# A NEW CRITERION FOR TIGHTNESS OF STOCHASTIC PROCESSES AND AN APPLICATION TO MARKOV PROCESSES

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**Abstract.** Let $U$ be an arbitrary stochastic process on the real line and $I = [a, b]$ an interval on $\mathbb{R}$. We prove a stochastic inequality for the modulus of continuity $\omega''(U, I)$ of $U$ on $I$:

$$\omega''(U,I) := \sup_{r \le s \le t} \{|U(r) - U(s)| \wedge |U(s) - U(t)| \, : r, s, t \in I\}.$$

Suppose that $U$ has righthand (or lefthand) continuous paths and that the increments of $U$ can be stochastically bounded in the following way:

$$\mathbb{P}\{|U(r) - U(s)| \wedge |U(s) - U(t)| \ge \lambda\} \le \lambda^{-\gamma} \eta(]r, t])$$

for all $r \le s \le t \in I$, for all $\lambda > 0$, with a real number $\gamma > 0$ and a setfunction $\eta$ which is subadditive in a certain sense. Then exists a constant $K = K(\gamma, \eta)$ with:

$$\mathbb{P}\{\omega''(U,I) > \lambda\} \le \frac{K}{\lambda^\gamma} \eta(]a, b]).$$

If in addition the paths of $U$ are cadlag and the jumps of $U$ can be bounded by a random variable $Z$ we can extend this result to a stochastic inequality for the modulus of continuity

$$\omega(U,I) := \sup_{s,t \in I} \{|U(s) - U(t)|\}$$

using that $\omega(U, I)$ is bounded in the following way: $\omega(U, I) \le 4\omega''(U, I) + Z$.

This result is used to prove the weak convergence of a goodness-of-fit test statistic for hypotheses about the conditional median function $f_0$ of a stationary, real-valued, Markovian time series.

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# 1. Introduction

This paper is concerned with stationary, real-valued, Markovian time series with stationary transition probabilities. Let $(X_t)_{t=1}^n$ be such a time series. The median function of the conditional law of $X_t$ given $\sigma(X_s : s < t)$ is implicitly defined through the following condition:

$$\operatorname{Med}\big(\mathcal{L}(X_t - f_0(X_{t-1})|X_{t-1})\big) = 0 \qquad (t = 1, \dots, n)$$

We propose a Kuiper type statistic to test the hypotheses $f = f_0$. It is based on the process

$$W_n(s, f) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} 1\{X_{t-1} \le s\} \operatorname{sign}\big(X_t - f(X_{t-1})\big)$$

and has the following form:

$$T_n(f) := \sup_{s,t \in \mathbb{R}} |W_n(s, f) - W_n(t, f)|.$$

The test will reject the hypotheses if $T_n(f)$ is too large.

In a simular context Hong-zhi and Bing (1991) have introduced a Kolmogorov-Smirnov type statistic to test hypotheses about the conditional mean function. But for the weak convergence of their statistic they need quite restrictive mixing-properties of the time series and also $(4 + \delta)$-moments of $X_1$. In my opinion the assumptions that are made here are much weaker. The idea of the proof of tightness is different and is due to Koul and Stute (1996). In their proof seems to be a gap which - under some slightly stronger assumptions - is closed here generalizing a result of Billingsley (1969, pp.98).

In section 2 we formulate this generalization and show how it can be used to prove tightness. In section 3 we apply this method to show the weak convergence of the process $W_n(\cdot)$ with $W_n(s) := W_n(s, f_0)$ under some quite weak assumptions. By the Continuos Mapping Theorem this result implies the weak convergence of $T_n(f)$. Most of the proofs are defered to section 4.

# 2. A Tightness Criterion

In this section we formulate a Proposition that generalizes a result of Billingsley. Given an arbitrary subset $A$ of $\mathbb{R}$ we want to state conditions under which we can get a stochastic inequation for the modulus of continuity $\omega''$ of a stochastic process $U$ on $A$. It is defined in the following way:

$$\omega''(U,A) := \sup_{r \le s \le t} \{|U(r) - U(s)| \wedge |U(s) - U(t)| \ : r,s,t \in A\}$$

If weak convergence is studied in the context of the function space $\mathcal{D}$ with the Skohorod-topology such inequations for $\omega''(U)$ are needed to prove tightness (see Billingsley, pp.109). Within the theory that is used here (see Pollard 1990) we have to treat with the following modulus of continuity:

$$\omega(U,A) := \sup_{s,t}\{|U(s) - U(t)| \ : s,t \in A\}$$

The stochastic inequation for $\omega(U,a)$ will be concluded from the inequality for $\omega''(U,A)$.

In order to formulate the conditions on the increments $U(s) - U(t)$ $(s,t \in I)$ that we need for the Proposition we have to make some preperations:

**Definition 1.** *(intervals in $\mathbb{R}$ with coordinates in $A$)*

*The set of all left open intervals with coordinates in $A$ will be denoted by $\mathcal{I}_A$:*

$$\mathcal{I}_A := \{I = ]a,b] : \ a,b \in A\}$$

*For $I := ]a,c]$ be an interval in $\mathcal{I}_A$. For every $b \in A$ with $a < b < c$ we define a subdivision $(I_b^1, I_b^2)$ of $I$ by:*

$$I_b^1 := ]a,b] \ \text{and} \ I_b^2 := ]b,c]$$

*Let $J := ]d,e]$ be another interval in $\mathcal{I}_A$. $I$ and $J$ are called neighbours if $c = d$.*

**Definition 2.** *(U as distribution function of a set function)*

*One can think of the increments of* $U$ *as a signed, additive set function* $\mu_U : \mathcal{I}_A \to \mathbb{R}$ *which maps intervals* $I = ]a, b]$ *to* $U(b) - U(a)$*. In order to simplify notation we write* $U(I) := \mu_U(I)$*.*

**Definition 3.** *(M - subadditive set functions)*

*Corresponding to the signed set functions* $\mu_U$ *we need nonnegative nondecreasing set functions* $\eta$ *on* $\mathcal{I}_A$*:*

$$\mathcal{M}(\mathcal{I}_A) := \{\eta : \mathcal{I}_A \to \mathbb{R} : 0 \leq \eta < \infty \, ; \, \eta \textit{ nondecreasing } \}$$

*There a set function is called nondecreasing if for arbitrary sets* $B, C \in \mathcal{I}_A$ $B \subset C$ *implies* $\eta(B) \leq \eta(C)$*.*

*Let now* $M$ *be a real number,* $M < 1$*. A set function* $\eta$ *is called* $M$ *- subadditive if for every interval* $I \in \mathcal{I}_A$ *with* $\eta(I) > 0$ *there exists a subdivision* $(I_b^1, I_b^2)$ *of* $I$ *so that the following holds:*

$$\eta(I_b^1) + \eta(I_b^2) \leq M\eta(I)$$

Now we can state the Proposition:

**Proposition 1.** *Let* $A$ *be a finite subset of the real line with smallest element* $a$ *and largest element* $b$*. Further let* $U$ *be a stochastic process on* $A$*. Suppose that there exists a real number* $M < 1$*, a* $M$*-subadditive set function* $\eta \in \mathcal{M}(\mathcal{I}_A)$ *and a real number* $\gamma > 0$ *which satisfie*

$$\mathbb{P}\{|U(I)| \wedge |U(J)| \geq \lambda\} \leq \lambda^{-\gamma}\eta(I \cup J)\} \qquad \textit{for all } \lambda > 0$$

*for all neighbouring intervalls* $I, J \in \mathcal{I}_A$*.*

*Then exists a constant* $K(M, \gamma)$ *with:*

$$\mathbb{P}\{\omega''(U, A) \geq \lambda\} \leq \frac{K(M, \gamma)}{\lambda^\gamma}\eta(]a, b])$$

The proof is defered to section 4.

**Remark 1.** *In Billingsley's result there is a more restrictive condition on the set function $\eta$, namely,that:*

$$\eta(\cdot) = \mu(\cdot)^{\alpha}$$

*with a real number $\alpha > 1$ and a finite measure $\mu(\cdot)$.*

**Remark 2.** *It follows immediately from the proof that the statement of the Proposition remains if we replace the functional $\omega''$ by the functional $\tilde{\omega}$ which is defined in the following way:*

$$\tilde{\omega}(U, A) := \min_{l \in A} \Big( \max_{i \in A, i < l} |U(i) - U(a)| \vee \max_{i \in A, i \geq l} |U(b) - U(i)| \Big)$$

*We will use this fact in one part of the proof of tightness for the process $W_n$.*

**Remark 3.** *Let $I = [a, b]$ be an arbitrary interval on the real line. Suppose that the paths of $U$ are righthand (or lefthand) continuous on $I$ and $U$ fullfills the conditions of the Proposition on every finite subset of $I$. Then the following inequality holds:*

$$\mathbb{P}\{\omega''(U, I) > \lambda\} \leq \frac{K(M, \gamma)}{\lambda^{\gamma}} \eta(]a, b])$$

*with the same constant $K(M, \gamma)$ as above.*

*Proof.* W.l.o.g. let $U$ have righthand continuous paths. We choose the following dyadic subsets $I_m$ of $I$:

$$I_m := \{a + \frac{i(b-a)}{2^m} : \ 0 \leq i \leq 2^m\}$$

For every $\epsilon > 0$ we find real numbers $r < s < t \in I$ with:

$$\omega''(U, I) \leq \big(|U(r) - U(s)| \wedge |U(s) - U(t)|\big) + \epsilon$$

Now we can approximate $r, s, t$ by elements $r_m, s_m, t_m \in I_m$: $r_m \downarrow r, s_m \downarrow s, t_m \downarrow t$. Using the righthand continuity of the paths of $U$ we find therefore that $\omega''(U, I_m)$ converges from below pathwise to $\omega''(U, I)$ $(m \to \infty)$. By the Monotone Convergence

Theorem we can conclude:

$$\begin{aligned}
\mathbb{P}\{\omega''(U,I) > \lambda\} &= \mathbb{E}\, 1\{\lim_{m\to\infty} \omega''(U,I_m) > \lambda\} \\
&= \mathbb{E} \lim_{m\to\infty} 1\{\omega''(U,I_m) > \lambda\} \\
&= \lim_{m\to\infty} \mathbb{E}\, 1\{\omega''(U,I_m) > \lambda\} \\
&\le \frac{K(M,\gamma)}{\lambda^\gamma} \eta(]a,b])
\end{aligned}$$

□

**Remark 4.** *For an arbitrary interval $I = [a,b]$ on $\mathbb{R}$ we define:*

$\mathcal{F}_{cadlag}(I) := \{f : I \to \mathbb{R} :$ *$f$ is righthand continuous and has existing lefthand limits* $\}$

*Suppose that the paths of $U$ are elements of $\mathcal{F}_{cadlag}(I)$ and that the jumps of $U$ are bounded uniformly on $I$ by a random variable $Z$.*

*Then we can conclude:*

$$\omega(U,I) \le 4\omega''(U,I) + Z$$

*Proof.* For arbitrary $s_1 < s_2 \in I$ we define:

$$\begin{aligned}
\sigma_1 &:= \sup\{\sigma \in I : \sup_{s_1 \le s \le \sigma} |U(s) - U(s_1)| \le 2\omega''(U,I)\} \\
\sigma_2 &:= \inf\{\sigma \in I : \sup_{\sigma \le t \le s_2} |U(t) - U(s_2)| \le 2\omega''(U,I)\}
\end{aligned}$$

If $\sigma_1 < \sigma_2$, then there exist real numbers $s, t$ $(\sigma_1 \le s < t \le < \sigma_2)$ with:

$$|U(s) - U(s_1)| > 2\omega''(U,I) \quad \text{and} \quad |U(t) - U(s_2)| > 2\omega''(U,I).$$

But from $|U(s) - U(s_1)| > \omega''(U,I)$ follows $|U(s) - U(s_2)| \le \omega''(U,I)$ and $|U(s) - U(t)| \le \omega''(U,I)$. Therefore we can conclude:

$$|U(t) - U(s_2)| \le |U(t) - U(s)| + |U(s) - U(s_2)| \le 2\omega''(U,I)$$

which contradicts the assumption made above and hence implies that $\sigma_2 \le \sigma_1$. Finally we get:

$$\begin{aligned}
|U(s_1) - U(s_2)| &\le |U(s_1) - U(\sigma_1-)| + |U(\sigma_1-) - U(\sigma_1+)| + |U(\sigma_1+) - U(s_2)| \\
&\le 4\omega''(U,I) + Z
\end{aligned}$$

$\square$

**Remark 5.** *There is also a multidimensional version of this Proposition, that deals with stochastic processes $U$ on finite subsets of $\mathbb{R}^q$. It can in a simulare way be extended to nonfinite subsets of the $\mathbb{R}^q$ (see Erlenmaier 1997).*

## 3. Weak Convergence of the Process $W_n$

We denote by $P_c(\cdot,\cdot)$ the kernel of the stationary transition probability of the time series and define:

$$P_c(A) := \sup_{x\in\mathbb{R}} P_c(A,x)$$

for all measurable sets $A \subset \mathbb{R}$ and $\hat{P}_n := 1/n \sum_{t=0}^{n-1} \delta_{X_t}$.

Now we are able to state the conditions under which the process $W_n$ converges weakly:

- **K**1 $\hat{P}_n \;\to_w\; P$ (in probability) where $P$ is the stationary measure of the time series $(X_t)$
- **K**2 There exists a real number $\beta_1 > 0$ so that $P_c(I) \le |I|^{\beta_1}$ for all intervalls $I \subset \mathbb{R}$.
- **K**3 $\sum_{i=-\infty}^{+\infty} \sqrt{P([i,i+1])} < \infty$

Condition **K**1 is needed for the convergence of the finite-dimensional distributions (fidis), **K**2 and **K**3 will be used to prove tightness.

The proof of tightness uses a functional CLT (see Pollard 1990). In order to apply this Theorem we first of all have to define a pseudometric $\rho$ on $[-\infty,\infty]$. We write $\rho_g(s,t) := |g(s) - g(t)|$, where $g$ is a continuously differentiable function on the compact real line with $g'(t) > 0$ for every $t \in \mathbb{R}$. That implies that $([-\infty,\infty],\rho_g)$ is a totally bounded pseudometric space.

**Theorem 1.** *If the conditions **K1** - **K3** are satisfied, the sequence $W_n(\cdot)$ of stochastic processes on $([-\infty,\infty],\rho_g)$ converges weakly to a pathwise uniformly continuous, centered Gaussian process $W$ with covariance function $K(s,t) := F(s \wedge t)$, where $F$ is the distribution function of $P$.*

*If condition **K**3 is not satisfied at least the sequence converges to $W$ on every finite intervall $I \subset \mathbb{R}$.*

*Proof.* The convergence of the fidis follows from condition **K**1 and the CLT for martingale difference arrays (see Pollard 1984). The proof of the stocastic equicontinuity is defered to section 4. □

**Corollary 1.** *If **K**1 - **K**3 hold the Continuous Mapping Theorem implies the weak convergence of the teststatistic to a functional of the standard Brownian motion $B$ on $[0,1]$:*

$$\begin{aligned} T_n(f_0) \ \to_w \ \sup_{s,t \in \mathbb{R}} |W(s) - W(t)| &= \sup_{s,t\in[0,1]} |B(F(s)) - B(F(t))| \\ &= \sup_{s,t\in[0,1]} |B(s) - B(t)| \qquad \textit{in law} \end{aligned}$$

**Remark 6.** *The stochastic equicontinuity of the more general marked empirical processes*

$$W_{n,Z}(s) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} 1\{X_{t-1} \le s\} Z_{n,t}$$

*which Koul and Stute investigate in section 3 of their paper can be proved in a similar way. There the series $(Z_{n,t})_{t=1}^n$ of random variables form a martingale difference array with respect to the series $\Big(\sigma(X_0, \dots X_t)\Big)_{t=0}^n$ of $\sigma$-algebra (for details see Koul and Stute 1996). The assumption about the random variable $Z_{n,t}$ can be replaced by the weaker assumption:*

$$\mathbb{E}\left[Z_{n,t}^2 | X_{t-1}\right] \le K < \infty \qquad \textit{for all } n \in \mathbb{N}; \quad t = 1, \dots, n \quad \textit{a.s}$$

*Proof.* Most of the argumentation is identical. The only difference is, that the jumps of their process $W_{n,Z}$ cannot be bounded by $1/\sqrt{n}$. But it is sufficient to observe that on an interval $I$ they are bounded by

$$Z_n := \max_{1\le t\le n} \frac{1}{\sqrt{n}} 1\{X_{t-1} \in I\} |Z_{n,t}|$$

Therefore we can conclude:

$$\begin{aligned}\mathbb{P}\{Z_n \geq \lambda\} &\leq \sum_{t=1}^{n} \mathbb{P}\{\frac{1}{\sqrt{n}} 1\{X_{t-1} \in I\} |Z_{n,t}| \geq \lambda\} \\ &\leq \sum_{t=1}^{n} \frac{1}{\lambda^2 n}\, \mathbb{E}\, 1\{X_{t-1} \in I\}\, \mathbb{E}\left[Z_{n,t}^2 | X_{t-1}\right] \\ &\leq \frac{1}{\lambda^2} K P(I)\end{aligned}$$

Therefore in this context we end up with the same tightness result as in the proof of Theorem 1. □

# 4. Some Proofs

Within some proofs there are statements which - for the sake of clearity - are proofed after the main argumentation is finished.

***Proof of Proposition 1.*** In most parts of the proof we follow the argumentation of Billingsley (1968).

W.l.o.g. suppose that $A = \{1, \ldots, m\}$. In the following we write $U_i := U(i)$ $(i = 1, \ldots, m)$.

First of all we recall the definition of $\tilde{\omega}(U)$:

$$\tilde{\omega}(U) := \min_{1 \leq l \leq m} \left( \max_{1 \leq i < l} |U_i - U_1| \vee \max_{l \leq i \leq m} |U_m - U_i| \right)$$

The following inequality is proofed below:

$$\omega''(U) \leq 2\tilde{\omega}(U) \tag{4.1}$$

Therefore it is sufficient to show the result of the proposition for $\tilde{\omega}$. We do this by induction over $m$.

For $m = 1, 2$ $\tilde{\omega}$ equals 0. Here is the induction step from $m - 1$ to $m$:

Choose an integer $h$ which satisfies $\eta(]1, h-1]) + \eta(]h+1, m]) \leq M\eta(]1, m])$ and define

$$Y := U|\{1, \ldots, h-1\} \quad , \quad Z := U|\{h+1, \ldots, m\}$$

Later on we want to apply the induction hypotheses to $\tilde{\omega}(Y)$ and $\tilde{\omega}(Z)$. Now we define $\tau(r,s,t) := |U_r - U_s| \wedge |U_s - U_t|$ and finally

$$B := \tau(1,h-1,m) \vee \tau(1,h-1,h+1) \vee \tau(h-1,h+1,m) \vee \tau(1,h+1,m)$$

to conclude:

$$\tilde{\omega}(U) \leq \tilde{\omega}(Y) \vee \tilde{\omega}(Z) + 2B \quad \text{(proofed below)} \tag{4.2}$$

Let $\zeta_1, \zeta_2 > 0$ be arbitrary numbers with $\zeta_1 + \zeta_2 = 1$ and define $\delta := 1/(1+\gamma)$. From condition (a) and induction hypotheses follows:

$$\begin{aligned}
\mathbb{P}\{\tilde{\omega}(U) \geq \lambda\} &\leq \mathbb{P}\{\tilde{\omega}(Y) \vee \tilde{\omega}(Z) \geq \lambda\zeta_1\} + \mathbb{P}\{2B \geq \lambda\zeta_2\} \\
&\leq \mathbb{P}\{\tilde{\omega}(Y) \geq \lambda\zeta_1\} + \mathbb{P}\{\tilde{\omega}(Z) \geq \lambda\zeta_1\} + \mathbb{P}\{B \geq \frac{\lambda\zeta_2}{2}\} \\
&\leq \frac{K(M,\gamma)}{\lambda^\gamma \zeta_1^\gamma}\Big(\eta(]1,h-1]) + \eta(]h+1,m])\Big) + \frac{2^{\gamma+2}}{\lambda^\gamma \zeta_2^\gamma}\eta(]a,b]) \\
&\leq \left\{\Big(\frac{K(M,\gamma)}{\lambda^\gamma} M\eta(]a,b])\Big)^\delta + \Big(\frac{2^{\gamma+2}}{\lambda^\gamma}\eta(]a,b])\Big)^\delta\right\}^{\frac{1}{\delta}} \\
&\leq \frac{\eta(]a,b])}{\lambda^\gamma}\left\{\Big(MK(M,\gamma)\Big)^\delta + 2^{(\gamma+2)\delta}\right\}^{\frac{1}{\delta}}
\end{aligned}$$

Choosing $K(M,\gamma) := 2^{\gamma+2}/(1-M^\delta)^{1/\delta}$ completes the proof. □

***Proof of inequality 4.1*** . We have to show that $\omega'' \leq 2\tilde{\omega}$. Let $r,s,t$ be the integers that minimize the expression for $\omega''$ and $l$ the one that maximizes the expression for $\tilde{\omega}$. We observe:

$$\begin{aligned}
|U_r - U_s| &\leq |U_r - U_1| + |U_1 - U_s| \leq 2\tilde{\omega} \quad (l > r,s) \\
|U_s - U_t| &\leq |U_s - U_m| + |U_m - U_t| \leq 2\tilde{\omega} \quad (l \leq s,t)
\end{aligned}$$

□

***Proof of inequality 4.2*** . We distinguish the following three cases

a) $|U_{h-1} - U_1| \leq B$ und $|U_m - U_{h+1}| \leq B$

b) $|U_{h-1} - U_1| > B$

c) $|U_m - U_{h+1}| > B$

In each case we want to choose a number $l$ which satisfies

$$\max_{1\le i<l}|U_i-U_1|\vee\max_{l\le i\le m}|U_m-U_i|\le\tilde\omega(Y)\vee\tilde\omega(Z)+2B$$

In case a) we can choose $l=h$, in b) $l=l_1$ and in case c) $l=l_2$, where $l_1$ and $l_2$ are supposed to be the integers that minimize the expressions for $\tilde\omega(Y)$ and $\tilde\omega(Z)$ respectively. Im case a) we prove this by the following lines:

$$\begin{aligned}
&|U_i-U_1|\le\tilde\omega(Y) && (1\le i<l_1)\\
&|U_i-U_1|\le|U_i-U_{h-1}|+|U_{h-1}-U_1|\le\tilde\omega(Y)+B && (l_1\le i<l=h)\\
&|U_m-U_i|\le|U_m-U_{h+1}|+|U_{h+1}-U_i|\le B+\tilde\omega(Z) && (h=l\le i<l_2)\\
&|U_m-U_i|\le\tilde\omega(Z) && (l_2\le i\le m)
\end{aligned}$$

Case b) implies:

$$|U_m-U_{h-1}|\le B\quad\text{and}\quad|U_{h+1}-U_{h-1}|\le B$$

Now we can argue as above:

$$\begin{aligned}
&|U_i-U_1|\le\tilde\omega(Y) && (1\le i<l=l_1)\\
&|U_m-U_i|\le|U_m-U_{h-1}|+|U_{h-1}-U_i|\le\tilde\omega(Y)+B && (l=l_1\le i<h)\\
&|U_m-U_i|\le|U_m-U_{h-1}|+|U_{h-1}-U_{h+1}|+|U_{h+1}-U_i|\le 2B+\tilde\omega(Z) && (h\le i<l_2)\\
&|U_m-U_i|\le\tilde\omega(Z) && (l_2\le i\le m)
\end{aligned}$$

Case c) can be treated simularly. □

***Proof of Theorem 1.*** We have to show that there is an integer $n_0$ and a real number $\delta_0>0$ so that the following holds:

$$\mathbb{P}\{\omega(W_n,\delta|\rho_g)>\lambda\}\le\epsilon\qquad\text{if } n\ge n_0\text{ and }\delta\le\delta_0$$

First of all we define intervals $I_k:=[k-1,k]$ and finite approximations $I_{k,m}$ of them $(k=1,2,\dots\ ;\ m=1,2\dots)$ , where for an arbitrary intervall $I=[a,b]$ its finite approximation $I_m$ is defined as follows:

$$I_m:=\{a+\frac{i(b-a)}{2^m}:\ 0\le i\le 2^m\}$$

With $a, l \in \mathbb{N}$ $(l > a)$ and $J_l := [a, l]$ one can show for all $n, m \in \mathbb{N}$:
(4.3)

$$\tilde{\omega}(W_n, J_{l,m}) \leq \Big(\max_{a+1 \leq k \leq l} \tilde{\omega}(W_n, I_{k,m})\Big) + \sum_{k=a+2}^{l} |W_n(]k-1, k])| \quad \text{(proofed below)}$$

and

$$\mathbb{P}\{\tilde{\omega}(W_n, I_{k,m}) \geq \lambda\} \leq \frac{KP(]k-1,k])}{\lambda^4} \quad \text{(proofed below)} \tag{4.4}$$

with a universal constant $K$. If we use that

$$\begin{aligned} \mathbb{E} \sum_{k=a+2}^{l} |W_n(]k-1,k])| &= \sum_{k=a+2}^{l} \mathbb{E}\,|W_n(]k-1,k])| \\ &\leq \sum_{k=a+2}^{l} \sqrt{\mathbb{E}\, W_n(]k-1,k])^2} \\ &= \sum_{k=a+2}^{l} \sqrt{P(]k-1,k])} \end{aligned}$$

we can conclude:

$$\begin{aligned} \mathbb{P}\{\omega''(W_n, J_l) > \lambda\} &\leq \mathbb{P}\{\omega''(W_n, J_{l,m}) \geq \lambda\} \\ &\qquad \text{(because } W_n \text{ has cadlag paths)} \\ &\leq \mathbb{P}\{2\tilde{\omega}(W_n, J_{l,m}) \geq \lambda\} \\ &\qquad \text{(see inequality 4.1)} \\ &\leq \mathbb{P}\{\max_{a+1 \leq k \leq l} \tilde{\omega}(W_n, I_{k,m}) \geq \frac{\lambda}{4}\} + \mathbb{P}\{\sum_{k=a+2}^{l} |W_n(]k-1,k])| \geq \frac{\lambda}{4}\} \\ &\leq \sum_{k=a+1}^{l} \mathbb{P}\{\omega''(W_n, I_{k,m}) \geq \frac{\lambda}{2}\} + \mathbb{P}\{\sum_{k=a+2}^{l} |W_n(]k-1,k])| \geq \frac{\lambda}{2}\} \\ &\leq \frac{4^4 KP(]a,\infty])}{\lambda^4} + \frac{4}{\lambda} \sum_{k=a+2}^{\infty} \sqrt{P(]k-1,k])} \end{aligned}$$

for all $n, m \in \mathbb{N}$.

Together with the fact, that the paths of $W_n$ has only a finite amount of values this implies that we can find a real number $a_0$ so that the following holds:

$$\mathbb{P}\{\omega''(W_n, [a_0, \infty]) \geq \lambda/5\} \leq \epsilon \qquad \forall n, m \in \mathbb{N}$$

Now we use the fact that the jumps of almost all paths of $W_n$ are uniformly bounded by $1/\sqrt{n}$ and apply Remark 4. That leads to the inequality:

$$\mathbb{P}\{\omega(W_n,[a_0,\infty])\geq\lambda\}\leq\epsilon \qquad \text{for all } n\geq n_0:=1/\lambda^2.$$

In the same way one can find a real number $a_1$ for the left edge with the property:

$$\mathbb{P}\{\omega(W_n,[-\infty,a_1])\geq\lambda\}\leq\epsilon \text{ for all } n\geq n_0$$

Therefore it just remains to treat the modulus of continuity on the compact $[a_0,a_1]$. We define: $\tilde{\delta}:=\big((\epsilon\lambda^4)/5^4K\big)^{1/\beta_1}\wedge 1$ and intervals

$$I_k^1:=[a+k\tilde{\delta},a+(k+1)\tilde{\delta}] \quad I_k^2:=[a+\frac{(2k-1)\tilde{\delta}}{2},a+\frac{(2k+1)\tilde{\delta}}{2}] \quad (0\leq k\leq u:=\lceil(a_1-a_0)/\tilde{\delta}\rceil)$$

We use Proposition 1 and the Remarks 3 and 4 to conclude:

$$\mathbb{P}\{\omega(W_n,I_k^j)\geq\lambda\}\leq\epsilon P(I_k^j) \quad \text{for all } n\geq n_0 \;\; (0\leq k\leq u\;;\; j=1,2)$$

(details can be found below). Now by the properties of $g$ we can find a constant $L$ with $\rho_g(s,t)\geq L|s-t|$ for all $s,t\in[a_0,a_1]$. If we define

$$\delta_0:=(\frac{L\tilde{\delta}}{2})\wedge\rho_g(-\infty,a_0)\wedge\rho_g(a_1,+\infty)$$

we know that two points $s,t\in\mathbb{R}$ with $\rho_g(s,t)\leq\delta_0$ lie in one of the intervals $I_k^j$ $(0\leq k\leq u)$, $[-\infty,a_0]$ or $[a_1,\infty]$. Therefore we can write:

$$\begin{aligned}\mathbb{P}\{\omega(W_n,\delta_0|\rho_g)\geq\lambda\}\leq&\sum_{k=0}^{u}\mathbb{P}\{\omega(W_n,I_k^1)\geq\lambda\}+\sum_{k=0}^{u}\mathbb{P}\{\omega(W_n,I_k^2)\geq\lambda\}\\&+\mathbb{P}\{\omega(W_n,[-\infty,a_0])\geq\lambda\}+\mathbb{P}\{\omega(W_n,[a_1,\infty])\geq\lambda\}\\\leq&\,4\epsilon\\&\text{(for all } n\geq n_0)\end{aligned}$$

which completes the proof. $\square$

***Proof of inequality 4.3*** . We use induction over $l$. The start of induction is trivial. Suppose now that the result holds for each integer up to $l$. That implies:

$$\begin{aligned}
\tilde\omega(W_n, J_{l+1,m}) &\le \tilde\omega(W_n, J_{l,m}) \vee \tilde\omega(W_n, I_{l+1,m}) + |W_n(]k-1,k])| \\
&\qquad \text{(see the proof of inequality 4.2)} \\
&\le \left\{ \Big( \max_{a+1\le k\le l} \tilde\omega(W_n, I_{k,m}) \Big) + \sum_{k=a+2}^{l} |W_n(]k-1,k])| \right\} \vee \tilde\omega(W_n, I_{l+1,m}) + |W_n(]k-1,k])| \\
&\le \Big( \max_{a+1\le k\le l+1} \tilde\omega(W_n, I_{k,m}) \Big) + \sum_{k=a+2}^{l+1} |W_n(]k-1,k])|
\end{aligned}$$

$\square$

***Proof of inequality 4.4*** . We know that the paths of $W_n$ are elements of $\mathcal{F}_{cadlag}$ and that the jumps of almost all of them are bounded by $1/\sqrt{n}$. Therefore we just have to show that Proposition 1 is applicable on arbitrary finite subsets of the real line.

Thus suppose that $I$ and $J$ are intervals in $\mathcal{I}_{\mathbb{R}}$. With

$$U_i := 1\{X_{i-1} \in I\}\,\mathrm{sign}\,\epsilon_i \qquad V_i := 1\{X_{i-1} \in J\}\,\mathrm{sign}\,\epsilon_i$$

we get:

$$\begin{aligned}
\mathbb{P}\{W_n(I) \wedge W_n(J) \ge \lambda\} &\le \frac{1}{\lambda^4}\, \mathbb{E}\, W_n(I)^2 W_n(J)^2 \\
&= \frac{1}{\lambda^4 n^2}\, \mathbb{E} \sum_{1\le i,j,k,l\le n} U_i U_j V_k V_l \\
&= \frac{1}{\lambda^4 n^2} \Big( \mathbb{E} \sum_{1\le i,j<k\le n} U_i U_j V_k^2 + \mathbb{E} \sum_{1\le i,j<k\le n} V_i V_j U_k^2 \Big)
\end{aligned}$$

For fixed $k \leq n$ we can conclude:

$$\begin{aligned}
\mathbb{E} \sum_{1 \leq i,j < k} U_i U_j V_k^2 &= \mathbb{E}\Big(\sum_{i=1}^{k-1} U_i\Big)^2 V_k^2 \\
&\leq 2\,\mathbb{E}\Big(\sum_{i=1}^{k-2} U_i\Big)^2 V_k^2 + 2\,\mathbb{E}\,U_{k-1}^2 V_k^2 \\
&= 2\,\mathbb{E}\Big(\sum_{i=1}^{k-2} U_i\Big)^2\, \mathbb{E}(V_k^2 | X_{k-2}) + 2\,\mathbb{E}\,U_{k-1}^2\, \mathbb{E}(V_k^2 | X_{k-2}) \\
&\leq 2P_c(J)\,\mathbb{E}\Big(\sum_{i=1}^{k-2} U_i\Big)^2 + 2P_c(J)\,\mathbb{E}\,U_{k-1}^2 \\
&\leq 2P_c(J) \sum_{i=1}^{k-2} \mathbb{E}\,U_i^2 + 2P_c(J)P(I) \\
&\leq 2(k-1)P_c(J)P(I)
\end{aligned}$$

The same argumentation applied to the second sum leads to:

$$\mathbb{E} \sum_{1 \leq i,j < k} V_i V_j U_k^2 \leq 2(k-1)P_c(I)P(J)$$

Summing over $k$ we find

$$\begin{aligned}
\mathbb{P}\{W_n(I) \wedge W_n(J) \geq \lambda\} &\leq \frac{1}{\lambda^4}\Big(P_c(I) \vee P_c(J)\Big)P(I \cup J) \\
&\leq \frac{1}{\lambda^4}\Big(P_c(J)P(I) + P_c(I)P(J)\Big) \\
&\leq \frac{1}{\lambda^4}|I \cup J|^{\beta_1} P(I \cup J)
\end{aligned}$$

The set function $|.|^{\beta_1}P(.)$ is $M$-subadditive with $M := 2^{-\beta_1}$. One gets this constant by halving the intervalls according to Lebesgues measure. $\square$

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