University of Heidelberg

Department of Economics

Discussion Paper Series | No. 600

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August 2015

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This paper provides nested sets and vector representations of knockout tournaments. The paper introduces classification of probability domain assumptions and a new set of axioms. Two new seeding methods are proposed: equal gap seeding and increasing competitive intensity seeding. Under different probability domain assumptions, several axiomatic justifications are obtained for equal gap seeding. A discrete optimization approach is developed. It is applied to justify equal gap seeding and increasing competitive intensity seeding. Some justification for standard seeding is obtained. Combinatorial properties of the seedings are studied.

JEL Classification: D01, Z2.

Keywords: elimination tournament, competitive intensity, fairness, economics of sport.

[1] The author would like to thank Fuad Aleskerov for valuable comments.

1. Introduction

A knockout tournament(also called a single elimination tournament or elimination tournament) is a system of selecting a single winner. Participants (teams or individuals) play *n* rounds of matches. After each round, the winners move on to the next round, the losers are dropped from the tournament, and the number of participants is reduced by half. Only balanced knockout tournaments without byes are considered in the paper. The number of participants in each round in such a tournament always equals a power of two. Knockout tournaments are widespread in sports competitions. They are frequently used as a playoff tournament in a larger tournament with several leagues, groups and conferences and/or as a qualifying tournament (Noll 2003).

The key feature of knockout tournaments is the importance of the seeding method, or simply the seeding. The seeding is a rule that, having information regarding the initial order of participants' strengths, determines the tournament bracket. There are several seedings used in different tournaments. The most popular seeding (called standard seeding) creates pairs in the first round of the strongest participant with the weakest participant, the second strongest participant with the second weakest participant, etc. The pairs in subsequent rounds are determined in a way that preserves the first two participants from the head-to-head match before the final and that delays the confrontations between strong participants until later rounds. Strong participants are rewarded for their success through such a seeding.

This paper is limited to seeding methods with predetermined tournament brackets, i.e., without random seeding (e.g., the method proposed in Schwenk (2000)) and without any reseeding methods (e.g., the method proposed in Hwang (1982)). This restriction comes from the tournament practice. The World Cup, NBA playoffs, NCAA basketball tournament, the NHL's Stanley Cup Playoffs and most professional tennis events are examples of tournaments without reseeding (Baumann et al 2010). Most professional tennis events are examples of tournaments without reseeding. Baumann et al. (2010) noted that reseeding causes teams and spectators to have to make last-minute travel plans, which both increases costs and potentially reduces demand, and that reseeded tournaments eliminate popular gambling options related to filling out full tournament brackets, potentially reducing fan interest in the tournament.

Seeding aims to ensure high competitive intensity, especially in the latter rounds of a tournament (Wright 2014), to provide incentives for the participants to maximize their performance both during the tournament and in the time period leading up to the tournament (Baumann et. al. 2010) to maximize spectator interest (Dagaev, Suzdaltsev 2015) and to ensure fairness (Horen, Riezman 1985). Such requirements create the theoretical problem of designing an ideal seeding method.

Standard seeding has various shortcomings. There is much evidence that, in the NCAA men's basketball tournament, weaker participants have a higher probability of winning the tournament and a better chances of advancing in some rounds than certain stronger participants (see Boulier, Stekler (1999), Baumann et al (2010), Jacobson et al (2011), Khatibi et al (2015)). It is an example of a violation of fairness and incentive compatibility.

Theoretical studies have attempted to find a best seeding and formalize different goals of seeding. Many studies contain impossibility results. Horen and Riezman (1985) showed that there is no "fair" seeding in the sense that a better participant has a higher probability of winning in the case of eight participants and a double monotonic domain. Schwenk (2000) proposed the

Delayed Confrontation, Sincerity Rewarded and Favoritism Minimized axioms. This set of axioms can be satisfied only within randomized seeding methods. Vu and Shoham (2010) investigated several impossibility results. They proved that there is no envy-free seeding under a double monotonic domain, there is no order-preserving seeding under a general domain, and there is no seeding satisfying the order preservation requirement and the condition for robustness against dropout.

Numerous statistical and simulation studies (Garry, Schutz 1997; Marchand 2002; Annis, Wu 2006; Glickman 2008; Scarf, Yusof 2011) did not find clear support for any seeding method with a predetermined tournament bracket. Computer science studies (Hazon et al 2008; Vu et. al. 2009; Vassilevska Williams 2010; Aziz et al 2014) aimed to find seedings with given characteristics. Basic problems, e.g., computing the maximum possible winning probability of a given player, are NP-hard.

The economic literature with game-theoretic analysis of knockout tournaments has mainly focused on incentives of exerting effort and selection properties. The problem of designing an optimal seeding is solved mainly for four-participant tournaments (Rosen 1986; Groh et al 2012; Kräkel 2014; Stracke et al 2015). For tournaments with an unrestricted number of participants, Ryvkin (2009) showed that it is not possible to manipulate the aggregate effort through seeding in the case of weakly heterogeneous participants.

The aim of this paper is to provide possibility results and give axiomatic justification for various seedings methods. The equal gap seeding and increasing competitive intensity seeding investigated in the paper were, to the author's best knowledge, never discussed in academic literature. Considering different domains and sets of axioms, several axiomatic justifications of equal gap seeding are proposed. A discrete optimization approach develops the axiomatic approach. It gives additional justification for the equal gap and increasing competitive intensity seedings and provides some support to standard seeding.

This paper is closely related to decision theory literature. Eliaz et al. (2011) axiomatized procedures of choosing the two finalists. Bajraj and Ülkü (2015) axiomatized procedures of choosing the two finalists and the winner. This paper axiomatizes procedures of choosing the winner, the two finalists, the four quarterfinalists, etc. within the specific domain of balanced elimination tournaments.

The structure of the paper is as follows. Section 2 describes seeding methods and probability domains. Section 3 presents axiomatics, combinatorial properties and representation theorems for the equal gap method. Section 4 presents a discrete optimization approach with justifications for the equal gap and the increasing competitive intensity seedings. Section 5 concludes the paper. Appendix 1 includes all proofs. Appendix 2 contains a table with comparisons of the investigated seedings under various domain assumptions.

2. Framework

Let $X = \{1,2,\dots,2^n\}$ be the set of alternatives (participants) and n be the number of rounds in the knockout tournament (later in the text tournament). The indices of the participants represent the order of the participants' strengths, where the first participant is the strongest. Let $T^1 = \{1,2\}$ be a tournament with one round and $T_i^{1,n} = \{a,b\}$, $a \neq b$; $a,b \in X$ be a subtournament with one round of a tournament with n rounds. Each tournament with $n \geq 2$

rounds consists of two subtournaments. Each subtournament with $n \geq 2$rounds also consists of two subtournaments. $T_i^{k,n}$ is a subtournament *i* with *k* rounds. It is part of a tournament with n rounds. For notational convenience, let $T^{n,n} = T^n$ and $T_i^{0,n} = \{i\}$. Let $\mathbb{T}^{k,n}$ be the set of all possible subtournaments with *k* rounds in a tournament with *n* rounds. Subtournaments $T_i^{k,n}, T_j^{k,n}$ are nonoverlapping if there is no participant that plays in both subtournaments. A tournament with *n* rounds is a set $T^n = \{T_i^{n-1,n}, T_j^{n-1,n}\}$, $T_i^{n-1,n}, T_j^{n-1,n} \in \mathbb{T}^{n-1,n}$, where subtournaments $T_i^{n-1,n}, T_j^{n-1,n}$ are nonoverlapping. A subtournament with *k* rounds is a set $T^{k,n} = \{T_i^{k-1,n}, T_j^{k-1,n}\}$, $T_i^{k-1,n}, T_j^{k-1,n} \in \mathbb{T}^{k-1,n}$, where subtournaments $T_i^{k-1,n}, T_j^{k-1,n}$ are nonoverlapping.

Tournaments $\{\{1,2\},\{3,4\}\}$ and $\{\{4,3\},\{1,2\}\}$ represent the same seeding. In this seeding, there are two matches, $\{1,2\},\{3,4\}$, in the first round. In the second round, the winners of each match meet. It is convenient to order subtournaments lexicographically. The subtournament with the strongest participant is the first. In each match, the strongest participant is also the first. Let s_l^n be the vector representation of a lexicographically ordered tournament. There are only three different two-round tournament seedings, $s_1^n = (1,2,3,4)$, $s_2^n = (1,3,2,4)$, and $s_1^n = (1,4,2,3)$. Participant *i* of subtournament *l* with *k* rounds in the vector representation is denoted as $s_{l,i}^{k,n}$.

2.1 Seedings

The total number of possible seedings in a tournament with *n* rounds is equal to

$$NS(n) = \frac{2^n!}{2^{2^n-1}}.$$

It is a fast-growing function with $NS(3) = 315$, $NS(4) = 638512875$, etc. It has the recurrence representation

$$NS(n) = (2^n - 1)!! \cdot NS(n-1),$$

$$NS(n) = \binom{2^n - 1}{2^{n-1} - 1} \cdot (NS(n-1))^2,$$

where $x!!$ is a double factorial and $\binom{x}{y}$ is a binomial coefficient.

Sports competitions and academic literature mainly address standard seeding. As an example, several eight-participant seedings are discussed in the literature, but only close seeding has a clear generalization. Three new seedings are proposed in this section.

2.1.1 Standard seeding

Standard seeding is defined recursively. The first rounds of the subtournaments are as follows

$$T_i^{1,n} = \{i, 2^n - i + 1\}, i = \overline{1, 2^{n-1}}.$$

For any *k* from 2 to n, we have

$$T_i^{k,n} = \{T_i^{k-1,n}, T_{2^{n-k+1}-i+1}^{k-1,n}\}, i = \overline{1, 2^{n-k}}.$$

Thus, for $n = 3$, we have

$$T_{standard}^3 = \{\{\{1,8\},\{4,5\}\},\{\{2,7\},\{3,6\}\}\}.$$

Standard seeding (also called seeded tournament, distant seeding (Dagaev, Suzdaltsev 2015), and balanced seeding (Ryvkin 2011)) is the most popular method. Generally, this method creates an advantage for strong participants, but Baumann et al. (2010) found that, in the NCAA men's basketball tournament, the 10th and 11th seeds have, on average, more wins and typically progress further in the tournament than the 8th and 9th seeds. The NCAA men's basketball tournament is a tournament with 16 participants

$$T^4_{standard} = \left\{\left\{\{\{1,16\},\{8,9\}\},\{\{4,13\},\{5,12\}\}\right\},\left\{\{\{2,15\},\{7,10\}\},\{\{3,14\},\{6,11\}\}\right\}\right\}.$$

Participants 8 and 9 have weaker opponents in the first round than participants 10 and 11 have. In the second round, the situation is reversed. Participants 8 and 9, with high probability, play against participant 1, but participants 10 and 11 play against, at the highest, participants 2 and 3. The advantages in the second round outweigh the disadvantages in the first round of the tournament.

Dagaev and Suzdaltsev (2015) showed that the standard seeding is a unique seeding that maximizes spectator interest when spectators care mainly about quality and later-round matches. Ryvkin (2011) found that the standard seeding, in conjunction with a uniform distribution of participants' relative abilities, can cancel out the dependence of a participant's equilibrium effort on her rivals' abilities.

2.1.2 Close seeding

Close seeding is defined recursively. The first rounds of the subtournaments are as follows

$$T^{1,n}_i = \{2i-1, 2i\}, i = \overline{1, 2^{n-1}},$$

For any k from 2 to n, we have

$$T^{k,n}_i = \{T^{k-1,n}_{2i-1}, T^{k-1,n}_{2i}\}, i = \overline{1, 2^{n-k}}.$$

Thus, for $n = 3$, we have

$$T^3_{close} = \left\{\{\{1,2\},\{3,4\}\},\{\{5,6\},\{7,8\}\}\right\}.$$

The vector representation of the close seeding is the following

$$s^n_{close} = (1,2,\dots,2^n-1,2^n).$$

This is the most preferable seeding to the weakest participant and a highly undesirable seeding for strong participants. Many strong participants are dropped in the first rounds of the tournament. This contradicts the main goal of seeding. Despite its simplicity, close seeding is not used in real tournaments.

Dagaev and Suzdaltsev (2015) showed that close seeding is a unique seeding that maximizes spectator interest when spectators care mainly about competitive intensity and do not prefer later-round matches.

2.1.3 Equal gap seeding

Equal gap seeding is defined recursively. The first rounds of the subtournaments are as follows

$$T^{1,n}_i = \{i, i+2^{n-1}\}, i = \overline{1, 2^{n-1}}.$$

For any k from 2 to n, we have

$$T^{k,n}_i = \{T^{k-1,n}_i, T^{k-1,n}_{i+2^{n-k}}\}, i = \overline{1, 2^{n-k}}.$$

Thus, for $n = 3$ and $n = 4$,we have

$$T_{equalgap}^{3} = \Big\{\{\{1,5\},\{3,7\}\},\{\{2,6\},\{4,8\}\}\Big\};$$

$$T_{equalgap}^{4} = \Big\{\Big\{\{\{1,9\},\{5,13\}\},\{\{3,11\},\{7,15\}\}\Big\},\Big\{\{\{2,10\},\{6,14\}\},\{\{4,12\},\{8,16\}\}\Big\}\Big\}.$$

To the author's best knowledge, equal gap seeding has never been discussed in academic literature, and it is only applied for the second life, known as the Process, in two-life croquet tournaments in England, New Zealand, Australia and the United States (The Laws of Association Croquet, 2000).

2.1.4 Increasing competitive intensity seeding I

Increasing competitive intensity seeding I is defined recursively with $\mathrm{s}^1 = (1,2)$. For any $n \geq 2$, we have

$$s_1^n = 1,$$
$$s_i^n = s_i^{n-1} + 2^{n-1}, \qquad i = \overline{2, 2^{n-1}},$$
$$s_i^n = s_{i-2^{n-1}}^{n-1} + 1, i = \overline{2^{n-1} + 1, 2^n}.$$

Thus, for $n = 3$ and $n = 4$, we have

$$T_{ici\ I}^{3} = \Big\{\{\{1,8\},\{6,7\}\},\{\{2,5\},\{3,4\}\}\Big\};$$

$$T_{ici\ I}^{4} = \Big\{\Big\{\{\{1,16\},\{14,15\}\},\{\{10,13\},\{11,12\}\}\Big\},\Big\{\{\{2,9\},\{7,8\}\},\{\{3,6\},\{4,5\}\}\Big\}\Big\}.$$

2.1.5 Increasing competitive intensity seeding II

Increasing competitive intensity seeding II is designed on the basis of increasing competitive intensity seeding I. For any $n < 3$, we have

$$\mathrm{s}^1 = (1,2), \mathrm{s}^2 = (1,4,2,3).$$

For any $n \geq 3$, we have

$$s_1^n = 1, s_{2^{n-1}+1}^n = 2,\ s_{2^{n-1}+2}^n = 2^{\mathrm{n}};$$
$$s_i^n = s_{ici\ I,i}^{n-1} + 2^{n-1} - 1, \qquad i = \overline{2, 2^{n-1}},$$
$$s_i^n = s_{ici\ I,i-2^{n-1}}^{n-1} + 1, i = \overline{2^{n-1} + 3, 2^n}.$$

Thus, for $n = 3$ and $n = 4$, we have

$$T_{ici\ II}^{3} = \Big\{\{\{1,7\},\{5,6\}\},\{\{2,8\},\{3,4\}\}\Big\};$$

$$T_{ici\ II}^{4} = \Big\{\Big\{\{\{1,15\},\{13,14\}\},\{\{9,12\},\{10,11\}\}\Big\},\Big\{\{\{2,16\},\{7,8\}\},\{\{3,6\},\{4,5\}\}\Big\}\Big\}.$$

The increasing competitive intensity seedings I and II favor participant 1.

2.2 Probability domains

The order of participants' strengths is known at the beginning of the tournament. Suppose that the strengths of participants remain unchanged during the tournament and that a probability p_{ij} that participant *i* wins against participant *j* depends only on the strengths of participants *i* and *j*. The organizers have some expectations (or assumptions) about p_{ij}. Because $p_{ij} + p_{ji}$ =1, we need only $2^n(2^n - 1)/2$ probabilities to describe all possible situations. These probabilities are also called the preference scheme (Hwang 1982). A set of all possible values $p_{ij},\ i,j \in X, i < j$, generates a probability domain $\mathcal{P} \subseteq [0,1]^{2^{n-1}(2^n-1)}$. Based on such domain, it is possible to derive properties of seedings. Properties of seedings help organizers choose the best one. Table 1 has collected several probability domains from knockout tournament literature.

Table 1. Probability domains.

Domain	Restrictions	Seeding studies	Comments
General domain $\mathcal{P}_g$	$0 \le p_{ij} \le 1$.	Vu, Shoham 2010.	Under the general domain assumption, a stronger participant can lose with a probability strictly greater than one half. There is no violation of the strength order. A stronger participant can win against some other participant with a higher probability.
Stochastically transitive domain $\mathcal{P}_{st}$	$0.5 \le p_{ij} \le 1$.	David 1959; Searls 1963.	It satisfies weak stochastic transitivity ($\forall i,j,k \in X$ if $p_{ij} \ge 0.5$ and $p_{jk} \ge 0.5$ then $p_{ik} \ge 0.5$).
Doubly monotonic domain $\mathcal{P}_{dm}$	$0.5 \le p_{ij} \le 1$, $p_{i'j} \ge p_{ij}, i' < i$, $p_{ij'} \ge p_{ij}, j' > j$.	Glenn 1960; Schwenk 2000; Vu, Shoham 2010; Dagaev, Suzdaltsev 2015.	It satisfies strong stochastic transitivity ($\forall i,j,k \in X$ if $p_{ij} \ge 0.5$, and $p_{jk} \ge 0.5$ then $p_{ik} \ge max(p_{ij}, p_{jk})$).
Linear domain $\mathcal{P}_l$	$p_{ij} = F(\theta_i - \theta_j)$, where $\theta_i > \theta_j$ are strengths of participants *i*, *j*, $F: (0,\infty) \longrightarrow [0.5,1]$ is nondecreasing concave function.	This domain was not applied for seedings studies.	In Bradley (1976) survey F is a distribution function for a symmetric distribution. Stern (1992) and Baker, McHale (2014) used gamma distribution.
Restricted linear domain $\mathcal{P}_{\mathrm{rl}}$	Linear domain with $\theta_i = 2^n + 1 - i$; F is strictly increasing concave function.	This domain was not applied for seedings studies.	Grifiths (2005) proposed to use $F(x) = 1 - a^x$, where $0 < a < \frac{1}{2}$, or $F(x) = \frac{1}{2} + \frac{x}{2^{n+1}}$.
Bradley-Terry domain $\mathcal{P}_{BT}$	$p_{ij} = \frac{\theta_i}{\theta_i + \theta_j}$, where $\theta_i > \theta_j > 0$ are strengths of participants *i*, *j*.	Bradley, Terry, 1952; Hartigan 1968; Prince et. al. 2013	Some generalizations of Bradley-Terry domain were proposed in (Koehler, Ridpath 1982; Rosen 1986; Hwang et al 1991).
Equal probabilities domain $\mathcal{P}_{\mathrm{ep}}$	$p_{ij} = p$, $0.5 \le p \le 1$.	-	$\mathcal{P}_{\mathrm{ep}} \not\subset \mathcal{P}_{BT}$ and $\mathcal{P}_{\mathrm{ep}} \not\subset \mathcal{P}_{rl}$.
Deterministic domain $\mathcal{P}_d$	$p_{ij} = 1$.	Wiorkowski, 1972; Vassilevska Williams 2010; Dagaev, Suzdaltsev 2015	$\mathcal{P}_d \subset \mathcal{P}_{\mathrm{ep}}$.

Having $\theta_i^l = ln(\theta_i^{BT}), \theta_j^l = ln(\theta_j^{BT})$ and $F = \frac{1}{1+e^{-x}}$, one can show that $\mathcal{P}_{BT} \subset \mathcal{P}_l$. $F = p$ leads to $\mathcal{P}_{ep} \subset \mathcal{P}_l$. Summarizing Table 1, we obtain

$$\mathcal{P}_d \subset (\mathcal{P}_{BT} \cup \mathcal{P}_{rl} \cup \mathcal{P}_{ep}) \subset \mathcal{P}_l \subset \mathcal{P}_{dm} \subset \mathcal{P}_{st} \subset \mathcal{P}_g.$$

Because F_{rl}, $F = \frac{1}{1+e^{-x}}$ are strictly increasing functions, $(\mathcal{P}_{rl} \cap \mathcal{P}_{ep}) \cup (\mathcal{P}_{BT} \cap \mathcal{P}_{ep}) = \emptyset$. The intersection of $\mathcal{P}_{rl}$ and $\mathcal{P}_{BT}$ is nonempty. It includes $\theta_i^{BT} = ca^{2^{n-i}}$, $a, c > 0$ in the Bradley-Terry domain and $F = \frac{1}{1+a^{-x}}$, $a > 0$ in the restricted linear domain. Appendix 2 contains many examples of various probability assumptions.

3. Axiomatic justification

Participants, organizers and spectators have their own interests. Participants want to play as many matches as possible. Organizers want to support spectator interest in all matches. Spectators want to see strong participants and matches with high competitive intensity. All of them are interested in a fair tournament with a strong winner. These ideas are represented in the following set of axioms.

A1.Delayed confrontation. (Schwenk 2000). Two participants rated among the top 2^k shall never meet until the number of participants has been reduced to 2^k or fewer.

This axiom represents the main idea of seeding. A strong participant should not be dropped at the beginning of the tournament. Spectators want to see strong participants. Strong participants should play more games. It supports spectator interest.

In the case of the deterministic domain, delayed confrontation means that, in any round, the strongest half of all participants wins, and in each round, the strongest loser participant is weaker than the weakest winner participant. We can define a new order of participants based on the number of wins (the number of rounds a participant plays). The more wins, the better a participant. A knockout tournament generates a weak order with several indifference sets: participants with no wins, with one win and so on. The initial order is linear order. In the case of the deterministic domain, we can interpret axiom A1 as the order-preserving property. If participant i is strictly better than participant j in terms of initial order, then participant i is at least as good as participant j in tournament wins ordering. If Axiom A1 holds in the deterministic domain, then it hold in any other domain in which a stronger participant wins against a weaker participant with a positive probability, i.e., in any reasonable probability domain.

There is another equivalent definition of axiom A1. The top 2^{k-1} participants of any subtournament with k rounds shall be divided equally between two subtournaments of this subtournament.

A2 Fairness. A seeding is fair if and only if, for any k, a participant's probability of winning k matches is no less than that of any weaker participant.

A2' Weak fairness. A seeding is weakly fair if and only if, for any k, a probability of winning k matches for any participant from the strongest 2^{n-k} participants is no less than that of any participant from the set of participants $\{2^{n-k}+1, \dots, 2^n\}$.

These axioms represent the same idea as monotonicity (Hwang 1982), fairness (Horen and Riezman 1985), sincerity rewarded (Schwenk 2000), envy-freeness, order preservation (Vu, Shoham 2011), and fairness (Prince et al. 2013). These axioms are also the incentive compatibility property. From this axiom, we conclude that the strongest participant has the highest probability of winning, the two strongest participants have the highest probabilities to reach the final, etc. These axioms prevent situations in which a participant will be strictly better off by losing a game in the qualification tournament and becoming weaker in the initial order. These incentives are widely discussed in the literature (Silverman, Schwartz 1973; Taylor, Trogdon 2002; Tang et al. 2010; Dagaev, Sonin 2014).

A3 Balance. Two subtournaments of any subtournament with more than one round shall have an equal sum of participants' ranks.

The sum of participants' ranks represents the strength (quality) of a subtournament. Axiom A3 equates the strengths (qualities) of all rival's subtournaments. Because the sum of all participants' ranks in the tournament equals $2^{n-1}(2^n+1)$, the sum of participants' ranks in any first round match equals 2^n+1. Except the first round match, this axiom does not create any constraints.

A4 Symmetry. A tournament designed for strength-ordered participants is equal to a tournament designed for weakness-ordered participants.

Tournaments $\{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ satisfy symmetry. Having reverse order $\{1,2,3,4\} \to \{4,3,2,1\}$, we obtain new tournaments $\{\{1,2\},\{3,4\}\} \to \{\{4,3\},\{2,1\}\} = \{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\} \to \{\{4,1\},\{3,2\}\} = \{\{1,4\},\{2,3\}\}$. All matches are the same. Tournament $\{\{\{1,8\},\{6,7\}\},\{\{2,5\},\{3,4\}\}\}$ violates symmetry. Having reverse order $\{1,2,3,4,5,6,7,8\} \to \{8,7,6,5,4,3,2,1\}$, we obtain a new tournament

$\{\{\{1,8\},\{6,7\}\},\{\{2,5\},\{3,4\}\}\} \to \{\{\{8,1\},\{3,2\}\},\{\{7,4\},\{6,5\}\}\} \neq \{\{\{1,8\},\{6,7\}\},\{\{2,5\},\{3,4\}\}\}$.

Pair $\{1,8\}$ plays with pair $\{6,7\}$ in the second round. Pair $\{1,8\}$ remains unchanged,but pair $\{6,7\}$ became stronger under the new order.

Axiom A4 is a strength/weakness invariance property. It does not matter whether strength or weakness is measured.

3.1 Deterministic domain assumption

The following three axioms are designed for the deterministic domain.

A5 Increasing competitive intensity. In each subsequent round, a winner faces a stronger participant than in the previous round.

This axiom supports spectator interest. It eliminates antagonism between the quality of the match (the sum of strengths) and its competitive intensity (the difference of strengths), discussed in (Dagaev, Suzdaltsev 2015). From round to round, the strength of rivals becomes higher and closer. The quality of the match and intensity of competition increase.

A6 Equal rank sums. For any match in round *k*, the sum of ranks of rivals is the same.

This axiom tries to equate all matches of one round by their quality. It supports spectator interest in all matches. Proposition 1 shows that only standard seeding satisfies this axiom. The proof for proposition 1 and subsequent propositions are given in the appendix.

Proposition 1. *Under the deterministic domain assumption, the standard seeding is a unique seeding that satisfies axiom A6.*

A7 Equal rank differences. For any match in round *k*, the difference in the ranks of rivals is the same.

Axiom A7 is a counterpart of axiom A6. This axiom tries to equate all matches of one round by competitive intensity. The rank difference varies from round to round. If $d \in \mathbb{N}^n$ is a vector of rank differences, then $\forall i \neq j, i, j \in K$, $d_i \neq d_j$.

Under the Bradley-Terry domain assumption, equal rank differences in the first round is a necessary condition for fairness (Prince et al. 2013, corollary 4.1). Close seeding and equal gap seeding satisfy axiom A7, and these seedings have the same set of rank differences {1,2,4}. Propositions 2 and 3 characterize seedings that satisfy axiom A7 and show that equality of rank difference sets is a general property.

Proposition 2. *If there exists a tournament with a vector of rank differences* $d \in \mathbb{N}^n$*, then for any permutation* σ*, there exists a tournament with a vector of rank differences* $\sigma d \in \mathbb{N}^n$*.*

Proposition 3. *If axiom A7 holds, then the rank difference in round k is equal to* $d_k = 2^{x_k}$*,* $x_k \in \mathbb{N}_0$*.*

Table 2 summarizes seedings properties under the deterministic domain assumption. Proposition 4 describe the relations between axioms.

Table 2. Seedings properties under the deterministic domain assumption

Seedings	A1	A2	A3	A4	A5	A6	A7
Standard	+	+	+	+	+	+	-
Close	-	-	-	+	-	-	+
Equal gap	+	+	-	+	+	-	+
Increasing competitive intensity I	-	-	-	-	+	-	-
Increasing competitive intensity II	-	-	-	-	+	-	-

Proposition 4. *Under the deterministic domain assumption, the following three statements hold*
(i) A1, A2, and A2' are equivalent;
(ii) A5 and A7 imply A1;
(iii) A1 implies A5.

Proposition 5 finds $NAi(n)$, the number of seedings that satisfies axiom Ai and characterizes the asymptotic properties of axioms.

Proposition 5. *Under the deterministic domain assumption, the following seven statements hold*
(i) $NA1(n) = NA2(n) = \prod_{k=1}^{n} 2^{k-1}!$;
(ii) $NA3(n) = \frac{2^{n-1}!}{2^{2^{n-1}-1}}$;
(iii) $NA4(n) = 2^{n-1}!\, y(n)$,
where $y(n) = 0.5(y(n-1))^2 + 1$ with $y(1) = 1$;
(iv) $NA5(n) = \frac{(2^n-2)!}{\prod_{k=1}^{n-2}(2^{n-k}-1)^{2^k}}$;
(v) $NA6(n) = 1$;
(vi) $NA7(n) = n!$;
(vii) $lim_{n\to\infty} \frac{NAi(n)}{NS(n)} = 0\,, i = \overline{1,7}$.

For $n = 3$, we have $NA1(3) = NA2(3) = 48$, $NA3(3) = 3$, $NA4(3) = 51$, $NA5(3) = 80$, $NA6(3) = 1$, and $NA7(3) = 6$. These combinatorial formulas help to analyze random seeding. The probability that random seeding satisfies any of axioms A1-A7 converges to zero as *n* goes to infinity. This is an argument against random seeding.

3.2 Representation theorems for the equal gap method.

This section includes two representation theorems for the equal gap method under various domain assumptions.

Proposition 6. *Under the deterministic domain assumption, the equal gap seeding is a unique seeding that satisfies one of three sets of axioms:*
(i) A1 and A7;
(ii) (A2 or A2') and A7;
(iii) A5 and A7.

From proposition 4, the set of axiomsA5 and A7 is the weakest requirement. All three justifications contain axiom A7, but it is not necessary for axiomatic justification under the restricted linear domain and for the discrete optimization justification presented in the following propositions. Proposition 7 uses a modification of axiom A2' and provides new justification for the equal gap method.

Proposition 7. *Under the restricted linear domain assumption, equal gap seeding is a unique seeding for which, for any k, the strongest* 2^{n-k} *participants have equal probability of winning* k *matches.*

Corollary. *Under the restricted linear domain assumption, equal gap seeding is weakly fair.*

Equal gap seeding is not weakly fair under the linear domain assumption. Consider an example for $n = 2$, $T^2_{equalgap} = \{\{1,3\},\{2,4\}\}$. Let $\theta_1 = 3, \theta_2 = 2, \theta_3 = 1, \theta_4 = -10^6$ and $F(x) = 1$ if $x \geq 10^6$, otherwise $F(x) = 0.5 + \frac{x}{2\cdot 10^6}$. F is a concave nondecreasing function. The probability that participant 1 wins the tournament is approximately equal to 0.25, and the probability that participant 2 wins the tournament is approximately equal to 0.5. This contradicts weak fairness.

4. Discrete optimization

The main purpose of seeding is to prevent strong participants from being eliminated at the beginning of a tournament. Many studies maximize the probability of the event "participant 1 wins a tournament."It is a narrow approach. Such events as "the strongest2^{n-k}participants win k matches" are also desirable for tournament design. Proposition 8 generalizes the discrete optimization approach, forming the general form of the objective function. This objective function represents an idea of weak fairness (axiom A2'). The probabilities that the top 2^{j-1} participants win $n - j + 1$ matches should be as high as possible.

Proposition 8. *Under the linear domain assumption, equal gap seeding is a unique seeding that maximizes*

$$\prod_{i=1}^{n} \left[Pr\left(\bigcup_{j=i}^{n} \{top\ 2^{j-1}\ participants\ win\ n-j+1\ matches\} \right) \right]^{\alpha_i},$$

where $\alpha_1 > 0, \alpha_i \geq 0, i = \overline{2,n}$.

The most meaningful corollary of proposition 8 is the case $\alpha_1 = 1, \alpha_i = 0,\ i = \overline{2,n}$. Under the linear domain assumption, equal gap seeding is the only seeding that maximizes the probability of advancing the strongest half of the participants in each round

$$\Pr\left(\bigcup_{i=0}^{n-1} \{top\ 2^i\ participants\ win\ n-i\ matches\} \right).$$

Equal gap seeding maximizes the probability that the strongest participant wins the tournament, the two strongest participants play in the final, the four strongest participants play in the quarterfinal, etc.

Axiom A5 is designed for the deterministic domain. Under other domains, axiom A5 holds with some probability. Maximizing such probability, it is possible to represent the idea of increasing the competitive intensity in various domains. Under the equal probabilities domain assumption, Proposition 9 represents optimal seedings.

Proposition 9. *Under equal probabilities domain assumption the increasing competitive intensity seedings I and II maximize probability of increasing competitive intensity (in each subsequent round a winner faces with a stronger participant than in previous round) and this probability equals to* $p_n^* = p^{2^{n-1}-1}$*, where* p *is a probability of stronger participant winning in any match.*

The increasing competitive intensity seedings I and II are not unique seedings that maximize the probability of increasing competitive intensity under the deterministic domain assumption. For example, $\{\{\{1,8\},\{5,6\}\},\{\{2,7\},\{3,4\}\}\}$ also has the highest probability of increasing competitive intensity. Under the linear domain assumption, $T^3_{ici\,II}$ is a unique seeding that maximizes the probability of increasing competitive intensity in the case of $n = 3$. The general solution of this discrete optimization problem is an issue for future research.

The discrete optimization problem from Proposition 8 is a counterpart of axiom A2'. The problem from Proposition 9 is a counterpart of axiom A5. It is possible to define a similar problem for axiom A7. Because weaker participants can win, the rank differences may not be equal to a power of two, and such a problem may be quite complicated. Because this problem has no rich interpretation, it is not solved. My conjecture is that, under the linear domain assumption, the equal gap method maximizes the probability of equal rank differences.

Appendix 2 compares equal gap seeding, standard seeding, and increasing competitive intensity seedings I and II. It illustrates how proposition 8 shows the probability that the strongest participant wins the tournament, the two strongest participants play in the final, the four strongest participants play in the quarterfinal, etc. Many examples confirm the superiority of equal gap seeding under the linear domain assumption. Appendix 2 also provides an example of the superiority of the standard seeding under the double monotonic domain assumption. The increasing competitive intensity seedings are the best seedings under the linear domain assumption, but under the double monotonic domain assumption, there exist better seedings. In our comparison, standard seeding is always in the middle. It is a compromise seeding.

5. Conclusion

The paper provides nested set and vector representations of tournament brackets. A new set of axioms and new methods (equal gap and increasing competitive intensity seedings) are designed. The paper provides several justifications of the equal gap method under deterministic, restricted linear and linear domain assumptions. It is unique seeding that, under the deterministic domain assumption, satisfies the delayed confrontation, fairness, increasing competitive intensity and equal rank differences axioms and that, under the linear domain assumption, maximizes the probability that the strongest participant is the winner, the strongest two participants are the finalists, the strongest four participants are the quarterfinalists, etc. Because of the simplicity and several justifications, equal gap seeding is a plausible alternative to standard seeding.

The increasing competitive intensity seedings I and II maximize the probability of increasing competitive intensity. Because these seedings have complicated designs, it is hard to implement them in real tournaments.

There is a trade-off between the weak fairness and increasing competitive intensity criteria. Under the linear domain assumption equal gap seeding is optimal with respect to the first criterion. Increasing competitive intensity seedings maximize the second criterion. Under the linear domain assumption, the standard seeding is always between these seedings in terms of the given criteria, but closer to the maximum than to the minimum. This is a trade-off justification of the standard seeding.

Appendix 1

Proof of proposition 1. If participant 1 is not paired with participant 2^{n}, then the sum of strengths is less than $2^{\mathrm{n}}+1$, and a pair, which includes participant 2^{n}, has a sum greater than $2^{\mathrm{n}}+1$. Therefore, $T_i^{1,n}=\{i,2^n-i+1\}$, $i=\overline{1,2^{n-1}}$, and the strongest $2^{\mathrm{n}-1}$ participants advance in the first round. For any round k from 2 to n, participant 1 should be paired with participant 2^{n-k+1}, and $T_i^{k,n}=\{T_i^{k-1,n},T_{2^{n-k+1}-i+1}^{k-1,n}\}$, $i=\overline{1,2^{n-k}}$.■

Proof of proposition 2. It is sufficient to prove the existence of a tournament with permuted two neighboring elements of a rank differences vector d. Having a tournament T with a vector of rank differences d, we will design a tournament T' with a vector of rank differences d' where, for some j, $0<j<n$, $d'_{j+1}=d_j$, $d'_j=d_{j+1}$, and $d'_i=d_i$ for all $i\neq j$, $i\neq j+1$. Let $S_k(T)\in\mathbb{N}^{2^{n-k}}$ be a set of the strongest participants of all subtournaments with k rounds of tournament T. If $i\in S_{k+1}(T)$, then $i\in S_k(T)$. If $i\in S_k(T)$, then $i+d_k\in S_{k-1}(T)$. All subtournaments of tournament T are numbered by the strongest participant of the subtournament; thus, we have $T_i^{1,n}=\{i,i+d_1\}, T_i^{k,n}=\{T_i^{k-1,n},T_{i+d_k}^{k-1,n}\}$.

Let $\forall k\colon 0<k<j, \forall i\in S_k(T), T'^{k,n}_i=T_i^{k,n}$.

There are 2^{n-j-1} pairs $a,b\in S_j(T)$ such that $b-a=d_{j+1}=d'_j$. Let $A=S_{j+1}(T)$, $B=\{a+d'_j|a\in A\}$. For all $a\in A$ and for all $b\in B$, we have$a,b\in S_{j-1}(T)$ and $a+d_j$, $b+d_j\in S_{j-1}(T)$. $A'=\{a+d_j|a\in A\}$, $B'=\{b+d_j|b\in B\}$. In round j of tournament T, $i\in A$ is paired with $i+d_j$, which belongs to A', and $i\in B$ is paired with $i+r_j$, which belongs to B'. There are 2^{n-j} such pairs. The union of all of these pairs generates set $S_{j-1}(T)$. In round j of tournament T', participant $i\in A$ is paired with participant $i+d'_j$, which belongs to B, and $i+d_j\in A'$ is paired with $b+d'_j\in B'$. Thus, $\forall i\in A\cup A'$, we have $T_i^{j,n}=\{T_i^{j-1,n},T_{i+d'_j}^{j-1,n}\}$, and $\forall i\in A$, we have $T_i^{j+1,n}=\{T_i^{j,n},T_{i+d'_{j+1}}^{j,n}\}$.

Because $S_{j+1}(T)=S_{j+1}(T')=A$, it is possible to define that, $\forall k\colon j+1<k<n$, $\forall i\in S_k(T)$, $T_i^{k,n}=\{T'^{k-1,n}_i,T'^{k-1,n}_{i+d'_k}\}$ and $T=\{T'^{n-1,n}_1,T'^{n-1,n}_{1+d'_n}\}$. ■

Proof of proposition 3. Because of proposition 2, it is sufficient to prove that $d_1$is equal to a power of two. In the first round, participants $\{1,\dots,d_1\}$ are paired with participants $\{d_1+1,\dots,2d_1\}$. Participants $\{1,\dots,2d_1\}$ play with each other, participants $\{2d_1+1,\dots,4d_1\}$ play with each other, etc. (if $d_1\leq 2^{n-2}$). d_1 is a divisor of the number of participants. Because the number of participants is equal to a power of 2, d_1 is equal to a power of 2.■

Proof of proposition 4. (i) Suppose that axiom A1 holds; then, in round k, 2^{n-k} winner participants are stronger than all of the loser participants. All of the loser participants are weaker and have zero probability of advancing. Therefore, axioms A2 and A2' hold.

Suppose that axiom A1 is violated; then, there exists a round $k$and participants i and j such that $2^{n-k} \geq i > j \geq 1$and participant i wins more matches than participant j. This violates axioms A2 and A2'.

(ii) Suppose that axioms A5 and A7 hold but that axiom A1 is violated. Because A1 is violated, there exists a participant $i \in \{1, \ldots, 2^{n-k}\}$ that wins *k-1* matches and loses the round *k* match. Let consider the lowest *k* for which such a participant exists. Because round *k* is the first round in which axiom A1 is violated, then $\{1, \ldots, 2^{n-k+1}\}$ is the set of round *k* participants. In round *k*, participant i plays with participant $j < i$. Because of axiom A7, the rank difference $i - j$ is common for all round *k* matches. From proposition 3, the rank difference is always equal to a power of 2. Because axiom A1 is violated, $i - j < 2^{n-k}$. Participants from the set $\{1, \ldots, 2^{n-k}\}$ play with each other, and participants from the set $\{2^{n-k} + 1, \ldots, 2^{n-k+1}\}$ play with each other. Half of the set of the strongest participants $\{1, \ldots, 2^{n-k}\}$ are is dropped. Because of axiom A5, in all subsequent rounds, participant from the set $\{1, \ldots, 2^{n-k}\}$ should play with each other. If this holds for all rounds before the final, then in the final, participant 1 play with participant $2^{n-k} + 1$ from the set of weaker participants. We have reached a contradiction.

(iii) Suppose that axiom A1 holds; then, in any round, all winner participants are stronger than all loser participants. Round *r+1* participants are winners of round *r*, and they play a game in round *r+1* with each other, whereas in the round *r*, they play with weaker participants. Therefore, axiom A5 holds.■

Proof of proposition 5. (i) From axiom A1, participants $\{2^{n-1} + 1, \ldots, 2^n\}$ should lose in round 1, participants $\{2^{n-2} + 1, \ldots, 2^{n-1}\}$ should lose in round 2, etc. Consider the vector representation of seeding. Participants in places with even numbers lose in the first round. There are $2^{\mathrm{n}-1}!$ ways to assign participants $\{2^{n-1} + 1, \ldots, 2^n\}$, there are $2^{\mathrm{n}-2}!$ ways to assign participants $\{2^{n-2} + 1, \ldots, 2^{n-1}\}$, etc. Therefore, we obtain $\mathrm{NA1(n)} = \prod_{\mathrm{k}=1}^{\mathrm{n}} 2^{\mathrm{k}-1}!$. From proposition 4(i), we have $\mathrm{NA1(n)} = \mathrm{NA2(n)}$.

(ii) Axiom A3 creates a rigid constraint for the first round. There is no constraint for other matches. For any tournament with *n* rounds, there exists a tournament with *n+1* rounds that satisfies axiom A3; hence, $NA3(n) = NS(n-1) = \frac{2^{\mathrm{n}-1}!}{2^{2^{\mathrm{n}-1}-1}}$.

(iii) A pair of sets $A, B \subseteq X$ are said to be symmetric if and only if $|A| = |B| = 2^{n-1}$, and if $x \in A$, then $2^n + 1 - x \in B$. A set $Y \subseteq X$ is called a symmetric set if and only if $|Y| = 2^{n-1}$, and if $x \in Y$, then $2^n + 1 - x \in Y$. Each symmetric subtournament has a symmetric set of participants. For each symmetric tournament, either the set of participants of each subtournament with n-1 rounds is symmetric or the sets of participants of subtournaments with n-1 rounds are symmetric. There are $NA4(n-1)$ different symmetric subtournaments with the given symmetric set of participants. Therefore, we obtain a recurrence relation

$$NA4(n) = \frac{(2^{n-1}-1)!}{(2^{n-2}-1)!2^{n-2}!}\left(NA4(n-1)\right)^2 + \frac{(2^n-2)!!}{(2^{n-1}-1)!}NS(n-1),$$

where $x!!$ is a double factorial. Having for odd x that $x!! = (0.5x)!\,2^{0.5x}$, we obtain

$$\frac{NA4(n)}{2^{n-1}!} = \frac{1}{2}\left(\frac{NA4(n-1)}{2^{n-2}!}\right)^2 + 1.$$

Defining $y(n) = \frac{NA4(n)}{2^{n-1}!}$, we have $y(1) = 1$ and

$$y(n) = 0.5(y(n-1))^2 + 1.$$

(iv) Consider the vector representation of seeding. Participant 2 should be in position $2^{n-1} + 1$; otherwise, he will play with participant 1 before the final, and participant 1 will play with a weaker participant in the subsequent round. There are two subtournaments, $(s_1, \dots, s_{2^{n-1}})$ and $(s_{2^{n-1}+1}, \dots, s_{2^n})$, that should satisfy axiom A5. There are $\mathrm{NA5(n-1)}$ ways to assign a given set of participants to each of these subtournaments. We have the recurrence relation

$$\mathrm{NA5(n)} = \frac{(2^n-2)!}{\left((2^{n-1}-1)!\right)^2}\left(\mathrm{NA5(n-1)}\right)^2.$$

Solving this equation with $\mathrm{NA5}(1) = 1$, we obtain

$$\mathrm{NA5(n)} = \frac{(2^{\mathrm{n}}-2)!}{\prod_{\mathrm{k=1}}^{\mathrm{n-2}}(2^{\mathrm{n-k}}-1)^{2^{\mathrm{k}}}}.$$

(v) $NA6(n) = 1$ follows from proposition 1.

(vi) From propositions 2 and 3, we have$\mathrm{NA7(n) = n!}$.

(vii) $\lim_{\mathrm{n}\to\infty}\left(\frac{\mathrm{NA5(n+1)}}{\mathrm{NS(n+1)}}\right)\Big/\left(\frac{\mathrm{NA5(n)}}{\mathrm{NS(n)}}\right) = \lim_{\mathrm{n}\to\infty}\left(\frac{2^{2^{\mathrm{n+1}}-\mathrm{n}-2}}{\prod_{\mathrm{k=0}}^{\mathrm{n-1}}(2^{\mathrm{n+1-k}}-1)^{2^{\mathrm{k}}}}\right)\Big/\left(\frac{2^{2^{\mathrm{n}}-\mathrm{n}-1}}{\prod_{\mathrm{k=0}}^{\mathrm{n-2}}(2^{\mathrm{n-k}}-1)^{2^{\mathrm{k}}}}\right) =$

$\lim_{\mathrm{n}\to\infty} 2^{2^{\mathrm{n}}-1}3^{-2^{\mathrm{n}-1}}\prod_{\mathrm{k=0}}^{\mathrm{n-2}}\left(\frac{2^{\mathrm{n-k}}-1}{2\cdot 2^{\mathrm{n-k}}-1}\right)^{2^{\mathrm{k}}} = \lim_{\mathrm{n}\to\infty} 2^{2^{\mathrm{n}}-1}3^{-2^{\mathrm{n}-1}}\prod_{\mathrm{k=0}}^{\mathrm{n-2}}\left(1-\frac{1}{2\cdot 2^{\mathrm{n-k}}-1}\right)^{2^{\mathrm{k}}} = 0.$

From this, it follows that $\lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA5(n)}}{\mathrm{NS(n)}} = 0$.

For $\mathrm{n} \geq 3$, we have $\mathrm{NA6(n)} < NA7(\mathrm{n}) < NA1(\mathrm{n}) < NA5(\mathrm{n})$.

Thus, $\lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA1(n)}}{\mathrm{NS(n)}} = \lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA2(n)}}{\mathrm{NS(n)}} = \lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA6(n)}}{\mathrm{NS(n)}} = \lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA7(n)}}{\mathrm{NS(n)}} = 0$.

$\lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA3(n)}}{\mathrm{NS(n)}} = \lim_{\mathrm{n}\to\infty}\left(\frac{2^{n-1}!}{2^{2^{n-1}-1}}\right)\Big/\left(\frac{2^n!}{2^{2^n-1}}\right) = \lim_{\mathrm{n}\to\infty}\frac{2^{2^{n-1}}}{\prod_{i=2^{n-1}+1}^{2^n} i} = \lim_{\mathrm{n}\to\infty}\frac{\prod_{i=1}^{2^{n-1}} 2}{\prod_{i=1}^{2^{n-1}}(i+2^{n-1})} = 0.$

For $n \geq 1$, we have $NA4(n) = 2^{\mathrm{n-1}}!\,y(n) < 2^{\mathrm{n-1}}!\,z(n)$, where $z(n) = (z(n-1))^2$, with $z(1) = 2$; hence, $NA4(n) = 2^{\mathrm{n-1}}!\,y(n) < 2^{\mathrm{n-1}}!\,2^{2^{n-1}}$. Thus, $\lim_{\mathrm{n}\to\infty}\frac{\mathrm{NA4(n)}}{\mathrm{NS(n)}} \leq \lim_{\mathrm{n}\to\infty}\frac{2^{\mathrm{n-1}}!2^{2^{n-1}}2^{2^{\mathrm{n}}-1}}{2^{\mathrm{n}}!} = \lim_{\mathrm{n}\to\infty} 0.5\frac{\prod_{\mathrm{i=1}}^{2^{\mathrm{n-1}}} 8}{\prod_{\mathrm{i=1}}^{2^{\mathrm{n-1}}}(2^{\mathrm{n-1}}+\mathrm{i})} = 0.$■

Proof of proposition 6. Because of proposition 4, it sufficient to prove part (*i*). From axiomA7, there are equal rank differences. $d_1 \leq 2^{n-1}$, otherwise it is impossible to find a pair for participant 2^{n-1}. From axiom A1, the rival of participant 1 is weaker than the participant 2^{n-1}, and then $d_1 \geq 2^{n-1}$. Therefore, we have $d_1 = 2^{n-1}$.

There are 2^{n-k} subtournaments with *k* rounds (*k* from 2 to n). Each of them has two subtournaments with *k-1* rounds. From A1$\{1, \dots, 2^{n-k+1}\}$ is the set of winners of these subtournaments. We numerate these subtournaments by the best participant from 1 to 2^{n-k+1}. From axiom A1, the rival of the subtournament 1 winner is weaker than the participant 2^{n-k}. The difference between the indexes of subtournament 1 and its rival subtournament is not less than 2^{n-k}. This difference is not higher than 2^{n-k},otherwise it is impossible to find a pair for the subtournament with index 2^{n-k}. Therefore, for any *k* from 2 to n, we have$d_k = 2^{n-k}$ and

$$T_i^{k,n} = \{T_i^{k-1,n}, T_{i+2^{n-k}}^{k-1,n}\}, i = \overline{1, 2^{n-k}}.■$$

Proof of proposition 7. Let us prove this proposition by induction. Consider the first round. The strongest 2^{n-1} participants should win with equal probability. Under the restricted linear domain assumption, all matches should have equal rank differences, and the strongest 2^{n-1} participants should not play each other. Thus, $d_1 = 2^{n-1}$ and $T_i^{1,n} = \{i, i + 2^{n-1}\}$, $i = \overline{1, 2^{n-1}}$. We have proved the proposition for $k = 1$. Suppose it is true for all k from1 to j ($1 \leq j \leq n-1$). Let us prove it for $j + 1$.

The strongest 2^{n-j} participants of the tournament are the strongest participants of the subtournaments $\left\{T_1^{j,n}, .., T_{2^{n-j}}^{j,n}\right\}$ numbered by the strongest participant.In round $j + 1$, there are four subtournaments with indexes $\alpha, \beta, \gamma, \delta$, such that α is paired with β ($\alpha < \beta$) and γ is paired with δ ($\gamma < \delta$). If $\beta - \alpha > \delta - \gamma$, then participant α has a grater probability of winning the next match than participant γ. The only way to equate the probabilities of winning $j + 1$ matches for the strongest 2^{n-j} participants is to equate the rank differences of subtournaments. Because the strongest 2^{n-j-1} participants should have equal probability, then $d_{j+1} = 2^{n-j-1}$ and

$$T_i^{j+1,n} = \left\{T_i^{j,n}, T_{i+2^{n-j-1}}^{j,n}\right\}, i = \overline{1, 2^{n-j-1}}. \blacksquare$$

Proof of proposition 8. If a seeding violates axiom A1, then the objective function equals 0. Optimal seeding satisfies axiom A1. Axiom A1 leads to the following structure of seedings. The strongest 2^{n-1} participants play with the weakest 2^{n-1} participants. In each subsequent round, winners of the strongest half of matches play with winners of the weakest half of matches. The strength of a match coincides with the strength of the best participant of the match.

Considering only seedings that satisfy axiom A1, we find the probability that, in round *k*, participants $W^k = \{1, \dots, 2^{n-k}\}$ win against participants $L^k = \{2^{n-k} + 1, \dots, 2^{n-k+1}\}$

$$Pr\{advancing\ in\ rou\ ndk\} = \prod_{j=1}^{2^{n-k}} p_{jr_j^k},$$

where $r_j^k \in L^k$ and $\forall a \neq b, r_a \neq r_b$.

Having $Pr\{advancing\ in\ rou\ ndk\}$, we rewrite the objective function

$$\prod_{i=1}^{n}\left[\Pr\left(\bigcup_{j=i}^{n}\{top\ 2^{j-1}\ participants\ win\ n-j+1\ matches\}\right)\right]^{\alpha_i} =$$

$$\prod_{k=1}^{n}(Pr\{advancing\ in\ rou\ ndk\})^{\sum_{i=1}^{n-k+1}\alpha_i} = \prod_{k=1}^{n}\left(\prod_{j=1}^{2^{n-k}} p_{jr_j^k}\right)^{\sum_{i=1}^{n-k+1}\alpha_i}.$$

Because $Pr\{advancing\ in\ rou\ ndk\}$ does not depend on $Pr\{advancing\ in\ rou\ ndk'\}$, this problem is separated on *n* independent problems of maximizing $Pr\{advancing\ in\ rou\ ndk\}$.

The solution of the problem of maximizing $Pr\{advancing\ in\ rou\ ndk\}$ is described by the set of pairs $S^k = \bigcup_{i=1}^{2^{n-k}}\{i, r_i^k\}$, where $r_i^k \in L^k$ and $\forall i \neq j, r_i \neq r_j$.

Let$\hat{S}^k$ be an optimal solution for the problem of maximizing $Pr\{advancing\ in\ rou\ ndk\}$. Assume there exist $i, j \in W^k$ and $s, t \in L^k$ such that $i < j < s < t$ and $\{i, t\} \cup \{j, s\} \in \hat{S}^k$. From the linear domain assumption, we have

$$p_{it}p_{js} = F(\theta_i - \theta_t)F(\theta_j - \theta_s),$$
$$\theta_i > \theta_j > \theta_s > \theta_t.$$

There exists a linear function $ax + b$ such that

$$F(\theta_i - \theta_s) = a(\theta_i - \theta_s) + b,$$
$$F(\theta_j - \theta_t) = a(\theta_i - \theta_s) + b.$$

Because F is an increasing concave function with $\lim_{x\to 0} F(x) \geq 0.5$, we have

$$F(\theta_i - \theta_t) \leq a(\theta_i - \theta_t) + b,$$
$$F(\theta_j - \theta_s) \leq a(\theta_j - \theta_s) + b.$$

Comparing $p_{it}p_{js}$ and $p_{is}p_{jt}$, we have

$$F(\theta_i - \theta_s)F(\theta_j - \theta_t) - F(\theta_i - \theta_t)F(\theta_j - \theta_s) \geq$$
$$(a(\theta_i - \theta_s) + b)(a(\theta_j - \theta_t) + b) - (a(\theta_i - \theta_t) + b)(a(\theta_j - \theta_s) + b) =$$
$$a^2(\theta_i\theta_s - \theta_j\theta_s - \theta_i\theta_t + \theta_j\theta_t) = a^2(\theta_i - \theta_j)(\theta_s - \theta_t) > 0.$$

$p_{is}p_{jt} > p_{it}p_{js}$contradicts the optimality of $\hat{S}^k$. Therefore, for all $i, j \in W^k$ and $s, t \in L^k$ such that $i < j < s < t$, we have $\{i, s\} \cup \{j, t\} \in \hat{S}^k$. 1 is paired with $2^{n-k} + 1$, 2 is paired with $2^{n-k} + 2$, etc. From this, we obtain the equal gap seeding.■

Proof of proposition 9. Let p_n^* be the highest possible probability of increasing competitive intensity. For $n = 2$, we have $p_2^* = p$, where p is the probability of the stronger participant winning in any match. The optimal tournament is $T_{ici}^2 = \{\{1,4\}, \{2,3\}\}$. The results of the match $\{2,3\}$ and the last round match have no impact on the event "increasing competitive intensity." Consider $n > 2$. The tournament that is generated by an optimal seeding consists of two subtournaments, which are also optimal in the sense of the probability of increasing competitive intensity. The results of the two last-round matches of these subtournaments have an impact on the event "increasing competitive intensity." If participant 1 loses in the first of these matches, then in the final, his/her rival meets with a weaker rival. If the rival of participant 1 is weaker than all possible rivals in the second subtournament, then the result of the second match may not be important as in $\{\{1,4\}, \{2,3\}\}$. In the increasing competitive intensity seedings I and II, all participants of subtournament 1 except participant 1 are weaker than any participant from subtournament 2 except participant 2^n. If the event "increasing competitive intensity" holds for the second subtournament, then participant 2^n loses in the first match. Therefore, for the increasing competitive intensity seedings I and II, the result of the last match of the second subtournament is not important, and for $n \geq 2$,we obtain the recurrence relation

$$p_{n+1}^* = p \cdot (p_n^*)^2.$$

Solving this equation, we obtain $p_n^* = p^{2^{n-1}-1}$ (which holds for $n \geq 1$).■

Appendix 2. Comparison of the equal gap seeding, the standard seeding, and the increasing competitive intensity seeding, *n=3*.

Domain	Assumption	$\Pr(\{top\ 4\ participants\ win\ 1\ matches\} \cup \{top\ 2\ participants\ win\ 2\ matches\} \cup \{top\ 1\ participant\ wins\ 3\ matches\})$				$\Pr(increasing\ competitive\ intensity)$			
		Equal gap seeding	Standard seeding	ICI seeding I	ICI seeding II	Equal gap seeding	Standard seeding	ICI seeding I	ICI seeding II
$\mathcal{P}_{BT}$	$\theta_i = 2^n + 1 - i$	0.052	0.045	0	0	0.097	0.284	0.411	0.467
$\mathcal{P}_{BT}$	$\theta_i = (2^n + 1 - i)^2$	0.140	0.104	0	0	0.248	0.524	0.651	0.738
$\mathcal{P}_{BT} \cap \mathcal{P}_{rl}$	$\theta_i = 2^{2^n - i}$	0.335	0.225	0	0	0.502	0.760	0.855	0.912
$\mathcal{P}_{rl}$	$F(x) = \frac{1}{2} + \frac{x}{2^{n+1}}$	0.070	0.064	0	0	0.124	0.360	0.524	0.574
$\mathcal{P}_{rl}$	$F(x) = 1 - 0.25^x$	0.649	0.408	0	0	0.865	0.968	0.983	0.996
$\mathcal{P}_{rl}$	$F(x)$ is CDF for standard normal distribution	0.803	0.594	0	0	0.955	0.997	0.999	1
$\mathcal{P}_{ep}$	$p_{ij} = p$	p^7	p^7	0	0	p^6	p^4	p^3	p^3
$\mathcal{P}_l$	$\theta_i = 5 - i, i = \overline{1,4}$ $\theta_i = -10^6 - i,\ i = \overline{5,8}$. $F(x) = 1$ if $x \geq 10^6$ $F(x) = 0.5 + \frac{x}{2 \cdot 10^6}$ otherwise	0.125	0.125	0	0	0.250	0.500	1	1
$\mathcal{P}_{dm}$	$p_{ij} = \frac{1}{2} + \frac{min(9-i;j)}{16}$	0.163	0.215	0	0	0.261	0.615	0.711	0.714
$\mathcal{P}_{dm}$	$p_{ij} = 1$ if $\frac{min(9-i;j)}{16} \geq \frac{7}{16}$; $p_{ij} = 0.5$ otherwise.	0.008	0.031	0	0	0.016	0.25	0.25	0.5
$\mathcal{P}_{dm}$	$p_{ij} = 1$ if $\frac{min(9-i;j)}{16} \geq \frac{1}{2}$; $p_{ij} = 0.5$ otherwise.	0.008	0.016	0	0	0.016	0.125	0.25	0.125

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