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Preference diversity orderings

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This paper surveys approaches to preference diversity measurement. Applying preference diversity axiomatics, a generalization of the Alcalde-Unzu and Vorsatz (2016) criterion, is developed. It is shown that all previously used indices violate this criterion. Two new indices (geometric mean based and leximax-based) are developed that satisfy a new criterion. Leximax-based orders act as a polarization index and are compared with Can et al.'s (2015) polarization index. The paper concludes by formulating a new open question: the preference profile reconstruction conjecture.

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1. Introduction

The diversity of preferences in a group is an aggregative concept representing the lack of coincidence among individual preferences, level of disagreement, polarization, difficulty of reaching an agreement, and so on. In some social choice studies (Alcalde-Unzu and Vorsatz 2013, 2016), the term group cohesiveness is used for the opposite situation, i.e., low preference diversity. This term is borrowed from social psychology, where it describes a broader concept comprising dynamic, emotional and other aspects (Carron, Brawley 2000; Cota et al. 1995). The term diversity is borrowed from studies by Hashemi, Endriss (2014), and Gehrlein et al. (2013). In social choice theory, we consider the terms diversity and cohesiveness as purely opposite concepts without nuance. In the opinion of the author, diversity is a more neutral term that does not create needless associations. For brevity, we use term 'preference diversity' rather than 'preference profile diversity.'

The degree of diversity of preferences in a group is a key parameter in social choice and matching theories. In social choice models, diverse preferences represent a high degree of conflict in a group. In general, a low diversity of preferences simplifies many voting problems, reduces the frequency of Condorcet's paradox, and prevents successful manipulation. Similarity among preferences induces higher competition in two-sided matching and house allocation problems. The degree of similarity of preferences also influences the stability and efficiency of matching mechanisms.

The importance of preference diversity was discovered in different models. Lepelley and Valognes (2003) found a positive relationship between strategic manipulation and diversity of preferences using a homogeneity parameter in the Pólya–Eggenberger model. Studying elections with three alternatives, Gehrlein et al. (2013) found that increasing the number of possible preference ranking types increases the probability of strategic manipulation. Gehrlein and Lepelley (2011, 2015) surveyed the relationship between diversity of preferences and the frequency of Condorcet's and other paradoxes. Hałaburda (2010) showed that in two-sided matching market, unravelling (contracting long before relevant information is available) is more likely to occur when participants have a higher degree of similar preferences. Boudreau and Knoblauch (2013) studied the connection between preference diversity and the price of stability in two-sided matching problems. Manea (2009) found that there is a relationship between the probability that the random serial dictatorship mechanism is ordinally efficient and the degree of similarity of preferences.

In economic theory, the diversity measurement problem is mainly associated with biodiversity measurement and other related problems (see survey Nehring, Puppe 2009). This framework applies a multi-attribute approach, uses only binary dissimilarity information, and requires an "acyclic" structure of attributes (Nehring, Puppe 2002). The preference diversity measurement problem has other characteristics and requires its own framework. Despite the importance of preference diversity measurement, it is novel area of research and still there is no consensus in such studies.

Our study focuses on ordinal measures of preference diversity. Preference diversity indices only represent the corresponding ordering of preference profiles. Preference profiles with different numbers of alternatives or agents are incompatible. There are several axiomatic justifications for certain preference diversity measures (Alcalde-Unzu and Vorsatz 2013, 2016; Can, Ozken and

Strocken 2015). All of these studies start from properties of preference diversity indices, but not from the properties of preference diversity orderings. The only information they utilize comes from a weighted tournament matrix. In this paper, we show that such a matrix is not enough for a well-discernible preference diversity index.

Alcalde-Unzu and Vorsatz's (2016) paper was motivated by the 3 agents, 3 alternatives conundrum. They showed that there is no distance-based preference diversity index with an arithmetic mean aggregator that can represent the correct diversity order on a pair of intuitively ordered preference profiles. We reinforce this conundrum, investigating a weak order of all 3 agents, 3 alternatives preference profiles, which we call the basic 3×3 order. We develop axiomatics from Hashemi and Endriss's (2014) survey, adding new axioms and justifying the basic 3×3 order by the set of axioms. The ability to represent this order is an aggregated condition for preference diversity indices. We show that all previously proposed indices fail to represent this order.

This study does not seek to find a unique preference diversity index that satisfies certain properties. Different indices are needed for different research goals, but some weak criteria should be satisfied for all diversity indices and the basic 3×3 order becomes this criterion.

We solve the 3 agents, 3 alternatives conundrum by proposing two new preference diversity orders. One of them is a distance-based preference diversity index with a geometric mean aggregator, and the second is based on the leximax comparison. The geometric mean-based index is an alternative to Alcalde-Unzu and Vorsatz's (2013, 2016) indices and is able to represent the basic 3×3 order. The family of leximax-based indices (and corresponding orders) do not belong to any class from Hashemi and Endriss's (2014) survey. These indices are alternatives to Can et al.'s (2015) polarization index because the maximally polarized preference profiles have the highest diversity with respect to leximax-based orders. Seeking to increase discernibility power (the number of anonymous and neutral classes of preference profiles that are not equivalent according to the index), we introduce iterative reinforcements of leximax orders. Potentially, there is an index from this family that uniquely characterizes each ANEC of the preference profiles. Preference profile reconstruction conjecture, which is discussed in the conclusion, uniquely defines preference profiles from the collection of preference deleted preference profiles. The proof of this conjecture is an important step in development of strongly discernible order using the leximax family. This conjecture links this study with graph theory, where the graph reconstruction conjecture remains one of the classical unsolved problems. In addition to the preference profile reconstruction conjecture, the domain reconstruction conjecture is discussed.

The remainder of the paper is organized as follows. Section 2 describes preference diversity axioms and analyzes the 3 agents, 3 alternatives case. Section 3 presents different preference diversity measures. Section 4 develops leximax preference diversity orderings. Section 5 concludes and describes reconstruction conjectures.

2. Framework

Let a finite set $X = \{1, \dots, m\}$, $m \geq 2$ be the set of alternatives and a finite set $\mathcal{N} = \{1, \dots, n\}$, $n \geq 2$ be the set of agents (voters). Each agent $i \in \mathcal{N}$ has a strict preference P_i over X (linear order). Let $\mathcal{L}(X)$ be the set of all possible linear orders over X. An n-tuple of preference

orders generates preference profile $\mathcal{P} = (P_1, \dots, P_n) \in \mathcal{L}(X)^n$. Function $pos(P_i, j) = |\{x \in X | x P_i j\}| + 1$ indicates the position of candidate j in preference profile P_i.

Within this model, names of voters (anonymity) and names of alternatives (neutrality) do not matter. An anonymous and neutral equivalence class (ANEC) is a set of preference profiles that could be obtained from each other by permuting preference orders and renaming alternatives. The permutation of preference orders is usually denoted by $\sigma: \mathcal{N} \to \mathcal{N}$, and the permutation of alternatives is usually denoted by $\tau: X \to X$. The image of profile $\mathcal{P}$ under permutations σ, τ is denoted by $(\mathcal{P}^\sigma)^\tau$. The image of preference order P_i under permutations σ, τ is denoted by $\left(P_{\sigma(i)}\right)^\tau$, $(P_i^\sigma)^\tau$ denotes the ith preference order in permuted profile. Preference profiles $\mathcal{P}, \mathcal{P}'$ belong to the same ANEC if and only if there are permutations $\sigma: \mathcal{N} \to \mathcal{N}, \tau: X \to X$, such that $(\mathcal{P}^\sigma)^\tau = \mathcal{P}'$. This relation, which is symmetrical, is denoted as $\mathcal{P} \sim_{ANEC} \mathcal{P}'$. The complementary binary relation is denoted as $\nsim_{ANEC}$.

A preference diversity relation (PDR) is a binary relation $\succcurlyeq$ on the space of preference profiles $\mathcal{L}(X)^n$. A strict relation $\succ$ means 'more diverse' and an indifference relation $\sim$ means 'equally diverse.'

2.1 Axiomatic analysis

Almost all diversity measurement studies are focused on diversity indices. Diversity indices are only representation of some PDR. This study starts from the axiomatics of PDR. Hashemi and Endriss (2014) investigated a set of axioms, six of which are presented here.

Axiom 1. A PDR $\succcurlyeq$ is **anonymous** if, for every permutation $\sigma: \mathcal{N} \to \mathcal{N}$, we have $\mathcal{P} \sim \mathcal{P}^\sigma$.

Axiom 2. A PDR $\succcurlyeq$ is **neutral** if, for every permutation $\tau: X \to X$, we have $\mathcal{P} \sim (\mathcal{P})^\tau$.

Axiom 3. A PDR $\succcurlyeq$ is **strongly discernible** if $\mathcal{P} \sim_{ANEC} \mathcal{P}'$ implies $\mathcal{P} \sim \mathcal{P}'$, and $\mathcal{P} \nsim_{ANEC} \mathcal{P}'$ implies $\mathcal{P} \nsim \mathcal{P}'$.

Axiom 4. A PDR $\succcurlyeq$ is **weakly discernible** if $\mathcal{P}$ being unanimous and $\mathcal{P}'$ not being unanimous together imply $\mathcal{P}' \succ \mathcal{P}$.

Axiom 5. A PDR $\succcurlyeq$ is **support-invariant** if $\text{Supp}(\mathcal{P}) = \text{Supp}(\mathcal{P}')$ implies $\mathcal{P} \sim \mathcal{P}'$.

Axiom 6. A PDR $\succcurlyeq$ is **nonlocal** if for every profile $\mathcal{P} = (P_1, \dots, P_n)$ and every voter $i \in \mathcal{N}$ there exists a preference order $P \in \mathcal{L}(X)$ such that $\mathcal{P} \nsim (P_1, \dots P_{i-1}, P, P_{i+1}, \dots, P_n)$.

Axiom 7. A PDR $\succcurlyeq$ is **independent** if it is the case that $\mathcal{P} \succcurlyeq \mathcal{P}'$ if and only if $\mathcal{P} \cup P \succcurlyeq \mathcal{P}' \cup P$ for every two profiles $\mathcal{P}, \mathcal{P}' \in \mathcal{L}(X)^n$ and every preference order $P \notin \text{Supp}(\mathcal{P}) \cup \text{Supp}(\mathcal{P}')$.

Proposition 1. *Binary relation* $\sim_{ANEC}$ *does not satisfy nonlocality and independence.*

Proof. For every m and $n = k \cdot m! + 1$, $k \in \mathbb{N}$, there is a preference profile that contains every possible preference order k times with one preference order being duplicated. Any preference profile obtained by changing the duplicated preference order belongs to the same ANEC. Binary relation $\sim_{ANEC}$ does not satisfy nonlocality.

Let P be a preference order with $x_1 P x_2 P x_3 P \dots P x_m$. Let $\tau = \begin{pmatrix} x_1 x_2 & x_{m-2} x_m \\ x_2 x_3 & \dots & x_{m-1} x_1 \end{pmatrix}$ be the permutation of alternatives. For any $m \geq 5$, there is $(P, (P)^\tau) \sim_{ANEC} ((P)^\tau, ((P)^\tau)^\tau)$, but $(P, (P)^\tau, (((P)^\tau)^\tau)^\tau) \nsim_{ANEC} ((P)^\tau, ((P)^\tau)^\tau, (((P)^\tau)^\tau)^\tau)$. Binary relation $\sim_{ANEC}$ does not satisfy independence. ∎

Because binary relation $\sim_{ANEC}$ does not satisfy nonlocality and independence, any anonymous and neutral PDR does not satisfy nonlocality and independence.

Strong discernibility is also a highly restrictive condition. For many aggregation rules, e.g., some voting methods, the top preferences are as important as the bottom preferences. This type of symmetry was studied by Bubbolini and Gori (2015). They introduced the reversal symmetry property for a majority voting rules framework. Applying the reversal symmetry principle to the diversity measurement problem leads to axiom 8. If every agent changes the order between any two alternatives, obtaining a completely reverse preference order, then preference diversity remains unchanged. If a preference profile belongs to a single-peaked domain, then its reversal belongs to the single-caved domain (also called a single-dipped domain). If a PDR satisfies reversal symmetry, then single-peaked and single-caved domains are equivalent in terms of preference diversity. The application of the reversal symmetry weakens strong discernibility axiom.

Axiom 8. A PDR $\succcurlyeq$ satisfies **reversal symmetry** if for every $\mathcal{P}, \mathcal{P}'$, such that $\forall x \in X, \forall i \in \mathcal{N}$, $pos(P_i, x) = m + 1 - pos(P'_i, x)$, we have $\mathcal{P} \sim \mathcal{P}'$.

Axiom 9. A PDR $\succcurlyeq$ satisfies **reverse invariant discernibility** if $\mathcal{P} \sim_{ANEC} \mathcal{P}'$ or $\forall x \in X, \forall i \in \mathcal{N}$, there are permutations $\sigma: \mathcal{N} \to \mathcal{N}$ and $\tau: X \to X$, such that $pos((P_i^\sigma)^\tau, x) = m - pos(P'_i, x)$ imply $\mathcal{P} \sim \mathcal{P}'$, otherwise $\mathcal{P} \nsim \mathcal{P}'$.

Preference diversity is a complex concept that can be formalized in different ways. Here, we present a set of axioms that formalize different aspects of preference diversity. Strong and weak monotonicity (Axioms 14 and 15) represent another view of the monotonicity properties from Hashemi and Endriss (2014). Weak monotonicity (axiom 15) coincides with the monotonicity property described in Alcalde-Unzu and Vorsatz (2013, 2016).

Axiom 10. A PDR $\succcurlyeq$ satisfies **top diversity** if $\{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, x P_i y\} \subset \{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, x P'_i y\}$ implies $\mathcal{P}' \succcurlyeq \mathcal{P}$.

Axiom 11. A PDR $\succcurlyeq$ satisfies **bottom diversity** if $\{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, y P_i x\} \subset \{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, y P'_i x\}$ implies $\mathcal{P}' \succcurlyeq \mathcal{P}$.

Axiom 12. A PDR $\succcurlyeq$ satisfies **top-bottom diversity** if $\{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, x P_i y\} \cap \{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, y P_i x\} \subset \{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, x P'_i y\} \cap \{x \in X | \exists i \in \mathcal{N} : \forall y \in X \backslash x, y P'_i x\}$ implies $\mathcal{P}' \succcurlyeq \mathcal{P}$.

Axiom 13. A PDR $\succcurlyeq$ satisfies **Condorcet property** if $\mathcal{P}' \succcurlyeq \mathcal{P}$ whenever preference profile $\mathcal{P}$ has an acyclic majority relation and preference profile $\mathcal{P}'$ does not have an acyclic majority relation.

Axiom 14. A PDR $\geqslant$ satisfies **strong monotonicity** if $\mathcal{P}' \geqslant \mathcal{P}$ whenever there exists preference order $P \in \mathcal{P}'$ such that $|\{i \in \mathcal{N}|P'_i = P\}| > n/2$, and $\{i \in \mathcal{N}|P'_i = P\} \subset \{i \in \mathcal{N}|P_i = P\}$, and for all $i \in \mathcal{N}$, such that $P_i \neq P$, $P_i = P'_i$.

Axiom 15. A PDR $\geqslant$ satisfies **weak monotonicity** if $\mathcal{P}' \geqslant \mathcal{P}$ whenever there exist alternatives $x, y \in X$ such that $|\{i \in \mathcal{N}|xP'_iy\}| > |\{i \in \mathcal{N}|yP'_ix\}|$, and $\{i \in \mathcal{N}|xP'_iy\} \subset \{i \in \mathcal{N}|xP_iy\}$, and for all $\{w, z\} \neq \{x, y\}$ we have $\{i \in \mathcal{N}|wP'_iz\} = \{i \in \mathcal{N}|wP_iz\}$.

Axiom 16. A PDR $\geqslant$ satisfies **Borda winner monotonicity** if $BordaWinnerScore(\mathcal{P}') < BordaWinnerScore(\mathcal{P})$ implies $\mathcal{P}' \geqslant \mathcal{P}$.

Axiom 17. A PDR $\geqslant$ satisfies **Borda loser monotonicity** if $BordaLoserScore(\mathcal{P}') > BordaLoserScore(\mathcal{P})$ implies $\mathcal{P}' \geqslant \mathcal{P}$.

Axiom 18. A PDR $\geqslant$ satisfies **Borda winner-loser difference monotonicity** if $BordaWinnerScore(\mathcal{P}') - BordaLoserScore(\mathcal{P}') < BordaWinnerScore(\mathcal{P}) - BordaLoserScore(\mathcal{P})$ implies $\mathcal{P}' \geqslant \mathcal{P}$.

Axiom 19. A PDR $\geqslant$ satisfies **number of unanimity pairs monotonicity** if $|\{(x, y) \in X \times X|\forall i \in \mathcal{N}xP_iy\}| > |\{(x, y) \in X \times X|\forall i \in \mathcal{N}xP'_iy\}|$ implies $\mathcal{P}' \geqslant \mathcal{P}$.

Axiom 20. A PDR $\geqslant$ satisfies **single-peakedness** if $\mathcal{P}' \geqslant \mathcal{P}$ whenever preference profile $\mathcal{P}$ belongs to a single-peaked domain and $\mathcal{P}'$ does not belong to a single-peaked domain.

Axiom 21. A PDR $\geqslant$ satisfies **single-caveness** if $\mathcal{P}' \geqslant \mathcal{P}$ whenever preference profile $\mathcal{P}$ belongs to a single-caved domain and $\mathcal{P}'$ does not belong to a single-caved domain.

The next axiom is designed only for preference profiles with an even number of agents. According to Can et al. (2015), preference profiles in which half of the individuals have a preference order P and the rest have the reverse of P are considered to be maximally polarized preference profiles. With polarization as one of aspects of preference diversity, we define the polarization axiom.

Axiom 22. A PDR $\geqslant$ satisfies **polarization** if for even n, $\mathcal{P}' \geqslant \mathcal{P}$ whenever preference profile $\mathcal{P}$ is not maximally polarized and $\mathcal{P}'$ is maximally polarized.

2.2 The 3 × 3 case

For the case of 3 agents and 3 alternatives, there are 10 ANECs (Veselova 2015). Table 1 presents one preference profile from each ANEC. Table 1 is an adjacency matrix. For each axiom (from 10 to 19) and for each pair of preference profiles, we check whether the axiom implies the preference diversity relation $\geqslant$ between two preference profiles. For example, axioms bottom diversity (BD) and Borda loser monotonicity (BL) imply that preference profile $\mathcal{P}_2$ is more diverse than $\mathcal{P}_3$.

Table 1. 3 agents 3 alternatives preference profiles

				$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$	$\mathcal{P}_8$	$\mathcal{P}_9$	$\mathcal{P}_{10}$
$\mathcal{P}_1$	x y z	x y z	x y z										
$\mathcal{P}_2$	x y z	x y z	x z y	BD; BL; BWL; NUP; SM; WM		BD; BL							
$\mathcal{P}_3$	x y z	x y z	y x z	TD; BW; BWL; NUP; SM; WM	TD; BW								
$\mathcal{P}_4$	x y z	x y z	z x y	TD; BD; TBD; BW; BL; BWL; NUP; SM	TD; TBD; BW: BL; BWL; NUP; WM	BD; TBD; BL; BWL; NUP		BL		BL; BWL			
$\mathcal{P}_5$	x y z	x y z	y z x	TD; BD; TBD; BW; BL; BWL; NUP; SM	TD; TBD; BW; BWL; NUP	BD; BW; TBD; BL; BWL; NUP; WM	BW			BW; BWL			
$\mathcal{P}_6$	x y z	x y z	z y x	TD; BD; TBD; BW; BL; BWL; NUP; SM	TD; TBD; BW; BL; BWL; NUP	BD; TBD; BW; BL; BWL; NUP	TBD; BW; BWL; NUP; WM	TBD; BL; BWL; NUP; WM		BW; BL; BWL; NUP			
$\mathcal{P}_7$	x y z	x z y	y x z	TD; BD; TBD; BW; BL; BWL; NUP	TD; TBD; BW; BWL; NUP; WM	BD; TBD; BL; BWL; NUP; WM							
$\mathcal{P}_8$	x y z	x z y	y z x	TD; BD; TBD; BW; BL; BWL; NUP; SP	TD; BD; TBD; BW; BL; BWL; NUP; SP	BD; TBD; BW; BL; BWL; NUP; SP	BD; BW; BWL; NUP; SP	BD; TBD; BL; BWL; NUP; WM; SP	BD; SP	BD; TBD; BW; BL; BWL; NUP; WM; SP		BD; SP	
$\mathcal{P}_9$	x y z	y x z	z x y	TD; BD; TBD; BW; BL; BWL; NUP; SC	TD; TBD; BW; BL; BWL; NUP; SC	TD; BD; TBD; BW; BL; BWL; NUP; SC	TD; TBD; BW; BWL; NUP; WM; SC	TD; BL; BWL; NUP; SC	TD; SC	TD; TBD; BW; BL; BWL; NUP; WM; SC	TD; SC		
$\mathcal{P}_{10}$	x y z	y z x	z x y	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; BD; TBD; BW; BL; BWL; SP; SC	C; TD; BD; TBD; BW; BL; BWL; NUP; SP; SC	C; TD; TBD; BW; BL; BWL; SC	C; BD; TBD; BW; BL; BWL; SP	

TD – top diversity (Axiom 10); BD – bottom diversity (Axiom 11); TBD – top-bottom diversity (Axiom 12); C – Condorcet (Axiom 13); SM – strong monotonicity (Axiom 14); WM – weak monotonicity (Axiom 15); BW - Borda winner monotonicity (Axiom 16); BL - Borda Loser Monotonicity (Axiom 17); BWL - Borda winner-loser monotonicity (Axiom 18); NUP - Number unanimity pairs (Axiom 19); SP – Single-peakedness (Axiom 20); SC – Single-caveness (Axiom 21).

Each axiom represents its own version of the preference diversity concept. Axioms 10-21 induce different binary relations. There are no two equal binary relations. For some pairs of preference profiles, axioms agree with each other, while others do not. There is no linear order satisfying all axioms 10-21; it is an impossibility result for strongly discernible aggregation. The most discernible weak order that satisfies axioms 10-21 is the order with 7 indifference sets:

$$\mathcal{P}_{10} \succ \mathcal{P}_9 \sim \mathcal{P}_8 \succ \mathcal{P}_6 \succ \mathcal{P}_5 \sim \mathcal{P}_4 \succ \mathcal{P}_7 \succ \mathcal{P}_3 \sim \mathcal{P}_2 \succ \mathcal{P}_1.$$

Let us call this PDR **the basic 3×3 PDR**. This order aggregates all constraints from axioms 10-21 for the 3×3 case. Because there is strong support from several axioms for each relation in this order, the basic 3×3 PDR is robust. Apart from satisfying axioms 10-21, this order satisfies nonlocality (axiom 6), and reverse invariant discernibility (axiom 9).

In one motivating example, Alcalde-Unzu and Vorsatz (2016) compared preference profiles $\mathcal{P}_5$ and $\mathcal{P}_7$. They argued that $\mathcal{P}_5 \succ \mathcal{P}_7$ and found that all distance based measures with an arithmetic mean aggregator violate $\mathcal{P}_5 \succ \mathcal{P}_7$. This simple condition helped considerably to reduce the set of reasonable preference diversity measures. The basic 3×3 PDR is a generalization of Alcalde-Unzu and Vorsatz's (2016) motivating example. In the next section, the ability to represent the basic 3×3 PDR is used as a robust criterion for preference diversity indices. If the preference diversity index fails to represent this order, then the PDR generated by this index violates one or more axioms from axioms 10-21. This criterion is weaker than the strong discernibility axiom, which requires 10 indifference sets for the 3×3 case.

Eliminating several axioms, it is possible to design a strongly discernible PDR. In many social choice and matching problems, top preferences are more important than bottom preferences. Hence, top diversity and Borda winner monotonicity are more important than bottom diversity and Borda loser monotonicity. Eliminating bottom diversity, Borda loser monotonicity, and single-peakness, we obtain:

$$\mathcal{P}_{10} \succ \mathcal{P}_9 \succ \mathcal{P}_8 \succ \mathcal{P}_6 \succ \mathcal{P}_5 \succ \mathcal{P}_4 \succ \mathcal{P}_7 \succ \mathcal{P}_3 \succ \mathcal{P}_2 \succ \mathcal{P}_1.$$

This order is not robust. Some relations in this order are supported by only one axiom. Changing one axiom to another or adding new axioms changes this order. For example, from single-peakedness we have $\mathcal{P}_8 \succcurlyeq \mathcal{P}_9$. We do not consider this order as unique and instead desire a strict order for the 3×3 case.

3. Preference diversity indices

The preference diversity index (PDI) is a real valued function $\Delta \colon \mathcal{L}(X)^n \to \mathbb{R}$ that respects $\Delta(P, \dots, P) = 0$. The preference diversity index represents PDR $\succcurlyeq$ if $\Delta(\mathcal{P}_a) \geq \Delta(\mathcal{P}_b) \Leftrightarrow \mathcal{P}_a \succcurlyeq \mathcal{P}_b$. We will say that PDI satisfies axiom x if the corresponding PDR satisfies axiom x.

This section follows Hashemi and Endriss's (2014) survey of PDI types, although it does not define disjoint classes. Some indices belong to the several classes. Hashemi and Endriss (2014) argued that non e of the PDIs considered in their paper satisfied strong discernibility. We generalize this result and focus on reverse invariant discernibility.

3.1 Support -based PDI (Hashemi, Endriss, 2014)

For a given $k \le m$, the support-based PDI $\Delta_{supp}^k = |\{T \in \mathcal{L}_k(X)|T \subseteq P_i \text{ for some } i \in N\}| - \binom{m}{k}$, where $\mathcal{L}_k(X)$ is the set of k alternatives of linear orders over X.

Propositions 2 and 3 shows poor discernibility properties of the support-based PDI.

Proposition 2. *For $n \ge 4$, support-based PDI does not satisfy reverse invariant discernibility.*

Proof. Consider two preference profiles $\mathcal{P}_a = \left(\underbrace{P, ..., P}_{n-1}, P'\right)$, $\mathcal{P}_b = \left(\underbrace{P, ..., P}_{n-2}, P', P'\right)$, with $P \ne P'$. These preference profiles do not belong to the same anonymous and neutral equivalence class, but they have identical support with $\Delta_{supp}^k(\mathcal{P}_a) = \Delta_{supp}^k(\mathcal{P}_b)$. ∎

Proposition 3. *The basic 3×3 PDR cannot be represented by support-based PDI.*

Proof. Δ_{supp}^3 generates 3 indifference classes (1, 2, or 3 types of different linear orders). Δ_{supp}^2 generates 4 indifference classes (3,4,5,6 types of different linear orders). The basic 3×3 PDR generates 7 indifference classes. Therefore, it cannot be represented by $\Delta_{supp}^3, \Delta_{supp}^2$. ∎

3.2 Distance based PDI (Hashemi, Endriss, 2014)

For a given distance $\delta: \mathcal{L}(X) \times \mathcal{L}(X) \to \mathbb{R}$ and aggregation operator $\Phi: \mathbb{R}^{n(n-1)/2} \to \mathbb{R}$, the **distance based PDI** $\Delta_{dist}^{\Phi,\delta}$ maps any given profile $\mathcal{P} \in \mathcal{L}(X)^n$ to the following value:
$$\Delta_{dist}^{\Phi,\delta}(\mathcal{P}) = \Phi(\delta(P_i, P_k)|i, k \in N \text{ with } i < k).$$

For every $P, P', P'' \in \mathcal{L}(X)$ a distance function satisfies the following four conditions:

1) $\delta(P, P') \ge 0$ (nonnegativity),
2) $\delta(P, P') = 0$ if and only if $P = P'$ (identity of indiscernibles),
3) $\delta(P, P') = \delta(P', P)$ (symmetry),
4) $\delta(P, P'') \le \delta(P, P') + \delta(P', P'')$ (triangle inequality),
5) for every permutation $\tau: X \to X$, we have $\delta(P, P') = \delta(\tau(P), \tau(P'))$ (neutrality).

Different examples of distances between preference orders are presented in Can (2014), Elkind et al. (2015), and Mescanen, and Nurmi (2008). Hashemi and Endriss (2014) also proposed a compromise-based PDI as an aggregation of distances between preference orders and a compromise order. Any compromise-based PDI can be represented in the form of a distance-based PDI, redefining the distance measure as:

$$\delta'(P, P') = 0 \text{ if } P = P',$$
$$\delta'(P, P') = \delta(P, P^{comp}) + \delta(P^{comp}, P').$$

Despite having a variety of distances and aggregators, the distance-based PDI does not satisfy reverse invariant discernibility.

Proposition 4. *For $m \geq 4$, the distance-based PDI does not satisfy reverse invariant discernibility.*

Proof. Let P be a preference order with $x_1 P x_2 P x_3 P \dots P x_m$. Let us define permutations as:

$$\tau_a = \begin{pmatrix} x_1 x_2 & \dots & x_{m-2} x_{m-1} x_m \\ x_2 x_3 & & x_{m-1} \ x_1 \ x_m \end{pmatrix}, \tau_b = \begin{pmatrix} x_1 & x_2 & \dots & x_{m-2} x_{m-1} x_m \\ x_{m-1} x_1 & & x_{m-3} x_{m-2} x_m \end{pmatrix}.$$

For $m \geq 4$, preference profile $\mathcal{P}_a = (P, \dots, P, (P)^{\tau_a})$, is not equivalent to $\mathcal{P}_b = (P, \dots, P, (P)^{\tau_b})$ with respect to anonymity, neutrality and reverse invariance. Because of distance neutrality, we obtain $\delta(P, (P)^{\tau_a}) = \delta((P)^{\tau_b}, ((P)^{\tau_a})^{\tau_b}) = \delta((P)^{\tau_b}, P) = \delta(P, (P)^{\tau_b})$, hence

$$\Delta_{dist}^{\Phi, \delta}(\mathcal{P}_a) = \Delta_{dist}^{\Phi, \delta}(\mathcal{P}_b). \blacksquare$$

Alcalde-Unzu and Vorsatz (2016) showed that the distance-based PDI with an arithmetic mean aggregator cannot represent relation $\mathcal{P}_5 \succ \mathcal{P}_7$. Other aggregators meet the challenge. Moreover, the basic 3×3 PDR can be represented by distance-based PDI, as shown in the next subsection with the new investigated index.

3.2.1 Geometric mean-based index (GM)

Let us define a slightly modified Kendall rank distance (swap distance):

$$\delta(P, P') = 0 \text{ if } P = P',$$
$$\delta(P, P') = |\{(x, y) \in X \times X | x P y, y P' x\}| + 0.5.$$

All five conditions for the distance measure are satisfied. For the 3 alternatives case, we have:

$$\delta \begin{pmatrix} x \ x \\ y, y \\ z \ z \end{pmatrix} = 0, \ \delta \begin{pmatrix} x \ x \\ y, z \\ z \ y \end{pmatrix} = \delta \begin{pmatrix} x \ y \\ y, x \\ z \ z \end{pmatrix} = 1.5, \ \delta \begin{pmatrix} x \ y \\ y, z \\ z \ x \end{pmatrix} = \delta \begin{pmatrix} x \ z \\ y, x \\ z \ y \end{pmatrix} = 2.5, \ \delta \begin{pmatrix} x \ z \\ y, y \\ z \ x \end{pmatrix} = 3.5.$$

Defining the aggregating operator as a small modification of geometric mean,

$$\Phi = \left(\prod_{i=1}^{n-1} \prod_{k=i+1}^{n} \max(\delta(P_i, P_k), 1) \right)^{\frac{2}{n(n-1)}} - 1,$$

we obtain $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_1) = 0$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_2) = \Delta_{GM}^{\Phi, \delta}(\mathcal{P}_3) = 0.310$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_7) = 0.778$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_4) = \Delta_{GM}^{\Phi, \delta}(\mathcal{P}_5) = 0.842$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_6) = 1.305$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_8) = \Delta_{GM}^{\Phi, \delta}(\mathcal{P}_9) = 1.358$, $\Delta_{GM}^{\Phi, \delta}(\mathcal{P}_{10}) = 1.5$, which represents the basic 3×3 PDR.

The geometric mean aggregator is a novelty of this index. Previous studies have mainly considered the arithmetic mean aggregator, but the arithmetic mean aggregator fails to represent relation $\mathcal{P}_5 \succ \mathcal{P}_7$ (Alcalde-Unzu and Vorsatz 2016). Taking the logarithm of the geometric mean aggregator, we obtain the arithmetic mean aggregator, but new distances violate triangle inequality: $ln(1.5) + ln(1.5) < ln(2.5)$. Apart from the ability to represent the basic 3×3 order, the geometric mean-based index satisfies anonymity, neutrality, and reversal symmetry.

For the case of 3 alternatives and 4, 5 or 6 agent preference profiles with the highest diversity, are the following:

$$\begin{matrix} x x y z & x x y y z & x x y y z z \\ y z z y, & y z z x y, & y z z x y x. \\ z y x x & z y x z x & z y x z x y \end{matrix}$$

According to the geometric mean-based index, the most diverse preference profiles show the lack of coincidence of individual preferences and a high level of disagreement for each pair of agents.

3.3 Weighted tournament-based PDI

For a given weighted tournament $WT: \mathcal{L}(X)^n \to \mathbb{N}_0^{m(m-1)/2}$, and aggregation operator $\Phi: \mathbb{N}_0^{m(m-1)/2} \to \mathbb{R}$, the **weighted tournament based PDI** Δ_{WT}^{Φ} maps any given profile $\mathcal{P} \in \mathcal{L}(X)^n$ to the following value:

$$\Delta_{WT}^{\Phi} = \Phi(n(x,y) - n(y,x)|x,y \in X),$$

where $n(x,y) = |\{i \in \mathcal{N}|xP_iy\}|$.

Indices from Alcalde-Unzu and Vorsatz (2013, 2016), Can, Ozken and Strocken (2015), and the Pareto measure from Bosch (2006) all belong to this class of PDI. The index proposed by Celik and Knoblauch (2007) utilizes information about the sums of Borda scores. Because only information about weighted tournament matrix is needed to calculate the sums of Borda scores (Fischer et al. 2016), this index also belongs to the class of weighted-tournament based PDIs.

Preference profiles $\mathcal{P}_{10}, \mathcal{P}_9, \mathcal{P}_8, \mathcal{P}_6$ have the same weighted tournament; hence the weighted tournament-based PDI do not satisfy reverse invariant discernibility and cannot represent the basic 3×3 PDR.

4. Leximax PDRs

All indices observed in the literature survey fail to represent PDR with good discernibility properties. This section starts from a PDR that solves the problem of the basic 3×3 PDR representation, whereas other PDRs creates more discernible orders. For each multidimensional characteristic of preference diversity, leximax orders are focused on the components that are associated with the highest diversity.

In this section, several binary relations are introduced. Strict and nonstrict relations are defined in the usual way. In some cases, unnecessarily detailed definitions determining strict and indifference relations are given for the sake of greater transparency.

4.1 PDR $\succcurlyeq_{Lmax\_1}^{\alpha}$

For a given scoring vector $\alpha \in \mathbb{R}^m$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, inequality in preferences between agents i and k relative to alternative j is measured by the absolute difference between their scores $|s_{i,j} - s_{k,j}|$, where $s_{i,j} = \alpha_{pos(P_i,j)}$. For the purpose of preference diversity ordering, we sum the absolute differences over all pairs of agents and then compare the sums lexicographically. The diversity of preference profile is associated with the alternative that has the highest sum of absolute difference between their scores. In the case of equality, we compare the whole vectors of differences.

Let a vector $\rho(\mathcal{P}) = \left(\left|s_{i,j} - s_{k,j}\right|\right)_{i,k\in\mathcal{N},i<k,j\in X}$ be the $mn(n-1)/2$-dimensional vector of absolute differences between individual scores. Vector $\rho(\mathcal{P})$ is a raw data for diversity measurement. Let a vector $\delta(\mathcal{P}) = \left(\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left|s_{i,j} - s_{k,j}\right|\right)_{j\in X}$ be the m-dimensional vector of the sums of absolute differences between individual scores. Vector $\delta(\mathcal{P})$ is the vector of diversities of alternatives' scores.

The leximax relation $\succcurlyeq_L$ on $\mathbb{R}^q$ is defined as follows. For any $x = \left(x_1,\ldots,x_q\right) \in \mathbb{R}^q$, let $x^* = \left(x_1^*,\ldots,x_q^*\right) \in \mathbb{R}^q$ be a permutation of the coordinates of vector x in the decreasing order: $x_1^* \geq \cdots \geq x_q^*$. If there is a $j \in \{1,\ldots,q\}$ such that $x_j^* > y_j^*$, while $x_i^* = y_i^*$ for all $i < j$, then $x \succ_L y$. If $x_j^* = y_j^*$ for all $j \in \{1,\ldots,q\}$, then $x \sim_L y$.

Leximax $PDR \succcurlyeq_{Lmax\_1}^{\alpha}$ (α is a given scoring vector) is defined by the following rule, which includes three conditions:

1. If $\delta(\mathcal{P}) \succ_L \delta(\mathcal{P}')$, then $\mathcal{P} \succ_{Lmax\_1}^{\alpha} \mathcal{P}'$;
2. If $\delta(\mathcal{P}) \sim_L \delta(\mathcal{P}')$ and $\rho(\mathcal{P}) \succ_L \rho(\mathcal{P}')$, then $\mathcal{P} \succ_{Lmax\_1}^{\alpha} \mathcal{P}'$;
3. If $\delta(\mathcal{P}) \sim_L \delta(\mathcal{P}')$ and $\rho(\mathcal{P}) \sim_L \rho(\mathcal{P}')$, then $\mathcal{P} \sim_{Lmax\_1}^{\alpha} \mathcal{P}'$.

First, preference profiles are compared using the diversity of an alternative with the most diverse scores. If the sums of the absolute differences between individual scores are equal for all alternatives, then preference profiles are compared using the highest absolute differences between individual scores. Vector $\delta(\mathcal{P})$ includes aggregated information. Leximax comparison of $\delta(\mathcal{P})$ is robust and has clear interpretation. Only in the case of equality should we consider raw data $\rho(\mathcal{P})$. Even considering $\rho(\mathcal{P})$, not all preference profiles would be ordered.

In the 3×3 case, $PDR \succcurlyeq_{Lmax\_1}^{\alpha}$ with Borda scores coincides with the basic 3×3 PDR. Other scores' vectors also do not generate any strongly discernible order. Preference profiles $\mathcal{P}_4, \mathcal{P}_5$ have permuted vectors $\delta(\mathcal{P}), \rho(\mathcal{P})$, then $\delta(\mathcal{P}_4) \sim_L \delta(\mathcal{P}_5)$ and $\rho(\mathcal{P}_4) \sim_L \rho(\mathcal{P}_5)$.

The following preference profiles are examples of preference profiles with the highest diversity according to $PDR \succcurlyeq_{Lmax\_1}^{\alpha}$, with Borda scores for different numbers of agents and alternatives:

$$\begin{matrix} xxyy & xxzyy & xxxyyy \\ zzzz, & yzxzz, & zzzzzz; \\ yyxx & zyyxx & yyyxxx \end{matrix}$$

$$\begin{matrix} xyz & xxyy & xxyyz & xxxyyy \\ zxy & zztt & yzttt & zzzttt \\ ttt, & ttzz, & ttzzx, & tttzzz. \\ yzx & yyxx & zyxxy & yyyxxx \end{matrix}$$

Proposition 5. $PDR \succcurlyeq_{Lmax\_1}^{\alpha}$ *with Borda scores satisfies anonymity, neutrality, reversal symmetry, and polarization.*

Proof. Because binary relation $\succcurlyeq_L$ and all functions defined above are anonymous and neutral, PDR $\succcurlyeq_{Lmax\_1}^{\alpha}$ is anonymous and neutral.

If $\mathcal{P}'$ is the reverse of $\mathcal{P}$, then $\delta(\mathcal{P}) \sim_L \delta(\mathcal{P}')$ and $\rho(\mathcal{P}) \sim_L \rho(\mathcal{P}')$, from which follows $\mathcal{P} \sim_{Lmax\_1}^{\alpha} \mathcal{P}'$.

Let n be even. Let $x_j = |\{i \in \mathcal{N}|pos(P_i, x) = j\}|$ be the number of preference orders, which have alternative x on position j. Let preference profile $\mathcal{P}$ be a preference profile with the

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highest diversity according to PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$. Let alternative $y \in \operatorname*{argmax}_{j \in X} \delta(\mathcal{P})_j$ have the highest diversity of scores; then we have $y_1 > 0$, and $y_m > 0$. Because of reversal symmetry, for every preference profile with $y_1 = a$, $y_m = b$ there is a preference profile for which $y_1 = b$, $y_m = a$. Without any loss of generality, specify $y_1 \leq y_m$.

Suppose that for alternative y, we have $\sum_{j=2}^{m-1} y_j \geq 1$. Then, we obtain:

$$\delta(\mathcal{P})_y = y_1 y_m (m-1) + \sum_{j=2}^{m-1}\left[y_j y_1 (j-1) + y_j y_m (m-j)\right] + \sum_{j=2}^{m-2}\sum_{k}^{m-1} y_j y_k (k-j).$$

Let $t \in \{2, \dots, m-1\}$ be a position such that $y_t > 0$ and for any $j \in \{2, \dots, t-1\}$ we have $y_j = 0$; then:

$$\delta(\mathcal{P})_y = y_1 y_m (m-1) + \sum_{j=t}^{m-1}\left[y_j y_1 (j-1) + y_j y_m (m-j)\right] + \sum_{j=t}^{m-2}\sum_{k=j+1}^{m-1} y_j y_k (k-j).$$

Let preference order P_i be a preference order for which we have $pos(P_i, y) = t$. Let us define preference profile $\mathcal{P}'$ such that for all preference orders except order i in preference profiles $\mathcal{P}', \mathcal{P}$ coincide and for order P'_i we have $pos(P'_i, y) = 1$; then:

$$\delta(\mathcal{P}')_y = (y_1 + 1) y_m (m-1) + (y_t - 1)(y_1 + 1)(t-1) + (y_t - 1) y_m (m-t) +$$
$$\sum_{j=t+1}^{m-1}\left[y_j (y_1 + 1)(j-1) + y_j y_m (m-j)\right] + \sum_{k=t+1}^{m-1}(y_t - 1) y_k (k-t) +$$
$$\sum_{j=t+1}^{m-2}\sum_{k=j+1}^{m-1} y_j y_k (k-j).$$

Subtracting, we obtain:

$$\delta(\mathcal{P}')_y - \delta(\mathcal{P})_y = y_m (m-1) + y_t (t-1) - y_1 (t-1) - y_m (m-t) + \sum_{j=t+1}^{m-1}\left[y_j (j-1)\right] -$$
$$\sum_{k=t+1}^{m-1} y_k (k-t) = (y_m + y_t - y_1)(t-1) + \sum_{j=t+1}^{m-1} y_j (t-1) > 0.$$

We derive a contradiction, therefore $\sum_{j=2}^{m-1} y_j = 0$. For alternative y, such that $y \in \operatorname*{argmax}_{j \in X} \delta(\mathcal{P})_j$ and $y_1 \leq y_m$, we have $\delta(\mathcal{P})_y = (m-1) y_1 (n - y_1)$, which reaches the highest value at $y_1 = y_m = n/2$. There are two alternatives x, y, such that $y_1 = y_m = x_1 = x_m = n/2$. Applying the same argument for a subprofile with $m - 2$ alternative, we find that there are two alternatives z, w, such that $z_2 = z_{m-1} = w_2 = w_{m-1} = n/2$. If m is even, then all alternatives are partitioned on pairs such that for each pair there is number $j \in \{1, \dots, m/2\}$ and in each preference order one alternative in each pair has position j and the other, position $m + 1 - j$. If m is odd, then all alternatives except one alternative are partitioned on pairs, such that for each pair there is number $j \in \{1, \dots, (m-1)/2\}$ and in each preference order one alternative in pair has position j and the other position $m + 1 - j$. An alternative without pairs has position $(m+1)/2$ in all preference orders. The maximally polarized preference profile has the same structure; therefore, the maximally polarized preference profile has the highest diversity. ∎

In all preference profiles, which are equivalent to the maximally polarized preference profile according to PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$, all alternatives have an equal sum of scores.

Can et al.'s (2015) polarization index also satisfies anonymity, neutrality, reversal symmetry, and polarization, but it is unable to represent the basic 3×3 order (it has worse discernibility power for this case). PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$ violates support independence and replication invariance properties from Can et al. (2015), but these properties are of a cardinal nature. In the ordinal framework, PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$ is superior to Can et al.'s (2015) polarization index.

4.2 PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$

Relaxing reversal symmetry, it is possible to design a more discernible order. In many decision-making problems, high diversity at or close to the top preferences induces a higher level of disagreement than does diversity near the bottom. Not all alternatives are equally relevant. Alternatives that are closer on average to the top choice are more relevant. The relevance of an alternative for preference diversity ordering is measured by the sum of scores. In the case of indifference induced by PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$, the diversity of the preference profile is associated with the vector of absolute differences between the individual scores of the most relevant alternative.

Let $\beta(\mathcal{P}) = (\beta(\mathcal{P})_1, \dots \beta(\mathcal{P})_m)$, with $\beta_j = \sum_{i=1}^{n} \alpha_{pos(P_i, j)}$ be the vector of scores sums. The score relation $\succcurlyeq_{S^{\alpha}}$ on X is defined as follows. If $\beta_x \geq \beta_y$, then $x \succcurlyeq_{S^{\alpha}(\mathcal{P})} y$. This relation generates the order of alternatives with respect to their relevance for the group of agents.

The leximax relation $\succcurlyeq_{L,S^{\alpha}(\mathcal{P})}$ on $\mathbb{R}^{qm}$ is defined as follows. For any $x = (x_1, \dots, x_m) \in \mathbb{R}^{qm}$, where x_j, with $j \in X$, is q-dimensional vector, let $x_j^* = x_{\tau(j)} \in \mathbb{R}^q$ be an element of the permuted vector x, where $\tau: X \to X$ is a permutation of alternatives with respect to the relevance and leximax orders: $\tau(1) \succcurlyeq_{S^{\alpha}(\mathcal{P})} \tau(2) \succcurlyeq_{S^{\alpha}(\mathcal{P})} \dots \succcurlyeq_{S^{\alpha}(\mathcal{P})} \tau(m)$ and if there is $j \in X \backslash \{m\}$, such that $\tau(j + 1) \succcurlyeq_{S^{\alpha}(\mathcal{P})} \tau(j)$, then we have $x_{\tau(j)} \succcurlyeq_L x_{\tau(j+1)}$. If there is a $j \in \{1, \dots, qm\}$ such that $x_j^* > y_j^*$, while $x_i^* = y_i^*$ for all $i < j$, then $x >_{L,S^{\alpha}(\mathcal{P}),} y$. If $x_j^* = y_j^*$ for all $j \in \{1, \dots, qm\}$, then $x \sim_{L,S^{\alpha}(\mathcal{P})} y$.

Leximax $PDR \succcurlyeq^{\alpha}_{Lmax\_2}$ (α is a given scoring vector) is defined as $>^{\alpha}_{Lmax\_1}$ extension:

1. If $\mathcal{P} >^{\alpha}_{Lmax\_1} \mathcal{P}'$, then $\mathcal{P} >^{\alpha}_{Lmax\_2} \mathcal{P}'$;
2. If $\mathcal{P} \sim^{\alpha}_{Lmax\_1} \mathcal{P}'$ and $\rho(\mathcal{P}) >_{L,S^{\alpha}(\mathcal{P})} \rho(\mathcal{P}')$, then $\mathcal{P} >^{\alpha}_{Lmax\_2} \mathcal{P}'$;
3. If $\mathcal{P} \sim^{\alpha}_{Lmax\_1} \mathcal{P}'$ and $\rho(\mathcal{P}) \sim_{L,S^{\alpha}(\mathcal{P})} \rho(\mathcal{P}')$, then $\mathcal{P} \sim^{\alpha}_{Lmax\_2} \mathcal{P}'$.

In the 3×3 case, $PDR >^{\alpha}_{Lmax\_2}$ with Borda scores generates a strongly discernible order:

$$\mathcal{P}_{10} > \mathcal{P}_8 > \mathcal{P}_9 > \mathcal{P}_6 > \mathcal{P}_5 > \mathcal{P}_4 > \mathcal{P}_7 > \mathcal{P}_3 > \mathcal{P}_2 > \mathcal{P}_1.$$

Because all PDI considered in section 3 fail to represent a strongly discernible order in the 3×3 case, $PDR >^{\alpha}_{Lmax\_2}$ has a higher discernibility power.

PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$ with Borda scores satisfies anonymity, neutrality, and polarization, but not reversal symmetry. PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$ has the same set of preference profiles, which are equivalent to the maximally polarized preference profile according to PDR $\succcurlyeq^{\alpha}_{Lmax\_1}$. In the general case, PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$ does not satisfy strong discernibility. The problem does not arise from Borda scores and arithmetic equality. The following preference profiles are equivalent under $\succcurlyeq^{\alpha}_{Lmax\_2}$ with any scores vector:

$$\begin{matrix} x\,x\,z \\ y\,y\,t \\ t\,t\,y \\ z\,z\,x \end{matrix} \sim^{\alpha}_{Lmax\_2} \begin{matrix} x\,\chi\,z \\ y\,t\,y \\ t\,y\,t \\ z\,z\,x \end{matrix}.$$

For each alternative, the scores vectors are leximax equal.

4.3 PDR $\succ^{\alpha}_{Lmax\_3}$

$PDR \succcurlyeq^{\alpha}_{Lmax\_2}$ uses an order of alternatives according to their relevance. $PDR \succcurlyeq^{\alpha}_{Lmax\_3}$ uses an order of agents. We define the collection of agent-deleted preference profiles. Preference profiles with high diversity should have a collection of highly diverse agent-deleted preference profiles. We do not compare preference profiles with different numbers of agents. Agent-deleting is a common approach to constructing nearby preference profiles that is successfully applied to determining near single-peaked and other domain extensions (Bredereck et al. 2016; Elkind, Lanckner 2014; Erdélyi et al. 2013). The collection of agent-deleted preference profiles is the closest representation of preference profiles in the simpler environment of *n-1* agents' preference profiles. In the most diverse case, there is no agent-deleted preference profile with low or moderately diversity according to PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$. In the case of indifference induced by PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$, the diversity of preference profiles is associated with the lowest diversity of the agent-deleted preference profiles.

Let $W(\mathcal{P}) = (W(\mathcal{P})_1, \ldots, W(\mathcal{P})_n)$ with $W(\mathcal{P})_i = (\mathcal{P} \backslash P_i)$ be the vector of preference profiles obtained by eliminating the corresponding preference order, $W(\mathcal{P})_i \in \mathcal{L}(X)^{n-1}$. Let $\Omega \subseteq \mathcal{L}(X)^{n(n-1)}$ be the set of all possible vectors $W(\mathcal{P})$.

The preference profile leximin relation $\succcurlyeq_{PPL}$ on Ω is defined as follows. Let $\pi(\mathcal{P}): \mathcal{N} \to \mathcal{N}$ be a permutation of preference profiles indices with respect to increasing the preference diversity order under PDR $\succcurlyeq^{\alpha}_{Lmax\_2}: W(\mathcal{P})^{\pi(\mathcal{P})}_{j+1} \succcurlyeq^{\alpha}_{Lmax\_2} W(\mathcal{P})^{\pi(\mathcal{P})}_j$. If there is a $j \in \mathcal{N}$ such that $W(\mathcal{P})^{\pi(\mathcal{P})}_j \succ^{\alpha}_{Lmax\_2} W(\mathcal{P}')^{\pi(\mathcal{P}')}_j$, while $W(\mathcal{P})^{\pi(\mathcal{P})}_i \sim^{\alpha}_{Lmax\_2} W(\mathcal{P}')^{\pi(\mathcal{P}')}_i$ for all $i < j$, then $W(\mathcal{P}) \succ_{PPL} W(\mathcal{P}')$. If $W(\mathcal{P})^{\pi(\mathcal{P})}_i \sim^{\alpha}_{Lmax\_2} W(\mathcal{P}')^{\pi(\mathcal{P}')}_i$ for all $j \in \mathcal{N}$, then $W(\mathcal{P}) \sim_{PPL} W(\mathcal{P}')$.

Leximax $PDR \succcurlyeq^{\alpha}_{Lmax\_3}$ (α is a given scoring vector) is defined as $\succ^{\alpha}_{Lmax\_2}$ extension:

1. If $\mathcal{P} \succ^{\alpha}_{Lmax\_2} \mathcal{P}'$, then $\mathcal{P} \succ^{\alpha}_{Lmax\_3} \mathcal{P}'$;
2. If $\mathcal{P} \sim^{\alpha}_{Lmax\_2} \mathcal{P}'$ and $W(\mathcal{P}) \succ_{PPL} W(\mathcal{P}')$, then $\mathcal{P} \succ^{\alpha}_{Lmax\_3} \mathcal{P}'$;
3. If $\mathcal{P} \sim^{\alpha}_{Lmax\_2} \mathcal{P}'$ and $W(\mathcal{P}) \sim_{PPL} W(\mathcal{P}')$, then $\mathcal{P} \sim^{\alpha}_{Lmax\_3} \mathcal{P}'$.

Defining PDR $\succcurlyeq^{\alpha}_{Lmax\_3}$ we order agent-deleted preference profiles according to PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$. Considering the example from the previous subsection, we have:

$$\begin{matrix} x\,x\,z \\ y\,t\,y \\ t\,y\,t \\ z\,z\,x \end{matrix} \succ^{\alpha}_{Lmax\_3} \begin{matrix} x\,x\,z \\ y\,y\,t \\ t\,t\,y \\ z\,z\,x \end{matrix},$$

where third agent-deleted preference profiles are the least diverse preference profiles in collections of agent-deleted preference profiles. All preference profiles that are equivalent to the maximally polarized preference profile according to PDR $\succcurlyeq^{\alpha}_{Lmax\_2}$ have collections of agent-deleted preference profiles consisting of equally diverse preference profiles.

PDR $\succcurlyeq^{\alpha}_{Lmax\_3}$ with Borda scores satisfies anonymity, neutrality, and polarization, but not strong discernibility. Because all corresponding agent-deleted preference profiles are equivalent, we have

$$\begin{matrix} x\,x\,z\,z \\ y\,y\,t\,t \\ t\,t\,y\,y \\ z\,z\,x\,x \end{matrix} \sim^{\alpha}_{Lmax\_3} \begin{matrix} x\,x\,z\,z \\ y\,t\,y\,t \\ t\,y\,t\,y \\ z\,z\,x\,x \end{matrix} \begin{matrix} x\,x\,z\,z \\ y\,y\,t\,t \\ z\,z\,x\,x \\ t\,t\,y\,y \end{matrix} \sim^{\alpha}_{Lmax\_3} \begin{matrix} x\,x\,z\,z \\ y\,t\,y\,t \\ z\,z\,x\,x \\ t\,y\,t\,y \end{matrix} \begin{matrix} x\,x\,z\,z \\ z\,z\,x\,x \\ y\,y\,t\,t \\ t\,t\,y\,y \end{matrix} \sim^{\alpha}_{Lmax\_3} \begin{matrix} x\,x\,z\,z \\ z\,z\,x\,x \\ y\,t\,y\,t \\ t\,y\,t\,y \end{matrix}.$$

In the 4×4 case, the geometric mean based index developed in section 3.2 is able to compare above-mentioned preference profiles, but the results are controversial:

$$\begin{matrix} xxzz \\ ytyt \\ tyty \\ zzxx \end{matrix} \succ_{GM} \begin{matrix} xx_{zz} & xxzz \\ yytt & ytyt \\ ttyy, & zzxx \\ zzxx & tyty \end{matrix} \succ_{GM} \begin{matrix} xxzz & xxzz \\ yytt & zzxx \\ zzxx, & yytt \\ ttyy & ttyy \end{matrix} \succ_{GM} \begin{matrix} xxzz \\ zzxx \\ ytyt \\ tyty \end{matrix}.$$

In two cases, the preference profiles with duplicated preference orders have lower diversity, while in another case, the preference profile with duplicated preference orders has higher diversity. Can et al.'s (2015) polarization index is as follows:

$$\Delta_{\text{Can et al.(2015)}} = \frac{\sum_{j=1}^{m-1} \sum_{k=j+1}^{m} (n - |n(j,k) - n(k,j)|)}{n \binom{m}{2}},$$

where $n(x, y) = |\{i \in \mathcal{N} | x P_i y\}|$, which implies also:

$$\begin{matrix} xxzz \\ yytt \\ ttyy \\ zzxx \end{matrix} \sim_{\text{Can et al.(2015)}} \begin{matrix} xxzz & xxzz \\ ytyt & yytt \\ tyty, & zzxx \\ zzxx & ttyy \end{matrix} \sim_{\text{Can et al.(2015)}} \begin{matrix} xxzz & xxzz \\ ytyt & zzxx \\ zzxx, & yytt \\ tyty & ttyy \end{matrix} \sim_{\text{Can et al.(2015)}} \begin{matrix} xxzz \\ zzxx \\ ytyt \\ tyty \end{matrix}.$$

It is possible to reinforce PDR $\succcurlyeq_{Lmax\_3}^{\alpha}$ e.g., defining order of alternative-deleted preference profiles[2], but the lack of rigorous definition of diversity implies the lack of reasonable criteria. Preference profiles with duplicated preference orders are more polarized, have a higher distance between preference orders, but preference profiles without duplicated preference orders have a higher number of preference types, a higher minimal distance between orders etc. Equivalence implied by PDR $\succcurlyeq_{Lmax\_3}^{\alpha}$ and Can et al. (2015) polarization index seems to be more reasonable in this case.

5. Conjectures and concluding remarks

Based on preference diversity axiomatics, the basic 3×3 order is developed. It generalizes the motivating example used by Alcalde-Unzu and Vorsatz (2016) and acts as an aggregated criterion for preference diversity relations. Two types of heuristic orders (and corresponding indices) are developed. The geometric mean-based index represents the basic 3×3 order and shows the possibility of applying distance-based measures. Leximax orders satisfy polarization and are able to represent the basic 3×3 order. Leximax PDR $\succcurlyeq_{Lmax\_3}^{\alpha}$ has quite good discernibility power. Because of these properties, leximax orders are reasonable alternatives to Can et al.'s (2015) polarization index, which fails to represent the basic 3×3 order and shows weak discernibility power.

The design of Leximax orders induces the problem of developing strongly discernible orders. Leximax PDR $\succcurlyeq_{Lmax\_3}^{\alpha}$ is not strongly discernible in the 4×4 case. If we design strongly discernible orders for $n = 2$ and $n = 3$, it is possible to construct an order for a higher number of

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[2] In this case, preference profiles $\begin{matrix} xxzz & xxzz \\ zzxx & zzxx \\ yytt, & ytyt \\ ttyy & tyty \end{matrix}$ remain equally diverse.

agents by iterative procedure. If the following conjecture holds, then this order would be strongly discernible.

Preference profile reconstruction conjecture. *For any two preference profiles $\mathcal{P}, \mathcal{P}' \in \mathcal{L}(X)^n$ with at least four agents, if agent-deleted preference profiles collections are equivalent with respect to $\sim_{ANEC}$, then $\mathcal{P} \sim_{ANEC} \mathcal{P}'$.*

Mathematically, this conjecture is closely related to the graph reconstruction conjecture formulated by P. Kelly and S. Ulam in 1941. There are two main versions of the conjecture: any two graphs with at least three vertices and the same vertex-deleted subgraph collections are isomorphic; and any two graphs with at least four edges and the same edge-deleted subgraph collections are isomorphic. A computer search has verified the reconstruction conjecture for graphs with nine or fewer vertices (Manvel 2000). The preference profile reconstruction conjecture can help to develop a strongly discernible index and deepen our knowledge of the structure of ANECs.

Because in the 3×3 case, agent-deleted preference profile collections for $\mathcal{P}_4, \mathcal{P}_5$ are equivalent, the preference profile reconstruction conjecture is false for $n = 3$. For the cases of 4 agents and 3 alternatives (24 ANECs) and 5 agents and 3 alternatives (42 ANECs), the preference profile reconstruction conjecture holds.

A similar conjecture for alternative-deleted preference profiles is false. In the following example, all alternative-deleted preference profiles are equivalent, but these preference profiles belong to different ANECs:

$$
\begin{matrix}
xxzz & xxzz \\
zzxx & zzxx \\
yytt & ytyt \\
ttyy & tyty
\end{matrix}
$$

Another interesting problem is to compare different domains by the diversity of preference profiles that belong to the domain. We introduce single-peakedness and single-caveness axioms but do not have general results. Apart from the preference profile reconstruction conjecture, a domain reconstruction conjecture is one possible step in finding a way to solve the problem.

Domain reconstruction conjecture. *For any two preference profile $\mathcal{P} \in \mathcal{L}(X)^n$ with at least k agents, if all preference profiles in an agent-deleted preference profiles collection belong to domain $\mathcal{D}$, then preference profile $\mathcal{P}$ belongs to domain $\mathcal{D}$.*

Ballester and Haeringer (2011) characterized the single-peaked and single-caved domain by forbidden substructures. Because these substructures include at most 3 preference orders, the single-peaked and single-caved domains reconstruction conjecture holds for *k=4*. Bredereck et al. (2013) characterized the single-crossing domain by forbidden substructures. Because these substructures include at most 4 preference profiles, the single-crossing domain reconstruction conjecture holds for *k=5*.

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