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Diplom-Mathematiker Martin Sigl
aus Deggendorf

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Ramification filtration by moduli in higher-dimensional global class field theory

Gutachter: Prof. Dr. Alexander Schmidt

Abstract

In their approach to higher-dimensional global class field theory, Kato and Saito define the class group of a proper arithmetic scheme $\bar{X}$ as an inverse limit $C_{KS}(\bar{X}) = \varprojlim_{\mathcal{I}} C_{\mathcal{I}}(\bar{X})$ of certain Nisnevich cohomology groups $C_{\mathcal{I}}(\bar{X})$ taken over all coherent ideal sheaves $\mathcal{I} \neq 0$ of $\mathcal{O}_{\bar{X}}$. The ideal sheaves $\mathcal{I}$ should be regarded as higher-dimensional analogues of the classical moduli $\mathfrak{m}$ on a global field K, which induce a filtration of the idele class group C_K of K by the ray class groups $C_K/C_K^{\mathfrak{m}}$. In higher dimensions however, it is not clear how the induced filtration of the abelian fundamental group can be interpreted in terms of ramification.
In view of Wiesend's class field theory, we define an easier notion of moduli in higher dimensions only involving curves on the scheme. We then show that both notions agree for moduli that correspond to tame ramification.

Zusammenfassung

Kato und Saito definieren in ihrem Zugang zur höherdimensionalen globalen Klassenkörpertheorie die Klassengruppe eines eigentlichen arithmetischen Schemas $\bar{X}$ als einen über sämtliche kohärenten Idealgarben $\mathcal{I} \neq 0$ von $\mathcal{O}_{\bar{X}}$ gebildeten inversen Limes $C_{KS}(\bar{X}) = \varprojlim_{\mathcal{I}} C_{\mathcal{I}}(\bar{X})$ gewisser Nisnevich-Kohomologiegruppen $C_{\mathcal{I}}(\bar{X})$. Die Idealgarben $\mathcal{I}$ sollten als höherdimensionale Analoga der klassischen Erklärungsmoduln $\mathfrak{m}$ eines globalen Körpers K verstanden werden, welche eine Filtrierung der Idelklassengruppe C_K von K durch die Strahlklassengruppen $C_K/C_K^{\mathfrak{m}}$ induzieren. Im Höherdimensionalen ist es jedoch nicht klar, wie die induzierte Filtrierung der abelschen Fundamentalgruppe bezüglich Verzweigung zu interpretieren ist.
In Hinblick auf Wiesends Klassenkörpertheorie definieren wir einen einfacheren Begriff von höherdimensionalen Erklärungsmoduln, der lediglich die Kurven auf dem Schema beinhaltet. Anschließend zeigen wir, dass beide Begriffe übereinstimmen, wenn der Erklärungsmodul zu zahmer Verzweigung korrespondiert.

Contents

Introduction

Background

It is content of classical one-dimensional global class field theory that the idele class group C_K of a global field K has a canonical filtration by *moduli.* A *modulus* is a formal product

$$\mathfrak{m} = \prod_v v^{n_v}$$

taken over all places v of K with integer exponents $n_v \geq 0$, which are zero for all but finitely many v, and $n_v \in \{0, 1\}$ for real v and $n_v = 0$ for complex v. To $\mathfrak{m}$ one associates the *congruence subgroup* $C_K^{\mathfrak{m}} \subset C_K$ *mod* $\mathfrak{m}$ of the idele class group C_K of K.
If K is a number field, the *ray class group* $C_K/C_K^{\mathfrak{m}}$ *mod* $\mathfrak{m}$ is finite and has a classical ideal-theoretic interpretation. It is isomorphic to the quotient $J^{\mathfrak{m}}/H_0^{\mathfrak{m}}$ of fractional ideals $J^{\mathfrak{m}}$ of K coprime to $\mathfrak{m}$ (where $\mathfrak{m}$ is viewed as an ideal of K by dropping the archimedean places) modulo the subgroup $H_0^{\mathfrak{m}}$ of principal ideals coprime to $\mathfrak{m}$.
The subgroup $C_K^{\mathfrak{m}}$ of C_K corresponds to a finite abelian extension $K^{\mathfrak{m}}$ of the number field K, called the *ray class field mod* $\mathfrak{m}$. To every finite abelian extension $L|K$ there exists a modulus $\mathfrak{m}$ such that $L \subset K^{\mathfrak{m}}$. In particular, one has a filtration

$$\mathrm{Gal}(K^{ab}|K) = \varprojlim_{\mathfrak{m}} \mathrm{Gal}(K^{\mathfrak{m}}|K)$$

of the Galois group of the maximal abelian extension $K^{ab}|K$. The modulus $\mathfrak{m}$ carries information about the ramification properties of $K^{\mathfrak{m}}|K$. Given a finite subset S of the set of all places of K containing all archimedean places, let $K_S^{ab}|K$ be the maximal abelian extension of K unramified outside

S. Then, by only taking the limit over those moduli $\mathfrak{m} = \prod_v v^{n_v}$ with $n_v = 0$ if $v \notin S$, one obtains

$$\mathrm{Gal}(K_S^{ab}|K) = \varprojlim_{\mathfrak{m}} \mathrm{Gal}(K^{\mathfrak{m}}|K).$$

The ray class field mod $\mathfrak{m} = 1$ (i.e. the modulus $\mathfrak{m} = \prod_v v^{n_v}$, with $n_v = 0$ for all v) is the maximal abelian unramified extension of K and is called the *Hilbert class field* of K. The ray class field mod $\mathfrak{m} = \prod_v v^{n_v}$, with $n_v = 0$ for $v \notin S$ and $n_v = 1$ for $v \in S$, is the maximal abelian extension of K unramified outside S and tamely ramified in S.

The goal of developing a description of abelian coverings of a higher dimensional arithmetic scheme solely by intrinsic information of the scheme (i.e. "higher-dimensional global class field theory") was reached by Kato and Saito in their work [KaS]. For a connected, normal and proper scheme $\bar{X}$ over $\mathrm{Spec}\,\mathbb{Z}$, they define a *class group* $C_{KS}(\bar{X})$ of $\bar{X}$ and a *reciprocity map*

$$\rho_{\bar{X}} \colon C_{KS}(\bar{X}) \to \mathrm{Gal}(K^{ab}|K)$$

into the Galois group of the maximal abelian extension of the function field K of $\bar{X}$ (which, for simplicity, is assumed to have no embeddings $K \hookrightarrow \mathbb{R}$). Notably, the class group $C_{KS}(\bar{X})$ is *by definition* an inverse limit

$$C_{KS}(\bar{X}) = \varprojlim_{\mathcal{I} \neq 0} C_{\mathcal{I}}(\bar{X})$$

taken over all non-zero coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$, and the reciprocity map is defined as the limit of reciprocity maps

$$C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(K^{\mathcal{I}}|K)$$

into suitable quotients $\mathrm{Gal}(K^{\mathcal{I}}|K)$ of $\mathrm{Gal}(K^{ab}|K)$. The $\mathcal{I}$ should be thought of as higher-dimensional moduli and the $C_{\mathcal{I}}(\bar{X})$ as generalized ray class groups. If $\bar{X}$ is flat over $\mathrm{Spec}\,\mathbb{Z}$, the reciprocity map $\rho_{\bar{X}}$ is an isomorphism and induces an isomorphism

$$\varprojlim_{\mathcal{I}|_X = \mathcal{O}_X} C_{\mathcal{I}}(\bar{X}) \to \pi_1^{ab}(X)$$

for any non-empty regular open subscheme $X \subset \bar{X}$ by restricting the limit to all ideal sheaves $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$. For $\dim X = 1$ one recovers the classical theory for number fields described above. However, in higher dimensions, the constructions of Kato and Saito are quite involved and based upon Kato's class field theory for higher-dimensional local fields. This makes it rather difficult to relate the properties of $\mathcal{I}$ at the boundary $\bar{X} \setminus X$ to ramification behavior outside X.

The theory of Wiesend, which he started developing in [Wi1] and [Wi2], and was later completed by Kerz-Schmidt in [KeS2], uses a comparatively easier approach. The class group of a regular arithmetic scheme X is built upon information only coming from closed points and curves on X. In view of this theory, we define a different notion of higher-dimensional moduli by starting with a given normal compactification $\bar{X}$ of X and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$, and saying that a finite étale abelian covering $Y \to X$ is *ramified with Wiesend modulus* $\mathcal{I}$, if for any curve C on X the induced covering of $k(C)$ is ramified with modulus $\mathcal{I}_C$, obtained by pulling back $\mathcal{I}$ to the regular compactification of the normalization $\tilde{C}$ of C.

Results

First, we show that this definition indeed gives us an exhaustive filtration of $\pi_1^{ab}(X)$.

Proposition. *Let $X \in Sch(\mathbb{Z})$ be a connected regular scheme and let $\bar{X}$ be a normal compactification of X. Let $Y \to X$ be an abelian étale covering. Then there exists an ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$ such that $Y \to X$ is ramified with Wiesend modulus $\mathcal{I}$.*

It is then natural to ask, whether one can compare this filtration of $\pi_1^{ab}(X)$ with the one induced by the class field theory of Kato-Saito

$$\varprojlim_{\mathcal{I}|_X = \mathcal{O}_X} C_{\mathcal{I}}(\bar{X}) \to \pi_1^{ab}(X).$$

It turns out that we can make a comparison of the respective quotients of $\pi_1^{ab}(X)$ corresponding to a "tame modulus" $\mathcal{I}$. On the side of Kato-Saito, the modulus $\mathcal{I}$ gives us a condition for the *codimension one* points and in case of a tame modulus this leads towards the notion of *divisor-tameness.*

Theorem. *Let $X \in Sch(\mathbb{Z})$ be a connected regular scheme which is flat over* $\operatorname{Spec}\mathbb{Z}$ *and let $\bar{X}$ be a normal compactification of X such that $D = \bar{X} \setminus X$ is a square-free divisor on $\bar{X}$. Assume that $X(\mathbb{R}) = \emptyset$. Let K_D be the maximal abelian extension $L|K$ such that $X_L \to X$ is tamely ramified along the generic points of D. Then for the ideal sheaf $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$ the reciprocity map of Kato-Saito induces an isomorphism*

$$C_{\mathcal{I}}(\bar{X}) \xrightarrow{\sim} \operatorname{Gal}(K_D|K).$$

On the other side, the tame Wiesend modulus condition is related to the notion of *curve-tameness*. Although the induced condition on curves might be weaker than just tame, we can show the existence of sufficiently many good curves.

Theorem. *Let $X \in Sch(\mathbb{Z})$ be a connected regular scheme and let $\bar{X}$ be a regular compactification of X such that $D = \bar{X} \setminus X$ is a normal crossing divisor on $\bar{X}$. Then for an abelian étale covering $Y \to X$ the following are equivalent:*

(i) *$Y \to X$ is curve-tame.*

(ii) *$Y \to X$ is tamely ramified along D.*

(iii) *$Y \to X$ is ramified with Wiesend modulus $\mathcal{O}_{\bar{X}}(-D)$.*

The equivalence (i) $\Leftrightarrow$ (ii) (for not necessarily abelian coverings) is part of the main result in [KeS1]. Finally, when D is a normal crossing divisor on $\bar{X}$, we can compare $C_{\mathcal{I}}(\bar{X})$ and the quotient $C^h_{W,\mathcal{I}}(X)$ of the henselian Wiesend class group $C^h_W(X)$ corresponding to the maximal abelian covering of X ramified with Wiesend modulus $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$.

Theorem. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular, flat scheme over* $\operatorname{Spec}\mathbb{Z}$ *and let $\bar{X}$ be a regular compactification of X such that $D = \bar{X} \setminus X$ is a normal crossing divisor on $\bar{X}$. Assume that $X(\mathbb{R}) = \emptyset$ and put $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$. Then there is a canonical isomorphism*

$$C^h_{W,\mathcal{I}}(X) \xrightarrow{\sim} C_{\mathcal{I}}(\bar{X}).$$

Structure of this thesis

In chapter one we begin by recalling the theory of moduli in dimension one. Then we present results which are significant for understanding the theory of Kato and Saito. This includes Nisnevich cohomology, Milnor K-theory and Kato's class field theory for higher-dimensional henselian local fields.

In the second chapter, after reviewing the definition of the Kato-Saito class group, we turn towards the notion of Wiesend moduli and show the proposition on existence of moduli.

The construction of the reciprocity map of Kato-Saito will be the content of the first section of chapter three. Then we recall Wiesend's class field theory and show that there is a canonical map from the henselian Wiesend class group to the Kato-Saito class group. In the last section we focus on tame coverings and prove the theorems relating the different notions of tameness to tame moduli. Finally, we show the isomorphism between the corresponding quotients of the respective class groups.

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Notation

The group of units of a ring R is denoted by $R^\times$ and in case R is an integral domain, we write $Q(R)$ for its field of fractions. If R is a local ring, then R^h is its henselization.

The structure sheaf of a scheme X is denoted by $\mathcal{O}_X$. The function field of X is denoted by $k(X)$ and the residue field of a point x on X by $k(x)$. For $n \geq 0$ the set of points of dimension n on X is denoted by X_n, and the set of points of codimension n by X^n. The set of regular points of X is X^{reg}. The normalization of X in a finite field extension of L of $k(X)$ is denoted by $X_L \to X$.

We write $Sch(\mathbb{Z})$ for the category of separated schemes of finite type over $\operatorname{Spec}\mathbb{Z}$.

Zariski-closed subsets are always endowed with the induced reduced scheme structure. By an *ideal sheaf* of $\mathcal{O}_X$ for a scheme X, we always mean a quasi-coherent ideal sheaf. An *étale covering* is a finite étale morphism.

Chapter 1

Preliminaries

In this chapter we recollect basic facts about the theory of moduli in one-dimensional class field theory. Furthermore, we gather some results concerning Nisnevich cohomology, Milnor K-theory and Kato's class field theory for higher-dimensional local fields that will be needed for studying the class field theory of Kato-Saito.

1.1 One-dimensional class field theory

We begin by recalling the classical theory of moduli and conductors on global fields as presented in [AT] Ch. 8, for instance.

Moduli

Let K be a global field. For a place v of K let K_v be the completion of K at v. If v is finite, let $\mathcal{O}_v \subset K_v$ be its ring of integers and let $U_v^n \subset \mathcal{O}_v^\times$ be the subgroup of n-th principal units, $n \geq 1$. We define $U_v^0 = \mathcal{O}_v^\times$ for finite and infinite v, and $U_v^1 = \mathbb{R}_{>0} \subset \mathbb{R}^\times$ if v is real. Denote by I_K (resp. C_K) the idele group (resp. idele class group) of K and let

$$\rho_K \colon C_K \to \mathrm{Gal}(K^{ab}|K)$$

be the reciprocity map.

Definition. A *modulus* $\mathfrak{m}$ on K is a formal product

$$\mathfrak{m} = \prod_v v^{n_v}$$

taken over all places v of K, where the n_v are integers ≥ 0, which are zero for all but finitely many v. We further require that $n_v \in \{0,1\}$ for real v and $n_v = 0$ for complex v.

So if K is a number field with ring of integers $\mathcal{O}_K$, a modulus $\mathfrak{m}$ on K can be viewed as an integral ideal of $\mathcal{O}_K$ together with a subset of the set of real places of K. If $K = k(X)$ for a smooth projective curve X over a finite field, then $\mathfrak{m}$ is the same as an effective divisor on X or equivalently, a non-zero ideal sheaf of $\mathcal{O}_X$.

To a modulus $\mathfrak{m}$ we associate the subgroup

$$I_K^{\mathfrak{m}} = \prod_v U_v^{n_v}$$

of the idele group I_K and denote its image $I_K^{\mathfrak{m}} K^\times / K^\times$ under the projection $I_K \to C_K$ by $C_K^{\mathfrak{m}}$. One has to distinguish between the number field and the function field case.

If K is a number field, the subgroup $C_K^{\mathfrak{m}} \subset C_K$ is of finite index and hence corresponds to a finite abelian extension $K^{\mathfrak{m}}$ of K. The reciprocity map ρ_K induces an isomorphism

$$C_K / C_K^{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathfrak{m}}|K).$$

To every finite abelian extension L of K there exists a modulus $\mathfrak{m}$ such that $L \subset K^{\mathfrak{m}}$ and so we obtain an isomorphism

$$\varprojlim_{\mathfrak{m}} C_K / C_K^{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Gal}(K^{ab}|K).$$

In the function field case the subgroup $C_K^{\mathfrak{m}} \subset C_K$ has infinite index. It is however contained as a subgroup of finite index in the degree zero part $C_K^0 = \ker(\deg\colon C_K \to \mathbb{Z})$ of C_K and we have a short exact sequence

$$0 \to C_K^0 / C_K^{\mathfrak{m}} \to C_K / C_K^{\mathfrak{m}} \to \mathbb{Z} \to 0.$$

Hence $C_K^{\mathfrak{m}}$ corresponds to an abelian extension $K^{\mathfrak{m}}$ of K which is finite over the $\hat{\mathbb{Z}}$-subextension $K\bar{\mathbb{F}}|K$, where $\bar{\mathbb{F}}$ denotes the algebraic closure of the constant field $\mathbb{F}$ of K. If $L|K$ is a finite abelian extension, then there exists a modulus $\mathfrak{m}$ such that $K^{\mathfrak{m}}$ contains $L\bar{\mathbb{F}}$ and, in particular, L. This is summed up by the short exact sequence

$$0 \to \varprojlim_{\mathfrak{m}} C_K^0/C_K^{\mathfrak{m}} \to \mathrm{Gal}(K^{ab}|K) \to \hat{\mathbb{Z}} \to 0.$$

The conductor
There is an obvious notion of divisibility for moduli and $\mathfrak{m}' | \, \mathfrak{m}$ implies that $K^{\mathfrak{m}'} \subset K^{\mathfrak{m}}$.

Definition. Let $L|K$ be a finite abelian extension. The *conductor* $\mathfrak{f}_{L|K}$ of $L|K$ is the g.c.d. of all moduli $\mathfrak{m}$ with $L \subset K^{\mathfrak{m}}$.

Thus we have $L \subset K^{\mathfrak{m}}$ if and only if $\mathfrak{f}_{L|K} | \, \mathfrak{m}$. Let v be a place of K. The following definition is independent of the choice of a place w of L above v.

Definition. If v is a finite place, the *local conductor* $\mathfrak{f}_{L|K,v}$ of $L|K$ at v is the smallest non-negative integer n such that U_v^n is contained in the norm group $N_{L_w|K_v}(L_w^\times)$. For infinite v we set $\mathfrak{f}_{L|K,v} = 0$ if $L_w = K_v$, and $\mathfrak{f}_{L|K,v} = 1$ if $L_w \neq K_v$.

Local and global conductor are related by the formula

$$\mathfrak{f}(L|K) = \prod_v v^{\mathfrak{f}_{L|K,v}},$$

which immediately follows from the equality

$$N_{L|K}(C_L) \cap K_v^\times = N_{L_w|K_v}(L_w^\times)$$

([AT] 8.1, Theorem 2). Now, by local class field theory we know that

$$\begin{aligned} v \text{ is unramified in } L_w \quad &\Leftrightarrow \quad U_v^0 \subset N_{L_w|K_v}(L_w^\times) \quad \Leftrightarrow \quad \mathfrak{f}_{L|K,v} = 0 \,, \\ v \text{ is tamely ramified in } L_w \quad &\Leftrightarrow \quad U_v^1 \subset N_{L_w|K_v}(L_w^\times) \quad \Leftrightarrow \quad \mathfrak{f}_{L|K,v} \leq 1 \end{aligned}$$

(cf. [Se] Ch. V). Hence we get the

Corollary 1.1.1. *A finite abelian extension $L|K$ of global fields is unramified if and only if $\mathfrak{f}_{L|K} = 1$. It is tamely ramified if and only if $\mathfrak{f}_{L|K}$ is square-free.*

1.2 Nisnevich cohomology

In this section we recall the existence of the coniveau spectral sequence for Nisnevich cohomology.

Let X be a finite-dimensional noetherian scheme and let $\mathcal{F}$ be a Nisnevich sheaf on X. For a closed subscheme Z of X let $H^n_Z(X_{Nis}, \mathcal{F})$ denote the n-th Nisnevich cohomology group of X with support in Z. For an arbitrary point $x \in X$ we define

$$H^n_x(X_{Nis}, \mathcal{F}) = \varinjlim_{U} H^n_{U\cap\overline{\{x\}}}(U_{Nis}, \mathcal{F}),$$

where the direct limit is taken over all Zariski-open subschemes U of X containing x.

Lemma 1.2.1. *For any $n \geq 0$ we have an isomorphism*

$$H^n_x(X_{Nis}, \mathcal{F}) \cong H^n_x((X^h_x)_{Nis}, \mathcal{F}),$$

where $X^h_x = \operatorname{Spec} \mathcal{O}^h_{X,x}$.

Proof. [Ni] 1.29.2. □

The local cohomology groups $H^*_x(X_{Nis}, \mathcal{F})$ satisfy certain axioms that imply the existence of the so-called coniveau spectral sequence:

Proposition 1.2.2. *Let $Z^p(X)$ be the set of closed subschemes of X of codimension $\geq p$. The filtration by coniveau on $H^*(X_{Nis}, \mathcal{F})$ defined by*

$$F^p H^*(X_{Nis}, \mathcal{F}) = \bigcup_{Z \in Z^p(X)} \ker\left[H^*(X_{Nis}, \mathcal{F}) \to H^*((X \setminus Z)_{Nis}, \mathcal{F})\right]$$

gives rise to a spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p} H^{p+q}_x(X_{Nis}, \mathcal{F}) \Rightarrow H^{p+q}(X_{Nis}, \mathcal{F}).$$

Proof. [Ni] 1.31. □

Corollary 1.2.3. *We have $H^n(X_{Nis}, \mathcal{F}) = 0$ for all $n > \dim X$, i.e. the cohomological dimension of X_{Nis} is at most* $\dim X$.

Proof. This follows by induction on $\dim X$ using the above spectral sequence and Lemma 1.2.1. (cf. [Ni] 1.32). □

By this vanishing result the coniveau spectral sequence yields the following isomorphism.

Corollary 1.2.4. *Let $d = \dim X$. Then we have an isomorphism*

$$H^d(X_{Nis}, \mathcal{F}) \simeq \operatorname{coker}\big[\bigoplus_{x\in X^{d-1}} H_x^{d-1}(X_{Nis}, \mathcal{F}) \to \bigoplus_{x\in X^d} H_x^d(X_{Nis}, \mathcal{F})\big].$$

1.3 Milnor K-theory

In this section we summarize some properties of (relative) Milnor K-groups and -sheaves, particularly the existence of the norm map and the tame symbol.

Milnor K-groups

For a commutative ring R with unit and $n \geq 1$ let $K_n^M(R)$ be the n-th Milnor K-group of R defined as the quotient of

$$\underbrace{R^\times \otimes \cdots \otimes R^\times}_{n \text{ times}}$$

by the subgroup generated by all elements $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_j = 1$ for some $i \neq j$. Set $K_0^M(R) = \mathbb{Z}$. For an ideal $I \subset R$ define

$$K_n^M(R, I) = \ker\big[K_n^M(R) \to K_n^M(R/I)\big].$$

The residue class of an element $a_1 \otimes \cdots \otimes a_n$ under $(R^\times \otimes \cdots \otimes R^\times) \to K_n^M(R)$ is denoted by $\{a_1, \ldots, a_n\}$.

Lemma 1.3.1. *Let R be a finite product of local rings and let $I \subset R$ be an ideal. Then $K_n^M(R, I)$ coincides with the subgroup of $K_n^M(R)$ generated by all symbols $\{a_1, \ldots, a_n\}$, with $a_i \in K_1^M(R, I) = \ker(R^\times \to (R/I)^\times)$ for some $1 \leq i \leq n$.*

Proof. [KaS] Lemma 1.3.1. □

For a noetherian scheme X and $n \geq 1$ define the Nisnevich sheaf $\mathcal{K}_n^M(\mathcal{O}_X)$ as the quotient of

$$\underbrace{\mathcal{O}_X^\times \otimes \cdots \otimes \mathcal{O}_X^\times}_{n \text{ times}}$$

by the subsheaf generated locally by sections $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_j = 1$ for some $i \neq j$. We define $\mathcal{K}_0^M(\mathcal{O}_X)$ as the constant sheaf $\mathbb{Z}$. For an ideal $\mathcal{I} \subset \mathcal{O}_X$ define

$$\mathcal{K}_n^M(\mathcal{O}_X, \mathcal{I}) = \ker \left[\mathcal{K}_n^M(\mathcal{O}_X) \to i_* \mathcal{K}_n^M(\mathcal{O}_Y)\right],$$

where Y denotes the closed subscheme of X defined by $\mathcal{I}$ and $i\colon Y \to X$ the closed immersion.

Norm map and tame symbol

Let $L|K$ be a finite extension of fields. It is shown in [Ka2] § 1.7, that there is a well-defined norm homomorphism

$$N_{L|K}\colon K_n^M(L) \to K_n^M(K),$$

which is the usual norm

$$N_{L|K}\colon L^\times \to K^\times$$

for $n = 1$ and satisfies the projection formula

$$\{a, N_{L|K}(b)\} = N_{L|K}(\{a, b\})$$

for $a \in K_{n-m}^M(K)$ and $b \in K_m^M(L)$.

Now let R be a discrete valuation ring with residue field k and fraction field K. By [BT] I, § 4, there is an epimorphism

$$\partial\colon K_n^M(K) \to K_{n-1}^M(k),$$

$n \geq 1$, called the *tame symbol*, which is the valuation for $n = 1$ and which is uniquely characterized by the property that for $a_1, \ldots, a_{n-1} \in R^\times$ we have

$$\partial(\{a_1, \ldots, a_{n-1}, \pi\}) = \{\overline{a}_1, \ldots, \overline{a}_{n-1}\},$$

where π is a uniformizer of R and $\overline{a}_i$ is the residue class of a_i in k. We define $U^0(K_n^M(K)) \subset K_n^M(K)$ as the subgroup generated by all symbols $\{a_1, \ldots, a_n\}$ with $a_i \in R^\times$ for $i = 1, \ldots, n$. Then we have

$$U^0(K_n^M(K)) = \ker(\partial).$$

([BT] I, §4, Proposition 4.5).

1.4 Class field theory of higher-dimensional henselian local fields

In the following we present Kato's class field theory for higher-dimensional complete local fields. It requires some extra effort to obtain the corresponding results for higher-dimensional excellent henselian local fields.

Higher dimensional henselian local fields

Definition. A valuation v on a field K is called a *discrete valuation of rank n* if its value group is isomorphic (as an ordered group) to $\mathbb{Z}^n$ (endowed with the lexicographic order).

If V is a discrete valuation ring of rank n of K, i.e. V is the valuation ring of a discrete valuation of rank n of K, then V has $n+1$ distinct prime ideals

$$\mathfrak{p}_0 \supsetneqq \cdots \supsetneqq \mathfrak{p}_n = 0.$$

For $0 \le i \le n$, the localization V_i of V at $\mathfrak{p}_i$ is itself a valuation ring of K of rank $n-i$, and we have an inclusion of valuation rings

$$V = V_0 \subset \cdots \subset V_n = K.$$

Let k_i be the residue field of V_i. Then for any $i < j \le n$ the image of V_i in k_j is a discrete valuation ring of k_j of rank $j-i$. The image $\bar{V}_i$ of V_i in k_{i+1} is a discrete valuation ring of rank 1 with field of fractions k_{i+1} and residue field k_i.

Definition. A field K is called *n-dimensional henselian local field* if it is the field of fractions of a henselian discrete valuation ring of rank n whose residue field is finite.

There is an inductive henselization process $V \mapsto V^h$ for a discrete valuation ring V of rank n (cf. [KaS], 3.1). If V has a finite residue field, the field of fractions of V^h is an n-dimensional henselian local field and the ring V is henselian if and only if for all $0 \le i < n$ the rings $\bar{V}_i$ are henselian ([Ri] F, Proposition 9). Hence one may also define higher-dimensional henselian local fields as follows: A 0-dimensional henselian local field is a finite field and an $(n+1)$-dimensional henselian local field is a field which is henselian under a discrete valuation of rank 1 whose residue field is an n-dimensional henselian local field. Therefore, to an n-dimensional henselian local field K we can associate a sequence

$$K = k_n, k_{n-1}, \dots, k_0$$

of residue fields and k_i is an i-dimensional henselian local field for each $0 \leq i \leq n$. By abuse of notation we will sometimes refer to k_{n-1} as the residue field of K. In the following, we call K an *excellent* henselian local field if for all i the discrete valuation rings $\bar{V}_i$ are excellent.

The reciprocity map

For an arbitrary field K, an integer m prime to the characteristic of K and $n \geq 0$ consider the Galois cohomology group

$$H^n(K, \mathbb{Z}/m\mathbb{Z}(n)),$$

where

$$\mathbb{Z}/m\mathbb{Z}(n) = \mu_m^{\otimes n}$$

denotes the n-th Tate twist of $\mathbb{Z}/m\mathbb{Z}$ and μ_m the group of m-th roots of unity of K^{sep}. We have a homomorphism

$$h^n_{m,K} \colon \underbrace{K^\times \times \cdots \times K^\times}_{n \text{ times}} \to H^n(K, \mathbb{Z}/m\mathbb{Z}(n)),$$

which for $n = 1$ is defined as the projection

$$K^\times \to K^\times/(K^\times)^m$$

followed by the identification

$$K^\times/(K^\times)^m \cong H^1(K, \mathbb{Z}/m\mathbb{Z}(1))$$

given by Kummer theory, and which for $n > 1$ is defined as the n-fold cup-product $h^1_{m,K} \cup \cdots \cup h^1_{m,K}$. We define $h^0_{m,K}$ to be the canonical projection

$$\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} = H^0(K, \mathbb{Z}/m\mathbb{Z}(0)).$$

For any $n \geq 0$ the map $h^n_{m,K}$ factors over $K^M_n(K)$ and the induced homomorphism

$$h^n_{m,K} \colon K^M_n(K) \to H^n(K, \mathbb{Z}/m\mathbb{Z}(n))$$

is called the *Galois symbol*. If $\mathrm{char}(K) = p > 0$ and $i \geq j$ we define

$$H^i(K, \mathbb{Z}/p^r\mathbb{Z}(j)) := H^{i-j}(K, W_r\Omega^j_{K,log}),$$

where $W_r\Omega^j_{K,log}$ is the logarithmic part of the de Rham-Witt complex $W_r\Omega^j_K$ (cf. [Il] I. 5.7). In the case $r = 1$ we have the description

$$H^n(K, \mathbb{Z}/p\mathbb{Z}(n)) = \ker\left[\Omega^n_K \stackrel{F-1}{\to} \Omega^n_K/d\Omega^{n-1}_K\right],$$

where

$$\Omega^n_K = \bigwedge\nolimits^n_K \Omega^1_{K|\mathbb{Z}}$$

is the n-th exterior power of the differential module $\Omega^1_{K|\mathbb{Z}}$,

$$d\colon \Omega^{n-1}_K \to \Omega^n_K$$

the differential, and

$$F\colon \Omega^n_K \to \Omega^n_K/d\Omega^{n-1}_K$$

is defined by

$$F(a\frac{a_1}{da_1} \wedge \ldots \wedge \frac{a_n}{da_n}) = a^p \frac{a_1}{da_1} \wedge \ldots \wedge \frac{a_n}{da_n},$$

$a \in K$, $a_1, \ldots, a_n \in K^\times$ (cf. [Ka3]). We have the *differential symbol*

$$h^n_{p^r,K}\colon K^M_n(K) \to H^n(K, \mathbb{Z}/p^r\mathbb{Z}(n)),$$

which for $r = 1$ is given by

$$\{a_1, \cdots, a_n\} \mapsto \frac{a_1}{da_1} \wedge \ldots \wedge \frac{a_n}{da_n}.$$

Finally, for $\mathrm{char}(K) = p > 0$ and $m = m'p^r$ with $(m', p) = 1$, define

$$H^n(K, \mathbb{Z}/m\mathbb{Z}(n)) := H^n(K, \mathbb{Z}/m'\mathbb{Z}(n)) \oplus H^n(K, \mathbb{Z}/p^r\mathbb{Z}(n))$$

and

$$h^n_{m,K}\colon K^M_n(K) \to H^n(K, \mathbb{Z}/m\mathbb{Z}(n))$$

as the map induced by $h^n_{m',K}$ and $h^n_{p^r,K}$. Define

$$H^i(K, \mathbb{Q}/\mathbb{Z}(j)) := \varinjlim_m H^i(K, \mathbb{Z}/m\mathbb{Z}(j)).$$

The cup-product induces a pairing

$$H^1(K,\mathbb{Q}/\mathbb{Z}) \times K_n^M(K) \to H^{n+1}(K,\mathbb{Q}/\mathbb{Z}(n)).$$

The fundamental property of higher dimensional excellent henselian local fields, which generalizes the isomorphism

$$Br(K) = H^2(K,\mathbb{Q}/\mathbb{Z}(1)) \cong \mathbb{Q}/\mathbb{Z}$$

for a one-dimensional henselian local field K, is as follows.

Theorem 1.4.1. *Let K be an n-dimensional excellent henselian local field. Then*

$$H^{n+1}(K,\mathbb{Q}/\mathbb{Z}(n)) \cong \mathbb{Q}/\mathbb{Z}.$$

Proof. Let $K = k_n, k_{n-1}, \ldots, k_0$ be the associated sequence of residue fields. One shows that

$$H^{n+1}(k_n,\mathbb{Q}/\mathbb{Z}(n)) \cong H^n(k_{n-1},\mathbb{Q}/\mathbb{Z}(n-1)).$$

This is done by Kato in [Ka2] § 1.1, Theorem 2, and § 3.2, Lemma 3, for the case of an n-dimensional *complete* local field (i.e. each k_i is complete with respect to the induced discrete valuation of rank 1). The excellent henselian case follows by an argument explained in [KaS] below Theorem 3.5. The result then follows by induction:

$$H^{n+1}(k_n,\mathbb{Q}/\mathbb{Z}(n)) \cong \cdots \cong H^2(k_1,\mathbb{Q}/\mathbb{Z}(1)) \cong H^1(k_0,\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

□

By this theorem, the pairing

$$H^1(K,\mathbb{Q}/\mathbb{Z}) \times K_n^M(K) \to H^{n+1}(K,\mathbb{Q}/\mathbb{Z}(n)) \cong \mathbb{Q}/\mathbb{Z}$$

induces a map

$$\rho_K \colon K_n^M(K) \to \mathrm{Hom}(H^1(K,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) = \mathrm{Gal}(K^{ab}|K),$$

called the *reciprocity map.* It has the following properties.

Theorem 1.4.2. *Let K be an n-dimensional excellent henselian local field with residue field $k = k_{n-1}$ and let $L|K$ be a finite extension.*

(i) *There is a commutative diagram*

$$\begin{array}{ccc} K_n^M(K) & \xrightarrow{\rho_K} & \mathrm{Gal}(K^{ab}|K) \\ {\scriptstyle\partial}\downarrow & & \downarrow \\ K_{n-1}^M(k) & \xrightarrow{\rho_k} & \mathrm{Gal}(k^{ab}|k) \end{array}$$

where the left vertical map is the tame symbol and the right vertical map is the restriction to the Galois group of the residue field.

(ii) *The diagram*

$$\begin{array}{ccc} K_n^M(L) & \xrightarrow{\rho_L} & \mathrm{Gal}(L^{ab}|L) \\ {\scriptstyle N_{L|K}}\downarrow & & \downarrow \\ K_n^M(K) & \xrightarrow{\rho_K} & \mathrm{Gal}(K^{ab}|K) \end{array}$$

commutes, where the left vertical map is the norm of Milnor K-theory and the right vertical map is the restriction.

(iii) *The diagram*

$$\begin{array}{ccc} K_n^M(K) & \xrightarrow{\rho_K} & \mathrm{Gal}(K^{ab}|K) \\ \downarrow & & \downarrow \\ K_n^M(L) & \xrightarrow{\rho_L} & \mathrm{Gal}(L^{ab}|L) \end{array}$$

commutes, where the right vertical map is the transfer map.

Proof. For higher-dimensional complete local fields this is shown in [Ka2] § 3.2, Corollary 1 and 2. Only the definition of the reciprocity map is needed and so the same arguments hold for the excellent henselian case. □

In order to deduce the *reciprocity law* for higher-dimensional excellent henselian local fields from the complete case, some additional arguments are necessary.

Theorem 1.4.3. *Let R be an excellent henselian discrete valuation ring and $\hat{R}$ its completion. Let $X \to \operatorname{Spec} R$ be a scheme of finite type. Then we have*

$$X(\hat{R}) \neq \emptyset \;\Rightarrow\; X(R) \neq \emptyset.$$

Proof. Since R is excellent, the extension $Q(\hat{R})|Q(R)$ is separable ([Liu] 8.2, Corollary 2.40) and the claim follows from [Gr] 1, Corollary 2 to Theorem 1. □

Corollary 1.4.4. *Let R be an excellent henselian discrete valuation ring and $\hat{R}$ its completion. Let $K = Q(R)$ and $\hat{K} = Q(\hat{R})$. Assume that $L|K$ is a finite separable extension and $\hat{L}$ the completion of L. Then for any $x \in K^\times$ we have*

$$x \in N_{\hat{L}|\hat{K}}(\hat{L}^\times) \;\Rightarrow\; x \in N_{L|K}(L^\times).$$

Proof. Let $e_{\hat{L}|\hat{K}}$ (resp. $e_{L|K}$) be the ramification index of $\hat{L}|\hat{K}$ (resp. $L|K$). Then, since R is excellent, we have $[\hat{L} : \hat{K}] = [L : K]$ and $e_{\hat{L}|\hat{K}} = e_{L|K}$. Choose a uniformizer π_L of L which also is a uniformizer of $\hat{L}$. After multiplying x with a suitable power of $N_{L|K}(\pi_L)$, we may assume that $x \in R^\times$ and that $x \in N_{\hat{L}|\hat{K}}(\hat{S}^\times) \subset \hat{R}^\times$, where $\hat{S}$ denotes the valuation ring of $\hat{L}$. Now let $\mathbb{G}_{m,R}$ (resp. $\mathbb{G}_{m,S}$) be the group scheme $\mathbb{G}_m$ over $\operatorname{Spec} R$ (resp. over $\operatorname{Spec} S$, where $S = \hat{S} \cap L$ is the valuation ring of L). Let $W_{S|R}(\mathbb{G}_{m,S})$ be the Weil-restriction of $\mathbb{G}_{m,S}$ from $\operatorname{Spec} S$ to $\operatorname{Spec} R$ and

$$N\colon W_{S|R}(\mathbb{G}_{m,S}) \to \mathbb{G}_{m,R}$$

the norm morphism. Define X as the base change

$$\begin{array}{ccc} X & \longrightarrow & W_{S|R}(\mathbb{G}_{m,S}) \\ \downarrow & & \downarrow N \\ \operatorname{Spec} R & \xrightarrow{\;x\;} & \mathbb{G}_{m,R} \end{array}$$

where the bottom arrow is induced by the multiplication by x. The claim then follows from the above theorem. □

In the following, for a discrete valuation ring R with field of fractions K, we denote the subgroup of n-th principal units of $R^\times$ by U_K^n, $n \geq 1$.

Corollary 1.4.5. *Let R be an excellent henselian discrete valuation ring and L a finite separable extension of $K = Q(R)$. Then for $n \gg 0$ we have $U_K^n \subset N_{L|K}(L^\times)$.*

Proof. We can pass to the completions $\hat{L}|\hat{K}$. The claim then follows from [Se] Ch. V and [Ka1] § 1. □

In [Ka1], Kato introduces the notion of a B_n-field:

Definition. Let $n \geq 0$. A field K is called *B_n-field* if for any finite extension $E|K$ and any finite extension $F|E$, the norm $N_{F|E}\colon K_n^M(F) \to K_n^M(E)$ is surjective.

Finite fields are obviously B_1-fields. Hence by the following proposition, an n-dimensional complete local field is a B_{n+1}-field.

Proposition 1.4.6. *Let K be a complete discretely valued field with residue field k. Then for $n \geq 0$ the following are equivalent.*

(i) *k is a B_n-field.*

(ii) *K is a B_{n+1}-field.*

Proof. [Ka2] § 3.3, Proposition 2. □

Lemma 1.4.7. *Let K be a B_n-field with* $\mathrm{char}(K) = p > 0$. *Then we have $[K : K^p] \leq p^n$.*

Proof. [Ka2] § 3.3, Lemma 11. □

Proposition 1.4.8. *Let K be the field of fractions of an excellent henselian discrete valuation ring with residue field k. Assume that k is a B_n-field. Then K is a B_{n+1}-field. In particular, an n-dimensional excellent henselian local field is a B_{n+1} field.*

Proof. Let $\hat{K}$ be the completion of K. By the above proposition, $\hat{K}$ is a B_{n+1}-field. We have to show that for finite extensions $F|E$ of K the norm $N_{F|E}\colon K_n^M(F) \to K_n^M(E)$ is surjective. We can assume that $E = K$ and that $[F : K]$ is a prime number. If $F|K$ is separable, the claim follows from Corollary 1.4.4. Assume that $\mathrm{char}(K) = p > 0$ and that $F|K$ is purely inseparable of degree p. Since $\hat{K}$ is a separable extension of K, we have

$$x \in \hat{K}^p \;\Rightarrow\; x \in K^p$$

for any $x \in K$. and therefore $[K : K^p] \leq [\hat{K} : \hat{K}^p]$. Lemma 1.4.7 implies that $[K : K^p] = p^m$ with $m \leq n+1$. By [Ka2] § 3.3, Lemma 12, the norm

$N_{F|K}\colon K_m^M(F) \to K_m^M(K)$ is surjective. Hence $N_{F|K}\colon K_i^M(F) \to K_i^M(K)$ is surjective for any $i \geq m$. □

With this in hand, we can prove the reciprocity law for higher excellent henselian local fields.

Theorem 1.4.9. *Let K be an n-dimensional excellent henselian local field. Then for any finite abelian extension $L|K$ the reciprocity map*

$$\rho_K\colon K_n^M(K) \to \mathrm{Gal}(L|K)$$

induces an isomorphism

$$K_n^M(K)/N_{L|K}(K_n^M(L)) \xrightarrow{\sim} \mathrm{Gal}(L|K).$$

Proof. One has to revisit Kato's proof of this statement in the complete case ([Ka2] § 3.1, Theorem 1 (1)). With Corollary 1.4.5 and Proposition 1.4.8 one shows the assertions (A) - (D) ([Ka2] § 3.3, proof of Proposition 2 and Theorem 1) which then imply assertion (H). □

Tamely ramified extensions

By Lemma 1.4.7 we have $[K : K^p] \leq p^{n+1}$ for an n-dimensional excellent henselian local field K of characteristic $p > 0$. In fact, we have:

Lemma 1.4.10. *Let K be an n-dimensional excellent henselian local field with $\mathrm{char}(K) = p > 0$. Then $[K : K^p] \leq p^n$.*

Proof. Finite fields are perfect, hence the statement is true for $n = 0$. Now assume that $n \geq 1$ and that the statement holds for the $(n-1)$-dimensional residue field k of K. Let us first consider the case when K is complete. It follows from the structure theory for equicharacteristic complete discretely valued fields that K is the field of formal Laurent series $k((t))$. If $x_1, \ldots, x_m$ is a system of representatives of k/k^p, $m \leq p^{n-1}$, then $(x_i t^j)_{i,j}$ with $1 \leq i \leq m$ and $0 \leq j \leq p-1$ is a system of representatives of K/K^p, hence $[K : K^p] \leq p^n$. The excellent henselian case follows from the inequality $[K : K^p] \leq [\tilde{K} : \tilde{K}^p]$, where $\tilde{K}$ is the completion (as n-dimensional local field) of K. □

Corollary 1.4.11. *Let K be an n-dimensional excellent henselian local field with $\mathrm{char}(K) = p > 0$. Then the following holds.*

(i) $K_{n+1}^M(K)$ *is* p*-divisible.*

(ii) *For a finite, purely inseparable extension* $E|K$ *the norm*

$$N_{E|K}\colon K_n^M(E) \to K_n^M(K)$$

is surjective.

Proof. Assertion (i) follows from [Ka2] § 1.3, Lemma 7, and (ii) from [Ka2] § 3.3, Lemma 12. □

Next, we give a rather self-contained proof of the fact that for tamely ramified extensions of henselian local fields the principal one-units are norm elements.

Lemma 1.4.12. *Let* K *be a field which is henselian under a discrete valuation (of rank* 1*). Let* $L|K$ *be a finite tamely ramified Galois extension. Then we have*

$$N_{L|K}(U_L^1) = U_K^1.$$

Proof. Let us first consider the extension $L|K'$, where $K'|K$ is the maximal unramified subextension of $L|K$. We want to show that the Tate cohomology group

$$\hat{H}^0(L|K', U_L^1) = U_{K'}^1/N_{L|K'}(U_L^1)$$

vanishes. Let $m = \#G_{L|K'}$. Since $\hat{H}^0(L|K', U_L^1)$ is an m-torsion group it is enough to show that the multiplication by m

$$\hat{H}^0(L|K', U_L^1) \xrightarrow{m} \hat{H}^0(L|K', U_L^1)$$

is an isomorphism. For any $n \geq 1$ (after choosing a uniformizer of L) we have an exact sequence

$$0 \to U_L^{n+1} \to U_L^n \to \ell^+ \to 0$$

of $\mathrm{Gal}(L|K')$-modules. Here ℓ^+ denotes the additive group of the residue field ℓ of L. Since m is prime to the characteristic of ℓ we have

$$\hat{H}^i(L|K', \ell^+) = 0$$

for any $i \in \mathbb{Z}$, hence we get an isomorphism

$$\hat{H}^i(L|K', U_L^n) \cong \hat{H}^i(L|K', U_L^1).$$

for any $n \geq 1$. By [Ne2] § 2, Lemma 3.5, (which only uses the fact that L is henselian and neither the completeness of L nor the finiteness of the residue field) for any $m \in \mathbb{N}$ there exists an $n \gg 0$ such that taking the m-th power induces an isomorphism

$$U_L^n \xrightarrow{\sim} U_L^{n+w(m)},$$

where w denotes the valuation on L. Hence the right vertical arrow in the commutative diagram

$$\begin{array}{ccc} \hat{H}^0(L|K', U_L^n) & \xrightarrow{\sim} & \hat{H}^0(L|K', U_L^1) \\ {\scriptstyle m}\downarrow\wr & & \downarrow{\scriptstyle m} \\ \hat{H}^0(L|K', U_L^{n+w(m)}) & \xrightarrow{\sim} & \hat{H}^0(L|K', U_L^1) \end{array}$$

is an isomorphism.

In order to show the claim for the unramified extension $K'|K$ we have to argue differently, because the above reasoning does not hold when K has positive characteristic dividing the order of $\mathrm{Gal}(K'|K)$.

Recall the following property of henselian local rings. Let X be a smooth scheme over a henselian local ring R and residue field k. Then the canonical map

$$X(R) \to X(k)$$

is surjective. This follows from [Mi] I, Proposition 3.42 (b) and Theorem 4.2 (d'). In particular, if $Y \to X$ is a smooth morphism of schemes over $\mathrm{Spec}\, R$, we have a surjection

$$Y_R(R) \to Y_R(k),$$

where $Y_R = Y \times_X \mathrm{Spec}\, R$.

Now let us return to the unramified extension $K'|K$ and consider the group schemes $\mathbb{G}_m$ and $W_{\mathcal{O}_{K'}|\mathcal{O}_K}(\mathbb{G}_m)$ over $\mathrm{Spec}\, \mathcal{O}_K$. Here $\mathcal{O}_K$ (resp. $\mathcal{O}_{K'}$) denotes the valuation ring of K (resp. K') and $W_{\mathcal{O}_{K'}|\mathcal{O}_K}(\mathbb{G}_m)$ is again the Weil-restriction of $\mathbb{G}_m$ from $\mathrm{Spec}\, \mathcal{O}_{K'}$ to $\mathrm{Spec}\, \mathcal{O}_K$. The norm

$$N_{K'|K}|_{\mathcal{O}_{K'}} \colon \mathcal{O}_{K'} \to \mathcal{O}_K$$

can be recovered from the norm

$$N \colon W_{\mathcal{O}_{K'}|\mathcal{O}_K}(\mathbb{G}_m) \to \mathbb{G}_m$$

by taking $\mathcal{O}_K$-rational points. The latter can be shown to be a smooth map since

$$\operatorname{Spec} \mathcal{O}_{K'} \to \operatorname{Spec} \mathcal{O}_K$$

is étale. The above remark then implies that for any $x \in \mathcal{O}_K$ whose reduction $\bar{x}$ to the residue field k is in the image of the norm map

$$N_{k'|k} \colon (k')^\times \to k^\times,$$

say $\bar{x} = N_{k'|k}(\bar{y})$, there exists a lift $y \in \mathcal{O}_{K'}$ of $\bar{y}$ with $N_{K'|K}(y) = x$. This applies in particular to the case where $x \in U_K^1$, for its image in k is $\bar{x} = 1 = N_{k'|k}(1)$. □

Let R be a discrete valuation ring with field of fractions K and residue field k. As mentioned in 1.3, the kernel of the tame symbol

$$\partial \colon K_n^M(K) \to K_{n-1}^M(k)$$

is equal to the subgroup $U^0(K_n^M(K))$ of $K_n^M(K)$ generated by all symbols $\{a_1, \ldots, a_n\}$ with $a_i \in R^\times$ for $i = 1, \ldots, n$. We define

$$U^1(K_n^M(K)) \subset K_n^M(K)$$

as the subgroup generated by all symbols $\{a_1, \ldots, a_n\}$ with $a_i \in K^\times$ for $i = 1, \ldots, n$ and $a_j \in U_K^1$ for some $1 \le j \le n$.

Remark. Let π be a uniformizer of R. Then we have

$$U^1(K_n^M(K)) = \operatorname{im}\big[K_n^M(R, (\pi)) \subset K_n^M(R) \to K_n^M(K)\big] \subset U^0(K_n^M(K)).$$

This can be seen as follows. The abelian group U_K^1 is generated by elements of the form $1 - \pi u$, $u \in R^\times$. We have $\{1 - \pi u, \pi u\} = 0$, hence

$$\{1 - \pi u, \pi\} = -\{1 - \pi u, u\},$$

which iplies the inclusion

$$U^1(K_n^M(K)) \subset \operatorname{im}\big[K_n^M(R, (\pi)) \subset K_n^M(R) \to K_n^M(K)\big].$$

By Lemma 1.3.1 this is an equality.

Proposition 1.4.13. *Let K be an n-dimensional excellent henselian local field, $n \geq 1$, and let $R \subset K$ be the induced discrete henselian valuation ring of rank 1. Then for a finite abelian extension $L|K$ the following holds.*

(i) *$L|K$ is unramified (w.r.t. R) $\Leftrightarrow U^0(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$.*

(ii) *$L|K$ is tamely ramified (w.r.t. R) $\Leftrightarrow U^1(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$.*

Proof. Denote the $(n-1)$-dimensional residue field of R by k. Let ℓ be the residue field of L and $k^s \subset \ell$ the maximal separable subextension of $\ell|k$. We have $\mathrm{Aut}(\ell|k) = \mathrm{Gal}(k^s|k)$ and a canonical isomorphism

$$\mathrm{Gal}(K^{nr}|K) \xrightarrow{\sim} \mathrm{Gal}(k^s|k),$$

where K^{nr} is the maximal unramified extension of K in L. Hence $L|K$ is unramified if and only if the canonical surjection

$$\mathrm{Gal}(L|K) \twoheadrightarrow \mathrm{Aut}(\ell|k)$$

is injective.

By Corollary 1.4.11 (ii) the norm $N_{\ell|k^s}\colon K_{n-1}^M(\ell) \to K_{n-1}^M(k^s)$ is surjective. Hence Theorem 1.4.9 gives us an isomorphism

$$K_{n-1}^M(k)/N_{\ell|k}(K_{n-1}^M(\ell)) \xrightarrow{\sim} \mathrm{Aut}(\ell|k)$$

fitting into the commutative diagram

$$\begin{array}{ccc} K_n^M(K)/N_{L|K}(K_n^M(L)) & \twoheadrightarrow & K_{n-1}^M(k)/N_{\ell|k}(K_{n-1}^M(\ell)) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Gal}(L|K) & \twoheadrightarrow & \mathrm{Aut}(\ell|k) \end{array}$$

(cf. Theorem 1.4.2). The upper horizontal map is induced by the tame symbol $\partial\colon K_n^M(K) \to K_{n-1}^M(k)$ and its kernel is

$$U^0(K_n^M(K))/U^0(K_n^M(K)) \cap N_{L|K}(K_n^M(L)).$$

Hence it is injective if and only if $U^0(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$. This implies assertion (i).

In order to show (ii), let us first assume that $\mathrm{char}(k) = p > 0$ and that

$[L:K] = p^r$ for some $r \geq 0$. Then $L|K$ is tamely ramified if and only if it is unramified. We have an exact sequence

$$0 \to U^1(K_n^M(K)) \to U^0(K_n^M(K)) \to K_n^M(k) \to 0$$

(cf. [BT] I, § 4, Proposition 4.3). By Corollary 1.4.11 $K_n^M(k)$ is p-divisible. Hence under the reciprocity map

$$K_n^M(K) \to \mathrm{Gal}(L|K)$$

into the finite p-group $\mathrm{Gal}(L|K)$, the subgroup $U^1(K_n^M(K))$ maps to zero if and only if $U^0(K_n^M(K))$ does and so the claim follows from (i).

Still in the case $\mathrm{char}(k) = p > 0$, let now $[L:K]$ be arbitrary. If $L|K$ is tamely ramified we have $N_{L|K}(U_L^1) = U_K^1$ by Lemma 1.4.12 and thus $U^1(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$ by the projection formula for the norm map. For the converse, let $K'|K$ be the subextension of $L|K$ such that $[L:K']$ is prime to p and $[K':K]$ is a p-power. If $U^1(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$ then in particular $U^1(K_n^M(K)) \subset N_{K'|K}(K_n^M(K'))$ and from the above it follows that $K'|K$ is tamely ramified. Hence $L|K$ is tamely ramified.

If $\mathrm{char}(k) = 0$ both conditions are always satisfied: $L|K$ is tamely ramified and $U^1(K_n^M(K)) \subset N_{L|K}(K_n^M(L))$ since $N_{L|K}(U_L^1) = U_K^1$. □

Chapter 2

Moduli in higher dimensions

In this chapter we explain the construction of the Kato-Saito class group, which by definition is a certain inverse limit taken over non-zero ideal sheaves that can be considered as moduli in higher dimensions. Then, following an idea of Alexander Schmidt, we present a different approach of defining moduli by using curves. This leads us to Wiesend's class field theory.

2.1 The class group of Kato-Saito

We briefly recall the definition of the class group of Kato-Saito and their main result for arithmetic schemes. In the one-dimensional case, the filtration by moduli agrees with the classical one described in 1.1.

Let $\bar{X} \in Sch(\mathbb{Z})$ be a connected normal scheme which is proper and flat over $\mathbb{Z}$. Let $K = k(\bar{X})$ be its function field and $d = \dim \bar{X}$. For simplicity we assume that K has no embeddings into the reals. For any non-zero ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ and any $n \geq 0$ let $\mathcal{K}_n^M(\mathcal{O}_{\bar{X}}, \mathcal{I})$ be the n-th relative Milnor K-sheaf in the Nisnevich topology defined in section 1.3.

Definition. The *Kato-Saito class group of modulus* $\mathcal{I}$ of $\bar{X}$ is defined as

$$C_{\mathcal{I}}(\bar{X}) = H^d(\bar{X}_{Nis}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})).$$

It is a finite group ([Ke] Theorem 10.2) and if $\mathcal{J} \subset \mathcal{I}$ is another non-zero ideal of $\mathcal{O}_{\bar{X}}$, the map

$$C_{\mathcal{J}}(\bar{X}) \to C_{\mathcal{I}}(\bar{X})$$

induced by the injection

$$\mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{J}) \hookrightarrow \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})$$

is surjective ([KaS] (1.4.4)).

Definition. The *Kato-Saito class group* of $\bar{X}$ is defined as

$$C_{KS}(\bar{X}) = \varprojlim_{\mathcal{I} \neq 0} C_{\mathcal{I}}(\bar{X})$$

where the limit is taken over all non-zero ideals $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$.

In section 3.1 we will recall the construction of the *reciprocity map*

$$\rho_{\bar{X}} : C_{KS}(\bar{X}) \to \mathrm{Gal}(K^{ab}|K).$$

It has the following properties.

Theorem 2.1.1. *Let $\bar{X} \in Sch(\mathbb{Z})$ be a connected normal scheme which is proper and flat over $\mathbb{Z}$. Assume that the function field $K = k(\bar{X})$ has no real embeddings. Then the following holds.*

(i) *The reciprocity map*

$$\rho_{\bar{X}} : C_{KS}(\bar{X}) \to \mathrm{Gal}(K^{ab}|K)$$

is an isomorphism.

(ii) *For any non-empty regular open subscheme $X \subset \bar{X}$ the reciprocity map induces an isomorphism*

$$C_{KS}(X) := \varprojlim_{\mathcal{I}|_X = \mathcal{O}_X} C_{\mathcal{I}}(\bar{X}) \xrightarrow{\sim} \pi_1^{ab}(X),$$

where $\mathcal{I}$ ranges over all ideal sheaves of $\mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$.

Proof. Cf. [KaS] § 9. □

By 1.2.4, setting $\mathcal{F} = \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})$, we have an isomorphism

$$C_{\mathcal{I}}(\bar{X}) \cong \mathrm{coker}\big[\bigoplus_{x \in \bar{X}_1} H_x^{d-1}(\bar{X}_{Nis}, \mathcal{F}) \to \bigoplus_{x \in \bar{X}_0} H_x^d(\bar{X}_{Nis}, \mathcal{F})\big]$$

and the groups

$$H_x^*(\bar{X}_{Nis}, \mathcal{F}) \cong H_x^*((\bar{X}_x^h)_{Nis}, \mathcal{F}),$$

where we put $\bar{X}_x^h = \mathrm{Spec}\, \mathcal{O}_{\bar{X},x}^h$, can be computed by using the localization sequence of cohomology.

Example. (cf. [Ra] 6.3.) Let K be a global field. If K is a number field with ring of integers $\mathcal{O}_K$, let $\bar{X} = \operatorname{Spec} \mathcal{O}_K$. In the function field case, let $\bar{X}$ be a smooth projective model of K. Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be a non-zero ideal sheaf and let $S = \operatorname{supp} \mathcal{O}_{\bar{X}}/\mathcal{I}$. Put $\mathcal{F} = \mathcal{K}_1^M(\mathcal{O}_{\bar{X}}, \mathcal{I})$. We have

$$H^1(\bar{X}_{Nis}, \mathcal{F}) = \operatorname{coker}\big[H^0(K, \mathcal{F}) \to \bigoplus_{x \in \bar{X}_0} H^1_x((\bar{X}^h_x)_{Nis}, \mathcal{F})\big]$$

and

$$H^0(K, \mathcal{F}) = K^\times.$$

Let K^h_x denote the quotient field of $\mathcal{O}^h_{\bar{X},x}$. Consider the localization sequence

$$H^0((\bar{X}^h_x)_{Nis}, \mathcal{F}) \to H^0(K^h_x, \mathcal{F}) \to H^1_x((\bar{X}^h_x)_{Nis}, \mathcal{F}) \to H^1((\bar{X}^h_x)_{Nis}, \mathcal{F}).$$

Henselian local rings have trivial Nisnevich cohomology, hence the right term vanishes and we have

$$\begin{aligned} H^1_x((\bar{X}^h_x)_{Nis}, \mathcal{F}) &\cong \operatorname{coker}\big[H^0((\bar{X}^h_x)_{Nis}, \mathcal{F}) \to H^0(K^h_x, \mathcal{F})\big] \\ &\cong \operatorname{coker}\big[K_1^M(\mathcal{O}^h_{\bar{X},x}, \mathcal{I}\mathcal{O}^h_{\bar{X},x}) \to (K^h_x)^\times\big]. \end{aligned}$$

For $x \notin S$ we have

$$K_1^M(\mathcal{O}^h_{\bar{X},x}, \mathcal{I}\mathcal{O}^h_{\bar{X},x}) = (\mathcal{O}^h_{\bar{X},x})^\times,$$

hence

$$H^1_x((\bar{X}^h_x)_{Nis}, \mathcal{F}) \cong (K^h_x)^\times / (\mathcal{O}^h_{\bar{X},x})^\times \cong \mathbb{Z}.$$

For $x \in S$ we have

$$K_1^M(\mathcal{O}^h_{\bar{X},x}, \mathcal{I}\mathcal{O}^h_{\bar{X},x}) = 1 + \mathcal{I}\mathcal{O}^h_{\bar{X},x} \subset (\mathcal{O}^h_{\bar{X},x})^\times,$$

and therefore

$$H^1_x((\bar{X}^h_x)_{Nis}, \mathcal{F}) \cong (K^h_x)^\times / (1 + \mathcal{I}\mathcal{O}^h_{\bar{X},x}).$$

We conclude that

$$C_{\mathcal{I}}(\bar{X}) \cong \operatorname{coker}\big[K^\times \to \bigoplus_{x \notin S} \mathbb{Z} \oplus \bigoplus_{x \in S} \big((K^h_x)^\times / (1 + \mathcal{I}\mathcal{O}^h_{X,x})\big)\big].$$

The latter group does not change when we replace henselization by completion, and is therefore isomorphic to the ray class group $C_K/C_K^{\mathcal{I}}$ of section

1.1, where $\mathcal{I}$ is considered as a modulus on K. So if K is a totally imaginary number field, Theorem 2.1.1 gives back the isomorphism

$$\varprojlim_{\mathcal{I}\neq 0} C_K/C_K^{\mathcal{I}} \xrightarrow{\sim} \mathrm{Gal}(K^{ab}|K)$$

of section 1.1, as well as the isomorphism

$$\varprojlim_{\mathcal{I}|_X=\mathcal{O}_X} C_K/C_K^{\mathcal{I}} \xrightarrow{\sim} \mathrm{Gal}(K_S^{ab}|K) \cong \pi_1^{ab}(X),$$

where X is the open complement of S in $\bar{X}$.

2.2 Moduli using curves

In this section we define a different notion of moduli for higher-dimensional arithmetic schemes. Essentially, a non-zero ideal sheaf is regarded as modulus on any curve not lying entirely on the closed subscheme defined by the ideal.

Let us introduce some notation:

- For a morphism $f\colon Y \to X$ of schemes and an ideal $\mathcal{I} \subset \mathcal{O}_X$ we let $\mathcal{I}\mathcal{O}_Y$ denote the image of $f^*\mathcal{I}$ under the canonical homomorphism
$$f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y.$$
- By a *curve on* a scheme $X \in Sch(\mathbb{Z})$ we mean an integral closed subscheme $C \subset X$ of Krull dimension one.
- The normalization of a curve $C \subset X$ in its function field is denoted by $\tilde{C}$ and the regular compactification of $\tilde{C}$ by $P(\tilde{C})$. It is a regular proper curve over $\mathrm{Spec}\,\mathbb{Z}$ containing $\tilde{C}$ as a dense open subscheme.
- Let $\mathfrak{m}$ be a modulus on a global field K and let $L|K$ be a finite abelian extension. Then we say that $L|K$ is *ramified with modulus* $\mathfrak{m}$, if L is contained in the field $K^{\mathfrak{m}}$ (cf. section 1.1).

We fix a connected, regular scheme $X \in Sch(\mathbb{Z})$ and a normal compactification $\bar{X} \in Sch(\mathbb{Z})$ of X.

For any curve C on X the morphism $\tilde{C} \to X$ extends uniquely to a morphism $P(\tilde{C}) \to \bar{X}$ by the valuative criterion of properness:

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ P(\tilde{C}) & \longrightarrow & \bar{X}. \end{array}$$

Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be an ideal such that $\mathcal{I}|_X = \mathcal{O}_X$. For any curve C on X we pull back $\mathcal{I}$ along the map $P(\tilde{C}) \to \bar{X}$ to obtain an ideal $\mathcal{I}_C := \mathcal{I}\mathcal{O}_{P(\tilde{C})}$ which we consider as a modulus on the global field $k(C)$ (ignoring the infinite part for horizontal curves). By construction we have $\mathcal{I}_C|_{\tilde{C}} = \mathcal{O}_{\tilde{C}}$. Hence coverings of $P(\tilde{C})$ that ramify with modulus $\mathcal{I}_C$ are unramified above $\tilde{C}$.

Definition. Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be an ideal such that $\mathcal{I}|_X = \mathcal{O}_X$. We say that an abelian étale covering $Y \to X$ is *ramified with Wiesend modulus* $\mathcal{I}$ if for any curve $C \subset X$ the associated finite abelian extension of $k(C)$ is ramified with modulus $\mathcal{I}_C$.

In the following, we show that there always exists a modulus that realizes a given covering.

Let X be a noetherian, connected scheme. Recall that for a finite, flat morphism $f\colon Y \to X$ of constant rank n, there is the *discriminant ideal* $\delta_{Y|X} \subset \mathcal{O}_X$ which is defined as follows. If $U = \operatorname{Spec} A$ is an affine open subset of X such that $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a free A-algebra with basis $(b_1, \dots, b_n)$, then $\Gamma(U, \delta_{Y|X})$ is the prinipal ideal generated by

$$\det(\mathrm{Tr}_{B|A}(b_i b_j))_{1 \le i,j \le n}).$$

The map $f\colon Y \to X$ is unramified if and only if $\delta_{Y|X} = \mathcal{O}_X$ (this follows from [Mi] I, Proposition 3.1 (d)). If X and Y are curves with $K = k(X)$ and $L = k(Y)$, the discriminant $\delta_{Y|X}$ coincides with the classical discriminant $\delta_{L|K}$ as considered in [Se] Ch. III, for instance.
If X is a normal scheme of dimension ≥ 2 and $f\colon Y \to X$ is the normalization of X in a finite separable extension of $k(X)$, then f is not flat in general. We want to extend the notion of the discriminant ideal to morphisms of this type.

Let A be a noetherian, integrally closed domain with field of fractions K. Let B be a finite A-algebra such that $L = B \otimes_A K$ is an étale K-algebra, i.e. a finite product of finite separable field extensions of K, and denote by $\tau\colon B \to L$ the canonical homomorphism. Since A is integrally closed, the trace

$$\mathrm{Tr}_{L|K}(\tau(b))$$

of an element $b \in B$ is contained in A. Let $n = \dim_K L$. For any $b_1, \ldots, b_n$ we define

$$\delta_{b_1,\ldots,b_n} = \det(\mathrm{Tr}_{L|K}((\tau(b_i b_j))_{1 \le i,j \le n})) \in A.$$

Definition. The discriminant $\delta_{B|A}$ is the ideal of A generated by all elements $\delta_{b_1,\ldots,b_n}$ with $b_1, \ldots, b_n \in B$.

Clearly, for any multiplicative subset $S \subset A$ we have

$$S^{-1}\delta_{B|A} = \delta_{S^{-1}B|S^{-1}A}.$$

Hence we can sheafify the discriminant for any normal, connected scheme X and any finite, generically étale morphism $f\colon Y \to X$.

Definition. The discriminant $\delta_{Y|X}$ is the coherent ideal on X defined by $\Gamma(U, \delta_{Y|X}) = \delta_{B|A}$ for any affine open subset $U = \operatorname{Spec} A$ of X and $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$.

If f is flat, then $\delta_{Y|X}$ coincides with the usual discriminant ideal.

Lemma 2.2.1. *Let $Y \to X$ be the normalization of X in a finite separable extension of $k(X)$. Then $Y \to X$ is étale if and only if $\delta_{Y|X} = \mathcal{O}_X$.*

Proof. The map $Y \to X$ is unramified at all points above a given point $x \in X$ if and only if the fiber $Y \times_X \operatorname{Spec} k(x)$ is a sum of spectra of finite separable field extensions of $k(x)$. Now a finite extension of fields is separable if and only if the trace-form is non-degenerate. It follows that $Y \to X$ is unramified at all points above x if and only if $(\delta_{Y|X})_x = \mathcal{O}_{X,x}$. If $Y \to X$ is unramified, it is étale by [Mi] I, Theorem 3.21. □

Lemma 2.2.2. *Let X and Z be normal, connected schemes. Let $Y \to X$ be a finite, generically étale morphism and let $Z \to X$ be a morphism, such that $Y \to X$ is étale above the image of the generic point of Z in X. Then*

$$\delta_{Y|X}\mathcal{O}_Z \subset \delta_{(Y \times_X Z)|Z}.$$

Proof. This follows immediately from the definition of the discriminant. □

Lemma 2.2.3. *Let X be a normal, connected scheme and let $Y \to X$ be a finite, generically étale morphism. Let η be the generic point of X. Assume that $Z \to Y$ is a finite morphism which induces an isomorphism*

$$Z \times_X \eta \to Y \times_X \eta.$$

Then we have

$$\delta_{Y|X} \subset \delta_{Z|X}.$$

Proof. This is again an immediate consequence from the definition of the discriminant. □

Proposition 2.2.4. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular scheme and let $\bar{X}$ be a normal compactification of X. Let $Y \to X$ be an abelian étale covering. Then there exists an ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$ such that $Y \to X$ is ramified with Wiesend modulus $\mathcal{I}$.*

Proof. Let $\bar{Y}$ be the normalization of $\bar{X}$ in $k(Y)$. We show that the discriminant ideal $\mathcal{I} := \delta_{\bar{Y}|\bar{X}}$ has the desired property. We clearly have $\mathcal{I}|_X = \delta_{Y|X} = \mathcal{O}_X$. Let C be a curve on X and let $P(\tilde{C}) \to \bar{X}$ be the induced morphism. By Lemma 2.2.2 we have

$$\mathcal{I}_C = \delta_{\bar{Y}|\bar{X}}\mathcal{O}_{P(\tilde{C})} \subset \delta_{(\bar{Y} \times_{\bar{X}} P(\tilde{C}))|P(\tilde{C})}.$$

Let D_i run through the connected components of $\bar{Y} \times_{\bar{X}} P(\tilde{C})$. Then it follows from Lemma 2.2.3, that we have an inclusion

$$\delta_{\bar{Y} \times_{\bar{X}} P(\tilde{C})|P(\tilde{C})} \subset \prod_i \delta_{P(\tilde{D}_i)|P(\tilde{C})}.$$

Using the notation

$$\delta_{k(D_i)|k(C)} = \delta_{P(\tilde{D}_i)|P(\tilde{C})}$$

the proposition now follows from the following Lemma. □

Lemma 2.2.5. *Let $L|K$ be a finite abelian extension of global fields and $\delta_{L|K}$ its discriminant. Then we have an inclusion of ideals*

$$\delta_{L|K} \subset \mathfrak{f}_{L|K}.$$

Proof. Recall the conductor-discriminant formula ([Ne1] VII, (11.9))

$$\delta_{L|K} = \prod_{\chi} \mathfrak{f}(\chi)^{\chi(1)},$$

where the product is taken over all irreducible characters χ of $\mathrm{Gal}(L|K)$ and where $\mathfrak{f}(\chi)$ is the global Artin conductor of χ. For any injective one-dimensional character $\chi\colon \mathrm{Gal}(L|K) \to \mathbb{C}^{\times}$ we have an equality $\mathfrak{f}(\chi) = \mathfrak{f}_{L|K}$ ([Ne1] VII, (11.10)). Hence the inclusion

$$\delta_{L|K} \subset \mathfrak{f}_{L|K}$$

holds if $L|K$ is cyclic. The general abelian case follows from the cyclic case taking into account that for any two finite abelian extensions L_1 and L_2 of K we have

$$\mathfrak{f}_{L_1L_2|K} = \mathrm{l.c.m.}(\mathfrak{f}_{L_1|K}, \mathfrak{f}_{L_2|K}),$$

which is an immediate consequence of the transitivity of the norm, and

$$\mathrm{l.c.m.}(\delta_{L_1|K}, \delta_{L_2|K}) \mid \delta_{L_1L_2|K}.$$

The latter holds, since for finite extensions $M|L|K$ of global fields we have

$$\delta_{M|K} = \delta_{L|K}^{[M:\,L]} N_{L|K}(\delta_{M|L}).$$

□

Definition. For an ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$ let $K_{\mathcal{I}}$ be the maximal abelian extension $L|K$ such that $X_L \to X$ is ramified with Wiesend modulus $\mathcal{I}$.

By Proposition 2.2.4 we get the

Corollary 2.2.6. *For $X \subset \bar{X}$ as above we have*

$$\pi_1^{ab}(X) = \varprojlim_{\mathcal{I}|_X = \mathcal{O}_X} \mathrm{Gal}(K_{\mathcal{I}}|K).$$

Chapter 3

Comparison results

We begin this chapter by describing the construction of the Kato-Saito reciprocity map. Then we recall the class field theory of Wiesend before we finally turn towards tame coverings. We will see that ramification with respect to a "tame modulus" in the sense of Kato-Saito corresponds to the notion of divisor-tameness, whereas for a Wiesend modulus, it means curve-tameness.

3.1 The reciprocity map of Kato-Saito

We need to recall the construction of the reciprocity map of Kato and Saito in detail. It is defined via the class field theory of the higher-dimensional henselian local fields using the formalism of Parshin chains.

Definition. Let X be a noetherian scheme.

(i) A *chain* on X is a sequence $P = (p_0, \ldots, p_r)$ of points $p_i \in X$ such that
$$\overline{\{p_0\}} \subset \overline{\{p_1\}} \subset \ldots \subset \overline{\{p_r\}}.$$

(ii) A chain $P = (p_0, \ldots, p_r)$ on X is called *Parshin chain* if $p_i \in X_i$ for $0 \le i \le r$. The integer r is called the *length* of P. The set of all Parshin chains of length r on X is denoted by $P_r(X)$.

(iii) Let $r \ge 1$ and $0 \le s \le r$. The set of all chains on X of the form
$$P = (p_0, \ldots, p_{s-1}, p_{s+1}, \ldots, p_r)$$

such that $p_i \in X_i$ for $i \in \{0, 1, \dots, s-1, s+1, \dots, r\}$ is denoted by $Q_{r,s}(X)$. We set $Q_{0,0}(X) = \emptyset$.

(iv) For $P = (p_0, \dots, p_{s-1}, p_{s+1}, \dots, p_r) \in Q_{r,s}(X)$ let $B(P)$ be the set of all points $x \in X_s$ such that the extended chain

$$P(x) = (p_0, \dots, p_{s-1}, x, p_{s+1}, \dots, p_r)$$

is a Parshin chain.

For a chain $P = (p_0, \dots, p_r)$ on X we define the semi-local ring $\mathcal{O}^h_{X,P}$ by iterated henselizations and localizations as follows. If $r = 0$ we put $\mathcal{O}^h_{X,P} = \mathcal{O}^h_{X,p_0}$. Let $r > 0$ and assume we have already defined $R = \mathcal{O}^h_{X,P'}$ for the chain $P' = (p_0, \dots, p_{r-1})$. Let T be the finite set of all prime ideals of R lying over p_r. We set

$$\mathcal{O}^h_{X,P} = \prod_{\mathfrak{p} \in T} (R_{\mathfrak{p}})^h.$$

Let $X^h_P = \operatorname{Spec} \mathcal{O}^h_{X,P}$. We define

$$k(P) = \prod_{x \in (X^h_P)_0} k(x)$$

as the finite product of the residue fields of $\mathcal{O}^h_{X,P}$. If P is a Parshin chain of length r on a scheme $X \in Sch(\mathbb{Z})$, the ring $k(P)$ is a finite product of r-dimensional henselian local fields (cf. 1.4).

For a Nisnevich sheaf $\mathcal{F}$ and a chain P on X we define the group

$$H^q_P(X_{Nis}, \mathcal{F}) = \bigoplus_{x \in (X^h_P)_0} H^q_x((X^h_P)_{Nis}, \mathcal{F}),$$

$q \geq 0$. If $P = (p_0, \dots, p_r)$ is a chain such that p_{r-1} is of codimension 1 in $\overline{\{p_r\}}$ the coniveau spectral sequence 1.2.2 gives us a homomorphism

$$\delta^q_P \colon H^q_P(X_{Nis}, \mathcal{F}) \to H^{q+1}_{P'}(X_{Nis}, \mathcal{F})$$

for any $q \geq 0$, where $P' = (p_0, \dots, p_{r-1})$. Composing the homomorphisms

$$H^q_P(X_{Nis}, \mathcal{F}) \xrightarrow{\delta^q_P} \dots \to H^{q+d}_{p_0}(X_{Nis}, \mathcal{F}) \to H^{q+d}(X_{Nis}, \mathcal{F})$$

for a Parshin chain $P \in P_d(X)$, $d = \dim X$, and putting $q = 0$ we obtain a homomorphism

$$c_P \colon H^0_P(X_{Nis}, \mathcal{F}) \to H^d(X_{Nis}, \mathcal{F}).$$

Repeatedly using the cokernel description of Corollary 1.2.4 and the localization sequence of cohomology, we get the important

Lemma 3.1.1. *Let X be a noetherian scheme of dimension d. For any Nisnevich sheaf $\mathcal{F}$ on X and any abelian group A the homomorphism*

$$\mathrm{Hom}(H^d(X_{Nis}, \mathcal{F}), A) \to \prod_{P \in P_d(X)} \mathrm{Hom}(H^0_P(X_{Nis}, \mathcal{F}), A)$$

defined by

$$g \mapsto (g \circ c_P)_{P \in P_d(X)}$$

maps $\mathrm{Hom}(H^d(X_{Nis}, \mathcal{F}), A)$ *isomorphically onto the subgroup of all families* $(g_P)_{P \in P_d(X)}$ *of homomorphisms*

$$g_P \colon H^0_P(X_{Nis}, \mathcal{F}) \to A$$

satisfying the following condition:
For any $0 \le s \le d$, any $P' \in Q_{d,s}(X)$ and any $c \in H^0_{P'}(X_{Nis}, \mathcal{F})$ we have $g_{P'(x)}(c_{P'(x)}) = 0$ for almost all $x \in B(P')$ and

$$\sum_{x \in B(P')} g_{P'(x)}(c_{P'(x)}) = 0.$$

Here $c_{P'(x)}$ denotes the image of c under the canonical map

$$H^0_{P'}(X_{Nis}, \mathcal{F}) \to H^0_{P'(x)}(X_{Nis}, \mathcal{F}).$$

Proof. [KaS] Lemma 1.6.3. □

In the following, let $\bar{X} \in Sch(\mathbb{Z})$ be a connected, normal and proper scheme over $\mathbb{Z}$ of dimension d with function field $K = k(\bar{X})$. Recall that a discrete valuation ring V of K of rank d comes together with a sequence

$$V = V_0 \subset \cdots \subset V_d = K.$$

of discrete valuation rings V_i of K rank $d - i$ (cf. 1.4).

Definition. Let $P = (p_0, \ldots, p_d)$ be a Parshin chain of length d on $\bar{X}$. We say that a discrete valuation ring V of K of rank d *dominates* P if for each $0 \le i \le d$ the valuation ring V_i dominates the local ring $\mathcal{O}_{\bar{X},p_i}$.

The number of discrete valuation rings V dominating a Parshin chain is finite and ≥ 1 (cf. the proposition below). The residue field of the valuation ring V_i is a finite extension of $k(p_i)$ for each i. Let V^h be the henselization of V and let

$$V^h = (V^h)_0 \subset \cdots \subset (V^h)_d = K_V$$

be the localizations at the prime ideals of V^h. For any $0 \le i \le d$ the residue field k_i^h of $(V^h)_i$ is the fraction field of an excellent henselian discrete valuation ring and k_0^h is a finite field. In particular, with the terminology of section 1.4, $k_d^h = K_V$ is a d-dimensional excellent henselian local field and we have the reciprocity map

$$\rho_{K_V} \colon K_d^M(K_V) \to \mathrm{Gal}(K_V^{ab}|K_V)$$

(cf. section 1.4). Now fix a finite abelian extension $L|K$. By composing ρ_{K_V} with the natural maps

$$\mathrm{Gal}(K_V^{ab}|K_V) \to \mathrm{Gal}(LK_V|K_V) \to \mathrm{Gal}(L|K)$$

we obtain homomorphisms

$$\rho_{K_V,L} \colon K_d^M(K_V) \to \mathrm{Gal}(L|K).$$

Proposition 3.1.2. *Let $P = (p_0, \ldots, p_d)$ be a Parshin chain of length d on $\bar{X}$ and let V range over all discrete valuation rings of rank d of K dominating P. Let K_V be the quotient field of V^h and let $R_V = (V^h)_{d-1}$ be the localization of V^h at the unique prime ideal of height one. Then we have*

$$k(P) = \prod_V K_V$$

and

$$\mathcal{O}^h_{X,P'} = \prod_V R_V,$$

where $P' = (p_0, \ldots, p_{d-1})$.

Proof. [KaS] Proposition 3.3 and its proof. □

Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be a non-zero ideal. By the Proposition above, taking the sum over all discrete valuation rings V of rank d dominating $P \in P_d(\bar{X})$ defines a homomorphism

$$\rho_{P,L} \colon H^0_P(\bar{X}_{Nis}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathcal{I})) \cong K^M_d(k(P)) \to \mathrm{Gal}(L|K).$$

In order to induce a homomorphism

$$\rho_{\bar{X},L} \colon H^d(\bar{X}_{Nis}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathcal{I})) \to \mathrm{Gal}(L|K),$$

the family $(\rho_{P,L})_{P \in P_d(\bar{X})}$ obtained this way has to satisfy the condition of Lemma 3.1.1 concerning the compositions

$$H^0_{P'}(\bar{X}_{Nis}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathcal{I})) \to H^0_{P'(x)}(\bar{X}_{Nis}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathcal{I})) \to \mathrm{Gal}(L|K)$$

for the chains $P' \in Q_{d,s}(\bar{X})$ and $x \in B(P')$. For the cases $0 \leq s < d$ this condition is always satisfied regardless of the modulus $\mathcal{I}$ ([KaS] (3.7.4)). The latter only has to be considered in the case $s = d$. For any $P' \in Q_{d,d}(\bar{X})$ the set $B(P')$ simply consists of the generic point η of $\bar{X}$. The condition of Lemma 3.1.1 means that

$$\rho_{P'(\eta),L} \colon K^M_d(k(P'(\eta))) \to \mathrm{Gal}(L|K)$$

has to annihilate the image of the canonical map

$$K^M_d(\mathcal{O}^h_{\bar{X},P'}, \mathcal{I}\mathcal{O}^h_{\bar{X},P'}) \to K^M_d(k(P'(\eta))).$$

By Proposition 3.1.2 we have the

Corollary 3.1.3. *The family $(\rho_{P,L})_{P \in P_d(\bar{X})}$ induces a homomorphism*

$$\rho_{\bar{X},L} \colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K),$$

if and only if for any discrete valuation ring V of rank d of K dominating a Parshin chain of length d on $\bar{X}$ the composition

$$K^M_d(R_V, \mathcal{I}R_V) \to K^M_d(K_V) \overset{\rho_{K_V}}{\to} \mathrm{Gal}(K_V^{ab}|K_V) \to \mathrm{Gal}(LK_V|K_V)$$

is zero, where K_V is the quotient field of the henselization V^h of V and R_V denotes the localization of V^h at the unique prime ideal of height one.

In the following, we simply say *the reciprocity map*

$$\rho_{\bar{X},L}\colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K)$$

exists if the above condition holds.

Proposition 3.1.4. *For any finite abelian extension $L|K$ there exists a non-zero ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ such that the reciprocity map*

$$\rho_{\bar{X},L}\colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K)$$

exists.

Proof. [KaS] 3.7.1 and 3.7.4. □

Definition. The *reciprocity map of Kato-Saito*

$$\rho_{\bar{X}}\colon C_{KS}(\bar{X}) \to \mathrm{Gal}(K^{ab}|K)$$

is defined by taking the limits over all finite abelian extensions $L|K$ and all non-zero ideals $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$.

The crucial point for using Nisnevich cohomology in the definition of the class group is the exactness of the direct image functor $f_*\colon Sh(Y_{Nis}) \to Sh(X_{Nis})$ for a finite morphism $f\colon Y \to X$ which allows us to define a norm map for the class group. Without going into futher detail, we just mention that for any finite surjective morphism $Y \to X$ of integral schemes $X, Y \in Sch(\mathbb{Z})$ and any non-zero ideal $\mathcal{I} \subset \mathcal{O}_X$ there exists a non-zero ideal $\mathcal{J} \subset \mathcal{O}_Y$, which can be chosen to be $\mathcal{J} = \mathcal{I}\mathcal{O}_Y$ if X is normal, such that the norm maps of Milnor K-theory induce a map

$$N_{Y|X}\colon C_{\mathcal{J}}(Y) \to C_{\mathcal{I}}(X)$$

(cf. [KaS] § 4). It has the following property.

Theorem 3.1.5. *Let $\bar{X}$ be as before and let $\bar{Y}$ be the normalization of $\bar{X}$ in a finite abelian extension $L|K$. Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be a non-zero ideal such that the reciprocity map*

$$\rho_{\bar{X},L}\colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K)$$

exists and let $\mathcal{J} \subset \mathcal{O}_{\bar{Y}}$ be a non-zero ideal such that the norm

$$N_{\bar{Y}|\bar{X}}\colon C_{\mathcal{J}}(\bar{Y}) \to C_{\mathcal{I}}(\bar{X})$$

is defined. Then the sequence

$$C_{\mathcal{J}}(\bar{Y}) \stackrel{N_{\bar{Y}|\bar{X}}}{\longrightarrow} C_{\mathcal{I}}(\bar{X}) \stackrel{\rho_{\bar{X},L}}{\longrightarrow} \mathrm{Gal}(L|K) \longrightarrow 0$$

is exact.

Proof. [KaS] Theorem 4.6. □

3.2 Class field theory of Wiesend

In this section we recall Wiesend's class field theory and show that there is a canonical map from the henselian Wiesend class group to the Kato-Saito class group compatible with the reciprocity maps.

Let $X \in Sch(\mathbb{Z})$ be a connected scheme. For any curve $C \subset X$ let C_∞ be the finite set of places of $k(C)$ corresponding to the closed points of $P(\tilde{C})\backslash\tilde{C}$ together with the archimedean places if $k(C)$ is of characteristic zero. For $v \in C_\infty$ let $k(C)_v$ be the completion of $k(C)$ with respect to v.

Definition. The *Wiesend idele group* $I_W(X)$ of X is defined as

$$I_W(X) = \bigoplus_{x \in X_0} \mathbb{Z} \oplus \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} k(C)_v^\times$$

endowed with the direct sum topology, where C ranges over all curves of X.

For any curve $C \subset X$, we obtain a natural map

$$k(C)^\times \to I_W(X)$$

induced by

- the embeddings $k(C)^\times \hookrightarrow k(C)_v^\times$ for $v \in C_\infty$,
- the discrete valuations $k(C)^\times \to \mathbb{Z}$ multiplied by $[k(\tilde{x}) : k(x)]$ for v corresponding to a closed point $\tilde{x}$ of $\tilde{C}$ mapping to $x \in C \subset X$.

Definition. The *Wiesend class group* $C_W(X)$ of X is defined as the cokernel of the map

$$\bigoplus_{C \subset X} k(C)^\times \to I_W(X)$$

and it is endowed with the quotient topology.

The maps

- $r_x \colon \mathbb{Z} \to \pi_1^{ab}(X), 1 \mapsto Frob_x$, for $x \in X_0$,
- $r_C \colon k(C)_v^\times \to \mathrm{Gal}(k(C)_v^{ab}/k(C)_v) \to \pi_1^{ab}(X)$, for a curve $C \subset X$ and $v \in C_\infty$, where the first arrow is the local reciprocity map,

induce a continuous homomorphism $r_X \colon I_W(X) \to \pi_1^{ab}(X)$ which factors through $C_W(X)$ and we call

$$\rho_X \colon C_W(X) \to \pi_1^{ab}(X)$$

the *reciprocity map*. The class group is functorial in X and for a finite morphism $Y \to X$ in $Sch(\mathbb{Z})$ we write

$$N_{Y|X} \colon C_W(Y) \to C_W(X)$$

for the induced map and call it the *norm map*. The main theorem of Wiesend for flat schemes over $\operatorname{Spec}\mathbb{Z}$ is as follows.

Theorem 3.2.1. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular scheme which is flat over* $\operatorname{Spec}\mathbb{Z}$.

(i) *The reciprocity map $\rho_X \colon C_W(X) \to \pi_1^{ab}(X)$ is surjective and its kernel is the connected component of the identity of $C_W(X)$.*

(ii) *For any connected étale abelian covering $Y \to X$ the reciprocity map induces an isomorphism of finite abelian groups*

$$C_W(X)/N_{Y|X}C_W(Y) \xrightarrow{\sim} G(Y|X).$$

(iii) *The open subgroups of $C_W(X)$ are precisely the groups $N_{Y|X}C_W(Y)$ for étale coverings $Y \to X$.*

Proof. [KeS2] Theorem 8.1. □

For any curve C on X and any valuation $v \in C_\infty$ let $k(C)_v^h$ be the henselization of $k(C)$ at v. We define the henselian version of the Wiesend idele group by replacing the complete local fields $k(C)_v$ by their henselian counterparts

$$I_W^h(X) = \bigoplus_{x \in X_0} \mathbb{Z} \oplus \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} (k(C)_v^h)^\times$$

(with the direct sum topology) and, analogously, we define the henselian class group $C^h_W(X)$ as the cokernel of the natural map

$$\bigoplus_{C \subset X} k(C)^\times \to I^h_W(X).$$

Again we have a canonical norm map

$$N_{Y|X} \colon C^h_W(Y) \to C^h_W(X)$$

for a finite morphism $Y \to X$ and there is the reciprocity map

$$\rho^h_X \colon C^h_W(X) \to \pi_1^{ab}(X)$$

defined in the same way as ρ_X. There is a natural continuous homomorphism $C^h_W(X) \to C_W(X)$ making the diagram

$$\begin{array}{ccc} C^h_W(X) & \longrightarrow & C_W(X) \\ & {\scriptstyle \rho^h_X} \searrow \quad \swarrow {\scriptstyle \rho_X} & \\ & \pi_1^{ab}(X) & \end{array}$$

commute.

Lemma 3.2.2. *The map $C^h_W(X) \to C_W(X)$ induces a bijection between the open subgroups of $C^h_W(X)$ and the open subgroups of $C_W(X)$.*

Proof. [Ke] Lemma 10.1. □

In [KaS] a local ring A is called *nice* if there is a smooth ring A' over a field or over an excellent Dedekind domain, such that A is ind-étale over A'. An excellent scheme X is called *nice* if all its local rings are nice (cf. [KaS], Definition 2.2). If $P = (p_0, \dots, p_r)$ is a Parshin chain of length r on a noetherian scheme $\bar{X}$ of dimension d such that p_r is contained in a nice open subscheme of $\bar{X}$, then for any non-zero ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ we have a canonical purity isomorphism

$$H^{d-r}_P(\bar{X}_{Nis}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathcal{I})) \cong K^M_r(k(P))$$

by [KaS] Corollary 2.4.1.

Proposition 3.2.3. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular scheme which is flat over* $\operatorname{Spec}\mathbb{Z}$*. Let $\bar{X}$ be a normal compactification of X and assume that X is nice and $X(\mathbb{R}) = \emptyset$. Let $\mathcal{I}$ be an ideal of $\mathcal{O}_{\bar{X}}$ such that $\mathcal{I}|_X = \mathcal{O}_X$. Then the following holds.*

(i) *There exists a natural continuous surjective homomorphism*

$$C^h_W(X) \to C_{\mathcal{I}}(\bar{X}),$$

where $C_{\mathcal{I}}(\bar{X})$ is given the discrete topology.

(ii) *Let L be a finite abelian extension of $K = k(\bar{X})$ such that the reciprocity map*

$$\rho_{\bar{X},L} \colon C_{\mathcal{I}}(\bar{X}) \to \operatorname{Gal}(L|K)$$

exists. Let $\bar{Y}$ be the normalization of $\bar{X}$ in L and assume that the base change $Y = \bar{Y} \times_{\bar{X}} X \to X$ is étale. Let $\mathcal{J} \subset \mathcal{O}_{\bar{Y}}$ be a non-zero ideal such that the norm

$$N_{\bar{Y}|\bar{X}} \colon C_{\mathcal{J}}(\bar{Y}) \to C_{\mathcal{I}}(\bar{X})$$

is defined. Then we have a commutative diagram

$$\begin{array}{ccccccc}
C^h_W(Y) & \xrightarrow{N_{Y|X}} & C^h_W(X) & \longrightarrow & \operatorname{Gal}(L|K) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
C_{\mathcal{J}}(\bar{Y}) & \xrightarrow{N_{\bar{Y}|\bar{X}}} & C_{\mathcal{I}}(\bar{X}) & \xrightarrow{\rho_{\bar{X},L}} & \operatorname{Gal}(L|K) & \longrightarrow & 0
\end{array}$$

with exact rows, where the second map in the upper row is induced by the reciprocity map ρ^h_X.

Proof. First observe that $I^h_W(X)$ and $C^h_W(X)$ may be expressed using Parshin chains as follows. Let $P_1(\bar{X}, X)$ be the set of Parshin chains on $\bar{X}$ of the form $P = (p_0, p_1)$ with $p_0 \in \bar{X} \setminus X$ and $p_1 \in X$. Denote the closed points of $\bar{X}$ lying on X temporarily by $P_0(\bar{X}, X)$. With this notation we have the equality

$$I^h_W(X) = \bigoplus_{P \in P_0(\bar{X},X)} K^M_0(k(P)) \oplus \bigoplus_{P \in P_1(\bar{X},X)} K^M_1(k(P)).$$

To see this, let $P = (p_0, p_1)$ be a chain in $P_1(\bar{X}, X)$ and define $C \subset X$ as the curve on X with generic point p_1. Let $\bar{C}$ be the closure of C in $\bar{X}$. Then $P(\tilde{C})$ is canonically isomorphic to the normalization of $\bar{C}$ and we have

$$K_1^M(k(P)) = \prod_{v \mapsto p_0} (k(C)_v^h)^\times,$$

where the product is taken over the finitely many valuations $v \in C_\infty$ which correspond to points in $P(\tilde{C})$ mapping to $p_0 \in \bar{C}$ under $P(\tilde{C}) \to \bar{C}$. Using the notation of the previous section, the curves on X correspond bijectively to the set of chains $Q_{1,0}(X)$. For $P' = (p_1) \in Q_{1,0}(X)$ we have

$$k(P') = k(C),$$

where C is the closure of p_1 in X. There is a natural map

$$K_1^M(k(P')) \to I_W^h(X)$$

and the henselian class group $C_W^h(X)$ is the cokernel of the induced map

$$\bigoplus_{P \in Q_{1,0}(X)} K_1^M(k(P)) \to I_W^h(X).$$

As mentioned above, for any Parshin chain $P = (p_0, \dots, p_r)$ of length r on $\bar{X}$ with $p_r \in X$ we have a canonical isomorphism

$$K_r^M(k(P)) \cong H_P^{d-r}(\bar{X}_{Nis}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I}))$$

by the assumption that X is nice. By composing this isomorphism with the canonical map

$$H_P^{d-r}(\bar{X}_{Nis}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})) \to H^d(\bar{X}_{Nis}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})) = C_{\mathcal{I}}(\bar{X})$$

we obtain a canonical homomorphism

$$K_r^M(k(P)) \to C_{\mathcal{I}}(\bar{X}).$$

By taking the sum over all Parshin chains $P \in P_0(\bar{X}, X) \cup P_1(\bar{X}, X)$, this gives us a continuous homomorphism

$$I_W^h(X) \to C_{\mathcal{I}}(\bar{X}).$$

We have to show that for any $P = (p_1) \in Q_{1,0}(X)$ the composition

$$K_1^M(k(P)) \to I_W^h(X) \to C_{\mathcal{I}}(\bar{X})$$

is zero. Let C be the closure of p_1 in X. By [KaS] Theorem 2.9, there exists an non-zero ideal $\mathcal{J}$ of $\mathcal{O}_{P(\tilde{C})}$ such that there is a map

$$C_{\mathcal{J}}(P(\tilde{C})) \to C_{\mathcal{I}}(\bar{X})$$

fitting into a commutative diagram

$$\begin{array}{ccccc} K_1^M(k(P)) & \longrightarrow & I_W^h(\tilde{C}) & \longrightarrow & C_{\mathcal{J}}(P(\tilde{C})) \\ \| & & \downarrow & & \downarrow \\ K_1^M(k(P)) & \longrightarrow & I_W^h(X) & \longrightarrow & C_{\mathcal{I}}(\bar{X}) \end{array}$$

where the composite of the upper horizontal maps is zero since $C_{\mathcal{J}}(P(\tilde{C}))$ is a quotient of the classical idele class group of the global field $k(C)$ (cf. section 2.1). The resulting map

$$C_W^h(X) \to C_{\mathcal{I}}(\bar{X})$$

is surjective since by [KaS] Theorem 2.5, this is already true for the map

$$\bigoplus_{P \in P_0(\bar{X},X)} \mathbb{Z} = \bigoplus_{P \in P_0(\bar{X},X)} K_0^M(k(P)) \to C_{\mathcal{I}}(\bar{X}).$$

This shows (i). By the compatibility of the norm $N_{\bar{Y}|\bar{X}} \colon C_{\mathcal{J}}(\bar{Y}) \to C_{\mathcal{I}}(\bar{X})$ with the norm on Milnor K-groups (cf. [KaS] (4.4.1)) and by [KaS] Lemma 4.8, we obtain that the square

$$\begin{array}{ccc} C_W^h(Y) & \xrightarrow{N_{Y|X}} & C_W^h(X) \\ \downarrow & & \downarrow \\ C_{\mathcal{J}}(\bar{Y}) & \xrightarrow{N_{\bar{Y}|\bar{X}}} & C_{\mathcal{I}}(\bar{X}) \end{array}$$

commutes. The commutativity of the diagram

$$\begin{array}{ccc} C_W^h(X) & \longrightarrow & \mathrm{Gal}(L|K) \\ \downarrow & & \| \\ C_{\mathcal{I}}(\bar{X}) & \xrightarrow{\rho_{\bar{X},L}} & \mathrm{Gal}(L|K) \end{array}$$

follows immediately from the definition of the reciprocity maps $\rho_{\bar{X},L}$ and ρ^h_X. In both cases the map

$$\mathbb{Z} = K_0^M(k(P)) \to \mathrm{Gal}(L|K)$$

for a closed Point $P = (p_0) \in P_0(\bar{X}, X)$ is defined by mapping $1 \in \mathbb{Z}$ to $Frob_x$, and for $P = (p_0, p_1) \in P_1(\bar{X}, X)$ the arrow

$$K_1^M(k(P)) \to \mathrm{Gal}(L|K)$$

is induced by the reciprocity map for one-dimensional henselian local fields.

□

Corollary 3.2.4. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular and flat scheme over* $\operatorname{Spec}\mathbb{Z}$. *Let $\bar{X}$ be a normal compactification of X and assume that $X(\mathbb{R}) = \emptyset$. Then there is a natural continuous homomorphism*

$$C^h_W(X) \to \varprojlim_{\mathcal{I}|_X = \mathcal{O}_X} C_{\mathcal{I}}(\bar{X}) = C_{KS}(X)$$

such that the diagram

$$\begin{array}{ccc} C^h_W(X) & \longrightarrow & C_{KS}(X) \\ & \searrow^{\rho^h_X} \quad \wr\swarrow^{\rho^{KS}_X} & \\ & \pi_1^{ab}(X) & \end{array}$$

commutes, where ρ^{KS}_X denotes the reciprocity map of Kato-Saito.

Proof. Let $U \subset X$ be a nice, dense open subscheme. For any ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$ we have a canonical surjection

$$C^h_W(U) \twoheadrightarrow C_{\mathcal{I}}(\bar{X})$$

by Proposition 3.2.3. It then follows from the construction that this map factors through the natural surjection $C^h_W(U) \twoheadrightarrow C^h_W(X)$. □

Definition. For an ideal $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ with $\mathcal{I}|_X = \mathcal{O}_X$ and a curve $C \subset X$ let $\mathcal{I}_C = \prod_v v^{n_v}$ be the induced modulus on $P(\tilde{C})$. Let $U_{\mathcal{I}}(X) \subset I^h_W(X)$ be the subgroup generated by the n_v-th group of principal units of all fields $k(C)^h_v$ for $v \in C_\infty$. Set $C^h_{W,\mathcal{I}}(X) = C^h_W(X)/\mathrm{im}(U_{\mathcal{I}}(X))$.

Remark. Let $K_{\mathcal{I}}$ be the maximal abelian extension $L|K$ such that $X_L \to X$ is ramified with Wiesend modulus $\mathcal{I}$. Then the reciprocity map induces an isomorphism

$$C^h_{W,\mathcal{I}}(X) \xrightarrow{\sim} \mathrm{Gal}(K_{\mathcal{I}}|K)$$

by Wiesend's class field theory.

In the next section we show that $C^h_{W,\mathcal{I}}(X)$ and $C_{\mathcal{I}}(\bar{X})$ are isomorphic when $\mathcal{I}$ is a tame modulus.

3.3 Tame moduli

In [KeS1], the authors discuss several notions of tameness for coverings of higher-dimensional schemes and show that these notions are basically equivalent. Let us recall some definitions.

Definition. Let $X \in Sch(\mathbb{Z})$ be a connected, normal scheme and let $X' \subset X$ be a dense open subscheme. Let $x \in X \setminus X'$ be a point of codimension 1. An étale covering $Y' \to X'$ is *unramified along* x if the discrete valuation of $k(X')$ associated to x is unramified in $k(Y')$. Otherwise $Y' \to X'$ *ramifies* along x.

In particular there is the notion of *tame* and *wild* ramification along a codimension 1 point $x \in X \setminus X'$.

Definition. Let $Y \to X$ be an étale covering of normal, connected schemes in $Sch(\mathbb{Z})$.

(i) Assume that X is regular and has an open embedding into a regular, proper scheme $\bar{X}$ such that $\bar{X} \setminus X$ is a normal crossing divisor (NCD) on $\bar{X}$. Then $Y \to X$ is called *tamely ramified along* $\bar{X} \setminus X$ if it is tamely ramified along the generic points of $\bar{X} \setminus X$.

(ii) The covering $Y \to X$ is *curve-tame* if for any curve $C \subset X$, the base change $Y \times_X \tilde{C} \to \tilde{C}$ is tamely ramified along $P(\tilde{C}) \setminus \tilde{C}$.

To this we will now add another notion of tameness for abelian coverings. Let $D = \sum n_i D_i$ be a divisor on a scheme X. We say that D is *square-free* if $n_i = 1$ for all i. In particular, a normal crossing divisor is square free.

Definition. Let $Y \to X$ be an abelian étale covering of regular, connected schemes in $Sch(\mathbb{Z})$ and let $\bar{X}$ be a normal compactification of X. We say that $Y \to X$ is *Wiesend modulus-tame* if there exists a square-free divisor D on $\bar{X}$ with $\operatorname{supp} D \subset \bar{X} \setminus X$ such that $Y \to X$ is ramified with Wiesend modulus $\mathcal{O}_{\bar{X}}(-D)$.

If $Y \to X$ is modulus-tame, the induced modulus condition on $k(C)$ for a curve C on X is not necessarily tame. However, the following proposition, which is a modified version of the "Key Lemma" 2.4 in [KeS1], ensures us a tame condition on sufficiently many curves.

Proposition 3.3.1. *Let $X \in Sch(\mathbb{Z})$ be a normal, connected scheme. Let $X' \subset X$ be a dense open subscheme and let $D \subset X \setminus X'$ be a prime divisor. Assume that $Y' \to X'$ and $Z' \to Y'$ are étale Galois coverings such that:*

- *$Y' \to X'$ is unramified along the generic point η of D.*
- *$Z' \to Y'$ is of prime degree p.*
- *The composition $Z' \to X'$ is ramified along η.*

Then there exists a curve C on X meeting X' and intersecting D transversely in a point $x \in C^{reg}$ such that $Z' \times_{X'} \tilde{C}' \to \tilde{C}'$ is ramified along x, where $C' = C \cap X$.

Proof. Let Y (resp. Z) be the normalization of X in $k(Y')$ (resp. $k(Z')$). Choose an affine open neighborhood $U = \operatorname{Spec} A$ of η on X on which D is given by the zero-set $V(\pi)$ of some irreducible element $\pi \in A$. Let $V = \operatorname{Spec} B$ (resp. $W = \operatorname{Spec} C$) be the preimage of U in Y (resp. Z). Note that there is no chance of confusing the algebra C with the curve we are looking for. Choose a point $\eta_V \in V$ above η and a point $\eta_W \in W$ above η_V. After restricting to smaller affine open subsets we may assume that U, V and W are regular, that η_V is defined by π (considered as an element of B) and that η_W corresponds to an irreducible element $\pi' \in C$. By assumption the extension of discrete valuation rings $B_{(\pi)}|A_{(\pi)}$ is unramified, whereas $C_{(\pi')}|B_{(\pi)}$ is ramified and of prime degree p. The following cases can occur:

1st case: $v_{C_{(\pi')}}(\pi) = p$.

We have $C_{(\pi')} \cong B_{(\pi)}[T]/(f)$ where $f \in B_{(\pi)}[T]$ is a monic Eisenstein polynomial of degree p, i.e. $f = T^p + b_{p-1}T^{p-1} + \cdots + b_0$ with $\pi | b_i$ for $0 \le i \le p-1$

and $\pi^2 \nmid b_0$. Again, by restricting to smaller affine open neighborhoods we may assume that already $f \in B[T]$, that $C \cong B[T]/(f)$ and that the coefficients b_i can be written as $b_i = \pi^{n_i} u_i$ where $u_i \in B^\times$, $n_i \geq 1$ for $i \neq 0$ and $n_0 = 1$. Now choose any closed point x of $U \cap D$ as well as a closed point $y \in V$ above x and a closed point $z \in W$ above y. Replace the extension $A \to B \to C$ by the localizations with respect to these points. Let d be the dimension of the regular local ring A. We can find elements $x_1, \ldots, x_{d-1} \in A$ such that $(x_1, \ldots, x_{d-1}, \pi)$ is a regular sequence in A. Let $(x'_1, \ldots, x'_{d-1}, \pi)$ be the image of $(x_1, \ldots, x_{d-1}, \pi)$ in B. It is again a regular sequence due to the flatness of $A \to B$. The $x'_1, \ldots, x'_{d-1}$ generate a prime ideal $\mathfrak{p}$ of B of height $d-1$ and the image $\bar{\pi}$ of π in $B/\mathfrak{p}$ is a uniformizer of the discrete valuation ring $B/\mathfrak{p}$ (cf. [Ma] §17, Theorem 36). By construction the polynomial $\bar{f} \in B/\mathfrak{p}[T]$ induced by f is a $\bar{\pi}$-Eisenstein polynomial. Hence, setting $K(\mathfrak{p}) = Q(B/\mathfrak{p})$, it follows that $K(\mathfrak{p}) \otimes_B C \cong K(\mathfrak{p})[T]/\bar{f}$ has ramification index p over $K(\mathfrak{p})$. Now let C be the integral curve on X corresponding to the prime ideal of A generated by the $x_1, \ldots, x_{d-1}$. It intersects D transversely in the point $x \in C^{reg}$ and $C' = C \cap X' \neq \emptyset$. The base change $Z' \times_{X'} \tilde{C}' \to \tilde{C}'$ is ramified along x.

2nd case: $v_{C_{(\pi')}}(\pi) = 1$.

After multiplication by a unit of $C^\times$ we can assume that $\pi' = \pi$. The residue field extension $k_C | k_B$ of $C_{(\pi)} | B_{(\pi)}$ has to be purely inseparable of degree p. We have $k_B = Q(B/(\pi))$ and $k_C = Q(C/(\pi))$ and may assume that $\operatorname{Spec} A/(\pi)$, $\operatorname{Spec} B/(\pi)$ and $\operatorname{Spec} C/(\pi)$ are again regular. Choose a codimension 1 point $\bar{y}$ of $\operatorname{Spec} B/(\pi)$ and a uniformizer $\bar{\lambda}$ of the local ring at $\bar{y}$. Let $\bar{z}$ be a point of $\operatorname{Spec} C/(\pi)$ above $\bar{y}$ and let $v_{\bar{z}}$ be the discrete valuation of k_C corresponding to $\bar{z}$. Again, there are two possibilities:

(i) $v_{\bar{z}}(\bar{\lambda}) = p$.

We may arrange the situation as follows:

- $\bar{\lambda}$ is already a uniformizer of the local ring of $\operatorname{Spec} A/(\pi)$ at the point $\bar{x} \in \operatorname{Spec} A/(\pi)$ below $\bar{y}$.
- There exists a uniformizer $\bar{\theta}$ of the local ring of $\operatorname{Spec} C/(\pi)$ at $\bar{z}$ such that $\bar{\lambda} = \bar{\theta}^p$.
- $\bar{\theta}$ is the reduction of an element $\theta \in C$ modulo π and the minimal polynomial $f = T^p + b_{p-1}T^{p-1} + \cdots + b_0 \in B[T]$ of θ over B has

constant coefficient $b_0 \in A$.

The reduction of f modulo π is equal to $T^p - \bar{\lambda}$, hence $\bar{\lambda}$ is the image of $\lambda := -b_0$ in B/π. After another reduction step, we may assume that for $1 \leq i \leq p-1$ the coefficients b_i can up to units of B be written as $b_i = \pi^{n_i}\lambda^{m_i}$ with $n_i > 0$ and $m_i \geq 0$. We may also assume that $B[T]/(f)$ and C are isomorphic. Now let z be a closed point of $W \cap \overline{\{\bar{z}\}}$. Let x (resp. y) be its image in U (resp. V). Localize the extension $A \to B \to C$ with respect to these points and keep the above notation in the local situation. Choose elements $x_1, \ldots, x_{d-2} \in A$, $d = \dim A$, that extend π and $\lambda - \pi$ to a regular sequence $(x_1, \ldots, x_{d-2}, \lambda - \pi, \pi)$ in A. Again by the flatness of $A \to B$ the image $(x'_1, \ldots, x'_{d-2}, \lambda - \pi, \pi)$ of $(x_1, \ldots, x_{d-2}, \lambda - \pi, \pi)$ is a regular sequence in B. Let $\mathfrak{p}$ be the prime ideal of B given by the elements $x'_1, \ldots, x'_{d-2}, \lambda - \pi$. Then the image $\bar{\pi}$ of π is a uniformizer of $B/(\mathfrak{p})$ and $\bar{f} \in B/(\mathfrak{p})[T]$ is a $\bar{\pi}$-Eisenstein polynomial. Put $K(\mathfrak{p}) = Q(B/\mathfrak{p})$. As in the first case, it follows that $K(\mathfrak{p}) \otimes_B C \cong K(\mathfrak{p})[T]/\bar{f}$ has ramification index p over $K(\mathfrak{p})$. So the curve C on X corresponding to the ideal generated by $x_1, \ldots, x_{d-2}, \lambda - \pi$ has the desired properties.

(ii) $v_{\bar{z}}(\bar{\lambda}) = 1$.

In this case the residue field extension of $\bar{z}$ over $\bar{y}$ is again purely inseparable of degree p. The residue fields of all points of codimension $d = \dim X$ are finite, hence perfect. Therefore, after finitely many iterations of the procedure above we have to end up with an extension that has ramification index p and inertia degree 1. To be more precise, we can restrict to smaller affine open subsets to obtain the following data:

- A sequence of points $y^{(0)}, y^{(1)}, \ldots, y^{(n)}$ in $\operatorname{Spec} B$, $n \leq d-1$, where $y^{(0)}$ is the point corresponding to (π) and $y^{(1)} := \bar{y}$, such that $y^{(i+1)}$ is regular point of codimension 1 in $\overline{\{y^{(i)}\}} \subset \operatorname{Spec} B$, $0 \leq i \leq n-1$.

- A sequence of points $z^{(0)}, z^{(1)}, \ldots, z^{(n)}$ in $\operatorname{Spec} C$, where $z^{(0)}$ is the point corresponding to (π) and $z^{(1)} := \bar{z}$, such that $z^{(i+1)}$ is a regular point of codimension 1 in $\overline{\{z^{(i)}\}} \subset \operatorname{Spec} C$ lying above $y^{(i+1)}$, $0 \leq i \leq n-1$. Moreover, the residue field extension of $z^{(i)}$ over $y^{(i)}$ is purely inseparable of degree p for $0 \leq i \leq n-1$, and $z^{(n)}$ has ramification index p over $y^{(n)}$.

- A sequence of elements $\lambda := \lambda_1, \lambda_2, \ldots, \lambda_n \in A$, where λ_i is a lift to B of a uniformizer of the local ring at $y^{(i)} \subset \overline{\{y^{(i-1)}\}}$, $1 \le i \le n$, which can be chosen to lie in A. Furthermore, λ_n can be chosen in a way such that its image in B/I_{n-1}, where I_{n-1} is the ideal corresponding to $y^{(n-1)}$, is the p-th root of a uniformizer $\theta_{z^{(n)}}$ of the local ring at $z^{(n)}$.
- A lift $\theta \in C$ of $\theta_{z^{(n)}}$ to C.
- The minimal polynomial $f = T^p + b_{p-1}T^{p-1} + \ldots + b_0 \in B[T]$ of θ over B.
- A closed point $x \in \operatorname{Spec} A \cap D$ and a closed point $y \in \operatorname{Spec} B$ above x, as well as elements $x_i \in A$, $1 \le i \le m := d - (n+1)$, and elements $x'_i \in B$, $1 \le i \le m$, such that $(x_1, \ldots, x_m, \lambda_n - \pi, \ldots, \lambda_1 - \pi, \pi)$ is a regular sequence at x and its image $(x'_1, \ldots, x'_m, \lambda_n - \pi, \ldots, \lambda_1 - \pi, \pi)$ in B is a regular sequence at y. In addition, these points can be chosen in a way such that f becomes an Eisenstein polynomial modulo the prime ideal corresponding to the ideal generated by the elements $x'_1, \ldots, x'_m, \lambda_n - \pi, \ldots, \lambda_1 - \pi$ (with respect to the uniformizer induced by π).

As in the previous cases, the curve C on X corresponding to the ideal generated by $x_1, \ldots, x_m, \lambda_n - \pi, \ldots, \lambda_1 - \pi$ satisfies the required conditions and the proof is finished. □

Theorem 3.3.2. *Let $X \in Sch(\mathbb{Z})$ be a regular scheme and let $\bar{X}$ be a regular compactification of X such that $D = \bar{X} \setminus X$ is a NCD on $\bar{X}$. Then for an abelian étale covering $Y \to X$ the following are equivalent:*

(i) *$Y \to X$ is curve-tame.*

(ii) *$Y \to X$ is tamely ramified along D.*

(iii) *$Y \to X$ is Wiesend modulus-tame.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) also holds for non-abelian coverings and is part of [KeS1] Theorem 4.4.
Assume that $Y \to X$ is curve-tame. Put $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$. For any curve C on $\bar{X}$ such that $C' := C \cap X \neq \emptyset$ let $\mathcal{I}_C = \prod_v v^{n_v}$ be the modulus on $k(C)$ induced by $\mathcal{I}$. For any place v of $k(C)$ we have $n_v \ge 1$ if the point of $\tilde{C}$

corresponding to v maps to $C \cap D$ under the normalization map $\tilde{C} \to C$, and $n_v = 0$ else. Since $Y \times_X \tilde{C}' \to \tilde{C}'$ is tamely ramified along $D \times_{\bar{X}} \tilde{C}$, we obtain that the corresponding extension of $k(C)$ is ramified with modulus $\mathcal{I}_C$. This shows that $Y \to X$ is ramified with Wiesend modulus $\mathcal{I}$.
We prove (iii) $\Rightarrow$ (ii). Assume that $Y \to X$ is not tamely ramified along a generic point η of D. In particular the residue charactaristic at η is a prime $p > 0$. In order to show that $Y \to X$ is not ramified with modulus $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$ (and hence with any tame modulus), it is sufficient to find a curve C on $\bar{X}$ meeting X and intersecting D in a point $x \in C^{reg}$ such that:

- The modulus-condition of $\mathcal{I}_C$ at x is tame.
- The base change $Y \times_X \tilde{C}' \to \tilde{C}'$ is wildly ramified along x, where $C' = C \cap C$.

The abelian group $G = \mathrm{Gal}(Y|X)$ equals the product of its Sylow subgroups and therefore it suffices to consider the case when G is a finite p-group. Let $\bar{Y}$ be the normalization of $\bar{X}$ in $k(Y)$. Let Y_G be the quotient of Y by the action of the inertia group of some point of $\bar{Y}$ above η and let $\bar{Y}_G$ be the normalization of $\bar{X}$ in $k(Y_G)$. Then $Y \to X$ factors through the étale covering $Y_G \to X$ and $Y \to Y_G$ is not tamely ramified along $\bar{Y}_G \times_{\bar{X}} D$. We may assume that the degree of $Y \to Y_G$ is p. Now we are in the situation to apply Proposition 3.3.1 which gives us the desired curve. □

Next, we show that the modulus condition induced by the class group $C_{\mathcal{I}}(\bar{X})$ for the ideal sheaf $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$ corresponds to the notion of tame ramification along the divisor D when D is square-free.

Theorem 3.3.3. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular scheme which is flat over* $\operatorname{Spec}\mathbb{Z}$ *and let $\bar{X}$ be a normal compactification of X such that $D = \bar{X} \setminus X$ is a square-free divisor on X. Assume that $X(\mathbb{R}) = \emptyset$. Let K_D be the maximal abelian extension $L|K$ such that $X_L \to X$ is tamely ramified along the generic points of D. Then for the ideal sheaf $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$ the reciprocity map induces an isomorphism*

$$\rho_{\bar{X},K_D} \colon C_{\mathcal{I}}(\bar{X}) \xrightarrow{\sim} \mathrm{Gal}(K_D|K).$$

Proof. Let $L|K$ be a finite abelian extension such that $X_L \to X$ is étale. Let V be a discrete valuation ring of rank d of K dominating a Parshin chain

$P = (p_0, \dots, p_d)$ of length d on X. Then V comes together with a sequence

$$V = V_0 \subset \cdots \subset V_n = K$$

of discrete valuation rings of K (cf. section 1.4) and by definition the discrete valuation ring (of rank 1) V_{d-1} dominates and hence is equal to $\mathcal{O}_{\bar{X},p_{d-1}}$. Let K_V be the quotient field of the henselization V^h of V and R_V the localization of V^h at the unique prime ideal of height one. By Corollary 3.1.3 the reciprocity map

$$\rho_{\bar{X},L} \colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K)$$

exists if and only if the composition

$$K_d^M(R_V, \mathcal{I}R_V) \to K_d^M(K_V) \overset{\rho_V}{\to} \mathrm{Gal}(K_V^{ab}|K_V) \to \mathrm{Gal}(LK_V|K_V)$$

is zero. For $p_{d-1} \in X$ we have

$$K_d^M(R_V, \mathcal{I}R_V) = K_d^M(R_V)$$

and $LK_V|K$ is unramified. By Proposition 1.4.13, we know that

$$\mathrm{im}\big[K_d^M(R_V) \to K_d^M(K_V)\big] \subset N_{LK_V|K_V}(K_d^M(L))$$

and so the composition

$$K_d^M(R_V) \to K_d^M(K_V) \overset{\rho_V}{\to} \mathrm{Gal}(K_V^{ab}|K_V) \to \mathrm{Gal}(LK_V|K_V)$$

is zero by Theorem 1.4.9. If $p_{d-1} \in D = \bar{X} \setminus X$ we have $\mathcal{I}R_V = (\pi_V)$, where π_V is a uniformizer of R_V. The morphism $X_L \to X$ is tamely ramified along p_{d-1} if and only if $LK_V|K_V$ is tamely ramified (w.r.t. R_V). By Proposition 1.4.13, this is equivalent to

$$\mathrm{im}\big[K_d^M(R_V, (\pi_V)) \to K_d^M(K_V)\big] \subset N_{LK_V|K_V}(K_d^M(L)).$$

Hence it follows from Theorem 1.4.9 that the reciprocity map

$$\rho_{\bar{X},L} \colon C_{\mathcal{I}}(\bar{X}) \to \mathrm{Gal}(L|K)$$

exists if and only if $X_L \to X$ is tamely ramified along the generic points of D.

Now by class field theory, the reciprocity map induces an isomorphism between $C_{\mathcal{I}}(\bar{X})$ and $\mathrm{Gal}(L|K)$ for a finite abelian extension $L|K$ such that

$X_L \to X$ is étale. It follows from the above that L has to be the function field of the maximal abelian covering of X which is tamely ramified along the generic points of D. □

We can now compare the modulus conditions of Kato-Saito and Wiesend corresponding to the ideal $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$ when D is a normal crossing divisor on $\bar{X}$.

Theorem 3.3.4. *Let $X \in Sch(\mathbb{Z})$ be a connected, regular, flat scheme over* $\operatorname{Spec}\mathbb{Z}$ *and let $\bar{X}$ be a regular compactification of X such that $D = \bar{X} \setminus X$ is a NCD on $\bar{X}$. Assume that $X(\mathbb{R}) = \emptyset$ and put $\mathcal{I} = \mathcal{O}_{\bar{X}}(-D)$. Then the canonical map*

$$C_W^h(X) \to C_{\mathcal{I}}(\bar{X})$$

induces an isomorphism

$$C_{W,\mathcal{I}}^h(X) \xrightarrow{\sim} C_{\mathcal{I}}(\bar{X}).$$

Proof. Let K_D be the function field of the maximal abelian extension of X which is ramified along D. By Corollary 3.2.4 we have a commutative diagram

$$\begin{array}{ccc} C_W^h(X) & \twoheadrightarrow & C_{\mathcal{I}}(\bar{X}) \\ & \searrow\!\!\!\!\searrow & \downarrow \wr\ \rho_{\bar{X},K_D} \\ & & \operatorname{Gal}(K_D|K), \end{array}$$

and $\rho_{\bar{X},K_D}$ is an isomorphism by Theorem 3.3.3. Let K_I be the maximal abelian extension $L|K$ such that $X_L \to X$ is ramified with Wiesend modulus $\mathcal{I}$. By Theorem 3.3.2 we have

$$\operatorname{Gal}(K_{\mathcal{I}}|K) = \operatorname{Gal}(K_D|K).$$

So it follows from the commutative diagram

$$\begin{array}{ccc} & C^h_W(X) & \\ \swarrow & & \searrow \\ C^h_{W,\mathcal{I}}(X) & \dashrightarrow & C_{\mathcal{I}}(\bar{X}) \\ \wr\downarrow & & \wr\downarrow \rho_{\bar{X},K_D} \\ \mathrm{Gal}(K_{\mathcal{I}}|K) & = & \mathrm{Gal}(K_D|K) \end{array}$$

that the map

$$C^h_W(X) \to C_{\mathcal{I}}(\bar{X})$$

factors through $C^h_{W,\mathcal{I}}(X)$ and that

$$C^h_{W,\mathcal{I}}(X) \to C_{\mathcal{I}}(\bar{X})$$

is an isomorphism. □

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