DISSERTATION

submitted

to the

Combined Faculties for the Natural Sciences and for Mathematics

of the

Ruperto-Carola University of Heidelberg, Germany

for the degree of

Doctor of Natural Sciences

Put forward by

Master of Science in Mathematics: Aprameyo Pal

Born in: Santiniketan, India

Oral examination:——————

Title

Functional Equation of Characteristic Elements of Abelian Varieties over Function Fields

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Abstract

In this thesis we apply methods from the Number field case of Perrin-Riou [PR03] & Zabradi [Záb08] in the Function field set up. In $\mathbb{Z}_l$- and GL_2-case ($l \neq p$), we prove algebraic functional equations of the Pontryagin dual of Selmer group which give further evidence of the Main conjectures of Iwasawa Theory. We also prove some parity conjectures in commutative and non-commutative cases. As consequence, we also get results on the growth behaviour of Selmer groups in commutative and non-commutative extension of Function fields.

Zusammenfassung

In dieser Arbeit wenden wir Methoden vom Zahlenkörperfall von Perrin-Riou [PR03] & Zabradi [Záb08] auf die Funktionenkörper-Situation an. Im $\mathbb{Z}_l$- and GL_2-Fall ($l \neq p$) beweisen wir algebraische Funktionalgleichungen des Pontryagin-duals der Selmer-gruppe. Dadurch gewinnen wir weitere Hinweise für Haupt-vermutungen der Iwasawa-Theorie. Ferner zeigen wir einige Paritäts-vermutung im kommutativen und nicht-kommutativen Fall. Dadurch bekommen wir Ergebnisse über das Wachstum von Selmer-gruppen in kommutativen und nicht-kommutativen Erweiterungen von Funktionenkörpern.

Contents

Acknowledgement

It is a pleasure to express my sincere gratitude to my advisor, Prof. Otmar Venjakob. Without his constant support and encouragement I would not have been able to write this thesis.

I want to acknowledge the support from ERC Iwasawa grant for giving me funding for the entire period. I sincerely thank Faculty of mathematics in Heidelberg University for all the hospitality, enriching seminars and talks. Special thanks are due to all the official staff especially Frau Schmoetten-Jonas for numerous official and informal help.

I want to thank ALGANT programme through which I entered the beautiful world of Number theory and all the Professors who taught me and helped to pursue a research career. Special thanks is due to my Master's thesis supervisor Dr. Jean-Robert Belliard who introduced me in this fascinating world of Iwasawa Theory.

I am thankful to all my colleagues for lots of inspiring discussions. Especially I want to thank PD Jakob Stix, Dr. Chandrakant Aribam, Dr. Somnath Jha, Dr. Peter Barth and Dr. Thanasis Bouganis for many helpful discussions. I would specially like to thank Dr. Malte Witte for correcting many stupid errors. All my friends in Heidelberg and outside and especially Samayita made my life in Heidelberg memorable which helped me to relax and focus on my work.

Finally I would like to thank my parents for their unconditional love, support and warmth. I could not have finished my PhD without their continuous support throughout my life.

0 Introduction

In Arithmetic Geometry one of the main themes has been always to understand the interplay between analytic invariants and algebraic invariants. One of the most famous examples of this interplay is Birch and Swinnerton-Dyer conjecture.

Iwasawa theory is one of the important tools which sheds some light on this issue. It provides a crucial link between the characteristic ideal of the Selmer groups (which are defined algebraically) and p-adic L-functions (which is defined analytically). There has been a large development of Iwasawa theory due to contribution of Mazur, Wiles, Greenberg, Rubin, Kato, Fukaya, Coates, Sujatha, Venjakob etc. Wiles, Mazur-Wiles, Skinner and Urban have proved some important commutative cases. Whereas Kakde, Ritter & Weiss, Hara, Burns etc have proved some important non-commutative cases.

But in most of these recent developments the global field has been taken as number field. The well known analogy with the function field gives us a hint that there should be an interesting Iwasawa theory over characteristic p also. In fact we already know quite a lot of good evidences of BSD conjectures in the function field case due to Ulmer, Kato, Trihan etc. There is already very well established theory over cylotomic extension of function field ([Tha94]).

In this thesis we start with Abelian varieties over a global function field and start asking the similar questions as over number fields like control theorems, defining characteristic elements and formulating Main conjectures of Iwasawa theory etc.

Let K be a global field of characteristic p and A be a non-isotrivial Abelian variety over K (i.e. $j(A) \notin \mathbb{F}_q$). Let K_∞/K be a Galois extension such that $G := Gal(K_\infty/K)$ is a p-adic Lie group. Suppose $\Lambda(G)$ is the associated Iwasawa algebra

$$\Lambda = \Lambda(G) := \mathbb{Z}_p[[G]] = \lim_{U \lhd G} \mathbb{Z}_p[G/U]$$

where U runs over all open normal subgroups of G and $X(A/K_\infty)$ is the Pontryagin dual of the Selmer group (See Definition 1.5.1). Assume there exists a closed normal subgroup H of G such that $G/H \cong \mathbb{Z}_p$. Let $\mathfrak{M}_H(G)$ be the category of finitely generated Λ-modules such that $M/M(p)$ is finitely generated $\Lambda(H)$-module[CFK$^+$05]. Let S be the canonical Ore set and

$$S^* = \cup_{n \geq 0} p^n S$$

For a module M in $\mathfrak{M}_H(G)$ we can define characteristic element (following [CFK$^+$05]) as a pre-image of the class $[M]$ under the connecting homomorphism

$$\delta_G : K_1(\Lambda(G)_{S^*}) \to K_0(\mathfrak{M}_H(G))$$

Now assuming $X(A/K_\infty) \in \mathfrak{M}_H(G)$, the Main conjectures requires the existence of a zeta element $\xi_{X(A/K_\infty)}$ (p-adic L-function) in $K_1(\Lambda(G)_{S^*})$ mapping to the class of $X(A/K_\infty)$ in $K_0(\mathfrak{M}_H(G))$ and satisfying a proper interpolation property.

When $G = \mathbb{Z}_p^d$, recently there has been progress due to King Fai Lai, Ignazio Longhi, Ki-Seng Tan and Fabien Trihan [KFL] and they have suggested a proof of the Main Conjecture. In non-commutative setup there has been much progress due to David Burns [Bur] and Malte Witte [Witb] using perfect complexes replacing the Selmer group. They also prove some Main conjectures relating Euler characteristic of the complex to the p-adic L function in equal and not-equal characteristic cases respectively. There is also progress due to Vuaclair and Trihan (Unpublished).

In general it is difficult to prove Main conjectures. In this work, we are more interested in how the characteristic element changes with respect to the involution of $\Lambda(G)$. There is a action of the group of order 2 on the localized K_1-group induced by the anti-isomorphism $\sharp$ of $\Lambda(G)$ and its opposite ring $\Lambda(G)^\circ$ which sends element of G to its inverse. Moreover, if M is a left $\Lambda(G)$-module then $M^\sharp$ denotes the right module defined by $mg := g^{-1}m$. Now, we know the zeta element ξ interpolates the complex L-function $L(A, \tau, s)$ where τ is an Artin representation of G. But the complex L-function satisfies a functional equation relating $L(A, \tau, s)$ and $L(A^t, \tau^{-1}, 2-s)$, where τ^{-1} is the contragradient representation and A^t is the dual Abelian variety. So conjecturally, $\xi_{X(A/K_\infty)}$ and $\xi_{X(A^t/K_\infty)^\sharp}$ are related by a functional equation. Thus assuming the Main Conjecture, we should get an algebraic functional equation relating the characteristic elements of $X(A/K_\infty)$ and $X(A^t/K_\infty)^\sharp$. But fortunately, this can be proved independently without assuming Main Conjecture. In turn it gives some evidence for the Main Conjecture.

On the other hand existence of this algebraic functional equation can also be thought in purely algebraic way. Set $X := X(A/K_\infty)$ and $X[0]$ denotes the complex centred at 0. Suppose $RHom_\Lambda(X[0], \Lambda)$ is the associated complex of Λ-modules and assume also that Λ is regular i.e. G does not contain any element of order p. Then $H^i(RHom_\Lambda(X[0], \Lambda)) = a^i_\Lambda(X)$, where $a^i_\Lambda(X) := Ext^i_\Lambda(X, \Lambda)$ is the extension group. For M in $\mathfrak{M}_H(G)$, if we denote its class in $K_0(\mathfrak{M}_H(G))$ by $[M]$, then

$$[RHom_\Lambda(X[0], \Lambda)] = \sum_i [a^i_\Lambda(X)]^{-1^i}$$

In some cases, it can be shown that characteristic element of $[RHom_\Lambda(X[0], \Lambda)]$ and characteristic element of $[X(A^t/K_\infty)]$ are the same up to $\Lambda^\times$. So by abuse of notation if we denote by ξ_M , the characteristic element of M then

$$\xi_X = \prod_i \xi_{a^i_\lambda(X)}^{(-1)^i}$$

But in nice cases, higher Ext terms vanish and then with some duality like Cassels-Tate, we should get the desired Functional equation. This principle is hidden behind our proofs although we shall not pursue this further.

Here, we first recall the notations and algebraic preliminaries. Then, in the commutative section we let K_∞ be a $\mathbb{Z}_l$-extensions of a global field K of characteristic p with Galois group $Gal(K_\infty/K) = \Gamma$ and $l \neq p$. We have the control theorem

(Theorem 4.5, [BL09b]) for Elliptic curve E over $\mathbb{Z}_l$-extensions over global field of characteristic p.

The Iwasawa algebra $\mathbb{Z}_l[|\Gamma|]$, denoted by $\Lambda = \Lambda(\Gamma)$, is non-canonically isomorphic to the power series ring $\mathbb{Z}_l[[T]]$. Then from the control theorem, we know that $X(E/L)$ is a finitely generated module over Λ. So we can define the characteristic ideal of $X(E/L)$.

Let M be any finitely generated module over Λ and $a^1_\Lambda(M) := Ext^1_\Lambda(M, \Lambda)$ (Definition 1.3.1). We define the Assumption (***Finite***) in Section 2.2. Then we get the following theorem,

Theorem 0.0.1. *(Theorem 2.3.12) Assume A satisfies the Assumption (**Finite**). Then we have a pseudo-isomorphism of Λ-modules*

$$X(A/K_\infty) \simeq a^1_\Lambda(X(A^t/K_\infty)^\sharp).$$

From this we can prove our main result

Theorem 0.0.2. *(Theorem 2.5.1) Assume A satisfies the Assumption (**Finite**). Then characteristic ideals of $X(A/K_\infty)^\sharp$ and $X(A^t/K_\infty)$, $Ch_\Lambda(X(A/K_\infty)^\sharp)$ and $Ch_\Lambda(X(A^t/K_\infty))$ respectively, satisfy the following functional equation*

$$Ch_\Lambda(X(A^t/K_\infty)^\sharp) = Ch_\Lambda(X(A/K_\infty)).$$

The algebraic functional equation can also be shown to be compatible with the analytic functional equation of L-functions.

In the non-commutative case, again let l be a rational prime number such that $l \neq p$. If E is an non-isotrivial Elliptic curve over K then $G := G_\infty := Gal(K(E[l^\infty])/K)$ is an open subgroup of $GL_2(\mathbb{Z}_l)$ [BLV09]. Let $K_\infty = K(E[l^\infty])$, K^{cyc} be the unramified $\mathbb{Z}_l$-extension of K and $\Gamma = Gal(K^{cyc}/K)$. Then the Galois group $H = Gal(K_\infty/K^{cyc})$ is a closed normal subgroup of G_∞ such that $\Gamma = G_\infty/H = \mathbb{Z}_l$.

Remark 0.0.3. By result of Gianluigi Sechi ([Sec]) $X(E/K_\infty)$ belongs to the category $\mathfrak{M}_H(G)$. Hence one can always associate a characteristic element to $X(E/K_\infty)$.

Now in this set up we again have a control theorem and similarly by Cassels-Tate pairing we will prove the following functional equation

Theorem 0.0.4. *(Theorem 3.1.5) Let E be a non-isotrivial Elliptic curve over K with good ordinary reduction at finite set of prime Σ. Then the characteristic element $\xi_{X(E/K_\infty)}$ of the $\Lambda(G)$-module $X(E/K_\infty)$ in the group $K_1(\Lambda(G)_{S^*})$ satisfies the functional equation*

$$\xi_{X(E/K_\infty)^\sharp} = \epsilon(X(E/K_\infty))\xi_{X(E/K_\infty)} \prod_{q \in R} \alpha_q$$

for some $\epsilon(X(E/K_\infty))$ in $K_1(\Lambda(G))$, α_q is defined in Lemma 3.3.1 and R denotes the set of primes of split multiplicative reductions.

We shall define $X(tw_\tau(E)/K)$ in Section 1 (Definition 1.5.3). Let $\mathcal{O}$ denote the ring of integers of some finite extension L of $\mathbb{Q}_p$. In the fourth section, we prove an important parity result

Theorem 0.0.5. *With notations as before, let τ be an irreducible orthogonal Artin representation of the group G. Then the $\mathcal{O}$-rank of $X(tw_\tau(E)/K^{cyc})$ has the same parity as the $\mathcal{O}$-rank of $X(tw_\tau(E)/K)$.*

Then we prove some results concerning parity conjectures in commutative and non-commutative setting. We also determine the growth of Selmer groups in these settings.

We can prove the following description of the sign of the functional equation when we substitute a self dual Artin representation (Section 1.2)

Lemma 0.0.6. *(Lemma 4.3.1) Let τ be a self dual Artin representation of the group G. Then we have*

$$\epsilon(X(E/K_\infty))(\tau) \equiv (-1)^{ord_{T=0}\xi_{X(tw_\tau(E)/K^{cyc})}} \equiv (-1)^{rank_{\mathcal{O}} X(tw_\tau(E)/K_{cyc}))} (mod \mathcal{M})$$

where $\xi_{X(tw_\tau(E)/K^{cyc})}$ *is the characteristic power series (in* $\mathcal{O}[[T]]$*) of* $X(tw_\tau(E)/K^{cyc})$ *and* $\mathcal{M}$ *is the maximal ideal of* $\mathcal{O}$*.*

This description has a nice consequence in terms of the parity of the sign in the functional equation. Let G_0 be the maximal pro-p normal subgroup of G then we can prove another important theorem which is very closely related to the Parity Conjecture

Theorem 0.0.7. *(Theorem 4.3.3) Let* E *be a non-isotrivial Elliptic curve defined over* K *with good ordinary reduction at finite set of prime* Σ *and* $\omega_E(\tau)$ *is the analytic root number. Then, if*

$$(-1)^{rank_{\mathcal{O}} X(tw_\tau(E)/K^{cyc})} = \omega_E(\tau)$$

holds for all self-dual Artin representations τ *of* G/G_0*, then this is also true for any self dual representation* τ *of* G*.*

1 Preliminaries

In this section we will fix the notations for the rest and recall some important algebraic lemmas.

1.1 Notation

Let K be a global field of characteristic p ($p \neq 2$) and K_∞/K is a Galois extension such that $G := Gal(K_\infty/K)$ is a l-adic Lie group ($l \neq p$, $l \neq 2$). Suppose $\Lambda(G)$ is the associated Iwasawa algebra

$$\Lambda = \Lambda(G) := \mathbb{Z}_l[[G]] = \lim_{U \lhd G} \mathbb{Z}_l[G/U]$$

where U runs over all open normal subgroups of G. Let M be a $\Lambda(G)$-module then its pontryagin dual $\widehat{M}$ of M is defined as $\widehat{M} := Hom(M, \mathbb{Q}_l/\mathbb{Z}_l)$.

Let Σ be a finite set of places of K and A be an Abelian variety over K with good ordinary reduction at every places in Σ. Suppose v is a place of of K then, G_v denotes the decomposition group at v and I_v denotes the inertia group at v. If L is any extension of K then $\Sigma(L)$ is the places of L above Σ.

Assume there exists a closed normal subgroup H of G such that $G/H \cong \mathbb{Z}_l$. Let $\mathfrak{M}_H(G)$ category be the category of finitely generated Λ-modules such that $M/M(l)$ is a finitely generated $\Lambda(H)$-module. Let S be the canonical ore set [CFK+05] and

$$S^* = \cup_{n \geq 0} l^n S$$

Then from [CFK+05], S^* is a multiplicatively closed subset and we can localize $\Lambda(G)$ at S^*. From loc. cit. We have a long exact sequence of K-groups

$$K_1(\Lambda(G)) \longrightarrow K_1(\Lambda(G)_{S^*}) \xrightarrow{\delta_G} K_0(\mathfrak{M}_H(G)) \longrightarrow 0.$$

For a module M in $\mathfrak{M}_H(G)$ we can define a characteristic element (following [CFK+05]) by taking a pre-image of $[M]$ under the connecting homomorphism

$$\delta_G : K_1(\Lambda(G)_{S^*} \to K_0(\mathfrak{M}_H(G))$$

1.2 l-adic representation

Let A be an Abelian variety of dimension g over global function field K. Let $\overline{K}$ denote a separable closure of K. If l is different from p, then the absolute Galois group $G_K = Gal(\overline{K}/K)$ acts on the group $A[l^n] \cong (\mathbb{Z}/l^n\mathbb{Z})^{2g}$ of l^n-torsion points of $A(\overline{K})$. Let T_l denote its Tate module, then $Gal(\overline{K}/K)$ acts continuously on the vector space $V_l := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Assume ρ is unramified at places v outside Σ whose residue characteristic is not l. By Serre [Ser98] this system of representations are compatible i.e. in particular, dimension of V_l is independent of l.

Let ρ denote the homomorphism $Gal(\overline{K}/K) \to Aut(V_l)$ and G_l denote the Zariski closure of $\rho(Gal(\overline{K}/K))$ in $GL_{2g}(\mathbb{Q}_l)$ (Algebraic monodromy group).

Now assume there exists a smooth geometrically connected algebraic curve X over $\mathbb{F}_q$ with function field K and U is a zariski open dense subset of X. Let $\bar{x}$ be a geometric point of U. Removing the finite set where ρ is ramified, then each representation comes from a representation of the *é*tale fundamental group $\pi_1(U, \bar{x})$. Every V_l is the stalk at $\bar{x}$ of a lisse l-adic sheaf $\mathcal{F}_l$ on U.

Remark 1.2.1. By Zahrin [Zar74], the action of $Gal(\overline{K}/K)$ on V_l is semisimple for all l. This implies G_l is reductive and canonical map $End_K(A) \times \mathbb{Q}_l \to End_{G_l}(V_l)$ is an isomorphism.

Remark 1.2.2. By Gajda et al [SAdR], G_l contains $Sp(2g, \mathbb{Z}_l)$, for certain Abelian variety of dimension g over global function field K.

1.3 L-Function

Let U be a zariski open dense subset of X with the inclusion map $j : U \hookrightarrow X$. let I_x denote the inertia subgroup and D_x denote the decomposition subgroup of $Gal(\overline{K}/K)$ at x respectively, where x runs over the closed points $|U|$ of U. Let $\pi_1^{arith}(U)$ denote the arithmetic fundamental group of U which is defined as follows

$$\pi_1^{arith}(U) = G_K / < I_x >_{x \in |U|}$$

We also have the following exact sequence from the definition

$$1 \to I_x \to D_x \to Gal(\overline{k_x}/k_x) \to 1$$

where k_x denotes the residue field of K at x. $Gal(\overline{k_x}/k_x)$ is topologically generated by the geometric Frobenius element $Frob_x$ defined by

$$Frob_x^{-1} : \alpha \to \alpha^{\sharp k_x}, \alpha \in \overline{k_x}$$

Let P_x denote the p-sylow subgroup of I_x. Then we have the following exact sequence

$$1 \to P_x \to I_x \to I_x^{tame} = \prod_{l \neq p} \mathbb{Z}_l(1) \to 1$$

Let $\mathcal{O}$ denote the ring of integers of some finite extension L of $\mathbb{Q}_l$ and V denote a finite dimensional vector space over L. Suppose we have a continuous representation ρ unramified on U

$$\rho : G_K \to GL(V)$$

equivalently, from the definition, ρ is a continuous representation of $\pi_1^{arith}(U)$

$$\rho : \pi_1^{arith}(U) \to GL(V)$$

The L-function of ρ at U is defined by

$$L(U, \rho) = \prod_{x \in |U|} \frac{1}{det(I - \rho(Frob_x)T^{deg(x)} \mid V)} \in 1 + \mathcal{O}[[T]]$$

Similarly, the L-function of ρ at X is defined by

$$L(X, \rho) = \prod_{x \in |X|} \frac{1}{det(I - \rho(Frob_x)T^{deg(x)} \mid V^{I_x})} \in 1 + \mathcal{O}[[T]]$$

Remark 1.3.1. As $l \neq p$, restriction of ρ to the sylow subgroup P_x has finite order and ρ is almost tame. In fact it is finite up to a twist if ρ has rank one (By class field theory). So there are not many l-adic representations.

Remark 1.3.2. By Grothendieck's trace formula [Gro95], $L(U,\rho)$ is always a rational function. Since,

$$L(U,\rho) = \prod_{i=0}^{2} det(1 - Frob_q T \mid H_c^i(U \otimes \overline{\mathbb{F}_q}, \mathcal{F}_\rho))^{(-1)^{i-1}} \in \mathcal{O}(T)$$

where $\mathcal{F}_\rho$ denote a lisse l-adic sheaf on U associated with ρ, and $H_c^i(U \otimes \overline{\mathbb{F}_q}, \mathcal{F}_\rho)$ denote finite dimensional cohomology with compact support.

Remark 1.3.3. Witte [Witb] has proved a Main conjceture interpolating $L(U,\rho)$ with some zeta element in $K_1(\Lambda(G)_S)$.

1.4 Twist

As defined in [CFKS10] there is an action of the group of order 2 on the localized K_1-group induced by the anti-isomorphism $\sharp$ of $\Lambda(G)$ and its opposite ring $\Lambda(G)^\circ$ which sends the element of G to its inverse. Moreover, if M is a left $\Lambda(G)$-module then $M^\sharp$ denotes the right $\Lambda(G)$-module defined by $mg := g^{-1}m$.

Let $\mathcal{O}$ denote the ring of integers of some finite extension L of $\mathbb{Q}_p$ and assume that we are given a continuous homomorphism

$$\rho : G \to GL_n(\mathcal{O})$$

where $n \geq 1$ is an integer. If M is a finitely generated $\Lambda(G)$-module, put $M_\mathcal{O} = M \otimes_{\mathbb{Z}_p} \mathcal{O}$ and define the twist of M with respect to ρ by endowing

$$tw_\rho(M) = M_\mathcal{O} \otimes_\mathcal{O} \mathcal{O}^n$$

with the diagonal G-action. Let $G/H = \Gamma \cong \mathbb{Z}_p$, $\Lambda_\mathcal{O}(\Gamma) := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$ and $Q_\mathcal{O}(\Gamma)$ denotes its field of fractions. Then as explained in [CFK+05], ρ induces a homomorphism

$$\Phi_\rho := K_1(\Lambda(G)_{S^*}) \to K_1(M_n(Q_\mathcal{O}(\Gamma))) \cong Q_\mathcal{O}(\Gamma)^\times$$

Let ϕ denote the augmentation map $\Lambda_\mathcal{O}(\Gamma) \to \mathcal{O}$ and $ker(\phi) =: \mathfrak{p}$. Then ϕ extends to the localization of $\Lambda_\mathcal{O}(\Gamma)$ at $\mathfrak{p}$ (defined as $\Lambda_\mathcal{O}(\Gamma)_\mathfrak{p} \subset Q_\mathcal{O}(\Gamma)$).

We get a map

$$\phi : \Lambda_{\mathcal{O}}(\Gamma)_{\mathfrak{p}} \to L$$

For $\xi \in K_1(\Lambda(G))_{S^*})$, we define $\xi(\rho) := \phi(\Phi_\rho(\xi))$, if $\Phi_\rho(\xi) \in \Lambda_{\mathcal{O}}(\Gamma)_{\mathfrak{p}}$, and ∞ otherwise.

If we identify $Q_{\mathcal{O}}(\Gamma)^\times \cong Q(\mathcal{O}[[T]])^\times$ sending the topological generator of Γ to the formal variable T, then the leading term $\xi^*(\rho)$ at ρ is defined to be the leading term of $\Phi_\rho(\xi)$ at $T = 0$.

Remark 1.4.1. Let M be a discrete $\Lambda(G)$-module, M_{div} denotes its maximal divisible subgroup and $M/M_{div} = M_{finite}$. Then there exists an exact sequence

$$0 \to M_{div} \to M \to M_{finite} \to 0.$$

Let ρ is given by a $\mathbb{Z}_p$-lattice T. Since, T is free, the exact sequence remains exact after taking tensoring with T. As M is discrete, M_{div} is some copies of $\mathbb{Q}_p/\mathbb{Z}_p$. Now $\mathbb{Q}_p/\mathbb{Z}_p \otimes T = (\mathbb{Q}_p/\mathbb{Z}_p)^{rank_{\mathbb{Z}_p} T}$, which is p-divisible. So we get that twist preserves the maximal divisible subgroup by flatness of T over $\mathbb{Z}_p$.

1.5 Selmer group

Let $A[l^m]$ denotes the kernel of the multiplication by l^m on A and F be any extension over K. Taking the following exact sequence

$$A[l^m] \hookrightarrow A \xrightarrow{l^m} A \to 0$$

and taking cohomology and direct limit, we get the Kummer homomorphism

$$A(F) \otimes \mathbb{Q}_l/\mathbb{Z}_l \hookrightarrow H^1(F, A[l^\infty])$$

Definition 1.5.1. We define l^∞-Selmer group $Sel_{l^\infty}(A/F)$ of A over any extension F over K as kernel of the homomorphism

$$H^1(F, A[l^\infty]) \to \bigoplus_v H^1(F_v, A[l^\infty])$$

where v runs through all places of F and F_v denotes the completion of F at v.

$X(A/F)$ is the Pontryagin dual of Selmer group.

Remark 1.5.2. Firstly, here we can use *é*tale (equivalently Galois) cohomology as $l \neq p$. This definition also makes sense as the image of the Kummer map at v vanishes as $l \neq p$ ([BL09b], Proposition 3.1).

We also define Selmer group twisted by a character of $G := \mathbb{Z}_l$ (Say ρ). As we get a l-adic representation space $V_l := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ from $E[l^\infty]$. We can define $E[l^\infty] \otimes \rho$ as a twist of representation by the character ρ.

Definition 1.5.3. We define the twisted l^∞-Selmer group as follows

$$Sel_{l^\infty}(tw_\rho(A)/F) = ker\{H^1(F, A[l^\infty] \otimes \rho) \to \bigoplus_v H^1(F_v, A[l^\infty] \otimes \rho)\}$$

where v runs through all places of F.

$X(tw_\rho(A)/F)$ denotes the Pontryagin dual of it.

Remark 1.5.4. This twist is compatible with the definition of twist defined in last subsection i.e. $X(tw_\rho(A)/K_\infty) = X(A/K_\infty) \otimes \rho$. (Page 735, [PR03]) .

1.6 Iwasawa Adjoints

Definition 1.6.1. Let M be a (left) Λ-module. Then we define the Iwasawa adjoints to be

$$a^i_\Lambda(M) := Ext^i_\Lambda(M, \Lambda)$$

which are a priori right Λ-module by functoriality and the right Λ-structure of the bi-module Λ.

Remark 1.6.2. Indeed, when $G \cong \mathbb{Z}_p$ then $a^1(M)$ is isomorphic to the Iwasawa's original adjoint upto inversion of group action.

Now we will recall another lemma from ([Záb10], Lemma 5.1) will be one of the important tools to form the pairing of the Pontryagin dual of Selmer group later.

Lemma 1.6.3. *Let M be a $\Lambda(G)$-module and $M \in \mathfrak{M}_H(G)$ then*

$$a^1_{\Lambda(G)}(M) \cong \varprojlim_L a^1_{\Lambda(\Gamma_L)}(M_{H_L})$$

where

$$K \subset_{finite} L \subset K_\infty, H_L = Gal(K_\infty/L^{cyc}), \Gamma_L = Gal(L^{cyc}/L).$$

Infact we can say more about the structure of Iwasawa Adjoints.

Proposition 1.6.4. *([Jan89]) Let G be a p-adic Lie group of dimension $= cd_p(G) = n$ and M a finitely generated $\Lambda(G)$-module, which is finitely generated as a $\mathbb{Z}_p$-module of rank r.*
(1) If M is finite and nonzero, then $a^i(M) = 0$ except $i \neq n+1$ and $a^{n+1}(M) \neq 0$.
(2) If M is a free $\mathbb{Z}_p$-module, then $a^i(M) = 0$ unless $i = n$ and $a^n(M)$ is a free $\mathbb{Z}_p$-module of rank r and there is an isomorphism $a^n(M)^\vee \cong M \otimes D^p_n$, where D^p_n denotes the p-dualizing module of G ([NSW00], p.149).

Now, if we fix $G \cong \mathbb{Z}_p^d$, then we have the following observation

Lemma 1.6.5. *Let $G \cong \mathbb{Z}_p^d$, $\Lambda = \Lambda(G)$ and M be a finitely generated torsion Λ-module. Then M is pseudo-isomorphic to $a^1(M)$.*

Proof. Suppose M is a module of the form Λ/f, $f \in \Lambda$ is non-zero. Then taking $Ext^*_\Lambda(-, \Lambda)$ of the following exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{f} \Lambda \longrightarrow \Lambda/f \longrightarrow 0$$

we get $a^1(\Lambda/f) \cong \Lambda/f$ as $Hom_\Lambda(\Lambda, \Lambda) \cong \Lambda$. Now as $G \cong \mathbb{Z}_p^d$, from structure theorem, we can say that there exists a pseudo-isomorphism from $M \to E$, where E is an elementary module i.e. product of the modules of the form Λ/f, $f \in \Lambda$. So for any elementary module E, we have $a^1(E) \cong E$. The pseudo-isomorphism from $M \to E$ induces pseudo-isomorphism

$$a^1(E) \to a^1(M)$$

since a^i of any pseudo-null module is pseudo null (In fact $a^1(N) = 0$, if N is pseudo-null). So we get M is pseudo-isomorphic to $a^1(M)$.

□

Exploiting similar methods, we recall another important Proposition ([Záb10], Proposition 6.1), when G is an open subgroup of $GL_2(\mathbb{Z}_p)$

Proposition 1.6.6. *Let $M \in \mathfrak{M}_H(G)$ and $\xi_M, \xi_{a^i(M)}$ denote the characteristic elements of M and $a^i(M)$ respectively for $1 \leq i \leq 5$. Then*

$$\xi_M = \prod_{i=1}^{5} \xi_{a^i(M)}^{(-1)^{i+1}}$$

modulo the image of $K_1(\Lambda(G))$ in $K_1(\Lambda(G)_{S^*})$

Remark 1.6.7. Note that here we take on the left hand side, the characteristic element of the left module M where on the right hand side we have characteristic elements of the right modules $a^i(M)$. Since all characteristic elements lie in $K_1(\Lambda(G)_{S^*})$, it nevertheless makes sense to compare them.

We will finish this section by recalling how the Twist defined in the previous section commutes with Iwasawa adjoints.

Proposition 1.6.8. *([Ven05], Proposition 7.3) For every $i \geq 0$ we have canonical isomorphism*

$$a^i(M(\rho)) \cong a^i(M)(\rho^{-1})$$

where ρ^{-1} denotes the contragredient representation i.e. $\rho^{-1}(g) = \rho(g^{-1})^t$ is the transpose matrix of $\rho(g^{-1})$.

2 Commutative case ($l \neq p$)

2.1 Notations

Let K be a function field in one variable over the finite field $\mathbb{F}_q$, where q is a power of p and let G_K denote its absolute Galois group. Let l be a prime different from p. We define $K_\infty^{(l)} := K\mathbb{F}_q^{(l)}$, where $\mathbb{F}_q^{(l)}$ is the unique subfield of $\overline{\mathbb{F}_q}$ such that $Gal(\mathbb{F}_q^{(l)}/\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}_l$. Let A be an Abelian variety over K with principle polarization.

Let $K_\infty^{(l)}$ be the unique $\mathbb{Z}_l$-extension of K and $\Gamma = Gal(K_\infty^{(l)}/K) = \mathbb{Z}_l$.

Remark 2.1.1. Let $\hat{F^*}$ denote the l-adic completion of a function field F and $V = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \hat{F^*}$. Then the set of $\mathbb{Z}_l^d$-extensions is in a bijection with Grassmanian $Grass_d(V)$
(Set of all $d-$dimensional subspaces of V) ([BL09b], Theorem A.1). But when, $F = K = \mathbb{F}_q(t)$, this forces $d = 1$.

Consider the sub-extensions $K_n := (K_\infty^{(l)})^{\Gamma_n}$, where $\Gamma_n = \Gamma^{l^n}$. For convenience, we will describe the whole situation in the following diagram:

$$\begin{array}{c} K_\infty^{(l)} \\ | \\ K_n \\ | \\ K \end{array}$$

Let $Gal(K_\infty/K_n) = \Gamma_n$ and $\Lambda_n = \mathbb{Z}_p[|Gal(K_\infty^{(l)}/K_n)|]$.

2.2 Control Theorem

We have a control theorem for Elliptic curve over $\mathbb{Z}_l$ extension over global field of characteristic p.

Theorem 2.2.1. *(Bandini-Longhi [BL09b], Theorem 4.5) Let $K_\infty^{(l)}$ be the $\mathbb{Z}_l$-extension of a global field K of characteristic p with Galois group $Gal(K_\infty/K) = \Gamma$. Let E be* *a non-isotrivial Elliptic curve over K. Then for every finite intermediate extension F of $K_\infty^{(l)}/K$, the restriction map*

$$res_{K_\infty^{(l)}/K_n} : Sel_{l^\infty}(E/K_n) \to Sel_{l^\infty}(E/K_\infty^{(l)})^{\Gamma_n}$$

has finite and bounded kernel and cokernel, where $\Gamma_n = \Gamma^{l^n}$.

Remark 2.2.2. This control theorem is also proved analogous to the number field case with local kummer map and bounding the first and second cohomology groups of the torsion points of the Elliptic curve.

Remark 2.2.3. It is well known that $\Lambda(\Gamma)$ is non-canonically isomorphic to the power series ring $\mathbb{Z}_l[[T]]$.

Corollary 2.2.4 ([BL09a]). *With the conditions from Theorem 2.2.1 we have that, $X(A/K_\infty)$ is a finitely generated module over $\Lambda(\Gamma)$.*

Proof. This follows easily from Theorem 2.2.1 and the Nakayama lemma. □

If we go through the proof of Theorem 2.2.1 carefully, then we can get more information. By entirely, similar proof the control theorem will hold for Abelian varieties also. We note the control theorem for Abelian variety in the following theorem with the following assumption.

Assumption (***Finite***)

An Abelian variety A satisfies both the following properties:

1) $A[l^\infty](K_\infty^{(l)})$ is finite.

2) If L is a complete DVR of characteristic p with $K_v \subset L \subset \overline{K_v}$, for some place v of K and its residue field does not contain $\mathbb{F}_q^{(l)}$, then $A[l^\infty](L)$ is finite.

Remark 2.2.5. For Elliptic curve, these conditions are satisfied by Lemma 3.2 and Lemma 3.3, [BL09a].

Theorem 2.2.6. *Let $K_\infty^{(l)}$ be the $\mathbb{Z}_l$-extension of a global field K of characteristic p with Galois group $Gal(K_\infty/K) = \Gamma$. Let A be a non-isotrivial Abelian variety over K and satisfies the Assumption (**Finite**). Then for every finite intermediate extension F of K_∞/K, the restriction map*

$$res_{K_\infty^{(l)}/K_n} : Sel_{l^\infty}(A/K_n) \to Sel_{l^\infty}(A/K_\infty^{(l)})^{\Gamma_n}$$

has finite and bounded kernel and cokernel, where $\Gamma_n = \Gamma^{l^n}$.

This statement will be very crucial in forming the relationship between the Pontryagin dual and its extension group. Now, we prove another version of control theorem for general twisted Selmer group, with the following assumption.

Assumption (***Finite-ρ***)
An Abelian variety A satisfies both the following properties:
1) $(A[l^\infty] \otimes \rho)(K_\infty^{(l)})$ is finite.
2) If L is a complete DVR of characteristic p with $K_v \subset L \subset \overline{K_v}$, for some place v of K and its residue field does not contain $\mathbb{F}_q^{(l)}$, then $(A[l^\infty] \otimes \rho)(L)$ is finite.

Lemma 2.2.7. *Let ρ be a character of Γ. Then, ignoring finitely many characters, we can always choose a character ρ such that part 2 of Assumption (**Finite-ρ**) is satisfied.*

Proof. Let K_0 be a local field of characteristic p. If we can show that $(A[l^\infty]\otimes\rho)(K_0)$ is finite, then following (Lemma 3.3, [BL09b]), we get that 2nd condition of Assumption (***Finite-ρ***) is true.

For convenience let us denote $A[l^\infty] \otimes \rho = A(\rho), T_l(\rho) = T_l(A) \otimes \rho$ and $V_l(\rho) = V_l(A)\otimes\rho$, where $T_l(A)$ is the associated Tate module and $V_l(A) = T_l(A)\otimes\mathbb{Q}_l$. From the short exact sequence

$$0 \to T_l(\rho) \to V_l(\rho) \to A(\rho) \to 0$$

we get the following exact sequence

$$0 \to T_l(\rho)^{G_{K_0}} \to V_l(\rho)^{G_{K_0}} \to A(\rho)^{G_{K_0}} \to H^1(G_{K_0}, T_l(\rho))$$

Now $A(\rho)^{G_{K_0}}$ is finite if and only if $T_l(\rho)^{G_{K_0}} = 0$ (Proposition 2.1, [Oze09]). So if $A(\rho)^{G_{K_0}}$ is infinite, $T_l(\rho)^{G_{K_0}} \neq 0$. This gives $V_l(\rho)^{G_{K_0}} \neq 0$. We now consider the action of G_{K_0} on vector space $V_l(\rho)$. After taking some finite base change of $V_l(\rho)$, action of G_{K_0} can be always made trianguline. Then we see that there are only finitely many choices of ρ such that $V_l(\rho)^{G_{K_0}} \neq 0$. Ignoring these finitely many choices, we can always get $(A[l^\infty] \otimes \rho)(K_0)$ is finite.

□

We have the following Theorem

Theorem 2.2.8. *Assume A satisfies the Assumption (**Finite-ρ**) and ρ is an character of Γ. The restriction map*

$$res_{K_\infty^{(l)}/K_n} : Sel_{l^\infty}(tw_\rho(A)/K_n) \to Sel_{l^\infty}(tw_\rho(A)/K_\infty^{(l)})^{\Gamma_n}$$

has finite and bounded kernel and cokernel.

Proof. For convenience let us denote $A[l^\infty] \otimes \rho = A(\rho)$.

We have the following commutative diagram by Hochschild-Serre

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^1(\Gamma_n, A(\rho)(K_\infty^{(l)})) & & \\
 & & \downarrow & & \\
Sel_{l^\infty}(tw_\rho(A)/K_n) & \hookrightarrow & H^1(K_n, A(\rho)) & \longrightarrow & \prod_{v \in \Sigma(K_n)} H^1((K_n)_v, A(\rho)) \\
\downarrow & & \downarrow & & \downarrow \\
Sel_{l^\infty}(tw_\rho(A)/K_\infty^{(l)})^{\Gamma_n} & \hookrightarrow & H^1(K_\infty^{(l)}, A(\rho))^{\Gamma_n} & \longrightarrow & \prod_{w \in \Sigma(K_\infty^{(l)})} H^1((K_\infty^{(l)})_w, A(\rho))^{\Gamma_n} \\
 & & \downarrow & & \\
 & & H^2(\Gamma_n, A(\rho)(K_\infty^{(l)})) & &
\end{array}$$

By (Lemma 4.1, [BL09a]) and Assumption (***Finite-ρ***), $H^1(\Gamma_n, A(\rho)(K_\infty^{(l)}))$ is finite and $H^2(\Gamma_n, A(\rho)(K_\infty^{(l)})) = 0$ (Since $\Gamma_n \cong \mathbb{Z}_l$). Using snake lemma, we find that coker($res_{K_\infty^{(l)}/K_n}$) is contained in $\prod_{w|v\in\Sigma(K_n)} H^1(\Gamma_{n,v}, A(\rho)(K_{\infty,w}^{(l)}))$, where $Gal(K_{\infty,w}^{(l)}/(K_n)_v) = \Gamma_{n,v}$.

We follow the proof of (Theorem 4.5, [BL09b]). Let v be a place of good reduction, then by criterion of Néron-Ogg-Shafarevich, $(K_n)_v)(A[l^\infty])$ is contained in $((K_n)_v)^{unr}$, maximal unramified extension of $(K_n)_v$. Now, $A(\rho)((K_n)_v^{unr}) = A(\rho)(\overline{(K_n)_v})^{I_v}$, where I_v is the inertia group at v. Since, ρ is an character of Γ and $K_\infty^{(l)}$ is unramified over K, $\rho_{|I_v} = 1$. So $A(\rho)(\overline{(K_n)_v})^{I_v} = A(\overline{(K_n)_v})^{I_v}(\rho) = A(\rho)(\overline{(K_n)_v})$, where the last equality comes from v being a place of good reduction.

Following the same proof of (Theorem 4.5, [BL09b]) and using Assumption (***Finite-***ρ), we get $H^1(Gal(((K_n)_v)^{unr}/(K_n)_v, A(\rho)) = 0$. Now, by inflation map, there is an injection between $H^1(\Gamma_{n,v}, A(\rho)(K_{\infty,w}^{(l)})) \hookrightarrow H^1(Gal(((K_n)_v)^{unr}/(K_n)_v, A(\rho))$. This gives $H^1(\Gamma_{n,v}, A(\rho)(K_{\infty,w}^{(l)})) = 0$.

If v is a place of bad reduction, then using $(A[l^\infty] \otimes \rho)((K_n)_v)$ is finite, we get $H^1(\Gamma_v, A(\rho)(K_{\infty,w}^{(l)}))$ is finite and bounded by $\mid (A[l^\infty] \otimes \rho)(K_{\infty,w}^{(l)})/((A[l^\infty] \otimes \rho)(K_{\infty,w}^{(l)}))_{div} \mid$ (Remark 3.5, loc. cit.).

□

2.3 Iwasawa Adjoint and Dualities

Suppose M is a finitely generated $\Lambda(\Gamma)$-module, where $\Gamma \cong \mathbb{Z}_l$. We make the following assumption.

Assumption 1

M_{Γ_n} is torsion as Λ_n-module for every n, where $\Lambda_n = \mathbb{Z}_l[|Gal(K_\infty^{(l)}/K_n)|]$ and $\Gamma_n = \Gamma^{l^n}$.

Remark 2.3.1. Let M be an finitely generated $\Lambda(\Gamma)$ module satisfying Assumption 1. Then we have

$$a^1_\Lambda(M) = \varprojlim_n a^1_{\Lambda_n}(M_{\Gamma_n})$$

Proof. This is a special case of Lemma 1.6.3.

□

Remark 2.3.2. When $\Gamma \cong \mathbb{Z}_l$. Let M be a finitely generated $\Lambda(\Gamma)$-module satisfying Assumption 1. Then we have $a^1_\Lambda(M) = \varprojlim_n \widehat{M^\sharp}^{\Gamma_n}$. It follows from the Remark from ([PR03], Proposition 1.3.1).

Using Tate module of the Abelian variety as a lisse sheaf, the following can be deduced from Corollary 5.24, [Wita]

Theorem 2.3.3. *Let A be an arbitrary Abelian variety over K. Then, $X(A/K^{(l)}_\infty)$ is a finitely generated torsion $\Lambda(\Gamma)$-module where $\Gamma = Gal(K^{(l)}_\infty/K)$.*

Remark 2.3.4. This is a strikingly different result from the number field case. In the number field case dual of Selmer group to be torsion module depends heavily on the reduction of the variety. But in contrast, in the function field case it is independent of the reduction type of the Abelian variety.

Definition 2.3.5. Let τ be an character of the absolute Galois group G_K with values in $\mathbb{Z}^*_l$. Then τ is called admissible for A, if $X(tw_\tau(A)/K^{(l)}_\infty)_{\Gamma_n}$ is finite for every n.

Remark 2.3.6. If τ is admissible, then $X(tw_\tau(A)/K^{(l)}_\infty)$ satisfies Assumption 1.

Lemma 2.3.7. *We can always choose an admissible character for A and for A^t.*

Proof. Let M be a finitely generated torsion $\Lambda(\Gamma)$-module and $P(T)$ be its characteristic polynomial. Now $(M\otimes\tau^k)_{\Gamma_n}$ is $(M\otimes\tau^k)/(\gamma^{l^n}-1)$, where k is an integer and γ is topological generator of Γ. So $(M\otimes\tau^k)_{\Gamma_n}$ is finite if and only if $P(u^k\zeta_n - 1)$ is nonzero, where ζ_n is a l^n-th root of unity and $u = \tau(\gamma)$. But by Weierstrass preparation theorem, $P(T)$ has only finitely many roots. So as $X(tw_\tau(A)/K^{(l)}_\infty)$ is finitely generated torsion $\Lambda(\Gamma)$-module, we can always choose an admissible character for A. Since, A has principle polarization, it can be chosen for for A^t.

□

Lemma 2.3.8. *Let* $\Gamma = Gal(K_\infty^{(l)}/K) \cong \mathbb{Z}_l$, $\Gamma_n = \Gamma^{l^n}$ *and* τ *be an admissible character. Then,*

$$\varprojlim_n Sel_{l^\infty}(tw_\tau(A)/K_\infty^{(l)})^{\Gamma_n} = a^1_{\Lambda(\Gamma)}(X(tw_\tau(A)/K_\infty^{(l)})^\sharp)$$

Proof. This follows from Remark 2.3.2.

□

Now we have the restriction maps

$$Sel_{l^\infty}(tw_\tau(A)/K_n) \to Sel_{l^\infty}(tw_\tau(A)/K_\infty^{(l)})^{\Gamma_n}$$

which induces after taking the projective limit with the previous lemma the following $\Lambda(\Gamma)$-homomorphism.

$$\mathcal{A}_{K_\infty^{(l)},\tau} : \varprojlim_n Sel_{l^\infty}(tw_\tau(A)/K_n) \to a^1_{\Lambda(\Gamma)}(X(tw_\tau(A)/K_\infty^{(l)})^\sharp)$$

Theorem 2.3.9. *Assume* A *satisfies the Assumption (**Finite-**τ) and* τ *is an character of* Γ. *Then* $ker\mathcal{A}_{K_\infty^{(l)},\tau}$ *and* $coker\mathcal{A}_{K_\infty^{(l)},\tau}$ *are finite and bounded by a bound independent of the base field* K.

Proof. Firstly, the maps

$$res_{K_\infty^{(l)}/K_n} : Sel_{l^\infty}(tw_\tau(A)/K_n) \to Sel_{l^\infty}(tw_\tau(A)/K_\infty^{(l)})^{\Gamma_n}$$

have finite and bounded kernel and cokernel. Now taking projective limit is a left exact functor. As projective limit of finite and bounded is still finite and bounded, from Theorem 2.2.8,

we get the following exact sequence

$$0 \longrightarrow \varprojlim_n ker(res(K_\infty^{(l)}/K_n)) \longrightarrow \varprojlim_n Sel_{l^\infty}(tw_\tau(A)/K_n) \longrightarrow a^1_{\Lambda(\Gamma)}(X(tw_\tau(A)/K_\infty^{(l)})^\sharp)$$

$$\longrightarrow \varprojlim_n coker(res(K_\infty^{(l)}/K_n))$$

$ker\mathcal{A}_{K_\infty^{(l)},\tau} = \varprojlim_n ker(res(K_\infty^{(l)}/K_n))$ and $coker\mathcal{A}_{K_\infty^{(l)},\tau} = \varprojlim_n coker(res(K_\infty^{(l)}/K_n))$ which are finite and bounded fom Theorem 2.2.8, which finishes the proof.

□

Remark 2.3.10. We can always choose an admissible character τ (Lemma 2.3.7). Then $Sel_{l^\infty}(tw_\tau(A^t)/F)$ is finite. From Kummer theory, $(A[l^\infty] \otimes \rho)(K_\infty^{(l)})$ is finite. From Lemma 2.2.7, 2nd condition of Assumption (***Finite-***τ) is true. So we can apply Theorem 2.3.9.

Now we describe the Cassels-Tate pairing in this case

Proposition 2.3.11. *Over any field extension of K, F, we have an isomorphism*

$$X(tw_\tau(A)/F)(l) \cong Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F)/div(Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F))$$

where div(.) denotes the maximal divisible subgroup and τ be a character of $Gal(F^{cyc}/F)$ with values in $\mathbb{Z}_l^*$ and A^t is the dual Abelian variety of A.

Proof. We use Cassels-Tate pairing for global fields from (Theorem 6.6, [GA09]). We have a canonical Galois-equivariant pairing $Sel_{l^\infty}(A/F) \times Sel_{l^\infty}(A^t/F) \to \mathbb{Q}_l/\mathbb{Z}_l$, whose left and right kernels are the maximal divisible subgroups of each group. Now from Remark 1.5.4, $Sel_{l^\infty}(tw_\tau(A)/F) = Sel_{l^\infty}(A/F) \otimes \tau$ and $Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F) = Sel_{l^\infty}(A^t/F) \otimes \tau^{-1}$. Since, the twist preserves the maximal divisible subgroup (Remark 1.4.1), the previous pairing induces a canonical Galois-equivariant pairing $Sel_{l^\infty}(tw_\tau(A)/F) \times Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F) \to \mathbb{Q}_l/\mathbb{Z}_l$, whose left and right kernels are the maximal divisible subgroups of each group.

This implies, $Hom(Sel_{l^\infty}(tw_\tau(A)/F)/div(Sel_{l^\infty}(tw_\tau(A)/F)), \mathbb{Q}_l/\mathbb{Z}_l)$
$\cong Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F)/div(Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F))$. Then we get the desired claim.

□

Corollary 2.3.12. *Assume A satisfies the Assumption (**Finite**). For the $\mathbb{Z}_l$-extension $K_\infty^{(l)}$ there is a pseudo-isomorphism*

$$X(A/K_\infty^{(l)}) \to a^1_{\Lambda(\Gamma)}(X(A^t/K_\infty^{(l)})^\sharp)$$

with kernel and cokernel, finite and bounded by a bound independent of the base field.

Proof. Let us assume $Gal(K_\infty^{(l)}/K)$ has an admissible character τ (Remark 2.3.7) for A and A^t. We can choose τ such that $Sel_{l^\infty}(tw_{\tau^{-1}}(A)/F)$ and $Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/F)$ is finite for any finite extension $K \subset F \subset K_\infty^{(l)}$. Then the divisible part vanishes. So we have an isomorphism from Proposition 2.3.11,

$$X(tw_\tau(A)/K_n)(l) \cong Sel_{l^\infty}(tw_{\tau^{-1}}(A^t)/K_n)$$

for every n. Now, as $\varprojlim_n X(tw_{\tau^{-1}}(A)/K_n)(l) \cong X(tw_{\tau^{-1}}(A)/K_\infty^{(l)})$, we have the following map using Theorem 2.3.9,

$$X(A/K_\infty^{(l)}) \otimes \tau^{-1} = X(tw_{\tau^{-1}}(A)/K_\infty^{(l)}) \longrightarrow a^1_{\Lambda(\Gamma)}(X(tw_\tau(A^t)/K_\infty^{(l)})^\sharp)$$

$a^1_{\Lambda(\Gamma)}(X(tw_\tau(A^t)/K_\infty^{(l)})^\sharp) = a^1_{\Lambda(\Gamma)}(X(A^t/K_\infty^{(l)})^\sharp) \otimes \tau^{-1}$, from Proposition 1.6.8. Taking tensor with τ we get the required homomorphism independent of the character and then we use Theorem 2.3.9.

□

2.4 Characteristic series

Recall that a finitely generated torsion $\Lambda(\Gamma)$-module is said to be pseudo-null if its annihilator ideal has height at least 2 (In other words, $E^1_\Lambda(M) = E^0_\Lambda(M) = 0$) i.e. finite, in this case.

If M is a finitely generated $\Lambda(\Gamma)$-module then there is a pseudo-isomorphism (i.e morphism with pseudo null kernel and cokernel)

$$M \sim \prod_{i=1}^{n} \Lambda(\Gamma)/(g_i^{e_i})$$

where g_i's are irreducible elements of $\Lambda(\Gamma)$ up to units and $e_i \geq 0$'s are uniquely determined by M.

Definition 2.4.1. In the above setting the characteristic ideal of M is defined by

$$Ch_{\Lambda(\Gamma)}(M) = (\prod_{i=1}^{n} g_i^{e_i}) \text{ if M is } \Lambda(\Gamma)\text{-torsion}$$

and zero otherwise.

Remark 2.4.2. From Theorem 2.2.4 we have that $X(A/L)$ is a finitely generated $\Lambda(\Gamma)$-module so we can define the characteristic ideal of $X(A/L)$.

Now we recall some important results related with the characteristic elements.

Remark 2.4.3. In particular, if M is finite then it represents a trivial element. One can also prove that, if M is a torsion $\Lambda(\Gamma)$-module then in the commutative case, $Ch_{\Lambda_\Gamma}(M) = 1$ if and only if M is pseudo-null.

Now we can prove an important lemma using the above structure theorem,

Lemma 2.4.4. *If M is a finitely generated torsion $\Lambda(\Gamma)$-module where $\Gamma = \mathbb{Z}_l$, then*

$$Ch_{\Lambda(\Gamma)}(a^1_\Lambda(M)) = Ch_{\Lambda(\Gamma)}(M)$$

Proof. From the Lemma 1.6.5, we have $a^1_\Lambda(M)$ is pseudo-isomorphic to M. That gives both to have identical Characteristic Ideal.

□

2.5 Functional equation

Now we will have the functional equation of the characteristic elements.

We do not need to assume additionally that $X(A/K_\infty^{(l)})$ is $\Lambda(\Gamma)$ torsion, as it is true by Theorem 2.3.3 Then we have the following theorem,

Theorem 2.5.1. *Assume A satisfies the Assumption (**Finite**). Then $Ch_{\Lambda_\Gamma}(X(A^t/K_\infty^{(l)})^\sharp)$ and $Ch_{\Lambda_\Gamma}(X(A/K_\infty^{(l)}))$ satisfies the following functional equation*

$$Ch_{\Lambda_\Gamma}(X(A^t/K_\infty^{(l)})^\sharp) = Ch_{\Lambda_\Gamma}(X(A/K_\infty^{(l)}))$$

Proof. Firstly, from Lemma 2.4.4, $Ch_{\Lambda(G)}(a^1_\Lambda(X(A^t/K^{(l)}_\infty)^\sharp)) = Ch_{\Lambda(G)}(X(A/K^{(l)}_\infty)^\sharp)$. Now two modules are pseudo-isomorphic then their characteristic polynomial differs by an unit in the Iwasawa algebra i.e. they define same ideal. Now Corollary 2.3.12 finishes the proof. □

Remark 2.5.2. In fact by entirely similar proof, we can prove there exists a functional equation relating Selmer group twisted by a character τ, if τ factors through $K^{(l)}_\infty$.

We can also explicitly determine the sign of the functional equation if we take the generators of the corresponding characteristic ideals.

Lemma 2.5.3. *Assume A satisfies the Assumption (**Finite**). Suppose f and $f^\sharp$ denote generators of the characteristic ideals of $Ch_\Lambda(X(A/K^{(l)}_\infty))$ and $Ch_\Lambda(X(A^t/K^{(l)}_\infty)^\sharp)$, where τ is a character. Then they satisfy the following equation*

$$f = \epsilon(X(A/K^{(l)}_\infty))f^\sharp$$

Where $\epsilon(X(A/K^{(l)}_\infty))$ is an unit of Λ and $\epsilon(X(A/K^{(l)}_\infty))(0) \equiv (-1)^{rank_{\mathcal{O}}(X(A/K^{(l)}_\infty))}$ (mod $\mathcal{M}$), where τ is realized over $\mathcal{O}$, ring of integers of finite extension of $\mathbb{Q}_l$ with maximal ideal $\mathcal{M}$.

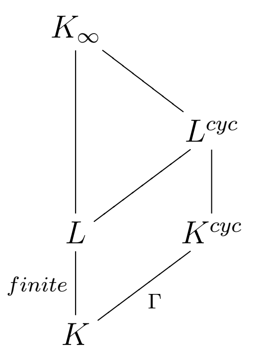
Proof. We have a functional equation $f(\frac{-T}{T+1}) = \epsilon_f(T)f(T)$ with $\epsilon_f(T) \in \mathcal{O}[[T]]^\times$. By Weierstrass-preparation theorem, let $f(T) = l^k u(T)g(T)$, with $k \in \mathbb{Z}, u(T) \in \mathcal{O}[[T]]^\times$ and $g(T)$ is a distinguished polynomial. Now, $g(\frac{-T}{T+1}) = u'(T)\epsilon_f(T)g(T)$, where $u(T) = u'(T).u(\frac{-T}{T+1})$. Clearly, the roots of $g(T)$ occur in pairs $(\alpha, \frac{-\alpha}{\alpha+1})$ excepts the roots $\alpha = 0$. So we can consider $g(T) = T^m.g_1(T)$, where $g_1(0) \neq 0$. Suppose, $(\frac{-T}{T+1})^m = \epsilon'_f(T).T^m$ and $g_1(\frac{-T}{T+1}) = \epsilon''_f(T).g_1(T)$, where $\epsilon_f(T) = \epsilon'_f(T).\epsilon''_f(T)$. Then $\epsilon'_f(0) = (-1)^m$ and $\epsilon''_f(0) = 1$ (Substituting $T = 0$). Since $deg(g_1)$ is even, $\epsilon''_f(0) = (-1)^{deg(g_1)}$. Combining we get $\epsilon_f(0) = (-1)^{deg(g)}$. By structure theory, it is clear that $deg(g) = rank_{\mathcal{O}}(X(A/K^{(l)}_\infty))$.

□

3 Non-commutative case ($l \neq p$)

3.1 Pairings

Firstly we will recall our set up in the GL_2-setting. Let $K_\infty = K(E[l^\infty])$ be a l-adic Lie extension of the global field K of char $p \neq l$, where E is an elliptic curve over K without complex multiplication. We know $G := G_\infty := Gal(K(E[l^\infty])/K)$ is an open subgroup of $GL_2(\mathbb{Z}_l)$(Theorem II.33, [Sec]). K^{cyc} is the cyclotomic $\mathbb{Z}_l$-extension of K and if L is any finite extension of K in K_∞ then L^{cyc} is the unique $\mathbb{Z}_l$-extension of L containing both L and K^{cyc}. So we get the following diagram.



Here, $H := Gal(K_\infty/K^{cyc}), \Gamma = G_\infty/H = \mathbb{Z}_l$. Again following Section 2, we can formulate the $\mathfrak{M}_H(G)$-conjecture and the Iwasawa main conjecture.
In this case, we have a result of Gianluigi Sechi (Theorem 4.18, [Sec])

Theorem 3.1.1. *$X(E/K_\infty)$ belongs to the category $\mathfrak{M}_H(G)$.*

Remark 3.1.2. Hence we can always associate a characteristic element to $X(E/K_\infty)$

Firstly, considering the extension L^{cyc}/L, we get the following

Theorem 3.1.3. *There exists a quasi-exact sequence (kernel and cokernel is finite with bounded number of generators)*

$$0 \to X(E/L^{cyc}) \xrightarrow{\phi} a^1_{\Lambda(Gal(L^{cyc}/L))}(X(E/L^{cyc})^\sharp) \to \prod_v M_v \to 0$$

where $M_v := H^1(L_v^{cyc}/L_v, E(L_v^{cyc}))^\vee$ and v runs through the set of primes in K where E has split multiplicative reductions.

Proof. This follows the same argument as Corollary 2.3.12. Similar to the commutative setting, first we define $\Gamma_n = \Gamma^{p^n}$ and $L_n = (L^{cyc})^{\Gamma_n}$. Now applying control theorem Theorem 2.2.6 over L_n, we get a homomorphism

$$Sel(E/L_n) \to Sel(E/L^{cyc})^{\Gamma_n}$$

with finite and bounded kernel and cokernel bounded by $\prod_v H^1(L_v^{cyc}/L_v, E(L_v^{cyc}))^\vee$ (Remark 4.6, [BL09b]. Also from the Cassels-Tate pairing we have a pairing of the Pontryagin dual to the Selmer group itself (Proposition 2.3.11). Then similar to the proof of Corollary **??**, taking projective limits we get the required quasi-exact sequence

$$\phi : X(E/L^{cyc}) \to a^1_{\Lambda(Gal(L^{cyc}/L))}(X(E/L^{cyc})^\sharp)$$

□

We claim the following

Theorem 3.1.4. *There is an exact sequence*

$$0 \to a^1_{\Lambda(\Gamma_L)}(X(E/L^{cyc}) \xrightarrow{\psi} a^1_{\Lambda(\Gamma_L)}(X(E/K_\infty)_{H_L}) \to coker(\psi) \to 0$$

where $coker(\psi)$ *induces the same element in* $K_0(\mathfrak{M}_H(G))$ *as* $\oplus_v(\Lambda(G) \otimes_{\Lambda(G_v)} \hat{B}(-1))$ *where* v *runs through primes where* E *has split multiplicative reduction and* B *is isomorphic to* $\mathbb{Q}_p/\mathbb{Z}_p$ *as* $Gal(K_\infty/L)_v$*-module.*

Proof. We will use the following fundamental diagram :

$$\begin{array}{ccccccc}
0 \longrightarrow & Sel(E/L^{cyc}) & \longrightarrow & H^1(G_R(L^{cyc}), E_{l^\infty}) & \xrightarrow{\lambda_R(L^{cyc})} & \oplus_{v\in R} H^1(L_v^{cyc}, E)(l) \\
& \downarrow f & & \downarrow g & & \downarrow h \\
0 \longrightarrow & Sel(E/K_\infty)^{H_L} & \longrightarrow & H^1(G_R(K_\infty), E_{l^\infty})^{H_L} & \xrightarrow{\Phi_\infty} & \oplus_{v\in R} J_v(K_\infty)^{H_L}
\end{array}$$

where $J_v(L) := \bigoplus_{w|v} H^1(L_w, E)(l)$ and R denotes the set of primes of potential multiplicative reduction and $G_R(L) := Gal(K_R/L)$, where K_R is the maximal unramified extension outside R.

From the diagram we claim the following:

i) $ker(g)$ and $coker(g)$ are finite.

ii) $ker(h)$ is cofinitely generated $\mathbb{Z}_p$-module.

iii) $ker(f)$ is finite and $coker(f)$ is $\mathbb{Z}_p$-cofinitely generated.

It is known that $\lambda_R(L^{cyc})$ and Φ_∞ is surjective from Gianluigi Sechi ([Sec], Lemma IV.16). Now we have from snake lemma

$$0 \longrightarrow ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \longrightarrow coker(f)$$

$$\longrightarrow coker(g) \longrightarrow coker(h)$$

We get $ker(g)$ and $coker(g)$ is bounded by $H^i(G_{L^{cyc}}, E[l^\infty])$ for $i = 1, 2$ respectively which are finite (Proposition III.13, [Sec]). This proves that $ker(f)$ is finite.

For $ker(h)$ it is easy to see that only interesting case is when v is place of split multiplicative reduction. Let E has potential bad reduction at prime v. Then there exists some finite subextension where it becomes split multiplicative. Now as we are taking projective limit over all finite subextensions, we may assume it becomes split multiplicative over L. Then we have a short exact sequence (from theory of Tate curve)(Section 3.3, [BLV09])

$$0 \to A \to E[l^\infty] \to B \to 0$$

where as $Gal(K_\infty/L)_v$ module A is isomorphic to μ_{l^∞} and B is isomorphic to $\mathbb{Q}_l/\mathbb{Z}_l$. Now $ker(h) = \bigoplus_v H^1(K_{\infty,v}/L_v^{cyc}, E(K_{\infty,v}[l^\infty])$ and from the above exact sequence

$H^1(K_{\infty,v}/L_v^{cyc}, E(K_{\infty,v}[l^\infty]) = B(-1)$

Then $ker(h) = \oplus_v B(-1)$, where v runs through places of split reduction. We have then, $coker(f)$ is $\mathbb{Z}_p$-cofinitely generated and bounded by $ker(h)$.

If we take the Pontryagin dual of the map f, then we get

$$\hat{f} : X(E/K_\infty)_{H_L} \longrightarrow X(E/L^{cyc})$$

with finite cokernel and $\mathbb{Z}_p$-finitely generated kernel.

Now $a^2_{\Lambda(\Gamma_L)}(X(E/L^{cyc})^\sharp)$ is finite with bounded number of generators. We have, $a^2_{\Lambda(\Gamma_L)}(X(E/L^{cyc})^\sharp) = a^2_{\Lambda(\Gamma_L)}(F)$, where F is the maximal finite submodule of $X(E/L^{cyc})^\sharp$ since finite modules correspond to pseudo-null modules for $\Lambda(\Gamma_L)$. By modifying the argument (Theorem 1 ,[HM00]) in the function field setting, F equals the projective limit of the kernels of the restrictions

$$Sel_{l^\infty}(E/L_n) \to Sel_{l^\infty}(E/L^{cyc})^{\Gamma_n}$$

where Γ_n is subgroup of Γ_L of index l^n and L_n is its fixed field (This works as all the ingredients are avaialble in this function field setting also). Now from the proof of Theorem 2.2.6, kernel of the above restriction is contained in the finite module $H^1(\Gamma_n, E(L^{cyc})[l^\infty]) = E(L^{cyc})[l^\infty]/(\gamma_L^{l^n} - 1)E(L^{cyc})[l^\infty]$ for big enough n. But as $E(L^{cyc})[l^\infty]$ is cogenerated by atmost 2 elements, we get $a^2_{\Lambda(\Gamma_L)}(X(E/L^{cyc})^\sharp)$ is finite with bounded number of generators.

Then we get the following quasi-exact sequence

$$0 \longrightarrow a^1_{\Lambda(\Gamma_L)}(X(E/L^{cyc})^\sharp) \xrightarrow{\psi} a^1_{\Lambda(\Gamma_L)}(X(E/K_\infty)^\sharp_{H_L}) \longrightarrow a^1_{\Lambda(\Gamma_L)}(\widehat{coker(f)}^\sharp) \to 0$$

Again as $X(E/K_\infty)$ belongs to the category $\mathfrak{M}_H(G)$, using Remark 2.3.2 again we get that $a^1_{\Lambda(\Gamma_L)}(\widehat{B(-1)}) = \hat{B}(-1)$. Now $\hat{B}(-1)$ has a class in $K_0(\mathfrak{M}_H(G_v))$. Now taking tensor with $\Lambda(G)$ over G_v, we get its image in $K_0(\mathfrak{M}_H(G))$ and we get the required result. □

Now we get one of our most important results.

Theorem 3.1.5. *There exists a quasi isomorphism*

$$\alpha : X(E/K_\infty) \to a^1_{\Lambda(G)}(X(E/K_\infty)^\sharp)$$

with $ker(\alpha)$ finitely generated over $\mathbb{Z}_p$ and $coker(\alpha)$ representing the same element in $K_0(\mathfrak{M}_H(G))$ as $\oplus_v(\Lambda(G) \otimes_{\Lambda(G_v)} T_l(E)^*)$ where $T_l(E)^* = Hom(T_l(E), \mathbb{Z}_l)$

Proof. First we take the composition of the following two quasi-exact sequences

$$0 \to X(E/L^{cyc}) \xrightarrow{\phi} a^1_{\Lambda(Gal(L^{cyc}/L))}(X(E/L^{cyc})^\sharp) \to \prod_v M_v \to 0$$

$$0 \to a^1_{\Lambda(\Gamma_L)}(X(E/L^{cyc})^\sharp) \xrightarrow{\psi^\sharp} a^1_{\Lambda(\Gamma_L)}((X(E/K_\infty)_{H_L})^\sharp) \to coker(\psi^\sharp) \to 0$$

The second map is obtained from Theorem 3.1.4 and taking involution. Composing the above maps gives us the map $X(E/L^{cyc}) \to a^1_{\Lambda(\Gamma_L)}(X(E/K_\infty)^\sharp_{H_L})$. Now we take projective limit over all finite extensions of K contained in K_∞ $K \subset_{finite} L \subset K_\infty$. Then using the Lemma 1.6.3, we get the following map

$$\alpha : X(E/K_\infty) \to a^1_{\Lambda(G)}(X(E/K_\infty)^\sharp)$$

So the only remaining part is to calculate the kernel and cokernel. Now $ker(\phi)$ is finitely generated over $\mathbb{Z}_l$ as it is projective limit of finite bounded (independent of K_∞) kernels. So $ker(\alpha)$ is finitely generated over $\mathbb{Z}_l$. For cokernel we have the following description of the $coker(\phi)$ and $coker(\psi^\sharp)$.

Now again over L (Assuming L large enough so that all potentially multiplicative reduction becomes split multiplicative), we have the exact sequence

$$0 \to B(1) \to E(L_v)[l^\infty] \to B[l^{d_L}] \to 0$$

where d_L is some nonnegative integer. Now from long exact sequence of cohomology from the previous sequence $H^1(L_v^{cyc}/L_v, E(L_v^{cyc})) \cong B[l^{d_L}]$. So we have from 3.1.3 $coker(\phi)$ is projective limit of $B[l^{d_L}]$ which is $\widehat{B}$. Now as we again have

$$0 \to \widehat{B} \to T_l(E)^* \to \widehat{B}(-1) \to 0$$

So we get that $coker(\alpha)$ is determined by class of $T_l(E)^*$. □

Remark 3.1.6. Initially, $coker(\phi)$ is calculated as $\Lambda(\Gamma_L)$-module. If $\Gamma_L^* = Gal(L^{cyc}/K)$, $coker(\phi)$ is a $\Lambda(\Gamma_L^*)$-module. As $\Lambda(G) = \varprojlim \Lambda(\Gamma_L^*)$, taking projective limit over L, we consider it as $\Lambda(G)$-module.

Remark 3.1.7. It can be easily seen that the characteristic element of $\Lambda(G) \otimes_{\Lambda(G_v)} T_l(E)^*$ is the same as the image of the characteristic element of $T_l(E)^*$ under the natural map

$$K_1(\Lambda(G_v)_{S_v}) \to K_1(\Lambda(G)_S)$$

where S_v is the canonical ore set in $\Lambda(G_v)$. Infact, we have a commutative diagram

$$\begin{array}{ccc} K_1(\Lambda(G_v)_{S_v}) & \longrightarrow & K_1(\Lambda(G)_S) \\ \downarrow & & \downarrow \\ K_0(\mathfrak{M}_H(G_v)) & \longrightarrow & K_0(\mathfrak{M}_H(G)) \end{array}$$

Then $coker(\psi)$ is also considered as $\Lambda(G)$-module.

3.2 Characteristic elements

Let ξ_M denote the characteristic element of the module M. From Theorem 3.1.5 we expect a functional equation relating $\xi_{X(E/K_\infty)}$ and $\xi_{a^1_{\Lambda(G)}(X(E/K_\infty)^\sharp)}$. In the commutative case, we obtained the functional equation almost directly from the homomorphism as in the previous theorem. But in the GL_2-case, things become complicated due to the non-commutativity (e.g. pseudo-null modules no longer represent trivial class in $K_0(\mathfrak{M}_H(G))$).

Firstly recall, as $\Lambda(G)$ is Auslander regular ring [Ven02], then every left or right $\Lambda(G)$-module satisfies Auslander condition [Ven02]. Let

$$0 \longrightarrow \Lambda(G) \xrightarrow{\mu_0} E_0 \xrightarrow{\mu_1} E_1 \xrightarrow{\mu_2} \ldots\ldots \xrightarrow{\mu_i} E_i$$

be the minimal injective resoulution of $\Lambda(G)$ ([Ste75]). Let us define a category $\mathcal{C}^n$:=full subcategory of all modules M such that $Hom_{\Lambda(G)}(M, E_0 \oplus \ldots.. \oplus E_n) = 0$.

Then we can collect the important technical results to prove our main result of this section.

Lemma 3.2.1. *([Záb10],Lemma 6.2)* $a^i_{\Lambda(G)}(a^1_{\Lambda(G)}(M, \Lambda(G)), \Lambda(G)) \in \mathcal{C}^3$ *for any* $M \in \mathfrak{M}_H(G)$, *for* $i \geq 2$

Lemma 3.2.2. *([Záb10], Lemma 6.3) Any M in $\mathfrak{M}_H(G) \cap \mathcal{C}^3$ represents the trivial class in $K_0(\mathfrak{M}_H(G))$*

Remark 3.2.3. Both these results of Zabradi remain true in the global field case also as they are only about abstract algebra.

Now we can prove the main result relating the characteristic elements of $\xi_{X(E/K_\infty)^\sharp}$ and $\xi_{a^1(X(E/K_\infty)^\sharp)}$

Proposition 3.2.4. *Let E be an elliptic curve without complex multiplication and with good ordinary reduction at finite set of prime Σ. Then characteristic elements $\xi^\sharp_{X(E/K_\infty)}$ and $\xi^\sharp_{a^1(X(E/K_\infty))}$ are the same modulo the image of $K_1(\Lambda(G))$ in $K_1(\Lambda(G)_{S^*})$.*

Proof. From Lemma 1.6.6, we need to prove that $a^i_{\Lambda(G)}(X(E/K_\infty)^\sharp)$ has trivial characteristic element for $i \geq 2$. Now from Theorem 3.1.5 after taking involution, we get the following homomorphism

$$\alpha^\sharp : X(E/K_\infty)^\sharp \to a^1_{\Lambda(G)}(X(E/K_\infty))$$

with $ker(\alpha)^\sharp$ finitely generated over $\mathbb{Z}_p$ and $coker(\alpha)^\sharp$ represents the same element in $K_0(\mathfrak{M}_H(G))$ as $\oplus_v \Lambda(G) \otimes_{\Lambda(G_v)} C_v$, where C_v represents the same element in $K_0(\mathfrak{M}_{H_v}(G_v))$ as $T_l(E)^*$. C_v is free of rank 2 over $\mathbb{Z}_p$, $\Lambda(G)$ is flat over $\Lambda(G_v)$ and $dim(G_v) = 2$ for $v \in R$ (Lemma 5.1, [CS]). So $coker(\alpha)^\sharp$ has only non-trivial Ext^2 (Proposition 1.6.4). Now if we take the long exact cohomological sequence induced from $\alpha^\sharp$, we obtain the following exact sequence

$$0 \to a^2_{\Lambda(G)}(coker(\alpha)^\sharp) \to a^2_{\Lambda(G)}(a^1_{\Lambda(G)}(X(E/K_\infty))) \to a^2_{\Lambda(G)}(X(E/K_\infty)^\sharp) \to 0$$

Here we are also using how a^i behaves under the base change $\Lambda(G) \otimes_{\Lambda(G_v)}$ (Lemma 5.5, [OV02]). If $M \in \mathfrak{M}_H(G)$ i.e. M is finitely generated over $\Lambda(H)$, then $a^i(M) \cong Ext^{i-1}(M, \Lambda(H))$ up to a twist (Theorem 3.1, [SV06]). So, $a^i(M)$ is finitely generated over $\Lambda(H)$ i.e. $a^i(M) \in \mathfrak{M}_H(G)$. The first two elements in the above sequence lie in $\mathfrak{M}_H(G) \cap \mathcal{C}^3$ by Lemma 3.2.1 and 3.2.2. Then $a^2_{\Lambda(G)}(X(E/K_\infty)^\sharp)$ has trivial characteristic element. Similarly (but easier), it follows $a^i_{\Lambda(G)}(X(E/K_\infty)^\sharp)$ has trivial characteristic element for $i \geq 3$.

□

3.3 Functional equation

Now from Theorem 3.1.5, if we can calculate the characteristic element of $\oplus_v \Lambda(G) \otimes_{\Lambda(G_v)} T_l(E)^*$ that will then finish the process of obtaining the required functional equation.

It is very interesting to note that the description of *coker*(α) coincides with description in Number field case [Záb10]. Now the structure of *coker*(α) as $\Lambda(G_v)$ module is purely algebraic and does not change in case of Function field setting. So we will omit the details of explicit calculation. Recall that as reduction type becomes split multiplicative over some finite extension of K, there exists an open subgroup $I_v^{'} \subset I_v$ of the inertia subgroup (can be chosen as pro-l Sylow subgroup) such that there exists

$$0 \to A_v \to T_l(E)^* \to B_v \to 0$$

where A_v and B_v are free $\mathbb{Z}_p$-modules of rank 1 and $I_v^{'}$ acts trivially on them. Then we have from (Lemma 6.5, [Záb10])

Lemma 3.3.1. *The characteristic element of* $\Lambda(G) \otimes_{\Lambda(G_v)} T_l(E)^*$ *is* $\alpha_v := \frac{(X_v \beta_v X_v^{-1)})^\sharp}{X_v \beta_v X_v^{-1)}}$, *where* $X_v + 1$ *is the topological generator of the* $I_v^{'}$ *which is isomorphic to* $\mathbb{Z}_p$ *and* β_v *is* $1 + e_v Frob_v$ *if* E *has non-split multiplicative reduction or* $1 - e_v Frob_v$ *otherwise.* e_v *is the central idempotent element in* $\Lambda(I_v) \subset \Lambda(G_v)$ *corresponding to the projective module*

$$P_q := \Lambda(G_v) \otimes_{\Lambda(I_v)} (A_v \otimes_{\mathbb{Z}_p} \Lambda(I_v^{'}))$$

Theorem 3.3.2. *Let* E *be an elliptic curve over* K *without complex multiplication and with good ordinary reduction at finite set of prime* Σ*. Then the characteristic* *element* $\xi_{X(E/K_\infty)}$ *of the* $\Lambda(G)$*-module* $X(E/K_\infty)$ *in the group* $K_1(\Lambda(G)_{S^*})$ *satisfies the functional equation*

$$\xi_{X(E/K_\infty)^\sharp} = \epsilon(X(E/K_\infty))\xi_{X(E/K_\infty)} \prod_{q \in R} \alpha_q$$

for some $\epsilon(X(E/K_\infty))$ *in* $K_1(\Lambda(G))$*,* α_q *defined above and* R *denotes the set of primes in* K *when* E *has split multiplicative reductions.*

Proof. Firstly note that two elements in $K_1(\Lambda(G)_{S^*})$ define the same class in $K_0(\mathfrak{M}_H(G))$ if and only if they differ by an element in $K_1(\Lambda(G))$. From Theorem 3.2.4, Theorem 3.1.5 and Lemma 3.3.1, we get the functional equation.

$\square$

Remark 3.3.3. It is interesting to note that it would be possible to generalize this functional equation in case of Abelian variety also. First of all, let A be an Abelian variety over K. If A is an Abelian variety of hall type, then $G = Gal(K_\infty/K)$ where $K_\infty := K(A[p^\infty])$ contains $Sp_{2g}(\mathbb{Z}_l)$ (Theorem 3.6, [SAdR]). Theorem 3.1.3 for Abelian varieties can be established using Corollary 2.3.12 and Theorem 3.1.4 also holds similarly. Then using the similar proof we expect the functional equation for $X(A/K_\infty)$. One important point is here we will have the assumption $X(A/K_\infty) \in \mathfrak{M}_H(G)$(which is true in certain cases, see (Corollary 4.8,[BV]). The second important point is here the structure theorems of Iwasawa algebras of Zabradi do not hold. So we have to check the structure of pseudo-null module and trivial characteristic elements etc.

Remark 3.3.4. In fact, from (Theorem 5.3, [BV]) we have control theorem in certain equal characteristic cases. In that set up again exploiting similar method we can prove Functional equation in Non-commutative equal characteristic cases. But unfortunately, there are no arithmetic situation known when this holds. The difficulty lies in the fact that the analogue of Serre's open image theorem is not known in equal-characteristic cases.

4 Root Numbers ($l \neq p$)

First we set some notations for this section. Let A be an Abelian variety over K with principle polarization. Assume A satisfies the Assumption (***Finite***). Let τ be a self dual Artin representation of the absolute galois group G_K such that A satisfies the Assumption (***Finite-τ***). It is realized over $\mathcal{O}$, the ring of integers of a finite extension L of $\mathbb{Q}_l$, with maximal ideal $\mathcal{M}$. Let W_τ be the $\mathcal{O}$-representation space of τ. Then we define the following

1. $r_A(\tau)$:= multiplicity of τ in $A(K) \otimes L$
2. $s_A(\tau)$:= $\mathcal{O}$-corank of $Sel(tw_\tau(A)/K)$
3. $\lambda_A(\tau)$:=$\mathcal{O}$-corank of $Sel(tw_\tau(A)/K^{cyc})$
4. $\omega_A(\tau)$:= the analytic root number associated with the complex L function $L(E, \tau, s)$

Remark 4.0.5. Multiplicity of τ in $X(A/K) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is given by $s_A(\tau)$ i.e. $\mathcal{O}$-corank of $Sel(tw_\tau(A)/K)$ (Remark 4.3.2, [Gre11]).

Suppose F is a finite extension of K. K^{cyc} and F^{cyc} denote the corresponding cyclotomic extension. Let us define the $\mathbb{Q}_l$-representation space $V := X(A/F^{cyc}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ for $G := Gal(F^{cyc}/K)$. Define $\Gamma_F := Gal(F^{cyc}/F)$, $D = Gal(F/K)$ and $\Delta = Gal(F^{cyc}/K^{cyc})$. If τ is an representation of the group G, then $\hat{\tau}$ denotes its contragradient representation.

First of all, it is important to have one theorem analogous to a duality theorem (In Number field case, due to Greenberg). We will prove this from the functional equation and pairing on the Selmer group, which will cover the next subsection.

Theorem 4.0.6. *Let τ be an self dual irreducible orthogonal Artin representation of the group G. Then*

$$s_A(\tau) \equiv \lambda_A(\tau) \ mod\ 2$$

4.1 Duality

Firstly we recall an important lemma

Lemma 4.1.1. *$X(A/F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is self dual.*

Proof. It is enough to show that if τ is an irreducible representation of $Gal(F/K)$, then τ and $\hat{\tau}$ have same multiplicity in $X(A/F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. As multiplicity in $X(A/F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is given by $\mathcal{O}$-corank of Selmer groups, it is enough to prove $s_A(\tau) = s_A(\hat{\tau})$. This follows the same argument as Theorem 1.1, [DD09], since $l \neq p$. Also see Proposition 4, [TW11].

□

Proposition 4.1.2. *$V := X(A/F^{cyc}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is a finite dimensional self dual-representation space for G. Furthermore, $W := Ker(V \to V_{\Gamma_{F_t}})$ admits a $\mathbb{Q}_l$-bilinear, non-degenerate, skew symmetric G-invariant pairing, where t is chosen such that $corank_{\mathbb{Z}_l} Sel_{l^\infty}(A/F_t)$ is maximal (Here F_n is the usual n-th layer in F^{cyc}).*

Proof. Main ingredient of the proof is, from control theorem (Theorem 2.2.6) we get that the restriction map

$$s_n : Sel_{l^\infty}(A/F_n) \to Sel_{l^\infty}(A/F^{cyc})^{\Gamma_{K_n}}$$

has finite, bounded kernel and cokernel, where F_n is the nth layer in F^{cyc}. This induces the dual map

$$\hat{s_n} : X(A/F^{cyc)})_{\Gamma_{F_n}} \to X(A/F_n)$$

which also has finite, bounded kernel and cokernel. Now $\hat{s_n}$ is $Gal(F_n/K)$-equivariant, so we obtain an isomorphism

$$V_{\Gamma_{F_n}} = X(A/F_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

From Lemma 4.1.1, $V_{\Gamma_{F_n}}$ is self dual. Then the rest of the argument follows (Proposition 10.1.1 [Gre11]).

Note that $Sel_{l^\infty}(A/F^{cyc})$ is cotorsion as $\mathbb{Z}_l[[\Gamma_F]]$ module in Function field setting from Theorem 2.3.3. So the extra assumption in Greenberg's proof of the number field case assumption is not needed here.

□

Corollary 4.1.3. *If σ is an irreducible representation of Δ, then $\lambda_A(\hat{\sigma}) = \lambda_A(\sigma)$.*

Proof. Since V is self-dual as Δ-representation space, we get $\lambda_A(\hat{\sigma}) = \lambda_A(\sigma)$.

□

We denote Orb_τ to be the set of irreducible constituents of $\tau \mid_\Delta$. Now, we can prove the main theorem in this subsection, which follows the same argument as(Theorem 10.2.1, [Gre11]

Theorem 4.1.4. *If τ is an orthogonal, self-dual, irreducible representation of D and σ is an irreducible constituent in $\tau \mid_\Delta$ and $l \geq 3$ then*

$$s_A(\tau) \equiv \lambda_A(\tau) \equiv \lambda_A(\sigma) \ mod\ 2$$

Proof. By definition, $\lambda_A(\tau) = \mid Orb_\tau \mid \lambda_A(\sigma)$. Here $\mid Orb_\tau \mid$ is odd as Orb_τ is power of l. We get $\lambda_A(\tau) \equiv \lambda_A(\sigma) \pmod 2$. Let t be as in the above Proposition 4.1.2 and τ' represents the representation space $X(A/F_t) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $D_t = Gal(F_t/K)$. From Lemma 4.1.1, τ' is self-dual. Now, if we consider τ as a representation of D_t then we get that the multiplicity of τ as a constituent in τ' is equal to $s_A(\tau)$. From Proposition 9.3.5 [Gre11],

$$s_A(\tau) = \langle \tau, \tau' \rangle_{D_t} \equiv \langle \tau \mid_{\Delta_t}, \tau' \mid_{\Delta_t} \rangle_{\Delta_t} \pmod 2$$

where Δ_t is the image of Δ in D_t, which is normal and of index of power of l. Now if we consider σ as a representation of Δ_t, then from assumption of the theorem, it is an irreducible constituent in $\tau \mid_{\Delta_t}$. So we have $s_A(\sigma) = \langle \sigma, \tau' \mid_{\Delta_t} \rangle_{\Delta_t}$. The orbit of σ under the action of D_t/Δ_t is Orb_τ. Moreover, if σ_1 and σ_2 lie in the same orbit, then their multiplicity are also the same i.e. $s_A(\sigma_1) = s_A(\sigma_2)$. So again using $\mid Orb_\tau \mid$ is odd, we have the following,

$$\langle \tau \mid_{\Delta_t}, \tau' \mid_{\Delta_t} \rangle_{\Delta_t} = \mid Orb_\tau \mid . \langle \sigma, \tau' \mid_{\Delta_t} \rangle_{\Delta_t} \equiv s_A(\sigma) \pmod 2$$

So we have

$$s_A(\tau) \equiv s_A(\sigma) \pmod 2$$

Let W be as defined in the previous Proposition 4.1.2. As, τ is orthogonal, from (Proposition 9.3.1, [Gre11]), σ is also orthogonal and multiplicity of σ in the Δ-representation space W is even. From the proof of the Proposition 4.1.3, $V = W \oplus V_{\Gamma_{F_t}}$, where $V_{\Gamma_{F_t}} \cong X(A/F_t) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then it follows,

$$\lambda_A(\tau) \equiv s_A(\sigma) \bmod 2$$

Combining with the previous observation, it finishes the proof. □

Proof. (Theorem 4.0.6) Let τ be an irreducible Artin representation of G. Then by definition it factors through some finite extension F/K. So we can think τ to be representation of $Gal(F/K)$. Then we have to show $\lambda_A(\tau) \equiv s_A(\tau) \pmod 2$, which is exactly Theorem 4.1.4.

□

Remark 4.1.5. It would be interesting to extend Theorem 4.1.1 to general l-adic Lie extension K_∞/K i.e to compare rank of $X(tw_\tau(A)/K_\infty)$ and rank of $X(tw_\tau(A)/K)$. The main difficulty lies in proving similar result to Proposition 4.1.2, as no longer the kernel and cokernel of the restriction map remains bounded.

4.2 Commutative Case

Here $K_\infty^{(l)}/K$ is a $\mathbb{Z}_l$ extension and $\Gamma = Gal(K_\infty^{(l)}/K) = \mathbb{Z}_l$. First we will recall the description of $\epsilon := \epsilon(X(A/K_\infty^{(l)}))$ from Lemma 2.5.3.

Lemma 4.2.1. *For each self-dual Artin representation τ of G over $\mathcal{O}$*

$$\epsilon(X(A/K_\infty^{(l)}))(\tau) \equiv (-1)^{\lambda_A(\tau)}$$

modulo $\mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$.

Proof. The proof follow the same argument as Lemma 2.5.3 □

Proposition 4.2.2. *Let A be an Abelian variety over K with good ordinary or split multiplicative reduction at all places in K. Then if $(-1)^{\lambda_A(\tau)} = \omega_A(\tau)$ holds for all self dual representation τ of G/G_0 then it is also true for any self dual representation τ of G, where G_0 denotes the maximal pro-l normal subgroup of G*

Proof. This follows by the same argument as Theorem 7.8, [Záb10] and using that both side depends only on semi-simplification of reduction of τ modulo the maximal ideal $\mathcal{M}$ of $\mathcal{O}$.

□

Now as a corollary, we get

Theorem 4.2.3. *For any self dual Artin representation τ of G we have*

$$(-1)^{s_A(\tau)} = \omega_A(\tau)$$

Proof. First of all, $K_\infty^{(l)}$ contains the l-th roots of unity. Then there is always a l-isogeny. Then arguing similar to (Proposition 7.9, [Záb10]) G does not have any symplectic self dual Artin representation and τ is orthogonal. Infact if τ factors through G/G_0 then τ is an one dimensional character and the theorem is true. Now by Theorem 4.0.6 and previous proposition, we get the result.

□

4.3 Non-commutative Case

Here we will prove the parity conjectures with some mild assumption on the elliptic curve. We will prove in using the functional equation and some explicit description of ϵ-factor as in the commutative case. Here also we will state the results (Proofs are same from Zabradi, Section 7, [Záb10], so we omit details) as they are again results on general algebra and will hold in our case also.

We can prove the following nice description of the sign of the functional equation when we evaluate a self dual Artin representation i.e. we evaluate $\epsilon(X(E/K_\infty))$ at τ.

Lemma 4.3.1. *Let τ be a self dual Artin representation of the group G and we have all the assumptions as in Section 3. Then we have*

$$\epsilon(X(E/K_\infty))(\tau)\prod_{q\in R}\alpha_q \equiv (-1)^{ord_{T=0}\xi_{X(tw_\tau(E)/K_{cyc})}} \equiv (-1)^{\lambda_E(\tau)}(mod\mathcal{M})$$

where $\xi_{X(tw_\tau(E)/K^{cyc})}$ is the characteristic power series (in $\mathcal{O}[[T]]$) of $X(tw_\tau(E)/K^{cyc})$, α_q is defined in Lemma 3.3.1 and $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$.

Let G_0 be the maximal pro-p normal subgroup of G then we can prove another important theorem which is very closely related with Parity conjectures.

Theorem 4.3.2. *Let E be a non-isotrivial Elliptic curve defined over K with good ordinary reduction at finite set of places Σ. Then if*

$$(-1)^{\lambda_E(\tau)} = \omega_E(\tau)$$

holds for all self-dual representations τ of G/G_0 then it is also true for any self dual representation τ of G.

Now as a immediate corollary we get one important case of Parity conjecture

Theorem 4.3.3. *Let E be a non-isotrivial Elliptic curve defined over K with good ordinary reduction at finite set of places Σ. Then if E has a l-isogeny over K then for any self-dual Artin representation τ, we have*

$$(-1)^{s_E(\tau)} = \omega_E(\tau)$$

Remark 4.3.4. Now we will use results from Rohrlich ([Roh08]). We would like to note that Rohrlich's proofs in ([Roh08], [Roh06]) are representation-theoretic and rest on papers of Deligne and Tate that are valid for local fields with any residual (or generic) characteristic.

Let us define

$$\theta_n^{-1} := \sum_{\tau} dimW_\tau$$

where the sum is over all irreducible self-dual representation of $Gal(K_n/K)$ where $K_n := K(E_{l^n})$ with $\omega_E(\tau) = -1$. Then from Rohrlich has shown only orthogonal τ occurs in the sum. From (Corollary 2, [Roh08]) follows that $\theta_n^{-1} \geq a.p^{2n}$ for some $a \gneq 0$ if and only if $l \equiv 3 \text{mod } 4$.

On the other hand, from Theorem 4.3.3, $s_{E/K_n} \geq \theta_n^{-1}$. So combining we get

Lemma 4.3.5. *Let Let E be a non-isotrivial Elliptic curve defined over K with good ordinary reduction at finite set of places Σ. Then if E has a l-isogeny over K and $l \equiv 3$ mod 4 then for any self-dual Artin representation τ, we have*

$$s_{E/K_n} \geq a.l^{2n}$$

Lemma 4.3.6. *Let Let E be a non-isotrivial Elliptic curve defined over K with good ordinary reduction at finite set of places Σ. Then if E has a l-isogeny over K and $l \equiv 3$ mod 4 then for any self-dual Artin representation τ, we have*

$$s_{E/K_n} = c.l^{2n}$$

for some c independent of n.

Proof. As $\mathfrak{M}_H(G)$ conjecture is true in this case, by (Page 52, [CFKS10]) we can get a upper bound also i.e. $s_{E/K_n} \leq b.l^{2n}$. With previous lemma, this gives

$$s_{E/K_n} = c.l^{2n}$$

for some c independent of n . □

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