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Thema

Quaternionic Drinfeld modular forms

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Abstract

Drinfeld modular forms were introduced by D. Goss in 1980 for congruence subgroups of $\mathrm{GL}_2(\mathbb{F}_q[T])$. They are a counterpart of classical modular forms in the function field world. In this thesis I study Drinfeld modular forms for inner forms of GL_2 that correspond to unit groups $\Lambda^\star$ of quaternion division algebras over $\mathbb{F}_q(T)$ split at the place $\infty = 1/T$. I show, following work of Teitelbaum for $\mathrm{GL}_2(\mathbb{F}_q[T])$, that these forms have a combinatorial interpretation as certain maps from the edges of the Bruhat-Tits tree $\mathcal{T}$ associated to $\mathrm{PGL}_2(K_\infty)$. Here K_∞ denotes the completion of K at ∞. A major focus of this thesis is on computational aspects: I present an algorithm for computing a fundamental domain for the action of $\Lambda^\star$ on $\mathcal{T}$ with an edge pairing, and describe how to obtain a basis of the space of these forms out of this fundamental domain. On this basis one can compute the Hecke action.

Zusammenfassung

Drinfeldsche Modulformen für Kongruenzuntergruppen von $\mathrm{GL}_2(\mathbb{F}_q[T])$ wurden von D. Goss 1980 eingeführt. Sie sind ein Gegenstück zu klassischen Modulformen für Funktionenkörper. In dieser Arbeit beschäftige ich mich mit Drinfeldschen Modulformen für innere Formen von GL_2 die zu Einheitengruppen $\Lambda^\star$ von Quaternionenalgebren über $\mathbb{F}_q(T)$ korrespondieren, die an der Stelle $\infty = 1/T$ unverzweigt sind. Ich zeige, analog zu Arbeiten von Teitelbaum für $\mathrm{GL}_2(\mathbb{F}_q[T])$, dass diese Formen eine kombinatorische Beschreibung als gewisse Abbildungen von den Kanten des Bruhat-Tits-Baums $\mathcal{T}$ zu $\mathrm{PGL}_2(K_\infty)$ haben, wobei K_∞ die Vervollständigung von K an ∞ ist. Ein Schwerpunkt der Arbeit liegt auf algorithmischen Aspekten: Ich beschreibe einen Algorithmus zum Berechnen eines Fundamentalbereichs für die Wirkung von $\Lambda^\star$ auf $\mathcal{T}$ mit Kantenpaarung, und zeige, wie man daraus eine Basis für den Raum der Modulformen konstruieren kann. Auf dieser Basis kann man Hecke-Operatoren berechnen.

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1 Introduction

Modular forms and more general automorphic forms play an important role in number theory. For example it was recently shown by Khare and Wintenberger in [KW] that all odd, irreducible mod p Galois representations of the absolute Galois group of $\mathbb{Q}$ arise from modular forms. This answers a famous conjecture by Serre from 1975 in the affirmative and demonstrates the importance of modular forms for arithmetic over $\mathbb{Q}$.
Consequently, in explicit number theory the development of algorithms for computing with modular forms over $\mathbb{Q}$ and automorphic forms over more general number fields became an important field of research [Cr, De, GV, GY, Ste].
We are interested in number theory over function fields. Let $\mathbb{F}_q[T]$ be the polynomial ring in one variable T over a finite field $\mathbb{F}_q$, with $q = p^r$ a prime power, and $K = \mathbb{F}_q(T)$ it's quotient field. They play the role of $\mathbb{Z}$ and $\mathbb{Q}$ respectivly. There are two distinct concepts that can be seen as analogs of modular forms in this setting. One are complex valued automorphic forms in the sense of Jacquet-Langlands [JL]. The others are Drinfeld modular forms, characteristic p valued functions on the Drinfeld upper half plane Ω introduced by Goss in 1980 [Go]. The relation between these two concepts was studied by Gekeler and Reveresat in [GR].
This thesis focusses on Drinfeld modular forms. For congruence subgroups inside $\mathrm{GL}_2(\mathbb{F}_q[T])$ there is an extensive theory for such forms: Goss showed in [Go] that, as in the classical case of number fields, the eigenvalues of Drinfeld Hecke eigenforms are algebraic over K. In [Bö] Böckle attached Galois representations to cuspidal eigenforms paralleling the construction of Deligne in the classical case. However, unlike in the case of classical modular forms, one obtains one-dimensional Galois representations. Teitelbaum in [Te2] gave a combinatorial describtion of Drinfeld cusp forms. He showed that the spaces of such forms are isomorphic to spaces of harmonic cocycles. These are certain maps from the edges of the Bruhat-Tits tree $\mathcal{T}$ associated to $\mathrm{PGL}_2(K_\infty)$ which satisfy a harmonicity condition and are equivariant under the action of the congruence subgroup at hand on the tree. Here K_∞ denotes the completion of K at the place $\infty = 1/T$. Such a combinatorial describtion makes Drinfeld cusp forms accessible for explicit computations. Namely, one can compute the Hecke action on quotients of the Bruhat-Tits tree. This approach is pursued in [GN, Te4].
In this thesis we study Drinfeld modular forms for inner forms of GL_2 that correspond to the unit group $\Lambda^\star$ of a quaternion division algebra D split at ∞, or sometimes more general to finite index subgroups of $\Lambda^\star$. In the classical setting the Jacquet-Langlands correspondence relates such modular forms to cusp forms for GL_2 which are newforms for $\Gamma_0(\mathfrak{n})$, where $\mathfrak{n}$ is the discriminant of D. In the case of Drinfeld modular forms, a Jacquet-Langlands correspondence has not been worked out yet. The analytic tools one has for automorphic forms are not available in positive characteristic. However,

it would be a surprise if such a correspondence would not hold in this setting too.
The goals of this thesis are: To develop a theory of Drinfeld modular forms for quaternion algebras; to relate them, analogous to the result by Teitelbaum, to spaces of harmonic cocycles on $\mathcal{T}$ and to make the Hecke action on these spaces accessible for explicit computations. As a prerequisite to this one needs an understanding of the quotient graph $\Lambda^\star\backslash\mathcal{T}$. A main difference to Drinfeld modular forms for congruence subgroups is that in our setting the quotient graph $\Lambda^\star\backslash\mathcal{T}$ is a finite graph.

The organisation is as follows:

In Chapter 2 we study the action of $\Lambda^\star$ on $\mathcal{T}$. We will describe an algorithm for computing a fundamental domain for this action together with an edge pairing. This consists of the following data:

(a) a finite subtree $\mathcal{Y}\subset\mathcal{T}$ whose image $\overline{\mathcal{Y}}$ in $\Lambda^\star\backslash\mathcal{T}$ is a maximal spanning tree, i.e., $\overline{\mathcal{Y}}$ is a tree such that adding any edge of $\Lambda^\star\backslash\mathcal{T}$ to it will create a cycle.

(b) for any edge $\bar{e}$ of $\Lambda^\star\backslash\mathcal{T}\smallsetminus\overline{\mathcal{Y}}$, an edge e of $\mathcal{T}$ connected to $\mathcal{Y}$ that maps to $\bar{e}$ and the glueing datum that connects the loose vertex of this edge via the action of $\Lambda^\star$ to a vertex of $\mathcal{Y}$. Let $\mathcal{Y}'$ be the union of $\mathcal{Y}$ with all such edges e.

This is an analog of a fundamental domain together with a side pairing in the sense of [Vo]. As explained in [Se1, Chapter I.4], this data yields a presentation of the group $\Lambda^\star$ in terms of explicit generators and relations. Moreover, the data provides a reduction algorithm from $\mathcal{T}$ to $\Lambda^\star\backslash\mathcal{T}$ and a solution to the word problem for $\Lambda^\star$.
Observing that a finite cover of $\Lambda^\star\backslash\mathcal{T}$ is a Ramanujan graph yields a bound on the diameter of $\Lambda^\star\backslash\mathcal{T}$. This in turn we use to bound the complexity of our algorithm, to bound the size of $\mathcal{Y}'$, and to bound the size of the representatives of $\mathcal{Y}'$ in terms of a natural height on the 2×2-matrices over K_∞. The main new result of this chapter is the existence of an effective algorithm together with precise complexity bounds. The results of Chapter 2 were published jointly with G. Böckle in [BB].

The purpose of Chapter 3 is to quickly recall some of the theory of Drinfeld modular forms for congruence subgroups. We claim no originality to the material covered in this chapter.

In Chapter 4 we study Drinfeld modular forms as well as harmonic cocycles for finite index subgroups Γ of $\Lambda^\star$. If Γ has no p'-torsion for $p'\neq p$ we can interpret these forms as sections of line bundles on the rigid analytic space $\Gamma\backslash\Omega$. Therefore we can compute the dimension of these spaces via the Riemann-Roch theorem.
We proceed by proving a dimension formula for Drinfeld modular forms for more general finite index subgroups $\Gamma\subseteq\Lambda^\star$ and for weight $n>2$. This is done following the classical case as in [Sh]. Namely we will first analyze under which conditions there are

non-trivial meromorphic functions on Ω fulfilling the modular transformation property. Using such a non-trivial meromorphic function, combined with an understanding of the elliptic points of $\Gamma\backslash\Omega$, we can again apply the Riemann-Roch theorem to obtain a dimension formula.
The main new results are Corollary 4.9 for the p'-torsion free case and the dimension formula in Theorem 4.19. Corollary 4.9 says, that in the p'-torsion free case dimensions of the spaces of modular forms equal the dimensions of the corresponding spaces of harmonic cocycles. The dimension formula in Theorem 4.19 is an explicit dimension formula in terms of the genus of $\Gamma\backslash\Omega$, for certain subgroups $\Gamma \subseteq \Lambda^\star$ and certain weights and types.

In Chapter 5 we construct, following Teitelbaum, an isomorphism from the spaces of our Drinfeld modular forms to spaces of harmonic cocycles. Sections 5.1 - 5.3 are an adaption of the work by Teitelbaum from [Te1] and [Te2], building up on work of Schneider [Sch2]. For the sake of completeness we give some proofs and computations adapted to our situation, which are either not present or sketchy in the work of Teitelbaum and Schneider. The argument for this homomorphism to be bijective uses the fact from Chapter 4 that in the p'-torsion free case the dimensions on both sides are equal.
We then proceed to introduce a Hecke action on both modular forms and harmonic cocycles compatible with this isomorphism. We will also make the Hecke action on the cocycles side explicit for computational purposes. The Hecke operators are not given through explicit formulas as in the GL_2-case. Instead we give an algorithm to compute the necessary double coset decomposition. The algorithm we give may not be the optimal one, it could be improved in the future.

Finally in Chapter 6 we give a construction of a basis for the space of harmonic cocycles for certain finite index subgroups of $\Lambda^\star$. This construction uses the fundamental domains with edge pairing from Chapter 2. From this basis we can also read of the dimension of spaces of harmonic cocycles. We receive the dimension formulas for spaces of modular forms from Chapter 4 in an independent way.

For technical reasons in Chapter 2, and at some other places, we restrict ourself to the case $p \neq 2$. In Appendix A we discuss aspects of the even characteristic case.

Implementations of the algorithms described in this thesis, based on the computer algebra system Magma [BCP], are available on request.

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2 Quaternion quotient graphs

This chapter studies the action of unit groups inside maximal orders of quaternion algebras over $\mathbb{F}_q(T)$ unramified at ∞ on the Bruhat-Tits tree. We will present an algorithm that computes a fundamental domain for this action together with an edge pairing. These are combinatorial data, that will be used in the computation of the Hecke action on modular forms for such groups.
The results of this chapter were published jointly with G. Böckle in [BB]. Most of Chapter 2 is almost identical with the article, although we present some things in greater detail here. The reason for the overlap is that the contens of this chapter was finished earlier then the rest of the thesis.
Let $k = \mathbb{F}_q, A = k[T]$ and let $K = k(T)$ be the rational function field over k. As usual, the infinite valuation v_∞ on K is given by

$$v_\infty(\frac{f}{g}) = \deg(g) - \deg(f)$$

for $f, g \in A, g \neq 0$ and $v_\infty(0) = \infty$. Let $\pi = 1/T$ be a uniformizer for this valuation and let $K_\infty = k((\pi))$ be the completion of K with respect to v_∞ and O_∞ its ring of integers. Let D be a quaternion algebra over K unramified at ∞ and Λ a maximal order of D, see Section 2.3 for details. Set $\Gamma := \Lambda^\star$ the group of units. Since D is unramified at ∞, the group Γ acts on the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$, see Section 2.2.
In this chapter we study the action of Γ on this tree. We exhibit an algorithm for computing a fundamental domain for the action of Γ on $\mathcal{T}$ and analyze its complexity by using the fact that the quotient graph $\Gamma\backslash\mathcal{T}$ is close to being a Ramanujan graph. The algorithm also yields a presentation of the group Γ and bounds on the size of a set of generators. Over number fields a similar algorithm was investigated by J. Voight in [Vo].

2.1 Notations from graph theory

We recall some definitions from the theory of graphs.

Definition 2.1 *(a) A (directed multi-)graph $\mathcal{G}$ is a pair $(\mathrm{V}(\mathcal{G}), \mathrm{E}(\mathcal{G}))$ where $\mathrm{V}(\mathcal{G})$ is a (possibly infinite) set and $\mathrm{E}(\mathcal{G})$ is a subset of $\mathrm{V}(\mathcal{G}) \times \mathrm{V}(\mathcal{G}) \times \mathbb{Z}_{\geq 0}$ such that*

- *(i) if $e = (v, v', i)$ lies in $\mathrm{E}(\mathcal{G})$, then so does its opposite $e^\star = (v', v, i)$,*
- *(ii) for any $(v, v') \in \mathrm{V}(\mathcal{G}) \times \mathrm{V}(\mathcal{G})$, the set $\{i \in \mathbb{Z}_{\geq 0} \mid (v, v', i) \in \mathrm{E}(\mathcal{G})\}$ is a finite initial segment of $\mathbb{Z}_{\geq 0}$ of cardinality denoted by $n_{v,v'}$,*
- *(iii) for any $v \in \mathrm{V}(\mathcal{G})$, the set $\mathrm{Nbs}(v) := \{v' \in \mathrm{V}(\mathcal{G}) \mid (v, v', 0) \in \mathrm{E}(\mathcal{G})\}$ is finite.*

(b) *A subgraph $\mathcal{G}' \subset \mathcal{G}$ is a graph $\mathcal{G}'$ such that* $\mathrm{V}(\mathcal{G}') \subseteq \mathrm{V}(\mathcal{G})$ *and* $\mathrm{E}(\mathcal{G}') \subseteq \mathrm{E}(\mathcal{G})$.

(c) *Suppose* $\mathrm{V}(\mathcal{G}) = \{v_1, \ldots, v_m\}$ *is finite. Then* $(n_{v_i,v_j})_{1 \leq i,j \leq m}$ *is called the adjacency matrix of* $\mathcal{G}$.

An element $v \in \mathrm{V}(\mathcal{G})$ is called a vertex, an element $e \in \mathrm{E}(\mathcal{G})$ is called an (oriented) edge and an element in $\mathrm{V}(\mathcal{G}) \sqcup \mathrm{E}(\mathcal{G})$ is called a simplex. The oriented edges (v, v', i) and (v', v, i) denote the same edge of $\mathcal{G}$ however with opposite orientation.

Definition 2.2 (a) *For each edge $e = (v, v', i) \in \mathrm{E}(\mathcal{G})$ we call $o(e) := v$ the origin of e and $t(e) := v'$ the target of e.*

(b) *Two vertices v, v' are called adjacent, if there is an edge e such that $\{v, v'\} = \{o(e), t(e)\}$.*

(c) *An edge e with $o(e) = t(e)$ is called a loop.*

For $e \in \mathrm{E}(\mathcal{G})$ we write $e^\star$ for the same edge with orientation reversed and for $v \in \mathrm{V}(\mathcal{G})$ we write $e \mapsto v$ if e is any edge with $t(e) = v$.
Let $v, v' \in \mathrm{V}(\mathcal{G})$. A path from v to v' is a finite sequence $(e_1, \ldots, e_k)$ in $\mathrm{E}(\mathcal{G})$ such that $t(e_i) = o(e_{i+1})$ for all $i = 1, \ldots, k-1$ and $o(e_1) = v, t(e_k) = v'$. The integer k is called the length of the path $(e_1, \ldots, e_k)$. A graph $\mathcal{G}$ is connected if for any two vertices $v, v' \in \mathrm{V}(\mathcal{G})$ there is a path from v to v'. A path $(e_1, \ldots, e_k)$ is a path without backtracking if for all $i = 1, \ldots, k-1$ we have $e_{i+1} \neq e_i^\star$. A geodesic from v to v' of $\mathcal{G}$ is a finite path from v to v' without backtracking. The distance from v to v', denoted $d(v, v')$, is the minimal length of all geodesics from v to v' or ∞ if there is no path from v to v'. Define the diameter of a graph $\mathcal{G}$ as

$$\mathrm{diam}(\mathcal{G}) := \max_{v,v' \in \mathrm{V}(G)} d(v, v').$$

A cycle of $\mathcal{G}$ is a geodesic from some vertex v to itself. A graph $\mathcal{G}$ is cycle-free if it contains no cycles. A tree is a connected, cycle-free graph. Note that if $\mathcal{G}$ is a tree, then for each two vertices $v, v' \in \mathrm{V}(\mathcal{G})$ there is exactly one geodesic between v and v'. Any subgraph $\mathcal{S} \subseteq \mathcal{G}$ which is a tree is called a subtree. A maximal subtree is a subtree which is maximal under inclusion among all subtrees of $\mathcal{G}$.

Definition 2.3 (a) *For $v \in \mathrm{V}(\mathcal{G})$ the degree of v is defined as*

$$\deg(v) := \#\{e \in \mathrm{E}(\mathcal{G}) \mid o(e) = v\}.$$

(b) *v is terminal if* $\deg(v) = 1$.

A graph $\mathcal{G}$ is finite, if $\#\,\mathrm{V}(\mathcal{G}) < \infty$. Then also $\#\,\mathrm{E}(\mathcal{G}) < \infty$ since $\deg(v)$ is finite for all vertices $v \in \mathrm{V}(\mathcal{G})$. A graph $\mathcal{G}$ is called k-regular if for all vertices $v \in \mathrm{V}(\mathcal{G})$ we have $\deg(v) = k$.

Definition 2.4 *(a) We define the first Betti number $h_1(\mathcal{G})$ of a finite connected graph to be*

$$h_1(\mathcal{G}) := \frac{\#\,\mathrm{E}(\mathcal{G})}{2} - \#\,\mathrm{V}(\mathcal{G}) + 1.$$

Any finite graph $\mathcal{G}$ can be viewed as an abstract simplicial complex, and one obtains in this way a topological space $|\mathcal{G}|$, the geometrical realization of $\mathcal{G}$. The first Betti number $h_1(\mathcal{G})$ is the dimension of $H^1(|\mathcal{G}|, \mathbb{Q})$, so the Betti number counts the number of independent cycles of $\mathcal{G}$.

2.2 The Bruhat-Tits tree

We recall the definition of the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$, which is an important combinatorial object for the arithmetic of K. The results in this section are well known and can be found in [Se1].

Definition 2.5 *The Bruhat-Tits tree $\mathcal{T} = (\mathrm{V}(\mathcal{T}), \mathrm{E}(\mathcal{T})$ of $\mathrm{PGL}_2(K_\infty)$ is defined as follows: Two $\mathcal{O}_\infty$-lattices $L, L' \subseteq K_\infty^2$ are called equivalent if there is a $\lambda \in K_\infty$ with $L' = \lambda L$. The set $\mathrm{V}(\mathcal{T})$ is the set of equivalence classes $[L]$ of such lattices. The set $\mathrm{E}(\mathcal{T})$ is the set of pairs $([L], [L'])$ such that L, L' are $\mathcal{O}_\infty$-lattices in K_∞^2 with $\pi L \subsetneq L' \subsetneq L$.*

In particular there is at most one edge between two vertices. By [Se1, Chapter II.1] $\mathcal{T}$ is the $(q+1)$-regular tree. The group $\mathrm{GL}_2(K_\infty)$ acts naturally on lattice classes by left multiplication $(g, [L]) \mapsto [gL]$. This induces an action on $\mathcal{T}$.
Let $e_1 = (1,0)^t, e_2 = (0,1)^t$ be the standard basis of K_∞^2. Write $\mathcal{O}_\infty^2$ for $\mathcal{O}_\infty e_1 \otimes \mathcal{O}_\infty e_2$. Since $\mathrm{GL}_2(K_\infty)$ acts transitivly on bases of K_∞^2 and the stabilizer of $[\mathcal{O}_\infty^2]$ is $\mathrm{GL}_2(O_\infty)K_\infty^\star$ one obtains:

Proposition 2.6 *The map*

$$\begin{array}{rcl} \varphi : \mathrm{GL}_2(K_\infty)/\mathrm{GL}_2(O_\infty)K_\infty^\star & \to & \mathrm{V}(\mathcal{T}) \\ A & \mapsto & \left[A\mathcal{O}_\infty^2\right] \end{array}$$

is a bijection.

Our next goal is to identify the vertices in the tree with explicitly given matrices and to see, which matrices correspond to adjacent vertices in the tree. The next two Lemmas will help us with that. The next Lemma is basically the row-reduction to the echelon form of a matrix in $\mathrm{GL}_2(K_\infty)$, we give a constructive proof that allows for an explicit algorithm to compute the vertex normal form of a matrix.

Lemma 2.7 *Every class of* $\mathrm{GL}_2(K_\infty)/\mathrm{GL}_2(O_\infty)K_\infty^\star$ *has a unique representative of the form*

$$\begin{pmatrix} \pi^n & g \\ 0 & 1 \end{pmatrix}$$

with $n \in \mathbb{Z}$ *and* $g \in K_\infty/\pi^n O_\infty$.

We call this representative the *vertex normal form* of a matrix $\gamma \in \mathrm{GL}_2(K_\infty)$ or of the corresponding vertex $\varphi(\gamma)$.

PROOF: Let $\gamma = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}_2(K_\infty)$. If $v_\infty(x_3) < v_\infty(x_4)$ we multiply from the right with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to swap the columns of γ. Hence we can assume $v_\infty(x_3) \geq v_\infty(x_4)$. Multiplying from the right with $\begin{pmatrix} 1 & 0 \\ -\frac{x_3}{x_4} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty)$ gives

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{x_3}{x_4} & 1 \end{pmatrix} = \begin{pmatrix} x_1 - \frac{x_2 x_3}{x_4} & x_2 \\ 0 & x_4 \end{pmatrix}.$$

Multiplying with $x_4^{-1} \in K_\infty^\star$ we obtain an equivalent matrix of the form

$$\begin{pmatrix} z_1 & z_2 \\ 0 & 1 \end{pmatrix}.$$

Write $z_1 = \pi^n \varepsilon$ with $\varepsilon \in \mathcal{O}_\infty^\star$ and multiply from the right with $\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty)$ to obtain a matrix of the form $\begin{pmatrix} \pi^n & y \\ 0 & 1 \end{pmatrix}$.
If we have

$$\begin{pmatrix} \pi^n & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi^m & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} \pi^m r + bt & \pi^m s + bu \\ t & u \end{pmatrix}$$

with $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL_2(\mathcal{O}_\infty)K_\infty^\star$, we conclude from the last row $u = 1, t = 0$ and hence $r = 1$ and $m = n$. The entry in the upper right corner is therefore only determined up to $\pi^n \mathcal{O}_\infty$. ∎

Lemma 2.8 *Consider the two matrices in vertex normal form*

$$A := \begin{pmatrix} \pi^n & g \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} \pi^{n+1} & g + \alpha\pi^n \\ 0 & 1 \end{pmatrix}$$

with $n \in \mathbb{Z}, \alpha \in k, g \in K_\infty/\pi^n O_\infty$ *and let* L_1 *and* L_2 *be the two lattices*

$$L_1 := A\mathcal{O}_\infty^2, L_2 := B\mathcal{O}_\infty^2.$$

Then $L_1 \supset L_2$ *and* $L_1/L_2 \cong k$.

Proof: Set

$$v_1 = \begin{pmatrix} \pi^n \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} g \\ 1 \end{pmatrix}.$$

Then $L_1 = \langle v_1, v_2 \rangle_{O_\infty}$ and $L_2 = \langle \pi v_1, v_2 + \alpha v_1 \rangle_{O_\infty}$. Hence

$$L_1 \supseteq L_2 \supseteq \pi L_1.$$

But $v_1 \notin L_2$ and $v_2 + \alpha v_1 \notin \pi L_1$, so

$$L_1 \supsetneq L_2 \supsetneq \pi L_1$$

and therefore $L_1/L_2 \cong k$. ∎

Remark 2.9 Lemma 2.8 only displays q vertices adjacent to $[L_1]$. The missing one is the class of $\begin{pmatrix} \pi^{n-1} & g \\ 0 & 1 \end{pmatrix} \mathcal{O}_\infty^2$ with g now being replaced by its class in $K_\infty/\pi^{n-1}\mathcal{O}_\infty$.

In Figure 1 below we have illustrated the tree together with the matrices in normal form corresponding to vertices. The identification is clear from the previous lemma. Note that each line in the picture symbolizes actually a whole fan expanding to the right. The elements $\alpha \in k^\star$, $\beta \in k$ agree on each fan.

Write $L(n, g)$ for the O_∞-lattice $\langle v_1, v_2 \rangle_{O_\infty}$ where $v_1 = \begin{pmatrix} \pi^n \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} g \\ 1 \end{pmatrix}$. Note that $L(n, g) = L(n, g')$ if and only if $g \equiv g' \pmod{\pi^n O_\infty}$.

Remark 2.10 For $n \in \mathbb{Z}$, $g \in K_\infty$ we define

$$\deg_n(g) := \min\{i \in \mathbb{N}_0 | g \in \pi^{n-i} O_\infty\}.$$

Then, setting $\delta := \deg_n(g)$, the path from $L(n, g)$ to $L(0, 0)$ in $\mathcal{T}$ is given as follows:

$$L(n,g) —— L(n-1,g) —— \ldots —— L(n-\delta,g) = L(n-\delta,0) —$$

$$— L(\operatorname{sign}(n-\delta)\cdot(|n-\delta|-1),0) —— \ldots —— L(\operatorname{sign}(n-\delta)\cdot 1,0) —— L(0,0)$$

In particular the distance between $L(n, g)$ and $L(0, 0)$ is $\deg_n(g) + |n - \deg_n(g)|$.

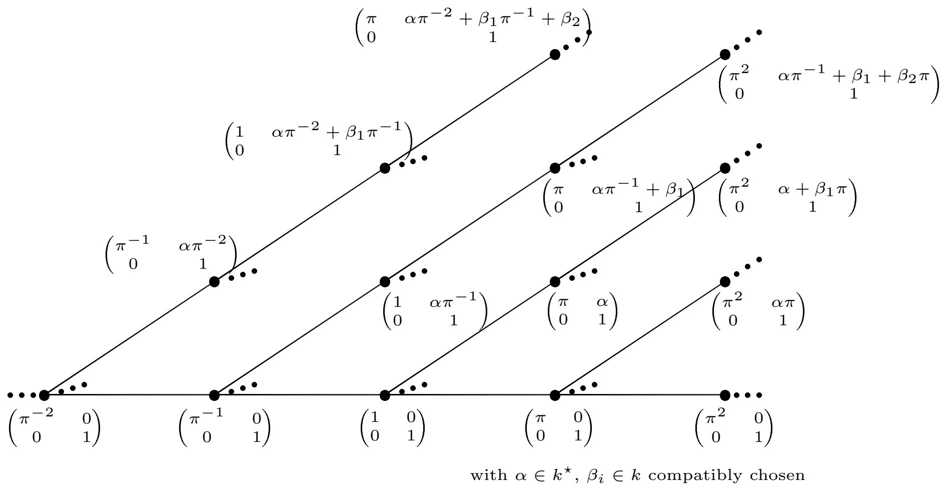

Figure 1: The tree $\mathcal{T}$ with the corresponding matrices

2.3 Quaternion algebras

We briefly recall some basics from the theory of quaternion algebras. All material covered in this section is well known. Standard references are [JS, Kap. IX] and [Vi]. Note that for 2.11 to 2.13, K could be any field.

Definition 2.11 *A quaternion algebra D over K is a central simple algebra of dimension 4 over K.*

There is a unique anti-involution $^- : D \to D$ such that $\gamma + \bar\gamma$ and $\gamma\bar\gamma$ are in K for all $\gamma \in D$. This map is called conjugation on D and we can use it to define the reduced trace and norm of an element $\gamma \in D$ as $\mathrm{trd}(\gamma) := \gamma + \bar\gamma$ and $\mathrm{nrd}(\gamma) := \gamma\bar\gamma$.

Definition 2.12 *Let $a, b \in K^\star$. We define $\left(\frac{a,b}{K}\right)$ to be the K-algebra with basis $1, i, j, ij$ and relations*

- $i^2 = a, j^2 = b, ij = -ji$ *for* $\mathrm{char}(K) \neq 2$ *and*
- $i^2 + i = a, j^2 = b, ij = j(i+1)$ *for* $\mathrm{char}(K) = 2$.

It is not hard to show, that any quaternion algebra over K is isomorphic to $\left(\frac{a,b}{K}\right)$ for some $a, b \in K$, see [Vi, Chapitere I.1] for arbitrary characteristic or [JS, Kapitel IX] for $\mathrm{char}(K) \neq 2$.
Write any $\gamma \in \left(\frac{a,b}{K}\right)$ uniquely as $\gamma = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 ij$ with $\lambda_i \in K$.
For $\mathrm{char}(K) \neq 2$ the anti-involution is given by $\bar\gamma = \lambda_1 - \lambda_2 i - \lambda_3 j - \lambda_4 ij$ and we compute $\mathrm{trd}(\gamma) = 2\lambda_1$ and $\mathrm{nrd}(\gamma) = \lambda_1^2 - a\lambda_2^2 - b\lambda_3^2 + ab\lambda_4^2$.

For $\mathrm{char}(K) = 2$ the anti-involution is given by $\bar{\gamma} = \lambda_1 + \lambda_2(i+1) + \lambda_3 j + \lambda_4 ij$ and we compute $\mathrm{trd}(\gamma) = \lambda_2$ and $\mathrm{nrd}(\gamma) = \lambda_1^2 + a\lambda_2^2 + b\lambda_3^2 + ab\lambda_4^2 + \lambda_1\lambda_2 + b\lambda_3\lambda_4$.
The norm map of the quaternion algebra $\left(\frac{a,b}{K}\right)$ gives us a quadratic form $Q_{a,b} : (\lambda_1, \ldots, \lambda_4) \mapsto \mathrm{nrd}(\lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 ij)$.
Any quaternion algebra is either a division algebra or isomorphic to $M_2(K)$ (see [JS, Satz 1.4, IX]) and we have the following proposition:

Proposition 2.13 *Suppose* $\mathrm{char}(K) \neq 2$. *Then the following are equivalent:*

(a) $D := \left(\frac{a,b}{K}\right) \cong M_2(K)$

(b) There is an $x \in D$, $x \neq 0$, *with* $\mathrm{nrd}(x) = 0$.

(c) The quadratic form $Q_{a,b}$ *is isotropic, i.e. there are* $(x, y, v, w) \in K^4 \setminus \{(0,0,0,0)\}$ *with* $Q_{a,b}(x, y, v, w) = 0$.

(d) The equation $Z^2 - aX^2 - bY^2 = 0$ *has a non-trivial solution over* K.

(e) $a \in \mathrm{Image}(\mathrm{Norm}(K(\sqrt{b})/K)$

(f) $b \in \mathrm{Image}(\mathrm{Norm}(K(\sqrt{a})/K)$

Proof: See [JS, Satz 1.9, Chapter IX]. ∎

For $\mathrm{char}(K) = 2$, the situation is different. Especially, one does not have a symmetry in the role of a and b as in the previous proposition. For more details we refer the reader to Appendix A where we treat the case of $\mathrm{char}(K) = 2$.
Let $\mathfrak{p}$ be a place of K.

Definition 2.14 *We say that a quaternion algebra* D *over* K *is ramified at* $\mathfrak{p}$ *if and only if* $D \otimes_K K_{\mathfrak{p}}$ *is a division algebra.*

Assumption 2.15 *For the remainder of the thesis, we assume that* D *is a division quaternion algebra which is unramified at* ∞, *i.e., that* D *is an indefinite quaternion algebra over* A. *We also fix an isomorphism* $D_\infty \cong M_2(K_\infty)$.

Lemma 2.16 $\left(\frac{a,b}{K}\right)$ *is unramified at* $\mathfrak{p}$ *if and only if* $Q_{a,b}$ *has a non-trivial solution over* $K_{\mathfrak{p}}$.

PROOF: Suppose that $Q_{a,b}$ has a non-trivial solution over $K_{\mathfrak{p}}$, i.e. there are elements $(0,0,0,0) \neq \lambda_1, \ldots, \lambda_4$ in $K_{\mathfrak{p}}$ such that $Q_{a,b}(\lambda_1, \ldots, \lambda_4) = 0$. But then $\gamma := \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 ij$ is a non-zero element of $\left(\frac{a,b}{K}\right) \otimes K_{\mathfrak{p}} \cong \left(\frac{a,b}{K_{\mathfrak{p}}}\right)$ with $\mathrm{nrd}(\gamma) = \gamma\bar{\gamma} = 0$ and hence$\left(\frac{a,b}{K}\right) \otimes K_{\mathfrak{p}}$ is not a division algebra.
On the other hand, suppose that $Q_{a,b}$ has no non-trivial solutions and let $0 \neq \gamma \in \left(\frac{a,b}{K_{\mathfrak{p}}}\right)$. Write $\gamma = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 ij$ with $\lambda_i \in K_{\mathfrak{p}}$. Then $Q_{a,b}(\lambda_1, \ldots, \lambda_4) = \mathrm{nrd}(\gamma) = \gamma\bar{\gamma}$ is in $K_{\mathfrak{p}}^{\star}$ and hence $\bar{\gamma}/(\gamma\bar{\gamma})$ is the multiplicative inverse of γ. ∎

We define the Hilbert symbol of a pair $(a,b) \in K^2$ at a place $\mathfrak{p}$ as follows:

Definition 2.17

$$(a,b)_{K_{\mathfrak{p}}} := \begin{cases} +1 & \left(\frac{a,b}{K}\right) \text{ is unramified at } \mathfrak{p} \\ -1 & \left(\frac{a,b}{K}\right) \text{ is ramified at } \mathfrak{p}. \end{cases}$$

Definition 2.18 *Let a, ϖ be in A with ϖ irreducible. Define the Legendre symbol of a and ϖ as*

$$\left(\frac{a}{\varpi}\right) := \begin{cases} 1 & a \neq 0 \text{ and } a \text{ is a square modulo } \varpi \\ -1 & a \text{ is a non-square modulo } \varpi \\ 0 & \varpi \text{ divides } a. \end{cases}$$

The next Proposition is proved by adaptating to the function-field situation the proof of [Se2, Chapter III, Theorem 1]:

Proposition 2.19 *Suppose* $\mathrm{char}(K) \neq 2$. *Write $\mathfrak{p} = (\varpi)$ and let $a = \varpi^{\alpha} u, b = \varpi^{\beta} v$ with $u, v \in O_{K_{\mathfrak{p}}}^{\star}, \alpha, \beta \in \mathbb{Z}$ and let $\varepsilon(\mathfrak{p}) := \frac{q-1}{2} \deg(\varpi) \pmod 2$. Then*

$$(a,b)_{K_{\mathfrak{p}}} = (-1)^{\alpha\beta\varepsilon(\mathfrak{p})} \left(\frac{u}{\varpi}\right)^{\beta} \left(\frac{v}{\varpi}\right)^{\alpha}.$$

PROOF: In the proof we write (a,b) for $(a,b)_{K_{\mathfrak{p}}}$.
The right hand side of the equation clearly depends only on $\alpha \pmod 2$ and $\beta \pmod 2$. If $Q_{a,b}$ is isotropic, then so are $Q_{\varpi^2 a,b}, Q_{a,\varpi^2,b}$ and vice versa. Hence the left hand side also only depends on $\alpha \pmod 2$ and $\beta \pmod 2$. Because of symmetry we only need to consider the three cases $(\alpha,\beta) = (0,0)$, $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (1,1)$.
Case one: $(\alpha,\beta) = (0,0)$: Here the right hand side is 1, so we have to show that $Z^2 - uX^2 - vY^2$ has a solution in $K_{\mathfrak{p}}$. But $Z^2 - uX^2 - vY^2$ has a solution modulo ϖ, since all quadratic forms in at least three variables over a finite field have a non-trivial solution (see [Se2, Chapter I.2, Cor. 2]). Since $\mathrm{disc}(Z^2 - uX^2 - vY^2) \in O_{K_{\mathfrak{p}}}^{\star}$, this solution lifts to $O_{K_{\mathfrak{p}}}$ by Hensel's Lemma.

Case two: $(\alpha, \beta) = (1, 0)$: We must check that $(\varpi u, v) = \left(\frac{v}{\varpi}\right)$, From case one we know that $(u, v) = 1$, hence $u \in \text{Image}(\text{Norm}(K_{\mathfrak{p}}(\sqrt{v})/K_{\mathfrak{p}})$ and we have $\varpi \in \text{Image}(\text{Norm}(K_{\mathfrak{p}}(\sqrt{v})/K_{\mathfrak{p}})$ if and only if $u\varpi \in \text{Image}(\text{Norm}(K_{\mathfrak{p}}(\sqrt{v})/K_{\mathfrak{p}})$.
So $(\varpi u, v) = (\varpi, v)$ and we may assume $u = 1$. If $v = (v')^2$ is a square in $K_{\mathfrak{p}}$, then clearly $\left(\frac{v}{\varpi}\right) = 1$ and also $(v', 0, 1)$ is a non-trivial solution of $Z^2 - \varpi X^2 - vY^2 = 0$, so $(\varpi, v) = 1$. Let v be a non-square in $K_{\mathfrak{p}}$. Since $v \in O^{\star}_{K_{\mathfrak{p}}}$, this is equivalent to $\left(\frac{v}{\varpi}\right) = -1$. Suppose $Z^2 - \varpi X^2 - vY^2$ has a non-trivial solution (z, x, y). By normalizing we can assume that (z, x, y) is primitive, i.e. $(z, x, y) \in O_{K_{\mathfrak{p}}}$, and at least one of them is in $O^{\star}_{K_{\mathfrak{p}}}$. Suppose either $z \equiv 0 \pmod{\varpi}$ or $y \equiv 0 \pmod{\varpi}$. Then since $z^2 - vy^2 \equiv 0 \pmod{\varpi}$ and $v \neq 0 \pmod{\varpi}$ we obtain both $z \equiv 0 \pmod{\varpi}$ and $y \equiv 0 \pmod{\varpi}$ and hence $\varpi x^2 \equiv 0 \pmod{\varpi^2}$, so $x \equiv 0 \pmod{\varpi}$. Therefore (z, x, y) was not primitive. So both z and y have to be non-zero modulo ϖ. Reducing $z^2 - \varpi x^2 - vy^2$ modulo ϖ we obtain $\left(\frac{v}{\varpi}\right) = 1$, which is a contradiction. So $Z^2 - \varpi X^2 - vY^2$ has no non-trivial solution, and hence $(\varpi, v) = -1$.
Case three: $(\alpha, \beta) = (1, 1)$:
We must check that $(\varpi u, \varpi v) = (-1)^{\varepsilon(\mathfrak{p})} \left(\frac{u}{\varpi}\right) \left(\frac{v}{\varpi}\right)$. But since $Q_{\varpi u, -\varpi u}$ is isotropic we have

$$\varpi v \in \text{Image}(\text{Norm}(K_{\mathfrak{p}}(\sqrt{\varpi u})/K_{\mathfrak{p}}) \Leftrightarrow -\varpi^2 uv \in \text{Image}(\text{Norm}(K_{\mathfrak{p}}(\sqrt{\varpi u})/K_{\mathfrak{p}})$$

and hence

$$(\varpi u, \varpi v) = (\varpi u, -\varpi^2 uv) = (\varpi u, -uv),$$

so we can apply case two and see that

$$(\varpi u, \varpi v) = (\varpi u, -uv) = \left(\frac{-uv}{\varpi}\right)$$

$$= \left(\frac{-1}{\varpi}\right) \left(\frac{u}{\varpi}\right) \left(\frac{v}{\varpi}\right) = (-1)^{\frac{q-1}{2} \deg(\varpi)} \left(\frac{u}{\varpi}\right) \left(\frac{v}{\varpi}\right). \blacksquare$$

Let D be an indefinite quaternion algebra over K. Indefinite means that D is unramified at the place ∞, i.e. $D \otimes_K K_\infty \cong M_2(K_\infty)$. Let R denote the set of all ramified places of D.

Proposition 2.20 *The number of places in R is finite and even and D is up to isomorphism uniquely determined by R.*

PROOF: See [Vi, Lemme III.3.1 and Theoreme III.3.1]. ∎

Let r be a monic generator of the ideal $\mathfrak{r} := \prod_{\mathfrak{p}\in R} \mathfrak{p}$. The ideal $\mathfrak{r}$ is called the discriminant of D.
An order of D is a free A-submodule of rank 4 in D that is also a ring. An order Λ of D is called maximal if it is not properly contained in any other order of D. For any 4 elements $\gamma_1, \ldots, \gamma_4 \in D$ let $\mathrm{disc}(\gamma_1, \ldots, \gamma_4) := \det(\mathrm{trd}(\gamma_i\gamma_j))_{i,j=1,\ldots,4}$. For any order Λ of D the ideal of A generated by the set $\{\mathrm{disc}(\gamma_1, \ldots, \gamma_4) \mid \gamma_i \in \Lambda\}$ is a square (see [Vi, Lemme I.4.7], and we define the reduced discriminant $\mathrm{disc}(\Lambda)$ to be the square root of this ideal. Since A is a principal ideal domain, for any A-basis $\{\gamma_1, \ldots, \gamma_4\}$ of Λ the element $\mathrm{disc}(\gamma_1, \ldots, \gamma_4)$ generates the ideal $\langle\{\mathrm{disc}(\gamma_1, \ldots, \gamma_4) \mid \gamma_i \in \Lambda\}\rangle_A$. An order Λ of D is maximal if and only if $\mathrm{disc}(\Lambda) = \mathfrak{r}$, see [Vi, Corollaire III.5.3]. Since D is split at infinity and since K has class number 1, a maximal order Λ of D is unique up to conjugation, i.e. for any other maximal order Λ' we have $\Lambda' = \gamma\Lambda\gamma^{-1}$ for an $\gamma \in D^\star$, see [Vi, Corollaire III.5.7].
Let $\Lambda = \langle\gamma_1, \ldots, \gamma_4\rangle_A$ and $\Gamma := \Lambda^\star$. Hence $\Gamma = \{\gamma \in \Lambda \mid \mathrm{nrd}(\gamma) \in k^\star\}$. Since D is unramified at K_∞ we have $D \otimes_K K_\infty \cong M_2(K_\infty)$ and we obtain an embedding $\iota : D \hookrightarrow M_2(K_\infty)$. Via this embedding Γ can be identified with a subgroup of $\mathrm{SL}_2(K_\infty)\begin{pmatrix} k^\star & 0 \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(K_\infty)$.
The following Proposition is well known. We give a proof for the sake of completeness.

Proposition 2.21 *$\iota(\Gamma)$ is a discrete subgroup of $\mathrm{GL}_2(K_\infty)$.*

Proof: The open sets $\{1+\pi^n M_2(O_\infty) \mid n \in \mathbb{N}\}$ form a basis of open neighbourhoods of 1 in $\mathrm{GL}_2(K_\infty)$. After shifting by 1 it suffices to show that $\iota(\Lambda) \cap M_2(O_\infty)$ is finite. To see this, let $\mathcal{D}$ be the unique locally free coherent sheaf of rings of rank 4 over $\mathbb{P}^1_k$ such that $\Lambda \cong \Gamma(\mathbb{A}^1_k, \mathcal{D})$ and such that the completed stalk at infinity satisfies $\mathcal{D}_\infty \cong M_2(O_\infty)$. Then $\iota(\Lambda) \cap M_2(O_\infty) = H^0(\mathbb{P}^1_k, \mathcal{D})$. By the Riemann-Roch Theorem this is a finite-dimesional k-vector space. ∎

2.4 Facts about quaternion quotient graphs

In Section 2.2 we have described the natural action of $\mathrm{GL}_2(K_\infty)$ on the Bruhat-Tits tree $\mathcal{T}$. In the previous section, starting from D as in Assumption 2.15, we have produced a discrete subgroup $\Gamma \subset \mathrm{GL}_2(K_\infty)$, the unit group of a maximal order. In this section we gather some known results about the induced action of Γ on $\mathcal{T}$ and the quotient graph $\Gamma\backslash\mathcal{T}$. We mainly follow [Pa1].

Lemma 2.22 *Let $v \in \mathrm{V}(\mathcal{T})$ and $\gamma \in \Gamma$. Than the distance $d(v, \gamma v)$ is even.*

Proof: See [Se1, Corollary of Proposition II.1]. ∎

Proposition 2.23 *$\Gamma\backslash\mathcal{T}$ is a finite graph.*

PROOF: See [Pa1, Lemma 5.1]. ∎

Proposition 2.24 *Let $v \in \mathrm{V}(\mathcal{T})$ and $e \in \mathrm{E}(\mathcal{T})$. Then $\Gamma_v := \mathrm{Stab}_\Gamma(v)$ is either isomorphic to $\mathbb{F}_q^\star$ or $\mathbb{F}_{q^2}^\star$ and $\Gamma_e := \mathrm{Stab}_\Gamma(e)$ is isomorphic to $\mathbb{F}_q^\star$.*

PROOF: See [Pa1, Proposition 5.2]. ∎

Note that the scalar matrices with diagonal in $\mathbb{F}_q^\star$ are precisly the scalar matrices in Γ. They act trivially on $\mathcal{T}$. Hence a stabilizer of a simplex is isomorphic to $\mathbb{F}_q^\star$ if and only if it is the set of scalar matrices with diagonal in $\mathbb{F}_q^\star$.

Definition 2.25 *We call a simplex s projectively stable if $\Gamma_s \cong \mathbb{F}_q^\star$ and projectively unstable if $\Gamma_s \cong \mathbb{F}_{q^2}^\star$.*

Let $\bar{\Gamma}$ be the image of Γ in $\mathrm{PGL}_2(K_\infty)$, hence $\bar{\Gamma} \cong \Gamma/\mathbb{F}_q^\star$. Then for vertices $v \in \mathrm{V}(\mathcal{T})$ the stabilizer $\bar{\Gamma}_v$ is either trivial or isomorphic to $\mathbb{F}_{q^2}^\star/\mathbb{F}_q^\star$ and for edges $e \in \mathrm{E}(\mathcal{T})$ the stabilizer $\bar{\Gamma}_e$ is always trivial. Thus t is $\bar{\Gamma}$-stable in the sense of [Se1, Definition II.2.9] for $\bar{\Gamma}$ the image of Γ in $\mathrm{PGL}_2(K_\infty)$ if and only if t is projectively stable.

Corollary 2.26 *Let $v \in \mathrm{V}(\mathcal{T})$ be projectively unstable. Then Γ_v acts transitively on the vertices adjacent to v.*

Let

$$\mathrm{odd}(R) := \begin{cases} 0 & \text{if some place in } R \text{ has even degree,} \\ 1 & \text{otherwise} \end{cases}$$

and let

$$g(R) := 1 + \frac{1}{q^2-1}\left(\prod_{\mathfrak{p}\in R}(q_\mathfrak{p}-1)\right) - \frac{q}{q+1}2^{\#R-1}\,\mathrm{odd}(R)$$

where $q_\mathfrak{p} = q^{\deg(\mathfrak{p})}$. Let $\pi : \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection.

Theorem 2.27 *(a) The graph $\Gamma\backslash\mathcal{T}$ has no loops.*

(b) $h_1(\Gamma\backslash\mathcal{T}) = g(R)$.

(c) For $\bar{v} \in \Gamma\backslash\mathcal{T}$ and $v \in \pi^{-1}(\bar{v})$ we have:

(i) $\bar{v}$ is a terminal vertex if and only if v is projectively unstable.

(ii) $\bar{v}$ *has degree* $q+1$ *if and only if* v *is projectively stable.*

(d) Let V_1 *(resp.* V_{q+1}*) be the number of terminal (resp. degree* $q+1$*) vertices of* $\Gamma\backslash\mathcal{T}$*. Then*

$$V_1 = 2^{\#R-1}\,\mathrm{odd}(R) \text{ and } V_{q+1} = \frac{1}{q-1}(2g(R) - 2 + V_1).$$

Proof: See [Pa1, Theorem 5.5] ∎

2.5 An algorithm to compute a fundamental domain

Let the notation $\mathcal{T}$, Γ be as in the previous section.

Definition 2.28 ([Se1, § I.3]) *Let* G *be a group acting on a graph* $\mathcal{X}$*. A* tree of representatives of $\mathcal{X}$ (mod G) *is a subtree* $\mathcal{S} \subset \mathcal{X}$ *whose image in* $G\backslash\mathcal{X}$ *is a maximal subtree.*

The following definition is basically [Se1, § I.4.1, Lem. 4], see also [Se1, § I.5.4, Thm. 13]. Note that (a) differs from [Se1, § I.4.1, Def. 7].

Definition 2.29 *Let* G *be a group acting on a tree* $\mathcal{X}$*.*

(a) A fundamental domain for $\mathcal{X}$ under G *is a pair* $(\mathcal{S}, \mathcal{Y})$ *of subgraphs* $\mathcal{S} \subset \mathcal{Y} \subset \mathcal{X}$ *such that*

  (i) $\mathcal{S}$ *is a tree of representatives of* $\mathcal{X}$ (mod G),

  (ii) the projection $\mathrm{E}(\mathcal{Y}) \to \mathrm{E}(G\backslash\mathcal{X})$ *is a bijection, and*

  (iii) any edge of $\mathcal{Y}$ *has at least one of its vertices in* $\mathcal{S}$*.*

(b) An edge pairing for a fundamental domain $\mathcal{Y}$ of $\mathcal{X}$ under G *is a map*

$$\mathrm{PE} := \mathrm{PE}_{(\mathcal{S},\mathcal{Y})} := \{e \in \mathrm{E}(\mathcal{Y}) \smallsetminus \mathrm{E}(\mathcal{S}) \mid o(e) \in \mathcal{S}\} \to G : e \mapsto g_e$$

such that $g_e t(e) \in \mathrm{V}(\mathcal{S})$*. We write* PE *for* paired edges*. To avoid cumbersome notation, we usually abbreviate* $\mathrm{PE}_{(\mathcal{Y},\mathcal{S})}$ *by* PE.

(c) An enhanced fundamental domain for $\mathcal{X}$ under G *consists of a fundamental domain, an edge pairing and simplex labels* $G_t := \mathrm{Stab}_G(t)$ *for all simplices* t *of* $\mathcal{Y}$*.*

An edge pairing encodes that under the G-action any $e = (v, v') \in \mathrm{PE}$ is identified (paired) with $ge = (g_e v, g_e v')$ when passing from $\mathcal{X}$ to $G\backslash\mathcal{X}$. Because $\mathcal{X}$ is a tree and the image of $\mathcal{S}$ in $G\backslash\mathcal{X}$ is a maximal subtree, each edge in $\mathrm{E}(\mathcal{Y}) \smallsetminus \mathrm{E}(\mathcal{S})$ has exactly one of its vertices in $\mathrm{V}(\mathcal{S})$ and therefore PE contains exactly those edges of $\mathrm{E}(\mathcal{Y}) \smallsetminus \mathrm{E}(\mathcal{S})$ pointing away from $\mathcal{S}$. An enhanced fundamental domain is a graph of groups in the sense of [Se1, I.4.4, Def. 8] realized inside $\mathcal{X}$. Given a fundamental domain with an edge pairing the tree $\mathcal{S}$ can be recovered from $\mathcal{Y}$ and PE.

Remark 2.30 If one barycentrically subdivides $\mathcal{T}$, an alternative way to think of an edge pairing is that it pairs the two half sides $[o(e), \frac{1}{2}o(e)+\frac{1}{2}t(e)]$ and $[g_e t(e), g_e(\frac{1}{2}o(e)+\frac{1}{2}t(e))]$.

It will be convenient to introduce the following notation:

Definition 2.31 *For any group G acting on a set X we define a category $\mathcal{C}_G(X)$ whose objects are the elements of X and whose morphism sets are defined as*

$$\mathrm{Hom}_G(x, y) := \{\gamma \in G \mid gx = y\} \subseteq G.$$

for $x, y \in X$. The composition of morphisms is given by multiplication in G.

In particular $\mathrm{End}_G(x) := \mathrm{Hom}_G(x, x) = \mathrm{Stab}_G(x)$.

For the remainder of this section, we assume that $\mathrm{Hom}_\Gamma(v, w)$ can be computed effectively for all $v, w \in \mathrm{V}(\mathcal{T})$. This will be verified in Section 2.7.

Algorithm 2.32 (Computation of the quotient graph)

Input: *A subgroup $\Gamma \subset \mathrm{GL}_2(K_\infty)$ for which there exists a routine for computing $\mathrm{Hom}_\Gamma(v, v')$ for all $v, v' \in \mathrm{V}(\mathcal{T})$ which are equidistant from $[L(0, 0)]$.*

Output: *A directed multigraph $\mathcal{G}$ with a label attached to each simplex. The label values on edges are either $(e, 1)$ (preset), or $(e, -1)$, or a pair (e, g) with $e \in \mathrm{E}(\mathcal{T})$, $g \in \Gamma$. The label values on vertices are either $(v, 1)$ (preset) or (v, G) for $v \in \mathrm{V}(\mathcal{T})$ and $G \subset \Gamma$ a finite subgroup.*

Algorithm:

(a) Set $v_0 = [L(0, 0)]$. If $\#\,\mathrm{End}_\Gamma(v_0) = q^2 - 1$, replace v_0 by $[L(1, 0)]$. If after replacement we still have $\#\,\mathrm{End}_\Gamma(v_0) = q^2 - 1$, then terminate the algorithm with the output the connected graph on 2 vertices and one edge and with vertex labels $\mathrm{End}_\Gamma(v)$ for each of the two vertices v.

(b) Initialize a graph $\mathcal{G}$ with $\mathrm{V}(\mathcal{G}) = \{v_0\}$ and $\mathrm{E}(\mathcal{G}) = \varnothing$. Also, initialize lists $L := (e \in \mathrm{E}(\mathcal{T}) \mid o(e) = v_0)$, the edges adjacent to v_0, and $L' := \varnothing$. All vertices v of $\mathcal{T}$ are given by a matrix in vertex normal form $\mathrm{vnf}(v)$.

(c) While L is not empty:

(i) For $i = 1$ to $\#L$ do:

i. Let $e = (v, v')$ be the ith element in L.
ii. Compute $\mathrm{End}_\Gamma(v')$.
iii. If $\#\,\mathrm{End}_\Gamma(v') = q^2 - 1$ then:
A. Add the vertex v' to $\mathrm{V}(\mathcal{G})$ and e and $e^\star$ to $\mathrm{E}(\mathcal{G})$.
B. Store $(v', \mathrm{End}_\Gamma(v'))$ as a vertex label for v'.
C. Remove e from L.
iv. If $\#\,\mathrm{End}_\Gamma(v') = q - 1$, then for all $j < i$ do the following:
A. Let $e' = (w, w')$ be the jth element in L.
B. Compute $\mathrm{Hom}_\Gamma(v', w')$.
C. If $\mathrm{Hom}_\Gamma(v', w') \neq \varnothing$, then do the following: • Add an edge e' from v to w' to $\mathrm{E}(\mathcal{G})$, as well as its opposite.
• Give e' the label (e, g_e) for some $g_e \in \mathrm{Hom}_\Gamma(v', w')$ and give $e'^\star$ the label $(e, -1)$.
• Remove (v, v') from L and set $j := i$.
• Remove $(w', \mathrm{vnf}(g_e v))$ from L'.
• If now $\mathrm{degree}_\mathcal{G}(w') = q + 1$, then remove (w, w') from L.
D. Continue with the next j.
v. If at the end of the j-loop we have $j = i$, then:
A. Add v' to $\mathrm{V}(\mathcal{G})$, add e and $e^\star$ to $\mathrm{E}(\mathcal{G})$.
B. For all adjacent vertices $w \neq v$ of v' in $\mathcal{T}$ add (v', w) to L'.

(ii) Set $L := L'$ and $L' := \varnothing$.

(d) If L is empty, return $\mathcal{G}$.

Remark 2.33 One could randomly choose a vertex $[L(n, g)]$ as v_0 and replace it by $[L(n + 1, g)]$, if it is projectively unstable. In this case, one would need to change the input of Algorithm 2.32 accordingly.

Remark 2.34 The vertex label $(v, 1)$ is used at all projectively stable vertices. For these, the stabilizer is the center of $\mathrm{GL}_2(K_\infty)$ intersected with Γ. There is no need to store this group each time. The same remark applies to all edges labeled $(e, 1)$.
A maximal subtree $\mathcal{S}$ of $\mathcal{G}$ consists of all vertices of $\mathcal{G}$ and those edges of $\mathcal{G}$ with edge label $(e, 1)$. It is completely realized within $\mathcal{T}$.
The edges with label (e, g) are the edges which occur (ultimately) in PE. The edge label $(e, -1)$ indicates that the opposite edge has a label (e, g). It is clear that the vertex and edge label allow one to easily construct an enhanced fundamental domain $(\mathcal{S}, \mathcal{Y})$ with an edge pairing and labels for the action of Γ on $\mathcal{T}$.

Theorem 2.35 *Suppose* Γ *from Algorithm 2.32 satisfies the following conditions:*

(a) $d(v, gv)$ *is even for all* $g \in \Gamma, v \in \mathrm{V}(\mathcal{T})$*,*

(b) for simplices t *of* $\mathcal{T}$ *either* $\bar{\Gamma}_t$ *is trivial, or* t *is a vertex and* $\bar{\Gamma}_t \cong \mathbb{Z}/(q+1)$*,*

(c) $\Gamma \backslash \mathcal{T}$ *is finite.*

Then Algorithm 2.32 terminates and computes an enhanced fundamental domain for $\mathcal{T}$ *under* Γ*.*

By the results in Section 2.4, hypotheses (a)–(c) are satisfied if Γ is the unit group of a maximal order of a quaternion algebra D as in Assumption 2.15.

Proof: Let $\mathcal{G}$ be the output of Algorithm 2.32. We show that any two distinct simplices of $\mathcal{G}$ have labels $(t, ?)$ and $(t', ?)$ with $t' \notin \Gamma t$ and that for all simplices t of $\mathcal{T}$ there is a simplex of $\mathcal{G}$ whose label is $(t', ?)$ for some $t' \in \Gamma t$.
For the first assertion, let $v_1, v_2 \in \mathrm{V}(\mathcal{T})$ be distinct first entries in labels of vertices of $\mathcal{G}$ and suppose that $\gamma v_1 = v_2$ for some $\gamma \in \Gamma \smallsetminus \Gamma_{v_1}$. We seek a contradiction. In a first reduction step we show that we may assume that v_1 is projectively stable: So suppose v_1 is projectively unstable. Then since

$$\mathrm{Stab}_\Gamma(v_2) = \gamma \,\mathrm{Stab}_\Gamma(v_1)\gamma^{-1}, \tag{1}$$

also v_2 has to be projectively unstable. Hence both v_1 and v_2 are terminal vertices in $\mathcal{G}$. Let v_1' and v_2' be their unique adjacent vertices in $\mathcal{G}$. Since v_1' is adjacent to v_1, it follows that $\gamma v_1'$ is adjacent to $\gamma v_1 = v_2$. By condition (b) the stabilizer $\mathrm{Stab}_\Gamma(v_2)$ acts transitively on the vertices adjacent to v_2. Hence there exists $\gamma' \in \mathrm{Stab}_\Gamma(v_2)$ with

$$\gamma'\gamma v_1' = v_2', \tag{2}$$

and so v_1' and v_2' are Γ-equivalent. If v_1' and v_2' were also projectively unstable and therefore terminal vertices in $\mathcal{G}$, then, since $\mathcal{G}$ is connected, $\mathcal{G}$ would have to be the graph consisting of the two vertices v_1, v_2 and one edge connecting them. This contradicts condition (a). Therefore v_1' and v_2' must be projectively stable and Γ-equivalent. To conclude the reduction, observe that we cannot have $v_1' = v_2'$, since in this case we must have $\gamma'\gamma \in \mathbb{F}_q^*$ from (2). But $\gamma'\gamma$ maps v_1 to v_2 and this would contradict $v_1 \neq v_2$.
Now suppose v_1 is projectively stable. Then by equation (1) so is v_2. Let v be the initial vertex of the algorithm and let $i_1 = d(v, v_1)$ and $i_2 = d(v, v_2)$. We prove the assertion by induction over i_1: If $i_1 = 1$ then also $i_2 = 1$ because of condition (a). Hence the vertices v_1 and v_2 both have the same distance 1 from v and since $\mathrm{Hom}_\Gamma(v_1, v_2) = q - 1$, Algorithm 2.32 with the first choice of L rules out that they

both lie in $\mathcal{G}$. This is a contradiction. The same reasoning rules out $i_1 = i_2$ for any $i_1, i_2 \geq 1$.
Suppose $i_1 > 1$. By condition (a) and the previous line we may assume $i_1 = i_2 + 2m$ for some $m \in \mathbb{Z}_{\geq 1}$. Let v_1' be the vertex on the geodesic from v_1 to v so that $d(v, v_1') = i_1 - 1$. Then by the construction of $\mathcal{G}$ we have $v_1' \in \mathcal{G}$. The vertex $\gamma v_1'$ is adjacent to $\gamma v_1 = v_2$. Now observe that $\gamma v_1'$ does not belong to $\mathcal{G}$ because otherwise we could apply the induction hypothesis to $v_1', \gamma v_1'$, using $d(v_1', v) = i_1 - 1$ and $d(\gamma v_1', v) \leq i_2 + 1$ to obtain a contradiction.
It follows that $v_2' := \gamma v_1' \notin \mathcal{G}$. Since by construction the geodesic from v to v_2 lies on $\mathcal{G}$, we have $d(v_2', v) = i_2 + 1$. Now by the algorithm that defines $\mathcal{G}$ the vertex v_2' must be equivalent to a vertex of distance $i_2 - 1$, i.e., there are $\gamma' \in \Gamma, v_2'' \in \mathcal{G}$ with $d(v_2'', v) = i_2 - 1$ such that $v_2'' = \gamma' v_2'$. But then we apply the induction hypothesis to v_1', v_2'' and again obtain a contradiction. This concludes the proof of the first assertion for vertices.
Suppose now that $e = (v_0, v_1), e' = (v_0', v_1')$ occur as first entries in $\mathrm{E}(\mathcal{G})$, lie in the same Γ-orbit, are distinct and occur in some edge labels of $\mathcal{G}$. Let γ be in Γ with $e' = \gamma e$. Note that not all the vertices v_i and v_i' must occur in vertex labels from $\mathcal{G}$ but each edge must at least have one vertex that does – see step (c)(i)4.C. Suppose after possibly changing the orientation of edges and the indices that v_0 has minimal distance from v. By construction of $\mathcal{G}$ the vertex v_0 occurs in a vertex label. If $v_0' = \gamma v_0$ occurs in a vertex label of $\mathcal{G}$, then by the case already treated, we must have $v_0 = v_0'$. Since $e \neq e'$ it follows that v_0 is projectively unstable. But then the algorithm does not yield an edge starting at v_0 and ending at a vertex v_1 with $d(v, v_1) > d(v, v_0)$. This is a contradiction.
It follows that $v_0' = \gamma v_0$ does not occur in a vertex label. Hence v_1' must occur in a vertex label. By essentially the argument just given, v_1 can also not occur in a vertex label. Hence (e, γ) must be an edge label and moreover $d(v, v_1') = d(v, v_0) + 1 = d(v, v_0') - 1$. But then in step (c)(i)4.C of Algorithm 2.32 the edge e' must have been removed from the list L' and so it cannot occur in a label of an edge of $\mathcal{G}$.

We finally come to the second assertion: By construction, $\mathcal{G}$ defines a connected graph. It is a subgraph of $\Gamma \backslash \mathcal{T}$, since we already showed that there are no Γ-equivalent simplices in $\mathcal{G}$. Moreover, at any vertex of this subgraph the degree within $\mathcal{G}$ and within $\Gamma \backslash \mathcal{T}$ is the same. Hence $\mathcal{G}$ defines a connected component of $\Gamma \backslash \mathcal{T}$. But $\mathcal{T}$ and hence $\Gamma \backslash \mathcal{T}$ are connected and thus $\mathcal{G} = \Gamma \backslash \mathcal{T}$. ∎

We further describe an algorithm to compute for any $v' \in \mathrm{V}(\mathcal{T})$ a Γ-equivalent vertex $v'' \in \mathcal{G}$. This can be done in time linear to the distance from v' to $\mathcal{G}$. For this algorithm we need the stabilizers of the terminal vertices of $\mathcal{G}$ and the elements $\gamma \in \mathrm{Hom}_\Gamma(v_i, v_j)$, which we both stored as vertex and edge labels during the computation of $\mathcal{G}$. We call this algorithm the reduction algorithm. We need to be able to do the following:

(a) Find the geodesic from v' to v. This was discussed in Remark 2.10.

(b) Determine the extremities of a given geodesic in $\mathcal{G}$. Since the vertices in $\mathcal{G}$ are all stored in the vertex normal form, this can be done in constant time.

Algorithm 2.36 (The reduction algorithm)
Input: *$v' \in \mathrm{V}(\mathcal{T})$ and $\mathcal{G}$ the output of Algorithm 2.32 with initial vertex v.*
Output: *A tuple $(w, \gamma) \in \mathrm{V}(\mathcal{G}) \times \Gamma$ with $v' = \gamma w$.*
Algorithm:

(a) Let $\mathcal{T}_0 : (v' = v_m, v_{m-1}, \ldots, v)$ be the geodesic from v' to v. Let v_i be the vertex of $\mathcal{T}_0 \cap \mathcal{G}$ closest to v'. Let $r = m - i$, this is the distance from v' to $\mathcal{G}$.

(b) If $r = 0$, we have $v' \in \mathcal{G}$. Then return $(v', 1)$.

(c) If $r > 0$, we distinguish two cases:

  (i) If v_i is projectively unstable, by a for-loop through the elements γ in $\mathrm{Stab}_\Gamma(v_i)$, find an element $\gamma \in \Gamma$ such that γv_{i+1} is a vertex of $\mathcal{G}$. Replace v' by $\gamma v'$ and apply the algorithm recursively to get some pair $(w, \tilde{\gamma})$ in $\mathrm{V}(\mathcal{G}) \times \Gamma$. Return $(w, \tilde{\gamma}\gamma)$.

  (ii) If v_i is projectively stable, run a for-loop through the vertices $\tilde{v}$ in $\mathcal{G}$ adjacent to v_i to find the unique $\tilde{v}$ such that either: (i), the edge label of the edge from $\tilde{v}$ to v_i is of the form (e, γ) for some $\gamma \in \Gamma$ with $\gamma t(e) = v_i$ and $\gamma o(e) = v_{i+1}$, or (ii), the edge label from v_i to $\tilde{v}$ is of the form (e, γ) for some $\gamma \in \Gamma$ with $o(e) = v_i$ and $t(e) = v_{i+1}$. In case (i), replace v' by $\gamma^{-1}v'$ and apply the algorithm recursively to get some pair $(w, \tilde{\gamma})$ for $\gamma^{-1}v'$. Return $(w, \tilde{\gamma}\gamma^{-1})$. In case (ii), replace v' by $\gamma v'$ and apply the algorithm recursively to get some pair $(w, \tilde{\gamma})$ for $\gamma v'$. Return $(w, \tilde{\gamma}\gamma)$.

Proposition 2.37 *Let v' in $\mathcal{T}$ and let $\mathcal{G}$ be the output of Algorithm 2.32 under the hypothesis of Theorem 2.35 with initial vertex v. Then Algorithm 2.36 computes a Γ-equivalent vertex w of v' and an element $\gamma \in \Gamma$ with $\gamma v' = w$. It requires $\mathcal{O}(n^3 \deg(r)^2)$ additions and multiplications in $\mathbb{F}_q$ where n is the distance of v' to $\mathcal{G}$.*

Proof: In both cases of the algorithm we find an edge label that moves v' closer to $\mathcal{G}$. Since each step of the algorithm decreases the distance $d(v', \mathcal{G})$, the algorithm terminates after at most n steps. From Corollary 2.63 and Proposition 2.46 it follows that at step j one multiplies a matrix of height $(j-1)\frac{5}{2}\deg(r)$ with one of height $\frac{5}{2}\deg(r)$. Further one has to compute the vertex normal form of a matrix of height at most $(j+1)\frac{5}{2}\deg(r)$. This takes at most $(8j + 8j^2)(\frac{5}{2})^2 \deg(r)^2$ operations in $\mathbb{F}_q$. Summing over j, the asserted bound follows. ∎

Example 2.38 In Figure 2 we give an example of the Algorithm 2.32, where $q = 5$ and $r = T(T+1)(T+2)(T+3)$. We start with $\begin{pmatrix} 1/T & 0 \\ 0 & 1 \end{pmatrix}$ as the initial vertex v. The adjacent vertices correspond to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is a terminal vertex, and the five matrices $\begin{pmatrix} 1/T^2 & \alpha 1/T \\ 0 & 1 \end{pmatrix}$ with $\alpha \in \mathbb{F}_5$. Using the algorithm described in Section 2.7 we compute that $\begin{pmatrix} 1/T^2 & 0 \\ 0 & 1 \end{pmatrix}$ is the only projectively unstable vertex and

$$\#\,\mathrm{Hom}_\Gamma(\begin{pmatrix} 1/T^2 & 1/T \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/T^2 & 41/T \\ 0 & 1 \end{pmatrix}) = 4,$$

$$\#\,\mathrm{Hom}_\Gamma(\begin{pmatrix} 1/T^2 & 2\pi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/T^2 & 3\pi \\ 0 & 1 \end{pmatrix}) = 4.$$

This finishes Step 1 of the algorithm, as depicted in Figure 2. In Step 2 we then continue with the eight indicated vertices of level 3. In this case, the algorithm terminates after 3 steps.

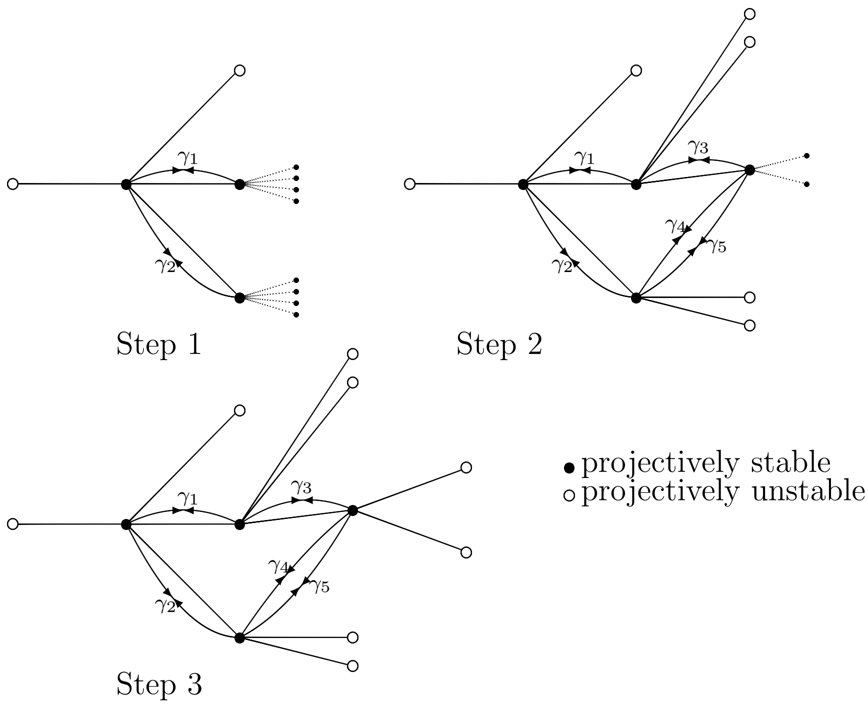

Figure 2: Example: $q = 5$, $r = T(T+1)(T+2)(T+3)$

Example 2.39 Consider $K = \mathbb{F}_5(T)$ and the two discriminants $r_1 = (T^2+T+1)\cdot T\cdot(T+1)\cdot(T+2)$ and $r_2 = (T^2+2)\cdot T\cdot(T+1)\cdot(T+2)$. Let Γ_i be the group of units of a maximal order of a quaternion algebra of discriminant r_i for $i \in \{1,2\}$. Then $\Gamma_1\backslash\mathcal{T}$ has 14 cycles of length 2, while $\Gamma_2\backslash\mathcal{T}$ has 10 cycles of length 2. Hence these two graphs are not isomorphic. This answers a question of Papikian who asked for an example in which the lists of degrees of the factors of r and r' are the same but where the graphs are non-isomorphic. This is similar to [GN, Rem 2.22] where congruence subgroups $\Gamma_0(\mathfrak{n})$ and $\Gamma_0(\mathfrak{n}')$ of $\mathrm{GL}_2(A)$ are considered.

2.6 Concrete models for D and Λ

In Algorithm 2.32 we assumed the existence of a routine for computing $\mathrm{Hom}_\Gamma(v,v')$ for all $v,v' \in \mathrm{V}(\mathcal{T})$ which are equidistant from $[L(0,0)]$. Such a routine will be described in Section 2.7. It is based on concrete models for the pair (D,Λ) from Section 2.3. We will describe such models here, assuming q odd. We will also describe explicit embeddings of D into $\mathrm{GL}_2(K_\infty)$.

First assume $\mathrm{odd}(R) = 1$.

Lemma 2.40 *Let $R = \{\mathfrak{p}_1,\dots,\mathfrak{p}_l\}$ be a set of finite places of K with $\deg(\mathfrak{p}_i)$ odd for all i and l even, let r be a monic generator of the ideal $\mathfrak{r} = \prod_{i=1}^{l}\mathfrak{p}_i$ and let $\xi \in k^\star\backslash(k^\star)^2$. Then $\left(\frac{\xi,r}{K}\right)$ is ramified exactly at the places $\mathfrak{p}_1,\dots,\mathfrak{p}_l$.*

Proof: We compute the Hilbert symbols using Proposition 2.19. For $(\varpi) = \mathfrak{p} \notin R$ we have

$$(\xi,r)_\mathfrak{p} = (-1)^0\left(\frac{\xi}{\varpi}\right)^0\left(\frac{r}{\varpi}\right)^0 = 1$$

and for $(\varpi) = \mathfrak{p} \in R$ we have

$$(\xi,r)_\mathfrak{p} = (-1)^0\left(\frac{\xi}{\varpi}\right)^1\left(\frac{r/\varpi}{\varpi}\right)^0 = \left(\frac{\xi}{\varpi}\right) = \xi^{\frac{q-1}{2}\deg(\varpi)} = -1,$$

since $\deg(\varpi)$ is odd. ∎

Set $D = \left(\frac{\xi,r}{K}\right)$ with $\xi \in k^\star\backslash(k^\star)^2$.

Lemma 2.41

$$\Lambda := \langle 1,i,j,ij\rangle_A$$

is a maximal order of D.

PROOF: Λ is clearly closed under multiplication and a lattice of rank 4. Hence Λ is an order. We have to check, that Λ is maximal. This is the ideal generated by

$$\det\begin{pmatrix} \mathrm{trd}(1) & \mathrm{trd}(i) & \mathrm{trd}(j) & \mathrm{trd}(ij) \\ \mathrm{trd}(i) & \mathrm{trd}(i^2) & \mathrm{trd}(ij) & \mathrm{trd}(i^2j) \\ \mathrm{trd}(j) & \mathrm{trd}(ji) & \mathrm{trd}(j^2) & \mathrm{trd}(jij) \\ \mathrm{trd}(ij) & \mathrm{trd}(iji) & \mathrm{trd}(ij^2) & \mathrm{trd}(ijij) \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\xi & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & -2\xi r \end{pmatrix} = -16\xi^2 r^2.$$

We see, that Λ has reduced discriminant $\mathfrak{r}$, and hence is a maximal order. ∎

Recall that r is monic and has even degree l. Hence there exists a square root of r in K_∞. We choose one and denote it by $\sqrt{r}$. The following Lemma shows that we can effectively compute $\sqrt{r} \in K_\infty$ to high precissions.

Lemma 2.42 *Let α be monic of even degree in A. To compute $\sqrt{\alpha}$ in $K_\infty = \mathbb{F}_q((\pi))$ to n digits of accuracy one requires $\mathcal{O}(n^3)$ additions and multiplications in $\mathbb{F}_q$.*

PROOF: Let $m = \deg(\alpha)$. It suffices to compute the square root u of the 1-unit $\pi^m\alpha$ to n digits accuracy. This can be done by the Newton iteration in n steps starting with the approximation $u_0 = 1$. The k-th approximation is $u_k = u_{k-1} - \frac{u_{k-1}^2 - \pi^m\alpha}{u_{k-1}}$. From the right hand expression one only needs to compute $u_{k-1}^2 - \pi^m\alpha$ which requires n^2 operations in $\mathbb{F}_q$. The k-th digit past the decimal point divided by 2 has then to be subtracted from u_{k-1}. ∎

Lemma 2.43 *The map $\iota : D \to M_2(K_\infty)$ defined by sending $i \mapsto \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix}$ and $j \mapsto \begin{pmatrix} \sqrt{r} & 0 \\ 0 & -\sqrt{r} \end{pmatrix}$ gives an isomorphism $D \otimes_K K_\infty \cong M_2(K_\infty)$.*

PROOF: Since $r = \prod_{i=1}^{l} \varpi_i$ with $\varpi_i \in A$, l even and all $\deg(\varpi_i)$ odd, the degree of r is even, hence we have $\sqrt{r} \in K_\infty$. One checks that the given matrices $\iota(i)$ and $\iota(j)$ fulfil the relations $\iota(i)^2 = \xi, \iota(j)^2 = r$ and $\iota(i)\iota(j) = -\iota(j)\iota(i)$. This easily yields $\left(\frac{\xi, r}{K_\infty}\right) \cong M_2(K_\infty)$ under ι. The isomorphism $D \otimes_K K_\infty \cong \left(\frac{\xi, r}{K_\infty}\right)$ is obvious by construction of D. ∎

Now let us drop the assumption $\mathrm{odd}(R) = 1$. Let $l \geq 2$ be even and let R be a set of l distinct prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ of A. Denote by ϖ_i the unique monic (irreducible) generator of $\mathfrak{p}_i$. Set $r := \prod_i \varpi_i$ and $\mathfrak{r} := \prod_i \mathfrak{p}_i$ where the index i ranges over $1, \ldots, l$.

Lemma 2.44 *There is an irreducible monic polynomial $\alpha \in A$ of even degree such that*

$$\left(\frac{\alpha}{\varpi_i}\right) = -1 \text{ for all } i. \tag{3}$$

Any such α also satisfies $\left(\frac{r}{\alpha}\right) = 1$.

PROOF: Choose any $a \in A$ such that

$$\left(\frac{a}{\varpi_i}\right) = -1$$

for all i. This can be done using the Chinese remainder theorem. By the strong form of the function field analogue of Dirichlet's theorem on primes in arithmetic progression, [Ro, Thm. 4.8], the set $\{a + rb \mid b \in A\}$ contains an irreducible monic polynomial α of even degree. Since $\alpha \equiv a \pmod{\varpi_i}$ we have

$$\left(\frac{\alpha}{\varpi_i}\right) = -1$$

for all i. By quadratic reciprocity, [Ro, Thm. 3.3], we deduce

$$\left(\frac{\varpi_i}{\alpha}\right) = (-1)^{\frac{q-1}{2} \deg \alpha \deg \varpi_i} \left(\frac{\alpha}{\varpi_i}\right) = -1$$

since $\deg(\alpha)$ is even. But then because l is even, we find

$$\left(\frac{r}{\alpha}\right) = \prod_{i=1}^{l} \left(\frac{\varpi_i}{\alpha}\right) = (-1)^l = 1. \ \blacksquare$$

Remark 2.45 In practice α is rapidly found by the following simple search:

Step 1: Start with $m = 2$.

Step 2: Check for all monic irreducible $\alpha \in A$ of degree m whether $\left(\frac{\alpha}{\varpi_i}\right) = -1$ for all $1 \leq i \leq l$.

Step 3: If we found an α then stop. Else increase m by 2 and go back to Step 2.

In the function field setting [MS] gives an unconditional effective version of the Čebotarov density theorem. This allows us to make Lemma 2.44 effective, i.e., to give explicit bounds on $\deg(\alpha)$ in terms of $\deg(r)$. The following result is from [BB] and was suggested by G. Böckle.

Proposition 2.46 *Abbreviate $d := \deg(r)$. The following table gives upper bounds on $d_\alpha := \deg(\alpha)$ depending on q and l:*

	$q=3$			$q=5,7$		$q=9$		$q\geq 11$	
	$l\leq 4$	$l=6$	$8\leq l$	$l\leq 6$	$8\leq l$	$l\leq 4$	$6\leq l$	$l=2$	$4\leq l$
$d_\alpha \leq$	$d+7$	$d+5$	$d+1$	$d+3$	$d+1$	$d+3$	$d+1$	$d+3$	$d+1$

A basic reference for the results on function fields used in the following proof is [Sti].

PROOF: Let $K' := K(\sqrt{\varpi_1},\ldots,\sqrt{\varpi_l})$. Then K'/K is a Galois extension with Galois group isomorphic to $\{\pm1\}^l$ with $\{\pm1\}\cong\mathbb{Z}/(2)$; it is branch locus in K is the divisor $\mathcal{D}$ consisting of the sum of the (ϖ_k) and (possibly) ∞; the constant field of K' is again $\mathbb{F}_q$. Denote by g' the genus of K' and by D' the ramification divisor of K'/K. The ramification degree at all places is 1 or 2 and hence tame because q is odd. It follows that $\deg(\mathcal{D}') = \#G/2\cdot\deg(\mathcal{D})$.
Let $\pi(k)$ denote the places of K of degree k; let $\pi_C(k)$ denote the places $\mathfrak{p}$ of K of degree k for which $\mathrm{Frob}_\mathfrak{p} = (-1,\ldots,-1)\in\{\pm1\}^l$. Note that the elements of $\pi_C(k)$ are in bijection to the monic irreducible polynomials α of degree k which satisfy the conditions (3). The following two inequalities are from [MS, Thm. 1 and (1.1)] and the Hurwitz formula, respectively:

$$\left|\pi_C(k) - \frac{1}{\#G}\pi(k)\right| \leq 2g'\frac{1}{\#G}\frac{q^{k/2}}{k} + 2\frac{q^{k/2}}{k} + \left(1+\frac{1}{k}\right)\deg(\mathcal{D}'). \tag{4}$$

$$\left|q^k + 1 - k\pi(k)\right| \leq 2g'\frac{q^{k/2}}{k}. \tag{5}$$

$$2g' = -2\#G + \deg(\mathcal{D}') + 2. \tag{6}$$

After some manipulations one obtains

$$\pi_C(k) \geq \frac{1}{\#G}\frac{q^k+1}{k} - \frac{q^{k/2}}{k}\left(\frac{\deg(\mathcal{D})}{2} + 2 + \frac{2}{\#G}\right) - \left(1+\frac{1}{k}\right)\#G\frac{\deg(\mathcal{D})}{2}.$$

To ensure that the right hand side is positive for some (even) k, it thus suffices that

$$f(k) := q^k - q^{k/2}\Big(2^{l-1}(\deg(r)+5)+2\Big) - \big(k+1\big)2^{2l-1}(\deg(r)+1) > 0. \tag{7}$$

We know that l is the number of prime factors of r and hence that $l\leq\deg(r)$. There are at most q places of degree 1 and so for small q such as $3,5,7$, already for small l

the degree of r must be quite a bit larger than l. For instance if $l \geq 7$ and $q = 3$, then $\deg(r) \geq 3l - 9$. Using these considerations and simple analysis on $f(k)$, it is simple if tedious to obtain the lower bounds in the table. We leave details to the reader. ∎

For the rest of this section let $D := \left(\frac{\alpha, r}{K}\right)$.

Proposition 2.47 *For α as in Lemma 2.44, the quaternion algebra D is ramified exactly at R.*

Proof: We compute the Hilbert symbols using Proposition 2.19. For $(\varpi) = \mathfrak{p} \notin R$ and ϖ not equal to α we have

$$(\alpha, r)_{\mathfrak{p}} = (-1)^0 \left(\frac{\alpha}{\varpi}\right)^0 \left(\frac{r}{\varpi}\right)^0 = 1$$

and for $\mathfrak{p} = (\alpha)$ we have

$$(\alpha, r)_{\mathfrak{p}} = (-1)^0 \left(\frac{1}{\alpha}\right)^0 \left(\frac{r}{\alpha}\right)^1 = 1.$$

Finally for $(\varpi) = \mathfrak{p} \in R$ we have

$$(\alpha, r)_{\mathfrak{p}} = (-1)^0 \left(\frac{\alpha}{\varpi}\right)^1 \left(\frac{r/\varpi}{\varpi}\right)^0 = -1. \blacksquare$$

Since r is a square modulo α, there are $\varepsilon, \nu \in A$ with $\deg(\varepsilon) < \deg(\alpha)$ and $\varepsilon^2 = r + \nu\alpha$.

Proposition 2.48 $\Lambda := \langle 1, i, j, \frac{\varepsilon i + ij}{\alpha} \rangle$ *is a maximal A-order of D.*

Proof: We first check, that Λ is an order. Let

$$\gamma = a + bi + cj + d\frac{\varepsilon i + ij}{\alpha} = a + (b + \frac{d\varepsilon}{\alpha})i + cj + \frac{d}{\alpha}ij$$

with $a, b, c, d \in A$ be any element of Λ. Then $\mathrm{trd}(\gamma) = 2a$ and

$$\mathrm{nrd}(\gamma) = a^2 - \alpha(b + \frac{d\varepsilon}{\alpha})^2 - rc^2 + \alpha r(\frac{d}{\alpha})^2 = a^2 - rc^2 + \frac{rd^2}{\alpha} - \frac{b^2\alpha^2 + 2bd\varepsilon\alpha + d^2\varepsilon^2}{\alpha}$$

$$= a^2 - rc^2 - b^2\alpha - 2bd\varepsilon + d^2\frac{r - \varepsilon^2}{\alpha} = a^2 - rc^2 - b^2\alpha - 2bd\varepsilon + d^2\nu$$

are both in A. Since $\{e_1, e_2, e_3, e_4\}$ is an K-basis of D, we conclude that Λ is an A-lattice of D. To check that Λ is a ring, we compute

$$e_1 e_i = e_i e_1 = e_i,$$

$$\begin{aligned}
e_2 e_3 &= -e_3 e_2 = ij = \alpha e_4 - \varepsilon e_2, \\
e_2 e_4 &= i\frac{\varepsilon i + ij}{\alpha} = \varepsilon + j = \varepsilon e_1 + e_3, \\
e_4 e_2 &= \frac{\varepsilon i + ij}{\alpha} i = \varepsilon - j = \varepsilon e_1 - e_3, \\
e_3 e_4 &= j\frac{\varepsilon i + ij}{\alpha} = -\frac{\varepsilon ij + ij^2}{\alpha} = -\frac{\varepsilon ij + ir}{\alpha} \\
&= -\frac{\varepsilon ij + i(\varepsilon^2 - \alpha\nu)}{\alpha} = \nu i - \varepsilon\frac{\varepsilon i + ij}{\alpha} = \nu e_2 - \varepsilon e_4, \\
e_4 e_3 &= \frac{\varepsilon i + ij}{\alpha} j = -j\frac{\varepsilon i + ij}{\alpha} = -e_3 e_4 = -\nu e_2 + \varepsilon e_4,
\end{aligned}$$

and

$$e_1^2 = e_1, e_2^2 = \alpha e_1, e_3^2 = r e_1, e_4^2 = \frac{(\varepsilon i + ij)^2}{\alpha^2} = \frac{\varepsilon^2 i^2 - i^2 j^2}{\alpha^2} = \frac{\varepsilon^2 - r}{\alpha} = \nu e_1.$$

So Λ is closed under multiplication and hence an order. To check that Λ is maximal, we compute

$$\det(\operatorname{trd}(e_i e_j)_{i,j=1,\ldots,4}) = \det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 2\varepsilon \\ 0 & 0 & 2r & 0 \\ 0 & 2\varepsilon & 0 & 2\nu \end{pmatrix} = 16r(\alpha\nu - \varepsilon^2) = -16r^2$$

and hence the ideal generated by $\det(\operatorname{trd}(e_i e_j)_{i,j=1,\ldots,4})$ is equal to (r^2). ∎

Since α has even degree and is monic, there exists a square root of α in K_∞. We choose one and denote it by $\sqrt{\alpha}$. Again Lemma 2.42 provides an effectiv way of computing $\sqrt{\alpha} \in K_\infty$ up to arbitrary precission.

Lemma 2.49 *The K-algebra homomorphism $\iota : D \to M_2(K_\infty)$ defined by $i \mapsto \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix}$ and $j \mapsto \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}$ induces an isomorphism $D \otimes_K K_\infty \cong M_2(K_\infty)$.*

PROOF: One verifies $\iota(i)^2 = \alpha, \iota(j)^2 = r$ and $\iota(i)\iota(j) = -\iota(j)\iota(i)$ by an explicit calculation. ∎

2.7 Computing $\mathrm{Hom}_\Gamma(v, w)$

In this section we finaly present a routine for computing $\mathrm{Hom}_\Gamma(v, v')$ for all $v, v' \in \mathrm{V}(\mathcal{T})$ which are equidistant from $[L(0,0)]$. We will also bound the size of the elements which show up in $\mathrm{Hom}_\Gamma(v, v')$. Let $r, \alpha, \varepsilon, \Lambda$ and $D = \left(\frac{\alpha}{r}\right) K$ and ι be as at the end of Section 2.6. To state our result, we define a (logarithmic) height $\| \ \|$ on elements of Λ.

Definition 2.50 *(a) For $(\lambda_1, \ldots, \lambda_4) \in A^4$ define*

$$\|\lambda_1 \cdot 1 + \lambda_2 \cdot i + \lambda_3 \cdot j + \lambda_4 \cdot \frac{\varepsilon i + ij}{\alpha}\| := \max_{i=1,\ldots,4} \deg(\lambda_i)\|.$$

(b) For M a matrix or a vector with entries in K_∞ define $v_\infty(M)$ to be the minimum of all the v_∞-valuations of all entries.

Theorem 2.51 *Suppose $v, v' \in \mathrm{V}(\mathcal{T})$ have distance n from $v_0 = [L(0,0)]$.*

(a) There is an algorithm that computes $\mathrm{Hom}_\Gamma(v, v')$ in time $\mathcal{O}(n^4)$ field operations over $\mathbb{F}_q$.

(b) All $\gamma \in \mathrm{Hom}_\Gamma(v, v')$ satisfy $\|\gamma\| \le n + \deg(\alpha)/2$.

PROOF: If $v = [L(l, g)]$ has distance n from v_0, then either

$$\begin{aligned} l = n \quad &\text{and} \quad \deg_l(g) \text{ lies in } \{0, \ldots, n\} \text{ or} \\ l \in \{-n, -n+2, -n+4, \ldots, n-2\} \quad &\text{and} \quad \deg_l(g) = \frac{n+l}{2}, \end{aligned}$$

see Figure 1 and Remark 2.10. Moreover the path from $[L(0,0)]$ to $[L(l,g)]$ is via $L(\frac{l-n}{2}, 0)$ if $l < n$ and via $L(n - \deg_l(g), 0)$ if $l = n$. Set $n_1 := \deg_l(g)$ and $n_2 := n - n_1$ if $l = n$ and $n_2 = n_1 - n$ if $l < n$. In Figure 1, the integers n_1 and $n_2 \in \mathbb{Z}$ are the coordinates of v from the baseline toward it and along the baseline, respectively. Moreover $l = n_1 + n_2$ and $g \in \pi^{l-n_1}\mathcal{O}_\infty = \pi^{n_2}\mathcal{O}_\infty$. Similarly we define n_1' and n_2' for $v' = [L(l', g')]$ which is also of distance n from $v_0 = [L(0,0)]$.

Let now $\gamma = \begin{pmatrix} \pi^l & g \\ 0 & 1 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} \pi^{l'} & g' \\ 0 & 1 \end{pmatrix}$ be the matrices in vertex normal form representing v and v' respectively. By definition of Hom_Γ we have

$$\mathrm{Hom}_\Gamma(v, v') = \gamma' \mathrm{GL}_2(\mathcal{O}_\infty) K_\infty^* \gamma^{-1} \cap \Gamma.$$

Because $v_\infty(\det(\gamma)) = l, v_\infty(\det((\gamma')^{-1}) = l'$, $v_\infty(\det(\sigma)) = 0$ for all $\sigma \in \mathrm{GL}_2(\mathcal{O}_\infty)$ and $v(\det \Gamma) = \{0\}$, we see that

$$\mathrm{Hom}_\Gamma(v, v') = \pi^{(l-l')/2} \gamma' \mathrm{GL}_2(\mathcal{O}_\infty) \gamma^{-1} \cap \Gamma,$$

where we simply write $\pi^{(l-l')/2}$ for the scalar matrix $\pi^{(l-l')/2} \cdot 1_2$. By taking determinants on both sides and using the fact that $\mathcal{O}_\infty \cap A = \mathbb{F}_q$, we finally obtain

$$\mathrm{Hom}_\Gamma(v, v') \ \dot{\cup} \ \{0\} \ = \ \pi^{(l-l')/2}\gamma' M_2(\mathcal{O}_\infty)\gamma^{-1} \ \cap \ \Lambda. \tag{8}$$

Set

$$C = \begin{pmatrix} 1 & \sqrt{\alpha} & 0 & \frac{\varepsilon}{\sqrt{\alpha}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{\alpha}} \\ 0 & 0 & r & \frac{-r}{\sqrt{\alpha}} \\ 1 & -\sqrt{\alpha} & 0 & \frac{-\varepsilon}{\sqrt{\alpha}} \end{pmatrix} \text{ and } B = \begin{pmatrix} \pi^{\frac{l'-l}{2}} & 0 & g'\pi^{\frac{-l'-l}{2}} & 0 \\ -g\pi^{\frac{l'-l}{2}} & \pi^{\frac{l'+l}{2}} & -gg'\pi^{\frac{-l'-l}{2}} & g'\pi^{\frac{l-l'}{2}} \\ 0 & 0 & \pi^{\frac{-l'-l}{2}} & 0 \\ 0 & 0 & -g\pi^{\frac{-l'-l}{2}} & \pi^{\frac{l-l'}{2}} \end{pmatrix}.$$

Observe that $v_\infty(g\pi^{-\frac{l}{2}}) \geq n_2 - \frac{n_1+n_2}{2} = \frac{n_2-n_1}{2} \geq -\frac{n}{2}$ and that $-|l| \geq -n$. This implies that $v_\infty(B) \geq -n$. Similarly, using $\deg(\varepsilon) \leq \deg(\alpha)$ and computing C^{-1} explicitly, one finds $v_\infty(C^{-1}) \geq -m$ where we abbreviate $m := \frac{\deg(\alpha)}{2} \in \mathbb{Z}_{\geq 1}$.
We now flatten 2×2-matrices in $M_2(K_\infty)$ to column vectors of length 4. Taking the explicit form of the A-basis of Λ from Lemma 2.49 into account, as well as the explicit forms of γ and γ', the solutions to (8) are the solution of the linear system of equations

$$C\underline{\lambda} = B\underline{x}, \tag{9}$$

where $\underline{\lambda}$ denotes a (column) vector in A^4 and $\underline{x}$ a (column) vector in $\mathcal{O}_\infty^4$. The equivalent form $\underline{\lambda} = C^{-1}B\underline{x}$ and the above estimates on the valuations of C^{-1} and B now immediately imply $v_\infty(\underline{\lambda}) \geq -(n+m)$. In other words, the components of $\underline{\lambda}$ are polynomials and $\|\underline{\lambda}\| \leq n+m$. This proves (b).
Next, consider (9) in the form $B^{-1}C\underline{\lambda} = \underline{x}$. Again by explicit computation, we have $v_\infty(B^{-1}) \geq -n$ and $v_\infty(C) \geq -\max\{\deg(r), m\} =: -d$. Writing $B^{-1}C = \sum_{k=-d}^{\infty} X_k\pi^k$ as a power series with $X_k \in M_4(\mathbb{F}_q)$ and using the bound from (b), equation (9) is equivalent to

$$\big(\textstyle\sum_{k=-(n+m)}^{n+d} X_k\pi^{-k}\big)\underline{\lambda} \equiv 0 \pmod{\mathcal{O}_\infty^4}.$$

We also expand $\underline{\lambda} = \sum_{k=0}^{n+m} \underline{\lambda}_k\pi^{-k}$ as a polynomial in π^{-1} with $\underline{\lambda}_k \in {\mathbb{F}_q}^4$ and let X_k and $\underline{\lambda}_k$ be zero outside the range of indices k indicated above. Then (9) becomes equivalent to the system of linear equations

$$\big(\textstyle\sum_{k=0}^{n+m} X_{h-k}\underline{\lambda}_k\big) = 0, \quad h = 0, \ldots, 2n+d+m$$

in the indeterminates $\underline{\lambda}_k$ and with coefficients in $\mathbb{F}_q$. (Each equation has 4 linear components.) On the one hand, this shows that we need to compute α to accuracy $n' = 2n+d+m+1$. On the other hand, we see that, using Gauss elimination, one can solve for the unknowns in $\mathcal{O}(n'^2)$ steps where each step consists of $(4n')^2$ additions and $(4n')^2$ multiplications in the field $\mathbb{F}_q$. Regarding $\deg(r)$ as a structural constant and applying Proposition 2.46, the complexity is thus $\mathcal{O}(n^4)$. ∎

Remark 2.52 Equation (8) can be interpreted in the following way: The intersection in (8) is up to change by conjugation the same as $M_2(\mathcal{O}_\infty) \cap \pi^{(l'-l)/2}\gamma'^{-1}\Lambda\gamma$. Here $M_2(\mathcal{O}_\infty)$ is the unit ball in $M_2(K_\infty)$, a K_∞-vector space of dimension 4 and $\pi^{(l'-l)/2}\gamma'^{-1}\Lambda\gamma$ is a discrete A-lattice (of rank 4) in this vector space. I.e., we need to compute the shortest non-zero vectors of the lattice $\pi^{(l'-l)/2}\gamma'^{-1}\Lambda\gamma$ with respect to the norm given by $M_2(\mathcal{O}_\infty)$. If these vectors have norm at most one, they form $\mathrm{Hom}_\Gamma(v, v')$. If their norm is larger than one, then $\mathrm{Hom}_\Gamma(v, v')$ is empty. In particular, the problem can in principle be solved by the function field version of the LLL algorithm.
However, the implemented versions of the LLL algorithm [He, Pau] need an a priori knowledge of the precision by which α has to be computed as an element in $\mathbb{F}_q((\pi))$. This in turn makes it necessary to find a bound on the height of the elements in $\mathrm{Hom}_\Gamma(v, v')$, if described as a linear combination in terms of our standard A-basis for Λ. Moreover, [He, Pau] do not give a complexity analysis for their algorithms.

Remark 2.53 We have chosen v_0 as a reference vertex in Theorem 2.51 for simplicity. Since $\mathrm{GL}_2(K_\infty)$ acts transitively on $\mathcal{T}$, one could work with any reference vertex. Also, if one chooses v_0 as the mid point of the geodesic from v to v', one sees that the complexity of an algorithm to compute $\mathrm{Hom}_\Gamma(v, v')$ is $\mathcal{O}(d^4)$ where $d = d(v, v')$. Note that only vertices that are an even distance apart can have non-trivial $\mathrm{Hom}_\Gamma(v, v')$, because $d(v, \gamma v)$ is even for all $\gamma \in \Gamma$ and $v \in \mathcal{T}$.

Remark 2.54 Our implementation of algorithm of Theorem 2.51 uses the Gauss algorithm and not LLL. The linear system that needs to be solved has $4n'$ equations in $4n + 2\deg(\alpha)$ variables with n' as in the above proof. In practice, $\deg(\alpha) \leq \deg(r)$, compare Proposition 2.46. As we shall see in Proposition 2.62, see also Remark 2.64, we have $n \leq 2\deg(r) - 2$ and typically $\leq 2\deg(r) - 4$. Therefore we have about $4n' \leq 22\deg(r)$ equations in about $10\deg(r)$ variables. Since the number of vertices of the quotient graph is essentially $q^{\deg(r)-3}$ (and $q \geq 3$), already $\deg(r) = 10$ is a large value to compute the entire graph. Over finite fields, systems of the size just described can be solved rather rapidly.

If $\mathrm{odd}(R) = 1$, we can use the first model of (D, Λ) described in Section 2.6, which is sometimes more convenient for computations. Let $\xi \in \mathbb{F}_q^\star/(\mathbb{F}_q^\star)^2$, $D = \left(\frac{\xi, r}{K}\right)$ and $\Lambda = \langle 1, i, j, ij\rangle_A$.
Let $v_0 = [L(0,0)]$. Note that $\mathrm{Stab}_{\mathrm{GL}_2(K_\infty)}(v_0) = \mathrm{GL}_2(O_\infty)K_\infty^\star$. Hence

$$\mathrm{Stab}_\Gamma(v_0) = \mathrm{GL}_2(O_\infty)K_\infty^\star \cap \Gamma = \mathrm{GL}_2(O_\infty) \cap \Gamma$$

and

$$\mathrm{GL}_2(O_\infty) \cap \Gamma \supseteq \mathrm{GL}_2(k) \cap \Gamma = \{a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix} \mid a, b \in k, (a, b) \neq (0, 0)\}.$$

Hence the vertex v_0 is projectively unstable, $\mathrm{GL}_2(k)\cap\Gamma = \mathrm{Stab}_\Gamma(v_0)$ and v_0 represents a terminal vertex of $\Gamma\backslash\mathcal{T}$.
Let v_1 be the vertex $[L(1,0)]$. If v_1 would be projectively unstable, then v_1 would also represents a terminal vertex of $\Gamma\backslash\mathcal{T}$, and since v_0 and v_1 are adjacent in $\mathcal{T}$ this would imply that $\Gamma\backslash\mathcal{T}$ is the graph containing two vertices and one edge connecting them. In that case $V_{q+1} = 0$. From the formulas in Theorem 2.27 one sees that this happens precisly when R consists of two degree 1 places, compare also [Pa1, Corollary 5.8].
If $V_{q+1} \neq 0$, then v_1 has to be projectively stable. Hence we can use v_1 as the initial vertex for the algorithm 2.32. We already checked that v_0 is unstable. The other vertices adjacent to v_1 are $[L(2,\alpha\pi)]$ for $\alpha \in k$, see Lemma 2.8. These are the vertices we need to compare in the first step of the algorithm 2.32. Generally, Lemma 2.8 implies that in the n-th step of the algorithm we need to compare vertices of the form $[L(n, g(\pi))]$, where $g \in k[T]$ with $\deg(g) < n$ and $g(0) = 0$. The next Proposition implies that we can do this in time $\mathcal{O}(n^2)$.

Proposition 2.55 *(a) Given $v = [L(n, g(\pi))]$ and $v' = [L(n, g'(\pi))]$ as above there is an algorithm that computes $\mathrm{Hom}_\Gamma(v', v)$ in time $\mathcal{O}(n^4)$ field operations over $\mathbb{F}_q$.*

(b) All $\gamma \in \mathrm{Hom}_\Gamma(v, v')$ satisfy $\|\gamma\| \leq n$.

Proof: Since this proposition is just a slight variant of Proposition 2.51 we ommit a proof here. ∎

2.8 Presentations of Γ and the word problem

From a fundamental domain for the action of Γ on $\mathcal{T}$ together with a side pairing one obtains a presentation of Γ as an abstract group. This has been explained in [Se1, Chapter I.4] interpreting Γ as the amalgam of the stabilizers of the vertices of $\Gamma\backslash\mathcal{T}$ along the stabilizers of the edges connecting them. Compare also [Pa1, Thm. 5.7].

Lemma 2.56 ([Se1, I.4.1, Lem. 4]) *Let G be a group acting on a connected graph $\mathcal{X}$ and $\mathcal{Y}$ a fundamental domain for the action of G on $\mathcal{X}$ with an edge pairing* PE*. Then G is generated by*

$$\{g_e \in e \in \mathrm{PE}\} \cup \{\mathrm{Stab}_G(v) \mid v \in \mathrm{V}(\mathcal{S})\}.$$

The relations among the generators of the previous lemma are given by [Se1, § I.5, Thm. 13] and are based on the construction of the fundamental group $\pi(\Gamma, \mathcal{Y}, \mathcal{S})$ in [Se1, p. 42]. For the group Γ considered here, all non-terminal vertices v of $\mathcal{S}$ have stabilizer $\mathbb{F}_q^*$ which lies in the center of Γ. The results just quoted therefore considerably simplify and yield:

Proposition 2.57 *Let $(\mathcal{Y}, \mathcal{S}, (g_e)_{e\in\mathrm{PE}})$ be a fundamental domain with an edge pairing for $(\Gamma, \mathcal{T})$ as provided by Algorithm 2.32. For each terminal vertex $v \in \mathrm{V}(\mathcal{S})$, let g_v be a generator of* $\mathrm{Stab}_\Gamma(v)$. *Then Γ is isomorphic to the group generated by*

$$\{g_0\} \cup \{g_v \mid v \textit{ terminal in } \mathrm{V}(\mathcal{S})\} \cup \{g_e \textit{ the edge-label} \mid e \in \mathrm{PE}\}$$

subject to the relations

$$g_0^{q-1} = 1,\ g_v^{q+1} = g_0 \textit{ for all terminal } v,\ [g_e, g_0] = 1 \textit{ for all } e \in \mathrm{PE}\,.$$

In particular g_0 lies in the center of Γ, as it should.

Example 2.58 In Example 2.38 the group Γ is generated by

$$\{g_0, g_{v_1}, \ldots, g_{v_8}, g_1, \ldots, g_5\}$$

with relations

$$g_0^4 = 1, g_{v_i}^6 = g_0, [g_0, g_i] = 1.$$

The word problem with respect to this set of generators was already solved by the reduction Algorithm 2.36, compare [Vo, Remark 4.6].

2.9 Complexity analysis and degree bounds

In this section we will analyze the complexity of Algorithm 2.32 and obtain some bounds on the size of generators of Γ. We start by bounding the diameter of the graph $\Gamma\backslash\mathcal{T}$. The idea of using the Ramanujan property to obtain complexity bounds was inspired by [KV, Conj. 6.6]. A standard reference is [Lu].

Definition 2.59 *A k-regular connected graph $\mathcal{G}$ is called a* Ramanujan graph *if for every eigenvalue λ of the adjacency matrix of $\mathcal{G}$ either $\lambda = \pm k$ or $|\lambda| \leq 2\sqrt{k-1}$.*

Proposition 2.60 ([Lu, Prop 7.3.11]) *Let $\mathcal{G}$ be a k-regular Ramanujan graph on $n \geq 3$ vertices.*[1] *Then*

$$\mathrm{diam}(\mathcal{G}) \leq \log_{k-1}(4n^2).$$

Let

$$\mathrm{one}(R) := \begin{cases} 1 & \text{if some place in } R \text{ has degree one,} \\ q(q-1) & \text{otherwise.} \end{cases}$$

[1] The proof in [Lu] requires at least one eigenvalue λ of the adjacency matrix with $|\lambda| \leq 2\sqrt{k-1}$ and hence $n \geq 3$. Also, the assertion is obviously wrong for $n = 2$ and k large.

Lemma 2.61 *There is a covering of $\mathcal{G} := \Gamma\backslash\mathcal{T}$ by a $q+1$-regular Ramanujan graph $\tilde{\mathcal{G}}$ with*

$$\#\,\mathrm{V}(\tilde{\mathcal{G}}) = \frac{2\,\mathrm{one}(R)}{(q-1)^2}\prod_{\mathfrak{p}\in R}(q_{\mathfrak{p}}-1).$$

Proof: Recall the definitions and formulas for V_1 and V_{q+1} from Theorem 2.27. If $\mathrm{one}(R) = 1$, we can choose a degree 1 place $\mathfrak{p}_0 \in R$. If not we choose an arbitray degree 1 prime $\mathfrak{p}_0$. Let $\Gamma(\mathfrak{p}_0)$ be the full level $\mathfrak{p}_0$ congruence subgroup in Γ. By [LSV, Thm. 1.2] we know that $\tilde{\mathcal{G}} := (\Gamma\cap\Gamma(\mathfrak{p}_0))\backslash\mathcal{T}$ is a Ramanujan graph. Observe that $(\Gamma\cap\Gamma(\mathfrak{p}_0))\backslash\Gamma \cong \mathbb{F}_{q^2}^*$ if $\mathfrak{p}_0 \in R$, which has cardinality q^2-1, and $(\Gamma\cap\Gamma(\mathfrak{p}_0))\backslash\Gamma \cong \mathrm{GL}_2(\mathbb{F}_q)$ otherwise, which has cardinality $\mathrm{one}(R)(q^2-1)$. By analyzing the growth of the stabilizers from $\Gamma\cap\Gamma(\mathfrak{p}_0)$ to Γ, we observe that

$$\frac{1}{\mathrm{one}(R)}\#\,\mathrm{V}(\tilde{\mathcal{G}}) = V_1 + (q+1)V_{q+1}$$

$$= \Big(V_1 + \frac{q+1}{q-1}V_1 + \frac{2(q+1)}{q-1}(g(R)-1)\Big) = \frac{2}{(q-1)^2}\prod_{\mathfrak{p}\in R}(q_{\mathfrak{p}}-1). \blacksquare$$

Proposition 2.62 *Suppose* $\mathrm{V}(\Gamma\backslash\mathcal{T}) \geq 3$. *Then*

$$\mathrm{diam}(\Gamma\backslash\mathcal{T}) \leq 2\deg(r) + 2(2\log_q(2) + 1 - \log_q(q-1)).$$

Proof: Let $\mathcal{G} = \Gamma\backslash\mathcal{T}$ and $\mathcal{G}'$ be the covering from Lemma 2.61. Then

$$\begin{aligned}
\mathrm{diam}(\mathcal{G}) \;\leq\; \mathrm{diam}(\mathcal{G}') &\overset{2.60}{\leq} 2\log_q(\#\,\mathrm{V}(\mathcal{G}')) + \log_q(4)\\
&\overset{2.61}{\leq} 2\log_q\Big(\frac{2q}{q-1}\prod_{\mathfrak{p}\in R}(q_{\mathfrak{p}}-1)\Big) + \log_q(4)\\
&\leq 4\log_q(2) + 2\log_q(\frac{q}{q-1}) + 2\log_q\big(\prod_{\mathfrak{p}\in R} q_{\mathfrak{p}}\big)\\
&= 2(2\log_q(2) + 1 - \log_q(q-1)) + 2\deg(r). \blacksquare
\end{aligned}$$

Corollary 2.63 *With $\|\;\|$ as in Definition 2.50, the group Γ is generated by the set*

$$\{\gamma\in\Gamma \mid \|\gamma\| \leq \deg(\alpha)/2 + 2\deg(r) + 2(2\log_q(2) + 1 - \log_q(q-1))\}.$$

Proof: By Proposition 2.57, the group Γ is generated by the vertex and edge labels of the quotient graph from Algorithm 2.32. By Proposition 2.51 these labels g_t have norm $\|g_t\| \leq \deg(\alpha)/2 + n$, where n is the distance in $\Gamma\backslash\mathcal{T}$ between the initial vertex and the labeled vertex. In particular, $n \leq \mathrm{diam}(\Gamma\backslash\mathcal{T})$. $\blacksquare$

Remark 2.64 If $\mathrm{one}(R) = 1$, we can obviously subtract $2 + 2\log_q(q-1)$ from the diameter in Proposition 2.62 and subsequently from the bounds in Corollary 2.63. In the other case we expect this to be possible as well. This should follow by replacing $\Gamma(\mathfrak{p}_0)$ by

$$\tilde{\Gamma}_1(\mathfrak{p}_0) := \{\gamma \in \Gamma \mid \gamma \equiv \begin{pmatrix} 1 & \star \\ 0 & \star \end{pmatrix} \pmod{\mathfrak{p}_0} \text{ in } \mathrm{GL}_2(\mathbb{F}_q)\}.$$

Unfortunately we could not find this analog of [LSV, Thm. 1.2] for a congruence subgroup other than $\Gamma(\mathfrak{p})$ in the literature although it seems likely to hold.
If this was indeed true, we would obtain the improved bound

$$\mathrm{diam}(\Gamma\backslash\mathcal{T}) \le 2\deg(r) + 4\log_q(2) - 4\log_q(q-1).$$

For $q > 19$ it gives $\mathrm{diam}(\Gamma\backslash\mathcal{T}) \le 2\deg(r) - 4$. The nice feature of this last bound is that it was assumed in many concrete examples that we have computed.

Proposition 2.65 *Algorithm 2.32 computes the quotient graph $\Gamma\backslash\mathcal{T}$ in time*

$$\mathcal{O}((\#\,\mathrm{V}(\Gamma\backslash\mathcal{T}))^2\,\mathrm{diam}(\Gamma\backslash\mathcal{T})^5) \overset{2.27}{=} \mathcal{O}(q^{2\deg(r)-6} \cdot \deg(r)^5)$$

in terms of operations over $\mathbb{F}_q$.

Proof: According to Prop 2.51, comparing two vertices in the algorithm can be done in time $\mathcal{O}(n^4)$, where n is always less or equal then $\mathrm{diam}(\Gamma\backslash\mathcal{T})$. The list of vertices in each step of the algorithm is always shorter than the cardinality of $\mathrm{V}(\Gamma\backslash\mathcal{T})$, so in each step the number of comparisons is bounded by $(\#\,\mathrm{V}(\Gamma\backslash\mathcal{T}))^2$. The number of steps is bounded by $\mathrm{diam}(\Gamma\backslash\mathcal{T})$ and the result follows. ∎

3 Modular forms for function fields

We want to quickly recall the theory of Drinfeld modular forms for congruence subgroups $\Gamma \subset \mathrm{GL}_2(A)$. By a theorem of Teitelbaum from [Te2] these forms can be related to harmonic cocycles, which are combinatorial objects defined on the tree $\mathcal{T}$. In this chapter we introduce the Drinfeld upper half plane, Drinfeld modular forms and harmonic cocycles. We will also define Hecke actions on both the sides of Drinfeld modular forms and of harmonic cocycles. The theorem of Teitelbaum then gives an isomorphism between the vector spaces of Drinfeld modular forms and that of harmonic cocycles. In [Bö] it is checked, that this isomorphism is also compatible with the Hecke actions on both sides.

3.1 Drinfeld modular curves

For $z \in K_\infty$ we write $|z|$ for the absolute value on K_∞ such that $|T| = q$. Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞. Then $| \, |$ extends uniquely to $\mathbb{C}_\infty$. The Drinfeld upper half plane Ω is defined as a set by $\Omega := \mathbb{P}_1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}_1(K_\infty)$. It is the analogue of the classical upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via fractional linear transformations. Let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$ from 2.5. There is a natural reduction map $\rho : \Omega \to \mathcal{T}$ compatible with the actions of $\mathrm{GL}_2(K_\infty)$ on Ω and $\mathcal{T}$, see [Bö, Prop. 3.7]. Via ρ we can equip Ω with the structure of a rigid analytic space such that for each edge $e = (v, v') \in \mathrm{E}(\mathcal{T})$ the inverse image $V(e) = \rho^{-1}(e \smallsetminus \{v, v'\})$ will be a rigid analytic open annulus. In particular, we have an open covering of Ω by open annuli of the form $V(e)$. See [Bö, Section 3] or [GR, Section 1] for more details.
Let $G := \mathrm{GL}_2(A)$. For $N \in A$ we define

$$\Gamma(N) := \{\gamma \in G \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}.$$

Definition 3.1 *(a) A subgroup $\Gamma \subseteq \mathrm{GL}_2(K_\infty)$ is called an arithmetic subgroup for G if there exists an $N \in A$ such that $\Gamma(N) \subseteq \Gamma$ and $[\Gamma : \Gamma(N)] < \infty$.*

(b) The level of an arithmetic subgroup Γ for G is the maximal $N \in A$ with the property from (a).

Note that an arithmetic subgroup for G is commensurable with G.

Proposition 3.2 *Let $v \in \mathrm{V}(\mathcal{T})$ and $\gamma \in \Gamma$ for Γ an arithmetic subgroup for G. Then $d(v, \gamma v)$ is even.*

PROOF: Choose a group $\Gamma(N)$ of finite index in Γ. Then $\{1\} = \det(\Gamma(N))$ is of finite index in $\det(\Gamma)$, hence $\det(\Gamma)$ is finite. Since all finite subgroups of $K_\infty^\star$ are contained in $k^\star$, we have $\det(\Gamma) \subseteq k^\star$, and hence for all $\gamma \in \Gamma$ we have $v_\infty(\gamma) = 0$. Then the result follows from [Se1, Corollary of Proposition II.1]. ∎

Proposition 3.2 implies that an arithmetic subgroup Γ of $\mathrm{GL}_2(K_\infty)$ acts without inversion on $\mathcal{T}$. The resulting quotient space $\Gamma\backslash\mathcal{T}$ will be a graph in the sense of Definition 2.1.

Definition 3.3 *(a) A half line of some graph $\mathcal{G}$ is a sequence $(v_i)_{i\in\mathbb{N}} \subseteq \mathrm{V}(\mathcal{G})$ such that for all $i \geq 1$ the vertices v_{i-1} and v_{i+1} are adjacent to v_i and for all $i \neq j$ we have $v_i \neq v_j$.*

(b) Two half lines $(v_i)_{i\in\mathbb{N}}, (v'_i)_{i\in\mathbb{N}}$ are equivalent if there exist $j, j' \geq 0$ such that $v_{i+j} = v'_{i+j'}$ for all $i \in \mathbb{N}$.

(c) An end of $\mathcal{G}$ is an equivalence-class of half lines.

In Lemma 2.8 we essentially showed that the ends of $\mathcal{T}$ are in bijection with $\mathbb{P}^1(K_\infty)$. To see this, let $v_0 = L(0,0)$. Then since $\mathcal{T}$ is a tree, each end has a unique representative starting with v_0. We showed in Lemma 2.8 that the vertices of $\mathcal{T}$ with distance n to v_0 are in bijection with $\mathbb{P}^1(O_\infty/\pi^n O_\infty)$. Hence the ends of $\mathcal{T}$ are in bijection with the projective limit $\varprojlim_{n\in\mathbb{N}} \mathbb{P}^1(O_\infty/\pi^n O_\infty) \cong \mathbb{P}^1(O_\infty) \cong \mathbb{P}^1(K_\infty)$.

Definition 3.4 *(a) An end of $\mathcal{T}$ is called rational if it corresponds to an element of $\mathbb{P}^1(K)$ under the above bijection.*

(b) The equivalence classes of rational ends of $\mathcal{T}$ modulo Γ are called the cusps of Γ.

Hence the cusps of Γ are in bijection with $\Gamma\backslash\mathbb{P}^1(K)$. In the following we identify the cusps with this set.
The following result describing the structure of $\Gamma\backslash\mathcal{T}$ is [Se1, Theorem II.2.9] or can also be found in this formulation as [Bö, Theorem 3.21].

Theorem 3.5 *Let Γ be an arithmetic subgroup for G. Then $\Gamma\backslash\mathcal{T}$ is the union of a finite connected subgraph $\mathbb{Y}$ and subgraphs Δ_x for each cusp x of Γ such that the following assertions hold:*

(a) Each Δ_x is a half line of $\Gamma\backslash\mathcal{T}$ and can be represented by a half line of $\mathcal{T}$ whose corresponding end is in the equivalence class of the cusp x.

(b) For cusps $x \neq x'$ the graphs Δ_x and $\Delta_{x'}$ are disjoint.

(c) Let $x = (v_{x,i})_{i\in\mathbb{N}}$. Then $\mathbb{Y} \cap \Delta_x$ consists only of the vertex $v_{x,0}$.

3.2 Drinfeld modular forms

Let Γ be an arithmetic subgroup for G. For $n, l \in \mathbb{N}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \mathrm{GL}_2(K_\infty)$ and $f : \Omega \to \mathbb{C}_\infty$ set

$$(f \underset{n,l}{|} \gamma)(z) := f(\gamma z)(\det(\gamma))^l (cz+d)^{-n}.$$

Following Gekeler [Ge3] and Teitelbaum [Te2] we define:

Definition 3.6 *A rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is a modular function of weight n and type l for Γ if $f \underset{n,l}{|} \gamma = f$ for all γ in Γ.*

As in the case of classical modular forms, we must require an additional condition on these functions regarding their behavior at the cusps to obtain an interesting arithmetic theory. To this end, we first need to define an analogue of $\exp(2\pi i z)$, which plays the role of a uniformizer at infinity.
Let $(0) \neq \mathfrak{a}$ be an ideal of K. Then

$$e_L(z) := z \prod_{\substack{a \in \mathfrak{a} \\ a \neq 0}} (1 - \frac{z}{a})$$

is a rigid analytic function on Ω, which is $\mathbb{F}_q$-linear and invariant under translations by $a \in \mathfrak{a}$, see [Ge, I.2.1]. Let x be a cusp for Γ. Since $\mathrm{GL}_2(K)$ acts transitively on $\mathbb{P}^1(K)$ there is a $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma\infty = x$. Let $U(x) := \mathrm{Stab}_\Gamma(x)$. Then $\gamma^{-1}U(x)\gamma$ fixes the cusp ∞. Let $U(x)'$ be the maximal p'-torsion free subgroup of $U(x)$, i.e. the maximal subgroup $H \subset U(x)$ such that all torsion elements in H have order a power of p. Then $\gamma^{-1}U(x)'\gamma$ consists of translations of the form $z \mapsto z + b$ with $b \in \mathfrak{a}$ for some fractional almost ideal $\mathfrak{a}$ of A, see [Bö, page 35]. Define $t(x, \Gamma) := e_{\mathfrak{a}}^{-1}$ for this ideal $\mathfrak{a}$.
If f is a modular function of weight n and type l for Γ, then by definition and since all $\lambda \in U(x)'$ have $\det(\lambda) = 1$, it follows that $(f \underset{n,l}{|} \gamma)(z) = (f \underset{n,l}{|} \gamma)(\lambda z)$ for all $\lambda \in U(x)'$. Hence the function $f \underset{n,l}{|} \gamma$ has a Laurent series expansion in the uniformizer $t(x, \Gamma)$ of the form

$$f \underset{n,l}{|} \gamma = \sum_{i \in \mathbb{Z}} a_i t(x, \Gamma)^i$$

with $a_i \in K$.

Definition 3.7 *(a) A modular function $f : \Omega \to \mathbb{C}_\infty$ is a Drinfeld modular form for Γ of weight n and type l if for all cusps x of Γ and $\gamma \in \mathrm{GL}_2(K)$ with $\gamma\infty = x$ the Laurent Series expansion of $f \underset{n,l}{|} \gamma$ has $a_i = 0$ for all $i < 0$.*

(b) *If additionaly* $a_0 = 0$ *in all expansions from (a), then* f *is called a Drinfeld cusp form for* Γ *of weight* n *and type* l*.*

We write $\mathcal{M}_{n,l}(\Gamma)$ for the space of Drinfeld modular forms for Γ of weight n and type l and $\mathcal{S}_{n,l}(\Gamma)$ for the space of cusp forms. These are vector spaces over $\mathbb{C}_\infty$ with pointwise addition and scalar multiplication.

One can define an action of Hecke operators $T_\mathfrak{p}$ for $\mathfrak{p}$ a maximal ideal in A. We will give an ad-hoc definition of such Hecke operators here. For simplicity we assume $\mathfrak{p} \neq (T)$. For a more systematic exposition see [Bö, 6.2]. Let $p_\mathfrak{p}$ be a generator of $\mathfrak{p}$ with $p_\mathfrak{p} \equiv 1 \pmod{T}$, define $y = \begin{pmatrix} p_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$ and let $E_\mathfrak{p}$ be the set of polynomials of degree less than $\deg(\mathfrak{p})$.

Definition 3.8 *For* $f \in \mathcal{M}_{n,l}(\Gamma)$ *define*

$$T_\mathfrak{p}(f)(z) = p_\mathfrak{p}^{l-n} \left(p_\mathfrak{p}^n f(p_\mathfrak{p} z) + \sum_{b \in E_\mathfrak{p}} f((z + b(1 - p_\mathfrak{p}))/p_\mathfrak{p}) \right).$$

By [Bö, Proposition 6.2 and Exampe 6.13] the operators $T_\mathfrak{p}$ are linear operators on the space $\mathcal{M}_{n,l}(\Gamma)$. Furthermore they preserve the subspace $\mathcal{S}_{n,l}(\Gamma)$ of cusp forms. Moreover, we have $T_\mathfrak{p} T_\mathfrak{q} = T_\mathfrak{q} T_\mathfrak{p}$ for all ideals $\mathfrak{p}$, $\mathfrak{q}$ in A. Hence there are modular forms which are simultanous eigenvectors for all operators $T_\mathfrak{p}, \mathfrak{p} \in A$. Such a form is called an eigenform.

3.3 Automorphic forms vs. modular forms

In the function field setting described here, there are two different concepts that replace classical modular forms, one being the rigid-analytic $\mathbb{C}_\infty$-valued functions described in Section 3.2. The other concept is that of $\mathbb{C}$ or $\overline{\mathbb{Q}}_l$-valued functions on some adele group, which can be interpreted as automorphic forms in the sense of Jacquet-Langlands as in [JL]. For classical modular forms there is no such distinction. The relation between these two concepts has been worked out in [GR, Section 6.5]. Loosely speaking, those Drinfeld modular forms of weight 2 and type 1, which are double cuspidal, meaning that in Definition 3.7 we additionaly require that $a_1 = 0$ for all such expansions, are the reduction modulo p of automorphic forms.
In this thesis we restrict ourself to working with Drinfeld modular forms. These objects are in some ways less rigid than their classical counterparts. For example there are counterexamples for multiplicity one, meaning that there are distinct eigenforms having the same system of eigenvalues. See [Go, Section 2] for an explicit example. It is an open question whether multiplicity one might hold for fixed weight or even only for weight 2.

Compare also the discussion in the introduction to [GR].

3.4 Harmonic cocycles

In [Te2] Teitelbaum gave a description of Drinfeld modular forms in terms of so called harmonic cocycles on $\mathcal{T}$. These are functions from the oriented edges of $\mathcal{T}$ taking values in certain vector spaces. We will start by introducing these spaces.
Let F/K_∞ be a field. We shall define for $n \geq 2$ a representation of $\mathrm{GL}_2(K_\infty)$ on the space

$$V_{n,l}(F) = \mathrm{Hom}(\mathrm{Sym}^{n-2}(\mathrm{Hom}(K_\infty^2, F)), K_\infty) \otimes_{K_\infty} F.$$

If X and Y denote the dual basis of the standard basis of K_∞^2, then the space $\mathrm{Sym}^{n-2}(\mathrm{Hom}(K_\infty^2, F))$ is the space of homogeneous polynomials over F in X, Y of degree $n-2$. Hence to define an action of $\mathrm{GL}_2(K_\infty)$ on an element $f \in V_{n,l}(F)$ explicitly, it suffices to define it at the values of f at monomials of the form $X^i Y^{n-2-i}$. Following Teitelbaum [Te3] we define

$$(\gamma \cdot_{n,l} f)(X^i Y^{n-2-i}) := \det(\gamma)^{1-l} \cdot f((aX+bY)^i (cX+dY)^{n-2-i})$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\infty)$. This extends by linearity to a well-defined action of $\mathrm{GL}_2(K_\infty)$ on $V_{n,l}(F)$.

Definition 3.9 *Let V be a vector space on which $\mathrm{GL}_2(K_\infty)$ acts and let Γ be any subgroup of $\mathrm{GL}_2(K_\infty)$.*

(a) A map $\kappa : \mathrm{E}(\mathcal{T}) \longrightarrow V$ is an V-valued harmonic cocycle, if

  (i) For all $v \in \mathrm{V}(\mathcal{T})$ we have $\sum_{e \mapsto v} \kappa(e) = 0$.

  (ii) $\kappa(e^\star) = -\,\kappa(e)$ for all $e \in \mathrm{E}(\mathcal{T})$.

(b) A map $\kappa : \mathrm{E}(\mathcal{T}) \longrightarrow V$ is called Γ-equivariant, if for all $\gamma \in \Gamma$ we have $\kappa(\gamma e) = \gamma\,\kappa(e)$.

(c) Let $C^{\mathrm{har}}(\Gamma, V)$ be the space of Γ-equivariant harmonic cocycles with values in V.

(d) Let $C^{\mathrm{har}}_{n,l}(\Gamma) := C^{\mathrm{har}}(\Gamma, V_{n,l}(K_\infty))$.

The set $C^{\mathrm{har}}_{n,l}(\Gamma)$ is a vector space over K_∞ with addition and scalar multiplication defined pointwise. We let $\mathrm{GL}_2(K_\infty)$ act on $C^{\mathrm{har}}_{n,l}(\Gamma)$ by

$$(\gamma, \kappa) \mapsto \gamma \cdot \kappa : \mathrm{E}(\mathcal{T}) \to V_{n,l}(K_\infty)\ ,\ e \mapsto \gamma^{-1} \cdot_{n,l} \kappa(\gamma e).$$

Let Γ be an arithmetic subgroup for G of level N. Our next goal is to define Hecke operators T_p on $C^{har}_{n,l}(\Gamma)$ for $p \in A$ irreducible with $(N,p) = 1$. Let

$$\Gamma^0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv 0 \pmod{p} \right\}$$

and let $\pi_p : (\Gamma \cap \Gamma^0(p))\backslash\mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection. For $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma)$ let $\pi_p^\star(\kappa) := \kappa \circ \pi_p$ be the pullback of c along π_p. For $\kappa : \mathrm{E}(\mathcal{T}) \to V$ let

$$\Phi_p(\kappa)(e) := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot_{n,l} \kappa(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} e)$$

and for $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma \cap \Gamma_0(p))$ let

$$\mathrm{tr}_p(\kappa)(e) := \sum_{\delta \in (\Gamma \cap \Gamma_0(p))\backslash\Gamma} \delta^{-1}\, \kappa(\delta e).$$

Note that tr_p is independent of the choice of representatives of $\Gamma \cap \Gamma_0(p))\backslash\Gamma$.

Definition 3.10 *For $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma)$ define $T_p\, \kappa(e) := \mathrm{tr}_p \circ \Phi_p \circ \pi_p^\star(\kappa)(e)$.*

One has to check that in this way one obtains a well-defined linear operator on $C^{\mathrm{har}}_{n,l}(\Gamma)$. For this we refer to [Bu, Section 3.1] for more details. The following diagram visualizes the definition of the operator T_p:

$$\begin{tikzcd}
C^{\mathrm{har}}_{n,l}(\Gamma \cap \Gamma^0(p)) \arrow[r, "\Phi_p"] \arrow[rrd, dotted, bend right, "\pi_p^\star(\kappa)"'] & C^{\mathrm{har}}_{n,l}(\Gamma \cap \Gamma_0(p)) \arrow[d, "\mathrm{tr}_p"] \arrow[rd, dotted, "\Phi_p \circ \pi_p^\star(\kappa)"] & \\
C^{\mathrm{har}}_{n,l}(\Gamma) \arrow[u, "\pi_p^\star"] \arrow[rr, dotted, bend right, "\kappa"'] & C^{\mathrm{har}}_{n,l}(\Gamma) \arrow[r, dotted, "T_p(\kappa)"] & V
\end{tikzcd}$$

An important result from [Te2] relates Drinfeld cusp forms with spaces of harmonic cocycles. It allows one to compute the Hecke action on cusp forms explicitly.

Theorem 3.11 *Suppose $n \geq 2$. Then there is an isomorphism of Hecke-modules $S_{n,l}(\Gamma) \to C^{\mathrm{har}}_{n,l}(\Gamma) \otimes \mathbb{C}_\infty$.*

Proof: As an isomorphism of $\mathbb{C}_\infty$-vector spaces, this is [Te2, Theorem 16]. The compatibility with the Hecke action is checked in [Bö, Proposition 6.15]. ∎

The aim of Chapter 5 will be to obtain an analogues result for the setting outlined in Chapter 4.

4 Quaternionic modular forms

In this section we develop the theory of Drinfeld modular forms for finite index subgroups Γ inside the unit group of a maximal order of a quaternion division algebra. The main goal of this section will be to compute the dimensions of the spaces of such modular forms. In the case of Γ being p'-torsion free we will show that they equal the dimensions of spaces of harmonic Γ-equivariant cocycles.

4.1 The setup

Let D be a quaternion algebra over K unramified at ∞ with discriminant R, Λ a maximal order and $\Gamma \subseteq \Lambda^\star$ a finite index subgroup. We identify Γ with its image under the embedding $\iota : \Lambda^\star \to \mathrm{GL}_2(K_\infty)$ from Proposition 2.21. For $n, l \in \mathbb{N}$ let $\mathcal{L}_{n,l}$ be the line bundle on Ω which is as a set $\mathbb{C}_\infty \times \Omega$ having a $\mathrm{GL}_2(K_\infty)$ action defined by

$$\gamma \cdot (w, z) := ((cz + d)^{-n} \det(\gamma)^l w, \gamma z)$$

for $\gamma \in \mathrm{GL}_2(K_\infty), w \in \mathbb{C}_\infty, z \in \Omega$. Let $\omega := \mathcal{L}_{2,1}$ and abbreviate $\omega_\Gamma = \Gamma\backslash\omega$ for the line bundle on $\Gamma\backslash\Omega$ obtained as the quotient by the action of Γ. For the existence of this quotient line bundle see Proposition 4.13.

Let $X = \Gamma\backslash\Omega$ and let $\pi : \Omega \to X$ denote the quotient map. Note that we also wrote π for the quotient map $\pi : \mathcal{T} \to \Gamma\backslash\mathcal{T}$ by abuse of notation. The quotient space X carries the structure of a rigid analytic space induced from the rigid analytic structure on Ω. X is the rigid-analytic space associated to a smooth, projective curve over K_∞, see [Pu, Theorem 3.3]. The reduction map $\rho : \Omega \to \mathcal{T}$ descends to a map $\pi^\star\rho : X \to \Gamma\backslash\mathcal{T}$ such that the diagram

$$\begin{tikzcd} \Omega \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathcal{T} \arrow[d, "\pi"] \\ X \arrow[r, "\pi^\star\rho"'] & \Gamma\backslash\mathcal{T} \end{tikzcd}$$

commutes. For the genus of X one has the formula

$$g(X) = h^1(\Gamma\backslash\mathcal{T}) = g(R), \tag{10}$$

see [Pa1, Theorem 2.7].

Let $\mathbf{\Omega}_\Omega$, respectivly $\mathbf{\Omega}_X$, be the sheaf of differentials on Ω, respectivly X. Since Ω is a rigid analytic space of dimension 1, every differential in $\Gamma(\Omega, \mathbf{\Omega}_\Omega) = \mathbf{\Omega}_\Omega(\Omega)$ is of the form $f(z)\mathrm{d}z$ with $f \in \mathcal{O}_\Omega$. Since

$$\mathrm{d}(\gamma z) = \frac{\det(\gamma)}{(cz + d)^2}\mathrm{d}z$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\infty)$, the action of $\mathrm{GL}_2(K_\infty)$ on Ω via fractional linear transformations induces an action on $\mathbf{\Omega}_\Omega(\Omega)$ given by the formula

$$\gamma^\star f(z)\mathrm{d}z = \frac{\det(\gamma) f(\gamma z)}{(cz+d)^2}\mathrm{d}z$$

for $f(z)\mathrm{d}z \in \mathbf{\Omega}_\Omega(\Omega)$.

Definition 4.1 *We define the space of quaternionic modular forms of weight n and type l for Γ as the set of rigid analytic holomorphic functions $f : \Omega \to \mathbb{C}_\infty$ such that*

$$f = f \underset{n,l}{|} \gamma$$

for all $\gamma \in \Gamma$.

Note that, unlike in the case of arithmetic subgroups for $\mathrm{GL}_2(A)$, we have no additional condition at cusps of Γ. This is due to the fact that X itself is already compact.

Lemma 4.2 *There is an isomorphism*

$$\mathbf{\Omega}_X \cong \omega_\Gamma.$$

PROOF: From the definition of ω we see that ω_Γ is isomorphic to the functions $f : \Omega \to \mathbb{C}_\infty$ such that $f(z) = \frac{\det(\gamma)}{(cz+d)^2} f(\gamma z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Sending f to $f(z)\mathrm{d}z$ and observing that Ω_X consists of those differentials $f(z)\mathrm{d}z$ such that $\gamma^\star f(z)\mathrm{d}z = f(z)\mathrm{d}z$ for all $\gamma \in \Gamma$ we obtain the claimed isomorphism. ∎

This lemma also implies that $\Gamma(X, \mathbf{\Omega}_X)$ is isomorphic to the space of quaternionic modular forms of weight 2 and type 1 for Γ.

4.2 Dimension formulas, the p'-torsion free case

Definition 4.3 *We say that Γ is p'-torsion free if for all $\gamma \in \Gamma$ the order of γ is either ∞ or a power of p.*

In the case of p'-torsion free subgroups of $\Lambda^\star$, the line bundles $\mathcal{L}_{n,l}$ descent to line bundles on X. This makes the computation of $\dim \mathcal{M}_{n,l}(\Gamma)$ a direct application of the Riemann-Roch theorem.

Lemma 4.4 *If Γ is p'-torsion free, then $\Gamma \backslash \mathcal{L}_{n,l}$ is a line bundle on X for all $n, l \in \mathbb{N}$.*

PROOF: Since the condition is local, we need to check it on open affinoids $U \subset \Omega$. Further, since there is a covering of Ω by open affinoids $U_v := \rho^{-1}(v)$ for $v \in \mathrm{V}(\mathcal{T})$ we can check the condition on some open affinoid U_v. By Proposition 2.24 and since $\Gamma \subseteq \Lambda^\star$ is p'-torsion free, we know that $\Gamma_v = \mathrm{Stab}_\Gamma(v) = \{1\}$. So locally at U_v the sheaf $\Gamma\backslash\mathcal{L}_{n,l}$ is $\Gamma_v\backslash(\mathbb{C}_\infty \times U_v) = \mathbb{C}_\infty \times U_v$ and hence a line bundle. ■

Note that $\det(\Gamma) \subseteq \det(\Lambda^\star) \subseteq \mathbb{F}_q^\star$ is a finite group of order coprime to p. This means that if Γ is p'-torsion free, then $\det(\Gamma) = \{1\}$. Hence in that case

$$\mathcal{L}_{n,l} = \mathcal{L}_{n,l'} \tag{11}$$

for all integers $l, l' \in \mathbb{N}$.
The following proposition follows directly from the definition of the line bundle $\mathcal{L}_{n,l}$ and from Lemma 4.4.

Proposition 4.5 *Let Γ be p'-torsion free. Then $\mathcal{M}_{n,l}(\Gamma)$ is isomorphic to the space of global sections $\Gamma(X, \Gamma\backslash\mathcal{L}_{n,l}) = H^0(X, \Gamma\backslash\mathcal{L}_{n,l})$.*

Using this interpretation of modular forms as sections of line bundels we can compute the dimensions of the spaces of modular forms directly via the Riemann-Roch theorem.

Proposition 4.6 *Suppose that Γ is p'-torsion free. Then for $n, l \in \mathbb{N}$ with $n \geq 2$ we have*

$$\dim \mathcal{M}_{n,l}(\Gamma) = \begin{cases} g(X) & \text{for } n = 2, \\ (g(X) - 1)(n - 1) & \text{for } n > 2. \end{cases}$$

PROOF: Denote $g := g(X)$. For any sheaf $\mathcal{L}$ on X let $\chi(\mathcal{L})$ denote the Euler-Poincare characterstic of $\mathcal{L}$, that is

$$\chi(\mathcal{L}) := h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}).$$

Let $\tilde{\omega} = \mathcal{L}_{1,0}$. By Lemma 4.4 $\tilde{\omega}$ descends to a line bundle $\tilde{\omega}_\Gamma = \Gamma\backslash\tilde{\omega}$ on X. By definition and Equation 11 we have $\tilde{\omega}_\Gamma^{\otimes 2} = \omega_\Gamma$. Then from Serre duality we know that

$$\chi(\tilde{\omega}_\Gamma) = h^0(X, \tilde{\omega}_\Gamma) - h^1(X, \tilde{\omega}_\Gamma) = h^0(X, \tilde{\omega}_\Gamma) - h^0(X, \tilde{\omega}_\Gamma^\vee \otimes \mathbf{\Omega}_X)$$

$$= h^0(X, \tilde{\omega}_\Gamma) - h^0(X, \tilde{\omega}_\Gamma^{-1} \otimes \tilde{\omega}_\Gamma^{\otimes 2}) = h^0(X, \tilde{\omega}_\Gamma) - h^0(X, \tilde{\omega}_\Gamma) = 0.$$

The Riemann-Roch theorem then implies for any line bundle $\mathcal{L}$ on X that $h^0(\mathcal{L}) = h^1(\mathcal{L}) + 1 - g + \deg(\mathcal{L})$, so that $\deg(\tilde{\omega}_\Gamma) = g - 1$ and hence

$$\deg(\tilde{\omega}_\Gamma^{\otimes n}) = n \deg(\tilde{\omega}_\Gamma) = n(g - 1).$$

So if $n \geq 3$ then $\deg(\tilde{\omega}_\Gamma^{\otimes n}) > \deg(\mathbf{\Omega}_X) = \deg(\tilde{\omega}_\Gamma^{\otimes 2})$. This implies $\deg((\tilde{\omega}_\Gamma^{\otimes n})^\vee \otimes \mathbf{\Omega}_X) < 0$ and hence by Serre duality

$$h^1(X, \tilde{\omega}_\Gamma^{\otimes n}) = h^0(X, (\tilde{\omega}_\Gamma^{\otimes n})^\vee \otimes \mathbf{\Omega}_X) = 0.$$

Then we compute

$$\dim \mathcal{M}_{n,l}(\Gamma) = h^0(X, \tilde{\omega}_\Gamma^{\otimes n}) = 0 + (1-g) + n(g-1) = (n-1)(g-1).$$

If $n = 2$ we obtain again by Serre duality $h^1(\tilde{\omega}_\Gamma^{\otimes 2}) = h^1(\mathbf{\Omega}_X) = h^0(\mathcal{O}_X) = 1$ and so

$$\dim \mathcal{M}_{2,l}(\Gamma) = h^0(X, \tilde{\omega}_\Gamma^{\otimes 2}) = 1 + 1 - g + 2(g-1) = 2 - g + 2g - 2 = g. \blacksquare$$

Our next goal is to show that this dimension equals the dimension of $C_{n,l}^{\mathrm{har}}(\Gamma)$. These spaces were definied in Definition 3.9.

Proposition 4.7 *Let V be a vector space on which $\mathrm{GL}_2(K_\infty)$ acts. Suppose that Γ is p'-torsion free. Then*

$$\dim C_{\mathrm{har}}(\Gamma, V) = \dim V \cdot (g(X) - 1) + \dim V_\Gamma$$

where $V_\Gamma = V/\{(\gamma - 1)V | \gamma \in \Gamma\}$.

Proof: Let $\mathcal{T}_0 := \mathrm{V}(\mathcal{T})$ and $\mathcal{T}_1 := \mathrm{E}(\mathcal{T})$ and $\delta : \mathcal{T}_1 \to \mathcal{T}_0, e \mapsto t(e)$. Since Γ is p'-torsion free, Proposition 2.24 implies that $\mathbb{Z}[\mathcal{T}_i]$ are free $\mathbb{Z}[\Gamma]$-modules for $i \in \{1, 2\}$. Then we have a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[\Gamma]$-modules

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}[\mathcal{T}_0] \xleftarrow{\delta} \mathbb{Z}[\mathcal{T}_1] \longleftarrow 0.$$

By tensoring this resolution over $\mathbb{Z}[\Gamma]$ with V we obtain an exact sequence

$$0 \longleftarrow H_0(\Gamma, V) \longleftarrow \mathbb{Z}[\mathcal{T}_0] \otimes_{\mathbb{Z}[\Gamma]} V \xleftarrow{\varphi} \mathbb{Z}[\mathcal{T}_1] \otimes_{\mathbb{Z}[\Gamma]} V \longleftarrow H_1(\Gamma, V) \longleftarrow 0.$$

By definiton $C_{\mathrm{har}}(\Gamma, V)$ can be identified with $\mathrm{Kern}(\varphi) = H_1(\Gamma, V)$. Since $\mathbb{Z}[\mathcal{T}_i]$ are free $\mathbb{Z}[\Gamma]$-modules for $i \in \{1, 2\}$ we have

$$\dim(\mathbb{Z}[\mathcal{T}_i] \otimes_{\mathbb{Z}[\Gamma]} V) = \dim V \cdot (\#(\Gamma \backslash \mathcal{T}_i)).$$

Since $H_0(\Gamma, V) \cong V_\Gamma$, we obtain

$$\dim C_{\mathrm{har}}(\Gamma, V) = \dim V_\Gamma + \dim V \cdot (\#(\Gamma \backslash \mathcal{T}_1) - \#(\Gamma \backslash \mathcal{T}_0))$$

which is equal to $\dim V_\Gamma + \dim V \cdot (g(X) - 1)$ by Euler's formula. $\blacksquare$

Next we will compute for $V = V_{n,l}(\mathbb{C}_\infty)$ the covariant space V_Γ explicitly. Let $\mathbb{C}_\infty[X,Y]_{n-2}$ be the space of homogeneous polynomials in X,Y of degree $n-2$ with $\mathrm{GL}_2(K_\infty)$ acting on $\mathbb{C}_\infty[X,Y]_{n-2}$ by

$$\gamma \cdot f(X,Y) = \det(\gamma)^{1-l} f((aX+cY),(bX+dY))$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\infty)$ and $f \in \mathbb{C}_\infty[X,Y]_{n-2}$. Then

$$(\det(\gamma)^{1-l} \otimes \mathrm{Sym}^{n-2}(\mathbb{C}_\infty^2))^\Gamma = \mathbb{C}_\infty[X,Y]_{n-2}^\Gamma.$$

Lemma 4.8 *Let Γ be p'-torsion free and $V = V_{n,l}(\mathbb{C}_\infty)$. Then*

$$\dim V_\Gamma = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{for } n > 2. \end{cases}$$

Proof: $V_{2,l}(\mathbb{C}_\infty)$ is by definition just $\mathbb{C}_\infty$ with $\mathrm{GL}_2(K_\infty)$-action given by multiplication with $\det(\gamma)^{1-l}$. Since $\det(\Gamma) = \{1\}$ if the group Γ is p'-torsion free, the action of Γ on $\mathbb{C}_\infty$ is trivial. This implies $(V_{2,l}(\mathbb{C}_\infty))_\Gamma = V_{2,l}(\mathbb{C}_\infty) = \mathbb{C}_\infty$.
Now let $n > 2, l \in \mathbb{N}$. By dualizing and [Bö, Lemma 5.21] we have $\dim V_\Gamma = \dim(V^\star)^\Gamma = \{v^\star \in V^\star \mid \gamma v^\star = v^\star \text{ for all } \gamma \in \Gamma\}$ for any $\mathbb{Z}[\Gamma]$-module V where $V^\star$ denotes the linear dual of V. Hence

$$\dim(V_{2,l}(\mathbb{C}_\infty)_\Gamma) = \dim((\det(\gamma)^{1-l} \otimes \mathrm{Sym}^{n-2}(\mathbb{C}_\infty^2))^\Gamma) = \mathbb{C}_\infty[X,Y]_{n-2}^\Gamma$$

By Proposition 2.57 and since Γ has finite index in $\Lambda^\star$ there exists non-torsion elements in Γ, and we can choose two such element γ_1 and γ_2 with $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$. After base change with an element of $\mathrm{GL}_2(K_\infty)$ we can assume that $\gamma_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for some $a \in K'_\infty \smallsetminus \bar{\mathbb{F}}_q$ where K'_∞/K_∞ is some quadratic extention. Let $B = (b_{i,j})_{i,j=0,\ldots,n-2} = \mathrm{Sym}^{n-2}(\gamma_1)$. Then $b_{i,j} = 0$ for $i \neq j$ and $b_{i,i} = a^{n-2-2i}$. There exists an $f \in \mathbb{C}_\infty[X,Y]_{n-2}^{\langle\gamma_1\rangle}$ with $f \neq 0$ if and only if there is an eigenvector of B with eigenvalue 1. Hence if n is odd, it follows that $\mathbb{C}_\infty[X,Y]_{n-2}^{\langle\gamma_1\rangle} = \{0\}$ and so also $\mathbb{C}_\infty[X,Y]_{n-2}^\Gamma = \{0\}$.
If n is even, then $a^{n-2-2(\frac{n-2}{2})} = 1$, so $X^{\frac{n-2}{2}}Y^{\frac{n-2}{2}}$ is an eigenvector for the eigenvalue 1 and

$$\mathbb{C}_\infty[X,Y]_{n-2}^{\langle\gamma_1\rangle} = \mathbb{C}_\infty X^{\frac{n-2}{2}}Y^{\frac{n-2}{2}}.$$

Let u,v be the two eigenvectors of γ_2. They correspond to linear forms $u(X,Y)$ and $v(X,Y) \in \mathbb{C}_\infty[X,Y]_1$. Let $C = \mathrm{Sym}^{n-2}(\gamma_2)$. Then the eigenvectors of C are $u^i v^{n-2-i}$ for $i \in \{0,\ldots,n-2\}$. Now if $\mathbb{C}_\infty[X,Y]_{n-2}^{\langle\gamma_1,\gamma_2\rangle} \neq \{0\}$, this would imply, that B and C

have a common eigenvector for the eigenvalue 1. Let λ, λ^{-1} be the eigenvalues of γ_2. Then as above we conclude that the eigenvalues of C are λ^{n-2-2i} for $0 \le i \le n-2$ and hence the eigenvector for the eigenvalue 1 has to be $u^{\frac{n-2}{2}} v^{\frac{n-2}{2}}$. Then $u^{\frac{n-2}{2}} v^{\frac{n-2}{2}}$ and $X^{\frac{n-2}{2}} Y^{\frac{n-2}{2}}$ are linear dependent. So

$$u(X,Y)^{\frac{n-2}{2}} v(X,Y)^{\frac{n-2}{2}} = \xi X^{\frac{n-2}{2}} Y^{\frac{n-2}{2}}$$

for some $\xi \in \mathbb{C}_\infty^\star$. This implies $u = \mu X$ and $v = \nu Y$ or $u = \mu Y$ and $v = \nu X$ for some $\mu, \nu \in \mathbb{C}_\infty^\star$. By changing the role of u and v we can assume that $u = \mu X$ and $v = \nu Y$. But then $\gamma_2 = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$ and hence γ_2 commutes with γ_1, which contradicts our choice of γ_1 and γ_2. Hence $\mathbb{C}_\infty[X,Y]_{n-2}^{\langle \gamma_1, \gamma_2 \rangle} = \{0\}$ and so also $\mathbb{C}_\infty[X,Y]_{n-2}^{\Gamma} = \{0\}$. ∎

Corollary 4.9 *Suppose Γ is p'-torsion free. Then for all $n, l \in \mathbb{N}$ we have*

$$\dim C_{\mathrm{har}}(\Gamma, V_{n,l}(\mathbb{C}_\infty)) = \dim \mathcal{M}_{n,l}(\Gamma).$$

Proof: If $n > 2$, then by Proposition 4.6 we have $\dim \mathcal{M}_{n,l}(\Gamma) = (n-1)(g(X)-1)$. By Proposition 4.7 and Lemma 4.8 and by observing that $\dim V_{n,l}(\mathbb{C}_\infty) = n-1$, we get

$$\dim C_{\mathrm{har}}(\Gamma, V_{n,l}(\mathbb{C}_\infty)) = \dim(V_{n,l}(\mathbb{C}_\infty)) \cdot (g(X)-1) + \dim V_{n,l}(\mathbb{C}_\infty)^\Gamma = (n-1)(g(X)-1).$$

If $n = 2$, then by Proposition 4.6 we have $\dim \mathcal{M}_{2,l}(\Gamma) = g(X)$. By Proposition 4.7 and Lemma 4.8 we have

$$\dim C_{\mathrm{har}}(\Gamma, V_{2,l}(\mathbb{C}_\infty)) = 1 \cdot (g(x)-1) + 1 = g(X).$$ ∎

4.3 Meromorphic modular functions

Next we need to study the general case of Γ possibly having p'-torsion. In this section we will answer the question, for which pairs of integers n, l there are non-trivial meromorphic functions from Ω to $\mathbb{C}_\infty$ having the right modular transformation property for the group Γ. As in the classical case carried out in [Sh], the existence of such a non-trivial function will be used for computing the dimensions of the spaces of modular forms. In this and the following section we follow ideas of Shimura.

Definition 4.10 *Let $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ be the space of rigid analytic meromorphic functions $f : \Omega \to \mathbb{C}_\infty$ such that*

$$f = f \underset{n,l}{|} \gamma$$

for all $\gamma \in \Gamma$.

The goal of this section is to give an answer to the question for which integers n, l the spaces $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ are non-trivial. Let $L := \#\det(\Gamma)$ and $w := \#(\mathbb{F}_q^\star \cap \Gamma)$. Hence $w = 1$ for a p'-torsion free group Γ and $w = q-1$ for $\Gamma = \Lambda^\star$. If w is even, then $\frac{w}{2}|L$, otherwise $w|L$. Note that if $l \equiv l' \pmod{L}$ then $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) = \mathcal{A}_{n,l'}(\Gamma, \mathbb{C}_\infty)$ and $\mathcal{M}_{n,l}(\Gamma) = \mathcal{M}_{n,l'}(\Gamma)$.

Lemma 4.11 *$\mathcal{A}_{0,l}(\Gamma, \mathbb{C}_\infty) \neq \{0\}$ if and only if $2l \equiv 0 \pmod{w}$.*

PROOF: If $l = 0$, then $\mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$ consists of the meromorphic functions on the algebraic curve $\Gamma\backslash\Omega$ and hence is non-trivial.
If $2l$ is not a multiple of w, then necessarily $w \neq 1$. Let $\gamma \in \mathbb{F}_q^\star \cap \Gamma$ with $\mathrm{ord}(\gamma) = w$. Then $\gamma = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ for some $d \in \mathbb{F}_q^\star$ and hence for all $f \in \mathcal{A}_{0,l}$ we have $f(z) = d^{-2l} f(z)$ for all $z \in \Omega$. If $f(z) \neq 0$, this would imply $d = 1$ which contradicts $\mathrm{ord}(\gamma) = w \neq 1$. Hence $f(z) = 0$ for all $z \in \Omega$.
Now let $2l$ be a multiple of w and let

$$\Gamma_0 := \{\gamma \in \Gamma \mid \det(\gamma) \in \det(\Gamma \cap \mathbb{F}_q^\star)\}.$$

Hence Γ/Γ_0 is cyclic of order $L/(\frac{w}{2})$ if w is even and of order L/w if w is odd. We have a covering of algebraic curves $\Gamma_0\backslash\Omega \to \Gamma\backslash\Omega$ with Galois group Γ/Γ_0. This implies that the space of meromorphic functions $\mathcal{A}_{0,0}(\Gamma_0, \mathbb{C}_\infty)$ is a Galois extention of $\mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$ with Galois group Γ/Γ_0. By the normal basis theorem of Galois theory it follows that

$$\mathcal{A}_{0,0}(\Gamma_0, \mathbb{C}_\infty) = \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)[\Gamma/\Gamma_0] = \mathbb{F}_q[\Gamma/\Gamma_0] \otimes_{\mathbb{F}_q} \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty).$$

Let $\widehat{\Gamma/\Gamma_0}$ be the group of all characters $\chi : \Gamma/\Gamma_0 \to \mathbb{F}_q^\star$. To $\chi \in \widehat{\Gamma/\Gamma_0}$ define as in [Wa, Section 6.3]

$$e_\chi := \frac{1}{|\Gamma/\Gamma_0|} \sum_{\sigma \in \Gamma/\Gamma_0} \chi(\sigma)\sigma^{-1} \in \mathbb{F}_q[\Gamma/\Gamma_0] \subseteq \mathcal{A}_{0,0}(\Gamma_0, \mathbb{C}_\infty).$$

Then the elements e_χ are idempotents of the group ring $\mathbb{F}_q[\Gamma/\Gamma_0]$ and as in *loc. cit.* by the properties of e_χ one obtains a decomposition

$$\mathbb{F}_q[\Gamma/\Gamma_0] = \bigoplus_{\chi \in \widehat{\Gamma/\Gamma_0}} e_\chi \mathbb{F}_q[\Gamma/\Gamma_0]$$

which induces a decomposition

$$\mathcal{A}_{0,0}(\Gamma_0, \mathbb{C}_\infty) = \bigoplus_{\chi \in \widehat{\Gamma/\Gamma_0}} e_\chi \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty).$$

By the definition of Γ_0, each character $\chi \in \widehat{\Gamma/\Gamma_0}$ is induced by a character

$$\det{}^i : \Gamma \xrightarrow{\det} \mathbb{F}_q^\star \xrightarrow{\alpha \mapsto \alpha^i} \mathbb{F}_q^\star$$

for some $i \in \{1, \ldots, L\}$ such that the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\det^i"] \arrow[d] & \mathbb{F}_q^\star \\
\Gamma/\Gamma_0 \arrow[ur, "\chi"'] &
\end{tikzcd}$$

commutes. Conversely, $\det^i$ defines such a χ if and only if $\det^i(\Gamma_0) = \{1\}$, so if and only if $\#\det(\Gamma_0)|i$. But $\#\det(\Gamma_0) = \#\det(\Gamma \cap \mathbb{F}_q^\star)$ equals $\frac{w}{2}$ if w is even or equals w if w is odd. In that case $e_\chi \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty) = \mathcal{A}_{0,i}(\Gamma, \mathbb{C}_\infty)$. In particular all components $\mathcal{A}_{0,i}(\Gamma, \mathbb{C}_\infty)$ of the decomposition are isomorphic and hence non-empty. ∎

Lemma 4.12 *If $f \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$, then for the derivative $f'(z) := \frac{\mathrm{d}f(z)}{\mathrm{d}z}$ one has $f' \in \mathcal{A}_{2,1}(\Gamma, \mathbb{C}_\infty)$.*

Proof: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ one computes $\frac{\mathrm{d}\gamma z}{\mathrm{d}z} = (cz+d)^{-2}\det(\gamma)$. Since for $f \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$ one has $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ it follows that

$$\frac{f'(z)}{f'(\gamma z)} = \frac{\mathrm{d}f(z)}{\mathrm{d}f(\gamma z)} \frac{\mathrm{d}\gamma z}{\mathrm{d}z} = \frac{\mathrm{d}f(z)}{\mathrm{d}f(z)}(cz+d)^{-2}\det(\gamma) = (cz+d)^{-2}\det(\gamma)$$

and hence $f' \in \mathcal{A}_{2,1}(\Gamma, \mathbb{C}_\infty)$. ∎

Proposition 4.13 *For even weight n one has $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \neq \{0\}$ if and only if $n \equiv 2l \pmod{w}$.*

Proof: From the definition of $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ it is clear that for $f_1 \in \mathcal{A}_{n_1,l_1}(\Gamma, \mathbb{C}_\infty)$ and $f_2 \in \mathcal{A}_{n_2,l_2}(\Gamma, \mathbb{C}_\infty)$ one has $f_1 \cdot f_2 \in \mathcal{A}_{n_1+n_2,l_1+l_2}(\Gamma, \mathbb{C}_\infty)$ and $\frac{1}{f_1} \in \mathcal{A}_{-n_1,-l_1}(\Gamma, \mathbb{C}_\infty)$. Now choose any non-constant meromorphic function $f \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$. If $f = g^p$ for some meromorphic function $g : \Gamma \backslash \Omega \to \mathbb{C}_\infty$, then also $g \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$. Hence w.l.o.g. we can assume that f is not a p-th power, so $f' \neq 0$. Note that $\mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$

is finitly generated over $\mathbb{C}_\infty$, so that for no non-constant $f \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$ we can have $f^{p^{-n}} \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$ for all $n \in \mathbb{N}$.
Then Lemma 4.12 implies that the spaces $\mathcal{A}_{2,1}(\Gamma, \mathbb{C}_\infty)$ is non-trivial and by the above this implies that the spaces $\mathcal{A}_{2i,i}(\Gamma, \mathbb{C}_\infty)$ are non-trivial for all $i \in \mathbb{Z}$.
Suppose that $n \equiv 2l \pmod{w}$. By the above we can choose a non-zero $f_1 \in \mathcal{A}_{n,\frac{n}{2}}(\Gamma, \mathbb{C}_\infty)$. Further since $2l - n \equiv 0 \pmod{w}$ by Lemma 4.11 we can choose a non-zero $f_2 \in \mathcal{A}_{0,l-\frac{n}{2}}(\Gamma, \mathbb{C}_\infty)$. Then $0 \neq f_1 \cdot f_2 \in \mathcal{A}_{n,\frac{n}{2}+l-\frac{n}{2}}(\Gamma, \mathbb{C}_\infty) = \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$. On the other hand suppose $n \not\equiv 2l \pmod{w}$ and $0 \neq f \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$. Choose a non-zero $g \in \mathcal{A}_{n,\frac{n}{2}}(\Gamma, \mathbb{C}_\infty)$. Then $0 \neq \frac{f}{g} \in \mathcal{A}_{0,l-\frac{n}{2}}(\Gamma, \mathbb{C}_\infty)$. This contradicts Lemma 4.12. ∎

This answers the question whether $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \neq \{0\}$ for even n. Next we will remark on the case of n odd.

Remark 4.14 $-\mathbb{1} := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$ if and only if w is even.

Proof: If $-\mathbb{1} \in \Gamma$, then by Lagranges theorem $2 = \operatorname{ord}(-\mathbb{1}) | \operatorname{ord}(\mathbb{F}_q^\star \cap \Gamma) = w$. If w is even choose γ a generator of $\mathbb{F}_q^\star \cap \Gamma$. Then $\gamma^{\frac{w}{2}} = -\mathbb{1}$. ∎

If $-\mathbb{1} \in \Gamma$, then for any $f \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ with n odd one has $f(z) = -f(z)$ for all $z \in \Omega$, hence $f = 0$. This implies that for w even and n odd we have $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) = \{0\}$ for all $l \in \mathbb{Z}$. If $\Gamma = \Lambda^\star$, then for odd q we have $w = q - 1$ is even. Hence for odd q if Γ is the unit group of a maximal order, then there are no non-trivial modular forms for Γ of odd weight. For $w = 1$, i.e. in the p'-torsion free case, we already saw in Proposition 4.6 that there are non-trivial modular forms for arbitrary $n \geq 2$ in $\mathcal{M}_{n,l}(\Gamma)$.
If $\Gamma \subset \Lambda^\star$ is more general or if q is even, the space $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ could be non-trivial. Alas we can not adapt the proof for odd n of Proposition [Sh, 2.15] from the classical theory of modular forms for the complex upper half plane $\mathbb{H}$. In this proof one makes use of the fact that if $f : \mathbb{H} \to \mathbb{C}$ is a meromorphic function such that for each $z \in \mathbb{H}$ the pole or vanishing order $\operatorname{ord}_f(z)$ is even, then f is a square, i.e. there is a meromorphic function $g : \mathbb{H} \to \mathbb{C}$ such that $g^2 = f$. This fact fails to be true for rigid analytic meromorphic functions from $\Omega \to \mathbb{C}_\infty$. Consider for example functions $f \in \mathcal{O}_\Omega(\Omega)^\star$, so meromorphic functions from $\Omega \to \mathbb{C}_\infty$ such that $\operatorname{ord}_f(z) = 0$ for all $z \in \Omega$. There is a canonical exact sequence

$$0 \to \mathbb{C}_\infty^\star \to \mathcal{O}_\Omega(\Omega)^\star \xrightarrow{r} \underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z}) \to 0$$

where $\underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z})$ denotes the space of $\mathbb{Z}$-valued harmonic cocycles with $\mathrm{GL}_2(K_\infty)$ acting trivially on $\mathbb{Z}$. The map r is a logarithm map so that for $f, g \in \mathcal{O}_\Omega(\Omega)^\star$ with $g^2 = f$ one has $r(f) = 2r(g)$, compare [GR, 1.7] where the map r is made explicit. As a

concrete example consider the harmonic cocycle $\kappa \in \underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z})$ which is zero outside of the line $\{[L(m,0)] \mid m \in \mathbb{Z}\}$, and ± 1 on $([L(m,0)], [L(m+1,0)])$ depending on wether m is even or odd. Then any preimage of κ under r has $\mathrm{ord}_f(z) = 0$ for all $z \in \Omega$ but is not a square of a meromorphic function $g : \Omega \to \mathbb{C}_\infty$.
We will postpone the question for which $n, l \in \mathbb{Z}$ with n odd we have $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \neq \{0\}$ to Corollary 6.12. There we will show that the statement of Proposition 4.13 is also true for n odd.

4.4 An explicit dimension formula

In this section, we give an explicit dimension formula for $\mathcal{M}_{n,l}(\Gamma)$ for certain $\Gamma \subset \Lambda^\star$ of finite index. Namely, we assume that $\Gamma \subseteq \Lambda^\star$ has the properties

$$\mathrm{Stab}_\Gamma(v) \cong \{1\} \text{ or } \mathbb{F}_q^\star \text{ or } \mathbb{F}_{q^2}^\star \text{ for all } v \in \mathrm{V}(\mathcal{T})$$

and thus

$$\mathrm{Stab}_\Gamma(e) \cong \{1\} \text{ or } \mathbb{F}_q^\star \text{ for all } e \in \mathrm{E}(\mathcal{T}).$$

This ensures, that only vertices of degree $q+1$ or degree 1 occur in the quotient graph $\Gamma \backslash \mathcal{T}$. This assumption holds for $\Gamma = \Lambda^\star$ by Proposition 2.24 and also for the p'-torsion free case. Note that our assumption on Γ implies $\omega = \#(\Gamma \cap \mathbb{F}_q^\star) \in \{1, q-1\}$. Furthermore, in this section we assume q to be odd and $n > 2$. We follow the methods for classical modular form as treated in [Sh, Chapter 2].

Definition 4.15 *(a) A point $\bar{z} \in X$ is called elliptic if for any $z \in \Omega$ with $\pi(z) = \bar{z}$ one has* $\mathrm{Stab}_\Gamma(z) \supsetneq (\mathbb{F}_q^\star \cap \Gamma)$.

(b) The order of an elliptic point $\bar{z}$ is defined as $\#\,\mathrm{Stab}_\Gamma(z)/w$ for any $z \in \Omega$ above $\bar{z}$.

Note that if two points z and z' are Γ-equivalent their stabilizers are conjugate. Hence the definition does not depend on the choice of a point z above $\bar{z}$. Before we can state and prove the dimension formula, we need two lemmata on elliptic points of X.

Lemma 4.16 *There are at most $2^{\#R}\,\mathrm{odd}(R)(\Lambda^\star : \Gamma)$ elliptic points of X.*

Proof: For each edge $e \in \mathrm{E}(\mathcal{T})$ we have $\mathrm{Stab}_{\Lambda^\star}(e) \cong \mathbb{F}_q^\star$. Hence via the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathcal{T} \arrow[d, "\pi"] \\ X \arrow[r, "\pi^\star \rho"'] \arrow[d, "\pi'"'] & \Gamma\backslash\mathcal{T} \arrow[d, "\pi'"] \\ \Lambda^\star\backslash\Omega \arrow[r, "\pi'^\star"] & \Lambda^\star\backslash\mathcal{T} \end{tikzcd}$$

an elliptic point $\bar{z} \in X$ gets send via $\pi^\star\rho$ to a vertex $\bar{v} \in \mathrm{V}(\Gamma\backslash\mathcal{T})$ with $\mathrm{Stab}_\Gamma(v) \cong \mathbb{F}_{q^2}^\star$ for any $v \in \pi^{-1}(\bar{v})$. These are the vertices $v \in \Gamma\backslash\mathcal{T}$ such that $\mathrm{degree}(\pi'(v)) = 1$. By Theorem 2.27 there are precisly $2^{\#R-1}\,\mathrm{odd}(R)$ many vertices of degree 1 in $\Lambda^\star\backslash\mathcal{T}$, and hence at most $2^{\#R-1}\,\mathrm{odd}(R)(\Lambda^\star : \Gamma)$ vertices in $\Gamma\backslash\mathcal{T}$ with $\mathrm{degree}(\pi'(v)) = 1$.
Let $\bar{v} \in \mathrm{V}(\Gamma\backslash\mathcal{T})$ be of degree 1 and $\bar{z} \in X$ an elliptic point with $\pi^\star\rho(\bar{z}) = \bar{v}$. Let $\langle\gamma\rangle = \mathrm{Stab}_\Gamma(v)$ for some $v \in \pi^{-1}(\bar{v})$. By conjugating with a matrix of $\mathrm{GL}_2(K_\infty)$ we can assume $\mathrm{Stab}_\Gamma(v) = \mathbb{F}_{q^2}^\star$. An element $a + b\gamma \in \mathrm{Stab}_\Gamma(v)$ is then mapped to the matrix

$$a\mathbb{1} + b\begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix}$$

under the embedding $\mathbb{F}_{q^2}^\star \hookrightarrow \mathrm{GL}_2(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_2(K_\infty)$, where ξ is some non-square element in $\mathbb{F}_q^\star$. Now if $b \neq 0$ the equation

$$\frac{az+b}{b\xi z + a} = \begin{pmatrix} a & b \\ b\xi & a \end{pmatrix} z = z$$

is equivalent to $\xi z^2 = 1$, hence $z \in \mathbb{F}_{q^2}^\star \smallsetminus \mathbb{F}_q^\star$. Since $K_\infty \cap \mathbb{F}_{q^2} = \mathbb{F}_q$ this implies $z \in \mathbb{C}_\infty \smallsetminus K_\infty$, and there are exactly two solutions to this equation in $\mathbb{C}_\infty$. ∎

Remark 4.17 The proof of Lemma 4.16 shows that there are exactly twice as many elliptic points of X than terminal vertices of $\Gamma\backslash\mathcal{T}$.

To a function $f \in \mathcal{A}_{n,l}(\Gamma)$ one can associate a divisor on X in the following way. For $P \in X$ we define $v_P(f)$ as follows: Choose any lift $\tilde{P}$ of P to Ω and set $e = \mathrm{Stab}_{\bar{\Gamma}}(\tilde{P}) = \mathrm{Stab}_\Gamma(\tilde{P})/\omega$. Choose a map $\lambda : \mathbb{P}^1(\mathbb{C}_\infty) \to \mathbb{P}^1(\mathbb{C}_\infty)$ sending $\tilde{P}$ to 0, let $t = \lambda(z)^e$ and put $v_P(f) = v_{t,\tilde{P}}(f) = v_{\tilde{P}}(f(t))$. Then we define $\mathrm{div}(f)$ to be the formal sum

$$\mathrm{div}(f) = \sum_{P \in X} v_P(f) P.$$

Lemma 4.18 *Let $P_1, \dots, P_r$ be the elliptic points of X of order $e_1, \dots, e_r$ and $0 \neq f \in \mathcal{A}_{2n,n}(\Gamma, \mathbb{C}_\infty)$. Set $\eta := f(z)\mathrm{d}z^n$. Then*

$$\mathrm{div}(f) = \mathrm{div}(\eta) + n(\sum_{i=1}^{r} 1 - \frac{1}{e_i})[P_i]$$

and

$$\deg(\mathrm{div}(f)) = n\lfloor (2g-2) + \sum_{i=1}^{r} (1 - \frac{1}{e_i}) \rfloor.$$

Proof: We have $\eta \in \mathcal{A}_{0,0}(\Gamma, \mathbb{C}_\infty)$, hence η can be viewed as a meromorphic function on X. Let $P \in X$ and $G = \mathrm{Stab}_\Gamma(\tilde{P})$ for $\tilde{P}$ any lift of P to Ω.
If P is an elliptic point of X of order e, then as in the proof of Lemma 4.16 there are two fixpoints $\{z_0, z_1\} \subset \mathbb{C}_\infty$ of G both not lying in K_∞. W.l.o.g. we can assume that z_0 is a lift of P. Hence the map

$$\begin{array}{rcl} \lambda_G := \lambda_{z_0,z_1} : \mathbb{P}^1(\mathbb{C}_\infty) & \to & \mathbb{P}^1(\mathbb{C}_\infty) \\ z & \mapsto & \dfrac{z - z_0}{z - z_1} \end{array}$$

is well-defined and has $\lambda_G(z_0) = 0$ and $\lambda_G(z_1) = \infty$. Therefore we can choose $t = \lambda_G(z)^e$. One computes

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{\mathrm{d}\lambda_G(z)^e}{\mathrm{d}z} = e\lambda_G(z)^{e-1}\frac{\mathrm{d}\lambda_G(z)}{\mathrm{d}z}.$$

and hence $v_{t,z_0}(\mathrm{d}t/\mathrm{d}z) = \frac{1}{e}(e-1) = 1 - e^{-1}$. Therefore $v_P(\eta) = v_P(f(z)\mathrm{d}z^n) = v_P(f) + n(e^{-1} - 1)$.
If P is a non-elliptic point of X, then $G = \mathbb{F}_q^\star \cap \Gamma$ and hence $e = 1$ and $v_P(f) = v_P(\eta)$.
Hence

$$\mathrm{div}(f) = \mathrm{div}(\eta) + n(\sum_{i=1}^{r} 1 - \frac{1}{e_i})[P_i]$$

and since $\deg(\mathrm{div}(\mathrm{d}z)) = 2g - 2$ the second formula also holds. ∎

We are now able to give an explicit dimension formula for certain pairs of n, l.

Theorem 4.19 *Let r be number of elliptic points of X and $g = g(X)$. Let $n > 2$ and $l \in \mathbb{Z}$ with $n \equiv 2l \pmod{w}$.*

(a) If $w = 1$ then $\dim \mathcal{M}_{n,l}(\Gamma) = (n-1)(g-1)$.

(b) If $w = q-1$ then n is even. If $l \equiv \frac{n}{2} \pmod{q-1}$ then

$$\dim \mathcal{M}_{n,l}(\Gamma) = (n-1)(g-1) + r \left\lfloor \frac{n}{2}(1 - \frac{1}{q+1}) \right\rfloor .$$

PROOF: (a) If $w = 1$, then $\mathrm{Stab}_\Gamma(s) = \{1\}$ for all simplices s of $\mathcal{T}$. Hence as in the proof of Lemma 4.4 we know that $\Gamma \backslash \mathcal{L}_{n,l}$ is a line bundle on X and we can compute the dimension of $\mathcal{M}_{n,l}(\Gamma)$ just as in the proof of Proposition 4.6.

(b) Since q is odd, the condition $n \equiv 2l \pmod{q-1}$ forces n to be even. By Proposition 4.13 we can choose $0 \neq F_0 \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$. Any $F \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ can then be written as $F = fF_0$ with some unique $f \in \mathcal{K} := \mathcal{A}_{0,0}(\Gamma)$. Set $B := \mathrm{div}(F_0)$. From the definition of $\mathcal{M}_{n,l}(\Gamma)$ and since $\mathrm{div}(f_1 f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$, we know that

$$\begin{aligned} \dim \mathcal{M}_{n,l}(\Gamma) &= \{f \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \mid \mathrm{div}(f) \geq 0\} \\ &= \{f \in \mathcal{K} \mid \mathrm{div}(f) \geq -B\} \\ &= \{f \in \mathcal{K} \mid \mathrm{div}(f) \geq -[B]\} \\ &= l([B]) \end{aligned}$$

where $[B]$ denotes the integral part of the divisor B, see [Sh, page 45] for a definition, and $l([B])$ denotes the dimension of the Riemann-Roch space of the divisor $[B]$.
As in [Sh, page 37-38] one constructs a graded algebra

$$\mathcal{D} = \sum_{n=-\infty}^{\infty} \mathrm{Dif}^n(X)$$

such that for each $\xi \in \mathrm{Dif}^n(X)$ one has naturally defined a divisor $\mathrm{div}(\xi)$ on X with $\deg(\mathrm{div}(\xi)) = n(2g-2)$ for $0 \neq \xi \in \mathrm{Dif}^n(X)$. If $l \equiv \frac{n}{2} \pmod{q-1}$ we have an isomorphism $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \to \mathrm{Dif}^{n/2}(X)$ given by $f \mapsto f(z)(\mathrm{d}z)^{n/2}$.
We can then apply Lemma 4.18 to obtain

$$\deg([B]) = \frac{n}{2} \left((2g-2) + r \left\lfloor \left(1 - \frac{1}{q+1}\right) \right\rfloor \right),$$

where we used our assumption on Γ, which implies that all elliptic fixpoints have order $q+1$. For $n > 2$ this shows that $\deg([B]) - (2g-2) > 0$ and hence we can use the Riemann-Roch theorem to conclude that

$$l([B]) = \deg([B]) - g + 1 = (n-1)(g-1) + r \left\lfloor \frac{n}{2}(1 - \frac{1}{q+1}) \right\rfloor . \blacksquare$$

Remark 4.20 If $l \equiv \frac{n}{2} + \frac{q-1}{2} \pmod{q-1}$, the map given in the proof of Theorem 4.19 between $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$ and $\mathrm{Dif}^{n/2}(X)$ is not well-defined. Hence the proof fails to work in this situation. In Corollary 6.14 plus Lemma 6.15 we will obtain an explicit dimension formula for the space of harmonic cocycles that holds also for $l \equiv \frac{n}{2} + \frac{q-1}{2}$. We will show in Chapter 5 that the spaces of harmonic cocycles and modular forms are isomorphic. Hence we obtain a dimension formula for $\mathcal{M}_{n,l}(\Gamma)$ with $l \equiv \frac{n}{2} + \frac{q-1}{2} \pmod{q-1}$ later at this point.

5 An analogue of a result of Teitelbaum

We keep the notation from Chapter 4. Recall that $\Gamma \subseteq \Lambda^\star$ is a finite index subgroup and $\mathcal{M}_{n,l}(\Gamma)$ is the space of quaternionic Drinfeld modular forms of weight n and type l for Γ. The goal of this chapter is to construct an isomorphism between $\mathcal{M}_{n,l}(\Gamma)$ and the space of harmonic cocycles $C^{\text{har}}_{n,l}(\Gamma)$. This is done via the residue map, studied in Section 5.1. In Section 5.2 and Section 5.3 we construct an explicit inverse of the residue map given by integration against the Poisson kernel. This is done in order to show that the residue map actually gives an isomorphism. Furthermore, in Section 5.4, we will introduce an action of Hecke operators on both sides and show that the residue map isomorphism is compatible with the action of Hecke operators. We also explain how to explicitly compute this Hecke action on cocycles.

The results and methods in Sections 5.1-5.3 are analogues to the work of Teitelbaum in [Te1] and [Te2] where he constructed similar isomorphisms for p-adic modular forms and Drinfeld modular forms for congruence subgroups. We follow his work quite closely, without always giving references. For the sake of completeness and readability we prove all statements and give the necessary computations. The results in [Te1] and [Te2] are sometimes rather sketchy.

Throughout this chapter n is an integer with $n \geq 2$.

5.1 The residue map

Let $e = (v, v')$ be an oriented edge of $\mathcal{T}$. Recall that the inverse image $V(e) := \rho^{-1}(e \smallsetminus \{v, v'\})$ under the reduction map $\rho : \Omega \to \mathcal{T}$ is an open annulus. In particular it is isomorphic to the standard open annulus

$$V := \{z \in \Omega \mid q > |z| > 1\}.$$

Definition 5.1 *An orientation of an open annulus W is an equivalence class of isomorphisms $w : W \to V$ where two such isomorphism w, w' are equivalent if we have $|w' \circ w^{-1}(z)| = |z|$ for all points $z \in V$.*

An open annulus has precisly two orientations. For example, on V one has the identity and the map $z \mapsto \frac{T}{z}$ representing the two different orientations of the standard annulus.

For $f\mathrm{d}z$ a rigid analytic differential on $V(e)$ we can choose an isomorphism v between $V(e)$ and V respecting the orientation on e. In this way one obtains an expansion of $f\mathrm{d}z$ as a rigid analytic differential of the form

$$f\mathrm{d}z = \sum_{n\in\mathbb{Z}} a_n v^n \mathrm{d}v$$

with $a_i \in \mathbb{C}_\infty$. The isomorphism v is not unique, however the coefficient a_{-1} in the above expansion does not depend on the choice of v [Se4, page 25].

Definition 5.2 *Define* $\operatorname{Res}_e f\mathrm{d}z = a_{-1}$ *in the above expansion of* $f\mathrm{d}z$.

Remark 5.3 If one changes the orientation on $V(e)$ then by [FvdP, page 23] the sign of $\operatorname{Res}_e f\mathrm{d}z$ changes. Hence

$$\operatorname{Res}_{e^\star} f\mathrm{d}z = -\operatorname{Res}_e f\mathrm{d}z. \tag{12}$$

Using the residue map one can construct harmonic cocycles in $C^{\text{har}}_{n,l}(\Gamma)$. Throughout this chapter let $n \geq 2$. Recall that an element $\kappa \in C^{\text{har}}_{n,l}(\Gamma)$ is a map

$$\kappa : \mathrm{E}(\mathcal{T}) \to V_{n,l}(\mathbb{C}_\infty) = (\det)^{l-1} \otimes_{K_\infty} \operatorname{Hom}(\operatorname{Sym}^{n-2}(\operatorname{Hom}(K_\infty^2, F)), K_\infty).$$

To specify such a map κ, it suffices to define its value $\kappa(e)$ evaluated at monomials of the form X^iY^{n-2-i} where i runs from 0 to $n-2$ and X, Y are the dual basis of the standard basis of K_∞^2.

Definition 5.4 *Let* $f \in \mathcal{M}_{n,l}(\Gamma)$. *Define for* $e \in \mathrm{E}(\mathcal{T})$ *and* $i \in \{0, \ldots, n-2\}$

$$\operatorname{Res}(f)(e)(X^iY^{n-2-i}) = \operatorname{Res}_e z^i f(z)\mathrm{d}z.$$

Following [Te2, Definition 10] we have the following proposition.

Proposition 5.5 *The assignment* $f \mapsto \operatorname{Res}(f)$ *gives a well-defined homomorphism from* $\mathcal{M}_{n,l}(\Gamma)$ *to* $C^{\text{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$.

Proof: By the rigid analytic residue theorem from [FvdP, Theorem 2.2.3] and since $z^i f(z)\mathrm{d}z$ is a rigid analytic differential on Ω it directly follows that for $v \in \mathrm{V}(\mathcal{T})$ we have

$$\sum_{e \mapsto v} \operatorname{Res}(f)(e)(X^iY^{n-2-i}) = \sum_{e \mapsto v} \operatorname{Res}_e z^i f(z)\mathrm{d}z = 0.$$

By Equation 12 one has $\operatorname{Res}(f)(e^\star) = -\operatorname{Res}(f)(e)$ for all $e \in \mathrm{E}(\mathcal{T})$.
To check the Γ-equivariance, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and note that $\operatorname{Res}_{\gamma e}(\omega) = \operatorname{Res}_e \gamma^\star \omega$ for any rigid analytic differential $\omega \in \Gamma(\Omega, \boldsymbol{\Omega}_\Omega)$. For ω of the form $f(z)\mathrm{d}z$ this action was given by

$$\gamma^\star f(z)\mathrm{d}z = \frac{\det(\gamma) f(\gamma z)}{(cz+d)^2}\mathrm{d}z.$$

Evaluating it at a monomial of the form X^iY^{n-2-i} and using that $f = f \underset{n,l}{|} \gamma$ one computes

$$\begin{aligned}
\mathrm{Res}(f)(\gamma e)(X^iY^{n-2-i}) &= \mathrm{Res}_{\gamma e}\, z^i f(z)\mathrm{d}z \\
&= \mathrm{Res}_e\, \gamma^\star z^i f(z)\mathrm{d}z \\
&= \mathrm{Res}_e \det(\gamma)(az+b)^i(cz+d)^{i-2} f(\gamma z)\mathrm{d}z \\
&= \mathrm{Res}_e \det(\gamma)^{1-l}(az+b)^i(cz+d)^{n-2-i} f(z)\mathrm{d}z \\
&= \det(\gamma)^{1-l}\,\mathrm{Res}_e(az+b)^i(cz+d)^{n-2-i} f(z)\mathrm{d}z \\
&= \det(\gamma)^{1-l}\,\mathrm{Res}(f)(e)((aX+bY)^i(cX+dY)^{n-2-i}) \\
&= (\gamma \cdot_{n,l} (\mathrm{Res}(f)(e))(X^iY^{n-2-i}).
\end{aligned}$$

The fact that Res is a homomorphism follows from the formulas $\mathrm{Res}_e(\omega_1+\omega_2) = \mathrm{Res}_e(\omega_1) + \mathrm{Res}_e(\omega_2)$ and $\mathrm{Res}_e(\lambda\omega_1) = \lambda\,\mathrm{Res}_e(\omega_1)$ for $\lambda \in \mathbb{C}_\infty$ and ω_i rigid analytic differentials in $\Gamma(\Omega, \mathbf{\Omega}_\Omega)$, see the remark on page 223 of [Sch2]. ∎

5.2 Measures

Let $\mathcal{A}_n$ be the ring of functions from $\mathbb{P}^1(K_\infty)$ to $\mathbb{C}_\infty$ which are locally analytic for all $x \neq \infty$ and have pole order at most $n-2$ at ∞ and let $\mathcal{P}_n \subseteq \mathcal{A}_n$ be the subring of locally polynomial functions of degree less than or equal $n-2$. in one variable x over K_∞. Here locally polynomial means that for every $f \in \mathcal{P}_n$ there is a finite open cover $\{U_i\}$ of $\mathbb{P}^1(K_\infty)$ such that on each U_i the function f can be expressed as a polynomial. We endow $\mathcal{A}_n$ with the *Fréchet* topology, see [Col, 1.8.1] for details. Finally let $P_n \subseteq \mathcal{P}_n$ be the space of globally polynomial functions on $\mathbb{P}^1(K_\infty)$ of degree less than or equal $n-2$.
Let $\mathcal{B} = \{U \subset \mathbb{P}^1(K_\infty) \mid U \text{ compact open}\}$. For $U \in \mathcal{B}$ we denote by χ_U the characteristic function for U that takes the value 1 on U and 0 on $\mathbb{P}^1(K_\infty) \setminus U$.

Definition 5.6 *(a) A measure on $\mathcal{P}_n$ is a linear map $\tilde{\mu} \in \mathrm{Hom}(\mathcal{P}_n, K_\infty)$. For $f \in \mathcal{P}_n, U \in \mathcal{B}$ write*

$$\int_U f(x)\tilde{\mu}(x) = \tilde{\mu}(f)\chi_U.$$

(b) A measure on $\mathcal{A}_n$ is a continous linear map $\mu \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}_n, K_\infty)$. For $f \in \mathcal{A}_n, U \in \mathcal{B}$ write

$$\int_U f(x)\mu(x) = \mu(f)\chi_U.$$

To each edge $e \in \mathrm{E}(\mathcal{T})$ let $U(e)$ be the set of ends of $\mathcal{T}$ which have a representative containing e. Then by the identification of the ends of $\mathcal{T}$ with $\mathbb{P}^1(K_\infty)$ we can identify $U(e)$ with a compact open subset of $\mathbb{P}^1(K_\infty)$. Clearly $U(e) \cap U(e^\star) = \varnothing$ and $U(e) \cup U(e^\star) = \mathbb{P}^1(K_\infty)$. The following Proposition is a direct consequence of the fact that the measures on $\mathcal{A}$ we consider are continuous. It can be found as [Te1, Proposition 9, (5)].

Proposition 5.7 *Let μ be a measure on $\mathcal{A}_n$.*

(a) Let $e \in \mathrm{E}(\mathcal{T})$ with $\infty \in U(e)$ and $f \in \mathcal{A}_n$ with Laurent expansion at ∞ of the form $f(x) = \sum_{i=-\infty}^{n-2} a_n x^n$ converging on $U(e) \smallsetminus \{\infty\}$. Then

$$\int_{U(e)} f(x) \mathrm{d}\mu = \sum_{i=-\infty}^{n-2} a_i \int_{U(e)} x^i \mathrm{d}\mu.$$

(b) Let $e \in \mathrm{E}(\mathcal{T})$ with $\infty \notin U(e)$, $r \in U(e)$ and $f \in \mathcal{A}_n$ with Taylor series expansion at r of the form $f(x) = \sum_{i=0}^{\infty} a_i (x-r)^i$ converging on $U(e)$. Then

$$\int_{U(e)} f(x) \mathrm{d}\mu = \sum_{i=0}^{\infty} a_i \int_{U(e)} (x-r)^i \mathrm{d}\mu.$$

Starting from harmonic cocycles we can construct measures. We will begin by constructing measures on $\mathcal{P}_n$ and show later that these measures extend in a unique way to measures on $\mathcal{A}_n$. Let $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)$ and define a measure $\tilde{\mu}_\kappa$ on $\mathcal{P}_n$ by setting formally

$$\int_{U(e)} x^i \mathrm{d}\tilde{\mu}_\kappa(x) = \kappa(e)(X^i Y^{n-2-i}) \tag{13}$$

for $e \in \mathrm{E}(\mathcal{T})$ and $i \in \{0, \ldots, n-2\}$. Extend this definition to $\mathcal{P}_n$ by linearity. This assignment completely defines a measure $\tilde{\mu}_\kappa$ on $\mathcal{P}_n$, since the open compact discs $U(e)$ form a basis for the topology on $\mathbb{P}^1(K_\infty)$.
For $v \in \mathrm{V}(\mathcal{T})$ let $\{e_1, \ldots, e_{q+1}\}$ be the $q+1$ edges e with $t(e) = v$. Then by the definition of $U(e)$ one has $U(e_1) = U(e_2^\star) \cup \cdots \cup U(e_{q+1}^\star)$. To see that $\tilde{\mu}_\kappa$ is a measure, one checks, using the harmonicity of κ, that

$$\begin{aligned} \int_{U(e_1)} x^i \mathrm{d}\tilde{\mu}_\kappa(x) &= \kappa(e_1)(X^i Y^{n-2-i}) = -\sum_{i=2}^{q+1} \kappa(e_i)(X^i Y^{n-2-i}) \\ &= \sum_{i=2}^{q+1} \kappa(e_i^\star)(X^i Y^{n-2-i}) = \sum_{i=2}^{q+1} \int_{U(e_i^\star)} x^i \mathrm{d}\tilde{\mu}_\kappa(x). \end{aligned}$$

Also since $\mathbb{P}^1(K_\infty) = U(e) \cup U(e^\star)$ for any $e \in \mathrm{E}(\mathcal{T})$, one has

$$\begin{aligned}\int_{\mathbb{P}^1(K_\infty)} x^i \mathrm{d}\tilde{\mu}_\kappa(x) &= \int_{U(e)} x^i \mathrm{d}\tilde{\mu}_\kappa(x) + \int_{U(e^\star)} x^i \mathrm{d}\tilde{\mu}_\kappa(x)\\ &= \kappa(e)(X^iY^{n-2-i}) + \kappa(e^\star)(X^iY^{n-2-i}) = 0\end{aligned}$$

and hence

$$\int_{\mathbb{P}^1(K_\infty)} f(x)\mathrm{d}\tilde{\mu}_\kappa(x) = 0 \text{ for all } f \in P_n. \tag{14}$$

Lemma 5.8 *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *and* $e \in \mathrm{E}(\mathcal{T})$ *one has*

$$\int_{U(\gamma e)} f(x)\mathrm{d}\tilde{\mu}_\kappa(x) = \det(\gamma)^{1-l} \int_{U(e)} f(\gamma x)(cx+d)^{n-2}\mathrm{d}\tilde{\mu}_\kappa(x)$$

for all $f \in \mathcal{P}_n$.

Note that since $(cx+d)$ occurs in the numerator of γx, the right hand side is always a polynomial of degree at most $n-2$.

Proof: One computes

$$\begin{aligned}\int_{U(\gamma e)} x^i \mathrm{d}\tilde{\mu}_\kappa(x) &= \kappa(\gamma e)(X^iY^{n-2-i}) = \gamma \cdot_{n,l} \kappa(e)(X^iY^{n-2-i})\\ &= \det(\gamma)^{1-l}\,\kappa(e)((aX+bY)^i(cX+dY)^{n-2-i})\\ &= \det(\gamma)^{1-l}\int_{U(e)} (\gamma x)^i (cx+d)^{n-2}\mathrm{d}\tilde{\mu}_\kappa(x)\end{aligned}$$

and then the claim follows by linearity. ∎

The following lemma is an adaption to our situation of the lemma on page 227 from [Sch].

Lemma 5.9 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $e \in \mathrm{E}(\mathcal{T})$ *and* $r \in K_\infty$ *such that* $\gamma r \neq \infty$, *i.e.* $cr + d \neq 0$. *Then for* $0 \le i \le n-2$ *one has*

$$\int_{U(\gamma e)} (x - \gamma r)^i \mathrm{d}\tilde{\mu}_\kappa(x)$$

$$= \begin{cases} \det(\gamma)^{i+1-l}(cr+d)^{n-2-2i}\sum_{j=0}^{n-2-i}\binom{n-2-i}{j}(r+\frac{d}{c})^{-j}\int_{U(e)}(x-r)^{i+j}\mathrm{d}\tilde{\mu}_\kappa(x) & \text{if } c \neq 0,\\ \det(\gamma)^{i+1-l}d^{n-2-2i}\int_{U(e)}(x-r)^i\mathrm{d}\tilde{\mu}_\kappa(x) & \text{if } c = 0.\end{cases}$$

Proof: First observe that

$$\gamma x - \gamma r = \frac{ax+b}{cx+d} - \frac{ar+b}{cr+d} = \frac{\det(\gamma)(x-r)}{(cx+d)(cr+d)}. \tag{15}$$

Using this we compute

$$\begin{aligned}\int_{U(\gamma e)} (x-\gamma r)^i \mathrm{d}\tilde{\mu}_\kappa(x) &\overset{5.8}{=} \det(\gamma)^{1-l} \int_{U(e)} (\gamma x - \gamma r)^i (cx+d)^{n-2} \mathrm{d}\tilde{\mu}_\kappa(x) \\ &\overset{(15)}{=} \det(\gamma)^{1-l} \int_{U(e)} \left(\frac{\det(\gamma)(x-r)}{(cx+d)(cr+d)}\right)^i (cx+d)^{n-2} \mathrm{d}\tilde{\mu}_\kappa(x) \\ &= \det(\gamma)^{1-l+i}(cr+d)^{-i} \int_{U(e)} (cx+d)^{n-2-i}(x-r)^i \mathrm{d}\tilde{\mu}_\kappa(x)\end{aligned}$$

For $c = 0$, this proves the claim. For $c \neq 0$ we use the binomial expansion.

$$\det(\gamma)^{1-l+i}(cr+d)^{-i} \int_{U(e)} (cx+d)^{n-2-i}(x-r)^i \mathrm{d}\tilde{\mu}_\kappa(x)$$

$$\begin{aligned}&= \det(\gamma)^{1-l+i}(cr+d)^{-i} \int_{U(e)} c^{n-2-i}((x-r)+(r+\frac{d}{c}))^{n-2-i}(x-r)^i \mathrm{d}\tilde{\mu}_\kappa(x) \\ &= \det(\gamma)^{1-l+i}(cr+d)^{-i}c^{n-2-i} \sum_{j=0}^{n-2-i} \binom{n-2-i}{j} (r+\frac{d}{c})^{n-2-i-j} \int_{U(e)} (x-r)^{i+j} \mathrm{d}\tilde{\mu}_\kappa(x) \\ &= \det(\gamma)^{1-l+i}(cr+d)^{n-2-2i} \sum_{j=0}^{n-2-i} \binom{n-2-i}{j} (r+\frac{d}{c})^{-j} \int_{U(e)} (x-r)^{i+j} \mathrm{d}\tilde{\mu}_\kappa(x) \quad \blacksquare\end{aligned}$$

The measures $\tilde{\mu}_\kappa$ constructed from harmonic cocycles fulfill a certain boundedness condition. This condition is crucial for extending $\tilde{\mu}_\kappa$ to a measure on $\mathcal{A}_n$. For $e \in \mathrm{E}(\mathcal{T})$ let

$$\tau(e) = \begin{cases} \operatorname{diam}(U(e)) = \sup_{x,y\in U(e)} |x-y| & \text{if } \infty \notin U(e) \\ \operatorname{diam}(U(e^\star))^{-1} & \text{if } \infty \in U(e). \end{cases}$$

Remark 5.10 If $\infty \notin U(e), x \in U(e)$ and $y \in \mathbb{P}^1(K_\infty) \smallsetminus U(e)$ then by the ultrametric triangle inequality we have $\tau(e) < |x-y|$.

Definition 5.11 *(a) A measure $\tilde{\mu}$ on $\mathcal{P}_n$ is called bounded if there exist a constant $C > 0$ such that for all $0 \leq i \leq n-2$ and all $e \in \mathrm{E}(\mathcal{T})$ with $\infty \notin U(e)$ and for all $r \in U(e)$ one has*

$$\left| \int_{U(e)} (x-r)^i \mathrm{d}\tilde{\mu}(x) \right| < C\tau(e)^{i-(n-2)/2}$$

and for all $e \in \mathrm{E}(\mathcal{T})$ with $\infty \in U(e)$, $0 \notin U(e)$ one has

$$\left| \int_{U(e)} x^i \mathrm{d}\tilde{\mu}(x) \right| < C\tau(e)^{-i+(n-2)/2}.$$

(b) Let $\widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ denote the set of bounded measures $\tilde{\mu}$ on $\mathcal{P}_n$ such that

$$\int_{\mathbb{P}^1(K_\infty)} f(x)\mathrm{d}\tilde{\mu}(x) = 0$$

for all $f \in P_n$ and such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, e \in \mathrm{E}(\mathcal{T})$ and all $f \in \mathcal{P}_n$ one has

$$\int_{U(\gamma e)} f(x)\mathrm{d}\tilde{\mu}(x) = \det(\gamma)^{1-l} \int_{U(e)} f(\gamma x)(cx+d)^{n-2}\mathrm{d}\tilde{\mu}(x).$$

Since the sum of two bounded measures and a scalar multiple of a bounded measure are still bounded, the set $\widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ is a subspace of the space of all measures on $\mathcal{P}_n$. In particular, it carries the structure of a K_∞-vector space. Following [Te2, Lemma 6] we prove the following proposition.

Proposition 5.12 *If $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma)$, then $\tilde{\mu}_\kappa \in \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$.*

Proof: The measure $\tilde{\mu}_\kappa$ has the correct transformation property by Lemma 5.8 and it vanishes on globally polynomial functions by Equation (14). It remains to show that $\tilde{\mu}_\kappa$ is bounded.
Let $\{e_1, \ldots, e_h\}$ be a set of representatives of the edges of $\Gamma\backslash\mathcal{T}$ with $\infty \notin U(e_i)$. This set is finite, since $\Gamma\backslash\mathcal{T}$ is a finite covering of $\Lambda^\star\backslash\mathcal{T}$ which is a finite graph by Proposition 2.23. W.l.o.g. we can assume that $\det(\Gamma) = \{1\}$. If not, we replace Γ by a finite index subgroup Γ' with $\det(\Gamma') = \{1\}$ and work with the representatives of the edges of $\Gamma'\backslash\mathcal{T}$. This is still a finite covering of $\Lambda^\star\backslash\mathcal{T}$.
Now let $e \in \mathrm{E}(\mathcal{T})$ with $\infty \notin U(e)$ and $r \in U(e)$. Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $1 \leq g \leq h$ such that $e = \gamma e_g$. Observe that, since $r \in U(e)$ but $\infty \notin U(e)$,

$$|\gamma^{-1}r + \frac{d}{c}| = |\gamma^{-1}r - \gamma^{-1}\infty| \geq \tau(e_g). \tag{16}$$

Next we will show that

$$\tau(e_g) = \tau(e)|c\gamma^{-1}r + d|^2. \tag{17}$$

For this choose $s \in U(e)$ with $|r-s| = \tau(e)$. Then

$$\begin{aligned}
\tau(e_g) &= \tau(\gamma^{-1}e) = |\gamma^{-1}r - \gamma^{-1}s| \\
&= \left|\frac{dr-b}{-cr+a} - \frac{ds-b}{-cs+a}\right| \\
&= \left|\frac{\det(\gamma)(r-s)}{(-cr+a)(-cs+a)}\right| \\
&= \frac{|r-s|}{|(-cr+a)^2|} \\
&= \tau(e)|c\gamma^{-1}r+d|^2
\end{aligned}$$

For $c \neq 0$ we deduce

$$\begin{aligned}
&\left|\int_{U(e)} (x-r)^i \mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
\overset{5.9}{=} &\left|\det(\gamma)^{i+l-1}(c\gamma^{-1}r+d)^{n-2-2i}\sum_{j=0}^{n-2-i}\binom{n-2-i}{j}(\gamma^{-1}r+\frac{d}{c})^{-j}\int_{U(e_g)}(x-\gamma^{-1}r)^{i+j}\mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
\leq \quad & |c\gamma^{-1}r+d|^{n-2-2i}\max_{j=0}^{n-2-i}\left|\gamma^{-1}r+\frac{d}{c}\right|^{-j}\left|\int_{U(e_g)}(x-\gamma^{-1}r)^{i+j}\mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
\overset{(16),(17)}{\leq} \quad & \tau(e_g)^{\frac{n-2}{2}-i}\tau(e)^{i-\frac{n-2}{2}}\max_{j=0}^{n-2-i}\tau(e_g)^{-j}\left|\int_{U(e_g)}(x-\gamma^{-1}r)^{i+j}\mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
= \quad & C\tau(e)^{i-\frac{n-2}{2}}
\end{aligned}$$

for a constant $C>0$ independent of the choice of γ and r.
For $c=0$ we deduce

$$\begin{aligned}
&\left|\int_{U(e)} (x-r)^i \mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
\overset{5.9}{=} \quad & |d^{n-2-2i}|\left|\int_{U(e_g)}(x-r)^i\mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
\overset{(17)}{\leq} \quad & \tau(e_g)^{\frac{n-2}{2}-i}\tau(e)^{i-\frac{n-2}{2}}\left|\int_{U(e_g)}(x-r)^i\mathrm{d}\tilde{\mu}_\kappa(x)\right| \\
= \quad & C\tau(e)^{i-\frac{n-2}{2}}
\end{aligned}$$

for a constant $C>0$ independent of the choice of γ and r.

This proves the first bound. For the second bound let $e \in \mathrm{E}(\mathcal{T})$ with $\infty \in U(e)$ and $0 \notin U(e)$. Then by Equation 14 and by what we have just shown we have

$$\left|\int_{U(e)} x^i \mathrm{d}\tilde{\mu}_\kappa(x)\right| = \left|\int_{U(e^\star)} x^i \mathrm{d}\tilde{\mu}_\kappa(x)\right| \leq C\tau(e^\star)^{i-(n-1)/2} = C\tau(e)^{-i+(n-1)/2}$$

for some constant $C > 0$. ∎

Proposition 5.13 *The map*

$$\widetilde{\psi} : C^{\mathrm{har}}_{n,l}(\Gamma) \to \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma), \kappa \mapsto \tilde{\mu}_\kappa$$

is an isomorphism of K_∞-vector spaces.

Proof: The well-definedness of $\widetilde{\psi}$ is Proposition 5.12. The injectivity of $\widetilde{\psi}$ and the assertion that $\widetilde{\psi}$ is a homomorphism are clear from the definition of $\tilde{\mu}_\kappa$ in Equation 13. For the surjectivity, let $\tilde{\mu} \in \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ and define for $e \in \mathrm{E}(\mathcal{T})$

$$\kappa_{\tilde{\mu}}(e)(X^iY^{n-2-i}) = \int_{U(e)} x^i \mathrm{d}\tilde{\mu}(x).$$

We will show that $\kappa_{\tilde{\mu}} \in C^{\mathrm{har}}_{n,l}(\Gamma)$. For the harmonicity observe that for $e \in \mathrm{E}(\mathcal{T})$ and $v \in \mathrm{V}(\mathcal{T})$ one has

$$\kappa_{\tilde{\mu}}(e^\star)(X^iY^{n-2-i}) = \int_{U(e^\star)} x^i \mathrm{d}\tilde{\mu}(x) = -\int_{U(e)} x^i \mathrm{d}\tilde{\mu}(x) = -\,\kappa_{\tilde{\mu}}(e)(X^iY^{n-2-i})$$

and

$$\sum_{e \mapsto v} \kappa_{\tilde{\mu}}(e)(X^iY^{n-2-i}) = \sum_{e \mapsto v} \int_{U(e)} x^i \mathrm{d}\tilde{\mu}(x) = \int_{\mathbb{P}^1(K_\infty)} x^i \mathrm{d}\tilde{\mu}(x) = 0.$$

For the Γ-equivariance one computes for $\gamma \in \Gamma$ and $e \in \mathrm{E}(\mathcal{T})$

$$\begin{aligned}
\kappa_{\tilde{\mu}}(\gamma e)(X^iY^{n-2-i}) &= \int_{U(\gamma e)} x^i \mathrm{d}\tilde{\mu}(x) \\
&= \det(\gamma)^{1-l} \int_{U(e)} (\gamma x)^i (cx+d)^{n-2} \mathrm{d}\tilde{\mu}(x) \\
&= (\gamma \cdot_{n,l} \kappa_{\tilde{\mu}}(e))(X^iY^{n-2-i}).
\end{aligned}$$

This implies $\kappa_{\tilde{\mu}} \in C^{\mathrm{har}}_{n,l}(\Gamma)$ and since by definition of $\widetilde{\psi}$ we have $\widetilde{\psi}(\kappa_{\tilde{\mu}}) = \tilde{\mu}$ it follows that $\widetilde{\psi}$ is surjectiv. ∎

We also need to integrate more general thean locally polynomial functions still having bounded pole order at infinity. For this we need to show that the measures $\tilde{\mu}_\kappa$ we constructed extend to certain measures on $\mathcal{A}_n$. We first introduce the space of measures to which $\tilde{\mu}_\kappa$ extends.

Definition 5.14 *Let* $\mathrm{Meas}_{n,l}(\Gamma)$ *be the* K_∞*-vector space of measures* μ *on* $\mathcal{A}_n$ *such that*

(a) For all $f \in P_n$ *one has*

$$\int_{\mathbb{P}^1(K_\infty)} f(x)\mathrm{d}\mu(x) = 0.$$

(b) For all $f \in \mathcal{A}_n$, $e \in \mathrm{E}(\mathcal{T})$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *one has*

$$\int_{U(\gamma e)} f(x)\mathrm{d}\mu = \det(\gamma)^{1-l} \int_{U(e)} f(\gamma x)(cx+d)^{n-2}\mathrm{d}\mu.$$

Remark 5.15 Both Definition 5.14(b) and Lemma 5.8 can be expressed as an equivariance of a measure. Namely in both cases one has

$$\mathrm{d}(\mu \cdot \gamma) = \det(\gamma)^{1-l}(cx+d)^{n-2}\mathrm{d}\mu.$$

Proposition 5.16 *Let* $\mu \in \mathrm{Meas}_{n,l}(\Gamma)$*. Then there is a constant* $C > 0$ *such that for all* $e \in \mathrm{E}(\mathcal{T})$ *with* $\infty \in U(e)$, $0 \notin U(e)$ *and for all* $-\infty < i \leq n-2$ *one has*

$$\left| \int_{U(e)} x^i \mathrm{d}\mu \right| \leq C\tau(e)^{-i+(n-2)/2}$$

and for all $e \in \mathrm{E}(\mathcal{T})$ *with* $\infty \notin U(e)$ *and for* $r \in U(e)$ *and all* $i \geq 0$ *one has*

$$\left| \int_{U(e)} (x-r)^i \mathrm{d}\mu \right| \leq C\tau(e)^{i-(n-2)/2}.$$

Proof: The proofs of Lemma 5.9 and Proposition 5.12 only used the invariance of the measure at hand under the Γ-action and the fact that the measure vanishes for global polynomials. These properties are just part (a) and (b) of Definition 5.14. Hence the proofs can be adapted to measures in $\mathrm{Meas}_{n,l}(\Gamma)$ and so we obtain the claimed bounds also for rational functions having pole order at most $n-2$ at ∞. ∎

The extention of bounded measures $\tilde{\mu} \in \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ to measures in $\mathrm{Meas}_{n,l}(\Gamma)$ is guaranteed by a theorem of Amice-Velu and Vishik in the form given in [Te2, Proposition 7]. We omit the proof here.

Theorem 5.17 *Let $\tilde{\mu} \in \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$. Then there is a unique measure $\mu \in \mathrm{Meas}_{n,l}(\Gamma)$ such that for all $f \in \mathcal{P}_n$ and $e \in \mathrm{E}(\mathcal{T})$ one has*

$$\int_{U(e)} f(x)\mathrm{d}\mu(x) = \int_{U(e)} f(x)\mathrm{d}\tilde{\mu}(x).$$

In the situation of Theorem 5.17 we say that μ is an extention of $\tilde{\mu}$ to $\mathcal{A}_n$. By the uniqueness asserted in the theorem and since by Proposition 5.16 every measure on $\mathcal{A}_n$ restricts to a well-defined measure on $\mathcal{P}_n$, we see that the spaces $\mathrm{Meas}_{n,l}(\Gamma)$ and $\widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ are isomorphic. Hence we obtain:

Corollary 5.18 *The isomorphism $\widetilde{\psi} : C^{\mathrm{har}}_{n,l}(\Gamma) \to \widetilde{\mathrm{Meas}}^{\mathrm{b}}_{n,l}(\Gamma)$ extends to an isomorphism $\psi : C^{\mathrm{har}}_{n,l}(\Gamma) \to \mathrm{Meas}_{n,l}(\Gamma)$.*

5.3 The Poisson Kernel

Lemma 5.19 *Let $z \in \Omega$, $x \in K_\infty$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\infty)$. Then*

$$\frac{(cx+d)^{n-2}}{\gamma z - \gamma x} = \frac{(cz+d)^n}{\det(\gamma)(z-x)} + P_{z,c,d}(x)$$

where $P_{z,c,d}(x) \in K_\infty[x,z]$ is a polynomial in x of degree $\deg_x(P_{z,c,d}(x)) \leq n-2$ depending on z, c and d.

Proof:

$$\frac{(cx+d)^{n-2}}{\gamma z - \gamma x} \overset{(15)}{=} \frac{(cx+d)^{n-1}(cz+d)}{\det(\gamma)(z-x)}.$$

To complete the proof substract

$$\frac{(cz+d)^n}{\det(\gamma)(z-x)}$$

and observe that the difference is equal to

$$\begin{aligned}
&\frac{(cz+d)}{\det(\gamma)(z-x)}((cx+d)^{n-1}-(cz+d)^{n-1})\\
=\;&\frac{(cz+d)}{\det(\gamma)(z-x)}\left(((cx+d)-(cz+d))\sum_{i=0}^{n-2}(cx+d)^i(cz+d)^{n-2-i}\right)\\
=\;&\frac{-c(cz+d)}{\det(\gamma)}\sum_{i=0}^{n-2}(cx+d)^i(cz+d)^{n-2-i}. \blacksquare
\end{aligned}$$

Following closely [Te1, Theorem 3] we proof the following theorem.

Theorem 5.20 *Let* $\mu \in \mathrm{Meas}_{n,l}(\Gamma)\otimes_{K_\infty}\mathbb{C}_\infty$. *Then*

$$f(z)=\int_{\mathbb{P}^1(K_\infty)}\frac{1}{z-x}\mathrm{d}\mu(x)\in\mathcal{M}_{n,l}(\Gamma).$$

PROOF: Since for any $z\in\Omega$ the function $x\mapsto\frac{1}{z-x}$ is in $\mathcal{A}_n$ for all $n\geq 2$ we can integrate $\frac{1}{z-x}$ against $\mathrm{d}\mu$. Hence we obtain a well-defined function $f:\Omega\to\mathbb{C}_\infty$ in this way. We have to show that f fulfills $f=f\underset{n,l}{|}\gamma$ for all $\gamma\in\Gamma$ and that f is rigid analytic.

For the first assertion let $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$. Then

$$\begin{aligned}
f(\gamma z)&=\int_{\mathbb{P}^1(K_\infty)}\frac{1}{\gamma z-x}\mathrm{d}\mu(x)\\
&=\int_{\mathbb{P}^1(K_\infty)}\frac{1}{\gamma z-\gamma x}\mathrm{d}\mu(\gamma x)\\
&\overset{5.15}{=}\det(\gamma)^{1-l}\int_{\mathbb{P}^1(K_\infty)}\frac{(cx+d)^{n-2}}{\gamma z-\gamma x}\mathrm{d}\mu(x)\\
&\overset{5.19}{=}\det(\gamma)^{-l}(cz+d)^n\left(\int_{\mathbb{P}^1(K_\infty)}\frac{1}{z-x}\mathrm{d}\mu(x)+\int_{\mathbb{P}^1(K_\infty)}\frac{\det(\gamma)P_{z,c,d}(x)}{(cz+d)^n}\mathrm{d}\mu(x)\right)\\
&\overset{5.14(a)}{=}\det(\gamma)^{-l}(cz+d)^n f(z)
\end{aligned}$$

and hence $f=f\underset{n,l}{|}\gamma$.

It remains to show that f is rigid analytic. For that let A be a connected affinoid domain in Ω. Then A is of the form

$$A=\mathbb{P}^1(\mathbb{C}_\infty)\smallsetminus\bigcup_{i=1}^m B_i$$

with finitely many open discs $B_i = B(a_i, r_i) = \{x \in \mathbb{C}_\infty \mid |x - a_i| < r_i\}$ with centers $a_i \in \mathbb{P}^1(K_\infty)$ and radii $r_i \in \mathbb{R}_{>0}$. We can assume $B_i \cap B_j = \varnothing$ for $i \neq j$, because if this is not the case, then either $B_i \subseteq B_j$ or $B_j \subseteq B_i$ and we can omit one of the discs. Let $U_i = B_i \cap \mathbb{P}^1(K_\infty)$. W.l.o.g. we choose $a_i = \infty$ for that disc B_i having $\infty \in B_i$. Then $U_i \subseteq \mathbb{P}^1(K_\infty)$ is compact open and $U_i \cap U_j = \varnothing$ for $i \neq j$. The collection $\{U_i \mid i = 1, \ldots, m\}$ is a covering of $\mathbb{P}^1(K_\infty)$ by distinct compact open subsets, since

$$\mathbb{P}^1(K_\infty) \subseteq (\mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus A) = \bigcup_{i=1}^{m} B_i.$$

Set

$$f_i(z) = \int_{U_i} \frac{1}{z - x} \mathrm{d}\mu(x).$$

Then by definition and since the U_i cover $\mathbb{P}^1(K_\infty)$ disjointly we have $f = \sum_{i=1}^m f_i$. Let e_i be the edge of $\mathcal{T}$ such that $U(e_i) = U_i$.
If $a_i \neq \infty$, then we have $\infty \notin U(e_i)$. We expand $\frac{1}{z-x}$ as a Taylor series around a_i in the form

$$\frac{1}{z - x} = \sum_{j=0}^{\infty} \frac{1}{(z - a_i)^{j+1}} (x - a_i)^j$$

converging for $z \in \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus B_i$ and $x \in U(e_i)$. Then by Proposition 5.7(b) one has

$$f_i(z) = \int_{U_i} \frac{1}{z - x} \mathrm{d}\mu(x) = \sum_{j=0}^{\infty} \frac{1}{(z - a_i)^{j+1}} \int_{U(e_i)} (x - a_i)^j \mathrm{d}\mu(x)$$

and by Proposition 5.16 this series converges uniformly on the complement of B_i. Hence f_i is rigid analytic on $\mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus B_i$.
If $a_i = \infty$, we have $0 \notin U(e_i)$. Expand $\frac{1}{z-x}$ as a Laurent series around ∞ as

$$\frac{1}{z - x} = \sum_{j=0}^{\infty} -z^j \frac{1}{x^{j+1}}$$

converging for $z \in \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus B_i$ and $x \in U(e_i)$. Then by Proposition 5.7(a) one has

$$f_i(z) = \int_{U_i} \frac{1}{z - x} \mathrm{d}\mu(x) = \sum_{j=-\infty}^{0} -z^{-j} x^{j+1}$$

and by Proposition 5.16 this series again converges uniformly on the complement of B_i. Again we see that f_i is rigid analytic on $\mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus B_i$.
Since $f = \sum_{i=1}^m f_i$, we see that f is rigid analytic on $A = \bigcap_{i=1}^m (\mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus B_i)$. Since Ω can be covered by connected affinoid domains, f is rigid analytic on all of Ω. ∎

Corollary 5.21 *The map*

$$\varphi : \mathrm{Meas}_{n,l}(\Gamma)\otimes\mathbb{C}_\infty \to \mathcal{M}_{n,l}(\Gamma),\ \mu\mapsto \left(f:\Omega\to\mathbb{C}_\infty,\ z\mapsto \int_{\mathbb{P}^1(K_\infty)} \frac{1}{z-x}\mathrm{d}\mu(x)\right)$$

is a well-defined homomorphism of $\mathbb{C}_\infty$-vector spaces.

The following diagram sums up the maps we constructed.

$$\begin{tikzcd}
\mathcal{M}_{n,l}(\Gamma) \arrow[rr, "\mathrm{Res}"] & & C^{\mathrm{har}}_{n,l}(\Gamma)\otimes_{K_\infty}\mathbb{C}_\infty \arrow[dl, "\psi\otimes\mathrm{id}"] \\
& \mathrm{Meas}_{n,l}(\Gamma)\otimes_{K_\infty}\mathbb{C}_\infty \arrow[ul, "\varphi"] &
\end{tikzcd}$$

Next, still following [Te1], we will observe that the residue map is a left inverse for $\varphi\circ\psi\otimes\mathrm{id}$. This will be essential for showing that the residue map is an isomorphism.

Proposition 5.22 *For $\kappa\in C^{\mathrm{har}}_{n,l}(\Gamma)$ we have*

$$\mathrm{Res}(\varphi(\psi\otimes\mathrm{id}(\kappa\otimes 1))) = \kappa\,.$$

Proof: Let $e\in\mathrm{E}(\mathcal{T})$ and $0\le i\le n-2$. We need to show that

$$\begin{aligned}
\kappa(e)(X^iY^{n-2-i}) &= \mathrm{Res}\left(\int_{\mathbb{P}^1(K_\infty)}\frac{1}{z-x}\mathrm{d}\mu_\kappa(x)\right)(e)(X^iY^{n-2-i}) \\
&= \mathrm{Res}_e\left(\int_{\mathbb{P}^1(K_\infty)}\frac{z^i}{z-x}\mathrm{d}\mu_\kappa(x)\right).
\end{aligned}$$

The residue on the right can be directly read of the Taylor series expansions in the proof of Theorem 5.20, in the particular case that A is the closure of the open annulus $V(e)$, distinguishing the case $\infty\notin U(e)$ or $\infty\in U(e)$. In both cases they equal to

$$\int_{U(e)} x^j\mathrm{d}\mu_\kappa(x) = \int_{U(e)} x^j\mathrm{d}\tilde{\mu}_\kappa(x) = \kappa(e)(X^iY^{n-2-i}). \ \blacksquare$$

We have an action of $\mathrm{GL}_2(K_\infty)$ both on $\mathcal{M}_{n,l}(\Gamma)$ and on $C^{\text{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$. The next lemma shows that the residue map is equivariant under this action.

Lemma 5.23 *Let $f : \Omega \to \mathbb{C}_\infty$ be a rigid analytic function and $\gamma \in \mathrm{GL}_2(K_\infty)$. Then*

$$\mathrm{Res}(f \underset{n,l}{|} \gamma)(e) = \gamma^{-1} \cdot_{n,l} \mathrm{Res}(f)(\gamma e).$$

Proof:

$$\begin{aligned}
& \gamma \cdot_{n,l} \mathrm{Res}(f \underset{n,l}{|} \gamma)(e)(X^i Y^{n-2-i}) \\
= \;& \det(\gamma)^{1-l} \mathrm{Res}(f \underset{n,l}{|} \gamma)(e)((aX+bY)^i (cX+dY)^{n-2-i}) \\
= \;& \det(\gamma)^{1-l} \mathrm{Res}_e(f \underset{n,l}{|} \gamma)(z)(az+b)^i (cz+d)^{n-2-i} \mathrm{d}z \\
= \;& \det(\gamma)^{1-l} \mathrm{Res}_e\, f(\gamma z) \det(\gamma)^l (cz+d)^{-n} (az+b)^i (cz+d)^{n-2-i} \mathrm{d}z \\
= \;& \mathrm{Res}_e\, f(\gamma z) \left(\frac{az+b}{cz+d}\right)^i \frac{\det(\gamma)}{(cz+d)^2} \mathrm{d}z \\
= \;& \mathrm{Res}_{\gamma e}\, f(z) z^i \mathrm{d}z \\
= \;& \mathrm{Res}(f)(\gamma e)(X^i Y^{n-2-i}) \;\blacksquare
\end{aligned}$$

Theorem 5.24 *For all $n \geq 2$ and for $l \in \mathbb{N}$ the residue map*

$$\mathrm{Res} : \mathcal{M}_{n,l}(\Gamma) \to C^{\text{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$$

is an isomorphism with inverse given by

$$\varphi \circ (\psi \otimes \mathrm{id}) : C^{\text{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty \to \mathcal{M}_{n,l}(\Gamma).$$

Proof: From Proposition 5.22 it follows that the residue map is surjective for all $\Gamma \subseteq \Lambda^\star$. If in addition Γ is p'-torsion free, then by Corollary 4.9 we have equal dimensions on both sides, and hence the residue map is an isomorphism with inverse given by $\varphi \circ (\psi \otimes \mathrm{id})$.
Now if Γ is not p'-torsion free, we can choose a finite index subgroup $\Gamma' \subseteq \Gamma$ which is p'-torsion free. E.g. choose any $\mathfrak{p} \in \mathbb{F}_q[T]$ with $\mathfrak{n} \notin R$ and such that under the embedding $\iota : \Gamma \hookrightarrow \mathrm{GL}_2(K_p)$ one has $\iota(\Gamma) \subseteq \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$. Then we let Γ' be the full level $\mathfrak{p}$ congruence subgroup in Γ, so the inverse image under ι of the kernel of the reduction modulo $\mathfrak{p}$ from $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ to $\mathrm{GL}_2(\mathbb{F}_\mathfrak{p})$. So

$$\Gamma' = \{\gamma \in \Gamma \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{p} M_2(\mathcal{O}_\mathfrak{p})}\}.$$

This is a p'-torsion free finite index subgroup of Γ. Then by what we have just shown $\mathcal{M}_{n,l}(\Gamma') \cong C^{\mathrm{har}}_{n,l}(\Gamma') \otimes_{K_\infty} \mathbb{C}_\infty$. Furthermore we have

$$\mathcal{M}_{n,l}(\Gamma) = \mathcal{M}_{n,l}(\Gamma')^{\Gamma/\Gamma'} = \{f \in \mathcal{M}_{n,l}(\Gamma') \mid f = f \underset{n,l}{|} \gamma \text{ for all } \gamma \in \Gamma/\Gamma'\}$$

and

$$C^{\mathrm{har}}_{n,l}(\Gamma) = C^{\mathrm{har}}_{n,l}(\Gamma')^{\Gamma/\Gamma'} = \{\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma') \mid \kappa(\gamma e) = \gamma \cdot \kappa(e) \text{ for all } \gamma \in \Gamma/\Gamma', e \in \mathrm{E}(\mathcal{T})\}.$$

Because $\mathrm{Res} : \mathcal{M}_{n,l}(\Gamma') \to C^{\mathrm{har}}_{n,l}(\Gamma') \otimes_{K_\infty} \mathbb{C}_\infty$ is Γ-equivariant by Lemma 5.23, it follows that it defines an isomorphism $\mathcal{M}_{n,l} \to C^{\mathrm{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$ with inverse given by $\varphi \circ (\psi \otimes \mathrm{id})$. ∎

5.4 Hecke operators

In this section we introduce an action of Hecke operators on $\mathcal{M}_{n,l}(\Gamma)$. This will be done in the usual way as a sum over certain double coset decompositions. We then translate this Hecke action to the side of harmonic cocycles, where one can explicitly compute these operators.

5.4.1 Hecke operators on $\mathcal{M}_{n,l}(\Gamma)$

Definition 5.25 *Let $\alpha \in \mathrm{GL}_2(K_\infty)$ and $f \in \mathcal{M}_{n,l}(\Gamma)$. Write the double coset $\Gamma\alpha\Gamma$ as a finite disjoint union $\cup_j \Gamma\beta_j$. Then we define the double coset operator as*

$$f[\Gamma\alpha\Gamma]_{n,l} = \sum_j f \underset{n,l}{|} \beta_j.$$

This operator does not depend on the choice of the decomposition of $\Gamma\alpha\Gamma$. For if $\cup\Gamma\beta'_j = \Gamma\alpha\Gamma = \cup\Gamma\beta_j$ are two distinct decompositions and suppose that $\Gamma\beta_j = \Gamma\beta'_j$, then there is an $\gamma \in \Gamma$ with $\beta_j = \gamma\beta'_j$ and hence

$$f \underset{n,l}{|} \beta_j = f \underset{n,l}{|} \gamma\beta'_j = f \underset{n,l}{|} \gamma \underset{n,l}{|} \beta'_j = f \underset{n,l}{|} \beta'_j.$$

Lemma 5.26 *For $f \in \mathcal{M}_{n,l}(\Gamma)$ and $\alpha \in \mathrm{GL}_2(K_\infty)$ the double coset operator $f \mapsto f[\Gamma\alpha\Gamma]_{n,l}$ induces an endomorphism of $\mathcal{M}_{n,l}(\Gamma)$.*

Proof: Let $\gamma \in \Gamma$ and $\Gamma\alpha\Gamma = \cup_j \Gamma\beta_j$. Then the set $\cup_j \Gamma\beta_j\gamma$ is another decomposition of $\Gamma\alpha\Gamma$ and hence by the above

$$f[\Gamma\alpha\Gamma]_{n,l} \underset{n,l}{|} \gamma = \sum_j f \underset{n,l}{|} \beta_j\gamma = f[\Gamma\alpha\Gamma]_{n,l}.$$

The linearity is clear from the definition of $[\Gamma\alpha\Gamma]_{n,l}$. ∎

Let $\mathcal{H}_{n,l}(\Gamma)$ be the subalgebra of $\mathrm{End}(\mathcal{M}_{n,l}(\Gamma))$ generated by all double coset operators for all $\alpha \in \mathrm{GL}_2(K_\infty)$. The Hecke operators we will consider in this section form a commutative subalgebra of $\mathcal{H}_{n,l}(\Gamma)$. To define them we need some further notation. Let $\mathfrak{n} \trianglelefteq A$ be an ideal with $(\mathfrak{n}, \mathfrak{r}) = 1$ and let $\Lambda(\mathfrak{n}) \subseteq \Lambda$ be an full level $\mathfrak{n}$ order in Λ. Since $K = \mathbb{F}_q(T)$ has class number 1, by [Vi, Corollaire 5.7] two different Eichler orders of level $\mathfrak{n}$ differ only by elements of $D^\star$. Suppose that $\Lambda(\mathfrak{n})^\star \subseteq \Gamma \subseteq \Lambda^\star$.
Let $\mathfrak{p} \trianglelefteq A$ be a prime ideal with $\mathfrak{p} \nmid \mathfrak{n}\mathfrak{r}$. By strong approximation [Vi, Theorème III.4.3], there is an element $\gamma_\mathfrak{p} \in \Lambda(\mathfrak{n})$ such that $\mathrm{nrd}(\gamma_\mathfrak{p})$ generates $\mathfrak{p}$. Via the embedding $\Lambda \hookrightarrow M_2(K_\infty)$ we view $\gamma_\mathfrak{p}$ as an element of $\mathrm{GL}_2(K_\infty)$.

Lemma–Definition 5.27 *The double coset operator $[\Gamma\gamma_\mathfrak{p}\Gamma]_{n,l}$ does not depend on the choice of $\gamma_\mathfrak{p} \in \Lambda(\mathfrak{n})$. We define $T_\mathfrak{p}$ to be the operator $[\Gamma\gamma_\mathfrak{p}\Gamma]_{n,l}$ for any $\gamma_\mathfrak{p} \in \Lambda(\mathfrak{n})$ with $(\mathrm{nrd}(\gamma_\mathfrak{p})) = \mathfrak{p}$.*

Proof: Let $\iota : D \hookrightarrow D \otimes_K K_\mathfrak{p} \cong M_2(K_\mathfrak{p})$ be the natural embedding. Since $\mathfrak{p}$ does not divide $\mathfrak{n}$, the completion of $\Lambda(\mathfrak{n})$ at $\mathfrak{p}$ is isomorphic to $M_2(\mathcal{O}_\mathfrak{p})$. If $\gamma_\mathfrak{p}$ and $\gamma'_\mathfrak{p}$ are in $\Lambda(\mathfrak{n})$ with $(\mathrm{nrd}(\gamma_\mathfrak{p})) = (\mathrm{nrd}(\gamma'_\mathfrak{p})) = \mathfrak{p}$, then $(\det(\iota(\gamma_\mathfrak{p}))) = (\det(\iota(\gamma_\mathfrak{p}))) = \mathfrak{p}$. So $\iota(\gamma'_\mathfrak{p}\gamma_\mathfrak{p}^{-1}) \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ and since $\gamma'_\mathfrak{p}\gamma_\mathfrak{p}^{-1}$ is in D, we have $\gamma'_\mathfrak{p}\gamma_\mathfrak{p}^{-1} \in D \cap \iota^{-1}(\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})) = \Lambda(\mathfrak{n})^\star$. That means there is a $\gamma \in \Lambda(\mathfrak{n})^\star \subseteq \Gamma$ with $\gamma_\mathfrak{p} = \gamma\gamma'_\mathfrak{p}$. This implies $[\Gamma\gamma_\mathfrak{p}\Gamma] = [\Gamma\gamma'_\mathfrak{p}\Gamma]$. ∎

As in [GV, page 6], the double coset space $\Gamma\gamma_\mathfrak{p}\Gamma$ decomposes into $\#\mathbb{P}^1(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) = \#\mathbb{P}^1(k_\mathfrak{p})$ many left Γ-cosets. By the above it follows that there are elements $\{\gamma_a \in \Gamma \mid a \in \mathbb{P}^1(k_\mathfrak{p})\}$ with

$$T_\mathfrak{p}(f) = \sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} f \underset{n,l}{|} \gamma_\mathfrak{p}\gamma_a \tag{18}$$

for $f \in \mathcal{M}_{n,l}(\Gamma)$.
The Hecke-algebra of $\mathcal{M}_{n,l}(\Gamma)$, denote by $\mathcal{H}'_{n,l}(\Gamma)$, is the subalgebra of $\mathcal{H}_{n,l}(\Gamma)$ generated by the operators $T_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in A$ coprime to $\mathfrak{n}\mathfrak{r}$.

Remark 5.28 Let $\mathfrak{p}, \mathfrak{p}' \trianglelefteq A$ be two different prime ideals coprime to $\mathfrak{n}\mathfrak{r}$. As in [Bö, Proposition 6.6] one shows that $T_\mathfrak{p}T_{\mathfrak{p}'} = T_{\mathfrak{p}'}T_\mathfrak{p}$. Hence $\mathcal{H}'_{n,l}(\Gamma)$ is a commutative subalgebra of $\mathcal{H}_{n,l}(\Gamma)$.

5.4.2 Hecke operators on $C^{\mathrm{har}}_{n,l}(\Gamma)$

We can formally translate the action of the Hecke algebra to $C^{\mathrm{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$ using the residue map and integration against the Poisson kernel.

Definition 5.29 *Let $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$ and $\mathfrak{p} \trianglelefteq A$ a prime ideal coprime to $\mathfrak{n}\mathfrak{r}$. We define*

$$T_\mathfrak{p}(\kappa) = \mathrm{Res}(T_\mathfrak{p}(\varphi(\psi \otimes \mathrm{id}(\kappa)))).$$

Remark 5.30 Since Res and $\varphi \circ (\psi \otimes \mathrm{id})$ are mutual inverses, for the above definition of a Hecke action on $C_{n,l}^{\mathrm{har}}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$, the map Res becomes an isomorphism of Hecke modules.

This means that in order to compute the action of Hecke operators on quaternionic modular forms, one can as well compute on the side of harmonic cocycles. To do this, we need to make the action of Hecke operators on harmonic cocycles more explicit.

Proposition 5.31 *Let $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)$, $\mathfrak{p} \trianglelefteq A$ a prime ideal coprime to $\mathfrak{nr}$ and $\{\gamma_a\} \subset \Gamma$ for $a \in \mathbb{P}^1(k_\mathfrak{p})$ the elements from Equation (18). Then*

$$T_\mathfrak{p}(\kappa)(e) = \sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} \gamma_a^{-1} \gamma_\mathfrak{p}^{-1} \kappa(\gamma_\mathfrak{p} \gamma_a e). \tag{19}$$

Proof: For $e \in \mathrm{E}(\mathcal{T})$ one computes

$$\begin{aligned}
T_\mathfrak{p}(\kappa)(e) &= \mathrm{Res}(T_\mathfrak{p}(\varphi(\psi(\kappa))))(e) \\
&= \mathrm{Res}\left(\sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} (\varphi(\psi(\kappa)) \underset{n,l}{|} \gamma_\mathfrak{p} \gamma_a) \right)(e) \\
&= \sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} \mathrm{Res}\left((\varphi(\psi(\kappa)) \underset{n,l}{|} \gamma_\mathfrak{p} \gamma_a) \right)(e) \\
&\overset{5.23}{=} \sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} \gamma_a^{-1} \gamma_\mathfrak{p}^{-1} \mathrm{Res}(\varphi(\psi(\kappa)))(\gamma_\mathfrak{p} \gamma_a e) \\
&\overset{5.24}{=} \sum_{a \in \mathbb{P}^1(k_\mathfrak{p})} \gamma_a^{-1} \gamma_\mathfrak{p}^{-1} \kappa(\gamma_\mathfrak{p} \gamma_a e). \quad \blacksquare
\end{aligned}$$

5.4.3 Explicit embeddings at $\mathfrak{p}$

Our next goal is, to describe how one can compute the elements $\gamma_\mathfrak{p} \gamma_a$ from the double coset decomposition of $\Gamma \gamma_\mathfrak{p} \Gamma$. For this, we need an explicit describtion of the embedding $D \hookrightarrow M_2(K_\mathfrak{p})$ for places $\mathfrak{p}$ where D splits.
Let $(\varpi) = \mathfrak{p} \notin R$ be a place of K. Then D splits at $\mathfrak{p}$, i.e. $D \otimes_K K_\mathfrak{p} \cong M_2(K_\mathfrak{p})$ where $K_\mathfrak{p}$ is the completion of K at the place $\mathfrak{p}$. Let $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $K_\mathfrak{p}$ and $k_\mathfrak{p}$ its residue field. In this section we shortly explain how to obtain an explicit embedding $\varphi_\mathfrak{p} : D \hookrightarrow M_2(K_\mathfrak{p})$ for such a place $\mathfrak{p}$. By explicit we mean that we give rules to compute approximations for the values of $\varphi(i)$ and $\varphi(j)$. The results in this section are well known. More details on this question can be found in [Vo2].

Definition 5.32 *The square symbol is defined as*

$$\left\{\frac{\alpha}{\mathfrak{p}}\right\} = \begin{cases} 1 & \alpha \in (K_{\mathfrak{p}}^{\star})^2, \\ -1 & \alpha \notin (K_{\mathfrak{p}}^{\star})^2 \text{ and } \mathrm{ord}_{\mathfrak{p}}(\alpha) \text{ even }, \\ 0 & \alpha \notin (K_{\mathfrak{p}}^{\star})^2 \text{ and } \mathrm{ord}_{\mathfrak{p}}(\alpha) \text{ odd }. \end{cases}$$

The square symbol can be computed effectivly by reducing to the Legendre symbol from Definition 2.18. See [Vo2, Section 5] for more details.

Lemma 5.33 *Suppose $D = \left(\frac{a,b}{K}\right)$. Then $\mathfrak{p} \notin R$ if and only if one of the following holds:*

$$\left\{\frac{a}{\mathfrak{p}}\right\} = 1 \text{ or } \left\{\frac{b}{\mathfrak{p}}\right\} = 1 \text{ or } \left\{\frac{-ab}{\mathfrak{p}}\right\} = 1 \text{ or } \left\{\frac{a}{\mathfrak{p}}\right\} = \left\{\frac{b}{\mathfrak{p}}\right\} = \left\{\frac{-ab}{\mathfrak{p}}\right\} = -1.$$

PROOF: See [Vo2, Proposition 5.5]. ∎

If $\mathfrak{p} \notin R$, then according to Proposition 2.13 the equation $Z^2 - aX^2 - bY^2 = 0$ has a non-trivial solution over $K_{\mathfrak{p}}$.

Lemma 5.34 *If $Z^2 - aX^2 - bY^2$ has a non-trivial solution (z_0, x_0, y_0) over $K_{\mathfrak{p}}$ with either z_0 or x_0 or y_0 equal to 0, than either $\left\{\frac{-ab}{\mathfrak{p}}\right\} = 1$ or $\left\{\frac{b}{\mathfrak{p}}\right\} = 1$ or $\left\{\frac{a}{\mathfrak{p}}\right\} = 1$ respectivly.*

PROOF: Suppose first that $z_0 = 0$, hence $-ax_0^2 - by_0^2 = 0$. Multiplying with b yields $-abx_0^2 - b^2y_0^2 = 0$ hence $-ab = \frac{y_0^2b^2}{x^2} = \sqrt{\frac{y_0b}{x}}$ is a square.
If $x_0 = 0$ we have $z_0^2 - by_0^2 = 0$, hence $b = \sqrt{\frac{z_0}{y_0}}$ is a square. Likewise if $y_0 = 0$ then $a = \sqrt{\frac{z_0}{x_0}}$ is a square. ∎

Lemma 5.35 *(a) If $\left\{\frac{a}{\mathfrak{p}}\right\} = 1$, then*

$$i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix} \text{ and } j \mapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$$

provides an embedding of $D \otimes K_{\mathfrak{p}} \hookrightarrow M_2(K_{\mathfrak{p}})$.

(b) If $\left\{\frac{b}{\mathfrak{p}}\right\} = 1$*, then*

$$i \mapsto \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \text{ and } j \mapsto \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}$$

provides an embedding of $D \otimes K_{\mathfrak{p}} \hookrightarrow M_2(K_{\mathfrak{p}})$*.*

(c) If $\left\{\frac{-ab}{\mathfrak{p}}\right\} = 1$*, then*

$$i \mapsto \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \text{ and } j \mapsto \begin{pmatrix} 0 & -\frac{\sqrt{-ab}}{a} \\ \sqrt{-ab} & 0 \end{pmatrix}$$

provides an embedding of $D \otimes K_{\mathfrak{p}} \hookrightarrow M_2(K_{\mathfrak{p}})$*.*

(d) If $\left\{\frac{a}{\mathfrak{p}}\right\} = \left\{\frac{b}{\mathfrak{p}}\right\} = \left\{\frac{-ab}{\mathfrak{p}}\right\} = -1$*, let* (z_0, x_0, y_0) *be a solution of* $Z^2 - aX^2 - bY^2$ *in* $K_{\mathfrak{p}}$ *with* $z_0, x_0, y_0 \neq 0$*. Let* $x := \frac{x_0}{z_0}$ *and* $y := \frac{y_0}{z_0}$*. Set* $e_0 := 1, e_1 := xi + yj, e_2 := k, e_3 := e_1e_2$*. Then* $e_0, \dots e_3$ *is a basis of* $D \otimes K_{\mathfrak{p}}$ *and*

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } e_2 \mapsto \begin{pmatrix} 0 & 1 \\ -ab & 0 \end{pmatrix}$$

provides an embedding of $D \otimes K_{\mathfrak{p}} \hookrightarrow M_2(K_{\mathfrak{p}})$*.*

PROOF: One has to check that the given matrices fulfill the relations from Definition 2.12. If all three symbols are -1, one first checks that $e_0, \dots, e_3$ form another basis of D with the relations $e_1^2 = 1, e_2^2 = -ab$ and $e_3 = e_1e_2 = -e_2e_1$. Note also that $ax^2 + by^2 = 1$. Hence $D \otimes K_{\mathfrak{p}} \cong \left(\frac{a,b}{K_{\mathfrak{p}}}\right) \cong \left(\frac{1,-ab}{K_{\mathfrak{p}}}\right)$. ∎

To obtain an explicit embedding, we first compute the symbols $\left\{\frac{a}{\mathfrak{p}}\right\}$, $\left\{\frac{b}{\mathfrak{p}}\right\}$ and $\left\{\frac{-ab}{\mathfrak{p}}\right\}$. If one of these symbols equals 1, we use Newton iteration to compute the square root in $K_{\mathfrak{p}}$ as a Laurent series in the variable $\mathfrak{p}$. For this, if $x \in K_{\mathfrak{p}}$ is a square root and $e = \mathrm{ord}_{\mathfrak{p}}(x)$, let $x_0 := x\varpi^{-e}$ and let y_0 be a solution of $y_0^2 \equiv x_0 \mod \mathfrak{p}$. Then we start with $y_0\varpi^{-e/2}$ as a first approximation.

If all three symbols are -1, we have to search for a non-trivial solution of $Z^2 - aX^2 - bY^2$ over the residue field $k_{\mathfrak{p}}$. By Lemma 5.34 we know that such a solution exists. To find it, we first compute a solution mod $\mathfrak{p}$. Then this solution can be lifted using Hensels Lemma. More explicitly, set $x_0 := \tilde{x}_0, y_0 := \tilde{y}_0$ and $z_0 := \tilde{z}_0u$ with $u = \sqrt{1 + (\frac{1}{\tilde{z}_0^2}(ax_0^2 + by_0^2) - 1)}$. The square-root exists in $K_{\mathfrak{p}}$ since $\sqrt{1+x}$ has a convergent Taylor series expansion for $\mathrm{char}(K_\mathfrak{p}) \neq 2$ and $v_\mathfrak{p}(x) \geq 1$. Note that u is a 1-unit in $K_\mathfrak{p}$. The images of i and j are then obtained from

$$\begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & xa & 0 & -y \\ 0 & yb & 0 & x \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Remark 5.36 Note that in all cases the embedding $\varphi_\mathfrak{p} : D \hookrightarrow M_2(K_\mathfrak{p})$ obtained from Lemma 5.35 has the property $\varphi_\mathfrak{p}(< 1, i, j, ij >_A) \subset M_2(\mathcal{O}_\mathfrak{p})$. Hence for $\mathrm{odd}(R) = 1$ this embedding sends Λ into $M_2(O_\mathfrak{p})$. Moreover, $\Lambda_\mathfrak{p} \cong M_2(\mathcal{O}_\mathfrak{p})$ where $\Lambda_\mathfrak{p}$ denotes the completion of Λ at $\mathfrak{p}$. If $\mathrm{odd}(R) = 0$ we can twist $\varphi_\mathfrak{p}$ with a suitable element γ of $\mathrm{GL}_2(K_\mathfrak{p})$ to obtain a concrete realization of the isomorphism $\Lambda_\mathfrak{p} \cong M_2(\mathcal{O}_\mathfrak{p})$ as well. For this, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the image of $\frac{\varepsilon i + ij}{\alpha}$ under $\varphi_\mathfrak{p}$, we can take $\gamma = \mathrm{diag}(\mathfrak{p}^{-\min(\mathrm{v}_\mathfrak{p}(a), \mathrm{v}_\mathfrak{p}(b), \mathrm{v}_\mathfrak{p}(c), \mathrm{v}_\mathfrak{p}(d))})$.

5.4.4 Computing the double coset decomposition

To evalute Equation (19) on cocycles, one needs to compute the elements $\gamma_\mathfrak{p}\gamma_a$ from the double coset decomposition of $\Gamma\gamma_\mathfrak{p}\Gamma$. As in [GV, Chapter 5] one has a bijection

$$\mathbb{P}^1(k_\mathfrak{p}) \to \{\gamma_\mathfrak{p}\gamma_a \mid a \in \mathbb{P}^1(k_\mathfrak{p})\}$$

given by sending the point $a = (x : y) \in \mathbb{P}^1(k_\mathfrak{p})$ to a generator λ_a of the left ideal

$$I_a := \Lambda(\mathfrak{n})\varphi_\mathfrak{p}^{-1}\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) + \Lambda(\mathfrak{n})\varpi$$

where ϖ is a generator of $\mathfrak{p}$. These generators then necessarily have $(\mathrm{nrd}(\lambda_a)) = \mathfrak{p}$. Using the explicit embedding $\varphi_\mathfrak{p}$ from Section 5.4.3 one finds generators of I_a as an A-module. As in [KV, Lemma 4.9] one shows that any element $\gamma \in I_a$ with $(\mathrm{nrd}(\gamma)) = \mathfrak{p}$ will generate I_a. One way to obtain such an element is to enumerate elements of I_a until we find one having the correct reduced norm.

Remark 5.37 The approach of enumerating elements of I_a works in practice for primes $\mathfrak{p}$ of small degree. For primes $\mathfrak{p}$ of larger degree, one could use a form of the LLL-algorithm. However, the LLL-algorithm presented in [He] or [Pau] computes a basis of I_a with minimal polynomial degree, where instead we need to minimize the reduced norm of the generators. It is unclear, how these two norms are related. In practice, minimizing the polynomial degree seems to work in most cases. We intend to investigate this further in the future.

6 A basis of the space of harmonic cocycles

In this chapter we present a construction of a K_∞-basis of the space $C_{n,l}^{\text{har}}(\Gamma)$, on which one can compute the Hecke action. The basis will be explicit for computations and we will construct it out of an enhanced fundamental domain for $\mathcal{T}$ under Γ. These fundamental domains are computed using Algorithm 2.32. As in Section 4.4 we need to assume that $\Gamma \subseteq \Lambda^\star$ has the property

$$\mathrm{Stab}_\Gamma(v) \cong \{1\} \text{ or } \mathbb{F}_q^\star \text{ or } \mathbb{F}_{q^2}^\star \text{ for all } v \in \mathrm{V}(\mathcal{T})$$

and hence

$$\mathrm{Stab}_\Gamma(e) \cong \{1\} \text{ or } \mathbb{F}_q^\star \text{ for all } e \in \mathrm{E}(\mathcal{T}).$$

This assumption was made to ensure that only vertices of degree $q+1$ or degree 1 occur in the quotient graph $\Gamma\backslash\mathcal{T}$. Recall that our assumption on Γ implies $\omega = \#(\Gamma \cap \mathbb{F}_q^\star) \in \{1, q-1\}$.

6.1 Preliminaries

Recall that an enhanced fundamental domain for the action of Γ on $\mathcal{T}$ is a triple $((\mathcal{Y}, \mathcal{S}), \mathrm{PE} = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{\mathrm{Stab}_\Gamma(t) \mid t \text{ a simplex of } \mathcal{Y}\})$ where $\mathcal{S} \subset \mathcal{Y} \subset \mathcal{T}$ are subtrees such that

(a) Under the projection $\pi : \mathcal{T} \to \Gamma\backslash\mathcal{T}$ the image $\pi(\mathcal{S})$ is a maximal subtree of $\Gamma\backslash\mathcal{T}$,

(b) $\mathrm{E}(\mathcal{Y}) \cong \mathrm{E}(\Gamma\backslash\mathcal{T})$ and

(c) any edge of $\mathcal{Y}$ has at least one of its vertices in $\mathcal{S}$.

The edge pairing PE is a map from $\{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$ to Γ such that $\mathrm{PE}(e)t(e) \in \mathrm{V}(\mathcal{S})$ for all e. In Figure 3 we give an example of an enhanced fundamental domain for $D/\mathbb{F}_5(T)$ the quaternion algebra ramified at $\{T, T+1, T+2, T+3\}$ and $\Gamma = \Lambda^\star$ for Λ a maximal order in D. This continues Example 2.38. In the figure the subtree $\mathcal{S} \subset \mathcal{Y}$ consists of those edges of $\mathcal{Y}$, which do not carry a name and are not labeled with a matrix of Γ.

Remark 6.1 If we have two different enhanced fundamental domains for the action of Γ on $\mathcal{T}$ given by $((\mathcal{Y}_i, \mathcal{S}_i), \mathrm{PE}_{(\mathcal{Y}_i,\mathcal{S}_i)}, \{\mathrm{Stab}_\Gamma(t) \mid t \text{ a simplex of } \mathcal{Y}_i\})$ for $i \in \{1, 2\}$, then there is a bijection $\varphi : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that all simplices t of $\mathcal{Y}_1$ are Γ-equivalent to its image $\varphi(t)$. Note that φ is not necessarily an isomorphism in the category of graphs. The stabilizers of the simplices of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are then Γ-conjugates. For $e \in \mathrm{E}(\mathcal{Y}_1 \smallsetminus \mathcal{S}_1)$ with $o(e) \in \mathcal{S}_1$ the value of the edge pairing $\mathrm{PE}_{(\mathcal{Y}_1,\mathcal{S}_1)}(e)$ is unique up to elements of the stabilizer of the edges, hence up to elements of $\mathbb{F}_q^\star \cap \Gamma$. So the values of $\mathrm{PE}_{(\mathcal{Y}_1,\mathcal{S}_1)}$ and $\mathrm{PE}_{(\mathcal{Y}_2,\mathcal{S}_2)}$ differ by elements of $\mathbb{F}_q^\star \cap \Gamma$.

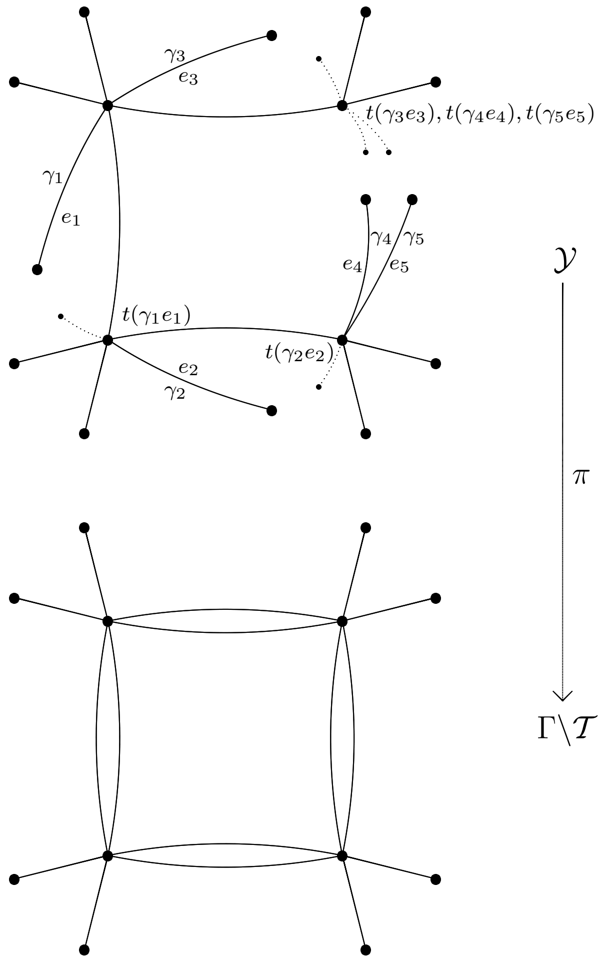

Figure 3: An example of an enhanced fundamental domain

Definition 6.2 *(a) Let* $\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{reg}} = \{e \in \mathrm{E}(\Gamma\backslash\mathcal{T}) \mid \deg(o(e)) = \deg(t(e)) = q+1\}$ *and* $\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}} = \mathrm{E}(\Gamma\backslash\mathcal{T}) \smallsetminus \mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{reg}}$.

(b) Let $C^{\mathrm{har}}_{n,l}(\Gamma)_{\mathrm{reg}} = \{\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma) \mid \kappa(e) = 0$ *for all* $e \in \pi^{-1}(\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}})\}$.

Note that in a lot of cases, for example if Γ is p'-torsion free or if $\mathrm{odd}(R) = 0$, then $\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}} = \varnothing$ and hence $C^{\mathrm{har}}_{n,l}(\Gamma) = C^{\mathrm{har}}_{n,l}(\Gamma)_{\mathrm{reg}}$. We start by observing that the values of Γ-equivariant cocycles have to be invariant under the action of $\mathbb{F}_q^\star \cap \Gamma$.

Lemma 6.3 *Let* $\kappa \in C^{\mathrm{har}}_{n,l}(\Gamma)$. *Then* $\kappa(e) \in V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$ *for all* $e \in \mathrm{E}(\mathcal{T})$.

PROOF: Let $e \in \mathrm{E}(\mathcal{T})$. Then by Proposition 2.24 one has $\mathrm{Stab}_\Gamma(e) = \mathbb{F}_q^\star \cap \Gamma$. So for $\gamma \in \mathbb{F}_q^\star \cap \Gamma$ and $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)$ one has $\gamma\,\kappa(e) = \kappa(\gamma e) = \kappa(e)$, hence the value of κ at e has to be invariant under the action of $\mathbb{F}_q^\star \cap \Gamma$. ∎

Lemma 6.4 *For all $n \geq 2$ and all $l \in \mathbb{Z}$ one has*

$$V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma} = \begin{cases} \{0\} & \text{if } n \not\equiv 2l \pmod{\omega}, \\ V_{n,l}(K_\infty) & \text{if } n \equiv 2l \pmod{\omega}. \end{cases}$$

where $\omega = \#(\mathbb{F}_q^\star \cap \Gamma)$.

PROOF: The group $\mathbb{F}_q^\star \cap \Gamma$ is cyclic, let $\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ be a generator. Then $\mathrm{ord}(\gamma) = \omega$ and

$$V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma} = V_{n,l}(K_\infty)^{\langle\gamma\rangle}$$

$$\begin{aligned}
&= \{\varphi \in V_{n,l}(K_\infty) \mid \gamma \cdot_{n,l} \varphi = \varphi\} \\
&= \{\varphi \in V_{n,l}(K_\infty) \mid \gamma \cdot_{n,l} \varphi(X^iY^{n-2-i}) = \varphi(X^iY^{n-2-i})\ \forall\ 0 \leq i \leq n-2\} \\
&= \{\varphi \in V_{n,l}(K_\infty) \mid \det(\gamma)^{1-l}\varphi((aX)^i(aY)^{n-2-i}) = \varphi(X^iY^{n-2-i})\ \forall\ 0 \leq i \leq n-2\} \\
&= \{\varphi \in V_{n,l}(K_\infty) \mid a^{n-2l}\varphi(X^iY^{n-2-i}) = \varphi(X^iY^{n-2-i})\ \forall\ 0 \leq i \leq n-2\} \\
&= \{\varphi \in V_{n,l}(K_\infty) \mid a^{n-2l}\varphi = \varphi\}.
\end{aligned}$$

If $n \not\equiv 2l \pmod{\omega}$, this set equals $\{0\}$. If $n \equiv 2l \pmod{\omega}$, we have no condition on φ. ∎

With regard to Theorem 5.24 the previous lemma can be viewed as the cocycle side version of Proposition 4.13. The assertion of Proposition 4.13 also holds for odd n. This will be shown in Corollary 6.12.

6.2 The case of weight $n > 2$

Let us assume $n > 2$. Our first goal is, to give an explicit description of $C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$ in this case. For this, we need an improved version of Lemma 4.8, where we showed that $V_{n,l}(\mathbb{C}_\infty)_\Gamma = \{0\}$. Let us start with a simple observation.

Lemma 6.5 *Let $\gamma \in \mathrm{GL}_m(V)$ for some m. Then $V^{\langle\gamma\rangle} = \{0\}$ if and only if $V_{\langle\gamma\rangle} = \{0\}$.*

PROOF: If $V^{\langle\gamma\rangle} \neq \{0\}$ then 1 is an eigenvalue for γ. Then 1 is also an eigenvalue of the transpose of γ, and hence also $V_{\langle\gamma\rangle} = (V^\star)^{\langle\gamma\rangle} \neq \{0\}$. ∎

In the proof of Lemma 4.8 we worked only with two non-commuting matrices $\gamma_1, \gamma_2 \in \Gamma$ of infinite order. The matrices used in the edge pairing of an enhanced fundamental domain have this property. Also, the proof was purely algebraic and works with $\mathbb{C}_\infty$ replaced by K_∞. Hence combined with Lemma 6.5, we obtain the following statement.

Lemma 6.6 *Let $((\mathcal{S},\mathcal{Y}), \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{G_t \mid t$ a simplex of $\mathcal{Y}\})$ be an enhanced fundamental domain for $\mathcal{T}$ under Γ. Let $g = g(\Gamma\backslash\Omega)$,*

$$\{e_i \mid i = 1, \dots, g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$.

(a) If n is odd, then $V_{n,l}(K_\infty)_{\langle\gamma_i\rangle}^{\mathbb{F}_q^\star\cap\Gamma} = \{0\} = V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma\times\langle\gamma_i\rangle}$ for all $1 \le i \le g$.

(b) If n is even, then $\dim V_{n,l}(K_\infty)_{\langle\gamma_i\rangle}^{\mathbb{F}_q^\star\cap\Gamma} = \dim V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma\times\langle\gamma_i\rangle} = 1$ for all $1 \le i \le g$.

Lemma 6.6 suggests that the construction of an explicit basis for $C_{n,l}^{\mathrm{har}}(\Gamma)$ will differ depending on n even or odd. We start with the easier case n odd.

Lemma 6.7 *Let $((\mathcal{S},\mathcal{Y}), \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{G_t \mid t$ a simplex of $\mathcal{Y}\})$ be an enhanced fundamental domain for $\mathcal{T}$ under Γ. Let $g = g(\Gamma\backslash\Omega)$,*

$$\{e_i \mid i = 1, \dots, g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$. Let $n > 2$ be odd. To each $v \in V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma}$ and each $e_i \in \{e_2, \dots, e_g\}$ there is a unique cocycle $\kappa_{(v,e_i)} \in C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$ with $\kappa_{(v,e_i)}(e_i) = v$ and $\kappa_{(v,e_i)}(e') = 0$ for all $e' \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S} \cup \{e_1, e_1^\star, e_i, e_i^\star\})$.

Proof: Note that a maximal subtree of $\Gamma\backslash\mathcal{T}$ necessarily contains $\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}}$, hence $e_i \in \pi^{-1}(\mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{reg}})$. We have $g(\Gamma\backslash\Omega) > 1$ by Theorem 2.27. By Lemma 6.6 we have $V_{n,l}(K_\infty)_{\langle\gamma_1\rangle}^{\mathbb{F}_q^\star\cap\Gamma} = \{0\}$. For $1 \le j \le g$ let $\mathcal{P}_j \subseteq \mathcal{S}$ be the unique path connecting $o(e_j)$ and $\gamma t(e_j)$ and $\mathcal{P}'_j \subseteq \mathcal{S}$ be the unique path connecting $o(e_1)$ and $o(e_j)$. For $\kappa_{(v,e_i)}$ to satisfy $\kappa_{(v,e_i)}(e') = 0$ for all $e' \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S} \cup \{e_1, e_1^\star, e_i, e_i^\star\})$ it is sufficient to require $\kappa_{(v,e_i)}(e_j) = 0$ for all $2 \le j \le g$ with $j \ne i$. By harmonicity this implies $\kappa_{(v,e_i)} \equiv 0$ on $\mathrm{E}(\mathcal{S} \smallsetminus (\mathcal{P}_1 \cup \mathcal{P}_i \cup \mathcal{P}'_i))$. Suppose that $\kappa_{(v,e_i)}(e_1) = w$. Then by the Γ-equivariance $\kappa_{(v,e_i)}(\gamma_1 e_1) = \gamma_1 w$ and $\kappa_{(v,e_i)}(\gamma_i e_i) = \gamma_i v$. Since by Lemma 2.22 the distance between $o(e_1)$ and $o(\gamma e_1)$ and the distance between $o(e_i)$ and $o(\gamma e_i)$ in $\mathcal{T}$ are even, for $\kappa_{(v,e_i)}$ to be harmonic v and w have to fulfil the relation $\gamma_i v - v + \gamma_1 w - w = 0$ if $d(o(e_1), o(e_i))$ is even or $\gamma_i v - v - \gamma_1 w + w = 0$ if $d(o(e_1), o(e_i))$ is odd. See Figure 4 for an illustration. Since by assumption $V_{n,l}(K_\infty)_{\langle\gamma_1\rangle}^{\mathbb{F}_q^\star\cap\Gamma} = \{0\}$ there is a unique w fulfilling this equation. Hence $\kappa_{(v,e_i)}$ is uniquely determined on all of $\mathcal{Y}$ and using $\mathrm{E}(\Gamma\backslash\mathcal{T}) \cong \mathrm{E}(\mathcal{Y})$ this extends to a unique Γ-equivariant cocycle from $\mathrm{E}(\mathcal{T})$ to $V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma}$. ∎

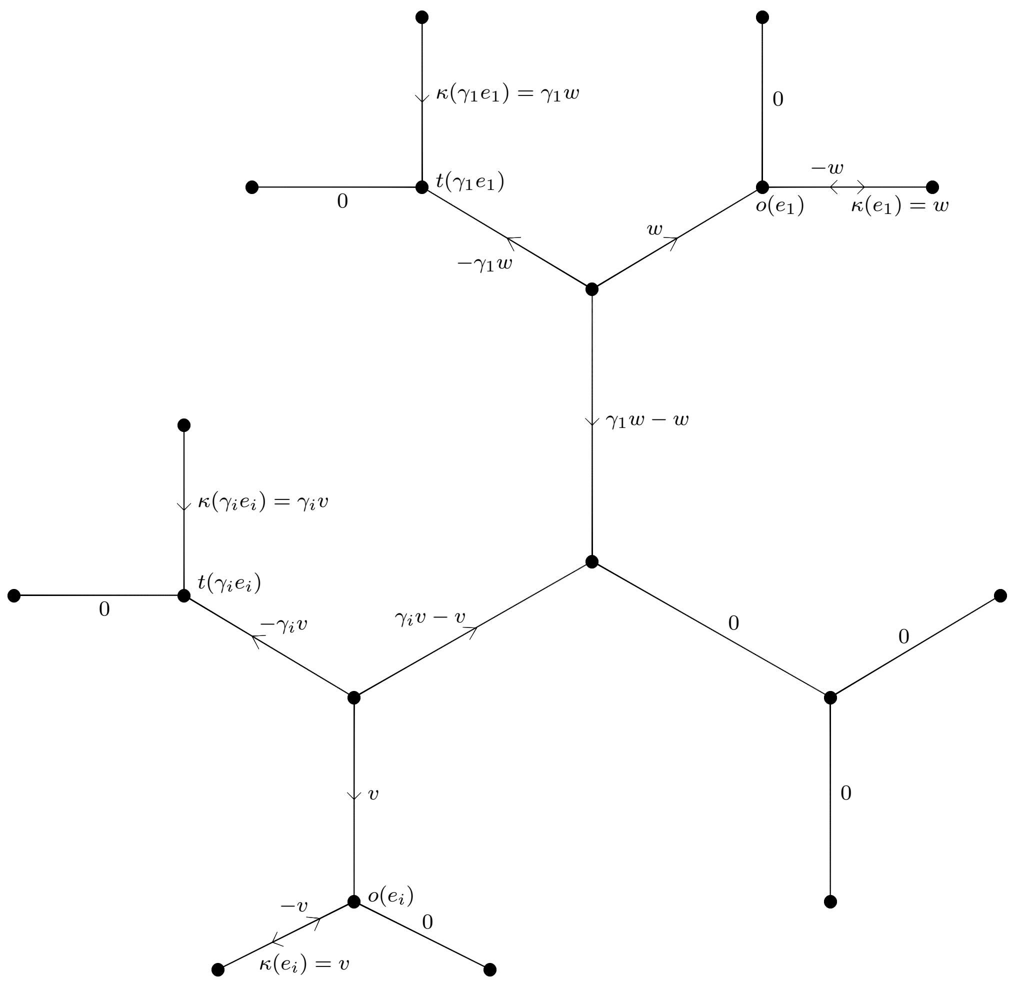

Figure 4: Illustration of a possible assingment of values for $\mathrm{E}(\mathcal{Y})$

Proposition 6.8 *Let* $((\mathcal{S}, \mathcal{Y}), \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{G_t \mid t \text{ a simplex of } \mathcal{Y}\})$ *be an enhanced fundamental domain for* $\mathcal{T}$ *under* Γ *and* $\mathcal{B}_{n,l}$ *a basis of* $V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$*. Let* $g = g(\Gamma \backslash \Omega)$,

$$\{e_i \mid i = 1, \ldots, g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$*. Let* $n > 2$ *be odd. The image of the map*

$$\psi : \{e_i \mid 2 \leq i \leq g\} \times \mathcal{B}_{n,l} \to C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}} : (e_i, v) \mapsto \kappa_{e_i,v}$$

is a basis of $C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$ *where* $\kappa_{e_i,v}$ *is the unique cocycle from Lemma 6.7.*

PROOF: Since the cocycles $\kappa_{e,v}$ for different e are linearly independent, the images $\psi(e,v)$ and $\psi(e',v')$ are linearly independent for all $e \in \{e_i \mid 2 \leq i \leq g\}$ and $v \in \mathcal{B}_{n,l}$. Let $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$. By Lemma 6.3 we have $\kappa(e) \in V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$ for all $e \in \mathrm{E}(\mathcal{T})$. Then using Lemma 6.7 we can substract linear combinations of cocycles from the set $\psi(\{e_i \mid 2 \leq i \leq g\} \times \mathcal{B}_{n,l})$ to assume that $\kappa(e) = 0$ for all $e \in \mathrm{E}(\mathcal{Y} \smallsetminus (\mathcal{S} \cup e_1, e_1^\star))$. Let $\kappa(e_1) = v$. Then as in the proof of Lemma 6.7 we must have $\gamma_1 v - v = 0$, hence $v \in V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma \times \langle \gamma_1 \rangle}$, which equals $\{0\}$ by Lemma 6.6. This implies $\kappa(e) = 0$ for all $e \in \mathrm{E}(\mathcal{Y})$ and so by the Γ-equivariance of κ we have $\kappa \equiv 0$. ∎

We now treat the case $n > 2$ even. Before we construct an explicit basis, we need the following lemma.

Lemma 6.9 *Let $((\mathcal{S},\mathcal{Y}), \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{G_t \mid t \text{ a simplex of } \mathcal{Y}\})$ be an enhanced fundamental domain for $\mathcal{T}$ under Γ and $\mathcal{B}_{n,l}$ a basis of $V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$. Let $g = g(\Gamma \backslash \Omega)$,*

$$\{e_i \mid i = 1, \ldots, g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$. Let n be even and $l \in \mathbb{Z}$ with $n \equiv 2l \pmod{\omega}$. Let $V = V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$. Then for all $i, j \in \{1, \ldots, g\}$ with $i \neq j$ the subspaces $(1-\gamma_i)V$ and $(1-\gamma_j)V$ of V are transversal.

PROOF: Let $\Gamma' = \langle \gamma_i, \gamma_i \rangle \subseteq \Gamma$. Then as in the proof of Lemma 4.8 one has

$$V_{\Gamma'} = V / \langle (\gamma - 1)v \mid v \in V, \gamma \in \Gamma' \rangle = \{0\}.$$

Now if $\gamma \in \Gamma'$ with $\gamma = \gamma'\gamma''$ then

$$(\gamma - 1) = (\gamma'\gamma'' - 1) = (\gamma'\gamma'' - \gamma'') + (\gamma'' - 1)$$

hence for $v \in V$ we have

$$(\gamma - 1)v = (\gamma' - 1)(\gamma'' v) + (\gamma'' - 1)v.$$

It follows, that

$$\langle (\gamma - 1)v \mid v \in V, \gamma \in \Gamma' \rangle = (\gamma_i - 1)V + (\gamma_j - 1)V.$$

By Lemma 6.6 we know that $(\gamma_i - 1)V$ and $(\gamma_j - 1)V$ are both $n-2$-dimensional. Since $V = (\gamma_i - 1)V + (\gamma_j - 1)V$ is $n-1$-dimensional, they have to be transversal. ∎

In the case of n odd, we could construct a basis out of cocycles that vanish on e_j for all $j \neq \{1, i\}$ for some fixed $i \in \{2, \ldots, g(\Gamma\backslash\Omega)\}$. In the case of n even, this construction does not work. Instead, we have to work with cocycles vanishing outside e_1, e_2 and e_i for some fixed $i \in \{3, \ldots, g(\Gamma\backslash\Omega)\}$. Note that $g(\Gamma\backslash\Omega)$ is always greater or equal 2. In the following Proposition we describe the construction of a basis for n even.

Proposition 6.10 *Let $((\mathcal{S}, \mathcal{Y}), \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}, \{G_t \mid t$ a simplex of $\mathcal{Y}\})$ be an enhanced fundamental domain for $\mathcal{T}$ under Γ and $\mathcal{B}_{n,l}$ a basis of $V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$. Let $g = g(\Gamma\backslash\Omega)$,*

$$\{e_i \mid i = 1, \ldots, g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$. Let $n > 2$ be even and $l \in \mathbb{Z}$ with $n \equiv 2l \pmod{\omega}$. Then there is an explicit basis of $C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$ with $(n-1)(g-1)$ many elements.

Proof: Let $V = V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$. Choose $V^1 \subseteq V$ such that $\mathrm{Kern}(\gamma_1 - 1)$ is complementary to V. Then by Lemma 6.9 we can choose $v_2 \in V$ such that $(\gamma_2 - 1)v_2 + (\gamma_1 - 1)V^1 = V$.
Then as in the proof of Lemma 6.7 one can show that for all $3 \leq i \leq g$ and for all $w \in V$ there exists an unique $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$ with $\kappa(e_j) = 0$ for all $j \neq \{1, 2, i\}$, $\kappa(e_i) = w$, $\kappa(e_2) \in K_\infty v_2$ and $\kappa(e_1) \in V^1$.
Let $V^2 = \{v \in V \mid (\gamma_2 - 1)v \in (\gamma_1 - 1)V\}$. This set has codimension 1 in V by Lemma 6.9. Then again as in the proof of Lemma 6.7 one shows that for all $w \in V^2$ there exist a unique $v \in V^1$ such that there is a unique $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)$ with $\kappa(e_1) = v$, $\kappa(e_2) = w$ and $\kappa(e_i) = 0$ for all $i \geq 3$.
Finally one shows that for all $v \in V^{\langle \gamma_1 \rangle}$ there is a unique $\kappa \in C_{n,l}^{\mathrm{har}}(\Gamma)$ with $\kappa(e_1) = v$ and $\kappa(e_i) = 0$ for all $i \geq 2$.
In total we constructed $(g-2)(n-1)$ plus $(n-2)$ plus 1 linear independent cocycles. These cocycles form a basis of $C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}}$. ∎

Since by Theorem 5.24 the residue map provides an isomorphism between $\mathcal{M}_{n,l}(\Gamma)$ and $C_{n,l}^{\mathrm{har}}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty$, we obtain a generalization of the dimension formula from Proposition 4.6, which was valid for the p'-torsion free case. Note that in that case one has $\mathbb{F}_q^\star \cap \Gamma = \{1\}$.

Corollary 6.11 *For $n > 2$ and $\Gamma \subseteq \Lambda^\star$ of finite index one has*

$$\dim C_{n,l}^{\mathrm{har}}(\Gamma)_{\mathrm{reg}} = \dim V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma} \cdot (g(\Gamma\backslash\Omega) - 1).$$

Using this we can now generalize to arbitrary $n \in \mathbb{Z}$ Proposition 4.13, which we previously could only proof for even n.

Corollary 6.12 *For all $n, l \in \mathbb{Z}$ one has $\mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty) \neq \{0\}$ if and only if $n \equiv 2l \pmod{\omega}$.*

Proof: It suffices by Proposition 4.13 to consider odd integers n. If $n > 2$ and $n \equiv 2l \pmod{\omega}$, then by Corollary 6.11 we have

$$\dim \mathcal{M}_{n,l}(\Gamma) \overset{\text{Thm. 5.24}}{=} \dim C^{\text{har}}_{n,l}(\Gamma) \otimes_{K_\infty} \mathbb{C}_\infty \geq \dim C^{\text{har}}_{n,l}(\Gamma)_{\text{reg}} \otimes_{K_\infty} \mathbb{C}_\infty > 0$$

and hence there are non-zero elements in $\mathcal{M}_{n,l}(\Gamma) \subset \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$. Now let $n = 1$ and $l \in \mathbb{Z}$ with $1 \equiv 2l \pmod{\omega}$. By Proposition 4.13 we know that there is an $0 \neq f_1 \in \mathcal{A}_{-2,-1}(\Gamma, \mathbb{C}_\infty)$ and by the above we know that there is an $0 \neq f_2 \in \mathcal{A}_{3,l+1}(\Gamma, \mathbb{C}_\infty)$. Hence $0 \neq f_1 \cdot f_2 \in \mathcal{A}_{-2+3,-1+l+1}(\Gamma, \mathbb{C}_\infty) = \mathcal{A}_{1,l}(\Gamma, \mathbb{C}_\infty)$.
If $n < 0$ and $l \in \mathbb{Z}$ with $n \equiv 2l \pmod{\omega}$. Then $-n \equiv -2l \pmod{\omega}$, so by the above there is an $0 \neq f \in \mathcal{A}_{-n,-l}(\Gamma, \mathbb{C}_\infty)$. Hence $0 \neq 1/f \in \mathcal{A}_{n,l}(\Gamma, \mathbb{C}_\infty)$.
On the other hand if $n \not\equiv 2l \pmod{\omega}$ and $f \in \mathcal{A}_{n,l}(\Gamma)$, choose $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ a generator of $\mathbb{F}_q^\star \cap \Gamma$. So $\operatorname{ord}(\alpha) = \operatorname{ord}(\gamma) = \omega$ and $f(z) = f \underset{n,l}{|} \gamma(z) = \alpha^{2l-n} f(z)$ and hence $f = 0$. ∎

What remains for the case $n > 2$ is to give a description of the non-regular harmonic cocycles. We sum this up in the following theorem. For $\gamma \in \Gamma$ let $N_\gamma : V_{n,l}(K_\infty) \to V_{n,l}(K_\infty)$ be the operator $v \mapsto \sum_{i=0}^{q} \gamma^i v$. If γ is a generator of $\operatorname{Stab}_\Gamma(v)$ for $\pi(v)$ a terminal vertex of $\Gamma\backslash\mathcal{T}$ then since γ has order $q+1$ on $V_{n,l}(\Gamma)^{\mathbb{F}_q^\star \cap \Gamma}$ we have

$$(\gamma - 1) V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma} \in \operatorname{Kern}(N_\gamma)$$

and

$$\operatorname{Kern}(N_\gamma)/(\gamma - 1) V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma} \cong \hat{H}_0(\operatorname{Stab}_{\bar{\Gamma}}(v), V_{n,l}(K_\infty))$$

where $\hat{H}_0$ denotes the 0-th Tate homology group, see [Se3, Chapter VIII] for a definition. Since $V_{n,l}(K_\infty)$ is a projective $\operatorname{Stab}_{\bar{\Gamma}}(v)$-module, by Proposition 1 of *loc. cit.* it follows that $\hat{H}_0(\operatorname{Stab}_{\bar{\Gamma}}(v), V_{n,l}(K_\infty)) = \{0\}$ and hence $\operatorname{Kern}(N_\gamma) = (1 - \gamma) V_{n,l}(K_\infty)$.

Theorem 6.13 *Let $n > 2$ and $\Gamma \subseteq \Lambda^\star$ of finite index. Then*

$$C^{\text{har}}_{n,l}(\Gamma) \cong C^{\text{har}}_{n,l}(\Gamma)_{\text{reg}} \oplus \bigoplus_{\substack{e \in \mathrm{E}(\Gamma\backslash\mathcal{T})_{\text{term}} \\ \deg(t(e))=1}} (1 - \gamma) V_{n,l}(K_\infty)^{\mathbb{F}_q^\star \cap \Gamma}$$

where e' is any edge in $\pi^{-1}(e)$ and γ is a generator of $\operatorname{Stab}_{\bar{\Gamma}}(e')$.

PROOF: Let $((\mathcal{S},\mathcal{Y}),\mathrm{PE}_{(\mathcal{S},\mathcal{Y})},\{G_t \mid t \text{ a simplex of } \mathcal{Y}\})$ be an enhanced fundamental domain for $\mathcal{T}$ under Γ, $g = g(\Gamma\backslash\Omega)$,

$$\{e_i \mid i = 1,\dots,g\} = \{e \in \mathrm{E}(\mathcal{Y} \smallsetminus \mathcal{S}) \mid o(e) \in \mathcal{S}\}$$

and $\gamma_i = \mathrm{PE}_{(\mathcal{S},\mathcal{Y})}(e_i)$. Let $e \in \mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}}$ with $\deg(t(e)) = 1$, $e' \in \pi^{-1}(e')$, γ a generator of $\mathrm{Stab}_{\bar\Gamma}(e')$ and let $v \in (1-\gamma)V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma} = \mathrm{Kern}(N_\gamma)$. We can suppose that $e' \in \mathcal{Y}$, otherwise we replace e' by its unique Γ-equivalent edge in $\mathcal{Y}$ and v by a Γ-translate of v. Then as in the proof of Lemma 6.7 for n odd or Proposition 6.10 for n even we can construct a unique cocycle $\kappa_{e,v} \in C^{\mathrm{har}}_{n,l}(\Gamma)$ with $\kappa_{e,v}(e') = v$ and $\kappa_{e,v}(e_i) = 0$ for all $i \geq 2$ if n is odd or $i \geq 3$ if n is even. For this cocycle to be harmonic it is necessary and sufficient that

$$0 = \sum_{\tilde e \mapsto t(e')} \kappa_{e,v}(\tilde e) = \sum_{\gamma' \in \mathrm{Stab}_{\bar\Gamma}(e')} \gamma' \kappa_{e,v}(e') = \sum_{i=0}^{q} \gamma^i \kappa_{e,v}(e') = N_\gamma(\kappa_{e,v}(e')) = N_\gamma(v). \blacksquare$$

To obtain a closed formula for the dimension of $C^{\mathrm{har}}_{n,l}(\Gamma)$, it remains to compute the dimension of $(1-\gamma)V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma}$ for γ a generator of $\mathrm{Stab}_{\bar\Gamma}(e')$. Clearly

$$\dim(1-\gamma)V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma} = \dim V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma} - \dim \mathrm{Kern}(1-\gamma)|_{V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma}}$$

and $\dim\mathrm{Kern}(1-\gamma)$ is the dimension of the eigenspace for the eigenvalue 1 in $\mathrm{Sym}^{n-2}(\gamma)$. To compute this, we can assume that after base change $\gamma \in \mathbb{F}_{q^2}^\star \hookrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_q) \subseteq \mathrm{GL}_2(\mathbb{C}_\infty)$ is of the form $\begin{pmatrix} z & 0 \\ 0 & z^q \end{pmatrix}$ for some $z \in \bar{\mathbb{F}}_q$ with $\mathrm{ord}(z) = q^2-1$ and hence the dimension depends on the multiplicity $s_{q,n,l}$ of the eigenvalue 1 of $\mathrm{Sym}^{n-2}(\gamma)\otimes\det^{1-l}$. Note also that by Lemma 6.4 $\dim V_{n,l}(K_\infty)^{\mathbb{F}_q^\star\cap\Gamma}$ equals 0 if $n \not\equiv 2l \pmod{\omega}$ and $n-1$ if $n \equiv 2l \pmod{\omega}$. We summarize this in the following:

Corollary 6.14 *Let $n > 2$ and set*

$$e_{\mathrm{reg}}(\Gamma) = \#\{e \in \mathrm{E}(\Gamma\backslash\mathcal{T})_{\mathrm{term}} \mid \deg(t(e)) = 1\}.$$

Then

$$\dim C^{\mathrm{har}}_{n,l}(\Gamma) = \begin{cases} 0 & \text{if } n \not\equiv 2l \pmod{\omega} \\ (n-1)(g(\Gamma\backslash\Omega)-1) + e_{\mathrm{reg}}(\Gamma)(n-1-s_{q,n,l}) & \text{if } n \equiv 2l \pmod{\omega}. \end{cases}$$

Recall that for $\Gamma = \Lambda^\star$ Theorem 2.27 implies $e_{\mathrm{reg}}(\Gamma) = 2^{\#R-1}\,\mathrm{odd}(R)$. If $\omega = 1$, then $e_{\mathrm{reg}}(\Gamma) = 0$. We will give an explicit formula for $s_{q,n,l}$ in the case q odd and $\omega = q-1$. In that case the condition $n \equiv 2l \pmod{q-1}$ implies that n is even.

Lemma 6.15 *Let q be odd and $n, l \in \mathbb{Z}$ with $n \geq 2$ and $2l \equiv n \pmod{q-1}$. Then*

$$s_{q,n,l} = \begin{cases} 1 + 2\left\lfloor \frac{n-2}{2(q+1)} \right\rfloor & \text{if } l \equiv \frac{n}{2} \pmod{q-1}, \\ 2\left\lfloor \frac{n-1+q}{2(q+1)} \right\rfloor & \text{if } l \equiv \frac{n}{2} + \frac{q-1}{2} \pmod{q-1}. \end{cases}$$

Proof: First note that $2l \equiv n \pmod{q-1}$ implies $l \equiv \frac{n}{2} \pmod{\frac{q-1}{2}}$, so l is either congruent $\frac{n}{2} \pmod{q-1}$ or congruent $\frac{n}{2} + \frac{q-1}{2} \pmod{q-1}$.
For $\gamma = \begin{pmatrix} z & 0 \\ 0 & z^q \end{pmatrix}$ we compute

$$\mathrm{Sym}^{n-2}(\gamma) \otimes \det{}^{1-l} = \mathrm{diag}(z^{n-2}, z^{n-3}z^q, \ldots, z^1 z^{(n-3)q}, z^{(n-2)q}) z^{(q+1)(1-l)}$$

and hence $s_{q,n,l}$ equals the number of $i \in \{0, \ldots, n-2\}$ with

$$z^i z^{(n-2-i)q} z^{(q+1)(1-l)} = 1.$$

Since z has order $q^2 - 1$, this amounts to counting the number of $i \in \{0, \ldots, n-2\}$ with

$$i + (n-2-i)q + (q+1)(1-l) \equiv 0 \pmod{q^2-1}.$$

Now

$$i + (n-2-i)q + (q+1)(1-l) = \left(-i + \frac{n-2}{2}\right)(q-1) + \left((1-l) + \left(\frac{n}{2} - 1\right)\right)(q+1).$$

If $l \equiv \frac{n}{2} \pmod{q-1}$, then $\left((1-l) + \left(\frac{n}{2} - 1\right)\right)(q+1) \equiv 0 \pmod{q^2-1}$ and hence we need to count the number of $i \in \{0, \ldots, n-2\}$ with $i \equiv \frac{n-2}{2} \pmod{q+1}$. There are precisely $1 + 2\left\lfloor \frac{n-2}{2(q+1)} \right\rfloor$ many such i.
If $l \equiv \frac{n}{2} + \frac{q-1}{2} \pmod{q-1}$, then $\left((1-l) + \left(\frac{n}{2} - 1\right)\right)(q+1) \equiv \frac{q-1}{2} \pmod{q^2-1}$ and hence we need to count the number of $i \in \{0, \ldots, n-2\}$ with $i \equiv \frac{n-2}{2} + \frac{q-1}{2} \pmod{q+1}$. There are $2\left\lfloor \frac{n-1+q}{2(q+1)} \right\rfloor$ many such i. ∎

By Theorem 5.24, the dimension formula obtained in this way should be equal to the one computed in Theorem 4.19 using divisors on $\Gamma\backslash\Omega$ and the Riemann-Roch theorem. To convince ourself, that these two formulas coincide, we need the following lemma.

Lemma 6.16 *For $m \geq 2$ an integer and $q \in \mathbb{N}$ we have*

$$\left\lfloor (m+1)(1 - \frac{1}{q+1}) \right\rfloor = m - \left\lfloor \frac{m}{q+1} \right\rfloor.$$

PROOF: If $m = j(q+1)$, then the right hand side equals $j(q+1) - j = jq = m - j$ and the left hand side equals

$$\left\lfloor \big(j(q+1)+1\big)\frac{q}{q+1} \right\rfloor = \left\lfloor \frac{jq(q+1)+q}{q+1} \right\rfloor = jq + \left\lfloor \frac{q}{q+1} \right\rfloor = jq.$$

Now for any m with $j(q+1) \leq m < (j+1)(q+1)$ the right hand side equals $m - j$. We also have $j(q+1) < m+1 \leq (j+1)(q+1)$ and hence $j < \frac{m+1}{q+1} \leq j+1$. So

$$\begin{aligned} \left\lfloor (m+1)(1-\frac{1}{q+1}) \right\rfloor - (m-j) &= \left\lfloor j+1-(m+1)(1-\frac{1}{q+1}) - (m+1) \right\rfloor \\ &= \left\lfloor j+1-\frac{m+1}{q+1} \right\rfloor = 0, \end{aligned}$$

and so the left hand side also equals $m - j$. ∎

By Remark 4.17 we saw that there are precisly twice as many elliptic points in $\Gamma\backslash\Omega$ than there are terminal vertices in $\Gamma\backslash\mathcal{T}$. Hence in order for our dimension formula from Theorem 4.19 to match the formula from Corollary 6.14 we have to show that

$$2e_{\text{reg}}(\Gamma)\left\lfloor \frac{n}{2}(1-\frac{1}{q+1}) \right\rfloor = e_{\text{reg}}(\Gamma)\left(n-2-2\left\lfloor \frac{n-2}{2(q+1)} \right\rfloor\right).$$

This follows from Lemma 6.16 with $m = \frac{n-2}{2}$.

6.3 The case of weight $n = 2$

In the case of weight $n = 2$, the coefficient module

$$V_{2,l}(K_\infty) = \text{Hom}(\text{Sym}^0(\text{Hom}(K_\infty^2, F)), K_\infty)$$

is isomorphic to K_∞ with the action of $\text{GL}_2(K_\infty)$ given by $(\gamma, z) \mapsto \det(\gamma)^{1-l} \cdot z$. For the case $l = 1$ we can give a description of a basis of $C_{2,1}^{\text{har}}(\Gamma)$. In that case $\text{GL}_2(K_\infty)$ acts trivially on K_∞.

Lemma 6.17 $C_{2,1}^{\text{har}}(\Gamma) = C_{2,1}^{\text{har}}(\Gamma)_{\text{reg}}$

PROOF: Let $\kappa \in C_{2,1}^{\text{har}}(\Gamma)$ and $e \in \pi^{-1}(\text{E}(\Gamma\backslash\mathcal{T})_{\text{term}})$. Let v be the extremity of e having $\deg(\pi(v)) = 1$. Then by Proposition 2.24 and by our assumption on $\Gamma_v = \text{Stab}_\Gamma(v)$ we know that $\bar{\Gamma}_v \cong \mathbb{F}_{q^2}^\star/\mathbb{F}_q^\star \cong \mathbb{Z}/(q+1)$ and that $\bar{\Gamma}_v$ operators transitivly on the edges leaving v. Hence

$$0 = \sum_{e' \mapsto v} \kappa(e') = \sum_{\gamma \in \bar{\Gamma}_v} \kappa(\gamma e) = \sum_{\gamma \in \bar{\Gamma}_v} \gamma\, \kappa(e) = (q+1)\,\kappa(e) = \kappa(e)$$

and so κ vanishes on all edges in $\pi^{-1}(\text{E}(\Gamma\backslash\mathcal{T})_{\text{term}})$. This implies $\kappa \in C_{2,1}^{\text{har}}(\Gamma)_{\text{reg}}$. ∎

Proposition 6.18 *Let* $((\mathcal{S},\mathcal{Y}),\mathrm{PE}_{\mathcal{S},\mathcal{Y}},\{G_t \mid t \text{ a simplex of } \mathcal{Y}\})$ *be an enhanced fundamental domain for* $\mathcal{T}$ *under* Γ, $g = g(\Gamma\backslash\Omega)$ *and*

$$\{e_i \mid i = 1,\dots,g\} = \{e \in \mathrm{E}(\mathcal{Y}\backslash\mathcal{S}) \mid o(e) \in \mathcal{S}\}.$$

(a) To each $v \in K_\infty$ *and for all* $1 \le i \le g$ *there is a unique cocycle* $\kappa_{e_i,v} \in C^{\mathrm{har}}_{2,1}(\Gamma)$ *with* $\kappa(e_i) = v$ *and* $\kappa(e_j) = 0$ *for all* $j \neq i$.

(b) The image of the map

$$\psi : \{e_i \mid 1 \le i \le g\} \to C^{\mathrm{har}}_{2,1}(\Gamma) : e_i \mapsto \kappa_{e_i,1}$$

is a basis of $C^{\mathrm{har}}_{2,1}(\Gamma)$.

PROOF: (a) If $\kappa \in C^{\mathrm{har}}_{2,1}(\Gamma)$ satisfies $\kappa(e_i) = v$ and $\kappa(e_j) = 0$ for all $j \neq i$, then κ is uniquely determined on all of $\mathcal{Y}$ and hence by Γ-equivariance on all of $\mathcal{T}$. This proves the uniqueness. κ is also well-defined, since

$$\kappa(\gamma e_i) - \kappa(e_i) = \gamma v - v = v - v = 0.$$

(b) Follows directly from (a). ∎

As an immediate consequence we obtain the following corollary.

Corollary 6.19 $\dim C^{\mathrm{har}}_{2,1}(\Gamma) = g(\Gamma\backslash\Omega)$

Remark 6.20 If $l \not\equiv 1 \pmod L$, the action of Γ on $V_{2,l}(K_\infty)$ by multiplication with $\det(\gamma)^{1-l}$ is non-trivial. We can not give an describtion of a basis of $C^{\mathrm{har}}_{2,l}(\Gamma)$ directly on $\Gamma\backslash\mathcal{T}$. One way to get by this problem is to replace Γ with $\Gamma' := \{\gamma \in \Gamma \mid \det(\gamma)^{1-l} = 1\}$. If $2 \equiv 2l \pmod \omega$, this is an index 2 subgroup of Γ. The subgroup Γ' acts trivially on $V_{2,l}(K_\infty)$ and hence on $\Gamma'\backslash\mathcal{T}$ one has an explicit description of $C^{\mathrm{har}}_{2,l}(\Gamma)$ similar to the one given in Proposition 6.18. One then obtains $C^{\mathrm{har}}_{2,l}(\Gamma)$ as the Γ/Γ'-invariant subspace of $C^{\mathrm{har}}_{2,l}(\Gamma')$.
To compute the dimension of the space of harmonic cocycles in this case one would need to analyze the ramification behaviour of the covering $\Gamma'\backslash\mathcal{T} \to \Gamma\backslash\mathcal{T}$ and than use the Hurwitz formula. We choose to omit the details here.

A Some remarks on $\mathrm{char}(K)$ even

In the case q even, the tools needed for the results in Section 2.3 and Section 2.6 are substantially different. Once one finds models for (D, Λ) as in Section 2.6, everything developed in Section 2.5 and Section 2.7 can be adapted. If q is even and $D = \left(\frac{a,b}{K}\right)$ with $a, b \in K^\star$ is a quaternion algebra, then the subfield $K(\sqrt{b}) \subseteq D$ is an inseparable extension of K once $b \notin (K^\star)^2$, but $D \supseteq K(i)$ with $i^2 + i + a = 0$ is a separable Artin-Schreier extension of K, compare Definition 2.12. This asymmetry in the role of a and b indicates that a formula like Proposition 2.19 for the ramification of D has to look quite different in the even characteristic case.

Division algebras over K can be constructed in a systematic way as cyclic algebras, we quickly recall this construction here, following mainly [Ja, Chapter 8].

Let $\mathrm{Br}(K)$ denote the Brauer group of K, that is the group of similarity-classes of finite-dimensional central simple algebras over K, where two such algebras A, B are similar if there are positive integers m and n such that $M_m(A) \cong M_n(B)$. We write $[A]$ for the similarity class of A. Multiplication in this group is defined by taking tensor products over K, the similarity class of K is the unit element, and since $A \otimes_K A^{\mathrm{op}} \cong M_n(K)$ with $n = \dim_K(A)$ we see that every element $[A]$ of $\mathrm{Br}(K)$ has $[A^{\mathrm{op}}]$ as an inverse.

Let F be an extension field of K. Then we have a natural map $\varphi : \mathrm{Br}(K) \to \mathrm{Br}(F)$ sending $[A]$ to $[A \otimes_K F]$. We define $\mathrm{Br}(K, F) := \mathrm{Kern}(\varphi)$.

Let F/K be a finite Galois extension of degree n with $G := \mathrm{Gal}(F/K)$ and let $\psi \in Z^2(G, F^\star)$ be a 2-cocycle. Let A be the F-algebra with basis $\{u_s \mid s \in G\}$ and multiplication

$$(\sum_{s\in G} \lambda_s u_s)(\sum_{s\in G} \mu_s u_s) = \sum_{s,t\in G} \psi(s,t)\lambda_s s(\mu_t) u_{st}. \tag{20}$$

We write $A = (F, G, \psi)$ and call A the crossed product of F and G with respect to ψ.

Theorem A.1 *$A = (F, G, \psi)$ is a central simple algebra over K of dimension n^2.*

PROOF: See [Ja, Theorem 8.7]. ∎

If F/K is a cyclic extension with $G = \langle s \rangle$, then we can choose ψ to be the map

$$\psi_\gamma(s^i, s^j) := \begin{cases} 1 & \text{if } 0 \le i + j < n \\ \gamma & \text{if } n \le i + j \le 2n - 2. \end{cases}$$

for some $\gamma \in K^\star$, see [Ja, Section 8.5]. We write $A = (F, s, \gamma)$ and call A the cyclic algebra defined by F/K, the generator s of G and $\gamma \in K^\star$.

Theorem A.2 *(a)* $[A] = [(F, s, \gamma)]$ *is independent of the choice of the element* γ *from* $K^\star/\operatorname{Norm}_{F/K}(F^\star)$.

(b) The map $\gamma : \operatorname{Norm}_{F/K}(F^\star) \mapsto [(F, s, \gamma)]$ *defines an isomorphism between the groups* $F^\star/\operatorname{Norm}_{F/K}(F^\star)$ *and* $\operatorname{Br}(K, F)$.

Proof: See [Ja, Theorem 8.14]. ∎

Example A.3 Let $a \in K^\star$ be any element such that $F := K[x]/(x^2 + x + a) \ncong K$. Then F/K is an Artin-Schreier extension, so it is cyclic of degree 2 and $G = \operatorname{Gal}(F/K)$ is generated by $s : r = \lambda_1 x + \lambda_2 \mapsto (\lambda_1 + 1)x + \lambda_2$. Choose any $b \in K^\star$. Then ψ_b is given by the following values:

	$(1,1)$	$(s,1)$	$(1,s)$	(s,s)
ψ_b	1	1	1	b

Let $A = (F, s, b)$. Then as an additive group

$$A = Fu_1 \oplus Fu_s \cong Ku_1 \oplus Ku_s \oplus Kxu_1 \oplus Kxu_s.$$

We compute a multiplication table using formula (20):

	u_1	u_s	xu_1	xu_s
u_1	u_1	u_s	xu_1	xu_s
u_s	u_s	bu_1	$xu_s + u_s$	$bxu_1 + bu_1$
xu_1	xu_1	xu_s	$xu_1 + au_1$	$xu_s + au_s$
xu_s	xu_s	bxu_1	au_s	abu_1

Hence the map given by $u_1 \mapsto 1$, $u_s \mapsto j$, $xu_1 \mapsto i$ and $xu_s \mapsto ij$ defines an isomorphism $A \cong \left(\frac{a,b}{K}\right)$.

We fix an $a \in K^\star$ such that $F := K[x]/(x^2 + x + a)$ is a cyclic degree 2 extension of K with Galois group $G = \{s\}$. Then by Example A.3 for any $b \in K^\star$ we obtain the quaternion algebra $\left(\frac{a,b}{K}\right)$ as the cyclic algebra (F, s, b).
Let v be a finite place of K and let ϖ_v denote the corresponding monic irreducible in $k[T]$.

Proposition A.4 *If* F/K *splits at* v*, then* (F, s, b) *is unramified at* v.

PROOF: By Example A.3 we have $D := (F, s, b) = \left(\frac{a,b}{K}\right)$. Hence $D_v := D \otimes_K K_v \cong \left(\frac{a,b}{K_v}\right)$. If F/K splits at v, then $x^2 + x + a$ has a solution over K_v. That means there is an $\alpha \in K_v$ such that $\alpha^2 + \alpha + a = 0$, and also $(\alpha + 1)^2 + \alpha + 1 + a = 0$. Hence the map $\varphi : \left(\frac{a,b}{K_v}\right) \to M_2(K_v)$ defined by $i \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha + 1 \end{pmatrix}$ and $j \mapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$ provides an embedding of D_v into $M_2(K_v)$. Since both D_v and $M_2(K_v)$ are of dimension 4 over K_v we have $D_v \cong M_2(K_v)$ and hence D is unramified at v. ∎

Proposition A.5 *If F/K is non-split at v, then (F, s, b) is unramified at v if and only if $b \in \mathrm{Norm}_{F_v/K_v}(F_v^\star)$.*

PROOF: Let $D := (F, s, b)$. Since F/K is non-split at v, the extension F_v/K_v is a degree 2 Galois extension and it is clear from the construction of D that $D_v = (F_v, s, b)$. Theorem A.2 applied to the extension F_v/K_v implies that $[D_v] = [K_v]$ if and only if $b \in \mathrm{Norm}_{F_v/K_v}(F_v^\star)$. If $[D_v] = [K_v]$ then there are $n, m \in \mathbb{N}$ such that $M_n(D_v) \cong M_m(K_v)$. But since D_v is central simple over K_v, we have $D_v \cong M_{n'}(\Delta)$ with Δ a division algebra over K_v. Hence $M_m(K_v) \cong M_n(D_v) \cong M_{nn'}(\Delta)$. This is only possible if $\Delta = K_v$. Since D_v is of dimension 4 over K_v this implies $D_v = M_2(K_v)$. On the other hand if $D_v \cong M_2(K_v)$ then clearly $[D_v] = [K_v]$. ∎

If F/K is non-split at v, then F_v/K_v is a degree 2 extension of local fields. If this extension is unramified, we have an easy criterion to decide whether some $b \in K_v^\star$ is in $\mathrm{Norm}_{F_v/K_v}(F_v^\star)$:

Proposition A.6 *Suppose F/K is unramified at v. Then $b \in \mathrm{Norm}_{F_v/K_v}(F_v^\star)$ if and only if $v(b) \equiv 0 \pmod 2$.*

PROOF: Since F_v/K_v is unramified it is an extension of residue fields. Hence we can assume w.l.o.g. that $K_v \cong \mathbb{F}_q((T))$ and $F_v \cong \mathbb{F}_{q^2}((T))$ for some prime power q. Then

$$\mathrm{Image}(\mathrm{Norm}_{F_v/K_v}(F_v^\star)) = T^{2\mathbb{Z}}\mathbb{F}_q[[T]]^\star$$

which implies the Lemma. ∎

As a consequence to the above propositions we see that a cyclic algebra is only ramified at a finite number of places, a fact we already know by Proposition 2.20:

Corollary A.7 *Let v be a finite place of K such that $v(a) = 0 = v(b)$. Then $\left(\frac{a,b}{K}\right)$ is unramified at v.*

Proof: Since $v(a) = 0$, the extension F_v/K_v either splits or is unramified. In the first case by Proposition A.4 we know that $\left(\frac{a,b}{K}\right)$ is unramified at v. In the second case we know by Proposition A.6 that $b \in \mathrm{Norm}_{F_v/K_v}(F_v^\star)$ and hence by Proposition A.5 that $\left(\frac{a,b}{K}\right)$ is unramified at v. ∎

Remark A.8 In fact, for $a \in K$ and v a place of K where $x^2 + x + a$ is non-split, we have F_v/K_v is unramified if and only if $v(a) \geq 0$.

Following [Con, Definition2.1] we define the Artin-Schreier symbol.

Definition A.9 *For $\varpi \in k[T]$ monic irreducible and $f \in k[T]$ we define the Artin-Schreier symbol*

$$[f, \varpi) := \begin{cases} 0 & \textit{if } f \equiv x^2 + x \pmod{\varpi} \textit{ for some } x \in k[T], \\ 1 & \textit{otherwise.} \end{cases}$$

For $f \in k$ it is an easy task to evaluate the Artin-Schreier symbol $[f, \varpi)$.

Proposition A.10 *[Con, Theorem 3.8] For $f \in k$, $\varpi \in k[T]$ monic irreducible we have*

$$[f, \varpi) \equiv \mathrm{Trace}_{k/\mathbb{F}_2}(f)\deg(\varpi) \pmod 2.$$

We need to be able to decide whether F/K is split or non-split at a given place v. We fix an uniformizer π_v of K_v. Let α denote the Laurent series expansion in π_v of a at the place v. Then F/K is split at v if and only if the equation $x^2 + x = \alpha$ has a solution in K_v. Before we can give a criterion we need a lemma.

Lemma A.11 *Let $k \in \mathbb{N}$. Then $x^2 + x = \alpha + \pi^{-2k}$ has a solution in K_v if and only if $y^2 + y = \alpha + \pi^{-k}$ has a solution in K_v.*

Proof: Suppose there is an $x \in K_v$ such that $x^2 + x = \alpha + \pi^{-2k}$. Set $y := x + \pi^{-k}$. Then

$$y^2 + y = (x + \pi^{-k})^2 + x + \pi^{-k} = x^2 + x + \pi^{-2k} + \pi^{-k} = \alpha + \pi^{-k}. \blacksquare$$

This Lemma allows us to replace α with an α' such that the principal part of α is zero or $v(\alpha)$ is odd. The next two propositions treat these cases.

Proposition A.12 *Suppose $\alpha \in K_v$ has principal part zero and let α_0 denote the constant coefficient of α. Then $x^2 + x = \alpha$ has a solution in K_v if and only if $[\alpha_0, \varpi_v) = 0$.*

PROOF: First suppose $\alpha_0 = 0$. Then by Proposition A.10 we have $[0, \varpi_v) = 0$, hence we have to show that $x^2 + x = \alpha$ has a solution in K_v. Set $x := \sum_{n\geq 0} \alpha^{2^n}$. Because $v(\alpha) > 0$ this sum converges and $x^2 = \sum_{n\geq 1} \alpha^{2^n}$. Hence $x^2 + x = \alpha^{2^0} = \alpha$.
Now suppose $\alpha_0 \neq 0$. Let $\alpha' = \alpha - \alpha_0$. By the above there is an $y \in K_v$ such that $y^2 + y = \alpha'$. Hence $x^2 + x = \alpha$ has a solution in K_v if and only if $x^2 + x = \alpha_0$ has a solution in K_v. But this is equivalent to $[\alpha_0, \varpi_v) = 0$. ∎

Proposition A.13 *Let $\alpha \in K_v$ with non-zero principal part and suppose $v(\alpha)$ is odd. Then $x^2 + x = \alpha$ has no solution in K_v.*

PROOF: Suppose there is an $x \in K_v$ with $x^2 + x + \alpha = 0$ and let $v(\alpha) = 2m + 1$ with $m \leq 0$. Then $v(x^2 + x) = 2m + 1$, hence $v(x) = m + \frac{1}{2} \notin \mathbb{Z}$, which is a contradiction. ∎

The previous propositions can be applied to obtain concrete models for (D, Λ) as in Section 2.6. In this appendix we restrict ourself to the case $\mathrm{odd}(R) = 1$.

Proposition A.14 *Let $R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ be a set of finite places of K with $\deg(\mathfrak{p}_i)$ odd for all i and l even, let r be a monic generator of the ideal $\mathfrak{r} = \prod_{i=1}^{l} \mathfrak{p}_i$ and let $\xi \in k^\star$ with $\mathrm{Trace}_{k/\mathbb{F}_2}(\xi) \neq 0$. Then $D := \left(\frac{\xi, r}{K}\right)$ is ramified exactly at the places $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$.*

PROOF: Let $F := K[x]/(x^2 + x + \xi)$. By Proposition A.12 and Proposition A.10 the extension F_v/K_v is split at every finite place v of even degree, hence by Proposition A.4 we know that D is unramified at every finite place of even degree. At a finite place v of odd degree we know that F_v/K_v is non-split, but since $v(\xi) = 0$ we also know that F_v/K_v is unramified. Hence by Proposition A.6 we have $r \notin \mathrm{Norm}_{F_v/K_v}(F_v^\star)$ if and only if v is one of the places $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$. So by Proposition A.5 D has the claimed ramification property at all finite places.
At the infinite place D has to be unramified since by Proposition 2.20 the number of ramified places of D is even. ∎

Let $D := \left(\frac{\xi, r}{K}\right)$ as in the previous Proposition and $\Lambda := \langle e_1 := 1, e_2 := i, e_3 := j, e_4 := ij\rangle_A$ throughout the remainder of this appendix.

Proposition A.15 *Λ is a maximal order of D.*

PROOF: It is clear that Λ is an order of D. We compute the square of the reduced discriminant of Λ as the ideal generated by

$$\det(\mathrm{trd}(e_ie_j)_{i,j=1,\dots,4}) = \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & r & 0 \end{pmatrix} = r^2.$$

Hence $\mathrm{disc}(\Lambda) = \mathfrak{r}$ and so Λ is maximal. ∎

Lemma A.16 *Set $\varepsilon = T^{\deg(r)/2}\xi^{-q/2}$. Then there is an $\alpha \in K_\infty$ such that*

$$(\frac{\alpha}{\varepsilon})^2 + \frac{\alpha}{\varepsilon} + (\xi + \frac{r}{\varepsilon^2}) = 0.$$

PROOF: Since $\deg(r)$ is even we have $\varepsilon \in K$. By the definition of ε we have $v_\infty(\xi + \frac{r}{\varepsilon^2}) \geq 1$, hence by Proposition A.12 there is an $x \in K_\infty$ such that $x^2 + x = \xi + \frac{r}{\varepsilon^2}$ and so $\alpha := x\varepsilon$ does the job. ∎

Fix α and ε as in the previous Lemma throughout the remainder.

Proposition A.17 *The map $\iota : D \to M_2(K_\infty)$ defined by $i \mapsto \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix}$ and $j \mapsto \begin{pmatrix} \alpha & \xi\varepsilon + \alpha \\ \varepsilon & \alpha \end{pmatrix}$ gives an isomorphism of $D \otimes_K K_\infty \cong M_2(K_\infty)$.*

PROOF: As in Lemma 2.43 we have to check that $\iota(i)$ and $\iota(j)$ fulfil the relations from Definition 2.12. We have

$$\iota(i)^2 = \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} + \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix} = \xi\iota(1) + \iota(i)$$

and

$$\iota(j)^2 = \begin{pmatrix} \alpha & \xi\varepsilon + \alpha \\ \varepsilon & \alpha \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 + \xi\varepsilon^2 + \alpha\varepsilon & 0 \\ 0 & \alpha^2 + \xi\varepsilon^2 + \alpha\varepsilon \end{pmatrix} = r\iota(j)$$

by the choice of α and ε. Finally we have

$$\begin{aligned} \iota(i)\iota(j) = \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \xi\varepsilon + \alpha \\ \varepsilon & \alpha \end{pmatrix} &= \begin{pmatrix} \xi\varepsilon & \xi\alpha \\ \alpha + \varepsilon & \xi\varepsilon \end{pmatrix} = \begin{pmatrix} \alpha & \xi\varepsilon + \alpha \\ \varepsilon & \alpha \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 1 & 0 \end{pmatrix} \\ &= \iota(j)(\iota(i) + \iota(1)). \end{aligned}$$ ∎

We can compute the first n coefficients of α in $K_\infty = k((\pi))$ in $\mathcal{O}(n^3)$ field operations over $\mathbb{F}_q$ by Newton iteration, or alternativly we can use the constructive proof of Proposition A.12.
Let $v_0 := [L(0,0)]$. We have

$$\mathrm{Stab}_\Gamma(v_0) = \mathrm{GL}_2(O_\infty)K_\infty^\star \cap \Gamma \supseteq \{a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & \xi \\ 1 & 1 \end{pmatrix} \mid a, b \in k, (a,b) \neq (0,0)\}.$$

Hence the vertex v_0 is projectively unstable and $\pi(v_0)$ is a terminal vertex of $\Gamma\backslash\mathcal{T}$. Let $v_1 := [L(1,0)]$. As in the case of q odd and $\mathrm{odd}(R) = 1$ we distinguish the cases $V_{q+1} = 0$ or $V_{q+1} \neq 0$. In the first case v_1 is also projectively unstable, $\pi(v_1)$ a terminal vertex of $\Gamma\backslash\mathcal{T}$ and $\Gamma\backslash\mathcal{T}$ consists of one edge connecting two terminal vertices. In the other case v_1 is projectively stable and we use it as the initial vertex for the algorithm 2.32. Hence Lemma 2.8 implies that in the n-th step of the algorithm we need to compare vertices of the form $[L(n, g(\pi))]$, where $g \in k[T]$ with $\deg(g) < n$ and $g(0) = 0$. As in Propostion 2.51 we can do this in $\mathcal{O}(n^4)$. We omit the proof here.

Proposition A.18 *(a) Given* $v = [L(n, g(\pi))]$ *and* $v' = [L(n, g'(\pi))]$ *as above there is an algorithm that computes* $\mathrm{Hom}_\Gamma(v', v)$ *in* $\mathcal{O}(n^4)$ *field operations over* $\mathbb{F}_q$.

(b) All $\gamma \in \mathrm{Hom}_\Gamma(v', v)$ *satisfy* $\|\gamma\| \leq n$.

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