lemma 5.22 $\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V) = [L : \mathbb{Q}_p]$.

Proof. Both, V and $V^*(1)$, are de Rham representations of G_L, thus from [BK, pp. 355-356] we have

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) + \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V^*(1)) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V); \tag{5.3}$$

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p}(t_V(L)) + \dim_{\mathbb{Q}_p} \mathrm{H}^0(L,V), \tag{5.4}$$

where $t_V(L) := \mathrm{D}^L_{dR}(V)/Fil^0\mathrm{D}^L_{dR}(V)$. The same is true for $V^*(1)$.

But $\mathrm{H}^0(L,V)$ and $\mathrm{H}^0(L,V^*(1))$ are zeros by the proof of Proposition 5.21, so that

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p}(t_V(L))$$

and

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V^*(1)) = \dim_{\mathbb{Q}_p}(t_{V^*(1)}(L)).$$

For a de Rham representation W by [FO, p. 148]

$$t_W(L) = gr^{-1}(W) \hookrightarrow (\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} W)^{G_L},$$

thus by Corollary 3.57 in (loc. cit.)

$$(\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} V)^{G_L} = (\mathbb{C}_p(\chi^{ur}))^{G_L} \cong L$$

and

$$(\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} V^*(1))^{G_L} = (\mathbb{C}_p(\chi^{ur})^{-1}(-1))^{G_L} = 0.$$

From the equality (5.3) we get

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V).$$

Finally, using the formula for the Euler characteristic

$$\sum_{i=0}^{\infty}(-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(L,V) = -[L:\mathbb{Q}_p]\cdot \dim_{\mathbb{Q}_p} V$$

we see that $\dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V) = [L:\mathbb{Q}_p]$, as $\mathrm{H}^i(L,V) = 0$ for $i \neq 1$ (cf. Proposition 5.21). □

Corollary 5.23 *From the above proposition it follows that*

- $\mathrm{H}^1_f(L,T) = \mathrm{H}^1(L,T)$ *is a* $\mathbb{Z}_p$*-module of rank* $[L:\mathbb{Q}_p]$,
- $\mathrm{H}^1_f(L,T^*(1)) = \mathrm{H}^1(L,T^*(1))_{tors} \cong \mathrm{H}^0(L,V^*(1)/T^*(1))$ *is a finite torsion group.*

Proof. The first part is obvious. By the definition of $\mathrm{H}^1_f(L,T^*(1))$ it contains the torsion subgroup of $\mathrm{H}^1(L,T^*(1))$ and, since the image of $\mathrm{H}^1_f(L,T^*(1))$ in $\mathrm{H}^1(L,V^*(1))$ is zero, they are equal. Consider an exact sequence of G_L-modules

$$0 \longrightarrow T^*(1) \longrightarrow V^*(1) \longrightarrow V^*(1)/T^*(1) \longrightarrow 0.$$

The associated long exact sequence in cohomology is

$$0 \longrightarrow \mathrm{H}^0(L,T^*(1)) \longrightarrow \mathrm{H}^0(L,V^*(1)) \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1)) \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(L,T^*(1)) \longrightarrow \mathrm{H}^1(L,V^*(1)) \longrightarrow \mathrm{H}^1(L,V^*(1)/T^*(1)) \longrightarrow \cdots$$

The groups $\mathrm{H}^0(L,T^*(1))$ and $\mathrm{H}^0(L,V^*(1))$ are zeros. Further, the group $\mathrm{H}^0(L,V^*(1)/T^*(1))$ is a finite torsion group, since $(\chi^{ur})^{-1} \equiv \mathrm{id} \pmod{p^k}$ for some $k >> 1$, so that we can replace $\mathrm{H}^1(L,T^*(1))$ by $\mathrm{H}^1_f(L,T^*(1))$ in the exact sequence above getting

$$0 \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1)) \xrightarrow{\cong} \mathrm{H}^1_f(L,T^*(1)) \longrightarrow 0.$$

□

5.3 Comparison isomorphism

The p-adic comparison isomorphism for a de Rham representation V

$$\begin{aligned} comp_{V,L/\mathbb{Q}_p} : B_{dR}\otimes_{\mathbb{Q}_p} \mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) &\overset{\cong}{\to} B_{dR}\otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p}V, \\ c\otimes x &\mapsto cx \end{aligned}$$

is a B_{dR}-linear map, which commutes with the action of $G_{\mathbb{Q}_p}$, where $G_{\mathbb{Q}_p}$ acts on $B_{dR}\otimes_{\mathbb{Q}_p} \mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ via $g(c\otimes x) = g(c)\otimes x$ and diagonally on $B_{dR}\otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p}V$.

We fix an element $t^{ur} \in (\widehat{\mathbb{Z}_p^{ur}}^{\times}(\chi^{ur}_{\mathbb{Q}_p}))^{Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$. Note that two such elements differ by an element of $\mathbb{Z}_p^\times$. Then $\mathcal{O}_L$ and $(\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}$ are isomorphic as Λ-modules by just sending $l \in \mathcal{O}_L$ to $t^{ur} \cdot l \in (\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}$. Thus every element $\tilde{l} \in (\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}$ can be written as

$$\tilde{l} = t^{ur} \cdot l = t^{ur} \cdot \sum_{g \in G} a_g g(b), \quad a_g \in \mathcal{O}_K.$$

Obviously, the same is true for L and $(\widehat{L^{ur}}(\chi^{ur}))^{G(L^{ur}/L)}$.

Let $t := log[\xi] \in B_{dR}$ denote the p-adic period analogous to $2\pi i$, then $g(t) = \chi^{cyc}(g) \cdot t$ for all $g \in G_{\mathbb{Q}_p}$. Denote by v a basis of $\mathbb{Z}_p(\chi^{ur}_{\mathbb{Q}_p})$, where $\chi^{ur}_{\mathbb{Q}_p}$ is a character of $G_{\mathbb{Q}_p}$ appearing as a twist $T_p\mathcal{F}_\pi(-1)$ of a p-adic Tate module $T_p\mathcal{F}_\pi$ for an appropriate $\pi \in \mathbb{Z}_p$, such that χ^{ur} is the restriction of $\chi^{ur}_{\mathbb{Q}_p}$ to G_K. Then the element $v \otimes \xi$ substitutes a basis of T. Moreover, $\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \cong \mathrm{D}^L_{dR}(V)$ (resp. $\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)) \cong \mathrm{D}^L_{dR}(V(-1))$ is a one-dimensional L-vector space with the basis $e_{\chi^{ur}_{\mathbb{Q}_p},1} := t^{ur} \cdot t^{-1} \otimes (v \otimes \xi)$ (resp. $e_{\chi^{ur}_{\mathbb{Q}_p},0} := t^{ur} \otimes v$). In particular, they are isomorphic as Ω-modules and we have a commutative diagram of $B_{dR}[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$\begin{tikzcd} B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V) \arrow[r, "comp_V", "\cong"'] \arrow[d] & B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V \arrow[d, "t \cdot \otimes f"] \\ B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V(-1)) \arrow[r, "comp_{V(-1)}", "\cong"'] & B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \end{tikzcd} \tag{5.5}$$

where the map $t\cdot$ is the multiplication with t and $f(v \otimes \xi) = v$.

Warning: the left vertical arrow in the above diagram is an isomorphism of Ω-modules induced by $e_{\chi^{ur}_{\mathbb{Q}_p},1} \mapsto e_{\chi^{ur}_{\mathbb{Q}_p},0}$, whereas the right vertical arrow is an isomorphism of $B_{dR}[G]$-modules with an action of $G_{\mathbb{Q}_p}$ and is responsible for the following normalization on K_1-groups.

We apply (not necessary commutative) determinant functor (see Appendix B) to $comp_{V,L/\mathbb{Q}_p}$ to obtain the map

$$\tilde{\alpha}_{V,L/K} \in \mathrm{Isom}(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \times^{K_1(\Omega)} K_1(B_{dR}[G])$$

Multiplying $\tilde{\alpha}_{V,L/K}$ with t^{-1} we get

$$\alpha_{V,L/K} = (x, y) \in \mathrm{Isom}(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \times^{K_1(\Omega)} K_1(\widehat{L^{ur}}[G])$$

with

$$g(y) = [\mathrm{Ind}_{L/\mathbb{Q}_p} V, g] \cdot y, \ \forall g \in G(L^{ur}/\mathbb{Q}_p)$$

(cf. Remark 2.17).

The maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$ is the composite of the maximal unramified extension $\mathbb{Q}_p^{ur}$ and the cyclotomic extension $\mathbb{Q}_{p,\infty}$, which is obtained by adjoining all p-power roots of 1. For $g \in G_{\mathbb{Q}_p}$ we define $g^{ur} \in Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ by $g^{ur}|_{\mathbb{Q}_p^{ur}} = g|_{\mathbb{Q}_p^{ur}}$ and $g^{ur}|_{\mathbb{Q}_p^{ram}} = \mathrm{id}$. We also define $g^{ram} \in Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ by $g^{ram}|_{\mathbb{Q}_p^{ur}} = \mathrm{id}$ and $g^{ram}|_{\mathbb{Q}_p^{ram}} = g|_{\mathbb{Q}_p^{ram}}$. Thus $g|_{\mathbb{Q}_p^{ab}} = g^{ur}g^{ram}$.

We set

$$\beta_{V,L/K} := \epsilon_D(L/K,V)^{-1} \cdot \alpha_{V,L/K} = (x, \epsilon_D(L/K,V)^{-1} \cdot t^{-1} \cdot y),$$

where

$$\epsilon_D(L/K,V) := \Big(\epsilon(\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V \otimes \rho_\chi^*)), \psi_{\mathbb{Q}_p}, dx)\Big)_{\chi \in Irr(G)} \in K_1(\overline{\mathbb{Q}_p}[G]).$$

The right hand side is defined similar to [BB, pp. 21-22] or [FK, 3.3.3] via the theory of local ϵ-constants à la Deligne. According to [BB, Lem. 2.4.3] $\beta_{V,L/K}$ is an element of

$$\mathrm{Isom}(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p}V) \times^{K_1(\Omega)} K_1(\tilde{\Omega})_{[\mathrm{Ind}_{L/\mathbb{Q}_p}V, \tau_p]}.$$

From the twisting property of the local ϵ-constants (see [Ta1, (3.4.5)]) we get the equality

$$\begin{aligned} &\epsilon_D(L/K,V) = \alpha \cdot \epsilon_D(L/K,V(-1)), \quad \alpha = (\alpha_\chi) \in K_1(\overline{\mathbb{Q}_p}[G]) \ \text{ with} \\ &\alpha_\chi = \big\| f(\mathrm{Ind}_{K/\mathbb{Q}_p}(V(-1) \otimes \rho_\chi^*)) \big\| \cdot p^{n(\psi_{\mathbb{Q}_p}) \cdot \dim_{\mathbb{Q}_p} \mathrm{Ind}_{K/\mathbb{Q}_p}(V(-1) \otimes \rho_\chi^*)}, \end{aligned} \tag{5.6}$$

where $f(-)$ is the local Artin conductor, $\|-\|$ denotes the absolute norm and $n(\psi_{\mathbb{Q}_p})$ is defined to be the largest integer n such that $\psi_{\mathbb{Q}_p}(p^{-n}\mathbb{Z}_p) = 1$. Using the induction property of local Artin conductors (see [Breu, Lem. 3.3]) and the assumption that the kernel of $\psi_{\mathbb{Q}_p}$ is equal to $\mathbb{Z}_p$ the last equality becomes

$$\alpha_\chi = N_{K/\mathbb{Q}_p}(f(\rho_\chi^*)) \cdot d_{K/\mathbb{Q}_p}^{\chi(1)} \tag{5.7}$$

$d_{K/\mathbb{Q}_p}$ being the discriminant of $K/\mathbb{Q}_p$.

Remark 5.24 *Let $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ be any embedding. We also write ι for the induced map $K_1(\overline{\mathbb{Q}}[G]) \to K_1(\overline{\mathbb{Q}_p}[G])$. Similar to [Breu, 3.4.4] we define an equivariant ϵ-constant à la Langland*

$$\epsilon_L(L/K,V) := \Big(\tau_{\mathbb{Q}_p}(Ind_{K/\mathbb{Q}_p}(V \otimes \chi^{-1}))\Big)_{\chi \in Irr(G)} \in K_1(\overline{\mathbb{Q}}[G]),$$

where $\tau_{\mathbb{Q}_p}$ is the local Galois Gauss sum. The image $\iota(\epsilon_L(L/K,V)) \in K_1(\overline{\mathbb{Q}_p}[G])$ depends on ι, but by [Breu, Thm. 3.8] two such images (for ι and ι') differ by an element of $K_1(\Lambda)$, thus $\partial(\iota(\epsilon_L(L/K,V)))$ is independent of ι.

From the relation between Deligne's and Langlands' local ϵ-constants (see (3.6.1) and (3.4.5) of [Ta1]) we deduce the equality up to an element of $K_1(\Lambda)$

$$\epsilon_D(L/K,V(-1)) = \vartheta \cdot \iota(\epsilon_L(L/K,V(-1))) \quad \textit{in } K_1(\overline{\mathbb{Q}_p}[G]),$$

where

$$\vartheta_\chi = p^{n(\psi_{\mathbb{Q}_p})\cdot\frac{1}{2}\cdot\dim(\mathrm{Ind}_{K/\mathbb{Q}_p}\rho_\chi^*)}.$$

The last are equal to 1 *for all* $\chi \in Irr(G)$, *as* $n(\psi_{\mathbb{Q}_p}) = 0$. *It follows, that*

$$\partial(\epsilon_D(L/K, V(-1))) = \partial(\iota(\epsilon_L(L/K, V(-1)))). \tag{5.8}$$

Finally, we remark that by [Ta2, Cor. 5]

$$\iota(\tau_{\mathbb{Q}_p}(Ind_{K/\mathbb{Q}_p}(\chi^{ur}\otimes\chi))) = \chi^{ur}_{\mathbb{Q}_p}\left(N_{K/\mathbb{Q}_p}(f(\chi))\cdot d_{K/\mathbb{Q}_p}^{\chi(1)}\right)\cdot\iota(\tau_{\mathbb{Q}_p}(Ind_{K/\mathbb{Q}_p}\chi)),$$

where the character $\chi^{ur}_{\mathbb{Q}_p}$ *is viewed as a character* $\mathbb{Q}_p^\times \to G^{ab}_{\mathbb{Q}_p} \overset{\chi^{ur}_{\mathbb{Q}_p}}{\to} \mathbb{Q}_p^\times$ *via the local reciprocity law.*

Remark 5.25 *An easy calculation shows that in our case the factor* $\Gamma_L(V)$ *of [FK, 3.3.4]: or* $\Gamma^*(V)$ *of [BB, 2.4] used for the correction of the comparison isomorphism is equal to* 1.

5.4 Bloch-Kato exponential map

In this subsection we follow closely the approach of [BB] to construct an isomorphism

$$\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) : \mathbf{d}_{\tilde{\Omega}}(0) \to \tilde{\Omega}\otimes_\Omega \left\{\mathbf{d}_\Omega(R\Gamma(L,V))\cdot\mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p}V)\right\}.$$

The constructed isomorphism will satisfy the condition $(\star)$, such that to prove $C^{na}_{ep}(L/K, V)$ it will be enough to find an isomorphism

$$\epsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbf{d}_{\tilde{\Lambda}}(0) \to \tilde{\Lambda}\otimes_\Lambda \Delta_{ep}(L/K, T)$$

with $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) = \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\epsilon_{\Lambda,\xi}(\mathbb{T})$.

Consider the exact sequence of $\mathbb{Q}_p[G]$-modules (see [BB, 2.5.1]):

$$\begin{gathered}0 \longrightarrow \mathrm{H}^0(L,V) \longrightarrow \mathrm{D}^L_{cris}(V) \longrightarrow \mathrm{D}^L_{cris}(V)\oplus t_V(L) \xrightarrow{exp_V} \mathrm{H}^1(L,V) \longrightarrow \\ \xrightarrow{exp^*_{V^*(1)}} \mathrm{D}^L_{cris}(V^*(1))^*\oplus t^*_{V^*(1)}(L) \longrightarrow \mathrm{D}^L_{cris}(V^*(1))^* \longrightarrow \mathrm{H}^2(L,V) \longrightarrow 0,\end{gathered} \tag{5.9}$$

where $exp_W : t_W(L) \to \mathrm{H}^1_f(L, W)$ is the Bloch-Kato exponential map for W. By the proof of Lemma 5.22 we know that $\mathrm{H}^0(L,V)$, $\mathrm{H}^2(L,V)$, $\mathrm{H}^1_f(L,V^*(1))$, $t_{V^*(1)}(L)$ are zeros, whence the exact sequence above degenerates to

$$0 \longrightarrow \mathrm{D}^L_{cris}(V^*(1))^* \xrightarrow{1-\phi^*} \mathrm{D}^L_{cris}(V^*(1))^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{D}^L_{cris}(V) \longrightarrow \mathrm{D}^L_{cris}(V)\oplus t_V(L) \xrightarrow{exp_V} \mathrm{H}^1(L,V) \longrightarrow 0,$$

where $\phi \in \mathrm{End}(\mathrm{D}^L_{cris}(-))$ is induced by an endomorphism of the ring B_{cris}. Further, using the exact sequence

$$0 \longrightarrow t^*_{V^*(1)}(L) \longrightarrow \mathrm{D}^L_{dR}(V) \longrightarrow t_V(L) \longrightarrow 0$$

($t^*_{V^*(1)}(L) \cong Fil^0\mathrm{D}^L_{dR}(V)$) and the isomorphism (of [BB, Lem. 1.4.1])

$$\frac{\mathrm{D}^L_{cris}(V)}{(1-\phi)\mathrm{D}^L_{cris}(V)} \cong \frac{\mathrm{H}^1_f(L,V)}{\mathrm{H}^1_e(L,V)} = 0$$

we deduce that

$$\mathrm{D}^L_{dR}(V) \xrightarrow{exp} \mathrm{H}^1(L,V) \text{ and } \mathrm{D}^L_{cris}(V) \xrightarrow{1-\phi} \mathrm{D}^L_{cris}(V)$$

are isomorphisms. The application of the determinant functor to (5.9) results in the isomorphism $\mathbf{d}_\Omega(1-\phi)\cdot\mathbf{d}_\Omega(exp^{-1})\cdot\mathbf{d}_\Omega((1-\phi^*)^{-1})$ sending

$$\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V))\cdot\mathbf{d}_\Omega(R\Gamma(L,V))^{-1}\cdot\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V^*(1))^*)$$

to

$$\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V))\cdot\mathbf{d}_\Omega(\mathrm{D}^L_{dR}(V))\cdot\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V^*(1))^*),$$

which after the composition with

$$\mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V))}\cdot\beta_{V,L/K}\cdot\mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V^*(1))^*)}$$

and multiplication with

$$\mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}^L_{cris}(V))^{-1}}\cdot\mathrm{id}_{\mathbf{d}_{\tilde\Omega}(R\Gamma(L,V))}\cdot\mathrm{id}_{\mathbf{d}_{\tilde\Omega}(\mathrm{D}^L_{cris}(V^*(1))^*)^{-1}}$$

gives the desired isomorphism $\tilde\epsilon_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$. Note that $\tilde\epsilon_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ satisfies the condition ($\star$) automatically by construction (cf. the construction of the ϵ-isomorphisms of de Rham representations in subsubsection 2.3.1 and Remark 5.25).

Let $\mathcal{G}$ be a commutative formal Lie group of finite height over $\mathcal{O}_K$ and W be a p-adic de Rham representation coming from the p-adic Tate module of $\mathcal{G}$. In [BK, pp. 359-360] is described a commutative diagram, which connects the Bloch-Kato exponential map with the classical exponential map of $\mathcal{G}$:

$$\begin{array}{ccc} \tan(\mathcal{G}_K)(L) & \xrightarrow{exp} & \mathcal{G}(\mathfrak{p}_L)\otimes\mathbb{Q}_p \\ \downarrow{=} & & \downarrow \\ t_W(L) & \xrightarrow{exp_W} & \mathrm{H}^1(L,W), \end{array}$$

where $t_W(L)$ is identified with the tangent space of $\mathcal{G}_K$, the upper (resp. lower) exp is the exponential map in the classical sense (resp. Bloch-Kato), and the right vertical map is the boundary map of the Kummer sequence (5.1).

By Proposition 5.21(3) and Lemma 5.22 the representation T being the p-adic Tate module of the formal group $\mathcal{F}$ (see Example 5.20) fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(\mathfrak{p}_L) \arrow[r] \arrow[d] & \mathrm{H}^1(L,T) \arrow[r] \arrow[d, "i"] & 0 \\ 0 \arrow[r] & t_V(L) \arrow[r, "exp"] & \mathrm{H}^1(L,V) \arrow[r] & 0, \end{tikzcd} \tag{5.10}$$

where the left vertical arrow is a Λ-homomorphism induced by the classical logarithm $log_{\mathcal{F}}$ of $\mathcal{F}$.

5.5 Reformulation of the ϵ-conjecture

In this subsection we follow closely the approach of [Breu] to reformulate conjecture $C_{ep}^{na}(L/K,V)$ in the language of relative K_0-groups instead of K_1-groups. In the previous subsection we constructed an isomorphism

$$\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) \in \mathrm{Isom}(\mathbf{d}_\Omega(0), \mathbf{d}_\Omega(R\Gamma(L,V)) \cdot \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p}V)) \times^{K_1(\Omega)} K_1(\tilde{\Omega})$$

satisfying condition $(\star)$. Therefore, $C_{ep}^{na}(L/K,V)$ states that there exists an element

$$\epsilon_{\Lambda,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}T) = (x,y) \in \mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T)) \times^{K_1(\Lambda)} K_1(\tilde{\Lambda}),$$

such that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \epsilon_{\Lambda,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}T) = \tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$. By Theorem 3.4 the set $\mathrm{Isom}(\mathbf{d}_\Lambda(0), \Delta_{ep}(L/K,T))$ is not empty, thus in view of the localization exact sequence for K-groups

$$1 \longrightarrow K_1(\tilde{\Lambda}) \longrightarrow K_1(\tilde{\Omega}) \xrightarrow{\partial} K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}}) \longrightarrow 0$$

$C_{ep}^{na}(L/K,V)$ is equivalent to saying that $\partial(y) = 0$ in $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}})$.

The complex $R\Gamma(L,T)$ is a perfect complex of Λ-modules and $\mathrm{Ind}_{L/\mathbb{Q}_p}T$ is a f.g. projective Λ-module, thus $M^\bullet := R\Gamma(L,T) \oplus \mathrm{Ind}_{L/\mathbb{Q}_p}T[0]$ is a perfect complex of Λ-modules with $\mathrm{H}^0(M^\bullet) \cong \mathrm{Ind}_{L/\mathbb{Q}_p}T$, $\mathrm{H}^1(M^\bullet) \cong \mathrm{H}^1(L,T)$ and $\mathrm{H}^i(M^\bullet) = 0$ for $i \geq 2$. There is an isomorphism

$$comp_V \circ exp^{-1} : \mathrm{H}^1(B_{dR} \otimes M^\bullet) \to \mathrm{H}^0(B_{dR} \otimes M^\bullet),$$

and we define $C_{L/K} := \chi(M^\bullet, comp_V \circ exp^{-1}) \in K_0(\tilde{\Lambda}, B_{dR})$ to be the refined Euler characteristic (see [Breu, 2.6]).

Set

$$U_{cris} := \partial([\mathrm{D}_{cris}^L(V), 1-\phi]) + \partial([\mathrm{D}_{cris}^L(V^*(1))^*, (1-\phi^*)^{-1}]) \in K_0(\Lambda, \mathbb{Q}_p).$$

Note that

$$\partial([\mathrm{D}_{cris}^L(V^*(1))^*, (1-\phi^*)^{-1}]) = -\partial([\mathrm{D}_{cris}^L(V^*(1)), 1-\phi]),$$

since $(-)^{-1}$ induces multiplication with -1 on relative K_0-groups (Appendix B), so that

$$U_{cris} = \partial([\mathrm{D}^L_{cris}(V), 1-\phi]) - \partial([\mathrm{D}^L_{cris}(V^*(1)), 1-\phi]).$$

Finally, the multiplication of $\tilde{\alpha}_{V,L/K}$ with t^{-1} and the equivariant ϵ-factor translates in the language of relative K_0-groups into the summation of their images under ∂.

Consider the class

$$C_{L/K} - \partial(t) - \partial(\epsilon_D(L/K, V)).$$

This belongs to $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}})$, because it is invariant under the action of $G_{\mathbb{Q}_p^{ur}}$ and $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}}) = K_0(\tilde{\Lambda}, B_{dR})^{G_{\mathbb{Q}_p^{ur}}}$. The conjecture $C^{na}_{ep}(L/K, V)$ takes the form:

$$C_{L/K} + U_{cris} + [\mathrm{H}^1_f(L, T^*(1))^*, exp^*_{V^*(1)}, 0] - \partial(t) - \partial(\epsilon_D(L/K, V)) = 0 \quad (5.11)$$

in $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}})$.

5.6 Tamely ramified extension

Let L/K be a tamely ramified extension, then $\mathcal{O}_L$ is a f.g. projective Λ-module (see [Fr, Cor. 1]). Set $K^\bullet := R\Gamma(L, T) \oplus \mathcal{O}_L e_{\chi_{\mathbb{Q}_p}^{ur},1}[0]$, a perfect complex of Λ-modules with $\mathrm{H}^0(K^\bullet) \cong \mathcal{O}_L e_{\chi_{\mathbb{Q}_p}^{ur},1}$, $\mathrm{H}^1(K^\bullet) \cong \mathrm{H}^1(L, T)$ and $\mathrm{H}^i(K^\bullet) = 0$ for $i \geq 2$. The composition rule for the refined Euler characteristic gives the equality

$$C_{L/K} = \chi(K^\bullet, exp^{-1}) + [\mathcal{O}_L e_{\chi_{\mathbb{Q}_p}^{ur},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] \quad (5.12)$$

in $K_0(\tilde{\Lambda}, B_{dR})$.

Recall that $\mathcal{F}(\mathfrak{p}_L)$ is a cohomologically trivial Λ-module (see [CG, Prop. 3.9]). Moreover, by [BKS, Lem. 1.1] $\mathcal{F}(\mathfrak{p}_L)[-1]$ is a perfect complex of Λ-modules. We set

$$E_{L/K}(\mathcal{F}(\mathfrak{p}_L)) := \mathcal{F}(\mathfrak{p}_L)[-1] \oplus \mathcal{O}_L[0],$$

a perfect complex of Λ-modules with

$$\mathrm{H}^0(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) \cong \mathcal{O}_L, \quad \mathrm{H}^1(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) \cong \mathcal{F}(\mathfrak{p}_L)$$

and $\mathrm{H}^i(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) = 0$ for $i \geq 2$. Using the identification

$$L e_{\chi_{\mathbb{Q}_p}^{ur},1} = \mathrm{D}^L_{dR}(V) = t_V(L) \cong \hat{\mathbb{G}}_a(L) = L,$$

the Λ-module isomorphism $\mathcal{O}_L e_{\chi_{\mathbb{Q}_p}^{ur},1} \cong \mathcal{O}_L$ and the diagram (5.10) we get the equality

$$\chi((K^\bullet), exp^{-1}) = \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) \text{ in } K_0(\Lambda, \mathbb{Q}_p) \quad (5.13)$$

exp^{-1} and $log_{\mathcal{F}}$ being Ω-modules isomorphisms.

Now let $n_0 \in \mathbb{N}$ be big enough such that $\mathfrak{p}_L^{n_0}$ is a projective Λ-module and

$$log_{\mathcal{F}} : \mathcal{F}(\mathfrak{p}_L^{n_0}) \xrightarrow{\cong} \hat{\mathbb{G}}_a(\mathfrak{p}_L^{n_0}) = \mathfrak{p}_L^{n_0}.$$

Then $\mathcal{F}(\mathfrak{p}_L^{n_0})$ is a projective Λ-submodule of finite index in $\mathcal{F}(\mathfrak{p}_L)$, hence we can define $E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0}))$ analogously to the previous consideration. But

$$[\mathcal{F}(\mathfrak{p}_L^{n_0}), log_{\mathcal{F}}, \mathfrak{p}_L^{n_0}] = 0 \text{ in } K_0(\Lambda, \mathbb{Q}_p),$$

so that

$$\begin{aligned} \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})), log_{\mathcal{F}}) &= [\mathcal{F}(\mathfrak{p}_L^{n_0}), log_{\mathcal{F}}, \mathfrak{p}_L^{n_0}] + [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] \\ &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L]. \end{aligned} \tag{5.14}$$

The exact sequence

$$0 \longrightarrow \mathcal{F}(\mathfrak{p}_L^{n_0}) \xrightarrow{s} \mathcal{F}(\mathfrak{p}_L) \longrightarrow \mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0}) \longrightarrow 0$$

gives rise to a distinguished triangle of perfect complexes of Λ-modules

$$E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})) \xrightarrow{j} E_{L/K}(\mathcal{F}(\mathfrak{p}_L)) \longrightarrow \mathrm{cone}(j),$$

where $j = s_* \oplus id_{\mathcal{O}_L}$, such that $\mathrm{H}^0(\mathrm{cone}(j)) = 0$, $\mathrm{H}^1(\mathrm{cone}(j)) \cong \mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})$ and $\mathrm{H}^i(\mathrm{cone}(j)) = 0$ for $i \geq 2$. This triangle together with (5.14) lead to the equalities

$$\begin{aligned} \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) &= \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})), log_{\mathcal{F}}) + \chi(\mathrm{cone}(j), 0) \\ &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] + \chi(\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})[-1], 0). \end{aligned} \tag{5.15}$$

The quotients $\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})$ and $\mathfrak{p}_L/\mathfrak{p}_L^{n_0}$ are filtered by the images of $\mathcal{F}(\mathfrak{p}_L^i)$ and $\mathfrak{p}_L^i$, respectively, for $i \geq 1$. The associated graded objects considered as complexes are canonically isomorphic perfect complexes of Λ-modules, thus by [BlB, Prop. 2.1(iii)] we have the equality

$$\begin{aligned} \chi(\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})[-1], 0) &= \chi(\mathfrak{p}_L/\mathfrak{p}_L^{n_0}[-1], 0) \\ &= [\mathfrak{p}_L, id, \mathfrak{p}_L^{n_0}] \\ &= [\mathcal{O}_L, id, \mathfrak{p}_L^{n_0}] - [\mathcal{O}_L, id, \mathfrak{p}_L]. \end{aligned} \tag{5.16}$$

Let $q_K = p^f := [\mathcal{O}_K : \mathfrak{p}_K]$ and $e_I := \frac{1}{|I|}\sum_{i \in I} i$ be the idempotent of Ω associated to the inertia subgroup I of G. Let ${}^\sharp x \in K_1(\Omega) \subset K_1(\tilde{\Omega})$ be defined for every element $x \in Cent(\Omega)$ as follows (cf. Appendix C). If $Cent(\Omega) = \prod F_i$ is the Wedderburn decomposition of $Cent(\Omega)$ into a product of fields and $x = (x_i)$ under this decomposition, then ${}^\sharp x = ({}^\sharp x_i)$ with ${}^\sharp x_i = x_i$ if $x_i \neq 0$ and ${}^\sharp x_i = 1$ if $x_i = 0$.

The normal basis theorem for $\mathcal{O}_L/\mathfrak{p}_L$ over $\mathbb{Z}_p/p\mathbb{Z}_p$ implies that there exists a short exact sequence of G/I-modules

$$0 \longrightarrow p \cdot \mathbb{Z}_p[G/I]^f \longrightarrow \mathbb{Z}_p[G/I]^f \longrightarrow \mathcal{O}_L/\mathfrak{p}_L \longrightarrow 0.$$

Using this sequence we compute that

$$[\mathcal{O}_L, id, \mathfrak{p}_L] = -\partial({}^\sharp(q_K e_I)). \tag{5.17}$$

Observing (5.16) and (5.17) the equality (5.15) becomes

$$\begin{aligned} \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] + [\mathcal{O}_L, id, \mathfrak{p}_L^{n_0}] + \partial({}^\sharp(q_K e_I)) \\ &= \partial({}^\sharp(q_K e_I)). \end{aligned} \tag{5.18}$$

Write $\Sigma(L)$ for the set of all embeddings $L \to \overline{\mathbb{Q}_p}$ fixing $\mathbb{Q}_p$. For each $\sigma \in \Sigma(K)$ we fix $\hat{\sigma} \in \Sigma(L)$ such that $\hat{\sigma}|_K = \sigma$. Let $b \in \mathcal{O}_L$ be a $K[G]$-basis of L and let χ be an irreducible $\overline{\mathbb{Q}_p}$-valued character of G. The norm resolvent is defined by

$$\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi) := \prod_{\sigma\in\Sigma(K)} \mathrm{Det}_\chi(\sum_{g\in G} \hat{\sigma}(g(b))g^{-1}) \in \overline{\mathbb{Q}_p}^\times,$$

where Det_χ is the homomorphism $\overline{\mathbb{Q}_p}[G]^\times \to \overline{\mathbb{Q}_p}^\times$ given by

$$\mathrm{Det}_\chi(\sum_{g\in G} a_g g) := \det(\sum_{g\in G} a_g \rho_\chi(g))$$

and $\rho_\chi : G \to GL_{\chi(1)}(\overline{\mathbb{Q}_p})$ is a matrix representation with character χ. Note that the definition of $\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)$ depends on the choice of the $\hat{\sigma}$. We also let $\{a_\sigma : \sigma \in \Sigma(K)\}$ be a fixed $\mathbb{Z}_p$-basis of $\mathcal{O}_K$ and define

$$\delta_K := det((\eta(a_\sigma))_{\eta,\sigma\in\Sigma(K)}) \in \overline{\mathbb{Q}_p}^\times.$$

This is a square root of the discriminant of K and depends on the choice of the a_σ.

Lemma 5.26 *There is an equality*

$$[\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] - \partial(t) = \partial(\theta) \text{ in } K_0(\tilde{\Lambda}, \widehat{L^{ur}}),$$

where $\theta = (\theta_\chi)_{\chi\in Irr(G)} \in K_1(\overline{\mathbb{Q}_p}[G])$ *with* $\theta_\chi = \delta_K^{\chi(1)}$.

Proof. First we remark that the class

$$[\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] - \partial(t)$$

is invariant under the action of $G_{L^{ur}}$, hence we can consider it in $K_0(\tilde{\Lambda}, \widehat{L^{ur}})$.

The unramified representation $V(-1)$ is $\mathbb{C}_p$-admissible (see [FO, Prop. 3.56]), thus we may replace the ring B_{dR} by $\mathbb{C}_p$ in the definition of the comparison isomorphism getting

$$\begin{array}{rcl} comp_{V(-1),L/\mathbb{Q}_p} : \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)) & \overset{\cong}{\to} & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \\ c \otimes x & \mapsto & cx \end{array}$$

a $\mathbb{C}_p$-linear map, which commutes with the action of $G_{\mathbb{Q}_p}$. Taking invariants under $G(\overline{\mathbb{Q}_p}/L^{ur})$ on both sides and using the theorem of Ax-Sen-Tate the isomorphism above becomes

$$comp_{V(-1),L/\mathbb{Q}_p} : \widehat{L^{ur}} \otimes_{\mathbb{Q}_p} Le_{\chi^{ur}_{\mathbb{Q}_p},0} \overset{\cong}{\to} \widehat{L^{ur}} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)$$

and is induced (via tensor product $\mathbb{Q}_p \otimes_{\mathbb{Z}_p}$) by

$$\mathcal{O}_{\widehat{L^{ur}}} \otimes_{\mathbb{Z}_p} \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0} \overset{\cong}{\to} \mathcal{O}_{\widehat{L^{ur}}} \otimes_{\mathbb{Z}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1).$$

From diagram (5.5) we deduce that

$$[\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] - \partial(t) = [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}, comp_{V(-1)}, \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1)] \tag{5.19}$$

in $K_0(\tilde{\Lambda}, \widehat{L^{ur}}) \subseteq K_0(\tilde{\Lambda}, \mathbb{C}_p)$, whence to prove the lemma we have to compute the last class.

Let $V_{triv} \cong \mathbb{Q}_p$ denote the trivial representation of G_K. There is a commutative diagram of $\mathbb{C}_p[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$\begin{tikzcd} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V_{triv}) \arrow[r, "comp_{V_{triv}}", "\cong"'] \arrow[d, "f_1"] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} \arrow[d, "f_2"] \\ \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V(-1)) \arrow[r, "comp_{V(-1)}", "\cong"'] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \end{tikzcd}$$

where

$$\mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} \cong \bigoplus_{\sigma \in \Sigma_K} \Omega, \qquad \mathrm{D}^L_{dR}(V_{triv}) \cong L,$$

$$\mathrm{Ind}_{L/\mathbb{Q}_p} V(-1) \cong \bigoplus_{\sigma \in \Sigma_K} \Omega v, \qquad \mathrm{D}^L_{dR}(V(-1)) \cong Le_{\chi^{ur}_{\mathbb{Q}_p},0};$$

and the maps are given by the formulas

$$comp_{V_{triv}}(q \otimes \textstyle\sum_{g\in G} a_g g(b)) = \Big(\big(q \cdot \sum_{g \in G} \sigma(a_g) g\big)_\sigma \Big)_{\sigma \in \Sigma_K},$$

$$comp_{V(-1)}(q \otimes \textstyle\sum_{g\in G} a_g g(b) e_{\chi^{ur}_{\mathbb{Q}_p},0}) = \bigg(\Big(\big(q \cdot t^{ur} \cdot \sum_{g\in G} \sigma(a_g) g\big) v \Big)_\sigma \bigg)_{\sigma \in \Sigma_K},$$

$$f_1\big(q \otimes \textstyle\sum_{g\in G} a_g g(b)\big) = q \otimes \sum_{g \in G} a_g g(b) e_{\chi^{ur}_{\mathbb{Q}_p},0},$$

$$f_2\big(q \otimes (\textstyle\sum_{g\in G} z_g g)_\sigma\big) = q \cdot t^{ur} \otimes \Big(\big(\sum_{g \in G} z_g g\big) v \Big)_\sigma, \quad q, q_g \in \mathbb{C}_p,\ z_g \in \mathbb{Q}_p.$$

It follows, that

$$\begin{aligned}&[\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] = [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}, comp_{V(-1)}, \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1)]\\ &+[\mathcal{O}_L, f_1, \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}] + [\mathrm{Ind}_{L/\mathbb{Q}_p} T, f_2^{-1}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}]\end{aligned} \tag{5.20}$$

in $K_0(\Lambda, \mathbb{C}_p)$. But the images of the last two classes in $K_0(\tilde{\Lambda}, \mathbb{C}_p)$ are zeros, as f_1 and f_2 are $\tilde{\Lambda}$-modules isomorphisms. Now we are reduced to computing $[\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}]$. For this we set

$$H_L := \underset{\eta\in\Sigma(L)}{\oplus} \mathbb{Z}_p,$$

a free Λ-module and consider the following commutative diagram of $\mathbb{C}_p[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$\begin{tikzcd} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V_{triv}) \arrow[r, "comp_{V_{triv}}", "\cong"'] \arrow[dr, "\rho_L", "\cong"'] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} \arrow[d, "\varphi_1"] \\ & \mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L, \end{tikzcd} \tag{5.21}$$

where the maps φ_1 and ρ_L are given by the formulas

$$\begin{aligned}&\rho_L(q\otimes l) = \big((q\cdot\eta(l))_\eta\big)_{\eta\in\Sigma(L)}, \quad q\in\mathbb{C}_p,\ l\in L;\\ &\varphi_1\big((\sum_{g\in G} q_g g)_\sigma\big) = \Big(\Big(\sum_{g\in G} q_g\cdot\eta(g(b))\Big)_\eta\Big)_{\eta\in\Sigma(\sigma)} \quad \text{for each } \sigma\in\Sigma(K),\ q_g\in\mathbb{C}_p\end{aligned}$$

and $\Sigma(\sigma) := \{\eta\in\Sigma(L) : \eta|_K = \sigma\}$. From the diagram (5.21) we deduce the equality

$$[\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] + [\mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}, \varphi_1, H_L] = [\mathcal{O}_L, \rho_L, H_L]$$

in $K_0(\Lambda, \mathbb{C}_p)$. Further, [Breu, Lem. 4.16] says that the last class is equal to $\partial(\theta')$, where $\theta' = (\theta'_\chi)_{\chi\in Irr(G)} \in K_1(\overline{\mathbb{Q}_p}[G])$ with $\theta'_\chi = \delta_K^{\chi(1)}\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)$, so that we have

$$[\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] = \partial(\theta') - [\mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}, \varphi_1, H_L]. \tag{5.22}$$

To compute the last class above we adopt the proof of [BlB, Prop. 3.4] to our case. We define a Λ-module isomorphism

$$\begin{aligned}\varphi_2 : H_L = \underset{\eta\in\Sigma(L)}{\oplus} \mathbb{Z}_p \quad &\to \quad \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv} \cong \underset{\sigma\in\Sigma(K)}{\oplus} \Lambda,\\ (z_\eta)_{\eta\in\Sigma(L)} \quad &\longmapsto \quad \Big(\Big(\sum_{\eta\in\Sigma(\sigma)} z_\eta(\eta^{-1}\circ\hat{\sigma})\Big)_\sigma\Big)_{\sigma\in\Sigma(K)},\ z_\eta\in\mathbb{Z}_p.\end{aligned}$$

Then there are equalities in $K_0(\Lambda, \mathbb{C}_p)$

$$\begin{aligned}[\mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}, \varphi_1, H_L] &= [H_L, \varphi_2, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] + [\mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}, \varphi_1, H_L] \\ &= [H_L, \varphi_1 \circ \varphi_2, H_L] \\ &= \partial([\mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L, \varphi_1 \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varphi_2)]) \\ &=: \partial(\theta''),\end{aligned} \tag{5.23}$$

i.e. θ'' is the reduced norm of $\varphi_1 \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varphi_2)$. Let W be an irreducible $\mathbb{C}_p[G]$-module with character χ, then θ''_χ is the determinant of the $\mathbb{C}_p$-linear automorphism ψ of $\mathrm{Hom}_{\mathbb{C}_p[G]}(W, \mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L)$ given by $f \mapsto \varphi_1 \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varphi_2) \circ f$.

We now choose a $\mathbb{C}_p$-basis $\{w_i \mid 1 \leq i \leq n\}$ of W and let $\{w_i^* \mid 1 \leq i \leq n\}$ denote the dual basis of $W^* := \mathrm{Hom}_{\mathbb{C}_p}(W, \mathbb{C}_p)$ with respect to the canonical evaluation pairing

$$\langle \cdot, \cdot \rangle : W \times W^* \to \mathbb{C}_p.$$

We observe that if $\rho_\chi : G \to GL_{\chi(1)}(\overline{\mathbb{Q}_p})$ is a matrix representation with respect to the basis $\{w_i \mid 1 \leq i \leq n\}$, then $\langle g w_i, w_j^* \rangle$ is equal to ρ_{ji} – the (j,i)-component of the matrix $\rho_\chi(g)$.

For each $\sigma \in \Sigma(K)$ and $j \in \{1, \ldots, n\}$ we define an element

$$\{\sigma, w_j^*\} \in \mathrm{Hom}_{\mathbb{C}_p[G]}(W, \mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L)$$

by setting, for each $w \in W$,

$$\{\sigma, w_j^*\}(w) := \Big(\sum_{g \in G} \langle g^{-1} w, w_j^* \rangle \, g \Big)_{\hat{\sigma}}.$$

Note that the image of $\{\sigma, w_j^*\}$ lies in the $\Sigma(\sigma)$-component of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L$, as the action of $g \in G$ takes the $\hat{\sigma}$-component to the $\hat{\sigma} g^{-1}$-component of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L$ and

$$\Sigma(\sigma) = \{\hat{\sigma} g\}_{g \in G}, \quad \forall \sigma \in \Sigma(K).$$

The set

$$\{\{\sigma, w_j^*\} \mid \sigma \in \Sigma(K), 1 \leq j \leq n\}$$

then constitutes a $\mathbb{C}_p$-basis of $\mathrm{Hom}_{\mathbb{C}_p[G]}(W, \mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L)$. As next we compute the matrix of ψ with respect to this basis. Without lost of generality we numerate the elements of the group G as follows $G = \{1_G = g_1, g_2, \ldots, g_m\}$. Then

$$\{\sigma, w_j^*\}(w_i) = \Big((\rho_{ji}(g_k^{-1}))_{\hat{\sigma} g_k^{-1}} \Big)_{k=1}^{m}.$$

The map $\varphi_1 \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varphi_2)$ is given by

$$(q_\eta)_{\eta \in \Sigma(L)} \longmapsto \bigg(\Big(\sum_{\eta \in \Sigma(\sigma)} q_\eta \cdot \eta' \big((\eta^{-1} \circ \hat{\sigma})(b) \big) \Big)_{\eta' \in \Sigma(\sigma)} \bigg)_{\sigma \in \Sigma(K)},$$

such that for a fixed $\eta_0 \in \Sigma(L)$, $\eta_0|_K =: \sigma_0 \in \Sigma(K)$

$$q_{\eta_0} \longmapsto \Big(\sum_{\eta \in \Sigma(\sigma_0)} q_\eta \cdot (\eta_0 \circ \eta^{-1} \circ \widehat{\sigma_0})(b) \Big)_{\eta_0}.$$

In particular, $\varphi_1 \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varphi_2)$ is an automorphism of the $\Sigma(\sigma)$-component of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L$ for each $\sigma \in \Sigma(K)$, whence

$$\psi(\{\sigma, w_j^*\}) = \sum_{k=1}^{n} \lambda_k^{(j)} \{\sigma, w_k^*\}, \ \lambda_k^{(j)} \in \mathbb{C}_p$$

or, equivalently,

$$\psi(\{\sigma, w_j^*\})(w_i) = \Big(\sum_{k=1}^{n} \lambda_k^{(j)} \{\sigma, w_k^*\} \Big)(w_i), \ \forall i \in \{1, \ldots, n\}.$$

An easy computation shows that

$$\Big(\psi(\{\sigma, w_j^*\})(w_i) \Big)_{\hat{\sigma}} = \sum_{g \in G} (\hat{\sigma} g)(b) \rho_{ji}(g^{-1})$$

and

$$\Bigg(\Big(\sum_{k=1}^{n} \lambda_k^{(j)} \{\sigma, w_k^*\} \Big)(w_i) \Bigg)_{\hat{\sigma}} = \lambda_i^{(j)}.$$

We deduce that ψ has the block matrix $A = \mathrm{diag}(\{A_\sigma\}_{\sigma \in \Sigma(K)})$, where

$$A_\sigma = \Big(\sum_{g \in G} (\hat{\sigma} g)(b) \rho_{ji}(g^{-1}) \Big)_{i,j=1}^{n} \in GL_n(\mathbb{C}_p).$$

Taking determinant of this matrix gives

$$\theta''_\chi = \prod_{\sigma \in \Sigma(K)} \det \Big(\Big(\sum_{g \in G} (\hat{\sigma} g)(b) \rho_{ji}(g^{-1}) \Big)_{i,j} \Big) = \mathcal{N}_{K/\mathbb{Q}_p}(b|\chi). \tag{5.24}$$

The equalities (5.19), (5.20), (5.22), (5.23) and (5.24) together prove the lemma. $\square$

Lemma 5.27 *Let L/K be (at most) tamely ramified. Then there exists $v \in \Lambda^\times$, such that $\mathrm{Det}_\chi(v) = \chi_{\mathbb{Q}_p}^{ur}\big(N_{K/\mathbb{Q}_p}(f(\chi)) \cdot d_{K/\mathbb{Q}_p}^{\chi(1)}\big)$ for all $\chi \in Irr(G)$, thus*

$$\partial(\iota(\epsilon_L(L/K, V(-1)))) = \partial(\iota(\epsilon_L(L/K, V_{triv})))$$

in $K_0(\tilde{\Lambda}, \mathbb{C}_p)$.

Proof. The character $\chi_{\mathbb{Q}_p}^{ur}:\mathbb{Q}_p^\times\to\mathbb{Z}_p^\times$ being a homomorphism we have

$$\chi_{\mathbb{Q}_p}^{ur}\big(N_{K/\mathbb{Q}_p}(f(\chi))\cdot d_{K/\mathbb{Q}_p}^{\chi(1)}\big)=\chi_{\mathbb{Q}_p}^{ur}(N_{K/\mathbb{Q}_p}(f(\chi)))\cdot\chi_{\mathbb{Q}_p}^{ur}(d_{K/\mathbb{Q}_p}^{\chi(1)}).$$

Let $\chi_{\mathbb{Q}_p}^{ur}(p)=:u\in\mathbb{Z}_p^\times$ and let $d_{K/\mathbb{Q}_p}=p^m$. Then for $u^m\in\mathbb{Z}_p^\times\subset\Lambda^\times$, $\chi\in Irr(G)$

$$\chi_{\mathbb{Q}_p}^{ur}(d_{K/\mathbb{Q}_p}^{\chi(1)})=\chi_{\mathbb{Q}_p}^{ur}(p)^{m\cdot\chi(1)}=u^{m\cdot\chi(1)}=\det(\rho_\chi(u^m1_G))=\mathrm{Det}_\chi(u^m).$$

Recall $q_K=p^f$ and $e_I=\frac{1}{|I|}\sum_{i\in I}i\in\Lambda$, since $(|I|,p)=1$. Let $v':=\big(\frac{u}{u\cdot e_I+(1_G-e_I)}\big)^f$. Then for $\chi\in Irr(G)$

$$\begin{aligned}\mathrm{Det}_\chi(v') &= \mathrm{Det}_\chi(u)^f\cdot\mathrm{Det}_\chi(u\cdot e_I+(1_G-e_I))^{-f}\\ &= u^{f\cdot\chi(1)}\cdot\det(u\cdot\rho_\chi(e_I)+\rho_\chi(1_G)-\rho_\chi(e_I))^{-f}.\end{aligned}$$

There exists a basis of $V_\rho=V_{\rho_\chi}$, such that

$$\rho_\chi(e_I)=\begin{pmatrix}1&*&\dots&*&&&\\0&1&*&\dots&*&&\\&&\ddots&&&&\\0&\dots&0&1&*&\dots&*\\0&0&0&\dots&0&*&\\&&&&&\ddots&\\0&0&0&0&0&\dots&0\end{pmatrix}$$

and

$$\rho_\chi(1_G)-\rho_\chi(e_I)=\begin{pmatrix}0&*&\dots&*&&&\\0&0&*&\dots&*&&\\&&\ddots&&&&\\0&\dots&0&0&*&\dots&*\\0&0&\dots&0&1&*&\\&&&&&\ddots&\\0&0&0&0&\dots&0&1\end{pmatrix},$$

as $e_I+1_G-e_I=1_G$. It follows, that $\det(u\cdot\rho_\chi(e_I)+\rho_\chi(1_G)-\rho_\chi(e_I))=u^{\mathrm{rank}(\rho_\chi(e_I))}$, hence

$$\mathrm{Det}_\chi(v')=u^{f\cdot(\chi(1)-\mathrm{rank}(\rho_\chi(e_I)))}.$$

But

$$\chi(1)=\dim V_\rho,\quad \mathrm{rank}(\rho_\chi(e_I))=\dim\mathrm{Im}(\rho_\chi(e_I))=\dim V_\rho^I,$$

so that $\chi(1)-\mathrm{rank}(\rho_\chi(e_I))=\mathrm{codim}\ V_\rho^I$. Further, since χ is a tamely ramified character, the Artin conductor $f(\chi)$ is equal to $\mathfrak{p}_K^{\mathrm{codim}\ V_\rho^I}$ (see [Ma, p. 22]), whence

$$\mathrm{Det}_\chi(v') = \chi^{ur}_{\mathbb{Q}_p}(p^{f\cdot\mathrm{codim}\,V_\rho^I}) = \chi^{ur}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\mathfrak{p}_K^{\mathrm{codim}\,V_\rho^I})) = \chi^{ur}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(f(\chi))).$$

Now we set $v := v' \cdot u$. It remains to prove $v \in \Lambda^\times$ and for this it is enough to show that $u \cdot e_I + (1 - e_I) \in \Lambda^\times$. But

$$(u \cdot e_I + (1_G - e_I)) \cdot (e_I + u \cdot (1_G - e_I)) = u1_G \in \Lambda^\times.$$

This finishes the proof of the first statement.

The second statement follows immediately from the first one by Remark 5.24. □

Theorem 5.28 *Let L/K be a Galois extension of p-adic fields which is (at most) tamely ramified and let $V = \mathbb{Q}_p(\chi^{ur})(1)$. Then $C^{na}_{ep}(L/K, V)$ is equivalent to the vanishing of*

$$\partial({}^\sharp(q_K e_I)) + \partial(\theta) + U_{cris} + [\mathrm{H}^1_f(L, T^*(1))^*, exp^*_{V^*(1)}, 0] - \partial(\alpha) - \partial(\iota(\epsilon_L(L/K, V_{triv}))) \tag{5.25}$$

in $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}})$.

Proof. The proof is given by (5.11), (5.6), (5.7), (5.8), (5.12), (5.13), (5.18), Lemma 5.26 and Lemma 5.27. □

Let $L = K$ and let K^0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in K, $[K^0 : \mathbb{Q}_p] = f$. Denote by Fr_K the arithmetic Frobenius of K, then $\tau_p^f = Fr_K$. We have

$$\mathrm{D}^K_{cris}(V) = K^0 e_{\chi^{ur}_{\mathbb{Q}_p},1} \text{ with } \phi(e_{\chi^{ur}_{\mathbb{Q}_p},1}) = p^{-1}\chi^{ur}_{\mathbb{Q}_p}(\tau_p^{-1}) e_{\chi^{ur}_{\mathbb{Q}_p},1}.$$

The map ϕ is $\mathbb{Q}_p$-linear but not K^0-linear, thus

$$\begin{aligned} \partial([\mathrm{D}^L_{cris}(V), 1-\phi]) &= \partial(1 - p^{-f}\chi^{ur}(Fr_K^{-1})) \\ &= \partial\Big(\tfrac{1}{p^f}(p^f - \chi^{ur}(Fr_K^{-1}))\Big) \\ &= -\partial(q_K), \end{aligned}$$

since $p^f - \chi^{ur}(Fr_K^{-1}) \in \mathbb{Z}_p^\times$. Analogously,

$$\partial([\mathrm{D}^L_{cris}(V^*(1)), 1-\phi]) = \partial(1 - \chi^{ur}(Fr_K)).$$

By Corollary 5.23

$$[\mathrm{H}^1_f(K, T^*(1))^*, exp^*_{V^*(1)}, 0] = [0, id, \mathrm{H}^0(K, V^*(1)/T^*(1))]. \tag{5.26}$$

Taking the Pontryagin dual of the exact sequence

$$0 \longrightarrow \mathrm{H}^0(K, V^*(1)/T^*(1)) \xrightarrow{1-Fr_K} \mathbb{Q}_p/\mathbb{Z}_p\big((\chi^{ur})^{-1}\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p\big((\chi^{ur})^{-1}\big) \longrightarrow 0$$

we obtain an exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow \mathbb{Z}_p(\chi^{ur}) \xrightarrow{1-Fr_K} \mathbb{Z}_p(\chi^{ur}) \longrightarrow \mathrm{H}^0(K, V^*(1)/T^*(1))^\vee \longrightarrow 0,$$

which shows that the last class in (5.26) is equal to $\partial(1 - \chi^{ur}(Fr_K))$. Now (5.25) takes the form

$$\partial(\delta_K) - \partial(d_{K/\mathbb{Q}_p}) - \partial(\iota(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\iota^{-1} \circ 1_K)))), \tag{5.27}$$

where 1_K is the unique (trivial) character of $G = \{e\}$.

Let $\tau' := \tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p} 1_K)$, then

$$\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\iota^{-1} \circ 1_K)) = \tau_K(\iota^{-1} \circ 1_K) \cdot \tau'.$$

Moreover, by [Breu, Lem. 4.29] $\delta_K/\iota(\tau') \in (\mathbb{Z}_p^{ur})^\times$, hence (5.27) becomes

$$-\partial(d_{K/\mathbb{Q}_p}) - \partial(\iota(\tau_K(\iota^{-1} \circ 1_K))). \tag{5.28}$$

Finally, setting $\psi_K := \psi_{\mathbb{Q}_p} \circ \mathrm{Tr}_{K/\mathbb{Q}_p}$ – the canonical additive character of K – and normalizing the Haar measure dx_K by

$$\int_{\mathcal{O}_K} dx_K = 1$$

we compute $\iota(\tau_K(\iota^{-1} \circ 1_K)) = d_{K/\mathbb{Q}_p}^{-1}$, such that the sum of the classes in (5.28) is equal to 0. By Theorem 5.28 this proves the conjecture $C_{ep}^{na}(K/K, V)$.

Remark 5.29 *Using Proposition 4.1 we can establish $C_{ep}^{na}(L/K, V)$ also for unramified extensions L/K of degree prime to p. Moreover, from Proposition 3.14(1) it follows that also $C_{ep}^{na}(L/K, V^*(1))$ is true for unramified extensions L/K with $(|G|, p) = 1$.*

Summarizing all above we get the main theorem of this section

Theorem 5.30 (Main Theorem) *Let K be a finite extension of $\mathbb{Q}_p$ and L/K be a finite unramified extension with $G = G(L/K)$ of order prime to p. Let $\chi^{ur} : G_K \to \mathbb{Z}_p^\times$ be a continuous unramified character with $\chi^{ur}(G_L) \neq 1$. Let V be either $\mathbb{Q}_p(\chi^{ur})$ or $\mathbb{Q}_p(\chi^{ur})(1)$ – a p-adic representation of G_K. Then the conjecture $C_{ep}^{na}(L/K, V)$ holds.*

6 Galois invariants of K_1-groups of Iwasawa algebras

This section is a joint work with my supervisor professor O. Venjakob. Explicitly, Theorem 6.29, Lemma 6.42, Theorem 6.43, Corollaries 6.44 and 6.45, Lemma 6.46 were proved by him.

6.1 The formulation of the problem

In [FK, Rem. 3.4.6] Fukaya and Kato formulate the following expectation, which plays an important role in the definition of ϵ-isomorphisms: For an adic ring Λ the sequence

$$1 \longrightarrow K_1(\Lambda) \longrightarrow K_1(\tilde{\Lambda}) \xrightarrow{1-\varphi} K_1(\tilde{\Lambda}) \longrightarrow 1$$

should be exact, where $1-\varphi$ denotes the map $x \mapsto x\varphi(x)^{-1}$ and φ denotes the Frobenius morphism acting on the ring of integers $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$ of $\widehat{\mathbb{Q}_p^{ur}}$. In particular, this implies, that for any $a \in K_1(\Lambda)$, the set $\left\{x \in K_1(\tilde{\Lambda}) \mid \varphi(x) = a \cdot x\right\}$ becomes a $K_1(\Lambda)$-torsor. Hence, if this were true for any adic ring Λ, Fukaya and Kato could (and still can) prove the uniqueness in $LTNC$.

For any finite group G and $\Lambda = \mathbb{Z}_p[G]$ this amounts to the statement:

$$i_* : K_1(\mathbb{Z}_p[G]) \cong K_1(\widehat{\mathbb{Z}_p^{ur}}[G])^{\varphi=id},$$

where i_* is induced by the inclusion $i : \mathbb{Z}_p \to \widehat{\mathbb{Z}_p^{ur}}$, and the Frobenius map φ acts coefficientwise on $\widehat{\mathbb{Z}_p^{ur}}[G]$ and hence on the K_1-group.

The original motivation of this section was to show Fukaya and Kato's expectation in this specific case. A more general question would rather be whether the following statement:

$$i_* : K_1(S^{\Delta}[G]) \cong K_1(S[G])^{\Delta} \tag{6.1}$$

holds whenever S is a ring and Δ is a group acting on S by ring automorphisms. But surprisingly, it turns out, that neither of the above statements does hold in general (see subsubsection 6.3.2). In this section we restrict our attention to the case, where Δ is the Galois group of some algebraic field extension related to the extension S over S^{Δ} of a specific class of rings S contained in the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}_p}$. We obtain partial results toward (a corrected version of) (6.1), see Theorem 6.52. In particular, we show, that in general the following sequence is exact

$$1 \longrightarrow SK_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{Z}_p[G]) \xrightarrow{i_*} K_1(\widehat{\mathbb{Z}_p^{ur}}[G])^{\varphi=id} \longrightarrow 1 \tag{6.2}$$

and induces an isomorphism of the rational K-groups

$$K_1(\mathbb{Z}_p[G])_{\mathbb{Q}} \cong K_1(\widehat{\mathbb{Z}_p^{ur}}[G])_{\mathbb{Q}}^{\varphi=id}.$$

If S is a finite algebraic extension of $\mathbb{Z}_p$ and $SK_1(S[G]) = 1$, the behavior of (6.1) is equivalent to the Galois descent property of the determinantal image:

$$i_* : \mathrm{Det}(S^\Delta[G]) \cong \mathrm{Det}(S[G])^\Delta.$$

The later has been proved by M. Taylor in the case, where S is unramified. But the case of infinite extensions of $\mathbb{Z}_p$ and infinite groups Δ seems not to be covered in the literature, not even by the fairly general recent treatment [CPT], where only finite group actions are considered, as was pointed out to us by M. Taylor. Actually one has to check that the techniques of integral group logarithms extend to this situation, either by extending Taylor's original definition as pursued in (loc. cit.) or by using Snaith's version in [Sn] - both in the case of p-groups and then use standard induction techniques to reduce the general case of finite groups to it, as e.g. in [Fr]. Both approaches work, and for the convenience of the reader we show, that the methods of [CPT] extend easily to our setting, recalling the main steps of their proof, but noting that for ramified extensions Snaith's construction might be better adapted.

The reason for the defect in (6.1) relies on the surprising vanishing

$$SK_1(\widehat{\mathbb{Z}_p^{ur}}[G]) = 1$$

for *all finite groups* G. In particular, SK_1 - in contrast to the Det-part - does not have good Galois descent in general, see Corollary 6.51 for a more precise statement.

This section is organized as follows: In the first subsection we recall for the convenience of the reader Galois descent results for group rings with coefficients in local or global fields using Fröhlich's Hom-description. In the second subsection, the heart of the section, we first concentrate on descent results for the Det-part. In particular, we obtain a rather general result not only for finite groups, but also for compact p-adic Lie groups and their Iwasawa algebras, which turns out to be quite useful in number theory, see [BV]. Then we deal with the SK_1-part recalling and generalizing results from [O 1]. Altogether both parts lead to the desired descent description (6.2) for K_1. Finally, in the third subsection we derive similar descent results over the corresponding residue class fields.

6.2 The case of "local" and "number" fields

The goal of this subsection is to prove the following theorem which is certainly known to experts but for lack of a reference we treat it here, because it forms the prototype for the descent results in the integral cases later.

We fix an embedding $\overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}_p$. Let L be a finite Galois extension of $\mathbb{Q}_p$ and M be either an arbitrary (possibly infinite) Galois extension M^0 of $\mathbb{Q}_p$ or the p-adic completion of a Galois extension M^0 of $\mathbb{Q}_p$, such that $\mathbb{Q}_p \subseteq L \subseteq M \subseteq \mathbb{C}_p$. Furthermore, we set $\Delta := Gal(M^0/L)$.

Remark 6.1 *In the following we shall several times use in the case of completions the theorem of Ax-Sen-Tate, which says that*

$$M^{\Delta} = (M^0)^{\Delta} = L \quad \text{and} \quad \mathcal{O}_M^{\Delta} = (\mathcal{O}_{M^0})^{\Delta} = \mathcal{O}_L.$$

Theorem 6.2 *In the situation as above let us assume that M^0 is of finite absolute ramification index over $\mathbb{Q}_p$ and let Γ be a finite group. Then*

$$i_* : K_1(L[\Gamma]) \cong K_1(M[\Gamma])^{\Delta}, \tag{6.3}$$

where Δ acts on the K_1-group coefficientwise.

For the proof of Theorem 6.2 we need the following

Proposition 6.3 *Let N be either an arbitrary (possibly infinite) algebraic extension of $\mathbb{Q}_p$ or the completion of an algebraic extension of finite absolute ramification index over $\mathbb{Q}_p$. Let Γ be a finite group. Then*
(i) The map $i_* : K_1(N[\Gamma]) \to K_1(\overline{N}[\Gamma])$ is injective,
(ii) If N is a finite Galois extension of $\mathbb{Q}_p$ and $G_N = Gal(\overline{N}/N)$ is the absolute Galois group, then

$$i_* : K_1(N[\Gamma]) \cong K_1(\overline{N}[\Gamma])^{G_N}.$$

Proof. The first statement is well known for the local fields, i.e. finite extensions of $\mathbb{Q}_p$, and more generally for the perfect discrete valued fields (see [Q, Prop. 2.8]; [MN, Thm. 1 and the Rem. after Thm. 2]). The infinite algebraic extensions can always be written as a direct limits of their finite subextensions. Since the direct limit is exact on the category of abelian groups and K_1 commutes with the direct limit (see [Ro, Exer. 2.1.9]), (i) is true for infinite algebraic extensions. This completes the proof of (i).

Let $R_\Gamma = R_\Gamma(\overline{N})$ denote the Grothendieck group of finitely generated $\overline{N}[\Gamma]$-modules. Alternatively, R_Γ will be viewed as the group of virtual $\overline{N}$-valued characters of Γ. Since $\overline{N}$ is algebraically closed, R_Γ is a free abelian group on the irreducible characters.

Using the Wedderburn decomposition of $\overline{N}[\Gamma]$ we get an isomorphism of G_N-modules

$$K_1(\overline{N}[\Gamma]) \cong \prod_{\chi} \overline{N}^{\times} \cong \mathrm{Hom}(R_\Gamma, \overline{N}^{\times}), \tag{6.4}$$

where χ are irreducible $\overline{N}$-valued characters and the action of G_N on the Hom-group is given by the actions on R_Γ and on $\overline{N}^{\times}$ in the standard way

$$f^g(\chi) = (f(\chi^{g^{-1}}))^g, \forall f \in \mathrm{Hom}(R_\Gamma, \overline{N}^{\times}),\ g \in G_N,\ \chi \in R_\Gamma.$$

From [T, part 1, §2] we get the corresponding Hom-description for $K_1(N[\Gamma])$

$$K_1(N[\Gamma]) \cong \mathrm{Hom}_{G_N}(R_\Gamma, \overline{N}^{\times}). \tag{6.5}$$

The second statement is now obvious, as i_* is a Galois homomorphism and commutes with the Hom-description. $\square$

Now let L and M be as in Theorem 6.2. We introduce a commutative diagram

$$\begin{tikzcd} K_1(L[\Gamma]) \arrow[r] \arrow[d, "i_*"] & K_1(\overline{L}[\Gamma]) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}(R_\Gamma(\overline{L}), \overline{L}^\times) \arrow[d] \\ K_1(M[\Gamma]) \arrow[r] & K_1(\overline{M}[\Gamma]) \arrow[r, "\cong"] & \mathrm{Hom}(R_\Gamma(\overline{M}), \overline{M}^\times), \end{tikzcd} \tag{6.6}$$

The rows are injective by the Proposition 6.3. The right hand side column is injective as $\overline{L}^\times \subseteq \overline{M}^\times$ and each $\psi \in R_\Gamma(\overline{M})$ being a character of a finite group is the composition of one of the characters $\chi \in R_\Gamma(\overline{L})$ with the inclusion map $i: \overline{L} \to \overline{M}$, so that

$$R_\Gamma(\overline{L}) \cong R_\Gamma(\overline{M}),$$

which we take as an identification henceforth. It follows, that the left hand side column is also injective.

Taking invariants under the action of Δ, which is a left exact functor, we obtain the inclusion of Theorem 6.2

$$i_* : K_1(L[\Gamma]) \subseteq (K_1(M[\Gamma]))^\Delta. \tag{6.7}$$

To prove the surjectivity of i_* we take invariants under the action of Δ of the following commutative diagram

$$\begin{tikzcd} K_1(L[\Gamma]) \arrow[r, "\cong"] \arrow[d, hook, "i_*"] & \mathrm{Hom}_{G_L}(R_\Gamma, \overline{L}^\times) \arrow[d, hook] \\ K_1(M[\Gamma]) \arrow[r, "\cong"] & \mathrm{Hom}_{G_{M^0}}(R_\Gamma, \overline{M}^\times), \end{tikzcd}$$

so that the right hand side injection becomes an isomorphism (cf. Remark 6.1), hence also the the left hand side map. This finishes the proof of Theorem 6.2.

Remark 6.4 *Unfortunately we cannot prove Theorem 6.2 in full generality, i.e. for M being the completion of an arbitrary Galois extension of $\mathbb{Q}_p$. For instance, it is not known to us, whether the theorem holds for M being the completion of the infinite purely ramified extension $\mathbb{Q}_p(\mu_{p^\infty})$ of $\mathbb{Q}_p$.*

Remark 6.5 *If M is a p-adically complete field, then from the* Hom*-description (resp. the Wedderburn decomposition) of $K_1(\overline{M}[\Gamma])$ we may obtain a* Hom*-description (resp. the Wedderburn decomposition) of $K_1(M[\Gamma])$ by taking the invariants under the action of G_{M^0}.*

Remark 6.6 *The proof above also works for "number" fields, i.e. algebraic (possible infinite) extensions of $\mathbb{Q}$. We just have to replace $\mathbb{Q}_p$ by $\mathbb{Q}$ in Theorem 6.2. Then, letting L be a finite Galois extension of $\mathbb{Q}$ and M be an arbitrary (possible infinite) Galois extension of $\mathbb{Q}$, we follow the proof of Theorem 6.2 using the same arguments and references to get*

$$i_* : K_1(L[\Gamma]) \cong K_1(M[\Gamma])^\Delta.$$

The only difference is, that the elements of Hom*-groups in the proof are to be totally positive on all symplectic representations.*

6.3 The case of rings of integers of "local" fields

Let G be a finite group. If S is an integral domain of characteristic zero with field of fractions L, then $\overline{L}$ will denote a chosen algebraic closure of L. We have a map induced by base extension $\overline{L} \otimes_S -$

$$\mathrm{Det}: \; K_1(S[G]) \to K_1(\overline{L}[G]) = \prod_{\chi} \overline{L}^{\times} \cong \mathrm{Hom}(R_G, \overline{L}^{\times}),$$

where the direct product extends over the irreducible $\overline{L}$-valued characters of G. We write $SK_1(S[G])$ for ker(Det). Since the Det-map factorizes over $K_1(L[G])$ and the map from $K_1(L[G])$ to $K_1(\overline{L}[G])$ induced by $\overline{L} \otimes_L -$ is injective (see Proposition 6.3 (i)), we have an exact sequence

$$1 \longrightarrow SK_1(S[G]) \longrightarrow K_1(S[G]) \longrightarrow \mathrm{Det}(K_1(S[G])) \longrightarrow 1. \tag{6.8}$$

Therefore we shall consider the two parts of K_1, namely the Det-part and the SK_1-part, separately.

Remark 6.7 *From the exact sequence* (6.8) *and the fact, that K_1 commutes with direct limits (see [Ro, Exer. 2.1.9]), we deduce, that Det and SK_1 also commute with direct limits.*

6.3.1 The Det-part

We keep the notation of the introduction at the beginning of this section. In this subsubsection let $S = \mathcal{O}_L$, where L is either an arbitrary (possible infinite) Galois extension L^0 of finite absolute ramification index over $\mathbb{Q}_p$ or the p-adic completion of such L^0. Then S is a Noetherian local ring, i.e. S has the unique maximal ideal, and $S[G]$ is semilocal, i.e. the quotient $S[G]/\mathrm{rad}(S[G])$ of the ring by its Jacobson radical is left Artinian (see [Lam, Prop. 20.6]). We have the following

Proposition 6.8 *Let Λ be a semilocal ring (for example $S[G]$). The maps*

$$\Lambda^{\times} = GL_1(\Lambda) \hookrightarrow GL(\Lambda) \twoheadrightarrow K_1(\Lambda)$$

induce an equality

$$\mathrm{Det}(\Lambda^{\times}) = \mathrm{Det}(GL(\Lambda)) = \mathrm{Det}(K_1(\Lambda)).$$

Proof. See [CR 2, Thm. 40.31]. $\square$

Conjecture 6.9 *Let $S = \mathcal{O}_L$ and G be as above. Let Δ be an open subgroup of $Gal(L^0/\mathbb{Q}_p)$ acting coefficientwise on $S[G]$ and hence on Det-groups. Then*

$$i_* : \mathrm{Det}(S^{\Delta}[G]^{\times}) \cong \mathrm{Det}(S[G]^{\times})^{\Delta}.$$

The proof of Conjecture 6.9 proceeds in two steps. At present we can prove step 2 and thus Conjecture 6.9 only under further assumptions on S (see Theorem 6.28). We first do the proof for finite extensions and completions of infinite extensions, since S is p-adically complete in these cases, and then we generalize the statement to infinite algebraic extensions using direct limits (see Remark 6.27).

Remark 6.10 *The map i_* in the conjecture is always a monomorphism, as the following diagram commutes and respects the action of Δ*

$$\begin{tikzcd}
\mathrm{Det}(S^{\Delta}[G]^{\times}) \arrow[r, hook] \arrow[d, "i_*"] & K_1(L^{\Delta}[G]) \arrow[d, "i_*"] \\
\mathrm{Det}(S[G]^{\times}) \arrow[r, hook] & K_1(L[G]),
\end{tikzcd}$$

and the right hand side map is injective (see subsection 6.2).

Step 1. Reduction of the general case to the p-group case. Let S and G be as in the conjecture (for infinite Galois extensions see Remark 6.27, so we assume, that S is p-adically complete). Let Δ be an open subgroup of $Gal(L^0/\mathbb{Q}_p)$, so that $R := S^{\Delta}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$. Then S is a local, Noetherian, normal ring satisfying
(i) S is an integral domain, which is torsion free as an abelian group,
(ii) the natural map $S \to \varprojlim S/p^n S$ is an isomorphism, so that S is p-adically complete,
(iii) S supports a lift of Frobenius, that is to say an endomorphism $F: S \to S$ with the property that for all $s \in S$

$$F(s) \equiv s^p \bmod \mathfrak{M},$$

where $\mathfrak{M}$ is the maximal ideal of S.

Note that with S also R satisfies (i)-(iii).

Remark 6.11 *The reduction step is based on the following conditions to be satisfied for every finite p-group G (S, R, Δ being as above); actually they are essential ingredients of step 2 and unfortunately are known to us at the present day only in the unramified case (see Remark 6.25) below.*

1. *There exists a homomorphism ν defined using the lift of Frobenius on S*

$$\nu : \mathrm{Det}(1 + I(S[G])) \to L[\mathcal{C}_G],$$

such that $\mathcal{L} = \nu \circ \mathrm{Det}$ (for the definition of $\mathcal{L}$ see pp. 12-13 in [CPT]). Here $I(S[G])$ denotes the augmentation ideal of the group ring $S[G]$ and $\mathcal{C}_G$ denotes the set of conjugacy classes of G.

2. *Let ν' denote the restriction of the homomorphism ν to* $\mathrm{Det}(1+\mathcal{A}(S[G]))$, *where $\mathcal{A}(S[G])$ is the kernel of the natural map from $S[G]$ to $S[G^{ab}]$, then ν' is an isomorphism*

$$\mathrm{Det}(1+\mathcal{A}(S[G])) \xrightarrow{\cong} p\phi(\mathcal{A}(S[G])),$$

where $\phi : L[G] \to L[\mathcal{C}_G]$ denotes the L-linear map obtained by mapping each element of G to its conjugacy class.

3. *We have the exact sequence*

$$0 \longrightarrow \phi(\mathcal{A}(S[G])) \xrightarrow{(\nu')^{-1}\circ(p\cdot)} \mathrm{Det}(S[G]^\times) \longrightarrow S[G^{ab}]^\times \longrightarrow 1.$$

4. *We have the isomorphism*

$$i_* : \mathrm{Det}(S^\Delta[G]^\times) \cong \mathrm{Det}(S[G]^\times)^\Delta,$$

where Δ acts coefficientwise on Det-groups.

From now we assume these conditions to be satisfied and describe the reduction step in this hypothetical generality, in the hope to prove Conjecture 6.9 some day in full generality.

Most ideas and techniques of the proof are contained in [CPT]. Thus, we only have to relax the condition (iii) in the Hypothesis on the ring in [CPT, p. 2]. So, let S be as above. The proof will now proceed in different stages, restricting G first to special types of groups.

$\mathbb{Q}$-p-elementary groups.

We begin with an algebraic result which we shall require later on this stage.

Suppose, that $\mathcal{O}_K$ is the ring of integers of a finite unramified extension K of $\mathbb{Q}_p$. Let A denote the ring $S \otimes_{\mathbb{Z}_p} \mathcal{O}_K$ and let M be the ring of fractions of A. Since M is a separable L-algebra, it can be written as a finite product of field extensions M_i of L:

$$M = \prod_{i=1}^{n} M_i.$$

Since K is a finite unramified extension of $\mathbb{Q}_p$, we know that A is étale over S and hence is normal (see [Mi, p. 27]). If A_i is the normalization of S in M_i, then

$$A = \prod_{i=1}^{n} A_i.$$

Lemma 6.12 *Let F denote the lift of Frobenius on A given by the tensor product of the lift of Frobenius on S with the Frobenius automorphism of $\mathcal{O}_K$; then $F(A_i) \subset A_i$.*

Proof. Let $\{e_i\}$ denote the system of primitive orthogonal idempotents associated to the above product decomposition of A. As F is a $\mathbb{Z}_p$-algebra endomorphism, we know, that $\{F(e_i)\}$ is a system of orthogonal idempotents with

$$1 = \sum_{i=1}^{n} F(e_i)$$

and so this system corresponds to a decomposition of the commutative algebra A into n components. Since the decomposition of Noetherian commutative algebras into indecomposable algebras is unique, we must have $F(e_i) = e_{\pi(i)}$ for some permutation π of $\{1, ..., n\}$. It will suffice to show, that the permutation π is the identity. Suppose for contradiction, that for some i, we have $\pi(i) = j \neq i$. We know by definition, that

$$F(e_i) \equiv e_i^p = e_i \pmod{\mathfrak{M}A}$$

and so

$$e_i \equiv F(e_i) \cdot e_i \equiv e_j \cdot e_i = 0 \pmod{\mathfrak{M}A}.$$

However, by [CR 1, Thm. 6.7 on p. 123] we know, that, since $\mathfrak{M}A$ is contained in the Jacobson radical of A, $e_i \bmod \mathfrak{M}A$ must be a primitive idempotent of $A/\mathfrak{M}A$, and so we have the desired contradiction. □

Suppose G is a $\mathbb{Q}$-p-elementary group, i.e. G may be written as a semidirect product $C \rtimes P$, where C is a cyclic normal subgroup of order s, which is prime to p, and where P is a p-group. We decompose the commutative group ring $\mathbb{Z}_p[C]$ according as the divisors m of s

$$\mathbb{Z}_p[C] = \prod_{m|s} \mathbb{Z}_p[m], \tag{6.9}$$

where $\mathbb{Z}_p[m]$ is the semilocal ring

$$\mathbb{Z}_p[m] = \mathbb{Z}[\zeta_m] \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

and where ζ_m is a primitive mth root of unity in $\overline{\mathbb{Q}_p}$. For brevity we set $S[m] = S \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[m]$, and we note, that by Lemma 6.12 $S[m]$ decomposes as a product of rings satisfying (i)-(iii), where the Frobenius is given by the restriction of the tensor product of the lift of Frobenius on S and the Frobenius automorphism of $\mathbb{Z}_p[m]$ to each component.

For each m the conjugation action of P on C induces a homomorphism

$$\alpha_m : P \to \mathrm{Aut}\,\langle\zeta_m\rangle$$

and we let $H_m = \mathrm{Ker}(\alpha_m)$ and $A_m = \mathrm{Im}(\alpha_m)$.

Tensoring the decomposition (6.9) with $-\otimes_{\mathbb{Z}_p[C]} S[G]$ affords a decomposition of S-algebras

$$S[G] = \prod_m S[m] \circ P, \tag{6.10}$$

where $S[m] \circ P$ is the free $S[m]$-module on the set of elements of P with the following multiplication $s_1p_1 \cdot s_2p_2 = s_1s_2^{p_1}p_1p_2$, P acting on $S[m]$ through A_m. $S[m] \circ P$ is also called the twisted group ring. We shall study the determinant group $\mathrm{Det}(GL(S[G]))$ by studying the various subgroups $\mathrm{Det}(GL(S[m] \circ P))$. Note that the twisted group ring $S[m] \circ P$ contains the standard group ring $S[m][H_m]$. We therefore have the inclusion map $i : S[m][H_m] \to S[m] \circ P$. We also have a restriction map defined by choosing a transversal $\{a_i\}$ of P/H_m. This induces a restriction homomorphism

$$\mathrm{res} : GL_n(S[m] \circ P) \to GL_{n|A_m|}(S[m][H_m]).$$

By Proposition 6.8 we know, that $\mathrm{Det}(GL(S[m][H_m])) = \mathrm{Det}(S[m][H_m]^\times)$, and so we have defined the composition:

$$r_m : \mathrm{Det}(GL_n(S[m] \circ P)) \to GL_{n|A_m|}(S[m][H_m]) \to \mathrm{Det}(S[m][H_m]^\times). \quad (6.11)$$

Since for $\pi \in P$, $x \in (S[m] \circ P)^\times$, we know, that $\mathrm{Det}(\pi x \pi^{-1}) = \mathrm{Det}(x)$, whence

$$r_m : \mathrm{Det}(GL_n(S[m] \circ P)) \to \mathrm{Det}(S[m][H_m]^\times)^{A_m}.$$

Here A_m acts via α_m on $S[m]$ and by conjugation on H_m. From [T, (3.8) on p. 69] we know, that r_m is injective. Note for future reference, that for $x \in S[m][H_m]^\times$, $i(x)$ is mapped by restriction to the diagonal matrix $\mathrm{diag}(x^{a_i})$; thus we write $\mathrm{Det}(x)$ for the usual element of $\mathrm{Det}(S[m][H_m]^\times)$ whereas $\mathrm{Det}(i(x))$ denotes an element of $\mathrm{Det}((S[m] \circ P)^\times)$. These two determinants are related by the identity

$$r_m(\mathrm{Det}(i(x))) = \prod_{a \in A_m} \mathrm{Det}(x^a) = N_{A_m}(\mathrm{Det}(x)).$$

Next we describe $\mathrm{Det}((S[m] \circ P)^\times)$, and more generally $\mathrm{Det}(GL(S[m] \circ P))$, and the maps i and r_m in terms of character functions. In Lemma 6.18 we shall see, that every irreducible character of G may be written in the form $\mathrm{Ind}_{H_m}^G(\phi_m)$ for some m, where ϕ_m is an abelian character of H_m with the property, that the restriction of ϕ_m to C has order m. With this notation the elements $\mathrm{Det}(i(x))$ in $\mathrm{Det}((S[m] \circ P)^\times)$ are character functions on such $\mathrm{Ind}_{H_m}^G(\phi_m)$ with

$$\mathrm{Det}(i(x))(\mathrm{Ind}_{H_m}^G(\phi_m)) = r_m(\mathrm{Det}(i(x)))(\phi_m) = \prod_{a \in A_m} \mathrm{Det}(x^a)(\phi_m).$$

It is also instructive to see the above in the context of K-theory. We then have induction and restriction maps

$$\begin{array}{ccccccc} ind & : & K_1(S[m][H_m]) & \leftrightarrows & K_1(S[m] \circ P) & : & r_m \\ ind & : & \mathrm{Det}(S[m][H_m]^\times) & \leftrightarrows & \mathrm{Det}((S[m] \circ P)^\times) & : & r_m. \end{array}$$

Similarly we have the corresponding maps on the representation rings

$$\mathrm{K}_0(\overline{\mathbb{Q}_p}[m][H_m]) \leftrightarrows \mathrm{K}_0(\overline{\mathbb{Q}_p}[m] \circ P)$$

and by Mackey theory the induction map $i = \mathrm{Ind}_{H_m}^P$ maps $\mathrm{K}_0(\overline{\mathbb{Q}_p}[m][H_m])$ onto $\mathrm{K}_0(\overline{\mathbb{Q}_p}[m] \circ P)$ (see [T, p. 68]).

Let I_{H_m}, resp. I_P, denote the augmentation ideal of $S[m][H_m]$, resp. the two sided $S[m] \circ P$-ideal generated by I_{H_m}. The main result on this stage is to show:

Theorem 6.13 *The map r_m defined in (6) gives an isomorphism*

$$r_m : \mathrm{Det}(GL(S[m] \circ P)) = \mathrm{Det}((S[m] \circ P)^\times) \to \mathrm{Det}(S[m][H_m]^\times)^{A_m}.$$

Proof. We have seen, that r_m is injective on $\mathrm{Det}(GL(S[m] \circ P))$; we now show, that r_m maps $\mathrm{Det}((S[m] \circ P)^\times)$ onto $\mathrm{Det}(S[m][H_m]^\times)^{A_m}$.

First we put $\widetilde{P}_m = P/[H_m, H_m]$, then the ring $S[m] \circ \widetilde{P}_m$ is isomorphic to the ring of $|A_m| \times |A_m|$ matrices over $(S[m][H_m^{ab}])^{A_m}$. To see it, we note, that by Lemma 6.12 $S[m] = \prod_i S_i$, where S_i are complete local rings. Further, by [Wall, Lem. 8.2 on p. 613] $S_i \circ \widetilde{P}_m$ is an Azumaya algebra over some complete local ring B_i. Thus, it represents a class in the Brauer group of B_i, which is the quotient group of the group of Azumaya algebras by the subgroup of full matrix algebras. But by [AG, Thm. 6.5] the Brauer group of B_i is isomorphic to the Brauer group of the residue class field b_i of B_i, and thus is trivial, as b_i is an algebraic extension of $\mathbb{F}_p$ and the Brauer group of a quasi-algebraically closed field is trivial (see [NSW, Thm. 6.5.4, Thm. 6.5.7 and Prop. 6.5.8]). Hence we see, that r_m induces an isomorphism

$$\mathrm{Det}((S[m] \circ \widetilde{P}_m)^\times) \cong ((S[m][H_m^{ab}])^{A_m})^\times. \tag{6.12}$$

From the conditions (2) and (3) of Remark 6.11 above (which trivially extends to products of rings, since formation of determinants commutes with ring products) and using (6.12), we have a commutative diagram with exact top row:

$$\begin{tikzcd}
\mathrm{Det}(1 + \mathcal{A}(S[m][H_m]))^{A_m} \arrow[r, hook] & \mathrm{Det}(S[m][H_m]^\times)^{A_m} \arrow[r] & ((S[m][H_m^{ab}])^{A_m})^\times \\
\mathrm{Det}(i(1 + \mathcal{A}(S[m][H_m]))) \arrow[r, hook] \arrow[u] & \mathrm{Det}((S[m] \circ P)^\times) \arrow[r, two heads] \arrow[u, "r_m"] & \mathrm{Det}((S[m] \circ \widetilde{P}_m)^\times). \arrow[u, "\cong"]
\end{tikzcd} \tag{6.13}$$

It will therefore suffice to show

$$r_m(\mathrm{Det}(i(1 + \mathcal{A}(S[m][H_m])))) \supseteq \mathrm{Det}(1 + \mathcal{A}(S[m][H_m]))^{A_m}$$

and this follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Det}(1 + \mathcal{A}(S[m][H_m])) \arrow[r, "\nu", "\cong"'] \arrow[d, "r_m"] & \phi(\mathcal{A}(S[H_m])) \otimes_S S[m] \arrow[d, "\mathrm{tr}_{A_m}"] \\
\mathrm{Det}(1 + \mathcal{A}(S[m][H_m]))^{A_m} \arrow[r, "\nu^{A_m}", "\cong"'] & (\phi(\mathcal{A}(S[H_m])) \otimes_S S[m])^{A_m}.
\end{tikzcd}$$

Recall that F is the tensor product of the lift of Frobenius on S with the Frobenius automorphism of $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$. Note also that A_m acts on $S[m] =$

$S \otimes \mathbb{Z}_p[m]$ via the second factor; so, because G is a $\mathbb{Q}$-p-elementary, the action of A_m on $\langle\chi(G)\rangle$ factors through $Gal(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p)$; this guarantees, that the actions of F and A_m commute; hence ν is an isomorphism of A_m-modules, and this gives the bottom row in the above diagram.

Since $S[m]$ is a free $S[A_m]$-module, it follows, that $\phi(\mathcal{A}(S[H_m])) \otimes_S S[m]$ is a projective $S[A_m]$-module (with diagonal action); and so tr_{A_m}, and therefore r_m, is surjective. $\square$

Finally we show:

Theorem 6.14 *Let G be a finite $\mathbb{Q}$-p-elementary group and let S, Δ and R be as above. Further, let Δ act coefficientwise on* $\mathrm{Det}(S[G]^\times)$*, then*

$$\mathrm{Det}(S[G]^\times)^\Delta = \mathrm{Det}(R[G]^\times).$$

Proof. By (6.10) together with Theorem 6.13

$$\mathrm{Det}(S[G]^\times)^\Delta = \bigoplus_m \mathrm{Det}((S[m] \circ P)^\times)^\Delta = \bigoplus_m (\mathrm{Det}(S[m][H_m]^\times)^{A_m})^\Delta.$$

Recall that Δ acts via the first term in $S[m] \circ P = S \otimes_R (R[m] \circ P)$ and that A_m acts via the second term; hence the actions of Δ and A_m commute on $S[m][H_m] = (S \otimes_R R[m])[H_m]$; hence we see, that

$$\mathrm{Det}(S[G]^\times)^\Delta = \bigoplus_m (\mathrm{Det}(S[m][H_m]^\times))^{A_m \times \Delta} = \bigoplus_m (\mathrm{Det}(S[m][H_m]^\times)^\Delta)^{A_m}$$

and so by the condition (4) of Remark 6.11 and Theorem 6.13 together with (6.10) we have equalities

$$\mathrm{Det}(S[G]^\times)^\Delta = \bigoplus_m (\mathrm{Det}(R[m][H_m]^\times))^{A_m} = \bigoplus_m \mathrm{Det}((R[m] \circ P)^\times) = \mathrm{Det}(R[G]^\times).$$

$\square$

Application. We conclude this stage by considering the implications of the above result for an arbitrary finite group G. From [S, 12.6] we know, that we can find an integer l prime to p, $\mathbb{Q}$-p-elementary subgroups H_i of G, integers n_i, and $\theta_i \in \mathrm{K}_0(\mathbb{Q}_p[H_i])$, such that

$$l \cdot 1_G = \sum_i n_i \cdot \mathrm{Ind}_{H_i}^G(\theta_i).$$

Thus, given $\mathrm{Det}(x) \in \mathrm{Det}(GL(S[G]))^\Delta$, then by the Frobenius structure of the module $GL(S[G])$ over $\mathrm{K}_0(\mathbb{Q}_p[H_i])$ (see [CPT, §5.a]) we have

$$\mathrm{Det}(x)^l = \prod_i \left(\mathrm{Ind}_{H_i}^G(\theta_i \cdot \mathrm{Res}_G^{H_i}(\mathrm{Det}(x)))\right)^{n_i}.$$

However, Theorem 6.14 above implies that

$$\theta_i \cdot \mathrm{Res}_G^{H_i}(\mathrm{Det}(x)) \in \mathrm{Det}(S[H_i]^\times)^\Delta = \mathrm{Det}(R[H_i]^\times).$$

Thus we have shown

Theorem 6.15 *For any finite group G each element in the quotient group*

$$\mathrm{Det}(GL(S[G]))^{\Delta}/\mathrm{Det}(GL(R[G]))$$

has finite order, which is prime to p.

$\mathbb{Q}$-l-elementary groups.

We consider a prime $l \neq p$ and a $\mathbb{Q}$-l-elementary group G, i.e G may be written as $(C \times C') \rtimes L$, where C is a cyclic p-group, C' is a cyclic group of order prime to pl and L is an l-group. On this stage we show:

Theorem 6.16 *If G is a $\mathbb{Q}$-l-elementary group, then*

$$\mathrm{Det}(S[G]^{\times})^{\Delta} = \mathrm{Det}(R[G]^{\times}).$$

Then, reasoning as in the Application on the previous stage, we can immediately deduce

Theorem 6.17 *For an arbitrary finite group G each element in the quotient group*

$$\mathrm{Det}(GL(S[G]))^{\Delta}/\mathrm{Det}(GL(R[G]))$$

has finite order, which is prime to l.

This, together with Theorem 6.15 and Proposition 6.8, will then establish Conjecture 6.9.

Prior to proving Theorem 6.16, we first need to recall four preparatory results:

Lemma 6.18 *Each irreducible character χ of G can be written in the form $\mathrm{Ind}_{\Omega}^{G}\phi$, where ϕ is an abelian character of a subgroup Ω, which contains $C \times C'$.*

Proof. See [S, 8.2] □

Proposition 6.19 *Let $\mathcal{O}$ denote the ring of integers of the finite extension of $\mathbb{Q}_p$ generated by the values of all characters of G, let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}$, and let $\mathcal{P}$ denote the $S \otimes_{\mathbb{Z}_p} \mathcal{O}$-ideal generated by $\mathfrak{m}$. With the notation of the previous lemma, we write $\phi = \phi'\phi_p$, where ϕ' (resp. ϕ_p) has order prime to p (resp. p-power order); and we put $\chi' = \mathrm{Ind}_{\Omega}^{G}\phi'$. Then for $r \in GL(S[G])$ we have the congruence*

$$\mathrm{Det}(r)(\chi - \chi') \equiv 1 \ (\mathrm{mod}\ \mathcal{P}).$$

Proof. It is an easy generalization of Lemma 1.3 on page 35 in [T]. □

Proposition 6.20 *Put $G' = G/C$. Then $\mathbb{Z}_p[G']$ is a split maximal $\mathbb{Z}_p$-order, i.e. it is a product of matrix rings*

$$\mathbb{Z}_p[G'] = \prod_i \mathrm{M}_{n_i}(\mathcal{O}_i)$$

over (local) rings of integers $\mathcal{O}_i$. Thus we have the equalities

$$\mathrm{Det}(GL(S[G'])) = \prod_i (S \otimes_{\mathbb{Z}_p} \mathcal{O}_i)^\times = \mathrm{Det}(S[G']^\times)$$

and

$$\mathrm{Det}(S[G']^\times)^\Delta = \prod_i \mathrm{Det}((S \otimes \mathcal{O}_i)^\times)^\Delta = \prod_i (S \otimes \mathcal{O}_i)^{\times\Delta} =$$

$$\prod_i (R \otimes \mathcal{O}_i)^\times = \prod_i \mathrm{Det}((R \otimes \mathcal{O}_i)^\times) = \mathrm{Det}(R[G']^\times).$$

Proof. See [Re, Thm. 41.1 and Thm. 41.7]. □

Lemma 6.21 *With the previous notation, the map $S[G]^\times \to S[G']^\times$ is surjective.*

Proof. Consider the canonical projection

$$A = S[G] \xrightarrow{\varphi} S[G'] = B.$$

Firstly, we have to prove, that $\mathrm{Ker}(\varphi)$ is contained in the Jacobson radical of A. We know, that $\mathrm{Ker}(\varphi) = I(C) \subset A$, where $I(C)$ is generated by $1-\sigma$, $\sigma \in C$. Thus there is a positive integer r, such that $(\mathrm{Ker}(\varphi))^{p^r} \subset \mathfrak{M}$, where $\mathfrak{M}$ is the maximal ideal of S. A positive power of an ideal is contained in the Jacobson radical of a ring if and only if the ideal is contained in the Jacobson radical (the image of such ideal would be a nilpotent ideal of $A/rad(A)$, hence it is contained in $rad(A/rad(A)) = (0)$, see [Re, §6]). Hence it is enough to prove, that $\mathfrak{M}$ is contained in $rad(A)$. But this follows from the more general lemma below (see Lemma 6.22), as the Jacobson radical of the ring A is the intersection of all maximal left ideals of A (see [Re, Thm. 6.3]).

Secondly, by [Re, Thm. 6.10] and the first step of the proof φ induces an isomorphism

$$\bar{\varphi} : A/rad(A) \xrightarrow{\cong} B/rad(B),$$

as $\varphi(rad(A)) \subset rad(B)$.

Finally, let $x \in B^\times$. We denote its image in $(B/rad(B))^\times$ by $\bar{x}$. Since φ is surjective, we can lift x to an element $y \in A$. The image of y in $A/rad(A)$ denoted $\bar{y}$ is then mapped under $\bar{\varphi}$ onto $\bar{x}$, thus it is contained in $(A/rad(A))^\times$ and hence $y \in A^\times$, as $1 + rad(A) \subset A^\times$ (see [Re, Thm. 6.5]). □

Lemma 6.22 *Let S be a discrete valuation ring, but not a field. Let $\mathfrak{M}$ denote its unique maximal (left) ideal. Assume, that we have an inclusion of the rings $i : S \to A$, where A is an arbitrary ring, then $i(\mathfrak{M}) \cdot S$ is contained in every maximal non-zero left ideal $\mathcal{M}$ of A.*

Proof. The preimage $i^{-1}(\mathcal{M})$ of $\mathcal{M}$ is a non-zero prime left ideal of S, thus it is a maximal left ideal, hence it coincides with $\mathfrak{M}$. Now applying i we get $i(\mathfrak{M}) \cdot S = i(i^{-1}(\mathcal{M})) \cdot S \subseteq \mathcal{M}$. □

Proof of Theorem 6.16. Suppose, that we are given $\mathrm{Det}(z) \in \mathrm{Det}(S[G]^\times)^\Delta$ and let z' denote the image of z in $S[G']$. Then by Proposition 6.20 we know, that we can find $x' \in R[G']^\times$ with $\mathrm{Det}(x') = \mathrm{Det}(z')$; moreover, by Lemma 6.21 we can find $x \in R[G]^\times$ with image x' in $R[G']^\times$. Thus, to conclude, it will be sufficient to show, that $\mathrm{Det}(zx^{-1})$ is in $\mathrm{Det}(R[G]^\times)$. However, by construction, $\mathrm{Det}(zx^{-1})$ is trivial on characters inflated from G', and so by Proposition 6.19 we see that

$$\mathrm{Det}(zx^{-1})(\chi) = \mathrm{Det}(zx^{-1})(\chi - \chi') \equiv 1 \pmod{\mathcal{P}}, \text{ for all } \chi.$$

Hence $\mathrm{Det}(zx^{-1})$ is in $\mathrm{Hom}(R_G, 1+\mathcal{P})$, where R_G denote the group of virtual $\overline{L}$-valued characters of G. Moreover, since $\mathrm{Det}(zx^{-1})$ is invariant under the action of Δ, $\mathrm{Det}(zx^{-1}) \in \mathrm{Hom}_\Delta(R_G, 1+\mathcal{P})$. In the case, where R is the ring of integers of a finite extension of $\mathbb{Q}_p$, the last Hom-group is isomorphic to the finite product of groups of higher principal units $U^{(1)}$ of finite extensions of $\mathbb{Q}_p$, and hence is a pro-p-group. Thus $\mathrm{Det}(zx^{-1})$ is a pro-p-element of $\mathrm{Det}(S[G]^\times)^\Delta$. But by Theorem 6.15 we know, that $\mathrm{Det}(zx^{-1})$ has image in the quotient group $\mathrm{Det}(S[G]^\times)^\Delta/\mathrm{Det}(R[G]^\times)$ of finite order, which is prime to p. Therefore we may deduce, that $\mathrm{Det}(zx^{-1})$ is in $\mathrm{Det}(R[G]^\times)$. □

Step 2. The p-group case. In contrast to the first step the second one can be proved only under further assumptions on S. In particular, we have to aggravate the condition (iii) of step 1. So let G be a finite p-group. Let S be a unitary ring satisfying the following conditions:
(i) S is an integral domain, which is torsion free as an abelian group,
(ii) the natural map $S \to \varprojlim S/p^nS$ is an isomorphism, so that S is p-adically complete,
(iii) S supports a lift of Frobenius, that is to say an endomorphism $F: S \to S$ with the property that for all $s \in S$

$$F(s) \equiv s^p \bmod pS.$$

For this step we generalize the ideas of [CPT] to the case of an infinite group Δ.

Remark 6.23 *Proposition 6.8 holds also for S satisfying the conditions (i)-(iii) above and G being a finite p-group (see [CPT, Thm. 1.2]).*

Now we are ready to formulate the main theorem of step 2.

Theorem 6.24 *Let G be a finite p-group. Let S be a unitary ring satisfying the conditions (i)-(iii) and Δ be a group acting on S by the ring automorphisms, such that $R = S^\Delta$ also satisfies the conditions (i)-(iii). We do not suppose, that the lift of Frobenius F_R is compatible with the lift of Frobenius F_S, so that $F_S|_R$ need not equal F_R. Then we have the isomorphism*

$$i_* : \mathrm{Det}(R[G]^\times) \cong \mathrm{Det}(S[G]^\times)^\Delta,$$

where Δ acts on Det-groups coefficientwise.

Proof. Since Δ acts on S by the ring automorphisms, we have the equality $(S^\times)^\Delta = R^\times$ and for any finitely generated free S-module $M = \bigoplus_i Se_i$, on which Δ acts coefficientwise, M^Δ is the finitely generated free R-module $M^\Delta = \bigoplus_i Re_i$.

S and R both satisfy the conditions (i)-(iii), which are precisely the Hypothesis in [CPT], thus the proof of the theorem is identical with the proof of Theorem 4.1 in [CPT]. Note that this proof does not depend on the condition whether Δ is a finite group or not, it only uses the equalities above. □

Remark 6.25 *In particular, Theorem 6.24 holds for $S = \mathcal{O}_L$, where L is either a finite unramified extension L^0 of $\mathbb{Q}_p$ or the completion of an infinite unramified extension L^0 of $\mathbb{Q}_p$ - in other words for the ring of Witt vectors $W(\kappa)$ of any algebraic extension κ of $\mathbb{F}_p$, and for Δ being an open subgroup of $Gal(L^0/\mathbb{Q}_p)$.*

Conjecture 6.9*: proved cases and one generalization.* Because of our restrictions in the p-group case we have proved Conjecture 6.9 only for $S = W(\kappa)$ the ring of Witt vectors of an algebraic extension κ of $\mathbb{F}_p$. There are some possible generalizations of this result as will be explained now.

Remark 6.26 *Using the reduction step described in [T, p. 92] we can prove Conjecture 6.9 for $S = \mathcal{O}_L$, where L is the completion of an at most tamely ramified extension L^0 of finite absolute ramification index over $\mathbb{Q}_p$, i.e., tamely ramified extension of the ring of Witt vectors of an algebraic extension κ of $\mathbb{F}_p$, and Δ being an open subgroup of $Gal(L^0/\mathbb{Q}_p)$ containing the inertia group.*

Remark 6.27 *The infinite Galois extensions can always be written as a direct limit (or simply union) of their finite subextensions, and so their rings of integers, too. Thus, if we have proved Conjecture 6.9 for some class of finite Galois extensions of $\mathbb{Q}_p$, we can obtain it also for the corresponding ind-objects (infinite extensions), i.e. direct limits of objects in the original class, as the Det-map commutes with direct limits (see Remark 6.7).*

Explicitly, let S be the ring of integers of an infinite extension L of $\mathbb{Q}_p$. Let $L = \bigcup_i L_i$ and $S = \bigcup_i S_i$, where L_i are finite extensions and S_i their rings of integers, and we have the statement of Conjecture 6.9 for all S_i. Further, let Δ be an open subgroup of $Gal(L/\mathbb{Q}_p)$, then

$$\mathrm{Det}(S[G]^\times)^\Delta = \mathrm{Det}\left(\bigcup_i S_i[G]^\times\right)^\Delta = \left(\bigcup_i \mathrm{Det}(S_i[G]^\times)\right)^\Delta = \bigcup_i \left(\mathrm{Det}(S_i[G]^\times)^\Delta\right) =$$
$$\bigcup_i \left(\mathrm{Det}(S_i^\Delta[G]^\times)\right) = \mathrm{Det}\left(\bigcup_i S_i^\Delta[G]^\times\right) = \mathrm{Det}(S^\Delta[G]^\times),$$

where Δ acts on S_i through the corresponding quotient group and the union commutes with such defined Δ-action. The maps between Det*-groups induced by inclusions of rings are inclusions by Remark 6.10.*

For example, Conjecture 6.9 holds for $S = \mathcal{O}_L$, where L is the maximal unramified extension of $\mathbb{Q}_p$ and Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ or using the previous remark L is the maximal tamely ramified extension of $\mathbb{Q}_p$ and Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ containing the inertia group.

Remarks 6.25, 6.26 and 6.27 imply

Theorem 6.28 *Let G be a finite group. Let $S = \mathcal{O}_L$, where L is either an arbitrary (possibly infinite) at most tamely ramified extension L^0 of $\mathbb{Q}_p$ (type 1) or the completion of an at most tamely ramified extension L^0 of finite absolute ramification index over $\mathbb{Q}_p$ (type 2), and let Δ be an open subgroup of $Gal(L^0/\mathbb{Q}_p)$ containing the inertia group. Then*

$$i_* : \mathrm{Det}(S^\Delta[G]^\times) \cong \mathrm{Det}(S[G]^\times)^\Delta.$$

We conclude this subsubsection with the following result generalizing Theorem 6.28 to the case of compact p-adic Lie groups and their Iwasawa algebras. Let $\mathcal{G}$ be a compact p-adic Lie group and let $S = \mathcal{O}_L$ be as in the theorem but unramified, as for the infinite and tamely ramified extensions we can use Remarks 6.26 and 6.27. We denote by $R = S^\Delta$ the ring of integers of a finite unramified extension K over $\mathbb{Q}_p$, where Δ is the Galois group $Gal(L^0/K)$, and write

$$\Lambda_S(\mathcal{G}) := S[[\mathcal{G}]] \quad (\text{resp. } \Lambda_R(\mathcal{G}) := R[[\mathcal{G}]])$$

for the Iwasawa algebra of $\mathcal{G}$ with coefficients in S (resp. in R). Note that it is a Noetherian pseudocompact ring (resp. a Noetherian compact ring). For the notion of pseudocompact rings and algebras see [Br]. Generalizing the ideas of Proposition 5.2.16 in [NSW] we deduce, that $\Lambda_R(\mathcal{G})$ and $\Lambda_S(\mathcal{G})$ are semilocal rings.

In the following we will use Froehlich's Hom-description as it has been adapted to Iwasawa theory by Ritter and Weiss in [RW]. We have the following commutative diagram

$$\begin{tikzcd} K_1(\Lambda_R(\mathcal{G})) \arrow[r, "\mathrm{Det}"] \arrow[d] & \mathrm{Hom}_{G_K}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times) \arrow[d, hook] \\ K_1(\Lambda_S(\mathcal{G})) \arrow[r, "\mathrm{Det}"] & \mathrm{Hom}_{G_{L^0}}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times), \end{tikzcd}$$

where $G_{L^0} = Gal(\overline{L}/L^0)$, $G_K = Gal(\overline{K}/K)$ and $R_\mathcal{G}$ as before is the free abelian group on the isomorphism classes of irreducible $\overline{\mathbb{Q}_p}$-valued Artin representations of $\mathcal{G}$.

Now we are ready to formulate

Theorem 6.29 *With the notation as above we have*

$$i_* : \mathrm{Det}(K_1(\Lambda_R(\mathcal{G}))) \cong \mathrm{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta,$$

where Δ acts on the K_1-groups coefficientwise.

Proof. From the diagram

$$\begin{tikzcd} \Lambda_R(\mathcal{G})^\times \arrow[r, "\mathrm{Det}"] \arrow[d, hook] & \mathrm{Hom}_{G_K}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times) \arrow[d, hook] \\ \Lambda_S(\mathcal{G})^\times \arrow[r, "\mathrm{Det}"] & \mathrm{Hom}_{G_{L^0}}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times), \end{tikzcd}$$

which is commutative by the construction, and from Proposition 6.8 we get the first obvious inclusion

$$\mathrm{Det}(\Lambda_R(\mathcal{G})^\times) = \mathrm{Det}(K_1(\Lambda_R(\mathcal{G}))) \subseteq \mathrm{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta = \mathrm{Det}(\Lambda_S(\mathcal{G})^\times)^\Delta.$$

For the opposite inclusion we use Theorem 6.28, then we only have to show, how the general case can be reduced to the case of finite groups. To this end write $\mathcal{G} = \varprojlim_n G_n$ as inverse limit of finite groups. By Theorem 6.28 we have compatible continuous maps

$$R[G_n]^\times \xrightarrow{\mathrm{Det}} \mathrm{Det}(K_1(\Lambda_S(G_n)))^\Delta \hookrightarrow \mathrm{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)^\Delta,$$

where the topology on $\mathrm{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)$ is induced from the valuation topology on $\mathbb{C}_p$. Taking the inverse limit yields, by the compactness of $\Lambda_R(\mathcal{G})^\times = \varprojlim_n (R/\pi^n[G_n])^\times$ and by letting $R_\mathcal{G} = \varinjlim_n R_{G_n}$, a factorization of the homomorphism Det into

$$\Lambda_R(\mathcal{G})^\times \xrightarrow{\mathrm{Det}} \left(\varprojlim_n \mathrm{Det}(K_1(\Lambda_S(G_n)))\right)^\Delta \hookrightarrow \mathrm{Hom}_{G_K}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times).$$

The claim follows, because denoting by

$$res_n : \mathrm{Hom}_{G_{L^0}}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times) \to \mathrm{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)$$

the restriction we obtain from the universal mapping property for

$$\varprojlim_n \mathrm{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times) \cong \mathrm{Hom}_{G_{L^0}}(R_\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times)$$

inclusions

$$\mathrm{Det}(K_1(\Lambda_S(\mathcal{G}))) \subseteq \varprojlim_n \mathrm{Im}(res_n \circ \mathrm{Det}) \subseteq \varprojlim_n \mathrm{Det}(K_1(\Lambda_S(G_n))),$$

whence

$$\mathrm{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta \subseteq \mathrm{Det}(\Lambda_R(\mathcal{G})^\times) = \mathrm{Det}(K_1(\Lambda_R(\mathcal{G}))).$$

□

For applications of the theorem above in number theory see [BV].

6.3.2 The SK_1-part

From [O 1] we have the following

Theorem 6.30 *Let R be the ring of integers in any finite extension K of $\mathbb{Q}_p$. Then for any p-group G, there is an isomorphism*

$$\Theta_{RG} : SK_1(R[G]) \xrightarrow{\cong} \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G),$$

where $\mathrm{H}_2^{ab}(G) = \mathrm{Im}\big[\sum\{\mathrm{H}_2(H) : \ H \subseteq G,\ H \ \textit{abelian}\} \xrightarrow{\sum \mathrm{Ind}} \mathrm{H}_2(G)\big]$

If $L \supseteq K$ is a finite extension, and if $S \subseteq L$ is the ring of integers , then
(i) $i_* : SK_1(R[G]) \to SK_1(S[G])$ (induced by inclusion) is an isomorphism, if L/K is totally ramified; and
(ii) $trf : SK_1(S[G]) \to SK_1(R[G])$ (the transfer) is an isomorphism, if L/K is unramified.

Proof. See [O 1, Thm. 8.7]. □

We see, that $SK_1(S[G])$ as an abstract finite group is independent of S. Note also, that i_* and trf are Galois homomorphisms, hence $SK_1(S[G])$ has trivial Galois action. In order to treat infinite algebraic extensions of $\mathbb{Q}_p$ and the analogous descent statement of the introduction for SK_1-groups we have to describe the maps i_* induced by inclusions, as they appear in the direct limits (see Remark 6.27 and Remark 6.7).

Now we assume that K is unramified over $\mathbb{Q}_p$, then from Proposition 21 (i) in [O 2], which is also valid for i_* by the same argument, we have commutative squares

$$\begin{tikzcd} SK_1(R[G]) \arrow[r, "\Theta_{RG}"] \arrow[d, "i_*"] & \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G) \arrow[d, "?"] \\ SK_1(S[G]) \arrow[r, "\Theta_{SG}"] & \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G) \end{tikzcd}$$

and

$$\begin{tikzcd} SK_1(R[G]) \arrow[r, "\Theta_{RG}"] & \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G) \\ SK_1(S[G]) \arrow[r, "\Theta_{SG}"] \arrow[u, "trf"] & \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G) \arrow[u, "id"] \end{tikzcd}$$

where Θ_{RG}, Θ_{SG} and trf are isomorphisms. To describe i_* and ? we need the following

Lemma 6.31 *With the previous notation the map*

$$trf \circ i_* : K_1(R[G]) \longrightarrow K_1(S[G]) \longrightarrow K_1(R[G])$$

is multiplication by $n = [L : K]$.

Proof. By [O 1, Prop. 1.18] the composite $trf \circ i_*$ is induced by tensoring over $R[G]$ with $S[G] \cong S \otimes_R R[G]$ regarded as an $(R[G], R[G])$-bimodule, where the bimodule structure on $S \otimes_R R[G]$ is given through the second factor in the natural way. Since S is a free R-modules of rank n, $S \otimes_R R[G] \cong R[G]^n$ as $(R[G], R[G])$-bimodules, and so $trf \circ i_*$ is multiplication by n on $K_1(R[G])$ (written additively). □

The map $trf \circ i_*$ on $K_1(R[G])$ corresponds via Θ_{RG} to the map

$$\mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G) \xrightarrow{n\cdot} \mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G),$$

and since trf corresponds to the identity map, i_* corresponds to the multiplication by n. From [O 1, Thm. 3.14] we know, that $SK_1(R[G])$ (hence $\mathrm{H}_2(G)/\mathrm{H}_2^{ab}(G)$) is a finite p-group, so that we have proved the

Theorem 6.32 *With the notation as above i_* is an isomorphism in the following two cases*
(i) if L/K is totally ramified,
(ii) if L/K is unramified and $p \nmid n$. In this case i_* corresponds via Θ_{RG} to the multiplication by n on the finite p-group (written additively).

If L/K is unramified and $p|n$, then i_* still corresponds via Θ_{RG} to the multiplication by n on the finite p-group, which is neither surjective nor injective, as finite p-groups always have p-torsion elements.

Corollary 6.33 *The groups $SK_1(R[G])$ and $SK_1(S[G])$ are always isomorphic (as abstract groups with the (trivial) action of $Gal(L/K)$), but the statement of the introduction for SK_1-groups, i.e.*

$$i_* : SK_1(R[G]) \cong SK_1(S[G])^{\Delta} \quad (\Delta = Gal(L/K)), \tag{6.14}$$

holds only in the cases (i) and (ii) of Theorem 6.32.

Corollary 6.34 *Let M be an infinite algebraic extension of $\mathbb{Q}_p$ and let M^0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in M. We write M as the direct limit (union) of its finite subextensions and use Remark 6.7 and the theorem above to get the following result:*
If p^∞ divides $[M^0 : \mathbb{Q}_p]$ (as supernatural numbers), then $SK_1(\mathcal{O}_M[G]) = 1$ for every finite p-group G.

Remark 6.35 *From Corollary 6.34 we can obtain a generalization of Corollary 6.33 for infinite extensions: Let L be an infinite algebraic extension of $\mathbb{Q}_p$ and let $K = L^{\Delta}$, where Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$. Then the statement* (6.14) *holds only in the cases (i) and (ii) of Theorem 6.32, here $p \nmid n$ as supernatural numbers. In the case, where p^∞ divides $[L^0 : \mathbb{Q}_p]$ and $SK_1(R[G]) \neq 1$, $SK_1(R[G])$ and $SK_1(S[G])$ are not isomorphic even as abstract groups. See [O 1, Exam. 8.11] for an example of a non-trivial $SK_1(\mathbb{Z}_p[G])$.*

Now we generalize our results to the case of an arbitrary finite group G. Note, that Theorem 3.14 in [O 1] and Lemma 6.31 still hold in this case. From [O 3] we have the

Theorem 6.36 *Let R be the ring of integers in any finite extension K of $\mathbb{Q}_p$ and let G be a finite group. Let $L \supseteq K$ be a finite extension, and let $S \subseteq L$ be the ring of integers , then*
(ii) $i_* : SK_1(R[G]) \to SK_1(S[G])$ is an isomorphism, if L/K is totally ramified;
(iii) $trf : SK_1(S[G]) \to SK_1(R[G])$ is onto, if L/K is unramified.

Proof. See [O 3, Thm. 1]. □

We need some more notation. For any finite group G and any fixed prime p G_r will denote the set of p-regular elements in G, i.e., elements of order prime to p. $\mathrm{H}_n(G, R(G_r))$ denotes the homology group induced by the conjugation action of G on the free R-module $R(G_r)$ on the set G_r.

When R is the ring of integers in a finite unramified extension of $\mathbb{Q}_p$, then Φ denotes the automorphism of $\mathrm{H}_n(G, R(G_r))$ induced by the map $\Phi(\sum_i r_i g_i) = \sum_i \varphi(r_i) g_i^p$ on coefficients. We set $\mathrm{H}_n(G, R(G_r))_\Phi = \mathrm{H}_n(G, R(G_r))/(1 - \Phi)$. In analogy with the p-group case, we define

$$\mathrm{H}_2^{ab}(G, R(G_r))_\Phi = \mathrm{Im}\big[\sum \{\mathrm{H}_2(H, R(H_r)) : \ H \subseteq G, \ H \text{ abelian}\} \xrightarrow{\sum \mathrm{Ind}} \mathrm{H}_2(G, R(G_r))_\Phi\big].$$

We use the notation above to formulate the

Theorem 6.37 *Let R be the ring of integers in a finite unramified extension of $\mathbb{Q}_p$. Then for any finite group G there is an isomorphism*

$$\Theta_G : SK_1(R[G]) \xrightarrow{\cong} (R/(1-\varphi)R) \otimes_{\mathbb{Z}_p} \mathrm{H}_2(G, \mathbb{Z}_p(G_r))_\Phi / \mathrm{H}_2^{ab}(G, \mathbb{Z}_p(G_r))_\Phi.$$

This new tensor product decomposition of $SK_1(R[G])$ in terms of $R/(1-\varphi)R$ and group cohomology comes from the results announced in [CPT 1] and we are very grateful to T. Chinburg, G. Pappas and M. J. Taylor for sharing this insight with us, which not least also influenced our results below.
Proof. See [O 1, Thm. 12.10] and use the facts

$$\mathrm{H}_2(G, R(G_r))_\Phi \cong (R/(1-\varphi)R) \otimes_{\mathbb{Z}_p} \mathrm{H}_2(G, \mathbb{Z}_p(G_r))_\Phi$$

and

$$\mathrm{H}_2^{ab}(G, R(G_r))_\Phi \cong (R/(1-\varphi)R) \otimes_{\mathbb{Z}_p} \mathrm{H}_2^{ab}(G, \mathbb{Z}_p(G_r))_\Phi.$$

□

Arguing as in the p-group case we deduce the following

Theorem 6.38 *With the notation as above* i_* *is*
(i) an isomorphism, if L/K *is totally ramified;*
(ii) a monomorphism, if L/K *is unramified and* $p \nmid n$.

Remark 6.39 *In the case (ii) we cannot say whether* i_* *is surjective or not, since, in general,* trf *is only an epimorphism and not an isomorphism.*

Corollary 6.40 *The statement* (6.14) *holds in the cases (i)-(ii) of Theorem 6.38.*

Proof. The statement is obvious in the case (i), since i_* is a Galois isomorphism. For (ii) we use the isomorphism of Theorem 6.37 for S and R noting that

$$R/(1-\varphi)R \cong \mathbb{Z}_p \cong S/(1-\varphi)$$

in this situation, hence $SK_1(S[G])$ is isomorphic to $SK_1(R[G])$ (as an abstract finite group). Since i_* is a Galois monomorphism in the case (ii), we get the statement. □

Remark 6.41 *Corollary 6.40 generalizes immediately to the case of infinite algebraic extensions* L.

To study more general rings (for example completions of infinite extensions of $\mathbb{Q}_p$) we need generalizations of Oliver's results on SK_1 to such rings as are announced to appear in [CPT 1] in a very general setting. Meanwhile we outline an ad hoc description sufficient for our purposes. We just note that the same arguments as used below also should work for any ring R as in the beginning of Step 1 of the subsubsection 6.3.1 and satisfying the surjectivity of $1-F$.

Let p be an odd prime number. For the rest of this subsubsection we assume that R is the ring of Witt vectors of a p-closed algebraic extension κ of $\mathbb{F}_p$, i.e. κ does not allow any extension of degree p. The main example we have in mind being $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$. We note, that such a ring satisfies Hypothesis in [CPT] and is a discrete valuation ring. We write $\mathfrak{m}$ for its maximal ideal pR and start with a crucial (certainly well-known) observation:

Lemma 6.42 *We have an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow R \xrightarrow{1-\varphi} R \longrightarrow 0,$$

where φ *denotes the Frobenius endomorphism of* R.

Proof. By Artin-Schreier theory and the p-closeness of κ we have the obvious exact sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \kappa \xrightarrow{1-\varphi} \kappa \longrightarrow 0,$$

because $(1-\varphi)(x) = x - x^p$. Inductively, one shows that, for all $n \geq 1$, also

$$0 \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p \longrightarrow R/p^nR \xrightarrow{1-\varphi} R/p^nR \longrightarrow 0$$

is exact. Thus for any given $r = (r_n)_n \in \operatorname{proj\,lim}_n R/p^n R = R$ the sets $S_n := \{s_n \in R/p^n R | (1-\varphi)(s_n) = r_n\}$ are finite and form an inverse system, whence $S := \operatorname{proj\,lim}_n S_n$ is non-empty and any $s \in S$ satisfies $(1-\varphi)(s) = r$ by construction. $\square$

Let G be a finite p-group. The split exact sequence

$$1 \longrightarrow I(R[G]) \longrightarrow R[G] \longrightarrow R \longrightarrow 1$$

induces isomorphisms

$$K_1(R[G]) \cong K_1(R[G], I(R[G])) \oplus R^\times$$

and

$$R[\mathcal{C}_G] \cong \phi(I(R[G])) \oplus R,$$

where $\phi : R[G] \twoheadrightarrow R[\mathcal{C}_G]$ denotes the canonical map, $\mathcal{C}_G$ denoting the conjugacy classes of G. By $\operatorname{Log}(1-x)$ we denote the logarithm series. Then the map $\frac{1}{p}\mathcal{L} = \phi(\frac{1}{p}(p-\Psi)(\operatorname{Log}(1-x)))$ defined on page 13 of [CPT] induces by [CPT, Cor. 3.3, Thm. 3.17] and the lemma above a surjective map

$$\Gamma_{I(R[G])} : K_1(R[G], I(R[G])) \twoheadrightarrow \phi(I(R[G])).$$

We use a generalization of Theorem 2.8 in [O 1] (for ideals contained in the Jacobson radical) in order to show, that this map is actually a group homomorphism, which together with the surjective homomorphism

$$\Gamma_R : R^\times \twoheadrightarrow R,$$

which sends $x \in 1 + \mathfrak{m}$ to $\frac{1}{p}(p-\varphi)\operatorname{Log}(x)$ and $x \in \kappa^\times$ to zero (note that $\operatorname{Log}(1+pR) = pR$ and that $p - \varphi$ is an isomorphism of R), defines a surjective group homomorphism

$$\Gamma_{R[G]} = \Gamma_{I(R[G])} \oplus \Gamma_R : K_1(R[G]) \twoheadrightarrow R[\mathcal{C}_G],$$

which factorizes over

$$\Gamma_{\operatorname{Wh}(R[G])} : \operatorname{Wh}(R[G]) := K_1(R[G])/(G^{ab} \times \mu_R) \twoheadrightarrow R[\mathcal{C}_G].$$

Setting

$$SK_1'(R[G]) := \ker(\Gamma_{\operatorname{Wh}(R[G])})$$

we obtain the following exact sequence

$$1 \longrightarrow SK_1'(R[G]) \longrightarrow \operatorname{Wh}(R[G]) \xrightarrow{\Gamma_{\operatorname{Wh}(R[G])}} R[\mathcal{C}_G] \longrightarrow 1. \tag{6.15}$$

The relation between $SK_1'(R[G])$ and the original $SK_1(R[G])$ will be cleared below.

Our goal is to prove the following

Theorem 6.43 *Let G be a p-group. Then $SK_1'(R[G]) = 1$. In particular,*

$$\mathrm{Wh}(R[G]) \cong R[\mathcal{C}_G]$$

is torsion free and a Hausdorff topological group (the second group being a pseudocompact R-module).

Proof. The proof proceeds by induction on the order of G. If G is trivial, it is well-known that the $SK_1'(R) = 1$, because the kernel of Γ_R is just μ_R. Now assume G to be non-trivial. Then there exists a central element $z \in G$ of order p. We set $\bar{G} := G/ < z >$ and write $\alpha : G \twoheadrightarrow \bar{G}$ for the canonical projection. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(\mathrm{Wh}(\alpha)) \arrow[r] \arrow[d] & (1-z)R[\mathcal{C}_G] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & SK_1'(R[G]) \arrow[r] \arrow[d, "SK_1'(\alpha)"] & \mathrm{Wh}(R[G]) \arrow[r, "\Gamma_{\mathrm{Wh}}"] \arrow[d, "\mathrm{Wh}(\alpha)"] & R[\mathcal{C}_G] \arrow[r] \arrow[d, "H_0(\alpha)"] & 0 \\
0 \arrow[r] & SK_1'(R[\bar{G}]) \arrow[r] & \mathrm{Wh}(R[\bar{G}]) \arrow[r, "\Gamma_{\mathrm{Wh}}"] & R[\mathcal{C}_{\bar{G}}] \arrow[r] & 0,
\end{tikzcd}$$

in which also the right column is exact by [CPT, Lem. 3.9].

Let I_z denotes the ideal $(1-z)R[G]$. An immediate generalization of [O 1, Prop. 6.4] to our setting tells us that the logarithm induces an exact sequence

$$1 \longrightarrow < z > \longrightarrow K_1(R[G], I_z) \xrightarrow{\log} \mathrm{H}_0(G, I_z) \longrightarrow 0$$

(note that τ in (loc. cit.) has to be replaced by the trivial map, because $1-\varphi$ in Lem. 6.3 is surjective on κ; also [O 1, Thm. 2.8] (for ideals contained in the Jacobson radical) needed for the proof generalizes immediately to our setting, because by [Lam, (20.4)] $S := M_n(R[G])$ is also semilocal and satisfies $J(S)^N \subseteq pS$ for N sufficiently big).

Thus, letting B denote the kernel of the third arrow in the upper line, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d] & K_1(R[G], I_z)/ < z > \arrow[r] \arrow[d, "\log", "\cong"'] & \ker(\mathrm{Wh}(\alpha)) \arrow[r] \arrow[d, "\Gamma_{R[G]}"] & 0 \\
0 \arrow[r] & \mathrm{H}_0(G, (1-z)R[\Omega]) \arrow[r] & \mathrm{H}_0(G, (1-z)R[G]) \arrow[r] & (1-z)R[\mathcal{C}_G] \arrow[r] & 0.
\end{tikzcd}$$

Here, following [O 1] we write

$$\Omega = \{g \in G |\ g \text{ is conjugate to } zg\}.$$

By the Snake-lemma we see that

$$
\begin{aligned}
\ker(SK_1'(\alpha)) &= \ker(\mathrm{Wh}(\alpha)) \cap SK_1'(R[G]) \\
&= \mathrm{H}_0(G, (1-z)R[\Omega])/\log B \\
&= R[\Omega]/\psi^{-1}(\log B),
\end{aligned}
$$

where $\psi : R[\Omega] \twoheadrightarrow \mathrm{H}_0(G, (1-z)R[\Omega])$ is induced by multiplication with $(1-z)$.

We note that our last term in the above equation corresponds to D/C in the proof of [O 1, Thm. 7.1]. Hence, copying literally the same arguments and noting again that $1-\varphi$ is surjective on R, we see from (c) on p. 176 in (loc. cit.) that $C = \psi^{-1}(\log B) = R[\Omega]$. In other words, $\ker(SK_1'(\alpha))$ is trivial. Since, by our induction hypothesis also $SK_1'(R[\bar{G}])$ vanishes, the theorem is proved. □

Let L^0 be the unique unramified algebraic extension of $\mathbb{Q}_p$ with residue field κ.

Corollary 6.44 *For any open subgroup $\Delta \subseteq Gal(L^0/\mathbb{Q}_p)$, there is an exact sequence*

$$1 \longrightarrow SK_1(R^\Delta[G]) \longrightarrow K_1(R^\Delta[G]) \longrightarrow K_1(R[G])^\Delta \longrightarrow 1,$$

and isomorphisms

$$\mathrm{H}^1(\Delta, \mu_R) \cong \mathrm{H}^1(\Delta, K_1(R[G])) \ \textit{ and } \ (\mu_R)_\Delta \cong K_1(R[G])_\Delta$$

of continuous cochain cohomology groups and coinvariants, respectively.

Proof. Taking Δ-invariants of the exact sequence (of topological Hausdorff modules, cp. Corollary 6.45 for K_1)

$$1 \longrightarrow G^{ab} \times \mu_R \longrightarrow K_1(R[G]) \xrightarrow{\Gamma_{R[G]}} R[\mathcal{C}_G] \longrightarrow 0$$

and noting the Galois invariance of $\Gamma_{R[G]}$ (if we choose the arithmetic Frobenius in its definition) we obtain the following commutative diagram with exact rows (the first of which is the standard exact sequence as proved in [O 1])

$$
\begin{tikzcd}
1 \arrow[r] & G^{ab} \times \mu_{R^\Delta} \times SK_1(R^\Delta[G]) \arrow[r] \arrow[d] & K_1(R^\Delta[G]) \arrow[r] \arrow[d] & R^\Delta[\mathcal{C}_G] \arrow[r] \arrow[d, "\cong"] & G^{ab} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G^{ab} \times {\mu_R}^\Delta \arrow[r] & K_1(R[G])^\Delta \arrow[r] & R[\mathcal{C}_G]^\Delta \arrow[r] & \mathrm{H}^1(\Delta, \mu_R) \times G^{ab} \arrow[r] & \cdots
\end{tikzcd}
$$

from which the claim follows using [NSW, Prop. 1.7.7]. Note that

$$
\begin{aligned}
\mathrm{H}^1(\Delta, R[\mathcal{C}_G]) &\cong \operatorname*{proj\,lim}_n \mathrm{H}^1(\Delta, R/p^nR)^{\#\mathcal{C}_G} \\
&\cong \operatorname*{proj\,lim}_n (R/p^nR)_\Delta^{\#\mathcal{C}_G} = 0
\end{aligned}
$$

by the straight forward generalization of [NSW, Thm. 2.7.5] to pseudocompact modules, again [NSW, prop. 1.7.7] and Lemma 6.42. Alternatively, we may replace the long exact cohomology sequence above by the kernel/cokernel exact sequence arising from the Snake-lemma associated to multiplication by $1-\tau$ for any topological generator τ of Δ. □

Corollary 6.45 $SK_1(R[G]) = 1$, *i.e.,* $K_1(R[G]) \cong \mathrm{Det}(R[G]^\times)$. *In particular,*

$$K_1(R[G]) \cong \operatorname{proj}\lim_n K_1(R/p^n R[G]).$$

The last claim follows from the first one using [CPT, Prop. 1.3]. For the proof of the first claim of the corollary consider the following diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & SK_1(R[G]) \arrow[d] & & \\
1 \arrow[r] & G^{ab}\times\mu_R \arrow[r] & K_1(R[G]) \arrow[r, "\Gamma_{R[G]}"] \arrow[d, "\mathrm{Det}"] & R[\mathcal{C}_G] \arrow[r] \arrow[d, hook, "\mathrm{Tr}"] & 1, \\
 & & \mathrm{Hom}(R_G,\mathbb{C}_p^\times) \arrow[r, "\Gamma_{\mathrm{Hom}}"] & \mathrm{Hom}(R_G,\mathbb{C}_p) &
\end{tikzcd} \tag{6.16}$$

where Γ_{Hom} is defined as follows: Choose any continuous lift $F : \mathbb{C}_p \to \mathbb{C}_p$ of the absolute Frobenius automorphism and denote by $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ the usual p-adic logarithm. We write ψ^p and ψ_p for the pth Adams operator, which is characterized by

$$tr(g, \psi^p\rho) = tr(g^p, \rho) \text{ for all } g \in G,$$

and its adjoint, respectively. Also the rule $f(\rho) \mapsto F(f(\rho^{F^{-1}}))$ induces an operator $\tilde{F}$ on $\mathrm{Hom}(R_G,\mathbb{C}_p^\times)$ and $\mathrm{Hom}(R_G,\mathbb{C}_p)$, which commutes obviously with ψ_p. Now we set

$$\Gamma_{\mathrm{Hom}}(f) = \frac{1}{p}(p - \tilde{F}\psi_p)(\log\circ f).$$

Finally, Tr, the additive analog of Det, is induced by

$$\mathrm{Tr}(\lambda)(\rho) = \mathrm{tr}(\rho(\lambda)),$$

where $\rho : R[G] \to M_n(\mathbb{C}_p)$ is the linear extension of $\rho : G \to GL_n(\mathbb{C}_p)$ keeping the same notation. One easily checks that

$$\mathrm{Det}(F(\lambda))(\rho) = \tilde{F}(\mathrm{Det}(\lambda))(\rho) = F\mathrm{Det}(\lambda)(\rho^{F^{-1}}) \tag{6.17}$$

and

$$\mathrm{Tr}(\Psi(\lambda))(\rho) = \tilde{F}(\mathrm{Tr}(\psi_p\lambda))(\rho) = F\mathrm{Tr}(\lambda)(\psi_p\rho^{F^{-1}}) \tag{6.18}$$

The Corollary will follow immediately from the following

Lemma 6.46 *The diagram* (6.16) *commutes and* Det *restricted to* $G^{ab} \times \mu_R$ *is injective.*

Proof. The injectivity being well-known we only check the commutativity similarly as in [Sn, Prop. 4.3.25]. Since $K_1(R[G], I(R[G]))$ and $K_1(R)$ generate $K_1(R[G])$ as has been observed above, it suffices to check this individually on each direct summand. The case of $K_1(R)$ being similar but easier, we assume that a belongs to $K_1(R[G], I(R[G]))$ and calculate using the definitions, (6.18), the continuity of ρ, the fact that log transforms det into tr and (6.17):

$$\begin{aligned}
(\mathrm{Tr} \circ \Gamma(a))(\rho) &= \mathrm{Tr}(\frac{1}{p}(p - \Psi) \log(a))(\rho) \\
&= \mathrm{Tr}(\log(a))(\rho) - \frac{1}{p} F \mathrm{Tr}(\log(a)(\psi_p \rho^{F^{-1}}) \\
&= \mathrm{tr}(\log \rho(a)) - \frac{1}{p} F \mathrm{tr} \log(\psi_p \rho^{F^{-1}}(a)) \\
&= \log(\det(\rho(a))) - \frac{1}{p} F \log(\det(\psi_p \rho^{F^{-1}}(a)) \\
&= \frac{1}{p}\{p \log \mathrm{Det}(a)(\rho) - F \log \mathrm{Det}(a)(\psi_p \rho^{F^{-1}})\} \\
&= \frac{1}{p}(p - \tilde{F}\psi_p) \log \mathrm{Det}(a)(\rho) \\
&= (\Gamma_{\mathrm{Hom}} \circ \mathrm{Det}(a))(\rho).
\end{aligned}$$

□

Remark 6.47 *The case* $p = 2$ *should be treated separately, since* $\mathrm{Log}(R^\times) = \mathrm{Log}(1+2R) = 4R + 2(1-\varphi)R \cong 2R$*, so that we have to replace* $R[\mathcal{C}_G]$ *by* $2R[\mathcal{C}_G]$ *in the exact sequence* (6.15)*, or, if we keep* $R[\mathcal{C}_G]$ *in* (6.15)*, then we get a finite cokernel, which we denote by* $\mu = \langle -1 \rangle$*. Doing required corrections we can proof Corollaries 6.44 and 6.45 also in this case. Note, that the finite cokernel* μ *also appears in the first row of the commutative diagram in the proof of Corollary 6.44 (see [O 1, Thm. 6.6]).*

Lemma 6.48 *The statement (i) of Theorem 6.30 is true also for* R *being the ring of Witt vectors of a* p*-closed algebraic extension* κ *of* $\mathbb{F}_p$*.*

Proof. The injectivity is obvious $SK_1(R[G])$ being trivial (see Corollary 6.45) and the surjectivity follows from the generalized Proposition 15 in [O 2]. □

Corollary 6.45 and the lemma above imply

Corollary 6.49 $SK_1(S[G]) = 1$ *for any totally ramified integral extension* S *of* R*, where* R *is the ring of Witt vectors of a* p*-closed algebraic extension* κ *of* $\mathbb{F}_p$*.*

Finally, we want to generalize Corollaries 6.34 and 6.49 to the case of an arbitrary finite group G. For this we need

Theorem 6.50 *Fix a prime p and a Dedekind domain R with field of fractions K, such that $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$. For any finite group G, let $g_1, \dots, g_k$ be K-conjugacy class representatives for elements in G of order $n_i = \operatorname{ord}(g_i)$ prime to p, and set*

$$N_i = N_G^K(g_i) = \{x \in G : \; x g_i x^{-1} = g_i^a, \text{ some } a \in Gal(K(\zeta_{n_i})/K)\}$$

and $Z_i = \mathcal{C}_G(g_i)$. Then $SK_1(R[G])$ is computable by induction from p-elementary subgroups of G and there is an isomorphism

$$SK_1(R[G]) \cong \bigoplus_{i=1}^{k} \mathrm{H}_0\big(N_i/Z_i; \varinjlim_{H \in \mathcal{P}(Z_i)} SK_1(R[\zeta_{n_i}][H])\big),$$

where $\mathcal{P}(Z_i)$ is the set of p-subgroups of Z_i and $R[\zeta_{n_i}]$ denotes the integral closure of R in $K(\zeta_{n_i})$.

Proof. See [O 1, Thm. 11.8 and Thm. 12.5] □

Corollary 6.51 *Let G be an arbitrary finite group. Let S be $\mathcal{O}_M$, with M as in Corollary 6.34, but of finite absolute ramification index over $\mathbb{Q}_p$, or S be a finite totally ramified extension of R, where R is the ring of Witt vectors of a p-closed algebraic extension κ of $\mathbb{F}_p$, then $SK_1(S[G]) = 1$.*

Proof. Use Corollaries 6.34, 6.49 and Theorem 6.50. Note also, that $R[\zeta_{n_i}]$ is a finite unramified extensions of R. □

End of the SK_1-part.

For a finite group G and S being the ring of integers of either an arbitrary tamely ramified extension L of $\mathbb{Q}_p$ (type 1) or the completion of a tamely ramified extension L, whose residue field is p-closed (type 2), both types having finite absolute ramification index over $\mathbb{Q}_p$. Let $R = S^{\Delta}$ be the fixed ring of S, where Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ containing the inertia group. We write K and C for the kernel and cokernel of the map

$$i_* : SK_1(R[G]) \to SK_1(S[G])^{\Delta},$$

respectively. Note that they are finite p-primary abelian groups. By $K_1(R[G])_{\mathbb{Q}}$ we denote rational K-groups $K_1(R[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Snake-lemma, Theorem 6.28 and the SK_1-part imply immediately

Theorem 6.52 *If L is of type 1, then we denote by L^0 the maximal unramified extension of L^{Δ} contained in L and let p^n be the p-part of $[L^0 : L^{\Delta}]$ $(0 \le n \le \infty)$. Then for both types:*

1. *The following sequence is exact*

$$1 \longrightarrow K \longrightarrow K_1(R[G]) \xrightarrow{i_*} K_1(S[G])^{\Delta} \longrightarrow C \longrightarrow 1$$

and induces

$$K_1(R[G])_{\mathbb{Q}} \cong K_1(S[G])^{\Delta}_{\mathbb{Q}}.$$

2. If S is of type 1 and $n = 0$, then

$$K = 1 \quad \text{and} \quad C = 1.$$

3. If S is either of type 2 or of type 1 with $n = \infty$, then we have

$$K \cong SK_1(R[G]) \quad \text{and} \quad C = 1.$$

4. Let G be a p-group and S be of type 1 with $0 < n < \infty$, then

$$K \cong SK_1(R[G])[p^n] \quad \text{and} \quad C \cong SK_1(R[G])/p^n.$$

6.4 The case of residue class fields

Let λ be an arbitrary (not necessary finite) Galois extension of $\mathbb{F}_p$ and G be a finite group. Let ϕ denote the Frobenius automorphism on λ, which takes $x \in \lambda$ to x^p, then $Gal(\lambda/\mathbb{F}_p) = \overline{\langle\phi\rangle}$. Moreover, if $\mathbb{F}_{p^n} \subset \lambda$, then $\mathbb{F}_{p^n} = \lambda^{\overline{\langle\phi^n\rangle}}$. We fix such an n, set $\kappa = \mathbb{F}_{p^n}$ and $\Delta = \overline{\langle\phi^n\rangle}$. We are going to prove the following

Theorem 6.53 *With the notation as above, we have an exact sequence*

$$1 \longrightarrow K \longrightarrow K_1(\kappa[G]) \xrightarrow{i_*} K_1(\lambda[G])^{\Delta} \longrightarrow C \longrightarrow 1,$$

which induces

$$K_1(\kappa[G])_{\mathbb{Q}} \cong K_1(\lambda[G])^{\Delta}_{\mathbb{Q}},$$

where K and C are as in Theorem 6.52 for R and S the unique unramified extensions of $\mathbb{Z}_p$ lifting κ and λ, respectively. As usual ϕ (resp. ϕ^n) acts on the K_1-groups coefficientwise.

Proof.

From [SV] we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\mathcal{C}_G] \longrightarrow K_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{F}_p[G]) \longrightarrow 1, \tag{6.19}$$

where $\mathbb{Z}_p[\mathcal{C}_G]$ is a finitely generated free $\mathbb{Z}_p$-module over the set of conjugacy classes in G.

With the same argument as in [SV], we can obtain (6.19) for finite unramified extensions of $\mathbb{Q}_p$ and their rings of integers, and, since K_1 commutes with the direct limit, also for infinite unramified extensions. Using this fact we get the following commutative diagram with exact rows

$$
\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & K \arrow[d] & & \\
0 \arrow[r] & R[\mathcal{C}_G] \arrow[r] \arrow[d] & K_1(R[G]) \arrow[r] \arrow[d] & K_1(\kappa[G]) \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & S[\mathcal{C}_G]^\Delta \arrow[r] & K_1(S[G])^\Delta \arrow[r] \arrow[d] & K_1(\lambda[G])^\Delta \arrow[r] & \mathrm{H}^1(\Delta, S[\mathcal{C}_G]), \\
 & & K \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}
$$

where the bottom row is the part of the long exact sequence in the group cohomology associated to the short exact sequence of Δ-modules, and Δ acts coefficientwise.

The left hand side vertical map is an isomorphism, as $R[\mathcal{C}_G]$ and $S[\mathcal{C}_G]$ are finitely generated free R- and S-modules respectively. The middle column is exact by Theorem 6.52. Thus, to prove the theorem, it is enough to show, that $\mathrm{H}^1(\Delta, S[\mathcal{C}_G]) = 1$.

From [FS, Prop. 2] we know, that $H^1(Gal(M_1/M_2), \mathcal{O}_{M_1}) = 1$ for every finite unramified extension M_1 of $\mathbb{Q}_p$. Since Δ can be written as the inverse limit of finite groups corresponding to the finite unramified subextensions of S, and $S[\mathcal{C}_G]$ as a Δ-module is isomorphic to the direct sum of copies of S, we get the statement above by using standard properties of group cohomology. □

7 Appendix A. Finiteness of projective resolutions for the $\mathbb{Z}_p[G]$- and $\mathbb{Q}_p[G]$-modules

Theorem 7.1 (Mashke) *The group ring $k[G]$ of a finite group G over a field k is semisimple if $char(k)$ does not divide the order of G.*

Theorem 7.2 *If R is a semisimple ring, then every R-module is projective.*

Proof. See [W 1, Thm. 4.2.2 on p. 95]. □

Corollary 7.3 *Let $\mathbb{Q}_p[G]$ be a group ring of a finite group G, then every $\mathbb{Q}_p[G]$-module is projective.*

For an arbitrary $\mathbb{Z}_p[G]$-module it is not true, that it admits a finite projective resolution. However, we show, that some special class of $\mathbb{Z}_p[G]$-modules has such a resolution.

Let $\mathbb{Q}_p \subseteq K \subseteq L$ and $[L : \mathbb{Q}_p] < \infty$. We denote by $G = G_K/G_L = Gal(L/K)$ the Galois group of L/K. Let M be a $\mathbb{Z}_p$-module endowed with a continuous G_K-action (i.e. a $\mathbb{Z}_p[G_K]$-module). We introduce $\mathrm{Ind}_{L/K}M = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_L]} M$. The goal is to prove, that this is a free $\mathbb{Z}_p[G]$-module, if M is a free $\mathbb{Z}_p$-module.

The action of G on $\mathrm{Ind}_{L/K}M$ is given in the standard way by

$$\overline{g}(\sigma \otimes m) = \sigma g^{-1} \otimes g(m), \forall \overline{g} \in G,$$

where $g \in G_K$ is some lift of $\overline{g}$.

Proposition 7.4 *There is a canonical isomorphism of G-modules*

$$\varphi : \mathrm{Ind}_{L/K}M \xrightarrow{\cong} \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} M, \quad \sigma \otimes m \mapsto \overline{\sigma} \otimes \sigma(m),$$

where G acts on $\mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} M$ by

$$\overline{g}(\overline{\sigma} \otimes m) = \overline{\sigma} \cdot (\overline{g})^{-1} \otimes m, \forall\, \overline{\sigma}, \overline{g} \in G.$$

Proof. φ is a homomorphism by the definition. The inverse homomorphism ψ is given by $\overline{\sigma} \otimes m \mapsto \sigma \otimes \sigma^{-1}(m)$, where $\sigma \in G_K$ is again some lift of $\overline{\sigma}$. Two lifts of $\overline{\sigma}$ differs by an element $t \in G_L$, thus

$$\psi(\varphi(\sigma \otimes m)) = \psi(\overline{\sigma} \otimes \sigma(m)) = \tilde{\sigma} \otimes \tilde{\sigma}^{-1}\sigma(m) = \tilde{\sigma} \otimes t^{-1}\sigma^{-1}\sigma(m) = \sigma t t^{-1} \otimes m = \sigma \otimes m,$$

$$\varphi(\psi(\overline{\sigma} \otimes m)) = \varphi(\sigma \otimes \sigma^{-1}(m)) = \overline{\sigma} \otimes \sigma\sigma^{-1}(m) = \overline{\sigma} \otimes m.$$

It remains to show, that φ is a G-isomorphism, i.e. $\overline{g}(\varphi(\sigma \otimes m)) = \varphi(\overline{g}(\sigma \otimes m)), \forall \overline{g} \in G$:

$$\varphi(\overline{g}(\sigma \otimes m)) = \varphi(\sigma g^{-1} \otimes g(m)) = \overline{\sigma} \cdot (\overline{g})^{-1} \otimes \sigma g^{-1} g(m) = \overline{\sigma} \cdot (\overline{g})^{-1} \otimes \sigma(m),$$

$$\overline{g}(\varphi(\sigma \otimes m)) = \overline{g}(\overline{\sigma} \otimes \sigma(m)) = \overline{\sigma} \cdot (\overline{g})^{-1} \otimes \sigma(m).$$

□

Corollary 7.5 *If M is a free $\mathbb{Z}_p$-module, then $\mathrm{Ind}_{L/K}M$ is a free $\mathbb{Z}_p[G]$-module.*

Remark 7.6 *The isomorphism φ is also a G_K-isomorphism, where the action of G_K on both sides is $\mathbb{Z}_p[G]$-linear.*

Proof. We define the action of G_K on both sides as follows:

- G_K acts on $\mathrm{Ind}_{L/K}M$ by $g(\sigma \otimes m) = g\sigma \otimes m,\ \forall g, \sigma \in G_K,\ m \in M$.
- G_K acts on $\mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} M$ by $g(\overline{\sigma} \otimes m) = \overline{g\sigma} \otimes g(m),\ \forall g \in G_K,\ \overline{\sigma} \in G,\ m \in M$.

Then, the action is $\mathbb{Z}_p[G]$-linear (notation as above, $\overline{\tau} \in G$):

$$g(\overline{\tau}(\sigma \otimes m)) = g(\sigma\tau^{-1} \otimes \tau(m)) = g\sigma\tau^{-1} \otimes \tau(m) \stackrel{!}{=} \overline{\tau}(g\sigma \otimes m) = \overline{\tau}(g(\sigma \otimes m)),$$

$$g(\overline{\tau}(\overline{\sigma} \otimes m)) = g(\overline{\sigma} \cdot (\overline{\tau})^{-1} \otimes m) = \overline{g\sigma} \cdot (\overline{\tau})^{-1} \otimes g(m) \stackrel{!}{=} \overline{\tau}(\overline{g\sigma} \otimes g(m)) = \overline{\tau}(g(\overline{\sigma} \otimes m));$$

and φ is a G_K-isomorphism:

$$\varphi(g(\sigma \otimes m)) = \varphi(g\sigma \otimes m) = \overline{g\sigma} \otimes g\sigma(m) \stackrel{!}{=} g(\overline{\sigma} \otimes \sigma(m)) = g(\varphi(\sigma \otimes m)).$$

□

Remark 7.7 $\mathrm{Ind}_{L/K}$ *is an exact functor, since $\mathbb{Z}_p[G_K]$ is a f.g. free $\mathbb{Z}_p[G_L]$-module, hence flat.*

Theorem 7.8 *Let R be a ring and $0 \to M \to M' \to M'' \to 0$ be an exact sequence of R-modules. If two of them admit finite projective resolutions, then the third also does.*

Proof. See [Lang, Ch. XX, Thm. 3.9]. □

8 Appendix B. Non-commutative determinants and K-groups

This appendix consists on the facts taken from [FK, §1] and [V, §1].

8.1 Reviews of K_0 and K_1

Let R be a ring. By an R-module we mean a left R-module.

Definition 8.1 *$K_0(R)$ is an abelian group, whose group law we denote additively, defined by the following generators and relations.*

- *Generators: $[P]$, where P is a f.g. (finitely generated) projective R-module.*
- *Relations:*
  1. *If $P \cong Q$, then $[P] = [Q]$.*
  2. *$[P \oplus Q] = [P] + [Q]$.*

As is seen easily, elements of $K_0(R)$ are expressed as $[P] - [Q]$ for f.g. projective R-modules P and Q. We have $[P] - [Q] = [P'] - [Q']$ if and only if there is a f.g. projective R-module T such that $P \oplus Q' \oplus T \cong P' \oplus Q \oplus T$.

Definition 8.2 *$K_1(R)$ is an abelian group, whose group law we denote multiplicatively, defined by the following generators and relations.*

- *Generators: $[P, \alpha]$, where P is a f.g. projective R-module and α is an automorphism of P.*
- *Relations:*
  1. *If there is an isomorphism $P \cong Q$ via which α corresponds to β, then $[P, \alpha] = [Q, \beta]$.*
  2. *$[P \oplus Q, \alpha \oplus \beta] = [P, \alpha] \cdot [Q, \beta]$.*
  3. *$[P, \alpha \circ \beta] = [P, \alpha] \cdot [P, \beta]$ for $\alpha, \beta \in \mathrm{Aut}(P)$.*

We have a canonical homomorphism $GL_n(R) \to K_1(R)$ defined by sending $\alpha \in GL_n(R)$ to $[R^n, \alpha]$. Here R^n is regarded as a set of row vectors and α acts on it from the right as an automorphism of the (left) R-module R^n.

Using the inclusion maps $GL_n(R) \hookrightarrow GL_{n+1}(R)$; $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, let

$$GL_\infty(R) = \bigcup_n GL_n(R).$$

Then the canonical homomorphisms $GL_n(R) \to K_1(R)$ induce an isomorphism

$$GL_\infty(R)/[GL_\infty(R), GL_\infty(R)] \xrightarrow{\cong} K_1(R), \tag{8.1}$$

where $[-,-]$ denote the commutator subgroup, for which we have

$$[GL_\infty(R), GL_\infty(R)] = E_\infty(R),$$

and $E_\infty(R)$ is the group of elementary matrices.

If R is commutative, the determinant map $\det : GL_n(R) \to R^\times$ induce the determinant map

$$\det : K_1(R) \to R^\times$$

via the isomorphism (8.1).

8.2 Non-commutative determinants

Let $\mathcal{P}(R)$ denote the category of f.g. projective R-modules and $(\mathcal{P}(R), is)$ its subcategory of isomorphisms, i.e. with the same objects, but whose morphisms are precisely the isomorphisms. Then there exists a category $\mathcal{C}_R$ and a functor

$$\mathbf{d}_R : (\mathcal{P}(R), is) \to \mathcal{C}_R,$$

which satisfies the following properties:

1. $\mathcal{C}_R$ has an associative and commutative product structure $(M, N) \to M \cdot N$ or written just MN with unit object $\mathbf{d}_R(0)$ and inverses. All objects are of the form $\mathbf{d}_R(P)\mathbf{d}_R(Q)^{-1}$ for some $P, Q \in \mathcal{P}(R)$.

2. all morphisms of $\mathcal{C}_R$ are isomorphisms, $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic if and only if their classes in $K_0(R)$ coincide. There is an identification of groups $\mathrm{Aut}(\mathbf{d}_R(0)) = K_1(R)$ and $\mathrm{Mor}(M, N)$ is either empty or an $K_1(R)$-torsor, where $\alpha : \mathbf{d}_R(0) \to \mathbf{d}_R(0) \in K_1(R)$ acts on $\phi : M \to N$ as $\alpha \cdot \phi : M = \mathbf{d}_R(0) \cdot M \to \mathbf{d}_R(0) \cdot N = N$.

3. $\mathbf{d}_R$ preserves the "product" structures: $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

We define the category $\mathcal{C}_R$ as follows. An object of $\mathcal{C}_R$ is a pair (P, Q) of f.g. projective R-modules and morphisms of $\mathcal{C}_R$ are as follows.

There exists a morphism $(P, Q) \to (P', Q')$ if and only if $[P]-[Q] = [P']-[Q']$ in $K_0(R)$. If $[P] - [Q] = [P'] - [Q']$, there is a f.g. projective R-module T such that $P \oplus Q' \oplus T \cong P' \oplus Q \oplus T$. Let

$$G_T = \mathrm{Aut}(P' \oplus Q \oplus T), \quad I_T = \mathrm{Isom}(P \oplus Q' \oplus T, P' \oplus Q \oplus T).$$

Then I_T is a G_T-torsor (that is, I_T is a non-empty set endowed with an action of G_T and for each $x, y \in I_T$, there exists an unique $g \in G_T$ such that $y = gx$). We define the set $\mathrm{Mor}((P, Q), (P', Q'))$ of morphisms $(P, Q) \to (P', Q')$ in $\mathcal{C}_R$ by

$$\mathrm{Mor}((P, Q), (P', Q')) = K_1(R) \times^{G_T} I_T.$$

Here $K_1(R) \times^{G_T} I_T$ denotes the quotient of $K_1(R) \times I_T$ by the action of G_T given by $(x, y) \mapsto (x\bar{g}, g^{-1}y)$ ($x \in K_1(R), y \in I_T, g \in G_T$ and $\bar{g}$ denotes the image of g in $K_1(R)$). It is the $K_1(R)$-torsor obtained from the G_T-torsor I_T

by the canonical homomorphism $G_T \to K_1(R)$. This set of morphisms does not depend on the choice of T (see [FK]). By definition any morphism in $\mathcal{C}_R$ is an isomorphism and

- For an object (P, Q) of $\mathcal{C}_R$ we denote the object (Q, P) of $\mathcal{C}_R$ by $(P, Q)^{-1}$ and call it the inverse of (P, Q) (with respect to the product structure).
- For objects (P, Q) and (P', Q') of $\mathcal{C}_R$ we denote the object $(P \oplus P', Q \oplus Q')$ of $\mathcal{C}_R$ by $(P, Q) \cdot (P', Q')$ and call it the product of (P, Q) and (P', Q').
- For a f.g. projective R-module P we denote the object $(P, 0)$ of $\mathcal{C}_R$ by $\mathbf{d}_R(P)$. Hence an object (P, Q) of $\mathcal{C}_R$ is described as

$$(P, Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)^{-1}.$$

Let R' be another ring and let Y be a f.g projective R'-module endowed with a structure of a right R-module such that the actions of R and R' on Y commute. Then we have a functor

$$Y \otimes_R : \mathcal{C}_R \to \mathcal{C}_{R'}, \quad (P, Q) \mapsto (Y \otimes_R P, Y \otimes_R Q).$$

For example, for a ring homomorphism $R \to R'$ we have a functor $R' \otimes_R : \mathcal{C}_R \to \mathcal{C}_{R'}$, by taking R' as Y.

Also, for example, for a ring homomorphism $R' \to R$ such that R is f.g. and projective as a (left) R'-module, we have a functor $R \otimes_R$ by taking R as Y, which is the functor to regard a R-module as a R'-module. The induced homomorphism $\mathrm{Aut}(\mathbf{d}_R(0)) \to \mathrm{Aut}(\mathbf{d}_{R'}(0))$ coincides with the norm homomorphism $K_1(R) \to K_1(R')$.

The functor $\mathbf{d}_R$ can be naturally extended to complexes. Let $\mathrm{C}^p(R)$ be the category of bounded complexes in $\mathcal{P}(R)$ and $(\mathrm{C}^p(R), quasi)$ its subcategory of quasi-isomorphisms. For $C \in \mathrm{C}^p(R)$ we set $C^+ = \oplus_{i \text{ even}} C^i$ and $C^- = \oplus_{i \text{ odd}} C^i$ and define $\mathbf{d}_R(C) := \mathbf{d}_R(C^+)\mathbf{d}_R(C^-)^{-1}$ and thus we obtain a functor

$$\mathbf{d}_R : (\mathrm{C}^p(R), quasi) \to \mathcal{C}_R$$

with the following properties $(C, C', C'' \in \mathrm{C}^p(R))$

4. If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of complexes, then there is a canonical isomorphism

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$$

which we take as an identification.

5. If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism

$$\mathbf{d}_R(0) \to \mathbf{d}_R(C).$$

6. $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$, where $C[r]$ denotes the r^{th} translate of C.

7. The functor $\mathbf{d}_R$ factorizes over the image of $\mathrm{C}^p(R)$ in $\mathrm{D}^p(R)$, the category of perfect complexes (as full triangulated subcategory of the derived category $\mathrm{C}^b(R)$ of the homotopy category of bounded complexes of R-modules), and extends to $(\mathrm{D}^p(R), is)$ (uniquely up to unique isomorphisms).

8. If $C \in \mathrm{D}^p(R)$ has the property, that all cohomology groups $\mathrm{H}^i(C)$ belong again to $\mathrm{D}^p(R)$, then there is a canonical isomorphism

$$\mathbf{d}_R(C) = \prod_i \mathbf{d}_R(\mathrm{H}^i(C))^{(-1)^i}.$$

9. Let R' be another ring and let Y be a f.g projective R'-module endowed with a structure of a right R-module such that the actions of R and R' on Y commute. Then we have a commutative diagram

$$\begin{tikzcd}
(\mathrm{D}^p(R), is) \arrow[r, "\mathbf{d}_R"] \arrow[d, "Y\otimes^{\mathbb{L}}_R -"'] & \mathcal{C}_R \arrow[d, "Y\otimes_R -"] \\
(\mathrm{D}^p(R'), is) \arrow[r, "\mathbf{d}_{R'}"] & \mathcal{C}_{R'}.
\end{tikzcd}$$

10. Let R^{o} be the opposite ring of R. Then the functor $\mathrm{Hom}_R(-, R)$ induces an anti-equivalence between $\mathcal{C}_R$ and $\mathcal{C}_{R^{\mathrm{o}}}$ with quasi-inverse induced by $\mathrm{Hom}_{R^{\mathrm{o}}}(-, R^{\mathrm{o}})$; both functors will be denoted by $-^*$. This extends to a commutative diagram

$$\begin{tikzcd}
(\mathrm{D}^p(R), is) \arrow[r, "\mathbf{d}_R"] \arrow[d, "R\mathrm{Hom}_R(-,R)"'] & \mathcal{C}_R \arrow[d, "-^*"] \\
(\mathrm{D}^p(R^{\mathrm{o}}), is) \arrow[r, "\mathbf{d}_{R^{\mathrm{o}}}"] & \mathcal{C}_{R^{\mathrm{o}}}
\end{tikzcd}$$

and similarly for $R\mathrm{Hom}_{R^{\mathrm{o}}}(-, R^{\mathrm{o}})$.

For the handling of the determinant functor in practice the following considerations are quite important:

Remark 8.3 *We have to distinguish two inverses of a map $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(D)$ with $C, D \in \mathrm{C}^p(R)$. The inverse with respect to the composition will be denoted by $\bar{\phi} : \mathbf{d}_R(D) \to \mathbf{d}_R(C)$. But due to the product structure in $\mathcal{C}_R$ and the identification $\mathbf{d}_R(C) \cdot \mathbf{d}_R(C)^{-1} = \mathbf{d}_R(0)$ the knowledge of ϕ is equivalent to that of*

$$\mathbf{d}_R(0) = \mathbf{d}_R(C) \cdot \mathbf{d}_R(C)^{-1} \xrightarrow{\phi\cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}}} \mathbf{d}_R(D) \cdot \mathbf{d}_R(C)^{-1}$$

or even

$$\phi^{-1} : \mathbf{d}_R(C)^{-1} \to \mathbf{d}_R(D)^{-1}$$

which is by definition $\overline{\mathrm{id}_{\mathbf{d}_R(D)^{-1}} \cdot \phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}}}$ *or in other words* $\phi \cdot \phi^{-1} = \mathrm{id}_{\mathbf{d}_R(0)}$*. In particular,* $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(C)$ *corresponds uniquely to* $\phi{\cdot}\mathrm{id}_{\mathbf{d}_R(C)^{-1}} : \mathbf{d}_R(0) \to \mathbf{d}_R(0)$*. Thus it can and will be considered as an element in* $K_1(R)$*. Note that under this identification the elements in* $K_1(R)$ *assigned to each of* ϕ^{-1} *and* $\bar{\phi}$ *is the inverse to the element assigned to* ϕ*. Furthermore, the following relation between* $\circ$ *and* $\cdot$ *is easily verified: Let* $A \overset{\phi}{\to} B$ *and* $B \overset{\psi}{\to} C$ *be morphisms in* $\mathcal{C}_R$*. Then the composite* $\psi \circ \phi$ *is the same as the product* $\psi \cdot \phi \cdot \mathrm{id}_{B^{-1}}$*.*
Convention: *If* $\phi : \mathbf{d}_R(0) \to A$ *is a morphism and* B *is an object in* $\mathcal{C}_R$*, then we write* $B \xrightarrow{\cdot\phi} B \cdot A$ *for the morphism* $\mathrm{id}_B \cdot \phi$*. In particular, any morphism* $B \xrightarrow{\phi} A$ *can be written as* $B \xrightarrow{\cdot(\mathrm{id}_{B^{-1}} \cdot \phi)} A$ *.*

Remark 8.4 *The determinant of the complex* $C = [P_0 \overset{\phi}{\to} P_1]$ *(in degrees* 0 *and* 1*) with* $P_0 = P_1 = P$ *is by definition* $\mathbf{d}_R(C) = \mathbf{d}_R(0)$ *and is defined even if* ϕ *is not an isomorphism (in contrast to* $\mathbf{d}_R(\phi)$*). But if* ϕ *happens to be an isomorphism, i.e. if* C *is acyclic, then by the property (5) there is also a canonical map* $\mathbf{d}_R(0) \xrightarrow{acyc} \mathbf{d}_R(C)$ *, which is in fact nothing else then*

$$\mathbf{d}_R(0) = \mathbf{d}_R(P_1) \cdot \mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}_R(\phi)^{-1} \cdot \mathrm{id}_{\mathbf{d}_R(P_1)^{-1}}} \mathbf{d}_R(P_0) \cdot \mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

(and which depends in contrast to the first identification on ϕ*). Hence, the composite* $\mathbf{d}_R(0) \xrightarrow{acyc} \mathbf{d}_R(C) = \mathbf{d}_R(0)$ *corresponds to* $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$ *according to the previous remark. In order to distinguish the above identifications between* $\mathbf{d}_R(0)$ *and* $\mathbf{d}_R(C)$ *we also say, that* C *is trivialized by the identity when we refer to* $\mathbf{d}_R(C) = \mathbf{d}_R(0)$ *(or its inverse with respect to composition). For* $\phi = \mathrm{id}_P$ *both identifications agree obviously.*

9 Appendix C. Wedderburn decomposition

Let $K[G]$ be the group ring of a finite group G, where K is a field of characteristic 0, and let $Irr(G)$ be the set of irreducible characters of G over $\overline{K}$ or $\mathbb{C}$. By Theorem 7.1 $K[G]$ is a semisimple ring, thus it admits a Wedderburn decomposition.

Let $\chi \in Irr(G)$, then there is a unique Wedderburn component $A(\chi, K)$ of $K[G]$ such that $\chi(A(\chi, K)) \neq 0$. The identity of the ring $A(\chi, K)$ is the primitive central idempotent $e(\chi, K)$ of $K[G]$ defined as

$$e(\chi, K) = \begin{cases} e(\chi) = \frac{\chi(1)}{|G|} \sum\limits_{g \in G} \chi(g^{-1})g, \text{ if } \chi(g) \in \mathbb{Q}_p, \ \forall g \in G; \\ \sum\limits_{\sigma \in Gal(K(\chi)/K)} e(\sigma \circ \chi), \text{ if } \chi(g) \notin K \text{ for some } g \in G. \end{cases}$$

$A(\chi, K)$ is a central simple algebra, which means that $A(\chi, K) \cong M_{n_\chi}(D_\chi)$, where D_χ is a division ring with

$$Cent(A(\chi, K)) = Cent(D_\chi) = K(\chi) = K(\chi(g), g \in G).$$

Here the field $K(\chi)$ is obtained by adjoining to K the values of χ.

We have surjections:

$$Irr(G) \twoheadrightarrow \{\text{Wedderburn components of } K[G]\}$$

$$Irr(G) \twoheadrightarrow \{\text{primitive central idempotents of } K[G]\}.$$

If $\chi, \chi' \in Irr(G)$, then $A(\chi, K) \cong A(\chi', K) \iff e(\chi, K) = e(\chi', K) \iff \chi' = \sigma \circ \chi$ for some $\sigma \in Gal(K(\chi)/K)$. In this case we say, that the characters χ and χ' are K-equivalent, and we denote by E the set of representatives of the K-equivalence classes of the irreducible characters of G. Then

$$K[G] = \prod_{\chi \in E} A(\chi, K) = \prod_{\chi \in E} e(\chi, K) \cdot K[G] \cong \prod_{\chi \in E} M_{n_\chi}(D_\chi). \tag{9.1}$$

Remark 9.1 $Cent(K[G]) = \prod\limits_{\chi \in E} Cent(A(\chi, K)) = \prod\limits_{\chi \in E} K(\chi)$.

Remark 9.2 *If K is a local field, then the reduced norm map is bijective and*

$$K_1(K[G]) = \prod_{\chi \in E} K_1(A(\chi, K)) = \prod_{\chi \in E} K_1(D_\chi) = \prod_{\chi \in E} K_1(K(\chi)) = \prod_{\chi \in E} (K(\chi))^\times.$$

Remark 9.3 *For all D_χ, there exists a finite extension F_χ of $K(\chi)$ such that $D_\chi \otimes_{K(\chi)} F_\chi \cong M_{d_\chi}(F_\chi)$.*

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