A GENERALIZED FRACTIONALLY DIFFERENCING APPROACH IN LONG-MEMORY MODELLING

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Abstract. We extend the class of known fractional ARIMA models to the class of generalized ARIMA models which allows the generation of long-memory time series with long-range periodical behaviour at a finite number of spectrum frequences. The exact asymptotics of the covariance function and the spectrum at the points of peaks and zeroes are given. For obtaining asymptotic expansions, Gegenbauer polynomials are used. Consistent parameter estimating is discussed using Whittle's estimate.

Key words. Time series, long-memory, fractional differencing, Gegenbauer polynomials, ARIMA models, Whittle's estimate.

1. Introduction

Most long-memory time series, i.e. whose covariance function is not absolutely summable, are characterized by the behaviour of a spectral density like $|\lambda|^{-d}$, $1 > d > 0$, as $\lambda \to 0$. This means that the spectral density is concentrated at low frequencies. One such sequence, called fractional differenced noise (or ARIMA (0, d, 0)), was introduced by Granger and Joyeux (1980) and Hosking (1981). They proposed the generation of long-memory time series via the fractional ARIMA(p,d,q) sequence, i.e. stationary solution $\{X_t, t \in \mathbf{Z}\}$ of the equation

$$\phi(B)\nabla^d X_t = \theta(B)\epsilon_t, \tag{1}$$

where $\{\epsilon_t\}$ is a white noise $(0, \sigma^2)$ sequence, and $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$ are p-th and q-th degree polynomials. The difference operator $\nabla^d = (1-B)^d$, $d > -1$ is defined by means of the binomial expansion

$$\nabla^d = \sum_{j=o}^{\infty} \tilde{\pi}_j B^j,$$

with

$$\tilde{\pi}_j = (-1)^j \begin{pmatrix} d \\ j \end{pmatrix} = \prod_{k=1}^{j} \frac{k-1-d}{k} = \frac{\Gamma(j-d)}{\Gamma(j+d)\Gamma(-d)},$$

where $BX_t = X_{t-1}$ is the backshift operator, $\Gamma(.)$ is the gamma function. In the case when $-1/2 < d < 1/2$ and $\phi(z) \neq 0$ for $|z| = 1$, a stationary solution of (1) exists (see Hosking (1981)).

A generalization of fractional ARIMA(p,d,q) models was proposed by Gray, Zhang and Woodward (1989). They discussed the fractional (GARMA) noise model

$$\phi(B)(1 - 2\cos\lambda \cdot B + B^2)^d X_t = \theta(B)\epsilon_t, \qquad (2)$$

($|\lambda| \leq \pi$, $|d| < 1/2$) suggested by Hosking (1981), which may exibit long-memory periodical behaviour at any frequency $0 \leq \lambda \leq \pi$ of the spectrum.

On the other hand, some authors (e.g. Anderson (1979)) introduced the so-called ARUMA(p, d, q), $d = 0, 1, \ldots$ sequences defined by the equation

$$\phi(B)U^d(B)X_t = \theta(B)\epsilon_t \qquad (3)$$

with a general nonstationary operator

$$U^d(B) = 1 - u_1 B - \ldots - u_d B^d,$$

with zeroes on the unit circle instead of the nonstationary operator $\nabla^d = (1 - B)^d$, $d = 1, 2, \ldots$ as in the ARIMA(p, d, q) equation.

In this paper we extend equations (2)-(3) for stationary operators having a finite number of zeroes or singularities of order $d_1, \ldots, d_m$ ($|d_j| < 1/2$) on the unit circle, which allow the modelling of long -short memory data containing seasonal periodicities. Such time series, as was mentioned by Hosking (1981), *exhibit both long-term persistence and quasiperiodic behaviour;* their correlation function resembles a superposition of hyperbolically damped sin waves. Definitions and main properties of generalized fractional noise are provided in Sections 2-3.

Spectral density of such a time series has a few peaks (zeroes) on $[-\pi, \pi]$. The asymptotical behaviour of corresponding covariance functions is treated in Section 5. Similar spectral densities and covariance functions of Gaussian stationary sequences have been discussed by Rosenblatt(1981) and Giraitis(1983).

Seasonal and non-seasonal models for economical time series based on modified fractional Gaussian noise were considered by Carlin and Dempster (1989). In fact, the analysis of dependent data exibiting seasonal behaviour is essential for many applications.

Concerning the literature about statistical inference for fractional ARIMA sequences (1) we refer to Beran (1992). The least squares and maximum likelihood estimator of differencing degree d at the zero frequency of the spectrum were treated by Yajima (1985). A class of estimators based on the logarithm of periodogram was proposed by Janacek (1982) and by Geweke and Porter-Hudak (1983) ($d > 0$). Whittle's (maximum likelihood) estimators for Gaussian and linear time-series were considered by Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990).

In Section 4 we discuss and prove consistency of Whittle's estimators for degrees $d_j, j = 1, \ldots, m$ and the location of peaks in the spectrum of the generalized fractional ARIMA model.

2.Generalized fractionally differenced noise

Introduce an operator

$$\begin{aligned} \nabla_{\lambda_1,\ldots,\lambda_m}^{d_1,\ldots,d_m} &= \prod_{j=1}^{m}(1 - e^{i\lambda_j}B)^{d_j}(1 - e^{-i\lambda_j}B)^{d_j} \\ &= \prod_{j=1}^{m}(1 - 2\cos\lambda_j \cdot B + B^2)^{d_j} \end{aligned}$$

by means of expansion

$$\prod_{j=1}^{m}(1 - 2\cos\lambda_j \cdot B + B^2)^{d_j} := \sum_{0}^{\infty} \pi_n B^n$$

in powers of the backshift operator, B. Here, $0 \le \lambda_1 < \ldots < \lambda_m \le \pi$ are some fixed frequences of the spectrum, $d_j \neq 0$ $(1 \le j \le m)$ are (fractional) differencing degrees and the coefficients π_n are found from the formula

$$\pi_n = \sum_{\substack{0 \le k_1, \ldots, k_m \le n, \\ k_1 + \ldots + k_m = n}} C_{k_1}^{(-d_1)}(\cos\lambda_1) \ldots C_{k_m}^{(-d_m)}(\cos\lambda_m), \tag{4}$$

where $C_k^{(d)}(x)$ are orthogonal Gegenbauer (or ultraspherical) polynomials (see Szegő (1959)) on $[-1, 1]$ with the weight function $(1-x^2)^{d-1/2}$, defined by their generating function

$$(1-2xz+z^2)^{-d} = \sum_{k=0}^{\infty} C_k^{(d)}(x) z^k,$$

$\mid z \mid \leq 1$, $d \neq 0$. Note that

$$C_k^{(d)}(\pm 1) = (\pm 1)^k \left(\begin{array}{c} 2d+k-1 \\ k \end{array} \right).$$

We formally define the generalized fractionally differenced noise $\{X_t\}$ or fractional ARIMA$(0, d_1, \ldots, d_m, 0)$ sequence with parameters $d_1, \ldots, d_m$; $\lambda_1, \ldots, \lambda_m$; σ as a stationary solution of the equation

$$\nabla_{\lambda_1,\ldots,\lambda_m}^{d_1,\ldots,d_m} X_t = \epsilon_t. \tag{5}$$

The possibility of the generalization (5) was also mentioned by Gray et al.(1989).

An important feature of model (5) is that, as a special case, it includes seasonal fractionally differenced models (see Porter-Hudak (1990)):

$$(1-B^s)^d X(t) = \epsilon_t, \quad \mid d \mid < 1/2.$$

This follows straightforwardly from the equality

$$1 - z^s = (1-z)(1+z)\prod_1^{s/2-1}(1-e^{2\pi k/s})(1-e^{-2\pi k/s}), \quad \text{if s is odd;}$$
$$= (1-z)\prod_1^{(s-1)/2}(1-e^{2\pi k/s})(1-e^{-2\pi k/s}), \quad \text{if s is even.}$$

We say that $\{X_t\}$ is *causal* if X_t can be represented as the one-sided moving-average sum

$$X_t = \sum_0^{\infty} \psi_j \epsilon_{t-j},$$

and *invertible* if the corresponding representation holds for

$$\epsilon_t = \nabla_{\lambda_1,\ldots,\lambda_m}^{d_1,\ldots,d_m} X_t \equiv \sum_{j=0}^{\infty} \pi_j X_{t-j}, \tag{6}$$

where

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty \qquad \text{and} \qquad \sum_{j=0}^{\infty} \pi_j^2 < \infty.$$

In the following theorem existence of a stationary solution of (5) , properties of the weights $\{\pi_n\}$, $\{\psi_n\}$, spectral density and covariance function are established.

Theorem 1. (i) *Let*

$$| d_j |< \begin{cases} 1/2 & \text{if } 0 < \lambda_j < \pi, \\ 1/4 & \text{if } \lambda_j = 0 \text{ or } \pi, \end{cases} \tag{7}$$

$(d_j \neq 0)$, $j = 1, \ldots, m$. Then there exists a unique stationary solution, $X(t)$ of (5) which is causal and invertible and has the moving-average representation

$$X_t = \sum_{n=0}^{\infty} \psi_n \epsilon_{t-n} \equiv \nabla_{\lambda_1,\ldots,\lambda_m}^{-d_1,\ldots,-d_m} \epsilon_t. \tag{8}$$

The coefficients ψ_n are given by

$$\psi_n = \sum_{\substack{0 \le k_1,\ldots,k_m \le n, \\ k_1+\ldots+k_m = n}} C_{k_1}^{(d_1)}(\cos\lambda_1)\ldots C_{k_m}^{(d_m)}(\cos\lambda_m) \tag{9}$$

and have the asymptotical expansion

$$\psi_n = 2 \sum_{k:\ 0<\lambda_k<\pi} D(k) \frac{\Gamma(n+d_k)}{\Gamma(n+1)\Gamma(d_k)} \cos(\lambda_k n + \nu_k)$$

$$+ \sum_{k:\ \lambda_k = 0 \text{ or } \pi} D(k) \frac{\Gamma(n+2d_k)}{\Gamma(n+1)\Gamma(2d_k)} \cos\lambda_k n + O(n^{-2+\max\{d_1^*,\ldots,d_m^*\}}) \tag{10}$$

as $n \to \infty$. Here $d_k^* = d_k$ if $0 < \lambda_k < \pi$; $d_k^* = 2d_k$ if $\lambda_k = 0$ or π,

$$\nu_k = \lambda_k \sum_1^m d_j - \pi \sum_1^{k-1} d_j - d_k \frac{\pi}{2},$$

and

$$\begin{aligned} D(k) &= | 2\sin\lambda_k |^{-d_k} \prod_{j\neq k} | 2(\cos\lambda_k - \cos\lambda_j) |^{-d_j}, \ \text{if } 0 < \lambda_k < \pi \ ; \\ &= \prod_{j\neq k} | 2(\cos\lambda_k - \cos\lambda_j) |^{-d_j}, \ \text{if } \lambda_k = 0 \text{ or } \pi. \end{aligned}$$

$\Gamma(n+d_k)/\Gamma(n+1) \sim n^{d_k-1}$ *in (9) as $n \to \infty$.*

The same expansion with $-d_1, \ldots, -d_m$ instead of $d_1, \ldots, d_m$ in (9) holds for the weights π_n, $n \ge 0$ in the invertible representation (6) of ϵ_t, which are given by (4).

(ii) The spectral density f_∇ of $\{X_t\}$ equals

$$f_\nabla(\lambda) = \frac{\sigma^2}{2\pi}\prod_{j=1}^{m} \mid 2\sin\frac{\lambda-\lambda_j}{2} 2\sin\frac{\lambda+\lambda_j}{2} \mid^{-2d_j}$$

$$= \frac{\sigma^2}{2\pi}\prod_{j=1}^{m} \mid 2(\cos\lambda - \cos\lambda_j) \mid^{-2d_j} \tag{11}$$

and has asymptotics

$$f_\nabla(\lambda) \sim \frac{\sigma^2}{2\pi} D(k)^2 \mid \lambda - \lambda_k \mid^{-2d_k^*} \quad as \quad \lambda \to \lambda_k, \quad k = 1, \ldots, m. \tag{12}$$

(iii) The covariance function $r(n) := EX_nX_0$, in the case $\max_{j=1,\ldots,m} d_j > 0$ has asymptotics

$$r(n) = \sum_{k=1,\ldots,m: d_k > 0} a_k \mid n \mid^{2d_k^*-1} (\cos n\lambda_k + o(1)) \tag{13}$$

as $n \to \infty$, with $a_k = a_k'$, if $\lambda_k = 0$ or π; $= 2a_k'$, if $0 < \lambda_k < \pi$,

$$a_k' = \frac{\sigma^2}{\pi}\Gamma(1-2d_k^*)\sin(d_k^*\pi)D^2(k).$$

PROOF of THEOREM 1. We shall prove assertion (10) by means of Darboux's method. Put $\lambda_{-j} = -\lambda_j$, $d_{-j} = d_j$, $j = 1, \ldots, m$. Let us consider the function

$$U^{d_1,\ldots,d_m}(z) := \prod_{j=1}^{m}(1 - e^{i\lambda_j}z)^{d_j}(1 - e^{-i\lambda_j}z)^{d_j},$$

with zero or singularity points $z_j = e^{i\lambda_j}$, $j = \pm1, \ldots, \pm m$ on the unit circle $\mid z \mid = 1$. Denote $\mathrm{I}(k) := \{j : 1 \leq \mid j \mid \leq m, \lambda_j \neq \lambda_{-k} \bmod 2\pi\}$. Because of the equality

$$1 - zz_j = (1 - z_jz_k)(1 + \frac{z_jz_k}{1 - z_jz_k}(1 - zz_{-k})), \quad z_jz_k \neq 1$$

the function $U^{d_1,\ldots,d_m}(z)$ has the representation

$$U^{d_1,\ldots,d_m}(z) = e_k h_k(1 - zz_{-k}),$$

in the neighbourhood of the point z_k $(1 \leq \mid k \mid \leq m)$, where

$$e_k = \prod_{j \in \mathrm{I}(k)} (1 - z_jz_k)^{d_j}$$

and

$$h_k(z) = z^{d_k^*} \prod_{j \in \mathrm{I}(k)} (1 + \frac{z_j z_k}{1 - z_j z_k} z)^{d_k}.$$

Expanding $h_k(z)$ in powers series about $z = 0$ we obtain

$$e_k h_k(z) = \sum_{\nu=0}^{\infty} c_\nu^{(k)} z^{d_k^* + \nu},$$

where

$$c_\nu^{(k)} = e_k \sum_{(s)} \prod_{j \in \mathrm{I}(k)} \binom{d_j^*}{s_j} (\frac{z_j z_k}{1 - z_j z_k})^{s_j}$$

and the sum $\sum_{(s)}$ is taken over all integers $0 \le s_j \le \nu$, $j \in \mathrm{I}(k)$ such that $\sum_{j \in \mathrm{I}(k)} s_j = \nu$. Note that $c_\nu^{(k)} = \overline{c_\nu^{(k)}}$, where $\overline{c}$ denotes the cojugate of complex number c. Therefore, by use of Darboux's method (see Theorem A, Appendix) the following general expansion for the weights π_n in (4) can be obtained:

$$\pi_n = \sum_{\nu=0}^{p-1} \sum_{k=1}^{m} \tilde{c}_\nu^{(k)} \binom{d_k^* + \nu}{n} + \mathrm{O}(n^{-p-\min\{d_1^*, \ldots, d_m^*\}-1}), \tag{14}$$

where $\tilde{c}_\nu^{(k)} = 2\mathrm{Re}(c_\nu^{(k)}(-e^{-i\lambda_k})^n)$ if $0 < \lambda_k < \pi$; $\tilde{c}_\nu^{(k)}=\mathrm{Re}(c_\nu^{(k)}(-e^{-i\lambda_k})^n)$ if $\lambda_k = 0$ or π.

If we stop the expansion (14) at the term $\nu = 0$ ($p = 1$), we obtain

$$\pi_n = \sum_{k=1}^{m} \binom{d_k^*}{n} \tilde{c}_0^{(k)} + O(n^{-2-\min\{d_1^*, \ldots, d_m^*\}})$$

as $n \to \infty$, what after easy calculations yields the expansion (10) for the weights π_n with $d_1, \ldots, d_m$ instead of $-d_1, \ldots, -d_m$ in (10). Clearly because of symmetry this yields also asymptotic expansion (10) for ψ_n.

An application of Stirling's formula $\Gamma(x) \sim (2\pi)^{1/2} e^{-x+1} (x-1)^{x-1/2}$, $x \to \infty$ leads to the asymptotics

$$\Gamma(n + d_k)/\Gamma(n + 1) \sim n^{d_k - 1} \qquad (n \to \infty)$$

in (10).

Now we prove that (8) is the unique stationary solution of (5). From (10) and condition (7) it follows that $\sum_{n=0}^{\infty} \psi_n^2 < \infty$ and $\sum_{n=0}^{\infty} \pi_n^2 < \infty$. In this case (see

Theorem B, Appendix and proof of Theorem 12.4.1, Brockwell and Davis (1987)), the application of the linear filter $(\psi_j, j \in \mathbf{Z})$ to a stationary white noise sequence $\{\epsilon_t\}$ with the spectral representation

$$\epsilon_t = \int_{(-\pi,\pi]} e^{it\lambda} dW(\lambda) \tag{15}$$

gives a well-defined stationary process

$$\tilde{X}_t \equiv \nabla_{\lambda_1,\ldots,\lambda_m}^{-d_1,\ldots,-d_m} \epsilon_t = \sum_{n=0}^{\infty} \psi_n \epsilon_{t-n}$$

with spectral representation

$$\tilde{X}_t = \int_{(-\pi,\pi]} e^{it\lambda} U^{-d_1,\ldots,-d_m}(e^{-i\lambda}) dW(\lambda)$$

and spectral density

$$f_\nabla(x) := \frac{\sigma^2}{2\pi} \mid U^{-d_1,\ldots,-d_m}(e^{-i\lambda}) \mid^2, \tag{16}$$

where

$$U^{-d_1,\ldots,-d_m}(z) = \prod_{j=1}^{m} (1 - e^{i\lambda_j} z)^{-d_j} (1 - e^{-i\lambda_j} z)^{-d_j} = \sum_{n=0}^{\infty} \psi_n z^n. \tag{17}$$

Further application of the operator $\nabla_{\lambda_1,\ldots,\lambda_m}^{d_1,\ldots,d_m}$ to $\tilde{X}_t$ shows that $\tilde{X}_t$ is the solution of (5).

We omit the proof of the uniqueness, which is based on Theorem B, Appendix and is the same as that of Theorem 12.4.1, Brockwell and Davis (1987).

Proof that $X(t)$ is invertible can be easily derived using similar arguments.

The asymptotics (13) for the covariance function $r(n) = EX_nX_0$ follows immediately from Lemma 3 and Corollary 1, Section 5. □

3. Fractional ARIMA$(p, d_1, \ldots, d_m, q)$ sequences

Let us formally define a fractional ARIMA$(p, d_1, \ldots, d_m, q)$ sequence as a stationary solution of the difference equation

$$\phi(B)\nabla^{d_1,\dots,d_m}_{\lambda_1,\dots,\lambda_m} X_t = \theta(B)\epsilon_t, \quad t \in \mathbf{Z}. \tag{18}$$

Clearly $\{X_t\}$ is a fractional ARIMA$(p, d_1, \dots, d_m, q)$ sequence if and only if the sequence $Z_t := \nabla^{d_1,\dots,d_m}_{\lambda_1,\dots,\lambda_m} X_t$ is an ARMA(p,q) sequence, i.e.

$$\phi(B)Z_t = \theta(B)\epsilon_t.$$

The solution of (18) yields similar results to that found by Hosking (1981, Theorem 5) or Brockwell and Davis (1987, Theorem 12.4.2).

Theorem 2. *Suppose that the degrees $d_j \neq 0$ $(j = 1, \dots, m)$ satisfy condition (7) and the polynomials $\phi(z)$ and $\theta(z)$ have no common zeroes.*

(a) If $\phi(z) \neq 0$ for $\mid z \mid = 1$, then there exists a unique stationary solution of (18) given by

$$X_t = \sum_{j \in \mathbf{Z}} \zeta_j \ \nabla^{-d_1,\dots,-d_m}_{\lambda_1,\dots,\lambda_m} \epsilon_{t-j},$$

where ζ_j are determined by the Laurent expansion

$$\sum_{j \in \mathbf{Z}} \zeta_j \ z^j = \frac{\theta(z)}{\phi(z)}$$

in some annulus of $\mid z \mid = 1$.

X_t has the spectral representation

$$X_t = \int_{(-\pi,\pi]} e^{it\lambda} \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} U^{-d_1,\dots,-d_m}(e^{-i\lambda}) dW(\lambda) \tag{19}$$

and the spectral density of $\{X_t\}$ is of the form

$$f(\lambda) = \mid \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \mid^2 f_\nabla(\lambda), \tag{20}$$

where $f_\nabla(\cdot)$ is the spectral density (16), the function $U^{-d_1,\dots,-d_m}(z)$ is defined by (17) and $dW(\lambda)$ denotes the random spectral measure from the spectral representation (15) of the WN sequence ϵ_t;

(b) $\{X_t\}$ is causal if and only if $\phi(z) \neq 0$ for $\mid z \mid \leq 1$;

(c) $\{X_t\}$ *is invertible if and only if* $\theta(z) \neq 0$ *for* $|z| \leq 1$*;*

(d) the covariance function of X_n *has the same asymptotics as in Theorem 1 with the weights* $\tilde{a}_k = \left| \frac{\theta(e^{-i\lambda_k})}{\phi(e^{-i\lambda_k})} \right|^2 a_k$ *instead of* a_k *(*$1 \leq k \leq m$*) in (13).*

PROOF of THEOREM 2. The statements (a),(b), (c) follow from the same arguments as in the proof of Theorem 12.4.2 by Brockwell and Davis (1987).

The spectral representation (19) for the solution $X_t = \phi(B)^{-1}\theta(B)\nabla_{\lambda_1,\ldots,\lambda_m}^{-d_1,\ldots,-d_m}\epsilon_t$ of (18) can be readily derived using Theorem B, Appendix. Obviously, (19) yields (20).

The asymptotics (d) of the covariance function can be obtained in the same way as (iii) in Theorem 1. □

4. Parameter estimators for ARIMA($p, d_1, \ldots, d_m, q$) model

In the following section we deal with parameter estimators for the fractional ARIMA($p, d_1, \ldots, d_m, q$) sequence (18) introduced in Section 3. The considered model includes (unknown) parameters σ (scale parameter); $\lambda = (\lambda_1, \ldots, \lambda_m)$, d $= (d_1, \ldots, d_m)$ (frequences λ_j of peaks and zeroes and differencing degrees d_j); coefficients $\phi = (\phi_1, \ldots, \phi_p)$ and $\theta = (\theta_1, \ldots, \theta_q)$ of the polynomials $\phi(\cdot)$ and $\theta(\cdot)$. We assume that the orders p and q of the polynomials and the number m of peaks (zeroes) are fixed. Partial cases of this problem are well investigated (see Introduction), but many aspects are yet to be explained.

In this section we shall turn our attention to Whittle's estimators for the mentioned parameters. Two important problems appear in this connection, namely: (a) consistency of the estimators , (b) description of their asymptotic distribution. We deal mainly with the first problem (when the spectrum has peaks), which is relatively simpler and comparatively general results can be obtained (an essential use is made of the method by Hannan(1973)). The problem (b) is currently under investigation. It is more complicated and involves most likely appearance of non-Gaussian limit distributions. A related question for Gaussian and moving-average sequences with the peak at the zero frequence in the spectrum was considered by Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990).

Let $\{X_n\}$ denote the fractional ARIMA$(p, d_1, \dots, d_m, q)$ sequence (18) with spectral density (20):

$$\sigma^2 f(x, \vartheta) := \left| \frac{\theta(e^{-ix})}{\phi(e^{-ix})} \right|^2 f_\nabla(x), \tag{21}$$

where σ and $\vartheta = (\lambda, \mathrm{d}, \phi, \theta) = (\lambda_1, \dots, \lambda_m;\ d_1, \dots, d_m;\ \phi_1 \dots, \phi_p;\ \theta_1, \dots, \theta_q)$ are unknown parameters lying in an open and relatively compact set $\Sigma \times \Theta_{m,p,q} \subset (0, \infty) \times \mathbf{R}^{\mathbf{2}m+p+q}$.

We assume that for any $\vartheta \in \Theta_{m,p,q}$ the following conditions are satisfied:

(w.1) the degrees $d_j > 0$, $j = 1, \dots, m$ are positive and satisfy condition (7);

(w.2) the polynomials $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeroes, $\phi(z) \neq 0$ and $\theta(z) \neq 0$ for $| z | \leq 1$;

(w.3) the polynomials $\phi(\cdot)$ and $\theta(\cdot)$ satisfy the normalization condition

$$\phi(0) = \theta(0) = 1.$$

In this section we suppose that the white noise sequence $\{\epsilon_n\}$ in (18) is an i.i.d. sequence with $E\epsilon_j = 0$, $E\epsilon_j^2 = \sigma^2$.

From assumptions (w.1) -(w.2) it follows that the solution $\{X_n\}$ of (18) is causal (see Theorem 2, Section 3). Therefore, it has a one-sided moving-average representation

$$X_n = \sum_0^\infty \alpha(j, \vartheta)\epsilon_{n-j}$$

with corresponding weights $\alpha(j, \vartheta)$, which in addition, satisfy the normalization condition introduced by Hannan (1973):

Lemma 1. *Under condition (w.3),*

$$\alpha(0, \vartheta) = 1, \tag{22}$$

and, equivalently,

$$\int_{-\pi}^{\pi} \log 2\pi f(x, \vartheta) dx = 0. \tag{23}$$

PROOF of LEMMA 1. The equivalence of (22) and (23) is well-known (see Hannan (1973), Fox and Taqqu (1986)). (22) means that the one-step prediction standard deviation of the sequence $\{\sigma^{-1} X_n\}$ does not depend on the parameter ϑ.

We prove (23). On account of (21),

$\int_{-\pi}^{\pi} \log 2\pi f(x,\vartheta) dx = \int_{-\pi}^{\pi} \log \mid \theta(e^{-ix}) \mid^2 dx - \int_{-\pi}^{\pi} \log \mid \phi(e^{-ix}) \mid^2 dx$

$$+ \int_{-\pi}^{\pi} \log(\frac{2\pi}{\sigma^2} f_{\nabla}(x)) dx. \tag{24}$$

We observe that under condition (w.1) the spectral density $f_{\nabla}(x)$ of the moving-average process (8) (see Theorem 1) is suitably normalized: from (9) it follows that $\psi_0 \equiv \psi(0,\vartheta) = 1$ in (8).

Therefore,

$$\int_{-\pi}^{\pi} \log(\frac{2\pi}{\sigma^2} f_{\nabla}(x)) dx = 0.$$

On the other hand, we have that the function $\frac{\sigma^2}{2\pi} \mid \theta(e^{-ix}) \mid^2$ is the spectral density of the sequence

$$Y_n = \theta(B)\epsilon_n \equiv \sum_{j=0}^{q} \theta_j \epsilon_{n-j},$$

and, moreover, $\theta_0 = 1$ according to (w.3). Hence, $\int_{-\pi}^{\pi} \log \mid \theta(e^{-ix}) \mid^2 dx = 0$. In a similar manner we obtain that $\int_{-\pi}^{\pi} \log \mid \phi(e^{-ix}) \mid^2 dx = 0$. Thus, the right hand side of (24) equals zero. □

As usual the Whittle's estimate $(\tilde{\sigma}_N, \tilde{\vartheta}_N)$ of the true parameter (σ_0, ϑ_0) is obtained by minimizing the quadratic form

$$L_N(X_1, \ldots, X_N; \sigma, \vartheta) = \frac{1}{2}\sigma^{-2} \sum_{t,s=1}^{N} b(t-s,\vartheta) X_t X_s + N \log \sigma$$

in σ and ϑ, where

$$b(t,\vartheta) = (2\pi)^{-2} \int_{-\pi}^{\pi} f^{-1}(x,\vartheta) e^{itx} dx.$$

We assume that

(w.4) the parameters (σ, ϑ) determine the spectral density (21) uniquely.

Concerning the notation we recall that

$\tilde{\vartheta}_N = (\tilde{\lambda}_1^{(N)}, \ldots, \tilde{\lambda}_m^{(N)}; \tilde{d}_1^{(N)}, \ldots, \tilde{d}_m^{(N)}; \tilde{\phi}_1^{(N)} \ldots, \tilde{\phi}_p^{(N)}; \tilde{\theta}_1^{(N)}, \ldots, \tilde{\theta}_q^{(N)}),$

$\vartheta_0 = (\lambda_1^0, \ldots, \lambda_m^0; d_1^0, \ldots, d_m^0; \phi_1^0 \ldots, \phi_p^0; \theta_1^0, \ldots, \theta_q^0)$, and assume that true parameter (σ_0, ϑ_0) lies in the interior of $\Sigma \times \Theta_{m,p,q,}$.

Theorem 3. *Let conditions (w.1)-(w.4) be satisfied.*

Then

$$\tilde{\sigma}_N \to \sigma_0 \quad and \quad \tilde{\vartheta}_N \to \vartheta_0$$

almost surely as $N \to \infty$.

PROOF THEOREM 3 is based on the general inferential theory for linear time-series constructed by Hannan (1973, Theorem 1). The conditions required by Hannan (1973) including (23) are naturally satisfied. □

An advantage of this approach is that besides of the long-term or short-term persistence, the parameter estimators for the generalized fractional ARIMA model can also be readily obtained.

When the spectrum contains both peaks and zeroes, problem (a) (and b) is more difficult and will be considered elsewhere.

5. The asymptotic covariances of a process with spectral density having peaks.

In this Section we present asymptotical formulas for $cov(X_t, X_0)$ of a stationary sequence $\{X_t\}$ with the spectral density of the form

$$s(\lambda) = f(\lambda) g(\lambda) \tag{25}$$

where

$$f(\lambda) \equiv f_\nabla(\lambda) = \frac{\sigma^2}{2\pi} \mid U^{-d_1, \ldots, -d_m}(e^{-it\lambda}) \mid^2 \tag{26}$$

is the spectral density (11) of the sequence (8). We deal with the case when f_∇ has peaks ($\max_{j=1,\ldots,m} d_j > 0$) and the powers $0 \neq d_i$, $i = 1, \ldots, m$ satisfy condition (7), and we assume that $g : [-\pi, \pi] \to [0, \infty]$ is a real even function, slowly varying at the peak- points λ_j (such that $d_j > 0$), $j = 1, \ldots, m$ in the manner of Zygmund, i.e.

(g.1) for any $\delta > 0$, $g(\lambda) \mid \lambda - \lambda_j \mid^{\delta}$ is an increasing (decreasing) and $g(\lambda) \mid \lambda - \lambda_j \mid^{-\delta}$ is a decreasing (increasing) function in some right-neighbourhood (left-neighbourhood) of λ_j,

and

(g.2) g has bounded variation on the set $[0,\pi]\setminus\bigcup_{i=1}^{m}[\lambda_j - \epsilon, \lambda_j + \epsilon]$ for any $\epsilon > 0$.

A function g slowly varying at 0 in Zygmund's sens satisfies the condition $g(tx)/g(t) \to 1$ $(t \to 0)$ $\forall x > 0$, i.e. it is slowly varying in the manner of Karamata.

We assume that the function g is extended with period 2π to the real line.

First we discuss the case when the spectral density f has one peak at the point $\lambda = 0$, i.e.

$$\tilde{f}(x) \equiv f(x,d) = \frac{\sigma^2}{2\pi} \mid 1 - e^{-ix} \mid^{-2d} = \frac{\sigma^2}{2\pi} \mid 2\sin\frac{x}{2} \mid^{-2d},$$

$0 < d < 1/2$. Then from Theorem 2.24 Zygmund(1959) (see also Corollary of Theorem 2, Samarov and Taqqu (1988)) it follows

Lemma 2. *Let*

$$s(x) = \tilde{f}(x) l(x) \quad \text{as } x \in [0,a],$$

$0 < a \le \pi$ *where* l *is slowly varying in Zygmund's sens and has bounded variation on* (ϵ, a) *for any* $0 < \epsilon < a$. *Then, as* $n \to \infty$,

$$\int_0^a s(x)\cos(xn)dx \sim \frac{\sigma^2}{2\pi} \mid n \mid^{2d-1} l(\frac{1}{n})\Gamma(1-2d)\sin(\pi d),$$

$$\int_0^a s(x)\sin(xn)dx \sim \frac{\sigma^2}{2\pi} \mid n \mid^{2d-1} l(\frac{1}{n})\Gamma(1-2d)\cos(\pi d).$$

In the case of a few peaks the following result may be stated:

Lemma 3. *Let* $s(\lambda) = f(\lambda)g(\lambda)$ *be the spectral density in (25), where the degrees* d_i *and the function* g *satisfy the conditions mentioned thereafter. Further suppose that*

$$g(-x + \lambda_k) \sim g(x + \lambda_k) \quad (x \to 0) \tag{27}$$

if $d_k > 0$, $1 \le k \le m$.

Then as $n \to \infty$

$$r(n) = \sum_{k=1,\dots,m:\ d_k>0} a_k \mid n \mid^{2d_k^*-1} g(-\frac{1}{n} + \lambda_k)(\cos n\lambda_k + o(1)) \tag{28}$$

where a_k and d_k^* are defined in Theorem 1.

PROOF of LEMMA 3. Assume without restriction of generality that degrees d_j are positive and $0 < \lambda_j < \pi$, $j = 1, \dots, m$, and split the interval $[0, \pi]$ into segments $[\lambda_k^{'}, \lambda_{k+1}^{'}]$, $k = 1, \dots, m$, where $\lambda_1^{'} = 0$; $\lambda_k^{'} = (\lambda_k + \lambda_{k+1})/2$, $k = 2, \dots, m-1$; $\lambda_m^{'} = \pi$.

Then

$$\int_{-\pi}^{\pi} s(\lambda) e^{i\lambda n} d\lambda = 2 \sum_{k=1}^{m} \int_{\lambda_k^{'}}^{\lambda_{k+1}^{'}} s(\lambda) \cos(\lambda n) d\lambda. \tag{29}$$

We separate the following integral into two parts:

$$\begin{aligned} \int_{\lambda_k^{'}}^{\lambda_{k+1}^{'}} s(\lambda)\cos(\lambda n) du &= \int_0^{(\lambda_{k+1}-\lambda_k)/2} s(u-\lambda_k)\cos n(u-\lambda_k) du \\ &+ \int_0^{(\lambda_k-\lambda_{k-1})/2} s(u+\lambda_k)\cos n(u+\lambda_k) du \\ &\equiv i_k^{'} + i_k^{''}. \end{aligned} \tag{30}$$

Since $\cos(\lambda - \lambda^{'}) = \cos\lambda\cos\lambda^{'} + \sin\lambda\sin\lambda^{'}$, Lemma 2 yields

$$\begin{aligned} i_k^{'} &= \int_0^{(\lambda_{k+1}-\lambda_k)/2} f(u-\lambda_k) g(u-\lambda_k)(\cos nu \cos n\lambda_k + \sin nu \sin n\lambda_k) du \\ &= \frac{\sigma^2}{2\pi} \mid n \mid^{2d_k-1} g(\frac{1}{n} - \lambda_k)\Gamma(1-2d_k)(\sin\pi d_k \cos n\lambda_k + \cos\pi d_k \sin n\lambda_k + o(1)) \end{aligned}$$

and

$$i_k^{''} = \frac{\sigma^2}{2\pi} \mid n \mid^{2d_k-1} g(\frac{1}{n} + \lambda_k)\Gamma(1-2d_k)(\sin\pi d_k \cos n\lambda_k - \cos\pi d_k \sin n\lambda_k + o(1))$$

as $n \to \infty$. Therefore, (28) holds because of (29), (30) and (27). □

It can be readily shown that a sufficient condition for the function g to be slowly varying at $x = 0$ in Zygmund's sense is the existence of a derivative $g^{'}$ such that

$$\frac{x g^{'}(x)}{g(x)} \to 0 \quad \text{as x} \to 0. \tag{31}$$

Moreover, it is well-known that the class of functions g slowly varying at infinity (Zygmund's), i.e. $g(1/x)$ is slowly varying at $x = 0$, coincides with a class of normalized functions which can be represented as

$$l(x) = a_0 \exp\{\int_c^x \frac{\epsilon(u)}{u} du\}$$

where $0 < a_0 < \infty$, $c > 0$ and $\epsilon(x) \to 0$ as $x \to \infty$; $\epsilon(x) = \frac{xl'(x)}{l(x)}$ almost surely. (see Bingham, Goldie and Teugels, 1987).

Therefore, the function

$$g(\lambda) \ := \mid \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \mid^2 \tag{32}$$

with polynomials θ and ϕ, where $\phi(z) \neq 0$ as $\mid z \mid = 1$, has bounded derivative and is slowly varying at any point $0 \leq \lambda \leq \pi$ such that $g(\lambda) \neq 0$.

This leads to the following result:

Corollary 1. *Suppose that in Theorem 1 the function g is defined by (32) and $\theta(e^{-i\lambda_k}) \neq 0$ if $d_k > 0$, $k = 1, \ldots, m$. Then*

$$cov(X_n, X_0) = \sum_{k=1,\ldots,m:\ d_k>0} \mid \frac{\theta(e^{-i\lambda_k})}{\phi(e^{-i\lambda_k})} \mid^2 a_k \mid n \mid^{2d_k^*-1} (\cos n\lambda_k + o(1))$$

as $n \to \infty$.

6. Remarks

6.1 The models with finite number of zeroes or peaks in the spectrum require a different approach in order to obtain the asymptotics (9) for the weigths ψ_j (and π_j) than the model with one peak/zero investigated by Gray et al. (1989). We have used the Darboux method (see Szegő (1959)).

6.2 Limit theorems for functionals of stationary Gaussian sequences with a similar covariance and spectral density function were studied by Rosenblatt (1981) and Giraitis (1983). The long-memory of these sequences means that the resulting limits can be non-Gaussian (self-similar processes).

6.3 The covariance asymptotics provided by Gray et al. (1989 , Theorem 3 (c)) (in case $m = 1$) is different from that in Lemma 3 because of a misprint of the sign on the third line, p. 241 of the paper mentioned.

6.4 The further properties of the estimators for $\lambda_1, \ldots, \lambda_m$ and $d_1, \ldots, d_m$ are currently investigated.

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APPENDIX

In this section we present a classical theorem or Darboux's method, which leads to asymptotical expansions for classical orthogonal polynomials (see Szegő (1959)), and a theorem concerning the spectral repesentation of stationary process transformed by linear filters (Brockwell, Davis (1987)).

Theorem A (Szegő (1959), Theorem 8.4). *Let $h(z)$ be regular for $\mid z \mid < 1$, and let it have a finite number of singularities*

$$e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_l}, \qquad e^{i\phi_k} \neq e^{i\phi_m} \quad (k \neq m)$$

on the unit circle $\mid z \mid = 1$. Let

$$h(z) = \sum_{\nu=0}^{\infty} c_\nu^{(k)} (1 - z e^{-i\phi_k})^{d_k + \nu b_k}, k = 1, \ldots, l$$

in the vicinity of $e^{i\phi_k}$, where $b_k > 0$. Then the expression

$$\sum_{\nu=0}^{\infty} \sum_{k=1}^{l} c_\nu^{(k)} \binom{d_k + \nu b_k}{n} (-e^{-i\phi_k})^n$$

furnishes an asymptotical expansion of the coefficient of z^n in $h(z)$ in the following sense: if Q is an arbitrary positive number , and if a sufficiently large number p of terms is taken in the sum $\sum_{\nu=0}^{\infty}$, we obtain an expression which approximates the *coefficient* q_n *in expansion* $h(z) = \sum_{n=0}^{\infty} q_n z^n$ *with an error equal to* $\mathrm{O}(n^{-Q})$*, i.e.*

$$q_n = \sum_{\nu=0}^{p-1} \sum_{k=1}^{l} c_\nu^{(k)} \begin{pmatrix} d_k + \nu b_k \\ n \end{pmatrix} (-e^{-i\phi_k})^n + O(n^{-Q}), \qquad n \to \infty.$$

A simple discussion shows that it suffices to stop at the term $\nu = p - 1$ of the sum, where p is a positive integer such that

$$p \geq \max_{1 \leq k \leq l} b_k^{-1} \{Q - \mathrm{Re}(d_k) - 1\}$$

(see Szegő (1959)).

Theorem B (Brockwell and Davis (1987), Theorem 4.10.1). *Let* $\{X_t\}$ *be a zero-mean stationary process with spectral representation*

$$X_t = \int_{(-\pi,\pi]} e^{it\nu} dZ_X(\nu)$$

and spectral distribution function $F_X(.)$*, where* $Z_X(.)$ *is an orthogonal-increment process,* $Z(-\pi) = 0$*. Suppose* $\{h_j, j \in \mathbf{Z}\}$ *is a sequence such that the series* $\sum_{j=-n}^{n} h_j e^{-ij\cdot}$ *converges in* $L^2(F_X)$ *norm to* $h(e^{-i\cdot})$ *as* $n \to \infty$*. Then the process*

$$Y_t = \sum_{j \in \mathbf{Z}} h_j X_{t-j}$$

is stationary with zero mean, spectral distribution function

$$F_Y(\lambda) = \int_{(-\pi,\lambda]} | h(e^{-i\nu}) |^2 \, dF_X(\nu)$$

and spectral representation

$$Y_t = \int_{(-\pi,\pi]} h(e^{it\nu}) dZ_X(\nu).$$

REFERENCES

Anderson, O.D. (1979). The autocovariance structures associated with general unit circle nonstationary factors in the autoregressive operators of otherwise stationary ARMA time series models. *Cahiers du CERO* **21**, 221-237.

Beran, J. (1992). Statistical methods for data with long-range dependence. *Statistical Science.* **7**, 404-427.

Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987). *Regular variation.* New York: Cambridge University Press.

Brockwell, P.J. and Davis, R.A. (1987). *Time series: Theory and Methods.* New York: Wiley.

Carlin, J.B. and Dempster, A.P. (1989). Sensitivity analysis of seasonal adjustments: empirical case studies. *JASA* **84**, 6-20.

Dahlhaus, R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Stat.* **17**, 1749-1766.

Fox, R. and Taqqu, M.S. (1986). Large sample properties of parameter estimates for strongly dependent stationary Gaussian time series *Ann. Stat.* **14**, 517-532.

Geweke, J. and Porter-Hudak, S. (1983). The estimation and application of long-memory time series models. *J.Time Series Anal.* **4**, 221-238.

Giraitis, L. (1983). On the convergence of some non-linear transformations of Gaussian sequence to self-similar prosess. *Lithuanian Math. J.* **23**, 58-68.

Giraitis, L. and Surgailis, D. (1990). A central limit theorem for quadratic forms in strongly deppendent random variables and its application to asymptotical normality of Whittle's estimate. *Probab.Th.Rel.Fields.* **86**, 87-104.

Granger, C.W. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *J.Time Series Anal.* **1**, 15-29.

Gray, H.,L., Zhang, N.-F. and Woodward, W.A. (1989). On generalized fractional processes. *J.Time Series Anal.* **10**, 233-257.

Hannan, E.J. (1973). The asymptotic theory of linear time-series models. *J. Appl. Prob.* **10**, 130-145.

Hosking, J.R.M. (1981). Fractional differencing. *Biometrika* **68**, 165-176.

Janacek, G.J. (1982). Determining the degree of differencing for time series via log spectrum. *J.Time Series Anal.* **3**, 177-183.

Rosenblatt, M. (1981). Limit theorems for Fourier transforms of functionals of Gaussian sequences. *Z. Wahrsch. verw. Geb.* **55**, 123-132.

Porter-Hudak, S. (1990). An application of seasonal fractionally differenced model to the monetary aggregates. *JASA* **85**, 338-344.

Samarov A., Taqqu M.S. (1988). On the efficiency of the sample mean in long-memory noise. *J.Time Series Anal.* **9**, 191-200.

Szegő, G. (1959). *Orthogonal polynomials.* AMS, New York.

Yajima, Y. (1985). On estimation of long-memory time-series models. *Austral.J.Stat.* **27**, 303-320.

Zygmund, A. (1959). *Trigonometric Series. I.* Cambridge University Press.