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On Perrin-Riou's exponential map and reciprocity laws for (φ, Γ)-modules

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Abstract

Let $K/\mathbb{Q}_p$ be a finite Galois extension and D a (φ, Γ)-module over the Robba-Ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and $\mathbf{N}_{\mathrm{dR}}(D)$ its associated p-adic differential equation.

In the first part we give a generalization of the Bloch-Kato exponential map for D using continuous Galois-cohomology groups $H^i(G_K, \mathbf{W}(D))$ for the B-pair $\mathbf{W}(D)$ associated to D. We construct a big exponential map $\Omega_{D,h}$ $(h \in \mathbb{N})$ for cyclotomic extensions of K for D in the style of Perrin-Riou extending the techniques of Berger, which interpolates the generalized Bloch-Kato exponential maps on the finite levels.

In the second part we extend two definitions for pairings on D and its dual $D^*(1)$ (resp. on $\mathbf{N}_{\mathrm{dR}}(D)$ and its dual $\mathbf{N}_{\mathrm{dR}}(D^*(1))$) and prove a generalization of the reciprocity law, which relates these pairings under the big exponential map.

Finally, we give some results on the determinant associated to $\Omega_{D,h}$ and formulate an integral version of a determinant conjecture in the semistable case. Further, we define i-Selmer groups and show under certain hypothesis a torsion property.

Zusammenfassung

Sei $K/\mathbb{Q}_p$ eine endliche Galoissche Erweiterung und D ein (φ, Γ)-Modul über $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ sowie $\mathbf{N}_{\mathrm{dR}}(D)$ die dazu assoziierte p-adische Differentialgleichung.

Im ersten Abschnitt definieren wir eine Verallgemeinerung der Bloch-Kato Exponentialabbildung für D, welche stetige Galois-Kohomologiegruppen $H^i(G_K, \mathbf{W}(D))$, die für das B-Paar $\mathbf{W}(D)$, welches zu D assoziiert ist, verwendet. Wir konstruieren eine grosse Exponentialabbildung $\Omega_{D,h}$ $(h \in \mathbb{N})$ für die zyklotomische Erweiterung von K für D im Stil von Perrin-Riou, wobei wir die Techniken von Berger verwenden, und zeigen, dass diese die verallgemeinerten Bloch-Kato Exponentialabbildungen auf allen endlichen Leveln interpoliert.

Im zweiten Abschnitt erweitern wir zwei Definitionen für Paarungen auf D und seinem Dual $D^*(1)$ (bzw. auf $\mathbf{N}_{\mathrm{dR}}(D)$ und seinem Dual $\mathbf{N}_{\mathrm{dR}}(D^*(1))$) und zeigen ein verallgemeinertes Reziprozitätgesetz, welches diese Paarungen mit Hilfe der grossen Exponentialabbildungen verbindet.

Schliesslich zeigen wir einige Ergebnisse hinsichtlich der Determinante assoziiert zu $\Omega_{D,h}$, und formulieren eine ganze Version einer Determinantenvermutung im semistabilen Fall. Letzlich definieren wir gewisse i-Selmer-Gruppen und zeigen under bestimmten Vorraussetzungen eine Torsionseigenschaft.

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Contents

Chapter 1

Introduction

In her seminal paper "Théorie d'Iwasawa des représentations p-adiques sur un corps local" ([33]) Perrin-Riou paved the way for a framework which should make it possible, by starting with a (crystalline) p-adic representation V and an Euler-system, to produce p-adic L-functions. We recall some notation. Let K be a finite extension of $\mathbb{Q}_p$ and denote by F the biggest subextension of K that is unramified over $\mathbb{Q}_p$. Let μ_{p^n} denote the roots of unity in a fixed algebraic closure $\overline{K}$ of K and set $K_n = K(\mu_{p^n})$ and $K_\infty = \bigcup_n K_n$. As usual G_K denotes the absolute Galois group of K, and we set $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = G_K/H_K$, so that $\Gamma_K = \mathrm{Gal}(K_\infty/K) \subset \mathbb{Z}_p^\times$. One has $\Gamma_K \cong \Delta_K \times \Gamma'_K$, where $\Gamma'_K = \mathbb{Z}_p$, and Δ_K is the torsion subgroup of Γ_K. One defines the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma_K]]$ which may be identified with $\mathbb{Z}_p[\Delta_K][[T]]$ for a variable T. Perrin-Riou considers a distribution algebra $\mathcal{H}(\Gamma_K)$ that contains Λ and may be described as a subalgebra of $\mathbb{Q}_p[\Delta_K][[T]]$ such that the power series satisfy a certain "growth" condition.

For a representation V, we may of course consider continuous cohomology $H^i(K,V)$. It is customary to set $H^1_{\mathrm{Iw}}(K,V) = (\varprojlim H^1(K_n,T)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where the transition morphisms are given by corestriction, and one checks that this is a finitely generated $\Lambda_{\mathbb{Q}_p} = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module.

By the theory of Fontaine one has certain $\mathbb{Q}_p$-algebras $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$ that come equipped with an action of G_K. One may then associate to V finite dimensional F-vector spaces

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \subset \mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

resp. a finite dimensional K-vector space $\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ with $\mathbf{D}_{\mathrm{st}}(V) \subset \mathbf{D}_{\mathrm{dR}}(V)$ where the first two come equipped with an action of a Frobenius φ and a nilpotent monodromy operator N, and the third one is equipped with a filtration coming from a filtration $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}$ on $\mathbf{B}_{\mathrm{dR}}$.

Bloch and Kato constructed, starting with the so-called "fundamental exact sequence" of G_K-modules

$$0 \longrightarrow V \longrightarrow \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V \longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \longrightarrow 0, \tag{1.1}$$

the exponential map $\exp : \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow H^1(K,V)$, which is nothing but the transition morphism arising from the long exact sequence of continuous Galois cohomology for (1.1).

They showed that there exists a deep connection between this map and the special values of the complex L-function attached to V.

Perrin-Riou set out to adapt this construction to the theory of p-adic L-functions. Explicitly, for $K/\mathbb{Q}_p$ unramified and V crystalline (i.e. $\dim_F \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$) she constructed a map $\Omega_{V(j),h}$ that fits into the following diagram

$$\begin{array}{ccc} \mathcal{H}(\Gamma_K)\otimes_{\mathbb{Q}_p}\mathbf{D}_{\mathrm{cris}}(V(j)) & \xrightarrow{\Omega_{V(j),h}} & \mathcal{H}(\Gamma_K)\otimes_\Lambda H^1_{\mathrm{Iw}}(K,V(j))/V(j)^{G_{\mathbb{Q}_{p,n}}} \\ \downarrow{\scriptstyle \Xi_{n,j}} & & \downarrow{\scriptstyle \mathrm{pr}_n} \\ K_n\otimes\mathbf{D}_{\mathrm{st}}(V(j)) & \xrightarrow{(h-1)!\exp_{K_n n,V(j)}} & H^1(K_n,V(j)) \end{array} \tag{1.2}$$

for $h \gg 0$, $j \gg 0$ and all n, where $\Xi_{n,j}$ and pr_n are certain canonical projections. The point here is that $\Omega_{V,h}$ interpolates infinitely many Bloch-Kato exponential maps on the finite levels.

In [36], Perrin-Riou extended her construction to semi-stable representations over unramified extensions. She gave a definition of a free $\mathcal{H}(\Gamma_K)$-module $\mathbf{D}_{\infty,g}(V)$ and a map

$$\Omega_{V,h} : \mathbf{D}_{\infty,g}(V) \longrightarrow \mathcal{H}(\Gamma_{\mathbb{Q}_p})\otimes_\Lambda H^1_{\mathrm{Iw}}(K,V)/V^{G_{K_\infty}}$$

that has a similar interpolation property as (1.2) for $j \gg 0$ and $n \gg 0$.

It was Berger who gave an explicit description of a "big exponential map" for crystalline representations using these modules not only on the finite level, but on the whole of $\mathcal{H}(\Gamma_K)\otimes\mathbf{D}_{\mathrm{cris}}(V)$ and $H^1_{\mathrm{Iw}}(K,V)$. His fundamental insight is the comparison isomorphism

$$\mathbf{B}^\dagger_{\log,K}[1/t]\otimes_F\mathbf{D}_{\mathrm{st}}(V) = \mathbf{B}^\dagger_{\log,K}[1/t]\otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}}\mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

(cf. Theorem 2.4.5). Let us briefly explain some notation. Basically, if we recall the definition of $\mathcal{H}(\Gamma_K)$, one has $\mathcal{H}(\Gamma_K) = \mathbb{Q}_p[\Delta_K]\otimes\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, where $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ corresponds to certain power series in $\mathbb{Q}_p[[T]]$ that satisfy a growth condition. Now one may think of $\mathbf{B}_{\mathrm{st}}$ as $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{cris}}[\log T]$, where the series $\log T$ is a transcendent element for the fraction field of certain analytic functions. Since one needs to employ the differential operator $\partial = (1+T)\frac{d}{dT}$, one sees that in the semi-stable case denominators $1/T^i$ should occur since $\partial\log T = 1 + 1/T$. Hence, one may define $\mathbf{B}^\dagger_{\mathrm{rig},K}$, also referred to as the Robba ring, as Laurent-series that satisfy certain growth conditions in both directions. One sets $\mathbf{B}^\dagger_{\log,K} = \mathbf{B}^\dagger_{\mathrm{rig},K}[\log T]$. Finally, the element $t\in\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ is defined as $\log(1+T)$ and is often referred to as the "period". All these rings come equipped with actions of Γ_K, a Frobenius φ, which induces a left-inverse ψ, and a monodromy operator N. We remark that there exists a bigger ring $\mathbf{B}^\dagger_{\mathrm{rig}}$ equipped with an action of G_K such that $\mathbf{B}^\dagger_{\mathrm{rig},K} = (\mathbf{B}^\dagger_{\mathrm{rig}})^{H_K}$. One sets $\mathbf{D}^\dagger_{\mathrm{rig}}(V) = (\mathbf{B}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{Q}_p} V)^{H_K}$, a so-called (φ,Γ)-module, i.e., a finitely generated free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module that comes equipped with commuting actions of a Frobenius φ and Γ_K.

The important part of the above comparison is that if V is a *positive* (cf. section 2.2.2) semi-stable representation then one has an inclusion

$$\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}^\dagger_{\log,K}\otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\Gamma_K} \subset \mathbf{B}^\dagger_{\log,K}\otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}}\mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

that is compatible with the actions of φ and N.

Returning to the crystalline case one obtains an inclusion

$$(t^h \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1} \subset \mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\psi=1}$$

for h large enough. By the work of Fontaine, Cherbonnier and Colmez one has the natural identification of $\mathcal{H}(\Gamma_K)$-modules

$$\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} = \mathcal{H}(\Gamma_K) \otimes_\Lambda H^1_{\mathrm{Iw}}(K,V).$$

Additionally, one has (under certain assumptions: cf. Proposition 3.3.2)

$$(\varphi-1)(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(V(j)))^{\psi=1} = ((\mathbf{B}^+_{\mathrm{rig},K})^{\psi=0} \otimes \mathbf{D}_{\mathrm{cris}}(V(j)))$$

and

$$(\varphi-1)\mathbf{D}^\dagger_{\mathrm{rig}}(V(j))^{\psi=1} = \mathcal{H}(\Gamma_{\mathbb{Q}_p}) \otimes_\Lambda H^1_{\mathrm{Iw}}(K,V(j))/V(j)^{G_{K_\infty}}.$$

Hence, one may hope to give a description of $\Omega_{V(j),h}$ in terms of these modules. Berger considered in the crystalline case the element $\nabla_{h-1} \circ \ldots \circ \nabla_0$, where $\nabla_i \in \mathcal{H}(\Gamma_K)$ is Perrin-Riou's differential operator, and showed that one obtains a map

$$\nabla_{h-1} \circ \ldots \circ \nabla_0 : (\varphi-1)(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(V(j)))^{\psi=1} \longrightarrow (\varphi-1)\mathbf{D}^\dagger_{\mathrm{rig}}(V(j))^{\psi=1}$$

that actually coincides with Perrin-Riou's $\Omega_{V(j),h}$ (see [5], Theorem II.13).

Since one has an embedding of the category of p-adic representations into the category of all (φ,Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ via the functor $\mathbf{D}^\dagger_{\mathrm{rig}}(\ )$, one might be inclined to generalize the framework of exponential maps to this setting. We recall the basic definitions in chapter two. If D is a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, one also has a notion of Galois cohomology groups $H^i(K,D)$, and one may consider projection maps $h^1 : D^{\psi=1} \longrightarrow H^1(K,D)$, so that once again $D^{\psi=1}$ takes the role of the Iwasawa cohomology. Similarly, one may define finite-dimensional vector spaces $\mathbf{D}_{\mathrm{cris}}(D)$, $\mathbf{D}_{\mathrm{st}}(D)$ and $\mathbf{D}_{\mathrm{dR}}(D)$, generalized Bloch-Kato exponential maps

$$\exp : \mathbf{D}_{\mathrm{dR}}(D) \to H^1(K,D),$$

and develop the notion of a (φ,Γ)-module being crystalline, semi-stable or de Rham. We define a $\mathcal{H}(\Gamma_K)$-module $\mathbf{D}_{\infty,g}(D)$ and show that there exists a map for $h \gg 0$

$$\Omega_{D,h} := \nabla_{h-1} \circ \ldots \circ \nabla_0 : \mathbf{D}_{\infty,g}(D) \longrightarrow (\varphi-1)D^{\psi=1}.$$

The main result of the third chapter is then the following interpolation property (see Theorem 3.2.21 for the precise statement):

Theorem. Let D be a de Rham (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, $g \in \mathbf{D}_{\infty,g}(D)$ and G a "complete solution" (cf. Definition 3.2.11) for g in L and let $h \gg 0$. Then for $k \geq 1-h$ and $n \gg 1$ one has

$$\begin{aligned} &h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k) \\ &\qquad = p^{-n(K_n)}(-1)^{h+k-1}(h+1-k)!\frac{1}{[L_n:K_n]}\mathrm{Cor}_{L_n/K_n}\exp_{K_n,D(k)}(\Xi_{n,k}(G)). \end{aligned}$$

Now if one is interested in the construction of p-adic L-functions, one needs to construct a certain "inverse" of the map Ω_h. This construction depends on the so-called reciprocity law. The general idea behind a reciprocity law (in our setting) for a (φ, Γ)-module D is to construct a pairing

$$[\ ,\ ]_{\mathrm{Iw},D} : \mathbf{D}_{\infty,g}(D) \times \mathbf{D}_{\infty,g}(D^*(1)) \longrightarrow \mathrm{Frac}(\mathcal{H}(\Gamma_K))$$

"coming from convolution of measures" and a pairing

$$\langle\ ,\ \rangle_{\mathrm{Iw},D} : (\varphi - 1)D^{\psi=1} \times (\varphi - 1)D^*(1))^{\psi=1} \longrightarrow \mathcal{H}(\Gamma_K)$$

"coming from cohomology" such that one has a relation of the form

$$\langle \Omega_{D,h}(y), \Omega_{D^*(1),1-h}(v)\rangle_{\mathrm{Iw},D} = [y, v]_{\mathrm{Iw},D}$$

for $y \in \mathbf{D}_{\infty,g}(D),\ v \in \mathbf{D}_{\infty,g}(D^*(1))$, modulo some conditions.

The main result of the fourth chapter is then (see Theorem 4.5.1):

Theorem. Assume $K/\mathbb{Q}_p$ is unramified and let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Assume that D is either semi-stable and étale or crystalline (cf. Definition 2.6.14). Let $y \in \mathbf{D}_{\infty,g}(D)$, $v \in \mathbf{D}_{\infty,g}(D^*(1))$. Then for every $h \geq 1$ one has

$$\langle \Omega_{V,h}(y), \sigma_{-1} \cdot \Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},D} = (-1)^{h+1}[y, \iota(v)]_{\mathrm{Iw},D}.$$

We remark that it is also possible to give a proof in the 2-dimensional semistable case withouth the étaleness assumption.

In chapter five we give some applications and examples concerning the exponential map and the reciprocity law. Namely, we consider the case of a two-dimensional p-adic representation V attached to an ordinary semi-stable elliptic curve E. We state a conjecture concerning the determinant of the map Ω_h which up until now was only formulated in the crystalline case, and prove it in the above example.

We mention that it is certainly important to consider the above constructions not only for (φ, Γ)-modules coming from p-adic representations: suppose for example that $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ for a representation V. Then, Kedlaya's slope filtration theorem ensures that there exists a filtration

$$0 = D_1 \subset \ldots \subset D_r = D$$

of (φ, Γ)-submodules such that the quotients D_i/D_{i-1} satisfy a certain condition on the slope. However, the single D_i need not come from a p-adic representation. By the method of dévissage, which we already employ here in proofs, one may then infer statements for $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ from the successive steps of the above filtration.

We remark that during this work learned of the results of Kentaro Nakamura, who gave a similar description of a "big exponential map" for (φ, Γ)-modules. However our definitions and proofs are different and use the more general notion of the $\mathcal{H}(\Gamma_K)$-modules $\mathbf{D}_{\infty,e}(D)$, $\mathbf{D}_{\infty,f}(D)$ and $\mathbf{D}_{\infty,g}(D)$ which are especially important if one wants to consider reciprocity laws and the connection of exponential maps with p-adic L-functions.

Chapter 2

Rings and Modules

2.1 General notations

The general strategy of Fontaine is to study p-adic representations by certain *admissibility* conditions. Recall that if V is a finite dimensional $\mathbb{Q}_p$-vectorspace endowed with a continuous action of a topological group G and if B is a topological $\mathbb{Q}_p$-algebra which also carries an action of G, then Fontaine considers the B^G-modules $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^G$. It inherits actions from B and V. One says that V is B**-admissible** if $B \otimes_{\mathbb{Q}_p} V \cong B^d$ as G-modules, i.e., B is "big enough" so that it trivializes the action of G.

Let k be a perfect field of characteristic p. We denote by $(W(k)$ the ring of Witt-vectors for k and set $F = \mathrm{Quot}(W(k))$. Let K/F be a totally ramified extension of F. Fix an algebraic closure $\overline{F}$ of F and denote by $\mathbb{C}_p = \widehat{\overline{F}}$ the p-adic completion of this closure. This is then again algebraically closed by Krasners Lemma. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the group of automorphisms of $\overline{K}$ which fix K. By continuity these are also the K-linear automorphisms of $\mathbb{C}_p$. Let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$ and $\mathfrak{m}_{\mathbb{C}_p}$ its maximal ideal. We have $\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} = \overline{k}$.

We denote by μ_{p^n} the group of roots of unity of p^n-order in $\mathbb{C}_p$ and set $K_n = K(\mu_{p^n})$. Further we pose $K_\infty = \bigcup_n K_n$. We fix once and for all a compatible set of primitive p-th roots of unity $\{\zeta_{p^n}\}_{n \geq 0}$ such that $\zeta_1 = 1$, $\zeta_p \neq 1$, $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. One has the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ which is defined by the formula $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for $n \geq 1$ and $g \in G_K$.We set $H_K = \ker(\chi)$ and $\Gamma_K = G_K/H_K$, which is the Galois group of K_∞/K. We know that this can also be identified via the cyclotomic character with an open subgroup of $\mathbb{Z}_p^\times$.

When we start with a finite extension $K/\mathbb{Q}_p$ we denote by $F = K_0$ the maximal unramified extension of $\mathbb{Q}_p$ in K. Further denote by K_0' the biggest unramified subextension of K_0 in K_∞.

By a p-adic representation we mean a finite dimensional $\mathbb{Q}_p$-vectorspace endowed with a continuous and linear action of G_K. A $\mathbb{Z}_p$-representation is a free $\mathbb{Z}_p$-module of finite rank equipped with a linear and continuous action of G_K. It is known that if V is a p-adic representation then there exists a $\mathbb{Z}_p$-lattice T in V that is stable under the action of G_K.

If $C^\bullet(-)$ denotes complex of R-modules for some commutative ring (for example,

$C^\bullet(G_K, M))$ R we denote as usual $R\Gamma(-)$ the complex which we regard as an object in the derived category of R-modules.

2.2 The rings of Fontaine

We recall certain rings constructed by Fontaine, see for instance [21].

2.2.1 Rings of characteristic p

Let

$$\widetilde{\mathbf{E}} = \varprojlim_{x\mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \ldots)|\ x^{(i)} \in \mathbb{C}_p, (x^{(i+1)})^p = x^{(i)}\ \forall i\}.$$

Similarly, let

$$\begin{aligned}\widetilde{\mathbf{E}}^+ = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} &= \{(x^{(0)}, x^{(1)}, \ldots)|\ x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}, (x^{(i+1)})^p = x^{(i)}\ \forall i\}\\ &\cong \{(x_n)_{n\in\mathbb{N}}|\ x_n \in \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}, x_{n+1}^p = x_n\ \forall n\}\end{aligned}$$

This is the set of elements of $\widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. One can define multiplication and addition on these sets in the following way. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are in $\widetilde{\mathbf{E}}$ then we define

$$\begin{aligned}(x+y)^{(i)} &= \lim_{j\to\infty} (x^{(i+j)} + y^{(i+j)})^{p^j}\\ (x\cdot y)^{(i)} &= x^{(i)}y^{(i)}.\end{aligned}$$

$\widetilde{\mathbf{E}}^+$ is a valuation ring with valuation

$$v_E(x) = v_p(x^{(0)})$$

and maximal ideal

$$\mathfrak{m}_{\widetilde{\mathbf{E}}^+} = \{x \in \widetilde{\mathbf{E}}^+|\ v_E(x) > 0\}.$$

One can show that $\widetilde{\mathbf{E}}$ is the fraction field of $\widetilde{\mathbf{E}}^+$.

With the choices of the primitive p^n-th roots of unity one defines the element

$$\varepsilon = (1, \zeta_p, \ldots) \in \widetilde{\mathbf{E}}^+.$$

We set $\overline{\pi} = \varepsilon - 1 \in \widetilde{\mathbf{E}}^+$. One has the following commuting actions on $\widetilde{\mathbf{E}}$, which restrict to actions of $\widetilde{\mathbf{E}}^+$:

a) A Frobenius φ, given by $\varphi((x^{(n)})) = ((x^{(n)})^p)$,

b) The action of $G_{\mathbb{Q}_p}$, given by $g((x^{(n)})) = ((gx^{(n)}))$ for $g \in G_{\mathbb{Q}_p}$.

For $K/\mathbb{Q}_p$ finite we set

$$\mathbf{E}_K^+ = \{(x_n) \in \widetilde{\mathbf{E}}^+ |\ x_n \in \mathcal{O}_{K_n}/p\mathcal{O}_{K_n}\ \forall n \geq n(K)\},$$

where $n(K)$ is some constant depending on K which arises in the fields of norm theory of Fontaine-Wintenberger (cf. [20]). We put $\mathbf{E}_K = \mathbf{E}_K^+[1/\overline{\pi}]$. One can show that that $\mathbf{E}_F = \kappa((\overline{\pi}))$ and one defines $\mathbf{E}$ as the seperable closure of $\mathbf{E}_F$ in $\widetilde{\mathbf{E}}$. Let $\mathbf{E}^+ = \mathbf{E} \cap \widetilde{\mathbf{E}}^+$ and $\mathfrak{m}_E = \mathbf{E} \cap \mathfrak{m}_{\widetilde{\mathbf{E}}}$. One can show that $\mathbf{E}_K = \mathbf{E}^{H_K}$ and one knows that $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K) \cong H_K$.

We need another description of $\mathbf{E}_K$. From [12], Proposition I.1.1 we know that we have an isomorphism $\iota_K : \varprojlim O_{K_n} \cong \mathbf{E}_K^+$ (here the limit is taken with respect to the norms). Let $m \in \mathbb{N}$ and $\omega_m \in K_m$ be a uniformizer and set $\omega_m^{(n)} = N_{K_m/K_n}(\omega_m)$ if $n \leq m$ and $\omega_m^{(n)} = 0$ if $n \geq m+1$. One sees that the sequence $((\omega_m^{(n)})_{n\in\mathbb{N}})_{m\in\mathbb{N}}$ has a limit $\omega = (\omega^{(n)})_{n\in\mathbb{N}}$ in $\varprojlim O_{K_n}$ such that $\omega^{(n)}$ is a uniformizer in K_n if n is big enough. Hence,

$$v_{\mathbf{E}}(\iota_K(\omega)) = \frac{1}{[K_\infty : F_\infty]} v_{\mathbf{E}}(\overline{\pi}), \tag{2.1}$$

which shows that E_K is a finite extension of E_{K_0} of degree $[E_K : E_{K_0}] = [K_\infty : K_{0,\infty}]$ and that the element $\overline{\pi}_K = \iota_K(\omega)$ is a uniformizer of E_K.

2.2.2 Rings of characteristic 0

Let W be the Witt functor. We set

$$\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+), \quad \widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}}) = W(\mathrm{Frac}(\widetilde{\mathbf{E}}^+)), \quad \widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p].$$

Every $x \in \widetilde{\mathbf{B}}^+$ can be written as

$$x = \sum_{k \gg -\infty}^{\infty} p^k [x_k],$$

where $x_k \in \widetilde{\mathbf{E}}^+$ and $[x_k]$ is its Teichmüller representative. The commuting actions of φ and $G_{\mathbb{Q}_p}$ on $\widetilde{\mathbf{E}}^+$ extend to an action of $\widetilde{\mathbf{B}}^+$ (and $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \ldots$):

a) $\varphi(\sum_{k\gg-\infty}^{\infty} p^k [x_k]) = \sum_{k\gg-\infty}^{\infty} p^k [x_k^p]$,

b) $g(\sum_{k\gg-\infty}^{\infty} p^k [x_k]) = \sum_{k\gg-\infty}^{\infty} p^k [g(x_k)]$ for $g \in G_{\mathbb{Q}_p}$.

We have a ring homomorphism

$$\theta : \widetilde{\mathbf{B}}^+ \longrightarrow \mathbb{C}_p$$
$$\sum_{k\gg-\infty}^{\infty} p^k [x_k] \longmapsto \sum_{k\gg-\infty}^{\infty} p^k x_k^{(0)}.$$

We set $\pi = [\overline{\pi}] = [\varepsilon] - 1$, $\pi_n = [\varepsilon^{p^{-n}}] - 1$, $\omega = \pi/\pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. Then $\ker(\theta)$ is a principal ideal generated by ω.

The ring $\mathbf{B}_{\mathrm{dR}}^+$ is defined by completing $\widetilde{\mathbf{B}}^+$ with the $\ker(\theta)$-adic topology, i.e.,

$$\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_{n\geq 0} \widetilde{\mathbf{B}}^+/(\ker(\theta)^n).$$

This gives a complete discrete valuation ring with maximal ideal $\ker(\theta)$. One can show that $\log([\varepsilon])$ converges in $\mathbf{B}_{\mathrm{dR}}^+$, and we denote this element by t. It is a generator of the maximal ideal, hence we can form the field $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$. This field is equipped with an action of $G_{\mathbb{Q}_p}$ and a canonical filtration defined by $\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}}) = t^i\mathbf{B}_{\mathrm{dR}}^+,\ i \in \mathbb{Z}$.

We say that a p-adic representation V of G_K is **de Rham** if it is $\mathbf{B}_{\mathrm{dR}}$-admissible. We put

$$\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V)(\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

From Fontaines theory it is known that $\mathbf{D}_{\mathrm{dR}}(V)$ is finite dimensional K-vectorspace which we endow with the above (exhaustive, seperated and decreasing) filtration.

We say that a p-adic representation V is **Hodge-Tate** with Hodge-Tate weights $h_1, \ldots, h_d$ if one has a decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{i=1}^d \mathbb{C}_p(h_i)$. We say that V is **positive** if its Hodge-Tate weights are negative. It is known that every de Rham representation is Hodge-Tate and that the Hodge-Tate weights are those integers h such that there is a jump in the filtration at $-h$, i.e. $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-h+1}\mathbf{D}_{\mathrm{dR}}(V)$. With this convention the representation $\mathbb{Q}_p(1)$ is of weight 1.

Let $\mathbf{A}_{\mathbb{Q}_p}^+ = \mathbb{Z}_p[[\pi]] \hookrightarrow \widetilde{\mathbf{A}}^+$, where the second arrow is an inclusion since $\widetilde{\mathbf{A}}^+$ is complete, which is stable under φ and $G_{\mathbb{Q}_p}$. Let

$$\mathbf{A}_{\mathbb{Q}_p} = \widehat{\mathbb{Z}[[\pi]][1/\pi]} = \left\{ \sum_{k\in\mathbb{Z}} a_k\pi^k \;\middle|\; a_k \in \mathbb{Z}_p, \lim_{k\to-\infty} v_p(a_k) = +\infty \right\} \hookrightarrow \widetilde{\mathbf{A}},$$

and set $\mathbf{B}_{\mathbb{Q}_p} = \mathbf{A}_{\mathbb{Q}_p}[1/p]$. Then $\mathbf{B}_{\mathbb{Q}_p}$ is a field, complete for the p-adic valuation with ring of integers $\mathbf{A}_{\mathbb{Q}_p}$ and residue field $\mathbf{E}_{\mathbb{Q}_p}$. Let now $\mathbf{B} = \bigcup_{K/\mathbb{Q}\text{finite}} B_K$ in $\widetilde{\mathbf{B}}$, which is a separable closure of $\mathbf{B}_{\mathbb{Q}_p}$. We define $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$, $\mathbf{A}^+ = \mathbf{A} \cap \widetilde{\mathbf{A}}^+$. These rings still have the commuting action of φ and $G_{\mathbb{Q}_p}$. We put $\mathbf{A}_K = A^{H_K}$ and $\mathbf{B}_K = \mathbf{A}_K[1/p]$. By Hensel's Lemma there exists a unique lift $\pi_K \in \mathbf{A}_K$ such that the reduction mod p is equal to $\overline{\pi}_K$, viewed as an element in $\widetilde{\mathbf{A}}$.

It is known that one may write

$$\mathbf{A} = \bigoplus_{i=0}^{p-1} (1+\pi)^i \varphi(\mathbf{A}), \tag{2.2}$$

see for instance [15], Lemma 5.3.1. A similar decomposition holds for $\mathbf{A}_K, \mathbf{E}_K, \mathbf{E}_K$ (with possibly $\overline{\pi}$ in place of π), so that one may define the (continuous) operator ψ by

$$\psi : \mathbf{A}_K \longrightarrow \mathbf{A}_K, \quad x = \sum_{i=0}^{p-1} (1+\pi)^i \varphi(x_i) \longmapsto x_0$$

which satisfies $\psi \circ \varphi = \mathrm{id}$ such that ψ is surjective and commutes with the action of G_K. An equivalent definition of ψ on $\mathbf{B}$ is given by the formula

$$\psi(x) = \frac{1}{p} \cdot \varphi^{-1}(\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x))$$

for $x \in \mathbf{B}$.

Definition 2.2.1. a) A (φ, Γ_K)**-module** D over $\mathbf{A}_K$ is a free, finitely generated $\mathbf{A}_K$-module with a semi-linear continuous map φ_D (i.e. $\varphi_D(\lambda x) = \varphi(\lambda)\varphi_D(x)$ for $\lambda \in K, x \in D$) and a continuous action of Γ_K which commutes with φ_D.

b) A (φ, Γ)**-module** D over $\mathbf{B}_K$ is a finite dimensional $\mathbf{B}_K$-vectorspace with a semi-linear map φ_D and commuting continuous action of Γ_K.

c) (φ, Γ)**-module** D over $\mathbf{A}_K$ is **étale** (or **of slope 0**) if $\varphi_D(D)$ generates D as an $\mathbf{A}_K$-module. Analoguously, a (φ, Γ_K)-module over $\mathbf{B}_K$ is étale if it has a $\mathbf{A}_K$-lattice which is étale.

φ_D will henceforth simply be denoted by φ.

Remark 2.2.2. If D is an étale (φ, Γ_K)-module over $\mathbf{A}_K$ the operator ψ extends uniquely to an operator $\psi : D \to D$ such that $\psi(a\varphi(d)) = \psi(a)d$, $\psi(\varphi(a)d) = a\psi(d)$ for all $a \in \mathbf{A}_K,\ d \in D$ and such that ψ commutes with the action of Γ_K.

The following theorem is due to Fontaine, cf. [18]:

Theorem 2.2.3. The functor $V \mapsto \mathbf{D}_K(V)$ is an equivalence of tensor categories from the category of $\mathbb{Z}_p$- (resp. $\mathbb{Q}_p$-)representations of G_K to the category of étale (φ, Γ_K)-modules over $\mathbf{A}_K$ (resp. $\mathbf{B}_K$). The inverse functor is given by $D \mapsto \mathbf{V}(D) = (\mathbf{A} \otimes_{\mathbf{A}_K} D)^{\varphi=1}$ (resp. $D \mapsto \mathbf{V}(D) = (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$).

2.3 The rings of Cherbonnier and Colmez

Colmez has defined the ring

$$\mathbf{B}^+_{\max} = \{\sum_{n \geq 0} a_n \frac{\omega^n}{p^n} \mid a_n \in \widetilde{\mathbf{B}}^+,\ a_n \to 0 \text{ for } n \to \infty\}$$

which is "very close" to $\mathbf{B}^+_{\mathrm{cris}}$. We set $\mathbf{B}_{\max} = \mathbf{B}^+_{\max}[1/t]$. There is a canonical injection of $\mathbf{B}_{\max}$ into $\mathbf{B}_{\mathrm{dR}}$ and it is therefore equipped with a canonical filtration. There are actions of φ and $G_{\mathbb{Q}_p}$ on $\mathbf{B}_{\max}$, which extend the actions on $\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$. Colmez puts

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}} = \bigcap_{n=0}^{\infty} \varphi^n(\mathbf{B}^+_{\max}).$$

and $\widetilde{\mathbf{B}}_{\mathrm{rig}} = \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$. We remark that one has

$$\widetilde{\mathbf{B}}_{\mathrm{rig}} = \bigcap_{n=0}^{\infty} \varphi^n(\mathbf{B}_{\mathrm{cris}})$$

and hence in particular

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\varphi=1} = \mathbf{B}_{\mathrm{max}}^{\varphi=1} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}.$$

We say that a representation is **crystalline** if it is $\mathbf{B}_{\mathrm{max}}$-admissible, which is the same as asking that it be $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$-admissible. We put

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{max}} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

This is a K_0-vectorspace of dimension d, equipped with a filtration induced by $\mathbf{B}_{\mathrm{dR}}$ and an action of Frobenius induced by $\mathbf{B}_{\mathrm{max}}$. If V is crystalline we have $\mathbf{D}_{\mathrm{dR}}(V) = K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V)$ which shows that a crystalline representation is also de Rham.

Following Berger the series $\log(\overline{\pi}^{(0)}) + \log(\overline{\pi}/\overline{\pi}^{(0)})$, after a choice of $\log p$, converges in $\mathbf{B}^+_{\mathrm{dR}}$, and we denote the limit by $\log[\overline{\pi}]$. This element is transcendent over $\mathrm{Frac}(\mathbf{B}^+_{\mathrm{max}})$, and we set $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{max}}[\log[\overline{\pi}]]$ and $\widetilde{\mathbf{B}}^+_{\log} = \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[\log[\overline{\pi}]]$. We say that a representation is **semistable** if it is $\mathbf{B}_{\mathrm{st}}$-admissible, which is the same as asking it being $\widetilde{\mathbf{B}}^+_{\log}[1/t]$-admissible. Similarly, as in the crystalline case we put

$$\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}^+_{\log}[1/t] \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Again this is a K_0-vectorspace of dimension d, equipped with a filtration and an action of Frobenius induced by $\mathbf{B}_{\mathrm{st}}$. As before we have in this case $\mathbf{D}_{\mathrm{dR}}(V) = K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V)$. Additionally one can define the monodromy operator $N = -d/d\log[\overline{\pi}]$ on $\mathbf{B}_{\mathrm{st}}$ which induces a nilpotent endomorphism on $\mathbf{D}_{\mathrm{st}}(V)$ and satisfies the relation $N\varphi = p\varphi N$. We also make use of the finite dimensional K_0-vectorrspace $\mathbf{D}^+_{\mathrm{st}}(V) = (\widetilde{\mathbf{B}}^+_{\log} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Recall that elements $x \in \widetilde{\mathbf{B}}$ may be written in the form $x = \sum_{k \gg -\infty} p^k [x_k]$ with $x_k \in \widetilde{\mathbf{E}}$. For $r > 0$ we set

$$\widetilde{\mathbf{B}}^{\dagger,r} = \left\{ x \in \widetilde{\mathbf{B}} \;\middle|\; \lim_{k \to +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{p-1} k = +\infty \right\}.$$

We note that x as above converges in $\mathbf{B}_{\mathrm{dR}}$ if and only if $\sum_{k \gg -\infty} p^k x_k^{(0)}$ converges in $\mathbb{C}_p$.

For $n \geq 0$ we set once and for all

$$r_n = (p-1)p^{n-1}.$$

Colmez and Cherbonnier showed that for n big enough such that $r_n \geq r$ there is an injection

$$\iota_n = \varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r} \longrightarrow \mathbf{B}^+_{\mathrm{dR}}, \qquad \sum_{k \gg -\infty} p^k [x_k] \longmapsto \sum_{k \gg -\infty} p^k [x_k^{p^{-n}}].$$

We put $\widetilde{\mathbf{B}}^{\dagger,n} = \widetilde{\mathbf{B}}^{\dagger,r_n}$. Let $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger} = \bigcup_{r\geq 0} \widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}^{\dagger} = \bigcup_{r\geq 0} \mathbf{B}^{\dagger,r}$. Let $\widetilde{\mathbf{A}}^{\dagger,r}$ be the elements of $\widetilde{\mathbf{B}}^{\dagger,r} \cap \widetilde{\mathbf{A}}$ such that $v_E(x) + \frac{pr}{p-1}k \geq 0$ for all $k \geq 0$. Let $\mathbf{A}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r} \cap \mathbf{A}$, $\mathbf{A}^{\dagger} = \widetilde{\mathbf{A}}^{\dagger} \cap \mathbf{A}$, $\widetilde{\mathbf{A}}^{\dagger} = \widetilde{\mathbf{B}}^{\dagger} \cap \widetilde{\mathbf{A}}$. Let $\mathbf{B}_K^{\dagger,r} = (\mathbf{B}^{\dagger,r})^{H_K}$, $\mathbf{A}_K^{\dagger,r} = (\mathbf{A}^{\dagger,r})^{H_K}$, $\widetilde{\mathbf{B}}_K^{\dagger,r} = (\widetilde{\mathbf{B}}^{\dagger,r})^{H_K}$, $\widetilde{\mathbf{A}}_K^{\dagger,r} = (\widetilde{\mathbf{A}}^{\dagger,r})^{H_K}$.

Proposition 2.3.1. Let $r \geq r_{n(K)}$, $\rho = p^{-1/e_K r}$ and

$$\mathcal{A}_K^{[\rho,1[}(x) = \left\{ \sum_{n\in\mathbb{Z}} a_n x^n | \ a_n \in \mathcal{O}_{K_0'}, \lim_{n\to-\infty} |a_n|\rho^n = 0 \right\},$$
$$\mathcal{B}_K^{[\rho,1[}(x) = \mathcal{A}_K^{[\rho,1[}(x)[1/p] \subset K_0'[[x,1/x]].$$

Then maps $\mathcal{A}_K^{[\rho,1[}(x) \to \mathbf{A}_K^{\dagger,r}$, $\mathcal{B}_K^{[\rho,1[}(x) \to \mathbf{B}_K^{\dagger,r}$, induced by $x \mapsto \pi_K$, are isomorphisms of topological rings.

Proof. See [4], Proposition 1.4. □

Proposition 2.3.2. If L/K be a finite extension then $\mathbf{B}_L^{\dagger}$ is a finite field extension of $\mathbf{B}_K^{\dagger}$ of degree $[L_\infty : K_\infty] = [H_K : H_L]$, and if L/K is Galois, then the same holds for $\mathbf{B}_L^{\dagger}/\mathbf{B}_K^{\dagger}$, which then has galois group $\mathrm{Gal}(L_\infty/K_\infty)$.

Proof. See [12], Proposition II.4.1. □

Definition 2.3.3. a) A (φ, Γ_K)**-module** D over $\mathbf{B}_K^{\dagger}$ is a free, finitely generated $\mathbf{B}_K^{\dagger}$-module with a semi-linear (i.e. $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$ for $\lambda \in K, x \in D$) continuous map φ and a continuous action of Γ_K which commutes with φ.

b) (φ, Γ_K)-module D over $\mathbf{B}_K^{\dagger}$ is **étale** (or of **slope 0**) if there exists a free $\mathbf{A}_K^{\dagger}$-submodule T of D which is stable under the actions of φ and Γ_K such that $\mathbf{B}_K^{\dagger} \otimes_{\mathbf{A}_K^{\dagger}} T = D$.

For a p-adic representation let us set $\mathbf{D}_K^{\dagger}(V) = (\mathbf{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_K}$ and $\mathbf{D}^{\dagger,r}(V) = (\mathbf{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V)^{H_K}$. We say that a p-adic representation is **overconvergent** if $\mathbf{D}_K(V)$ has a basis over $\mathbf{B}_K$ consisting of elements in $\mathbf{D}_K^{\dagger}(V) = (\mathbf{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_K}$. The main result of [12] is then:

Theorem 2.3.4. Every p-adic representation V of G_K is overconvergent, i.e., there exists an $r = r(V)$ such that $D(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^{\dagger,r}} \mathbf{D}_K^{\dagger,r}(V)$. Hence, the functor $V \mapsto \mathbf{D}_K^{\dagger}(V)$ is an equivalence from the category of p-adic representations of G_K to the category of étale (φ, Γ_K)-modules.

2.4 The rings of Berger

If A is a ring which is complete for the p-adic topology and X, Y are indeterminantes we let

$$A\{X,Y\} = \varprojlim_n A[X,Y]/p^n A[X,Y],$$

that is, $A\{X,Y\}$ is the p-adic completion of $A[X,Y]$. Every element of $A\{X,Y\}$ can be written as $\sum_{i,j\geq 0} a_{ij}X^iY^j$ where a_{ij} is a sequence in A tending to 0 in the p-adic topology. We let $r,s \in \mathbb{N}[1/p]\cup\{+\infty\}$ such that $r\leq s$. By definition one has (in $\mathrm{Fr}(\widetilde{B})$) $p/[\overline{\pi}]^{+\infty} = 1/[\overline{\pi}]$ and $[\overline{\pi}]^{+\infty}/p = 0$. Let

$$\begin{aligned}\widetilde{\mathbf{A}}_{[r;s]} &= \widetilde{\mathbf{A}}^+\{\frac{p}{[\overline{\pi}]^r}, \frac{[\overline{\pi}]^s}{p}\}\\ &= \widetilde{\mathbf{A}}^+\{X,Y\}/([\overline{\pi}]^rX-p, pY-[\overline{\pi}]^s, XY-[\overline{\pi}]^{s-r})\\ \widetilde{\mathbf{B}}_{[r;s]} &= \widetilde{\mathbf{A}}_{[r;s]}[1/p].\end{aligned}$$

If I is any interval of $\mathbb{R}\cup\{+\infty\}$ we let

$$\widetilde{\mathbf{B}}_I = \bigcap_{[r;s]\subset I} \widetilde{\mathbf{B}}_{[r;s]}.$$

It is clear that if $I\subset J$ are two closed intervals then $\widetilde{\mathbf{B}}_J \subset \widetilde{\mathbf{B}}_I$. One has a p-adic valuation V_I on $\widetilde{\mathbf{B}}_I$ defined by the condition $V_I(x)=0$ if and only if $x\in\widetilde{\mathbf{A}}_I\setminus p\widetilde{\mathbf{A}}_I$ and such that the image of V_I is $\mathbb{Z}$. With this valuation $\widetilde{\mathbf{B}}_I$ becomes a p-adic Banach space.

The action of G_F on $\widetilde{\mathbf{A}}^+$ extends to $\widetilde{\mathbf{A}}^+[p/[\overline{\pi}]^r, [\overline{\pi}]^s/p]$ and by continunity further extends to $\widetilde{\mathbf{A}}_I$ and $\widetilde{\mathbf{B}}_I$. The Frobenius φ extends to a morphism

$$\varphi : \widetilde{\mathbf{A}}^+[\frac{p}{[\overline{\pi}]^r}, \frac{[\overline{\pi}]^s}{p}] \longrightarrow \widetilde{\mathbf{A}}^+[\frac{p}{[\overline{\pi}]^{pr}}, \frac{[\overline{\pi}]^{ps}}{p}]$$

and finally to a map $\varphi:\widetilde{\mathbf{A}}_I\to\widetilde{\mathbf{A}}_{pI}$ for every I. We recall some rings with this notation, see [4].

Example 2.4.1. a) $\mathbf{B}^+_{\max} = \widetilde{\mathbf{B}}_{[0,r_0]}$,

b) $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} = \widetilde{\mathbf{B}}_{[0,+\infty[}$,

c) $\widetilde{\mathbf{A}}^+ = \widetilde{\mathbf{A}}_{[0,+\infty]}$, $\widetilde{\mathbf{A}}^+ = \widetilde{\mathbf{A}}_{[0,+\infty]}$,

d) $\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}_{[+\infty,+\infty]}$, $\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}_{[+\infty,+\infty]}$,

e) $\widetilde{\mathbf{A}}^{\dagger,r} = \widetilde{\mathbf{A}}_{[r,+\infty]}$, $\widetilde{\mathbf{B}}^{\dagger,r} = \widetilde{\mathbf{B}}_{[r,+\infty]}$.

Berger defines

$$\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} = \widetilde{\mathbf{B}}_{[r,+\infty[}, \qquad \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}.$$

$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is endowed with the Frechet topology defined by the family of valuations V_I for closed subsets $I\subset[r,+\infty[$. One can define $\widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}$ as the ring of integers of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ with respect to the valuation $V_{[r;r]}$. We put $\widetilde{\mathbf{A}}^{\dagger}_{\mathrm{rig}} = \bigcup_{r\geq 0}\widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}$.

One defines $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ to be the completion of $\mathbf{B}^{\dagger}_K$ with respect to the Fréchet topology induced by the V_I. A more hands-on description is given by the next Proposition:

Proposition 2.4.2. Let $r \geq r_{n(K)}$ and $\rho = p^{-1/e_K r}$

$$\mathcal{B}_K^{[\rho,1[}(x) = \left\{ \sum_{n\in\mathbb{Z}} a_n x^n |\ a_n \in K_0',\ \lim_{n\to\pm\infty} |a_n| r^n = 0,\ \forall r \in [\rho, 1[\right\},$$

Then the map $\mathcal{B}_K^r(x) \to \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$, induced by $x \mapsto \pi_K$, is an isomorphism of topological rings.

Proof. See [4], Proposition 2.31. $\square$

Let $f \in \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ be represented by an element $f(\pi_k) = \sum_{n\in\mathbb{Z}} a_n \pi_K^n$ and $I_{K,r} = [p^{-1/e_K r}, 1[$. We define

$$\|f\|_\rho := \sup_n |a_n| \rho^n.$$

From the above discussions it is clear that a sequence $(f_n)_{n\in\mathbb{N}}$ converges to an element $f \in \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ if and only if $\lim_{n\to\infty} \|f_n - f\|_\rho = 0$ for all $\rho \in I_{K,r}$.

If we put $\mathbf{B}_{\mathrm{rig}}^\dagger = \mathbf{B}_{\mathrm{rig},F}^\dagger \otimes_{\mathbf{B}_F^\dagger} \mathbf{B}^\dagger$ then

Lemma 2.4.3. a) $\mathbf{B}_{\mathrm{rig},K}^\dagger = \mathbf{B}_{\mathrm{rig},F}^\dagger \otimes_{\mathbf{B}_F^\dagger} \mathbf{B}_K^\dagger$.

b) $\mathbf{B}_{\mathrm{rig}}^\dagger = \mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}^\dagger$.

c) $(\mathbf{B}_{\mathrm{rig}}^\dagger)^{H_K} = \mathbf{B}_{\mathrm{rig},K}^\dagger$.

Proof. See [4], section 3.4. $\square$

Berger has shown the existence of unique map $\log : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger[X]$ such that $\log([x]) = \log[x]$, $\log(p) = 0$ and $\log(xy) = \log(x) + \log(y)$. Hence one defines $\log\pi := \log(\pi)$ and sets $\widetilde{\mathbf{B}}_{\log}^\dagger = \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger[\log\pi]$, $\mathbf{B}_{\log}^\dagger = \mathbf{B}_{\mathrm{rig}}^\dagger[\log\pi]$ and $\mathbf{B}_{\log,K}^\dagger = \mathbf{B}_{\mathrm{rig},K}^\dagger[\log\pi]$. One defines a monodromy operator N on $\widetilde{\mathbf{B}}_{\log}^\dagger$ by extending $N\log\pi := -1$ in the usual way.

As before, for a p-adic representation V we set $\mathbf{D}_{\mathrm{rig},K}^\dagger(V) := (\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$. Furthermore, let us define

$$\mathbf{D}_{\log,K}^\dagger(V) := (\mathbf{B}_{\log}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}, \quad \mathbf{D}_{\mathrm{rig},K}^+ := (\mathbf{B}_{\mathrm{rig},K}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$$

Most of the time we shall simply drop the K from the notation. We collect some more facts from [4], [6], which cover the étale case. Later we shall generalize these to arbitrary (φ, Γ_K)-modules.

Lemma 2.4.4. The following maps are surjective with kernel $\mathbb{Q}_p$:

$$1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger,\quad 1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^+,\quad 1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger,\quad 1-\varphi : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,n} \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,n+1}.$$

Theorem 2.4.5. Let V be a p-adic representation.

a) If V is positive then $\mathbf{D}_{\mathrm{st}}(V) = (\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $\mathbf{D}_{\mathrm{cris}}(V) = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

b) If V is semistable then $\mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V) = \mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes_F \mathbf{D}_{\mathrm{st}}(V)$.

c) If V is crystalline then $\mathbf{B}^{\dagger}_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V) = \mathbf{B}^{\dagger}_{\mathrm{rig},K}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$.

All morphisms are compatible with the actions of G_K, N, φ.

Proposition 2.4.6. Let $h \geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}_{\mathrm{st}}(V)$.

a) If V is semistable and $y \in \mathbf{B}^{\dagger}_{\log,F} \otimes_F \mathbf{D}_{\mathrm{st}}(V)$, then $t^h y \in \mathbf{D}^{\dagger}_{\log}(V)$.

b) If V is crystalline and $y \in \mathbf{B}^{\dagger}_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then $t^h y \in \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$.

c) If V is crystalline and $y \in \mathbf{B}^{+}_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then $t^h y \in \mathbf{D}^{+}_{\mathrm{rig}}(V)$.

Proof. The proof is the same as [6], II.3. We shall sketch it in the semistable case. Since $\mathbf{D}_{\mathrm{st}}(V(-h))$ has negative Hodge-Tate weights, we have

$$\mathbf{D}_{\mathrm{st}}(V(-h)) \subset \mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V(-h)).$$

Since $\mathbf{D}_{\mathrm{st}}(V(-h)) = t^h\mathbf{D}_{\mathrm{st}}(V) \otimes e_{-h}$ we have $t^h\mathbf{D}_{\mathrm{st}}(V) \subset \mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V)$, whence the claim. ☐

2.5 The ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$

Let us collect some facts about φ-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$.

Definition 2.5.1. Let $h \geq 1$ and $a \in \mathbb{Z}$. The **elementary φ-module** $M_{a,h}$ is the φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ with basis $e_0, \ldots, e_{h-1}$ and $\varphi(e_0) = e_1, \ldots, \varphi(e_{h-2}) = e_{h-1}, \varphi(e_{h-1}) = p^a e_0$.

Proposition 2.5.2. If M is a φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ then there exist integers a_i, h_i such that $M \cong \bigoplus_i M_{a_i,h_i}$.

Proof. See [24], Theorem 4.5.7. ☐

Definition 2.5.3. Let M be a φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$. If $M = M_{a,h}$ is elementary one defines the **slope** of M as $\mu(M) = a/h$ and one says that M is **pure** of this slope. In general if $M \equiv \bigoplus M_{a_i,h_i}$ one define $\mu(M) = \sum \mu(M_{a_i,h_i})$, so that μ is compatible with short exact sequences.

Let D now be a (φ,Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. One sets $\mathbf{B}_e := (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi=1}$. From [7], Proposition 2.2.6, we know that

a) $\mathbf{W}_e(D) := (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{\varphi=1}$ is a free $\mathbf{B}_e$-module of rank d which inherits an action of G_K,

b) $\mathbf{W}^+_{\mathrm{dR}}(D) := \mathbf{B}^+_{\mathrm{dR}} \otimes_{\iota_n, \mathbf{B}^{\dagger, r_n}_{\mathrm{rig},K}} D^{(n)}$ does not depend on $n \gg 0$ and is a free $\mathbf{B}^+_{\mathrm{dR}}$-module of rank d which inherits an action of G_K.

With this in mind, Berger defined:

Definition 2.5.4. A tuple $W = (W_e, W^+_{\mathrm{dR}})$, where W_e is a free $\mathbf{B}_e$-module of finite rank equipped with an semi-linear action of G_K and W^+_{dR} is a $\mathbf{B}^+_{\mathrm{dR}}$-lattice in $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ that is stable under the action of G_K, is called a **B-pair**.

From [7], Proposition 2.2.6 it follows that the tuple $\mathbf{W}(D) = (\mathbf{W}_e(D), \mathbf{W}^+_{\mathrm{dR}}(D))$ actually is a B-pair. Furthermore, Berger proved:

Theorem 2.5.5. The functor $D \mapsto \mathbf{W}(D)$ gives rise to an equivalence of categories between the category of (φ, Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and the category of B-pairs.

One knows (cf. [8], section 2.2.) how to construct a functor $\widetilde{\mathbf{D}}$ from the category of B-pairs to the category of (φ, G_K)-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ such that there exists a unique (φ, Γ_K)-module $\mathbf{D}(W)$ over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ with $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}(W) = \widetilde{\mathbf{D}}(W)$. Hence, one has, similarly as in the preceding theorem:

Theorem 2.5.6. The functor $D \mapsto \widetilde{D} := \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$ gives rise to an equivalence of categories between the category of (φ, Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and the category of (φ, G_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig}}$.

We shall also abbreviate $\widetilde{D}_{\log} = \widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$ and $\mathbf{W}_{\mathrm{dR}}(D) := \mathbf{B}_{\mathrm{dR}} \otimes_{\iota_n \mathbf{B}^{\dagger, r_n}_{\mathrm{rig},K}} D^{(n)}$, which is independent of the choice of n for $n \gg 0$.

It is known that the canonical map

$$\widetilde{\mathbf{B}}^{\dagger, r_n}_{\mathrm{rig}} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D) \to \widetilde{\mathbf{B}}^{\dagger, r_n}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger, r_n}_{\mathrm{rig},K}} D^{(n)},$$

induced by $a \otimes x \mapsto ax$, is an isomorphism of G_K-modules for every $n \geq n(D)$. One defines the following map of G_K-modules:

$$\beta : \mathbf{W}_e(D) \hookrightarrow \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D) \cong \mathbf{B}_{\mathrm{dR}} \otimes_{\iota_n, \mathbf{B}^{\dagger, r_n}_{\mathrm{rig},K}} (\widetilde{\mathbf{B}}^{\dagger, r_n}_{\mathrm{rig}} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D)) \cong \mathbf{W}_{\mathrm{dR}}(D). \quad (2.3)$$

We use the same symbol for the map $\beta : \mathbf{W}_e(D) \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{B}_e} \mathbf{W}_e(D)$. Set $\mathbf{W}^+_e(D) = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{\varphi=1}$.

Let now W be a B-pair and set $X^0(W) = W_e \cap W^+_{\mathrm{dR}} \subset W_{\mathrm{dR}}$ and $X^1(W) = W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}})$, which are nothing but the kernel and cokernel respectively of the natural map $W_e \to W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$. Hence, one has ([9], Theorem 3.1):

Theorem 2.5.7. If W is a B-pair and $\widetilde{D} = \widetilde{\mathbf{D}}(W)$, there are natural identifications

a) $X^0(W) \cong \mathbf{W}^+_e(D)$ and $X^1(W) \cong \widetilde{D}/(1 - \varphi)$,

b) $X^0(W) = 0$ if and only if all slopes of $\widetilde{D}$ are > 0; $X^1(W) = 0$ if and only if all slopes of $\widetilde{D}$ are ≤ 0.

We recall the following definition, introduced by Fontaine (see [19]):

Definition 2.5.8. An **almost $\mathbb{C}_p$-representation** is a p-adic Banach space X equipped with a linear and continuous action of G_K such that there exists a $d \geq 0$ and two (finite-dimensional) p-adic representations $V_1 \subset X,\ V_2 \subset \mathbb{C}_p^d$ such that $X/V_1 \cong \mathbb{C}_p^d/V_2$.

Berger has shown that $X^0(W)$ and $X^1(W)$ are almost $\mathbb{C}_p$-representations, cf. [9].

2.6 (φ, Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$

2.6.1 Basic definitions

We describe how to extend certain results of [4] to (in general non-étale) (φ, Γ_K)-modules, cf. also [8].

We make use of the following notation: Suppose R is a commutative ring equipped with an endomorphism $f : R \to R$, and M is a R-module. We may then consider the R-module $R \otimes_{f,R} M$, where R is considered as an R-module via $r \cdot s := f(r)s$ $(r, s \in R)$.

a) A (φ, Γ_K)**-module** D over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is a free, finitely generated $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-module with a semi-linear continuous map φ_D (i.e. $\varphi_D(\lambda x) = \varphi(\lambda)\varphi_D(x)$ for $\lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},K}, x \in D$) and a continuous action of Γ_K which commutes with φ_D, such that the map

$$\varphi^* : \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\varphi, \mathbf{B}^{\dagger}_{\mathrm{rig},K}} D \longrightarrow D, \quad a \otimes x \longmapsto a\varphi(x)$$

is an isomorphism of $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-modules.

b) (φ, Γ_K)-module D over $\mathbf{B}^{\dagger}_{K}$ is **étale** (or **of slope 0**) if there exists an étale $\mathbf{B}^{\dagger}_{K}$-submodule D' of D which is stable under the actions of φ and Γ_K such that $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger}_{K}} D' = D$.

φ_D will henceforth simply be denoted by φ. Let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

For the ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ we have, analogously as in (2.2), a decomposition $\mathbf{B}^{\dagger}_{\mathrm{rig},K} = \bigoplus_{i=0}^{p-1}(1+\pi)^i\varphi(\mathbf{B}^{\dagger}_{\mathrm{rig},K})$ so that one may define an operator ψ (by the same formula) on $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ that extends the operator ψ on $\mathbf{B}^{\dagger}_{K}$. More generally, if D is a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ we have thanks to condition a) in the definition that there exists a unique operator ψ on D that satisfies analogous properties as in Remark 2.2.2 and commutes with the action of Γ_K.

Proposition 2.6.1. If $0 \to D' \to D \to D'' \to 0$ is an exact sequence of (φ, Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ then $0 \to D'^{\psi=0} \to D^{\psi=0} \to D''^{\psi=0} \to 0$ is an exact sequence of Γ_K-modules.

Proof. For the proof of the right-exactness one just uses the fact that if $x \in D^{\psi=0}$ then (uniquely) $x = \sum_{i=1}^{p-1}(1+\pi)^i\varphi(x_i)$ with $x_i \in D$. The compatibility with the action of Γ_K is clear since it commutes with ψ. □

If L/K is a finite extension, we denote the **restriction** $D|_L$ by

$$D|_L := \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D,$$

with actions of φ and Γ_L defined diagonally. Hence, $D|_L$ is a (φ,Γ_L)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$.

The **dual** D^* of a (φ,Γ_K)-module D over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is defined by

$$D^* := \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}}(D, \mathbf{B}^{\dagger}_{\mathrm{rig},K}),$$

where for $f \in D^*$ the actions of Γ_K and φ are defined via

$$\gamma(f)(x) := \gamma(f(\gamma^{-1}x)),\ \gamma \in \Gamma_K, x \in D, \quad \varphi(f)(x) := \sum a_i\varphi(f(x_i)),\ x = \sum a_i\varphi(x_i) \in D.$$

If D_1, D_2 are two (φ,Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ then the **tensor product** of D_1 and D_2 is defined by

$$D_1 \otimes D_2 := D_1 \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D_2,$$

where φ and Γ_K act diagonally. Note that this does *not* imply that ψ acts diagonally.

Let D be a (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ of rank d. By [8], Theorem I.3.3 there exists an $n(D)$ and a unique finite free $\mathbf{B}^{\dagger,r_{n(D)}}_{\mathrm{rig},K}$-module $D^{(n(D))} \subset D$ of rank d with

a) $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r_{n(D)}}_{\mathrm{rig},K}} D^{(n(D))} = D$,

b) Let $D^{(n)} = \mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r_{n(D)}}_{\mathrm{rig},K}} D^{(n(D))}$ for each $n \geq n(D)$. Then $\varphi(D^{(n)}) \subset D^{(n+1)}$ and the map
$$\mathbf{B}^{\dagger,r_{n+1}}_{\mathrm{rig},K} \otimes_{\varphi,\mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}} D^{(n)} \to D^{(n+1)}, \quad a \otimes x \mapsto a\varphi(x),$$
is an isomorphism.

2.6.2 Cohomology of (φ,Γ_K)-modules

Liu (cf. [29]) has worked out reasonable definitions for cohomology of (in general non-étale) (φ,Γ_K)-modules over $\mathbf{B}_K, \mathbf{B}^{\dagger}_K$ and $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

Let D be a (φ,Γ_K)-module over one of these rings and let Δ_K be a torsion subgroup of Γ_K. Γ_K is an open subgroup of $\mathbb{Z}_p^\times$ and Δ_K is a finite group of order dividing $p-1$ (or 2 if $p=2$). Define the idempotent operator p_{Δ_K} by $p_{\Delta_K} = (1/|\Delta_K|)\sum_{\delta\in\Delta_K}\delta$, so that p_{Δ_K} is the projection from D to $D' := D^{\Delta_K}$. If $\Gamma'_K := \Gamma_K/\Delta_K$ is procyclic with generator γ_K define the exact sequence

$$C^{\bullet}_{\varphi,\gamma_K}(D):\ 0 \longrightarrow D' \xrightarrow{d_1} D' \oplus D' \xrightarrow{d_2} D' \longrightarrow 0 \tag{2.4}$$

with

$$d_1(x) = ((\varphi - 1)x, (\gamma_K - 1)x), \quad d_2(x, y) = (\gamma_K - 1)x - (\varphi - 1)y.$$

Define for $i \in \mathbb{Z}$

$$H^i(K, D) := H^i(C^\bullet_{\varphi,\gamma_K}(D)),$$

which is, up to canonical isomorphism, independent of the choice of γ_K (cf. [29], section 2), so that we shall now fix a choice of Δ_K and γ_K.

For applications in Iwasawa-theory one also considers the following complex:

$$C^\bullet_{\psi,\gamma_K}(D): \ 0 \longrightarrow D' \xrightarrow{d_1} D' \oplus D' \xrightarrow{d_2} D' \longrightarrow 0 \tag{2.5}$$

with

$$d_1(x) = ((\psi - 1)x, (\gamma_K - 1)x), \quad d_2(x, y) = (\gamma_K - 1)x - (\psi - 1)y.$$

If D_1 and D_2 are two (φ, Γ_K)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ one may, following Herr ([22]), define the following cup products (we always mean classes where appropriate):

$$\begin{array}{ll} H^0(K, D_1) \times H^0(K, D_2) \longrightarrow H^0(K, D_1 \otimes D_2), & (x, y) \mapsto (x \otimes y), \\ H^0(K, D_1) \times H^1(K, D_2) \longrightarrow H^1(K, D_1 \otimes D_2), & (x, (y, z)) \mapsto (x \otimes y, x \otimes z), \\ H^0(K, D_1) \times H^2(K, D_2) \longrightarrow H^2(K, D_1 \otimes D_2), & (x, y) \mapsto (x \otimes y), \\ H^1(K, D_1) \times H^1(K, D_2) \longrightarrow H^2(K, D_1 \otimes D_2), & ((x, y), (w, v)) \mapsto y \otimes \gamma_K(w) - x \otimes \varphi(v). \end{array} \tag{2.6}$$

We note that some authors swap the maps of the sequence $C^\bullet_{\varphi,\gamma_K}(D)$ so that of course one has to adjust the definition of the cup-product. We adhere to the conventions made in [22].

Liu's result is then ([29], Theorem 0.1 and Theorem 0.2):

Theorem 2.6.2. Let D be a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$.

a) If $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is étale one has canonical functorial isomorphisms $H^i(K, \mathbf{D}^\dagger_{\mathrm{rig}}(V)) \cong H^i(G_K, V)$ for all $i \in \mathbb{Z}$ that are compatible with cup-products.

b) $H^i(K, D)$ is a finite dimensional $\mathbb{Q}_p$-vectorspace and vanishes for $i \neq 0, 1, 2$.

c) For $i = 0, 1, 2$ the pairing

$$\begin{aligned} H^i(K, D) \times H^{2-i}(K, D^*(1)) \longrightarrow H^2(K, D \otimes D^*(1)) &= H^2(K, \mathbf{B}^\dagger_{\mathrm{rig},K}(1)) \\ &= H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \end{aligned}$$

where $D \otimes D^*(1) \to \mathbf{B}^\dagger_{\mathrm{rig},K}(1)$ is the map $x \otimes f \mapsto f(x)$, is perfect.

Similarly, if one is interested in Iwasawa-theoretic applications one has the following setting, as developed in [37]: Let as usual denote $\Lambda = \Lambda_K = \mathbb{Z}_p[[\Gamma_K]]$ and $\Lambda' = \mathbb{Z}_p[[\Gamma'_K]]$ so that $\Lambda = \mathbb{Z}_p[\Delta_K] \otimes_{\mathbb{Z}_p} \Lambda'$, the Iwasawa algebra for Γ_K and Γ'_K. It is a complete noetherian semi-local ring, and we denote by $\mathfrak{m}$ the Jacobson radical of of Γ'_K. Then one may consider $\Lambda'[\mathfrak{m}^n/p]$ for every n, which is the Λ'-submodule of $\Lambda'[1/p]$ generated by elements m/p, where $m \in \mathfrak{m}^n$. Denote by $\Lambda'_n = \Lambda'[\mathfrak{m}^n/p]^\wedge = \varprojlim_k \Lambda'[\mathfrak{m}^n/p]/p^k\Lambda'[\mathfrak{m}^n/p]$ the p-adic completion of $\Lambda'[\mathfrak{m}^n/p]$ and write $\Lambda_n = \Lambda'_n[\Delta_K]$. One has an identification of Λ'_n with $\mathbb{Z}_p[T, T^n/p]^\wedge$ via $\gamma \mapsto 1+T$. The natural maps of Λ'-modules $\mathfrak{m}^n \hookrightarrow \mathfrak{m}^m$ for $n \geq m$ induce inclusions $\Lambda_n \hookrightarrow \Lambda_m$ so that one may form $\Lambda_\infty = \varprojlim_n (\Lambda_n[1/p])$.

On the other hand, for Γ_K as above, Perrin-Riou defined the algebra $\mathcal{H}(\Gamma_K)$ in the following way. First consider $\mathcal{H}(\Gamma'_K)$, which is defined as the image of $\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}$ by the substitution $\pi \mapsto \gamma_K - 1$. Let $\mathcal{H}(\Gamma_K) = \mathbb{Q}_p[\Delta_K] \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma'_K)$. Analogously as in 2.4.1, a), one has

$$\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p} = \bigcap_{n\geq 0} \mathbb{Z}_p[[\pi]][\pi^n/p]^\wedge[1/p] \subset \mathbb{Q}_p[[\pi]],$$

so that the identification $\mathbb{Z}_p[[\Gamma'_K]] \cong \mathbb{Z}_p[[\pi]]$ extends to an identification $\Lambda_\infty = \mathcal{H}(\Gamma_K)$.

In the same vein for $m \geq 0$ and $l \geq m$ one may define $\Lambda'_{[m,l]} = \mathbb{Z}_p[T, p/T^m, T^l/p]^\wedge$ which one may consider as a continuous Λ'-algebra via $\gamma \mapsto T+1$. We set $\Lambda_{[m,l]} = \Lambda'_{[m,l]}[\Delta_K]$. Then for $m' \leq m \leq l \leq l'$ one has canonical maps $\Lambda_{[m',l']} \to \Lambda_{[m,l]}$ so that one may form $\Lambda_{\pm\infty} = \varinjlim_{m\geq 0} \varprojlim_{l\geq 0} \Lambda_{[m,l]}[1/p]$.

Again, for Γ_K as above, Perrin-Riou defined the algebra $\mathcal{B}(\Gamma_K)$ in the following way. Consider $\mathcal{B}(\Gamma'_K)$, which is defined as the image of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}$ by the substitution $\pi \mapsto \gamma_K - 1$. Let $\mathcal{B}(\Gamma_K) = \mathbb{Q}_p[\Delta_K] \otimes_{\mathbb{Q}_p} \mathcal{B}(\Gamma'_K)$. As before one has

$$\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} = \bigcup_{m\geq 0} \bigcap_{l\geq m} \mathbb{Z}_p[[\pi]][p/\pi^m, \pi^l/p]^\wedge[1/p] \subset \mathbb{Q}_p[[\pi, 1/\pi]],$$

so that the identification $\mathbb{Z}_p[[\Gamma'_K]] \cong \mathbb{Z}_p[[\pi]]$ extends to an identification $\Lambda_{\pm\infty} = \mathcal{B}(\Gamma_K)$.

Definition 2.6.3. (Cf. [38], §3) A Λ_∞-module M is called **coadmissible** if the following holds: there exists a family $(M_n)_n$ of modules M_n such that M_n is a finitely generated $\Lambda_n[1/p]$-module with the property $\Lambda_n[1/p] \otimes_{\Lambda_{n+1}[1/p]} M_{n+1} \xrightarrow{\cong} M_n$ and $M \cong \varprojlim_n M_n$.

We recall some structure theory for $\Lambda_\infty = \mathcal{H}(\Gamma_K)$-modules (see also [30], sections 3.1, 3.4). Let $\widehat{\Delta}_K$ be the character group of Δ_K and for any $\eta \in \widehat{\Delta}_K$ denote by e_η the corresonding idempotent. Then one has a canonical ring-isomorphism $\mathbb{Q}_p[\Delta_K] \cong \bigoplus_{\eta\in\widehat{\Delta}_K} \mathbb{Q}_p e_\eta$. This extends to an isomorphism $\mathcal{H}(\Gamma_K) \cong \bigoplus_{\eta\in\widehat{\Delta}_K} \mathcal{H}(\Gamma'_K)e_\eta$. From this it follows that for the total ring of fractions $\mathcal{K}(\Gamma_K) := \mathrm{Frac}(\mathcal{H}(\Gamma_K))$ of $\mathcal{H}(\Gamma_K)$ one has $\mathcal{K}(\Gamma_K) \cong \bigoplus_{\eta\in\widehat{\Delta}_K} \mathrm{Frac}(\mathcal{H}(\Gamma'_K)e_\eta)$. Now if M is any $\mathcal{H}(\Gamma_K)$-module one obtains a decomposition (we follow the usual convention and write $M_\eta = Me_\eta$) $M \cong \bigoplus_{\eta\widehat{\Delta}_K} M_\eta$, where each M_η is a $\mathcal{H}(\Gamma'_K)e_\eta$-module (which is as a ring isomorphic to $\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}$, hence a Bézout-domain, cf. [27]). We call a $\mathcal{H}(\Gamma_K)$-module M **torsion** if each M_η is torsion as a $\mathcal{H}(\Gamma'_K)e_\eta$-module. By the above decomposition this is equivalent to the property that $\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)} M = 0$. Of course, analogous considerations hold for $\mathcal{B}(\Gamma_K)$-modules, where again each factor $\mathcal{B}(\Gamma'_K)e_\eta$ is a Bézout domain (cf. [27], [24]).

One has the following (see [37], Proposition 6.1):

Proposition 2.6.4. Let M be a coadmissible $\mathcal{H}(\Gamma_K)$-module. Then M_{tor} is also coadmissible and M/M_{tor} restricts to a finitely generated free module over each integral factor $\mathcal{H}(\Gamma'_K)e_\eta$ of $\mathcal{H}(\Gamma_K)$.

Let now D be a (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Define $\Lambda_n[\Gamma_K]$-modules $\widetilde{\Lambda}_n$ (resp. $\widetilde{\Lambda}^{\iota}_n$) by $\widetilde{\Lambda}_n = \widetilde{\Lambda}^{\iota}_n = \Lambda_n$ as Λ_n-modules and $\gamma_K(\lambda) = [\gamma]\lambda$ (where $[\,] : \Gamma_K \to \Lambda_K^{\times}$ is the natural group homomorphism) for $\lambda \in \widetilde{\Lambda}_n$ (resp. $\gamma_K(\lambda) = [\gamma^{-1}]\lambda$ for $\lambda \in \widetilde{\Lambda}^{\iota}_n$). Observe that $\widetilde{\Lambda}^{\iota}_n[1/p]$ and $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ are complete $\mathbb{Q}_p$-Banach vector spaces, so that the completed tensor product in the following definition makes sense:

Definition 2.6.5. If D is a (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and $n \in \mathbb{N}$ one defines the **cyclotomic deformation** $\overline{D}_n$ of D as

$$\overline{D}_n = D \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\Lambda}^{\iota}_n[1/p],$$

which is a $\mathbf{B}^{\dagger}_{\mathrm{rig},K}\widehat{\otimes}\Lambda_n[1/p]$-module, such that φ,ψ act via the first factor and Γ_K acts diagonally.

With this definition one may consider complexes $C^{\bullet}_{\varphi,\gamma_K}(\overline{D}_n)$, $C^{\bullet}_{\psi,\gamma_K}(\overline{D}_n)$ defined exactly as in (2.4) and (2.5), and cohomology groups $H^1(K,\overline{D}_n)$, resp. cup-products as in (2.6), with $\overline{D}_n$ in place of D. One checks that one has a canonical morphism of complexes $C^{\bullet}_{\varphi,\gamma_K}(\overline{D}_{n+1}) \to C^{\bullet}_{\varphi,\gamma_K}(\overline{D}_n)$ which inducess a map $H^i(K,\overline{D}_{n+1}) \to H^i(K,\overline{D}_n)$ of Λ_{n+1}-modules, so that we define the $\mathcal{H}(\Gamma_K)$-module

$$H^1_{\mathrm{Iw}}(K,\overline{D}) := \varprojlim_n H^1(K,\overline{D}_n)$$

One of the main theorems of [37] is the following (see loc.cit., Theorem 6.8):

Theorem 2.6.6. Let D be a (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

a) The map $H^i(K,\overline{D}_{n+1}) \otimes_{\Lambda_{n+1}} \Lambda_n \cong H^i(K,\overline{D}_n)$ of Λ_n-modules is an isomorphism.

b) The $H^i_{\mathrm{Iw}}(K,\overline{D})$ are coadmissible $\mathcal{H}(\Gamma_K)$-modules, zero for $i \neq 1,2$, torsion for $i = 2$ and of rank equal to $\mathrm{rank}(D)\cdot[K:\mathbb{Q}_p]$ for $i = 1$.

c) $\gamma_K - 1$ acts invertibly on $D'^{\psi=0} = D^{\Delta_K,\psi=0}$ and $\overline{D}'^{\psi=0}_n$, and the morphism of complexes

$$\begin{array}{ccccccccc}
C^{\bullet}_{\varphi,\gamma_K}(D): 0 & \longrightarrow & D' & \xrightarrow{d_1} & D'\oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\psi\oplus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
C^{\bullet}_{\psi,\gamma_K}(D): 0 & \longrightarrow & D' & \xrightarrow{d_1} & D'\oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
C^{\bullet}_{\varphi,\gamma_K}(\overline{D}_n): 0 & \longrightarrow & \overline{D}'_n & \xrightarrow{d_1} & \overline{D}'_n\oplus \overline{D}'_n & \xrightarrow{d_2} & \overline{D}'_n & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\psi\oplus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
C^{\bullet}_{\psi,\gamma_K}(\overline{D}_n): 0 & \longrightarrow & \overline{D}'_n & \xrightarrow{d_1} & \overline{D}'_n\oplus \overline{D}'_n & \xrightarrow{d_2} & \overline{D}'_n & \longrightarrow & 0
\end{array}$$

are quasi-isomorphisms.

d) One has canonical isomorphisms of $\mathcal{H}(\Gamma_K)$-modules

$$D^{\psi=1} \cong H^1_{\mathrm{Iw}}(K,\overline{D}), \quad (D/(\psi-1)D) \cong H^2_{\mathrm{Iw}}(K,\overline{D}).$$

Let $\Gamma_K^n = \mathrm{Gal}(K_\infty/K_n)$. When $p \neq 2$ and $n \geq 1$ (or $p = 2$ and $n \geq 2$), Γ_K^n is torsion free. We put $\log_0(a) = \frac{\log a}{p^{v_p(\log(a))}} \in \mathbb{Z}_p^\times$ for $a \in \mathbb{Z}_p^\times$. Consider now $D^{\psi=1}$ for some (φ,Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. For every $n \gg 0$ such that Γ_{K_n} is torsion free we have a canonical map $h^1_{K_n,D} : D^{\psi=1} \to H^1(K_n,D)$ (by taking into account that D is also a (φ,Γ_{K_n})-module over $\mathbf{B}^\dagger_{\mathrm{rig},K_n} = \mathbf{B}^\dagger_{\mathrm{rig},K}$), given by the following construction: if $y \in D^{\psi=1}$ then $p_\Delta((\varphi-1)y) \in D'^{\psi=0}$, so that by Theorem 2.6.6 there exists an $x \in D'^{\psi=0}$ with $(\gamma_{K_n}-1)x = (\varphi-1)y$. We may then put $h^1_{K_n,D}(y) = |\Delta_K|\log_0(\chi(\gamma_K))\overline{(x,y)}$.

In the same way we have for $n \gg 0$ a canonical map $h^1_{\overline{D}_n} : D^{\psi=1} \to H^1(K,\overline{D}_n)$ given by the following construction: consider $p_\Delta(y\widehat{\otimes}1) \in \overline{D}'_n$ so that by Theorem 2.6.6 there exists a unique $x_n \in \overline{D}'_n$ such that $(\varphi-1)y\widehat{\otimes}1 = (\gamma_{K_n}-1)x$, so that we may put $h^1_{\overline{D}_n}(y) = |\Delta_K|\log_0(\chi(\gamma_K))\overline{(x_n,y\widehat{\otimes}1)}$. One checks that for $m \geq n$ these elements are compatibel with canonical projections $H^1(K,\overline{D}_m) \to H^1(K,\overline{D}_n)$, so that the isomorphism $D^{\psi=1} \to H^1_{\mathrm{Iw}}(K,\overline{D})$ is explicitly described via $y \mapsto (|\Delta_K|\log_0(\chi(\gamma_K))\overline{(x_n,y\otimes 1)})_n$.

Analogously to the étale case one may define induced modules of (φ,Γ_K)-modules, restriction and corestriction for the cohomology of (φ,Γ_K)-modules as follows: Let L/K be a finite extension and D a (φ,Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$. Let

$$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D = \{f : \Gamma_K \longrightarrow D |\ f(hg) = hf(g) \text{ for } h \in \Gamma_L\}.$$

$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ has the structure of a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module via $(af)(g) = g(a)f(a)$ for $f \in \mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$, $a \in \mathbf{B}^\dagger_{\mathrm{rig},K}$, $g \in \Gamma_K$. Additionally, φ and Γ_K-actions may be defined via

$$(\varphi f)(g) = \varphi(f(g)), \quad (\sigma f)(g) = f(g\sigma).$$

Note that since $\mathbf{B}^\dagger_{\mathrm{rig},L}/\mathbf{B}^\dagger_{\mathrm{rig},K}$ is an extension of degree $[H_K : H_L]$ and $[L:K] = [\Gamma_K : \Gamma_K]\cdot[H_K : H_L]$ the rank of $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ is equal to $[L:K]\cdot \mathrm{rank} D$ and $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ is called the **induced module**. In the case $L = K_n$ for $n \geq 0$ one may identify the induced module with the following one: consider $D\otimes_{\mathbb{Q}_p}\mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]}$, where $\mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]} = \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$ as a $\mathbb{Q}_p$-vectorspace and $\gamma(\lambda) = [\overline{\gamma}]\lambda$ for $\gamma \in \Gamma_K, \lambda \in \mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]}$, where $[\ ] : \Gamma_K \to \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$ is defined similarly as before. Then

$$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D \xrightarrow{\cong} D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]}, \quad f \longmapsto \sum_{\gamma\in\Gamma_K/\Gamma_{K_n}} f(\gamma)\otimes\gamma^{-1}$$

gives an isomorphism that is compatible with all the given actions. Recall that one can define the $\mathbb{Q}_p$-linear involution $\iota : \mathcal{H}(\Gamma_K) \to \mathcal{H}(\Gamma_K)$, which is defined by the property that it sends $\sigma \in \Gamma_K$ to σ^{-1}. Similarly, we denote by the same letter the analogous map on Λ, $\mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$, etc. Shapiro's Lemma implies an identification

$$H^1(K_n,D) \xrightarrow{\sim} H^1(K, D\otimes_{\mathbb{Q}_p}\mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]^\iota})$$

via the map on representatives induced by the map

$$\begin{aligned} D \oplus D &\longrightarrow D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}]^\iota \oplus D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}]^\iota, \\ (x,y) &\longmapsto \left(\sum_{\sigma \in \Gamma_K/\Gamma_{K_n}} (\sigma^{-1}x \otimes \overline{\sigma}), y \otimes 1 \right), \end{aligned}$$

where we have fixed a system of representatives Γ_K/Γ_{K_n} (see [29], Theorem 2.2).

We defined $D|_L = \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D$. Let $m = [\Delta_K : \Delta_L]$ and n be such that $\gamma_K^{p^n} = \gamma_L$. Define $\tau_{L/K} = \sum_{i=0}^{p^n-1} \gamma_K^i$ and $\sigma_{L/K} = \sum_{g \in \Gamma_K/\Gamma_L} g$

We define the restriction maps Res : $H^i(K,D) \to H^i(L, D|_L)$ via the map induced by the following map on complexes (where $*'$ means the invariants with respect to the "right" Δ):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D' & \xrightarrow{d_1} & D' \oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\oplus(m\cdot\tau_{L/K})} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & D|_L' & \xrightarrow{d_1} & D|_L' \oplus D|_L' & \xrightarrow{d_2} & D|_L' & \longrightarrow & 0 \end{array}$$

Similarly, we define the corestriction map Cor : $H^i(K,D) \to H^i(L, D|_L)$ via the map induced by the following map on complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D|_L' & \xrightarrow{d_1} & D|_L' \oplus D|_L' & \xrightarrow{d_2} & D|_L' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_{L/K}} & & \downarrow{\scriptstyle \sigma_{L/K}\oplus\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & D' & \xrightarrow{d_1} & D' \oplus D' & \xrightarrow{d_2} & D' & \longrightarrow & 0 \end{array}$$

Proposition 2.6.7. The map Cor $\circ$ Res on $H^i(K,D)$ is nothing but multiplication by $[L:K]$.

Proof. It is clear that on $H^0(K,D) = D^{\varphi=1,\gamma_K=1}$ (thus γ_K acts trivially) the map Cor$\circ$Res is just the trace map and equal to multiplication by $[L:K]$. Since the $H^i(K,D)$ are cohomological δ-functors (see [26], Theorem 8.1) we get the claim. □

2.6.3 $(\varphi, N, \mathbf{Gal}(L/K))$-modules associated to (φ, Γ_K)-modules

We begin with a series of definitions (see [4], section 5, and [8]).

Definition 2.6.8. Let D be (φ, Γ_K)-module and $n \geq n(D)$. Set

$$\mathbf{D}^+_{\mathrm{dif},n}(D) := K_n[[t]] \otimes_{\iota_n, \mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}} D^{(n)}, \quad \mathbf{D}_{\mathrm{dif},n}(D) := K_n((t)) \otimes_{\iota_n, \mathbf{B}^{\dagger,r_n}_{\mathrm{rig},K}} D^{(n)}$$

and, via the transition maps $\mathbf{D}^+_{\mathrm{dif},n}(D) \hookrightarrow \mathbf{D}^+_{\mathrm{dif},n+1}$, $f(t) \otimes x \mapsto f(t) \otimes \varphi(x)$ (and similarly for $\mathbf{D}_{\mathrm{dif},n}(D) \hookrightarrow \mathbf{D}_{\mathrm{dif},n+1}$)

$$\mathbf{D}^+_{\mathrm{dif}}(D) := \varinjlim_n \mathbf{D}^+_{\mathrm{dif},n}(D), \quad \mathbf{D}_{\mathrm{dif}}(D) := \varinjlim_n \mathbf{D}_{\mathrm{dif},n}(D).$$

Note that $\mathbf{D}^{+}_{\mathrm{dif}}(D)$ (resp. $\mathbf{D}_{\mathrm{dif}}(D)$) is a free $K_{\infty}[[t]] := \bigcup_{n=1}^{\infty} K_n[[t]]$- (resp. $K_{\infty}((t)) = K_{\infty}[[t]][1/t]$-)module of rank d with a semi-linear action of Γ_K. One defines a Γ_K-equivariant injection

$$\iota_n : D^{(n)} \hookrightarrow \mathbf{D}^{+}_{\mathrm{dif},n}(D), \quad x \mapsto 1 \otimes x.$$

Definition 2.6.9. Let D be a (φ, Γ_K)-module. Set

$$\begin{aligned} \mathbf{D}^{K}_{\mathrm{cris}}(D) &:= (\mathbf{B}^{\dagger}_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{\Gamma_K}, \\ \mathbf{D}^{K}_{\mathrm{st}}(D) &:= (\mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{\Gamma_K}, \\ \mathbf{D}^{K}_{\mathrm{dR}}(D) &:= (\mathbf{D}_{\mathrm{dif}}(D))^{\Gamma_K}, \end{aligned}$$

and

$$\mathrm{Fil}^i \mathbf{D}^{K}_{\mathrm{dR}}(D) := \mathbf{D}^{K}_{\mathrm{dR}}(D) \cap t^i \mathbf{D}^{+}_{\mathrm{dif}}(D) \subset \mathbf{D}_{\mathrm{dif}}(D), \ i \in \mathbb{Z}.$$

We set $\mathbf{D}^{K,+}_{\mathrm{dR}}(D) := \mathrm{Fil}^0(\mathbf{D}^{K}_{\mathrm{dR}}(D)) = \mathbf{D}^{+}_{\mathrm{dif}}(D)^{\Gamma_K}$.

One has canonical maps which we will denote by α_* for $* \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, induced by $a \otimes d \mapsto ad$:

$$\begin{aligned} \mathbf{B}^{\dagger}_{\mathrm{rig},K}[1/t] \otimes \mathbf{D}^{K}_{\mathrm{cris}}(D) &\to D[1/t] \\ \mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes \mathbf{D}^{K}_{\mathrm{st}}(D) &\to \mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes D, \\ K_{\infty}((t)) \otimes \mathbf{D}^{K}_{\mathrm{dR}}(D) &\to \mathbf{D}_{\mathrm{dif}}(D). \end{aligned}$$

Note that

$$\begin{aligned} \mathbf{D}^{K}_{\mathrm{cris}}(D) &= (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{G_K}, \\ \mathbf{D}^{K}_{\mathrm{st}}(D) &= (\widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{G_K}, \\ \mathbf{D}^{K}_{\mathrm{dR}}(D) &:= (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D))^{G_K}, \end{aligned}$$

where the first two equalities are due to Proposition 2.6.18, and the last one will be proved in Proposition 3.1.18, so that one may also consider maps α for $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes \mathbf{D}^{K}_{\mathrm{cris}}(D) \to \widetilde{D}[1/t]$, etc.

Proposition 2.6.10. All maps α_* above are injective. Hence, one always has inequalities

$$\dim_{K_0} \mathbf{D}^{K}_{\mathrm{cris}}(D) \leq \dim_{K_0} \mathbf{D}^{K}_{\mathrm{st}}(D) \leq \dim_{K} \mathbf{D}^{K}_{\mathrm{dR}}(D) \leq \mathrm{rank}_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D,$$

and equalities $\dim \mathbf{D}^{K}_{*}(D) = \mathrm{rank}_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$ for $* \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$ if and only if the corresponding α is an isomorphism.

Proof. Standard proof. □

The filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}^K(D)$ is decreasing, separated and exhaustive, i.e.,

a) $\mathrm{Fil}^{i+1}\mathbf{D}_{\mathrm{dR}}^K(D) \subset \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}^K(D)$,

b) $\bigcap_i \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}^K(D) = 0$

c) $\bigcup_i \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$.

Definition 2.6.11. The **Hodge-Tate weights** of a (φ, Γ_K)-module are those integers h such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}^K(D) \neq \mathrm{Fil}^{-h+1}\mathbf{D}_{\mathrm{dR}}^K(D)$. We say that D is **positive** if $h \leq 0$ for all weights h, and that D is **negative** if $h \geq 0$ for all weights h.

Proposition 2.6.12. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$. If D is positive then $\mathbf{D}_{\mathrm{dR}}^{K,+}(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$. More generally, let $h \geq 0$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^K(D)$. Then $t^h\mathbf{D}_{\mathrm{dR}}^K(D) = \mathbf{D}_{\mathrm{dR}}^{K,+}(D(-h))$ (in $\mathbf{D}_{\mathrm{dif}}(D)$).

Proof. The first part is obvious from the definitions and can be shown the same way as in the étale case. The second follows similarly from Lemma 2.6.13. □

One can define the **Tate-twist** for a (φ, Γ_K)-module D: if $k \in \mathbb{Z}$, then $D(k)$ is the (φ, Γ_K)-module with D as $\mathbf{B}_{\mathrm{rig},K}^\dagger$-module, but with

$$\varphi|_{D(k)} = \varphi|_D, \quad \gamma x = \chi^k(\gamma)\gamma x,\ x \in D.$$

Analoguously one define a **Tate-twist** for a filtered (φ, N)-module D over K_0. If $k \in \mathbb{Z}$, then $D[k]$ is the filtered (φ, N)-module with D as K_0-vectorspace and filtration $\mathrm{Fil}^r(D[k])_K = \mathrm{Fil}^{r-k}D_K$ and

$$N|_{D[k]} = N|_D, \quad \varphi|_{D[k]} = p^k\varphi|_D.$$

Lemma 2.6.13. One has $\mathbf{D}_{\mathrm{st}}^K(D(k)) = \mathbf{D}_{\mathrm{st}}^K(D)[-k]$.

Proof. One has $D(k) = D \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$, and if e_k is a generator of $\mathbb{Z}_p(k)$, the isomorphism

$$(\mathbf{B}_{\log,K}^\dagger[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D)^{\Gamma_K}[-k] \to (\mathbf{B}_{\log,K}^\dagger[1/t] \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D(k))^{\Gamma_K}$$

is given by

$$d = \sum a_n \otimes d_n \mapsto \sum a_n e_{-k} \otimes (d_n \otimes e_k) = (e_{-k} \otimes e_k)d.$$

□

Definition 2.6.14. A (φ, Γ_K)-module D is defined to be **crystalline** (resp. **semi-stable**, resp. **de Rham**) if $\dim_{K_0} \mathbf{D}_{\mathrm{cris}}^K(D) = \mathrm{rank}_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$ (resp. $\dim_{K_0} \mathbf{D}_{\mathrm{st}}^K(D) = \mathrm{rank}_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$, resp. $\dim_K \mathbf{D}_{\mathrm{dR}}^K(D) = \mathrm{rank}_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$).

Similarly, we define D to be **potentially crystalline** (resp. **potentially semi-stable**) if there exists a finite extension L/K such that $D|_L$ is cristalline (resp. semistable).

Definition 2.6.15. Let D be a de Rham (φ,Γ_K)-module of rank d. If $n \geq n(D)$, set

$$\mathbf{N}^{(n)}_{\mathrm{dR}}(D) := \{x \in D^{(n)}[1/t] |\ \iota_m(x) \in K_m[[t]] \otimes_K \mathbf{D}^K_{\mathrm{dR}}(D) \text{ for any } m \geq n\}$$

and $\mathbf{N}_{\mathrm{dR}}(D) = \varinjlim_n \mathbf{N}^{(n)}_{\mathrm{dR}}(D)$.

Definition 2.6.16. a) For a torsion free element γ_K of Γ_K Perrin-Riou's differential operator ∇ is defined as

$$\nabla = -\frac{\log(\gamma)}{\log_p(\chi(\gamma_K))} = -\frac{1}{\log_p(\chi(\gamma_K))}\sum_{n\geq 1}\frac{(1-\gamma_K)^n}{n} \in \mathcal{H}(\Gamma_K).$$

b) The operator ∂ (on $\mathbf{B}^\dagger_{\mathrm{rig},K}[1/t]$) is defined as $\partial := 1/t \cdot \nabla$.

We remark that ∇ is independent of the choice of γ, which may be checked with the series properties of log. The module $\mathbf{N}_{\mathrm{dR}}(D)$ is denoted by $\mathbf{D}$ in [10], Theorem III.2.3. This theorem also implies:

Theorem 2.6.17. Let D be a de Rham (φ,Γ_K)-module of rank d. Then $\mathbf{N}_{\mathrm{dR}}(D)$ is a (φ,Γ_K)-module of rank d with the following properties:

a) $\mathbf{N}_{\mathrm{dR}}(D)[1/t] = D[1/t]$,

b) $\nabla_0(\mathbf{N}_{\mathrm{dR}}(D)) \subset t\mathbf{N}_{\mathrm{dR}}(D)$.

The following proposition is analoguous to [4], Theorem 3.6.

Proposition 2.6.18. Let D be a semistable (φ,Γ_K)-module. Then one has

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = D^{\Gamma_K},$$
$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = (\mathbf{B}^\dagger_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_K},$$
$$(\widetilde{\mathbf{B}}^\dagger_{\log}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} = (\mathbf{B}^\dagger_{\log,K}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_K}.$$

Proof. We only treat the first case, as the proof of the others is similar.

One has $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K} \subset (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{H_K} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D$ since H_K acts trivially on D (it is a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module). Let $\{e_i\}_{1\leq i\leq d}$ be a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-basis of D and $\{d_i\}_{1\leq i\leq r}$ be a K_0-basis for $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{G_K}$ and $M \in M_{r\times d}(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})$ defined by the relation $(d_i) = M(e_i)$. M has rang r (that is, the image of a basis of D under M form a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module of rank r) and satisfies $\gamma_K(M)G = M$ (since the elements d_i are fixed under γ_K), where $G \in \mathrm{GL}_d(\mathbf{B}^\dagger_{\mathrm{rig},K})$ is the matrix of γ_K with respect to the basis $\{e_i\}$. The operator R_m of Colmez/Berger (cf. loc.cit., §2.6) give $\gamma_K(R_m(M))G - R_m(M) = 0$ for every $m \in \mathbb{N}$. Further $R_m(M) \overset{m\to\infty}{\longrightarrow} M$ and $N = \varphi^m(R_m(M)) \in \mathbf{B}^\dagger_{\mathrm{rig},K}$ since R_m

is a section of $\varphi^{-m}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K}) \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$. Hence, $\gamma(N)\varphi^m(G) = N$ and since the actions of φ and Γ_K commute on D one has $\varphi(G)P = \gamma_K(P)G$, where $P \in M_d(\mathbf{B}^{\dagger}_{\mathrm{rig},K})$ is the matrix of φ with respect to the basis $\{e_i\}$. If one sets $Q = \varphi^{m-1}(P)\ldots\varphi(P)P$ then $\varphi^m(G)Q = \gamma_K(Q)G$ and hence $\gamma_K(NQ)G = NQ$, so that NQ determines r elements in D that are fixed under Γ_K. But since for m big enough the matrix M has rank r and P has full rank, since it is an injection and $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \cdot \varphi(D) = D$, one sees that these elements give a rank r-submodule of D. Hence, the K_0-vectorspace generated by these elements is also of dimension r, whence the claim. □

Before stating the next result we recall the notion of a p-adic differential equation. If D is any (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ it is known that the same definition as for ∇ gives rise to differential operator $\nabla_D : D \to D$ that commutes with the action of φ and Γ_K such that $\nabla_D(\lambda x) = \nabla(\lambda)x + \lambda\nabla_D(x)$ (see [8], Proposition III.1.1). With this one may also consider the operator $\partial_D = 1/t \cdot \nabla_D$ on $D[1/t]$. A **p-adic differential equation** is a (φ, Γ_K)-module D over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ that is stable under the operator ∂_D.

If there is no confusion we will drop the index D of the operators ∇_D and ∂_D.

Theorem 2.6.19. Let M be a p-adic differential equation equipped with a Frobenius. Then there exists a finite extension L/K such that the natural map

$$\mathbf{B}^{\dagger}_{\log,L} \otimes_{L_0'} (\mathbf{B}^{\dagger}_{\log,L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D)^{\partial=0} \to \mathbf{B}^{\dagger}_{\log,L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D.$$

is an isomorphism.

Proof. [1]. □

Recall that a **∇-crystal** over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is a free $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-module equipped with an action of a Frobenius and a connection (also denoted by ∇), compatible with ∇ on $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, that commutes with the Frobenius. A ∇-crystal over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is called **unipotent** if it admits a filtration of sub-crystals such that each successive quotient has a basis consisting of elements in the kernel of ∇. More generally, a ∇-crystal M is called **quasi-unipotent** if there exists a finite extension L/K such that $\mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} M$ (which is a ∇-crystal over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ in a natural way) is unipotent.

We note the following result, which is known by the experts and may be proved as in the étale case ([4], Proposition 5.6):

Proposition 2.6.20. Every de Rham (φ, Γ_K)-module is potentially semi-stable.

Proof. One defines the (faithful, exact, ...) functor $D \mapsto \mathbf{N}_{\mathrm{dR}}(D)$ from the category of de Rham (φ, Γ_K)-modules into the category of p-adic differential equations equipped with a Frobenius. Since by André's theorem 2.6.19 one knows that any such equation is quasi-unipotent, it suffices to show that D is potentially semistable if and only if $\mathbf{N}_{\mathrm{dR}}(D)$ is quasi-unipotent.

Now D is potentially semistable if and only if there exists a finite extension L/K such that

$$\dim_{L_0}(\mathbf{B}^\dagger_{\log,L}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_L} = \mathrm{rank}_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D =: d.$$

This gives via [4], Proposition 5.5 a unipotent ∇-subcrystal of $D|_L[1/t]$, which is nothing else but $\mathbf{N}_{\mathrm{dR}}(D|_L) \cong \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{N}_{\mathrm{dR}}(D)$.

Conversely if $D|_{L'}[1/t]$ contains a unipotent ∇-subcrystal of rank d for some finite extension L'/K then the again by loc.cit. there exist elements $e_0, \dots, e_{d-1}$ which generate an L'_0-vectorspace of dimension d on which $\log(\gamma)$ acts trivially. Hence, there exists a finite extension L/L' such that Γ_L acts trivially on this basis, so that we obtain a basis of $(\mathbf{B}^\dagger_{\log,L}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)^{\Gamma_L}$ of the right dimension, i.e. D is potentially semistable. □

We briefly review the slope theory of φ-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ or $\mathbf{B}^\dagger_K$.

Definition 2.6.21. Let M is a φ-module over one of these rings. If M is of rank 1 and v a generator, then $\varphi(v) = \lambda v$ for some $\lambda \in (\mathbf{B}^\dagger_{\mathrm{rig},K})^\times = (\mathbf{B}^\dagger_K)^\times$ (cf. [24]; see also [25], Hypothesis 1.4.1. resp. Example 1.4.2). We define the **degree** $\deg(M)$ of M to be $w(\lambda)$, where w is the p-adic valuation of $\mathbf{B}_K$. If M is of rank n then $\bigwedge^n M$ has rank 1. We define the **slope** $\mu(M)$ of M as $\mu(M) = \deg(M)/\mathrm{rk}M$.

We remark that the definition of the degree (hence the slope) is independent of the choice of the generator. Under the equivalence of Theorem 2.5.6 we have the following correspondence of the slope theory: If D is a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, one may consider the φ-module $\widetilde{D}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$. Then the two definitions of the slope for D coincide. Hence, we have the notion of a (φ, Γ_K)-module that is **pure** of some slope. The fundamental theorem is the following result by Kedlaya:

Theorem 2.6.22. (Slope filtration theorem) Let M be a φ-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then there exists a unique filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$ by saturated φ-submodules whose successive quotients are pure with $\mu(M_1/M_0) < \ldots < \mu(M_{l-1}/M_l)$. If M is a (φ, Γ_K)-module all M_i are (φ, Γ_K)-submodules.

Proof. See [25]. □

We recall that Berger has constructed an exact $\otimes$-functor $\mathcal{M}$ from the category of filtered (φ, N, G_K)-modules to the category of (φ, Γ_K)-modules such that the associated connection is locally trivial (see [8]). This functor allows for the following construction. Assume D is a semi-stable (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then one has a sequence of filtered (φ, N)-modules (since $p\varphi N = N\varphi$)

$$0 = N^j \mathbf{D}^K_{\mathrm{st}}(D) \subset N^{j-1}\mathbf{D}^K_{\mathrm{st}}(D) \subset \ldots \subset N\mathbf{D}^K_{\mathrm{st}}(D) \subset \mathbf{D}^K_{\mathrm{st}}(D)$$

determined by the monodromy operator N such that each quotient $N^i\mathbf{D}^K_{\mathrm{st}}(D)/N^{i+1}\mathbf{D}^K_{\mathrm{st}}(D)$ is also a filtered (φ, N)-module. Hence, we often reduce to the case of an exact sequence

$$0 \longrightarrow N\mathbf{D}^K_{\mathrm{st}}(D) \longrightarrow \mathbf{D}^K_{\mathrm{st}}(D) \longrightarrow \mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D) \longrightarrow 0 \qquad (2.7)$$

which induces an exact sequence

$$0 \longrightarrow \mathcal{M}(N\mathbf{D}_{\mathrm{st}}^K(D)) \longrightarrow D \longrightarrow \mathcal{M}(\mathbf{D}_{\mathrm{st}}^K(D)/N\mathbf{D}_{\mathrm{st}}^K(D)) \longrightarrow 0.$$

We want to consider the slope filtration on $\mathbf{N}_{\mathrm{dR}}(D)$. If D is crystalline then this comes from a filtration of vector spaces on $\mathbf{D}_{\mathrm{cris}}^K(D)$. If D is semi-stable we can actually show that Kedlaya's slope filtration is compatible with (2.7), i.e., one may assume that one has a filtraton on $(\mathbf{B}_{\log,K}^\dagger \otimes N\mathbf{D}_{\mathrm{st}}^K(D))^{N=0}$ and $(\mathbf{B}_{\log,K}^\dagger \otimes \mathbf{D}_{\mathrm{st}}^K(D)/N\mathbf{D}_{\mathrm{st}}^K(D))^{N=0}$ such that the slopes of the former are all strictly smaller than the slopes of the latter (since $N\varphi = p\varphi N$). This induces then the slope filtration on $\mathbf{N}_{\mathrm{dR}}(D)$. However, we shall not make use of this fact.

One has the following result (cf. [7], Theorem 3.1.5):

Theorem 2.6.23. Let D be a φ-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$. Then there exists an étale φ-module $D' \subset D[1/t]$ such that $D'[1/t] = D[1/t]$.

The proof uses the technique of a modification of a (φ, Γ)-module, cf. loc.cit., section 3.1, for the definition and notation. We note that if M is any modification of D one has an inclusion of φ-modules $tD = D[0] \subset D[M] \subset D$ and hence $D[M][1/t] = D[1/t]$. Similarly, we can prove the following:

Theorem 2.6.24. Let D be a de Rham (φ, Γ)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ that is pure of some weight s. Then there exists a finite extension L/K and an étale (φ, Γ)-module D' over $\mathbf{B}_{\mathrm{rig},L}^\dagger$ with $D' \subset D|_L[1/t]$ such that $D'[1/t] = D|_L[1/t]$.

Proof. We may assume that D is semistable and further that $D = \mathbf{N}_{\mathrm{dR}}(D) = (\mathbf{B}_{\log,K}^\dagger \otimes_{K_0} \mathbf{D}_{\mathrm{st}}^K(D))^{N=0}$. Then D is pure of slope $s = a/h$.

We may modify D by an M of codimension 1 which gives a module $D[M] \subset D$ of degree $\deg D + 1$. Further we may choose this M in such a way that it is stable by the action of Γ_K, noting that the action of Γ_K occurs on the $\mathbf{B}_{\log,K}^\dagger$-part of D. Since $\mu(D) = \deg D/\mathrm{rk}\, D$ after a finite number of these modifications we obtain that the slope of $D[M']$ is an integer, so we may modify it by a power of t to obtain an étale (φ, Γ)-module which gives us the solution. □

Remark 2.6.25. Note that the statement for general (φ, Γ)-modules is false, see for instance [7], Remark 3.1.7. This is even true in the case of a de Rham (φ, Γ_K)-module as the example in loc.cit. shows.

Chapter 3

Exponential maps

3.1 Bloch-Kato exponential maps for (φ, Γ_K)-modules

In this section we define short exact sequences associated to (φ, Γ_K)-modules, generalizing the "classical" Bloch-Kato sequence (see [11]) which one may use to study cohomological questions relating to p-adic representations (i.e. the slope zero case). One interesting phenomenon that occurs in this more general setting is that, in order to get the general versions of the exponential maps, it is necessary to distinguish between the slope ≤ 0-case and the slope > 0-case.

We are interested in the long exact sequences for continuous Galois-cohomology induced by these sequences. Let us briefly recall the machinery. Let M be continuous G_K-module and define the continuous imhomogeneous cochains in the usual way ($q \geq 0$):

$$C^q_{\mathrm{cont}}(G_K, M) := C^q_{\mathrm{cont}}(K, M) := \{x : G^n \longrightarrow M \mid x \text{ continuous}\}$$

with differential $\delta^q : C^q_{\mathrm{cont}}(K, M) \to C^{q+1}_{\mathrm{cont}}(K, M)$ defined by

$$\begin{aligned}\delta^q(x)(g_1, \ldots, g_{q+1}) = g_1 x(g_2, \ldots, g_{q+1}) + (-1)^{q+1} x(g_1, \ldots, g_q) \\ + \sum_{i=1}^{q} (-1)^i x(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{q+1}).\end{aligned}$$

By convention $C^{-i}(G_K, M) = 0$ for $i > 1$. The continuous cochain complex is then defined via

$$C^\bullet_{\mathrm{cont}}(K, M) := \left[C^0_{\mathrm{cont}}(K, M) \xrightarrow{\delta^0} C^1_{\mathrm{cont}}(K, M) \xrightarrow{\delta^1} \ldots\right],$$

and one defines continuous cohomology via

$$H^q_{\mathrm{cont}}(K, M) := H^q(C^\bullet_{\mathrm{cont}}(K, M)).$$

Lemma 3.1.1. If $0 \to M' \longrightarrow M \xrightarrow{f} M'' \longrightarrow 0$ is an exact sequence of G_K-modules such that f admits a continuous (but not necessarily G_K-equivariant) splitting, then continuous cohomology induces a long exact sequence

$$\ldots \to H^i_{\mathrm{cont}}(K, M') \to H^i_{\mathrm{cont}}(K, M) \to H^i_{\mathrm{cont}}(K, M'') \to H^{i+1}_{\mathrm{cont}}(K, M') \to \ldots$$

Proof. This is standard, see for example [40], §2. □

If there is no possibility of confusion we will drop the subscript "cont". The splitting property in our setting will be granted by the following

Proposition 3.1.2. If $f : B_1 \longrightarrow B_2$ be a linear continuous surjective map of p-adic Banach spaces, there exists a continuous splitting $s : B_2 \longrightarrow B_1$ of f, i.e. $f \circ s = \mathrm{id}_{B_2}$.

Proof. See [14], Proposition I.1.5, (iii). □

We define the following set X, which will be used in the next few statements:

$$X := \{(x, y, z) \in \widetilde{D}_{\log}[1/t] \oplus \widetilde{D}_{\log}[1/t] \oplus \mathbf{W}_e(D)/\mathbf{W}^+_{\mathrm{dR}}(D) |\ N(y) = (p\varphi - 1)(x)\}.$$

Lemma 3.1.3. Let D be a (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. We assume D is pure of slope $\mu(D) \leq 0$. Then one has the following exact sequences of G_K-modules (cf. (2.3) for the definition of β):

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \mathbf{W}_e(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \longrightarrow 0$$
$$x \longmapsto \beta(x)$$

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \widetilde{D}[1/t] \xrightarrow{g} \widetilde{D}[1/t] \oplus \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \longrightarrow 0$$
$$x \longmapsto ((\varphi - 1)(x), \beta(x))$$

$$0 \longrightarrow \mathbf{W}^+_e(D) \xrightarrow{f} \widetilde{D}_{\log}[1/t] \xrightarrow{g} X \longrightarrow 0$$
$$x \longmapsto (N(x), (\varphi - 1)(x), \beta(x))$$

Additionally, each g above admits a continuous (not necessarily G_K-equivariant) splitting.

Proof. The exactness of the first sequence is tautological, see Theorem 2.5.7. For the second recall that for a φ-module M over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ the map $\varphi - 1 : M[1/t] \to M[1/t]$ is surjective. This implies the exactness of the second sequence. For the exactness of the last sequence first observe that the map g is well-defined. Recall that $N : \widetilde{D}_{\log} \to \widetilde{D}_{\log}$ is extended linearly from the operator N on $\widetilde{\mathbf{B}}^\dagger_{\log}$, so that

$$N(\sum_{i \geq 0} d_i \log^i \pi) = -\sum_{i \geq 1} i \cdot d_i \log^{i-1} \pi \tag{3.1}$$

for $\sum_{i\geq 0} d_i \log^i \pi \in \widetilde{D}_{\log}$. The exactness at $\widetilde{D}_{\log}[1/t]$ is clear since from (3.1) one has $(\widetilde{D}_{\log}[1/t])^{N=0} = \widetilde{D}[1/t]$, so we only have to check the exactness at X. The surjectivity of $N : \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \to \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]$, which again follows from (3.1), implies that it is enough to check that if $(0, y, z) \in X$ then there exists $x' \in \widetilde{D}[1/t]$ such that $g(x') = (0, y, z)$, which is nothing but exactness of the second sequence.

The splitting property follows from Proposition 3.1.2, where we remark that X is a complete space since by definition it is a closed subspace of the complete Banach space $\widetilde{D}_{\log}[1/t] \oplus \widetilde{D}_{\log}[1/t] \oplus \mathbf{W}_e(D)/\mathbf{W}^+_{\mathrm{dR}}(D)$. □

Lemma 3.1.4. Let D be a (φ, Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. We assume D is pure of slope $\mu(D) > 0$. Then one has the following exact sequences of G_K-modules (cf. (2.3) for the definition of β):

$$0 \longrightarrow \mathbf{W}_e(D) \xrightarrow{f} \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$x \longmapsto \overline{x}$$

$$0 \longrightarrow \widetilde{D}[1/t] \xrightarrow{f} \widetilde{D}[1/t] \oplus \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$f : x \longmapsto ((1-\varphi)(x), \overline{x})$$
$$g : (x, y) \longmapsto \overline{y}$$

$$0 \longrightarrow \widetilde{D}_{\log}[1/t] \xrightarrow{f} X \xrightarrow{g} \mathbf{W}_{\mathrm{dR}}(D)/(\mathbf{W}_e(D) + \mathbf{W}^+_{\mathrm{dR}}(D)) \longrightarrow 0$$
$$f : x \longmapsto (N(x), (\varphi - 1)(x), \overline{x})$$
$$g : (x, y, z) \longmapsto \overline{z}$$

Additionally, each g above admits a continuous (not necessarily G_K-equivariant) splitting.

Proof. The exactness of the first sequence is again tautological by Theorem 2.5.7. The rest of the proof follows analoguously to the previous proposition. □

Putting everything together, we also see:

Corollary 3.1.5. Let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Then one has the following exact sequence of G_K-modules:

$$0 \longrightarrow X^0(\widetilde{D}) \xrightarrow{i} \widetilde{D}_{\log}[1/t] \xrightarrow{f} X \xrightarrow{p} X^1(\widetilde{D}) \longrightarrow 0$$
$$i : x \longmapsto x$$
$$f : x \longmapsto (N(x), (\varphi - 1)(x), \overline{x})$$
$$p : (x, y, z) \longmapsto \overline{z}$$

Following Nakamura, we now define for a B-pair $W = (W_e, W^+_{\mathrm{dR}})$ the following complex:

$$C^{\bullet}(G_K, W) := \mathrm{cone}(C^{\bullet}(G_K, W_e) \longrightarrow C^{\bullet}(G_K, W_{\mathrm{dR}}/W^+_{\mathrm{dR}})),$$

which is induced by the canonical inclusion $W_e \xrightarrow{i} W_{\mathrm{dR}}$. That is, we have

$$C^i(G_K, W) = C^i(G_K, W_e) \oplus C^{i-1}(G_K, W_{\mathrm{dR}}/W_{\mathrm{dR}}^+)$$

with differentials

$$\delta_C^i : C^i(G_K, W) \ni (a, b) \mapsto (\delta^i_{C^i(G_K,W_e)}(a), i(a) - \delta^{i-1}_{C^i(G_K,W_e)}(b))$$

More generally, one may define the following complexes:

$$C^\bullet(G_K, W') := \mathrm{cone}(C^\bullet(G_K, \widetilde{D}[1/t]) \xrightarrow{(1-\varphi,i)} C^\bullet(G_K, \widetilde{D}[1/t] \oplus W_{\mathrm{dR}}/W_{\mathrm{dR}}^+)),$$
$$C^\bullet(G_K, W'') := \mathrm{cone}(C^\bullet(G_K, \widetilde{D}_{\log}[1/t]) \xrightarrow{(N,1-\varphi,i)} C^\bullet(G_K, X)),$$

We recall:

Lemma 3.1.6. Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be a short exact sequence of continuous G_K-modules such that g admits a continuous, but not necessarely G_K-equivariant, splitting. We write (by abuse of notation)

$$\mathrm{cone}(g) := \mathrm{cone}(C^\bullet(G_K, B) \xrightarrow{g_*} C^\bullet(G_K, C))$$
$$\mathrm{cone}(f) := \mathrm{cone}(C^\bullet(G_K, A) \xrightarrow{f_*} C^\bullet(G_K, B)).$$

a) The natural map of complexes

$$\begin{array}{cccccc}
C^\bullet(G_K, A): & C^0(G_K, A) & \longrightarrow & C^1(G_K, A) & \longrightarrow & \cdots \\
\downarrow & \downarrow f & & \downarrow (f,0) & & \\
\mathrm{cone}(g): & C^0(G_K, B) & \longrightarrow & C^1(G_K, B) \oplus C^0(G_K, C) & \longrightarrow & \cdots
\end{array}$$

is a quasi-isomorphism that is compatible with the long exact sequence, i.e. the following diagram is commutative:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i(G_K, A) & \longrightarrow & H^i(G_K, B) & \longrightarrow & H^i(G_K, C) & \xrightarrow{\delta} & H^{i+1}(G_K, A) & \longrightarrow \cdots \\
& \downarrow & & \| & & \| & & \downarrow & \\
\cdots \longrightarrow & H^i(\mathrm{cone}(g)) & \longrightarrow & H^i(G_K, B) & \longrightarrow & H^i(G_K, C) & \xrightarrow{\delta} & H^{i+1}(\mathrm{cone}(g)) & \longrightarrow \cdots
\end{array}$$

b) The natural map of complexes

$$\begin{array}{cccccc}
C^\bullet(G_K, C)[-1]: & 0 = C^{-1}(G_K, C) & \longrightarrow & C^0(G_K, C) & \longrightarrow & \cdots \\
\uparrow & 0 \uparrow & & (0,g) \uparrow & & \\
\mathrm{cone}(f): & C^0(G_K, A) & \longrightarrow & C^1(G_K, A) \oplus C^0(G_K, B) & \longrightarrow & \cdots
\end{array}$$

is a quasi-isomorphism that is compatible with the long exact sequence, i.e. the following diagram is commutative:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i(G_K,A) & \longrightarrow & H^i(G_K,B) & \longrightarrow & H^i(G_K,C) & \xrightarrow{\delta} & H^{i+1}(G_K,A) & \longrightarrow \cdots \\
 & \| & & \| & & \uparrow & & \| & \\
\cdots \longrightarrow & H^i(G_K,A) & \longrightarrow & H^i(G_K,B) & \xrightarrow{\delta} & H^{i+1}(\mathrm{cone}(f)) & \xrightarrow{\delta} & H^{i+1}(G_K,A) & \longrightarrow \cdots
\end{array}$$

Proof. This is left as an exercise, see for example [41], 1.5.8. □

Lemma 3.1.7. We have canonical quasi-isomorphisms $C^\bullet(G_K,W) \cong C^\bullet(G_K,W') \cong C^\bullet(G_K,W'')$.

Proof. Let $W = \mathbf{W}(D)$. Observe that the inclusions $\mathbf{W}_e(D) \subset \widetilde{D}[1/t] \subset \widetilde{D}_{\log}[1/t]$ and $\mathbf{W}_{\mathrm{dR}}(D)$ induce canonical maps on these complexes. If $W = \mathbf{W}(D)$ with D pure of some slope the statement then follows from Lemmas 3.1.3, 3.1.4 and 3.1.6.

For general D we are by Kedlaya's slope filtration theorem reduced to the case of an exact sequence $0 \to D_1 \to D \to D_2 \to 0$ such that the statement is true for D_1, D_2, hence the claim follows by considering the long exact sequences associated to this. □

With this statement and the properties of the cone we obtain a long exact sequence of cohomology groups:

$$\ldots \to H^i(G_K,W) \to H^i(G_K,\widetilde{D}_{\log}[1/t]) \to H^i(G_K,X) \xrightarrow{\delta} H^{i+1}(G_K,W) \to \ldots$$

With these exact sequences in mind we suggest the following

Definition 3.1.8. Let D be a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. The transition map

$$\exp_{K,D} : H^0(K,X) \to H^1(K,\mathbf{W}(D))$$

from the exact sequence above is called **generalized Bloch-Kato exponential map** for D.

Remark 3.1.9. Let D be an étale (φ,Γ_K)-module, so that $D = \mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ for some p-adic representation V of Γ_K. Then since the slope of D is equal to zero, the first exact sequence in Lemma 3.1.3 computes to

$$0 \longrightarrow V \to \mathbf{B}_e \otimes_{\mathbb{Q}_p} V \longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \longrightarrow 0$$

This is nothing but the usual Bloch-Kato short exact sequence associated to the p-adic representation V.

Recall that if D is any (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ the map $\varphi - 1 : \widetilde{D}[1/t] \to \widetilde{D}[1/t]$ is surjective. If $x \in \widetilde{D}$ we write $(\varphi-1)^{-1}(x)$ for a choice of an element $y \in \widetilde{D}[1/t]$ such that $(\varphi - 1)(y) = x$. We want to consider the following maps:

$$\alpha : \widetilde{D} \longrightarrow \mathbf{W}_e(D), \quad x \mapsto \begin{cases} x, & \varphi(x) = x, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta : \widetilde{D} \longrightarrow \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D), \quad x \mapsto \iota_n((\varphi-1)^{-1}(x)),$$

where the second map is well-defined due to the discussion in [9], Remark 3.4. α and β are continuous and fit into the following commutative diagram of G_K-modules:

$$\begin{array}{ccccccccccc}
0 & \to & \widetilde{D}^{\varphi=1} & \to & \widetilde{D} & \xrightarrow{\varphi-1} & \widetilde{D} & \to & \widetilde{D}/(\varphi-1)\widetilde{D} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \to & X^0(\widetilde{D}) & \to & \mathbf{W}_e(D) & \to & \mathbf{W}_{\mathrm{dR}}(D)/\mathbf{W}^+_{\mathrm{dR}}(D) & \to & X^1(\widetilde{D}) & \to & 0,
\end{array}$$

where we use the identifications for X^0 and X^1 from Theorem 2.5.7.

Proposition 3.1.10. One has a quasi-isomorphism

$$\mathrm{cone}(C^\bullet(G_K,\widetilde{D}) \xrightarrow{\varphi-1} C^\bullet(G_K,\widetilde{D})) \cong C^\bullet(G_K,\mathbf{W}(D))$$

that is functorial in D.

Proof. We denote by $A^\bullet$ the complex on the left hand side of the statement. One checks that the commutativity of the preceeding diagram and the cohomological version of [41], Exercise 1.5.9 show that one has a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \to & H^n(G_K,\widetilde{D}^{\varphi=1}) & \to & H^n(A^\bullet) & \to & H^{n-1}(G_K,\frac{\widetilde{D}}{(\varphi-1)\widetilde{D}}) & \to & H^{n+1}(G_K,\widetilde{D}^{\varphi=1}) & \to & \cdots \\
 & & \| & & \downarrow & & \| & & \| & & \\
\cdots & \to & H^n(G_K,X^0(\widetilde{D})) & \to & H^n(G_K,\mathbf{W}(D)) & \to & H^{n-1}(G_K,X^1(\widetilde{D})) & \to & H^{n+1}(G_K,X^0(\widetilde{D})) & \to & \cdots
\end{array}$$

which gives the proof. □

Recall the following property of continuous cohomology: If $f: M^\bullet \to N^\bullet$ is map of complexes of continuous G-modules for some profinite group G one has an identification of complexes

$$C^\bullet_{\mathrm{cont}}(G,\mathrm{cone}(M^\bullet \xrightarrow{f} N^\bullet)) = \mathrm{cone}\left(C^\bullet_{\mathrm{cont}}(G,M^\bullet) \xrightarrow{f_*} C^\bullet_{\mathrm{cont}}(G,N^\bullet)\right) \tag{3.2}$$

(cf. the discussion in [31], 3.4.1.3, 3.4.1.4; it holds in this general setting).

We recall that in the derived category of $\mathbf{B}^\dagger_{\mathrm{rig},K}$-modules, the complex $C^\bullet_{\varphi,\gamma}$ is also represented by

$$R\Gamma(K,D) = R\Gamma_{\mathrm{cont}}(\Gamma_K,\mathrm{cone}\left[D \xrightarrow{\varphi-1} D\right]) \cong \mathrm{cone}\left[R\Gamma_{\mathrm{cont}}(\Gamma_K,D) \xrightarrow{\varphi-1} R\Gamma_{\mathrm{cont}}(\Gamma_K,D)\right],$$

cf. [37], section 3.3, where the last identification is due to (3.2).

The following is then a generalization of Proposition 2.6.18:

Proposition 3.1.11. One has an isomorphism

$$R\Gamma(K,D) \cong R\Gamma(K,\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D)$$

that is functorial in D.

Proof. The proof is similar to [37], Proposition 3.8. It suffices to show that that the natural map

$$R\Gamma_{\rm cont}(\Gamma_K, D) \longrightarrow R\Gamma_{\rm cont}(\Gamma_K, \widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D)$$

is an isomorphism, since applying cone $\left[\bullet \xrightarrow{\varphi-1} \bullet\right]$ induces the morphism in the statement again due to (3.2). We apply the techniques of [2], Appendix I and use the notation there, as follows: Let $\widetilde{\Lambda} := \widetilde{\mathbf{B}}^{\dagger,r}_{\rm rig}$, $\mathcal{G} = G_K$, $\mathcal{H} = H_K$ so that $d = 0$. Further, $\mathcal{H}' = \mathcal{H}$, $\Lambda^{(i)}_{m,\mathcal{H}'} = \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_{{\rm rig},K})$ (since $i = 0$ is the only possible choice) and the maps $\tau^{(i)}_{m,\mathcal{H}'}$ correspond to the maps $R_m : \widetilde{\mathbf{B}}^{\dagger,r}_{{\rm rig},K} \to \varphi^{-m}(\mathbf{B}^{\dagger,p^m r}_{{\rm rig},K})$ (cf. [4], Proposition 2.32). As in [2], section 7.6, the maps R_m induce maps (by the usual process of taking the direct limit over all sufficiently big r) $R_m : \widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D \to \widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D$ for $m \geq 0$, and as in loc.cit. one obtains a decomposition of Γ_K-modules

$$\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D \cong (1 - R_m)(\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D) \oplus (\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D)^{R_m=1}.$$

By construction of the map R_m it is clear that $(\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D)^{R_0=1} = D$. Furthermore, as in the proof of loc.cit., Proposition 7.7, one may infer that $\gamma_K - 1$ acts invertibly on $(1 - R_0)(\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D)$, so that $R\Gamma_{\rm cont}(\Gamma_K, (1 - R_0)(\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes_{\mathbf{B}^{\dagger}_{{\rm rig},K}} D)) = 0$, which gives the claim. □

Putting everything together, we see:

Corollary 3.1.12. One has an isomorphism

$$R\Gamma(K, D) \cong R\Gamma(G_K, \mathbf{W}(D)).$$

that is functorial in D.

Proof. We observe that the natural map

$$\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \cong R\Gamma_{\rm cont}(H_K, \widetilde{\mathbf{B}}^{\dagger}_{\rm rig}) \tag{3.3}$$

is a quasi-isomorphism. This, together with the preceeding isomorphisms implies

$$\begin{aligned} R\Gamma(K, D) &\cong R\Gamma_{\rm cont}(\Gamma_K, {\rm cone}(D \xrightarrow{\varphi-1} D)) \\ &= R\Gamma_{\rm cont}(\Gamma_K, {\rm cone}(\widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes D \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}^{\dagger}_{{\rm rig},K} \otimes D)) \\ &= R\Gamma_{\rm cont}(\Gamma_K, R\Gamma_{\rm cont}(H_K, {\rm cone}(\widetilde{D} \xrightarrow{\varphi-1} \widetilde{D}))) \\ &\overset{(*)}{=} R\Gamma_{\rm cont}(G_K, {\rm cone}(\widetilde{D} \xrightarrow{\varphi-1} \widetilde{D})) \\ &= R\Gamma(G_K, \mathbf{W}(D)). \end{aligned}$$

where $(*)$ holds since the natural map $H^i(G_K/H_K, \widetilde{D}^{H_K}) \to H^i(G_K, \widetilde{D})$ is an isomorphism, since again $H^n(H_K, \widetilde{D}) = 0$ for $n > 0$ due to (3.3): for $i = 1$ this follows from the five term exact sequence in low degree, which extends in this case for continuous cohomology similarly as in e.g. [32], §6, to higher degrees by induction. □

Corollary 3.1.13. $H^i(G_K, \mathbf{W}(D)) = 0$ for $i \neq 0, 1, 2$ and $H^i(G_K, \mathbf{W}(D))$ is a finite-dimensional $\mathbb{Q}_p$-vectorspace.

Proof. This follows from the preceeding Corollary and [26], Theorem 8.1. □

We wish to give a more explicit description of the isomorphisms on cohomology which we will need in the characterizing property of the big exponential map, where actually only the map for the H^1's will be important for us. Hence, we may only sketch certain steps for the higher cohomology groups (that is, H^2).

We briefly describe how one may interpret, in the slope ≤ 0-case, the cohomology group $H^1(G_K, \mathbf{W}_e^+(D))$ as extensions of $\mathbb{Q}_p$ by $\mathbf{W}_e^+(D)$. So let $c \in H^1(G_K, \mathbf{W}_e^+(D))$ and consider the exact sequence of G_K-modules

$$0 \longrightarrow \mathbf{W}_e^+(D) \longrightarrow E_c \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

where $E_c = \mathbb{Q}_p \oplus \mathbf{W}_e^+(D)$ as $\mathbb{Q}_p$-vectorspace and G_K acts on E_c via

$$\sigma(a, m) = (a, \sigma m + a c_\sigma).$$

Since c is a 1-cocycle one has

$$\sigma(\tau(a, x)) = \sigma(a, \tau x + a c_\tau) = (a, \sigma\tau x + a\sigma c_\tau + c_\sigma) = \sigma\tau(a, x),$$

so that one has a well-defined map $Z^1(K, D) \to \mathrm{Ext}(\mathbb{Q}_p, \mathbf{W}_e^+(D))$. E_c is trivial if and only if there exists an element $1 \in E_c$ such that $g1 = 1$ for all g, i.e.

$$1 = (1, x), \quad g1 - 1 = (0, gx - x + c_g) = 0,$$

so that $c_g = (1-g)x$ is a coboundary, which implies that the above map factors through $B^1(K, D)$. The fact that this map is an isomorphism can be checked as in the p-adic representation case.

Proposition 3.1.14. Suppose we are in the situation of Lemma 3.1.3. Then the complex $C^\bullet_{\varphi,\gamma_K}(K, D)$ (functorially) computes the cohomology of $C^\bullet_{\mathrm{cont}}(G_K, X^0(D))$.

Proof. We may assume that Γ_K is pro-cyclic with generator γ_K. First we have

$$H^0(K, D) = D^{\Gamma_K, \varphi=1} = \widetilde{D}^{G_K, \varphi=1} = X^0(D)^{G_K} = H^0(G_K, X^0(D)).$$

thanks to Proposition 2.6.18.

For H^1 we apply the construction of Cherbonnier/Colmez ([13]). To wit, let $(x, y) \in H^1(K, D)$ and pick $b \in \widetilde{D}$ such that $(\varphi - 1)b = x$. Then

$$h^1_{K,D}((x, y)) = \log_p^0(\chi(\gamma)) \cdot \left(\sigma \longmapsto \frac{\sigma - 1}{\gamma_K - 1} y - (\sigma - 1)b\right)$$

defines a 1-cocycle with values in $\widetilde{D}$ but one easily checks that $(\varphi - 1)h^1_{K,D}((x, y)) = 0$ so that we actually have a cocycle in $H^1(G_K, X^0(D))$. Injectivity and surjectivity now follow in the same way as in loc.cit. if one uses the description of extensions of $\mathbb{Q}_p$ by $X^0(D)$ given above, so that we obtain the isomorphism in the H^1-case.

For H^2 one can show that since $X^0(D)$ is an almost $\mathbb{C}_p$-representation that one has a Hochschild-Serre spectral sequence $H^i(\Gamma_K, H^j(H_K, X^0(D))) \Rightarrow H^{i+j}(G_K, X^0(D))$ associated to the exact sequence $1 \to H_K \to G_K \to \Gamma_K \to 1$. Since the cohomology on the left vanishes for j or i greater or equal to 2 one has with the fact that $H^3(G_K, X^0(D)) = 0$

$$H^2(G_K, X^0(D)) \cong H^1(\Gamma_K, H^1(H_K, X^0(D))).$$

Now the exact sequence $0 \to X^0(D) \to \widetilde{D} \overset{\varphi-1}{\to} \widetilde{D} \to 0$ of G_K-modules gives rise to a sequence

$$\ldots \longrightarrow \widetilde{D}^{H_K} \overset{\varphi-1}{\longrightarrow} \widetilde{D}^{H_K} \longrightarrow H^1(H_K, X^0(D)) \longrightarrow 0,$$

since $H^1(H_K, \widetilde{D}) = H^1(H_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes D) \cong H^1(H_K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^d = 0$. Hence, by Iwasawa theory

$$H^2(G_K, X^0(D)) \cong \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1).$$

Looking at the quasi-isomorphisms in Corollary 3.1.12 one sees that using Lemma 3.1.6, since we are in the $X^1(D) = 0$-case, the map $H^2(K, D) \to H^2(G_K, X^0(D))$ is given by the canonical inclusion of finite-dimensional $\mathbb{Q}_p$-vectorspaces

$$H^2(K, D) = D/(\varphi - 1, \gamma_K - 1) \subset \widetilde{D}^{H_K}/(\varphi - 1, \gamma_K - 1) = H^2(G_K, X^0(D)),$$

that are of the same dimension. This gives the description of the map for H^2. □

Lemma 3.1.15. Let D be a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and assume that Γ_K is pro-cyclic with generator γ_K. Then one has an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{D^{\varphi=1}}{(\gamma_K-1)} & \overset{f}{\longrightarrow} & H^1(K, D) & \overset{g}{\longrightarrow} & \left(\frac{D}{\varphi-1}\right)^{\Gamma_K} & \longrightarrow & 0 \\ & & y & \longmapsto & (0, y) & & & & \\ & & & & (x, y) & \longmapsto & x & & \end{array}$$

Proof. Recall that by definition

$$H^1(K, D) = \{(x, y) \in D \oplus D \mid (\gamma_K - 1)x = (\varphi - 1)y\}/\{((\varphi - 1)z, (\gamma_K - 1)z) \mid z \in D\},$$

so that the first map is well-defined an injective. One checks that the map g is well-defined and if $x \in D/(\varphi - 1)$ such that $(\gamma_K - 1)x \in (\varphi - 1)D$ then there exists an $y \in D$ such that $(x, y) \in H^1(K, D)$ and $g(x, y) = x$. Obviously $g \circ f = 0$. Let $g(x, y) = 0$ so that $x = (\varphi - 1)z$ for some $z \in D$, so that $(x, y) \sim (0, y - (\gamma_K - 1)z)$ in $H^1(K, D)$. Hence, (x, y) is in the image of f. □

We remark that this sequence is nothing but the short exact sequence associated to the inflation-restriction sequence if D is étale, i.e.,

$$0 \longrightarrow H^1(\Gamma_K, V^{H_K}) \longrightarrow H^1(G_K, V) \longrightarrow H^1(H_K, V^{\Gamma_K}) \longrightarrow 0,$$

see for example [15], section 5.2.

Proposition 3.1.16. Suppose we are in the situation of Lemma 3.1.4. Then the complex $C^\bullet_{\varphi,\gamma_K}(K,D)$ computes the cohomology of $C^\bullet_D := C^\bullet_{\text{cont}}(G_K, X^1(D))[1]$

Proof. We may assume that Γ_K is procyclic with generator γ_K. Since the slope of D is > 0 one has $X^0(D) = 0$, so that $D^{\varphi=1} = 0$ since $D^{\varphi=1} \subset \widetilde{D}^{\varphi=1} = 0$, so that $H^0(K, D) = 0$. The same holds tautologically for $H^0(C^\bullet_D)$.

For the case of the H^1's observe that since $X^0(D) = 0$ Lemma 3.1.15 implies that the canonical map $H^1(K, D) \to (D/(\varphi-1))^{\Gamma_K})$, $\overline{(x, y)} \mapsto \overline{x}$, is an isomorphism. From Theorem 2.5.7 we also know that $X^1(D) = \widetilde{D}/(\varphi - 1)$. Hence, from Corollary 3.1.12 and Lemma 3.1.6 we have that the map

$$H^0(G_K, X^1(D)) = \left(\frac{\widetilde{D}}{\varphi-1}\right)^{G_K} = \left(\frac{D}{\varphi-1}\right)^{\Gamma_K} \cong\ H^1(K, D).$$

gives the identification.

For H^2 one has similarly as in the slope ≤ 0-case a Hochschild-Serre spectral sequence $H^i(\Gamma_K,\ H^j(H_K, X^1(D))) \Rightarrow H^{i+j}(G_K, X^1(D))$. From the exact sequence in low degree terms one then has

$$0 \to H^1(\Gamma_K, H^0(H_K, \widetilde{D}/(\varphi-1)) \to H^1(G_K, \widetilde{D}/(\varphi-1)) \to H^0(\Gamma_K, H^1(H_K, \widetilde{D}/(\varphi-1)).$$

From the sequence $0 \longrightarrow \widetilde{D} \overset{\varphi-1}{\longrightarrow} \widetilde{D} \to X^1(D) \longrightarrow 0$ one infers the vanishing of $H^1(H_K, X^1(D))$ since $H^1(H_K, \widetilde{D}) = H^2(H_K, TD) = H^2(H_K, \widetilde{\mathbf{B}}^\dagger_{\text{rig}})^d = 0$. Hence, we see

$$H^1(G_K, X^1(D)) = H^1(\Gamma_K, H^0(H_K, \widetilde{D}/(\varphi-1)) = \widetilde{D}^{H_K}/(\varphi-1, \gamma_K - 1).$$

so that again by Corollary 3.1.12 and Lemma 3.1.6 the canonical inclusion of finite-dimensional $\mathbb{Q}_p$-vectorspaces

$$H^2(K, D) = D/(\varphi-1, \gamma_K - 1) \subset \widetilde{D}^{H_K}/(\varphi-1, \gamma_K-1) = H^1(G_K, X^1(D)),$$

gives the description of the map for H^2. □

Finally we describe how one may piece together the isomorphisms $H^i(K, D) \overset{h^1}{\longrightarrow} H^i(K, \mathbf{W}(D))$ in the general case (where we only make the case H^1 explicit, which is all we need for the application to Perrin-Riou's exponential map): If $(x, y) \in H^1(K, D)$ write $x = (\varphi-1)(b') + s(b'')$, where $s : \widetilde{D}/(\varphi-1)\widetilde{D} \to \widetilde{D}$ is a continuous splitting of the natural projection (which exists thanks to Proposition 3.1.2), $b' \in \widetilde{D}$ and $b'' \in \widetilde{D}/(\varphi-1)\widetilde{D}$. Putting the two constructions together, we may consider the tuple

$$\begin{aligned} h^1(x, y) := \Big(\log^0_p(\chi(\gamma)) \cdot \Big(\sigma \longmapsto \tfrac{\sigma-1}{\gamma_K-1}y - (\sigma-1)b'\Big), (0, 0, \varphi^{-n}((\varphi-1)^{-1}(s(b'')))\Big) \\ \in C^1(G_K, \widetilde{D}_{\log}) \oplus C^0(G_K, X), \end{aligned} \tag{3.4}$$

and one sees that actually $h^i((x, y)) \in H^1(K, \mathbf{W}(D))$, which gives the description of the isomorphism in the general case by the properties of the mapping cone.

We will briefly describe, similarly as in the slope ≤ 0-case before, how one may interpret the cohomology group $H^1(G_K, \mathbf{W}_e(D))$ as extensions of $\mathbf{B}_e$ by $\mathbf{W}_e(D)$ (note however that we do not make any assumptions about the slopes of D). So let $c \in H^1(G_K, \mathbf{W}_e(D))$ and consider the exact sequence of G_K-modules

$$0 \longrightarrow \mathbf{W}_e(D) \longrightarrow E_c \longrightarrow \mathbf{B}_e \longrightarrow 0,$$

where $E_c = \mathbf{B}_e \oplus \mathbf{W}_e(D)$ as a $\mathbf{B}_e$-module with G_K-action $\sigma(a,x) = (\sigma a, \sigma x + \sigma a \cdot c_\sigma)$. One has

$$\sigma(\tau(a,x)) = \sigma(\tau a, \tau x + \tau a \cdot c_\tau) = (\sigma\tau a, \sigma\tau x + \sigma\tau a \cdot \sigma c_\tau + \sigma\tau a \cdot c_\sigma) = \sigma\tau(a,x),$$

so that one has a well-defined map $Z^1(K, \mathbf{W}_e(D)) \to \mathrm{Ext}(\mathbb{Q}_p, \mathbf{W}_e^+(D))$. E_c is trivial if and only if there exists an element $1 \in E_c$ such that $g1 = 1$ for all g, i.e.

$$1 = (1,x), \quad g1 - 1 = (0, gx - x + c_g) = 0,$$

so that $c_g = (1-g)x$ is a coboundary, which implies that the above map factors through $B^1(K, \mathbf{W}_e(D))$. The fact that this map is an isomorphism can be checked as before.

Proposition 3.1.17. Let D be a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then the complex $C^\bullet_{\varphi,\gamma_K}(K, D[1/t])$ computes the cohomology of $C^\bullet_{\mathrm{cont}}(G_K, \mathbf{W}_e(D))$.

Proof. The proof is similar to the ones before; in fact, one may reduce to the case of Corollary 3.1.5 by taking direct limits (see also [30], Theorem 4.5). We are interested in the explicit description of the maps. From Proposition 2.6.18 again we have:

$$H^0(K, D[1/t]) = D[1/t]^{\varphi=1,\Gamma_K} = \widetilde{D}[1/t]^{\varphi=1,G_K} = H^0(G_K, \mathbf{W}_e(D)).$$

For H^1 we apply the same construction as in Proposition 3.1.14. So let $(x,y) \in H^1(K, D[1/t])$ and pick $b \in \widetilde{D}[1/t]$ such that $(\varphi-1)b = x$. Then

$$h^1_{K,D}((x,y)) = \log_p^0(\chi(\gamma)) \cdot \left(\sigma \longmapsto \frac{\sigma-1}{\gamma_K-1}y - (\sigma-1)b\right)$$

defines a 1-cocycle with values in $\widetilde{D}[1/t]$ which lies actually in $\mathbf{W}_e(D)$. Injectivity and surjectivity now follow in the same way as in loc.cit. if one uses the description of extensions of $\mathbf{B}_e$ by $\mathbf{W}_e(D)$ given above, so that we obtain the isomorphism in the H^1-case.

The case of the H^2's follows in the same way as in Proposition 3.1.14. □

Proposition 3.1.18. One has an identification $H^0(K, \mathbf{W}_{\mathrm{dR}}(D)) = \mathbf{D}^K_{\mathrm{dR}}(D)$

Proof. From [13], Proposition IV.1.1 (i) we know that $K_\infty[[t]]$ is dense in $(\mathbf{B}^+_{\mathrm{dR}})^{H_K}$, and the inclusion is compatible the action of Γ_K. Also one has $(\mathbf{B}^+_{\mathrm{dR}})^{G_K} = K_\infty[[t]]^{\Gamma_K} = K$. Since D is free as a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module with trivial H_K-action, we see that $(\mathbf{B}^+_{\mathrm{dR}} \otimes D)^{G_K} = ((\mathbf{B}^+_{\mathrm{dR}})^{H_K} \otimes D)^{\Gamma_K} = \mathbf{D}^+_{\mathrm{dif}}(D)^{\Gamma_K}$. Since $\mathbf{B}_{\mathrm{dR}} = \varinjlim_{n\geq 0} 1/t^n \cdot \mathbf{B}^+_{\mathrm{dR}}$ and $K_\infty((t)) = \varinjlim_{n\geq 0} 1/t^n \cdot K_\infty[[t]]$ the claim follows, since taking invariants is compatible with direct limits.

Alternatively, the claim also follows from [19], Theorem 2.14, B) i). □

We shall make use of the following considerations. Let D be a semi-stable (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and consider the following complex $\mathfrak{C}_{\mathrm{st}}(K,D)$ (concentrated in degrees 0, 1, 2):

$$\begin{array}{ccccc} \mathbf{D}^K_{\mathrm{st}}(D) & \to & \mathbf{D}^K_{\mathrm{st}}(D)\oplus\mathbf{D}^K_{\mathrm{st}}(D)\oplus\mathbf{D}^K_{\mathrm{dR}}(D)/\mathrm{Fil}^0\mathbf{D}^K_{\mathrm{dR}}(D) & \to & \mathbf{D}^K_{\mathrm{st}}(D) \\ x & \mapsto & (N(x),(\varphi-1)(x),\beta(x)) & & \\ & & (x,y,z) & \mapsto & N(x)-(p\varphi-1)(y). \end{array} \tag{3.5}$$

Then an element in $H^1(\mathfrak{C}_{\mathrm{st}}(K,D))$ can be considered as an element in $H^0(K,X)$ and hence be mapped via the exponential map to $H^1(K,\mathbf{W}(D))$.

We shall give two maps which will be important in the construction of the dual exponential map for de Rham (φ,Γ_K)-modules.

First we remark that the canonical inclusion $D\to\mathbf{W}_{\mathrm{dR}}(D)$ factors via $D\to D[1/t]$. This allows us to describe a map $H^1(K,D)\to H^1(G_K,\mathbf{W}_{\mathrm{dR}}(D))$ explicitly via the composition of the canonical map $H^1(K,D)\to H^1(K,D[1/t])$, the identification $H^1(K,D[1/t])\xrightarrow{\sim} H^1(G_K,\mathbf{W}_e(D))$ (cf. Proposition 3.1.17) and the canonical map $H^1(K,\mathbf{W}_e(D))\longrightarrow H^1(K,\mathbf{W}_{\mathrm{dR}}(D))$.

Secondly, we show that the map

$$\mathbf{D}^K_{\mathrm{dR}}(D)\longrightarrow H^1(G_K,\mathbf{W}_{\mathrm{dR}}(D)),\quad x\longmapsto[g\mapsto\log(\chi(\overline{g}))x] \tag{3.6}$$

which generalizes Kato's formula of [23], §II.1, is an isomorphism, which may be proved as follows. First observe that

$$H^1(G_K,\mathbf{B}_{\mathrm{dR}}\otimes D)\cong H^1(G_K,\mathbf{B}_{\mathrm{dR}}\otimes_K\mathbf{D}^K_{\mathrm{dR}}(D))=H^1(G_K,\mathbf{B}_{\mathrm{dR}})\otimes_K\mathbf{D}^K_{\mathrm{dR}}(D).$$

From [21], Proposition 5.25, one knows that $K=H^0(G_K,\mathbf{B}_{\mathrm{dR}})\to H^1(G_K,\mathbf{B}_{\mathrm{dR}})$, $x\mapsto x\cdot\log\chi$ is an isomorphism. This gives the claim.

Definition 3.1.19. The **generalized Bloch-Kato dual exponential map** $\exp^*_{K,D^*(1)}$ is the composition of the above maps $H^1(K,D)\to H^1(G_K,\mathbf{W}_{\mathrm{dR}}(D))$ with the inverse of the isomorphism $\mathbf{D}^K_{\mathrm{dR}}(D)\xrightarrow{\sim} H^1(G_K,\mathbf{W}_{\mathrm{dR}}(D))$.

Of course, in the étale case this is nothing but the dual exponential map considered by Kato in [23]. But even in this more general case this map has the desired property with respect to adjunction via pairings. First recall that one may define the K-bilinear perfect pairing $[\ ,\ ]_{K,D}$ by the natural map

$$[\ ,\ ]_{K,D}:\mathbf{D}^K_{\mathrm{dR}}(D)\times\mathbf{D}^K_{\mathrm{dR}}(D^*(1))\xrightarrow{\mathrm{ev}}\mathbf{D}^K_{\mathrm{dR}}(\mathbf{B}^{\dagger}_{\mathrm{rig},K}(1))\longrightarrow K.$$

For the next proposition we note that Nakamura uses a different definition of the dual exponential map (see [30], section 2.4), which we briefly recall (we refer to loc.cit for the proofs): one may define the cohomology groups $H^i(K,\mathbf{D}_{\mathrm{dif}}(D))$ by $H^i_{\mathrm{cont}}(\Gamma_K,\mathbf{D}_{\mathrm{dif}}(D))$, which is computed by the complex

$$C^{\bullet}_{\gamma,\Delta}(\mathbf{D}_{\mathrm{dif}}(D)):\ \mathbf{D}_{\mathrm{dif}}(D)\xrightarrow{\gamma-1}\mathbf{D}_{\mathrm{dif}}(D).$$

Since the natural map $K_\infty((t)) \otimes_K \mathbf{D}^K_{\mathrm{dR}}(D) \to \mathbf{D}_{\mathrm{dif}}(D)$ is an isomorphism one has an identification

$$g_D : \mathbf{D}^K_{\mathrm{dR}}(D) \xrightarrow{\sim} H^1(K, \mathbf{D}_{\mathrm{dif}}(D)), \quad x \mapsto \overline{(\log \chi(\gamma))1 \otimes x}.$$

The second definition of $\exp^*_{K,D}$ is then given by the composition of the map $H^1(K, D) \to H^1(K, \mathbf{D}_{\mathrm{dif}}(D))$, $[(x, y)] \mapsto \iota_n(y)$ (for n big enough) and the inverse of g_D. Since $H^i(H_K, \mathbf{B}_{\mathrm{dR}}) = 0$ for $i > 0$ the five term exact sequence gives $H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D)) \cong H^1(\Gamma_K, \mathbf{B}^{H_K}_{\mathrm{dR}} \otimes D)$. Using the same argument as in Proposition 3.1.18 one sees that the natural map $H^1(K, \mathbf{D}_{\mathrm{dif}}(D)) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ is an isomorphism. Further, the natural map $H^1(K, D) \to H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D))$ defined before is also given by $[(x, y)] \mapsto \iota_n(y)$. Hence, using all these identifications one obtains a commutative diagram

$$\begin{array}{ccccc}
H^1(K, D) & \longrightarrow & H^1(K, \mathbf{D}_{\mathrm{dif}}(D)) & \xleftarrow{\sim} & \mathbf{D}^K_{\mathrm{dR}}(D) \\
\| & & \downarrow\wr & & \downarrow\wr \\
H^1(K, D) & \longrightarrow & H^1(G_K, \mathbf{W}_{\mathrm{dR}}(D)) & \xleftarrow{\sim} & H^0(G_K, \mathbf{W}_{\mathrm{dR}}(D)),
\end{array}$$

which shows that the two definitions of $\exp^*$ coincide.

Proposition 3.1.20. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and let $x \in \mathbf{D}^K_{\mathrm{dR}}(D)$ and $y \in H^1(K, D^*(1))$. Then

$$\langle \exp_{K,D}(x), y \rangle_{K,D} = \mathrm{Tr}_{K/\mathbb{Q}_p}[x, \exp^*_{K,D}(y)]_{K,D}$$

Proof. See [30], Proposition 2.16. □

Proposition 3.1.21. Let D be a semi-stable (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Let $y \in D^{\psi=1}$ and consider y as $y \in (\mathbf{B}^\dagger_{\log,K}[1/t] \otimes_F \mathbf{D}^K_{\mathrm{st}}(D))^{N=0,\psi=1}$ via the comparison isomorphism. Then for $n \gg 0$

$$\exp^*_{V^*(1)}(h^1_{D,K_n}(y)) = p^{-n} \varphi^{-n}(y)(0).$$

Proof. As before we have

$$h^1_{D,K_n}(y)(\sigma) = \frac{\sigma - 1}{\gamma_{K_n} - 1} y - (\sigma - 1)b,$$

with $(\gamma_{K_n} - 1)(\varphi - 1)b = (\varphi - 1)y$ for some $b \in \widetilde{D}[1/t]$. Further Let n be big enough so that we may embed this cocycle into $\mathbf{B}_{\mathrm{dR}} \otimes D$, hence $\varphi^{-n}(y) \in K_n((t)) \otimes \mathbf{D}^K_{\mathrm{st}}(D)$ and we may consider $\varphi^{-n}(b)$ as an element in $\mathbf{B}_{\mathrm{dR}} \otimes D$. Since $\gamma_{K_n} t = \chi(\gamma_{K_n})t$ the action of $\gamma_{K_n} - 1$ is invertible on $t^k K_n \otimes \mathbf{D}^K_{\mathrm{st}}(D)$ for every $k \neq 0$. Putting this together we see that h^1_{D,K_n} is equivalent in $H^1(K_n, \mathbf{B}_{\mathrm{dR}} \otimes D)$ to

$$\sigma \longmapsto \frac{\sigma - 1}{\gamma_{K_n} - 1} (\varphi^{-n}(y))(0).$$

σ acts via its image $\overline{\sigma} \in \Gamma_K^n$ (trivially) on K_n. Furthermore, if $n_i \in \mathbb{Z}$ is a sequence such that $\overline{\sigma} = \lim_{i\to\infty} \gamma_{K_n}^{n_i}$ one checks by going to the limit that

$$\frac{\sigma - 1}{\gamma_{K_n} - 1} \frac{\log_p \chi(\gamma_{K_n})}{\log_p \chi(\overline{\sigma})}$$

acts trivially on K_n. Hence, the above cycle is equivalent to

$$\sigma \longmapsto p^{-n} \log(\chi(\overline{\sigma}))(\varphi^{-n}(y))(0)$$

The claim follows now from formula (3.6). □

3.2 Perrin-Riou exponential maps for (φ, Γ_K)-modules

We make the following definitions:

Definition 3.2.1. Let M be a (φ, N)-module over F. Define $\mathbf{N}_{\mathrm{dR}}(M) = (\mathbf{B}^{\dagger}_{\log,K} \otimes_F M)^{N=0}$, where $N = 1 \otimes N + N \otimes 1$ on $\mathbf{B}^{\dagger}_{\log,K} \otimes_F M$.

If D is a semi-stable (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ then $\mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D)) = \mathbf{N}_{\mathrm{dR}}(D)$ (see Definition 2.6.15).

Definition 3.2.2. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

a) Let $\mathbf{D}_{\infty,g}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that there exists an $r \in \mathbb{Z}$ such that the equation $(1-p^r\varphi)G = \partial^r(g)$ has a solution in $G \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=p^r}$.

b) Let $\mathbf{D}_{\infty,f}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that there exists a family $(G_k)_{k\in\mathbb{Z}}$ of elements $G_k \in \mathbf{N}_{\mathrm{dR}}(D)$ with $\partial(G_k) = G_{k+1}$ and an $r \in \mathbb{Z}$ such that $(1 - p^r\varphi)G = \partial^r(g)$

c) Let $\mathbf{D}_{\infty,e}(D)$ be the submodule of elements $g \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ such that the equation $(1 - p^r\varphi)G = \partial^r(g)$ has a solution in $G \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=p^r}$ for every $r \in \mathbb{Z}$.

We first note that if $D \longrightarrow D'$ is a morphism of two de Rham (φ, Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ then this induces a map of Γ_K-modules $\mathbf{D}_{\infty,*}(D) \to \mathbf{D}_{\infty,*}(D')$. Also, one clearly has

$$\mathbf{D}_{\infty,e}(D) \subset \mathbf{D}_{\infty,f}(D) \subset \mathbf{D}_{\infty,g}(D) \subset \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}.$$

By the above definition one may also define the modules $\mathbf{D}_{\infty,*}(\ )$ by starting with a $(\varphi, \mathbf{N})$-module.

We note that we shall define another module $\mathcal{D}_{\Lambda_\infty}(D)$ in Definition 5.1.1 which is "very close" to $\mathbf{D}_{\infty,f}(D)$ (and in certain cases coincides with it) of which we think that it is the "right" generalization of $\mathbf{D}_{\infty,f}(D)$ in the crystalline case.

Definition 3.2.3. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. We say that D is of **Perrin-Riou-type** (or of PR-type) if D is semistable and $K_0 = K_0'$.

Lemma 3.2.4. The map $\partial : \mathbf{B}^\dagger_{\log,K} \to \mathbf{B}^\dagger_{\log,K}$ is surjective.

Proof. This amounts to an integration of power-series, cf. [4], Proposition 4.4. □

Lemma 3.2.5. Suppose $K_0 = K_0'$. Then the kernel of ∂ on $\mathbf{B}^\dagger_{\log,K}$ is equal to K_0.

Proof. Let $f \in \mathbf{B}^\dagger_{\mathrm{rig},K}$. Due to Proposition 2.3.2 and Lemma 2.4.3 there is a polynomial P in $\mathbf{B}^\dagger_{\mathrm{rig},F}$ such that $P(f) = 0$ and $P'(f) \neq 0$. Then $\partial(f) = -(\partial P)(f)/P'(f)$, so that $\partial(f) = 0$ if and only if $f \in K_0$.

Now suppose $f = \sum_{i=1}^r f_i \log^i \pi \in \mathbf{B}^\dagger_{\log,K}$ and $\partial(f) = 0$. Since $\log \pi$ is a transcendent element over any $\mathbf{B}^\dagger_{\mathrm{rig},K}$ this gives rise to relations $\partial(f_i) + (j+1)\frac{\pi+1}{\pi} f_{i+1} = 0$ with $f_{r+1} = 0$. For $i = r$ this implies $f_r = \lambda \in K_0$, hence $\partial(f_{r-1}) = -\lambda r \frac{\pi+1}{\pi}$. Suppose there exists an $f \in \mathbf{B}^\dagger_{\mathrm{rig},K}$ with $\partial(f) = \frac{1+\pi}{\pi}$. Then $\partial(\log \pi - f) = 0$, so that $\log \pi = f + a$ with $a \in K_0$, a contradiction to the transcendency property of $\log \pi$. Hence, $\frac{\pi+1}{\pi}$ is not an element in the image of ∂ on $\mathbf{B}^\dagger_{\mathrm{rig},K}$, and we obtain $\lambda = 0$. By recurrence this shows that the kernel of ∂ on $\mathbf{B}^\dagger_{\log,K}$ is contained in K_0. □

Let again D be a de Rham (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$.

Lemma 3.2.6. Let D be of PR-type. Then the map $\partial : \mathbf{B}^\dagger_{\log,K} \otimes \mathbf{N}_{\mathrm{dR}}(D) \to \mathbf{B}^\dagger_{\log,K} \otimes \mathbf{N}_{\mathrm{dR}}(D)$ is surjective.

Proof. We have
$$\mathbf{B}^\dagger_{\log,K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{N}_{\mathrm{dR}}(D) = \mathbf{B}^\dagger_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D),$$
whence the claim follows from the Lemma above. □

Proposition 3.2.7. Let D be of PR-type. The map
$$\partial : \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} \longrightarrow \mathbf{N}_{\mathrm{dR}}(D[1])^{\psi=0}(1)$$
is an isomorphism of Γ_K-modules.

Proof. With our preparations, namely, Lemma 3.2.4 and Lemma 3.2.5, this proof works the same as in [35], Proposition 2.2.3. □

Obviously the operator ∂ induces a map of Γ_K-modules
$$\partial : \mathbf{N}_{\mathrm{dR}}(D)^{\psi=1} \to \mathbf{N}_{\mathrm{dR}}(D[1])^{\psi=1}(1)$$
which however is in general neither injective nor surjective. This should be contrasted with the étale case where $D^{\psi=1} = \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} = H^1(K, V \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma_K))$ and the fact that ∂ in this setting corresponds to the Tate-twist isomorphism.

For a semistable (φ, Γ_K)-module consider the following complex:
$$\mathfrak{C}_K(D) : \quad 0 \to \mathbf{D}^K_{\mathrm{st}}(D) \xrightarrow{\delta_0} \mathbf{D}^K_{\mathrm{st}}(D) \times \mathbf{D}^K_{\mathrm{st}}(D) \xrightarrow{\delta_1} \mathbf{D}^K_{\mathrm{st}}(D) \to 0$$

with

$$\delta_0(\nu) = (N\nu, (1-\varphi)\nu),$$
$$\delta_1(\lambda, \mu) = N\mu - (1-p\varphi)\lambda.$$

Hence,

$$H^0(\mathfrak{C}_K(D)) = \mathbf{D}_{\mathrm{st}}^K(D)^{\varphi=1, N=0},$$
$$H^1(\mathfrak{C}_K(D)) = \{(\lambda, \mu) \in \mathbf{D}_{\mathrm{st}}^K(D) \times \mathbf{D}_{\mathrm{st}}^K(D) |\ N\mu = (1-p\varphi)\lambda\}/\delta_0(\mathbf{D}_{\mathrm{st}}^K(D)),$$
$$H^2(\mathfrak{C}_K(D)) = \mathbf{D}_{\mathrm{st}}^K(D)/(N, 1-p\varphi)\mathbf{D}_{\mathrm{st}}^K(D).$$

One also checks that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\mathbf{D}_{\mathrm{st}}^K(D)^{N=0}}{(\varphi-1)\mathbf{D}_{\mathrm{st}}^K(D)^{N=0}} & \longrightarrow & H^1(\mathfrak{C}_K(D)) & \longrightarrow & \frac{\mathbf{D}_{\mathrm{st}}^K(D)}{N\mathbf{D}_{\mathrm{st}}^K(D))^{\varphi=p^{-1}}} & \longrightarrow & 0 \\ & & \mu & \longmapsto & (0,\mu) & & & & \\ & & & & (\lambda,\mu) & \longmapsto & \lambda & & \end{array} \tag{3.7}$$

furnishes an exact sequence for $H^1(\mathfrak{C}(D))$.

We see that $H^0(\mathfrak{C}(D(k))) = 0$ for $k \gg 0$ resp. $k \ll 0$ since the groups $\mathbf{D}_{\mathrm{st}}(D(k))^{\varphi=1}$ and $(\varphi-1)\mathbf{D}_{\mathrm{st}}(D(k))$ vanish for those k. Similarly, $H^1(\mathfrak{C}(D(k))) = 0$ for $k \gg 0$ resp. $k \ll 0$.

Now let D be a de Rham (φ, Γ_K)-module and fix a finite extension L/K such that $D|_L$ is semistable with $L_0 = L_0'$.

Lemma 3.2.8. Let $k \in \mathbb{N}$. Then one has an exact sequence of Γ_K-modules

$$0 \to \bigoplus_{-k \le i < 0} H^0(\mathfrak{C}(D|_L(-i)))(i) \cap \mathbf{N}_{\mathrm{dR}}(D(k))^{\psi=1}(-k)) \to \mathbf{N}_{\mathrm{dR}}(D(k))^{\psi=1}(-k)$$
$$\xrightarrow{\partial^k} \mathbf{N}_{\mathrm{dR}}(D)^{\psi=1} \xrightarrow{\widetilde{\mathcal{R}}_D} \bigoplus_{-k \le i < 0} H^1(\mathfrak{C}(D|_L(-i)))(i)$$

Proof. The proof may be done in an analogous way as in [35], Lemma 2.2.5. We give a description of the map $\widetilde{\mathcal{R}}_D$ following the definition of a map $\mathcal{R}_D$ (cf. equation (3.10)) since the constructions which give rise to it will be important later on. We just briefly mention that this map depends on the inclusion $\mathbf{N}_{\mathrm{dR}}(D) \subset \mathbf{N}_{\mathrm{dR}}(D|_L)$ which is induced by the inclusion $D \subset D|_L$. □

From the lemma we see that, by considering the possible eigenvalues for φ,

$$\mathbf{D}_{\infty,e}(D) = \partial^h(1-p^{-h}\varphi)\mathbf{N}_{\mathrm{dR}}(D(h))^{\psi=1}, \tag{3.8}$$
$$\mathbf{D}_{\infty,g}(D) = \partial^{-h}(1-p^h\varphi)\mathbf{N}_{\mathrm{dR}}(D(-h))^{\psi=1} \tag{3.9}$$

for $h \gg 0$ since the $H^i(\mathfrak{C}(D))$, $i = 0, 1$, vanish in this case. More precisely, for étale (φ, Γ_K)-module one has the following:

Lemma 3.2.9. Let $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ for a p-adic representation V that is de Rham. Let $h \geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}^K_{\mathrm{dR}}(D) = \mathbf{D}^K_{\mathrm{dR}}(D)$. Then $\mathbf{D}_{\infty,g}(D) = \partial^{-(h+1)}(1-p^{h+1}\varphi)\mathbf{N}_{\mathrm{dR}}(D(-(h+1)))^{\psi=1}$.

Proof. We may reduce to the case that D is semi-stable with $K_0 = K_0'$ and further by twisting that $h = 1$. We have to check that $\partial : \mathbf{N}_{\mathrm{dR}}(D(-2))^{\psi=1} \to \mathbf{N}_{\mathrm{dR}}(D(-3))^{\psi=1}(1)$ is an isomorphism, i.e., we have to check the vanishing of $H^0(\mathfrak{C}(D(-2)))$ and $H^1(\mathfrak{C}(D(-2)))$. For the first this is obvious since for an admissible filtered (φ, N)-module that is positive the eigenvalues of the Frobenius are positive. Similarly, thanks to the exact sequence (3.7), we see that the H^1-part vanishes. □

Remark 3.2.10. We suspect that in the cases where V is as above and does not contain the subrepresentation $\mathbb{Q}_p(h)$ one actually has $\mathbf{D}_{\infty,g}(D) = \partial^{-h}(1 - p^h\varphi)\mathbf{N}_{\mathrm{dR}}(D(-h))^{\psi=1}$. This would fit in with the characterizing description of the big exponential map in the étale case; cf. also the discussion in [34], section 5.1.

We recall the application $\mathcal{R}_D$. For our purposes (since we may restrict/corestrict) it will be enough for this part to assume that D of PR-type over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

Definition 3.2.11. Let $g \in \mathbf{D}_{\infty,g}(D)$ and r be big enough such that $\mathbf{D}_{\infty,g}(D)$ admits the description in (3.9). A family of elements $(G_k)_{k\in\mathbb{Z}}$ in $\mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{N}_{\mathrm{dR}}(D)$ is called a **complete solution** for $(1-\varphi)G = g$ if $\partial(G_k) = G_{k+1}$ (cf. 3.2.6) and $\partial^r(g) = (1-p^r\varphi)G_r$ for r big enough.

If $G = (G_k)$ is a complete solution of $g \in \mathbf{D}_{\infty,g}(D)$ we also write $\partial^{-k}(G) = G_k$ by abuse of notation. Let $s \gg 0$ such that $(1 - p^s\varphi)G_s = \partial^s(g)$. Then one sees inductively thanks to Lemma 3.2.8 that

$$N(G_k) = \sum_{j\geq -k} \lambda_j \frac{t^{j+k}}{(j+k)!} =: L_k, \quad \lambda_j \in \mathbf{D}^K_{\mathrm{st}}(D)$$

$$(\psi \otimes 1 - p^{-k} \otimes \varphi)(G_k) = p^{-k} \sum_{j\geq -k} \mu_j \frac{t^{j+k}}{(j+k)!} =: (\psi \otimes 1)(M_k), \quad \mu_j \in \mathbf{D}^K_{\mathrm{st}}(D),$$

where for almost all j one has $\lambda_j = \mu_j = 0$. On $\mathbf{B}^{\dagger}_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D)$, as one checks easily, we have the identity of operators

$$(pN\otimes 1+1\otimes N)(\psi\otimes 1-p^{-k}\otimes\varphi) = (\psi\otimes 1-p^{-k+1}\otimes\varphi)(N\otimes 1+1\otimes N) = (\psi\otimes 1-p^{-k+1}\otimes\varphi)N,$$

hence

$$N((\psi \otimes 1)(M_k)) = (\psi \otimes 1 - p^{-k+1} \otimes \varphi)(L_k),$$

since $N\otimes 1$ vanishes on elements of $\sum t^i \cdot \mathbf{D}^K_{\mathrm{st}}(D)$, hence the relation (by applying $(\psi^{-1}\otimes 1 = \varphi \otimes 1$, which we may since ψ acts invertibly on $\sum t^i \cdot \mathbf{D}^K_{\mathrm{st}}(D))$

$$N(M_k) = (1 - p^{-k+1}\varphi)(L_k).$$

On the coefficients this implies the relation

$$N\mu_j = (1 - p^{-j+1}\varphi)\lambda_j.$$

If $A = \sum_{j\geq -k} \nu_j/(j+k)! \cdot t^{j+k}$ and if one changes G_k to $G'_k = G_k + A$ so that still $\partial^k(G'_k) = \partial^k(G_k)$, then λ_j is changed to $\lambda_j + N(\nu_j)$ and μ_j is changed to $\mu_j + (1-p^j\varphi)\nu_j$. Hence, L_k is changed to $L_k + N(A)$ and M_k is changed to $M_k + (1-\varphi)(A)$, so that the class of (λ_i, μ_i) is well-defined in $H^1(\mathfrak{C}(D|_L(-i)))$. The tupel (λ_j, μ_j) may be considered as an element of $H^1(\mathfrak{C}(D|_L(-i)))(i)$, and we denote the collection of these elements element by $\mathcal{R}_D(g)$, i.e. one has a Γ_K-equivariant map

$$\mathcal{R}_D : \mathbf{D}_{\infty,g}(D) \longrightarrow \bigoplus_{i\in\mathbb{Z}} H^1(\mathfrak{C}(D|_L(-i)))(i). \tag{3.10}$$

We note that the map $\widetilde{\mathcal{R}}_D$ in Lemma 3.2.8 is the composition of $(1-\varphi)$ with $\mathcal{R}_D$ and the natural projection to the sum $\bigoplus_{-k\leq i<0} H^1(\mathfrak{C}(D|_L(-i)))(i)$.

Define for all $k \in \mathbb{Z}$

$$N(G_k) = L_k =: \partial^{-k}(L)$$
$$(\psi\otimes 1 - 1\otimes\varphi)(G_k) = \psi\otimes 1(M_k) =: \psi\otimes 1(\partial^{-k}(M)).$$

These definitions imply that (calculating again in $\mathbf{B}^\dagger_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D)$)

$$\psi((1-\varphi)(G_k) - M_k) = (\psi\otimes 1)((1-\varphi)(G_k) - M_k)) = 0,$$

hence, since ∂ acts invertibly on $(\mathbf{B}^\dagger_{\log,K} \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D))^{\psi=0}$,

$$\partial^k(g) = (1-p^k\varphi)G_k - M_k.$$

Of course, $M_k = L_k = 0$ for k big enough. We will also refer to the system $H = (L_k^{[1]}, M_k, G_k)$ as a **complete solution** for $g \in \mathbf{D}_{\infty,g}(D)$, where by $L_k^{[1]}$ we mean that the action of φ is multiplied by p. This extra factor is introduced so that the interpolation property holds.

Following Perrin-Riou, we set

$$\mathcal{U}(D) := \bigoplus_{i\in\mathbb{Z}} t^i \cdot \mathbf{D}_{\mathrm{st}}(D)$$

and

$$\mathbf{D}^2_{\infty,g}(D) := \mathcal{U}(D)/(1-p\varphi, N)\mathcal{U}(D).$$

Proposition 3.2.12. One has the following exact sequences of Γ_K-modules:

$$0 \longrightarrow \mathbf{D}_{\infty,e}(D) \longrightarrow \mathbf{D}_{\infty,g}(D) \xrightarrow{\mathcal{R}_D} \bigoplus_{i\in\mathbb{Z}} H^1(\mathfrak{C}_L(D|_L(-i)))(i)$$

$$0 \longrightarrow \mathbf{D}_{\infty,f}(D) \longrightarrow \mathbf{D}_{\infty,g}(D) \xrightarrow{\overline{\mathcal{R}}_D} (\mathcal{U}(D|_L)/N\mathcal{U}(D|_L))^{\varphi=p^{-1}}$$

$$0 \longrightarrow \mathbf{D}_{\infty,e}(D) \longrightarrow \mathbf{D}_{\infty,f}(D) \xrightarrow{\mathcal{R}_D} (\mathcal{U}(D|_L))^{N=0}/(1-\varphi)(\mathcal{U}(D))^{N=0}.$$

Proof. See [35], Proposition 2.3.4. □

We remark that in the case where $K/\mathbb{Q}_p$ is unramified one can show all the right-most maps in the preceeding Proposition are actually surjective. This can be deduced as in [35], Proposition 4.1.1.

The following Lemma will show in an example that $\mathbf{D}_{\infty,f}(\ )$ need not be exact:

Lemma 3.2.13. Assume $K/\mathbb{Q}_p$ is unramified. Let $0 \to D_1 \to D_2 \to D_3 \to 0$ be an exact sequence of (φ, N)-modules over K. Then one has a commutative diagram of Γ_K-modules with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathbf{D}_{\infty,f}(D_2)}{\mathbf{D}_{\infty,f}(D_1)} & \longrightarrow & \mathbf{D}_{\infty,f}(D_3) & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathbf{D}_{\infty,g}(D_2)}{\mathbf{D}_{\infty,g}(D_1)} & \longrightarrow & \mathbf{D}_{\infty,g}(D_3) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & \frac{(\mathcal{U}(D_2)/N\mathcal{U}(D_2))^{\varphi=p^{-1}}}{(\mathcal{U}(D_1)/N\mathcal{U}(D_1))^{\varphi=p^{-1}}} & \xrightarrow{h} & (\mathcal{U}(D_3)/N\mathcal{U}(D_3))^{\varphi=p^{-1}} & \longrightarrow & P & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array},$$

where $N = \ker(\mathbf{D}^2_{\infty,g}(D_1) \to \mathbf{D}^2_{\infty,g}(D_2))$, $P = \operatorname{coker}(h)$, $M = \ker(g)$.

Proof. The diagram may be constructed from Proposition 3.2.12 and [36], Proposition 4.3.2. The surjectivity of f follows from the snake lemma. □

Definition 3.2.14. a) For a torsion free element γ of Γ_K and $i \in \mathbb{Z}$ Perrin-Riou's differential operator $\nabla_i = l_i$ is defined as

$$\nabla_i = \frac{\log(\gamma)}{\log_p(\chi(\gamma))} - i = \nabla_0 - i$$

b) The operator $\nabla_0/(\gamma_n - 1)$ for n such that Γ_n is cyclic is defined as

$$\frac{\nabla_0}{\gamma_n - 1} := \frac{\log(\gamma_n)}{\log_p(\chi(\gamma))(\gamma_n - 1)} := \frac{1}{\log_p(\chi(\gamma_n))} \sum_{i=1}^{\infty} \frac{(1-\gamma_n)^{i-1}}{i}.$$

First, we remark that the second operator is *not* a quotient of two operators, although it behaves as one would like. To clarify we observe that the first definition is independent of the choice of γ since $\log(\gamma^m)/\log_p(\chi(\gamma^m)) = m/m \cdot \log(\gamma)/\log_p(\chi(\gamma))$. Hence, if $\nabla_0(y)$

for some $y \in D$ (for instance, $y \in D^{\psi=0}$) is such that $\gamma_n - 1$ acts invertibly on it we see that $(\gamma_n - 1)^{-1}\nabla_0(y) = \frac{\nabla_0}{\gamma_n - 1}(y)$. From this it also follows that $(\gamma_n - 1)\frac{\nabla_0}{\gamma_n - 1} = \nabla_0$. Secondly we observe that

$$\nabla_i = \frac{\log(\chi(\gamma)^{-i}\cdot\gamma)}{\log_p(\chi(\gamma))} = \mathrm{Tw}^{-i}\left(\frac{\log(\gamma)}{\log_p(\chi(\gamma))}\right)$$

where Tw^i is the operator on $\mathcal{B}(\Gamma_K)$ which sends γ to $\chi(\gamma)^k\gamma$.

Definition 3.2.15. If $h \geq 1$ we define $\Omega_h := \nabla_{h-1} \cdot \ldots \cdot \nabla_0 \in \mathcal{H}(\Gamma_K)$.

Lemma 3.2.16. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and let $h \in \mathbb{N}$ such that $\mathrm{Fil}^{-h}\mathbf{D}^K_{\mathrm{dR}}(D) = \mathbf{D}^K_{\mathrm{dR}}(D)$. Then $\Omega_h(\mathbf{N}_{\mathrm{dR}}(D)) \subset D$.

Proof. Since $\Omega_h = \nabla_{h-1} \circ \nabla_{h-2} \circ \ldots \circ \nabla_0 = t^h\partial^h$ it suffices to show that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$. First assume that D is semi-stable. We know from Proposition 2.6.12 that if D is positive, then $\mathbf{D}^K_{\mathrm{st}}(D) = (\mathbf{B}^{\dagger}_{\log,K}[1/t] \otimes D)^{\Gamma_K} \subset \mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$, so that $\mathbf{N}_{\mathrm{dR}}(D) = (\mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}^K_{\mathrm{st}}(D))^{N=0} \subset D$. For general D if $h \geq 1$ is as in the statement then $D(-h)$ is positive, so that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$. Now if D is de Rham and L/K a finite extension such that $D|_L$ is semi-stable, then we have that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset t^h\mathbf{N}_{\mathrm{dR}}(D|_L) \subset D|_L$ and $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D[1/t]$, so that $t^h\mathbf{N}_{\mathrm{dR}}(D) \subset D$ as required. □

Definition 3.2.17. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and $h \geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}^K_{\mathrm{dR}}(D) = \mathbf{D}^K_{\mathrm{dR}}(D)$. We define Perrin-Riou's **big exponential map** by

$$\begin{aligned} \Omega_{D,h} : \mathbf{D}_{\infty,g}(D) &\longrightarrow D^{\psi=0} \\ g &\longmapsto \nabla_{h-1} \circ \ldots \circ \nabla_0(g) \end{aligned}$$

Lemma 3.2.18. One has the following commutative diagram:

$$\begin{array}{ccc} \mathbf{D}_{\infty,g}(D) & \xrightarrow{\partial^{-k}} & \mathbf{D}_{\infty,g}(D(k)) \\ \downarrow{\scriptstyle \Omega_h} & & \downarrow{\scriptstyle \Omega_{h+k}} \\ D^{\psi=0} & \xrightarrow{t^k} & D(k)^{\psi=0} \end{array}$$

Proof. This is clear from the fact that $\Omega_h = t^h\partial^h$. □

Lemma 3.2.19. Let D be as before and assume that K is such that Γ_K is torsion free. Then one has a canonical map $h^1_{K,D} : (\varphi - 1)D^{\psi=1} \to H^1(K, D)/(D^{\varphi=1}/(\gamma_K - 1))$ such that the diagram

$$\begin{array}{ccc} (\varphi - 1)D^{\psi=1} & \xleftarrow{\varphi-1} & D^{\psi=1} \\ \downarrow{\scriptstyle \tilde{h}^1_{K,D}} & & \downarrow{\scriptstyle h^1_{K,D}} \\ H^1(K, D)/(D^{\varphi=1}/(\gamma_K - 1)) & \longleftarrow & H^1(K, D) \end{array}$$

is commutative.

Proof. Obviously $D^{\psi=1}/D^{\varphi=1} \cong (\varphi - 1)D^{\psi=1}$. It is clear that the map $h^1_{K_n,D}$ factorizes over $D^{\psi=1}_{\Gamma_K}$. The claim follows. $\square$

Remark 3.2.20. If D is of PR-type and let h be such that (3.9) is satisfied. If $g \in \mathbf{D}_{\infty,g}(V)$ and $k \geq 1 - h$ we actually have $\Omega_h(g) \otimes e_k \in (1 - \varphi)D(k)^{\psi=1}$.

Proof. Let $\partial^{-k}(g) = (1 - \varphi)\partial^{-k}(G) - \partial^{-k}(M)$. Then

$$\partial^{-k}(M) = \sum_{j \geq 0}^{h+k-1} \mu_{j-k} \frac{t^j}{j!} \in \mathcal{H} \otimes \mathbf{D}_{\mathrm{st}}(V(k)).$$

Since $\nabla_{h+k-1} \circ \ldots \circ \nabla_0 = t^{h+k}\partial^{h+k}$ the $\partial^{-k}(M)$-part of $\partial^{-k}(g)$ is killed by Ω_h. $\square$

Hence, we see that if h is such that (3.9) is satisfied and $h - r > 0$ the diagram

$$\begin{array}{ccc}
(\mathbf{B}^{\dagger}_{\log,K} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D(-r))^{N=0,\psi=1} & \xrightarrow{\Omega_{h-r}} & D(-r)^{\psi=1} \\
\downarrow{\scriptstyle 1-p^r\varphi} & & \downarrow{\scriptstyle 1-p^r\varphi} \\
(1-p^r\varphi)(\mathbf{B}^{\dagger}_{\log,K} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D(-r))^{N=0,\psi=1} & \xrightarrow{\Omega_{h-r}} & (1-p^r\varphi)D(r)^{\psi=1} \\
\downarrow{\scriptstyle \partial^{-r}} & & \downarrow{\scriptstyle \mathrm{Tw}^r} \\
\mathbf{D}_{\infty,g}(D) & \xrightarrow{\Omega_h} & (1-\varphi)D^{\psi=1}
\end{array}$$

commutes.

Let D be of PR-type, $g \in \mathbf{D}_{\infty,g}(D)$ and $G = (L_k, M_k, G_k)$ be a complete solution for g. Then for each k and $n \gg 0$ one has that the element

$$\Xi_{n,k}(G) := p^{n(k-1)}\varphi^{-n}\partial^{-k}(H)(0) := p^{n(k-1)}(p^{-n}\varphi^{-n}\partial^{-k}(L)(0), \varphi^{-n}\partial^{-k}(M)(0), \varphi^{-n}\partial^{-k}(G)(0))$$

may be viewed as an element in $H^1(\mathfrak{C}_{\mathrm{st}}(K, D(k)))$ (see (3.5)).

Theorem 3.2.21. Let D be a de Rham (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, $g \in \mathbf{D}_{\infty,g}(D)$ and G a complete solution for g in L. Let h be such that (3.9) is satisfied. Then for $k \geq 1 - h$ and $n \gg 1$ one has

$$\begin{aligned}
&h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(g) \otimes e_k) \\
&\qquad = p^{-n(K_n)}(-1)^{h+k-1}(h + 1 - k)! \frac{1}{[L_n : K_n]} \mathrm{Cor}_{L_n/K_n} \exp_{K_n,D(k)}(\Xi_{n,k}(G)),
\end{aligned}$$

where we consider the elements on both sides in $H^1(K_n, D)/(D^{\varphi=1}/(\gamma_{K_n} - 1))$.

Proof. The proof is divided into several parts. The first general assumption is that D is of PR-type.

Let D be pure of slope ≤ 0. Then the exponential map has the description given in Proposition 3.1.14. We may assume n big enough so that Γ^n_K is torsion free. Recall the relation

$$\Omega_{D(k),h+k}(\partial^{-k}(G)) = \Omega_{D,h}(G) \otimes e_k$$

Hence, for the $k \geq 1-h$ we have

$$h^1_{K_n,D(k)}(\nabla_{h-1} \circ \ldots \circ \nabla_0(G) \otimes e_k) = h^1_{K_n,D(k)}(\nabla_{h+k-1} \circ \ldots \circ \nabla_0(\partial^{-k}(G))).$$

Let $y_h = \nabla_{h+k-1} \circ \ldots \nabla_0(\partial^{-k}(G))$ and $w_{n,h} = \nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(\partial^{-k}(G))$. Then in this case

$$h^1_{K_n,D(k)}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1) b_{n,h} \in H^1(K_n, D(k)),$$

where $b_{n,h} \in \widetilde{D}$ is such that $(\gamma_n - 1)(\varphi - 1) b_{n,h} = (\varphi - 1) y_h$. Recall that $\partial^{-k}(g) = (1-\varphi)\partial^{-k}(G) - \partial^{-k}(M)$ and $\Omega_{D(k),h+k}(\partial^{-k}(g)) = (1-\varphi)\Omega_{D(k),h+k}(\partial^{-k}(G))$, hence

$$\nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}(\partial^{-k}(g)) = \nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}((1-\varphi)G_{-k}) - \nabla_{h+k-1} \circ \ldots \circ \frac{\nabla_0}{\gamma_n - 1}(M_{-k}).$$

With this we may choose

$$b_{n,h} = (\varphi - 1)^{-1} \left(\frac{\Omega_{D(k),h+k}}{\gamma_n - 1}((1-\varphi)G_{-k}) - \frac{\Omega_{D(k),h+k}}{\gamma_n - 1}(M_{-k}) \right) \in \widetilde{D}.$$

Now for $n \gg 0$ we have $g \in \mathbf{B}^{\dagger,n}_{\log,K} \otimes \mathbf{D}^K_{\mathrm{st}}(D)$, hence the cocycle $h^1_{K_n,V(k)}(y_h)(\sigma) = (\sigma - 1)(w_{n,h} - b_{n,h})$ is cohomologuous to

$$h^1_{K_n,V(k)}(y_h)(\sigma) = (\sigma - 1)\left(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})\right)$$

since $(\varphi - 1)(w_{n,h} - b_{n,h}) \in \mathbf{D}^K_{\mathrm{st}}(D(k))$ so that G_K acts trivially (and φ acts as usual invertibly on $\mathbf{D}^K_{\mathrm{st}}(D(k))$). We use the exact sequences from the generalized Bloch-Kato map from Proposition 3.1.14. By the general properties of the connecting homomorphism for continuous cohomology we have the following: if $(x, y, z) \in H^1(\mathfrak{C}_{\mathrm{st}}(K, D(k)))$ and $\tilde{x} \in \widetilde{D}_{\log}[1/t]$ is such that $g(\tilde{x}) = (x, y, z)$ then $\exp_{K_n,D(k)}((x, y, z))(\sigma) = (\sigma - 1)\tilde{x}$. First one has

$$\varphi^{-n}(y) - \varphi^{-n}(y)(0) \in tK_0[[t]] \otimes_{K_0} \mathbf{D}^K_{\mathrm{st}}(D),$$

hence

$$\frac{\nabla_0}{\gamma_n - 1}\varphi^{-n}(y) = p^{-n}\varphi^{-n}(y)(0) + tz_1.$$

The same recursion as in [4], Theorem II.3 shows that

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)! p^{-n}\varphi^{-n}(y)(0) \in \mathbf{B}^+_{\mathrm{dR}} \otimes D.$$

Next we have

$$N(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})) = p^{-n}\varphi^{-n}(\nabla_{h+k-1} \circ \ldots \frac{\nabla_0}{\gamma_n - 1}(N\partial^{-k}(G))).$$

Again we see by recursion with our choice of h that since $N\partial^{-k}(G) = L_{-k}$ and

$$L_{-k} = \sum_{i=0}^{h-1} \lambda_i \cdot t^i / i!,$$

that we obtain an equality

$$p^{-n}\varphi^{-n}(\nabla_{h+k-1} \circ \dots \frac{\nabla_0}{\gamma_n - 1}(L_{-k})) = (-1)^{h-1}(h-1)! p^{-2n}\varphi^{-n}(L_{-k})(0).$$

Finally one has

$$(\varphi - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})) = \varphi^{-n}(\nabla_{h+k-1} \circ \dots \frac{\nabla_0}{\gamma_n - 1}(M_{-k})).$$

Similarly, as before we have

$$M_{-k} = \sum_{i=0}^{h-1} \mu_i \cdot t^i / i!,$$

so that the recursion shows

$$\varphi^{-n}(\nabla_{h+k-1} \circ \dots \frac{\nabla_0}{\gamma_n - 1}(L_{-k})) = (-1)^{h-1}(h-1)! p^{-n}\varphi^{-n}(M_{-k})(0).$$

Altogether this shows that

$$(-1)^{h-1}(h-1)! p^{-n} \exp_{K_n, D(k)}(\Xi_{n,k}(G))(\sigma) = (\sigma - 1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h})),$$

which is the claim in this case.

Next assume D is pure of slope > 0. Then the exponential map has the description given in Proposition 3.1.16. First we note that $h^1_{K_n, D(k)}(\Omega_{D,h}(g) \otimes e_k) = (x, y)$ with

$$y = \Omega_{D(k), h+k}(G_{-k}), \quad x = \nabla_{h+k-1} \circ \dots \circ \frac{\nabla_0}{\gamma_K - 1}((\varphi - 1)(G_{-k})).$$

The exponential map sends $\Xi_{n,k}(G)$ to $\varphi^{-n}(G_{-k})(0) \in X^1(\widetilde{D})^{G_K}$. The identification $\widetilde{D}/(\varphi - 1) \xrightarrow{\sim} X^1(\widetilde{D})$ is given by the following construction (see [7], Remark 3.4): If $x \in \widetilde{D}/(\varphi - 1)$ and $y \in \widetilde{D}[1/t]$ is chosen so that $(\varphi - 1)y = x$ then for $n \gg 0$ the image of x is $\varphi^{-n}(y)$. With this we see that under these identifications the class of $h^1_{K_n, D(k)}(\Omega_{D,h}(g) \otimes e_k)$ is send to

$$\varphi^{-n}(\nabla_{h+k-1} \circ \dots \circ \frac{\nabla_0}{\gamma_K - 1}(G_{-k})) \equiv (-1)^{h-1}(h-1)! p^{-n}\varphi^{-n}(G_{-k})(0) \mod \mathbf{B}^+_{\mathrm{dR}} \otimes D$$

where we use the same recursion as before, hence the claim in this case.

In the general case of semistable a D of PR-type one may use the exact $0 \to D_{\leq 0} \to D \to D_{>0} \to 0$, where $\mathbf{D}_{\leq 0}$ is the biggest submodule of D with slopes ≤ 0, and $D_{>0} = D/D_{\leq 0}$, which is a (φ, Γ_K)-module with slopes > 0. By using the description of the isomorphism (3.4) and the explicit description of the transition morphism for the cone one is reduced, since all maps are compatible with exact sequences, to the case of a module with all slopes ≤ 0 or all slopes > 0. But in these cases we have just verified that the statement holds.

Now assume D is de Rham and let L/K be a finite extension such that D is of PR-type over L. Then for $y \in \mathbf{D}_{\infty,g}(D)$ one has, if we consider $y \in \mathbf{D}_{\infty,g}(D|_L)$

$$\mathrm{Res}_{L_n/K_n}(h^1_{K_n,D(k)}(\Omega_{D,h}(y))) = h^1_{L_n,D|_L(k)}(\Omega_{D,h}(y)),$$

so that the claim follows from Proposition 2.6.7. □

For the record we state the next proposition in case D is semi-stable. As before, let $h \geq 1$ be such that (3.9) is satisfied for D, and dually let $h^* \geq 1$ be such that (3.9) is satisfied for $D^*(1)$

Proposition 3.2.22. a) If $k \geq 1-h$ and $n \geq 1$ then

$$h^1_{K_n,D(k)}(\nabla_{h-1}\circ\ldots\circ\nabla_0(g)\otimes e_k) = p^{-n(K_n)}(-1)^{h+k-1}(h+1-k)!\exp_{K_n,D(k)}(\Xi_{n,k}(G))$$

b) If $k \leq -h^*$ and $n \geq 1$ then

$$\exp^*_{K_n,D^*(1)}(h^1_{K_n,D(k)}(\nabla_{h-1}\circ\ldots\circ\nabla_0(g)\otimes e_k)) = p^{-n(K_n)}\frac{1}{(-h-k)!}\varphi^{-n}(\partial^{-k}g\otimes t^{-j}e_j)(0)$$

Proof. The first part is just the preceding theorem. For the second observe that due to Proposition 3.1.21 one has

$$\exp^*_{K_n,D^*(1)}(h^1_{K_n,D(k)}(\nabla_{h-1}\circ\ldots\circ\nabla_0(g)\otimes e_k)) = p^{-n(K_n)}\varphi^{-n}(\nabla_{h-1}\circ\ldots\circ\nabla_0(g)\otimes e_k)(0).$$

A computation with the Taylor series shows that

$$p^{-n(K_n)}\varphi^{-n}(\nabla_{h-1}\circ\ldots\circ\nabla_0(g)\otimes e_k)(0) = p^{-n(K_n)}\frac{1}{(-h-j)!}\varphi^{-n}(\partial^{-k}g\otimes t^{-k}e_k)(0),$$

hence the claim. □

In [35], Perrin-Riou shows how to construct an "inverse" to $\Omega_{D,h}$ in the case where D is étale. First, let us define the following:

Definition 3.2.23. If $h^* \geq 1$ we define $\mathcal{L}_{h^*} = \nabla_{-h^*+1}\circ\ldots\circ\nabla_{-1} \in \mathcal{H}(\Gamma_{\mathbb{Q}_p})$.

Returning to Perrin-Riou's setting, let V be semistable (over an unramified extension) and let $h, h^* \geq 1$ be so that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}_{\mathrm{st}}(V)$ and $\mathrm{Fil}^{-h^*}\mathbf{D}_{\mathrm{st}}(V^*(1)) = \mathbf{D}_{\mathrm{st}}(V^*(1))$. For $x \in H^1_{\mathrm{Iw}}(K,V) = \mathbf{D}^\dagger(V)^{\psi=1}$ she shows that (using properties of the determinant of $\Omega_{V,h}$) there exists an $y \in \mathbf{D}_{\infty,f}(V)$ such that

$$\prod_{-h<j<h^*}\nabla_{-j}((\varphi-1)x) = \Omega_{V,h}(y).$$

With our description of the map $\Omega_{V,h}$ this implies that, by calculating by extending scalars to the total ring of fractions of $\mathcal{H}(\Gamma_K)$, that $\mathcal{L}_{h^*}((\varphi-1)x) \in \mathbf{D}_{\infty,f}(V)$, and Perrin-Riou denotes this map by $\mathcal{L}_{V,h}$. More generally, we have

Proposition 3.2.24. Let D be a de Rham (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and $h^*\geq 1$ such that (3.9) is satisfied for $D^*(1)$. If $x\in D^{\psi=1}$ then $\mathcal{L}_{h^*}((\varphi-1)x)\in \mathbf{D}_{\infty,g}(D)$.

Proof. We first assume that D is semi-stable so that $D\subset D[1/t]=\mathbf{N}_{\mathrm{dR}}(D)[1/t]=(\mathbf{B}^\dagger_{\log,K}\otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}}\mathbf{D}^K_{\mathrm{st}}(D))^{N=0}[1/t]$. Recall that on $\mathbf{B}^\dagger_{\mathrm{rig},K}[1/t]$ one has $\nabla_i(t^ix)=t^i\nabla_0(x)$ for all $i\in\mathbb{Z}$. Hence, if $x=t^{-1}x'\in t^{-1}\mathbf{N}_{\mathrm{dR}}(D)$ then $\nabla_{-1}(x)=\partial(x')\in\mathbf{N}_{\mathrm{dR}}(D)$. A recursion argument then shows that if $n>1$, $x=t^{-n}x'$ and $\nabla_{-n+1}\circ\ldots\circ\nabla_{-1}(x)=\partial^{n-1}(t^{-1}x')$ then $\nabla_{-n}\circ\ldots\circ\nabla_{-1}(x)=\partial^n(x')$. This shows that a base for D lies in $\mathbf{D}_{\infty,g}(D)$ for h^* big enough under $\mathcal{L}_{h^*}$, hence $\mathcal{L}_{h^*}((\varphi-1)x)\in\mathbf{D}_{\infty,g}(D)$ since $(\varphi-1)D^{\psi=1}$, hence the claim in this case. The general case may be deduced as in the proof of Lemma 3.2.16. □

Hence, we may define:

Definition 3.2.25. Let D be a de Rham (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and choose L/K so that $D|_L$ is semi-stable over $\mathbf{B}^\dagger_{\mathrm{rig},L}$. Let $h^*\geq 1$ such that (3.9) is satisfied for $D^*(1)$. We define Perrin-Riou's **Logarithm map** by

$$\mathcal{L}_{D^*(1),h^*}: D^{\psi=1}\longrightarrow \mathbf{D}_{\infty,g}(D)$$

Dual to the statement of Lemma 3.2.9 we have:

Remark 3.2.26. If $D=\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ is étale then one may choose h^* greater or equal to h' such that $\mathrm{Fil}^{-h'}\mathbf{D}^K_{\mathrm{dR}}(D^*(1))=\mathbf{D}^K_{\mathrm{dR}}(D^*(1))$.

3.3 The crystalline case

Let D be a crystalline (φ,Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, that is, $\mathbf{D}^K_{\mathrm{cris}}(D)$ is a K_0-vectorspace of dimension $d=\mathrm{rank}(D)$, equipped with an action of a Frobenius φ. We want to give a short description of the module $\mathbf{D}_{\infty,f}(D)$. Recall that one may define the ring $\mathbf{B}^+_{\mathrm{rig},K}=\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\cap\mathbf{B}^\dagger_{\mathrm{rig},K}$. If one fixes a choice of an element π_K as before the identification in Proposition 2.4.2 for $\mathbf{B}^\dagger_{\mathrm{rig},K}$ gives then rise to an identification of the set of power-series

$$\left\{\sum_{n\in\mathbb{N}}a_nx^n|\ a_n\in K_0',\ \lim_{n\to\infty}|a_n|\rho^n=0\ \forall 0\leq\rho<1\right\}$$

with $\mathbf{B}^+_{\mathrm{rig},K}$ via the map $f\mapsto f(\pi_K)$. It is clear that one has an identification $\mathbf{B}^+_{\mathrm{rig},K}=\mathbf{B}^+_{\mathrm{rig},F}\otimes_{\mathbf{B}^+_F}\mathbf{B}^+_K$, and since $(\mathbf{B}^+_K)^{\psi=0}$ has a structure of a $\mathcal{H}(\Gamma_K)$-module (cf. Proposition 4.0.3) one sees that $(\mathbf{B}^+_{\mathrm{rig},K})^{\psi=0}\otimes_{K_0}\mathbf{D}^K_{\mathrm{cris}}(D)$ is a free module over this ring.

Proposition 3.3.1. With the assumptions above one has $(\mathbf{B}^+_{\mathrm{rig},K}\otimes\mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=0}\subset\mathbf{D}_{\infty,f}(D)$.

Proof. Let $g \in (\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=0}$. We may assume that $\mathbf{D}^K_{\mathrm{cris}}(D)$ is "positive enough" so that $\sum_{n\geq 0}\varphi^n(g)$ converges and gives an element in $(\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}^K_{\mathrm{cris}}(D))^{\psi=1}$. Then from the computations of the surjectivity for ∂ it is clear that there exists a family (G_k) with $G_k \in \mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}^K_{\mathrm{cris}}(D)$ such that $\partial(G_k) = G_{k+1}$, i.e. $g \in \mathbf{D}_{\infty,f}(D)$. □

In fact, we conjecture that equality holds in the above proposition. We sketch a proof in the case that there exists an r such that $(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1} \subset \frac{1}{\pi_K^r}\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D)$ (which holds for example in the unramified case). Choose $f \in \mathbf{D}_{\infty,f}(D)$ such that $f \notin (\mathbf{B}^+_K \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=0}$. Again by twisting D to be positive enough we may assume that there exists a $G \in (\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1}$ such that $(1-\varphi)G = f$. Then obviously G has only finitely many terms $a_n\pi_K^n$ in its development as a Laurent-series for $n \leq 0$, and let m be the smallest such that $a_m \neq 0$. Now, $\pi_K \in \mathbf{B}^+_K$ and ∇_0 leaves $\mathbf{B}^\dagger_{\mathrm{rig},K}$ stable, so that $\partial(\pi_K) \in \mathbf{B}^+_{\mathrm{rig},K}$. Further, by looking at the development of $\partial(\pi_K) = -\frac{(\partial P)(f)}{P'(f)}$ we may choose π_K in the beginning in such a way such that $\partial(\pi_K) \notin \pi_K \cdot \mathbf{B}^+_{\mathrm{rig},K}$. Hence, the development of $G/\partial(\pi_K)$ has a smallest term a'_m. The lift under partial hence has a term $a''_{m+1} \neq 0$. Repeating this step a finite number of times, we see that eventually there exists a smallest term $b_{-1}\pi_K^{-1}$ in the development, which contradicts the choice $f \in \mathbf{D}_{\infty,f}(D)$, hence the claim.

For the rest of this section we assume that $K/\mathbb{Q}_p$ is *unramified*. We show that one has the desired equality in Proposition 3.3.1 and collect some facts and notation from [35], section 2.4.

For every $i \in \mathbb{Z}$ let

$$\begin{aligned}\Delta_i : (\mathbf{B}^+_{\mathrm{rig},K})^{\psi=0} \otimes_{K_0} \mathbf{D}^K_{\mathrm{cris}}(D) &\longrightarrow (\mathbf{D}^K_{\mathrm{cris}}(D)/(1-p^i\varphi)\mathbf{D}^K_{\mathrm{cris}}(D))(i)\\ f &\longmapsto \Delta_i(f) = \partial^i(f)(0) \mod (1-p^i\varphi)\mathbf{D}^k_{\mathrm{cris}}(D),\end{aligned}$$

where $g(0)$ for $g \in \mathbf{B}^\dagger_{\mathrm{rig},K}$ means $\pi \mapsto 0$, which is well-defined since ∂ is an isomorphism on $(\mathbf{B}^+_{\mathrm{rig},K})^{\psi=0}$.

Proposition 3.3.2. One has exact sequences

$$\begin{aligned}0 \longrightarrow (\varphi-1)(\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1} &\longrightarrow (\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=0} \longrightarrow\\ &\longrightarrow \bigoplus_{i\geq 0} \frac{(\mathbf{D}_{\mathrm{cris}}(D)}{(1-p^i\varphi)\mathbf{D}_{\mathrm{cris}}(D))}(i) \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}0 \longrightarrow (\varphi-1)(\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1} &\longrightarrow (\varphi-1)(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1} \xrightarrow{(\partial_j)}\\ &\xrightarrow{(\partial_j)} \bigoplus_{j<0} \mathbf{D}_{\mathrm{cris}}(D)^{\varphi=p^{-j-1}}(j) \longrightarrow 0,\end{aligned}$$

for some map ∂_j, $j<0$ (cf. [35], section 2.4). If $\mathbf{D}_{\mathrm{cris}}(D)$ is "positive enough" the map ∂_j coincides with $\overline{\mathcal{R}}_{D,j}$, so that

$$\mathbf{D}_{\infty,f}(D) = (\varphi-1)(\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=1} = (\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=0}.$$

Proof. In the unramified case this may be done exactly as in loc.cit., Proposition 2.4.1., cf. also (2.4.1), (2.4.3) there. □

Chapter 4

Reciprocity laws

If D is a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ we denote by $\mathcal{C}(D)$ the finitely generated $\mathcal{H}(\Gamma_K)$-module $(\varphi - 1)D^{\psi=1}$ (see Theorem 2.6.6). Further we use the notation $\mathcal{C}_{/\mathrm{tor}}(D) := D^{\psi=1}/(D^{\psi=1})_{\mathrm{tor}}$. Similarly, if T is a $\mathbb{Z}_p$-representation of G_K we set

$$\mathcal{C}(T) = (\varphi - 1)\mathbf{D}(T)^{\psi=1}, \quad \mathcal{C}^{\dagger}_{\mathrm{rig}}(T) = (\varphi - 1)\mathbf{D}^{\dagger}_{\mathrm{rig}}(T)^{\psi=1},$$

so that $\mathcal{C}(T)$ is a finitely generated Λ-module and $\mathcal{H}(\Gamma_K) \otimes_{\Lambda} \mathcal{C}(T) = \mathcal{C}^{\dagger}_{\mathrm{rig}}(T)$.

Let us first recall some facts about the interpolation properties of elements of $(\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$.

Proposition 4.0.3. The continuous map

$$\begin{aligned} \mathcal{B}(\Gamma_{\mathbb{Q}_p}) &\longrightarrow (\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \\ f &\longmapsto f \cdot (1+\pi) \end{aligned}$$

is an isomorphism of $\mathbb{B}(\Gamma_{\mathbb{Q}_p})$-modules. It restricts to an isomorphism $\mathcal{H}(\Gamma_{\mathbb{Q}_p}) \to (\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$.

Proof. See [35], Corollary B.2.8. □

Proposition 4.0.4. Let $f, g \in (\mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ and suppose that $\partial^k(f)(0) = \partial^k(g)(0)$ for all $k \gg 0$. Then $f = g$.

Proof. Recall that $\chi(\gamma) = 1 + pu$ with some $u \in \mathbb{Z}_p^{\times}$. By the preceeding proposition proposition we may assume $f = \lambda(\gamma-1)\cdot(1+\pi)$ and $g = \mu(\gamma-1)\cdot(1+\pi)$ with $\lambda, \mu \in \mathbf{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}$, where $\lambda = \sum_{n\geq 0} a_n \pi^n$ and $\mu = \sum_{n\geq 0} b_n \pi^n$. Then $\partial^{p^k}(f)(0) = \sum_{n\geq 0} a_n(\chi(\gamma)^{p^k} - 1)^n$. By assuming $a_0, b_0 \in \mathbb{Z}_p$ (by multiplying with an appropriate power of p) and by noticing that $\chi(\gamma)^{p^k} \equiv 1 \mod p^k$ one shows via $\mod p^i$ considerations that $a_0 = b_0$, then $a_1 = b_1$, and so on. □

4.1 The pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ in the étale case

For this section let V be a p-adic representation of G_K. Choose a G_K-stable $\mathbb{Z}_p$-lattice T of V. Perrin-Riou defined a pairing of Λ-modules

$$\langle\ ,\ \rangle_V : H^1_{\mathrm{Iw}}(K,T) \times H^1_{\mathrm{Iw}}(K,T^*(1)) \longrightarrow \Lambda(G_K) = \Lambda$$

that is induced by the local Tate-pairings

$$\langle\ ,\ \rangle_{K_n,V} : H^1(K_n,T) \times H^1(K_n,T^*(1)) \longrightarrow H^2(K_n,\mathbb{Z}_p(1)) \overset{\mathrm{inv}}{\cong} \mathbb{Z}_p$$

and the isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(M_{\Gamma_n},\mathbb{Z}_p) = \mathrm{Hom}_\Lambda(M,\mathbb{Z}_p[\Gamma/\Gamma_n])$, which holds for any Λ-module M of finite type. In the same vein Colmez defined a pairing

$$\langle\ ,\ \rangle_{\mathrm{Iw}} : \mathcal{C}(T) \times \mathcal{C}(T^*(1)) \longrightarrow \Lambda$$

via the formula (cf. [17], Proposition VI.1.2)

$$\langle x,y\rangle_{\mathrm{Iw}} = \lim_{n\to\infty} \sum_{\sigma\in\Gamma_K/\Gamma_{K_n}} \mathrm{inv}\left(\frac{\sigma^{-1}}{\gamma_n-1}\cdot x\otimes y\right)\cdot\sigma \tag{4.1}$$

for $x\in\mathcal{C}(T)$ and $y\in\mathcal{C}(T^*(1))$.

To be able to further relate to Colmez' work, we recall the following

Definition 4.1.1. Assume that $\Gamma_K \cong \mathbb{Z}_p$. If $i\in\mathbb{Z}_p$ we write σ_i for the element $\sigma_i\in\Gamma_{\mathbb{Q}_p}$ such that $\chi_{\mathrm{cyc}}(\sigma_i) = i$.

We remark that Colmez considers the case $\Gamma_K = \mathbb{Z}_p^\times$, but his definition extends to the general case. We note that we have switched in (4.1) the σ to σ^{-1} in Colmez' definition in loc.cit. to be consistent with the "classical" definition given by Perrin-Riou. Also, we have dropped the additional operator σ_{-1} in the x-component (cf. [17], I.2, section 4) which is needed only later when one formulates reciprocity laws. Additionally, if we assume $\log_p(\chi(\gamma_n)) = p^n$ we may drop the factor $\tau_n(\gamma_n)$.

For T as above one has an exact sequence

$$0\longrightarrow \mathbf{D}^\dagger(T)^{\varphi=1} \longrightarrow \mathbf{D}^\dagger(T)^{\psi=1} \overset{\varphi-1}{\longrightarrow} \mathcal{C}(T)\longrightarrow 0$$

(see for instance [15], Proposition 6.3.2). Hence, one has an identification

$$\varphi-1 : H^1_{\mathrm{Iw}}(K,T)/H^1_{\mathrm{Iw}}(K,T)_{\mathrm{tor}} \cong \mathcal{C}(T)$$

and an identification of pairings $\langle\ ,\ \rangle_V = \langle\ ,\ \rangle_{\mathrm{Iw}}$ on $\mathbf{D}(T)^{\psi=1}\times\mathbf{D}(T^*(1))^{\psi=1}$. Further, the pairing $\langle\ ,\ \rangle_{\mathrm{Iw}}$ has the following properties:

Proposition 4.1.2. (See [33], Lemma 3.6.1)

a) For all $\lambda \in \Lambda$ one has

$$\langle \lambda \cdot x, y\rangle_{\mathrm{Iw}} = \lambda \cdot \langle x, y, \rangle_{\mathrm{Iw}} = \langle x, \iota(\lambda)\cdot y\rangle_{\mathrm{Iw}},$$

where ι is defined as in section 2.6.2.

b) For every $j \in \mathbb{Z}$

$$\langle x \otimes e_j, y \otimes e_{-j}\rangle_{\mathrm{Iw}} = \partial^j(\langle x, y\rangle_{\mathrm{Iw}}).$$

We recall that the maps

$$h^1_{K_n,\mathbf{D}^\dagger(T)} : \mathbf{D}^\dagger(T)^{\psi=1} \to H^1(K_n, \mathbf{D}^\dagger(T)) \cong H^1(K_n, T)$$

give rise to an isomorphism $\mathbf{D}^\dagger(T)^{\psi=1} \cong H^1_{\mathrm{Iw}}(K,T)$ of Γ_K-modules. This implies the following equalities,

$$\mathbf{D}^\dagger(T)^{\psi=1} \otimes_\Lambda \mathcal{H}(\Gamma_K) \cong H^1_{\mathrm{Iw}}(K,T) \otimes_\Lambda \mathcal{H}(\Gamma_K) = H^1_{\mathrm{Iw}}(K, T \otimes_\Lambda \mathcal{H}(\Gamma_K)) = \mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\psi=1},$$

where for the two last identities we refer to the discussion in [37], section 6.2. Hence, by the Λ-bi-semilinearity of $\langle\ ,\ \rangle_{\mathrm{Iw}}$ one may extend this pairing to $\mathcal{C}^\dagger_{\mathrm{rig}}(V) \times \mathcal{C}^\dagger_{\mathrm{rig}}(V^*(1))$. With the description of $\mathcal{H}(\Gamma_K)$ as in section 2.6.2 one also has a natural extension of ι to $\mathcal{H}(\Gamma_K)$.

If we now assume $K/\mathbb{Q}_p$ to be unramified then Colmez proved the following ([17], Proposition VI.1.2):

Proposition 4.1.3. $\langle\ ,\ \rangle_{\mathrm{Iw}}$ is a perfect pairing, i.e., it induces a Λ-equivariant isomorphism $\mathcal{C}(T) \cong \mathrm{Hom}_\Lambda(\mathcal{C}(T^*(1)), \Lambda)^\iota$.

Hence:

Proposition 4.1.4. If $K/\mathbb{Q}_p$ is unramified, the pairing $\langle\ ,\ \rangle_{\mathrm{Iw}}$ extends to a perfect pairing of $\mathcal{H}(\Gamma_K)$-modules on $\mathcal{C}^\dagger_{\mathrm{rig}}(V) \times \mathcal{C}^\dagger_{\mathrm{rig}}(V^*(1))$.

4.2 The pairing $\langle\ ,\ \rangle_{\mathbf{Iw},D}$ in the general case

Recall (cf. (2.6)) that we have cup product pairings for $\mathbf{B}^\dagger_{\mathrm{rig},K}\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\Lambda}^\iota_n[1/p]$-modules. We are especially interested in the following case:

$$\langle\ ,\ \rangle_{K_n,D} : H^1(K_n, D) \times H^1(K_n, D^*(1)) \longrightarrow H^2(K_n, \mathbf{B}^\dagger_{\mathrm{rig},K}(1)) \cong H^2(K_n, \mathbb{Q}_p) \overset{\mathrm{inv}}{\cong} \mathbb{Q}_p \tag{4.2}$$

resp.

$$\langle\ ,\ \rangle_{\overline{D}_n} : H^1(K, \overline{D}_n) \times H^1(K, \overline{D^*(1)}_n) \longrightarrow H^2(K, \overline{(\mathbf{B}^\dagger_{\mathrm{rig},K}(1))}_n) \tag{4.3}$$

which are induced by the map (using representatives as in (2.6))

$$((x,y),(w,v)) \longmapsto \overline{(\gamma_K(w))(y) - (\varphi(v))(x)}.$$

resp.

$$((x \otimes \lambda, y \otimes \mu), (w \otimes \alpha, v \otimes \beta)) \longmapsto \overline{(\gamma_K(w))(y) \otimes \mu\alpha - (\varphi(v))(x) \otimes \lambda\beta}.$$

By abuse of notation we also denote by

$$\langle\ ,\ \rangle_{K_n,D} : H^1(K, D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}]^\iota) \times H^1(K, D^*(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}]^\iota)^\iota \longrightarrow \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$$

the pairing induced by (4.2) and use the identification

$$\mathrm{Hom}_{\mathbb{Q}_p}(M, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]}(M, \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}])^\iota$$

which holds for any $\mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$-module M.

With these preparations one is tempted to define a pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ on $D^{\psi=1} \times D^*(1)^{\psi=1}$ via the formula

$$\langle x, y\rangle_{\mathrm{Iw},D} = \lim_{n\to\infty} \sum_{\sigma\in\Gamma_K/\Gamma_{K_n}} \langle \sigma^{-1} \cdot h^1_D(x), h^1_{D^*(1)}(y)\rangle_{K_n,D} \cdot \sigma,$$

but it is a priori not clear whether this element will appear in $\mathcal{H}(\Gamma_K)$ (it will however converge to an element in $\mathbb{Q}_p[[\Gamma_K]] = \varprojlim_n \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$). It is possible to show the convergence with a version of [33], Lemma 1.2.2. We will however give an alternate definition and show that this definition has the right "interpolation property".

Recall that we have projection maps $h^1_{K_n,D} : D^{\psi=1} \to H^1(K_n, D)$ and $h^1_{\overline{D}_n} : D^{\psi=1} \to H^1(K, \overline{D}_n)$. We observe the following

Lemma 4.2.1. Let $y \in D$ and $v \in D^*(1)^{\psi=1}$. The elements $\langle h^1_{K_n,D}(y), h^1_{K_n,D^*(1)}(v)\rangle_{K_n,D} \in H^2(K_n, \mathbf{B}^\dagger_{\mathrm{rig},K}(1))$ and $\langle h^1_{\overline{D}_n}(y), h^1_{\overline{D^*(1)}_n}(v)\rangle_{\overline{D}_n} \in H^2(K, \overline{D}_n)$ are computed via $(\gamma_{K_n} - 1)^{-1}(\varphi - 1)y \otimes (\varphi - 1)(v)$.

Proof. For notational purposes we only treat the first case. Let (x, y) (resp. (w, v)) be the tuples obtained by the projection $h^1_{K_n,D}$ (resp. $h^1_{K_n,D^*(1)}$) so that $\langle h^1_{K_n,D}(y), h^1_{K_n,D^*(1)}(v)\rangle_{K_n,D}$ is the class of $y \otimes \gamma_{K_n}(w) - x \otimes \varphi(v)$ in $\mathbf{B}^\dagger_{\mathrm{rig},K}(1)/(\varphi - 1, \gamma_K - 1)$. Since $x \in D^{\psi=0}$ one easily checks that $\psi(\varphi(v))(x) = 0$, so that this class is equivalent under the isomorphism in Theorem 2.6.6 to the class of $(\gamma_K(w))(y)$. Similarly, one sees that $\varphi(y) \otimes \gamma_K(w)$ is the trivial class, so that we may compute

$$\begin{aligned}\langle h^1_{K_n,D}(y), h^1_{K_n,D^*(1)}(v)\rangle_{K_n,D} &\equiv (1 - \varphi)(y) \otimes \gamma_{K_n}(w) \equiv (1 - \gamma_{K_n})(x) \otimes \gamma_{K_n}(w) \\ &\equiv x \otimes \gamma_{K_n}(w) - x \otimes w \equiv (\gamma_{K_n} - 1)^{-1}(\varphi - 1)(y) \otimes (\varphi - 1)(v)\end{aligned}$$

in $H^2(K_n, \mathbf{B}^\dagger_{\mathrm{rig},K}(1))$, which concludes the proof. □

With this we make the following definition: first assume that Γ_K is torsion free. If $y \in D^{\psi=1}$, then $(\varphi - 1)y \in \overline{D}_n^{\psi=0}$ and $(\gamma_K - 1)$ acts invertibly on $\overline{D}_n^{\psi=0}$ so that there exists a unique element $x = (\gamma_K - 1)^{-1}(\varphi - 1)y \in \overline{D}_n^{\psi=0}$. We define for every n the pairing

$$\langle\ ,\ \rangle_{\mathrm{Iw},D} : D^{\psi=1} \times (D^*(1)^{\psi=1})^\iota \longrightarrow \Lambda_n[1/p]$$

via the projections $D^{\psi=1} \to \overline{D}_n,\ y \mapsto x, D^*(1)^{\psi=1} \to \overline{D^*(1)}_n,\ t \mapsto t \otimes 1$ with the composition with the natural map

$$\overline{D}_n \times \overline{D^*(1)}_n \longrightarrow \overline{(D \otimes D^*(1))}_n \xrightarrow{\text{ev}} \overline{\mathbf{B}^{\dagger}_{\mathrm{rig},K}(1)}_n \xrightarrow{\text{inv}} \mathbb{Q}_p \otimes_{\mathbb{Q}_p} \Lambda_n[1/p].$$

Here, inv is the invariant map

$$\mathbf{B}^{\dagger}_{\mathrm{rig},K}(1) \twoheadrightarrow H^2(K, \mathbf{B}^{\dagger}_{\mathrm{rig},K}(1)) \cong \mathbb{Q}_p.$$

That is, we obtain a compatible family of maps of $\Lambda_n[1/p]$-modules

$$D^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p] \longrightarrow \mathrm{Hom}_{\Lambda_n[1/p]}((D^*(1)^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p])^{\iota}, \Lambda_n[1/p]).$$

Taking the limit over n one obtains the desired bi-linear pairing of $\mathcal{H}(\Gamma_K)$-modules

$$\langle\ ,\ \rangle_{\mathrm{Iw},D} : D^{\psi=1} \times (D^*(1)^{\psi=1})^{\iota} \longrightarrow \Lambda_\infty = \mathcal{H}(\Gamma_K).$$

Now if Γ_K decomposes as $\Delta_K \times \Gamma'_K$ with Γ'_K torsion-free we know that there exists an n such that $\Gamma_{K_n} \subset \Gamma_K$ is torsion free. Since one has $\mathbf{B}^{\dagger}_{\mathrm{rig},K_n} = \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ one may consider D and $D^*(1)$ as (φ, Γ_{K_n})-modules and one obtains the above pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D|_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_n}}}$ whose image lies in $\mathcal{H}(\Gamma_{K_n}) \subset \mathcal{H}(\Gamma_K)$. Define

$$\langle y, t\rangle_{\mathrm{Iw},D} = \sum_{\sigma \in \Gamma_K/\Gamma_{K_n}} \langle \sigma^{-1} y, t\rangle_{\mathrm{Iw},D|_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_n}}} \cdot \sigma \in \mathcal{H}(\Gamma_K).$$

for a choice of representatives σ.

Now we can prove:

Proposition 4.2.2. If $y \in D^{\psi=1}$ and $t \in D^*(1)^{\psi=1}$ then

$$\partial^k(\langle y, t\rangle_{\mathrm{Iw},D}) \equiv \sum_{\sigma \in \Gamma_K/\Gamma_{K_n}} \langle \sigma^{-1}(y \otimes e_k), (t \otimes e_{-k})\rangle_{K_n,D} \cdot \sigma \mod (\gamma_K^{p^n} - 1)$$

for every $k \geq 0, n \geq 0$.

Proof. We first look at the case of $k = 0$ and assume Γ_K to be torsion-free. Recall that $D^{\psi=1} \cong H^1_{\mathrm{Iw}}(K, \overline{D})$ and $\Lambda_\infty = \varprojlim_n (\Lambda_n[1/p])$ may be considered as an intersection of all $\Lambda_n[1/p]$. The projection maps $h^1_{K_n,D}$ and ρ_n are compatible in the following way:

$$\begin{array}{ccc} D^{\psi=1} & \xrightarrow{\ \rho_n\ } & \overline{D}_n^{\psi=0} \\ \downarrow{\scriptstyle h^1_{K_n,D}} & & \downarrow \\ Z^1(K_n, D) & \xrightarrow{\ \mathrm{pr}_1\ } & (D \otimes \mathbb{Q}_p\widetilde{[\Gamma_K/\Gamma_{K_n}]})^{\psi=0} \end{array}$$

where $\rho_n(y) = (\gamma_K - 1)^{-1}((\varphi - 1)(y) \otimes 1)$ and the vertical arrow on the right is induced by the projection (via division with remainder) $\mathcal{H}(\Gamma_K) \to \mathbb{Q}_p[[\Gamma_K]] \to \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}]$ which factors over $\Lambda_n[1/p]$. Similarly, we have for $D^*(1)$

$$\begin{array}{ccc} D^*(1)^{\psi=1} & \xrightarrow{\rho_n^*} & \overline{D^*(1)}_n^{\psi=0} \\ \downarrow{\scriptstyle h^1_{K_n,D^*(1)}} & & \downarrow \\ Z^1(K_n, D^*(1)) & \xrightarrow{(\varphi-1)\circ \mathrm{pr}_2} & (D^*(1) \otimes \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}])^{\psi=0} \end{array}$$

where $\rho_n^*(t) = (\varphi - 1)(t) \otimes 1$. Hence, we see thanks to Lemma 4.2.1 and the following commutative diagram

$$\begin{array}{ccccc} \overline{D}_n & \times & \overline{D^*(1)}_n & \longrightarrow & \Lambda_n[1/p] \\ \downarrow & & \downarrow & & \downarrow \\ D \otimes \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}] & \times & D^*(1) \otimes \mathbb{Q}_p[\widetilde{\Gamma_K/\Gamma_{K_n}}] & \to & \mathbb{Q}_p[\Gamma_K/\Gamma_{K_n}] \end{array}$$

where the vertical maps are all the canonical projections that the claim follows in this case. If Γ_K has torsion then the claim follows by considering the single $\langle \sigma^{-1}y, t\rangle_{\mathrm{Iw},D|_{\mathbf{B}^\dagger_{\mathrm{rig},K_n}}}$. The claim for general k can then be derived from the $k = 0$ case by the properties of the cup-product pairing for D and $D^*(1)$ by using the analogue of Proposition 4.1.2, b). □

Lemma 4.2.3. Let $f : M \to N$ be a map of (φ, Γ_K)-modules which induces a map $f^* : N^*(1) \to M^*(1)$. One has a commutative diagram of $\Lambda_n[1/p]$-modules

$$\begin{array}{ccccc} M^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p] & \times & M^*(1)^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p] & \to & \Lambda_n[1/p] \\ {\scriptstyle f\otimes 1}\downarrow & & \uparrow{\scriptstyle f^*\otimes 1} & & \| \\ N^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p] & \times & N^*(1)^{\psi=1} \otimes_{\Lambda_\infty} \Lambda_n[1/p] & \to & \Lambda_n[1/p] \end{array}$$

where the horizontal arrows are induced by the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$.

Proof. This may be derived directly from the definition of the pairing resp. the definition of f^*. □

Corollary 4.2.4. Let $f : M \to N$ with induced f^* be as above. The diagrams

$$\begin{array}{ccc} M^{\psi=1} & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(\Gamma_K)}(M^*(1)^{\psi=1}, \mathcal{H}(\Gamma_K)) \\ \downarrow & & \downarrow \\ N^{\psi=1} & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(\Gamma_K)}(N^*(1)^{\psi=1}, \mathcal{H}(\Gamma_K)) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)} M^{\psi=1} & \longrightarrow & \mathrm{Hom}_{\mathcal{K}(\Gamma_K)}(\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)} M^*(1)^{\psi=1},\mathcal{K}(\Gamma_K)) \\ \downarrow & & \downarrow \\ \mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)} N^{\psi=1} & \longrightarrow & \mathrm{Hom}_{\mathcal{K}(\Gamma_K)}(\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)} N^*(1)^{\psi=1},\mathcal{K}(\Gamma_K)) \end{array}$$

are commutative.

Proof. The first is a direct consequence of the preceeding lemma by taking the limit over n The commutativity of the last diagram is clear after tensoring with $\mathcal{K}(\Gamma_K)$. □

We note that since the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ by definition factors over $\mathcal{C}_{/\mathrm{tor}}(D)\times\mathcal{C}_{/\mathrm{tor}}(D^*(1))$ for a (φ,Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, so that one may replace the $(\ )^{\psi=1}$-part for all the modules in the preceding Lemma and Corollary by $\mathcal{C}_{/\mathrm{tor}}(\ )$.

We want to show that the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ is perfect. For this we need to extend it to a bigger module and assume that $K/\mathbb{Q}_p$ is *unramified* for the rest of this section.

Proposition 4.2.5. Let D be a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then there exists a finite extension L/K such that $(L\otimes_{\mathbb{Q}_p} D)^{\psi=0}$ is a finite free $L\otimes_{\mathbb{Q}_p}\mathcal{B}(\Gamma_K)$-module. As a consequence, $D^{\psi=0}$ is torsion-free as a $\mathcal{H}(\Gamma_K)$-module.

Proof. See [16], V.1.19. □

Proposition 4.2.6. If $0\to D'\to D\to D''\to 0$ is an exact sequence of (φ,Γ_K)-modules then

$$0\to\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}_{/\mathrm{tor}}(D')\to\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}_{/\mathrm{tor}}(D)\to\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}_{/\mathrm{tor}}(D'')\to 0$$

is an exact sequence of $\mathcal{K}(\Gamma_K)$-modules.

Proof. Since $(D'^{\psi=1})_{\mathrm{tor}}=(D^{\psi=1})_{\mathrm{tor}}\cap D'^{\psi=1}$ and since taking $\psi=1$-invariants is left exact, the exact sequence $0\to D'\to D\to D''\to 0$ furnishes an exact sequence $0\to\mathcal{C}_{/\mathrm{tor}}(D')\to\mathcal{C}_{/\mathrm{tor}}(D)\to\mathcal{C}_{/\mathrm{tor}}(D'')$ of torsion-free $\mathcal{H}(\Gamma_K)$-modules. Hence, one obtains an injection $\mathcal{C}_{/\mathrm{tor}}(D)/\mathcal{C}_{/\mathrm{tor}}(D')\hookrightarrow\mathcal{C}_{/\mathrm{tor}}(D'')$ of $\mathcal{H}(\Gamma_K)$-modules of the same rank equal to $\mathrm{rk}_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D''\cdot[K:\mathbb{Q}_p]$ (cf. Theorem 2.6.6). By Proposition 2.6.4 one also sees that $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ has a theory of elementary divisors (cf. also [4], Proposition 4.2). Hence, the quotient of the last injection is torsion and is killed by tensoring with the total ring of fractions $\mathcal{K}(\Gamma_K)$ of $\mathcal{H}(\Gamma_K)$. □

For the next theorem we remark that if D is an étale (φ,Γ_K)-module then $\varphi-1$ induces an isomorphism $\mathcal{C}_{/\mathrm{tor}}(D)\xrightarrow{\sim}\mathcal{C}(D)$ (see e.g. [15], Proposition 6.3.2).

Theorem 4.2.7. Let D be a (φ,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then the pairing $\langle\ ,\ \rangle_{\mathrm{Iw,\,D}}$ is perfect on $\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}_{/\mathrm{tor}}(D)\times(\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}_{/\mathrm{tor}}(D^*(1)))^\iota$ as a pairing of $\mathcal{K}(\Gamma_K)$-modules.

Proof. We first prove the theorem in the case of a pure module. Let $d = \mathrm{rk}D$, $\deg(D) = s$ so that $\mu(D) = s/d$. Since $\deg(D^*) = -\deg(D)$ we may assume that $\deg(D) \geq 0$. If D is étale (i.e. pure of slope 0) then the statement holds thanks to Proposition 4.1.4. So let D be such that $s > 0$ and assume that the statement is true for all pure modules of degree ≥ 0 and $< s$. As in the proof of [29], Theorem 4.7, one has an exact sequence $0 \to D \to E \to t^{-1}\mathbf{B}^{\dagger}_{\mathrm{rig},K} \to 0$ with E a (φ, Γ_K)-module of rank $d+1$. E posesses a unique slope-filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_l = E$ such that E_i/E_{i-1} pure of positive slope with degree $< s$, cf. loc.cit.. Hence, one is reduced to the case of an exact sequence $0 \to E' \to E \to E'' \to 0$ such that the statement holds for E', E''. Due to Proposition 4.2.6 and Corollary 4.2.4 we obtain a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E') & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{K}(\Gamma_K)}(\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E'^*(1))^{\iota}, \mathcal{K}(\Gamma_K)) \\
\downarrow & & \downarrow \\
\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E) & \longrightarrow & \mathrm{Hom}_{\mathcal{K}(\Gamma_K)}(\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E^*(1))^{\iota}, \mathcal{K}(\Gamma_K)) \\
\downarrow & & \downarrow \\
\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E'') & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{K}(\Gamma_K)}(\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} \mathcal{C}_{/\mathrm{tor}}(E''^*(1))^{\iota}, \mathcal{K}(\Gamma_K)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

which shows that $\langle\ ,\ \rangle_{\mathrm{Iw,\,D}}$ is perfect for E. Now, using the same diagram with D, E, $t^{-1}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ in place of E', E, E'' we know that the middle horizontal arrow is an isomorphism. An easy calculation shows that

$$\mathcal{C}_{/\mathrm{tor}}((t^{-1}\mathbf{B}^{\dagger}_{\mathrm{rig},K})) \longrightarrow \mathrm{Hom}_{\mathcal{H}(\Gamma_K)}(\mathcal{C}_{/\mathrm{tor}}(t\mathbf{B}^{\dagger}_{\mathrm{rig},K})(1)), \mathcal{H}(\Gamma_K))$$

is injective: both are free $\mathcal{H}(\Gamma_K)$-modules of rank one, so it is enough to show that a basis for $\mathcal{C}_{/\mathrm{tor}}(t^{-1}\mathbf{B}^{\dagger}_{\mathrm{rig},K})$ is sent to a non-trivial homomorphism, which may be checked with Proposition 4.2.2 and the result that the Tate pairing is perfect on $t^{-1}\mathbf{B}^{\dagger}_{\mathrm{rig},K} \times t\mathbf{B}^{\dagger}_{\mathrm{rig},K}(1)$ (see [29], Lemma 4.5). Hence, the corresponding map for $\mathcal{K}(\Gamma_K)$-modules is injective, so that the pairing is also perfect on D.

For general D one is reduced by using Kedlaya's slope filtration theorem to the case of an exact sequence $0 \to D' \to D \to D'' \to 0$ of (φ, Γ_K)-modules such that the statement holds for D', D''. By the same argument as before we obtain that it must also hold for D. □

We shall also simply write that the above pairing is perfect on D.

Corollary 4.2.8. Let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Then (as $\mathcal{H}(\Gamma_K)$-modules) $D^{\psi=1}_{\mathrm{tor}} = D^{\varphi=1}$.

Proof. Looking at description of the pairing $\langle\ ,\ \rangle_{\overline{D}_n}$ in Proposition 4.2.2 we see that an element $y \in D^{\varphi=1}$ is send to the trivial homomorphism in $\mathrm{Hom}_{\Lambda_n[1/p]}(H^1(K, \overline{D^*(1)}_n), \Lambda_n[1/p])$. Hence, the isomorphism in the previous theorem shows that $\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} D^{\varphi=1} = 0$, so that $D^{\varphi=1}$ is torsion. Since $D^{\psi=1}/D^{\varphi=1} \cong (\varphi - 1)D^{\varphi=1} \subset D^{\psi=0}$ and the latter is torsion-free as a $\mathcal{H}(\Gamma_K)$-module, we get the claim. □

4.3 The pairing [,]$_{\mathbf{Iw},D}$

For this whole section we assume that $K/\mathbb{Q}_p$ is unramified, hence $\Gamma_K = \Gamma_{\mathbb{Q}_p}$. The following is inspired by Colmez' approach to build the "correct" convolution on $(\mathbf{B}^\dagger_{\log,K})^{\psi=0}$ which gives rise to reciprocity laws. Let us recall the construction (cf. [17], V.4).

If μ is a measure on $\mathbb{Z}_p^\times$ then the **Mahler transform** is defined as $A_\mu = \int_{\mathbb{Z}_p^\times} \phi(x)\mu(x)$. If μ_1, μ_2 are two measures on $\mathbb{Z}_p^\times$ and A_{μ_1}, A_{μ_2} the respective Mahler-transforms, one has the convolution $\mu_1 * \mu_2$, which is a measure defined via

$$\int_{\mathbb{Z}_p^\times} \phi\mu_1 * \mu_2 = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \phi(xy)\mu_1(x)\mu_2(y).$$

The Mahler-transform hence takes the form

$$\begin{aligned} A_{\mu_1 * \mu_2} &= \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (1+T)^{xy}\mu_1(x)\mu_2(y) \\ &= \lim_{n\to+\infty} \sum_{i,j \in \mathbb{Z}_p^\times \bmod p^n} \int_{(j+p^n\mathbb{Z}_p)\times(i+p^n\mathbb{Z}_p)} (1+T)^{xy}\mu_1(x)\mu_2(y). \end{aligned}$$

If one puts $xy = ij + i(x-j) + j(y-i) + (x-j)(y-i)$ and uses the fact that $(x-j)(y-i)$ is small in $(j + p^n\mathbb{Z}_p) \times (i + p^n\mathbb{Z}_p)$ one obtains

$$\begin{aligned} A_{\mu_1 * \mu_2} &= \lim_{n\to+\infty} \sum_{i,j \in \mathbb{Z}_p^\times \bmod p^n} (1+T)^{ij} \int_{(j+p^n\mathbb{Z}_p)\times(i+p^n\mathbb{Z}_p)} (1+T)^{i(x-j)+j(y-i)}\mu_1(x)\mu_2(y) \\ &= \lim_{n\to+\infty} \sum_{i,j \in \mathbb{Z}_p^\times \bmod p^n} (1+T)^{ij}\sigma_i((1+T)^{-j}\mathrm{Res}_{j+p^n\mathbb{Z}_p} A_{\mu_1})\sigma_j((1+T)^{-i}\mathrm{Res}_{i+p^n\mathbb{Z}_p} A_{\mu_2}). \end{aligned}$$

One has

$$(1+T)^{-k}\mathrm{Res}_{k+p^n\mathbb{Z}_p} A_{\mu_l} = \mathrm{Res}_{p^n\mathbb{Z}_p}((1+T)^{-k}A_l) = \varphi^n\psi^n((1+T)^{-k}A_{\mu_l}), \quad l = 1, 2,$$

and thus finally

$$A_{\mu_1 * \mu_2} = \lim_{n\to+\infty} \sum_{i,j \in \mathbb{Z}_p^\times \bmod p^n} (1+T)^{ij}\varphi^n((\sigma_i\psi^n((1+T)^{-j}A_{\mu_1}))(\sigma_j\psi^n((1+T)^{-i}A_{\mu_2}))).$$

Colmez proved:

Proposition 4.3.1. Let D_1, D_2, D_3 be étale (φ, Γ_K) over $\mathbf{B}_K$ and let $M : D_1 \times D_2 \longrightarrow D_3$ be a $\mathbf{B}_K$-bilinear form that commutes with the action of φ and Γ_K. Let $y \in D_1^{\psi=0}$ and $v \in D_2^{\psi=0}$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ defined via

$$u_n = \sum_{i,j\in\Gamma_K \bmod p^n} (1+\pi)^{ij}\mathrm{Tr}_{K/\mathbb{Q}_p}\varphi^n(M(\sigma_i \cdot \psi^n((1+\pi)^{-j}y), \sigma_j \cdot \psi^n((1+\pi)^{-i}v))) \quad (4.4)$$

converges to a limit $M(y, v) \in D_3^{\psi=0}$. The limit does not depend on the choice of representatives $\bmod p^n$, and the resulting pairing

$$M(\, , \,) : D_1^{\psi=0} \times D_2^{\psi=0} \longrightarrow D_3^{\psi=0}$$

is $\Lambda(\Gamma_K)_{\mathbb{Q}_p}$-bilinear.

Proof. See [17], Proposition V.4.1 where the proof is done in the case $K = \mathbb{Q}_p$. Observe that the trace is continous (coefficient-wise), and one may deduce the convergence in an analogous manner. □

Recall that if D is a (φ, Γ_K)-module we have the canonical pairing $D \times D^*(1) \to D \otimes D^*(1) \cong \mathbf{B}^\dagger_{\mathrm{rig},K}(1)$ (also refered to as the **Tate pairing**) which we simply denote by "$\otimes_D$" or "$\otimes$". If D is an étale (φ, Γ_K)-module over $\mathbf{B}_K$ we denote the resulting pairing

$$D^{\psi=0} \times D^*(1)^{\psi=0} \longrightarrow \mathbf{B}_{\mathbb{Q}_p}(1)^{\psi=0}$$

by $[\, , \,]'_{\mathrm{Iw},D}$.

One may now proceed as in the $\langle \, , \, \rangle_{\mathrm{Iw},D}$-case to extend the above pairing for an étale (φ, Γ_K)-module D over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ by $\mathcal{B}(\Gamma_K)$-linearity to a pairing

$$[\, , \,]'_{\mathrm{Iw},D} : D^{\psi=0} \times D^*(1)^{\psi=0} \longrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(1)^{\psi=0}.$$

We now want to define a related pairing $[\, , \,]_{\mathrm{Iw},D}$, specifically for the (in general) non-étale $\mathcal{B}(\Gamma_K)$-module $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ and its dual. First observe that the Tate-pairing (up to a twist) induces a pairing of (φ, Γ_K)-modules

$$[\, , \,]_{\mathbf{N}_{\mathrm{dR}}(D)} : \mathbf{N}_{\mathrm{dR}}(D) \times \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{N}_{\mathrm{dR}}(D^*(1)) \to \mathbf{N}_{\mathrm{dR}}(D) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{N}_{\mathrm{dR}}(D)^*[-1] \to \mathbf{B}^\dagger_{\mathrm{rig},K}[-1].$$

We first point out the following relation:

Lemma 4.3.2. One has the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{N}_{\mathrm{dR}}(D) & \otimes & \mathbf{N}_{\mathrm{dR}}(D^*(1)) & \xrightarrow{[\, , \,]_{\mathbf{N}_{\mathrm{dR}}(D)}} & \mathbf{B}^\dagger_{\mathrm{rig},K}[-1] \\
\downarrow \mathrm{id} & & \downarrow \cdot t & & \downarrow \cdot t \\
\mathbf{N}_{\mathrm{dR}}(D) & \otimes & \mathbf{N}_{\mathrm{dR}}(D)^*(1) & \xrightarrow{\otimes_{\mathbf{N}_{\mathrm{dR}}(D)}} & \mathbf{B}^\dagger_{\mathrm{rig},K}(1)
\end{array}$$

Proof. Since both pairings are $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-bilinear and $\mathbf{N}_{\mathrm{dR}}(D^*(1)) = \mathbf{N}_{\mathrm{dR}}(D)^*[-1]$, this is clear from the definitions. □

If D is semi-stable the following method of **induction by the degree of nilpotence** of N is crucial, which we describe next.

So assume first that D is crystalline so that $\mathbf{N}_{\mathrm{dR}}(D) = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_F \mathbf{D}^K_{\mathrm{cris}}(D)$ and $\mathbf{N}_{\mathrm{dR}}(D^*(1)) = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_F \mathbf{D}^K_{\mathrm{cris}}(D^*(1))$. One has the perfect pairing $[\ ,\ ]_{K,D} : \mathbf{D}^K_{\mathrm{cris}}(D) \times \mathbf{D}^K_{\mathrm{cris}}(D^*(1)) \to F$, such that if $f \otimes d \in \mathbf{N}_{\mathrm{dR}}(D)$ and $g \otimes d^* \in \mathbf{N}_{\mathrm{dR}}(D^*(1))$ then

$$[f \otimes d, g \otimes d^*]_{\mathbf{N}_{\mathrm{dR}}(D)} = f \cdot g \cdot [d, d^*]_{K,D}.$$

In general, one may then assume $N \neq 0$ on $\mathbf{D}^K_{\mathrm{st}}(D)$ and use dévissage on the F-dimension of D and the exact sequences

$$\begin{gathered} 0 \to N\mathbf{D}^K_{\mathrm{st}}(D) \to \mathbf{D}^K_{\mathrm{st}}(D)) \to \mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D) \to 0 \\ 0 \to \mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0} \to \mathbf{D}^K_{\mathrm{st}}(D^*(1)) \to \mathbf{D}^K_{\mathrm{st}}(D^*(1))/\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0} \to 0. \end{gathered} \tag{4.5}$$

One checks that the functor $\mathbf{N}_{\mathrm{dR}}(-)$ on (φ, N)-modules leaves these sequences exact, for example by fixing a basis of $\mathbf{D}^K_{\mathrm{st}}(D)$ adapted to the nilpotency operator N and using the operator $\mathcal{E}$ (cf. (4.8)). Hence, one obtains exact sequences

$$\begin{gathered} 0 \to \mathbf{N}_{\mathrm{dR}}(N\mathbf{D}^K_{\mathrm{st}}(D)) \to \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D)) \to \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D))) \to 0 \\ 0 \to \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0}) \to \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))) \to \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))/\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0}) \to 0 \end{gathered} \tag{4.6}$$

In fact, $\mathbf{N}_{\mathrm{dR}}(-)$ is an exact $\otimes$-functor. This may be checked by using these exact sequences and a 9-term diagram as in the proof of [35], Proposition 4.3.2. Similarly, one obtains exact sequences

$$\begin{gathered} 0 \to \mathcal{M}(N\mathbf{D}^K_{\mathrm{st}}(D)) \to \mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D))) \to \mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D)) \to 0 \\ 0 \to \mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0}) \to \mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))) \to \mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))/\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0}) \to 0 \end{gathered} \tag{4.7}$$

of semi-stable (φ, Γ_K)-modules (note that $\mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D)) = D$ and $\mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))) = D^*(1)$). Hence, if D is semi-stable and $f \otimes d \in \mathbf{N}_{\mathrm{dR}}(N\mathbf{D}^K_{\mathrm{st}}(D))$ and $g \otimes d^* \in \mathbf{N}_{\mathrm{dR}}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))))$ then

$$[f \otimes d, g \otimes d^*]_{\mathbf{N}_{\mathrm{dR}}(D)} = [f \otimes d, \overline{g \otimes d^*}]_{\mathbf{N}_{\mathrm{dR}}(N\mathbf{D}_{\mathrm{st}}(D))},$$

where $[\ ,\ ]_{\mathbf{N}_{\mathrm{dR}}(N\mathbf{D}_{\mathrm{st}}(D))}$ is the pairing on $\mathbf{N}_{\mathrm{dR}}(N\mathbf{D}_{\mathrm{st}}(D)) \times \mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}(D^*(1))/\mathbf{D}_{\mathrm{st}}(D^*(1)))^{N=0}$. Similarly the pairing $[\ ,\ ]_{\mathbf{N}_{\mathrm{dR}}(D)}$ factorizes if one starts with $g \otimes d^* \in \mathbf{N}_{\mathrm{dR}}(D^*(1)^{N=0})$.

Recall that if D is a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ we know that $D^{\psi=0}$ is a free $\mathcal{B}(\Gamma_K)$-module (cf. [16], Proposition V.1.19). More precisely for étale modules have the following explicit description of a basis:

Proposition 4.3.3. Let D be an étale (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ of rank d. Then $D^{\psi=0}$ is a free $\mathcal{B}(\Gamma_K)$-module of rank $[K : \mathbb{Q}_p] \cdot \mathrm{rk}_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$.

Proof. Let $s \geq 0$ be such that $\Gamma_{K_s} \subset \Gamma_K \subset \Gamma_{\mathbb{Q}_p}$. We know that D is a free $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$-module of rank $[H_{\mathbb{Q}_p} : H_K] \cdot \mathrm{rk}_{\mathbf{B}^\dagger_{\mathrm{rig},K}} D$, and we may choose a $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$-basis (d_i) of D. Colmez has shown (cf. [16], section V.1.4) that $(1+\pi)\varphi^s(D) = \bigoplus \mathcal{B}(\Gamma_s) \cdot (1+\pi)\varphi^s(d_i)$. Using the fact that $D = \bigoplus_{i=0}^{p-1}(1+\pi)^i\varphi(D)$, hence $D^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+\pi)^i\varphi(D)$ and inductively $D^{\psi=0} = \bigoplus_{i=1}^{p^s-1}(1+\pi)^i\varphi^s(D)$, one obtains the claim from the fact that $[K : \mathbb{Q}_p] = [H_{\mathbb{Q}_p} : H_K] \cdot [\Gamma_{\mathbb{Q}_p} : \Gamma_K]$ □

Corollary 4.3.4. Let $K/\mathbb{Q}_p$ be unramified with basis $f_1, \ldots, f_n$. Then $(\mathbf{B}^\dagger_{\mathrm{rig},K})^{\psi=0}$ is a $\mathcal{B}(\Gamma_K)$-module with basis $f_i(1+\pi)$.

Proof. Since $\Gamma_K = \Gamma_{\mathbb{Q}_p}$ and $\mathbf{B}^\dagger_{\mathrm{rig},K} = F \otimes_{\mathbb{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ the claim may be deduced from the proof of the previous proposition. □

Let α be an element of $\mathbf{B}^\dagger_{\log,K}$ such that $N\alpha = 1$ and $\alpha \in \varphi(\mathbf{B}^\dagger_{\log,K})$. For example, with our conventions one may choose $-1/p \cdot \varphi(\log \pi)$. Perrin-Riou considered the following map (see [35], 2.2):

$$\begin{aligned} \mathcal{E}_\alpha = \mathcal{E} : \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D) &\longrightarrow \mathbf{B}^\dagger_{\log,K} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D) \\ f \otimes d &\longmapsto \exp(-\alpha)(f \otimes d) \\ &:= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \alpha^k \cdot f \otimes N^k(d) \end{aligned} \tag{4.8}$$

A simple calculation shows that for $d \in \mathbf{D}^K_{\mathrm{st}}(D)$ one actually has $\mathcal{E}(f \otimes d) \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$.

Lemma 4.3.5. Suppose $K/\mathbb{Q}_p$ is unramified. Let $d_1, \ldots, d_n$ be a basis for $\mathbf{D}^K_{\mathrm{st}}(D)$ adapted to the monodromy operator N and $f_1, \ldots, f_m$ be a basis for $F/\mathbb{Q}_p$. Then the $\mathcal{E}(f_i \cdot (1+\pi) \otimes d_j)$ form a basis of the $\mathcal{B}(\Gamma_K)$-module $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$.

Proof. We prove the statement for a basis $(d_i)_i$ adapted to the nilpotent operator N. For a crystalline (φ, Γ_K)-module the statement follows from Corollary 4.3.4. If $N \neq 0$ on $\mathbf{D}^K_{\mathrm{st}}(D)$ one obtains the result by considering the exact sequence

$$0 \to (\mathbf{B}^\dagger_{\log,K} \otimes N\mathbf{D}^K_{\mathrm{st}}(D))^{N=0,\psi=0} \to \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} \to (\mathbf{B}^\dagger_{\log,K} \otimes \mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D))^{N=0,\psi=0} \to 0$$

of $\mathcal{B}(\Gamma_K)$-modules, where by assumption the left and the right module are free with basis vectors given by $f_1, \ldots f_r$ resp. $\overline{f_{r+1}}, \ldots, \overline{f_m}$, so that the claim follows. □

Having established the pairings $\otimes_D$ and $[\ ,\ ]_D$ and an explicit basis for $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ we make (for not necessarely étale, but semi-stable (φ, Γ_K)-modules D) the following

Assumption 4.3.6. Let D be (φ, Γ_K)-module. If $y \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ and $v \in \mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0}$, the formula in (4.4) with $M = [\ ,\ ]_{\mathbf{N}_{\mathrm{dR}}(D)}$ defines a sequence $(u_n)_{n\in\mathbb{N}}$ which converges in $(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}[-1])^{\psi=0}$ which does not depend on the choices of the representative $\mod p^n$.

Definition 4.3.7. Let D be such that Assumption 4.3.6 holds for $\mathbf{N}_{\mathrm{dR}}(D)$. Then the formula (4.4) defines a pairing

$$[\ ,\ ]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)} : \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} \times \mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0} \longrightarrow (\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \cong \mathcal{H}(\Gamma_K).$$

Of course, we suspect that the assumption should hold for (at least) all de Rham (φ,Γ)-modules.

If the assumption holds, since

$$\sigma_a \cdot u_n = \sum_{i,j\in\mathbb{Z}_p^\times} (1+\pi)^{aij}\varphi^n([(\sigma_{ai}\cdot\psi^n((1+\pi)^{-j}y), \sigma_j\cdot\psi^n((1+\pi)^{-ai}\sigma_a\cdot v))]_{\mathbf{N}_{\mathrm{dR}}(D)}, \quad (4.9)$$

one has, by going to the limit,

$$\sigma_a \cdot [y,v]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)} = [\sigma_a\cdot y, v]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)} = [y, \sigma_a\cdot v]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)}.$$

Hence, the pairing is also $\mathcal{B}(\Gamma_K)$-bilinear since $[y,v]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)} \in (\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$.

Proposition 4.3.8. Assumption 4.3.6 holds in the following cases:

a) D is étale.

b) D is crystalline.

c) D is semi-stable and two-dimensional.

Proof. We first consider the étale case. Since we extended the pairing in Proposition 4.3.1 to $\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0} \times \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))^{\psi=0}$ by bilinearity one sees that u_n in loc.cit. actually converges over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ (with respect to the Fréchet topology). We may choose $h \geq 1$ such that for $\Omega_h = \nabla_{h-1}\circ\ldots\circ\nabla_0 \in \mathcal{H}(\Gamma_K)$ we have $\Omega_h(y) \in D$, $\Omega_h(v) \in D^*(1)$. Since $D[1/t] = \mathbf{N}_{\mathrm{dR}}(D)[1/t]$, it follows from Lemma 4.3.2 that for $r \in \mathbf{N}_{\mathrm{dR}}(D), s \in \mathbf{N}_{\mathrm{dR}}(D^*(1))$ such that additionally $r \in D, s \in D^*(1)$,

$$[r,s]_D = [r,s]_{\mathbf{N}_{\mathrm{dR}}(D)}.$$

Hence, using (4.9) we infer that by going to the limit

$$\Omega_h \cdot \Omega_h \cdot \lim_n u_n(y,v) = \lim_n \cdot u_n(\Omega_h\cdot y, \Omega_h\cdot v),$$

where on the left hand side we mean u_n with respect to $[\ ,\ ]_{\mathbf{N}_{\mathrm{dR}}(D)}$, and on the right hand side with respect to $[\ ,\ ]_D$. Since Ω_h is a product of non-zero divisors of $\mathcal{H}(\Gamma_K)$, $u_n(y,v)$ converges by Proposition 4.3.1.

Let now D be crystalline. Fix a basis $\{d_i\}$ of $\mathbf{D}_{\mathrm{cris}}(D)$ with corresponding dual basis $\{d_i^*\}$ under $[\ ,\ ]_{K,D}$.

$$[\sigma_i\cdot\psi^n((1+\pi)^{-j}(1+\pi))\otimes d_i, \sigma_j\cdot\psi^n((1+\pi)^{-i}(1+\pi))\otimes d_i^*)]_{\mathbf{N}_{\mathrm{dR}}(D)}$$

for $i, j \in \mathbb{Z}_p^\times$. If $i = j = 1$, then this equals 1. If one of i or j does not equal 1 then one checks easily that ψ^n anihilates the term $(1+\pi)^{-j+1}$ (resp. for i). Hence, by the $\mathcal{B}(\Gamma_K)$-bilinearity:

$$\lim_{n\to\infty} u_n(\lambda \cdot (1+\pi) \otimes d_i, \mu \cdot (1+\pi) \otimes d_j^*) = \delta_{ij} \cdot \lambda \cdot \mu \cdot (1+\pi),$$

so that the pairing converges everywhere and has the required properties.

In the two-dimensional semi-stable case, we may fix a basis d_1, d_2 such that $Nd_1 = d_2$, and dually a basis d_1^*, d_2^* (which is dual to d_1, d_2) such that $Nd_2^* = -d_1^*$. Since $\mathcal{E}((1+\pi)\otimes d_1), \mathcal{E}((1+\pi)\otimes d_2)$ resp. $\mathcal{E}((1+\pi)\otimes d_1^*), \mathcal{E}((1+\pi)\otimes d_2^*)$ form a basis for $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ resp. $\mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0}$, it suffices to check (by $\mathcal{B}(\Gamma_K)$-bilinearity) that $u_n(\mathcal{E}((1+\pi)\otimes d_1), \mathcal{E}((1+\pi)\otimes d_2^*)$ converges (the other cases are handled in the same way as in the crystalline case). Since again $\psi^n((1+\pi)^{-i+1})$ vanishes for $i \neq 1$ we need to consider the terms (recall that $\alpha = 1/p \cdot \varphi(\log \pi))$

$$[1 \otimes d_1 - \psi^n(\alpha \otimes d_2), 1 \otimes d_2^* + \psi^n(\alpha \otimes d_1^*)]_D$$

which by the definition of $[\ ,\ ]_D$ vanishes, so that $u_n(\mathcal{E}((1+\pi)\otimes d_1), \mathcal{E}((1+\pi)\otimes d_2^*) = 0$ for all n, which shows the claim. □

Further one may show:

Proposition 4.3.9. Assume D is semi-stable and Assumption 4.3.6 holds. Then $[\ ,\ ]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)}$ is a perfect pairing of $\mathcal{B}(\Gamma_K)$-modules.

Proof. In the crystalline case this follows from the proof of Proposition 4.3.8: basically, if $d_1, \dots, d_r$ is a basis of $\mathbf{D}_{\mathrm{cris}}(D)$ with dual basis $d_1^*, \dots, d_r^* \in \mathbf{D}_{\mathrm{cris}}(D^*(1))$ with respect to the pairing $[\ ,\ ]_{K,D}$, then $[\lambda \cdot d_i, \mu \cdot d_j^*]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)} = \delta_{ij} \cdot \mu \cdot \lambda \cdot (1+\pi)$.

By dévissage we obtain, similarly as in the case for the cohomology pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$, a commutative diagram of $\mathcal{B}(\Gamma_K)$-modules

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbf{N}_{\mathrm{dR}}(ND)^{\psi=0} & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{B}(\Gamma_K)}(\mathbf{N}_{\mathrm{dR}}(D^*(1)/D^*(1)^{N=0}), \mathcal{B}(\Gamma_K)) \\
\downarrow & & \downarrow \\
\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} & \longrightarrow & \mathrm{Hom}_{\mathcal{B}(\Gamma_K)}(\mathbf{N}_{\mathrm{dR}}(D^*(1)), \mathcal{B}(\Gamma_K)) \\
\downarrow & & \downarrow \\
\mathbf{N}_{\mathrm{dR}}(D/ND)^{\psi=0} & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{B}(\Gamma_K)}(\mathbf{N}_{\mathrm{dR}}(D^*(1)^{N=0}), \mathcal{B}(\Gamma_K)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the isomorphisms in the top and bottom row are by assumption. Hence, the claim. □

Now one may define the pairing on $\mathbf{N}_{\mathrm{dR}}(D)$ also as follows. Consider the sequences (4.5) and assume that the pairing is already defined on

$$\mathbf{N}_{\mathrm{dR}}(N\mathbf{D}_{\mathrm{st}}^K(D))^{\psi=0}\times\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D)/\mathbf{D}_{\mathrm{st}}^K(D)^{N=0})^{\psi=0} \tag{4.10}$$

and

$$\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D)/N\mathbf{D}_{\mathrm{st}}^K(D))^{\psi=0}\times\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D)^{N=0})^{\psi=0}. \tag{4.11}$$

After a choice of a $\mathcal{B}(\Gamma_K)$-basis one has isomorphisms of $\mathcal{B}(\Gamma_K)$-modules

$$\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}\cong\mathbf{N}_{\mathrm{dR}}(N\mathbf{D}_{\mathrm{st}}^K(D))^{\psi=0}\oplus\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D)/N\mathbf{D}_{\mathrm{st}}^K(D))^{\psi=0}$$

and

$$\mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0}\cong\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D^*(1))^{N=0})^{\psi=0}\oplus\mathbf{N}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{st}}^K(D^*(1))/\mathbf{D}_{\mathrm{st}}^K(D^*(1))^{N=0})^{\psi=0}$$

so that the $\mathcal{B}(\Gamma_K)$-bilinear pairing by defining it on the corresponding factors.

Remark 4.3.10. Assume we are in the situation of Proposition 4.3.9 such that D is étale. Let $y\in\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ and $v\in\mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0}$ and assume that $y\in D^{\psi=0}$ and $v\in D^*(1)$. Then

$$[y,v]'_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)}=[y,v]'_{\mathrm{Iw},D}.$$

4.4 Reciprocity for étale (φ,Γ_K)-modules à la Colmez

We keep the assumption that $K/\mathbb{Q}_p$ is unramified. To formulate reciprocity laws we first need to recall some more notation introduced by Colmez. Firstly (see [17], section III.1.2), if D is an étale (φ,Γ_K)-module over $\mathbf{B}_K$ or $\mathbf{B}^\dagger_{\mathrm{rig},K}$, for $a\in\mathbb{Z}_p$ and $k\in\mathbb{N}$, we denote by $\mathrm{Res}_{a+p^k\mathbb{Z}_p}$ the operator

$$\mathrm{Res}_{a+p^k\mathbb{Z}_p}:D\longrightarrow D,\quad x\longmapsto(1+\pi)^a\varphi^k(\psi^k((1+\pi)^{-a}x)).$$

Of course, this is a natural generalization of the concept of a "restriction of a measure". Now if D is étale over $\mathbf{B}_K$ and $x\in D$ consider the general term

$$u_n(x)=\sum_{i\in\mathbb{Z}_p^\times \bmod p^n}(1+\pi)^{i^{-1}}\sigma_{-i^{-2}}(\mathrm{Res}_{p^n\mathbb{Z}_p}\left((1+\pi)^{-i}x\right)). \tag{4.12}$$

Colmez has shown (see loc.cit., Lemma V.1.2) that the limit $\lim_{n\to\infty}u_n(x)$ exists in D, so that one obtains a $\mathbf{B}_K$-linear map $w_*:D\to D$. Further:

Lemma 4.4.1. If $\sigma\in\Gamma_K$ then $w_*(\sigma\cdot x)=\sigma^{-1}w_*(x)$.

Proof. Say $\sigma=\sigma_j$ for $j\in\mathbb{Z}_p^\times$. Then on the level of the u_n's one has

$$\begin{aligned}\sigma_j\cdot&\sum_{i\in\mathbb{Z}_p^\times \bmod p^n}(1+\pi)^{i^{-1}}\sigma_{-i^{-2}}(\mathrm{Res}_{p^n\mathbb{Z}_p}\left((1+\pi)^{-i}\sigma_jx\right))\\&=\sum_{i\in\mathbb{Z}_p^\times \bmod p^n}(1+\pi)^{i^{-1}j}\sigma_{-i^{-2}j^2}(\mathrm{Res}_{p^n\mathbb{Z}_p}\left((1+\pi)^{-ij^{-1}}x\right))\end{aligned}$$

so that the substitution $i\mapsto ji$ gives the claim. ∎

Thanks to the previous lemma it is possible to extend w_* uniquely to $\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)^{\psi=0}$ in the following way. Recall that $\mathcal{H}(\Gamma_K)\otimes_\Lambda \mathbf{D}(T)^{\psi=1} = \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)^{\psi=1}$ and further $\mathcal{B}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathcal{C}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)^{\psi=0}$. If $x\in \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)^{\psi=0}$ we may write it as $x=\sum_i \lambda_i\otimes x_i$ with $x_i\in(\varphi-1)\mathbf{D}(T)^{\psi=1}$ and $\lambda_i\in\mathcal{B}(\Gamma_K)$, so that one may define $w_*(x)=\sum_i \iota(\lambda_i)\otimes w_*(x_i)$. We note that since

$$\mathbf{D}_{\infty,f}(V)\subset \mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathbf{D}_{\infty,f}(V)=\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$$

one also has a natural extension of w_* to $\mathbf{D}_{\infty,f}(V)$.

Colmez proved the following reciprocity law:

Theorem 4.4.2. Let $D=\mathbf{D}(V)$ be an étale (φ,Γ_F)-module over $\mathbf{B}_K$ and $y\in D$, $v\in D^*(1)$. Then

$$\partial(\langle y,\sigma_{-1}\cdot v\rangle_{\mathrm{Iw},D}=-[y,w_*(v)]'_{\mathrm{Iw},D}.$$

Proof. The only thing to check is that the proof of [17], section VI.2 extends to the case $K/\mathbb{Q}_p$ unramified. With our definition one has (with the notation of loc.cit., Lemma VI.2.3, VI.2.4, VI.2.5)

$$\partial\langle y,\sigma_{-1}v\rangle_{\mathrm{Iw},D}=\lim_{n\to+\infty}\sum_{j\in\mathbb{Z}_p^\times \bmod p^n}\mathrm{Tr}_{K/\mathbb{Q}_p}\mathrm{res}\left(\frac{j}{(1+\pi)^{-j}-1}u_{n,j}(y,v)\right)(1+\pi)^j.$$

and

$$[y,w_*(v)]_{\mathrm{Iw},D}=\sum_{j\in\mathbb{Z}_p^\times \bmod p^n}(1+\pi)^j\mathrm{Tr}_{K/\mathbb{Q}_p}\varphi^n(u_{n,j}(y,v))$$

so that thanks to the properties of the trace the very last equation in loc.cit. still holds (up to a trace) and shows the desired equality. □

Corollary 4.4.3. Let $D=\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)$ be an étale (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and $y\in D$, $v\in D^*(1)$. Then

$$\partial(\langle y,\sigma_{-1}\cdot v\rangle_{\mathrm{Iw},D}=-[y,w_*(v)]'_{\mathrm{Iw},D}.$$

In the following theorem the equality is meant to be understood via the natural map $\mathbf{B}^{\dagger}_{\mathrm{rig},K}[-1]^{\psi=0}\longrightarrow \mathbf{B}^{\dagger}_{\mathrm{rig},K}(1)^{\psi=0}$, since the pairing $[\ ,\ ]_{\mathrm{Iw},\mathbf{N}_{\mathrm{dR}}(D)}$ lands in former and the pairing $[\ ,\ ]'_{\mathrm{Iw},D}$ lands in the latter, cf. also Lemma 4.3.2. Since assumption 4.3.6 holds in the étale case, we may prove:

Theorem 4.4.4. Let V be a semi-stable representation of G_K, $y\in\mathbf{N}_{\mathrm{dR}}(V)^{\psi=0}$, $v\in\mathbf{N}_{\mathrm{dR}}(V^*(1))$. Then for every $h\geq 1$ one has

$$\langle\Omega_{V,h}(y),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},V}=(-1)^{h+1}[y,w_*(v)]_{\mathrm{Iw},V}.$$

Proof. Let $h,h^*\geq 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{st}}(V)=\mathbf{D}_{\mathrm{st}}(V)$ and $\mathrm{Fil}^{-h^*}\mathbf{D}_{\mathrm{st}}(V^*(1))=\mathbf{D}_{\mathrm{st}}(V^*(1))$. Formally, the expression on the left is defined via

$$\left\langle\Omega_{V,h}(y),\sigma_{-1}\left(\prod_{-h<j<h^*}\nabla_j\right)^{-1}\Omega_{V^*(1),h^*}(y)\right\rangle_{\mathrm{Iw},V},$$

where we work in $\mathcal{K}(\Gamma_K)$.

We add a $\nabla_0 \cdot \nabla_0^{-1}$ in the first argument and pull out the $\nabla_0 = t\partial$ to obtain by Colmez' theorem resp. Corollary 4.4.3 an equality

$$\begin{aligned}\langle \Omega_{V,h}(y),\ \sigma_{-1} \cdot \Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)} \\ &= t\partial \langle \nabla_0^{-1}\Omega_{V,h}(y), \sigma_{-1} \cdot \Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)} \\ &= -t[\nabla_0^{-1}\Omega_{V,h}(y), w_*(\Omega_{V^*(1),1-h}(v))]'_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)}.\end{aligned}$$

We explicitely calculate for general $a \in \mathbf{N}_{\mathrm{dR}}(V)^{\psi=0}$ and $b \in \mathbf{N}_{\mathrm{dR}}(V^*(1))^{\psi=0}$, such that additionally $a \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ and $b \in \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$, that $t[a,b]'_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)}$ is equal to

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i,j\in\mathbb{Z}_p^\times \bmod p^n} (1+T)^{ij}\mathrm{Tr}_{K/\mathbb{Q}_p} p^{-n}\varphi^n(\sigma_i \cdot \psi^n((1+T)^{-j}y) \otimes t \cdot \sigma_j \cdot \psi^n((1+T)^{-i}v)) \\ = &\lim_{n\to\infty} \sum_{i,j\in\mathbb{Z}_p^\times \bmod p^n} (1+T)^{ij}\mathrm{Tr}_{K/\mathbb{Q}_p}\varphi^n([\sigma_i \cdot \psi^n((1+T)^{-j}y), \sigma_j \cdot \psi^n((1+T)^{-i}v)]_{\mathbf{D}^\dagger_{\mathrm{rig}}(V)})\end{aligned}$$

thanks to Lemma 4.3.2. Applying this to the previous equation and recalling that $\Omega_{V,h} = \nabla_{h-1} \cdot \ldots \cdot \nabla_0$ and $w_*(\nabla_j(x)) = (-1) \cdot \nabla_{-j} w_*(x)$ for $j \in \mathbb{Z}$ we obtain

$$\begin{aligned}\langle \Omega_{V,h}(y),\ \sigma_{-1} \cdot \Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)} \\ &= -[\nabla_0^{-1}\Omega_{V,h}(y), w_*(\Omega_{V^*(1),1-h}(v))]_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)} \\ &= (-1)^{h+1}[y, w_*(v)]_{\mathrm{Iw},\mathbf{D}^\dagger_{\mathrm{rig}}(V)},\end{aligned}$$

which gives the claim. □

4.5 Reciprocity for crystalline and semi-stable (φ, Γ_K)-modules

We now concern ourselves with the general case of semi-stable (φ, Γ_K)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and prove a reciprocity law following Berger's proof for the crystalline étale case. So assume D is such a module that is semi-stable over an *unramified* extension $K/\mathbb{Q}_p$.

We first describe briefly how to extend the operator ι to $\mathbf{N}_{\mathrm{dR}}(D)$. Of course, one should expect that the formula (4.12) which defined the map w_* makes sense in the general case of a (φ, Γ_K)-module and use this in place of ι. We do not give a proof here, but we believe that the two definitions should actually coincide.

Recall that if D is crystalline then

$$\mathbf{D}_{\infty,f}(D) = (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D) \subset (\mathbf{B}^\dagger_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}^K_{\mathrm{st}}(D) = (\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes_{\mathbb{Q}_p} \mathbf{D}^K_{\mathrm{st}}(D),$$

and $(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ is a free $\mathcal{H}(\Gamma_K)$-module of rank 1 via the isomorphism $\lambda \mapsto \lambda \cdot (1+\pi)$. Hence, ι is defined naturally as

$$\iota(\sum \lambda_i \cdot (1+\pi) \otimes d_i) = \sum \iota(\lambda_i) \cdot (1+\pi) \otimes d_i.$$

Theorem 4.5.1. Let $K/\mathbb{Q}_p$ be unramified and D be a crystalline (φ,Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Let $y\in\mathbf{D}_{\infty,g}(D)$, $v\in\mathbf{D}_{\infty,g}(D^*(1))$. Then for every $h\geq 1$ one has

$$\langle\Omega_{D,h}(y),\sigma_{-1}\cdot\Omega_{D^*(1),1-h}(v)\rangle_{\mathrm{Iw},D}=(-1)^{h+1}[y,\iota(v)]_{\mathrm{Iw},D}.$$

in $\mathcal{K}(\Gamma_K)$.

Proof. Thanks to Proposition 3.2.12 we may assume $y\in\mathbf{D}_{\infty,e}(D)$, $v\in\mathbf{D}_{\infty,e}(D^*(1))$ since $\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathbf{D}_{\infty,e}(D)=\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}\mathbf{D}_{\infty,g}(D)$ By p-adic interpolation (cf. 4.0.4) it suffices to show that

$$\partial^j(\langle\Omega_{V,h}(y),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},D})(0)=\partial^j((-1)^{h+1}[y,\iota(v)]_{\mathrm{Iw},D})(0).$$

for $j\gg 0$. This is equivalent to

$$(-1)^{-j}\langle h^1_{K,D(j)}\Omega_{h+j}(\partial^{-j}y\otimes t^{-j}e_j),h^1_{K,D(1-j)}\Omega_{1-h-j}(\partial^{j}v\otimes t^{j}e_{-j})\rangle_{K,D(j)} \tag{4.13}$$
$$=(-1)^{h+1}[(\partial^{-j}y\otimes t^{-j}e_j)(0),(\partial^{j}v\otimes t^{j}e_{-j})(0)]_{K,D(j)}. \tag{4.14}$$

Let y' and v' such that $(\varphi-1)y'=y$ and $(\varphi-1)v'=v$. Then by Proposition 3.2.22 we see that

$$h^1_{K,D(j)}\Omega_{h+j}(\partial^{-j}y\otimes t^{-j}e_j)=(-1)^{h+j-1}(h+j-1)!\exp_{K,D(j)}((1-p^{-1}\varphi^{-1})(\partial^{-j}y'\otimes t^{-j}e_j)(0))$$

and

$$\exp^*_{K,D(j)}(h^1_{K,D(1-j)}\Omega_{1-h-j}(\partial^{j}v\otimes t^{j}e_{-j}))=\frac{1}{(h+j-1)!}(1-p^{-1}\varphi^{-1})(\partial^{j}v'\otimes t^{j}e_{-j})(0).$$

Using Proposition 3.1.20 we see that (4.13) is equal to

$$(-1)^{h+1}[(1-p^{-1}\varphi^{-1})(\partial^{-j}y'\otimes t^{-j}e_j)(0),(1-p^{-1}\varphi^{-1})(\partial^{j}v'\otimes t^{j}e_{-j})(0)]_{K,D(j)}.$$

Since $(1-\varphi)$ is the adjoint of $(1-p^{-1}\varphi^{-1})$ under the pairing $[\ ,\ ]_{K,D(j)}$ and $(1-\varphi)$ commutes with taking $\partial^k(-)(0)$, we get the claim in this case. □

In the general semistable case one possible idea is to use dévissage ((4.5) and (4.7)), although we currently we can only give a complete proof in the 2-dimensional case:

Let $D_1=\mathcal{M}(N\mathbf{D}^K_{\mathrm{st}}(D))$ and $D_2=\mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D)/N\mathbf{D}^K_{\mathrm{st}}(D))$ so that we have an exact sequence $0\to D_1\to D\to D_2\to 0$ of semi-stable (φ,Γ_K)-modules. Then for the dual exact sequence we have $D_1^*(1)=\mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))/\ \mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0})$ and $D_2^*(1)=\mathcal{M}(\mathbf{D}^K_{\mathrm{st}}(D^*(1))^{N=0})$ and an exact sequence $0\to D_2^*(1)\to D^*(1)\to D_1^*(1)\to 0$ of (φ,Γ_K)-modules. Hence, one has corresponding pairings $\langle\ ,\ \rangle_{\mathrm{Iw},D_1}$ and $\langle\ ,\ \rangle_{\mathrm{Iw},D_2}$. We may assume that the statement of the theorem holds for D_1 and D_2.

Suppose $a\in\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D_1^{\psi=1}$ and $b\in\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D^*(1)^{\psi=1}$. Then the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ factorizes as

$$\langle a,b\rangle_{\mathrm{Iw},D}=\langle a,\overline{b}\rangle_{\mathrm{Iw},D_1}$$

where $\bar{b}$ is the image of b in the canonical projection

$$\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D^*(1)^{\psi=1}\to\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D_1^*(1)^{\psi=1}$$

(cf. Proposition 4.2.6 for the exactness of $\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)(-)^{\psi=1}$) Otherwise there would exist a $b'\in\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D_2^*(1)^{\psi=1}$ such that $\langle a,b'\rangle_{\mathrm{Iw},D}\neq 0$. Hence, for some $k\gg 0$ one has $0\neq\partial^k\langle a,b'\rangle_{\mathrm{Iw},D}(0)$ which is not possible since the pairing $\otimes$ factorizes in the same way we want $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ to factorize. Analogously if $b\in\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D_2^*(1)^{\psi=1}$ and $a\in\mathcal{K}(\Gamma_K)\otimes_{\mathcal{H}(\Gamma_K)}(\varphi-1)D^{\psi=1}$. Then the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ factorizes as

$$\langle a,b\rangle_{\mathrm{Iw},D}=\langle\bar{a},b\rangle_{\mathrm{Iw},D_2}.$$

Hence, suppose $y=y_1+y_2$ with $y_1\in\mathbf{D}_{\infty,f}(N\mathbf{D}_{\mathrm{st}}(D)),y_2\in\mathbf{D}_{\infty,f}(\mathbf{D}_{\mathrm{st}}(D)/N\mathbf{D}_{\mathrm{st}}(D))$ and $v=v_1+v_2$ with $v_1\in\mathbf{D}_{\infty,f}(D^*(1)/D^*(1)^{N=0}),v_2\in\mathbf{D}_{\infty,f}(D^*(1)^{N=0})$. Since the pairings $\langle\ ,\ \rangle_{\mathrm{Iw},-}$ are perfect (cf. Theorem 4.2.7) and w_* is compatible with the above decomposition for any such choice in the 2-dimensional case in this case one computes

$$\begin{aligned}
&\langle\Omega_{V,h}(y),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v)\rangle_{\mathrm{Iw},D}\\
&\qquad=\langle\Omega_{V,h}(y_1),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v_1)\rangle_{\mathrm{Iw},D}+\langle\Omega_{V,h}(y_1),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v_2)\rangle_{\mathrm{Iw},D}\\
&\qquad\quad+\langle\Omega_{V,h}(y_2),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v_1)\rangle_{\mathrm{Iw},D}+\langle\Omega_{V,h}(y_2),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v_2)\rangle_{\mathrm{Iw},D}\\
&\qquad=\langle\Omega_{V,h}(y_1),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(\overline{v_1})\rangle_{\mathrm{Iw},D_1}+\langle\Omega_{V,h}(\overline{y_2}),\sigma_{-1}\cdot\Omega_{V^*(1),1-h}(v_2)\rangle_{\mathrm{Iw},D_2}\\
&\qquad=(-1)^{h+1}[y_1,\iota(\overline{v_1})]_{\mathrm{Iw},D_1}+(-1)^{h+1}[\overline{y_2},\iota(v_2)]_{\mathrm{Iw},D_2}\\
&\qquad=(-1)^{h+1}[y,\iota(v)]_{\mathrm{Iw},D},
\end{aligned}$$

since one easily checks in a similar fashion that a factorization similar as for the pairing $\langle\ ,\ \rangle_{\mathrm{Iw},D}$ holds for the pairing $[\ ,\ ]_{\mathrm{Iw},D}$, which finishes the proof.

Chapter 5

Applications and Prospects

5.1 Determinant of $\Omega_{D,h}$

In this section we assume that $K/\mathbb{Q}_p$ is unramified. In [33], Perrin-Riou formulated conjectures "$\delta_{\mathbb{Q}_p}(V)$" and "$\delta_{\mathbb{Z}_p}(V)$" for a *crystalline* p-adic representation V of $G_{\mathbb{Q}_p}$, which are closely connected with Tamagawa numbers and ε-factors associated to V. Benois and Berger ([3]) proved these conjectures (in fact, $\delta_{\mathbb{Q}_p}(V)$ is known to be a consequence of the reciprocity law, and hence already known) using the theory of Wach-modules.

In the more general semi-stable case, since for example $\mathbf{D}_{\infty,f}(V)$ somehow lies "diagonally" in $(\mathbf{B}^{\dagger}_{\log,K} \otimes_F \mathbf{D}_{\mathrm{st}}(V))^{N=0,\psi=0}$ and there are denominators, it is not sufficient to work with $\mathbf{B}^{+}_{\mathrm{rig},K}$.

In the de Rham case Nakamura ([30], Theorem 3.14) proved a version of a $\delta(D)$-conjecture over $\mathcal{H}(\Gamma_K)$ using the modules $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=1}$ and $\mathbf{N}_{\mathrm{dR}}(D)/(\psi-1)$. One already has (by twisting appropriately) $\mathbf{D}_{\infty,g}(D) = (\varphi-1)\mathbf{N}_{\mathrm{dR}}(D)^{\psi=1}$, and exact sequences (arising from an $0 \to D' \to D \to D'' \to 0$ of semistable (φ,Γ)-modules)

$$0 \to \mathbf{D}_{\infty,g}(D') \to \mathbf{D}_{\infty,g}(D) \to \mathbf{D}_{\infty,g}(D'') \to \mathbf{D}^2_{\infty,g}(D') \to \mathbf{D}^2_{\infty,g}(D') \to \mathbf{D}^2_{\infty,g}(D') \to 0$$

(cf. [35], 4.3) resp.

$$0 \to D'^{\psi=1} \to D^{\psi=1} \to D''^{\psi=1} \to D'/(\psi-1)D' \to D/(\psi-1)D \to D''/(\psi-1)D'' \to 0.$$

We believe that it should similarly be possible to relate the $\mathbf{D}^2_{\infty,g}(-)$ in a functorial way to the $\mathbf{N}_{\mathrm{dR}}(-)/(\psi-1)\mathbf{N}_{\mathrm{dR}}(-)$ (which are both finite-dimensional $\mathbb{Q}_p$-vector spaces), which would relate [30], Theorem 3.14 to [35], Theorem 5.4.4.

Next, we want to formulate a version of a $\delta(D)$-conjecture (and in certain cases even an integral one) which is closer to the one originally proposed by Perrin-Riou and proved by Benois/Berger.

Suppose that D is a crystalline (φ,Γ_K)-module. We define $\mathcal{D}'_{\Lambda_\infty}(D) := \mathbf{D}_{\infty,f}(D)$. If D is semistable we apply as usual the induction by the degree of the nilpotence. Hence we may suppose that we are given an exact sequence of semistable (φ,Γ_K)-modules $0 \to D' \to D \to D'' \to 0$ such that there are free $\mathcal{H}(\Gamma_K)$-submodules $\mathcal{D}'_{\Lambda_\infty}(*) \subset \mathbf{N}_{\mathrm{dR}}(*)^{\psi=0}$ for

$* \in \{D', D''\}$. One has the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}_{\infty,f}(D') & \longrightarrow & \mathbf{D}_{\infty,f}(D) & \xrightarrow{\mathrm{pr}} & \mathbf{D}_{\infty,f}(D'') & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathbf{N}_{\mathrm{dR}}(D')^{\psi=0} & \longrightarrow & \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0} & \xrightarrow{\mathrm{pr}} & \mathbf{N}_{\mathrm{dR}}(D'')^{\psi=0} & \longrightarrow & 0 \end{array}$$

with exact lines. Also we know that the quotient $\mathbf{D}_{\infty,f}(D'')/\mathrm{pr}(\mathbf{D}_{\infty,f}(D))$ is $\Lambda_{\mathbb{Q}_p}$-torsion (cf. [35], 4.3). Hence, if one chooses a $\mathcal{H}(\Gamma_K)$-basis of $\mathcal{D}'_{\Lambda_\infty}(D'')$ there exist lifts in $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$.

Inductively, together with a $\mathcal{H}(\Gamma_K)$-basis for $\mathcal{D}'_{\Lambda_\infty}(D')$, this defines a free $\mathcal{H}(\Gamma_K)$-submodule $\mathcal{D}'_{\Lambda_\infty}(D)$ of $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ that contains $\mathbf{D}_{\infty,f}(D)$ and

fits into an exact sequence $0 \to \mathcal{D}'_{\Lambda_\infty}(D') \to \mathcal{D}'_{\Lambda_\infty}(D) \to \mathcal{D}'_{\Lambda_\infty}(D'') \to 0$.

Since we have not found a more canonical construction of such a module we try to give a first step of a definition of a related $\mathcal{H}(\Gamma_K)$-module $\mathcal{D}_{\Lambda_\infty}(D)$ below (resp. a Λ-module $\mathcal{D}_\Lambda(D) \subset \mathcal{D}_{\Lambda_\infty}(D)$) with explicit basis which also occurs when producing p-adic L-functions and which in certain cases coincides with $\mathcal{D}'_{\Lambda_\infty}(D)$.

We recall Perrin-Riou's construction of a free $\mathcal{H}(\Gamma_K)$-basis contained in $\mathbf{D}_{\infty,g}(D)$ (see [35], Theorem 4.2.1). Since the modules $\mathbf{D}_{\infty,*}(D)$ are isomorphic under twisting by ∂, one may assume $\|\psi\| < 1$ on $\mathbf{D}_{\mathrm{st}}(D)$. As usual we choose a basis $\{d_1, \dots d_n\}$ adapted to the operator N. Let S be the finite set of integers such that $D^{\varphi=p^{-k}} \neq 0$ for $k \in S$. Then [33], Proposition 2.2.1 shows that for

$$R(d_i) = \prod_{k\in S}(\chi(\gamma)^k\gamma - 1)\cdot(1+\pi)\otimes d_i \in (\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes \mathbf{D}_{\mathrm{st}}(D))^{\psi=0}$$

there exists a $\widetilde{R}(d_i) \in (\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes \mathbf{D}_{\mathrm{st}}(D))^{\psi=1}$ such that $(1-\varphi)\widetilde{R}(d_i) = R(d_i)$. Perrin-Riou's operator $\widetilde{N}_D$ (see [36], Theorem 3.2.1) allows to produce elements $\widetilde{N}_D(\widetilde{R}(d_i)) \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=1}$, and one shows by a recurrence argument that

$$(1-\varphi)(\widetilde{N}_D(\widetilde{R}(d_i))) \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$$

are actually contained in $\mathbf{D}_{\infty,g}(D)$ and form a free (as $\mathcal{H}(\Gamma_K)$-modules) system of rank $\dim_{\mathbb{Q}_p}\mathbf{D}_{\mathrm{st}}(D)$. By twising back with ∂ one obtains a basis $\{f_i\}$ for a free $\mathcal{H}(\Gamma_K)$-module for our original D.

Since $\mathrm{Frac}(\mathbb{Z}_p[[\Gamma_K]]) \subset \mathcal{B}(\Gamma_K)$ we may define:

Definition 5.1.1. Let D be semi-stable and $\{f_i\}$ as above. Let S be a finite set of integers that contains those integers k such that $\mathbf{D}_{\mathrm{st}}(D)^{\varphi=p^{-k}} \neq 0$. We define

$$g_i := \prod_{k\in S}(\chi(\gamma)^k\gamma - 1)^{-1} f_i \in \mathbf{N}_{\mathrm{dR}}(D)^{\psi=0},$$

so that $\{g_i\}$ form a free $\mathcal{H}(\Gamma_K)$-module $\mathcal{D}_{\Lambda_\infty}(D)$ contained in $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$.

Of course, the above construction highly depends on the choices of S and d_i (and the twist by ∂) which we implicitly suppress. Also, one may define $\mathcal{D}_{\Lambda_\infty}(D)$ for any (φ, N)-module D module and a choice of K.

One checks that if D is crystalline, then

$$\mathcal{D}_{\Lambda_\infty}(D) = \mathbf{D}_{\infty,f}(D) = (\mathbf{B}^+_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(D))^{\psi=0}, \tag{5.1}$$

so that the definition is independent of the choices.

For certain "compatible choices"[1] of S, d_i and so on one can state the following: Let $0 \to D' \to D \to D'' \to 0$ be an exact sequence of semistable (φ, Γ_K)-modules. Then the sequence

$$0 \longrightarrow \mathcal{D}_{\Lambda_\infty}(D') \longrightarrow \mathcal{D}_{\Lambda_\infty}(D) \longrightarrow \mathcal{D}_{\Lambda_\infty}(D'') \longrightarrow 0 \tag{5.2}$$

is exact. To prove this, we may assume by twisting that $\|\psi\| < 1$ on each $\mathbf{D}_{\mathrm{st}}(-)$. Also note that by the assumption on the modules one has an exact sequence $0 \to \mathbf{D}_{\mathrm{st}}(D') \to \mathbf{D}_{\mathrm{st}}(D) \to \mathbf{D}_{\mathrm{st}}(D'') \to 0$. We choose a compatible basis for $\mathbf{D}_{\mathrm{st}}(D)$ and S big enough and obtain a basis g_i of $\mathcal{D}_{\Lambda_\infty}(D)$. By a dévissage argument analoguous as in Lemma 4.3.5, since (5.1) holds for any crystalline (φ, Γ_K)-module, this basis shows the exactness of (5.2).

We give a formulation of the conjectures of [3], section 4.1., in our setting. One sets

$$\begin{aligned}\Delta_{\mathrm{PR}}(K_\infty/K, D) &= \det{}_{\mathcal{H}(\Gamma_K)} R\Gamma_{\mathrm{Iw}}(K, \overline{D}) \otimes \det{}_{\mathcal{H}(\Gamma_K)} \mathcal{D}_{\Lambda_\infty}(D) \\ &\cong \bigotimes_{i=1}^{2} (\det{}_{\mathcal{H}(\Gamma_K)} H^i_{\mathrm{Iw}}(K, \overline{D}))^{(-1)^i} \otimes \det{}_{\mathcal{H}(\Gamma_K)} \mathcal{D}_{\Lambda_{\mathbb{Q}_p}}(D),\end{aligned}$$

so that the big exponential map $\Omega_{D,h}$ for each suitable h induces a map $\delta'_{D,K_\infty/K,h} : \Delta_{\mathrm{Iw}}(K_\infty/K, T) \to \mathcal{H}(\Gamma_K)$. Hence, if one sets

$$\Gamma(i) = \begin{cases} \nabla_1^{-1} \cdot \nabla_2^{-1} \cdot \ldots \nabla_i^{-1}, & \text{if } i \geq 0 \\ \nabla_0 \cdot \nabla_{-1} \cdot \ldots \nabla_{i+1}, & \text{if } i < 0 \end{cases}$$

and $\Gamma(D) = \Gamma(h_1) \cdot \ldots \cdot \Gamma(h_d)$, one may consider the map

$$\delta'_{D,K_\infty/K} := \Gamma(D) \cdot \delta'_{D,K_\infty/K,h}$$

which one checks (as is done in [33], 3.3.2) is independent of the choice of h.

Proposition 5.1.2. One has $\delta'_{D,K_\infty/K}(\Delta_{\mathrm{PR}}(K_\infty/K, D)) = \mathcal{H}(\Gamma_K)$.

Proof. If D is crystalline, the claim follows from [30], Theorem 3.21. For general semistable D the claim follows from the compatibility with exact sequences for $\mathcal{D}_{\Lambda_\infty}(D)$ (5.2), $R\Gamma_{\mathrm{Iw}}(K, \overline{D})$ and the map $\Omega_{D,h}$. □

To proceed further, we assume for the rest of the section that D *is étale*. We expect that one can extend certain results to general (φ, Γ_K)-modules. So let $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ for some representation V of G_K with $K/\mathbb{Q}_p$ unramified.

[1] i.e., if one is given a choice for D' and D'' there exists a choice for D such that (5.2) holds

Next, we define the following $\mathcal{O}_F$-lattice $\mathbf{M}(D)$ of $\mathbf{D}_{\mathrm{st}}(V)$ for $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, where V is a *positive* semistable representation with lattice T, as

$$\mathbf{M}(D) = \{x \in (\mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{A}^{\dagger}_K} \mathbf{D}^{\dagger}(T))^{\Gamma_K} |\ \varphi^{-n}(x)(0) \in \mathcal{O}_{K_n} \otimes_{\mathbf{A}^{\dagger}_K, \varphi^{-n}} \mathbf{D}^{\dagger}(T)\ \forall n \gg 0\}.$$

Define the following subset $\mathcal{D}_{\Lambda}(D)$ of $\mathcal{D}_{\Lambda_\infty}(D)$:

$$\mathcal{D}_{\Lambda}(D) = \{x \in \mathcal{D}_{\Lambda_\infty}(D)|\ \varphi^{-n}(x)(0) \in \mathcal{O}_{K_n} \otimes_{\mathcal{O}_F} \mathbf{M}(D)\ \forall n \gg 0\}$$

The injection $\varphi^{-n} : \mathbf{B}^{\dagger, r_n}_{\mathrm{rig},K} \hookrightarrow K_n[[t]]$ is compatible with Γ_K, i.e. $\gamma a t^n = \gamma(a)\chi(\gamma)t^n$. We see that $\mathcal{D}_{\Lambda}(D)$ is a free Λ-submodule of $\mathcal{D}_{\Lambda_\infty}(D)$ of the same rank, by looking at the $\mathcal{H}(\Gamma_K)$-basis of $\mathcal{D}_{\Lambda_\infty}(D)$.

We immediatly have:

Lemma 5.1.3. If V is a positive crystalline representation then $\mathbf{M}(D)$ coincides with the lattice M of [3], section 3.2. As a consequence, one has $\mathcal{D}_{\Lambda}(T) = \Lambda \otimes_{\mathbb{Z}_p} M$.

We also set

$$\mathcal{D}_{\Lambda_{\mathbb{Q}_p}}(D) = \Lambda_{\mathbb{Q}_p} \otimes_{\Lambda} \mathcal{D}_{\Lambda}(D).$$

which is then a free $\Lambda_{\mathbb{Q}_p}$-module. It is clear that then $\mathcal{D}_{\Lambda_\infty}(D) = \mathcal{H}(\Gamma_K) \otimes_{\Lambda_{Q_p}} \mathcal{D}_{\Lambda_{\mathbb{Q}_p}}(D)$.

Similarly as before, one sets

$$\begin{aligned}\Delta_{\mathrm{PR}}(K_\infty/K, V) &= \det_{\Lambda_{\mathbb{Q}_p}} R\Gamma_{\mathrm{Iw}}(K, V) \otimes \det_{\Lambda_{\mathbb{Q}_p}} \mathcal{D}_{\Lambda_{\mathbb{Q}_p}}(D) \\ &\cong \bigotimes_{i=1}^{2} (\det_{\Lambda_{\mathbb{Q}_p}} H^i_{\mathrm{Iw}}(K, V))^{(-1)^i} \otimes \det_{\Lambda_{\mathbb{Q}_p}} \mathcal{D}_{\Lambda_{\mathbb{Q}_p}}(D),\end{aligned}$$

Conjecture 5.1.4. ($\delta_{\mathbb{Q}_p}(D)$) One has $\delta'_{D,K_\infty/K}(\Delta_{\mathrm{PR}}(K_\infty/K, D)) = \Lambda_{\mathbb{Q}_p}$.

If $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ is étale with lattice $T \subset V$ one sets for the integral version

$$\begin{aligned}\Delta_{\mathrm{Iw}}(K_\infty/K, T) &= \det_{\Lambda} R\Gamma_{\mathrm{Iw}}(K, T) \otimes \det_{\Lambda}(\mathrm{Ind}_{K_\infty/\mathbb{Q}_p} T) \\ &\cong \bigotimes_{i=1}^{2} (\det_{\Lambda} H^i_{\mathrm{Iw}}(K, T))^{(-1)^i} \otimes \det_{\Lambda}(\mathrm{Ind}_{K_\infty/\mathbb{Q}_p} T)\end{aligned}$$

We note that we have not yet found a good description of a conjecture in the style of conjecture 4.1.3. in [3] for semistable V. However, the conjecture in loc.cit. implies an integral version of $\delta_{\mathbb{Q}_p}$ (see also [33], section 3.4.8, specifically equation $\delta(\Omega^{\varepsilon}_V(\Delta_\infty(\underline{T}, M)^{-1}) = \Lambda)$, which we may state in our setting as follows:

Conjecture 5.1.5. ($\delta_{\mathbb{Z}_p}(T)$) One has $\delta'_{D,K_\infty/K}(\Delta_{\mathrm{Iw}}(K_\infty/K, T)) = \Lambda$.

We shall establish these conjectures in the example of an ordinary semistable elliptic curve defined over $\mathbb{Q}_p$ after the next section.

5.2 Coleman maps

Let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. For $y \in \mathbf{N}_{\mathrm{dR}}(D)$ Perrin-Riou considers the following map:

$$\mathcal{L}_{h,y} : D^*(1)^{\psi=1} \longrightarrow \mathcal{H}(\Gamma_K)$$
$$v \longmapsto \langle \Omega_{D,h}(y), v \rangle_{\mathrm{Iw},D}$$

If h is big enough we know that $\mathcal{L}_{D,h}(v) \in \mathbf{N}_{\mathrm{dR}}(D^*(1))$, hence in the cases where the reciprocity law (cf. Theorem 4.4.4 and Theorem 4.5.1) holds this may be rewritten as follows:

$$\mathcal{L}_{h,z}(y) = \langle \Omega_{D,h}(z), \Omega_{D^*(1),1-h}(\mathcal{L}_{D,h}(y)) \rangle_{\mathrm{Iw},D} = (-1)^{h+1}[z, \iota(\mathcal{L}_{D,h}(\sigma_{-1}y))]_{\mathrm{Iw},D}. \quad (5.3)$$

Also we know that $\mathcal{L}_h((\varphi-1)\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))^{\psi=1})$ is contained in $\mathbf{D}_{\infty,g}(V^*(1))$. The general process of how one would like to be able to produce p-adic L-functions in the case of a p-adic representation then may be summed up as follows:

a) Find a "good" Euler-system $y \in H^1_{\mathrm{Iw}}(K, T^*(1))$,

b) Fix a $\mathcal{H}(\Gamma_K)$-basis (y_i) of $\mathcal{D}_{\Lambda_\infty}(V)$,

c) Project the element $\mathcal{L}_{D,h}(v)$ in $\mathcal{D}_{\Lambda_\infty}(V^*(1))$ along the basis z_i to produce p-adic L-functions (up to the operators ι and σ_{-1}).

Recall that $(\varphi - 1)\mathbf{D}(T)^{\psi=1}$ is a free Λ-module of rank $[K : \mathbb{Q}_p] \cdot \dim V$. Suppose we have fixed a finitely generated free Λ-module $\mathcal{D} \subset \mathcal{D}_{\Lambda_\infty}(V^*(1))$ with basis $(z_i)_i$ such that $\mathcal{L}_h((\varphi - 1)\mathbf{D}(T^*(1))^{\psi=1}) \subset \mathcal{D}$. Define the projection

$$\underline{\mathrm{Col}}_i : \mathbf{D}(T^*(1))^{\psi=1} \overset{\mathcal{L}_h\circ(\varphi-1)}{\longrightarrow} \mathcal{D} \cong \bigoplus_i \Lambda \cdot z_i \overset{\mathrm{pr}_i}{\longrightarrow} \Lambda$$

We use the same notation for the corresponding map $\mathbf{D}(V^*(1))^{\psi=1} \to \Lambda_{\mathbb{Q}_p}$, which is obtained by base-change.

5.3 Musings in dimension 2

5.3.1 The setup

We shall give some illustrations in the dimension 2 case, which is the smallest case where it is possible to have something semi-stable that is not crystalline. So assume $K = \mathbb{Q}_p$ and let D be a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ of rank two. Then $\mathbf{D}^K_{\mathrm{st}}(D) = M$ is a filtered (φ, N)-module of dimension two over F. We assume $N \neq 0$ on $\mathbf{D}^K_{\mathrm{st}}(D)$ and fix a basis $d_1,\ d_2$ of $\mathbf{D}^K_{\mathrm{st}}(D)$ that is adapted to the operator N, i.e.

$$N(d_1) = d_2,\ N(d_2) = 0 \implies \varphi(d_1) = \alpha d_1, \varphi(d_2) = p^{-1}\alpha d_2, \quad \alpha \in \mathbb{Q}_p^\times$$

The dual basis d_1^*, d_2^* for $\mathbf{D}_{\mathrm{st}}^K(D^*(1))$ therefore has the properties (recall that $N\log\pi = -1$)

$$N(d_2^*) = -d_1^*,\ N(d_1^*) = 0 \implies \varphi(d_1^*) = \alpha^{-1}d_1^*,\ \varphi(d_2^*) = p\alpha^{-1}d_2^*.$$

With this one has canonical $\mathbf{B}^\dagger_{\mathrm{rig},K}$-bases

$$1 \otimes d_1 - \log\pi \otimes d_2,\quad 1 \otimes d_2$$

for $\mathbf{N}_{\mathrm{dR}}(D)$ resp.

$$1 \otimes d_2^* + \log\pi \otimes d_1^*,\quad 1 \otimes d_1^*$$

for $\mathbf{N}_{\mathrm{dR}}(D^*(1))$. Similarly, one has canonical $\mathcal{B}(\Gamma_K)$-bases

$$\mathcal{E}((1+\pi)\otimes d_1) = (1+\pi)\otimes d_1 - (1+\pi)p^{-1}\varphi\log\pi\otimes d_2,\quad \mathcal{E}((1+\pi)\otimes d_2) = (1+\pi)\otimes d_2$$

for $\mathbf{N}_{\mathrm{dR}}(D)^{\psi=0}$ resp.

$$\mathcal{E}((1+\pi)\otimes d_2^*) = (1+\pi)\otimes d_2^* + (1+\pi)p^{-1}\varphi\log\pi\otimes d_1^*,\quad \mathcal{E}((1+\pi)\otimes d_1^*) = (1+\pi)\otimes d_1^*$$

for $\mathbf{N}_{\mathrm{dR}}(D^*(1))^{\psi=0}$.

The setting where we shall look at explicit examples comes from an *ordinary* semi-stable elliptic curve E with bad reduction. Hence, we are to consider the cases of split resp. non-split multiplicative reduction. It is known that in this case that for the $\mathbb{Z}_p$ representation T associated to E one already has an exact sequence

$$0 \longrightarrow T_1 \longrightarrow T \longrightarrow T_2 \longrightarrow 0$$

with T_i one-dimensional $\mathbb{Z}_p$ representations. The dual sequence is written as follows:

$$0 \longrightarrow T_2^*(1) \longrightarrow T^*(1) \longrightarrow T_1^*(1) \longrightarrow 0. \tag{5.4}$$

As usual if T' is a p-adic representation over $\mathbb{Z}_p$ we write $V' = T'\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ for the associated p-adic representation. One has a decomposition $\mathbb{C}_p\otimes_{\mathbb{Q}_p} V = \mathbb{C}_p \oplus \mathbb{C}_p(1)$ as representations, so that V has Hodge-Tate weights $0, 1$. The same holds for the dual representation $V^*(1)$. For the (φ, N)-modules this implies the following exact sequence:

$$0 \longrightarrow \mathbb{Q}_p\cdot d_2 \longrightarrow \mathbf{D}_{\mathrm{st}}(V) \longrightarrow \mathbb{Q}_p\cdot d_1 \longrightarrow 0, \tag{5.5}$$

such that $\alpha = 1$ in the split-case, and $\alpha = -1$ in the non-split case. For the $\mathbf{A}_{\mathbb{Q}_p}$-modules this induces an exact sequence

$$0 \longrightarrow \mathbf{A}_{\mathbb{Q}_p}(1) \longrightarrow \mathbf{D}(T) \longrightarrow \mathbf{A}_{\mathbb{Q}_p} \longrightarrow 0$$

and by the property that $\mathbf{A}_{\mathbb{Q}_p}$ is a discrete valuation ring we see that a basis of $\mathbf{D}(\mathbb{Z}_p(1))$ lifts to a basis of $\mathbf{D}(T)$. Thus there exists a basis $(d_i)_{i=1,2}$ of $\mathbf{D}(T)$ such that the following is satisfied:

a) d_1 is a $\mathbf{A}_{\mathbb{Q}_p}$-basis for $\mathbf{D}(\mathbb{Z}_p(1))$,

b) $((1+\pi)\varphi(d_i))$ is a Λ-basis for $(\varphi-1)\mathbf{D}(T)^{\psi=1}$,

c) $(1+\pi)\varphi(d_1)$ is a Λ-basis for $(\varphi-1)\mathbf{D}(T_1)^{\psi=1}$.

We shall illustrate that $\mathbf{D}_{\infty,f}$ is not exact in general. Assume that we are in the case of a representation arising from a split-multiplicative semi-stable elliptic curve, so that $\alpha=1$ and the sequence (5.5) translates to

$$0 \longrightarrow \mathbf{D}_{\mathrm{st}}(\mathbb{Q}_p(1)) \longrightarrow \mathbf{D}_{\mathrm{st}}(V) \longrightarrow \mathbf{D}_{\mathrm{st}}(\mathbb{Q}_p) \longrightarrow 0.$$

We apply Lemma 3.2.13 and use the notation there: one easily checks that $N=\mathbb{Q}_p t^{-1}$, $P=0$ so that M is a one-dimensional $\mathbb{Q}_p$-vectorspace. This implies that in this case one has an exact sequence

$$0 \longrightarrow \mathbf{D}_{\infty,f}(\mathbb{Q}_p(1)) \longrightarrow \mathbf{D}_{\infty,f}(V) \longrightarrow \mathbf{D}_{\infty,f}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p \longrightarrow 0,$$

(cf. also [36]), and we see that $\mathbf{D}_{\infty,f}$ is not exact.

Let us describe an explicit $\mathcal{H}(\Gamma_{\mathbb{Q}_p})$-basis (following [35]) for $\mathcal{D}_{\Lambda_\infty}(V)$ in the cases $\alpha=\pm 1$ which we shall use in the example of an ordinary elliptic curve. Let $f=(\gamma-1)\cdot(1+\pi)\in\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and consider $f\otimes\overline{d}_1\in\mathbf{D}_{\infty,f}(\mathbb{Q}_p\cdot\overline{d}_1)$. Then there exists an $F_1\in\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ such that $(1-\varphi)F_1=f$. Further, Perrin-Rious operator $\widetilde{\mathcal{N}}_M$ (see [35], Theorem 3.2.1) shows that there exists an $F_2\in\mathbf{B}^\dagger_{\log,\mathbb{Q}_p}$ such that $\psi(F_2)=p^{-1}F_2$ and $NF_2=F_1$. Hence, the element

$$(1-\varphi)(F_1\cdot d_1-F_2\cdot d_2)=(\gamma-1)\cdot(1+\pi)\cdot d_1-G\cdot d_2$$

is a lifting of $f\otimes\overline{d}_1$ in $\mathbf{D}_{\infty,f}(V)$. With this, the elements $(1+\pi)\otimes d_2$, $(1+\pi)\otimes d_1-(\gamma-1)^{-1}\cdot G\otimes d_2$ form a $\mathcal{H}(\Gamma_K)$-basis for $\mathcal{D}_{\Lambda_\infty}(V)$.

Returning to the properties of the determinant for V, we see that Conjectures 5.1.4 and 5.1.5 are by the properties of the determinant compatible with exact sequences of semi-stable representations, since our module $\mathcal{D}(T)$ is defined precisely with such compatibility in mind. Since these are known to be true for T_1 and T_2 by the work of Benois and Berger ([3]) (since these representations are crystalline), the conjectures are also true for T and V.

5.3.2 i-Selmer groups and torsion property

Let l,p be prime numbers and suppose $K/\mathbb{Q}_l$ is finite. Let V be p-adic representation of G_K and fix a $\mathbb{Z}_p$-lattice T that is stable under this action. One defines $\mathbb{Q}_p$-subspaces

$$H^1_e(K,V)\subset H^1_f(K,V)\subset H^1_g(K,V)\subset H^1(K,V)$$

and $\mathbb{Z}_p$-modules

$$H^1_e(K,T)\subset H^1_f(K,T)\subset H^1_g(K,T)\subset H^1(K,T)$$

depending on whether $l\neq p$ or $l=p$, cf. [11], section 3, (3.7.1), (3.7.2), (3.7.3). Note that then $H^1_*(K,T)$ always contains the torsion subgroup of $H^1(K,T)$.

We now switch to a different notation. Let K be a number field and let V be a p-adic representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Suppose T is a $\mathbb{Z}_p$-lattice stable under $G_{\mathbb{Q}}$. One has the usual definitions for the (continuous) Galois cohomology for G_K and G_{K_ν} for every place ν of K.

Consider the perfect pairing

$$\cup : H^1(K_{\nu,n}, T) \times H^1(K_{\nu,n}, V^*/T^*(1)) \to \mathbb{Q}_p/\mathbb{Z}_p,$$

where T^* is the set of elements $f \in V^*$ such that $f(T) \subset \mathbb{Z}_p$. By going to the projective limit one obtains a perfect duality

$$\cup_{\mathrm{Iw}} : H^1_{\mathrm{Iw}}(K_\nu, T) \times H^1(K_{\nu,\infty}, V^*/T^*(1)) \to \mathbb{Q}_p/\mathbb{Z}_p. \tag{5.6}$$

With $\cup$ one defines

$$H^1_f(K_{\nu,n}, V^*/T^*(1)) := \{x \in H^1(K_{\nu,n}, V^*/T^*(1))|\ x \cup y = 0\ \forall y \in H^1_f(K_{\nu,n}, T)\}.$$

The p-Selmer group of V over K is defined via

$$\mathrm{Sel}_p(K, V) = \mathrm{Ker}\left(H^1(K, V^*/T^*(1))) \to \prod_\nu \frac{H^1(K_\nu, V^*/T^*(1)}{H^1_f(K_\nu, V^*/T^*(1)}\right),$$

where ν runs over all places of K. One defines the p^∞-Selmer group as $\mathrm{Sel}_p(K_\infty, V) = \varinjlim_n \mathrm{Sel}_p(K_n, V)$.

Following [28], we now define the i-th Selmer groups corresponding to the projection $\underline{\mathrm{Col}}_i$. Similarly as before one defines

$$H^1_f(K_n, V^*/T^*(1))^i := \{x \in H^1(K_n, V^*/T^*(1))|\ x \cup y = 0\ \forall y \in \ker(\mathrm{pr}_n(\underline{\mathrm{Col}}_i))\}$$

and we set

$$\mathrm{Sel}^i_p(K_n, V) := \mathrm{Ker}\left(\mathrm{Sel}_p(K_n, V) \longrightarrow \frac{H^1(K_n, V^*/T^*(1))}{H^1(K_n, V^*/T^*(1))^i}\right),$$

so that one may form $\mathrm{Sel}^i_p(K_\infty, V) := \varinjlim_n \mathrm{Sel}^i_p(K_n, V)$. By using the pairing $\cup_{\mathrm{Iw}}$ of (5.6) one sees that if one forms

$$H^1_f(K_\infty, V^*/T^*(1))^i = \{x \in H^1(K_\infty, V^*/T^*(1))|\ x \cup_{\mathrm{Iw}} y = 0\ \forall y \in \ker(\underline{\mathrm{Col}}_i)\},$$

then

$$\mathrm{Sel}^i_p(K_\infty, V) = \left(\mathrm{Sel}_p(K_\infty, V) \longrightarrow \frac{H^1(K_\infty, V^*/T^*(1))}{H^1(K_\infty, V^*/T^*(1))^i}\right).$$

We wish to apply this to the example of an ordinary semi-stable elliptic curve. So let V be as in section 5.2.1 and consider the representation $\overline{T} = T^*(1)$, and write $\overline{T}_1 = T^*_1(1)$, $\overline{T}_2 = T^*_2(1)$ (cf. (5.4)).

For $\overline{T}_2$ one has $\mathbf{D}_{\mathrm{st}}(\overline{T}_2(-1)) = (\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}^\dagger(\overline{T}_2(-1)))^{\Gamma_K}$, hence, thanks to the identification $t^h\mathbf{D}_{\mathrm{st}}(W) = \mathbf{D}_{\mathrm{st}}(W(-h))(h)$, which holds for any semi-stable representation

W, and the fact that $\overline{T}_2(-1)$ is a $\mathbb{C}_p$-admissible representation, an injection $\mathbf{D}^\dagger_{\mathrm{rig}}(\overline{V}_2) \subset \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes \mathbf{D}_{\mathrm{cris}}(\overline{V}_2) = \mathbf{N}_{\mathrm{dR}}(\overline{V}_2)$. The second exact sequence of Proposition 3.3.2 shows that $(\varphi-1)\mathbf{N}_{\mathrm{dR}}(\overline{V}_2)^{\psi=1} \subset (\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbf{D}_{\mathrm{cris}}(\overline{V}_2)$. Hence, one may choose d_1^* in such a way that there exists an $f_1 \in (\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ such that $(\varphi-1)\mathbf{D}(\overline{T}_2)^{\psi=1}$ is contained in the free rank one Λ-module $\Lambda \cdot f \otimes d_1^* \subset \mathcal{H}(\Gamma_K) \cdot (1+\pi) \otimes d_1^*$.

As for $\overline{T}_1$, a similar argument as above shows that $(\varphi-1)\mathbf{D}(\overline{T}_1)^{\psi=1}$ is contained in $\mathbf{D}_{\infty,g}(\overline{V}_1)$. Hence, one may choose d_2^* in such a way that there exists an $f_2 \in (\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ and $\lambda \in \mathrm{Frac}(\Lambda)$ such that $(\varphi-1)\mathbf{D}(\overline{T}_1)^{\psi=1}$ is contained in the free rank-one Λ-module $\Lambda \cdot (\lambda \cdot f_2) \otimes \overline{d}_2^* \subset \mathbf{D}_{\infty,g}(\overline{V}_1)$.

The 5-lemma and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\varphi-1)\mathbf{D}(\overline{T}_2)^{\psi=1} & \longrightarrow & (\varphi-1)\mathbf{D}(\overline{T})^{\psi=1} & \longrightarrow & (\varphi-1)\mathbf{D}(\overline{T}_1)^{\psi=1} & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \Lambda \cdot f_1 \otimes d_1^* & \longrightarrow & \mathcal{D}'_\Lambda(\overline{T}) & \longrightarrow & \Lambda \cdot (\lambda \cdot f_2) \otimes \overline{d}_2^* & \longrightarrow & 0,
\end{array}$$

where $\mathcal{D}'_\Lambda(\overline{T})$ is the free Λ-submodule of $\mathcal{K}(\Gamma_K) \otimes_{\mathcal{H}(\Gamma_K)} (\varphi-1)\mathbf{N}_{\mathrm{dR}}(\overline{T})^{\psi=1}$ determined by the above exact sequence, show that $(\varphi-1)\mathbf{D}(\overline{T}_1)^{\psi=1}$ injects into $\mathcal{D} := \mathcal{D}'_\Lambda(\overline{T})$. Thus we may consider the Coleman-map $\underline{\mathrm{Col}}_i$ for a lift of the basis vector $\lambda \cdot f_2 \otimes \overline{d}_2^*$. Let us relate this map to the map $\mathcal{L}_{h,(1+\pi)\otimes d_2}$. By (5.3), since $h=1$, $\underline{\mathrm{Col}}_2$ equals $\mathcal{L}_{h,(1+\pi)\otimes d_2}$ up to a -1, and an application of $\sigma_{-1} \cdot f'$ for some non-zero $f' \in \mathrm{Frac}(\Lambda)$ and the involution ι on Λ.

Lemma 5.3.1. If $y \in \mathbf{D}(\overline{T})^{\psi=1}$ then $\langle \Omega_h((1+\pi)\otimes d_2), y\rangle_{\mathrm{Iw},V} \in \Lambda$.

Proof. One has $\Omega_1((1+\pi)\otimes d_2) = t(1+\pi)\otimes d_2$. With our choice of d_2 this implies, similarly as in the discussion for $\overline{T}$, that $\Omega_1((1+\pi)\otimes d_2) \in (\varphi-1)\mathbf{D}(T_1)^{\psi=1}$. Since $\langle\ ,\ \rangle_{\mathrm{Iw},V}$ is induced as a pairing on the Λ-modules $\mathcal{C}(T) \times \mathcal{C}(T^*(1))$, the claim follows. □

Lemma 5.3.2. One has the inclusion $\mathrm{pr}_n(\ker(\underline{\mathrm{Col}}_2)) \subset H^1_f(\mathbb{Q}_{p,n}, T)$.

Proof. Since $\overline{T}_2$ is the biggest subrepresentation of $\overline{T}$ such that the Hodge-Tate weights of it are greater or equal to 1 (it is in fact a one-dimensional representation with Hodge-Tate weight 1), one has $\mathrm{Fil}^1\overline{T} = \overline{T}_2$.

It then follows from the definition that $\ker(\underline{\mathrm{Col}}_2) = \mathbf{D}(\overline{T})^{\varphi=1} + \mathbf{D}(\mathrm{Fil}^1\overline{T})^{\psi=1}$. The torsion part $\mathbf{D}(T)^{\varphi=1}$ is mapped to the trivial cocycle in $H^1(\mathbb{Q}_{p,n}, \overline{T})$. Thus we need to concern ourselves with the $\mathbf{D}(\mathrm{Fil}^1\overline{T})^{\psi=1}$-part. [6], Theorem A shows that $H^1_{\mathrm{Iw}}(K, \mathrm{Fil}^1\overline{V}) \subset H^1_{\mathrm{Iw},f}(K, \overline{V})$. The commutativity of the diagram

$$\begin{array}{ccc}
H^1_{\mathrm{Iw}}(K, \overline{T}) & \xrightarrow{\mathrm{pr}_n} & H^1(K_n, \overline{T}) \\
\big\downarrow \otimes\mathbb{Q}_p & & \big\downarrow \otimes\mathbb{Q}_p \\
H^1_{\mathrm{Iw}}(K, \overline{V}) & \xrightarrow{\mathrm{pr}_n} & H^1(K_n, \overline{V}).
\end{array}$$

implies that one has $H^1_{\mathrm{Iw}}(K, \mathrm{Fil}^1\overline{T}) \subset H^1_{\mathrm{Iw},f}(K, \overline{T})$, hence the claim. □

Lemma 5.3.3. One has $H^1_f(\mathbb{Q}_{p,n}, V^*/T^*(1)) \subset H^1_f(\mathbb{Q}_{p,n}, V^*/T^*(1))^2$ for every n.

Proof. One sees that

$$H^1_f(\mathbb{Q}_{p,n}, V^*/T^*(1)) = \ker(H^1(\mathbb{Q}_{p,n}, V^*/T^*(1)) \to \mathrm{Hom}(H^1_f(\mathbb{Q}_{p,n}, T), \mathbb{Q}_p/\mathbb{Z}_p))$$
$$y \mapsto (- \cup y)$$

and similarly for $H^1_f(\mathbb{Q}_{p,n}, V^*/T^*(1))^2$ with $\mathrm{pr}_n(\ker(\underline{\mathrm{Col}}_2))$ in place of $H^1_f(\mathbb{Q}_{p,n}, T)$. Hence, the claim follows from the previous lemma. □

Putting it all together we have shown that $\mathrm{Sel}^2_p(K_\infty, V) = \mathrm{Sel}_p(K_\infty, V)$, the usual Selmer group associated to V.

Proposition 5.3.4. Assume we are in the above setting. Let $\mathbf{z}$ be Kato's zeta element and assume further that $\underline{\mathrm{Col}}_2(\mathbf{z})^\eta \neq 0$ for every character η of Δ. Then $\mathrm{Sel}_p(K_\infty, V)$ is Λ-cotorsion.

Proof. As in [28], section 6.1. (60), we have an exact sequence

$$\mathbb{H}^1(\overline{T}) \longrightarrow \Lambda \longrightarrow \mathrm{Sel}_p(K_\infty, V)^\vee \longrightarrow \mathbb{H}^2(\overline{T}) \longrightarrow 0$$

Since $\underline{\mathrm{Col}}_2(\mathbf{z})^\eta \neq 0$ for η of $\widehat{\Delta}$ the cokernel of the first map in the above sequence is Λ-torsion. Since $\mathbb{H}^2(\overline{T})$ is also Λ-torsion, the same holds for $\mathrm{Sel}_p(K_\infty, V)$. □

5.4 p-adic Lie-group case

In the last section we wish to give a short description of generalizations of the algebras $\mathcal{H}(\Gamma_K)$ and $\mathcal{B}(\Gamma_K)$ inspired by the work of Schneider/Venjakob ([39]). We recently learned that Zábrádi ([42]) has defined (φ, Γ)-modules over non-commutative Robba rings.

Let G be a compact p-adic Lie group, $\mathcal{O}$ the ring of integers of any finite extension of $\mathbb{Q}_p$ with residue field κ. Write

$$\Lambda(G) = \varprojlim_{U \subset G \text{ open}} \mathcal{O}[G/U]$$

for the completed group algebra of G with coefficients in $\mathcal{O}$. We assume that G has a closed normal subgroup H such that $G/H =: \Gamma \cong \mathbb{Z}_p$. We fix once and for all a topological generator γ of Γ. It is then known that the group algebra $R = \Lambda(H)$ is compact, and we set $X = \gamma - 1$, $\sigma(r) := \gamma r \gamma^{-1}$ for $r \in R$ and $\delta = \sigma - \mathrm{id}$. The σ-derivation δ is topologically nilpotent and hence σ-nilpotent. Hence, for any $k \geq 1$ there exists an $m \geq q$ such that $\delta^m(R) \subset \mathrm{Jac}(R)^k$. The topological ring

$$B = R\ll X; \sigma, \delta]]$$

exists (for all this, confer [39]).

We can fix a norm on R by setting

$$|a| := p^{-k} \text{ if } a \in \mathrm{Jac}(R)^k \setminus \mathrm{Jac}(R)^{k+1}$$

which defines the pseudocompact topology, i.e. it is a function $|\ | : R \to \mathbb{R}_{\geq 0}$ which satisfies the axioms

i) $|a - b| \leq \max(|a|, |b|)$,

ii) $|a| = 0 \iff a = 0$,

iii) $|ab| \leq |a||b|$,

iv) $|1| = 1$

v) $|a| \leq 1$

for $a, b \in R$. Additionally we assume our norm satisfies

vi) $|\sigma(a)| = a$ for $a \in R$,

vii) there exists a $0 \leq D < 1$ such that $|\delta(a)| \leq D|a|$ for $a \in R$.

With this the ring B can explicitely be described as

$$B = \left\{ \sum_{n\in\mathbb{Z}} a_n X^n \;\middle|\; a_n \in R, \lim_{n\to-\infty} |a_n| = 0 \right\}$$

We shall also extend the above norm to $R[1/p]$ via $|1/p| = |p|^{-1}$. This gives a ring equipped with a norm that still satisfies the above axioms except v).

Schneider and Venjakob define a left R-submodule of $B^\dagger(|\ |) = B^\dagger$ of B (we usually supress the dependency on the norm in the notation since all norms defined via "ideal-norms" are equivalent), defined as follows. First for any $D < u < 1$ let

$$B^{\dagger,u} := \left\{ \sum_{i\in\mathbb{Z}} a_i X^i \in B \;\middle|\; \lim_{i\to-\infty} |a_i| u^i = 0 \right\}.$$

On the rings $B^{\dagger,u}$ one can define a norm $|\cdot|_u$ for $f = \sum_n a_n X^n \in B^{\dagger,u}$

$$|f|_u := \sup_n |a_n| u^n.$$

Next, let $B^\dagger = \bigcup_{D<u<1} B^{\dagger,u}$.

Definition 5.4.1. a) For $D < u < 1$, let

$$\mathcal{B}^{[u,1[}(G) := B^{\dagger,u}_{\mathrm{rig}} := \left\{ \sum_{n\in\mathbb{Z}} a_n X^n \;\middle|\; a_n \in R[1/p], \lim_{n\to\pm\infty} |a_n| r^n = 0 \ \forall u \leq r < 1 \right\}.$$

Further we define

$$\mathcal{B}(G) := B^{\dagger}_{\mathrm{rig}} := \bigcup_{D<u<1} B^{\dagger,u}_{\mathrm{rig}}.$$

b) Similarly, define

$$\mathcal{H}(G) := \left\{ \sum_{n\in\mathbb{N}} a_n X^n \;\middle|\; a_n \in R[1/p], \lim_{n\to+\infty} |a_n| r^n = 0 \ \forall 0 \leq r < 1 \right\}.$$

On the rings $\mathcal{B}^{[u,1[}(G)$ one can define a family of norms $|\cdot|_r$ for $r \in [u,1[$ and $f = \sum_n a_n X^n \in \mathcal{B}^{[u,1[}(G)$ via

$$|f|_r := \sup_n |a_n| r^n.$$

We equip $\mathcal{B}^{[u,1[}(G)$ with the Fréchet topology with respect to these norms.

As in the paper of Schneider and Venjakob one can work out the following formulas: For this let $M_{k,l}(Y,Z)$ be the sum of all noncommutative monomials in two variables Y, Z with k factors Y and l factors Z. Then

$$\left(\sum_{j\in\mathbb{Z}} a_j X^j\right) \cdot \left(\sum_{l\in\mathbb{Z}} b_l X^l\right) = \sum_{m\in\mathbb{Z}} c_m X^m \tag{5.7}$$

with

$$c_m := c_m^+ + c_m^-, \tag{5.8}$$

$$c_m^+ := \sum_{j\geq n\geq 0} a_j M_{j-n,n}(\delta,\sigma)(b_{m-n}) \tag{5.9}$$

$$c_m^- := \sum_{n\leq j<0} a_j \sigma' M_{j-n,-1-j}(\delta',\sigma')(b_{m-n}). \tag{5.10}$$

Lemma 5.4.2. Let $x = \sum_{j\in\mathbb{Z}} a_j X^j$, $y = \sum_{l\in\mathbb{Z}} b_l X^l$ in $\mathcal{B}^{[u,1[}(G)$ and put $xy = \sum_{m\in\mathbb{Z}} c_m X^m$ with $c_m = c_m^+ + c_m^-$ as in (5.7) - (5.10). Then $xy \in \mathcal{B}(G)$.

Proof. Let $u \leq r < 1$ and suppose $\varepsilon > 0$. We have to check that $|c_m| r^m \leq \varepsilon$ for $m \gg 0$ and $m \ll 0$. We have

$$|c_m^+| r^m \leq \sup_{j\geq n\geq 0} |a_j| r^j \cdot |b_{m-n}| r^{m-n} \cdot \left(\frac{D}{r}\right)^{j-n}$$

$$|c_m^-| r^m \leq \sup_{n\leq j<0} |a_j| r^j \cdot |b_{m-n}| r^{m-n} \cdot \left(\frac{D}{r}\right)^{j-n}$$

a) The case $c_m^+, m \to -\infty$. One has

$$|c_m^+| r^m \leq \sup_{j\geq n\geq 0} |a_j| r^j \cdot |b_{m-n}| r^{m-n} \leq \sup_{n\geq 0} |x|_r \cdot |b_{m-n}| r^{m-n}$$

From this it is clear that $\lim_{m\to-\infty} |c_m^+| r^m = 0$.

b) The case $c_m^-, m \to -\infty$. There exist $N_0, N_1, N_2 > 0$ such that for $j \leq n$

$$|a_j| r^j \cdot |b_{m-n}| r^{m-n} \cdot \left(\frac{D}{r}\right)^{j-n} \leq$$

$$\begin{cases} |x|_r |y|_r (D/r)^{j-n} \leq \varepsilon & \text{for } j-n \geq N_1 & \text{since } D/r < 1, \\ |a_j| u^j \cdot |y|_r \leq \varepsilon & \text{for } j \leq -N_0 & \text{since } \lim_{j\to-\infty} |a_j| r^j = 0 \\ |x|_r \cdot |b_{m-n}| u^{m-n} \leq \varepsilon & \text{for } m-n \leq -N_2 & \text{since } \lim_{j\to-\infty} |b_j| r^j = 0 \end{cases}$$

Now $j-n \leq N_1$, $j \leq -N_0$, $m-n \leq -N_2$ imply $m \leq -N_0 - N_1 - N_2$, hence $|c_m^{\pm}| < \varepsilon$.

c) The case $c_m^+, m \to +\infty$. One has

$$|c_m^+|r^m \leq \sup_{j \geq n \geq 0} |a_j|r^j \cdot |b_{m-n}|r^{m-n}.$$

Now if m is big enough and n small enough we can estimate $|a_j|r^j \cdot |b_{m-n}|r^{m-n} < \varepsilon$ since $\lim_{j\to+\infty} |b_j|r^j = 0$. Analoguously if n is big enough since this forces j big and since $\lim_{j\to-\infty} |a_j|r^j = 0$.

d) The case $c_m^-, m \to +\infty$. One has

$$|c_m^-|r^m \leq \sup_{n<0} |x|_r \cdot |b_{m-n}|r^{m-n}$$

It is clear that $\lim_{m\to+\infty} |c_m^+|r^m = 0$.

□

We have defined a ring structure on $\mathcal{B}(G)$. We now check that the topology behaves as one would hope. First note that the inclusions $\mathcal{B}^{[u,1[}(G) \subset \mathcal{B}^{[v,1[}(G)$ for $u \leq v$ are compatible with the Fréchet topology, hence $\mathcal{B}(G)$ is equipped with the natural inductive limit topology, which we shall also refer to as the Fréchet topology.

We remark that it is possible to prove that $B^{\dagger,u}[1/p]$ as a subring of $B_{\mathrm{rig}}^{\dagger,u}$ is dense with respect to the Fréchet topology. One can also show that $\mathcal{B}^{[u,1[}(G)$ is complete for the Fréchet topology.

Finally we mention that with these definitions in mind the goal is to define completions Λ_n for G as in section 2.6.2 and cohomology groups $H^1(K, D\widehat{\otimes}_{\mathbb{Q}_p}\Lambda_n[1/p])$ following Definition 2.6.5 so that one may form $H^1_{\mathrm{an}}(K,D) = \varprojlim_n H^1(K, D\widehat{\otimes}_{\mathbb{Q}_p}\Lambda_n[1/p])$ and an exponential map

$$\mathcal{H}(G) \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{st}}(D) \longrightarrow H^1_{\mathrm{an}}(K,D)/H^1_{\mathrm{an}}(K,D)_{\mathrm{tor}}$$

that interpolates Bloch-Kato exponential maps in the finite levels.

List of notations

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