INAUGURAL-DISSERTATION

zur
Erlangung der Doktorwürde
an der
Naturwissenschaftlich-Mathematischen Gesamtfakultät
der Ruprecht-Karls-Universität Heidelberg

TANNAKIAN CATEGORIES OF PERVERSE SHEAVES ON ABELIAN VARIETIES

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Tag der mündlichen Prüfung: ____________

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Zusammenfassung. Die vorliegende Dissertation beschäftigt sich mit Tannaka-Kategorien, welche durch Faltung perverser Garben auf abelschen Varietäten entstehen. Die Konstruktion dieser Kategorien ist eng verbunden mit einem kohomologischen Verschwindungssatz, der ein Analogon zum Satz von Artin-Grothendieck darstellt und als Spezialfall die generischen Verschwindungssätze von Green und Lazarsfeld enthält. Die erhaltenen Tannaka-Gruppen bilden interessante geometrische Invarianten, welche in vielen Situationen eine Rolle spielen — nachdem die Grundlagen für das Studium dieser Gruppen bereitgestellt sind, wird als ein wichtiges Beispiel der Thetadivisor einer prinzipal polarisierten komplexen abelschen Varietät behandelt. Durch Entartungsargumente wird die Tannaka-Gruppe für eine generische abelsche Varietät bestimmt, welche wenigstens in Dimension 4 eine neue Antwort auf das Schottky-Problem liefert. Das Faltungsquadrat des Thetadivisors in Dimension 4 hängt eng zusammen mit einer Familie glatter Flächen vom allgemeinen Typ, und ein Studium dieser Familie führt auf eine Variation von Hodge-Strukturen mit Monodromiegruppe $W(E_6)$, die mit der Prym-Abbildung verbunden ist. Die Dissertation schließt ab mit einer Rekursionsformel für den generischen Rang der durch Faltungen von Kurven entstehenden Brill-Noether-Garben auf Jacobischen Varietäten.

Abstract. We study Tannakian categories that arise from convolutions of perverse sheaves on abelian varieties. The construction of these categories is closely intertwined with a cohomological vanishing theorem which is an analog of the Artin-Grothendieck theorem and contains as a special case the generic vanishing theorems of Green and Lazarsfeld. The arising Tannaka groups form a powerful new tool applicable in many different geometric contexts — after providing the framework for the study of these groups, we consider as an important example the theta divisor of a complex principally polarized abelian variety. Using degeneration techniques we determine the associated Tannaka group for a generic abelian variety, and we show that in dimension 4 this yields a Tannakian solution to the Schottky problem. The convolution square of the theta divisor in dimension 4 is closely related to a family of surfaces of general type, and a detailed study of this family leads to a variation of Hodge structures with monodromy group $W(E_6)$ which is connected with the Prym map. To conclude the dissertation, we provide a recursive formula for the generic rank of Brill-Noether sheaves which arise from the convolution of curves on Jacobian varieties.

Acknowledgements

It is a pleasure for me to thank my advisor, Professor R. Weissauer, for sharing his beautiful ideas in many productive mathematical discussions, for his general open-mindedness in all occuring questions, for his careful reading of the manuscript and for various helpful suggestions concerning the exposition. I am also grateful to my roommates Johannes Schmidt, Thorsten Heidersdorf and Uwe Weselmann and to the other members of the Mathematical Institute at Heidelberg for the pleasant environment which they created during the past years. Furthermore I would like to thank several people outside Heidelberg who enlivened the atmosphere on conferences and expressed their interest in my work, among them Anders Lundman, Bashar Dudin, Christian Schnell, Dan Petersen, Dmitry Zakharov, Giovanni Mongardi, Katharina Heinrich, Nicola Tarasca and Paul Hacking. I am also indebted to the *Studienstiftung des deutschen Volkes* and to all members of its staff for their flexibility and for the financial and ideal support from which I profited a lot. Finally, my most special thanks go to my family and friends for their encouragement and patience during my work, but perhaps even more for the multifarious, pleasant and interesting distractions from it without which I would not have been able to think freely.

Contents

Introduction

The geometry of a smooth complex projective curve C with Jacobian variety $Jac(C)$ is governed, after the choice of a base point on the curve, by the Abel-Jacobi morphism. In particular, the latter defines for each $n \in \mathbb{N}$ a morphism $C^n = C \times \cdots \times C \longrightarrow Jac(C)$ whose fibres correspond to linear series of divisors on the curve, and the loci where the fibre dimension jumps are the subvarieties $W_n^r \subseteq Jac(C)$ of special divisors that have been studied extensively in classical Brill-Noether theory [**6**].

For smooth complex projective varieties Y of higher dimension one still has the Albanese morphism

$$f_n: \quad Y^n = Y \times \cdots \times Y \longrightarrow X = Alb(Y),$$

but here the situation is much harder to describe in general. For example, the image $f_1(Y) \hookrightarrow X$ can have any dimension between zero and $\dim(X)$, and there is no obvious substitute for Brill-Noether theory in general. In some sense, the first three chapters of this thesis provide a possible framework for such a substitute. More specifically, the properties of f_n we are interested in (such as the loci on X where the fibre dimension jumps) can be studied via the higher direct images

$$R^i f_{n*}(\mathbb{C}_{Y^n}) \quad \text{for} \quad i \geq 0,$$

which are constructible sheaves in the sense that there exists a stratification of X into finitely many locally closed subvarieties over which they restrict to locally constant sheaves. In general the geometry of these higher direct image sheaves is rather involved, as we will see in a particular example in chapter 5. However, we propose a simple description of such direct images in terms of the representation theory of a reductive algebraic group that can be attached to Y via the Tannakian formalism [**33**].

The basis for this Tannakian description will be a vanishing theorem for constructible sheaves on abelian varieties which can be seen as an analog of the Artin-Grothendieck affine vanishing theorem [**7**, exp. XIV, cor. 3.2] and contains as a special case the generic vanishing theorems of Green and Lazarsfeld [**47**]. For any character $\chi: \pi_1(X,0) \longrightarrow \mathbb{C}^*$ of the fundamental group, let us denote by L_χ the corresponding local system on X. Then in chapters 1 and 2 which are based on joint work with R. Weissauer [**68**], we show that for any constructible sheaf F on X and a sufficiently general character χ one has

$$H^i(X, F \otimes_{\mathbb{C}} L_\chi) = 0 \quad \text{for all} \quad i > \dim(\mathrm{Supp}(F)).$$

An independent proof of this result has also been given by C. Schnell [**94**], using the Fourier-Mukai transform for holonomic $\mathscr{D}$-modules. Our proof is of a different flavour and is closely intertwined with the definition of the Tannakian categories we are interested in. It is based on an abstract quotient construction for semisimple tensor categories — we only use $\mathscr{D}$-modules at a single place in section 1.4 where we classify all perverse sheaves of Euler characteristic zero (which will be precisely the objects which become isomorphic to zero under the above quotient construction). At present we do not know whether this classification extends to algebraically closed base fields of positive characteristic $p > 0$, but otherwise all our arguments also work for l-adic constructible sheaves on abelian varieties over the algebraic closure of a finite field $\mathbb{F}_p$ for prime numbers $l \neq p$.

Our Tannakian results are best formulated in the framework of perverse sheaves which has its historic roots in the theory of $\mathscr{D}$-modules [**57**] and in the sheaf-theoretic construction of intersection cohomology for singular varieties [**10**]. Let us for convenience briefly recall some basic definitions and notations from loc. cit. For any complex algebraic variety Z, we will denote by $D^b_c(Z,\mathbb{C})$ the derived category of bounded $\mathbb{C}$-sheaf complexes whose cohomology sheaves are constructible for some stratification of Z into Zariski-locally closed subsets. This is a triangulated category, and one can define the full abelian subcategory

$$\mathrm{Perv}(Z,\mathbb{C}) \subset D^b_c(Z,\mathbb{C})$$

of perverse sheaves to be the core of the middle perverse t-structure. More explicitly, a sheaf complex K is said to be semi-perverse if its cohomology sheaves $\mathscr{H}^{-i}(K)$ satisfy the support estimate $\dim(\mathrm{Supp}\,\mathscr{H}^{-i}(K)) \leq i$ for all $i \in \mathbb{Z}$, and it is a perverse sheaf iff both K and its Verdier dual $D(K)$ are semi-perverse. For example, the perverse intersection cohomology sheaf is by definition the intermediate extension

$$\delta_Z = j_{!*}(\mathbb{C}_U[\dim(Z)]) \in \mathrm{Perv}(Z,\mathbb{C})$$

where $j : U \hookrightarrow Z$ denotes the inclusion of a smooth open dense subset. This perverse sheaf is self-dual with respect to Verdier duality, and it does not depend on the choice of the smooth open dense subset. Thus for smooth varieties Z the perverse intersection cohomology sheaf δ_Z coincides with the constant sheaf up to a degree shift. If Z is a closed subvariety of X with embedding $i : Z \hookrightarrow X$, then the direct image $i_* : \mathrm{Perv}(Z,\mathbb{C}) \hookrightarrow \mathrm{Perv}(X,\mathbb{C})$ is a fully faithful functor, and in this case we will by abuse of notation also write δ_Z for the perverse sheaf $i_*(\delta_Z) \in \mathrm{Perv}(X,\mathbb{C})$.

In the context of perverse sheaves, our above vanishing theorem can be reformulated as the statement that for any perverse sheaf $P \in \mathrm{Perv}(X,\mathbb{C})$ and a general character $\chi : \pi_1(X,0) \longrightarrow \mathbb{C}^*$ the hypercohomology groups satisfy the property

$$H^i(X, P\otimes_{\mathbb{C}} L_\chi) \;=\; 0 \quad \text{for all} \quad i \neq 0,$$

as we will explain in more detail in chapter 1. Our methods also allow to determine the precise locus $\mathscr{S}(P)$ of characters χ for which the above vanishing property fails. In section 1.5 we will see that this locus is a finite union of translates of algebraic subtori of the character torus

$$\Pi(X) \;=\; Hom(\pi_1(X,0),\mathbb{C}^*),$$

and that for perverse sheaves $P \in \mathrm{Perv}(X,\mathbb{C})$ of geometric origin in the sense of [**10**, sect. 6.2.4] the occuring translations can be taken to be torsion points. Again this has been shown independently by C. Schnell in [**94**] with a different argument. The two approaches seem to be complementary in some sense — for example, whereas in the framework of loc. cit. the proof of the statement about torsion points requires a deep result of C. Simpson, the methods to be explained below allow to reduce it to a statement over a finite field which can be checked directly by looking at the eigenvalues of the Frobenius operator, see lemma 1.11.

We now come back to the Albanese morphism $f_n : Y^n \longrightarrow X = Alb(Y)$ of a smooth complex projective variety Y of arbitrary dimension. The group law $a : X\times X \longrightarrow X$ of the Albanese variety defines a convolution product on the bounded derived category $D^b_c(X,\mathbb{C})$ by the formula

$$K*M \;=\; Ra_*(K\boxtimes M) \quad \text{for} \quad K,M \;\in\; D^b_c(X,\mathbb{C}),$$

and in these terms the direct image of the constant sheaf can be written (up to a degree shift) as the n-fold convolution

$$Rf_{n*}(\delta_{Y^n}) \;=\; \underbrace{Rf_{1*}(\delta_Y)*\cdots*Rf_{1*}(\delta_Y)}_{n\text{ times}}.$$

Our goal is to interpret this convolution as a tensor product in the category of representations of some reductive algebraic group. To motivate this, let us take a look at the case where $Y=C$ is a curve. Then $f_1 : C \hookrightarrow X$ is the Abel-Jacobi embedding, and we are interested in the convolution powers of the perverse sheaf $\delta_C \in \mathrm{Perv}(X,\mathbb{C})$. Although in general these convolution powers are not perverse, it turns out that the failure of perversity only comes from locally constant sheaves: In view of Gabber's decomposition theorem we can write

$$(\delta_C)^{*n} \;=\; \delta_n \oplus \tau_n$$

where τ_n denotes the maximal direct summand all of whose cohomology sheaves are locally constant on X, and then by [**106**, th. 7] the remaining direct summand δ_n is a perverse sheaf. Let us say that a perverse sheaf is a Brill-Noether sheaf if it is a subquotient of δ_n for some $n \in \mathbb{N}$. Then for a suitable normalization of the Abel-Jacobi morphism, it has been shown in theorem 14 of loc. cit. that the category of all Brill-Noether sheaves is equivalent to the category of algebraic representations of the group

$$G(\delta_C) \;=\; \begin{cases} Sp_{2g-2}(\mathbb{C}) & \text{if } C \text{ is hyperelliptic,} \\ Sl_{2g-2}(\mathbb{C}) & \text{otherwise,} \end{cases}$$

and that the tensor product of representations is induced by the convolution product of perverse sheaves. This Tannakian description reduces classical geometric problems to simple computations in representation theory. For example, if g denotes the genus of C, then the perverse sheaf δ_Θ attached to the theta divisor $\Theta = W_{g-1} \subset X$ corresponds via the above equivalence to the $g-1^{\text{st}}$ fundamental representation of $G(\delta_C)$. Using this one may decompose the convolution $\delta_\Theta * \delta_\Theta$ into its irreducible constituents, which leads to a new proof of Torelli's theorem [**111**].

Alongside our proof of the vanishing theorem and still based on [**68**], in chapter 2 we generalize the Tannakian constructions from above to the case of semisimple sheaf complexes on arbitrary complex abelian varieties X. In particular, we show that every convolution of semisimple perverse sheaves is a direct sum of a semisimple perverse sheaf and a further sheaf complex which is negligible in a suitable sense (for example, on a simple abelian variety a complex is negligible iff all its cohomology sheaves are locally constant). To any semisimple perverse sheaf $P \in \mathrm{Perv}(X, \mathbb{C})$ we then attach a reductive algebraic group $G(P)$ whose representation theory governs the decomposition of the convolution powers $P^{*n} = P * \cdots * P$ up to negligible direct summands, see corollary 2.14.

We remark that for algebraic tori in place of abelian varieties, similar results have been obtained by Gabber and Loeser in [**41**]. However, in their case the essential problem is to define the correct notion of convolution in the non-proper case, and once this has been done, the required properties follow from Artin's vanishing theorem for affine morphisms. By way of contrast, in the case of abelian varieties the main point of the construction is to find a proper analog of Artin's theorem, which will be precisely the vanishing theorem that we stated earlier.

The Tannaka groups $G(P)$ attached to perverse sheaves $P \in \mathrm{Perv}(X, \mathbb{C})$ as above form an interesting family of new invariants which contain much information about the geometry and the moduli of the underlying abelian variety. So far the most efficient method to determine such Tannaka groups has been to study degenerations of the abelian variety (an example for this can be found in chapter 4). Even if one is only interested in semisimple perverse sheaves on abelian varieties, such degenerations naturally lead to non-semisimple perverse sheaves on semiabelian varieties. In chapter 3 we extend our Tannakian constructions to this more general case, combining the results of Gabber and Loeser for tori with our vanishing theorem for abelian varieties. We also give a Tannakian description for the functor of nearby cycles for a family of abelian varieties which degenerates into a semiabelian variety. Even though in the non-proper case the functor of nearby cycles does not preserve the Euler characteristic, we will show that the degenerate Tannaka group is a subgroup of the generic one whenever one can possibly expect this to hold (see theorem 3.15).

In the remaining chapters we study in more detail the case where $P = \delta_\Theta$ is the perverse intersection cohomology sheaf associated with a symmetric theta divisor of a principally polarized abelian variety X. In chapter 4 we use a degeneration argument to show that for a general principally polarized abelian variety (ppav) of dimension g one has

$$G(\delta_\Theta) \;=\; \begin{cases} SO_{g!}(\mathbb{C}) & \text{if } g \text{ is odd,} \\ Sp_{g!}(\mathbb{C}) & \text{if } g \text{ is even,} \end{cases}$$

and that δ_Θ corresponds to the standard representation of this group, a result that was conjectured in [**67**]. As an application we deduce that for $g = 4$ the invariant $G(\delta_\Theta)$ solves the Schottky problem in the sense that it detects the locus of Jacobian varieties inside the moduli space $\mathscr{A}_4$ of ppav's.

Once we know the Tannaka group $G(\delta_\Theta)$, we can use representation theory to decompose arbitrary convolution powers of the perverse sheaf δ_Θ into their simple constituents. This produces a plethora of interesting simple perverse sheaves which describe the Hodge theory of various subvarieties of X. For example, consider the translate $\Theta_x = \Theta + x$ of the theta divisor by a point $x \in X(\mathbb{C})$. The geometry of the intersections

$$Y_x \;=\; \Theta \cap \Theta_x$$

is closely connected with the moduli of the underlying ppav and has been studied classically in relation with Torelli's theorem [**25**], with the Schottky problem [**27**] and with the Prym map [**61**]. The involution $\sigma = -id_X$ acts on these intersections, and if the theta divisor is smooth, then for a general point $x \in X(\mathbb{C})$ the quotient variety

$$Y_x^+ \;=\; Y_x/\sigma$$

will be smooth as well. We will see in chapter 5 that for varying x, the variable part of the cohomology $H^\bullet(Y_x^+, \mathbb{Q})$ can be identified naturally with the stalk cohomology of the alternating resp. symmetric convolution square of the perverse sheaf δ_Θ, depending on whether the dimension $g = \dim(X)$ is even or odd. For a general ppav the Tannakian result of chapter 4 implies that this alternating resp. symmetric convolution square is irreducible up to a skyscraper sheaf and negligible terms.

So for x varying in a Zariski-open dense subset of $X(\mathbb{C})$, the variable part of the cohomology of Y_x^+ defines a variation of Hodge structures whose underlying local system is irreducible. The main topic of chapter 5 is a study of this variation in the case where $g = 4$. In this case we are dealing with a family of smooth surfaces of general type with a surprisingly explicit and beautiful geometry. On the level of integral cohomology, we will determine the Néron-Severi lattices

$$H^2(Y_x, \mathbb{Z}) \quad \text{and} \quad H^2(Y_x^+, \mathbb{Z})$$

of the respective surfaces and deduce that the local system underlying the considered variation of Hodge structures essentially has the monodromy group $W(E_6)$, see theorem 5.1. We will also see that the appearance of this Weyl group in the present context is closely related to the ubiquitous 27 lines on a cubic surface — the link is provided by the Prym-embedded curves on Y_x which have been studied by E. Izadi in [**61**] and appear here as glue vectors for the above Néron-Severi lattices.

In view of this example, it remains an intriguing question to ask for the general relationship between the Tannaka group $G(P)$ attached to a perverse sheaf $P \in \mathrm{Perv}(X, \mathbb{C})$ and the monodromy groups defined by the simple perverse sheaves in the corresponding Tannakian category. At present this relationship remains mysterious even for Brill-Noether sheaves on Jacobian varieties. As a first step in this direction, we conclude this dissertation with a recursive formula for the generic rank of Brill-Noether sheaves — in the hope that this formula may be given a representation-theoretic and more conceptual interpretation at some future time.

Some commonly used notations

$D_c^b(X,\Lambda)$	derived category of bounded constructible sheaf complexes on X as defined in [**10**], in the analytic sense if $\Lambda = \mathbb{C}$ and in the l-adic sense if Λ is an extension of $\mathbb{Q}_l$
$\mathrm{Perv}(X,\Lambda)$	the abelian subcategory of perverse sheaves in $D_c^b(X,\Lambda)$
$H^i(X,K)$	i^{th} hypercohomology group of $K \in D_c^b(X,\Lambda)$
$H_c^i(X,K)$	i^{th} compactly supported hypercohomology group of K
$\mathscr{H}^i(K)$	i^{th} cohomology sheaf of K
${}^pH^i(K)$	i^{th} perverse cohomology sheaf of K
$\chi(K)$	Euler characteristic $\sum_{i\in\mathbb{Z}}(-1)^i \dim H^i(X,K)$ of K
$\mathrm{Supp}(K)$	support of K
$D(K)$	Verdier dual of K
$K(i)$	i-fold Tate twist of K
$\delta_Y = \mathrm{IC}_Y[d]$	perverse intersection cohomology sheaf attached to a closed subvariety $Y \hookrightarrow X$ of dimension d
Λ_Y	constant sheaf with coefficients in the field Λ and support on a closed subvariety $Y \hookrightarrow X$
$\Pi(X)$	group of all characters $\chi : \pi_1(X,0) \longrightarrow \Lambda^*$ of the topological fundamental group, resp. of all continuous characters of the tame étale fundamental group in the l-adic setting
$\Pi(X)_l$	maximal pro-l-subgroup of $\Pi(X)$
L_χ	local system attached to a character $\chi \in \Pi(X)$
$K_\chi = K \otimes_\Lambda L_\chi$	twist of a complex $K \in D_c^b(X,\Lambda)$ by a character $\chi \in \Pi(X)$
$\mathscr{S}(P)$	spectrum of $P \in \mathrm{Perv}(X,\Lambda)$ as defined in section 1.5
$\Psi,\ \Psi_1$	functor of nearby cycles resp. its unipotent part
sp	specialization functor as defined in section 4.3
$G(P)$	Tannaka group attached to $P \in \mathrm{Perv}(X,\Lambda)$ in corollary 3.10
$\mathfrak{A}_n,\ \mathfrak{S}_n$	alternating resp. symmetric group of degree n

CHAPTER 1

Vanishing theorems on abelian varieties

As we mentioned in the introduction, our construction of Tannakian categories will be closely intertwined with a vanishing theorem for perverse sheaves on abelian varieties. In the first two chapters of this dissertation which are based on joint work with R. Weissauer [**68**], we discuss various incarnations of this vanishing theorem and explain how it may be proved in an abstract Tannakian framework.

1.1. The main result

Let X be a complex abelian variety. For characters $\chi : \pi_1(X,0) \longrightarrow \mathbb{C}^*$ of the fundamental group we denote by L_χ the corresponding local system of complex vector spaces on X of rank one, and for a bounded constructible sheaf complex $K \in D^b_c(X,\mathbb{C})$ we consider the twist $K_\chi = K \otimes_{\mathbb{C}} L_\chi$. With these notations, our vanishing theorem can be formulated as

THEOREM 1.1. *Let $P \in \mathrm{Perv}(X,\mathbb{C})$ be a perverse sheaf. Then for most characters χ we have*

$$H^i(X,P_\chi) = 0 \quad \textit{for} \quad i \neq 0.$$

Here we use the following terminology: For abelian subvarieties $A \subseteq X$ let $K(A)$ be the group of characters $\chi : \pi_1(X,0) \longrightarrow \mathbb{C}^*$ whose restriction to the subgroup $\pi_1(A,0) \subseteq \pi_1(X,0)$ is trivial. We then say that a statement holds for most characters χ if it holds for all χ in the complement of a thin set of characters, where by a thin set of characters we mean a finite union of translates $\chi_i \cdot K(A_i)$ for certain non-zero abelian subvarieties $A_i \subseteq X$ and suitable characters χ_i of $\pi_1(X,0)$. The same terminology will be used in theorems 1.4 and 1.5 below for line bundles $\mathscr{L} \in Pic^0(X)$.

The statement of theorem 1.1 can be sharpened as follows. Let $\Pi(X)$ denote the group of characters χ of the fundamental group $\pi_1(X,0)$; this is a complex algebraic torus. For a perverse sheaf $P \in \mathrm{Perv}(X,\mathbb{C})$ we define the spectrum

$$\mathscr{S}(P) \;=\; \left\{\chi \in \Pi(X) \mid H^i(X,P_\chi) \neq 0 \text{ for some } i \neq 0\right\}$$

to be the locus of all characters $\chi \in \Pi(X)$ for which the vanishing in the above theorem fails in some cohomology degree. We will see in section 1.5

below that the spectrum $\mathscr{S}(P)$ is not just contained in a thin subset but actually equal to such a set, in other words

$$\mathscr{S}(P) = \bigcup_{i=1}^{n} \chi_i \cdot K(A_i)$$

is a finite union of translates of algebraic subtori $K(A_i) \subset \Pi(X)$ for certain abelian subvarieties $A_i \subseteq X$ and $\chi_i \in \Pi(X)$. Furthermore, if the perverse sheaf P is of geometric origin in the sense of [**10**, sect. 6.2.4], we will see that the characters χ_i can be chosen to be of finite order.

Our proof of theorem 1.1 is based on two ingredients. The main part is an abstract Tannakian argument to be given in chapter 2 below: Using a general construction of André and Kahn [**1**] we show that a certain quotient of the category of semisimple perverse sheaves on X is a rigid symmetric monoidal semisimple abelian category, and via a criterion by Deligne [**30**] we deduce that this category is super Tannakian in a sense to be explained below. To see that in the case at hand the construction leads to a Tannakian category in the usual sense, we require the second ingredient of the proof which is of a more geometric flavour: A classification of perverse sheaves on X with Euler characteristic zero. This classification is independent from the Tannakian arguments and uses the theory of $\mathscr{D}$-modules, extending the results of Franecki and Kapranov [**37**]. We will discuss it in section 1.4, and this is the only place where we need to work over the base field $\mathbb{C}$. The arguments of chapter 2 also work for l-adic perverse sheaves on abelian varieties over the algebraic closure of a finite field.

Our theorem easily generalizes to a relative vanishing theorem for a homomorphism of abelian varieties as we explain in section 1.2. From a different point of view, it can also be reformulated as a statement about constructible sheaves: Indeed, using dévissage for the perverse t-structure together with Verdier duality one checks that the theorem is equivalent to the statement that any semi-perverse complex K satisfies $H^i(X, K_\chi) = 0$ for all $i > 0$ and most characters χ. Note that for any constructible sheaf F the degree shift $K = F[\dim(\operatorname{Supp} F)]$ is a semi-perverse sheaf complex, so we in particular obtain

THEOREM 1.2. *Let F be a constructible sheaf on X. Then for most characters χ we have*

$$H^i(X, F_\chi) = 0 \quad \textit{for} \quad i > \dim(\operatorname{Supp} F).$$

This can be viewed as an analog of the Artin-Grothendieck affine vanishing theorem in the same way as one can consider the generic vanishing theorem of Green and Lazarsfeld [**47**, th. 1] as an analog of the Kodaira-Nakano vanishing theorem. Motivated by this observation, in section 1.3 we explain how to recover a stronger version of the Green-Lazarsfeld theorem as a special case of our result. In the remaining sections of this chapter we then discuss perverse sheaves with Euler characteristic zero and, as a first application, deduce from the obtained classification the statement about the spectrum of a perverse sheaf mentioned above.

1.2. A relative generic vanishing theorem

Let $A \subseteq X$ be an abelian subvariety and $f : X \longrightarrow B = X/A$ the quotient morphism. Assuming theorem 1.1 only for the abelian variety A, we obtain the following relative generic vanishing theorem. We remark that here the quantifier *most* can be read in the slightly stronger sense that it does not refer to the characters of $\pi_1(X,0)$ but only to their pull-back to $\pi_1(A,0)$ as we will explain in more detail at the end of section 1.5 below.

THEOREM 1.3. *Let* $P \in \mathrm{Perv}(X,\mathbb{C})$*. Then for most characters* χ *the direct image* $Rf_*(P_\chi)$ *is a perverse sheaf on* B*.*

Proof. By Verdier duality it will be enough to show that for most χ the direct image complex $Rf_*(P_\chi)$ satisfies the semi-perversity condition

$$\dim\bigl(\mathrm{Supp}\,\mathscr{H}^{-k}(Rf_*(P_\chi))\bigr) \;\leq\; k \quad \text{for all} \quad k \in \mathbb{Z}.$$

To check this condition, note that by lemma 2.4 and section 3.1 in [**12**] we can find Whitney stratifications $X = \sqcup_\beta X_\beta$ and $B = \sqcup_\alpha B_\alpha$ such that the following properties hold:

(a) for all β, i, χ the cohomology sheaves $\mathscr{H}^{-i}(P_\chi) = \mathscr{H}^{-i}(P) \otimes_{\mathbb{C}} L_\chi$ are locally constant on the strata X_β,

(b) each $f(X_\beta)$ is contained in some B_α,

(c) for all α, β with $f(X_\beta) \subseteq B_\alpha$ the morphism $f : X_\beta \to B_\alpha$ is smooth.

By theorem 4.1 of loc. cit. then the restriction $\mathscr{H}^{-k}(Rf_*(P_\chi))|_{B_\alpha}$ is locally constant for all α, k and χ. Now there are only finitely many strata B_α, and we have $\mathscr{H}^{-k}(Rf_*(P_\chi)) \neq 0$ for only finitely many k. Hence it follows that if the direct image complex $Rf_*(P_\chi)$ were not semi-perverse for most characters χ, then we could find α and k such that

(d) $\dim(B_\alpha) > k$ (where as usual by the dimension of a constructible subset we mean the maximum of the dimensions of the irreducible components of its closure), and

(e) $\mathscr{H}^{-k}(Rf_*(P_\chi))_b \;\neq\; 0$ for all points $b \in B_\alpha(\mathbb{C})$ and all χ in a set of characters which is not thin in the sense of the introduction.

Indeed, if a property does not hold for most characters, then by definition it fails on a set of characters which is not thin. Fixing α and k as above, we now argue by contradiction.

Choose a point $b \in B_\alpha(\mathbb{C})$. Consider the fibre $F_b = f^{-1}(b)$, and for any character χ denote by $M_b = P_\chi|_{F_b}$ the restriction of P_χ to F_b (in what follows we suppress the character twist in this notation). For the perverse cohomology sheaves

$$M_b^r \;=\; {}^pH^{-r}(M_b)$$

we have the spectral sequence

$$E_2^{rs} \;=\; H^{-s}(F_b, M_b^r) \;\Longrightarrow\; H^{-(r+s)}(F_b, M_b) \;=\; \mathscr{H}^{-(r+s)}(Rf_*(P_\chi))_b.$$

Theorem 1.1 for $F_b \cong A$ shows that for most χ we have $H^{-s}(F_b, M_b^r) = 0$ for all $s \neq 0$ and all $r \in \mathbb{Z}$. For such χ the spectral sequence degenerates, i.e.

$$\mathscr{H}^{-k}(Rf_*(P_\chi))_b \;=\; H^0(F_b, M_b^k).$$

On the other hand, by (e) we can assume $\mathscr{H}^{-k}(Rf_*P_\chi)_b \neq 0$. By the above then $M_b^k \neq 0$. Since $M_b^k = {}^pH^0(M_b[-k])$, it follows by definition of the perverse t-structure that

$$\dim\big(\operatorname{Supp} \mathscr{H}^{-i}(M_b)\big) \;=\; i-k \;\geq\; 0 \quad \text{for some} \quad i \in \mathbb{Z}.$$

Now by (a) the support of $\mathscr{H}^{-i}(P_\chi)$ is a union of certain strata X_β, so using the above dimension estimate and the definition of $M_b = P_\chi|_{F_b}$ we find a stratum $X_\beta \subseteq \operatorname{Supp} \mathscr{H}^{-i}(P_\chi)$ with $\dim(F_b \cap X_\beta) = i-k$. Since by (b), (c) the stratum X_β is equidimensional over B_α, it follows that

$$\dim\big(\operatorname{Supp} \mathscr{H}^{-i}(P_\chi)\big) \;\geq\; \dim(X_\beta) \;=\; i-k+\dim(B_\alpha).$$

But $\dim(B_\alpha) > k$ by property (d), so it follows that the perverse sheaf P_χ is not semi-perverse, a contradiction. □

Note that in the proof of theorem 1.3 we have only used theorem 1.1 for the fibres $f^{-1}(b) \cong A$ but not for X itself. Indeed, using this observation and assuming theorem 1.1 only for simple abelian varieties, one can by induction on the dimension deduce for arbitrary abelian varieties a slightly weaker version of theorem 1.1 where *most* is replaced by *generic* [**107**].

1.3. Kodaira-Nakano-type vanishing theorems

From theorem 1.1 one easily recovers stronger versions of the generic vanishing theorems of Green and Lazarsfeld as follows. Let Y be a compact connected Kähler manifold of dimension d whose Albanese variety $\operatorname{Alb}(Y)$ is algebraic, and denote by

$$f:\; Y \longrightarrow X \;=\; \operatorname{Alb}(Y)$$

the Albanese morphism. To pass from coherent to constructible sheaves, recall that every line bundle $\mathscr{L} \in Pic^0(X)$ admits a flat connection; the horizontal sections with respect to such a connection form a local system corresponding to a character $\chi \in \Pi(X)$ such that $\mathscr{L} \cong L_\chi \otimes_{\mathbb{C}} \mathscr{O}_X$.

For a given line bundle $\mathscr{L} \in Pic^0(X)$, the set of all characters χ with the above property is a torsor under the group $H^0(X, \Omega^1_X)$. Indeed, this follows from the truncated exact cohomology sequence

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^1(X, \mathbb{C}^*) \longrightarrow Pic^0(X) \longrightarrow 0$$

attached to the exact sequence $0 \to \mathbb{C}^*_X \to \mathscr{O}^*_X \to \Omega^1_{X,cl} \to 0$ where $\Omega^1_{X,cl}$ denotes the sheaf of closed holomorphic 1-forms. On the other hand, from the point of view of Hodge theory it is better to restrict our attention to unitary characters $\chi : \pi_1(X,0) \to U(1) = \{z \in \mathbb{C}^* \mid |z| = 1\}$, which has the extra benefit of making the passage from coherent to constructible sheaves unique: Comparing the exponential sequences $0 \to \mathbb{Z}_X \to \mathbb{R}_X \to U(1)_X \to 0$ and $0 \to \mathbb{Z}_X \to \mathscr{O}_X \to \mathscr{O}^*_X \to 0$ one sees that the morphism

$$H^1(X, U(1)) \stackrel{\cong}{\longrightarrow} Pic^0(X)$$

is an isomorphism, so for every line bundle $\mathscr{L} \in Pic^0(X)$ there is a unique unitary character χ with $\mathscr{L} \cong L_\chi \otimes_{\mathbb{C}} \mathscr{O}_X$. Concerning the applicability of theorem 3.1 in this unitary context, we remark that the intersection of any thin subset of $\Pi(X)$ with the set of unitary characters is mapped via the above isomorphism to a thin subset of $Pic^0(X)$.

In what follows, for $n \in \mathbb{N}_0$ we put $X_n = \{x \in X \mid \dim(f^{-1}(x)) = n\}$ and consider the number

$$w(Y) \;=\; \min\bigl\{2d - (\dim(X_n) + 2n) \mid n \in \mathbb{N}_0, X_n \neq \varnothing\bigr\}.$$

Notice that $w(Y) \leq d$ (indeed, for some n the preimage $f^{-1}(X_n)$ is dense in Y so that $d = \dim(f^{-1}(X_n)) = \dim(X_n) + n$, hence $2d - (\dim(X_n) + 2n)$ is equal to $2d - (d+n) = d - n \leq d$). Furthermore, by base change one checks that for any local systems E of complex vector spaces on Y the direct image

$$Rf_*E[2d - w(Y)] \quad \text{is semi-perverse,}$$

because for $x \in X_n(\mathbb{C})$ the fibre $F = f^{-1}(x) \subseteq Y$ satisfies $H^i(F, E|_F) = 0$ for all $i > 2n$. This being said, theorem 1.1 implies the following version of the vanishing theorem given in [**47**, th. 2].

THEOREM 1.4. *Let E be a unitary local system on Y. Then for most $\mathscr{L}$ in $Pic^0(Y)$ we have*

$$H^p(Y, \Omega^q_Y(E \otimes_{\mathbb{C}} \mathscr{L})) \;=\; 0 \quad \textit{if} \quad p + q \;<\; w(Y).$$

Proof. The morphism $f^* : Pic^0(X) \longrightarrow Pic^0(Y)$ is an isomorphism by construction of the Albanese variety [**48**, p. 553], so every coherent line bundle $\mathscr{L} \in Pic^0(Y)$ arises as the pull-back of some $\mathscr{M} \in Pic^0(X)$. As explained above, there is a unitary character χ such that

$$\mathscr{M} \;\cong\; \mathscr{O}_X \otimes_{\mathbb{C}} L_\chi.$$

Then $\mathscr{L} \cong f^*(\mathscr{M}) \cong \mathscr{O}_Y \otimes_{\mathbb{C}} f^*(L_\chi)$. Since all the occuring local systems are unitary, Hodge theory says that

$$\bigoplus_{p+q=k} H^p(Y, \Omega_Y^q(E \otimes_{\mathbb{C}} \mathscr{L})) \cong H^k(Y, E \otimes_{\mathbb{C}} f^*(L_\chi)).$$

Putting $K = Rf_*E[2d - w(Y)]$ we can identify the cohomology group on the right hand side with the group $H^{k-2d+w(Y)}(X, K_\chi)$. Since the direct image complex K_χ is semi-perverse, theorem 1.1 shows that for $k > 2d - w(Y)$ and most characters χ the above group vanishes. The theorem now follows by an application of Serre duality. □

For a similar result in this direction, consider for $n \in \mathbb{N}_0$ the closed analytic subsets

$$\overline{X}_n = \{x \in X \mid \dim(f^{-1}(x)) \geq n\} \quad \text{and} \quad \overline{Y}_n = f^{-1}(\overline{X}_n),$$

and put $d_n = \dim(\overline{Y}_n)$ with the convention that $d_n = -\infty$ for $\overline{Y}_n = \emptyset$. Then our vanishing theorem implies the following

THEOREM 1.5. *Suppose that $p + q = d - n$ for some $n \geq 1$. Then for most line bundles $\mathscr{L} \in Pic^0(Y)$,*

$$H^p(Y, \Omega_Y^q(\mathscr{L})) = 0 \quad \textit{unless} \quad d - d_n \leq p, q \leq d_n - n.$$

Proof. By Serre duality the claim is equivalent to the statement that if $p + q = d + n$ for some $n \geq 1$, then $H^p(Y, \Omega_Y^q(\mathscr{L})) = 0$ for most $\mathscr{L}$ unless the Hodge types satisfy the estimates

$$d + n - d_n \leq p, q \leq d_n.$$

In fact it will suffice to establish the upper estimate $p, q \leq d_n$, the lower estimate is then automatic since $p + q = d + n$ by assumption.

The decomposition theorem for compact Kähler manifolds [**90**, th. 0.6] says that $Rf_*\mathbb{C}_Y[d] \cong \bigoplus_m M_m[-m]$ where each M_m is a pure Hodge module on X of weight $m + d$ in the sense of [**89**]. Furthermore, for any unitary character χ with complex conjugate $\bar{\chi}$ the local system $L_\chi \oplus L_{\bar{\chi}}$ of rank two has an underlying real structure and hence can be viewed as a real Hodge module of weight zero in a natural way. So for any real Hodge module M on X also $M_{\chi,\bar{\chi}} = M_\chi \oplus M_{\bar{\chi}}$ is a real Hodge module. This being said, by theorem 1.1 we have

$$H^{d+n}(Y, f^*(L_\chi \oplus L_{\bar{\chi}})) \cong H^n(X, (Rf_*\mathbb{C}_Y[d])_{\chi,\bar{\chi}}) \cong H^0(X, (M_n)_{\chi,\bar{\chi}})$$

for most unitary characters χ. The formalism of Hodge modules equips the cohomology group on the right hand side with a pure $\mathbb{R}$-Hodge structure of weight $n + d$ compatible with the natural one on the left hand side. We are looking for bounds on the types (p, q) in this Hodge structure.

One easily checks that $\mathrm{Supp}(M_n) \subseteq \overline{X}_n$, so $M_n[-n]$ is a direct summand of $Rf_*\mathbb{C}_{\overline{Y}_n}[d]$ by base change. To control the Hodge structure on twists of the cohomology of this direct image, let $\pi : \tilde{Y} \to Y$ be a composition of blow-ups in smooth centers that gives rise to an embedded resolution of singularities $\tilde{Y}_n = \pi^{-1}(\overline{Y}_n) \to \overline{Y}_n$, see [**56**] or [**11**, th. 10.7]. Then $\mathbb{C}_Y[d]$ occurs as a direct summand of the complex $R\pi_*\mathbb{C}_{\tilde{Y}}[d]$ by the decomposition theorem, so the restriction $\mathbb{C}_{\overline{Y}_n}[d]$ is a direct summand of $R\pi_*\mathbb{C}_{\tilde{Y}_n}[d]$. It then follows that $M_n[-n]$ is a direct summand of $Rf_*R\pi_*\mathbb{C}_{\tilde{Y}_n}[d]$, and we get an embedding

$$H^0(X,(M_n)_{\chi,\bar{\chi}}) \hookrightarrow H^{d+n}(\tilde{Y}_n, \pi^* f^*(L_\chi \oplus L_{\chi^{-1}})).$$

But the Hodge types (p,q) on the right hand side satisfy $p,q \leq \dim(\tilde{Y}_n) = d_n$ as one may check from the Hodge theory of compact Kähler manifolds with coefficients in unitary local systems. □

The above result contains the generic vanishing theorem of Green and Lazarsfeld [**47**, second part of th. 1] as the special case $q = 0$. Indeed, for any $p < \dim(f(Y))$ the number $n = d - p$ is larger than the dimension of the generic fibre of the Albanese morphism, hence $d_n < d$ so that $H^p(Y,\mathscr{L}) = 0$ for most $\mathscr{L}$ by theorem 1.5. If Y is algebraic, the theorem also holds more generally for $H^p(Y,\Omega^q_Y(E \otimes_{\mathbb{C}} \mathscr{L}))$ with a unitary local system E on Y.

In general the bounds in the above theorem are strict: If $d = 4$ and if Y is the blow-up of X along a smooth algebraic curve $C \subset X$ of genus ≥ 2, then one has $w(Y) = d_1 = 3$ but $H^2(Y,\Omega^1_Y(\mathscr{L})) \neq 0$ for all non-trivial line bundles $\mathscr{L}$ as explained in [**47**, top of p. 402].

1.4. Negligible perverse sheaves

A crucial ingredient for the proof of our vanishing theorem to be given in section 2.7 will be a classification of all perverse sheaves on X of Euler characteristic zero. To obtain this classification we use an index formula for $\mathscr{D}$-modules, extending the arguments of [**37**, cor. 1.4] which have shown that on a complex abelian variety X every perverse sheaf $P \in \mathrm{Perv}(X,\mathbb{C})$ has Euler characteristic $\chi(P) \geq 0$. As we mentioned earlier, this is the only part of our proof of theorem 1.1 which so far only works over $\mathbb{C}$.

PROPOSITION 1.6. *Let $P \in \mathrm{Perv}(X,\mathbb{C})$ be a simple perverse sheaf.*

(a) One has $\chi(P) = 0$ iff there exists an abelian subvariety $A \hookrightarrow X$ of positive dimension with quotient morphism $q : X \twoheadrightarrow B = X/A$ such that

$$P \cong L_\varphi \otimes q^*(Q)[\dim(A)]$$

for some $Q \in \mathrm{Perv}(B,\mathbb{C})$ and some character $\varphi \in \Pi(X)$.

(b) One has $\chi(P) = 1$ iff P is a skyscraper sheaf on X of rank one.

Proof. Via the Riemann-Hilbert correspondence [**57**] we can consider P as a $\mathscr{D}$-module. For $Z \subseteq X$ closed and irreducible, let $\Lambda_Z \subseteq T^*X$ be the closure in T^*X of the conormal bundle in X to the smooth locus of Z. As in loc. cit. we write the characteristic cycle of P as a finite formal sum

$$CC(P) \;=\; \sum_{Z \subseteq X} n_Z \cdot \Lambda_Z \quad \text{with} \quad n_Z \in \mathbb{N}_0,$$

where Z runs through all closed irreducible subsets of X. From $CC(P)$ the support of P can be recovered via $\operatorname{Supp} P = \bigcup_{n_Z \neq 0} Z$. Furthermore, by the microlocal index formula [**44**, th. 9.1] we have

$$\chi(P) \;=\; \sum_{Z \subseteq X} n_Z \cdot d_Z \quad \text{with} \quad d_Z = [\Lambda_X] \cdot [\Lambda_Z] \in \mathbb{Z}.$$

The intersection numbers d_Z are well-defined even though Λ_Z is not proper for $Z \neq X$, see loc. cit. for details. Now if X is a simple abelian variety, then lemma 1.8 below implies *(a)*, and if we additionally assume $\dim(X) > 1$, also *(b)* follows in view of lemma 1.9 below. The non-simple case can be reduced to the simple case, see [**107**]. □

The reduction to the case of simple abelian varieties in loc. cit. works for ground fields k of characteristic $p > 0$ as well, but we do not know how to deal with simple abelian varieties in that case. For $k = \mathbb{C}$, Christian Schnell has given in [**94**, cor. 3.11] a different proof of proposition 1.6(a) via the Fourier-Mukai transform for $\mathscr{D}$-modules.

COROLLARY 1.7. *A simple perverse sheaf* $P \in \operatorname{Perv}(X, \mathbb{C})$ *has Euler characteristic zero iff* $H^\bullet(X, P_\chi) = 0$ *for most characters* χ.

Proof. "$\Leftarrow$" holds by corollary 2.2. For "$\Rightarrow$" take a positive-dimensional abelian subvariety $A \hookrightarrow X$ with quotient $q: X \twoheadrightarrow B = X/A$ and a character φ such that $P \cong L_\varphi \otimes q^*(Q)[\dim(A)]$ for some perverse sheaf Q on B as in proposition 1.6(a). We can assume that the Euler characteristic of Q is not zero. Then we claim that

$$H^\bullet(X, P_\chi) = H^\bullet(B, Rq_*(P_\chi)) = H^\bullet(B, Rq_*(L_{\varphi\chi}) \otimes Q[\dim(A)])$$

vanishes iff the restriction of the local system $L_{\varphi\chi}$ to $A = \ker(q) \subseteq X$ is not trivial. Indeed, if this restriction is non-trivial, then by base change one checks that $Rq_*(L_{\varphi\chi}) = 0$ and hence a fortiori $H^\bullet(X, P_\chi) = 0$. On the other hand, if this restriction is trivial, then $L_{\varphi\chi} = q^*(L_\psi)$ for some character ψ and then $H^\bullet(X, P_\chi) = H^\bullet(A, \mathbb{C}) \otimes H^\bullet(B, Q_\psi)[\dim(A)]$ is non-zero since the Euler characteristic of Q_ψ is not zero. □

For completeness we include the following two elementary lemmas about characteristic cycles that have been used, in the case of simple abelian varieties, in the proof of proposition 1.6.

LEMMA 1.8. *With notations as above, $d_Z \geq 0$ for all Z. One has $d_Z = 1$ iff Z is reduced to a single point. If X is simple, then $d_Z = 0$ iff $Z = X$.*

Proof. Put $g = \dim(X)$. The cotangent bundle $T^*X = X \times \mathbb{C}^g$ is trivial, and projecting from $\Lambda_Z \subseteq T^*X$ onto the second factor $\mathbb{C}^g$ induces the Gauß mapping $p : \Lambda_Z \to \mathbb{C}^g$. By [**37**, prop. 2.2] the intersection number d_Z is the generic degree of p. In particular $d_Z \geq 0$ for all Z.

If $d_Z = 1$, then Λ_Z is birational to $\mathbb{C}^g$, so by [**75**, cor. 3.9] there does not exist a non-constant map from Λ_Z to an abelian variety. So the image Z of the composite map $\Lambda_Z \subseteq T^*X \longrightarrow X$ consists of a single point.

If $d_Z = 0$, then p is not surjective, so $\dim(p(\Lambda_Z)) < g$. Then for some cotangent vector $\omega \in p(\Lambda_Z)$ the fibre $p^{-1}(\omega)$ is positive-dimensional, and for $Z \neq X$ we can assume $\omega \neq 0$. Let $Y \subseteq X$ be the image of $p^{-1}(\omega) \subseteq T^*X$ under the map $T^*X \longrightarrow X$. Then $\dim(Y) > 0$, and up to a translation we can assume $0 \in Y$. By construction ω is normal to Y in every smooth point of Y, so the preimage of Y under the universal covering $\mathbb{C}^g \longrightarrow X = \mathbb{C}^g/\Lambda$ lies in the hyperplane of $\mathbb{C}^g$ orthogonal to ω. Accordingly the abelian subvariety of X generated by Y is strictly contained in X, but not zero. This contradicts our assumption that X is simple. □

LEMMA 1.9. *Put $g = \dim(X)$, and assume $P \in \mathrm{Perv}(X, \mathbb{C})$ is simple. If there is a closed subset $Y \subset X$ with $\dim(Y) \leq g - 2$ such that*

$$CC(P) \; = \; n_X \Lambda_X + \sum_{Z \subseteq Y} n_Z \Lambda_Z \quad \textit{and} \quad n_X > 0,$$

then $P = L_\chi[g]$ for some character χ, hence $n_X = 1$ and $n_Z = 0$ for $Z \neq X$.

Proof. Consider the open embedding $j : U = X \setminus Y \hookrightarrow X$. Since open embeddings are non-characteristic for any $\mathscr{D}_X$-module, by [**57**, sect. 2.4] we have $CC(j^*(P)) \; = \; CC(P) \cap T^*U \; = \; n_X \cdot \Lambda_U$. So we get $j^*(P) = L_U[g]$ for a local system L_U on U by prop. 2.2.5 in loc. cit. Since X is smooth, the purity of the branch locus and $\dim(Y) \leq g - 2$ implies $L_U = j^*(L)$ for a local system L on X. By simplicity of P it follows that $P = L[g]$, and as an irreducible representation of the abelian group $\pi_1(X, 0)$ the local system L must have rank one. □

1.5. The spectrum of a perverse sheaf

Using the classification from the previous section, we now explain how to determine for $P \in \mathrm{Perv}(X, \mathbb{C})$ the spectrum

$$\mathscr{S}(P) \; = \; \big\{ \chi \in \Pi(X) \mid H^i(X, P_\chi) \; \neq \; 0 \text{ for some } i \neq 0 \big\}.$$

As a refinement of theorem 1.1 we will show that $\mathscr{S}(P)$ is a finite union of translates of proper algebraic subtori of $\Pi(X)$ and that furthermore for P of geometric origin each of the occuring translations can be taken to be torsion points of $\Pi(X)$, see lemma 1.11 below. More generally, for semisimple sheaf complexes $K = \bigoplus_{n\in\mathbb{Z}} {}^pH^{-n}(K)[n]$ we define

$$\mathscr{S}(K) \;=\; \bigcup_{n\in\mathbb{Z}} \mathscr{S}({}^pH^{-n}(K)).$$

The definitions imply that $\mathscr{S}(K_\chi) = \chi^{-1}\cdot\mathscr{S}(K)$ for all $\chi\in\Pi(X)$, and for all semisimple K_1, K_2 we have

$$\mathscr{S}(K_1 * K_2) \;\subseteq\; \mathscr{S}(K_1)\cup\mathscr{S}(K_2) \;=\; \mathscr{S}(K_1\oplus K_2),$$

where $K_1 * K_2$ denotes the convolution of K_1 and K_2 as defined in section 2.1 below. Note that the equality displayed on the right hand side reduces the computation of the spectrum of semisimple sheaf complexes to the case of simple perverse sheaves. Of course the spectrum $\mathscr{S}(P)$ may be empty, for example if P is a skyscraper sheaf or, more interestingly, if $P = i_*E[1]$ where $i: C\hookrightarrow X$ is the embedding of a smooth curve in X and where E is an irreducible local system on C of rank at least two.

Since $\pi_1(X,0)\cong\mathbb{Z}^{2g}$, the character group $\Pi(X)\cong\mathbb{C}^{2g}$ is a complex algebraic torus of rank $2g$. Furthermore Π is a contravariant functor: For any homomorphism $h:\ X\longrightarrow B$ of complex abelian varieties, the pull-back of characters gives rise to a homomorphism $\Pi(h):\ \Pi(B)\longrightarrow\Pi(X)$ between the corresponding algebraic tori.

REMARK 1.10. *The functor* Π *has the following properties.*

(a) Let $h: X\to B$ *be an isogeny with kernel* F*. Then we have an exact sequence*

$$0\longrightarrow Hom(F,\mathbb{C}^*)\longrightarrow\Pi(B)\xrightarrow{\Pi(h)}\Pi(X)\longrightarrow 0.$$

For $P\in\mathrm{Perv}(X,\mathbb{C})$ *we have* $h_*(P)\in\mathrm{Perv}(B,\mathbb{C})$*, and* $\Pi(h)$ *induces a surjection*

$$\mathscr{S}(h_*(P))\twoheadrightarrow\mathscr{S}(P).$$

(b) Let $i: A\hookrightarrow X$ *be an embedding of abelian varieties with quotient morphism* $q: X\to B = X/A$*. Then we have an exact sequence*

$$0\longrightarrow\Pi(B)\xrightarrow{\Pi(q)}\Pi(X)\xrightarrow{\Pi(i)}\Pi(A)\longrightarrow 0.$$

In this situation we denote by $K(A)\subseteq\Pi(X)$ *the image of* $\Pi(q)$*.*

Proof. The exactness of the considered sequences can be seen from the description of a complex abelian variety as the quotient of a complex vector space modulo a lattice. For the surjectivity $\mathscr{S}(g_*(P))\twoheadrightarrow\mathscr{S}(P)$ in part *(a)* one can use that $H^i(X,P_\varphi) = H^i(B,g_*(P)_\chi)$ for the pull-back $\varphi = \Pi(g)(\chi)$ and that $\Pi(g)$ is surjective. □

In what follows, let $E(X)$ be the class of all semisimple perverse sheaves on X with Euler characteristic zero. We say that a perverse sheaf is clean if it does not contain constituents from $E(X)$. For $x \in X(\mathbb{C})$ we denote by $t_x : X \to X$ the translation morphism $y \mapsto x+y$, and for $K \in D_c^b(X,\mathbb{C})$ we consider the stabilizer

$$Stab(K) \ = \ \{x \in X(\mathbb{C}) \mid t_x^*(K) \cong K\}.$$

By constructibility of K this is an algebraic subset of X, and its connected component $Stab(K)^0 \subseteq Stab(K)$ is an abelian subvariety of X. We can now formulate the main result of the present section.

LEMMA 1.11. *(a) Let $P \in E(X)$ be a semisimple perverse sheaf of Euler characteristic zero. Then a character χ lies in $\mathscr{S}(P)$ iff $H^\bullet(X,P_\chi) \neq 0$. In particular, if P is simple, there exists a character φ such that*

$$\mathscr{S}(P) \ = \ \varphi^{-1} \cdot K(A) \quad \textit{for} \quad A = Stab(P)^0$$

where $K(A) \subset \Pi(X)$ is the algebraic subtorus from remark 1.10(b).

(b) For any semisimple perverse sheaf P on X there are non-zero abelian subvarieties $A_1,\dots,A_n \subseteq X$ and $\chi_1,\dots,\chi_n \in \Pi(X)$ such that

$$\mathscr{S}(P) \ = \ \bigcup_{i=1}^{n} \chi_i \cdot K(A_i).$$

(c) If in part (b) the semisimple perverse sheaf P is of geometric origin in the sense of [**10**, 6.2.4]*, then the χ_i can be chosen to be torsion characters.*

Proof. (a) The first statement is obvious from proposition 1.6(a), and the second one follows easily from the proof of corollary 1.7.

(b) The proof of this part is most conveniently formulated in terms of the convolution product $*$ to be defined in section 2.1 below. For the time being it suffices to know that the convolution of semisimple sheaf complexes is again a semisimple sheaf complex by Gabber's decomposition theorem, that for $K,M \in D_c^b(X,\mathbb{C})$ we have the Künneth formula

$$H^\bullet(X,K*M) \ = \ H^\bullet(X,K) \otimes_{\mathbb{C}} H^\bullet(X,M)$$

and that convolution is compatible with character twists in the sense that for all $\chi \in \Pi(X)$ we have $(K*M)_\chi \cong K_\chi * M_\chi$, see proposition 2.1. Hence for the g-fold convolution power of a semisimple perverse sheaf $P \in \mathrm{Perv}(X,\mathbb{C})$ it follows from theorem 1.1 that $P^{*g} = Q \oplus \bigoplus_{\nu\in\mathbb{Z}} N_\nu[\nu]$ where Q is a clean semisimple perverse sheaf and the N_ν are semisimple perverse sheaves in $E(X)$. So for any $\chi \in \Pi(X)$ we obtain

$$(\star) \qquad H^\bullet(X,P_\chi)^{\otimes g} \ = \ H^\bullet(X,Q_\chi) \oplus \bigoplus_{\nu\in\mathbb{Z}} H^\bullet(X,(N_\nu)_\chi)[\nu].$$

If $\chi \in \mathscr{S}(P)$, then $H^\bullet(X,P_\chi)$ is not concentrated in degree zero, so $(\star)$ is non-zero in some degree d with $|d| \geq g$. But Q is a clean perverse sheaf and as such it does not contain the constant perverse sheaf, hence

$$H^d(X,Q_\chi) \;=\; 0 \quad \text{for} \quad |d| \geq g$$

by the adjunction property in [**10**, prop. 4.2.5]. So it follows from the above that $H^\bullet(X,(N_\nu)_\chi) \neq 0$ for some index ν and hence that $\chi \in \mathscr{S}(N_\nu)$ by part *(a)*. Conversely, $\chi \in \mathscr{S}(N_\nu)$ implies by proposition 1.6*(a)* that the hypercohomology $H^\bullet(X,(N_\nu)_\chi)$ is non-zero in more than one cohomology degree; then by $(\star)$ the same holds for $H^\bullet(X,P_\chi)$, so $\chi \in \mathscr{S}(P)$. Altogether this shows

$$\mathscr{S}(P) \;=\; \bigcup_{\nu \in \mathbb{Z}} \mathscr{S}(N_\nu),$$

and part *(b)* of our claim follows by applying part *(a)* to the $N_\nu \in E(X)$.

(c) First we claim that a local system L_χ is of geometric origin iff χ is a torsion character. For the non-trivial direction note that if L_χ is of geometric origin, then X has a model X_A over a subring $A \subset \mathbb{C}$ of finite type over $\mathbb{Z}$ such that L_χ descends to a local system on X_A, corresponding to a continuous character

$$\chi_A: \;\; \pi_1(X_A) \;\longrightarrow\; \overline{\mathbb{Q}}_l^*.$$

Take a closed point of $Spec(A)$ with finite residue field κ. Let $V \subset \mathbb{C}$ be a strictly Henselian ring with $A \subset V$ whose residue field is an algebraic closure $\bar{\kappa}$ of κ. For $X_V = X_A \times_A V$ the inclusion of the special fibre $X_{\bar{\kappa}}$ induces an epimorphism $\pi_1(X_{\bar{\kappa}}) \twoheadrightarrow \pi_1(X_V)$ by [**51**, exp. X, 1.6]. In view of the commutative diagram

$$\begin{tikzcd}
 & \pi_1(X) \arrow[d] \arrow[ddr, bend left, "\chi"] & \\
\pi_1(X_{\bar{\kappa}}) \arrow[r, two heads] \arrow[d] & \pi_1(X_V) \arrow[d] & \\
\pi_1(X_\kappa) \arrow[r] \arrow[rr, bend right, "\chi_\kappa"'] & \pi_1(X_A) \arrow[r, "\chi_A"] & \overline{\mathbb{Q}}_l^*
\end{tikzcd}$$

it then suffices to show that the pull-back $\chi_\kappa : \pi_1(X_\kappa) \longrightarrow \overline{\mathbb{Q}}_l^*$ is a torsion character. But since κ is a finite field, this follows by a consideration of the eigenvalues of the Frobenius operator as in [**28**, 1.3.4(i)]. This proves our claim that L_χ is of geometric origin iff χ is a torsion character.

Now for P of geometric origin the perverse sheaves $N_\nu \in E(X)$ in *(b)* and hence also all their simple constituents N are of geometric origin. Each such constituent has the form $N \cong L_\varphi \otimes q^*(Q)[\dim(A)]$ for suitable φ and some abelian subvariety $A \subseteq X$ as in proposition 1.6*(a)*. In this situation the pullback $i^*(N)$ to A is an isotypic multiple of $i^*(L_\varphi)$ and of geometric origin. Hence $\Pi(i)(\varphi)$ is a torsion character. Writing $\mathscr{S}(N) = \chi \cdot K(A)$ we can take for χ^{-1} any torsion character in $\Pi(i)^{-1}(\Pi(i)(\varphi))$. $\square$

The above arguments can also be generalized to the relative setting of theorem 1.3. For a homomorphism $f : X \to B$ of abelian varieties, define the relative spectrum $\mathscr{S}_f(P)$ of a perverse sheaf $P \in \mathrm{Perv}(X, \mathbb{C})$ to be the set of all $\chi \in \Pi(X)$ such that $Rf_*(P_\chi)$ is not perverse. By abuse of notation, for $\chi \in \Pi(X)$ and $\psi \in \Pi(B)$ we write $\chi\psi = \chi \cdot (\Pi(f)(\psi)) \in \Pi(X)$. Then the projection formula shows

$$Rf_*(P_{\chi\psi}) \;=\; (Rf_*(P_\chi))_\psi,$$

hence $\mathscr{S}_f(P)$ is invariant under $\Pi(B)$. In particular, if $B = X/A$ for an abelian subvariety $A \subseteq X$, then $\mathscr{S}_f(P)$ is determined by its image $\overline{\mathscr{S}}_f(P)$ in $\Pi(A) = \Pi(X)/\Pi(B)$. Furthermore, in the situation of theorem 1.3 the assertion *for most characters* can be read in $\Pi(A)$, i.e. in the stronger sense that $\overline{\mathscr{S}}_f(P)$ is contained in a finite union of translates of proper algebraic subtori of $\Pi(A)$. Indeed we have

LEMMA 1.12. $\mathscr{S}_f(P) \subseteq \mathscr{S}(P) \cdot \Pi(B)$.

Proof. If a character χ is not in $\mathscr{S}(P) \cdot \Pi(B)$, then for any $\psi \in \Pi(B)$ we have $\chi\psi \notin \mathscr{S}(P)$ and hence $H^n(X, P_{\chi\psi}) = H^n(B, (Rf_*(P_\chi))_\psi) \neq 0$ for some $n \neq 0$. By theorem 1.1 then $Rf_*(P_\chi)$ is not perverse. $\square$

CHAPTER 2

Proof of the vanishing theorem(s)

In this chapter we give an abstract Tannakian proof of theorem 1.1 based on joint work with R. Weissauer [**68**]. As a geometric input we only require the classification of perverse sheaves with Euler characteristic zero from section 1.4. For this classification we have used the theory of $\mathscr{D}$-modules, and at present we do not know whether it also holds over a ground field of positive characteristic. However, since this is the only reason why for the time being the proof only applies to complex abelian varietes, in what follows we formulate our arguments in a uniform way including also the case of l-adic perverse sheaves in positive characteristic.

2.1. The setting

Let X be an abelian variety over an algebraically closed field k which has characteristic zero or is the algebraic closure of a finite field. As in [**10**] we consider the derived category $D_c^b(X,\Lambda)$ of complexes of Λ-sheaves with bounded constructible cohomology sheaves, where Λ is either a subfield of $\overline{\mathbb{Q}}_l$ for some fixed prime number $l \neq char(k)$ or a subfield of $\mathbb{C}$ if we are working over the base field $k = \mathbb{C}$. We denote by $\mathrm{Perv}(X,\Lambda) \subset D_c^b(X,\Lambda)$ the full abelian subcategory of perverse sheaves as defined in loc. cit., and we write $\pi_1(X,0)$ for the étale fundamental group, resp. for the topological fundamental group in the complex case. By a character $\chi : \pi_1(X,0) \longrightarrow \Lambda^*$ we mean in the étale setting a continuous character with image in a finite extension field of $\mathbb{Q}_l$. Any such character defines a local system L_χ, and as in the previous chapter we denote by $K_\chi = K \otimes_\Lambda L_\chi$ the corresponding twist of a complex $K \in D_c^b(X,\Lambda)$.

Our Tannakian proof of theorem 1.1 relies on the notion of convolution of sheaf complexes. Let $a : X \times X \longrightarrow X$ denote the group law, and consider the convolution product

$$* : \quad D_c^b(X,\Lambda) \times D_c^b(X,\Lambda) \longrightarrow D_c^b(X,\Lambda), \quad K_1 * K_2 = Ra_*(K_1 \boxtimes K_2).$$

It has been shown in [**106**, sect. 2.1] that the category $D_c^b(X,\Lambda)$ equipped with this convolution product is a symmetric monoidal category in the sense of [**73**, sect. VII.7]. In other words, there exists a unit object $\mathbf{1}$ in $D_c^b(X,\Lambda)$

such that we have natural unity constraints

$$K * \mathbf{1} \xleftarrow{\cong} K \xrightarrow{\cong} \mathbf{1} * K$$

and natural commutativity and associativity constraints

$$S_{K,L}: \ K * L \xrightarrow{\cong} L * K \quad \text{and} \quad A_{K,L,M}: \ K * (L * M) \xrightarrow{\cong} (K * L) * M$$

for all objects K, L, M in $D_c^b(X, \Lambda)$, and these constraints satisfy the usual compatibilities, in particular the pentagon and hexagon axioms as well as the symmetry property $S_{L,K} \circ S_{K,L} = id_{K*L}$. The commutativity constraint is defined as follows. The involution $\sigma : X \times X \longrightarrow X \times X, (x,y) \mapsto (y,x)$ together with the usual commutativity constraint for the tensor product of complexes induces a natural isomorphism $K \boxtimes L \cong \sigma^*(L \boxtimes K)$, and $S_{K,L}$ is defined by the following commutative diagram.

$$\begin{tikzcd}[column sep=small]
& K * L \arrow[rr, "S_{K,L}"] \arrow[dl, equal] & & L * K \arrow[dr, equal] & \\
Ra_*(K \boxtimes L) \arrow[dr, "\cong"] & & & & Ra_*(L \boxtimes K) \arrow[dl, "\cong"'] \\
& Ra_*(\sigma^*(L \boxtimes K)) \arrow[rr, equal] & & Ra_*(\sigma_* \sigma^*(L \boxtimes K)) &
\end{tikzcd}$$

The definition of the associativity constrains $A_{K,L,M}$ is similar, and the unit object $\mathbf{1}$ of the category $D_c^b(X, \Lambda)$ is the skyscraper sheaf δ_0 of rank one with support in the zero point of the abelian variety.

Furthermore, if we define the adjoint dual of a complex $K \in D_c^b(X, \Lambda)$ in terms of the Verdier dual $D(K)$ by

$$K^\vee = (-id_X)^* D(K),$$

then the symmetric monoidal category $D_c^b(X, \Lambda)$ has been shown in [**105**] to be rigid in the following sense: We have natural evaluation and coevaluation morphisms

$$ev_K: \ K^\vee * K \longrightarrow \mathbf{1} \quad \text{and} \quad coev_K: \ \mathbf{1} \longrightarrow K * K^\vee$$

such that the composite morphisms

$$K \xrightarrow{coev_K * id_K} (K * K^\vee) * K \xrightarrow{A^{-1}_{K,K^\vee,K}} K * (K^\vee * K) \xrightarrow{id_K * ev_K} K$$

$$K^\vee \xrightarrow{id_{K^\vee} * coev_{K^\vee}} K^\vee * (K * K^\vee) \xrightarrow{A_{K^\vee,K,K^\vee}} (K^\vee * K) * K^\vee \xrightarrow{ev_K * id_{K^\vee}} K^\vee$$

are the identity morphisms. In passing we remark that every skyscraper sheaf $K = \delta_x$ of rank one, supported in a point $x \in X(\mathbb{C})$, is an invertible object in the sense that the morphism ev_K is an isomorphism. Over a base field k of characteristic zero every invertible object has this form, indeed from the Künneth formula [**7**, exp. XVII, sect. 5.4]

$$H^\bullet(X, K * L) \;\cong\; H^\bullet(X,K) \otimes_\Lambda H^\bullet(X,L)$$

one easily checks that every invertible object has Euler characteristic one, so proposition 1.6 *(b)* applies. In what follows, if we want to stress the rigid symmetric monoidal structure on $D^b_c(X,\Lambda)$ we write $(D^b_c(X,\Lambda), *)$.

The most prominent examples of rigid symmetric monoidal categories are the abelian categories $(Vect_\Lambda, \otimes)$ of finite-dimensional vector spaces and more generally the abelian categories $(Rep_\Lambda(G), \otimes)$ of representations of an affine group scheme G over Λ. Indeed, at the heart of the Tannakian formalism lies the fact that a rigid symmetric monoidal Λ-linear abelian category $(\mathbf{C}, *)$ with $End_{\mathbf{C}}(\mathbf{1}) = \Lambda$ is of the form $(Rep_\Lambda(G), \otimes)$ provided that it admits a fibre functor, by which we mean a faithful exact Λ-linear functor

$$\omega : \quad (\mathbf{C}, *) \longrightarrow (Vect_\Lambda, \otimes)$$

which is a tensor functor ACU in the sense that it is compatible with the symmetric monoidal structures on both sides [**33**, th. 2.11].

Now the Λ-linear rigid symmetric monoidal category $D^b_c(X,\Lambda)$ is not an abelian category but only triangulated; on the other hand, its full abelian subcategory $\mathrm{Perv}(X,\Lambda)$ does not inherit the structure of a rigid symmetric monoidal category since in general the convolution of two perverse sheaves is no longer perverse. However, for the full subcategory $\mathbf{D} \subseteq D^b_c(X,\Lambda)$ of all direct sums of degree shifts of semisimple perverse sheaves, we will obtain in section 2.4 by a general quotient construction due to André and Kahn [**1**] a symmetric monoidal quotient category $\bar{\mathbf{D}}$ which is indeed a semisimple abelian category. Under the quotient morphism

$$\mathbf{D} \longrightarrow \bar{\mathbf{D}}$$

an sheaf complex $K \in \mathbf{D}$ becomes isomorphic to zero in $\bar{\mathbf{D}}$ iff all its simple constituents have Euler characteristic zero. Over $k = \mathbb{C}$ the classification in proposition 1.6 shows that this is the case if and only if $H^\bullet(X, K_\chi) = 0$ for most characters χ. Furthermore, via a characterization of semisimple rigid symmetric Λ-linear abelian categories given by Deligne [**30**] we will see in section 2.5 that the category $\bar{\mathbf{D}}$ is a limit of representation categories of reductive algebraic super groups over Λ.

It then turns out that the non-perversity of convolution products can be controlled by central characters of these groups. This will imply via an argument from representation theory that the semisimple perverse sheaves define a full subcategory of $\bar{\mathbf{D}}$ that is stable under convolution. So for any semisimple perverse sheaf P and $r \in \mathbb{N}$ the convolution powers P^{*r} split by Gabber's decomposition theorem into a sum of a perverse sheaf and a complex N_r which is isomorphic to zero in the quotient category $\bar{\mathbf{D}}$. From the Künneth isomorphism

$$H^\bullet(X, P^{*r}) \;\cong\; (H^\bullet(X,P))^{\otimes r}$$

we will then deduce in lemma 2.11 the vanishing in theorem 1.1 for any character χ with the property that $H^\bullet(X,(N_r)_\chi) = 0$ for some $r > g$. Indeed the vanishing theorem is closely connected to the Tannakian property of the above categories — for the rigid symmetric abelian subcategory $\mathbf{C} = \langle P \rangle$ generated inside $\bar{\mathbf{D}}$ by a semisimple perverse sheaf P, we will see later on that the functor

$$\omega: \quad \mathbf{C} \longrightarrow Vect_\Lambda, \quad K \mapsto H^0(X, K_\chi)$$

is a fibre functor for any character χ with the property in theorem 1.1. As a first step in this direction, we will show in the next section that the twist functor $K \mapsto K_\chi$ is a tensor functor ACU.

2.2. Character twists and convolution

In this section we show that twisting by a character χ defines a tensor functor ACU in the sense of [**86**, chapt. I, sect. 4.2.4] on the symmetric monoidal category $D_c^b(X,\Lambda)$ with respect to convolution. Recall that this means that we have natural isomorphisms $(K * L)_\chi \cong K_\chi * L_\chi$ for all K, L in $D_c^b(X,\Lambda)$, compatible with the associativity, commutativity and unity constraints. This observation will be crucial for lemma 2.11 below, but its proof is rather formal and may be skipped at a first reading.

PROPOSITION 2.1. *For any character χ, the auto-equivalence $K \mapsto K_\chi$ of the category $D_c^b(X,\Lambda)$ defines a tensor functor ACU which is compatible with degree shifts and perverse truncations.*

Proof. The functor $K \mapsto K_\chi$ preserves semi-perversity, so it is t-exact with respect to the perverse t-structure since $D(K_\chi) \cong D(K)_{\chi^{-1}}$. It remains to check tensor functoriality. Clearly $\mathbf{1}_\chi \cong \mathbf{1}$.

Depending on the context, put $R = \mathbb{Z}_l$, $R = \mathbb{Z}$ or $R = \mathbb{Z}/n\mathbb{Z}$, including the case where $p = char(k)$ divides n. We claim that in all these cases the group law $a: X \times X \longrightarrow X$ induces on the first cohomology groups the diagonal morphism

$$a^*: \; H^1(X,R) \longrightarrow H^1(X \times X, R) = H^1(X,R) \oplus H^1(X,R), \; x \mapsto (x,x).$$

For $R = \mathbb{Z}_l$ or $R = \mathbb{Z}$ this follows from the formula preceding lemma 15.2 in [**75**]. For a finite coefficient ring $R = \mathbb{Z}/n\mathbb{Z}$ with $p \nmid n$ we have $R \cong \mu_n$ since the ground field k is algebraically closed; then our claim follows from the identification $H^1(X,\mu_n) \cong Pic^0(X)[n]$ in [**74**, cor. III.4.18], since for coherent line bundles $\mathscr{L} \in Pic^0(X)$ one has $a^*(\mathscr{L}) \cong pr_1^*(\mathscr{L}) \otimes pr_2^*(\mathscr{L})$

by [**75**, prop. 9.2]. It remains to deal with $R = \mathbb{Z}/n\mathbb{Z}$ when $n = p^r$ for some integer $r \in \mathbb{N}$. In that case $H^1(X, \mathbb{Z}/n\mathbb{Z}) \cong H^1(X, W_r)^F$ by [**95**, prop. 13], and our claim then follows by taking Frobenius invariants and using the isomorphism $H^1(X \times X, W_r) \cong H^1(X, W_r) \oplus H^1(X, W_r)$ of [**98**, p. 136].

Now, using [**75**, rem. 15.5] for $R = \mathbb{Z}_l$ and [**95**, p. 50] for $R = \mathbb{Z}/n\mathbb{Z}$ we have in all cases a natural identification

$$H^1(X, R) \;=\; Hom(\pi_1(X, 0), R),$$

where in the étale setting homomorphisms are required to be continuous. If we write the group structure on fundamental groups additively, it follows that

$$a_* : \; \pi_1(X,0) \times \pi_1(X,0) \;=\; \pi_1(X \times X, 0) \;\longrightarrow\; \pi_1(X,0)$$

is the addition morphism $(x,y) \mapsto x+y$. For $\psi \in Hom(\pi_1(X,0), R)$ this implies $\psi(a_*(x,y)) = \psi(x+y) = \psi(x) + \psi(y)$, i.e. $\psi \circ a_* = \psi \boxtimes \psi$ as an additive character on $\pi_1(X,0) \times \pi_1(X,0) = \pi_1(X \times X, 0)$. For multiplicative characters $\chi : \pi_1(X,0) \to \Lambda^*$ this implies

$$\chi(a_*(x,y)) \;=\; \chi(x+y) = \chi(x) \cdot \chi(y), \quad \text{i.e.} \quad \chi \circ a_* \;=\; \chi \boxtimes \chi.$$

Indeed, for $\Lambda \subseteq \mathbb{C}$ one has $Hom(\pi_1(X,0), R) \otimes_R \mathbb{C}^* = Hom(\pi_1(X,0), \mathbb{C}^*)$ taking $R = \mathbb{Z}$. For $\Lambda \subseteq \overline{\mathbb{Q}}_l$ any multiplicative character χ takes values in $E^* \cong \mathbb{Z} \times F^* \times U$, where F is the residue field of a finite extension field E of $\mathbb{Q}_l$ and U is its group of 1-units. By continuity we have $\chi = \chi_F \cdot \chi_U$ for certain characters χ_F and χ_U with values in F^* resp. U. The character χ_U can be handled as above, and the discussion for the character χ_F is covered by the case $R = \mathbb{Z}/n\mathbb{Z}$ with $n = \#F^*$.

For the local system $L = L_\chi$ defined by a character $\chi : \pi_1(X,0) \to \Lambda^*$ this gives an isomorphism of local systems on $X \times X$

$$\varphi : \; a^* L \;\xrightarrow{\sim}\; L \boxtimes L$$

which is uniquely determined up to multiplication by an scalar in Λ^*. In what follows, we fix a choice of φ once and for all. The choice of φ will not matter for the commutativity of the diagrams below, as long as we use the same φ consistently. However, we remark that the datum of a tensor functor consists not only of the underlying functor but also comprises the isomorphisms that describe the compatibility of the functor with the tensor product. In this sense, different choices of φ will lead to different (though of course isomorphic) tensor functors. For us, it will be most convenient to fix a trivialization $\lambda : L_0 \cong \Lambda$ of the stalk L_0 at the origin 0 of X, and to require that the stalk morphism $\varphi_0 : a^* L_{(0,0)} \longrightarrow (L \boxtimes L)_{(0,0)} = L_0 \otimes_\Lambda L_0$ at the origin $(0,0)$ of $X\times X$ makes the following diagramm commutative:

$$\begin{tikzcd}
(a^*L)_{(0,0)} \arrow[r, "\varphi_0"] \arrow[d, equal] & L_0\otimes_\Lambda L_0 \arrow[d, "\lambda\otimes\lambda"] \\
L_0 \arrow[r, "\lambda"] & \Lambda
\end{tikzcd}$$

Here $L_0 = e_X^*(L) = e_{X\times X}^*(a^*(L)) = (a^*L)_{(0,0)}$ since $a\circ e_{X\times X} = e_X$ holds for the unit sections

$$e_X:\ \{0\}\longrightarrow X \quad\text{and}\quad e_{X\times X}:\ \{(0,0)\}\longrightarrow X\times X.$$

With the above normalization, the unique element $v\in L_0$ with $\lambda(v)=1$ then satisfies $\varphi_0^{-1}(\alpha\cdot v\otimes\beta\cdot v)=\alpha\beta\cdot v$ for all $\alpha,\beta\in\Lambda$.

Let $A,B\in D_c^b(X,\Lambda)$, and let $p_1,p_2: X\times X\to X$ be the projections onto the two factors. Using our fixed choice of φ, we get an isomorphism

$$\psi:\ (A*B)_\chi \xrightarrow{\sim} A_\chi * B_\chi$$

defined by the commutative diagram

$$\begin{tikzcd}
(A*B)_\chi \arrow[r, dotted, "\psi"] \arrow[d, equal] & A_\chi * B_\chi \arrow[d, equal] \\
(Ra_*(A\boxtimes B))\otimes L \arrow[d, equal] & Ra_*((A\otimes L)\boxtimes(B\otimes L)) \\
Ra_*((A\boxtimes B)\otimes a^*L) \arrow[r, "Ra_*(id\otimes\varphi)"] & Ra_*((A\boxtimes B)\otimes(L\boxtimes L)) \arrow[u, "\cong"', "Ra_*(id\otimes S'\otimes id)"]
\end{tikzcd}$$

where by $S': p_2^*(B)\otimes p_1^*(L)\xrightarrow{\sim} p_1^*(L)\otimes p_2^*(B)$ we denote the symmetry constraint of the usual tensor product on the derived category.

The isomorphisms ψ are compatible with the symmetry constraint S of the symmetric monoidal category $(D_c^b(X,\Lambda),*)$, i.e. for all A,B in $D_c^b(X,\Lambda)$ the diagram

$$\begin{tikzcd}
(A*B)_\chi \arrow[r, "\psi"] \arrow[d, "S_\chi"'] & A_\chi * B_\chi \arrow[d, "S"] \\
(B*A)_\chi \arrow[r, "\psi"] & B_\chi * A_\chi
\end{tikzcd}$$

is commutative. Indeed, unravelling the definition of S in [**106**] one sees that the commutativity of the above diagram is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
a^*L \arrow[r, "\varphi"] \arrow[d, equal] & L \boxtimes L \arrow[r, equal] & p_1^*L \otimes p_2^*L \\
\sigma^* a^* L \arrow[r, "\sigma^*(\varphi)"] & \sigma^*(L \boxtimes L) \arrow[r, equal] & p_2^*L \otimes p_1^*L \arrow[u, "\cong"', "S'"]
\end{tikzcd}$$

where $\sigma : X \times X \to X \times X$ is the morphism $(x,y) \mapsto (y,x)$ and S' is the symmetry constraint of the ordinary tensor product. Since in any case our diagram commutes up to a scalar in Λ^*, it suffices to check commutativity on the stalks at the origin $(0,0)$. There everything boils down to the fact that $(\lambda \otimes \lambda)(u \otimes v) = (\lambda \otimes \lambda)(v \otimes u)$ for all $u, v \in L_0$.

Next we claim that the isomorphisms ψ are also compatible with the associativity constraint of the symmetric monoidal category $(D_c^b(X,\Lambda), *)$. Indeed, by strictness [**106**, p. 11] the associativity constraints are the identity morphisms, so it suffices to show that the diagram

$$\begin{tikzcd}
((A*B)*C)_\chi \arrow[r, "\psi"] \arrow[d, equal] & ((A*B)_\chi) * C_\chi \arrow[r, "\psi * id"] & (A_\chi * B_\chi) * C_\chi \arrow[d, equal] \\
(A*(B*C))_\chi \arrow[r, "\psi"] & A_\chi * ((B*C)_\chi) \arrow[r, "id * \psi"] & A_\chi * (B_\chi * C_\chi)
\end{tikzcd}$$

commutes for all $A, B, C \in D_c^b(X,\Lambda)$. Writing

$$((A*B)*C)_\chi = Ra_* R(a \times id)_*(((A \boxtimes B) \boxtimes C) \otimes (a \times id)^* a^* L)$$

and similarly for the other convolutions, the commutativity of the diagram becomes equivalent to the commutativity of the diagram on $X \times X \times X$

$$\begin{tikzcd}
(a \times id)^* a^* L \arrow[r, "(a\times id)^*\varphi"] \arrow[d, equal] & (a \times id)^* L \boxtimes L = a^* L \boxtimes L \arrow[r, "\varphi \boxtimes id"] & (L \boxtimes L) \boxtimes L \arrow[d, equal] \\
(id \times a)^* a^* L \arrow[r, "(id\times a)^*\varphi"] & (id \times a)^* L \boxtimes L = L \boxtimes a^* L \arrow[r, "id \boxtimes \varphi"] & L \boxtimes (L \boxtimes L)
\end{tikzcd}$$

Again it suffices to check the commutativity on stalks at $(0,0,0)$. The upper arrow becomes the composition

$$(\varphi \otimes id) \circ \varphi : \ L_0 \longrightarrow L_0 \otimes_\Lambda L_0 \longrightarrow (L_0 \otimes_\Lambda L_0) \otimes_\Lambda L_0.$$

The inverse morphism maps $(\alpha \cdot v \otimes \beta \cdot v) \otimes \gamma \cdot v$ to $(\alpha\beta)\gamma \cdot v$. By a similar computation for the lower row, the commutativity of the diagram hence boils down to the associativity law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ of the field Λ. □

As a trivial by-product, the tensor functoriality provides a simple proof of the following result of [**59**].

COROLLARY 2.2. *For $K \in D_c^b(X,\Lambda)$ the Euler characteristic of K_χ does not depend on the character χ.*

Proof. In [**106**, lemma 8 on p. 28] it has been deduced from the Künneth formula that hypercohomology defines a tensor functor ACU

$$H^\bullet(X,-): \ (D_c^b(X,\Lambda),*) \longrightarrow (Vect_\Lambda^s, \otimes^s)$$

where the right hand side denotes the rigid symmetric monoidal category of super vector spaces over Λ, i.e. $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces where the symmetry constraint is twisted by the usual sign rule. Hence the Euler characteristic of K is equal, as an element of $End_{D_c^b(X,\Lambda)}(\mathbf{1}) = \Lambda$, to the composite morphism

$$\mathbf{1} \xrightarrow{coev_K} K * K^\vee \xrightarrow{S_{K,K^\vee}} K^\vee * K \xrightarrow{ev_K} \mathbf{1},$$

and as such it is invariant under character twists by proposition 2.1. $\square$

2.3. An axiomatic framework

Since the Tannakian constructions to be given below are of independent interest also in more general situations than in the proof of theorem 1.1, for the rest of this chapter we work in the following axiomatic setting. Consider a Λ-linear rigid symmetric monoidal category $(\mathbf{D},*)$ whose unit object $\mathbf{1}$ satisfies $End_{\mathbf{D}}(\mathbf{1}) \cong \Lambda$, and let

$$rat: \ (\mathbf{D},*) \longrightarrow (D_c^b(X,\Lambda),*)$$

be a faithful Λ-linear tensor functor ACU. The notation *rat* is motivated by the case where $k=\mathbb{C}$, $\Lambda=\mathbb{Q}$ and where $\mathbf{D}=D^b(\mathrm{MHM}(X))$ is the bounded derived category of the category $\mathrm{MHM}(X)$ of mixed Hodge modules [**89**], but our formulation also applies to other situations.

For $K \in \mathbf{D}$ we denote by $H^\bullet(X,K)$ resp. by $\chi(K)$ the hypercohomology resp. the Euler characteristic of the sheaf complex $rat(K)$. Similarly we use the notation $H^\bullet(X,K_\chi) = H^\bullet(X, rat(K)_\chi)$ for twists by characters χ. Notice however that we do not assume the twisting functor lifts to the category $\mathbf{D}$, so the formal token K_χ does not refer to an object in $\mathbf{D}$. Depending on the context, we require the first four or all of the following axioms.

(D1) *Degree shifts*. We have an auto-equivalence $K \mapsto K[1]$ on $\mathbf{D}$ which induces the usual shift functor on $D_c^b(X,\Lambda)$.

(D2) *Perverse truncations*. For $n \in \mathbb{Z}$ we have endofunctors ${}^p\tau_{\leq n}$, ${}^p\tau_{\geq n}$ on $\mathbf{D}$ and natural morphisms

$${}^p\tau_{\leq n}(K) \longrightarrow K \quad \text{and} \quad K \longrightarrow {}^p\tau_{\geq n}(K) \quad \text{for} \quad K \in \mathbf{D}$$

which induce on $D_c^b(X,\Lambda)$ the usual perverse truncations.

Furthermore, the essential image $\mathbf{P}$ of the perverse cohomology functors

$$ {}^pH^n \;=\; {}^p\tau_{\leq n} \circ {}^p\tau_{\geq n} : \quad \mathbf{D} \longrightarrow \mathbf{P} $$

is a full abelian subcategory of $\mathbf{D}$, and $rat : \mathbf{P} \longrightarrow \mathrm{Perv}(X, \Lambda)$ is an exact functor between the respective abelian categories.

(D3) *Perverse decomposition*. For all $K \in \mathbf{D}$ we have a (non-canonical) isomorphism

$$ K \;\cong\; \bigoplus_{n \in \mathbb{Z}} {}^pH^{-n}(K)[n]. $$

(D4) *Semisimplicity*. In (D2) the abelian category $\mathbf{P}$ is semisimple.

(D5) *Hard Lefschetz*. In $\mathbf{D}$ there exists an invertible object $\mathbf{1}(1)$ whose image in $\mathrm{Perv}(X, \Lambda)$ under *rat* is the Tate twist of $\mathbf{1}$. For all K, L in $\mathbf{D}$ and all $i \geq 0$ we have functorial Lefschetz isomorphisms

$$ {}^pH^{-i}(K * L) \;\cong\; {}^pH^{i}(K * L)(i), $$

where the Tate twist (i) means i-fold convolution with $\mathbf{1}(1)$.

We do not assume $\mathbf{D}$ to be triangulated, indeed we will later deal with the following non-triangulated categories.

EXAMPLE 2.3. *The axioms (D1) – (D5) hold if $\mathbf{D} \subseteq D^b_c(X, \Lambda)$ is the full subcategory of all direct sums of degree shifts of semisimple perverse sheaves which in case $char(k) > 0$ are defined over some finite field.*

For $k = \mathbb{C}$ this follows from [**35**] together with [**14**], [**43**], or alternatively from [**87**] and [**78**]. On the other hand, for $char(k) > 0$ one can invoke the mixedness results of [**69**] and [**10**]. Note that in the above example we could also replace the category $\mathbf{D}$ by the full subcategory of objects of geometric origin in the sense of [**10**, sect. 6.2.4].

EXAMPLE 2.4. *The axioms (D1) – (D5) hold for $k = \mathbb{C}$ and $\Lambda = \mathbb{Q}$ if one takes $\mathbf{D}$ to be the full subcategory of $D^b(\mathrm{MHM}(X))$ consisting of all direct sums of degree shifts of semisimple Hodge modules.*

In the above setting we consider a full subcategory $\mathbf{N}$ of $\mathbf{D}$ consisting of objects that are negligible for our purposes. In our later application this will be the category of objects which become isomorphic to zero in the André-Kahn quotient category $\overline{\mathbf{D}}$. Again, since we want to proceed as far as possible over a base field of arbitrary characteristic, we formulate the required properties as the following axioms.

(N1) *Stability*. We have $\mathbf{N} * \mathbf{D} \subseteq \mathbf{N}$, and $\mathbf{N}$ is stable under taking retracts, degree shifts, perverse truncations and adjoint duals.

(N2) *Twisting*. Every object $K \in \mathbf{N}$ satisfies $H^\bullet(X, K_\chi) = 0$ for most characters χ of the fundamental group.

(N3) *Acyclicity*. The category $\mathbf{N}$ contains all $K \in \mathbf{D}$ with $H^\bullet(X, K) = 0$.

(N4) *Euler characteristics*. The category $\mathbf{N}$ contains all simple objects of $\mathbf{P}$ whose Euler characteristic vanishes.

The meaning of these axioms will become clear later on. For the time being we content ourselves with the following

REMARK 2.5. *Let Π be a set of characters of $\pi_1(X,0)$, and $\mathbf{N} \subseteq \mathbf{D}$ the full subcategory of all $K \in \mathbf{D}$ such that $rat(K)$ is a direct sum of degree shifts of local systems L_χ with $\chi \in \Pi$. Then axioms (N1) and (N2) hold.*

Proof. For any $M \in D_c^b(X, \Lambda)$ we have $L_\chi * M = L_\chi \otimes_\Lambda H^\bullet(X, M_{\chi^{-1}})$ by [**106**, p. 20], which in particular implies the stability property $\mathbf{N} * \mathbf{D} \subseteq \mathbf{N}$ so that axiom (N1) holds. For (N2) use that $H^\bullet(X, L_\chi) = 0$ if and only if the character χ is non-trivial. □

2.4. The André-Kahn quotient

For the Tannakian arguments to be given below, we want to work with rigid symmetric monoidal categories which are at the same time semisimple abelian categories. To obtain such a category $\overline{\mathbf{D}}$ that is as close as possible to the triangulated category $\mathbf{D}$, we will apply a general quotient construction introduced by André and Kahn in [**1**]. In what follows we always assume that axioms (D1) – (D4) from the previous section, i.e. all axioms except for the hard Lefschetz axiom, are satisfied.

The construction of André and Kahn works as follows. By rigidity we have for each $K \in \mathbf{D}$ an isomorphism $Hom_{\mathbf{D}}(K,K) \xrightarrow{\sim} Hom_{\mathbf{D}}(\mathbf{1}, K * K^\vee)$ which in what follows we denote by $f \mapsto f^\sharp$. We then define the categorical trace $tr(f) \in End_{\mathbf{D}}(\mathbf{1}) = \Lambda$ of an endomorphism $f \in Hom_{\mathbf{D}}(K,K)$ to be the composite morphism

$$tr(f): \qquad \mathbf{1} \xrightarrow{f^\sharp} K * K^\vee \xrightarrow{S_{K,K^\vee}} K^\vee * K \xrightarrow{ev_K} \mathbf{1}$$

where as usual $S_{K,K^\vee}$ denotes the symmetry constraint and ev_K denotes the evaluation morphism. Following section 7.1 of loc. cit. we then define the André-Kahn radical $N_{\mathbf{D}}$ on objects $K, L \in \mathbf{D}$ by

$$N_{\mathbf{D}}(K,L) \;=\; \big\{ f \in Hom_{\mathbf{D}}(K,L) \mid \forall g \in Hom_{\mathbf{D}}(L,K):\; tr(g \circ f) = 0 \big\}.$$

This is an ideal of $\mathbf{D}$ in the sense that for all objects $K, K', L, L' \in \mathbf{D}$ and all morphisms $h_1 \in Hom_{\mathbf{D}}(K', K)$, $h_2 \in Hom_{\mathbf{D}}(L, L')$ one can show

$$h_2 \circ N_{\mathbf{D}}(K,L) \circ h_1 \;\subseteq\; N_{\mathbf{D}}(K', L').$$

For any ideal one has the notion of the corresponding quotient category. By definition, the quotient

$$\bar{\mathbf{D}} \;=\; \mathbf{D}/N_{\mathbf{D}}$$

has the same objects as $\mathbf{D}$, but the morphisms in $\bar{\mathbf{D}}$ between two objects K, L are defined by

$$Hom_{\bar{\mathbf{D}}}(K,L) \;=\; Hom_{\mathbf{D}}(K,L)/N_{\mathbf{D}}(K,L).$$

We have a natural quotient functor $q: \mathbf{D} \longrightarrow \bar{\mathbf{D}}$ that is given by the identity on objects and by the quotient map on morphisms, and in what follows we denote by $\bar{\mathbf{P}}$ the essential image of $\mathbf{P}$ under this quotient functor. Recall that ultimately we want to construct a semisimple abelian category; as a first step towards this goal we have

LEMMA 2.6. *The quotient functor $q: \mathbf{D} \longrightarrow \bar{\mathbf{D}}$ preserves direct sums, and the category $\bar{\mathbf{P}}$ is pseudo-abelian in the sense that every idempotent morphism in it splits as the projection onto a direct summand.*

Proof. The functor q preserves direct sums since it is Λ-linear. To see that idempotents in $\bar{\mathbf{P}}$ split, let P be an object of $\mathbf{P}$. Since $\mathbf{P}$ is an abelian category and hence in particular pseudo-abelian, it suffices to show that every idempotent in

$$End_{\bar{\mathbf{P}}}(P) \;=\; End_{\mathbf{P}}(P)/N_{\mathbf{D}}(P,P)$$

lifts to an idempotent in $End_{\mathbf{P}}(P)$. Since $\mathbf{P}$ is semisimple by axiom (D4), passing to isotypic components we can assume $P = Q^{\oplus r}$ for some simple object Q of $\mathbf{P}$ and $r \in \mathbb{N}$. Then $End_{\mathbf{P}}(P)$ is the ring of $r \times r$ matrices over the skew field $End_{\mathbf{P}}(Q)$. Since matrix rings over skew fields do not have proper two-sided ideals, it follows that either $N_{\mathbf{D}}(P,P) = 0$ or $N_{\mathbf{D}}(P,P) = End_{\mathbf{P}}(P)$, and in both cases the lifting of idempotents is obvious. $\square$

It turns out that this is already enough to conclude that the quotient category $\bar{\mathbf{D}}$ has the desired properties:

PROPOSITION 2.7. *The quotient category $\bar{\mathbf{D}}$ is a semisimple abelian Λ-linear rigid symmetric monoidal category.*

Proof. By [**1**, lemma 7.1.1] the André-Kahn radical $N_{\mathbf{D}}$ is a monoidal ideal in the sense that $id_M * N_{\mathbf{D}}(K,L) \subseteq N_{\mathbf{D}}(M*K, M*L)$ for all $K, L, M \in \mathbf{D}$, hence by sorite 6.1.4 of loc. cit. the quotient $\bar{\mathbf{D}} = \mathbf{D}/N_{\mathbf{D}}$ inherits from $\mathbf{D}$ the structure of a Λ-linear rigid symmetric monoidal category whose unit element satisfies $End_{\bar{\mathbf{D}}}(\mathbf{1}) = \Lambda$.

The main part of the proof is to check that the category $\bar{\mathbf{D}}$ is semisimple abelian. For this we first claim that

$$(\star) \qquad Hom_{\bar{\mathbf{D}}}(P[m], Q[n]) \;=\; 0 \quad \text{for all objects } P, Q \text{ in } \mathbf{P} \text{ and } m \neq n.$$

Indeed, for $m > n$ we even have $Hom_{\mathbf{D}}(P[m], Q[n]) = 0$ since under the faithful functor *rat* this *Hom*-group injects into

$$Hom_{D_c^b(X,\Lambda)}(rat(P)[m], rat(Q)[n]) \;=\; Ext^{n-m}_{\mathrm{Perv}(X,\Lambda)}(rat(P), rat(Q))$$

which vanishes for $m > n$ (for the above identification as an *Ext*-group recall that $D_c^b(X,\Lambda)$ is the derived category of $\mathrm{Perv}(X,\Lambda)$). On the other hand, in the case $m < n$ we have $Hom_{\mathbf{D}}(Q[n], P[m]) = 0$, and then the definition of the André-Kahn radical implies that

$$Hom_{\mathbf{D}}(P[m], Q[n]) \;=\; N_{\mathbf{D}}(P[m], Q[n])$$

which is mapped to zero under the quotient functor $q: \mathbf{D} \longrightarrow \bar{\mathbf{D}}$. In both cases the claim $(\star)$ follows.

Now by the perverse decomposition axiom (D3) every object K of $\bar{\mathbf{D}}$ can be written as $K = \bigoplus_{n\in\mathbb{Z}} K_n[n]$ with certain K_n in $\bar{\mathbf{P}}$. For such K the vanishing in $(\star)$ implies

$$(\star\star) \qquad End_{\bar{\mathbf{D}}}(K) \;=\; \bigoplus_{n\in\mathbb{Z}} End_{\bar{\mathbf{D}}}(K_n[n]) \;=\; \bigoplus_{n\in\mathbb{Z}} End_{\bar{\mathbf{P}}}(K_n).$$

In particular, every idempotent endomorphism of K in the category $\bar{\mathbf{D}}$ is a direct sum of idempotent endomorphisms of the summands $K_n[n]$, and by lemma 2.6 it follows that $\bar{\mathbf{D}}$ is pseudo-abelian. Hence to show that $\bar{\mathbf{D}}$ is a semisimple abelian category, it will suffice by [**1**, A.2.10] to show that it is a semisimple Λ-linear category in the sense of section 2.1.1 in loc. cit. For this we use the following general result [**2**, th. 1].

Let F be a field and $\mathbf{A}$ an F-linear rigid symmetric monoidal category with $End_{\mathbf{A}}(\mathbf{1}) = F$. Suppose there exists an F-linear tensor functor ACU from $\mathbf{A}$ to an abelian F-linear rigid symmetric monoidal category $\mathbf{V}$ such that $\dim_\Lambda(Hom_{\mathbf{V}}(V_1, V_2)) < \infty$ for all $V_1, V_2 \in \mathbf{V}$. Then the quotient of $\mathbf{A}$ by its André-Kahn radical $N_{\mathbf{A}}$ is a semisimple F-linear category, and $N_{\mathbf{A}}$ is the unique monoidal ideal of $\mathbf{A}$ with this property.

In our case this applies for $F = \Lambda$, $\mathbf{A} = \mathbf{D}$ and for the functor $H^\bullet(X, -)$ from $\mathbf{D}$ to the abelian category $\mathbf{V}$ of super vector spaces over Λ. $\square$

COROLLARY 2.8. *The functors $\mathbf{P} \to \bar{\mathbf{P}}$ and $\bar{\mathbf{P}} \hookrightarrow \bar{\mathbf{D}}$ are exact functors between semisimple abelian categories. The image of a simple object $P \in \mathbf{P}$ inside $\bar{\mathbf{P}}$ is either simple or isomorphic to zero, and if Λ is algebraically closed, then the latter case occurs if and only if $\chi(P) = 0$.*

Proof. By proposition 2.7, $\bar{\mathbf{D}}$ is a semisimple abelian category, and it also follows from the proof of the proposition that $\bar{\mathbf{P}}$ is a semisimple abelian subcategory of $\bar{\mathbf{D}}$. Since the considered functors are additive, they are exact by semisimplicity. If P is a simple object of $\mathbf{P}$, then $End_{\mathbf{P}}(P)$ is a skew field, hence $End_{\bar{\mathbf{P}}}(P)$ is a skew field or zero, and P is simple or zero in $\bar{\mathbf{P}}$. Over an algebraically closed field Λ there exist no skew fields other than Λ itself, hence in this case we have $End_{\mathbf{P}}(P) = \Lambda$, and it follows that $id_P \in N_{\mathbf{D}}(P,P)$ iff $tr(id_P) = 0$, which is the case iff $\chi(P) = 0$. □

COROLLARY 2.9. *Let* $\mathbf{N} \subseteq \mathbf{D}$ *be the full subcategory of all objects which become isomorphic to zero in the quotient category* $\bar{\mathbf{D}}$. *If* Λ *is algebraically closed, then* $\mathbf{N}$ *satisfies the stability axiom (N1), the acyclicity axiom (N3) and the Euler axiom (N4), and an object* $K \in \mathbf{D}$ *lies in the subcategory* $\mathbf{N}$ *iff all simple constituents of all* ${}^pH^n(K)$ *have Euler characteristic zero.*

Proof. Property (N1) is obvious, property (N3) follows from (N4), and the latter is immediate from corollary 2.8 by the perverse decomposition axiom (D3) and the semisimplicity axiom (D4). □

2.5. Super Tannakian categories

Using a criterion of Deligne, we now show that the semisimple abelian rigid symmetric monoidal category $\bar{\mathbf{D}}$ from the previous section is almost Tannakian: It is an inductive limit of super Tannakian categories, a notion that we will recall below and in appendix A. In the case $k = \mathbb{C}$ we will see later (in corollary 2.14, using that in this case the Euler characteristic of perverse sheaves is non-negative) that $\bar{\mathbf{D}}$ is in fact an inductive limit of ordinary Tannakian categories, and this is closely related to our proof of theorem 1.1. One may conjecture the same also for $char(k) > 0$.

Throughout this section we will always assume that Λ is algebraically closed and that axioms (D1) – (D4) of section 2.3 hold. By semisimplicity the convolution functor

$$* : \ \bar{\mathbf{D}} \times \bar{\mathbf{D}} \longrightarrow \bar{\mathbf{D}}$$

is exact in each variable, and $End_{\bar{\mathbf{D}}}(\mathbf{1}) = \Lambda$ (this latter property is inherited from $\mathbf{D}$ where it can be checked by applying the faithful functor *rat*). So $\bar{\mathbf{D}}$ is a *catégorie* Λ*-tensorielle* in the sense of [**30**, sect. 0.1].

Recall that a full subcategory of $\bar{\mathbf{D}}$ is said to be finitely tensor generated if it is the category $\langle K \rangle$ consisting of all subquotients of convolution powers of $K \oplus K^\vee$ for some fixed object $K \in \bar{\mathbf{D}}$. Our next goal is to show that any such subcategory is super Tannakian in the following sense.

The framework of algebraic geometry is generalized to super algebraic geometry by replacing the category of commutative rings with the category of $\mathbb{Z}/2\mathbb{Z}$-graded super commutative rings. In particular one has the notion of an algebraic resp. reductive super group and its super representations, see appendix A. For an algebraic super group G over Λ and a point $\varepsilon \in G(\Lambda)$ with $\varepsilon^2 = 1$ such that $\mathrm{int}(\varepsilon)$ is the parity automorphism of G, let us denote by $Rep_\Lambda(G,\varepsilon)$ the category of all super representations $V = V_0 \oplus V_1$ of G over Λ such that ε acts by $(-1)^a$ on V_a for $a = 0, 1$. Categories of this form will be called super Tannakian, with Tannaka super group G.

THEOREM 2.10. *Any finitely generated full tensor subcategory $\langle K \rangle \subset \bar{\mathbf{D}}$ is super Tannakian, and its Tannaka super group $G = G(K)$ is reductive.*

Proof. Since $\bar{\mathbf{D}}$ is a catégorie Λ-tensorielle, for the first claim it suffices by [**30**, th. 0.6] to see that for any object $K \in \bar{\mathbf{D}}$ and all $n \in \mathbb{N}$ the number of simple constituents of $(K \oplus K^\vee)^{*n}$ in the semisimple abelian category $\bar{\mathbf{D}}$ is at most N^n for some constant $N = N(K)$ depending only on K. In the case at hand one can take

$$N(K) \;=\; \sum_{i \in \mathbb{Z}} \dim_\Lambda(H^i(X, M))$$

for any object $M \in \mathbf{D}$ that becomes isomorphic to $K \oplus K^\vee$ in $\bar{\mathbf{D}}$, using the argument given in [**110**, top of p. 5]. Concerning reductivity, note that a category $Rep_\Lambda(G,\varepsilon)$ is semisimple iff G is reductive [**108**]. □

2.6. Perverse multiplier

We now introduce the notion of a multiplier. This will play an important role in our proof of theorem 1.1, indeed the theorem essentially amounts to the statement that any perverse sheaf on a complex abelian variety is a multiplier. In this section Λ need not be algebraically closed, but we assume as always that axioms (D1) – (D4) from section 2.3 hold.

Let $\mathbf{N}$ be a full subcategory of $\mathbf{D}$ satisfying the stability axiom (N1). We then define an $\mathbf{N}$-multiplier to be an object $K \in \mathbf{D}$ such that for all $r \in \mathbb{N}_0$ and all $n \neq 0$ the subquotients of ${}^pH^n((K \oplus K^\vee)^{*r})$ all lie in $\mathbf{N}$. The relevance of this notion for theorem 1.1 is clear from

LEMMA 2.11. *For $P \in \mathbf{P}$ the following holds.*

- *(a) If the subcategory $\mathbf{N}$ satisfies the twisting axiom (N2) and if P is an $\mathbf{N}$-multiplier, then $H^\bullet(X, P_\chi)$ is concentrated in degree zero for most characters χ.*
- *(b) Conversely, suppose that $\mathbf{N}$ satisfies the acyclicity axiom (N3) and that the hard Lefschetz axiom (D5) holds. Then P is an $\mathbf{N}$-multiplier if $H^\bullet(X, P)$ is concentrated in degree zero.*

Proof. *(a)* Put $g = \dim(X)$. The perverse decomposition axiom (D3) shows that

$$P^{*(g+1)} \;=\; \bigoplus_{m \in \mathbb{Z}} P_m[m] \quad \text{for suitable} \quad P_m \in \mathbf{P}\,.$$

By assumption P is an $\mathbf{N}$-multiplier, hence $P_m \in \mathbf{N}$ for all $m \neq 0$. Via the twisting axiom (N2) it follows that for most characters χ and all $n \in \mathbb{Z}$,

$$H^n(X, P_\chi^{*(g+1)}) \;=\; H^n(X, (P_0)_\chi).$$

The right hand side vanishes for $|n| > g$ since $rat(P_0)_\chi$ is perverse. But for the left hand side we have

$$H^\bullet(X, P_\chi^{*(g+1)}) \;=\; (H^\bullet(X, P_\chi))^{\otimes(g+1)}$$

by proposition 2.1 and since $H^\bullet(X,-)$ is a tensor functor by the Künneth theorem. So the above vanishing statement for $|n| > g$ implies that $H^\bullet(X, P_\chi)$ is concentrated in degree zero.

(b) Put $Q = (P \oplus P^\vee)^{*r}$ for any $r \in \mathbb{N}$. Since hypercohomology is a tensor functor by the Künneth theorem, with $H^\bullet(X, P_\chi)$ also $H^\bullet(X, Q)$ is concentrated in degree zero. Using the hard Lefschetz axiom (D5), one then deduces that for all $n \neq 0$ one has $H^\bullet(X, {}^pH^n(Q)) = 0$ so that by (N3) the subcategory $\mathbf{N}$ contains ${}^pH^n(Q)$. Since this holds for arbitrary $r \in \mathbb{N}$, it follows that P is an $\mathbf{N}$-multiplier. $\square$

In view of part *(a)* of the lemma, to prove theorem 1.1 we want to show that for a suitable subcategory $\mathbf{N}$ every object of $\mathbf{P}$ is an $\mathbf{N}$-multiplier. For this we will argue by contradiction, using the following

LEMMA 2.12. *Suppose that* $\mathbf{N}$ *satisfies the stability axiom (N1) and the Euler axiom (N4), that* $\mathbf{D}$ *satisfies all axioms (D1) – (D5) and that* $P \in \mathbf{P}$ *is not an* $\mathbf{N}$*-multiplier. Then for some* $r \in \mathbb{N}$ *the convolution power*

$$(P * P^\vee)^{*r} \;=\; (P * P^\vee) * \cdots * (P * P^\vee)$$

admits a direct summand of the form $\mathbf{1}[2i](i)$ *with an integer* $i \neq 0$.

Proof. If P is not an $\mathbf{N}$-multiplier, we can find integers $a, b \in \mathbb{N}$ such that $P^{*a} * (P^\vee)^{*b}$ admits a direct summand $Q[i]$ for some $i \neq 0$ and some simple object $Q \in \mathbf{P}$ which is not in $\mathbf{N}$. By the hard Lefschetz axiom (D5) then $Q[-i](-i)$ is a direct summand of $P^{*a} * (P^\vee)^{*b}$ as well. It then follows that also the dual $Q^\vee[i](i)$ is a direct summand of $P^{*b} * (P^\vee)^{*a}$. Altogether then the convolution product $Q[i] * Q^\vee[i](i) = Q * Q^\vee[2i](i)$ will be a direct summand of $(P * P^\vee)^{*r}$ for the exponent $r = a + b$.

It remains to show that $\mathbf{1}$ is a direct summand of $Q * Q^\vee$. For this note that the trace map $tr(Q) : \mathbf{1} \longrightarrow Q * Q^\vee \cong Q^\vee * Q \longrightarrow \mathbf{1}$ is non-zero since we have $\chi(Q) \neq 0$ by axiom (N4). Now $tr(Q)$ factors over ${}^pH^0(Q * Q^\vee)$, indeed $Hom_{\mathbf{D}}(\mathbf{P}, {}^p\tau_{>0}\mathbf{P}) = Hom_{\mathbf{D}}({}^p\tau_{<0}\mathbf{P}, \mathbf{P}) = 0$. So $tr(Q)$ exhibits $\mathbf{1}$ as a retract of ${}^pH^0(Q * Q^\vee)$ in the abelian category $\mathbf{P}$, and we are done. $\square$

2.7. The main argument

To finish the proof of theorem 1.1 we will control the non-perversity of convolution products such as in lemma 2.12 via central characters of the Tannaka super group introduced in theorem 2.10. By dévissage we can restrict ourselves to semisimple perverse sheaves as in example 2.3. So suppose that $\Lambda = \mathbb{C}$ or $\Lambda = \overline{\mathbb{Q}}_l$ and that $\mathbf{D}$ satisfies all axioms (D1) – (D5) of section 2.3. Consider the semisimple abelian rigid symmetric monoidal quotient category $\bar{\mathbf{D}}$ from section 2.4.

For the full subcategory $\mathbf{N} \subseteq \mathbf{D}$ of all objects that become isomorphic to zero in $\bar{\mathbf{D}}$, the stability axiom (N1), the acyclicity axiom (N3) and the Euler axiom (N4) are satisfied as we have seen in corollary 2.9. In the setting of example 2.3 presumably also the twisting axiom (N2) always holds; at least over the base field $k = \mathbb{C}$ we have shown this in corollary 1.7. Hence in the complex case part *(a)* of lemma 2.11 can be applied using the twisting axiom (N2), and then theorem 1.1 follows from axioms (N1) and (N4) via the following argument from representation theory.

THEOREM 2.13. *Let $\mathbf{N} \subseteq \mathbf{D}$ be a full subcategory satisfying the stability axiom (N1) and the Euler axiom (N4). Then every $P \in \mathbf{P}$ is an $\mathbf{N}$-multiplier.*

Proof. Suppose that $P \in \mathbf{P}$ is simple and not an $\mathbf{N}$-multiplier. Then for some integer $r \in \mathbb{N}$ the convolution $(P * P^\vee)^{*r}$ contains by lemma 2.12 a direct summand $L = \mathbf{1}[2i](i)$ with $i \neq 0$. In particular, it follows that the full rigid symmetric monoidal subcategory $\bar{\mathbf{D}}_1 \subset \bar{\mathbf{D}}$ generated by the object P contains the full rigid symmetric monoidal subcategory $\bar{\mathbf{D}}_0 \subset \bar{\mathbf{D}}$ generated by the invertible object L.

Theorem 2.10 shows that for certain reductive super groups G_i over Λ we have tensor equivalences $\omega_i : \bar{\mathbf{D}}_i \xrightarrow{\sim} Rep_\Lambda(G_i, \varepsilon_i)$ for $i \in \{0,1\}$, and by the Tannakian formalism the inclusion of categories $\bar{\mathbf{D}}_0 \subseteq \bar{\mathbf{D}}_1$ defines an epimorphism of reductive super groups

$$h: \; G_1 \twoheadrightarrow G_0 .$$

The category $\bar{\mathbf{D}}_0$ consists of all direct sums of skyscraper sheaves L^{*n} with integers $n \in \mathbb{Z}$, where by a negative power we mean the corresponding power of the dual. Since $L^{*n} \cong \mathbf{1}[2ni](ni)$ and $i \neq 0$, equation ($\star\star$) in the proof of proposition 2.7 implies that one has an isomorphism $L^{*n} \cong \mathbf{1}$ in $\bar{\mathbf{D}}$ only if $n = 0$. Thus the tensor equivalence ω_0 between $\bar{\mathbf{D}}_0$ and $Rep_\Lambda(G_0, \varepsilon_0)$ is realized explicitly with $G_0 = \mathbb{G}_m$ and $\varepsilon_0 = -1$ via

$$L^{*n} \;\mapsto\; (\text{the character } z \mapsto z^n \text{ of } \mathbb{G}_m).$$

In particular, the representation $W_0 = \omega_0(L)$ is non-trivial, corresponding to the identity character $z \mapsto z$ of the multiplicative group $\mathbb{G}_m$.

But proposition A.3 in the appendix applies to the torus $T_0 = G_0 = \mathbb{G}_m$, so there exists a central torus $T_1 \cong \mathbb{G}_m$ in G_1 such that $h : G_1 \twoheadrightarrow G_0$ restricts to an isogeny $T_1 \to T_0$. By Schur's lemma the central torus T_1 acts via some character on the irreducible super representation $W_1 = \omega_1(P)$, so T_1 acts trivially on $W_1 \otimes W_1^\vee = \omega_1(P * P^\vee)$. Hence T_1 acts trivially on the direct summand

$$W_0 \subseteq (W_1 \otimes W_1^\vee)^{\otimes r}$$

and therefore also T_0 acts trivially on $W_0 = \omega_0(L)$, a contradiction. □

COROLLARY 2.14. *If the base field k is $\mathbb{C}$, then the super group $G(K)$ in theorem 2.10 is a classical reductive algebraic group.*

Proof. Corollary 2.9 and theorem 2.13 show that the category $\bar{\mathbf{P}}$ is stable under convolution. Using this one easily reduces our claim to the special case where K lies inside $\bar{\mathbf{P}}$. In this case, all objects of $Rep_\Lambda(G(K), \varepsilon)$ have non-negative dimension because over the ground field $k = \mathbb{C}$ we have seen in section 1.4 that $\chi(P) \geq 0$ for all perverse sheaves $P \in \mathrm{Perv}(X, \Lambda)$. The assertion then follows from [**31**, th. 7.1]. □

CHAPTER 3

Perverse sheaves on semiabelian varieties

The Tannaka groups in corollary 2.14 provide a new tool for the study of smooth projective varieties with non-trivial Albanese morphism. Their representation theory can be considered as a substitute for Brill-Noether theory in higher dimensions, and they are also closely related to the moduli of abelian varieties as we will discuss in chapter 4. However, in general these Tannaka groups are hard to compute. At present the most effective tool to determine them is to study degenerations of the underlying abelian variety; even if one is only interested in semisimple perverse sheaves on abelian varieties, this naturally leads to non-semisimple perverse sheaves on semiabelian varieties. In this chapter we extend our previous Tannakian constructions to this more general case, combining arguments of Gabber and Loeser for tori [**41**] with the generic vanishing theorem 1.1 for abelian varieties. For degenerations of abelian varieties into semiabelian varieties we then show that the nearby cycles functor induces an embedding of the degenerate Tannaka group into the generic one whenever one can possibly expect this (see theorem 3.15). Finally, we also discuss how the obtained Tannaka groups vary in families on abelian schemes.

Before we come to the details, let us begin with a brief overview over the constructions that follow. Throughout we work over an algebraically closed field k of any characteristic $p \geq 0$. Let X be a semiabelian variety, a commutative group variety which is an extension

$$1 \longrightarrow T \longrightarrow X \longrightarrow A \longrightarrow 1$$

of an abelian variety A by an algebraic torus T over k. Fixing a prime $l \neq p$, we put $\Lambda = \overline{\mathbb{Q}}_l$ and denote by

$$\mathbf{D} = \mathbf{D}(X) = D^b_c(X,\Lambda) \quad \text{and} \quad \mathbf{P} = \mathbf{P}(X) = \mathrm{Perv}(X,\Lambda)$$

the triangulated category of bounded constructible Λ-sheaf complexes resp. its full abelian subcategory of perverse sheaves. Let $m : X \times_k X \longrightarrow X$ be the group law. In the non-proper case we have two different notions of convolution which are given by

$$K *_! L = Rm_!(K \boxtimes L) \quad \text{and} \quad K *_* L = Rm_*(K \boxtimes L)$$

for $K, L \in \mathbf{D}$. In general the abelian subcategory $\mathbf{P} \subset \mathbf{D}$ will of course not be stable under these two convolution products. Motivated by the results of the previous chapter, we consider the full subcategory $\mathbf{T} \subset \mathbf{D}$ of all sheaf complexes $K \in \mathbf{D}$ which are negligible in the sense that for all $n \in \mathbb{Z}$ the perverse cohomology sheaves ${}^pH^n(K)$ have Euler characteristic zero. Over the base field $k = \mathbb{C}$ we will see that

(a) the triangulated quotient category $\bar{\mathbf{D}} = \mathbf{D}/\mathbf{T}$ exists (corollary 3.6),

(b) both $*_!$ and $*_*$ descend to the same bifunctor $* : \bar{\mathbf{D}} \times \bar{\mathbf{D}} \longrightarrow \bar{\mathbf{D}}$ which preserves the essential image $\bar{\mathbf{P}} \subset \bar{\mathbf{D}}$ of $\mathbf{P}$ (theorem 3.8),

(c) with $*$ as its tensor product, the category $\bar{\mathbf{P}}$ is an inductive limit of neutral Tannakian categories (corollary 3.10).

Recall [**33**, th. 2.11] that a neutral Tannakian category is a category which is equivalent to the category $Rep_\Lambda(G)$ of linear representations of an affine group scheme G over Λ. In particular, to any perverse sheaf $P \in \mathbf{P}$ we will attach an affine algebraic group $G = G(P)$ over Λ which corresponds to the tensor subcategory $\langle P \rangle \subset \bar{\mathbf{P}}$ generated by P under convolution. Since Λ is algebraically closed, the group $G(P)$ is determined uniquely by P up to isomorphism. These are the Tannaka groups we are interested in and whose degeneration behaviour will be studied in theorem 3.15.

For algebraic tori $X = T$ the properties (a) – (c) have been obtained in [**41**, sect. 3.6 – 3.7] via the Mellin transform as a consequence of Artin's affine vanishing theorem, and our arguments in sections 3.1 through 3.4 are heavily indebted by loc. cit. However, for abelian varieties $X = A$ we no longer dispose of Artin's vanishing theorem. Over the complex numbers we can instead apply our vanishing theorem 1.1 which is in fact closely related with the Tannakian property as we have seen in chapter 2. Even though at present we can prove this vanishing theorem only over $k = \mathbb{C}$, it is likely to hold in positive characteristic as well. So in what follows we formulate it as an axiomatic assumption under which our arguments work over an algebraically closed field k of arbitrary characteristic.

3.1. Vanishing theorems revisited

To formulate the generic vanishing axiom that generalizes theorem 1.1 to the non-proper case over a field k of arbitrary characteristic, we consider characters of the tame fundamental group. As a base point we will always take the neutral element $0 \in X(k)$. Recall from [**97**, sect. 1.3] that every semiabelian variety X has a smooth compactification with a normal crossing boundary divisor. The tame fundamental group $\pi_1^t(X,0)$ classifies the finite étale covers of X that are tamely ramified along each component of such a boundary divisor, see [**51**, exp. XIII, sect. 2 and 5], [**52**] and [**93**]. In particular $\pi_1^t(X,0)$ is a quotient of the étale fundamental group $\pi_1(X,0)$, and equal to it if the field k has characteristic zero or if X is proper.

The group $\Pi(X)$ of continuous characters $\chi : \pi_1^t(X,0) \longrightarrow \Lambda^*$ admits a natural product decomposition

$$\Pi(X) \;=\; \Pi(X)_{l'} \times \Pi(X)_l ,$$

where

$$\Pi(X)_{l'} \;=\; \{\, \chi \in \Pi(X) : \; \chi^n = 1 \text{ for some } n \text{ with } l \nmid n \,\}$$

denotes the subgroup of all characters of finite order prime to l and where by definition $\Pi(X)_l$ is the group of all characters that factor over the maximal pro-l-quotient $\pi_1(X,0)_l = \pi_1^t(X,0)_l$. The latter is a free $\mathbb{Z}_l$-module of finite rank [**17**]. So with the same arguments as in [**41**, sect. 3.2] the set $\Pi(X)_l$ of pro-l-characters can be identified with the set of Λ-valued points of a scheme in a natural way (even though as in loc. cit. the multiplication of characters does not come from a group scheme structure). This being said, we consider the set of all tame characters

$$\Pi(X) \quad = \coprod_{\chi \in \Pi(X)_{l'}} \chi \cdot \Pi(X)_l$$

as the disjoint union of the infinitely many components $\chi \cdot \Pi(X)_l$ indexed by the torsion characters $\chi \in \Pi(X)_{l'}$, and we say that a statement holds for a generic character if it holds for all characters in an open subset of $\Pi(X)$ that is dense in each of these components.

For sheaf complexes $K \in \mathbf{D}(X)$ and characters $\chi \in \Pi(X)$ we consider as in the previous chapters the twist $K_\chi = K \otimes_\Lambda L_\chi$ by the local system L_χ. Our Tannakian constructions will be based on the following generic vanishing assumption for such character twists.

ASSUMPTION GV(X). *Let $P \in \mathbf{P}(X)$. Then for generic $\chi \in \Pi(X)$ the forget support map*

$$H_c^\bullet(X, P_\chi) \;\longrightarrow\; H^\bullet(X, P_\chi)$$

is an isomorphism, and

$$H^i(X, P_\chi) \;=\; H_c^i(X, P_\chi) \;=\; 0 \quad \textit{for all} \quad i \neq 0.$$

For complex abelian varieties this is precisely the content of the generic vanishing theorem 1.1, the claim about the forget support map being trivial in that case. At present we do not know whether the assumption $GV(X)$ also holds for abelian varieties over an algebraically closed field k of positive characteristic $p = char(k) > 0$, but in any case the semiabelian version can be deduced from the abelian one as follows.

THEOREM 3.1. *If the maximal abelian variety quotient $A = X/T$ of X satisfies the assumption $GV(A)$, then also $GV(X)$ holds.*

Proof. We first claim that for any given perverse sheaf $P \in \mathbf{P}(X)$, the set of all characters χ which violate the conditions in our assumption $GV(X)$ forms a closed subset of the scheme $\Pi(X)$. To see this, consider the Mellin transforms $\mathscr{M}_!(P)$ and $\mathscr{M}_*(P)$ as defined in [**41**, sect. 3.3] (in loc. cit. the definition is only given for algebraic tori, but it carries over verbatim to the case at hand). These Mellin transforms are objects of the bounded derived category of coherent sheaves on $\Pi(X)$ with the property that for all $i \in \mathbb{Z}$ and any character $\chi \in \Pi(X)$ we have isomorphisms

$$H^i_c(X, P_\chi) \;\cong\; \mathscr{H}^i(Li^*_\chi \mathscr{M}_!(P)),$$
$$H^i(X, P_\chi) \;\cong\; \mathscr{H}^i(Li^*_\chi \mathscr{M}_*(P)),$$

where $i_\chi : \{\chi\} \hookrightarrow \Pi(X)$ denotes the embedding of the closed point given by the chosen character. Furthermore, we have a morphism $\mathscr{M}_!(P) \longrightarrow \mathscr{M}_*(P)$ which induces via the above identifications the forget support morphism on cohomology. Since on a Noetherian scheme the support of any coherent sheaf is a closed subset, it follows that the locus of all characters χ which violate $GV(X)$ forms a closed subset $\mathscr{S}(P) \subseteq \Pi(X)$ as claimed.

We must show that under the assumption $GV(A)$ the complement of this closed subset $\mathscr{S}(P)$ meets every irreducible component of $\Pi(X)$. So we must see that for at least one character χ in each component the properties in $GV(X)$ hold. To this end consider the exact sequence

$$0 \longrightarrow T \xrightarrow{\;i\;} X \xrightarrow{\;f\;} A \longrightarrow 0$$

which defines the given semiabelian variety. Lemma 3.2 below will show that every component of $\Pi(X)$ contains a character χ such that the forget support morphism

$$Rf_!(P_\chi) \;\longrightarrow\; Rf_*(P_\chi)$$

is an isomorphism. In this case Artin's vanishing theorem for the affine morphism f also shows that these two isomorphic direct image complexes are perverse [**41**, lemma 2.4]. For all $\varphi \in \Pi(A)$ and $\psi = f^*(\varphi) \in \Pi(X)$ the projection formula furthermore says

$$(Rf_!(P_\chi))_\varphi \;=\; Rf_!(P_{\chi\psi}) \quad \text{and} \quad (Rf_*(P_\chi))_\varphi \;=\; Rf_*(P_{\chi\psi}),$$

hence we are finished by an application of the vanishing assumption $GV(A)$ to these perverse direct images. □

To fill in the missing statement in the above proof, we consider the given semiabelian variety as a torsor $f : X \longrightarrow A$ under the torus T in the sense of [**74**, sect. III.4]. The condition of being a torsor is satisfied since clearly the morphism $X \times_k T \longrightarrow X \times_A X$, $(x,t) \mapsto (x, xt)$ is an isomorphism. The result we are looking for only uses the structure as a torsor, and for its proof it will be convenient to forget that our base A is an abelian variety. So we will work in the following setting.

Let B be an arbitrary variety over the field k, and let $f : Y \longrightarrow B$ be a torsor under the torus T. By remark 4.8 *(a)* in loc. cit. there exists an étale covering $B' \longrightarrow B$ (not necessarily finite) over which our torsor admits a trivialization.

$$\begin{tikzcd} Y \times_B B' \arrow[rr, "\cong"] \arrow[dr] & & T \times_k B' \arrow[dl] \\ & B' & \end{tikzcd}$$

Let us denote by $i : Y_b = f^{-1}(b) \longrightarrow Y$ the fibre over some chosen geometric point $b \in B(k)$. In what follows we fix a base point $y \in Y_b(k)$ and denote the group of all continuous characters of the tame fundamental group $\pi_1^t(Y_b, y)$ as above by $\Pi(Y_b)$, and similarly for $\Pi(Y)$. Since Y_b is isomorphic to the torus T, the character group $\Pi(Y_b)$ can be identified with the set of closed points of a scheme as explained above. In particular, it makes sense to speak about its irreducible components.

LEMMA 3.2. *In the above setting, for every perverse sheaf $P \in \mathbf{P}(Y)$ there exists a subset $U \subseteq \Pi(Y_b)$ which meets every irreducible component and has the property that the forget support morphism*

$$Rf_!(P_\chi) \longrightarrow Rf_*(P_\chi)$$

is an isomorphism for all characters $\chi \in \Pi(Y)$ with pull-back $i^*(\chi) \in U$.

Proof. Let $g : B' \longrightarrow B$ be an étale covering which trivializes the given torsor in the sense that the pull-back $Y' = Y \times_B B'$ is isomorphic over B' to the trivial torsor $T \times_k B'$. Choosing a geometric point $b' \in B'(k)$ with image $g(b') = b$, we have a commutative diagram

$$\begin{tikzcd} Y_b \arrow[r, "i'"] \arrow[d] \arrow[rr, bend left, "i"] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Spec(k) \arrow[r, "b'"] \arrow[rr, bend right, "b"'] & B' \arrow[r, "g"] & B \end{tikzcd}$$

where the two squares are Cartesian. Since g is an étale covering, the forget support morphism $Rf_!(P_\chi) \longrightarrow Rf_*(P_\chi)$ is an isomorphism iff its pull-back under g^* is an isomorphism. For the pull-back $P' = g'^*(P)$ and $\chi' = g'^*(\chi)$

we have a commutative diagram

$$\begin{tikzcd} g^*(Rf_!(P_\chi)) \arrow[r] \arrow[d, "\cong"'] & g^*(Rf_*(P_\chi)) \arrow[d, "\cong"] \\ Rf'_!(P'_{\chi'}) \arrow[r] & Rf'_*(P'_{\chi'}) \end{tikzcd}$$

where the vertical arrows are isomorphisms due to the smooth base change theorem [**38**, th. I.7.3]. Furthermore, by construction $i'^*(\chi') = i^*(\chi)$. Hence it will be enough to prove the lemma in the case of the trivial torsor, and this has been done in [**41**, cor. 2.3.2]. □

Let us now return to our semiabelian variety X. For later reference it will be convenient to reformulate the observation in theorem 3.1 in the following equivalent way.

COROLLARY 3.3. *Suppose that $A = X/T$ satisfies axiom $GV(A)$, and let $K \in \mathbf{D}(X)$ be a constructible sheaf complex. For all degrees $r \in \mathbb{Z}$ and a generic character $\chi \in \Pi(X)$ then*

$$H^r_c(X, K_\chi) \;\cong\; H^r(X, K_\chi) \;\cong\; H^0(X, {}^pH^r(K)_\chi),$$

where the first isomorphism is induced by the forget support morphism.

Proof. For the perverse cohomology groups $P_r = {}^pH^r(K)$ and $\chi \in \Pi(X)$ we have $(P_r)_\chi = {}^pH^r(K_\chi)$ since twisting by χ is a t-exact functor for the perverse t-structure. Only finitely many P_r are non-zero, so for generic χ and all r theorem 3.1 says that the hypercohomology groups $H^s_c(X, (P_r)_\chi)$ and $H^s(X, (P_r)_\chi)$ will be isomorphic to each other via the forget support morphism, and vanish for all $s \neq 0$. Then the spectral sequences

$$\begin{aligned} E_2^{rs} &= H^s(X, (P_r)_\chi) \Longrightarrow H^{r+s}(X, K_\chi) \\ E_2^{rs} &= H^s_c(X, (P_r)_\chi) \Longrightarrow H^{r+s}_c(X, K_\chi) \end{aligned}$$

degenerate, and our claim follows since the forget support map between the limit terms is induced from the one between the E_2-terms. □

3.2. Some properties of convolution

Before proceeding further, it will be convenient to collect some basic properties of convolution for later reference. As in [**106**, sect. 2.1] one sees that the derived category $\mathbf{D} = \mathbf{D}(X)$ is a symmetric monoidal category with respect to the convolution product $*_!$ and also with respect to $*_*$. In both cases the unit object $\mathbf{1}$ is the rank one skyscraper sheaf supported in the neutral element of the group variety X. For $K \in \mathbf{D}$ we consider the Verdier dual $D(K)$ and define the adjoint dual by the formula $K^\vee = (-id)^* D(K)$ where as usual $-id : X \longrightarrow X$ denotes the inversion morphism.

LEMMA 3.4. *For all $K, M \in \mathbf{D}$ and $\chi \in \Pi(X)$ one has the following natural isomorphisms.*

(a) Adjoint duality:

$$Hom_{\mathbf{D}}(\mathbf{1}, K^\vee *_* M) \cong Hom_{\mathbf{D}}(K, M) \cong Hom_{\mathbf{D}}(K *_! M^\vee, \mathbf{1}).$$

(b) Character twists:

$$(K_\chi)^\vee \cong (K^\vee)_\chi,$$

$$(K *_! M)_\chi \cong K_\chi *_! M_\chi \quad \text{and} \quad (K *_* M)_\chi \cong K_\chi *_* M_\chi.$$

(c) Verdier duality:

$$D(K *_! M) \cong D(K) *_* D(M), \quad D(K *_* M) \cong D(K) *_! D(M).$$

(d) Künneth formulae:

$$\begin{aligned} H_c^\bullet(X, K) \otimes_\Lambda H_c^\bullet(X, M) &\xrightarrow{\sim} H_c^\bullet(X, K *_! M), \\ H^\bullet(X, K *_* M) &\xrightarrow{\sim} H^\bullet(X, K) \otimes_\Lambda H^\bullet(X, M). \end{aligned}$$

Proof. Part *(a)* follows from adjunction as in [**41**, p. 533]. In *(b)* the first identity comes from $RHom(L_\chi, \Lambda_X) = L_{\chi^{-1}}$ and $(-id)^*(L_{\chi^{-1}}) = L_\chi$, and the other two follow as in proposition 2.1. Part *(c)* follows from the compatibility of Verdier duality with exterior tensor products. Part *(d)* is the Künneth isomorphism [**7**, exp. XVII.5.4] resp. its Verdier dual. □

3.3. The thick subcategory of negligible objects

In this and in the next section we always assume that for the abelian variety $A = X/T$ the vanishing assumption $GV(A)$ of section 3.1 holds. We have seen in theorem 3.1 that $GV(X)$ then holds as well. Let $\mathbf{S}(X) \subset \mathbf{P}(X)$ be the full subcategory of all perverse sheaves of Euler characteristic zero, and $\mathbf{T}(X) \subset \mathbf{D}(X)$ the full subcategory of all sheaf complexes K whose perverse cohomology sheaves ${}^pH^n(K)$ lie in $\mathbf{S}(X)$ for all degrees $n \in \mathbb{Z}$. For brevity we put

$$\mathbf{P} = \mathbf{P}(X), \quad \mathbf{D} = \mathbf{D}(X), \quad \mathbf{S} = \mathbf{S}(X) \quad \text{and} \quad \mathbf{T} = \mathbf{T}(X).$$

The following explains why we only use characters of the tame fundamental group — we want the Euler characteristic $\chi(K) = \sum_{i \in \mathbb{Z}} \dim_\Lambda(H^i(X, K))$ of any complex $K \in \mathbf{D}$ to be invariant under character twists.

LEMMA 3.5. *Let $K \in \mathbf{D}$. Then $\chi(K) = \chi(K_\varphi)$ for all $\varphi \in \Pi(X)$, hence the following three conditions are equivalent:*

(a) The complex K lies in the full subcategory $\mathbf{T}$.

(b) We have $H^\bullet(X, K_\varphi) = 0$ for generic $\varphi \in \Pi(X)$.

(c) We have $H_c^\bullet(X, K_\varphi) = 0$ for generic $\varphi \in \Pi(X)$.

Proof. To prove the twist invariance of the Euler characteristic, we can by dévissage assume that the complex K is a constructible sheaf placed in a single cohomology degree. Then [**59**, cor. 2.9] says that if $j : X \hookrightarrow \bar{X}$ denotes a smooth compactification, then $j_!(K)$ and $j_!(K_\varphi)$ have the same Euler characteristic (the assumptions of the corollary are satisfied since χ is a tame character, see sect. 2.6 of loc. cit.). Since for the Euler characteristic it makes no difference whether we use compactly supported or ordinary hypercohomology [**70**], it follows that $\chi(K) = \chi(K_\varphi)$. The equivalence of the three conditions *(a) – (c)* then follows via corollary 3.3. □

Recall that a full subcategory of an abelian category is said to be a Serre subcategory if it is stable under the formation of arbitrary subquotients and extensions. More generally, a full triangulated subcategory of a triangulated category is called a thick subcategory if it has the following property: For any morphism $f : K \to L$ which factors over an object of the subcategory and has its cone in the subcategory, the objects K and L must both belong to the subcategory as well.

COROLLARY 3.6. *The subcategory* $\mathbf{S} \subset \mathbf{P}$ *is Serre, and* $\mathbf{T} \subset \mathbf{D}$ *is thick.*

Proof. Let $0 \longrightarrow P \longrightarrow Q \longrightarrow R \longrightarrow 0$ be a short exact sequence in the abelian category $\mathbf{P}$. Theorem 3.1 says that for a generic character $\chi \in \Pi(X)$ the hypercohomology of the perverse sheaves P_χ, Q_χ and R_χ is concentrated in degree zero so that the long exact sequence in hypercohomology reduces to a short exact sequence

$$0 \longrightarrow H^0(X, P_\chi) \longrightarrow H^0(X, Q_\chi) \longrightarrow H^0(X, R_\chi) \longrightarrow 0.$$

Therefore lemma 3.5 shows that Q lies in $\mathbf{S}$ if and only if both P and R are in $\mathbf{S}$. Hence $\mathbf{S} \subset \mathbf{P}$ is a Serre subcategory, and then the subcategory $\mathbf{T} \subset \mathbf{D}$ is thick by [**41**, prop. 3.6.1(i)]. □

In particular, localizing the abelian category $\mathbf{P}$ at the class of morphisms whose kernel and cokernel lie in the Serre subcategory $\mathbf{S}$ we can form the abelian quotient category $\bar{\mathbf{P}} = \mathbf{P}/\mathbf{S}$ in the sense of [**42**, chap. III]. Note that the category $\bar{\mathbf{P}}$ has the same objects as $\mathbf{P}$ and that for any objects P_1, P_2 the elements of $Hom_{\bar{\mathbf{P}}}(P_1, P_2)$ can be represented by equivalence classes of diagrams in $\mathbf{P}$ of the form

$$\begin{tikzcd} & Q \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

where the kernel and cokernel of the morphism f_1 lie in $\mathbf{S}$. In the same way, localizing the triangulated category $\mathbf{D}$ at the class of morphisms whose cone lies in the thick subcategory $\mathbf{T}$ we can form the triangulated quotient category $\bar{\mathbf{D}} = \mathbf{D}/\mathbf{T}$ as described in [**82**, sect. 2.1].

It has been shown in [**41**, prop. 3.6.1] that these quotient constructions are compatible with each other in the sense that the perverse t-structure on $\mathbf{D}$ induces on the triangulated quotient category $\bar{\mathbf{D}}$ a t-structure whose core is naturally equivalent to the abelian quotient category $\bar{\mathbf{P}}$. So we have a commutative diagram

$$\begin{tikzcd} \mathbf{P} \arrow[r, hook] \arrow[d, two heads] & \mathbf{D} \arrow[d, two heads] \\ \bar{\mathbf{P}} \arrow[r, hook] & \bar{\mathbf{D}} \end{tikzcd}$$

where the rows are embeddings of full subcategories and the columns are the quotient functors. It turns out that convolution behaves well with respect to the above quotient constructions in the following sense.

LEMMA 3.7. *Let $K, M \in \mathbf{D}$.*

(a) The cone of the morphism $K *_! M \longrightarrow K *_* M$ lies in $\mathbf{T}$.

(b) If K or M lies in $\mathbf{T}$, then $K *_! M$ and $K *_* M$ are also in $\mathbf{T}$.

(c) If both K and M are perverse, then ${}^pH^n(K *_! M)$ and ${}^pH^n(K *_* M)$ lie in the Serre subcategory $\mathbf{S}$ for all $n \neq 0$.

Proof. (a) Let C be the cone of the morphism $K *_! M \to K *_* M$. We want to show that $H^\bullet(X, C_\chi) = 0$ for a generic character $\chi \in \Pi(X)$. Lemma 3.4 implies that for any character χ the twist C_χ is isomorphic to the cone of the twisted morphism

$$K_\chi *_! M_\chi \;\longrightarrow\; K_\chi *_* M_\chi,$$

so it suffices to check that the induced morphism

$$H^\bullet(X, K_\chi *_! M_\chi) \;\longrightarrow\; H^\bullet(X, K_\chi *_* M_\chi)$$

is an isomorphism for generic χ. By corollary 3.3 we can replace the left hand side by the compactly supported hypercohomology of $K_\chi *_! M_\chi$. So we must see that the forget support map

$$f_{K_\chi \boxtimes L_\chi}: \quad H_c^\bullet(X \times_k X, K_\chi \boxtimes M_\chi) \;\longrightarrow\; H^\bullet(X \times_k X, K_\chi \boxtimes M_\chi)$$

is an isomorphism for generic χ. For this we go back to the definition of compactly supported hypercohomology. For simplicity of notation we will suppress the character twist in what follows, replacing K and L by their twists K_χ and L_χ. Let $j : X \hookrightarrow \bar{X}$ be a compactification, and consider the following diagram where all the horizontal arrows are forget support morphisms (the vertical arrows will be discussed below).

$$\begin{tikzcd}
H_c^\bullet(X\times_k X, K\boxtimes L) \arrow[r, "f_{K\boxtimes L}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & H^\bullet(X\times_k X, K\boxtimes L) \arrow[d, equal] \\
H^\bullet(\overline{X}\times_k\overline{X}, (j,j)_!(K\boxtimes L)) \arrow[r] \arrow[dr, phantom, "(2)"] & H^\bullet(\overline{X}\times_k\overline{X}, R(j,j)_*(K\boxtimes L)) \\
H^\bullet(\overline{X}\times_k\overline{X}, j_!(K)\boxtimes j_!(L)) \arrow[r] \arrow[u] \arrow[dr, phantom, "(3)"] & H^\bullet(\overline{X}\times_k\overline{X}, Rj_*(K)\boxtimes Rj_*(L)) \arrow[u] \\
H^\bullet(\overline{X}, j_!(K))\otimes H^\bullet(\overline{X}, j_!(L)) \arrow[r] \arrow[u] \arrow[d, equal] \arrow[dr, phantom, "(4)"] & H^\bullet(\overline{X}, Rj_*(K))\otimes H^\bullet(\overline{X}, Rj_*(L)) \arrow[u] \arrow[d, equal] \\
H_c^\bullet(X,K)\otimes H_c^\bullet(X,L) \arrow[r, "f_K\otimes f_L"] & H^\bullet(X,K)\otimes H^\bullet(X,L)
\end{tikzcd}$$

The horizontal arrow $f_K\otimes f_L$ on the bottom line is the tensor product of the forget support morphisms for K and L, hence by corollary 3.3 it is an isomorphism for generic χ. On the other hand, the horizontal arrow $f_{K\boxtimes L}$ on the top line is the forget support morphism we are interested in. So we will be finished if we can show that the above diagram commutes and that all the vertical arrows (to be defined yet) are isomorphisms.

The squares (1) and (4) are commutative by the very definition of $f_{K\boxtimes L}$ and $f_K\otimes f_L$. The vertical arrows in (3) are the Künneth isomorphisms for the proper morphism $\overline{X}\to Spec(k)$, and the commutativity of this square follows from the fact that the Künneth morphisms are natural in the involved complexes. Finally, the square (2) is induced by the square

$$\begin{tikzcd}
(j,j)_!(K\boxtimes L) \arrow[r] & R(j,j)_*(K\boxtimes L) \\
j_!(K)\boxtimes j_!(L) \arrow[r] \arrow[u, "ad_!^{-1}"] & Rj_*(K)\boxtimes Rj_*(L) \arrow[u, "ad_*"']
\end{tikzcd}$$

where the morphisms

$$\begin{aligned}
ad_!: &\quad (j,j)_!(K\boxtimes L) \longrightarrow j_!(K)\boxtimes j_!(L) \\
ad_*: &\quad Rj_*(K)\boxtimes Rj_*(L) \longrightarrow R(j,j)_*(K\boxtimes L)
\end{aligned}$$

are the natural morphisms which correspond via adjunction to the identity morphism of

$$K\boxtimes L \;=\; (j,j)^!(j_!(K)\boxtimes j_!(L)) \;=\; (j,j)^*(Rj_*(K)\boxtimes Rj_*(L)).$$

Note that $ad_!$ is an isomorphism and that over $U = X\times_k X \hookrightarrow \overline{X}\times_k\overline{X}$ all the morphisms in the above diagram restrict to the identity. In particular, the diagram is commutative because by adjunction there exists only one morphism $j_!(K) \boxtimes j_!(L) \longrightarrow R(j,j)_*(K \boxtimes L)$ which restricts over U to the identity. By the same argument, the diagram

$$\begin{array}{ccc} Rj_*(K) \boxtimes Rj_*(L) & \xrightarrow{\quad ad_* \quad} & R(j,j)_*(K \boxtimes L) \\ \| & & \| \\ D(j_!(D(K)) \boxtimes j_!(D(L))) & \xrightarrow{D(ad_!)} & D((j,j)_!(D(K) \boxtimes D(L))) \end{array}$$

commutes. Since the lower row is an isomorphism (being the Verdier dual of an isomorphism), it follows that ad_* is an isomorphism as well, and this finishes the proof of part *(a)* of the lemma.

(b) Suppose that $K \in \mathbf{T}$, and take a generic character $\chi \in \Pi(X)$. We know from lemma 3.5 that $H_c^\bullet(X, K_\chi) = 0$, and the Künneth formula in lemma 3.4 then implies that

$$H_c^\bullet(X, (K *_! M)_\chi) \;=\; H_c^\bullet(X, K_\chi) \otimes_\Lambda H_c^\bullet(X, M_\chi) \;=\; 0$$

as well. So $K *_! M$ lies in $\mathbf{T}$ as required. The statement for $K *_* M$ follows in the same way or can be deduced via Verdier duality.

(c) For generic $\chi \in \Pi(X)$ theorem 3.1 says that $H_c^\bullet(X, K_\chi) \cong H^\bullet(X, K_\chi)$ and that this hypercohomology is concentrated in degree zero. The same also holds with M in place of K. So lemma 3.4 shows that $H_c^\bullet(X, K *_! M)$ and $H^\bullet(X, K *_* M)$ are concentrated in degree zero for generic χ, and our claim follows from corollary 3.3. □

3.4. Tannakian categories

In this section we assume as before that for the semiabelian variety X and its quotient $A = X/T$ the axiom $GV(A)$ and hence also $GV(X)$ from section 3.1 holds. The results of the previous section then imply

THEOREM 3.8. *The two convolution products $*_!$ and $*_*$ descend to the same well-defined bifunctor*

$$* : \bar{\mathbf{D}} \times \bar{\mathbf{D}} \longrightarrow \bar{\mathbf{D}}$$

which satisfies $\bar{\mathbf{P}} * \bar{\mathbf{P}} \subset \bar{\mathbf{P}}$ and with respect to which both $\bar{\mathbf{D}}$ and $\bar{\mathbf{P}}$ become symmetric monoidal Λ-linear triangulated categories.

Proof. By lemma 3.7 *(b)* both $*_!$ and $*_*$ descend to a bifunctor on $\bar{\mathbf{D}}$, and part *(a)* of the lemma shows that these two bifunctors coincide. It follows as in [**106**, sect. 2.1] that $(\bar{\mathbf{D}}, *)$ is a symmetric monoidal category. Part *(c)* of lemma 3.7 furthermore shows that $\bar{\mathbf{P}} * \bar{\mathbf{P}} \subseteq \bar{\mathbf{P}}$ so that $(\bar{\mathbf{P}}, *)$ inherits the structure of a symmetric monoidal category as well. □

It is sometimes convenient to have a lift of the quotient category $\bar{\mathbf{P}}$ to a category of true perverse sheaves. To obtain such a lift, consider the full subcategory $\mathbf{P}_{int} \subset \mathbf{P}$ of all perverse sheaves without subquotients in $\mathbf{S}$. By the same arguments as in [**41**, sect. 3.7] the functor $\mathbf{P} \longrightarrow \bar{\mathbf{P}}$ restricts to an equivalence of categories between $\mathbf{P}_{int}$ and $\bar{\mathbf{P}}$, and via this equivalence the convolution product $*$ induces a bifunctor

$$*_{int}: \quad \mathbf{P}_{int} \times \mathbf{P}_{int} \longrightarrow \mathbf{P}_{int}$$

with respect to which $\mathbf{P}_{int}$ becomes a Λ-linear symmetric monoidal category equivalent to $\bar{\mathbf{P}}$. The unit object $\mathbf{1}$ of $\mathbf{P}_{int}$ is the rank one skyscraper sheaf supported in the neutral element of the group variety X.

As an application we show that the category $\bar{\mathbf{P}}$ is rigid in the sense of section 2.1, i.e. that we have a notion of duality which involves evaluation and coevaluation morphisms with the properties familiar from the case of finite-dimensional vector spaces or group representations.

THEOREM 3.9. *The symmetric monoidal abelian category $\bar{\mathbf{P}}$ is rigid.*

Proof. Let $P \in \mathbf{P}_{int}$ be a perverse sheaf without constituents in $\mathbf{S}$. Via the adjunction property in lemma 3.4 the identity morphism $id_P : P \longrightarrow P$ defines two morphisms $\mathbf{1} \longrightarrow P^\vee *_* P$ and $P *_! P^\vee \longrightarrow \mathbf{1}$ in $\mathbf{D}$. Under the quotient functor $\mathbf{D} \longrightarrow \bar{\mathbf{D}}$ these induce morphisms in the full subcategory $\bar{\mathbf{P}}$ by lemma 3.7. Let us denote by

$$coev: \ \mathbf{1} \longrightarrow P^\vee *_{int} P \qquad \text{and} \qquad ev: \ P *_{int} P^\vee \longrightarrow \mathbf{1}$$

the corresponding morphisms in $\mathbf{P}_{int}$. By definition, rigidity means that for all $P \in \mathbf{P}_{int}$ the composite morphism

$$\gamma: \quad P = P *_{int} \mathbf{1} \xrightarrow{id *_{int} coev} P *_{int} P^\vee *_{int} P \xrightarrow{ev *_{int} id} \mathbf{1} *_{int} P = P$$

and its counterpart $(id *_{int} ev) \circ (coev *_{int} id): \ P^\vee \longrightarrow P^\vee$ are the identity morphisms. Since the argument is the same in both cases, we will only deal with the morphism γ in what follows.

We must show $\gamma - id_P = 0$. This assertion is invariant under character twists, so by the generic vanishing property $GV(X)$ we can assume that for all subquotients Q of the perverse sheaf P the hypercohomology $H^\bullet(X,Q)$ is concentrated in degree zero. Then $H^\bullet(X,-)$ behaves like an exact functor on all short exact sequences which only involve subquotients of P. After a suitable character twist we can furthermore assume that

$$H^\bullet(X, P^\vee *_! P) \ = \ H^\bullet(X, P^\vee *_{int} P) \ = \ H^\bullet(X, P^\vee *_* P)$$

and that the forget support morphism for these hypercohomology groups is an isomorphism. Then for $H = H^\bullet(X,P)$ and $H^\vee = Hom_\Lambda(H,\Lambda)$ the diagrams in [**41**, appendix A.5.4] show that the morphism

$$\Lambda \;=\; H^\bullet(X,\mathbf{1}) \xrightarrow{coev_*} H^\bullet(X, P^\vee *_{int} P) \;=\; H^\bullet(X, P^\vee *_* P) \;=\; H^\vee \otimes_\Lambda H$$

is the coevaluation in the category of vector spaces, and dually for ev. Since the category of vector spaces of finite dimension over Λ is rigid, it follows that on hypercohomology $\gamma - id_P$ induces the zero morphism. Accordingly the perverse subquotient $Q = P/\ker(\gamma - id_P)$ of P has $H^\bullet(X,Q) = 0$. By definition of $\mathbf{P}_{int}$ it follows that $Q = 0$ and hence that $\gamma - id_P = 0$. $\square$

This in particular allows to define the Tannaka groups we are interested in, generalizing those in corollary 2.14. For $P \in \bar{\mathbf{P}}$ we denote by $\langle P \rangle \subset \bar{\mathbf{P}}$ the full subcategory of all objects which are isomorphic to subquotients of convolution powers of $P \oplus P^\vee$. Recall that full subcategories of this form are said to be finitely tensor generated. By construction they inherit from $\bar{\mathbf{P}}$ the structure of a rigid symmetric monoidal Λ-linear abelian category, and we claim that they are neutral Tannakian in the following sense.

COROLLARY 3.10. *For every $P \in \bar{\mathbf{P}}$ there exists an affine algebraic group $G = G(P)$ over Λ and an equivalence*

$$\omega:\ \langle P \rangle \stackrel{\sim}{\longrightarrow} Rep_\Lambda(G)$$

with the rigid symmetric monoidal Λ-linear abelian category $Rep_\Lambda(G)$ of finite-dimensional algebraic representations of G.

Proof. Consider P as an object of the lifted category $\mathbf{P}_{int} \subset \mathbf{P}$. By theorem 3.1 there exists a character $\chi \in \Pi(X)$ such that all constituents of all convolution powers

$$(P \oplus P^\vee)_\chi *_{int} \cdots *_{int} (P \oplus P^\vee)_\chi$$

have their hypercohomology concentrated in degree zero (indeed these are countably many conditions, and identifying $\Pi(X)_l$ as in [**41**, sect. 3.2] with the set of Λ-valued points of the affine scheme $Spec(\Lambda \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[t_1,\dots,t_n]])$ for some $n > 0$ one checks that $\Pi(X)_l$ cannot be covered by countably many proper closed subsets — for this one may proceed by induction on n, using the description of ideals given in proposition A.2.2.3 of loc. cit.). For such a character χ the functor $Q \mapsto H^0(X, Q_\chi)$ is a fibre functor from $\langle P \rangle$ to the category of finite-dimensional vector spaces, and our corollary follows from theorem 3.9 via the Tannakian formalism [**33**, th. 2.11]. $\square$

The above fibre functor depends on the chosen character twist (though all choices lead to isomorphic Tannaka groups), and this is the reason why we have restricted ourselves to finitely tensor generated subcategories. For a more canonical construction one may as in [**41**, th. 3.7.5] take the functor which to a perverse sheaf assigns the generic fibre of its Mellin transform, but then the fibre functor is only defined over a large extension field of Λ, and the descent to Λ again involves non-canonical choices.

REMARK 3.11. *If $P \in \mathbf{P}(X)$ is a semisimple perverse sheaf, then $G(P)$ is a reductive algebraic group, and over $k = \mathbb{C}$ this group coincides with the one that has been defined in corollary 2.14.*

Proof. The group $G(P)$ acts faithfully on the representation $V = \omega(P)$ which in the case at hand is semisimple, whereas the unipotent radical of an algebraic group acts trivially on every irreducible representation. So the unipotent radical must be trivial, i.e. $G(P)$ is reductive. The remaining statement follows easily from the universal property of the localization $\bar{\mathbf{D}}$ together with the last statement of corollary 2.9. $\square$

3.5. Nearby cycles

To describe the behaviour of the above Tannaka groups with respect to degenerations, we work over the spectrum S of a strictly Henselian discrete valuation ring with closed point s and generic point η. We assume that the prime l is invertible on S. Fix a geometric point $\bar\eta$ over η, and denote by $\bar S$ the normalization of S in the residue field of $\bar\eta$.

Let $X \to S$ be a semiabelian scheme, i.e. a smooth commutative group scheme over S whose fibres X_s and X_η are semiabelian varieties. In what follows we always assume the generic fibre X_η is an abelian variety. We put $\bar X = X \times_S \bar S$ and write $X_{\bar s} = X_s$ and $X_{\bar\eta}$ for its geometric fibres.

$$\begin{tikzcd}
& X_{\bar s} \arrow[rr, "\bar i"] \arrow[dl, equal] \arrow[dd] & & \bar X \arrow[dl] \arrow[dd] & & X_{\bar\eta} \arrow[ll, "\bar j"'] \arrow[dl] \arrow[dd] \\
X_s \arrow[rr, "i"] \arrow[dd] & & X \arrow[dd] & & X_\eta \arrow[ll, "j"'] \arrow[dd] & \\
& \bar s \arrow[rr] \arrow[dl, equal] & & \bar S \arrow[dl] & & \bar\eta \arrow[ll] \arrow[dl] \\
s \arrow[rr] & & S & & \eta \arrow[ll] &
\end{tikzcd}$$

Degenerations of constructible sheaves can in this setting be studied via the functor of nearby cycles [**32**, exp. XIII-XIV]

$$\Psi: \quad \mathbf{D}(X_{\bar\eta}) \longrightarrow \mathbf{D}(X_s), \quad \Psi(K) = \bar i^*(R\bar j_*(K)).$$

By [**60**, th. 4.2 and cor. 4.5] this functor commutes with Verdier duality and restricts to an exact functor between the abelian categories $\mathbf{P}(X_{\bar\eta})$ and $\mathbf{P}(X_s)$ of perverse sheaves. However, the following example illustrates that unlike for abelian schemes, in the case of semiabelian schemes the functor Ψ in general does not preserve the Euler characteristic.

EXAMPLE 3.12. *For $n \in \mathbb{N}$ consider the kernel $X[n]$ of the multiplication morphism $[n] : X \longrightarrow X$. This kernel is a quasi-finite group scheme over S and decomposes as a disjoint union*

$$X[n] \;=\; Y \amalg Z$$

where Y is finite over S and where $Z \cap X_s = \varnothing$. In general $Z \neq \varnothing$, but the perverse skyscraper sheaf

$$\delta_{Z_{\bar\eta}} \;\in\; \mathbf{P}(X_{\bar\eta}) \quad \textit{satisfies} \quad \Psi(\delta_{Z_{\bar\eta}}) \;=\; 0$$

since the support of the nearby cycles must be contained in $Z \cap X_s = \varnothing$.

For the rest of this section we will always assume that for all geometric points t of S the fibre X_t satisfies the generic vanishing assumption $GV(X_t)$ from section 3.1. We can then consider the quotient categories

$$\bar{\mathbf{D}}(X_t) = \mathbf{D}(X_t)/\mathbf{T}(X_t) \quad \text{and} \quad \bar{\mathbf{P}}(X_t) = \mathbf{P}(X_t)/\mathbf{S}(X_t)$$

as defined in section 3.4. Note that the Euler characteristic is well-defined on objects of these quotient categories.

LEMMA 3.13. *The functor Ψ descends to a functor $\overline{\Psi} : \bar{\mathbf{P}}(X_{\bar\eta}) \to \bar{\mathbf{P}}(X_s)$, for all $P, Q \in \bar{\mathbf{P}}(X_{\bar\eta})$ we have*

$$(i) \qquad 0 \;\leq\; \chi(\overline{\Psi}(P)) \;\leq\; \chi(P),$$

$$(ii) \qquad 0 \;\leq\; \chi(\overline{\Psi}(P) * \overline{\Psi}(Q)) \;\leq\; \chi(\overline{\Psi}(P * Q)),$$

and $\overline{\Psi}(P) * \overline{\Psi}(Q)$ is a direct summand of $\overline{\Psi}(P * Q)$ in the category $\bar{\mathbf{P}}(X_s)$.

Proof. We begin with some preliminary remarks. Let $f : Y \to Z$ be a homomorphism of semiabelian S-schemes. By abuse of notation we again write f for any base change of it. Consider for $\bar{Y} = Y \times_S \bar{S}$ and $\bar{Z} = Z \times_S \bar{S}$ the commutative diagram

$$\begin{tikzcd}
Y_s \arrow[r, "\bar i"] \arrow[d, "f"'] & \bar{Y} \arrow[d, "f"'] & Y_{\bar\eta} \arrow[l, "\bar j"'] \arrow[d, "f"'] \\
Z_s \arrow[r, "\bar i"] & \bar{Z} & Z_{\bar\eta} \arrow[l, "\bar j"']
\end{tikzcd}$$

over $\bar{S}$. For $K \in \mathbf{D}(\bar{Y})$ this gives a commutative diagram

$$\begin{tikzcd}
\bar{i}^*(Rf_!(K)) \arrow[r, "bc_!"] \arrow[d, "\alpha"'] & Rf_!(\bar{i}^*(K)) \arrow[d, "\beta"] \\
\bar{i}^*(Rf_*(K)) \arrow[r, "bc_*"] & Rf_*(\bar{i}^*(K))
\end{tikzcd}$$

where α and β denote the forget support morphisms and where $bc_!$ and bc_* are the base change morphisms. Let us briefly recall how these latter two are defined — the direct image of the adjunction morphism $K \longrightarrow i_*(i^*(K))$ yields two morphisms

$$\begin{aligned} Rf_!(K) &\longrightarrow Rf_!(\bar{i}_*(\bar{i}^*(K))) &= \bar{i}_*(Rf_!(\bar{i}^*(K))), \\ Rf_*(K) &\longrightarrow Rf_*(\bar{i}_*(\bar{i}^*(K))) &= \bar{i}_*(Rf_*(\bar{i}^*(K))), \end{aligned}$$

and again by adjunction these give rise to the base change morphisms $bc_!$ and bc_* from above. We also remark [**7**, exp. XVII, th. 5.2.6] that as a result of the proper base change theorem, the base change morphism $bc_!$ is always an isomorphism. Putting $K = R\bar{j}_*(L)$ with $L \in \mathbf{D}(Y_{\bar{\eta}})$, we obtain a factorization

$$\begin{tikzcd} Rf_!(\Psi_Y(L)) \arrow[rr, "\beta"] \arrow[dr, "\alpha\circ(bc_!)^{-1}"'] & & Rf_*(\Psi_Y(L)) \\ & \Psi_Z(Rf_*(L)) \arrow[ur, "bc_*"'] & \end{tikzcd}$$

where Ψ_Y and Ψ_Z denote the nearby cycles for $Y \to S$ resp. $Z \to S$. We will apply this in the following two situations.

First we take $f : X \to S$ to be the structure morphism with $Y = X$, $Z = S$ and $L = P$. In this case the above diagram more explicitly says that the forget support morphism β for the cohomology of the nearby cycles $\Psi(P)$ admits the following factorization.

$$\begin{tikzcd} H_c^\bullet(X_s, \Psi(P)) \arrow[rr, "\beta"] \arrow[dr] & & H^\bullet(X_s, \Psi(P)) \\ & H^\bullet(X_{\bar{\eta}}, P) \arrow[ur] & \end{tikzcd}$$

Now consider a character χ of the tame fundamental group of $\overline{X}$, and denote by $\chi_t \in \Pi(X_t)$ its pull-back to the fibre X_t over the special point $t = s$ or over the geometric generic point $t = \bar{\eta}$. The projection formula (see section 3.6) implies that we have

$$\Psi(P_{\chi_{\bar{\eta}}}) \;=\; (\Psi(P))_{\chi_s}$$

for any such character. In lemma 3.16 we will see that χ can be chosen in such a way that the characters χ_s and $\chi_{\bar{\eta}}$ are both generic. By the generic vanishing axiom $GV(X_s)$ we can therefore assume that the forget support morphism β is an isomorphism, in which case the above factorization shows that $H^\bullet(X_s, \Psi(P))$ is a direct summand of $H^\bullet(X_{\bar{\eta}}, P)$. Furthermore, by the generic vanishing axiom $GV(X_{\bar{\eta}})$ we can also assume that all the occuring hypercohomology groups are concentrated in cohomology degree zero, in which case it follows that $0 \le \chi(\Psi(P)) \le \chi(P)$. In particular, the nearby cycles functor Ψ then sends the Serre subcategory $\mathbf{S}(X_{\bar\eta}) \subset \mathbf{P}(X_{\bar\eta})$ into the Serre subcategory $\mathbf{S}(X_s) \subset \mathbf{P}(X_s)$ and hence induces a functor

$$\overline{\Psi}: \bar{\mathbf{P}}(X_{\bar\eta}) \longrightarrow \bar{\mathbf{P}}(X_s)$$

between the quotient categories. The property (i) of $\overline{\Psi}$ is inherited from the corresponding estimate for Ψ shown above.

Secondly we take $f = m$ to be the group law, with $Y = X \times_k X$, $Z = X$ and $L = P \boxtimes Q$. In $\bar{\mathbf{D}}(X_s)$ we can identify $\Psi_Z(Rf_*(L))$ with $\overline{\Psi}(P * Q)$. Since by [**60**, th. 4.7] the exterior tensor product $\boxtimes$ commutes with nearby cycles, we can also identify $Rf_*(\Psi_Y(L))$ with $\overline{\Psi}(P) * \overline{\Psi}(Q)$. Then the factorization of β from above shows that the nearby cycles $\overline{\Psi}(P * Q)$ admit $\overline{\Psi}(P) * \overline{\Psi}(Q)$ as a direct summand in $\bar{\mathbf{P}}(X_s)$. Hence (ii) follows. $\square$

In general we cannot expect $\overline{\Psi}$ to be a tensor functor since it does not preserve the Euler characteristic (see example 3.12). In what follows we will call a perverse sheaf P on $X_{\bar\eta}$ admissible if

$$\chi(\overline{\Psi}(P)) \;=\; \chi(P).$$

Note that for an abelian scheme $X \to S$ every perverse sheaf is admissible since the nearby cycles are compatible with proper morphisms.

LEMMA 3.14. *The admissible objects form a rigid symmetric monoidal full abelian subcategory*

$$\bar{\mathbf{P}}(X_{\bar\eta})^{ad} \;\subset\; \bar{\mathbf{P}}(X_{\bar\eta})$$

which is stable under the formation of extensions and subquotients.

Proof. Any short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ in $\bar{\mathbf{P}}(X_{\bar\eta})$ is mapped under the nearby cycles functor to a short exact sequence in the category $\bar{\mathbf{P}}(X_s)$. Then in particular

$$\chi(P_2) \;=\; \chi(P_1) + \chi(P_3) \quad \text{and} \quad \chi(\overline{\Psi}(P_2)) \;=\; \chi(\overline{\Psi}(P_1)) + \chi(\overline{\Psi}(P_3))$$

because the Euler characteristic is additive in short exact sequences. But on the other hand $\chi(P_i) \geq \chi(\overline{\Psi}(P_i))$ for $i = 1,2,3$ by lemma 3.13 *(i)*. Hence it follows that

$$\chi(P_i) \;=\; \chi(\overline{\Psi}(P_i)) \quad \text{holds for } i = 2 \text{ iff it holds for } i = 1 \text{ and } i = 3.$$

In other words, the perverse sheaf P_2 is admissible if and only if both P_1 and P_3 are admissible. So the category of admissible objects is stable under the formation of extensions and subquotients.

It remains to show that for admissible $P_1, P_2 \in \bar{\mathbf{P}}(X_{\bar\eta})$ also $P_1 * P_2$ is admissible. This follows from

$$\begin{aligned}
\chi(P_1 * P_2) &= \chi(P_1)\cdot\chi(P_2) && \text{(part } (d) \text{ of lemma 3)}\\
&= \chi(\overline{\Psi}(P_1))\cdot\chi(\overline{\Psi}(P_2)) && \text{(admissibility of } P_1, P_2)\\
&= \chi(\overline{\Psi}(P_1) * \overline{\Psi}(P_2)) && \text{(part } (d) \text{ of lemma 3)}\\
&\le \chi(\overline{\Psi}(P_1 * P_2)) && \text{(part } (ii) \text{ of lemma 12)}\\
&\le \chi(P_1 * P_2) && \text{(part } (i) \text{ of lemma 12)}
\end{aligned}$$

which implies that the last two estimates must in fact be equalities. □

Concerning the failure of tensor functoriality it now follows that the situation is as good as one could possibly hope for.

THEOREM 3.15. *On the rigid symmetric monoidal abelian subcategory of admissible objects, the nearby cycles define a tensor functor ACU*

$$\overline{\Psi}: \quad \bar{\mathbf{P}}(X_{\bar\eta})^{ad} \longrightarrow \bar{\mathbf{P}}(X_s)$$

so that for $P \in \bar{\mathbf{P}}_{\bar\eta}^{ad}$ we obtain a closed immersion $G(\overline{\Psi}(P)) \hookrightarrow G(P)$.

Proof. Let $P_1, P_2 \in \bar{\mathbf{P}}(X_{\bar\eta})^{ad}$. The last statement in lemma 3.13 *(ii)* gives a split monomorphism

$$\overline{\Psi}(P_1) * \overline{\Psi}(P_2) \hookrightarrow \overline{\Psi}(P_1 * P_2)$$

in the abelian category $\bar{\mathbf{P}}(X_s)$. But $\chi(\overline{\Psi}(P_1) * \overline{\Psi}(P_2)) = \chi(\overline{\Psi}(P_1 * P_2))$ as we have seen in the proof of lemma 3.14. So the above monomorphism is an isomorphism in $\bar{\mathbf{P}}(X_s)$, and hence $\overline{\Psi}$ is a tensor functor on $\bar{\mathbf{P}}(X_{\bar\eta})^{ad}$. For the statement about the Tannaka groups, recall that by definition every object of $\langle\overline{\Psi}(P)\rangle$ is a subquotient of the image under $\overline{\Psi}$ of some object in $\langle P\rangle$. So our claim follows from the Tannakian formalism [**33**, prop. 2.21b)]. □

3.6. Specialization of characters

Let $X \longrightarrow S$ be a semiabelian scheme as above. Since we assumed the prime number l to be invertible on S, we know from [**17**] that the (tame) pro-l fundamental groups

$$\pi_1(X_{\bar\eta}, 0)_l = \pi_1^t(X_{\bar\eta}, 0)_l \quad \text{and} \quad \pi_1(X_s, 0)_l = \pi_1^t(X_s, 0)_l$$

are free $\mathbb{Z}_l$-modules of finite rank. However, the basic example of abelian varieties with semiabelian reduction illustrates that in general $\pi_1(X_s, 0)_l$ will have strictly smaller rank than $\pi_1(X_{\bar\eta}, 0)$. To complete the proof of lemma 3.13 we thus need to justify why in passing from the generic to the special fibre, we retain enough characters to apply the generic vanishing assumption from section 3.1 on both fibres. To this end, let $\Pi(\overline{X})_l$ denote the group of all continuous characters of $\pi_1(\overline{X}, x)_l$ for any chosen geometric point x in $\bar{X}$. The passage to a different choice of x corresponds to an inner automorphism of the fundamental group, which is not seen on the level of characters. So we have well-defined restriction homomorphisms

$$\bar{i}^*: \quad \Pi(\bar{X})_l \longrightarrow \Pi(X_s)_l, \qquad \chi \mapsto \chi_s,$$
$$\bar{j}^*: \quad \Pi(\bar{X})_l \longrightarrow \Pi(X_{\bar{\eta}})_l, \qquad \chi \mapsto \chi_{\bar{\eta}}.$$

Any character $\chi \in \Pi(\bar{X})_l$ defines a local system L_χ on $\bar{X}$, and for $K \in \mathbf{D}(\bar{X})$ the projection formula shows that

$$\Psi(K_{\chi_{\bar{\eta}}}) \;=\; \bar{i}^*(R\bar{j}_*(K \otimes_\Lambda \bar{j}^*(L_\chi))) \;=\; \bar{i}^*(R\bar{j}_*(K) \otimes_\Lambda L_\chi) \;=\; \Psi(K)_{\chi_{\bar{s}}}.$$

To apply this formula in relation with the vanishing axiom from section 3.1, one can use the following well-known result.

LEMMA 3.16. *The pull-back homomorphisms $\bar{i}^*$ and $\bar{j}^*$ on characters are an epimorphism resp. an isomorphism*

$$\Pi(X_s)_l \twoheadleftarrow \Pi(\bar{X})_l \xrightarrow{\sim} \Pi(X_{\bar{\eta}})_l.$$

In particular, for all open dense subsets $U_s \subseteq \Pi(X_s)_l$ and $U_{\bar{\eta}} \subseteq \Pi(X_{\bar{\eta}})_l$ one can find $\chi \in \Pi(\bar{X})_l$ such that

$$\chi_s \in U_s \quad \text{and} \quad \chi_{\bar{\eta}} \in U_{\bar{\eta}}.$$

Proof. Choose any path from the point 0 in X_s to the point x in $\bar{X}$. We first claim that the corresponding homomorphism

$$\bar{i}_*: \quad \pi_1(X_s, 0)_l \longrightarrow \pi_1(\bar{X}, x)_l$$

is injective. To check this, we must by the criterion in [**51**, exp. V, cor. 6.8] show that every finite étale cover

$$f: \quad Y \longrightarrow X_s$$

of l-primary degree is dominated by a connected component $Z_s^0 \subseteq Z_s$ of the special fibre Z_s of some finite étale cover $\bar{h}: \bar{Z} \longrightarrow \bar{X}$ of l-primary degree as indicated in the following diagram.

$$\begin{tikzcd}
Z_s^0 \arrow[r, hook, dotted] \arrow[d, dotted] & Z_s \arrow[r, hook, dotted] & \bar{Z} \arrow[dd, dotted, "\bar{h}"] \\
Y \arrow[d, "f"'] & & \\
X_s \arrow[rr, hook] & & \bar{X}
\end{tikzcd}$$

We can assume that Y is connected and that f is a Galois cover. Since the prime l is invertible on S, it then follows that Y can be made into a semiabelian variety such that f is an isogeny [**17**]. So by [**15**, th. 7.3.5] there exists an isogeny $g : X_s \longrightarrow Y$ such that $g \circ f$ and hence also $f \circ g$ is the multiplication by $n = l^\nu$ for some ν. Thus it suffices to treat the case where $Y = X_s$ and where $f = [n] : X_s \longrightarrow X_s$ is the multiplication by n.

If we were dealing with an abelian scheme, we could now take $\bar{Z} = \bar{X}$ and define $\bar{h} = [n] : \bar{X} \longrightarrow \bar{X}$ to be the multiplication by n. However, for a semiabelian scheme the multiplication by n is in general not a finite étale cover. To get around this problem we consider the Néron model of the generic fibre X_η, which will allow to find a finite étale cover $h : Z \longrightarrow X$ whose base change under $\bar{S} \longrightarrow S$ gives the desired $\bar{h}$. But before we can carry out this idea, we need to perform a base change to make sure that the Néron model does what we want.

Note that $\bar{X}$ and the special fibre X_s do not change if we replace the given semiabelian scheme $X \longrightarrow S$ by a base change $X' = X \times_S S' \longrightarrow S'$ where S' denotes the normalization of S in a finite extension of the quotient field $\kappa(\eta)$. After such a base change we can assume that all n-torsion points of the generic fibre X_η are defined over $\kappa(\eta)$. Then the universal property of the Néron model

$$N \longrightarrow S$$

of X_η implies that the multiplication morphism $[n] : N \longrightarrow N$ has constant fibre degree over its image and thus restricts to a finite étale cover over the image. This image contains the connected component N^0 which by prop. 7.4.3 of loc. cit. is isomorphic to X. Hence if we define $h : Z \longrightarrow X$ via the fibre product in the Cartesian diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "h"'] & N \arrow[d, "{[n]}"] \\ X \arrow[r, hook] & N \end{tikzcd}$$

then h will be a finite étale cover of X. The special fibre Z_s will in general not be connected, but on the connected component

$$Z_s^0 \;=\; Z_s \cap N_s^0 \;=\; N_s^0 \;=\; X_s$$

the cover h restricts to the multiplication morphism $f = [n] : X_s \longrightarrow X_s$. So we can put $\bar{Z} = Z \times_S \bar{S}$ and take $\bar{h}$ to be the base change of h.

Summing up, this shows that the homomorphism $\bar{i}_* : \pi_1(X_s)_l \longrightarrow \pi_1(\bar{X})_l$ is an embedding. With the same arguments as above one also checks that the homomorphism $\bar{j}_* : \pi_1(X_{\bar{\eta}})_l \longrightarrow \pi_1(\bar{X})_l$ is an isomorphism (surjectivity is in this case clear because a finite étale cover of $\bar{X}$ is connected iff it restricts to a connected cover of $X_{\bar{\eta}}$). From this one deduces that the induced homomorphisms

$$\Pi(\bar{X})_l \;\longrightarrow\; \Pi(X_s)_l \quad \text{and} \quad \Pi(\bar{X})_l \;\longrightarrow\; \Pi(X_{\bar{\eta}})_l$$

are an epi- resp. an isomorphism. The final statement then follows from the observation that these homomorphisms come from morphisms between the underlying schemes and that the intersection of any two Zariski-open dense subsets of the irreducible scheme $\Pi(\bar{X})_l \cong \Pi(X_{\bar{\eta}})_l$ is non-empty. $\square$

3.7. Variation of Tannaka groups in families

In this section we turn the local result from theorem 3.15 into a more global statement. Under some mild finiteness assumptions we will see that for a smooth family of varieties, the Tannaka groups of the fibres define a constructible stratification of the base space. This will be used in chapter 4 to determine the Tannaka group for the theta divisor of a general principally polarized complex abelian variety.

For technical reasons we will restrict ourselves to abelian rather than semiabelian schemes. In what follows $X \longrightarrow S$ always denotes an abelian scheme whose base scheme S is an algebraic variety over an algebraically closed field k of characteristic zero. Let

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

be a closed subscheme which is smooth of relative dimension d over the base S, and for geometric points $\bar{s}$ in S consider the perverse intersection cohomology sheaf

$$\delta_{Y_{\bar{s}}} \;=\; \Lambda_{Y_{\bar{s}}}[d] \;\in\; \mathbf{P}(Y_{\bar{s}}) \;=\; \mathrm{Perv}(Y_{\bar{s}}, \Lambda)$$

on the fibre $Y_{\bar{s}}$. We want to understand how the Tannaka groups $G(\delta_{Y_{\bar{s}}})$ vary with the chosen point $\bar{s}$. The following examples illustrate that in spite of our smoothness assumption, this is in general a non-trivial problem which involves the symmetry properties of the fibres $Y_{\bar{s}}$.

EXAMPLE 3.17. *(a) If $S = E$ is an elliptic curve and $X = S \times E$, then for the tautological subvariety $Y = \{(e,e) \in X \mid e \in E\}$ we have*

$$G(\delta_{Y_{\bar{s}}}) \;\cong\; \begin{cases} \mathbb{Z}/n\mathbb{Z} & \textit{if } \bar{s} \textit{ is a torsion point in } E \textit{ of precise order } n, \\ \mathbb{G}_m & \textit{if } \bar{s} \textit{ is a point of infinite order in } E. \end{cases}$$

(b) If $Y \to S$ is a smooth, projective family of curves of genus $g \geq 3$ and is embedded into its relative Picard scheme $X = Pic^0_{Y/S}$ in a suitable way, then by [**106**, th. 14] *we have*

$$G(\delta_{Y_{\bar{s}}}) \;\cong\; \begin{cases} Sp_{2g-2}(\Lambda) & \textit{if } Y_{\bar{s}} \textit{ is hyperelliptic}, \\ Sl_{2g-2}(\Lambda) & \textit{otherwise}. \end{cases}$$

In particular, example *(a)* shows that without further assumptions the Tannaka groups do not always define a constructible stratification of S. In general we only have the following semicontinuity property.

LEMMA 3.18. *Let $\eta \in S$ be any scheme-theoretic point, $s \in \overline{\{\eta\}}$ a point in its closure, and consider geometric points $\bar{\eta}$ and $\bar{s}$ above these. Then we have an embedding*

$$G(\delta_{Y_{\bar{s}}}) \hookrightarrow G(\delta_{Y_{\bar{\eta}}}).$$

Proof. The statement is only concerned with the geometric fibres of the families $Y \longrightarrow S$ and $X \longrightarrow S$, so we can replace these families by their base change under any quasi-finite morphism $S' \longrightarrow S$ whose image contains the points s and η as indicated in the following diagram.

$$\begin{tikzcd}
& Y' \arrow[r] \arrow[dl] \arrow[ddr] & X' \arrow[dl] \arrow[dd] & & \\
Y \arrow[r] \arrow[ddr] & X \arrow[dd] & & & \\
& & S' \arrow[dl] & \exists s' \arrow[dl, mapsto] & \exists \eta' \arrow[dl, mapsto] \\
& S & s & \eta &
\end{tikzcd}$$

In particular, replacing S by the reduced closed subscheme $S' = \overline{\{\eta\}}_{\mathrm{red}}$ we can assume S is irreducible with generic point η. By normalization we can also achieve that S is a normal variety. Furthermore, passing to an open neighborhood of the point s we can assume that

$$S = Spec(A)$$

is affine. Let $\mathfrak{p} \lhd A$ be the prime ideal corresponding to the point s. Without loss of generality $s \neq \eta$, so we can find a non-zero element $f \in \mathfrak{p} \setminus \{0\}$ in our prime ideal. Then the vanishing locus

$$V(f) = Spec(A/(f)) \hookrightarrow S = Spec(A)$$

is a proper closed subscheme of codimension one and contains s. Now take any irreducible component $D \hookrightarrow V(f)_{\mathrm{red}}$ of the underlying reduced closed subscheme such that the point s lies on the Weyl divisor D, and let $\bar{\eta}_D$ be a geometric generic point of this irreducible divisor. By induction on the dimension $\dim(S)$ we have an embedding

$$G(\delta_{Y_{\bar{s}}}) \hookrightarrow G(\delta_{Y_{\bar{\eta}_D}}),$$

hence it will be enough to prove our claim when $s = \eta_D$ is the generic point of an irreducible Weyl divisor D on S.

This being said, recall from the above that we can assume S to be normal and hence regular in codimension one. In this case the one-dimensional local ring $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ is a discrete valuation ring. Its residue field $k(s)$ can be identified naturally with the function field $k(D)$ of the reduced irreducible subvariety $D \hookrightarrow S$ via the diagram

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d] & A/\mathfrak{p} \arrow[r, hook] \arrow[d] & k(D) \\ A_{\mathfrak{p}} \arrow[r, two heads] \arrow[urr, dotted] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r, equal] & k(s) \arrow[u, dotted, "\cong"'] \end{tikzcd}$$

where the dotted arrows exist by the universal property of the localization functor. Hence we can replace S by the strict Henselization $Spec(A_{\mathfrak{p}}^{sh})$ which still has $\bar{s}$ and $\bar{\eta}$ as geometric points. From theorem 3.15 we then obtain an embedding

$$G(\Psi(\delta_{Y_{\bar{\eta}}})) \hookrightarrow G(\delta_{Y_{\bar{\eta}}})$$

for the nearby cycles on the corresponding abelian scheme, and our claim follows because for a smooth morphism $Y \to S$ one has $\Psi(\delta_{Y_{\bar{\eta}}}) = \delta_{Y_{\bar{s}}}$. □

To obtain a constructibility statement for the stratifications defined by the Tannaka groups, we need to impose some finiteness conditions on the perverse sheaves on the fibres. To simply the notation, let us temporarily assume that X is an abelian variety over $S = Spec(k)$. Let $P \in \mathbf{P}(X)$ be a perverse sheaf, and consider the equivalence

$$\omega : \quad \langle P \rangle \xrightarrow{\sim} Rep_{\Lambda}(G(P))$$

from corollary 3.10. In what follows we will be particularly interested in the determinant character

$$\det(\omega(P)).$$

Like any character of the Tannaka group, this determinant corresponds by part *(b)* of proposition 1.6 to a perverse skyscraper sheaf $\delta_x \in \mathbf{P}(X)$ which is supported on a closed point $x \in X$. In this context the following two conditions are equivalent:

- the determinant $\det(\omega(P))$ is a torsion character of order n,
- the corresponding point $x \in X$ is a torsion point of order n.

In many applications P is isomorphic to its adjoint dual $P^{\vee} = (-id_X)^* D(P)$ so that $\omega(P)$ is self-dual and the above conditions hold with $n = 2$.

LEMMA 3.19. *Let $d, n \in \mathbb{N}$. Then there exists a finite set of subgroups of $Gl_d(\Lambda)$ with the property that for every simple perverse sheaf $P \in \mathbf{P}(X)$ of Euler characteristic $\chi(P) = d$ with*

$$\det(\omega(P))^{\otimes n} = 1,$$

the Tannaka group $G(P)$ is isomorphic to one of these finitely many groups.

Proof. Lemma B.1 and B.2 in the appendix show that up to conjugation there are only finitely many possibilities for the Tannaka group $G = G(P)$ as a subgroup of $Gl_d(\Lambda)$. $\square$

Let us now return to the case of an abelian scheme $p : X \longrightarrow S$ whose base scheme is an arbitrary variety S over the algebraically closed field k of characteristic zero. For any sheaf complex $P \in \mathbf{D}(X)$ and any geometric point $\bar{s}$ of S we will denote by

$$P_{\bar{s}} \;=\; i_{\bar{s}}^*(P) \;\in\; \mathbf{D}(X_{\bar{s}})$$

the restriction to the geometric fibre $i_{\bar{s}} : X_{\bar{s}} \longrightarrow X$. Then the constructibility result we are interested in can be formulated as follows.

PROPOSITION 3.20. *Let $n \in \mathbb{N}$, and consider a complex $P \in \mathbf{D}(X)$ such that for each geometric point $\bar{s}$ of S, the restriction $P_{\bar{s}}$ is a simple perverse sheaf with*

$$\det(\omega(P_{\bar{s}}))^{\otimes n} \;=\; 1.$$

Then there are reductive algebraic groups $G_1, \dots, G_m$ and a stratification into locally closed subsets

$$S \;=\; \coprod_{i=0}^{m} S_i \quad \text{such that} \quad G(P_{\bar{s}}) = G_i \quad \text{for all geometric points } \bar{s} \text{ in } S_i.$$

Proof. Recall from [**29**, prop. 3.1(c)] that if V is a finite-dimensional vector space over Λ, then any reductive algebraic subgroup of $Gl(V)$ is determined uniquely by its subspaces of invariants in the tensor powers $W^{\otimes r}$ of $W = V \oplus V^\vee$ with $r \in \mathbb{N}$. Furthermore, if we only want to distinguish between finitely many given reductive subgroups up to conjugacy, then it suffices to look at exponents $r \leq r_0$ for some fixed $r_0 \in \mathbb{N}$.

Hence in view of lemma 3.19 our claim will follow if we can show that in a suitable sense, the tensor functor $\omega : \langle P_{\bar{s}} \rangle \longrightarrow Rep_\Lambda(G(P_{\bar{s}}))$ in the proof of corollary 3.10 varies in a constructible way with the geometric point $\bar{s}$ and that the subspaces of invariants — corresponding to the direct sum of all direct summands $\mathbf{1} = \delta_0$ in the convolution powers of $P_{\bar{s}} \oplus P_{\bar{s}}^\vee$ — also vary in a constructible way. More precisely, we would be done if we could show that there exist

(a) a complex $P^\vee \in \mathbf{D}(X)$ which on each geometric fibre $X_{\bar{s}}$ restricts to the adjoint dual

$$P_{\bar{s}}^\vee \;=\; (-id)^* D(P_{\bar{s}}),$$

(b) a character $\chi \in \Pi(X)$ such that for the twist $Q = P_\chi \oplus P_\chi^\vee$ the direct image complex

$$\mathscr{W} \;=\; Rp_*(Q)$$

is a constructible sheaf on S which is concentrated in degree zero and has geometric stalks

$$\mathscr{W}_{\bar{s}} \;=\; H^0(X_{\bar{s}}, Q_{\bar{s}}) \;=\; \omega(P_{\bar{s}} \oplus P_{\bar{s}}^\vee),$$

(c) for each $r \leq r_0$ a constructible subsheaf of $\mathscr{W}^{\otimes r}$ whose stalks at any geometric point $\bar{s}$ coincide under the above identification with the cohomology of the direct sum of all skyscraper summands $\mathbf{1} = \delta_0$ in the convolution $Q_{\bar{s}}^{*r} = Q_{\bar{s}} * \cdots * Q_{\bar{s}}$.

In general this is too much to expect. However, since the groups $G_{\bar{s}} = G(P_{\bar{s}})$ only depend on the geometric fibres of our abelian scheme and since for the proof of constructibility we can argue by Noetherian induction on the dimension $\dim(S)$, it will do no harm if we replace our original abelian scheme by the base change

$$X' \;=\; X \times_S S' \;\longrightarrow\; S'$$

under a quasi-finite étale morphism $S' \longrightarrow S$ and also replace $P \in \mathbf{D}(X)$ by its pull-back $P' \in \mathbf{D}(X')$. In particular, for the proof of the proposition we are free to replace our base scheme S by arbitrary Zariski-open dense subsets of S and also by finite étale covers of such subsets.

With this extra freedom, property *(a)* can be obtained from the general fact [**64**, prop. 1.1.7] that for any morphism $p: X \to S$ of varieties with smooth target variety S and any $P \in \mathbf{D}(X)$, there is a Zariski-open dense subset $S' \hookrightarrow S$ such that the formation of the relative dual

$$D_{X/S}(P) \;=\; R\mathscr{H}om(P, p^!\Lambda_S)$$

commutes with every base change that factors over $S' \hookrightarrow S$. Property *(b)* will be obtained in lemma 3.21 after a base change to a finite étale cover of some Zariski-open dense subset of S, which is needed to make sure that there are sufficiently many characters available. For part *(c)* consider the relative convolution power $Q^{*r} = Ra_*(Q \boxtimes_S \cdots \boxtimes_S Q)$ where $a: X \times_S \cdots \times_S X \longrightarrow X$ is the group law. We have a relative Künneth isomorphism

$$Rp_*(Q^{*r}) \;\xrightarrow{\cong}\; (Rp_*(Q))^{\otimes r} \;=\; \mathscr{W}^{\otimes r}$$

as in the absolute case. Let $Z \subset X$ be the zero section of our abelian scheme, and consider the morphism

$$\varphi: \quad Rp_*(R\mathscr{H}om(\Lambda_Z, Q^{*r})) \;\longrightarrow\; R\mathscr{H}om(\Lambda_S, \mathscr{W}^{\otimes r}) \;\cong\; \mathscr{W}^{\otimes r}$$

which corresponds by adjunction to the composite morphism ψ given by the following diagram (where the diagonal arrow on the left comes from the adjoint of the relative Künneth isomorphism).

$$\begin{tikzcd} R\mathscr{H}om(\Lambda_Z, Q^{*r}) \arrow[rr, "\psi"] \arrow[dr] & & p^!(R\mathscr{H}om(\Lambda_S, \mathscr{W}^{\otimes r})) \\ & R\mathscr{H}om(p^*(\Lambda_S), p^!(\mathscr{W}^{\otimes r})) \arrow[ur, "\cong"'] & \end{tikzcd}$$

As a candidate for the constructible subsheaf in part *(c)* we take the image of the induced morphism $\mathscr{H}^0(\varphi)$ on the stalk cohomology sheaves in degree zero. After shrinking S we can by loc. cit. assume that the occuring $R\mathscr{H}om$ commute with base change. If we put

$$K \;=\; (P_{\bar{s}} \oplus P_{\bar{s}}^{\vee})^{*r}$$

and denote by $\chi_{\bar{s}}$ the restriction of the character χ to the geometric fibre $X_{\bar{s}}$, then on stalks we obtain a commutative diagram

$$\begin{tikzcd} H^0(X_{\bar{s}}, R\mathscr{H}om(\delta_0, K_{\chi_{\bar{s}}})) \arrow[r, "\mathscr{H}^0(\varphi)_{\bar{s}}"] \arrow[d, equal] & H^0(X_{\bar{s}}, K_{\chi_{\bar{s}}}) \\ \bigoplus_{i\in\mathbb{Z}} Ext^i(\delta_0, {}^pH^{-i}(K)_{\chi_{\bar{s}}}) \arrow[r] & Hom(\delta_0, {}^pH^0(K)_{\chi_{\bar{s}}}) \arrow[u] \end{tikzcd}$$

where the *Ext* and *Hom* in the lower row refer to the abelian category $\mathbf{P}(X_{\bar{s}})$ and where the vertical identifications come from Gabber's decomposition theorem by which $K_{\chi_{\bar{s}}} \cong \bigoplus_{i\in\mathbb{Z}} {}^pH^i(K)_{\chi_{\bar{s}}}[-i]$. The fact that $\mathscr{H}^0(\varphi)_{\bar{s}}$ factors over the projection in the lower row follows from the observation that by part *(b)* we have $H^\bullet(X_{\bar{s}}, {}^pH^{-i}(K)_{\chi_{\bar{s}}}) = 0$ for all $i \neq 0$. Hence *(c)* follows as well, and we are done. $\square$

In the above proof we have used that after a suitable quasi-finite étale base change there exist enough characters $\chi \in \Pi(X)$ so that we can apply the vanishing theorem from section 3.1 fibre by fibre. More precisely we have the following result.

LEMMA 3.21. *Let $P \in \mathbf{D}(X)$ be a complex which restricts to a perverse sheaf on each geometric fibre of the abelian scheme $X \to S$, and suppose that S is irreducible. Then there exists*

- *a quasi-finite étale morphism $f : S' \to S$,*
- *a character $\chi \in \Pi(X')$ for the abelian scheme $X' = X \times_S S'$,*

such that the direct image of $P' = (id_X, f)^*(P) \in \mathbf{D}(X')$ under the structure morphism $p : X' \to S'$ satisfies

$$\mathscr{H}^i(Rp_*(P' \otimes_\Lambda L_\chi)) \;=\; 0 \quad \textit{for all} \quad i \neq 0.$$

Proof. Take a geometric generic point $\bar{\eta}$ of S. The set of all torsion characters $\chi_{\bar{\eta}} : \pi_1(X_{\bar{\eta}}, 0) \longrightarrow \Lambda^*$ is dense in the character torus $\Pi(X_{\bar{\eta}})$, so by the generic vanishing theorem from chapter 1 we can find a character $\chi_{\bar{\eta}}$ of some finite order $N \in \mathbb{N}$ with the property that $H^i(X_{\bar{\eta}}, P_{\bar{\eta}} \otimes_\Lambda L_{\chi_{\bar{\eta}}}) = 0$ for all $i \neq 0$. If we could extend this character to a global character $\chi \in \Pi(X)$, then the above vanishing property would mean that for the direct image under $p : X \to S$ the generic stalk cohomology satisfies

$$\mathscr{H}^i(Rp_*(P \otimes_\Lambda L_\chi))_{\bar{\eta}} \;=\; 0 \quad \text{for all} \quad i \neq 0.$$

Since the cohomology sheaves are constructible, the vanishing of their stalks at the geometric generic point $\bar{\eta}$ would imply their vanishing over some open dense subset $U \subseteq S$, and we would be done even without the need to pass to a finite étale cover of the base variety S and without using that $\chi_{\bar{\eta}}$ is a torsion character of order N.

However, in general it may be impossible to find $\chi_{\bar{\eta}} \in \Pi(X_{\bar{\eta}})$ with the required property which lifts to some $\chi \in \Pi(X)$ (for example, think of the universal abelian scheme X over a finite level cover S of the moduli space of principally polarized abelian varieties — here the abelianized fundamental group $\pi_1^{ab}(X)$ and hence also the group $\Pi(X)$ is finite). To get around this problem we perform a base change under a finite étale cover depending on the chosen torsion character $\chi_{\bar{\eta}}$. Consider the split exact fibration sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(X_{\bar{\eta}}, 0) \arrow[r] & \pi_1(X, 0) \arrow[r] & \pi_1(S, \bar{\eta}) \arrow[r] \arrow[l, dotted, bend right] & 0 \end{tikzcd}$$

from [**51**, exp. XIII, prop. 4.3 and ex. 4.4], where the splitting comes from the zero section of our abelian scheme. Via the splitting, the group $\pi_1(S, \bar{\eta})$ acts by conjugation on $\pi_1(X_{\bar{\eta}}, 0)$ and hence on the corresponding character torus $\Pi(X_{\bar{\eta}})$. Since the group of torsion points of order N in this torus is finite, we can find a normal subgroup of finite index in $\pi_1(S, \bar{\eta})$ which fixes every such torsion point. This normal subgroup defines a finite étale cover $S' \longrightarrow S$ for which the character $\chi_{\bar{\eta}} \in \Pi(X_{\bar{\eta}})$ can be extended to a character $\chi \in \Pi(X')$. After a base change by this finite étale cover we can then proceed as in the first part of the proof. $\square$

CHAPTER 4

The Tannaka group of the theta divisor

The geometric relevance of the constructions in the previous chapters is illustrated by the following example. Let X be a principally polarized abelian variety (ppav) of dimension g over the complex numbers. The given polarization determines an ample theta divisor $\Theta \subset X$ up to a translation by a point in $X(\mathbb{C})$. For us it will be most convenient to choose a translation such that the theta divisor is symmetric, i.e. stable under $-id_X$. Then the perverse intersection cohomology sheaf

$$\delta_\Theta \;=\; \mathrm{IC}_\Theta[g-1]$$

will be isomorphic to its adjoint dual as defined in section 2.1. With this normalization, consider the Tannaka group $G(\delta_\Theta)$ which describes the rigid symmetric monoidal abelian category generated by the convolution powers of δ_Θ as in corollary 2.14 or equivalently in corollary 3.10. This Tannaka group is a new invariant of the ppav, a reductive complex algebraic group which behaves in a very different and much more rigid way than classical invariants such as Mumford-Tate groups do.

For example, for the Jacobian variety $X = JC$ of a smooth projective curve C it has been shown in [**106**, p. 124 and th. 14] that

$$G(\delta_\Theta) \;=\; \begin{cases} Sp_{2g-2}(\mathbb{C})/\varepsilon_{g-1} & \text{if } C \text{ is hyperelliptic,} \\ Sl_{2g-2}(\mathbb{C})/\mu_{g-1} & \text{otherwise,} \end{cases}$$

where $\mu_{g-1} \subset Sl_{2g-2}(\mathbb{C})$ denotes the central subgroup of $g-1^{\text{st}}$ roots of unity in the special linear group and $\varepsilon_{g-1} = \mu_{g-1} \cap \{\pm 1\}$ is its intersection with the symplectic group. In a nutshell, this behaviour can be explained as follows. For a suitable choice of the Abel-Jacobi embedding $C \hookrightarrow X$ one can use Brill-Noether theory and arguments from representation theory to show that the Tannaka group of the perverse sheaf $\delta_C \in \mathrm{Perv}(X, \mathbb{C})$ is

$$G(\delta_C) \;=\; \begin{cases} Sp_{2g-2}(\mathbb{C}) & \text{if } C \text{ is hyperelliptic,} \\ Sl_{2g-2}(\mathbb{C}) & \text{otherwise,} \end{cases}$$

and that δ_C corresponds to the standard representation of this group. Since on a Jacobian variety $X = JC$ the theta divisor Θ is birational to the $g-1^{\text{st}}$ symmetric product of the curve C, one deduces that the perverse sheaf δ_Θ corresponds on the Tannakian side to the $g-1^{\text{st}}$ fundamental representation of the group $G(\delta_C)$, and this implies that the Tannaka group $G(\delta_\Theta)$ of the theta divisor has the form given above.

The high fundamental representation in the above example is accounted for by the particular symmetry of the theta divisor of Jacobian varieties, and in general it is reasonable to expect the Tannaka groups $G(\delta_\Theta)$ to be much larger. One may speculate [**110**] whether among all ppav's of dimension g only Jacobian varieties have the above Tannaka groups — this would give a Tannakian answer for the Schottky problem to characterize the locus of Jacobian varieties in the moduli space

$$\mathscr{A}_g = \mathscr{H}_g / Sp_{2g}(\mathbb{Z}),$$

where $\mathscr{H}_g$ denotes the Siegel upper half space. More generally, one may ask for the stratification of $\mathscr{A}_g$ defined by the invariant $G(\delta_\Theta)$.

In this context, let us say that a statement holds for a general ppav of dimension g if it holds for every ppav in a Zariski-open dense subset of the moduli space $\mathscr{A}_g$. For example, the theta divisor of a general ppav is smooth by the results of Andreotti and Mayer [**4**]. For the open stratum in the above-mentioned stratification of $\mathscr{A}_g$ we discuss in this chapter the following conjecture which goes back to the preprint [**67**].

CONJECTURE 4.1. *If X is a complex ppav of dimension g equipped with a smooth symmetric theta divisor $\Theta \subset X$, then*

$$G(\delta_\Theta) = \begin{cases} SO_{g!}(\mathbb{C}) & \text{for } g \text{ odd}, \\ Sp_{g!}(\mathbb{C}) & \text{for } g \text{ even}, \end{cases}$$

and δ_Θ corresponds to the standard representation of this group.

Intuitively, this conjecture says that in marked contrast with the singular situation for Jacobian varieties, for a smooth theta divisor Θ the Tannaka group $G(\delta_\Theta)$ should be as large as possible — see lemma 4.6 below. Let us take a look at some small examples to begin with. The case $g = 1$ is of course trivial. For $g = 2$ every ppav with a smooth theta divisor is the Jacobian of a smooth hyperelliptic curve; then our conjecture holds by the result of [**106**] quoted above. Similarly, for $g = 3$ every ppav with a smooth theta divisor is the Jacobian of a smooth non-hyperelliptic curve, and in this case the conjecture holds since the second fundamental representation of $Sl_4(\mathbb{C})/\mu_2$ corresponds to the standard representation of $SO_6(\mathbb{C})$ via the exceptional isomorphism between these two groups. So the first open case of our conjecture occurs for $g = 4$. This is the first case where the realm of Jacobian varieties is left, hence also the first non-trivial case of the Schottky problem, and indeed we will see in corollary 4.4 below that in this case the invariant $G(\delta_\Theta)$ determines the locus of Jacobian varieties in $\mathscr{A}_4$.

To avoid case distinctions, from now on we put $G(g) = SO_{g!}(\mathbb{C})$ if g is odd and $G(g) = Sp_{g!}(\mathbb{C})$ if g is even. The main goal of this chapter is to prove the following weaker version of our conjecture, extending ideas that have been used in dimension $g = 4$ in the earlier preprint [**67**].

THEOREM 4.2. *For a general complex ppav X of dimension g with a symmetric theta divisor Θ one has*

$$G(\delta_\Theta) \ = \ G(g),$$

and δ_Θ corresponds to the standard representation of this group.

As we have remarked above, this essentially means that for a general ppav the Tannaka group of the theta divisor is as large as possible — the main task will be to find sufficiently large lower bounds on this group. By a constructibility argument it suffices to do this for the generic fibre of a suitable family of ppav's over a smooth complex algebraic curve S. So we can apply degeneration methods: By theorem 3.15 the Tannaka group of the generic fibre admits those of the perverse sheaves of nearby cycles on the special fibres as subquotients. With this in mind, we collect in section 4.3 some general properties of the monodromy filtration that allow to control the nearby cycles in concrete geometric situations. The crucial part of our argument is then given in section 4.4 where we consider three degenerations of a generic ppav — in one of them the theta divisor becomes nodal, and in the other two the ppav degenerates into a product of ppav's resp. into a Jacobian variety. In each case the nearby cycles contain a large irreducible constituent that can be controlled geometrically, and this will reduce the proof of our theorem to arguments in representation theory to be worked out in section 4.5.

4.1. The Schottky problem in genus 4

Before we come to the proof of theorem 4.2, let us briefly discuss its connection with the Schottky problem. Recall that in dimension $g \geq 4$ the Andreotti-Mayer locus $\mathscr{N}_g \subset \mathscr{A}_g$ of ppav's with a singular theta divisor is itself a divisor with precisely two irreducible components [**26**]. One of the components is the locus $\theta_{null,g}$ of all ppav's with a vanishing theta null; the other component contains the closure $\mathscr{J}_g$ of the locus of Jacobian varieties and for $g = 4$ is equal to it [**8**]. This being said, in dimension $g = 4$ we can sharpen the formulation of theorem 4.2 as follows.

THEOREM 4.3. *For $g = 4$ the locus of all ppav's with $G(\delta_\Theta) \neq Sp_{24}(\mathbb{C})$ is contained in a Zariski-closed subset of $\mathscr{A}_4$ whose divisorial components are precisely $\mathscr{J}_4$ and $\theta_{null,4}$.*

The proof of this will be given along with our proof of theorem 4.2, based on the observation that every divisor on $\mathscr{A}_4$ intersects the locus of Jacobian varieties — on the Tannakian side this information is encoded in part (3) of proposition 4.12 below. Assuming the theorem, we recover the following result of [**67**].

COROLLARY 4.4. *For $g = 4$ the invariant $G = G(\delta_\Theta)$ determines both the Jacobian locus and the theta null locus*

$$\mathscr{J}_4 \subset \mathscr{A}_4 \quad \textit{and} \quad \theta_{null,4} \subset \mathscr{A}_4.$$

Proof. Theorem 4.3 says that the closure of the locus of all ppav's in $\mathscr{A}_4$ with $G \neq Sp_{24}(\mathbb{C})$ is a proper Zariski-closed subset, with $\mathscr{J}_4$ and $\theta_{null,4}$ as its only divisorial components. By the result quoted at the beginning of this chapter, the group G is equal to $Sl_6(\mathbb{C})/\mu_3$ for a general ppav in the Jacobian locus $\mathscr{J}_4$. By way of contrast, for a general ppav in $\theta_{null,4}$ the degeneration argument in part (2) of proposition 4.12 below shows that the irreducible representation $W \in Rep_{\mathbb{C}}(G)$ corresponding to the simple perverse sheaf δ_Θ has dimension $\dim(W) = g! - 2 = 22$. In particular, for such a ppav the group G is different from $Sl_6(\mathbb{C})/\mu_3$ since by [**3**] the latter does not admit an irreducible representation of dimension 22. □

In this context, we remark that the possible singularities of theta divisors in dimension $g = 4$ can be described explicitly as follows. Let $\mathscr{J}_{4,hyp}$ be the closure of the locus of Jacobian varieties of hyperelliptic curves, and denote by $\mathscr{A}_{4,dec} \subset \theta_{null,4} \cap \mathscr{J}_4$ the locus of decomposable ppav's. Then the following types of singularities occur.

LEMMA 4.5. *a) For ppav's in $\mathscr{J}_4 \setminus (\theta_{null,4} \cap \mathscr{J}_4)$ the theta divisors have precisely two singularities, both ordinary double points. b) For ppav's in $\theta_{null,4} \setminus (\theta_{null,4} \cap \mathscr{J}_4)$ they have only one singularity, again an ordinary double point. c) On $(\theta_{null,4} \cap \mathscr{J}_4) \setminus (\mathscr{J}_{4,hyp} \cup \mathscr{A}_{4,dec})$ they have only one singular point, and there the Hesse matrix of the Riemann theta function has rank three. d) On $\mathscr{J}_{4,hyp} \setminus (\mathscr{J}_{4,hyp} \cap \mathscr{A}_{4,dec})$ the singular locus of each theta divisor has dimension one. e) On $\mathscr{A}_{4,dec}$ it has dimension two.*

Proof. It has been shown in [**54**, thm. 10 and cor. 15] that for $a)$, $b)$ and $c)$ the rank of the Hesse matrix at any singular point of the theta divisor is the given one. In particular, in cases $a)$ and $b)$ only isolated singularities occur. This also holds for $c)$ since by Riemann's singularity theorem the singular locus of the theta divisor on a Jacobian variety is the Brill-Noether subvariety W^1_{g-1}, which in the non-hyperelliptic case has dimension $g-4$ by Martens' theorem [**6**, th. 4.5.1]. The number of singularities can be obtained from [**20**, sect. 10] and from [**8**] and [**81**, case b2, p. 55f]. Part $d)$ follows again from Martens' theorem, and $e)$ is trivial. □

4.2. Nondegenerate bilinear forms

In the introduction to this chapter we have remarked that conjecture 4.1 essentially says that for a smooth theta divisor the associated Tannaka group is as large as possible. To justify this interpretation, let X be a complex ppav of dimension $g \geq 2$ and $\Theta \subset X$ a symmetric theta divisor. For simplicity we assume that the ppav X is indecomposable, in which case Θ is irreducible and hence in particular normal by [**36**].

Then δ_Θ is a simple perverse sheaf, and by symmetry of the theta divisor it is isomorphic to its adjoint dual $(\delta_\Theta)^\vee = (-id)^* D(\delta_\Theta)$. On the Tannakian side, if we denote by

$$\omega : \quad \langle \delta_\Theta \rangle \stackrel{\sim}{\longrightarrow} Rep_{\mathbb{C}}(G(\delta_\Theta))$$

the tensor functor in corollary 3.10, then it follows that $V = \omega(\delta_\Theta)$ is an irreducible self-dual representation of $G(\delta_\Theta)$. Any representation with these properties is either orthogonal or symplectic, depending on whether the unit object $\mathbf{1} = \delta_0$ of our Tannakian category lies in the alternating square $\Lambda^2(V)$ or in the symmetric square $S^2(V)$. To decide which of the two possibilities occurs, consider the commutativity constraint

$$S \;=\; S_{\delta_\Theta, \delta_\Theta} : \quad \delta_\Theta * \delta_\Theta \stackrel{\sim}{\longrightarrow} \delta_\Theta * \delta_\Theta$$

as defined in section 2.1. We claim that this constraint S acts by $(-1)^{g-1}$ on the stalk cohomology group $\mathscr{H}^0(\delta_\Theta * \delta_\Theta)_0$ at the origin. Indeed we have $\delta_\Theta = \mathrm{IC}_\Theta[g-1]$, so our claim amounts to the statement that S acts trivially on the stalk cohomology group $\mathscr{H}^{2g-2}(\mathrm{IC}_\Theta * \mathrm{IC}_\Theta)_0$. For this latter statement one can use that since by assumption the theta divisor is normal, we have a natural identification

$$\mathscr{H}^{2g-2}(\mathrm{IC}_\Theta * \mathrm{IC}_\Theta)_0 \;\cong\; \mathscr{H}^{2g-2}(\mathbb{C}_\Theta * \mathbb{C}_\Theta)_0$$

which allows to replace the intersection cohomology sheaf with the constant sheaf. Then our claim easily follows via base change.

For the representation $V = \omega(\delta_\Theta)$ it follows from the above that the unit object $\mathbf{1}$ lies in the alternating square $\Lambda^2(V)$ if g is even, resp. in the symmetric square $S^2(V)$ if g is odd. This implies

LEMMA 4.6. *If the theta divisor $\Theta \subset X$ is smooth, then there exists an embedding*

$$G(\delta_\Theta) \;\hookrightarrow\; G(g) \;=\; \begin{cases} Sp_{g!}(\mathbb{C}) & \text{for even } g \\ SO_{g!}(\mathbb{C}) & \text{for odd } g \end{cases}$$

such that $\omega(\delta_\Theta)$ is the restriction of the standard representation of $G(g)$.

Proof. Since Θ is smooth, the irreducible representation $V = \omega(\delta_\Theta)$ is of dimension $\dim_{\mathbb{C}}(V) = \chi(\delta_\Theta) = g!$ by the Gauss-Bonnet formula, and we have noted above that V is a symplectic resp. orthogonal representation if g is even resp. odd. This proves the lemma with the full orthogonal group in place of the special orthogonal group. It remains to show that $\det(V)$ is the trivial character of $G(\delta_\Theta)$. If not, the character $\det(V)$ would by part *(b)* of proposition 1.6 correspond to a skyscraper sheaf δ_x supported in a 2-torsion point $x \in X(\mathbb{C}) \setminus \{0\}$. For monodromy reasons it is impossible to select such a 2-torsion point naturally on every ppav on an Zariski-open dense subset of the moduli space $\mathscr{A}_g$. So we can finish the proof by a specialization argument, using that the Tannaka group defines a constructible stratification of the moduli space (see the remarks preceding proposition 4.12 below) and that by theorem 3.15 it only becomes smaller on closed subsets. □

4.3. Local monodromy

To find sufficiently large lower bounds on the Tannaka group of the theta divisor on a general ppav, we will study degenerations of ppav's in terms of the nearby cycles functor. For this purpose we gather in the present section some general facts about the monodromy filtration that allow to get hold on the nearby cycles in concrete geometric situations. Throughout we work in the following algebraic setting.

Let S be the spectrum of a strictly Henselian discrete valuation ring (in our applications in proposition 4.12 it will be the strict Henselization of a smooth complex algebraic curve in the moduli space of ppav's). We denote by s the unique closed point of S and fix a geometric point $\bar{\eta}$ above the generic point η of S. For a separated S-scheme of finite type

$$f: \quad Y \longrightarrow S$$

we consider as in section 3.5 the functor of nearby cycles

$$\Psi: \quad \mathbf{P}(Y_\eta) \;=\; \mathrm{Perv}(Y_\eta, \Lambda) \;\longrightarrow\; \mathbf{P}(Y_s) \;=\; \mathrm{Perv}(Y_s, \Lambda)$$

on perverse sheaves with coefficients in $\Lambda = \overline{\mathbb{Q}}_l$ for some fixed prime l which is invertible on S. Here we consider perverse sheaves on X_η and not just perverse sheaves on the geometric fibre $X_{\bar{\eta}}$ since we want to keep track of the monodromy operation. By construction [**32**, exp. XIII], for any perverse sheaf $\delta \in \mathbf{P}(Y_\eta)$ the local monodromy group $G = Gal(\bar{\eta}/\eta)$ acts naturally on the nearby cycles $\Psi(\delta) \in \mathbf{P}(Y_s)$. Since we are working over a strictly Henselian base, this group coincides with the inertia group.

If $p \geq 0$ denotes the residue characteristic of the point s, the inertia group sits in an exact sequence $1 \to P \to G \to \prod_{l' \neq p} \mathbb{Z}_{l'}(1) \to 1$ where the pro-p-group P is the wild inertia group and where l' runs through the set of all prime numbers other than p, including our fixed prime l. In what follows we will always assume that the nearby cycles $\Psi(\delta)$ are tame in the sense that P acts trivially on them (in the application in proposition 4.12 we only work in characteristic $p=0$ anyway). The perverse sheaf $\Psi(\delta)$ is then equipped with a natural action of quotient $\mathbb{Z}_l(1)$ of the tame inertia group, and we denote by $T:\Psi(\delta)\longrightarrow\Psi(\delta)$ the endomorphism induced by a topological generator $2\pi i$ of $\mathbb{Z}_l(1)$. In the abelian category $\mathrm{Perv}(Y_s,\Lambda)$ we have as in [**85**, lemma 1.1] a Jordan decomposition

$$\Psi(\delta) \;=\; \Psi_1(\delta)\oplus\Psi_{\neq 1}(\delta) \quad\text{with}\quad \Psi_{\neq 1}(\delta) \;=\; \bigoplus_{\alpha\neq 1}\Psi_\alpha(\delta),$$

where for each $\alpha\in\Lambda$ the perverse subsheaf $\Psi_\alpha(\delta)$ is stable under the action of T and killed by a power of $T-\alpha\cdot id$. We are particularly interested in the case of the perverse intersection cohomology sheaf $\delta=\delta_{Y_\eta}$.

REMARK 4.7. *Suppose that the nearby cycles $\Psi(\delta_{Y_\eta})$ are tame. If the morphism $f:Y\to S$ is proper and if the geometric generic fibre $Y_{\bar\eta}$ is smooth, then after replacing f by its base change under some finite branched covering of S we can assume that*

$$H^\bullet(Y_s,\Psi_{\neq 1}(\delta_{Y_\eta})) \;=\; 0.$$

Proof. Let $S'\to S$ be the normalization of S in a finite extension of the residue field of η with generic point $\eta'\mapsto\eta$, and denote by $\bar S\to S$ the normalization in the residue field of $\bar\eta$. For the base changes $Y'=Y\times_S S'$ and $\bar Y=Y\times_S\bar S$ we then have a commutative diagram

$$\begin{array}{ccccc}
\bar Y_s & \longrightarrow & \bar Y & \longleftarrow & Y_{\bar\eta} \\
\| & & \downarrow & & \downarrow \\
Y'_s & \longrightarrow & Y' & \longleftarrow & Y'_{\eta'} \\
\| & & \downarrow & & \downarrow \\
Y_s & \longrightarrow & Y & \longleftarrow & Y_\eta
\end{array}$$

where the vertical identifications on the left hand side come from the fact that we are working over a strictly Henselian base. In particular, as an object of $\mathrm{Perv}(Y_s,\Lambda)=\mathrm{Perv}(Y'_s,\Lambda)$ the nearby cycles $\Psi(\delta_{Y_\eta})$ do not change if we replace our original family $Y\to S$ by the base change $Y'\to S'$, though of course the local monodromy operation and hence in the tame case the Jordan decomposition is modified under this replacement. This being said, our claim can be checked as follows.

For each $\alpha \in \Lambda$ some power of the endomorphism $T - \alpha \cdot id$ acts trivially on the perverse sheaf $\Psi_\alpha(\delta_{Y_\eta})$ and hence also on its hypercohomology. In particular, the Jordan decomposition

$$H^\bullet(Y_s, \Psi(\delta_{Y_\eta})) \;=\; \bigoplus_\alpha H^\bullet(Y_s, \Psi_\alpha(\delta_{Y_\eta}))$$

shows that $H^\bullet(Y_s, \Psi_{\neq 1}(\delta_{Y_\eta})) = 0$ iff the action of T on $H^\bullet(Y_s, \Psi(\delta_{Y_\eta}))$ is unipotent. But this latter condition can be achieved after a finite branched base change $S' \to S$ as above: Indeed we have

$$H^\bullet(Y_s, \Psi(\delta_{Y_\eta})) \;=\; H^\bullet(Y_{\bar\eta}, \delta_{Y_{\bar\eta}})$$

by proper base change, and if the geometric generic fibre $Y_{\bar\eta}$ is smooth, then Grothendieck's local monodromy theorem [**60**, th. 1.4] implies that on this cohomology group the generator T acts quasi-unipotently. □

Returning to an arbitrary perverse sheaf $\delta \in \mathbf{P}(Y_\eta)$ with tame nearby cycles, to get hold on $\Psi_1(\delta)$ we consider the nilpotent operator

$$N \;=\; \tfrac{1}{2\pi i}\log(T): \quad \Psi_1(\delta) \;\longrightarrow\; \Psi_1(\delta)(-1).$$

Here $\frac{1}{2\pi i} \in \mathbb{Z}_l(-1)$ denotes the dual of our chosen generator $2\pi i \in \mathbb{Z}_l(1)$ so that the morphism N is well-defined and equivariant under the Galois action. By [**28**, sect. 1.6] there is a unique finite increasing filtration $F_\bullet$ of the perverse sheaf $\Psi_1(\delta)$ such that

$$N(F_i(\Psi_1(\delta))) \;\subseteq\; F_{i-2}(\Psi_1(\delta))(-1)$$

and such that for each $i \geq 0$ the i-fold iterate $N^i : \Psi_1(\delta) \longrightarrow \Psi_1(\delta)(-i)$ induces an isomorphism

$$Gr_i(\Psi_1(\delta)) \;\xrightarrow{\cong}\; Gr_{-i}(\Psi_1(\delta))(-i)$$

of the graded pieces with respect to the filtration. Furthermore, if for $i \geq 0$ we denote by $P_{-i}(\delta)$ the kernel of $N : Gr_{-i}(\Psi_1(\delta)) \to Gr_{-i-2}(\Psi_1(\delta))(-1)$, then by loc. cit. we have a decomposition

$$Gr_{-i}(\Psi_1(\delta)) \;\cong\; \bigoplus_{k \geq 0} P_{-i-2k}(-k).$$

We can represent this situation by a diagram of the following shape, where each horizontal line of the triangle contains the composition factors of the corresponding graded piece shown on the left (for the arrows labelled N one must of course ignore the Tate twists in the diagram, which have only been inserted to be conform with the graded pieces on the left).

$$
\begin{array}{ccccc}
\vdots & & & & \cdots \\
Gr_2(\Psi_1(\delta)) & & & P_{-2}(\delta)(-2) & \\
Gr_1(\Psi_1(\delta)) & & P_{-1}(\delta)(-1) & \cong \Big\downarrow {\scriptstyle N} & \cdots \\
Gr_0(\Psi_1(\delta)) & P_0(\delta) & \cong \Big\downarrow {\scriptstyle N} & P_{-2}(\delta)(-1) & \\
Gr_{-1}(\Psi_1(\delta)) & & P_{-1}(\delta) & \cong \Big\downarrow {\scriptstyle N} & \cdots \\
Gr_{-2}(\Psi_1(\delta)) & & & P_{-2}(\delta) & \\
\vdots & & & & \cdots
\end{array}
$$

The lower boundary entries $P_0(\delta), P_{-1}(\delta), P_{-2}(\delta), \dots$ of the triangle are the graded pieces of the specialization

$$sp(\delta) \;=\; \ker(N : \Psi_1(\delta) \to \Psi_1(\delta)(-1)),$$

with $P_0(\delta)$ as the top quotient. Accordingly the graded pieces of $sp(\delta)$ determine those of all the $Gr_i(\Psi_1(\delta))$. Furthermore, in the case of mixed perverse sheaves of geometric origin in the sense of [**10**, chapt. 6] we have the following result due to O. Gabber [**9**, th 5.1.2].

REMARK 4.8. *If δ is pure of weight w, then each $Gr_i(\Psi_1(\delta))$ is pure of weight $w+i$ so that the monodromy filtration coincides with the weight filtration up to an index shift.*

In this case, to compute the graded pieces $Gr_i(\Psi_1(\delta))$ we only need to determine the specialization $sp(\delta)$ and its weight filtration. Returning again to the general case, let us denote by $j : Y_{\bar\eta} \hookrightarrow Y$ resp. $i : Y_s \hookrightarrow Y$ the embedding of the generic resp. special fibre. Recall [**60**, p. 48] that the perverse t-structure on Y is defined by

$$
\begin{aligned}
K \in {}^pD^{\leq 0}(Y) &\iff i^*K \in {}^pD^{\leq 0}(Y_s) \text{ and } j^*K \in {}^pD^{\leq -1}(Y_{\bar\eta}), \\
K \in {}^pD^{\geq 0}(Y) &\iff i^!K \in {}^pD^{\geq 0}(Y_s) \text{ and } j^*K \in {}^pD^{\geq -1}(Y_{\bar\eta}).
\end{aligned}
$$

We denote by $\mathbf{P}(Y)$ the abelian category of perverse sheaves which is the core of the above perverse t-structure. By abuse of notation we also write δ for the pull-back to $\mathbf{P}(Y_{\bar\eta})$ of the perverse sheaf $\delta \in \mathbf{P}(Y_\eta)$. Then Artin's affine vanishing theorem implies that the direct image complexes $Rj_!(\delta[1])$ and $Rj_*(\delta[1])$ are perverse, see loc. cit. This being said, we define the intermediate extension

$$j_{!*}(\delta[1]) \;=\; im\big(Rj_!(\delta[1]) \longrightarrow Rj_*(\delta[1])\big)$$

to be the image of the natural morphism between these two direct image complexes in the abelian category $\mathbf{P}(Y)$.

LEMMA 4.9. *With notations as above, the specialization of δ is given by the formulae*

$$sp(\delta) \;=\; {}^pH^0(i^*Rj_*(\delta)) \;=\; i^*(j_{!*}\delta[1])[-1].$$

Proof. We first claim that in the triangulated category $\mathbf{D}(Y_s) = D^b_c(Y_s, \Lambda)$, the cone of the morphism N is given by

$$(\star) \qquad Cone\big(\Psi_1(\delta) \xrightarrow{N} \Psi_1(\delta)(-1)\big) \;=\; i^*Rj_*(\delta[1]).$$

Indeed, if we forget about weights, the cone of N on $\Psi_1(\delta)$ is isomorphic to the cone of $T-1$ on $\Psi(\delta)$ because $T-1$ is an isomorphism on $\Psi_{\neq 1}(\delta)$ and on $\Psi_1(\delta)$ its kernel and cokernel are isomorphic to those of N. Hence $(\star)$ follows by the same argument as in [**60**, eq. (3.6.2) and thereafter], using that by assumption the wild inertia group P acts trivially on $\Psi(\delta)$. Now if for $n \in \{0,1\}$ we write

$$sp^n(\delta) = {}^pH^n(i^*Rj_*(\delta)),$$

we obtain from $(\star)$ an exact sequence of perverse sheaves

$$0 \longrightarrow sp^0(\delta) \longrightarrow \Psi_1(\delta) \xrightarrow{N} \Psi_1(\delta)(-1) \longrightarrow sp^1(\delta) \longrightarrow 0.$$

Hence the first equality in the lemma follows. For the second equality note that we always have $i^*j_{!*}(\delta[1]) = {}^p\tau_{<0}i^*Rj_*(\delta[1])$ by the basic properties of the intermediate extension functor shown as in [**65**, sect. III.5.1], and that again by $(\star)$ the sheaf complex $i^*Rj_*(\delta[1])$ is concentrated in perverse cohomology degrees -1 and 0. $\square$

To connect these local algebraic results to the more global setting of section 4.4 below, it will be convenient to reset our notation. So let S be a smooth complex algebraic curve and $f: Y \longrightarrow S$ a morphism of complex algebraic varieties which is smooth over the complement of some given point $s \in S(\mathbb{C})$. Consider the strict Henselization $\tilde{S} = Spec(\mathscr{O}^{sh}_{S,s}) \longrightarrow S$ of S at the point s, and let η be the generic point of the strictly Henselian local scheme $\tilde{S}$. Referring to the base change

$$\tilde{f}: \quad \tilde{Y} \;=\; Y \times_S \tilde{S} \;\longrightarrow\; \tilde{S}$$

of the morphism f under this strict Henselization, we can form as above the nearby cycles and specialization functors

$$\begin{tikzcd} \mathbf{P}(Y_\eta) \arrow[r, bend left, "\Psi"] \arrow[r, bend right, "sp"'] & \mathbf{P}(Y_s) \end{tikzcd}$$

where Y_η and Y_s denote the generic resp. special fibres of $\tilde{f}$. Note that Y_s can be naturally identified with the fibre $f^{-1}(s) \subset Y$. In the case of the perverse intersection cohomology sheaf the passage between the global and the local picture is provided by the following observation, where we now denote by $i : Y_s = f^{-1}(s) \hookrightarrow Y$ the global embedding.

COROLLARY 4.10. *In the above situation,*

$$sp(\delta_{Y_\eta}) \;=\; i^*(\delta_Y[-1]).$$

Proof. Our smoothness assumption on f implies that the generic fibre Y_η is smooth so that the perverse intersection cohomology sheaf $\delta = \delta_{Y_\eta}$ is the constant sheaf up to a degree shift. From this one easily deduces that the intermediate extension $j_{!*}(\delta[1])$ which occurs in lemma 4.9 arises from the perverse intersection cohomology sheaf δ_Y on the total space Y via the Henselization morphism $\tilde{Y} \longrightarrow Y$. □

4.4. Degenerations of abelian varieties

To find sufficiently large lower bounds on the general Tannaka group in theorem 4.2 we consider certain families of ppav's whose theta divisor degenerates. To construct these we fix an integer $n \geq 3$ and work over the moduli space $\mathscr{A}_{g,n}$ of ppav's of dimension g with level n structure. It has been shown in chapter 7.3 of [**80**] that this moduli space is representable by a smooth quasi-projective variety over $\mathbb{Q}$. Analytically it can be written as the quotient

$$\mathscr{A}_{g,n}(\mathbb{C}) \;=\; \mathscr{H}_g/\Gamma_g(n)$$

of the Siegel upper half space by the action of the principal congruence subgroup $\Gamma_g(n) = \ker(Sp_{2g}(\mathbb{Z}) \longrightarrow Sp_{2g}(\mathbb{Z}/n\mathbb{Z}))$. Recall that for $n \geq 3$ this action is free, which again explains why in this case the moduli space is smooth. We denote by

$$p: \quad \mathscr{X} \;=\; \mathscr{H}_g \times \mathbb{C}^g \,/\, \Gamma_g(n) \ltimes \mathbb{Z}^{2g} \;\longrightarrow\; \mathscr{A}_{g,n}$$

the universal abelian scheme and by $\Theta \subset \mathscr{X}$ the divisor which is defined on the universal covering $\mathscr{H}_g \times \mathbb{C}^g$ by the zero locus of the Riemann theta function $\vartheta(\tau, z)$. Thus for each point $\tau \in \mathscr{A}_{g,n}(\mathbb{C})$ the fibre $X_\tau = p^{-1}(\tau)$ is a complex ppav with $\Theta_\tau = \Theta \cap X_\tau$ as a symmetric theta divisor. More generally, for any complex algebraic variety S with a morphism $S \longrightarrow \mathscr{A}_{g,n}$ we denote by

$$X_S \;=\; \mathscr{X} \times_{\mathscr{A}_{g,n}} S \quad \text{and} \quad \Theta_S \;=\; \Theta \times_{\mathscr{A}_{g,n}} S$$

the corresponding abelian scheme resp. its relative theta divisor. For the construction of families of ppav's with degenerating theta divisors we will use the following general observation.

LEMMA 4.11. *For every point $s \in \mathscr{A}_{g,n}(\mathbb{C})$ there is a smooth complex quasi-projective curve $S \hookrightarrow \mathscr{A}_{g,n}$, passing through the point s in a general tangent direction, such that*

(a) the generic fibre of the family $\Theta_S \to S$ is smooth,

(b) the singular loci of the total space and of the special fibre of this family satisfy

$$Sing(\Theta_S) \subseteq Sing(\Theta_s).$$

If the theta divisor Θ_s contains a singular point of precise multiplicity two, then the inclusion in part (b) is strict for a suitable choice of S.

Proof. Since the moduli space $\mathscr{A}_{g,n}$ is smooth and quasi-projective, for any $s \in \mathscr{A}_{g,n}(\mathbb{C})$ we can find a smooth quasi-projective curve $S \hookrightarrow \mathscr{A}_{g,n}$ passing through the point s in a general tangent direction. We can assume that our general curve S is not contained in the locus of ppav's with singular theta divisor, indeed by [**4**] this locus is itself a divisor in $\mathscr{A}_{g,n}$. So after shrinking S we can assume that for all $t \in S(\mathbb{C}) \setminus \{s\}$ the theta divisor Θ_t is smooth. Then property *(a)* is clearly satisfied, and property *(b)* easily follows from the fact that the total space Θ_S is given locally in the smooth variety X_S as the zero locus of a single analytic function.

Explicitly, let $\Delta \subset S$ be an analytic coordinate disk with coordinate w centered at the given point s, and consider a local lift $h : \Delta \to \mathscr{H}_g$ of the embedding $\Delta \hookrightarrow S \hookrightarrow \mathscr{A}_{g,n}$. On the universal covering the divisor Θ_S is described as the locus

$$\big\{(w,z) \in \Delta \times \mathbb{C}^g \mid F(w,z) = 0\big\} \ \subset \ \Delta \times \mathbb{C}^g$$

where the analytic function $F(w,z) = \vartheta(h(w),z)$ vanishes. If a point (w,z) on this locus defines a singular point of the relative theta divisor Θ_S, then the gradient of F must vanish at this point. For the gradient in the variable z this implies

$$0 \ = \ (\nabla_z F)(w,z) \ = \ (\nabla_z \vartheta)(\tau,z) \quad \text{for} \quad \tau \ = \ h(w).$$

Hence (τ,z) defines a singular point of the fibre Θ_t where $t \in S(\mathbb{C})$ denotes the image of the point τ. By our choice of the curve S the only singular fibre Θ_t is the one over the point $t = s$, hence claim *(b)* follows.

This being said, if some point $(\tau,z) \in \mathscr{H}_g \times \mathbb{C}^g$ defines a singular point of the theta divisor Θ_s with precise multiplicity two, then by definition the

Hesse matrix

$$\Big(\frac{\partial^2\vartheta}{\partial z_\alpha \partial z_\beta}(\tau,z)\Big)_{\alpha,\beta=1,\dots,g}$$

is non-zero. Then the heat equation

$$\frac{\partial\vartheta}{\partial \tau_{\alpha\beta}}(\tau,z) \;=\; 2\pi i\cdot(1+\delta_{\alpha\beta})\cdot\frac{\partial^2\vartheta}{\partial z_\alpha\partial z_\beta}(\tau,z)$$

implies for the gradient with respect to the variable τ that $(\nabla_\tau\vartheta)(\tau,z)\neq 0$ as well. Hence, taking S to be a curve which passes through the point s in a sufficiently general tangent direction u, we obtain with notations as in the preceding paragraph that

$$(\partial F/\partial w)(0,z) \;=\; (u\cdot\nabla_\tau\vartheta)(\tau,z) \;\neq\; 0$$

where $F(w,z)=\vartheta(h(w),z)$ is defined for w in some coordinate disk $\Delta\subset S$ by a suitable local lift $h:\Delta\to\mathscr{H}_g$ with $h(0)=\tau$. In particular, since the gradient of F does not vanish at the considered point, it follows that (τ,z) defines a smooth point of the total space Θ_S of our family. This shows that the inclusion in claim *(b)* is strict. $\square$

We will now use the above construction to obtain information about the Tannaka groups $G(\delta_{\Theta_\tau})$ by varying the base point $\tau\in\mathscr{A}_{g,n}(\mathbb{C})$. In order to apply the algebraic formalism of nearby cycles developed in section 4.3 we consider perverse sheaves with coefficients in $\Lambda=\overline{\mathbb{Q}}_l$ throughout, but the final result may as well be read in the category of analytic perverse sheaves with coefficients in $\Lambda=\mathbb{C}$. Anyway all the occuring perverse sheaves will be of geometric origin in the sense of [**10**, sect. 6.2.4].

In general it is not clear how to put the perverse sheaves δ_{Θ_τ} for the various points $\tau\in\mathscr{A}_{g,n}(\mathbb{C})$ into a global family because there is no good relative notion of a perverse intersection cohomology sheaf. However, over the Zariski-open dense locus $U\subset\mathscr{A}_{g,n}$ of ppav's with a smooth theta divisor this is no issue — the shifted constant sheaf $\Lambda_\Theta[g-1]$ restricts to δ_{Θ_τ} on the fibre X_τ for each geometric point τ in U. This being said, proposition 3.20 shows that there are finitely many reductive algebraic groups $G_1,\dots,G_m$ over Λ and a constructible stratification

$$U \;=\; \bigsqcup_{i=0}^{m} U_i \quad\text{with}\quad G(\delta_{\Theta_\tau})=G_i \quad\text{for all geometric points } \tau \text{ in } U_i.$$

The hypotheses of the proposition are satisfied because for each geometric point τ in U the divisor $\Theta_\tau\subset X_\tau$ is irreducible and symmetric. Indeed these two conditions precisely say that the perverse sheaf δ_{Θ_τ} is simple and isomorphic to its adjoint dual.

Let $U_0 \subseteq U$ be the open dense stratum so that $G_0 \subseteq G(g)$ is the Tannaka group of the theta divisor on a generic ppav. By lemma 4.6 the perverse intersection cohomology sheaf of such a theta divisor corresponds on the Tannakian side to the restriction $V|_{G_0}$ of the standard representation V of the special orthogonal or symplectic group $G(g)$.

PROPOSITION 4.12. *With notations as above, the following properties hold for the Tannaka groups of smooth theta divisors.*

(1) *For $g = g_1 + g_2$ the generic Tannaka group G_0 has a subquotient which is isogenous to*

$$G(g_1) \times G(g_2),$$

provided that theorem 4.2 holds for the dimensions g_1 and g_2.

(2) *There exists a connected subgroup $H \hookrightarrow G_0$ and an irreducible representation W of H such that*

$$V|_H = \begin{cases} W \oplus \mathbf{1} \oplus \mathbf{1} & \textit{if } g \textit{ is even}, \\ W \oplus \mathbf{1} & \textit{if } g \textit{ is odd}, \end{cases}$$

where $\mathbf{1}$ denotes the one-dimensional trivial representation.

(3) *For $g \geq 4$ and any stratum U_i of codimension at most one in U there exists a homomorphism*

$$f : \ Sl_{2g-2}(\Lambda) \longrightarrow G_i$$

such that $f^*(V)$ contains the $g-1^{st}$ fundamental representation of the special linear group as a direct summand.

Proof. Consider a morphism from a smooth irreducible quasi-projective complex curve S to the moduli space $\mathscr{A}_{g,n}$ such that the generic point of S is mapped into some stratum U_i in the above stratification, and fix a geometric generic point $\bar{\eta}$ of S. Passing to the strict Henselization of S at $s \in S(\mathbb{C})$ we can form the perverse sheaf $\Psi(\delta_{\Theta_{\bar{\eta}}}) \in \mathrm{Perv}(X_s, \mathbb{C})$ of nearby cycles. By theorem 3.15 we have an embedding

$$G(\Psi(\delta_{\Theta_{\bar{\eta}}})) \hookrightarrow G(\delta_{\Theta_{\bar{\eta}}}) = G_i \hookrightarrow G(g)$$

such that $\Psi(\delta_{\Theta_{\bar{\eta}}})$ corresponds on the Tannakian side to the restriction of the standard representation V to the subgroup $G(\Psi(\delta_{\Theta_{\bar{\eta}}}))$. We will apply this remark in the following situations.

For part (1) take $S \hookrightarrow \mathscr{A}_{g,n}$ to be an embedding of a smooth curve which meets the locus of decomposable ppav's in a single point $s \in S(\mathbb{C})$ but which is otherwise contained in the open dense stratum U_0 and has the properties in lemma 4.11. We choose the point s such that the corresponding ppav and its theta divisor have the form

$$X_s \,=\, X_1 \times X_2 \quad \text{resp.} \quad \Theta_s \,=\, (\Theta_1 \times X_2) \cup (X_1 \times \Theta_2)\,,$$

where for $\alpha \in \{1,2\}$ the X_α are general complex ppav's of dimension g_α with a symmetric theta divisor Θ_α. In particular, like for any divisor with two components which intersect each other transversally along a smooth subvariety, we have an exact sequence

$$0 \longrightarrow \delta_{\Theta_1 \times \Theta_2} \longrightarrow \Lambda_{\Theta_s}[g-1] \longrightarrow \delta_{\Theta_1 \times X_2} \oplus \delta_{X_1 \times \Theta_2} \longrightarrow 0$$

of perverse sheaves. Restricting this short exact sequence to the open dense subset

$$V \,=\, X_s \setminus Sing(\Theta_S) \,\subset\, X_s$$

we get a monomorphism

$$\delta_{\Theta_1 \times \Theta_2}|_V \,\hookrightarrow\, \Lambda_{\Theta_s}[g-1]|_V \,=\, sp(\delta_{\Theta_\eta})|_V$$

of perverse sheaves, where the last equality holds by the formula for the specialization in corollary 4.10. Now by the last statement in lemma 4.11 we can assume that the singular locus $Sing(\Theta_S)$ is a proper closed subset of $Sing(\Theta_s) = \Theta_1 \times \Theta_2$, and in this case the open dense subset V will have non-empty intersection with $\Theta_1 \times \Theta_2$. Then via intermediate extension it follows from the above that the semisimplification of $sp(\delta_{\Theta_\eta})$ contains a direct summand $\delta_{\Theta_1 \times \Theta_2}$. By the properties of the monodromy filtration in section 4.3 the same then a fortiori holds for the semisimplification of the perverse sheaf of nearby cycles $\Psi(\delta_{\Theta_{\bar{\eta}}})$. This being said, our claim follows from the elementary observation that the Tannaka group $G(\delta_{\Theta_1 \times \Theta_2})$ is isogenous to $G(\delta_{\Theta_1}) \times G(\delta_{\Theta_2})$.

For part (2) let $s \in \mathscr{A}_{g,n}(\mathbb{C})$ be a point which corresponds to a general ppav X_s with a vanishing theta null, and let $S \hookrightarrow \mathscr{A}_{g,n}$ be a general curve which passes through s but is otherwise contained in the open stratum U_0 and has the properties in lemma 4.11. For the special fibre with a vanishing theta null we know from [**26**] with the correction given in [**53**, rem. 4.5], or alternatively also from theorem 4.2 of loc. cit., that the theta divisor Θ_s has an isolated ordinary double point e as its only singularity. Hence the Picard-Lefschetz formula [**32**, exp. XV, th. 3.4] says that

$$\chi(\Theta_s) \,=\, \chi(\Theta_{\bar{\eta}}) \,+\, (-1)^g \,=\, (-1)^{g-1} \cdot (g! - 1)$$

because the generic theta divisor $\Theta_{\bar{\eta}}$ is smooth of dimension $g-1$ with Euler characteristic $(-1)^{g-1} \cdot g!$ by the Gauss-Bonnet theorem. Now the perverse intersection cohomology sheaf near an ordinary double point can be controlled explicitly, indeed lemma 4.16 below gives an exact sequence of perverse sheaves

$$0 \longrightarrow \kappa_g \longrightarrow \Lambda_{\Theta_s}[g-1] \longrightarrow \delta_{\Theta_s} \longrightarrow 0$$

for the skyscraper sheaf

$$\kappa_g = \begin{cases} \delta_e(-\frac{g-2}{2}) & \text{if } g \text{ is even,} \\ 0 & \text{if } g \text{ is odd.} \end{cases}$$

Hence it follows that

$$\chi(\delta_{\Theta_s}) = \chi(\Lambda_{\Theta_s}[g-1]) - \chi(\kappa_g) = \begin{cases} g!-2 & \text{if } g \text{ is even,} \\ g!-1 & \text{if } g \text{ is odd.} \end{cases}$$

On the other hand, by remark 4.7 we can assume that the nearby cycles for our degeneration coincide with the unipotent nearby cycles. Then from the formula for the specialization in corollary 4.10 one deduces that the semisimplification of the nearby cycles must have the form

$$\Psi(\delta_{\Theta_{\bar\eta}})^{ss} = \delta_{\Theta_s} \oplus \gamma$$

for some perverse skyscraper sheaf γ supported on the singular point e of the special fibre. Looking at the above Euler characteristics, one sees that this skyscraper sheaf γ must have rank two if g is even, resp. rank one if g is odd. Now take a Levi splitting

$$G_s = G(\delta_{\Theta_s}) \hookrightarrow G(\Psi(\delta_{\Theta_{\bar\eta}})),$$

and let W be the irreducible representation of G_s which corresponds to the simple perverse sheaf δ_{Θ_s}. In lemma 4.13 below we will see as a general fact about irreducible theta divisors that the restriction of W to the connected component $H = G_s^0 \subseteq G_s$ remains irreducible, so our claim follows.

For part (3) consider the locus $J \subset \mathscr{A}_{g,n}$ of Jacobian varieties of smooth projective curves of genus g. If a stratum $U_i \subseteq U$ has codimension at most one in the open subset U, then the closure $\overline{U}_i \subseteq \mathscr{A}_{g,n}$ has codimension at most one in $\mathscr{A}_{g,n}$. In this case it has been observed in [**83**, cor. (0.7)] that the intersection $J \cap \overline{U}_i$ is nonempty and hence of codimension at most one in J. We can then find a point $t \in \overline{U}_i(\mathbb{C})$ which corresponds to the Jacobian variety of a non-hyperelliptic smooth curve (indeed the hyperelliptic locus has codimension greater than one in J for $g \geq 4$). Now consider in the closure $\overline{U}_i$ a curve which meets the Jacobian locus J in the chosen point t and is otherwise contained in the open dense subset $U_i \subset \overline{U}_i$. Define

$$\varphi: \ S \longrightarrow \mathscr{A}_{g,n}$$

to be the normalization of this curve, and choose $s \in S(\mathbb{C})$ to be any point with $\varphi(s) = t$. Then $X_S \to S$ is a family of ppav's whose geometric generic fibre $X_{\bar\eta}$ is a ppav in the stratum U_i and whose special fibre X_s is the Jacobian of a smooth non-hyperelliptic curve. As in lemma 4.11 the singular loci of the relative theta divisor Θ_S and of the fibre Θ_s satisfy

$$Sing(\Theta_S) \subseteq Sing(\Theta_s),$$

so it follows from corollary 4.10 that the specialization $sp(\delta_{\Theta_\eta})$ admits the simple perverse sheaf δ_{Θ_s} as a subquotient. Then the same holds for the nearby cycles $\Psi(\delta_{\Theta_{\bar\eta}})$ as well, and this proves our claim since by the result for Jacobian varieties in [**106**], the Brill-Noether sheaf δ_{Θ_s} corresponds to the $g-1^{\mathrm{st}}$ fundamental representation of its Tannaka group which in the non-hyperelliptic case is $G(\delta_{\Theta_s}) = Sl_{2g-2}(\Lambda)/\mu_{g-1}$. □

4.5. Proof of the main theorem

Let X be a complex ppav of dimension g and $\Theta \subset X$ a symmetric theta divisor defining the polarization. Consider the Tannaka group $G = G(\delta_\Theta)$ and its representation

$$V \;=\; \omega(\delta_\Theta) \;\in\; Rep_{\mathbb{C}}(G)$$

which corresponds to the perverse intersection cohomology sheaf δ_Θ under the equivalence $\omega : \langle\delta_\Theta\rangle \longrightarrow Rep_{\mathbb{C}}(G)$ in corollary 3.10. The Tannaka group does not have to be connected, but fortunately it is not very far from being connected either. Indeed we have the following

LEMMA 4.13. *If the theta divisor is irreducible, then V restricts to an irreducible representation $V|_{G^0}$ of the connected component $G^0 \subseteq G$.*

Proof. The irreducibility of the theta divisor implies that δ_Θ is a simple perverse sheaf, and accordingly V is an irreducible representation of the Tannaka group G. But by the fundamental result of [**109**] the group G/G^0 is abelian, hence if the claim of our lemma were not true, then by lemma B.4 in the appendix we would have an isomorphism $V \otimes \chi \cong V$ for a suitable non-trivial character $\chi : G \longrightarrow \mathbb{C}^*$. The classification of perverse sheaves with Euler characteristic one in proposition 1.6 shows that χ corresponds to a skyscraper sheaf δ_x of rank one, supported in some closed point $x \neq 0$ of the ppav X. Hence the above isomorphism would on the geometric side correspond to an isomorphism

$$\delta_\Theta * \delta_x \;\cong\; \delta_\Theta,$$

meaning that $t_x(\Theta) = \Theta$ for the translation $t_x : X \to X, y \mapsto y + x$. But this is impossible for any point $x \neq 0$, indeed the morphism

$$X \;\longrightarrow\; Pic^0(X), \quad x \;\mapsto\; \left[\mathscr{O}_X(\Theta - t_x(\Theta))\right]$$

is an isomorphism since the theta divisor gives a principal polarization. □

Now suppose that X is a general ppav. By lemma 4.6 the corresponding Tannaka group admits an embedding

$$G \;=\; G(\delta_\Theta) \;\hookrightarrow\; G(g) \;=\; \begin{cases} Sp_{g!}(\mathbb{C}) & \text{for even } g, \\ SO_{g!}(\mathbb{C}) & \text{for odd } g, \end{cases}$$

and the representation $V = \omega(\delta_\Theta)$ arises as the restriction of the standard representation of the classical group on the right hand side. For the proof of theorem 4.2 we will show that for a general ppav the group G must be as large as possible, using the lower bounds in proposition 4.12. We begin with the following observation.

LEMMA 4.14. *For a general ppav X with Tannaka group $G = G(\delta_\Theta)$, the connected component $G^0 \subseteq G$ is simple modulo its center.*

Proof. Let us first introduce some notations. For any reductive complex algebraic group H consider the derived group $H^0_{der} = [H^0, H^0]$. This is a connected semisimple group, and we denote by

$$\tilde{H} \;\twoheadrightarrow\; H^0_{der}$$

its simply connected cover. The covering group $\tilde{H}$ is a product of simply connected covers of simple algebraic groups. Furthermore, since by the theory of reductive groups [**100**, cor. 8.1.6] the connected component H^0 is the product of its derived group and its center (with finite intersection), any irreducible representation of H^0 is also irreducible under $\tilde{H}$.

We now return to the Tannaka group $G = G(\delta_\Theta)$ for a general ppav. To prove the lemma we argue by contradiction. If the claim of the lemma is not true, then the simply connected cover of this Tannaka group can be written in the form

$$\tilde{G} \;=\; G_1 \times G_2$$

for certain non-trivial simply connected groups G_1 and G_2. Furthermore, for the representation $V = \omega(\delta_\Theta)$ we have seen in lemma 4.13 that $V|_{G^0}$ is irreducible. Hence by the above remarks also $\tilde{G} = G_1 \times G_2$ acts irreducibly on V so that we can write

$$V|_{\tilde{G}} \;=\; V_1 \otimes V_2$$

with certain irreducible representations $V_i \in Rep_{\mathbb{C}}(G_i)$. Note that since V is a faithful representation of G, it follows from the definition of the simply connected covering group that the representations V_i are non-trivial and in particular that $\dim(V_i) > 1$. Now consider a connected subgroup $H \hookrightarrow G$ as in part (2) of proposition 4.12 so that

$$V|_H \;=\; W \oplus \begin{cases} \mathbf{1} \oplus \mathbf{1} & \text{if } g \text{ is even}, \\ \mathbf{1} & \text{if } g \text{ is odd}, \end{cases}$$

where W is some irreducible representation of the group H. Then via the commutative diagram

$$\begin{tikzcd}
\tilde{H} \arrow[r, dotted, "\exists"] \arrow[d] & \tilde{G} \arrow[rr] \arrow[d] & & Gl(V_1)\times Gl(V_2) \arrow[d] \\
H \arrow[r, hook] & G \arrow[r, hook] & Gl(V) \arrow[r, equal] & Gl(V_1\otimes V_2)
\end{tikzcd}$$

we can consider the restrictions of V_1 and V_2 to the covering group $\tilde{H}$, and by construction we have

$$V_1|_{\tilde{H}}\otimes V_2|_{\tilde{H}} \;=\; V|_{\tilde{H}} \;=\; W|_{\tilde{H}}\oplus\begin{cases}\mathbf{1}\oplus\mathbf{1} & \text{if } g \text{ is even,}\\ \mathbf{1} & \text{if } g \text{ is odd,}\end{cases}$$

where $W|_{\tilde{H}}$ is irreducible by the general remarks from the beginning of the proof. This implies that both $V_i|_{\tilde{H}}$ are irreducible because otherwise more than one non-trivial direct summand would occur. But then, since

$$Hom_{\tilde{H}}(\mathbf{1}, V_1|_{\tilde{H}}\otimes V_2|_{\tilde{H}}) \;\neq\; 0,$$

adjunction shows $V_1|_{\tilde{H}}$ is isomorphic to the dual of $V_2|_{\tilde{H}}$. In particular V_1 and V_2 have the same dimension. This is impossible since $\dim(V)=g!$ is not the square of a natural number for $g>1$.

For $g=4$ we can alternatively argue as follows, using only part (3) of proposition 4.12. Here both V_1 and V_2 have dimension at most 12 since their dimensions must be non-trivial divisors of $\dim(V)=24$. So if instead of our previous choice we now take the subgroup $H\hookrightarrow G$ to be isogenous to $Sl_{2g-2}(\mathbb{C})=Sl_6(\mathbb{C})$ with the property in part (3) of the proposition, then by the classification of small representations in [**3**] each $V_i|_{\tilde{H}}$ can contain only trivial representations and 6-dimensional standard representations as irreducible constituents. In particular, looking at $V_1|_{\tilde{H}}\otimes V_2|_{\tilde{H}}$ one sees that the third fundamental representation of $Sl_6(\mathbb{C})$ cannot occur as a constituent of $V|_H$, and this contradicts our choice of the subgroup H. $\square$

Now recall that the Tannaka group $G=G(\delta_\Theta)$ of a general ppav is in any case a subgroup of the symplectic or special orthogonal group $G(g)$, hence to prove theorem 4.2 it will be enough to show that the connected component G^0 must be the full group $G(g)$. Note that by lemma 4.13 this connected component is an irreducible subgroup of $G(g)$ in the sense that for the standard representation $V\in Rep_{\mathbb{C}}(G(g))$, the restriction $V|_{G^0}$ is still irreducible. So we are in the situation of the following general lemma.

LEMMA 4.15. *Let H be an irreducible connected subgroup of $G(g)$ which is simple modulo its center. Then*

$$\dim(H) \leq g! \quad or \quad H = G(g).$$

Proof. Let V denote the standard representation of the symplectic or special orthogonal group $G(g)$, and suppose $\dim(H) > g! = \dim(V)$. Then by irreducibility the restriction of V to H must be one of the representations in table 1 of [**3**]. Since $\dim(V)$ is the factorial of a natural number, one of the following cases must occur for some $r \in \mathbb{N}$.

(a) The group H is of type A_r, C_r or D_r and acts on V via its standard representation.

(b) The group H is of type A_r and acts on V via the symmetric or via the alternating square of its standard representation.

Case *(a)* can only occur for $H = G(g)$ since $G(g)$ is itself the symplectic or special orthogonal group with V as its standard representation. To deal with case *(b)*, note that by the criterion in [**46**, th. 3.2.14] the symmetric square of the standard representation of A_r is self-dual only if $r = 1$, but then it has dimension $3 \neq \dim(V)$. In the same vein, the alternating square of the standard representation of A_r is self-dual of dimension $g!$ only in the case $(r, g) = (3, 3)$ where $H = G(g) = SO_6(\mathbb{C})$ for dimension reasons. □

To complete the proof of theorem 4.2, all that remains to be done is to show that on a general ppav X we have

$$\dim(G) > g!$$

for the group $G = G(\delta_\Theta)$ attached to the theta divisor. But this follows by induction on the dimension $g = \dim(X)$. Indeed, we already know that the theorem holds for $g \leq 3$, so we can assume $g \geq 4$. To start the induction, for $g = 4$ we can use the subgroup in part (3) of proposition 4.12 since in that case $\dim(Sl_{2g-2}(\mathbb{C})) = 35 > g! = 24$. This being settled, the induction step from dimension $g-1$ to dimension g is provided by part (1) of the proposition because $\dim(G(g-1)) > g!$ for $g \geq 5$. This finishes the proof of the theorem. Going through the above arguments, one furthermore checks that for $g = 4$ all we need is part (3) of proposition 4.12. Hence theorem 4.3 follows with the same proof.

4.6. Singularities of type A_k

For completeness we include in this section some remarks about double points that have been used in the proof of part (2) of proposition 4.12. In that proof we have only dealt with ordinary double points, but since this is hardly more work, in what follows we place ourselves in a slightly more general setting. Let $Y \subset \mathbb{C}^{n+1}$ be a complex analytic hypersurface. For $k \geq 1$, we say that a point e in Y is a singularity of type A_k if in a neighborhood of e the hypersurface Y is defined by an equation of the form

$$(\star) \qquad z_0^{k+1} + z_1^2 + z_2^2 + \cdots + z_n^2 \;=\; 0$$

in suitable local analytic coordinates $(z_0,\dots,z_n)$ on $\mathbb{C}^{n+1}$ centered at e. So a singularity of type A_k is just an ordinary double point for $k=1$ whereas for $k>1$ it is a double point of corank one.

Working as usual with coefficients in $\Lambda=\mathbb{C}$ or $\Lambda=\overline{\mathbb{Q}}_l$ in the algebraic case, we want to describe the perverse intersection cohomology sheaf near a singularity of type A_k.

LEMMA 4.16. *Let $Y\subset\mathbb{A}^{n+1}(\mathbb{C})$ be a complex analytic hypersurface whose only singular point e is a singularity of type A_k.*

(a) If k or n is even, then $\delta_Y=\Lambda_Y[n]$.

(b) Otherwise we have an exact sequence of perverse sheaves

$$0\longrightarrow\delta_e(-\tfrac{n-1}{2})\longrightarrow\Lambda_Y[n]\longrightarrow\delta_Y\longrightarrow 0.$$

Proof. We can assume Y is defined by an equation $(\star)$. To compute the blowup $p:\tilde{Y}\longrightarrow Y$ in the singular point e we choose $z=(z_0,\dots,z_n)$ as coordinates on the affine space $\mathbb{A}^{n+1}=\mathbb{A}^{n+1}(\mathbb{C})$. The corresponding coordinates on $\mathbb{P}^n=\mathbb{P}^n(\mathbb{C})$ will be denoted $u=[u_0:\cdots:u_n]$. Then the blowup

$$\tilde{Y}\;\subset\;\tilde{\mathbb{A}}^{n+1}\;=\;\big\{(z,u)\in\mathbb{A}^{n+1}\times\mathbb{P}^n\mid z\in u\big\}$$

can be computed explicitly in the affine charts $U_i=\{(z,u)\in\tilde{\mathbb{A}}^{n+1}\mid u_i\neq 0\}$ with the standard coordinates $(u_0,\dots,u_{i-1},z_i,u_{i+1},\dots,u_n)$ by putting $u_i=1$ and $z_j=z_i\,u_j$ for $j\neq i$. Thus

$$\tilde{Y}\cap U_0\;=\;\big\{z_0^{k-1}+u_1^2+\cdots+u_n^2=0\big\}$$

whereas for $i>0$ one obtains that

$$\tilde{Y}\cap U_i\;=\;\big\{u_0^{k+1}z_i^{k-1}+u_1^2+\cdots+u_{i-1}^2+1+u_{i+1}^2+\cdots+u_n^2=0\big\}.$$

In particular, for $k=1$ it follows that the blowup $\tilde{Y}$ is smooth. For $k>1$ it has an isolated singularity $\tilde{e}\in\tilde{Y}\cap U_0$ of type A_{k-2}, and we get by induction on k an exact sequence

$$0\to\kappa_{nk}\to\Lambda_{\tilde{Y}}[n]\to\delta_{\tilde{Y}}\to 0\quad\text{where}\quad\kappa_{nk}=\begin{cases}0 & \text{if } 2\mid nk,\\ \delta_{\tilde{e}}(-\frac{n-1}{2}) & \text{if } 2\nmid nk.\end{cases}$$

Furthermore, by purity we can write

$$Rp_*(\delta_{\tilde{Y}}) \;=\; \delta_Y \oplus \varepsilon$$

where ε is a skyscraper complex supported on e and self-dual with respect to Verdier duality up to a Tate twist. We now distinguish two cases.

If $k=1$, the above calculations show that the exceptional divisor of the blowup $\tilde{Y}$ is the smooth quadric $Q_n = \{u \in \mathbb{P}^n \mid u_0^2 + u_1^2 + \cdots + u_n^2 = 0\}$. The cohomology of such a quadric has been computed in [**32**, exp. XII, th. 3.3],

$$H^i(Q_n, \Lambda) \;=\; \begin{cases} \Lambda(-\frac{i}{2}) & \text{if } i \in \{0,2,\dots,2n-2\} \setminus \{n-1\}, \\ \Lambda^2(-\frac{i}{2}) & \text{if } i = n-1 \text{ and } n \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Since by base change $\mathscr{H}^{i-n}(\delta_Y \oplus \varepsilon)_0 = \mathscr{H}^{i-n}(Rp_*(\delta_{\tilde{Y}}))_0 = H^i(Q_n, \Lambda)$, one then easily concludes the proof of the lemma by an application of the hard Lefschetz theorem to the skyscraper summand ε.

If $k>1$, the above calculations show that the exceptional divisor of the blowup $\tilde{Y}$ is the singular quadric $Q_n^\star = \{u \in \mathbb{P}^n \mid u_1^2 + u_2^2 + \cdots + u_n^2 = 0\}$. In this case we will see below that

$$H^i(Q_n^\star, \Lambda) \;=\; \begin{cases} \Lambda(-\frac{i}{2}) & \text{if } i \in \{0,2,\dots,2n-2\} \setminus \{n\}, \\ \Lambda^2(-\frac{i}{2}) & \text{if } i = n \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Now by base change $\mathscr{H}^{i-n}(Rp_*(\Lambda_{\tilde{Y}}[n]))_0 = H^i(Q_n^\star, \Lambda)$, and combining this with the distinguished triangle

$$Rp_*(\kappa_{nk}) \longrightarrow Rp_*(\Lambda_{\tilde{Y}}[n]) \longrightarrow \underbrace{Rp_*(\delta_{\tilde{Y}})}_{=\,\delta_Y \oplus \varepsilon} \longrightarrow \cdots$$

one concludes the proof of the lemma as before by checking that the only two possibly non-zero cohomology sheaves of δ_Y are $\mathscr{H}^{-1}(\delta_Y) = Rp_*(\kappa_{nk})$ and $\mathscr{H}^{-n}(\delta_Y) = \Lambda_Y$ in this case.

It remains to check that the cohomology of the singular quadric $Q_n^\star$ has the form given above. Consider the singular point $p = [1:0:\cdots:0]$ of this quadric, with smooth complement $E = Q_n^\star \setminus \{p\}$. By the excision sequence for compactly supported cohomology we have $H_c^i(E,\Lambda) = H^i(Q_n^\star, \Lambda)$ for all $i>1$, and in low degrees we have an exact sequence

$$0 \to H_c^0(E,\Lambda) \to H^0(Q_n^\star,\Lambda) \to \Lambda \to H_c^1(E,\Lambda) \to H^1(Q_n^\star,\Lambda) \to 0.$$

So it suffices to show

$$H_c^i(E,\Lambda) \;=\; \begin{cases} \Lambda(-\frac{i}{2}) & \text{if } i \in \{2,4,\dots,2n-2\} \setminus \{n\}, \\ \Lambda^2(-\frac{i}{2}) & \text{if } i = n \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

For this we use that the projection $[u_0 : u_1 : \cdots : u_n] \mapsto [u_1 : \cdots : u_n]$ with center p defines a rank one vector bundle

$$\pi: \; E \longrightarrow Q_{n-1} \subset \mathbb{P}^{n-1}$$

over the smooth quadric $Q_{n-1} = \{[u_1 : \cdots : u_n] \in \mathbb{P}^{n-1} \mid u_1^2 + \cdots + u_n^2 = 0\}$ of dimension $n-2$. Using Poincaré duality and the homotopy invariance of cohomology, we thus obtain

$$H_c^i(E,\Lambda) \;\cong\; (H^{2n-2-i}(E,\Lambda)(n-1))^\vee \;\cong\; (H^{2n-2-i}(Q_{n-1},\Lambda)(n-1))^\vee$$

and the result follows by plugging in on the right hand side the cohomology of the smooth quadric Q_{n-1}. $\square$

CHAPTER 5

A family of surfaces with monodromy $W(E_6)$

Let X be a complex principally polarized abelian variety (ppav for short) of dimension $g \geq 2$, and fix a theta divisor $\Theta \subset X$ which defines the given polarization and which is symmetric in the sense that it is stable under the inversion morphism $-id_X : X \longrightarrow X$. For points $x \in X(\mathbb{C})$ let $\Theta_x = \Theta + x$ denote the corresponding translate of the theta divisor. The geometry of the intersections

$$Y_x \;=\; \Theta \cap \Theta_x$$

is closely connected with the moduli of the given ppav and has been studied in relation with Torelli's theorem [**25**], with the Schottky problem [**27**] and with the Prym map [**61**]; these intersections are also among the few concrete examples for varieties with ample cotangent bundle [**24**]. One of our goals is to understand how their cohomology varies with the point x. As we will see later on, this is closely related to the convolution square of the theta divisor (in the sense of the previous chapters).

To set the stage for what follows, in section 5.1 we discuss some general features of the intersections Y_x. If the theta divisor is smooth or has at most isolated singularities, then for general $x \in X(\mathbb{C})$ a Bertini-type argument will show that Y_x is smooth and that the involution $\sigma_x : Y_x \longrightarrow Y_x$, $y \mapsto x - y$ is étale so that the quotient

$$Y_x^+ \;=\; Y_x/\sigma_x$$

is again a smooth variety. After a survey of the cases $g \leq 3$ where a close connection with the moduli of Prym varieties emerges, we focus on the dimension $g = 4$. Here we are dealing with a family of smooth surfaces of general type with a beautiful explicit geometry. We will see that the varying part of the cohomology of these surfaces defines over a Zariski-open dense subset $U \subset X$ a variation of Hodge structures $\mathbb{V}_+$ of rank six whose fibres are of pure Hodge type $(1,1)$. One of the main results of this chapter will be the proof of the following conjecture of R. Weissauer.

THEOREM 5.1. *For a general complex ppav X of dimension $g = 4$, the monodromy group G of the local system underlying $\mathbb{V}_+$ is either the Weyl group $W(E_6)$ or its unique simple subgroup of index two which is the kernel of the sign homomorphism $sgn : W(E_6) \longrightarrow \{\pm 1\}$.*

Here by the monodromy group $G = \mathrm{Im}(\pi_1(U,x) \longrightarrow Aut_{\mathbb{C}}(\mathbb{V}_{+,x}))$ we mean the image of the monodromy representation of the fundamental group on the stalk of $\mathbb{V}_+$ at some chosen point $x \in U(\mathbb{C})$. Up to isomorphism this image is of course independent of our choices.

We remark that the occurance of the Weyl group $W(E_6)$ in the present context can be related to the 27 lines on a cubic surface via the work of E. Izadi on Prym-embedded curves [**61**], though this relationship will not be used in what follows (for more about this see section 5.5 below). For the proof of theorem 5.1 a detailed guideline will be provided in section 5.2, so let us here only mention the most important points. To show that the monodromy group G must be a subgroup of $W(E_6)$, we give in section 5.3 and 5.4 a detailed study of the integral cohomology and of the Néron-Severi lattices of the smooth surfaces Y_x and Y_x^+ for general $x \in X(\mathbb{C})$. Once this has been done, the main task will be to find a sufficiently large lower bound on the monodromy group. For this we use two ideas. On the one hand, in section 5.6 we relate the variation of Hodge structures $\mathbb{V}_+$ to the convolution square of the theta divisor — this allows to apply the Tannakian formalism from chapters 2 and 3 and theorem 4.2 to conclude that for a general ppav the monodromy group G acts irreducibly on the stalks. On the other hand, a degeneration argument will show that in dimension $g = 4$ this monodromy group contains as a subgroup the corresponding monodromy group for the Jacobian variety JC of a general curve C.

For Jacobian varieties we have the following result. Let C be a smooth complex projective curve of even genus $g = 2n$, and identify the symmetric product

$$C_n \;=\; (C \times \cdots \times C)/\mathfrak{S}_n$$

with the set of effective divisors of degree n. We then have the difference morphism

$$d_n : \quad C_n \times C_n \longrightarrow JC \;=\; Pic^0(C)$$

which sends a pair (D,E) of effective divisors of degree n on the curve C to the isomorphism class of the line bundle $\mathscr{O}_C(D-E)$. Using the Poincaré formula we will see in section 5.8 that the morphism d_n is generically finite of degree $N = \binom{2n}{n}$. In particular, over some Zariski-open dense subset of X the difference morphism restricts to a finite étale cover, and it makes sense to speak of its Galois group $G(d_n)$. An application of Brill-Noether sheaves [**106**] to be explained in proposition 5.24 *(b)* shows that if C is a non-hyperelliptic curve of genus $g = 4$, the Galois group $G(d_2)$ can be identified in a natural way with the monodromy group of the local system underlying $\mathbb{V}_+$ on the Jacobian variety. The desired lower bound for the proof of theorem 5.1 is then given by the following observation.

THEOREM 5.2. *If C is a general curve of genus $g = 2n$, then $G(d_n)$ is either the alternating group $\mathfrak{A}_N$ or the full symmetric group $\mathfrak{S}_N$.*

The proof of this in section 5.8 again uses a degeneration, this time from a general curve into a hyperelliptic one. We also remark that if the Galois group $G(d_n)$ in the above theorem is the full symmetric group $\mathfrak{S}_N$, then also the monodromy group G in the previous theorem has to be the full Weyl group $W(E_6)$ because it then contains a reflection. However, at present we do not know how to decide whether this is indeed the case.

5.1. Intersections of theta divisors

Consider a complex ppav X of dimension $g \geq 2$ with a symmetric theta divisor $\Theta \subset X$ as above. In this section we will always assume that Θ is smooth or has at most isolated singularities, a situation in which we have the following Bertini-type result.

LEMMA 5.3. *If the theta divisor Θ is smooth or has at most isolated singularities, then over some Zariski-open dense subset $U \subset X$ there exists a smooth proper family*

$$f_U : \quad Y_U \longrightarrow U$$

whose fibre over any point $x \in U(\mathbb{C})$ is isomorphic to $Y_x = \Theta \cap \Theta_x$.

Proof. Recall the general fact [**55**, cor. III.10.7] that if $f : V \longrightarrow W$ is a morphism of complex algebraic varieties and if V is smooth, then over some Zariski-open dense $U \subseteq W$ the restriction $f_U = f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is a smooth morphism. Since we can replace W by any Zariski-open dense subset, this remains true if the smoothness assumption on V is replaced by the weaker assumption that the singular locus $Sing(V)$ is mapped via f into a proper closed subset of W. In the case at hand, we apply this to the composite morphism

$$f : \quad V = \Theta \times \Theta \hookrightarrow X \times X \xrightarrow{a} X = W$$

where $a : X \times X \longrightarrow X$ denotes the group law. If the theta divisor has at most isolated singularities, then $Sing(V) = (Sing(\Theta) \times \Theta) \cup (\Theta \times Sing(\Theta))$ has dimension $g-1$ or is empty. In both cases the general fact from above shows that over some Zariski-open dense subset $U \subset X$ the restriction f_U is smooth of relative dimension $g-2$. This being said, our claim follows from the observation that the projection onto the first factor $\Theta \times \Theta \longrightarrow \Theta$ induces an isomorphism $f^{-1}(x) \cong \Theta \cap \Theta_x$ for $x \in X(\mathbb{C})$. $\square$

We want to study the fibres Y_x of the above family. In this context an important role is played by the involution $\sigma_x : X \longrightarrow X, y \mapsto x - y$. Notice that the intersection $Y_x = \Theta \cap \Theta_x$ is mapped onto itself by this involution because we assumed the theta divisor to be symmetric. The above Bertini argument yields the following

COROLLARY 5.4. *If the theta divisor is smooth or has only isolated singularities, then there exists a Zariski-open dense subset $U \subset X$ such that for all $x \in U(\mathbb{C})$ the quotient morphism*

$$Y_x \longrightarrow Y_x^+ = Y_x/\sigma_x$$

is an étale double covering between smooth varieties of dimension $g-2$.

Proof. From lemma 5.3 we get a Zariski-open dense subset $U \subset X$ such that Y_x is smooth of dimension $g-2$ for all $x \in U(\mathbb{C})$. Shrinking this open dense subset we can furthermore assume that it does not contain points of the form $x = 2y$ with $y \in \Theta(\mathbb{C})$. Then the involution $\sigma_x : Y_x \longrightarrow Y_x$ is étale for all points $x \in U(\mathbb{C})$, and our claim follows. $\square$

For the rest of this section we always fix a point x with the properties in the above corollary, and we put

$$Y = Y_x, \quad Y^+ = Y_x^+ \quad \text{and} \quad \sigma = \sigma_x$$

for brevity. Like for any étale double cover of smooth complex varieties, the rational cohomology of the quotient Y^+ coincides with the eigenspace for the eigenvalue $+1$ in

$$H^\bullet(Y,\mathbb{Q}) = H^\bullet(Y,\mathbb{Q})^+ \oplus H^\bullet(Y,\mathbb{Q})^-$$

where the upper index $\pm$ indicates that the involution σ acts by ± 1 on the respective two eigenspaces. Now recall the following version of the weak Lefschetz theorem [**71**, rem. 3.1.29]: If W is a smooth complex projective variety of dimension d, then for all ample effective divisors $D_1, \dots, D_r$ the restriction map $H^n(W,\mathbb{Z}) \longrightarrow H^n(D_1 \cap \cdots \cap D_r, \mathbb{Z})$ is an isomorphism in degrees $n < d-r$ and a monomorphism in degree $n = d-r$. Note that the intersecting divisors are not required to be smooth or transverse to each other. In our case, since Y is the intersection of two ample divisors on X it follows that the restriction morphism

$$H^n(X,\mathbb{Q}) \longrightarrow H^n(Y,\mathbb{Q})$$

is an isomorphism for $n < g-2$ and a monomorphism for $n = g-2$. So the interesting part of the cohomology of Y sits in the middle cohomology degree $n = g-2$ and can be defined as the orthocomplement

$$V = H^{g-2}(X,\mathbb{Q})^\perp \subseteq H^{g-2}(Y,\mathbb{Q})$$

with respect to the intersection form. The involution σ acts by $\varepsilon = (-1)^g$ on $H^{g-2}(X,\mathbb{Q})$, hence

$$H^{g-2}(X,\mathbb{Q}) \subseteq H^{g-2}(Y,\mathbb{Q})^\varepsilon = \begin{cases} H^{g-2}(Y,\mathbb{Q})^+ & \text{for } g \text{ even,} \\ H^{g-2}(Y,\mathbb{Q})^- & \text{for } g \text{ odd.} \end{cases}$$

So the orthocomplement from above admits a decomposition $V = V_+ \oplus V_-$ into the eigenspaces

$$V_{-\varepsilon} \;=\; H^{g-2}(Y,\mathbb{Q})^{-\varepsilon} \quad \text{and} \quad V_\varepsilon \;=\; H^{g-2}(X,\mathbb{Q})^\perp \;\subseteq\; H^{g-2}(Y,\mathbb{Q})^\varepsilon$$

where now $\perp$ denotes the orthocomplement inside the ε-eigenspace. Since the restriction morphism in the weak Lefschetz theorem is a morphism of Hodge structures, V_- and V_+ are Hodge substructures of $H^{g-2}(Y,\mathbb{Q})$. To compute their Hodge numbers one can use the Hirzebruch-Riemann-Roch theorem together with the following result, where $[\Theta] \in H^2(X,\mathbb{Q})$ denotes the fundamental class of the theta divisor.

LEMMA 5.5. *In terms of the restriction $\theta = [\Theta]|_Y \in H^2(Y,\mathbb{Q})$, the Chern classes of Y are given by*

$$c_i(Y) \;=\; (-1)^i \cdot (i+1) \cdot \theta^i \;\in\; H^{2i}(Y,\mathbb{Q}).$$

In particular, the variety Y is of general type with canonical class $K_Y = 2\theta$.

Proof. By definition we have $Y = \Theta \cap \Theta_x$ for some point $x \in X(\mathbb{C})$. The embedding $Y \hookrightarrow \Theta$ gives an adjunction sequence

$$0 \longrightarrow \mathscr{T}_Y \longrightarrow \mathscr{T}_\Theta|_Y \longrightarrow \mathscr{O}_X(\Theta_x)|_Y \longrightarrow 0$$

where $\mathscr{T}_Y$ and $\mathscr{T}_\Theta$ are the tangent bundles to Y resp. Θ (note that this makes sense also if the theta divisor has isolated singularities, provided that x is chosen in such a way that Y does not meet the singularities). By restriction of the adjunction sequence for the embedding $\Theta \hookrightarrow X$ we also get an exact sequence

$$0 \longrightarrow \mathscr{T}_\Theta|_Y \longrightarrow \mathscr{T}_X|_Y \longrightarrow \mathscr{O}_X(\Theta)|_Y \longrightarrow 0.$$

Since the tangent bundle $\mathscr{T}_X$ is trivial, it follows from these two adjunction sequences that the Chern polynomial $c_t(Y) = 1 + c_1(Y)\cdot t + c_2(Y)\cdot t^2 + \cdots$ satisfies

$$1 \;=\; c_t(Y)\cdot(1+\theta t)^2$$

since the Chern polynomial is multiplicative in short exact sequences. From this identity our claim about the Chern classes follows by a comparison of coefficients. By definition the canonical sheaf ω_Y is the top exterior power of the cotangent bundle $\mathscr{T}_Y^*$, so for the canonical class we get

$$K_Y \;=\; c_1(\omega_Y) \;=\; c_1(\Lambda^{g-2}(\mathscr{T}_Y^*)) \;=\; c_1(\mathscr{T}_Y^*) \;=\; -c_1(Y) \;=\; 2\theta$$

where in the third equality we have used that by the splitting principle the first Chern class of a bundle coincides with the first Chern class of any of its exterior powers [**40**, rem. 3.2.3(c)]. In particular, the canonical class K_Y is ample and therefore also big in the sense of [**71**, sect. 2.2], so the smooth variety Y is of general type. $\square$

With notations as in the above lemma, to compute intersection numbers between powers of Chern classes we only need to determine the image of the top power θ^{g-2} under the degree map $\deg_Y : H^{g-2}(Y,\mathbb{Q}) \longrightarrow \mathbb{Q}$. To achieve this recall that Y is the intersection of two general translates of the theta divisor. Since the intersection of cycles corresponds to the cup product on cohomology, it follows that the fundamental class of Y is the cup product square $[Y] = [\Theta]^2 \in H^4(X,\mathbb{Q})$. Hence

$$\deg_Y(\theta^{g-2}) \;=\; \deg_X([\Theta]^{g-2} \cdot [Y]) \;=\; \deg_X([\Theta]^g) \;=\; g!$$

where the last equality holds by the Poincaré formula [**13**, sect. 11.2.1]. As a direct application we have the following

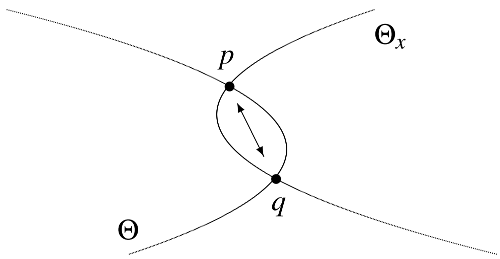
COROLLARY 5.6. *For $g \geq 2$ the topological Euler characteristic of Y is given by the formula*

$$\chi(Y) \;=\; (-1)^g \cdot (g-1) \cdot g!\,.$$

Proof. By the Gauss-Bonnet theorem [**48**, p. 416] the topological Euler characteristic of any compact complex manifold is equal to the degree of its top Chern class. Hence $\chi(Y) = \deg_Y(c_{g-2}(Y))$, and the claim follows from lemma 5.5 and from the above formula for the degree $\deg_Y(\theta^{g-2})$. □

To get a feeling for the geometry involved in the above constructions, let us take a more explicit look at the situation for some small values of the dimension $g = \dim(X)$ of the underlying ppav.

The case $g = 2$. Here Y consists of two points which are interchanged by the involution σ so that we have the following picture.



The fundamental classes p, q of the two interchanged points of Y form a basis of $H^0(Y,\mathbb{Q})$, and the eigenspace

$$H^0(Y,\mathbb{Q})^+ \;=\; \mathbb{Q} \cdot (p+q) \;=\; H^0(X,\mathbb{Q})$$

equals the image of the weak Lefschetz embedding. Thus $V_+ = 0$, and the Hodge structure $V_- = \mathbb{Q} \cdot (p-q)$ of weight zero has rank one.

The case $g = 3$. In this case the morphism $Y \longrightarrow Y^+$ is an étale double covering of smooth projective curves. To any such covering one can attach its Prym variety [**79**] [**13**, ch. 12]. Before we come to the specific example at hand, let us briefly recall the definition and some basic features of Prym varieties in general, referring to loc. cit. for more details.

For any covering $\tilde{C} \longrightarrow C$ of smooth projective curves we have a norm homomorphism

$$N_{\tilde{C}/C}: \quad J\tilde{C} \longrightarrow JC$$

between the corresponding Jacobian varieties, induced by the pushforward morphism on divisors. This norm homomorphism is always surjective. For an étale double covering its kernel has precisely two connected components, and one defines the Prym variety

$$P = Prym(\tilde{C}/C) = (\ker(N_{\tilde{C}/C}))^0 \subset J\tilde{C}$$

to be the connected component which contains the origin. By construction this is an abelian subvariety of $J\tilde{C}$. The special thing about étale double covers of curves is that this Prym variety also comes along with a principal polarization, indeed by loc. cit. the principal polarization on $J\tilde{C}$ restricts on the Prym variety P to twice a principal polarization. Prym varieties are useful in the study of ppav's since they are more general than Jacobian varieties but nevertheless still accessible in terms of algebraic curves. Note that if the curve C has genus $g+1$ (say), then $\tilde{C}$ has genus $2g+1$ due to the Riemann-Hurwitz formula [**55**, cor. IV.2.4], and then the Prym variety has dimension

$$\dim(P) = (2g+1)-(g+1) = g.$$

Hence if we denote by $\mathscr{M}_{g+1}$ the coarse moduli space of smooth projective curves of genus $g+1$ and define $\mathscr{R}_{g+1}$ to be the coarse moduli space of étale double covers of such curves, then we have a diagram

$$\begin{tikzcd} & \mathscr{R}_{g+1} \arrow[dl, "\varphi"'] \arrow[dr, "\pi"] & \\ \mathscr{M}_{g+1} & & \mathscr{A}_g \end{tikzcd}$$

where φ is the forgetful morphism and where π is the morphism that assigns to a double covering its Prym variety. Here the morphism φ is generically finite of degree $2^{g+1}-1$ because the étale double covers of a given curve correspond bijectively to the non-trivial two-torsion points on its Jacobian variety. The Prym morphism π plays an important role for the moduli of abelian varieties and has been studied a lot in the literature; as the most important results, we only mention that for $g \geq 6$ it is generically finite of degree one [**39**] but never injective [**34**]. In dimensions $g < 6$ the Prym morphism π is no longer generically finite but its fibres have a surprisingly rich geometric structure which is discussed in loc. cit.

This brings us back to the étale double covering $Y \longrightarrow Y^+$ defined by the intersection of two general translates of the theta divisor on a complex ppav X of dimension $g = 3$. Here the curve Y has genus 7 by corollary 5.6, hence the Prym variety $P = Prym(Y/Y^+)$ is a complex ppav of dimension 3 as well. The following result has been obtained in [**84**] and [**66**].

THEOREM 5.7. *For a general complex ppav X of dimension $g = 3$ the following properties are satisfied.*

(a) For general $x \in X(\mathbb{C})$ the Prym variety P of the cover $Y_x \longrightarrow Y_x^+$ is isomorphic to the ppav X.

(b) Every étale double covering of smooth curves with Prym variety isomorphic to X arises as above for some point $x \in X(\mathbb{C})$, and the coverings for two points x_1, x_2 are isomorphic iff $x_1 = \pm x_2$.

In other words, the above construction identifies the fibre of the Prym morphism $\pi : \mathscr{R}_4 \longrightarrow \mathscr{A}_3$ over a general ppav $X \in \mathscr{A}_4(\mathbb{C})$ with an open dense subset of the Kummer variety

$$K_X \;=\; X/\langle \pm id_X \rangle.$$

Fixing a general point $x \in X(\mathbb{C})$, we now again use the notation $Y = Y_x$ etc. The definition of the Prym variety P as a component of the kernel of the norm epimorphism $N : JY \twoheadrightarrow JY^+$ shows that the Jacobian variety JY is isogenous to $P \times JY^+$, which together with the identification in part *(a)* of the above theorem implies

$$H^1(Y, \mathbb{Q}) \;=\; H^1(X, \mathbb{Q}) \oplus H^1(Y^+, \mathbb{Q}).$$

Hence in this case we have $V_- = 0$, and $V_+ = H^1(Y^+, \mathbb{Q})$ is a pure Hodge structure of weight one and rank eight.

The case $g = 4$. Here $Y \longrightarrow Y^+$ is an étale double covering of smooth projective surfaces that will occupy us for the rest of this chapter. We claim that V_- has Hodge numbers $h^{2,0} = h^{0,2} = 11$ and $h^{1,1} = 30$ whereas V_+ is of pure Hodge type $(1,1)$ and rank six. Indeed, bearing in mind the natural identifications

$$H^2(Y^+, \mathbb{Q}) \;=\; H^2(Y, \mathbb{Q})^+ \;=\; H^2(X, \mathbb{Q}) \oplus V_+$$

this is a direct consequence of the numerical data in the following table.

LEMMA 5.8. *For a complex ppav X of dimension $g = 4$ we have the following Hodge numbers.*

	$h^{2,0} = h^{0,2}$	$h^{1,1}$	$h^{1,0} = h^{0,1}$
Y	17	52	4
Y^+	6	22	0
X	6	16	4

Proof. The last row follows from the Hodge decomposition for a ppav of dimension $g=4$ and has only been included for reference. Similarly, the last column is clear because $H^1(Y,\mathbb{Q}) = H^1(Y,\mathbb{Q})^- = H^1(X,\mathbb{Q})$ by the weak Lefschetz theorem. It remains to compute the Hodge numbers of the smooth surfaces Y and Y^+ in cohomology degree two. Corollary 5.6 gives the topological Euler characteristic $\chi(Y) = 72$, which implies $h^2(Y) = 86$ because $h^0(Y) = h^4(Y) = 1$ and $h^1(Y) = h^3(Y) = 8$. In the same way, via lemma 5.5 the Hirzebruch-Riemann-Roch theorem gives the holomorphic Euler characteristic

$$\chi(\mathscr{O}_Y) \;=\; \deg_Y(c_1^2(Y) + c_2(Y))/12 \;=\; \deg_Y((-2\theta)^2 + 3\theta^2)/12 \;=\; 14.$$

Plugging in the values $h^{0,0}(Y) = 1$ and $h^{0,1}(Y) = 4$ we get $h^{0,2}(Y) = 17$, and then $h^{1,1}(Y) = 52$ because the Hodge numbers in degree two must sum up to the second Betti number $h^2(Y) = 86$.

To obtain from these numbers also the Hodge numbers of the quotient surface $Y^+ = Y/\sigma$ we use that the quotient morphism $q: Y \to Y^+$ is an étale double covering. In particular we have a commutative diagram

$$\begin{tikzcd}
H^2(Y^+,\mathbb{Q}) \times H^2(Y^+,\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "q^*"'] & H^4(Y^+,\mathbb{Q}) \arrow[r, "\deg_{Y^+}"] \arrow[d, "q^*"'] & \mathbb{Q} \arrow[d, "2\cdot"] \\
H^2(Y,\mathbb{Q}) \times H^2(Y,\mathbb{Q}) \arrow[r, "\cup"] & H^4(Y,\mathbb{Q}) \arrow[r, "\deg_Y"] & \mathbb{Q}
\end{tikzcd}$$

where the vertical arrow on the right hand side is multiplication with the degree $\deg(q) = 2$, and the Chern classes satisfy $c_i(Y) = q^*(c_i(Y^+))$ for all i. This being said, the Gauss-Bonnet and Hirzebruch-Riemann-Roch theorem show

$$\chi(Y^+) \;=\; \chi(Y)/2 \;=\; 36 \quad \text{and} \quad \chi(\mathscr{O}_{Y^+}) \;=\; \chi(\mathscr{O}_Y)/2 \;=\; 7.$$

Hence it follows that $h^2(Y^+) = 36-1-1 = 34$ and $h^{0,2}(Y^+) = 7-1 = 6$ by similar computations as above, and we are done. □

As an immediate corollary we obtain that the intersection form has the following signatures on the various occuring subspaces of $H^2(Y,\mathbb{Q})$, where we denote by s_+ and s_- the dimension of maximal subspaces on which the form is positive resp. negative definite.

COROLLARY 5.9. *For $g=4$ the intersection form has the following signatures on the various subspaces of $H^2(Y,\mathbb{Q})$.*

	$H^2(Y,\mathbb{Q})$	$H^2(Y^+,\mathbb{Q})$	$H^2(X,\mathbb{Q})$	V_-	V_+
s_+	35	13	13	22	0
s_-	51	21	15	30	6

Proof. The Hodge index theorem [**103**, th. 6.33] says that for a smooth compact Kähler surface S with Kähler class $\omega \in H^{1,1}(X) \cap H^2(X,\mathbb{R})$ the intersection form is

- positive definite on the subspace $(H^{2,0}(S) \oplus H^{0,2}(S)) \cap H^2(S,\mathbb{R})$ and also on the line $\mathbb{R} \cdot \omega$,
- negative definite on the orthocomplement of the Kähler class ω inside $H^{1,1}(S) \cap H^2(S,\mathbb{R})$.

In our case, taking $S = Y$ and using the Hodge numbers in lemma 5.8 one sees that on $H^2(Y,\mathbb{Q}) = H^2(Y,\mathbb{Q})^+ \oplus V_-$ and $H^2(Y,\mathbb{Q})^+ = H^2(X,\mathbb{Q}) \oplus V_+$ the intersection form has the given signatures. □

In particular, it follows that on the subspace V_+ the intersection form is negative definite. In sections 5.3 and 5.4, working on the level of integral cohomology we will determine the structure of the lattices whose signatures have been listed above. But let us first say a few words about the variations of Hodge structures defined by the above subspaces.

5.2. Variations of Hodge structures

So far we have mostly fixed a general point $x \in X(\mathbb{C})$. Now let us see what happens when this point varies. For the moment the dimension g can be arbitrary, but as before we always assume that the theta divisor $\Theta \subset X$ is smooth or has only isolated singularities. To begin with, we put the Hodge structures $V_\pm$ from section 5.1 into a family as follows.

LEMMA 5.10. *Over some Zariski-open dense subset $U \subset X$ there exists for each integer $n \geq 0$ a polarized variation of pure Hodge structures $\mathbb{H}^n$ with stalks*

$$\mathbb{H}^n_x \;=\; H^n(Y_x,\mathbb{Q}),$$

and there are subvariations

$$\mathbb{V}_+ \;\subset\; \mathbb{H}^{g-2} \quad \textit{and} \quad \mathbb{V}_- \;\subset\; \mathbb{H}^{g-2}$$

whose stalks are precisely the subspaces V_+ and V_- defined in section 5.1.

Proof. Consider the Zariski-open dense subset $U \subset X$ and the smooth proper family $f_U : Y_U \longrightarrow U$ from lemma 5.3 and corollary 5.4. We can then define

$$\mathbb{H}^n \;=\; R^n f_{U*}(\mathbb{Q}_{Y_U}) \quad \text{for} \quad n \geq 0$$

since for any smooth proper family of complex varieties the direct images define variations of Hodge structures whose stalks are the cohomology of the fibres [**103**, chapt. III]. It remains to construct the subvariations $\mathbb{V}_\pm$. For this we consider the commutative diagram

$$\begin{tikzcd} Y_U \arrow[r, hook, "i"] \arrow[dr, "f_U"'] & U \times X \arrow[d, "p_U"] \\ & U \end{tikzcd}$$

where p_U denotes the projection onto the first factor and where the closed immersion i is defined by $i(x,y) = (x+y,y)$, recalling that $Y_U \subset \Theta \times \Theta$ by construction. The adjunction morphism $\mathbb{Q}_{U\times X} \longrightarrow i_*(\mathbb{Q}_{Y_U})$ defines in each degree $n \geq 0$ a morphism

$$H^n(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_U \;=\; R^n p_{U*}(\mathbb{Q}_{U\times X}) \;\longrightarrow\; R^n f_{U*}(\mathbb{Q}_{Y_U})$$

of polarized variations of pure Hodge structures, and this morphism induces on each fibre the restriction morphism that occurs in the weak Lefschetz theorem. Furthermore, the involution

$$\sigma : \quad U \times X \longrightarrow U \times X\,, \quad (x,y) \mapsto (x, x-y)$$

preserves Y_U and restricts on each fibre Y_x to the involution σ_x. This being said, we can perform the constructions from section 5.1 fibre by fibre to obtain the subvariations $\mathbb{V}_\pm \subset \mathbb{H}^{g-2}$ with the desired property. □

As we mentioned in the introduction, one of the principal goals of this chapter will be the proof of theorem 5.1. We now explain how this will follow from the results to be discussed in the next sections. Let us for convenience first recall the statement of the theorem.

THEOREM 5.1. *For a general complex ppav X of dimension $g = 4$ the monodromy group G of the local system $\mathbb{V}_+$ is either the Weyl group $W(E_6)$ or its unique simple subgroup of index two which is the kernel of the sign homomorphism $sgn : W(E_6) \longrightarrow \{\pm 1\}$.*

Plan of the proof. To begin with, in sections 5.3 and 5.4 we give a detailed study of the integral cohomology and the Néron-Severi lattices of the smooth surfaces $Y_x = \Theta \cap \Theta_x$ and $Y_x^+ = Y_x/\sigma_x$ for $x \in U(\mathbb{C})$. As a result of the lattice computations it will in particular follow that if we view $\mathbb{V}_+$ as a polarized variation of $\mathbb{Z}$-Hodge structures, then its stalks can be identified with the E_6-lattice up to a rescaling of the bilinear form. Coming back to the monodromy group G of the underlying local system, we then deduce in proposition 5.18 that

$$G \;\leq\; W(E_6)$$

is a subgroup of the Weyl group. Once we have this upper bound, we can proceed as follows. If equality holds, then we are done. If not, then G must be contained in some maximal proper subgroup $M < W(E_6)$. Now the table of maximal subgroups in [**23**, p. 26] shows that any such subgroup M must be conjugate to one of the following subgroups:

(a) the simple subgroup $W^+(E_6)$ of index two which is the kernel of the sign homomorphism $sgn : W(E_6) \longrightarrow \{\pm 1\}$,

(b) the stabilizer of a line through some minimal vector in the dual lattice $E_6^* = \{v \in E_6 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle v, w\rangle \in \mathbb{Z} \ \forall w \in E_6\}$,

(c) the stabilizer of a line through some root vector inside E_6, with the symmetric group $\mathfrak{S}_6$ as a subgroup of index two,

(d) three other subgroups, two of order $2^4 \cdot 3^4$ and one of order $2^7 \cdot 3^9$.

Furthermore, any proper subgroup of the simple group in *(a)* is by the table of loc. cit. also contained in some of the maximal subgroups described in cases *(b), (c)* or *(d)*. So for the proof of theorem 5.1 it will suffice to show that for a general ppav the monodromy group G cannot be contained in any of the subgroups listed in *(b), (c)* and *(d)*.

One tool for this will be the Tannakian formalism of convolution given in chapters 2 and 3. Indeed, consider the perverse intersection cohomology sheaf $\delta_\Theta = \mathbb{C}_\Theta[g-1] \in \mathrm{Perv}(X, \mathbb{C})$. Over the open dense subset $U \subset X$ it turns out that the local system underlying $\mathbb{V}_+$ can be identified with a direct summand of the convolution

$$\delta_\Theta * \delta_\Theta \ \in \ D_c^b(X, \mathbb{C}),$$

as we will show in section 5.6. Since for a general ppav of dimension $g = 4$ we know the Tannaka group $G(\delta_\Theta) = Sp_{24}(\mathbb{C})$ from theorem 4.2, it will then follow via the representation theory of the symplectic group that the local system underlying $\mathbb{V}_+$ is irreducible — in other words, the monodromy group G acts irreducibly on the stalks $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$ of this local system, see proposition 5.23. Hence G cannot be contained in the stabilizer of a line as described in the above cases *(b)* or *(c)*.

The remaining cases cannot be excluded with the same argument, in fact some of the subgroups in case *(d)* act irreducibly on $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$. An example is given in appendix C. However, a look at the group orders shows that the maximal subgroups in case *(d)* do not contain the alternating group $\mathfrak{A}_6$ as a subgroup, so the proof of theorem 5.1 will be finished if we can show that for a general ppav there exists an embedding

$$\mathfrak{A}_6 \hookrightarrow G.$$

For this we use a degeneration argument to fill in the dotted lines in the following Hasse diagram, where $G(d_2)$ is the Galois group of the difference morphism for a general curve as in theorem 5.2.

$$
\begin{tikzcd}[column sep=small]
 & & W(E_6) \arrow[dl, no head] \arrow[dr, no head] \arrow[dll, no head, bend right] & \\
 & G \arrow[dl, no head, dotted] \arrow[dr, no head, dotted] & & \mathfrak{S}_6 \arrow[dl, no head] \arrow[dll, no head, bend left, shift left] \\
W^+(E_6) \arrow[dr, no head] & & G(d_2) \arrow[dl, no head, dotted] & \\
 & \mathfrak{A}_6 & &
\end{tikzcd}
$$

More specifically, in lemma 5.28 we construct over some smooth complex quasi-projective curve S a principally polarized abelian scheme $X_S \longrightarrow S$ with a relative theta divisor $\Theta_S \subset X_S$ such that the following properties are satisfied.

(1) For some point $s_0 \in S(\mathbb{C})$ the fibre X_{s_0} is the Jacobian variety of a general curve of genus $g = 4$, but for all other points $s \neq s_0$ the theta divisor $\Theta_s \subset X_s$ is smooth.

(2) The relative addition morphism $f: \; Y_S = \Theta_S \times_S \Theta_S \longrightarrow X_S$ restricts to a smooth proper morphism

$$f_{U_S}: \quad f^{-1}(U_S) \longrightarrow U_S$$

over some Zariski-open dense subset $U_S \subset X_S$ that surjects onto S.

We remark that the statement in (2) is nothing but the relative version of the Bertini-type lemma 5.3. The proof of lemma 5.10 carries over verbatim to this relative setting, so over the Zariski-open dense subset $U_S \subset X_S$ we obtain two variations of Hodge structures

$$\mathbb{V}_{S+} \quad \text{and} \quad \mathbb{V}_{S-}$$

which on the fibre over each point $s \in S(\mathbb{C})$ restrict to the variations $\mathbb{V}_\pm$ from lemma 5.10. If for each such point we denote by $G(s)$ the monodromy group of the local system

$$\mathbb{V}_{S+}|_{U_s} \quad \text{on the fibre} \quad U_s \subset X_s,$$

then a general result about degenerating monodromy representations, to be recalled in lemma 5.27 below, says that for general $s \in S(\mathbb{C})$ we have an embedding

$$G(s_0) \hookrightarrow G(s).$$

So we will be finished if we can show that the monodromy group $G(s_0)$ for the Jacobian variety $X_{s_0} = JC$ of a general curve C of genus $g = 4$ contains the group $\mathfrak{A}_6$ as a subgroup. To achieve this we show in proposition 5.24

via the theory of Brill-Noether sheaves that $G(s_0)$ coincides with the Galois group of the difference morphism

$$d_2: \quad C_2 \times C_2 \longrightarrow Pic^0(C), \quad (D,E) \mapsto [\mathscr{O}_C(D-E)]$$

where C_2 denotes the second symmetric product of the curve, identified with the group of effective divisors of degree two. By theorem 5.2 (to be proven in section 5.8 below) the Galois group $G(d_2)$ of this difference morphism is either the alternating group $\mathfrak{A}_6$ or the full symmetric group $\mathfrak{S}_6$. This fills in the dotted lines in the above Hasse diagram and thereby completes the proof of theorem 5.1. $\square$

5.3. Integral cohomology

Let X be a complex ppav of dimension $g = 4$ with a smooth symmetric theta divisor $\Theta \subset X$. We fix a general point $x \in X(\mathbb{C})$ and consider the étale double covering of smooth surfaces

$$Y \;=\; \Theta \cap \Theta_x \;\longrightarrow\; Y^+ = Y/\sigma$$

for the involution $\sigma = \sigma_x$ as in corollary 5.4. In this section we determine the cohomology of these surfaces with integral coefficients, refining the computations that we did with rational coefficients in section 5.1. The result can be summarized in the following way.

PROPOSITION 5.11. *The smooth projective surfaces Y and Y^+ have the following integral cohomology groups.*

n	0	1	2	3	4
$H^n(Y,\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^8$	$\mathbb{Z}^{86}$	$\mathbb{Z}^8$	$\mathbb{Z}$
$H^n(Y^+,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^{34} \oplus (\mathbb{Z}/2\mathbb{Z})^9$	$(\mathbb{Z}/2\mathbb{Z})^9$	$\mathbb{Z}$

Moreover, the quotient morphism $q: Y \longrightarrow Y^+$ induces an epimorphism onto the σ-invariants

$$q^*: \; H^2(Y^+,\mathbb{Z}) \twoheadrightarrow H^2(Y,\mathbb{Z})^+ \subset H^2(Y,\mathbb{Z})$$

and the kernel of this epimorphism is the torsion subgroup of $H^2(Y^+,\mathbb{Z})$.

Proof. We will use the weak Lefschetz theorem in the following integral form, see [**76**, cor. 7.3] and [**104**, th. 1.23]. Let V be a complex projective variety of dimension n with an ample effective divisor $W \subset V$ with smooth complement $V \setminus W$. Then the restriction map

$$H^k(V,\mathbb{Z}) \longrightarrow H^k(W,\mathbb{Z})$$

is an isomorphism for $k < n-1$ and a monomorphism for $k = n-1$, and the pushforward map

$$H_k(W,\mathbb{Z}) \longrightarrow H_k(V,\mathbb{Z})$$

is an isomorphism for $k < n-1$ and an epimorphism for $k = n-1$. Applying this first to the ample divisor $\Theta \subset X$ and then to the ample divisor $Y \subset \Theta$, we in particular obtain that

$$H^n(Y,\mathbb{Z}) \;\cong\; H^n(X,\mathbb{Z}) \quad \text{and} \quad H_n(Y,\mathbb{Z}) \;\cong\; H_n(X,\mathbb{Z}) \quad \text{for} \quad n < 2.$$

In particular, the weak Lefschetz theorem for cohomology implies that the groups $H^n(Y,\mathbb{Z})$ have the claimed form for $n < 2$. Similarly, for $n > 2$ the corresponding statement follows from the weak Lefschetz theorem for homology because $H^n(Y,\mathbb{Z}) \cong H_{4-n}(Y,\mathbb{Z})$ by Poincaré duality. For any n we furthermore have an exact sequence

$$0 \longrightarrow \underbrace{Ext(H_{n-1}(Y,\mathbb{Z}),\mathbb{Z})}_{\text{torsion group}} \longrightarrow H^n(Y,\mathbb{Z}) \longrightarrow \underbrace{Hom(H_n(Y,\mathbb{Z}),\mathbb{Z})}_{\text{torsion-free}} \longrightarrow 0$$

by the universal coefficient theorem. Putting $n = 2$ and using that by the weak Lefschetz theorem the group $H_1(Y,\mathbb{Z}) \cong H_1(X,\mathbb{Z})$ is torsion-free so that $Ext(H_1(Y,\mathbb{Z}),\mathbb{Z}) = 0$, we obtain that

$$H^2(Y,\mathbb{Z}) \;\cong\; Hom(H_2(Y,\mathbb{Z}),\mathbb{Z})$$

is a free abelian group. The rank of this group is equal to $h^2(Y) = 86$ by lemma 5.8. Altogether it then follows that the integral cohomology groups of Y are those given in the table.

It remains to show that $q^* : H^2(Y^+,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$ is an epimorphism onto the σ-invariants whose kernel is the torsion subgroup, and to determine the groups $H^n(Y^+,\mathbb{Z})$. In the computation of these cohomology groups we can in fact restrict our attention to degrees $n \leq 2$. Indeed, Poincaré duality says that

$$H_n(Y^+,\mathbb{Z}) \;\cong\; H^{4-n}(Y^+,\mathbb{Z})$$

for all n, hence by the universal coefficient theorem we have (non-canonical) isomorphisms

$$\begin{aligned} H^n(Y^+,\mathbb{Z}) \;&\cong\; H_{n-1}(Y^+,\mathbb{Z})_{tor} \oplus H_n(Y^+,\mathbb{Z})_{free} \\ &\cong\; H^{5-n}(Y^+,\mathbb{Z})_{tor} \oplus H^{4-n}(Y^+,\mathbb{Z})_{free} \end{aligned}$$

where the subscripts *tor* and *free* refer to the maximal torsion subgroup and the maximal free abelian subgroup, respectively. So for the computation of the groups $H^n(Y^+,\mathbb{Z})$ it suffices to deal with degrees $n \leq 2$. This being said, we can argue as follows.

Recall from topology [**18**, ex. VII.4(b)] that for every regular covering map $W \longrightarrow W^+ = W/\Gamma$ between connected, locally pathwise connected topological spaces with Galois group $\Gamma = Aut(W/W^+)$, one has a spectral sequence

$$E_2^{pq} \;=\; H^p(\Gamma, H^q(W,\mathbb{Z})) \;\Longrightarrow\; H^{p+q}(W^+,\mathbb{Z})$$

where the left hand side denotes group cohomology with coefficients in the Γ-module $H^q(W,\mathbb{Z})$. We apply this for $W = Y$ and $\Gamma = \langle\sigma\rangle$. To compute in this case some of the E_2-terms, recall that the group cohomology of any module M under the cyclic group $\langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$ is given by

$$H^p(\langle\sigma\rangle, M) \;=\; \begin{cases} \ker(\sigma - 1) & \text{for } p = 0, \\ \ker(\sigma + 1)/\mathrm{im}(\sigma - 1) & \text{for } p > 0 \text{ odd}, \\ \ker(\sigma - 1)/\mathrm{im}(\sigma + 1) & \text{for } p > 0 \text{ even}. \end{cases}$$

Let us plug in $M = H^q(Y,\mathbb{Z})$ with $q \in \{0,1,2\}$. Clearly on $H^0(Y,\mathbb{Z}) \cong \mathbb{Z}$ the involution σ acts trivially, whereas by the weak Lefschetz theorem we know that on $H^1(Y,\mathbb{Z}) \cong H^1(X,\mathbb{Z}) \cong \mathbb{Z}^8$ the involution σ acts by -1. Hence in this case we have

$$\frac{\ker(\sigma + 1 \mid H^q(Y,\mathbb{Z}))}{\mathrm{im}(\sigma - 1 \mid H^q(Y,\mathbb{Z}))} \;\cong\; \begin{cases} 0 & \text{for } q = 0, \\ (\mathbb{Z}/2\mathbb{Z})^8 & \text{for } q = 1, \end{cases}$$

$$\frac{\ker(\sigma - 1 \mid H^q(Y,\mathbb{Z}))}{\mathrm{im}(\sigma + 1 \mid H^q(Y,\mathbb{Z}))} \;\cong\; \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } q = 0, \\ 0 & \text{for } q = 1. \end{cases}$$

In other words, in the case at hand the E_2-tableau of the spectral sequence takes the following form.

$$\begin{tikzcd}
q & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \\
2 & H^2(Y,\mathbb{Z})^+ \arrow[drr] & * \arrow[drr] & * & * & \cdots \\
1 & 0 \arrow[drr] & (\mathbb{Z}/2\mathbb{Z})^8 \arrow[drr] & 0 & (\mathbb{Z}/2\mathbb{Z})^8 & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \cdots \\
 & 0 & 1 & 2 & 3 & p
\end{tikzcd}$$

In particular it follows that $H^0(Y^+,\mathbb{Z}) = \mathbb{Z}$ and $H^1(Y^+,\mathbb{Z}) = 0$. The zeroes for $(p,q) = (2,1),(3,0)$ furthermore show that for the graded object with respect to the limit filtration we have

$$Gr_i(H^2(Y^+,\mathbb{Z})) \;=\; \begin{cases} H^2(Y,\mathbb{Z})^+ & \text{for } i = 2, \\ (\mathbb{Z}/2\mathbb{Z})^8 & \text{for } i = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } i = 0, \end{cases}$$

and the top quotient defines a surjection

$$H^2(Y^+,\mathbb{Z}) \;=\; E_\infty^2 \;\twoheadrightarrow\; E_\infty^{02} \;=\; E_2^{02} \;=\; H^2(Y,\mathbb{Z})^+.$$

By construction of the spectral sequence this surjection coincides with the map q^* induced by the quotient morphism $q: Y \longrightarrow Y^+$.

The spectral sequence also shows that the kernel K of this surjection is an extension of $(\mathbb{Z}/2\mathbb{Z})^8$ by $\mathbb{Z}/2\mathbb{Z}$. In fact $K = (\mathbb{Z}/2\mathbb{Z})^9$ because K must be a 2-torsion group: If we denote by

$$q_!: \quad H^2(Y,\mathbb{Z}) \;\longrightarrow\; H^2(Y^+,\mathbb{Z})$$

the Gysin map (i.e. the Poincaré dual to the pushforward on homology), then $q_!q^*$ is multiplication by the degree $\deg(q) = 2$, hence for any $\alpha \in K$ it follows that $2\alpha = q_!q^*(\alpha) = q_!(0) = 0$ as claimed.

To conclude the proof, note that $F = H^2(Y^+,\mathbb{Z})/K$ is a subgroup of the free abelian group $H^2(Y,\mathbb{Z})$. As such the group F is torsion-free and therefore equal to the free part of $H^2(Y^+,\mathbb{Z})$, whose rank is $h^2(Y^+) = 34$ by the computation of Hodge numbers in lemma 5.8. $\square$

5.4. Néron-Severi lattices

Recall that in the setting of the previous section we have embeddings of free abelian groups

$$H^2(X,\mathbb{Z}) \;\subset\; H^2(Y,\mathbb{Z})^+ \;\subset\; H^2(Y,\mathbb{Z}).$$

In what follows we equip these groups with the integral bilinear form that is given by the intersection form on Y. The goal of the present section is to determine the structure of the obtained integral lattices with respect to this bilinear form, for which we first briefly recall from [**22**] some basic notions from lattice theory that will be used throughout.

By a lattice we mean a pair consisting of a finitely generated free abelian group L and a non-degenerate symmetric bilinear form $b: L \times L \longrightarrow \mathbb{Z}$, though we will usually suppress the bilinear form in the notation. Such a lattice is called even if $b(\lambda,\lambda) \in 2\mathbb{Z}$ for all $\lambda \in L$. For integers $n \neq 0$ we denote by $L(n)$ the lattice with the same underlying free abelian group as L but with the bilinear form $n \cdot b$ in place of b. As usual, by an embedding of lattices we mean an embedding of free abelian groups which respects the bilinear forms, and similarly for isomorphisms of lattices. For example, the diagonal embedding

$$diag: \quad L(2) \;\hookrightarrow\; L^2 \;=\; L \oplus L$$

is given by $\lambda \mapsto (\lambda,\lambda)$. We also remark that any lattice can be defined with respect to a chosen basis $e_1,\ldots,e_n$ of the underlying free abelian group by the corresponding Gram matrix — the non-degenerate integral symmetric matrix $(b(e_i, e_k))_{i,k=1,\dots,n}$ of size $n \times n$ whose entries are the scalar products between the basis vectors. The discriminant of a lattice is defined as the determinant of its Gram matrix with respect to any chosen basis, and the lattice is called unimodular if its discriminant is ± 1. In what follows we consider the lattice

$$\Lambda \;=\; \mathbb{Z}^4 \quad \text{with Gram matrix} \quad \begin{pmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{pmatrix}$$

and the standard hyperbolic lattice $U = \mathbb{Z}^2$ with Gram matrix $\left(\begin{smallmatrix} 0&1 \\ 1&0 \end{smallmatrix}\right)$. Note that both Λ and U are even lattices and that U is unimodular whereas a direct calculation shows that Λ has discriminant -3. This being said, consider the even lattices

$$K \;=\; U^{12} \oplus \Lambda \quad \text{and} \quad L \;=\; U^{13} \oplus E_8(-1).$$

In terms of these two lattices, the main result of the present section can be formulated as follows.

PROPOSITION 5.12. *There exists an isomorphism $H^2(Y,\mathbb{Z}) \cong L^2 \oplus U^9$ of lattices which gives a commutative diagram*

$$\begin{tikzcd} H^2(X,\mathbb{Z}) \arrow[r, hook] \arrow[d, "\cong"] & H^2(Y,\mathbb{Z})^+ \arrow[r, hook] \arrow[d, "\cong"] & H^2(Y,\mathbb{Z}) \arrow[d, "\cong"] \\ K(2) \arrow[r, hook, "i(2)"] & L(2) \arrow[r, hook, "(diag{,}0)"] & L^2 \oplus U^9 \end{tikzcd}$$

for some lattice embedding $i : K \hookrightarrow L$ with orthocomplement $K^\perp \cong E_6(-1)$.

To complete the picture, we remark that with the above identifications it follows that on the lattice $L^2 \oplus U^9$ the involution $\sigma : Y \longrightarrow Y$ acts by the formula

$$\sigma(\lambda_1, \lambda_2, u) \;=\; (\lambda_2, \lambda_1, -u) \quad \text{for} \quad u \in U^9 \quad \text{and} \quad \lambda_1, \lambda_2 \in L.$$

Indeed, looking at the eigenspace decomposition over the rational numbers one sees that the involution σ must act by -1 on the orthocomplement of the σ-invariant part $H^2(Y,\mathbb{Z})^+$ inside $H^2(Y,\mathbb{Z})$.

The proof of proposition 5.12 will occupy the rest of this section and will also involve a study of the intersection form on $H^2(Y^+,\mathbb{Z})$. Note that since this latter cohomology group contains torsion, it is not a lattice in the above sense. So let us introduce the following notations. For any smooth complex projective surface S let

$$L_S \;=\; H^2(S,\mathbb{Z})/\mathit{Torsion}$$

be the quotient of the finitely generated abelian group $H^2(S,\mathbb{Z})$ by its torsion subgroup. Since the intersection form

$$b: \quad H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

takes its values in the torsion-free group $\mathbb{Z}$, it is clear that the intersection pairing between any torsion class and any other class vanishes. Hence the intersection form of the surface factors over a non-degenerate symmetric bilinear form $b: L_S \times L_S \longrightarrow \mathbb{Z}$, and we view L_S as a lattice with respect to this bilinear form. Note that by Poincaré duality this lattice is always unimodular. We also have the following well-known

LEMMA 5.13. *If for some integral class $\alpha \in H^2(S,\mathbb{Z})$ the first Chern class satisfies $c_1(S) \equiv 2\alpha$ modulo torsion, then L_S is an even lattice.*

Proof. By naturality, the reduction map $r: \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ gives rise to a commutative diagram

$$\begin{tikzcd}
H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "(r_*,r_*)"'] & H^4(S,\mathbb{Z}) \arrow[r, "\deg_S"] \arrow[d, "r_*"'] & \mathbb{Z} \arrow[d, "r"] \\
H^2(S,\mathbb{Z}/2) \times H^2(S,\mathbb{Z}/2) \arrow[r, "\cup"] & H^4(S,\mathbb{Z}/2) \arrow[r, "\deg_S"] & \mathbb{Z}/2
\end{tikzcd}$$

where the degree maps $\deg_S$ are defined by evaluation on the fundamental homology class of S. The composite of the upper horizontal arrows is the intersection pairing we are interested in. Now the image of the first Chern class $c_1(S)$ under the reduction map $r_*: H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z}/2\mathbb{Z})$ is the Stiefel-Whitney class $w_2(S)$ by [**77**, p. 171], and for the intersection form modulo two we have

$$\deg_S(w_2(S) \cup r_*(\beta)) = r(\deg_S(\beta^2)) \quad \text{for all} \quad \beta \in H^2(S,\mathbb{Z})$$

by the Wu formula [**45**, prop. 1.4.18]. Hence the claim follows. ☐

After this topological digression, let us now come back to the smooth surfaces Y and Y^+ defined by a general intersecion of translates of a smooth theta divisor on a complex ppav X of dimension $g = 4$. Let us first determine the abstract isomorphism type of the corresponding lattices.

LEMMA 5.14. *With notations as above,*

$$L_Y \cong L^2 \oplus U^9 \quad \textit{and} \quad L_{Y^+} \cong L.$$

Proof. It follows from corollary 5.9 that the intersection form on the lattices L_Y and L_{Y^+} is indefinite of signature $(35,51)$ resp. $(13,21)$. An even unimodular lattice is determined uniquely by its rank and signature [**96**], so it only remains to show that both L_Y and L_{Y^+} are even lattices. For this we can apply lemma 5.13. Indeed, the condition of the lemma holds for L_Y because

$$c_1(Y) \;=\; 2\alpha \quad \text{for the class} \quad \alpha \;=\; -[\Theta]|_Y \;\in\; H^2(Y,\mathbb{Z})$$

by lemma 5.5. Furthermore, by proposition 5.11 we can find $\alpha^+ \in L_{Y^+}$ such that $q^*(\alpha^+) = \alpha$ where $q: Y \longrightarrow Y^+$ is the quotient morphism. It then follows that

$$q^*(c_1(Y^+) - 2\alpha^+) \;=\; q^*(c_1(Y^+)) - 2q^*(\alpha^+) \;=\; c_1(Y) - 2\alpha \;=\; 0$$

so that by proposition 5.11 the class $c_1(Y^+) - 2\alpha^+$ is a torsion class. Hence by lemma 5.13 also L_{Y^+} is an even lattice. □

Now that we have found the abstract isomorphism type of the lattices L_Y and L_{Y^+}, the next thing to be done is to determine the relationship between these two lattices. For this we denote by

$$L_Y^+ \;=\; H^2(Y,\mathbb{Z})^+ \;\subset\; L_Y \;=\; H^2(Y,\mathbb{Z})$$

the invariants of L_Y under the action of the involution σ, i.e. the lattice which occurs in the middle part of the diagram in proposition 5.12. The relationship we are looking for is then given by

LEMMA 5.15. *The pull-back under the quotient morphism $q: Y \longrightarrow Y^+$ gives rise to a lattice isomorphism*

$$q^*: \; L_{Y^+}(2) \xrightarrow{\;\cong\;} L_Y^+ \;\subset\; L_Y\,.$$

Proof. By proposition 5.11 the pull-back q^* induces an isomorphism of the underlying abelian groups. Via this isomorphism, the intersection form on L_Y induces twice the intersection form on L_{Y^+} since $q: Y \to Y^+$ is an étale double cover: The degree maps for the top cohomology of our surfaces satisfy

$$\deg_Y(q^*(a) \cup q^*(b)) \;=\; \deg_Y(q^*(a \cup b)) \;=\; 2\deg_{Y^+}(a \cup b)$$

for all cohomology classes a, b in $H^2(Y^+,\mathbb{Z})$. □

From the above two lemmas we obtain the middle and the right vertical isomorphism in the diagram of proposition 5.12. These isomorphisms can be chosen in such a way that the square on the right hand side of the diagram commutes, indeed we have the following

LEMMA 5.16. *Let $L_1, L_2 \subset L_Y$ be sublattices of rank $r \leq 34$ which are primitive in the sense that the quotients L_Y/L_1 and L_Y/L_2 are torsion-free groups. Then any isomorphism $\varphi_0 : L_1 \longrightarrow L_2$ of lattices extends to a lattice automorphism $\varphi : L_Y \longrightarrow L_Y$.*

Proof. This is a direct application of the criterion for the extension of lattice isomorphisms in [**62**]. Let us briefly check that the two conditions in loc. cit. are satisfied in this case: The first condition is that the rank r of the two primitive sublattices and the signature $(s_+, s_-) = (35, 51)$ of the ambient lattice L_Y satisfy

$$2(r+1) \ \leq \ (s_+ + s_-) - |s_+ - s_-|,$$

which in the present situation is granted by our assumption that $r \leq 34$. The second condition concerns primitive vectors $v \in L_Y$ which are characteristic in the sense that $b(v,w) \equiv b(w,w) \bmod 2\mathbb{Z}$ for all vectors $w \in L_Y$. But in the case at hand this condition is void because an even unimodular lattice does not contain any primitive characteristic vectors. So by loc. cit. the given isomorphism φ_0 can be extended to an automorphism φ as claimed. □

To finish the proof of proposition 5.12 it remains to deal with the square on the left hand side in the diagram of that proposition. For this we need to compute the image of the weak Lefschetz embedding, by which we mean the sublattice

$$L_X \ = \ H^2(X, \mathbb{Z}) \ \subset \ L_Y^+ \ = \ H^2(Y, \mathbb{Z})^+.$$

Note that $K = U^{12} \oplus \Lambda$ has signature $(13, 15)$ and discriminant -3 whereas the even unimodular lattice $L = U^{13} \oplus E_8(-1)$ has signature $(13, 21)$. So the orthocomplement $K^\perp$ of any embedding

$$i: \ K \hookrightarrow L$$

must be a negative definite even lattice of rank six with discriminant 3, hence abstractly isomorphic to $E_6(-1)$ by the classification of lattices with small discriminant in [**21**]. Thus the proof of proposition 5.12 is completed by the following computation.

LEMMA 5.17. *With notations as above, there exists an isomorphism of lattices*

$$L_X \ \xrightarrow{\cong} \ K(2).$$

Proof. Consider the embedding $i: Y = \Theta \cap \Theta_x \hookrightarrow X$. By definition the Gysin morphism $i_!$ on cohomology is the Poincaré dual of the pushforward morphism i_* on homology, so we have a commutative diagram

$$\begin{tikzcd}
H^n(Y, \mathbb{Z}) \arrow[r, "i_!"] \arrow[d, "\cong"'] & H^{n+4}(X, \mathbb{Z}) \arrow[d, "\cong"] \\
H_{4-n}(Y, \mathbb{Z}) \arrow[r, "i_*"] & H_{4-n}(X, \mathbb{Z}).
\end{tikzcd}$$

If $\mathbf{1} \in H^0(Y,\mathbb{Z})$ denotes the unit element of the cohomology ring of Y, the fundamental cohomology class of Y in X is $i_!(\mathbf{1}) = [Y] \in H^4(X,\mathbb{Z})$. So the projection formula shows

$$i_!(i^*(\gamma)) \;=\; i_!(i^*(\gamma)\cup \mathbf{1}) \;=\; \gamma \cup i_!(\mathbf{1}) \;=\; \gamma \cup [Y]$$

for all cohomology classes $\gamma \in H^4(X,\mathbb{Z})$. This being said, it follows from the commutative diagram

$$\begin{tikzcd}
H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "i^*"] & H^4(X,\mathbb{Z}) \arrow[r, "-\cup[Y]"] \arrow[d, "i^*"] & H^8(X,\mathbb{Z}) \arrow[d, "\deg_X"] \\
H^2(Y,\mathbb{Z}) \times H^2(Y,\mathbb{Z}) \arrow[r, "\cup"] & H^4(Y,\mathbb{Z}) \arrow[r, "\deg_Y"] \arrow[ur, "i_!"] & \mathbb{Z}
\end{tikzcd}$$

that the intersection form on $H^2(Y,\mathbb{Z})$ restricts on the subspace $H^2(X,\mathbb{Z})$ to the bilinear form

$$b:\; H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z},\; (\alpha,\beta) \mapsto \deg_X(\alpha \cup \beta \cup [Y]).$$

We want to compute this bilinear form. Since the intersection product in homology corresponds to the cup product in cohomology, the fundamental class of Y in cohomology is

$$[Y] \;=\; [\Theta \cap \Theta_x] \;=\; [\Theta] \cup [\Theta_x] \;=\; [\Theta]^2.$$

To compute this class explicitly, we choose a basis of the cohomology ring of X as follows. The principal polarization on X is given by an alternating bilinear form on $H_1(X,\mathbb{Z})$. Take an integral basis

$$\lambda_1,\ldots,\lambda_4,\mu_1,\ldots,\mu_4 \;\in\; H_1(X,\mathbb{Z}) \;\cong\; \mathbb{Z}^8$$

with respect to which this alternating form is given by $(\lambda_i,\lambda_k) = (\mu_i,\mu_k) = 0$ and $(\lambda_i,\mu_k) = -(\mu_k,\lambda_i) = \delta_{ik}$ for all i, k, and denote by

$$x_1,\ldots,x_4,y_1,\ldots,y_4 \;\in\; H^1(X,\mathbb{Z}) \;=\; Hom(H_1(X,\mathbb{Z}),\mathbb{Z})$$

the dual basis. Since the Chern class $[\Theta] = c_1(\mathscr{O}_X(\Theta))$ defines the principal polarization, it follows that in the exterior algebra $H^\bullet(X,\mathbb{Z}) = \Lambda^\bullet(H^1(X,\mathbb{Z}))$ this class is given by

$$[\Theta] \;=\; \sum_{i=1}^{4} x_i \wedge y_i \;\in\; H^2(X,\mathbb{Z}).$$

The cup product in cohomology corresponds to the wedge product in the exterior algebra, so we get

$$[Y] \;=\; [\Theta]^2 \;=\; \Big(\sum_{i=1}^{4} x_i \wedge y_i\Big) \wedge \Big(\sum_{j=1}^{4} x_j \wedge y_j\Big) \;=\; 2 \cdot \sum_{1 \le i < j \le 4} x_i \wedge y_i \wedge x_j \wedge y_j.$$

In particular, it follows that $v \wedge [Y] = 0$ for all vectors $v \in H^4(X,\mathbb{Z})$ of the form $v = x_k \wedge x_l \wedge x_m \wedge x_n$ or $v = x_k \wedge x_l \wedge x_m \wedge y_n$ or $v = x_k \wedge y_l \wedge y_m \wedge y_n$ or $v = y_k \wedge y_l \wedge y_m \wedge y_n$ with arbitrary indices k,l,m,n, and also for all vectors of the form $v = x_k \wedge x_l \wedge y_m \wedge y_n$ with $\{k,l\} \neq \{m,n\}$. However, for $k \neq l$ one easily checks that

$$x_k \wedge x_l \wedge y_k \wedge y_l \wedge [Y] \;=\; -2 \cdot \omega$$

for the cohomology class

$$\omega \;=\; x_1 \wedge y_1 \wedge \cdots \wedge x_4 \wedge y_4 \;=\; \tfrac{1}{4!} \cdot [\Theta]^4$$

mapped to 1 under the degree isomorphism $\deg_X : H^8(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$.

Now consider the basis of the lattice $L_X = \Lambda^2(H^1(X,\mathbb{Z}))$ consisting of the vectors $u_{ik} = x_i \wedge y_k$ with $i,k \in \{1,2,3,4\}$ and of the vectors $v_{ik} = x_i \wedge x_k$ and $w_{ik} = y_i \wedge y_k$ with $1 \leq i < k \leq 4$. It follows from the above that the bilinear form b is non-zero only on those pairs of the above vectors whose wedge product has the form $\pm x_l \wedge x_m \wedge y_l \wedge y_m$ with $l \neq m$. So the only non-zero scalar products between our basis vectors are

$$\begin{aligned}
&b(u_{ii},u_{kk}) = x_i \wedge y_i \wedge x_k \wedge y_k \wedge [Y] = 2 \text{ for } i \neq k,\\
&b(u_{ik},u_{ki}) = x_i \wedge y_k \wedge x_k \wedge y_i \wedge [Y] = -2 \text{ for } i \neq k, \text{ and}\\
&b(v_{ik},w_{ik}) = b(w_{ik},v_{ik}) = y_i \wedge y_k \wedge x_i \wedge x_k \wedge [Y] = -2 \text{ for } i < k.
\end{aligned}$$

Hence we obtain an orthogonal sum decomposition

$$L_X \;=\; \langle u_{11},\dots,u_{44}\rangle \;\oplus\; \bigoplus_{1 \leq i < k \leq 4} \langle u_{ik},u_{ki}\rangle \oplus \langle v_{ik},w_{ik}\rangle$$

where $\langle u_{11},\dots,u_{44}\rangle \cong \Lambda(2)$ and where $\langle u_{ik},u_{ki}\rangle \cong \langle v_{ik},w_{ik}\rangle \cong U(2)$. □

5.5. The upper bound $W(E_6)$

Let us now come back to the local system which underlies the variation of Hodge structures $\mathbb{V}_+$ on the open dense subset $U \subset X$ in lemma 5.10 for $g = 4$, assuming that the theta divisor of our ppav is smooth. Since all constructions in the proof of that lemma make sense on the level of integral cohomology, we can in fact consider $\mathbb{V}_+$ as a polarized variation of $\mathbb{Z}$-Hodge structures. Proposition 5.12 then allows to identify the fibres of this variation of Hodge structures with the lattice $K^\perp \cong E_6(-1)$ up to a rescaling of the bilinear form by a factor two.

This being said, consider the monodromy group G of the local system underlying $\mathbb{V}_+$. Since the monodromy action for any smooth proper family of varieties preserves the intersection form on the cohomology of the fibres, it follows from the above that G must be contained in the automorphism group $Aut(E_6)$ of the E_6-lattice. Recall from [**22**, p. 126] that we have a product decomposition

$$Aut(E_6) \;=\; W(E_6) \times \{\pm 1\}$$

where $\{\pm 1\}$ acts via multiplication by ± 1 on the E_6-lattice. It turns out that the monodromy group G is already contained in the Weyl group itself.

PROPOSITION 5.18. *With notations as above, the monodromy group G is contained in the subgroup*

$$W(E_6) \;<\; Aut(E_6).$$

Proof. Let K and L be as in proposition 5.12, and for some $x \in U(\mathbb{C})$ consider the monodromy operation of the group $\pi_1(U,x)$ on the σ-invariant lattice $H^2(Y,\mathbb{Z})^+ \cong L(2)$. Notice that on the sublattice $H^2(X,\mathbb{Z}) \cong K(2)$ this monodromy operation is trivial. Dividing the bilinear forms by two, we obtain from proposition 5.12 an embedding

$$M \;=\; K \oplus E_6(-1) \;\hookrightarrow\; L \;=\; U^{13} \oplus E_8(-1)$$

with an action of the fundamental group $\pi_1(U,x)$ on L which is trivial on the sublattice K and which preserves the orthocomplement $E_6(-1) = K^\perp$ on which it defines the quotient morphism

$$\pi_1(U,x) \;\twoheadrightarrow\; G \;\hookrightarrow\; Aut(E_6)$$

we are interested in. We must show that the image G is already contained in the subgroup $W(E_6) < Aut(E_6)$. To this end we will consider the induced action of G on the discriminant group of the E_6-lattice.

Since the lattices L and M have the same rank, L is obtained from M by adjoining certain glue vectors, by which we mean as in [**22**] vectors from the dual lattice

$$M^* \;=\; \big\{ m \in M \otimes_{\mathbb{Z}} \mathbb{Q} \mid b(m,n) \in \mathbb{Z} \;\; \forall n \in M \big\} \;=\; K^* \oplus E_6(-1)^*$$

where b denotes the bilinear form of the lattice M, extended to $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Now the lattice $E_6(-1)$ has discriminant 3 whereas L is unimodular, so there exists in the sublattice $L \subseteq M^* = K^* \oplus E_6(-1)^*$ at least one glue vector of the form

$$\lambda \;=\; k + e \quad \text{with} \quad k \in K^*,\, e \in E_6(-1)^* \quad \text{but} \quad e \notin E_6(-1).$$

Fixing k as above, consider the set $S = \{ f \in E_6(-1)^* \mid k + f \in L \}$. This set is stable under the monodromy operation, indeed this operation fixes k because $H^2(X,\mathbb{Z}) \cong K(2)$ is contained in the monodromy invariant part of the cohomology. By construction S contains the coset $e + E_6(-1)$. On the other hand, for any $f \in S$ we have

$$f - e \;=\; (k+f) - (k+e) \;\in\; E_6(-1)^* \cap L \;=\; E_6(-1)$$

where the last equality uses that $E_6(-1) = K^{\perp}$ is primitive in L, being the orthocomplement of a sublattice. Altogether then

$$S \;=\; e + E_6(-1)$$

is a non-trivial coset of $E_6(-1)$ in $E_6(-1)^*$ which is preserved by G. Since the discriminant group

$$E_6(-1)^*/E_6(-1) \;\cong\; E_6^*/E_6 \;\cong\; \mathbb{Z}/3\mathbb{Z}$$

is generated by any non-trivial coset, it follows that G acts trivially on this discriminant group. Now recall from [**23**, p. 27] that as a subgroup of $Aut(E_6) = W(E_6) \times \{\pm 1\}$ we have

$$W(E_6) \;=\; \ker(Aut(E_6) \longrightarrow Aut(E_6^*/E_6)),$$

so the fact that the monodromy group G act trivially on the discriminant group implies that $G \leq W(E_6)$ as required. □

At this point let us briefly explain the connection of the above result with the 27 Prym-embedded curves that have been studied by E. Izadi in [**61**], although this will not be used in the sequel. For the basic definitions and facts about Prym varieties we refer to the brief survey in section 5.1 and to the references given there. Suppose that X is a general complex ppav of dimension $g = 4$. Then by [**8**, prop. 6.4] it can be written as the Prym variety

$$X \;\cong\; Prym(\tilde{C}/C)$$

for some étale double cover of smooth projective curves of genus 9 and 5, respectively. Consider then the covering involution $\iota : \tilde{C} \to \tilde{C}$, and choose any translate $\alpha : \tilde{C} \hookrightarrow J\tilde{C}$ of the Abel-Jacobi map. Then the morphism

$$\tilde{C} \;\longrightarrow\; J\tilde{C}, \quad p \;\mapsto\; \alpha(p) - \alpha(\iota(p))$$

factors over the kernel of the norm homomorphism $N_{\tilde{C}/C} : J\tilde{C} \to JC$, hence the image of some translate of this morphism is contained in the connected component

$$Prym(\tilde{C}/C) \;=\; (\ker(N_{\tilde{C}/C}))^0$$

of the identity. Via the chosen isomorphism $X \cong Prym(\tilde{C}/C)$ this defines an embedding $\tilde{C} \hookrightarrow X$ which up to a translation is determined uniquely by the étale double cover. Embeddings of this form are known as Abel-Prym embeddings, and by a Prym-embedded curve in X we mean the image of any such embedding. Note that the étale double covering $\tilde{C} \to C$ is by no means uniquely determined by the ppav X, indeed a dimension count shows that the general fibre of the Prym morphism

$$\pi : \quad \mathscr{R}_5 \;\longrightarrow\; \mathscr{A}_4$$

has dimension two. In studying this Prym morphism, E. Izadi has observed in [**61**, cor. 4.9] that for general $x \in X(\mathbb{C})$, among all the Prym-embedded curves precisely 27 are contained in the surface $Y = Y_x = \Theta \cap \Theta_x$. This number suggests a relationship with the E_6-lattice in the cohomology of this surface. Recall from [**22**] that the dual lattice E_6^* has precisely 27 pairs of minimal vectors, with norm $4/3$. It turns out that the fundamental classes of Prym-embedded curves define such minimal vectors and are related to the glue vector $\lambda = k + e \in L$ with components $k \in K^*$ and $e \in E_6(-1)^*$ that we studied in the proof of proposition 5.18.

LEMMA 5.19. *The above glue vector $\lambda = k + e$ can be taken to be the fundamental class*

$$[\tilde{C}] \in H^2(Y,\mathbb{Z})^+ \cong L(2)$$

of any Prym-embedded curve $\tilde{C} \subset Y$, and in this case the vector $e \in E_6(-1)^*$ is a minimal vector with the norm $b(e,e) = -4/3$.

Proof. Any cohomology class $\lambda \in H^2(Y,\mathbb{Z})^+$ can be written uniquely in the form

$$\lambda \;=\; \alpha + \beta \quad \text{with} \quad \alpha \in H^2(X,\mathbb{Q}),\ \beta \in H^2(X,\mathbb{Q})^\perp \subset H^2(Y,\mathbb{Q})^+$$

and such a class defines a glue vector iff $\alpha \notin H^2(X,\mathbb{Z})$. The integrality of α can be checked via the Gysin morphism for the embedding $i : Y \hookrightarrow X$ since $i_!(\lambda) = \alpha \cup [\Theta]^2 \in H^6(X,\mathbb{Q})$ is the image of α under the Lefschetz isomorphism in the following commutative diagram.

$$\begin{tikzcd}
H^2(X,\mathbb{Q}) \arrow[rr, "\cong"] \arrow[dr, "i^*"'] & & H^6(X,\mathbb{Q}) \\
& H^2(Y,\mathbb{Q})^+ \arrow[ur, "i_!"'] & \\
H^2(X,\mathbb{Z}) \arrow[uu, hook] \arrow[rr] \arrow[dr] & & H^6(X,\mathbb{Z}) \arrow[uu, hook] \\
& H^2(Y,\mathbb{Z})^+ \arrow[uu, hook] \arrow[ur] &
\end{tikzcd}$$

Now by [**13**, sect. 12.2] the fundamental class $\lambda = \alpha + \beta \in H^2(Y,\mathbb{Q})^+$ of a Prym-embedded curve satisfies

$$\alpha \cup [\Theta]^2 \;=\; i_!(\lambda) \;=\; [\Theta]^3/3 \;\notin\; H^6(X,\mathbb{Z})$$

and hence indeed provides a glue vector for our lattices. It also follows from the above that

$$\deg_Y(\alpha^2) \;=\; \deg_X(\alpha^2 \cup [\Theta]^2) \;=\; \deg_X([\Theta]^4/9) \;=\; 8/3$$

where the last equality is due to the Poincaré formula. Furthermore, by definition $\lambda \in H^2(Y,\mathbb{Q})$ is the class of a Prym-embedded curve $\tilde{C} \subset Y$ and any such curve has genus $g_{\tilde{C}} = 2g+1 = 9$. Hence the adjunction formula for the genus of a smooth curve on a smooth projective surface [**55**, prop. V.1.5] gives the intersection number

$$\deg_Y(\lambda^2) \;=\; 2g_{\tilde{C}} - 2 - \deg_Y(K_Y|_{\tilde{C}}) \;=\; 16 - 16 \;=\; 0,$$

indeed for the canonical class K_Y we have

$$\deg_{\tilde{C}}(K_Y|_{\tilde{C}}) \;=\; \deg_Y(\lambda \cup K_Y) \;=\; \deg_X(2\cdot[\Theta]^4/3) \;=\; 16$$

by the formula $i_!(\lambda) = [\Theta]^3/3$ from above and by lemma 5.5. We then also obtain the intersection number $\deg_Y(\beta^2) = \deg_Y(\lambda^2) - \deg_Y(\alpha^2) = -8/3$, and to see that the vector e is a minimal vector all that remains to be done is to divide the bilinear form by two. $\square$

It seems a tempting idea to prove theorem 5.1 by looking directly at the monodromy operation on the fundamental classes of the Prym-embedded curves from above, but we have not seen how to do this. In what follows we give a different proof which does not use the Prym construction, and from this point of view the theorem can be seen as an independent result about the monodromy operation on the 27 Prym-embedded curves.

5.6. Negligible constituents

We now justify our earlier statement that for a general ppav the local systems underlying the variations of Hodge structures $\mathbb{V}_+$ and $\mathbb{V}_-$ from lemma 5.10 are irreducible. In fact we will obtain stronger results since we will work with perverse sheaves on the whole ppav rather than with local systems on a Zariski-open dense subset. This shift in perspective reveals a close connection with the Tannakian formalism of convolution as developed in chapters 2 and 3, which is of interest in its own right.

Recall that for any complex abelian variety X, the convolution product on the derived category $D^b_c(X,\mathbb{C})$ of bounded complexes of $\mathbb{C}$-sheaves with constructible cohomology sheaves in the sense of [**10**] is defined by

$$K * L \;=\; Ra_*(K \boxtimes L) \;\in\; D^b_c(X,\mathbb{C}) \quad \text{for} \quad K, L \in D^b_c(X,\mathbb{C}),$$

where $a: X\times X \longrightarrow X$ denotes the group law. One of the main results of chapter 2 was that the full abelian subcategory $\mathrm{Perv}(X,\mathbb{C}) \subset D^b_c(X,\mathbb{C})$ of perverse sheaves is almost stable under the convolution product: Let us say that a perverse sheaf is negligible if all its perverse subquotients have Euler characteristic zero; then theorem 2.13 shows that for all $P, Q \in \mathrm{Perv}(X,\mathbb{C})$ and all $n \neq 0$ the perverse cohomology sheaves

$${}^pH^n(X, P*Q) \;\in\; \mathrm{Perv}(X,\mathbb{C})$$

are negligible. At the other end of the scale, we will say that a perverse sheaf is clean if it does not have any negligible subquotients.

Suppose now that X has dimension $g \geq 2$ and is principally polarized with a symmetric theta divisor $\Theta \subset X$, and consider the perverse intersection cohomology sheaf

$$\delta_\Theta \;=\; \mathrm{IC}_\Theta[g-1] \;\in\; \mathrm{Perv}(X,\mathbb{C}).$$

This is a semisimple perverse sheaf and arises via intermediate extension from the constant sheaf on the smooth locus of the theta divisor. Thus we have $\delta_\Theta = \mathbb{C}_\Theta[g-1]$ if the theta divisor is smooth, but later on we will also deal with isolated singularities.

LEMMA 5.20. *If the theta divisor Θ has at most isolated singularities, then for every clean perverse sheaf P and all $n \in \mathbb{Z}$, any simple negligible subquotient of ${}^pH^n(\delta_\Theta * P)$ is isomorphic to $\delta_X = \mathbb{C}_X[g]$.*

Proof. By Verdier duality it suffices to show this for $n \leq 0$. Since the theta divisor is a local complete intersection, by [**65**, lemma III.6.5] the shifted constant sheaf $\lambda_\Theta = \mathbb{C}_\Theta[g-1]$ is perverse, so we have an exact sequence of perverse sheaves

$$0 \longrightarrow \kappa \longrightarrow \lambda_\Theta \longrightarrow \delta_\Theta \longrightarrow 0$$

where κ is a skyscraper sheaf supported on the finite set of singular points of the theta divisor. Convolution with the clean perverse sheaf P yields a distinguished triangle

$$\kappa * P \longrightarrow \lambda_\Theta * P \longrightarrow \delta_\Theta * P \xrightarrow{[+1]} \cdots$$

where $\kappa * P$ is again a clean perverse sheaf. By the corresponding long exact sequence of perverse cohomology groups, it will therefore suffice to show that every simple negligible subquotient of ${}^pH^n(\lambda_\Theta * P)$ for $n \leq 0$ must be isomorphic to δ_X. But this follows as in [**110**, ex. 3] from the excision sequence for the ample divisor $\Theta \subset X$ with affine complement $X \setminus \Theta$, using Artin's vanishing theorem. $\square$

Let us now consider the convolution square $\delta_\Theta * \delta_\Theta$. Due to Gabber's decomposition theorem this is a direct sum of degree shifts of semisimple perverse sheaves. So we can write

$$\delta_\Theta * \delta_\Theta \;=\; S^2(\delta_\Theta) \oplus \Lambda^2(\delta_\Theta)$$

where the symmetric square and the alternating square on the right hand side are by definition the maximal direct summands on which the commutativity constraint

$$S \;=\; S_{\delta_\Theta,\delta_\Theta}: \quad \delta_\Theta * \delta_\Theta \;\longrightarrow\; \delta_\Theta * \delta_\Theta$$

from section 2.1 acts by $+1$ resp. by -1. We want to relate the above sheaf complexes to the variations of Hodge structures $\mathbb{V}_\pm$ in lemma 5.10. In this context, the contribution from the weak Lefschetz theorem will be described by the complexes

$$\mathbb{L}_{\pm} \;=\; \bigoplus_{\mu\in I_{\pm}} H^{g-2-|\mu|}(X,\mathbb{C})\otimes_{\mathbb{C}}\delta_X[\mu] \quad \text{with} \quad \begin{cases} I_+ = 1+2\mathbb{Z}, \\ I_- = 2\mathbb{Z}, \end{cases}$$

where by convention $H^i(X,\mathbb{C}) = 0$ for $i<0$. Strictly speaking certain Tate twists should be inserted in the above definition so that $\mathbb{L}_{\pm}$ become pure complexes of weight $2g-2$ like $\delta_\Theta * \delta_\Theta$, but to keep the notations simple we will in what follows suppress all occuring Tate twists.

LEMMA 5.21. *If Θ has at most isolated singularities, then there are semisimple perverse sheaves $\delta_{\pm}\in \mathrm{Perv}(X,\mathbb{C})$ such that*

$$S^2(\delta_\Theta) \;\cong\; \mathbb{L}_+\oplus\delta_+ \quad \textit{and} \quad \Lambda^2(\delta_\Theta) \;\cong\; \mathbb{L}_-\oplus\delta_-,$$

and over the Zariski-open dense subset $U\subset X$ from lemma 5.10 we have an isomorphism

$$\delta_{\pm}|_U \;\cong\; \mathbb{V}_{\pm\varepsilon}[g] \quad \textit{with a sign twist by} \quad \varepsilon = (-1)^{g-1}.$$

Proof. Consider the smooth proper family $f_U : Y_U \longrightarrow U$ that we defined in lemma 5.3. By the construction in the proof of that lemma, we have a cartesian diagram

$$\begin{tikzcd} Y_U = f^{-1}(U) \arrow[r, hook] \arrow[d, "f_U"'] & \Theta^{sm}\times\Theta^{sm} \arrow[r, hook] & \Theta\times\Theta \arrow[d, "f"] \\ U \arrow[rr, hook] & & X \end{tikzcd}$$

where $\Theta^{sm}\subseteq\Theta$ is the smooth locus of the theta divisor and where f is induced by the group law. From the definition of convolution we therefore obtain natural identifications

$$(\delta_\Theta*\delta_\Theta)|_U \;\cong\; (\mathbb{C}_\Theta[g-1]*\mathbb{C}_\Theta[g-1])|_U \;\cong\; Rf_{U*}(\mathbb{C}_{Y_U}[2g-2]).$$

By Gabber's decomposition theorem this is a direct sum of degree shifts of semisimple perverse sheaves, hence

$$(\delta_\Theta*\delta_\Theta)|_U \;\cong\; \bigoplus_{n=0}^{2g-4} \mathbb{H}^n[2g-2-n]$$

for the local systems $\mathbb{H}^n = R^n f_{U*}(\mathbb{C}_{Y_U})$ with stalks $\mathbb{H}^n_x = H^n(Y_x,\mathbb{C})$. In the proof of lemma 5.10 we have observed that these local systems contain constant subsheaves whose stalks are the subspaces which are the images of the restriction morphisms $H^n(X,\mathbb{C})\longrightarrow H^n(Y_x,\mathbb{C})$. This being said, we obtain that

$$(\delta_\Theta*\delta_\Theta)|_U \;=\; (\mathbb{L}_+\oplus\mathbb{L}_-)|_U\oplus\mathbb{V}[g]$$

where $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_-$ denotes the local system underlying the variations of Hodge structures in lemma 5.10. Via Gabber's decomposition theorem one then deduces that

$$\delta_\Theta * \delta_\Theta \;=\; \mathbb{L}_+ \oplus \mathbb{L}_- \oplus \delta \quad \text{for some} \quad \delta \in D^b_c(X, \mathbb{C}).$$

But we know from theorem 2.13 that for all $n \neq 0$ the perverse cohomology sheaves ${}^pH^n(\delta_\Theta * \delta_\Theta)$ are negligible and are hence by lemma 5.20 multiples of the constant perverse sheaf δ_X. Since over the open dense subset $U \subset X$ we have $\delta|_U = \mathbb{V}[g]$, it then follows that δ is a perverse sheaf.

Now consider the direct sum decomposition $\delta = \delta_+ \oplus \delta_-$ where $\delta_\pm \subset \delta$ are defined as the maximal perverse subsheaves on which the commutativity constraint $S : \delta_\Theta * \delta_\Theta \longrightarrow \delta_\Theta * \delta_\Theta$ acts by ± 1. Going back to the definitions, one checks that this commutativity constraint S induces on the stalks the involution $\varepsilon \cdot \sigma_x^* : H^n(Y_x, \mathbb{Q}) \longrightarrow H^n(Y_x, \mathbb{Q})$ where $\sigma_x : Y_x \longrightarrow Y_x$ denotes the involution that we defined in section 5.2 and where the twist by the extra sign $\varepsilon = (-1)^{g-1}$ comes from the degree shift in $\delta_\Theta = \mathrm{IC}_\Theta[g-1]$ because of the Koszul rule for the tensor product of complexes. □

Recall that a perverse sheaf is said to be clean if it does not admit any perverse subquotients which are negligible. For a smooth theta divisor we claim that $\mathbb{L}_\pm$ are the only negligible constituents in $\delta_\Theta * \delta_\Theta$.

LEMMA 5.22. *If the theta divisor $\Theta \subset X$ is smooth, then $\delta_\pm$ are clean.*

Proof. If this were not true, then by lemma 5.20 the semisimple perverse sheaf $\delta_+ \oplus \delta_-$ would contain δ_X as a summand. Then $H^{-g}(X, \delta_+ \oplus \delta_-) \neq 0$ and hence the inclusion

$$H^{-g}(X, \mathbb{L}_+ \oplus \mathbb{L}_-) \;\hookrightarrow\; H^{-g}(X, \delta_\Theta * \delta_\Theta)$$

would be strict. However, the Künneth formula in lemma 3.4 and the weak Lefschetz theorem for the smooth ample divisor $\Theta \subset X$ show that

$$\begin{aligned} H^{-g}(X, \delta_\Theta * \delta_\Theta) \;&=\; \bigoplus_{n=1}^{g-1} H^{-n}(X, \delta_\Theta) \otimes H^{n-g}(X, \delta_\Theta) \\ &=\; \bigoplus_{n=1}^{g-1} H^{g-1-n}(X, \mathbb{C}) \otimes H^{n-1}(X, \mathbb{C}) \end{aligned}$$

which by direct inspection is equal to $H^{-g}(X, \mathbb{L}_+ \oplus \mathbb{L}_-)$. □

Note that for a general complex ppav of dimension g the theta divisor is smooth by the work of Andreotti and Mayer [**4**], so the above lemma together with theorem 4.2 leads to the final result of this section.

PROPOSITION 5.23. *If X is a general complex ppav of dimension g, then the local systems underlying $\mathbb{V}_\pm$ are irreducible.*

Proof. For a general ppav we know from theorem 4.2 that the Tannaka group $G(\delta_\Theta)$ is either a symplectic or a special orthogonal group and that the perverse intersection cohomology sheaf δ_Θ corresponds to the standard representation of this group. Now the symmetric and alternating square of the standard representation are irreducible up to a trivial one-dimensional representation, and this trivial representation corresponds to the skyscraper sheaf δ_0 of rank one supported in the origin. Furthermore, by corollary 2.9 each representation of the Tannaka group corresponds to a unique clean perverse sheaf. Hence from lemma 5.22 we obtain that there exist clean and simple perverse sheaves $\gamma_\pm \in \mathrm{Perv}(X,\mathbb{C})$ such that

$$\delta_+ \;=\; \begin{cases} \gamma_+ \oplus \delta_0 \\ \gamma_+ \end{cases} \quad \text{and} \quad \delta_- \;=\; \begin{cases} \gamma_- \\ \gamma_- \oplus \delta_0 \end{cases} \quad \text{if } g \text{ is} \quad \begin{cases} \text{odd}, \\ \text{even}, \end{cases}$$

and by lemma 5.21 the simplicity of $\gamma_\pm$ in particular implies that the local systems $\mathbb{V}_\pm$ are irreducible. ☐

5.7. Jacobian varieties

Let us now see what happens if $X = JC$ is the Jacobian variety of a smooth complex projective curve C of genus $g \geq 2$. To formulate our results in this case we first need to introduce some more notations. For $n \in \mathbb{N}$ we will denote by

$$C_n = (C \times \cdots \times C)/\mathfrak{S}_n$$

the n-fold symmetric product of the curve and by $W_n \subset X$ the image of C_n under a suitable translate of the Abel-Jacobi map, normalized such that the theta divisor $\Theta = W_{g-1}$ becomes symmetric. We write $f_n : C_n \longrightarrow W_n$ for the Abel-Jacobi map and $g_n : W_n \times W_n \longrightarrow X$ for the difference morphism with $g_n(x,y) = y - x$. Then the composite morphism

$$\begin{tikzcd} C_n \times C_n \arrow[r, "(f_n{,}f_n)"] \arrow[rr, bend right, "d_n"'] & W_n \times W_n \arrow[r, "g_n"] & X \cong Pic^0(C) \end{tikzcd}$$

does not depend on the chosen normalization of the Abel-Jacobi map, and it sends a pair $(D,E) \in C_n \times C_n$ of effective divisors of degree n on the curve to the isomorphism class of the line bundle $\mathscr{O}_C(D-E)$.

In what follows we will be especially interested in non-hyperelliptic curves of genus $g = 4$. For these we know from lemma 4.5 that the theta divisor has only isolated singularities, so we can consider the variations of Hodge structures $\mathbb{V}_\pm$ from lemma 5.10 and the perverse sheaves $\delta_\pm$ from lemma 5.21. Here we have the following situation.

PROPOSITION 5.24. *Let $X = JC$ be the Jacobian variety of a smooth non-hyperelliptic complex projective curve C of genus $g = 4$.*

(a) The above semisimple perverse sheaves δ_+ and δ_- each contain the negligible constituent δ_X precisely once.

(b) After shrinking the open dense subset $U \subset X$ in lemma 5.10 there exists an isomorphism

$$\mathbb{V}_+ \cong d_{2*}(\mathbb{C}_{C_2 \times C_2})|_U.$$

Proof. (a) We must check for the lowest hypercohomology degree that $\dim(H^{-g}(X, \delta_+)) = \dim(H^{-g}(X, \delta_-)) = 1$. For this we can use the same computation as in lemma 5.22, the only difference is that for a Jacobian variety the hypercohomology $H^\bullet(X, \delta_\Theta)$ is larger than the one for a general ppav. Indeed it has been shown in [**106**, cor. 13(iii)] that if the curve C is not hyperelliptic, then

$$H^i(X, \delta_\Theta) = \begin{cases} H^{i+3}(X, \mathbb{C}) & \text{for } i \in \{-3, -2\}, \\ H^{i+3}(X, \mathbb{C}) \oplus H^{i+1}(X, \mathbb{C}) & \text{for } i \in \{-1, 0\}. \end{cases}$$

Compared with the computation in lemma 5.22, for $i = -1$ the extra direct summand $H^0(X, \mathbb{C}) \subset H^{-1}(X, \delta_\Theta)$ gives a two-dimensional extra term

$$H^0(X, \mathbb{C}) \otimes H^{-3}(X, \delta_\Theta) \oplus H^{-3}(X, \delta_\Theta) \otimes H^0(X, \mathbb{C}) \subset H^{-4}(X, \delta_\Theta * \delta_\Theta).$$

This extra term splits into two one-dimensional contributions, one in the symmetric and one in the alternating tensor square. Hence

$$H^{-4}(X, S^2(\delta_\Theta)) = H^{-4}(X, \mathbb{L}_+) \oplus \mathbb{C},$$
$$H^{-4}(X, \Lambda^2(\delta_\Theta)) = H^{-4}(X, \mathbb{L}_-) \oplus \mathbb{C},$$

and our claim follows because $S^2(\delta_\Theta) = \mathbb{L}_+ \oplus \delta_+$ and $\Lambda^2(\delta_\Theta) = \mathbb{L}_- \oplus \delta_-$.

(b) This is most conveniently checked in the setting of Brill-Noether sheaves and will be done after the proof of lemma 5.25 below. $\square$

Before we proceed further, let us recall from [**106**] some facts about Brill-Noether sheaves. Here we are mainly interested in their Tannakian description — more about their geometric construction can be found in the brief survey in section 6.1 below. Let $X = JC$ be the Jacobian variety of a smooth non-hyperelliptic complex projective curve C of genus $g \geq 2$, and let $C \hookrightarrow X$ be a translate of the Abel-Jacobi map such that the corresponding theta divisor $\Theta = W_{g-1}$ is symmetric. Then by loc. cit. the convolutions of the perverse intersection cohomology sheaf $\delta_C \in \mathrm{Perv}(X, \mathbb{C})$ are described via corollary 2.14 by the Tannaka group

$$G(\delta_C) = Sl_{2g-2}(\mathbb{C}).$$

For partitions $\alpha = (\alpha_1, \alpha_2, \dots)$ such that $2g-2 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq 0$, one can define the Brill-Noether sheaf

$$ {}^p\delta_\alpha \, \in \, \mathrm{Perv}(X, \mathbb{C}) $$

as the simple perverse sheaf corresponding to the irreducible representation of $Sl_{2g-2}(\mathbb{C})$ whose highest weight is given by the conjugate partition α^t in the basis of the weights defined by the diagonal entries of matrices. For instance, the singleton partitions $\alpha = (\alpha_1)$ correspond to the fundamental representations, and for these it has been shown in loc. cit. that

$$ {}^p\delta_i \cong \delta_{W_i} \quad \text{and} \quad {}^p\delta_{2g-2-i} \, \cong \, \delta_{-W_i} \quad \text{for} \quad 0 \leq i \leq g-1, $$

where the second equality is due to the Riemann-Roch theorem. Now the Tannakian formalism provides a means to decompose arbitrary convolution products between these perverse intersection cohomology sheaves via the combinatorial Littlewood-Richardson rule for the decomposition of tensor products of irreducible representations of $Sl_{2g-2}(\mathbb{C})$, see [**46**, sect. 9.3.5] and [**106**, sect. 5.4]. In the special case of fundamental representations this gives the formula

$$ {}^p\delta_m * {}^p\delta_n \;\cong\; \tau_{m,n} \oplus \bigoplus_{i=0}^{n} {}^p\delta_{(m+i,n-i)} \quad \text{for} \quad m \geq n, $$

where $\tau_{m,n} \in D^b_c(X, \mathbb{C})$ is a negligible sheaf complex in the sense that all its perverse cohomology sheaves have Euler characteristic zero. In fact it has been shown in lemma 27 of loc. cit. that $\tau_{m,n}$ (and more generally any negligible direct summand of a convolution of Brill-Noether sheaves) is a direct sum of degree shifts of $\delta_X = \mathbb{C}_X[g]$. As an application we have the following result concerning the difference morphism.

LEMMA 5.25. *Let $X = JC$ be the Jacobian variety of a non-hyperelliptic smooth projective curve of even genus $g = 2n$. Then over some Zariski-open dense subset $U \subset X$ there exists an irreducible local system L_U of complex vector spaces such that*

$$ d_{n*}(\mathbb{C}_{C_n \times C_n})|_U \;\cong\; \mathbb{C}_U \oplus L_U\,. $$

Proof. A Bertini-type argument like the one in lemma 5.3 shows that over some Zariski-open dense subset $U \subset X$ the morphism d_n restricts to a finite étale cover. Furthermore, if S denotes the singular locus of W_n, then for dimension reasons the closed subset $(S \times W_n) \cup (W_n \times S) \subset W_n \times W_n$ is mapped under the difference morphism

$$ g_n : \quad W_n \times W_n \longrightarrow X $$

to a proper closed subset of X, so we can assume that $g_n^{-1}(U)$ is contained in $(W_n \setminus S) \times (W_n \setminus S)$. The Riemann-Kempf singularity theorem [**48**, p. 348] says that the Abel-Jacobi morphism $f_n : C_n \longrightarrow W_n$ restricts over the smooth locus to an isomorphism

$$f_n^{-1}(W_n \setminus S) \xrightarrow{\cong} W_n \setminus S.$$

Putting everything together and using the commutative diagram

$$\begin{tikzcd} C_n \times C_n \arrow[r, "(f_n,f_n)"] \arrow[dr, bend right, "d_n"'] & W_n \times W_n \arrow[r, hook, "(-i_n,i_n)"] \arrow[d, "g_n"] & X \times X \arrow[dl, bend left, "a"] \\ & X & \end{tikzcd}$$

for the closed embedding $i_n : W_n \hookrightarrow X$, we therefore obtain by the definition of convolution an isomorphism

$$d_{n*}(\mathbb{C}_{C_n \times C_n}[g])|_U \;\cong\; (\delta_{W_n} * \delta_{-W_n})|_U .$$

To control the convolution on the right hand side, we use the formalism of Brill-Noether sheaves. Recall that $\delta_{W_n} = {}^p\delta_n$ and $\delta_{-W_n} = {}^p\delta_{3n-2}$, where the second equality is due to the Riemann-Roch theorem. Hence we get from the Littlewood-Richardson rule that

$$\delta_{W_n} * \delta_{-W_n} \;\cong\; {}^p\delta_n * {}^p\delta_{3n-2} \;\cong\; \tau \,\oplus\, \bigoplus_{i=0}^{n} {}^p\delta_{3n-2+i,n-i}$$

where τ is a direct sum of degree shifts of δ_X. In fact $\tau = \delta_X$ as one may see from the above isomorphism $d_{n*}(\mathbb{C}_{C_n \times C_n}[g])|_U \cong (\delta_{W_n} * \delta_{-W_n})|_U$ and from the fact that d_n is generically finite. Therefore our claim will follow for the cohomology sheaf

$$L_U \;=\; \mathscr{H}^{-g}({}^p\delta_{3n-2,n})|_U ,$$

provided that for all $i > 0$ the perverse sheaves ${}^p\delta_{3n-2+i,n-i}$ are supported on a proper closed subset of X. But this is indeed the case because by the Littlewood-Richardson rule the perverse sheaf

$$\bigoplus_{i=1}^{n} {}^p\delta_{3n-2+i,n-i} \;\cong\; {}^p\delta_{n-1} * {}^p\delta_{2g-(n-1)} \;\cong\; \delta_{W_{n-1}} * \delta_{-W_{n-1}}$$

has the support $d_{n-1}(C_{n-1} \times C_{n-1}) \subset X$ of dimension $2(n-1) < g$. □

Proof of part (b) in proposition 5.24. For this we again use Brill-Noether sheaves. The perverse sheaf $\delta_\Theta = \delta_{W_3} = {}^p\delta_3$ corresponds by definition to the third fundamental representation of the Tannaka group $Sl_6(\mathbb{C})$. One can check that the alternating square of this representation decomposes into two irreducible pieces of highest weight α^t for $\alpha = (4,2)$ and $\alpha = (6)$. Hence it follows that

$$\delta_- \;\cong\; {}^p\delta_{4,2} \oplus {}^p\delta_6 \oplus \tau$$

where $\tau \in \mathrm{Perv}(X,\mathbb{C})$ is some negligible perverse sheaf. In fact $\tau = \delta_X$ by part *(a)* of the proposition. Furthermore ${}^p\delta_6 = \delta_0$ is a skyscraper sheaf, hence for $U \subset X$ sufficiently small we have

$$\mathbb{V}_+ \;\cong\; \mathscr{H}^{-4}(\delta_-)|_U \;\cong\; \mathscr{H}^{-4}({}^p\delta_{4,2} \oplus \delta_X)|_U \;\cong\; d_{2*}(\mathbb{C}_{C_2\times C_2})$$

where the first isomorphism is due to lemma 5.21 and where the last one comes from the proof of lemma 5.25 for $n = 2$. $\square$

5.8. The difference morphism

Motivated by the previous section, let C be a smooth complex projective curve of genus $g = 2n$ with $n \in \mathbb{N}$, and put

$$C_n \;=\; (C\times\cdots\times C)/\mathfrak{S}_n.$$

We want to study the difference morphism $d_n : C_n \times C_n \longrightarrow X = JC$ from above. To begin with, we claim that this morphism is generically finite and has generic degree

$$N \;=\; \deg(d_n) \;=\; \binom{2n}{n}.$$

Indeed, by birationality of the Abel-Jacobi map $C_n \to W_n$ it suffices to check that the difference morphism $g_n : W_n \times W_n \to X$ is generically finite of the given degree. Note that the fibre of g_n over a point $x \in X(\mathbb{C})$ is isomorphic to the intersection $Z = W_n \cap (W_n + x)$. Now a Bertini-type argument and the Poincaré formula [**13**, sect. 11.2.1] for the fundamental classes

$$[W_n] \;=\; [W_n + x] \;=\; \tfrac{1}{n!}\cdot[\Theta]^n \;\in\; H^n(X,\mathbb{Z})$$

show that for a sufficiently general point $x \in X(\mathbb{C})$ the intersection Z is finite of cardinality

$$|Z| \;=\; \deg_X\big([W_n]\cup[W_n+x]\big) \;=\; \tfrac{g!}{n!\cdot n!} \;=\; N.$$

So over some Zariski-open dense subset $U \subset X$ the morphisms g_n and d_n restrict to a finite étale cover of degree N as claimed.

Now pick $x \in U(\mathbb{C})$, and consider the monodromy action of $\pi_1(U,x)$ on the N distinct points of the fibre $d_n^{-1}(x)$. Labelling these points in any chosen order, we get a homomorphism

$$\pi_1(U,x) \;\longrightarrow\; \mathfrak{S}_N$$

to the symmetric group, and we define the monodromy group $G(d_n)$ to be the image of this homomorphism. A different choice of the labelling only changes this subgroup by an inner automorphism of $\mathfrak{S}_n$. Our goal is to determine the monodromy group $G(d_n)$ for a general curve C of even genus $g = 2n$. For this we will study a degeneration into a hyperelliptic curve, where we have the following situation.

LEMMA 5.26. *If C is a hyperelliptic curve of genus $g = 2n$, then the above monodromy group $G(d_n)$ is isomorphic to the symmetric group $\mathfrak{S}_{2n}$, and its action on the N points of a general fibre of d_n can be identified with the action of $\mathfrak{S}_{2n}$ on the set of all n-element subsets of $\{1,2,\dots,2n\}$.*

Proof. If g^1_2 denotes the hyperelliptic linear series on C, then for every effective divisor $D \in C_n$ the linear series $n \cdot g^1_2 - D$ contains an effective divisor. Hence W_n is a translate of its negative $-W_n$, and it follows that up to a translation the difference morphism d_n coincides with the addition morphism $C_n \times C_n \to X$. The latter factors as $C_n \times C_n \to C_{2n} \to X$ where the Abel-Jacobi morphism $C_{2n} \to X$ is birational. So we must determine the monodromy group H of the finite branched cover $C_n \times C_n \to C_{2n}$. This cover is not Galois, but it is dominated by the Galois cover with group $\mathfrak{S}_{2n}$ in the following diagram.

$$\begin{tikzcd} C^{2n} \arrow[r, equal] \arrow[ddr, bend right, "\mathfrak{S}_{2n}"'] & C^n \times C^n \arrow[d, "\mathfrak{S}_n \times \mathfrak{S}_n"] \\ & C_n \times C_n \arrow[d] \\ & C_{2n} \end{tikzcd}$$

Take a point $p_1 + \cdots + p_{2n} \in C_{2n}$ with the $p_i \in C$ all distinct. Via the first projection $C_n \times C_n \to C_n$, the fibre F of the cover $C_n \times C_n \to C_{2n}$ over this point is identified with the set of all n-element subsets of $\{p_1, \dots, p_{2n}\}$, and the monodromy group H is the image of the homomorphism

$$\varphi : \mathfrak{S}_{2n} \longrightarrow Aut(F)$$

which is given by the action of the Galois group $\mathfrak{S}_{2n}$ of $C^{2n} \to C_{2n}$ on the fibre F via permutation of $p_1, \dots, p_{2n}$. Note that φ is injective: The identity is the only permutation in $\mathfrak{S}_{2n}$ that fixes all n-element subsets of the set $\{p_1, \dots, p_{2n}\}$. Hence the claim of the lemma follows. $\square$

We can now also deal with a general curve, proving theorem 5.2 from the introduction. Here the monodromy group $G(d_n)$ turns out to be almost as large as possible — with the only drawback that we do not know whether it contains an odd permutation.

THEOREM 5.2. *If C is a general curve of genus $g = 2n$, then $G(d_n)$ is either the alternating group $\mathfrak{A}_N$ or the full symmetric group $\mathfrak{S}_N$.*

Proof. By lemma 5.26 we can assume that $g > 2$. In particular then a general curve C of genus g is non-hyperelliptic. Hence lemma 5.25 shows that the monodromy representation of $G(d_n)$ splits into a direct sum of the one-dimensional trivial representation plus only a single further irreducible representation. Now recall that by definition $G(d_n)$ is a subgroup of the symmetric group $\mathfrak{S}_N$. If $V = \mathbb{C}^N$ denotes the permutation representation of $\mathfrak{S}_N$, then the monodromy representation of the subgroup $G(d_n)$ is simply the restriction of V to this subgroup. Note that $V = \mathbf{1} \oplus W$ splits as the direct sum of the one-dimensional trivial representation plus an irreducible representation W of $\mathfrak{S}_N$ with dimension $N-1$. In these terms, the remarks at the beginning of the proof amount to the statement that the restriction of W to the subgroup $G(d_n) \leq \mathfrak{S}_N$ remains irreducible.

It is a general fact from the theory of permutation groups [**91**, th. 1(b)] that any subgroup of $\mathfrak{S}_N$ with this irreducibility property is a 2-transitive subgroup. So by lemma D.1 in the appendix it will suffice to show that for a general curve C of genus $g = 2n$ the monodromy group $G(d_n)$ contains the hyperelliptic one from lemma 5.26. For this we consider a degeneration of a general curve into a hyperelliptic curve.

Let $p : \mathscr{C} \to S$ be a flat, projective family of smooth curves of genus g over a smooth quasi-projective complex base curve S, and assume that the fibre $\mathscr{C}_s = p^{-1}(s)$ is hyperelliptic for some special point $s = s_0 \in S(\mathbb{C})$ but non-hyperelliptic for all $s \neq s_0$. The existence of such families follows e.g. from [**5**, th. XII.9.1]. To rephrase our previous constructions in this relative setting, let $\mathscr{X}$ be the relative Picard scheme of $\mathscr{C}$ over S as defined in [**15**, sect. 8.2], and denote by $\mathscr{C}_n = (\mathscr{C} \times_S \cdots \times_S \mathscr{C})/\mathfrak{S}_n$ the n-th relative symmetric product. Let $\mathscr{L}$ be a Poincaré bundle on $\mathscr{C} \times_S \times \mathscr{C}_n$ so that for all $s \in S(\mathbb{C})$ and every divisor $D \in \mathscr{C}_{n,s}(\mathbb{C})$ we have $\mathscr{L}|_{\mathscr{C}_s \times \{D\}} \cong \mathscr{O}_{\mathscr{C}_s}(D)$. If $p_1, p_2 : \mathscr{C} \times_S \mathscr{C}_n \times_S \mathscr{C}_n \to \mathscr{C} \times_S \mathscr{C}_n$ denote the projections, then by the universal property of the Picard scheme the line bundle $p_1^*(\mathscr{L}) \otimes p_2^*(\mathscr{L}^{-1})$ determines a morphism $d_n : \mathscr{C}_n \times_S \mathscr{C}_n \to \mathscr{X}$ which on the fibres over each point $s \in S(\mathbb{C})$ restricts to the difference morphism from above.

This being said, take an open subset $U \subseteq \mathscr{X}$ over which d_n is finite étale of degree N and such that over every point $s \in S(\mathbb{C})$ the fibre $U_s = U \cap \mathscr{X}_s$ is dense in $\mathscr{X}_s$. Applying the semicontinuity lemma 5.27 below to the local system $\mathbb{L} = d_{n*}(\mathbb{C}_{\mathscr{C}_n \times_S \mathscr{C}_n})|_U$, we obtain that the monodromy group of d_n for a general curve of genus $g = 2n$ contains the hyperelliptic one from lemma 5.26 as a subgroup. This finishes the proof. $\square$

For the above degeneration argument we have made use of the following semicontinuity property for monodromy groups in families. Let $f : U \to S$ be a smooth morphism of complex algebraic varieties whose target is a quasi-projective curve S, and consider a local system $\mathbb{L}$ of complex vector spaces on U with finite monodromy group. For any point $u \in U(\mathbb{C})$ with image $s = f(u)$ we denote by

$$G(u) \;=\; \mathrm{Im}\big(\pi_1(U_s, u) \;\longrightarrow\; Aut_{\mathbb{C}}(\mathbb{L}_u)\big)$$

the monodromy group of the restriction $\mathbb{L}|_{U_s}$ of the given local system $\mathbb{L}$ to the fibre $U_s = f^{-1}(s)$. Up to isomorphism the group $G(u)$ of course only depends on the image point $s = f(u)$.

LEMMA 5.27. *Let $u_0 \in U(\mathbb{C})$. Then there is a non-empty Zariski-open subset $S' \subseteq S$ with the following property: For all points $u \in f^{-1}(S'(\mathbb{C}))$, any identification $\mathbb{L}_{u_0} \cong \mathbb{L}_u$ of the stalks gives rise to an embedding of the monodromy groups such that the following diagram commutes.*

$$\begin{tikzcd}
G(u_0) \arrow[r, hook, dashed, "\exists"] \arrow[d, hook] & G(u) \arrow[d, hook] \\
Aut_{\mathbb{C}}(\mathbb{L}_{u_0}) \arrow[r, "\cong"] & Aut_{\mathbb{C}}(\mathbb{L}_u)
\end{tikzcd}$$

Proof. The inclusion $U_{s_0} \hookrightarrow U$ of the special fibre, followed by a path from u_0 to u to change the base points, induces on fundamental groups a pushforward homomorphism

$$\pi_1(U_{s_0}, u_0) \longrightarrow \pi_1(U, u_0) \xrightarrow{\cong} \pi_1(U, u).$$

For any Zariski-open dense subset $S' \subset S$ with preimage $U' = f^{-1}(S')$ and all $s \in S'(\mathbb{C})$ this leads to a commutative diagram

$$\begin{tikzcd}
\pi_1(U_{s_0}, u_0) \arrow[r] \arrow[d] & \pi_1(U, u) \arrow[d] & \pi_1(U', u) \arrow[l, two heads] \arrow[d] & \pi_1(U_s, u) \arrow[l] \arrow[d] \\
G(u_0) \arrow[r, hook] \arrow[d, hook] & G_U \arrow[r, equal] \arrow[d, hook] & G_U \arrow[d, hook] & G(u) \arrow[l, hook', "i"'] \arrow[d, hook] \\
Aut_{\mathbb{C}}(\mathbb{L}_{u_0}) \arrow[r, "\cong"] & Aut_{\mathbb{C}}(\mathbb{L}_u) \arrow[r, equal] & Aut_{\mathbb{C}}(\mathbb{L}_u) \arrow[r, equal] & Aut_{\mathbb{C}}(\mathbb{L}_u)
\end{tikzcd}$$

where $G_U \subseteq Aut_{\mathbb{C}}(\mathbb{L}_u)$ denotes the image of $\pi_1(U, u)$ under the monodromy representation. Note that G_U coincides with the image of $\pi_1(U', u)$ because the embedding of any Zariski-open dense subset induces an epimorphism on fundamental groups. The proof of the lemma would be finished if we could show that the embedding i in the above diagram is an isomorphism for a suitable choice of the Zariski-open dense subset $S' \subseteq S$.

This need not be the case in general. However, since the statement of the lemma only concerns the fibres of the family $f : U \longrightarrow S$ and not the total space, we can for the purpose of the proof replace the given family by its base change under any quasi-finite branched cover of a Zariski-open dense subset of S containing s_0, accordingly replacing the local system $\mathbb{L}$

by its pull-back under this base change. In particular, since étale-locally any smooth morphism admits a section, we can assume that there exists a section $\sigma : S \longrightarrow U$ of our family $f : U \longrightarrow S$. Since by assumption $\mathbb{L}$ has finite monodromy, we can after passing to a further branched cover assume that the local system $\sigma^*(\mathbb{L})$ is trivial. Now recall from [**102**, cor. 5.1] that for any (not necessarily proper) morphism $f : U \longrightarrow S$ of complex algebraic varieties, there exists a Zariski-open dense subset $S' \subseteq S$ over which the restriction $U' \longrightarrow S'$ is a topologically locally trivial fibration in the analytic topology. We then in particular have the exact fibration sequence

$$\begin{tikzcd} \cdots \arrow[r] & \pi_1(U_s,u) \arrow[r] & \pi_1(U',u) \arrow[r] & \pi_1(S,s) \arrow[r] \arrow[l, bend right, "\sigma_*"'] & \cdots \end{tikzcd}$$

for $s \in S'(\mathbb{C})$ and $u = \sigma(s)$. Hence the triviality of $\sigma^*(\mathbb{L})$ implies that the monodromy group

$$G(u) \;=\; \mathrm{Im}\big(\pi_1(U_s,u) \;\longrightarrow\; Aut_{\mathbb{C}}(\mathbb{L}_u)\big)$$

is mapped isomorphically onto the group

$$G_{U'} \;=\; \mathrm{Im}\big(\pi_1(U',u) \;\longrightarrow\; Aut_{\mathbb{C}}(\mathbb{L}_u)\big)$$

via the homomorphism i in the diagram from the beginning of the proof, and hence the lemma follows. $\square$

5.9. A smooth global family

To complete the proof of theorem 5.1 it remains to construct an abelian scheme $X_S \longrightarrow S$ with the properties discussed in section 5.2, which will allow to relate the monodromy group of the local system $\mathbb{V}_+$ for a general ppav of dimension $g = 4$ to the corresponding monodromy group for the Jacobian variety of a smooth non-hyperelliptic curve. To this end we fix an integer $n \geq 3$ and work as in section 4.1 over the moduli space $\mathscr{A}_{4,n}$ of ppav's of dimension 4 with a full level n structure, which by [**80**, chapt. 7.3] is represented by a smooth quasi-projective variety. For subvarieties $S \hookrightarrow \mathscr{A}_{4,n}$ we denote by

$$\begin{tikzcd} \Theta_S \arrow[r, hook] \arrow[dr] & X_S \arrow[d] \\ & S \end{tikzcd}$$

the corresponding abelian scheme with its relatively ample divisor which on the universal covering is defined by the Riemann theta function. We are interested in the composite morphism

$$f: \quad Y_S = \Theta_S \times_S \Theta_S \hookrightarrow X_S \times_S X_S \xrightarrow{a} X_S$$

where a denotes the group law of the abelian scheme. For each $s \in S(\mathbb{C})$ the restriction $f|_{X_s} : f^{-1}(X_s) \longrightarrow X_s$ is the family of the intersections of translates of the theta divisor for the corresponding ppav X_s as in the proof of the Bertini-type lemma 5.3.

LEMMA 5.28. *There exists a smooth quasi-projective curve $S \subset \mathscr{A}_{4,n}$ with the following properties.*

(a) For some $s_0 \in S(\mathbb{C})$ the fibre X_{s_0} is the Jacobian variety of a general curve, whereas for all other points $s \neq s_0$ the theta divisor $\Theta_s \subset X_s$ is smooth.

(b) The family $f: Y_S \longrightarrow S$ of intersecting theta divisors restricts to a smooth proper morphism

$$f|_{U_S}: \quad f^{-1}(U_S) \longrightarrow U_S$$

over some Zariski-open subset $U_S \subset X_S$ which surjects onto S.

Proof. The locus of all ppav's with a singular theta divisor is itself a divisor in the moduli space $\mathscr{A}_{4,n}$, and it contains the locus of Jacobian varieties as a component [**4**]. Hence if we take $S \hookrightarrow \mathscr{A}_{4,n}$ to be a smooth quasi-projective complex curve which meets the locus of Jacobian varieties in a general point $s_0 \in S(\mathbb{C})$ and is otherwise contained in the Zariski-open dense locus of ppav's with a smooth theta divisor, then *(a)* holds.

Consider then the corresponding morphism $f: Y_S \longrightarrow X_S$. By the generic flatness theorem in the form of lemma 5.29 below, we find a Zariski-open dense subset $U_S \subset X_S$ which maps surjectively onto S and furthermore has the property that the restriction

$$f|_{U_S}: \quad f^{-1}(U_S) \longrightarrow U_S$$

is a flat morphism. Now recall from lemma 4.5 that the theta divisor of the non-hyperelliptic Jacobian variety X_{s_0} has only isolated singularities, so by the Bertini-type lemma 5.3 we can find a point u in $U_{s_0} = U_S \cap X_{s_0}$ such that the fibre $Y_u = f^{-1}(u)$ is smooth. Furthermore, for any proper and flat morphism of varieties over a field which has at least one smooth fibre, there exists by [**55**, ex. III.10.2] a non-empty Zariski-open subset of the target over which the morphism is smooth. In our case therefore f restricts to a smooth morphism over some Zariski-open neighborhood of the point u in U_S. We can replace U_S by such a neighborhood and accordingly shrink the curve S in such a way that the special point s_0 still lies in $S(\mathbb{C})$ but that now also the properties in claim *(b)* are satisfied. $\square$

For the sake of completeness we include the following version of the generic flatness theorem that has been used in the above lemma.

LEMMA 5.29. *Let V and W be two integral schemes which are locally of finite type over a locally Noetherian, integral, regular base scheme S of dimension one. Suppose that for every point s in S the scheme-theoretic fibre V_s is non-empty and W_s is reduced. Let*

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr] & & W \arrow[dl] \\ & S & \end{tikzcd}$$

be an S-morphism of finite type. Then the flat locus

$$U_S \ = \ \{w \in W \mid f \text{ is flat at } v \text{ for all } v \in V \text{ with } f(v) = w\}$$

is a Zariski-open dense subset of W which maps surjectively onto S.

Proof. Since V and W are integral schemes which dominate the integral regular one-dimensional scheme S, they are automatically flat over S in view of [**55**, prop. III.9.7]. So the fibre-wise flatness criterion [**50**, th. 11.3.10] shows that the morphism f is flat over a point w in W if and only if the restriction $f_s : V_s \to W_s$ to the scheme-theoretic fibres over the image $s \in S$ of w is flat over w. Now the generic flatness theorem [**49**, th. 6.9.1] applies on the one hand to the finite type morphism $f : V \to W$ and on the other hand to its restriction $f_s : V_s \to W_s$. Since by assumption both W and W_s are integral, it follows that the flat locus U_S is on the one hand a Zariski-open dense subset of W and on the other hand meets every fibre W_s. $\square$

CHAPTER 6

The generic rank of Brill-Noether sheaves

As the example in the previous chapter illustrates, one can consider the Tannakian formalism as a device to construct interesting perverse sheaves on abelian varieties. However, so far the connection between the Tannaka group and the geometric properties of the arising perverse sheaves remains rather mysterious even for Brill-Noether sheaves on Jacobian varieties. As a first step towards a better understanding of this connection, we provide in this chapter a recursion formula for the generic rank of Brill-Noether sheaves — hoping for a future combinatorial interpretation of this formula in terms of representation theory.

6.1. Brill-Noether sheaves

To set the scenery for what follows, let us briefly recall the construction of Brill-Noether sheaves from [**106**]. Let C be a smooth complex projective curve of genus $g \geq 2$ with Jacobian variety $X = JC$, and let $f: C \hookrightarrow X$ be a translate of the Abel-Jacobi map by some point in $X(\mathbb{C})$. Via addition we then obtain a morphism

$$f_d: \quad C^d \longrightarrow X, \quad (p_1, \dots, p_d) \mapsto f(p_1) + \cdots + f(p_d)$$

for each $d \in \mathbb{N}$. The symmetric group $\mathfrak{S}_d$ acts on $C^d = C \times \cdots \times C$ by permutation of the factors, and the morphism f_d factors over the smooth quotient variety $C_d = C^d / \mathfrak{S}_d$ so that we get a commutative diagram

$$\begin{tikzcd} C^d \arrow[rr, "f_d"] \arrow[dr, "q_d"'] & & X \\ & C_d \arrow[ur, "h_d"'] & \end{tikzcd}$$

where q_d denotes the quotient morphism. By the Riemann-Roch theorem we can choose the translate f of the Abel-Jacobi morphism in such a way that the Brill-Noether subvarieties $W_d^r = \{x \in X(\mathbb{C}) \mid \dim(f_d^{-1}(x)) \geq r\}$ have the symmetry property

$$W_d^r \;=\; -W_e^s \quad \text{for} \quad d + e \;=\; 2g - 2 \quad \text{and} \quad r - s \;=\; d + 1 - g.$$

With these normalizations, the following explicit description has been given in loc. cit. for the Tannakian category generated by the convolution powers of the perverse sheaf $\delta_C = \delta_{W_1} \in \mathrm{Perv}(X,\mathbb{C})$. A complete set of irreducible representations of the symmetric group $\mathfrak{S}_d$ up to isomorphism is given by the Specht modules σ_α where α runs through the partitions of degree d, see [**88**]. So the direct image of the constant perverse sheaf under the branched Galois cover $q_d : C^d \longrightarrow C_d$ splits up under the action of $\mathfrak{S}_d$ as a direct sum

$$q_{d*}(\delta_{C^d}) \;=\; \bigoplus_{\deg(\alpha)=d} \sigma_\alpha \boxtimes \gamma_\alpha \quad \text{with certain} \quad \gamma_\alpha \in \mathrm{Perv}(C_d,\mathbb{C}).$$

Consider now the direct image complexes $\delta_\alpha = Rh_{d*}(\gamma_\alpha)$. Each of these is a retract of $Rf_{d*}(\delta_{C^d}) = \delta_C * \cdots * \delta_C$ as one easily checks from the above decomposition. By theorem 10 in loc. cit. we can write

$$\delta_\alpha \;=\; {}^p\delta_\alpha \oplus {}^c\delta_\alpha$$

where ${}^p\delta_\alpha$ is a semisimple perverse sheaf without constituents of Euler characteristic zero and where the sheaf complex ${}^c\delta_\alpha$ is a sum of degree shifts of the constant perverse sheaf $\delta_X = \mathbb{C}_X[g]$. In particular, ${}^c\delta_\alpha$ is the maximal direct summand of δ_α which lies in the full subcategory $\mathbf{T}(X)$ of negligible objects from section 3.3. One can show that ${}^p\delta_\alpha \neq 0$ iff the partition $\alpha = (d_1,\dots,d_n)$ satisfies $d_1 \leq 2g-2$. Let us call such partitions admissible. The constituents of the perverse sheaves ${}^p\delta_\alpha$ for the admissible partitions α are called Brill-Noether sheaves. They are the simple objects of the Tannakian category generated by δ_C in the sense of corollary 3.10, and by loc. cit. the corresponding Tannaka group is

$$G(\delta_C) \;=\; \begin{cases} Sp_{2g-2}(\mathbb{C}) & \text{if } C \text{ is hyperelliptic,} \\ Sl_{2g-2}(\mathbb{C}) & \text{otherwise.} \end{cases}$$

If the curve C is not hyperelliptic, then for any admissible partition α it turns out that the perverse sheaf ${}^p\delta_\alpha$ corresponds to the irreducible representation of $Sl_{2g-2}(\mathbb{C})$ whose highest weight is given by the conjugate partition α^t in the basis defined by the diagonal entries of matrices. For example, the fundamental representations of $Sl_{2g-2}(\mathbb{C})$ correspond to partitions $\alpha = (d)$ with a single part, and for these one has

$${}^p\delta_d \;=\; \begin{cases} \delta_{W_d} & \text{for} \quad 0 \leq d \leq g-1, \\ \delta_{-W_{2g-2-d}} & \text{for} \quad g \leq d \leq 2g-2. \end{cases}$$

A similar interpretation holds in the hyperelliptic case, but here the ${}^p\delta_\alpha$ are in general no longer irreducible; their decomposition is obtained by applying the restriction functor from representations of the group $Sl_{2g-2}(\mathbb{C})$ to representations of the subgroup $Sp_{2g-2}(\mathbb{C})$.

6.2. A formula for the generic rank

The main goal of the present chapter is to obtain a recursive formula for the generic rank r_α of the perverse sheaves ${}^p\delta_\alpha$, by which we more precisely mean the rank

$$r_\alpha \;=\; \dim_{\mathbb{C}}\big(\mathscr{H}^{-g}({}^p\delta_\alpha)_\eta\big)$$

of the lowest degree cohomology sheaf at the generic point $\eta \in X$. Note that $r_\alpha = 0$ if and only if the perverse sheaf ${}^p\delta_\alpha$ is supported on a proper closed subset of X (indeed any simple perverse sheaf arises via intermediate extension from a local system on an open dense subset of its support).

THEOREM 6.1. *For partitions $\alpha = (d_1,\dots,d_n)$ of degree $d = \deg(\alpha)$ the generic rank r_α is given by the recursion*

$$r_\alpha \quad = \sum_{\substack{c_1+\cdots+c_n=g\\ 0\le c_i\le d_i}} \Big[\tfrac{g!}{c_1!\cdots c_n!}-1\Big]\prod_{k=1}^{n}\binom{2(g-1-c_k)}{d_k-c_k} \quad - \sum_{\substack{\beta>\alpha\\ \deg(\beta)=d}} K_{\beta\alpha}\cdot r_\beta.$$

where the $K_{\beta\alpha}$ are the Kostka numbers defined combinatorially below.

Here we use the following conventions: For any integers $a,m\in\mathbb{Z}$ the binomial coefficient $\binom{m}{a}$ is defined as the coefficient of t^a in the expansion of $(1+t)^m$ as a power series, hence it vanishes for $a<0$ and is given for exponents $m<0$ by the formula $\binom{m}{a} = (-1)^a\cdot\binom{a-m-1}{a}$. For partitions α and β the notation $\beta>\alpha$ refers to the lexicographic ordering. Finally, we recall that the Kostka numbers $K_{\beta\alpha}$ are non-negative integers which can be defined combinatorially in the following way.

By a Young diagram of shape $\beta = (e_1,\dots,e_m)$ we mean a diagram of m left-aligned rows such that for each i the i^{th} row consists of e_i cells. If the cells of the diagram are filled with the entries $1,2,\dots,n$ and each entry j occurs precisely d_j times, the resulting array is called a Young tableau of content $\alpha = (d_1,\dots,d_n)$. A Young tableau is said to be semistandard if the entries of the rows are weakly increasing from left to right and those of the columns are strictly increasing from top to bottom; by definition the Kostka number $K_{\beta\alpha}$ is the number of such tableaux of shape β and content α. For example, we have $K_{(3,2),(2,2,1)} = 2$ since

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$$

are the only semistandard tableaux of shape $(3,2)$ and content $(2,2,1)$. The above definition also easily implies that the Kostka numbers satisfy $K_{\alpha\alpha}=1$ and $K_{\beta\alpha}=0$ for all partitions $\beta<\alpha$.

The Kostka numbers arise in the representation theory of the symmetric group: Young's rule [**88**, th. 2.11.2] says that for partitions $\alpha = (d_1, \ldots, d_n)$ of degree $d = \deg(\alpha)$ we have

$$(\star) \qquad \operatorname{Ind}_{\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n}}^{\mathfrak{S}_d}(\mathbf{1}) \;=\; \bigoplus_{\beta \geq \alpha} K_{\beta\alpha} \cdot \sigma_\beta$$

where the left hand side denotes the representation induced from the trivial representation $\mathbf{1}$ of the subgroup $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n} \leq \mathfrak{S}_d$. Since $K_{\alpha\alpha} = 1$, it follows that the dimensions of the Specht modules satisfy

$$(\star\star) \qquad \dim(\sigma_\alpha) \;=\; \tfrac{d!}{d_1! \cdots d_n!} \;-\; \sum_{\beta > \alpha} K_{\beta\alpha} \cdot \dim(\sigma_\beta).$$

This being said, let us take a look at the recursion formula in theorem 6.1 in some explicit examples.

EXAMPLE 6.2. *Let α be a partition of degree d. For $d < g$ the first sum in theorem 6.1 is empty so that $r_\alpha = 0$. More interestingly, in degree $d = g$ the recursion of the theorem says*

$$r_\alpha \;=\; \tfrac{g!}{d_1! \cdots d_n!} \;-\; 1 \;-\; \sum_{\beta > \alpha} K_{\beta\alpha} \cdot r_\beta$$

which together with $(\star\star)$ *implies*

$$r_\alpha \;=\; \begin{cases} \dim(\sigma_\alpha) & \text{if } \alpha \neq (g), \\ 0 & \text{if } \alpha = (g). \end{cases}$$

We remark that this can also be checked directly as follows. For $d < g$ the subvariety W_d is a proper closed subset of X, and by construction it contains the support of ${}^p\delta_\alpha$. Similarly, in degree $d = g$ the Abel-Jacobi morphism $h_d : C_d \to X$ is birational, and in this case our claim follows from the observation that the perverse sheaf $\gamma_\alpha \in \operatorname{Perv}(C_d, \mathbb{C})$ from section 6.1 has generic rank $\dim(\sigma_\alpha)$ and contains the constant perverse sheaf if and only if $\alpha = (g)$ is a singleton partition. However, for partitions of higher degree the situation becomes increasingly complicated.

EXAMPLE 6.3. *For $g = 4$ theorem 6.1 predicts that $r_6 = r_{5,1} = 0$ and more interestingly that*

$$r_{4,2} \;=\; [\tfrac{4!}{2! \cdot 2!} - 1] \cdot \tbinom{2}{2} \cdot \tbinom{2}{0} \;=\; 5,$$

$$r_{3,3} \;=\; 2 \cdot [\tfrac{4!}{3! \cdot 1!} - 1] \cdot \tbinom{0}{0} \cdot \tbinom{4}{2} + [\tfrac{4!}{2! \cdot 2!} - 1] \cdot \tbinom{2}{1} \cdot \tbinom{2}{1} - r_{4,2} \;=\; 51.$$

In this case the vanishing $r_6 = r_{5,1} = 0$ can again be checked directly, using the observation that

$$ {}^p\delta_6 \oplus {}^p\delta_{5,1} \;=\; {}^p\delta_1 * {}^p\delta_5 \;=\; \delta_{W_1} * \delta_{-W_1} $$

must be supported on the two-dimensional closed subset $W_1 - W_1 \subset X$. The values $r_{4,2} = 5$ and $r_{3,3} = 51$ are less obvious from a geometric point of view, but notice that they are one less than the generic ranks of the local systems $\mathbb{V}_\pm$ in chapter 5, in accordance with part (b) of proposition 5.24.

More numerical values for partitions of small degree can be found in section 6.6 below. After these examples, let us now come to the proof of the recursion formula in theorem 6.1. Consider the stalk cohomology of the convolution product $\delta^\alpha = \delta_{d_1} * \cdots * \delta_{d_n}$. For each $i \in \{1,2,\dots,n\}$ the Specht module σ_{d_i} is the trivial representation of the symmetric group $\mathfrak{S}_{d_i}$, hence it follows from equation ($\star$) above and from the construction of Brill-Noether sheaves in section 6.1 that

$$ \delta^\alpha \;=\; \bigoplus_{\beta\geq\alpha} K_{\beta\alpha}\cdot\delta_\beta \;=\; {}^p\delta^\alpha \oplus {}^c\delta^\alpha $$

where

$$ {}^p\delta^\alpha \;=\; \bigoplus_{\beta\geq\alpha} K_{\beta\alpha}\cdot{}^p\delta_\beta \quad\text{and}\quad {}^c\delta^\alpha \;=\; \bigoplus_{\beta\geq\alpha} K_{\beta\alpha}\cdot{}^c\delta_\beta. $$

The stalk cohomology of a perverse sheaf at a general point $x \in X(\mathbb{C})$ is concentrated in degree $-g$, so to prove theorem 6.1 we can replace the generic rank by $(-1)^g$ times the Euler characteristic of the generic stalk cohomology. Thus it will suffice to show that for general $x \in X(\mathbb{C})$,

$$ \sum_{m\in\mathbb{Z}} (-1)^{m+g}\dim_{\mathbb{C}}(\mathscr{H}^m(\delta^\alpha)_x) \;=\; \sum_{\substack{c_1,\dots,c_n\geq 0\\ c_1+\cdots+c_n=g}} \tfrac{g!}{c_1!\cdots c_n!}\prod_{k=1}^n \tbinom{2(g-1-c_k)}{d_k-c_k}, $$

$$ \sum_{m\in\mathbb{Z}} (-1)^{m+g}\dim_{\mathbb{C}}(\mathscr{H}^m({}^c\delta^\alpha)_x) \;=\; \sum_{\substack{c_1,\dots,c_n\geq 0\\ c_1+\cdots+c_n=g}} \prod_{k=1}^n \tbinom{2(g-1-c_k)}{d_k-c_k}. $$

The first of these equalities will follow from theorem 6.6 below since by base change $\mathscr{H}^\bullet(\delta^\alpha)_x$ is equal to the cohomology of the fibre $F = f_\alpha^{-1}(x)$ of the Abel-Jacobi morphism $f_\alpha : C_{d_1}\times\cdots\times C_{d_n} \to X$. The second equality rests on a computation of Betti polynomials, see corollary 6.8.

6.3. Computations in the symmetric product

In this section we compute some intersection numbers of cycle classes in the cohomology ring of the symmetric products of C that will be needed in the proof of theorem 6.6 below. For $d \in \mathbb{N}$ consider the Abel-Jacobi morphism $h_d :\ C_d \to X = JC$ as before. We will always work with rational cohomology unless otherwise stated, so we put $H^\bullet(-) = H^\bullet(-,\mathbb{Q})$. Fixing a point $p \in C(\mathbb{C})$ we have embeddings $\{p\} \hookrightarrow C \hookrightarrow C_2 \hookrightarrow \cdots \hookrightarrow C_{d-1} \hookrightarrow C_d$ given by addition of p. Via the last embedding, let

$$x \;=\; [C_{d-1}] \;\in\; H^2(C_d)$$

be the fundamental class of C_{d-1}. Since the cup product in cohomology corresponds to the intersection of cycles, we have $x^a = [C_{d-a}]$ in $H^{2a}(C_d)$ for $a = 1,\dots,d$. We also put

$$\theta \;=\; h_d^*[\Theta] \;\in\; H^2(C_d)$$

where $[\Theta] \in H^2(X)$ denotes the class of a theta divisor $\Theta \subset X$. Since $[\Theta]$ is a principal polarization, the integral cohomology $H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{Q})$ admits a symplectic basis $\lambda_1,\dots,\lambda_{2g}$, i.e. a basis such that using the cup product on $H^\bullet(X)$ we have

$$[\Theta] \;=\; \sum_{i=1}^{g} \lambda_i \cdot \lambda_{g+i}\,.$$

Note that the construction of Jacobian varieties and the Künneth theorem provide natural isomorphisms $H^1(X) \cong H^1(C) \cong H^1(C_d)$. So in the sequel we will not distinguish notationally between the classes $\lambda_1,\dots,\lambda_{2g} \in H^1(X)$ and their images inside $H^1(C)$ or $H^1(C_d)$.

For any complex variety Z we denote by $\deg_Z : H^{2\dim(Z)}(Z) \longrightarrow \mathbb{Q}$ the degree isomorphism given by evaluation on the fundamental homology class of Z. For example, the classes

$$\eta_i \;=\; \lambda_i \cdot \lambda_{g+i} \quad \text{for} \quad 1 \le i \le g$$

satisfy $\deg_X(\eta_1 \cdots \eta_g) = \deg_X([\Theta]^g/g!) = 1$ by the Poincaré formula. In what follows it will be convenient to use the shorthand notation

$$\eta_I \;=\; \eta_{i_1} \cdots \eta_{i_c}$$

for subsets $I = \{i_1,\dots,i_c\} \subset \{1,\dots,g\}$ of cardinality $|I| = c$, noting that the ordering of the factors does not matter for η_I since the cup product is commutative in even degrees.

LEMMA 6.4. *Let $a,b,c \in \mathbb{N}_0$ with $a+b+c = d$, and let $\eta \in H^{2c}(C_d)$ be a monomial in the classes $\lambda_1,\dots,\lambda_{2g}$. Then*

$$\deg_{C_d}\bigl(x^a \cdot \theta^b \cdot \eta\bigr) \;=\; \begin{cases} \binom{g-c}{b} \cdot b! & \textit{if } \eta = \eta_I \textit{ for some } I, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since $H^1(C_d) \cong H^1(X)$, any monomial in $\lambda_1,\dots,\lambda_{2g} \in H^1(C_d)$ can be written as the pull-back $\eta = h_d^*(\mu)$ of some cohomology class μ

in $H^{2c}(X)$. Then $\theta^b \cdot \eta = h_d^*([\Theta]^b \cdot \mu)$, so from the projection formula we obtain the degree

$$\deg_{C_d}(x^a \cdot \theta^b \cdot \eta) \;=\; \deg_X(h_{d!}(x^a) \cdot [\Theta]^b \cdot \mu)$$

where $h_{d!} : H^{2a}(C_d) \longrightarrow H^{2(a+g-d)}(X)$ denotes the Gysin morphism. We can assume without loss of generality that $b+c \leq g$. For the fundamental class $x^a = [C_{b+c}]$ then

$$h_{d!}(x^a) \cdot [\Theta]^b \;=\; [W_{b+c}] \cdot [\Theta]^b \;=\; \tfrac{1}{(g-b-c)!}\,[\Theta]^{g-c}$$

by the Poincaré formula. This easily implies the claim of the lemma if one uses that $[\Theta]^{g-c} = (g-c)!\sum_{|J|=g-c} \eta_J$. □

COROLLARY 6.5. *Let $\eta \in H^{2c}(C_d)$ be a monomial in $\lambda_1, \dots, \lambda_{2g}$ for some $c \in \mathbb{N}_0$. Then*

$$\deg_{C_d}\big(c_{d-c}(C_d) \cdot \eta\big) \;=\; \begin{cases} (-1)^{d-c} \cdot \binom{2(g-1-c)}{d-c} & \textit{if } \eta = \eta_I \textit{ for some } I, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By [**6**, sect. VII.5] the Chern polynomial of C_d can be written as the power series $c_t(C_d) = (1+tx)^{d-g+1} \cdot e^{-t\theta/(1+tx)}$. Writing out the exponential series and then expanding the occuring powers of $1+tx$ via the binomial series, we obtain

$$\begin{aligned} c_t(C_d) \;&=\; \sum_{b\geq 0} \tfrac{(-1)^b}{b!}\,(t\theta)^b\,(1+tx)^{d-g+1-b} \\ &=\; \sum_{b\geq 0}\sum_{a\geq 0} \tfrac{(-1)^b}{b!}\,\tbinom{d-g+1-b}{a} \cdot x^a \cdot \theta^b \cdot t^{a+b}\,. \end{aligned}$$

In particular, looking at the coefficient of t^{d-c} in the above series and using lemma 6.4 we obtain that $c_{d-c}(C_d) \cdot \eta = 0$ unless $\eta = \eta_I$ for some I, in which case

$$\begin{aligned} \deg_{C_d}\big(c_{d-c}(C_d) \cdot \eta\big) \;&=\; \sum_{a+b=d-c} \tfrac{(-1)^b}{b!}\,\tbinom{d-g+1-b}{a} \cdot x^a \cdot \theta^b \cdot \eta \\ &=\; \sum_{a+b=d-c} (-1)^b\,\tbinom{d-g+1-b}{a}\,\tbinom{g-c}{b} \\ &=\; (-1)^{d-c} \sum_{a+b=d-c} \tbinom{g-2-c}{a}\,\tbinom{g-c}{b}. \end{aligned}$$

Up to the sign this last expression coincides with the coefficient of t^{d-c} in the power series

$$\sum_{a\geq 0} \tbinom{g-2-c}{a} t^a \cdot \sum_{b\geq 0} \tbinom{g-c}{b} t^b \;=\; (1+t)^{g-2-c}(1+t)^{g-c} \;=\; (1+t)^{2(g-1-c)}$$

so we are done by binomial expansion of the right hand side. □

6.4. The fibres of multiple Abel-Jacobi maps

For a partition $\alpha = (d_1, \ldots, d_n)$ of degree d consider now the partially symmetrized product $C_\alpha = C_{d_1} \times \cdots \times C_{d_n}$, and denote by $q_\alpha : C_\alpha \longrightarrow C_d$ the quotient morphism. We have a commutative diagram

$$\begin{tikzcd} C_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "q_\alpha"'] & & X \\ & C_d \arrow[ur, "h_d"'] & \end{tikzcd}$$

given by the Abel-Jacobi morphism, and our goal is to compute the Euler characteristic of the fibre $F = f_\alpha^{-1}(x)$ for general $x \in X(\mathbb{C})$. For $k = 1, \ldots, n$ let $p_k : C_\alpha = C_{d_1} \times \cdots \times C_{d_n} \longrightarrow C_{d_k}$ denote the projection onto the k-th factor. By the Künneth theorem we have an isomorphism

$$H^\bullet(C_{d_1}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H^\bullet(C_{d_n}) \xrightarrow{\cong} H^\bullet(C_\alpha)$$

sending an element $\alpha_1 \otimes \cdots \otimes \alpha_n$ to the cup product $p_1^*(\alpha_1) \cdots p_n^*(\alpha_n)$. In degree one this gives a commutative diagram

$$\begin{tikzcd} H^1(X) \arrow[d, "diag"'] \arrow[dr, "f_\alpha^*"] & \\ \bigoplus_{k=1}^n H^1(C_{d_k}) \arrow[r, "\cong"] & H^1(C_\alpha) \end{tikzcd}$$

where the vertical arrow corresponds to the diagonal embedding *diag* under the natural identifications $H^1(X) \cong H^1(C_{d_k})$ which as before we suppress in the notation. In this sense we write

$$f_\alpha^*(\lambda) \;=\; \sum_{k=1}^n p_k^*(\lambda) \quad \text{for cohomology classes} \quad \lambda \in H^1(X).$$

The Künneth isomorphism is also compatible with the degree maps on top cohomology, i.e. the diagram

$$\begin{tikzcd} H^{2d_1}(C_{d_1}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H^{2d_n}(C_{d_n}) \arrow[r, "\cong"] \arrow[dr, "\deg\otimes\cdots\otimes\deg"'] & H^{2d}(C_\alpha) \arrow[d, "\deg"] \\ & \mathbb{Q} \end{tikzcd}$$

commutes. Thus degree computations in the cohomology ring $H^\bullet(C_\alpha)$ can be done in terms of the individual factors $H^\bullet(C_{d_k})$ to which we can apply the results of the previous section. This being said, we proceed as follows.

THEOREM 6.6. *There exists a Zariski-open dense subset $U \subseteq X$ such that for $u \in U(\mathbb{C})$ the fibre $F = f_\alpha^{-1}(u)$ is smooth of Euler characteristic*

$$\chi(F) \;=\; (-1)^{d-g} \sum_{\substack{c_1,\dots,c_n \geq 0 \\ c_1+\dots+c_n=g}} \tfrac{g!}{c_1!\cdots c_n!} \prod_{k=1}^{n} \tbinom{2(g-1-c_k)}{d_k-c_k}.$$

Proof. Since C_α is smooth, the morphism $f_\alpha : C_\alpha \longrightarrow X$ restricts to a smooth morphism over some Zariski-open dense subset $U \subseteq X$. Then in particular the fibre $F = f_\alpha^{-1}(u)$ is smooth for all $u \in U(\mathbb{C})$. Furthermore, if $i : F \hookrightarrow C_\alpha$ denotes the embedding of such a fibre, we have an exact sequence of tangent bundles

$$0 \longrightarrow \mathscr{T}_F \longrightarrow i^*(\mathscr{T}_{C_\alpha}) \longrightarrow i^*(f_\alpha^*(\mathscr{T}_X)) \longrightarrow 0.$$

Since $\mathscr{T}_X$ is the trivial bundle, it follows that $c_t(F) = i^*(c_t(C_\alpha))$. Hence the Gauss-Bonnet formula and the projection formula show that

$$(i) \qquad \chi(F) \;=\; \deg_F(c_{d-g}(F)) \;=\; \deg_{C_\alpha}([F]\cdot c_{d-g}(C_\alpha))$$

where $[F] \in H^{2g}(C_\alpha)$ denotes the fundamental class of F. To compute the Chern polynomial of $C_\alpha = C_{d_1} \times \cdots \times C_{d_n}$, note that the tangent bundle of this direct product splits as

$$\mathscr{T}_{C_\alpha} \;=\; p_1^*(\mathscr{T}_{C_{d_1}}) \oplus \cdots \oplus p_n^*(\mathscr{T}_{C_{d_n}})$$

so that $c_t(C_\alpha) = \prod_{k=1}^n p_k^*(c_t(C_{d_k}))$ in the ring $H^\bullet(C_\alpha)[t]$. For the Chern class in question this implies

$$(ii) \qquad c_{d-g}(C_\alpha) \;=\; \sum_{\substack{c_1,\dots,c_n \geq 0 \\ c_1+\dots+c_n=g}} \prod_{k=1}^{n} p_k^*(c_{d_k-c_k}(C_{d_k}))\,.$$

Now consider the fundamental class $[F]$ of the fibre $F = f_\alpha^{-1}(u)$. The class of a point is $[u] = \frac{1}{g!}[\Theta]^g = \eta_1 \cdots \eta_g$ by the Poincaré formula, using the notations of the previous section. By the above remarks about the Künneth isomorphism we have $f_\alpha^*(\eta_i) = f_\alpha^*(\lambda_i)\cdot f_\alpha^*(\lambda_{g+i}) = \sum_{k,l=1}^n p_k^*(\lambda_i)\cdot p_l^*(\lambda_{g+i})$ so that

$$(iii) \qquad [F] \;=\; f_\alpha^*[u] \;=\; \sum_{\substack{k_1,\dots,k_g=1 \\ l_1,\dots,l_g=1}}^{n} \prod_{i=1}^{g} p_{k_i}^*(\lambda_i)\cdot p_{l_i}^*(\lambda_{g+i})\,.$$

In view of corollary 6.5 the only summands in (*iii*) which possibly have a non-zero cup product with (*ii*) are those with $k_i = l_i$ for all i. By the same corollary, each of these summands gives in (*i*) the contribution

$$\prod_{k=1}^{n} \tbinom{2(g-1-c_k)}{d_k-c_k} \quad \text{where} \quad c_k = \#\{i \mid k_i = k\}\,.$$

Finally, for $c_1, \dots, c_n \in \mathbb{N}_0$ with $c_1 + \cdots + c_n = g$ there are precisely $\frac{g!}{c_1! \cdots c_n!}$ partitions of $\{1, \dots, g\}$ into n disjoint subsets of size $c_1, \dots, c_n$, so there are precisely as many corresponding choices of $k_1, \dots, k_g$. □

6.5. Betti polynomials

To finish the proof of theorem 6.1 it remains to compute the contribution of the negligible complex ${}^c\delta^\alpha$ from the end of section 6.2. Recall that this complex is a direct sum of degree shifts of δ_X, which will allow to control it in terms of hypercohomology. More specifically, for any $K \in D^b_c(X, \mathbb{C})$ we have the Betti polynomial

$$h(K,t) \;=\; \sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}}(H^i(X,K)) \cdot t^i \;\in\; \mathbb{C}[t, t^{-1}].$$

If $K \cong D(K)$ is isomorphic to its Verdier dual, then the polynomial $h(K,t)$ is invariant under the coordinate transformation $t \mapsto 1/t$ and hence can be written as a polynomial $p(K, \xi) \in \mathbb{C}[\xi]$ in the new variable

$$\xi \;=\; t^{-1/2} + t^{1/2}$$

where of course only even powers of ξ occur. Note that $p(\delta_X, \xi) = \xi^{2g}$ so that this new variable is particularly useful for the detection of the direct summands δ_X we are interested in.

LEMMA 6.7. *For any $d \in \mathbb{N}_0$ we have*

$$p(\delta_d, \xi) \;=\; \sum_{c=0}^{d} \binom{2(g-1-c)}{d-c} \cdot \xi^{2c}.$$

Proof. By definition $\delta_d = Rh_{d*}(\delta_{C_d}) = Rh_{d*}(\mathbb{C}_{C_d}[d])$ is the direct image of the constant perverse sheaf under the Abel-Jacobi morphism $h_d : C_d \to X$, so its hypercohomology is

$$H^\bullet(X, \delta_d) \;=\; H^{\bullet + d}(C_d, \mathbb{C}).$$

Now the cohomology of symmetric products of a curve has been computed in [**72**, eq. 4.2]. For the Betti polynomial this gives

$$h(\delta_d, t) \;=\; \sum_{m=0}^{d} \binom{2g}{d-m} P_m(t) \quad \text{where} \quad P_m(t) \;=\; \sum_{c=0}^{m} t^{2c-m}.$$

Hence it will suffice to check that

$$(i) \qquad P_m \;=\; \sum_{c \geq 0} (-1)^{m-c} \binom{m+1+c}{m-c} \cdot \xi^{2c} \qquad \text{for all } m \geq 0,$$

$$(ii) \qquad \binom{2(g-1-c)}{d-c} \;=\; \sum_{m \geq 0} (-1)^{m-c} \binom{2g}{d-m} \binom{m+1+c}{m-c} \qquad \text{for all } c \geq 0.$$

We first deal with (*i*). For $m \in \{0,1\}$ this is easily verified. In general one computes that $(t^{-1}+t)\cdot P_m = P_{m+1}+P_{m-1}$ and $t^{-1}+t = \xi^2-2$, so we have the recursion formula

$$P_{m+1} \;=\; (\xi^2-2)\cdot P_m \;-\; P_{m-1}.$$

Assuming by induction that we already know the lemma for P_m and P_{m-1}, the coefficient of ξ^{2c} on the right hand side is

$$(-1)^{m+1-c}\left[\tbinom{m+c}{m-c+1} + 2\tbinom{m+1+c}{m-c} - \tbinom{m+c}{m-1-c}\right].$$

The expression in square brackets is equal to $\binom{m+2+c}{m+1-c}$ due to the binomial addition formulae

$$\tbinom{m+1+c}{m-c} - \tbinom{m+c}{m-1-c} \;=\; \tbinom{m+c}{m-c}, \qquad \tbinom{m+c}{m-c+1} + \tbinom{m+c}{m-c} \;=\; \tbinom{m+1+c}{m+1-c},$$

$$\tbinom{m+1+c}{m+1-c} + \tbinom{m+1+c}{m-c} \;=\; \tbinom{m+2+c}{m+1-c}.$$

This finishes the induction step and thereby proves equation (*i*). For (*ii*) note that

$$(-1)^{m-c}\tbinom{m+1+c}{m-c} \;=\; \tbinom{-2c-2}{m-c}$$

and that for $m<0$ this binomial coefficient vanishes if $c \geq 0$. So it follows that

$$\sum_{m\geq 0}(-1)^{m-c}\tbinom{2g}{d-m}\tbinom{m+1+c}{m-c} \;=\; \sum_{m\in\mathbb{Z}}\tbinom{2g}{d-m}\tbinom{-2c-2}{m-c}.$$

Now the sum on the right hand side is the coefficient of t^{d-c} in the power series expansion of

$$(1+t)^{2g}(1+t)^{-2c-2} \;=\; (1+t)^{2(g-c-1)}$$

so that equation (*ii*) follows by direct inspection. □

Using the above lemma we can compute the Euler characteristic of the stalk cohomology of ${}^c\delta^\alpha$ for any partition α and thereby conclude our proof of the recursion formula in theorem 6.1 as follows.

COROLLARY 6.8. *For any $x \in X(\mathbb{C})$ and any partition $\alpha = (d_1,\dots,d_n)$ we have the formula*

$$\chi\big(\mathscr{H}^\bullet({}^c\delta^\alpha)_x\big) \;=\; (-1)^g \sum_{\substack{c_1,\dots,c_n\geq 0\\ c_1+\cdots+c_n=g}} \prod_{k=1}^{n} \tbinom{2(g-c_k-1)}{d_k-c_k}.$$

Proof. We have already remarked above that the negligible complex ${}^c\delta^\alpha$ is a direct sum of degree shifts of δ_X. Hence for any $x \in X(\mathbb{C})$ we can write it in the form

$${}^c\delta^\alpha \;=\; \mathscr{H}^\bullet({}^c\delta^\alpha)_x[-g]\otimes_{\mathbb{C}}\delta_X.$$

Since the Betti polynomial of the constant perverse sheaf is $p(\delta_X,\xi)=\xi^{2g}$ it follows that

$$p({}^c\delta^\alpha,\xi) \;=\; \xi^{2g}\cdot q(\xi),$$

where $q\in\mathbb{C}[\xi]$ is determined by

$$q(t^{1/2}+t^{-1/2}) \;=\; \sum_{m\in\mathbb{Z}}(-1)^{m+g}\dim_{\mathbb{C}}(\mathscr{H}^m({}^c\delta^\alpha)_x)\cdot t^m\,.$$

In particular, the Euler characteristic we are interested in is $(-1)^g$ times the coefficient of ξ^{2g} in $p({}^c\delta^\alpha,\xi)$. To compute this coefficient, recall that by definition we have a direct sum decomposition $\delta^\alpha={}^p\delta^\alpha\oplus{}^c\delta^\alpha$ where ${}^p\delta^\alpha$ is a perverse sheaf without constituents δ_X. The latter property implies that the coefficient of t^e in $h({}^p\delta^\alpha,t)$ vanishes for all $|e|\geq g$, hence $p({}^p\delta^\alpha,\xi)$ is a polynomial of degree less than $2g$ in ξ. In other words,

$$p(\delta^\alpha,\xi) \;=\; p({}^c\delta^\alpha,\xi) \;+\; \text{terms of degree} < 2g,$$

and it remains to note that in view of lemma 6.7 the coefficient of ξ^{2g} in the polynomial $h(\delta^\alpha,\xi)=h(\delta_{d_1},\xi)\cdots h(\delta_{d_n},\xi)$ is given by the sum displayed on the right hand side of the corollary. $\square$

6.6. Some numerical values

The recursion formula in theorem 6.1 determines the generic rank r_α for any partition α. For future reference we include below a list of these generic ranks for all partitions with $3\leq\deg(\alpha)\leq 9$. For each degree $d=\deg(\alpha)$ the values of r_α are listed in a table whose rows are labeled by the partition α and whose columns are labeled by the genus g. For the sake of brevity we denote multiplicities in a partition by exponents, so the notation `[3^2,1]` refers to the partition $(3,3,1)$. Of course it suffices to deal with genera $g\leq d$ since otherwise $r_\alpha=0$ by example 6.2.

In the same example we have seen that in degree $d=\deg(\alpha)=g$ the generic rank r_α is the dimension of Specht module σ_α. This allows to check the last column in our tables by hand via the hook formula for the dimension of Specht modules [**88**, sect. 3.10]. It seems reasonable to expect a similar, though more complicated, combinatorial rule for the computation of the generic ranks r_α in general. For example, the tables suggest that

$$r_\alpha \;=\; \binom{3g-3+d}{d-g} \quad \text{for the partition} \quad \alpha \;=\; (d)^t \;=\; \underbrace{(1,1,\dots,1)}_{d\text{ parts}}.$$

However, unfortunately such a formula cannot be proved directly from the recursion in theorem 6.1 without a guess about the ranks r_α also for all other partitions, indeed the partition $(1,1,\dots,1)$ is minimal with respect to the lexicographic ordering.

alpha \ g	2	3
[3]	0	0
[2, 1]	0	2
[1^3]	6	1

alpha \ g	2	3	4
[4]	0	0	0
[3, 1]	0	0	3
[2^2]	0	8	2
[2, 1^2]	1	18	3
[1^4]	21	10	1

alpha \ g	2	3	4	5
[5]	0	0	0	0
[4, 1]	0	0	0	4
[3, 2]	0	2	22	5
[3, 1^2]	0	3	36	6
[2^2, 1]	0	69	38	5
[2, 1^3]	6	93	40	4
[1^5]	56	55	14	1

alpha \ g	2	3	4	5	6
[6]	0	0	0	0	0
[5, 1]	0	0	0	0	5
[4, 2]	0	0	5	42	9
[4, 1^2]	0	0	6	60	10
[3^2]	0	0	51	30	5
[3, 2, 1]	0	26	264	128	16
[3, 1^3]	0	28	246	100	10
[2^3]	0	132	153	50	5
[2^2, 1^2]	1	336	339	102	9
[2, 1^4]	21	360	291	70	5
[1^6]	126	220	105	18	1

alpha \ g	2	3	4	5	6	7
[7]	0	0	0	0	0	0
[6, 1]	0	0	0	0	0	6
[5, 2]	0	0	0	9	68	14
[5, 1^2]	0	0	0	10	90	15
[4, 3]	0	0	22	177	92	14
[4, 2, 1]	0	2	96	645	290	35
[4, 1^3]	0	1	76	510	200	20
[3^2, 1]	0	3	580	583	198	21
[3, 2^2]	0	69	824	751	222	21
[3, 2, 1^2]	0	162	1772	1485	410	35
[3, 1^4]	0	147	1252	905	210	15
[2^3, 1]	0	759	1454	849	188	14
[2^2, 1^3]	6	1218	2024	1062	212	14
[2, 1^5]	56	1155	1526	653	108	6
[1^7]	252	715	560	171	22	1

alpha \ g	2	3	4	5	6	7	8
[8]	0	0	0	0	0	0	0
[7, 1]	0	0	0	0	0	0	7
[6, 2]	0	0	0	0	14	100	20
[6, 1^2]	0	0	0	0	15	126	21
[5, 3]	0	0	3	72	408	196	28
[5, 2, 1]	0	0	8	230	1272	544	64
[5, 1^3]	0	0	4	160	915	350	35
[4^2]	0	0	2	304	302	112	14
[4, 3, 1]	0	0	315	2680	2280	700	70
[4, 2^2]	0	8	414	2800	2228	616	56
[4, 2, 1^2]	0	18	843	5520	4140	1080	90
[4, 1^4]	0	10	533	3220	2165	490	35
[3^2, 2]	0	18	2101	3568	2112	504	42
[3^2, 1^2]	0	36	3662	5776	3228	728	56
[3, 2^2, 1]	0	468	6665	9480	4780	980	70
[3, 2, 1^3]	0	696	8744	11344	5272	992	64
[3, 1^5]	0	570	5249	5876	2343	378	21
[2^4]	0	1056	3386	3280	1302	224	14
[2^3, 1^2]	1	2844	8393	7712	2908	476	28
[2^2, 1^4]	21	3660	9323	7600	2580	380	20
[2, 1^7]	126	3234	6440	4308	1227	154	7
[1^8]	462	2002	2380	1140	253	26	1

alpha \ g	2	3	4	5	6	7	8	9
[9]	0	0	0	0	0	0	0	0
[8, 1]	0	0	0	0	0	0	0	8
[7, 2]	0	0	0	0	0	20	138	27
[7, 1^2]	0	0	0	0	0	21	168	28
[6, 3]	0	0	0	9	160	774	352	48
[6, 2, 1]	0	0	0	20	448	2205	910	105
[6, 1^3]	0	0	0	10	290	1491	560	56
[5, 4]	0	0	0	177	1276	1092	364	42
[5, 3, 1]	0	0	58	1395	7704	5904	1692	162
[5, 2^2]	0	0	60	1385	7080	5160	1360	120
[5, 2, 1^2]	0	0	114	2620	13236	9219	2310	189
[5, 1^4]	0	0	54	1405	6900	4410	980	70
[4^2, 1]	0	0	60	4345	6544	3822	952	84
[4, 3, 2]	0	2	1404	14055	18746	9681	2128	168
[4, 3, 1^2]	0	3	2352	22325	28326	13983	2928	216
[4, 2^2, 1]	0	69	3972	29485	35094	16071	3120	216
[4, 2, 1^3]	0	93	4956	34565	38544	16527	2982	189
[4, 1^5]	0	55	2724	16705	16766	6363	1008	56
[3^3]	0	1	2634	7346	7220	3066	588	42
[3^2, 2, 1]	0	162	16344	41343	37034	14553	2576	168
[3^2, 1^3]	0	214	16980	39490	32920	12120	2000	120
[3, 2^3]	0	759	14160	30865	24536	8694	1400	84
[3, 2^2, 1^2]	0	1977	34314	71097	54216	18432	2844	162
[3, 2, 1^4]	0	2372	35262	66080	46080	14385	2030	105
[3, 1^6]	0	1815	19068	30553	18424	4998	616	28
[2^4, 1]	0	4931	23532	33955	20644	5964	812	42
[2^3, 1^3]	6	8530	36912	49654	28380	7734	992	48
[2^2, 1^5]	56	9636	35700	42256	21660	5331	618	27
[2, 1^6]	252	8151	23184	22515	9746	2061	208	8
[1^7]	792	5005	8568	5985	2024	351	30	1

The tables listed above have been obtained with the computer algebra system SAGE [**101**], using for each table the function `GenusList(d)` which is defined by the following source code.

```
def Initial(g, alpha):
    n = len(alpha)
    sum = 0
    for c in IntegerVectors(g, n, outer = alpha):
        product = [binomial(2*(g-1-c[k]), alpha[k]-c[k]) for k in range(n)]
        product.append(multinomial(c)-1)
        sum = sum + mul(product)
    return sum


def GenericRank(g, d):
    P = Partitions(d)
    N = P.cardinality()
    R = [Initial(g, beta) for beta in P]
    for a in range(N):
        for b in range(a):
            Kostka = (SemistandardTableaux(P[b],P[a])).cardinality()
            R[a] = R[a] - Kostka*R[b]
    return R


def GenusList(d):
    GR = [GenericRank(g,d) for g in range(2,d+1)]
    P = Partitions(d).list()
    N = len(P)
    print '%30s' %'alpha \ g',
    for g in range(2,d+1):
        print '%8s' %g,
    print "\n"
    for a in range(N):
        print '%30s' %P[a],
        for i in range(0,d-1):
            print '%8s' %GR[i][a],
        print "\n"
```

Here the function `Initial(g, alpha)` computes the leading term in the recursion formula, i.e. the value

$$\texttt{sum} \quad = \sum_{\substack{c_1+\cdots+c_n=g \\ 0\le c_i\le d_i}} \underbrace{\left[\tfrac{g!}{c_1!\cdots c_n!}-1\right] \prod_{k=1}^{n} \tbinom{2(g-1-c_k)}{d_k-c_k}}_{\texttt{mul(product)}} .$$

The function `GenericRank(g,d)` then does the main work: It starts with an array `R` consisting of the above initial values for all partitions α of the given degree `d`, and using the recursion in theorem 6.1 it then step by step transforms this array into an array of the generic ranks r_α. Finally, the function `GenusList(d)` prints out the obtained values on the screen in a convenient form like in the above tables.

APPENDIX A

Reductive super groups

In this appendix we recall the definition of an algebraic super group and collect some basic facts about these in the reductive case. Working over an algebraically closed field Λ of characteristic zero, algebraic super groups are built from algebraic groups by a “super commutative thickening” of their Lie algebra: Let $\mathbf{G} = (G, \mathfrak{g}_1, Q)$ be a triple consisting of

- a classical algebraic group G over Λ, for which we consider the adjoint representation on the Lie algebra $\mathfrak{g}_0 = Lie(G)$,
- a finite-dimensional algebraic representation $\mathfrak{g}_1$ of the group G over Λ, given by a homomorphism $Ad_1 : G \to Gl(\mathfrak{g}_1)$,
- a G-equivariant quadratic form $Q : \mathfrak{g}_1 \to \mathfrak{g}_0$, for which we denote by $b : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ the corresponding symmetric Λ-bilinear form with the property that $Q(v) = b(v, v)$ for all $v \in \mathfrak{g}_1$.

The “super commutative thickening” we are interested in is the super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. As usual for super vector spaces, the vectors $x \in \mathfrak{g}_\alpha$ are called homogenous of degree $|x| = \alpha$ for $\alpha \in \{0, 1\}$. The Lie bracket on $\mathfrak{g}_0$ together with the bilinear form b and with the differential $ad_1 = Lie(Ad_1)$ of Ad_1 define a Λ-bilinear map

$$[\cdot, \cdot] : \quad \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{such that} \quad [x, y] = (-1)^{|x||y|} \cdot [y, x]$$

for all homogenous $x, y \in \mathfrak{g}$. The triple $\mathbf{G}$ is called an algebraic super group over Λ if for all homogenous $x, y, z \in \mathfrak{g}$ the super Jacobi identity

$$(-1)^{|x||z|} \cdot [x, [y, z]] \; + \; (-1)^{|y||x|} \cdot [y, [z, x]] \; + \; (-1)^{|z||y|} \cdot [z, [x, y]] \; = \; 0$$

holds, so that $\mathfrak{g}$ becomes a super Lie algebra in the sense of [**92**]. In the present setting this is equivalent to the requirement that

$$ad_1(Q(v))(v) = 0 \quad \text{for all} \quad v \in \mathfrak{g}_1,$$

as one may check by plugging in $v = ax + by + cz$ with fixed $x, y, z \in \mathfrak{g}_1$ and with indeterminate coefficients a, b, c. Of course any classical algebraic group G over Λ can be considered as the super group $\mathbf{G} = (G, 0, 0)$. A more interesting class of examples can be constructed from affine super Hopf algebras as we will explain below.

The algebraic super groups over Λ form a category in a natural way: By definition, a homomorphism

$$h:\ (G,\mathfrak{g}_1,Q) \longrightarrow (H,\mathfrak{h}_1,P)$$

of algebraic super groups over Λ is a pair $h=(f,g)$ where $f:G\longrightarrow H$ is a homomorphism of algebraic groups and $g:\mathfrak{g}_1\longrightarrow\mathfrak{h}_1$ is a Λ-linear map which is equivariant with respect to f and satisfies $P\circ g = Lie(f)\circ Q$. Such a homomorphism is a mono- resp. an epimorphism of algebraic super groups iff both f and g are mono- resp. epimorphisms. The parity automorphism of a super group $\mathbf{G}=(G,\mathfrak{g}_1,Q)$ is defined as $(id_G,-id_{\mathfrak{g}_1}):\mathbf{G}\longrightarrow\mathbf{G}$. These notions are motivated by the following example.

Let $A=A_0\oplus A_1$ be an affine super Hopf algebra over Λ, in other words, a graded-commutative $\mathbb{Z}/2\mathbb{Z}$-graded Hopf algebra of finite type over Λ. A linear map $\partial : A\to A$ is called a super derivation if it satisfies the product rule $\partial(a\cdot b)=\partial(a)\cdot b+(-1)^{|a|}\cdot a\cdot\partial(b)$ for all homogenous $a,b\in A$. This being said, let $J\trianglelefteq A$ be the ideal generated by A_1. Then the quotient A/J is an algebra of finite type over Λ, the spectrum

$$G \;=\; Spec(A/J)$$

is a classical algebraic group, and one can check that the left invariant super derivations of A form a super Lie algebra $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ with a natural action of G extending the adjoint action on $\mathfrak{g}_0=Lie(G)$. It has been shown in [**108**] that the triple $\mathbf{G}=(G,\mathfrak{g}_1,Q)$ is an algebraic super group over Λ and that this realizes the opposite of the category of affine super Hopf algebras as a full subcategory of the category of algebraic super groups. Hence for algebraic super groups associated to affine super Hopf algebras, the notions introduced here are compatible with those in [**30**].

A particular instance is the general linear super group $\mathbf{G}=\mathbf{Gl}(V)$ on a super vector space $V=V_0\oplus V_1$ of finite dimension over Λ. In this case the underlying classical group is defined to be the group $G=Gl(V_0)\times Gl(V_1)$ of all Λ-linear automorphisms of V that preserve the grading, whereas

$$\mathfrak{g}_1 \;=\; Hom_\Lambda(V_0,V_1)\oplus Hom_\Lambda(V_1,V_0) \;\subset\; End_\Lambda(V)$$

consists of all endomorphisms that switch the grading. Here $\mathfrak{g}_1$ is equipped with the adjoint action of G, and one takes $Q(A\oplus B)=AB+BA$. The affine super Hopf algebra corresponding to $\mathbf{Gl}(V)$ can be defined as in the classical case, with the appropriate sign modifications.

For an algebraic super group $\mathbf{G}=(G,\mathfrak{g}_1,Q)$ over Λ let $\mathbf{G}^0=(G^0,\mathfrak{g}_1,Q)$ denote its Zariski connected component, and define its super center to be the classical group $Z(\mathbf{G})=(Z,0,0)$ where $Z\subseteq Z(G)$ is the largest central subgroup of G that acts trivially on $\mathfrak{g}_1$. Note that every element $g\in G$

induces an interior automorphism $\mathrm{int}(g) = (g^{-1}(-)g, Ad_1(g))$ of $\mathbf{G}$ and that $Z \subseteq G$ is the subgroup of all $g \in G$ with $\mathrm{int}(g) = id_{\mathbf{G}}$.

By a super representation of an algebraic super group $\mathbf{G}$ over Λ we mean an algebraic homomorphism $\rho_V : \mathbf{G} \to \mathbf{Gl}(V)$ for a super vector space V of finite dimension over Λ. By definition, a homomorphism between super representations ρ_V and ρ_W is a homomorphism $V \to W$ of super vector spaces which preserves the gradings and intertwines ρ_V with ρ_W. One can show that the category $Rep_\Lambda(\mathbf{G})$ of all super representations of $\mathbf{G}$ over Λ is an abelian Λ-linear rigid symmetric monoidal category with respect to the super tensor product. We also have Schur's lemma:

LEMMA A.1. *For an irreducible super representation $\rho_V : \mathbf{G} \to \mathbf{Gl}(V)$, every endomorphism of ρ_V has the form $\lambda \cdot id_V$ with some scalar $\lambda \in \Lambda$.*

Proof. The proof works as in the classical case. Note that by definition we only consider endomorphisms which preserve the grading; otherwise Schur's lemma would have to be modified, see [**92**, prop. 2, p. 46]. $\square$

In particular, the super center $Z(\mathbf{G})$ must act on any irreducible super representation of $\mathbf{G}$ by a character $\chi : Z(\mathbf{G}) \to \Lambda^*$ (recall that $Z(\mathbf{G})$ is a classical commutative algebraic group and that each of its elements defines an endomorphism of any given super representation of $\mathbf{G}$).

An algebraic super group $\mathbf{G} = (G, \mathfrak{g}_1, Q)$ is called reductive if the abelian category $Rep_\Lambda(\mathbf{G})$ is semisimple. The reductive algebraic super groups have been classified in [**108**]. Every classical reductive algebraic group G is also reductive when viewed as the super group $\mathbf{G} = (G, 0, 0)$. Another example of reductive super groups is given by the orthosymplectic super groups $\mathbf{Spo}_\Lambda(2r, 1)$ for $r \in \mathbb{N}$ which can be defined as follows: Let J be a non-degenerate antisymmetric $2r \times 2r$ matrix $J = (J_{ik})$ over Λ. In terms of the classical symplectic group $Sp_\Lambda(2r, J) = \{g \in Gl_\Lambda(2r) \mid g^t J g = J\}$ we then put

$$\mathbf{Spo}_\Lambda(2r, 1) \; = \; (Sp_\Lambda(2r, J), \Lambda^{2r}, Q),$$

where Λ^{2r} is equipped with the standard action of $Sp_\Lambda(2r, J)$ and where the quadratic map $Q : \Lambda^{2r} \to Lie(Sp_\Lambda(2r, J))$ is defined by $Q(v)_{ik} = \sum_{l=1}^{2r} v_i v_l J_{lk}$ in the standard coordinates. In particular, we have a natural representation of the super group $\mathbf{Spo}_\Lambda(2r, 1)$ on the super vector space $V = V_0 \oplus V_1$ with even part $V_0 = \Lambda^{2r}$ and odd part $V_1 = \Lambda$. On the odd part of the super Lie algebra this representation is given by

$$\Lambda^{2r} \ni v \; \mapsto \; \big((1 \mapsto v) \oplus (w \mapsto w^t J v)\big) \in Hom_\Lambda(V_1, V_0) \oplus Hom_\Lambda(V_0, V_1).$$

Of course a different choice of the non-degenerate antisymmetric matrix J will result in an isomorphic super group $\mathbf{Spo}_\Lambda(2r, 1)$.

It turns out that the above examples essentially exhaust all reductive algebraic super groups over Λ. Indeed, by theorem 6 of loc. cit. an algebraic super group $\mathbf{G}$ over Λ is reductive iff there are a classical reductive group H and integers $N \in \mathbb{N}_0$, $n_i, r_i \in \mathbb{N}$ such that $\mathbf{G}$ is isomorphic to a semidirect product

$$\mathbf{G} \;=\; \Big(\prod_{i=1}^{N} \big(\mathbf{Spo}_\Lambda(2r_i,1)\big)^{n_i}\Big) \rtimes H$$

with respect to a homomorphism $\pi_0(H) \to \prod_{i=1}^{N} \mathfrak{S}_{n_i}$ where each symmetric group $\mathfrak{S}_{n_i}$ acts on $\big(\mathbf{Spo}_\Lambda(2r_i,1)\big)^{n_i}$ by permutation of the factors.

COROLLARY A.2. *For a reductive algebraic super group $\mathbf{G}$ over Λ, the underlying classical group G is reductive as well; furthermore, the super center $Z(\mathbf{G})$ is a subgroup of finite index in the classical center $Z(G)$.*

Proof. By the above it suffices to show this in case $\mathbf{G} = \mathbf{Spo}_\Lambda(2r,1)$ for some $r \in \mathbb{N}$. But then $G = Sp_\Lambda(2r)$, and $Z(G) = \mu_2$ is finite. □

PROPOSITION A.3. *Let $h : \mathbf{G}_1 \to \mathbf{G}_0$ be a homomorphism of reductive super groups over Λ which induces an epimorphism*

$$f:\ G_1 \twoheadrightarrow G_0$$

on the underlying classical groups. If the super center $Z(\mathbf{G}_0)$ contains a classical torus T_0, then also $Z(\mathbf{G}_1)$ contains a classical torus T_1 such that the epimorphism f restricts to an isogeny $p : T_1 \to T_0$.

Proof. The category of tori (or diagonalizable groups) over Λ, up to isogeny, is equivalent to the category of finite vector spaces over $\mathbb{Q}$ via the cocharacter functor $T \mapsto X(T) = Hom(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If π is a finite group acting on T, then we have

$$X((T^\pi)^0) = (X(T))^\pi$$

for the invariants of the action on the torus resp. on its cocharacters.

For reductive super groups $\mathbf{G}$ we have $Z(\mathbf{G})^0 = Z(G)^0$ by corollary A.2. On $Z(G^0)$ the group G acts by conjugation, and this action factors over the finite group $\pi = \pi_0(G)$. By definition we have $Z(G)^0 \subset Z(G^0)^\pi$, and this is a subgroup of finite index: This follows by an application of the cocharacter functor since $X(Z(G)^0) = X(Z(G^0))^\pi$.

For the proof of the proposition it suffices to show that T_0 is contained in the image of $Z(G_1)^0$. By assumption h induces an epimorphism of the underlying classical groups, hence an epimorphism $f : (G_1)^0 \to (G_0)^0$ of their connected components. By the theory of connected reductive classical groups, the torus $T = Z((G_0)^0)^0$ is the image of $S = Z((G_1)^0)^0$. Now f restricts to a homomorphism

$$f: \ S \longrightarrow T$$

and this homomorphism is equivariant for the action of $\pi = \pi_0(G_1)$ on S and T, where by the action on T we mean the one induced by the natural homomorphism $\pi_0(G_1) \to \pi_0(G_0)$. We claim that $f : (S^\pi)^0 \to (T^\pi)^0$ is surjective: Indeed, the functor of invariants under a finite group π is right exact on finite-dimensional vector spaces over $\mathbb{Q}$. Since $(S^\pi)^0 \subset Z(G_1)^0$ and $T_0 \subset (T^\pi)^0$ this completes the proof. $\square$

APPENDIX B

Irreducible subgroups with bounded determinant

In this appendix we discuss the facts from representation theory that have been used to obtain the finiteness result in lemma 3.19. The basis for this is the following extended version of the fundamental result of [**109**], where by $\mathbf{P}(X) = \mathrm{Perv}(X, \mathbb{C})$ we denote as before the category of perverse sheaves on a complex abelian variety X.

LEMMA B.1. *Let $P \in \mathbf{P}(X)$ be a semisimple perverse sheaf, $G = G(P)$ the associated Tannaka group and $H = G^0$ its connected component. Then the group $\pi = G/H$ of connected components is a finite abelian group whose conjugation action is trivial on the torus $H^{ab} = H/[H,H]$.*

Proof. By loc. cit. the group $\pi = G/H$ of connected components is a finite abelian group, and its characters correspond to skyscraper sheaves δ_x supported on torsion points $x \in X$. Thus the group $Hom(\pi, \mathbb{C}^*)$ of characters can be identified in a natural way with a finite subgroup $K \subseteq X$ of torsion points. Consider then the étale isogeny

$$f: \quad X \longrightarrow Y = X/K.$$

Since $|K| = [G:H]$, lemma 1 in loc. cit. shows that the component $H = G^0$ coincides with the Tannaka group $H = G(f_*(P))$ of the direct image and that the adjoint functors

$$\begin{tikzcd} \mathbf{P}(X) \arrow[r, bend left, "f_*"] & \mathbf{P}(Y) \arrow[l, bend left, "f^*"] \end{tikzcd} \quad \text{correspond to} \quad \begin{tikzcd} Rep_{\mathbb{C}}(G) \arrow[r, bend left, "\mathrm{Res}_H^G"] & Rep_{\mathbb{C}}(H). \arrow[l, bend left, "\mathrm{Ind}_H^G"] \end{tikzcd}$$

Now put $m = |K|$. For any closed point $y \in Y$ the pull-back $f^*(\delta_y)$ is a direct sum of rank one skyscraper sheaves supported on the m distinct point of the fibre $f^{-1}(y)$, hence $f_*(f^*(\delta_y)) = (\delta_y)^{\oplus m}$ is an isotypic multiple of δ_y. In other words

$$\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\chi)) \quad \text{is isotypic for all} \quad \chi \in Hom(H, \mathbb{C}^*).$$

Going back to the construction of induced representations, it follows that every character $\chi : H \longrightarrow \mathbb{C}^*$ is invariant under conjugation by G. Via the equivalence between connected tori and their character groups this implies that G/H acts trivially on the torus $H^{ab} = H/[H,H]$. $\square$

With this input from geometry, the rest of the argument for the desired finiteness result will be an exercise in representation theory. Recall that a complex algebraic group G is reductive iff the abelian category $Rep_{\mathbb{C}}(G)$ of its finite-dimensional representations over $\mathbb{C}$ is semisimple, which is the case iff G admits a faithful irreducible representation. This is in particular the case for any irreducible subgroup $G \hookrightarrow Gl(V)$ of the general linear group attached to a finite-dimensional complex vector space V, where as usual by an irreducible subgroup we mean a subgroup which acts irreducibly on the given vector space. The result we are looking for is the following.

LEMMA B.2. *Fix an integer $n \in \mathbb{N}$, and let V be a finite-dimensional complex vector space. Then up to conjugacy there exist only finitely many irreducible algebraic subgroups $G \hookrightarrow Gl(V)$ such that*

(a) the quotient $\pi = G/H$ by the connected component $H = G^0$ is a finite abelian group, and

(b) via conjugation π acts trivially on $H^{ab} = H/[H,H]$, and

(c) the determinant $\det(V)$ is a character of G of order at most n.

Proof. Using the structure theory of connected reductive groups and the fact that every such group has at most finitely many representations of any given dimension, one checks that up to conjugation there are only finitely many connected reductive algebraic subgroups $H \hookrightarrow Gl(V)$. So it will be enough to show that for any given H there exist only finitely many irreducible subgroups $G \hookrightarrow Gl(V)$ with connected component $G^0 = H$ such that *(a) – (c)* hold. For this we will show that every such subgroup G can be recovered from the connected component $G^0 = H$ together with certain finite data. We proceed in the following steps.

Step 1. Let $G^* \subseteq G$ be a normal subgroup of finite index, and decompose the restriction of the given representation as $V|_{G^*} = \bigoplus_{i=1}^m V_i$ with irreducible representations $V_i \in Rep_{\mathbb{C}}(G^*)$. Then we claim that for each index i the determinant character

$$\det(V_i): \quad G^* \longrightarrow \mathbb{C}^*$$

is a character of order at most $m \cdot n$. Indeed, by the bound in *(c)* it will be enough to show that $\det(V_i)$ does not depend on i. Note that each $g \in G$ defines an automorphism $int(g): G^* \longrightarrow G^*$, and since V is an irreducible representation of G, all the irreducible constituents V_i of the restriction $V|_{G^*}$ are conjugate to each other by such inner automorphisms. Hence it follows from assumption *(b)* that the restriction $\det(V_i)|_H$ does not depend on the index i. Again by the bound in *(c)* this restriction is then a torsion character of the connected group H^{ab} and as such it must be trivial. Thus all the $\det(V_i)$ are characters of the quotient group G^*/H. By assumption *(a)* the adjoint action of G on this quotient group is trivial, so our claim follows.

Step 2. We now apply step 1 as follows. Let $U \subseteq V|_H$ be an irreducible constituent of the restriction of V to the connected component $G^0 = H$, and consider the stabilizer

$$G^* \;=\; \big\{ g \in G \mid gU \cong U \;\text{ in }\; Rep_{\mathbb{C}}(H) \big\} \;\subseteq\; G.$$

We define the isotypic part $W \subseteq V|_H$ to be the sum of all subrepresentations of $V|_H$ which are isomorphic to the chosen representation U. Then W is a representation of G^*, and a version of Mackey's lemma shows that

$$V \;\cong\; \mathrm{Ind}_{G^*}^{G}(W)$$

is isomorphic to the representation induced from W. Since $V \in Rep_{\mathbb{C}}(G)$ is irreducible, it follows that $W \in Rep_{\mathbb{C}}(G^*)$ is irreducible as well. But the group $\pi^* = G^*/H$ of connected components is abelian by *(a)* and the restriction $W|_H$ is isotypic, hence lemma B.3 below shows that even $W|_H$ is irreducible. Hence in step 1 we can take $V_1 = W$, and accordingly each of the conjugate representations $V_1, \dots, V_m$ is stable under the group G^* and irreducible. For the centralizer of the subgroup H in G then

$$Z_G(H) \;\hookrightarrow\; Z_{G^*}(H) \;\hookrightarrow\; \prod_{i=1}^{m} \mathbb{C}^* \cdot id_{V_i},$$

where the first inclusion comes from the fact that the centralizer preserves each isotypic component and is therefore contained in G^* and where the second inclusion comes from Schur's lemma. So each element of $Z_G(H)$ acts on each V_i by a scalar, and by step 1 this scalar is a root of unity of bounded order. Hence $Z_G(H)$ is a finite group of bounded order.

Step 3. This being said, let us fix as before a connected and reductive subgroup $H \hookrightarrow Gl(V)$. By [**99**, cor. 2.14] the short exact sequence of inner and outer automorphism groups admits a splitting

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Int}(H) \arrow[r] & \mathrm{Aut}(H) \arrow[r] & \mathrm{Out}(H) \arrow[r] \arrow[l, bend right, dotted, "\exists s"'] & 1 \end{tikzcd}$$

with some section s. In what follows we will fix such a section once and for all. If $G \hookrightarrow Gl(V)$ is an irreducible algebraic subgroup with $G^0 = H$ and such that *(a) – (c)* hold, we want to show that G can be recovered from certain finite data involving H and s. To achieve this, consider the conjugation action

$$int: \quad G \longrightarrow \mathrm{Aut}(H)$$

and denote by $F = int^{-1}(\mathrm{Im}(s)) \subset G$ the preimage of the section s. By construction we have an exact sequence

$$1 \longrightarrow Z_G(H) \longrightarrow F \xrightarrow{\;\varphi\;} \mathrm{Out}(H)$$

where $Z_G(H) = \ker(\varphi)$ denotes the centralizer of H in G. By Step 2 this centralizer and hence also F is finite of bounded order. Furthermore, from the definitions one checks that the homomorphism

$$\pi: \quad H \rtimes_\varphi F \longrightarrow G, \quad (h,f) \mapsto hf$$

is an epimorphism and that $\ker(\pi) \subseteq Z(H) \rtimes_\varphi F$ where $Z(H)$ denotes the center of H. Note that by Schur's lemma this center acts by a scalar on each isotypic component of $V|_H$, in particular $Z(H)$ is a finite group of bounded order in view of the bound for the determinant in step 1.

Thus any irreducible algebraic subgroup $G \hookrightarrow Gl(V)$ with $G^0 = H$ and with properties *(a) – (c)* is isomorphic to a quotient

$$G \cong (H \rtimes_\varphi F)/K$$

where F is a finite group of bounded order, acting on the group H via a homomorphism $\varphi: F \to \mathrm{Aut}(H)$ with image in the finite group $s(\mathrm{Out}(H))$, and where K is a subgroup of the finite group $Z(H) \rtimes_\varphi F$. Hence for the isomorphism type of G there are only finitely many possibilities.

Finally, for (F, φ, K) as above, any embedding $(H \rtimes_\varphi F)/K \hookrightarrow Gl(V)$ which extends the given embedding of H is determined uniquely by the homomorphism $\rho: F \longrightarrow Gl(V)$ it induces. We claim that under the given conditions there exist only finitely many such ρ. Indeed, if $\tilde{\rho}$ is any other such homomorphism, then for any $f \in F$ one computes that

$$\rho(f) \cdot \tilde{\rho}(f)^{-1} \in Z_{Gl(V)}(H) = \prod_{i=1}^{m} \mathbb{C}^* \cdot id_{V_i}$$

where the last equality holds by Schur's lemma, and in view of the bound on the determinant in step 1 the centralizer $Z_{Gl(V)}(H)$ is finite. □

In step 2 of the above proof we have used the following fact, where we reset our notations and replace G^* by G for simplicity.

LEMMA B.3. *Let W be a finite-dimensional complex vector space, G an irreducible subgroup of $Gl(W)$ whose group G/G^0 of components is abelian, and $H \hookrightarrow G$ a subgroup of finite index such that the restriction $W|_H$ is isotypic. Then in fact $W|_H$ is irreducible.*

Proof. By dévissage for the finite abelian group G/H it will be enough to show this when $G/H \cong \mathbb{Z}/m\mathbb{Z}$ is a finite cyclic group. Furthermore, by the assumption of being isotypic we can write

$$W|_H = U \otimes \mathbb{C}^n$$

for some irreducible representation $U \in Rep_{\mathbb{C}}(H)$ and some $n \in \mathbb{N}$. We must show $n = 1$. Since G/G^0 is abelian, the finite index subgroup $H \subseteq G$

is automatically a normal subgroup. Hence

$$G \hookrightarrow N_{Gl(W)}(H) = N_{Gl(U)}(H) \otimes Gl_n(\mathbb{C})$$

where the last equality follows via a block matrix calculation from Schur's lemma; here the tensor product on the right hand side denotes the image of the natural homomorphism $f : N_{Gl(U)}(H) \times Gl_n(\mathbb{C}) \longrightarrow Gl(U \otimes \mathbb{C}^n)$. Now let $\tilde{G} = f^{-1}(G)$. If $\mathbf{1}$ denotes the trivial subgroup of $Gl_n(\mathbb{C})$, then we have an epimorphism

$$\tilde{G}/(H \times \mathbf{1}) \twoheadrightarrow G/H$$

with central kernel (this follows from the fact that f has central kernel). But by assumption the group G/H is cyclic, and any central extension of a cyclic group is an abelian group — being generated by a central subgroup together with a single further element. Hence $\tilde{G}/(H \times \mathbf{1})$ is an abelian group. On the other hand this abelian group must act irreducibly on $\mathbb{C}^n$ because G acts irreducibly on W. So it follows that $n = 1$. □

We conclude this appendix with the following result which does not directly depend on the above but also involves a Mackey-type argument and has been used in lemma 4.13 to control the possible non-connectedness of the Tannaka group attached to the theta divisor.

LEMMA B.4. *Let G be a complex reductive group whose group G/G^0 of connected components is abelian, and let $V \in Rep_{\mathbb{C}}(G)$ be an irreducible representation. If the restriction $V|_{G^0}$ is reducible, then for some non-trivial character $\chi : G \longrightarrow \mathbb{C}^*$ there exists an isomorphism $V \cong V \otimes_{\mathbb{C}} \chi$.*

Proof. Consider the tensor product $W = V \otimes_{\mathbb{C}} V^{\vee} \in Rep_{\mathbb{C}}(G)$ of the given representation with its dual. We must show that this tensor product contains a non-trivial one-dimensional representation $\chi : G \longrightarrow \mathbb{C}^*$ as a direct summand. Note that since $H = G^0$ is a normal subgroup of G, the space of H-invariants

$$W^H = \{w \in W \mid hw = w \, \forall h \in H\} \subset W$$

is stable under the action of G and hence defines a representation of the finite abelian group G/H. As such it splits into a direct sum of one-dimensional representations. Among these the trivial representation $\mathbf{1}$ occurs precisely once because by adjunction $\dim_{\mathbb{C}}(Hom_G(\mathbf{1}, W)) = \dim_{\mathbb{C}}(Hom_G(V, V)) = 1$, using that V is an irreducible representation of G so that Schur's lemma applies. So it will be enough to show that

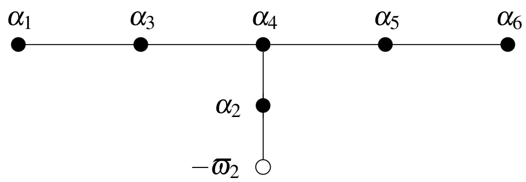
$$\dim_{\mathbb{C}}(W^H) > 1 \quad \text{if} \quad V|_H \quad \text{is reducible.}$$

But this follows from the observation that if the restriction $U = V|_H$ splits into n direct summands, then $W|_H = U \otimes_{\mathbb{C}} U^{\vee}$ contains at least n copies of the one-dimensional trivial representation. □

APPENDIX C

An irreducible subgroup of $W(E_6)$

In the computation of the monodromy group in section 5.2, we have remarked that the Weyl group $W(E_6)$ has subgroups $M < W(E_6)$ of index greater than two which act irreducibly on $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$. An example of such a subgroup can be obtained as follows. Take a system $\alpha_1, \dots, \alpha_6$ of simple roots for E_6 such that as in [**16**, ch. VI, no. 4.12] we have the extended Dynkin diagram



where $\varpi_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the highest root. Then we have an embedding

$$A_2 \oplus A_2 \oplus A_2 \;\cong\; \langle \alpha_1, \alpha_3 \rangle \oplus \langle \alpha_6, \alpha_5 \rangle \oplus \langle -\varpi_2, \alpha_2 \rangle \;\subset\; E_6$$

of three mutually orthogonal copies of the root system A_2. The symmetric group $\mathfrak{S}_3$ acts on the left hand side by permutation of these three copies via the identifications $\alpha_1 \longleftrightarrow \alpha_6 \longleftrightarrow -\varpi_2$ and $\alpha_3 \longleftrightarrow \alpha_5 \longleftrightarrow \alpha_2$. We also remark that the root system E_6 is obtained from $A_2 \oplus A_2 \oplus A_2$ by adjoining the root vector

$$\alpha_2 \;=\; -\tfrac{1}{3}\big((\alpha_1 + 2\alpha_3) + (\alpha_6 + 2\alpha_5) + (-\varpi_2 + 2\alpha_2)\big).$$

Hence the action of $\mathfrak{S}_3$ extends to the whole root system E_6, and we get an embedding

$$M \;=\; \prod_{i=1}^{3} W(A_2) \rtimes \mathfrak{S}_3 \;\hookrightarrow\; W(E_6)$$

where the semidirect product on the left hand side is formed with respect to the action of $\mathfrak{S}_3$ which permutes the three factors. Each factor $W(A_2) \cong \mathfrak{S}_3$ acts irreducibly on the corresponding direct summand in

$$\bigoplus_{i=1}^{3} A_2 \otimes_{\mathbb{Z}} \mathbb{C} \;\cong\; E_6 \otimes_{\mathbb{Z}} \mathbb{C},$$

so M acts irreducibly on $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$. But clearly $|M| = 2^4 \cdot 3^4 < |W(E_6)|$.

APPENDIX D

Doubly transitive permutation groups

In this appendix we verify the fact about 2-transitive permutation groups that has been used in the proof of theorem 5.2. For $n \in \mathbb{N}$ consider the set of all n-element subsets of $\{1,2,\dots,2n\}$. The symmetric group $\mathfrak{S}_{2n}$ permutes these $N = \binom{2n}{n}$ subsets transitively, so we have an embedding

$$\varphi : \mathfrak{S}_{2n} \hookrightarrow \mathfrak{S}_N$$

as a transitive subgroup. Consider the stabilizer of $\{1,2,\dots,n\}$, i.e. the subgroup $\mathfrak{S}_n \times \mathfrak{S}_n \subset \mathfrak{S}_{2n}$. This stabilizer fixes precisely two n-element subsets, viz. $\{1,2,\dots,n\}$ and $\{n+1,n+2,\dots,2n\}$. The same holds for the subgroup $A = \mathfrak{A}_n \times \{1\} \subset \mathfrak{S}_n \times \mathfrak{S}_n$.

LEMMA D.1. *Let G be a 2-transitive subgroup of $\mathfrak{S}_N$ containing $\varphi(A)$. Then either $G = \mathfrak{A}_N$ or $G = \mathfrak{S}_N$.*

Proof. By [**19**, prop. 5.2] the 2-transitive permutation group G has a unique minimal normal subgroup H which is either elementary abelian or simple. In the elementary abelian case the degree N of the permutation representation would be a prime power by theorem 4.1 of loc. cit., which is not the case for $N = \binom{2n}{n}$ with $n > 1$. So H is simple, and by loc. cit.

$$H \subseteq G \subseteq Aut(H)$$

where H is one of the finite simple groups in the following list.

	H	N
$a)$	$\mathfrak{A}_N$	arbitrary
$b)$	$PSl_d(q)$	$(q^d-1)/(q-1)$
$c)$	$PSU_3(q)$	q^3-1
$d)$	$\mathrm{Sz}(q)$	q^2+1
$e)$	$\mathrm{R}(q)$	q^3+1
$f)$	$Sp_{2d}(q)$	$2^{d-1}(2^d \pm 1)$
$g)$	further cases	$11,12,15,22,23,24,176,276$

We must show that under the assumptions of our lemma, the only possible case is $a)$. Let us assume first that $n > 4$. Clearly $f)$ and $g)$ cannot occur since in these cases N is not a binomial coefficient $\binom{2n}{n}$, see remark D.2 below. To exclude the remaining cases we use that the group $\varphi(A) \cong \mathfrak{A}_n$ is simple for $n > 4$. Since the outer automorphism group $Out(H) = Aut(H)/H$

is known to be solvable for H a finite simple group [**23**], it follows that the composite $\varphi(A) \hookrightarrow G \hookrightarrow Aut(H) \twoheadrightarrow Out(H)$ is zero. In other words, $\varphi(A)$ is already contained in the subgroup $H \subseteq G$.

Now recall that the permutation group $\varphi(A) \subset \mathfrak{S}_N$ stabilizes precisely two distinct points, corresponding to $\{1,\ldots,n\}$ and $\{n+1,\ldots,2n\}$. Thus cases $c)$, $d)$ and $e)$ cannot occur – in these cases the stabilizer in H of two distinct points is a cyclic group [**63**] and hence cannot contain $\varphi(A) \cong \mathfrak{A}_n$.

In case $b)$ the group $H = PSl_d(q)$ acts as a permutation group via its natural action on the points or on the hyperplanes of $\mathbb{P}^{d-1}(\mathbb{F}_q)$. By duality it suffices to deal with the action on points. Recall that $\varphi(A)$ fixes precisely two distinct points. Let $v_1, v_2 \in \mathbb{F}_q^d$ be vectors whose classes $[v_i] \in \mathbb{P}^{d-1}(\mathbb{F}_q)$ are these two points. Consider the preimage $B \subset Sl_d(q)$ of $\varphi(A)$ under the epimorphism $Sl_d(q) \twoheadrightarrow PSl_d(q) = H$. Since $\varphi(A)$ fixes the points $[v_i]$ in projective space, the group B acts by scalars on the vectors v_i. So the commutator subgroup $C = [B,B]$ acts trivially on the v_i. In particular, C fixes the whole plane spanned by v_1 and v_2. However, since $\varphi(A) \cong \mathfrak{A}_n$ and $n > 4$ we have $\varphi(A) = [\varphi(A), \varphi(A)]$, so with B also the commutator subgroup C surjects onto $\varphi(A)$. Hence $\varphi(A)$ fixes the whole projective line through $[v_1]$ and $[v_2]$, which is impossible since $\varphi(A)$ fixes only two points.

It remains to deal with the cases $n \leq 4$. For $n \in \{3,4\}$ the corresponding value $N \in \{20,70\}$ does not occur in cases $b)$ - $g)$ of the above list, so we can assume $n = 2$. Then $N = 6$ and hence G is a 2-transitive subgroup of the symmetric group $\mathfrak{S}_6$. By [**58**, th. II.4.7] the only such subgroups other than $\mathfrak{S}_6$ and $\mathfrak{A}_6$ are the exceptionally embedded subgroups $\mathfrak{S}_5$ and $\mathfrak{A}_5$. The exceptional embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ is given by the permutation action of $\mathfrak{S}_5$ on the set of its six 5-Sylow groups. Suppose that an element $\sigma \neq 1$ of the exceptionally embedded $\mathfrak{S}_5$ fixes one of these 5-Sylow groups, say the 5-group generated by the cycle $\alpha = (12345) \in \mathfrak{S}_5$. Then there exists an $r \in \{1,2,\ldots,5\}$ such that $\sigma(i) \equiv i + r \mod 6$ for all i, i.e. $\sigma = \alpha^r$. In particular, σ lies inside the 5-Sylow subgroup which it stabilizes. Since the intersection of any two distinct 5-Sylow subgroups is trivial, it follows that every non-trivial element of $\mathfrak{S}_5$ fixes at most one 5-Sylow subgroup. But we assumed that G contains the non-trivial subgroup $\varphi(A) \cong \mathfrak{A}_2$ which fixes two distinct points! So this case does not occur either. $\square$

REMARK D.2. *$2^{d-1}(2^d \pm 1) \neq \binom{2n}{n}$ for all $(d,n) \in \mathbb{N}^2 \setminus \{(2,2)\}$.*

Proof. For $n \in \mathbb{N}$ and a prime p, let $a = a_p(n)$ denote the biggest natural number such that p^a divides $n!$. Then $a_p(n) = \sum_{e \geq 1} \lfloor n/p^e \rfloor$ where the Gauss bracket $\lfloor n/p^e \rfloor$ counts how many of the numbers $1,2,\ldots,n$ are divisible by p^e. In particular, for $p = 2$ this formula implies $a_2(2n) = n + a_2(n)$.

Now suppose that $2^{d-1}(2^d \pm 1) = \binom{2n}{n}$. Then 2^{d-1} is the highest power of two dividing this binomial coefficient. Since this binomial coefficient equals $(2n)!/(n!)^2$, it follows that $d-1 = a_2(2n) - 2a_2(n)$, so from the above formulae for $a_2(2n)$ and $a_2(n)$ we obtain

$$d-1 \;=\; a_2(2n) - 2a_2(n) \;=\; n - a_2(n) \;\leq\; n - \lfloor n/2 \rfloor - \lfloor n/4 \rfloor .$$

For $n \geq 8$ it would follow that $d-1 < n/2-1$ and hence $2d < n$, giving the contradiction

$$2^{d-1}(2^d \pm 1) \;\leq\; 2^{2d} \;<\; 2^n \;<\; \prod_{i=0}^{n-1} \tfrac{2n-i}{n-i} \;=\; \binom{2n}{n}.$$

So we must have $n < 8$. Then $\binom{2n}{n} \in \{2, 20, 70, 252, 924, 3432\}$. Since 2^{d-1} must be the maximal power of two dividing this binomial coefficient, it follows by direct inspection that $d = n = 2$ with $2 \cdot (2^2 - 1) = \binom{4}{2}$ is the only possibility. $\square$

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