Dissertation
submitted to the
Combined Faculties of the Natural Sciences and Mathematics
of the Ruperto-Carola-University of Heidelberg. Germany
for the degree of
Doctor of Natural Science

Put forward by
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born in: Erfurt

Oral examination: March 30th, 2017

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Cosmological Viability of Theories with Massive Spin-2 Fields

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“Denn alle synthetischen Grundsätze des Verstandes sind von immanentem Gebrauch; zu der Erkenntnis eines höchsten Wesens aber wird ein transzendenter Gebrauch derselben erfordert, wozu unser Verstand gar nicht ausgerüstet ist.”

Immanuel Kant · *Kritik der reinen Vernunft*

Abstract (German)

Theorien von Spin-2 Feldern nehmen eine besondere Rolle in der modernen Physik ein. Sie beschreiben nicht nur die Vermittlung von Gravitation, die einzige Theorie fundamentaler Wechselwirkung, welche keine quantenfeldtheoretische Beschreibung besitzt, es wurde weiterhin angenommen, dass sie notwendigerweise masselose Eichbosonen vorhersagen. Erst kürzlich konnte eine Theorie massiver Gravitonen konstruiert werden und wurde anschließend zu einer bimetrischen Theorie zweier interagierender Spin-2 Felder verallgemeinert. Diese Dissertation untersucht die Gültigkeit und Konsquenzen auf kosmologischen Skalen sowohl in massiver als auch bimetrischer Gravitation. Wir zeigen, dass sich alle konsistenten und von Gradienten- sowie Geistinstabilitäten freie Modelle wie das kosmologische Standardmodell, ΛCDM, verhalten. Zudem entwickeln wir eine neue Theorie einer massiven Gravitation, welche, obgleich von einem Boulware-Deser Geist geplagt, stabil im klassischen Hintergrund und auf Quantenebene ist.

Abstract (English)

Theories of spin-2 fields take on a particular role in modern physics. They do not only describe the mediation of gravity, the only theory of fundamental interactions of which no quantum field theoretical description exists, it furthermore was thought that they necessarily predict massless gauge bosons. Just recently, a consistent theory of a massive graviton was constructed and, subsequently, generalized to a bimetric theory of two interacting spin-2 fields. This thesis studies both the viability and consequences at cosmological scales in massive gravity as well as bimetric theories. We show that all consistent models that are free of gradient and ghost instabilities behave like the cosmological standard model, ΛCDM. In addition, we construct a new theory of massive gravity which is stable at both classical background and quantum level, even though it suffers from the Boulware-Deser ghost.

List of Accompanying Publications

Publication 1

Frank Könnig, Aashay Patil, and Luca Amendola. *Viable cosmological solutions in massive bimetric gravity*, JCAP 1403 (2014) 029, arXiv:1312.3208

Principal author: Frank Könnig
Personal Contribution: The comparison of the models with a SNe Ia catalog was done by me. All remaining parts were equally split among Luca Amendola and me.

Publication 2

Frank Könnig and Luca Amendola. *Instability in a minimal bimetric gravity model*, Physical Review D 90, 044030 (2014), arXiv:1402.1988

Principal author: Frank Könnig
Personal Contribution: I have played a major role in the derivation of the quasi-static limit and in the comparison of the model to observational data. The analytical analysis of the instability was first performed by Luca Amendola and independently carried out be me. Additionally, I have written a large part of the text.

Publication 3

Frank Könnig, Yashar Akrami, Luca Amendola, Mariele Motta, and Adam R. Solomon. *Stable and unstable cosmological models in bimetric massive gravity*, Physical Review D 90, 124014 (2014), arXiv:1407.4331

Principal author: Frank Könnig
Personal Contribution: All analytical results are based on calculations which were carried out by Luca Amendola and me. Furthermore, I have developed the proof to generalize the existence of gradient instabilities to all finite-branch models and their absence in all infinite-branch solutions. The main part of the text was written by me.

Publication 4

Adam R. Solomon, Jonas Enander, Yashar Akrami, Tomi S. Koivisto, Frank Könnig, and Edvard Mörtsell. *Cosmological viability of massive gravity with generalized matter coupling*, JCAP 1504 (2015) 027, arXiv:1409.8300

Principal author: Adam R. Solomon
Personal Contribution: I have contributed to the discussion of the cosmological viability

and the importance of the lapse-dependence in the presence of an additional scalar field.

Publication 5

Luca Amendola, Frank Könnig, Matteo Martinelli, Valeria Pettorino, and Miguel Zumalacárregui. *Surfing gravitational waves: can bigravity survive growing tensor modes?*, JCAP 1605 (2015) 052, arXiv:1503.02490

Principal author: Luca Amendola
Personal Contribution: Together with Luca Amendola, I have worked out the tensor perturbation equations and analyzed them analytically. Furthermore, I was involved in all discussions.

Publication 6

Frank Könnig. *Higuchi ghosts and gradient instabilities in bimetric gravity*, Physical Review D 91, 104019 (2015), arXiv:1503.07436

Principal author: Frank Könnig
Personal Contribution: This project was done by myself.

Publication 7

Yashar Akrami, S. F. Hassan, Frank Könnig, Angnis Schmidt-May, Adam R. Solomon. *Bimetric gravity is cosmologically viable*, Phys. Letters B B748 (2015), arXiv:1503.07521

Principal author: Adam R. Solomon
Personal Contribution: Adam Solomon and I have computed the eigenfrequencies and the effective Cosmological Constants independently. Moreover, I have checked all calculations myself and participated in all discussions.

Publication 8

Frank Könnig, Henrik Nersisyan, Yashar Akrami, Luca Amendola, and Miguel Zumalacárregui. *A spectre is haunting the cosmos: Quantum stability of massive gravity with ghosts*, JHEP 11 (2016) 118, arXiv:1605.08757

Principal author: Frank Könnig
Personal Contribution: Working out the idea and performing all calculations, except the computation of the decay rate which was performed by Henrik Nersisyan and checked by me, as well as writing almost the entire text has been done by me.

Acknowledgements

"Physicists are machines that convert coffee into papers." After having worked in Heidelberg for many years, I can safely say that this common saying is absolutely wrong. It takes much more than just that. Smart colleagues, great friends and a supporting family are indispensable.

I am extremely grateful to my supervisor Luca Amendola, not only for supporting me in any situation but for all the discussions, his patience, and spending so much time for answering all the questions I had; and, of course, for guiding me in the fascinating world of two metrics!

My warmest gratitude goes to Björn Malte Schäfer. Thank you for all your selfless efforts to help me and for taking your time to redraw my picture of gravity.

Not even a single line of this thesis would have been written without my other former and current collaborators: Yashar Akrami, Jonas Enander, Fawad Hassan, Nima Khosravi, Tomi Koivisto, Matteo Martinelli, Mariele Motta, Edvard Mörtsell, José Navarro, Hendrik Nersisyan, Aashay Patil, Valeria Pettorino, Angnis Schmidt-May, Adam Solomon, and Miguel Zumalacárregui. In particular, I am very grateful to Fawad for sharing his knowledge on bimetric theories and, especially, ghosts in many inspiring discussions. Additionally, thank you, Tomi, for hosting me in Stockholm and taking me into the exciting realm of gauge gravity. I would also like to thank you, Nima, for all the stimulating discussions and sharing your great ideas! Of course, I also have to thank you, Adam. Not only for proving that this amazing English grass really exists, but also working with you was so much fun and enlightening, especially when combining it with your thoughts about What CosmologY really IS. Many thanks also to Miguel, I always have enjoyed our discussions. I would also like to thank you, Hendrik. I cannot remember that there was a single boring day in the office; and I was really happy for having you in our ghost project, thanks for all your help! Furthermore, I am thankful to Jonas, for the great time we have had in Stockholm and all the speculations about what gravity truly is. Special thanks also deserve Yashar and Matteo for answering all the Big Bang Questions.

I am also grateful for all the fruitful conversations with many other colleagues around the world, from which I have greatly benefited. Here, I especially have to mention Giulia Cusin, Claudia de Rham, Guillermo Ballesteros, Arthur Hebecker, Matteo Fasiello, Mikica Kocic, Macarena Lagos, Valerio Marra, Johannes Noller, Malin Renneby, Javier Rubio, and Mikael von Strauss.

My thanks also go to my office mate Santiago Casas. Thank you for contributing to this creative atmosphere and, very important, for co-organizing our statistic seminar.

Additionally, I am grateful to Florian Führer who saved me so often, when he was the only one who has not had enough coffee. Furthermore, I would like to thank Luisa Jaime, Marvin Lüben, Alessio Spurio Mancini, and Janina Renk for a great time in the office(s).

I also greatly benefited from the support of Tilman Plehn within his graduate college.

Additionally, I need to thank Christian Angrick for keeping the coffee machine, and with this the entire building, alive.

One person must not be unmentioned: Eduard Thommes. Thank you for organizing the entire institute and your help in any situation.

Last, but certainly not least, I wish to acknowledge with deep thanks Caroline Heneka. Thank you for your encouragement, support, all the beautiful memories, and just making me happy every single day. Finally, I would like to thank my entire family, especially my sister, who has shown the beauty of maths and physics to me, as well as my parents for their constant support in all my pursuits.

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Part I

Introduction

Chapter 1

WHY TO MODIFY GRAVITY?

OR: BREAKING FROM JAIL OF THE EFFECTIVE FIELD THEORY

What is *gravity*? A question that was already raised when no elementary particle had been discovered, no neighboring planet observed, and still, even after the detection of gravitational waves, no convincing answer exists. Newton's picture of gravity as a force between massive objects is certainly the most commonly used theory of gravity today. Not only in everyday life to describe simple processes on Earth, it is even often used to compute the gravitational interactions between galaxies in huge cosmological simulations. The reason of the success is merely its simplicity; it surely does not reflect any conviction that gravitational interactions are based on Newton's law. In fact, various measurements on scales of our Solar System conflict with Newtonian gravity. One of the most prominent one is the measurement of Mercury's orbit indicating that its perihelion, i.e., its closest point to our Sun, is shifting less than a tenth of what is predicted by Newtonian gravity.

Several possibilities to rescue the classical theory were suggested. Could there be an unobserved planet that influences the motion of Mercury? This is the crossroad at which scientists stood, and are still standing today: Contradictions with theoretical predictions may indicate that there are additional, unobserved objects or could point to a failure of the theory itself. More than a century after the prediction of an additional dark planet *Vulcan* in order to reconcile Newtonian gravity with observations, the modern cosmological framework again predicts not only a yet unobserved *Dark Matter* (DM) component, but, additionally, a *Dark Energy* (DE) whose origin is not understood either. Both together seem to account for around 95 percent of the energy content of our Universe [1]. Predicting such a dark sector is certainly not illegitimate in order to approach a better understanding of the physics at the largest scales, but at the same time the question of the fundamental theory of gravity has to be not forgotten about.

Today, a more modern way to describe gravity is favored, Einstein's *General Relativity* (GR). But it was not the number of discrepancies that arise when using Newton's law to describe the physics in our Solar System that initiated Einstein's belief in the necessity to revise the picture of gravity. GR was originally constructed from the equivalence principle, i.e., the space-time measured by a freely falling observer can locally be described by a flat Minkowski metric, together with the assumption of the invariance under diffeomorphisms. This concept of an underlying symmetry as a starting point to construct a theory is a rather modern approach. The construction of the theory on the basis of General Covariance is perhaps the main reason for the success of GR. While most theories of modified gravity try to add or adjust specific properties of the standard model of gravity, e.g., a change in the number of space-time dimensions [2] or the usage of an additional field [3, 4], GR is rather based on fundamental principles. Ironically, in most theories of modified gravity the additional freedom introduces new challenges or even renders the theory nonviable. A prominent example, one in which the graviton is allowed to carry a mass, will be investigated in chapter 3. But despite these setbacks, the continuation of finding alternatives to GR remains an important task. Not only to solve open questions in modern cosmology, but also in order to better understand gravity by itself. And if all modifications are theoretically or observationally ruled out, one can even more appreciate the success of GR.

However, the minimal freedom in GR also implies its rigidity. Recent observations indicate an accelerating expansion of our Universe [5, 6]. A behavior that cannot be explained with ordinary baryonic matter and requires an energy component whose pressure is negative, a DE. This discovery has completely knocked the view of cosmic evolution on the head and was consequently honored by the award of the Nobel prize in 2011. Although GR exhibits the freedom to add a bare Cosmological Constant (CC) in order to obtain an accelerating epoch and, thus, evades any conflict with observations, it introduces various theoretical problems rendering GR with a CC less appealing (see section 2.4). Its most problematic property is the *technically unnatural* connection of ultraviolet (UV) with infrared (IR) physics. One could similarly think of a theory of thermodynamics where the precise values of the macroscopic quantities (pressure, volume, and temperature) strongly depend on how every single molecule moves. The beauty of symmetries and simplicity that originally characterized GR is now challenged by modifications to fit observational data. While the perihelion shift of Mercury can nowadays be well understood in a different theoretical framework that generalizes Newtonian physics without any requirement of an additional unobserved planet, the origin of DE is still a mysterious puzzle and could either be a consequence of a CC or an additional field, or hints towards new physics beyond the cosmological standard model.

Finding a satisfying answer to explain recent observations is surely one of the main reasons to think about modifications of the theory of gravity. But the search for new theories is not only restricted to the field of cosmology, it is rather of great importance to properly understand the gravitational sector in order to draw conclusions about a fundamental theory which combines Quantum Field Theory (QFT) and gravity. Due to the nonrenormalizability of GR, it can just be seen as an Effective Field Theory (EFT) and, thus, an IR limit of a, though not yet existing, quantum gravity. Without any doubts, the great success of GR should not be denied. Just like Newtonian physics is a great choice to measure the movement of many objects on Earth, GR with a CC explains most astrophysical and cosmological processes with sufficiently high accuracy. However, to answer the question of a fundamental field theory it is inevitable to modify the theory of gravity.

A common field theoretical approach to understand gravity is often the analysis of the full class of theories that are allowed and obey various assumptions, e.g., on the number of fields or the exhibition of an invariance under symmetries. In fact, there are a number of theorems that seem to indicate that GR is the unique theory of gravity (see section 2.1); the most famous one is the so-called *Lovelock theorem*. Although there are many, partly strong, assumptions that render the theorem less significant, it simplifies the search for possible modifications of gravity by softening exactly these assumptions. Asking for viable modifications of GR is then often translated to questions for the, under certain assumptions, most general theory that, e.g., includes an additional scalar or tensor field.

In this thesis we are especially concentrating on modifications that render the graviton, the gauge boson mediating gravitational interactions, massive. Equivalently, we are asking for modifications of gravity with an additional tensor field. While this formulation sounds like a rather less motivated approach, a very old underlying question stays behind: Can a spin-2 field be massive?

Our Standard Model of particle physics predicts particles of spin 0, $\frac{1}{2}$, and 1. While the Higgs boson, a massive spin-0 field, was just recently discovered, massive fermions with spin $\frac{1}{2}$, like electrons or quarks, are known for many decades. Furthermore, several spin-1 bosons can acquire a mass by the Higgs mechanism, too, e.g., the W- or Z-bosons. While all these particles are allowed to be massive, the only known spin-2 field, the graviton, seems to be massless, as predicted by GR. Naturally, one asks the question whether a spin-2 field is theoretically allowed to be massive. And, if the answer is positive, how does a theory with two interacting spin-2 fields look like? First attempts to answer this question had been as negative as old: In the seventies, Boulware and Deser claimed to prove that such theories are ill behaved [7]. They would necessarily contain a *ghost* field, a scalar field with the wrong sign in front of its kinetic term. A field that would not only be responsible for a spontaneous emission of other particles, but would even cause an immediate decay of the vacuum state. Fortunately, several decades after their conjecture a loophole in their argument was found [8, 9] and has paved the way towards a theory describing a massive graviton.

With all the questions that have been arising in our cosmological standard model and the knowledge of the theoretical possibility to allow the graviton to be massive, it is a justified hope that these theories manifest themselves in the observation of an accelerated expansion of our Universe.

This introduction to the papers that have been written over the period of the PhD studies continues with chapter 2, which provides a brief review of the cosmological standard model to lay the foundations for an understanding of both the necessity of modifications and its success to describe the cosmic evolution with which every other theory has to compete. Subsequently, an introduction into massive and bimetric gravity together with an elaboration of the results that have been obtained during the doctoral studies is presented in chapter 3 and 4, respectively.

Notations and Conventions

Throughout the thesis, we will set the speed of light as well as Planck's constant to unity: $c \equiv \hbar \equiv 1$. Furthermore, a dot will denote a derivative with respect to cosmic time whereas a prime indicates a derivative with respect to e-folding time. A trace of a tensor field will either be indicated by an absence of indices or, in case of matrices, by squared brackets $[\cdot]$. For the metric we will use the signature $(-,+,+,+)$. Finally, the spatial derivatives $\sum_j X_{,j} Y_{,j}$ are often shortened with $X_i Y_i$.

Chapter 2

GENERAL RELATIVITY

OR: THE STANDARD PICTURE OF GRAVITY

Even if observations would be able to rule out GR, the current standard model of gravity, it would certainly survive as a useful recipe to construct new theories of gravity. A proper understanding of its peculiarities as well as the way GR can be derived from basic principles can provide an idea how to find similarly elegant alternatives.

The following chapter aims to provide an insight into the theoretical fundament on which GR is based on and, subsequently, sketches both its success in describing the cosmic evolution as well as open problems that might indicate a failure of GR to describe gravity on all scales.

2.1 Uniqueness of General Relativity

Standard gravity with a CC is not only well accepted due to its success to describe physical phenomenons even on the largest scales we have observed, it is also its simplicity that makes the theory attractive. Its field equation, the Einstein equation,

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = M_{\mathrm{P}}^{-2}\, T_{\mu\nu}, \tag{2.1}$$

combines the Planck mass (M_{P}) suppressed energy momentum tensor $T_{\mu\nu}$ with a cosmological constant Λ and the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}, \tag{2.2}$$

which depends on the curvature of the space-time and the metric $g_{\mu\nu}$. In the geometrical picture, the curvature tensor

$$R\,(x, y)\, v = \nabla_x \nabla_y v - \nabla_y \nabla_x v - \nabla_{[x,y]} v \tag{2.3}$$

measures the change of a vector v after parallel transporting it along an infinitesimal closed curve. Since only its contractions affect the Equations of Motion (EoM), the Einstein equation is only sensitive to the change of a volume, described by the Ricci tensor $R_{\mu\nu}$, and the Ricci scalar R which is closely related to the Gaussian curvature of a hyperspace embedded in a Euclidean space, i.e., the ratio of a surface of a sphere in the curved manifold and its surface in a flat space.

Einstein himself had started to search for the field equations by assuming a Newtonian limit. While this approach seems to look arbitrary and not unique, a modern approach is the usage of *Lovelock*'s theorem which shows that, under certain assumptions, the Einstein equation is the only allowed field equation, without the need of demanding a Newtonian limit. With this level of simplicity, a minimal set of fields and parameters, it challenges all competitors.

Before stating Lovelock's theorem explicitly, we will discuss its descent in more detail. A proper analysis could then enable us to explicitly see which assumptions enter and could, or even should, potentially be lifted.

Throughout this thesis, we implicitly assume that for every function f, therewith any tensor component, living on a manifold X it holds $f \in C^\infty(X)$. Moreover, we presume any dependence to be *natural*, i.e., [1]

Definition 1. ***(Naturalness)*** *Let V be an open set on X and $x \in V$. A tensor T is naturally constructed from a metric g if the following properties hold:*

1. *Restriction: T is compatible with restriction, i.e., $T\left(g(x|_V)\right) = T\left(g(x)\right)\big|_V$.*
2. *Regularity: If g is smooth in x, then $T\left(g(x)\right)$ is smooth in x, too.*

With this, we can formulate the theorem of *Vermeil-Cartan*, who independently proved [10, 11]:

Theorem 1. ***(Vermeil-Cartan)*** *Let K be a natural tensor that*

1. *is symmetric,*
2. *is divergence-free,*
3. *has rank-2,*
4. *is of second order in the derivatives of the coefficients of the pseudo-Riemannian metric $g_{\mu\nu}$, and*
5. *is linear in these second derivatives.*

Then K is a linear combination of the Einstein tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$, i.e.,

$$K_{\mu\nu} = \alpha\, G_{\mu\nu} + \beta\, g_{\mu\nu} \equiv \alpha\, R_{\mu\nu} + \left(\beta - \frac{\alpha}{2} R\right) g_{\mu\nu}, \qquad \alpha, \beta \in \mathbb{R}. \tag{2.4}$$

[1] This condition is often equated with being *local*. However, this terminology carries an ambiguity since an increased interest in so-called *non-local theories* has been arising which still satisfy the conditions of naturalness.

Lovelock's theorem is now a specialization in four dimensions [12]:

Theorem 2. ***(Lovelock)*** *In four dimensions, every divergence-free, rank-2 tensor that is of second order in the derivatives of the metric is symmetric and linear in the second derivatives.*

Since Einstein's idea to relate a tensor originated from geometry with the rank-2 energy-momentum tensor, which is assumed be conserved, i.e.,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.5}$$

the only possible field equation that is compatible with the assumptions of naturalness and those that enter in Theorem 1 is indeed the Einstein equation (2.2).[2]

A strong restriction is the dependency on the metric tensor only. The presence of additional fields can not only enlarge the class of viable theories significantly but, at the same time, is often required to manifest specific properties of a theory. For instance, a massive graviton has to be described by a Lagrangian that contains a mass term which, as will be motivated in section 3.1, needs to be build by an additional tensor field. Therefore, the assumption of the absence of additional, even non-dynamical, fields automatically implies the assumption of a massless graviton.

Furthermore, the metric entering in the theorems above describes a (pseudo-) Riemannian manifold. This implies a space-time without torsion T, which defined by

$$T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y] \,, \tag{2.6}$$

as well as a metric compatible connection, i.e.,

$$\nabla_c \, g_{ab} = 0. \tag{2.7}$$

Since gravity is a *gauge theory*, GR is, in fact, not unique. Theories in which only the torsion tensor is the non-vanishing gauge field and, thus, the manifold is not curved and has a metric-compatible connection or, equivalently, neither curvature nor torsion but the non-metricity tensor is non-zero can be formulated such that they are indistinguishable from GR [13]. Therefore, the *Einstein-Hilbert action* leading to the Einstein equation (2.2) has to read

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x \, \sqrt{-g} \, \left(R + g_{\mu\nu} \, \Lambda + \rho^{\mu}_{ab} \, Q^{ab}{}_{\mu} + \sigma^{\alpha\beta}{}_{\gamma} \, T^{\gamma}{}_{\alpha\beta} \right). \tag{2.8}$$

Here, the Lagrange multipliers ρ^{μ}_{ab} and $\sigma^{\alpha\beta}_{\gamma}$ are introduced by hand to enforce a vanishing torsion T and non-metricity tensor Q.

Even though the analysis of the theorem of Vermeil-Cartan makes several strong assumptions and the non-uniqueness of GR apparent, Einstein's theory is nevertheless an incredibly successful framework that still withstands all observational challenges.

[2]This corollary is often attributed to Lovelock's theorem.

2.2 Ghost Instabilities

All Lovelock assumptions except one seem to be unbreakable: The absence of third- or higher-order derivatives in the EoM. It is not only a violation of the compatibility with Newtonian gravity in the weak field limit. As will be discussed in the following, a theory with higher-order EoM necessarily leads to a ghost instability.

While GR is ghost-free, many theories of modified gravity do not preserve this property. Because this problem occurs in almost all theories of massive gravity, we will dedicate this section to all ghosts, discuss their potential to render a theory unphysical and comment on possibilities to evade ghost instabilities.

2.2.1 Ostrogradsky Ghost

Generally, a *non-degenerated* Lagrangian $L\left(x, \dot{x}, \ddot{x}, \ldots, x^{(n)}\right)$, i.e.,

$$\det\left(\frac{\partial^2 L}{\partial x_i^{(n)}\, \partial x_j^{(n)}}\right) \neq 0, \tag{2.9}$$

where $x \equiv (x_1, \ldots, x_N)$, leads to EoM with derivatives of order $2n$. Thus, $4n$ initial conditions are needed to be set in order to solve the full system. In a specific example with $n = 2$, a Legendre transform of the Lagrangian results in a Hamiltonian

$$H = P_1\, \dot{Q}_1 + P_2\, \dot{Q}_2 - L \tag{2.10}$$

that requires four canonical variables, which can be chosen to be

$$Q_1 = x, \qquad Q_2 = \dot{x}, \tag{2.11}$$

$$P_1 = \frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}}, \qquad P_2 = \frac{\partial L}{\partial \ddot{x}}. \tag{2.12}$$

Since for $n = 2$ one has $L = L(x, \dot{x}, \ddot{x})$, all terms with at most second derivatives can be replaced by a combination of Q_1, Q_2, and P_2. Remarkably, all higher derivatives only appear in P_1 which is not constrained and, therefore, can take arbitrary values. Since the Hamiltonian depends linearly on P_1, it becomes unbounded. The system contains a degree of freedom that can even carry negative energy values! Such a mode is called *ghost*. In general, Ostrogradsky showed:

Theorem 3. ***(Ostrogradsky theorem)*** *Every non-degenerated Lagrangian that contains derivatives of order two or higher describes a theory that propagates a least one ghost degree of freedom.*

The consequences are indeed fatal. If a ghost degree of freedom interacts with other particles, then the ghost can excite them to arbitrary high energies. The system will become unstable. Even worse: In a quantum mechanical system with a ghost, the vacuum is able to decay into ghosts by emitting other particles. Because interactions at higher energies are entropically favored, the decay of the vacuum will happen almost instantaneously.

Even though Ostrogradsky's theorem only holds for non-degenerated Lagrangians, it was recently shown that the same conclusions can be generalized to theories that are described by degenerated Lagrangians leading to third-order derivatives in the EoM [14].

2.2.2 Ghosts in EFTs

Ghosts are often seen to render every physical theory nonviable. However, their most dangerous consequence is the vacuum decay; a catastrophe in the UV which is, in fact, a régime outside the validity of the EFT. Is this argument enough to justify the ignoring of ghosts? The EFT limitation indeed opens two possible paths to cure a ghost mode, but both should be taken with care.

If the theory itself is only an EFT, then the unknown fundamental theory might predict new operators that break either Lorentz Invariance (LI) or locality and thereby modify the decay time of the vacuum [15, 16]. The viability of a toy model in which a Lorentz breaking (LB) above the cutoff is assumed is discussed in section 3.5.

Moreover, the ghost mode itself is often thought to be harmless if its mass lies above the cutoff of the EFT. However, this conclusion dangerously exploits the limits of an EFT. In this framework, a field can be neglected if it is not excitable below the cutoff. This could be the case if the mass is very large. It is often commonly said that everything above the cutoff of an EFT can be ignored. But if a ghost is present then this statement is not correct anymore. Since a ghost does not require positive energy to become excited, any interaction with a ghost can and will occur at all energy scales, even if its mass lies above the cutoff. A scattering with a ghost is rather more likely if the mass is huge because the entropy of the final state is larger [16]. Therefore, any propagating ghost, regardless of its mass, invalidates the consistency of the EFT and destabilizes the theory [17].

Fortunately, some, but not all, theories benefit from a loophole to evade the fatal instability with ghosts above the cutoff. In ref. [18], the author made it explicit by studying a simple scalar field φ of mass m that contains a higher-derivative operator suppressed by a high mass M and an external current J:

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu\varphi\,\partial^\mu\varphi + m^2\varphi^2 + \frac{1}{M^2}\left(\Box\varphi\right)^2\right] + J\varphi. \tag{2.13}$$

The vacuum persistance amplitude for this Lagrangian

$$\Gamma = i\int \mathrm{d}^4k \frac{\left|J\right|^2}{k^2 + m^2 + \frac{k^4}{M^2}} \tag{2.14}$$

suggests that there might be another particle state with mass $M + \mathcal{O}\left(m^2/M\right)$. On the other hand, if we treat the dangerous term $M^{-2}\left(\Box\varphi\right)^2$ as just a first-order perturbation, then the effective Lagrangian (2.13) can be expressed as an expansion of

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu\varphi\,\partial^\mu\varphi + m^2\varphi^2 + m^4 M^{-2}\varphi^2\right] + \left(1 + \frac{m^2}{M^2}\right) J\varphi - \frac{1}{2M^2} J \tag{2.15}$$

and the vacuum persistance amplitude

$$\Gamma = i\int \mathrm{d}^4k \left|J\right|^2 \left[\frac{1}{k^2 + m^2} - \frac{k^4}{M^2\left(k^2 + m^2\right)^2} + O\left(M^{-4}\right)\right] \tag{2.16}$$

is indeed compatible with only one particle with mass $m + O\left(m^3/M^2\right)$. This example demonstrates that a ghost with mass above a cutoff might be ignorable if the corresponding theory can be written as a first-order expansion of a ghost-free Lagrangian. Equivalently, if higher-order derivatives introduce a ghost above the EFT cutoff then they have to be eliminated, with, e.g., the leading-order field equations [18]. If this procedure is not successful then the theory is very likely not viable.

2.2.3 Putting on the Weights

A proof of the absence of ghosts does often require a rather complicated counting of degrees of freedom in the Hamiltonian. The analysis of higher-order derivatives in the action alone is not sufficient as the EoM might still be of second-order. Already the quite simple Einstein-Hilbert Lagrangian introduces non-trivial terms with derivatives of order two. It has become a challenging task to find alternative ways of proving the viability of a Lagrangian.

One way is to find an equivalent, trivially ghost-free formulation. In fact, the reason why the Einstein equation does not contain higher-order derivatives is the choice of the connection to be Levi-Civita, the unique metric-compatible connection in a pseudo-Riemannian space-time without torsion. If one expresses the curvature tensor in terms of an a priori unknown connection,

$$R^m_{\ ikp} = \Gamma^m_{\ ip,k} - \Gamma^m_{\ ik,p} + \Gamma^a_{\ ip}\Gamma^m_{\ ak} - \Gamma^a_{\ ik}\Gamma^m_{\ ap}, \tag{2.17}$$

then the variation with respect to the connection provides a constraint that enforces Γ to be the Levi-Civita connection. In this so-called *Palatini* approach, the field equations are, indeed, the Einstein equations but the Lagrangian does not contain any dangerous derivative terms.

Despite the quite simple analysis that avoids the need of a Hamiltonian analysis this approach is rather uncommon since a search for an equivalent theory is often not successful. A promising alternative is the analysis of intrinsic properties of the tensors present in the Lagrangian that are related to the number of propagating degrees of freedom. Specifically, the *weight* w of a homogeneous tensor field K, defined through the dependence on the metric g via

$$K\left(\lambda^2 g\right) = \lambda^w K(g) \quad \forall\, g\, \forall \lambda \in \mathbb{R}^+, \tag{2.18}$$

seems to play such a role. Recently, an alternative to Lovelock's theorem was presented in ref. [19] that has all assumptions but one, the absence of higher-order derivatives, and adds an additional requirement for the weight:

Theorem 4. ***(Navarro-Sancho)*** *In four dimensions, every rank-2 tensor K with weight $w > -2$ that naturally depends on a pseudo-Riemannian metric g and is both symmetric and divergence-free can be written as a linear combination of the Einstein tensor and the metric itself.*

Is the requirement of having the highest weight equivalent to a minimization of the number of degrees of freedom and therefore the absence of ghosts? This question cannot be safely answered yet and formulating its proof is still an ongoing work, but preliminary results indeed indicate that even the ghost-free action describing a massive graviton naturally arises which support the conjecture of a relation between the weight of a Lagrangian and its viability [20].

2.3 Cosmological Solutions

Especially the cosmological picture has undergone a revolution. Although it has changed quite recently, its phenomenological consequences, inter alia a Big Bang singularity, an inflationary epoch, and the composition of the total energy content which results in a late-time acceleration, are widely accepted and serve as important consistency tests for a comparison with alternative models.

To analyze the evolution of our Universe at large scales, the background can very well be approximated by a flat, homogeneous, and isotropic background described by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N(t)^2\,\mathrm{d}t^2 + a(t)^2\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{2.19}$$

Due to GR's invariance under diffeomorphisms, a gauge can be chosen to set the lapse function N to unity such that the Universe is solely describable by the scale factor $a(t)$ that measures the spatial size of the Universe. If it is filled with just a single homogeneously distributed fluid with density ρ and pressure p, then the components of the Energy-Momentum (EM) tensor reduce to

$$T^\mu{}_\nu = \mathrm{diag}\,(-\rho, p, p, p) \tag{2.20}$$

and the only non-redundant components in the EoM (2.2) lead to the *Friedmann equations*, which read

$$H^2 \equiv \left(\frac{\dot a}{a}\right) = M_\mathrm{P}^{-2}\,\rho + \frac{\Lambda}{3}, \tag{2.21}$$

$$\dot H + H^2 = \frac{\ddot a}{a} = -\frac{1}{2}M_\mathrm{P}^{-2}\,(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.22}$$

The presence of a constant term $\Lambda/3$ is, in fact, a possible explanation of DE that drives an acceleration. Recent measurements have shown that it does, in fact, dominate over the gravitational attraction due to ordinary matter today [5, 6]. Observations of distant Supernovae of Type Ia (SNe Ia) appear to be much fainter than expected in a universe without DE and require a dynamical space-time that acceleratetly expands, while the photons emitted from SNe Ia have been traveling towards us.

The first hurdle of describing the recent acceleration has quite simply been taken by the consideration of just an additional constant term in the action. However, our Universe is only be describable as a homogeneous and isotropic fluid on its largest scales. In fact, all structure in the Universe, including galaxy clusters and even our Earth, perturb the FLRW metric (2.19). The distribution of matter on scales larger than roughly some Mpc (1 Mpc $\simeq 3\times10^6$ light years) today can be described by linear scalar perturbations around an FLRW background $\bar{g}$:

$$g = \bar{g} + \delta g, \tag{2.23}$$

where the corresponding line element for δg is built out of the four scalar potentials $\Psi, \Phi, B,$ and E and reads

$$\mathrm{d}s^2_{\delta g} = a^2 \begin{pmatrix} -2\,\Psi & E_{,i} \\ E_{,i} & 2\,\Phi\,\delta_{ij} + \left(\partial_i\,\partial_j - \frac{1}{3}\delta_{ij}\,\nabla^2\right)B \end{pmatrix}. \tag{2.24}$$

Here, and from now on, the components belong to a frame in which the time is measured by a *conformal time* $\mathrm{d}\eta = a\mathrm{d}t$ and a dot denotes the derivative with respect to it.

Fortunately, we are again able to benefit from the gauge invariance of GR and are allowed to fix two potentials suitably. In the following, the *Newtonian gauge* $E = B = 0$ will be chosen. The potentials are generated by perturbations of the EM tensor that are, if we assume dark pressureless matter only, induced by the density contrast $\delta \equiv \delta\rho/\rho$ and the peculiar velocity divergence $\vartheta \equiv v^i_{;i}$, and reads

$$\delta T^0_0 = -\delta\rho, \quad \text{and} \quad \delta T^0_i = -\delta T^i_0 = \rho\, v^i. \tag{2.25}$$

The linear perturbation equations correspond to the (0,0)-,(0,i)-, (i,i)-, and (i,j)-components of the Einstein equations and become in Fourier space

$$3\mathcal{H}\left(\mathcal{H}\,\Psi - \dot{\Phi}\right) - k^2\,\Phi = -\frac{1}{2}M_\mathrm{P}^{-2}\,a^2\,\delta\rho, \tag{2.26}$$

$$-k^2\left(\dot{\Phi} - \mathcal{H}\,\Psi\right) = \frac{1}{2}M_\mathrm{P}^{-2}\,a^2\,\rho\,\vartheta, \tag{2.27}$$

$$\ddot{\Phi} + 2\mathcal{H}\,\dot{\Phi} - \mathcal{H}\,\dot{\Psi} - \left(\mathcal{H}^2 + 2\,\dot{\mathcal{H}}\right)\Psi = 0, \tag{2.28}$$

$$\Psi + \Phi = 0, \tag{2.29}$$

where k denotes the wave number and $\mathcal{H} \equiv H/a$ the conformal Hubble function.

Since GR describes a massless spin-2 field that only carries two helicity-2 degrees of freedom, the only propagating scalar degree of freedom comes from the matter perturbation $\delta\rho$. In fact, eq. (2.29) indicates that a change in the EM tensor induces just one effective potential. Moreover, the velocity divergence ϑ is only auxiliary and does not correspond to a dynamical field. Therefore, the set of scalar perturbation equations can be reduced to a single differential equation describing one propagating scalar degree of freedom. On scales that are much smaller then the Hubble radius, i.e., on *sub-horizon scales*, $k/\mathcal{H} \gg 1$, one obtains

$$\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\delta = 0, \tag{2.30}$$

or, in e-folding time parametrization $N \equiv \log a$ and $X' \equiv \mathrm{d}X/\mathrm{d}N$,

$$\delta'' + \left(\mathcal{H}^{-1}\mathcal{H}' + 1\right)\delta' - \frac{3}{2}\delta = 0. \tag{2.31}$$

As previously mentioned, the inclusion of a positive CC in the Friedmann equations (2.21, 2.22) is necessary in order to be compatible with observational data. However, this modification will also propagate into a change of the evolution of the density contrast. Surprisingly, this was measured by analyzing a catalog of galaxy distributions and found to be compatible with a DE due to a CC [21].

2.4 The Achilles' heel(s) of the Cosmological Standard Model

The standard model of cosmology has not only been found to be compatible with various observations but is at the same time a rather minimal and well motivated theory. However, its framework is built on GR and the limited freedom in Einstein gravity requires additional ingredients. Since many of them come along with new problems, they are often seen as signatures of modified gravity theories.

2.4.1 Dark Matter

The main part of the energy content of our Universe today, DE, is responsible for its acceleration and is, at least in the standard picture, constant in time. While the universe expands, the density of matter and radiation gets diluted. Hence, when going back in time, the CC becomes less dominant. But even all observable matter in our Universe could not counteract the CC enough to produce the distribution of galaxies that we can observe today.

A second, yet unknown, matter component has indirectly been observed in the seventies by Vera Rubin [22] who found that most stars in almost all galaxies rotate around their center with roughly the same speed. A surprising discovery since most visible matter is expected to be localized near the galaxy's core and stars in the outer region should move with

$$v \simeq \sqrt{\frac{G\,M_{\mathrm{core}}}{r}}. \tag{2.32}$$

Instead, the velocities seem to be constant. If the theory of gravity is assumed to work properly on these scales, then this behavior can only be explained if much more gravitating matter is present outside the center of the galaxy, a huge amount of unobservable

DM. Later, this conclusion was supported by independent indirect observations, like the deflection of light due to the gravitational potential of DM.

Observations on cosmological scales require an amount of DM that is even larger than the amount of all baryonic matter. Its origin is still completely unknown and the list of candidates is growing constantly. The only knowledge about this type of matter is its weak, or even absent, interaction with baryonic matter and photons, as well as is presumably non-relativistic motion. The cosmological standard model including a CC and cold DM (CDM) is therefore often referred to ΛCDM.

Instead of predicting new particles, the indirect discovery of DM could also be a signature of a modification of gravity. Several candidates already exist, e.g., mimetic gravity [23, 24], gravitons with high-masses [25], and doubly-coupled bimetric gravity [26]. And as long as no DM particle has been directly detected, the list of alternatives will keep growing.

2.4.2 Inflation

When the Friedmann equations (2.21,2.22) were discussed, the geometry of the Universe was assumed to be flat. This is indeed in great agreement with current observations of the Cosmic Microwave Background (CMB) [1], albeit surprising because due to the different scaling of curvature and radiation with the scale factor the Universe should have been extremely flat at very early times. The necessity of a fine-tuned curvature contribution is known as *flatness problem*.

The analysis of the CMB has further shown that the measured temperature is almost the same in every direction. This would not be a surprising observation if all parts in the Universe are causally connected. But the expansion of the space-time influences the causal connection, which is limited due to the speed of light, between two points in the sky and the entire Universe should not necessarily be thermalized. In fact, one would expect that just a region with an angular size of order 1°, which roughly corresponds to the area on the sky covered by the moon, has been in causal contact; a *horizon problem*!

Both problems can be tackled with one extension of ΛCDM: an additional inflationary epoch at very early times. One (or even more) additional scalar field with a suitable potential could drive an acceleration period right after the Big Bang and this would quickly flatten the Universe and freeze tiny quantum fluctuations, the seeds of all structure in our present Universe. Many models were suggested, but all require the usage of at least one additional degree of freedom (see, e.g., ref. [27] for a comprehensive list of models).

Despite the success of inflationary models, it should not be left unsaid that they are not free from criticism. Especially the probability with which an inflationary epoch could produce suitable initial values and whether the quantum fluctuations are stable are still controversially discussed. As a result, different scenarios were suggested. One possibility, a bouncing model in which the universe collapses and undergoes a bounce before it expands again, even arises automatically in theories of modified gravity. In ref. [28] we could show that solutions of bimetric gravity, a generalization of a massive gravity that is discussed in more detail in chapter 4, exist that predict such a bouncing behavior.

2.4.3 The Cosmological Constant Problem

Many, perhaps all but one, challenges in the standard picture can be solved elegantly, often by utilizing additional fields. The most prominent exception is the unsatisfactory interpretation of the CC, the *Cosmological Constant problem*.

At first glance, a non-vanishing CC is an auspicious step towards an understanding of a theory of quantum gravity. In QFT, the vacuum state is expected to carry a vacuum energy

$$\langle 0 \,|\, T_{\mu\nu} \,|\, 0 \rangle = -\rho_{\mathrm{vac}}\, g_{\mu\nu}, \tag{2.33}$$

which contributes to a CC in the field equations. Therefore, observations are sensible to the sum of the vacuum contribution and the bare value from the Einstein-Hilbert Lagrangian,

$$\Lambda_{\mathrm{obs}} = \Lambda_{\mathrm{bare}} + M_{\mathrm{P}}^{-2}\, \rho_{\mathrm{vac}}. \tag{2.34}$$

To approximate the value of the vacuum energy one can consider a canonical scalar field with mass m and integrate over all modes

$$\rho_{\mathrm{vac}} \propto \int \mathrm{d}^3 k\, \sqrt{k^2 + m^2}. \tag{2.35}$$

Regularizing this integral leads to [29]

$$\rho_{\mathrm{vac}} \simeq \frac{m^4}{64\pi^2} \log\left(\frac{m^2}{\mu^2}\right), \tag{2.36}$$

where μ denotes the renormalization scale. If the mass of the scalar field corresponds to the mass of the heaviest particle in the Standard Model of particle physics, then the vacuum energy value will become around 55 orders of magnitudes larger than the observed one [29]. Therefore, the bare CC then has to be extremely fine-tuned! This already seems to be unappealing but acceptable. However, the quartic dependence on the field's mass makes the CC *technically unnatural*: Since the dominant contribution comes from the field with the highest mass, one should fine-tune the CC again whenever a new particle with a higher mass will be detected. In the extreme case, the masses could become of order the Planck mass and one should expect $\rho_{\mathrm{vac}} = \mathcal{O}\left(10^{120}\rho_{\mathrm{obs}}\right)$. If the CC would be technically natural, then a change in the cut-off would not cause such a huge correction by many orders of magnitudes.

Chapter 3

MASSIVE GRAVITY

OR: THE UNIQUE THEORY OF A MASSIVE SPIN-2 FIELD

Rarely before have two different camps in physics worked together to find a theory of a massive graviton. Particle physicist have searched for a proper understanding of the theory describing spin-2 fields, while cosmologists realized its potential and saw an elegant theory solving the puzzles of cosmology. The search for a massive gravity received renewed interest when SNe Ia have been discovered and appeared to be fainter than expected in a universe that solely consists of ordinary matter [5, 6]. A gauge boson with mass m was expected to cause an additional Yukawa suppression of the gravitational potential $V(r)$,

$$V(r) \propto \frac{1}{r} e^{-mr^2}, \tag{3.1}$$

which could explain the weakening of gravity on large scales. Even better, such a suppression would screen a large CC that naturally arises due to a vacuum energy. Unfortunately, all viable theories of massive gravity turned out to not be able to solve the CC problem. Nonetheless, the discovery of a ghost-free massive gravity has initiated an exciting search for alternative cosmological models to finally understand the origin of DE.

3.1 Linear Theory of a Massive Spin-2 Field

Finding a viable non-linear theory of a massive graviton turned out to become quite challenging. Much simpler is the concentration on a linear version, though, which has already been found by Markus Fierz and Wolfgang Pauli in 1939 [30]. Considering small fluctuations around a Minkowski background,

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}, \tag{3.2}$$

limits the numer of possible mass terms drastically. As the action requires the construction of a scalar, the only non-trivial potential term without derivatives is a linear combination of all possible contractions of $h_{\mu\nu}$. Thus, the linear *Fierz-Pauli* (FP) theory of a massive spin-2 field simply reads

$$S_{\mathrm{FP}} = S_{\mathrm{GR}}^{(\mathrm{lin})} - \frac{1}{2} m^2 \int \mathrm{d}^4 x \left(h_{\mu\nu} h^{\mu\nu} - (1-a)\, h^2 \right), \qquad a \in \mathbb{R}. \tag{3.3}$$

Let us first assume the special choice $a = 0$, the *Fierz-Pauli tuning*. Although the EoM for $h_{\mu\nu}$ are lengthy and not much illuminating, an equivalent reformulation into three equations can be found which especially simplifies the counting of propagating degrees of freedom enormously [31]:

$$\left(\Box - m^2 \right) h_{\mu\nu} = 0, \tag{3.4}$$

$$\partial^\mu h_{\mu\nu} = 0, \tag{3.5}$$

$$h = 0. \tag{3.6}$$

In d dimensions, the first equation describes the propagation of $\frac{1}{2}\left(d^2 + d\right)$ degrees of freedom, while the second and third one add $d + 1$ constraints. In a four dimensional space-time, this leaves in total five degrees of freedom. A result which is indeed expected for a massive spin-2 field, as it can carry at most one helicity-0 (scalar), two helicity-1 (vector), and two helicity-2 (tensor) modes.

Interestingly, it turns out that eq. (3.6) is not present for the choice $a \neq 0$. In this case, the loss of one constraint implies an additional degree of freedom, which turns out to be a ghost. Hence, the requirement of stability uniquely fixes the linear theory for a spin-2 field with mass m.

Although the FP theory is only a linear version of a massive gravity, it should already be sufficient to analyze phenomenons in the weak field limit, e.g., in our Solar System. In fact, one of the first major successes of GR was the prediction and observational confirmation of the correct light deflection around the Sun using the linear theory. For this, one can use the Newtonian limit for the metric,

$$g_{\mu\nu} = \mathrm{diag}\left[-(1+2\psi)\,, 1+2\phi, 1+2\phi, 1+2\phi \right], \tag{3.7}$$

to obtain a solution of the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta = 0. \tag{3.8}$$

Assuming a photon ($\mathrm{d}s^2 = 0$) with energy E that moves along the x^3 direction and is deflected in x^1 direction due to an object with mass M, then the momentum will gain a contribution

$$p^1 = -\int \mathrm{d}x^3 \left(1 - \frac{\phi}{\psi} \right) M\, E\, \frac{x^1}{b^3}, \tag{3.9}$$

where b denotes the photon's impact factor. An integration then leads to the deflection angle

$$\alpha = 2\left(1 - \frac{\psi}{\phi} \right) \frac{GM}{b}. \tag{3.10}$$

Since the perturbation equations for GR (2.29) imply $\phi = -\psi$, the bending of light in Einstein's theory is found to be twice as large as in Newtonian gravity.

For a massive FP theory the metric potentials become [31]

$$h_{00} = \frac{2M}{3M_{\rm P}} \frac{1}{4\pi r} e^{-mr}, \tag{3.11}$$

$$h_{ij} = \frac{M}{3M_{\rm P}} \frac{1}{4\pi r} e^{-mr} \delta_{ij}, \tag{3.12}$$

and indicate an $O(1)$ difference in their ratio compared to the massless GR solution. An extremely surprising result – shouldn't the ratio reduce to the standard value in limit of a vanishing graviton mass m? Because of the highly accurate measurements of the light bending in our Solar System, such a large deviation would immediately rule out a theory of massive gravity.

That a measurement could distinguish between an incredibly tiny graviton mass and an exactly vanishing mass indicates the presence of a discontinuity in the theory. In massive gravity, this is often referred as van Dam-Veltman-Zakharov (vDVZ) discontinuity [32, 33]. The graviton in the FP theory propagates five modes instead of two in the massless case. The additional scalar helicity-0 degree of freedom gets strongly coupled and therefore does not vanish in the smooth limit $m \to 0$. However, in a strong coupling régime the linear theory looses its predictability and higher orders necessarily play a significant role. Specifically, Vainshtein has found the breakdown of the linear FP massive gravity for regions inside a sphere with radius [34]

$$r_V \equiv \left(\frac{M}{m^4 M_{\rm P}^2} \right)^{1/5} \tag{3.13}$$

around an object of mass M. The knowledge of a non-linear theory is therefore not only demanded by curiosity or consistency, but required in order to obtain valid predictions with which observations can be compared.

3.2 Ghost-free Non-Linear Massive Gravity

Constructing a ghost-free non-linear theory has turned out to be a complicated challenge. For decades it has been thought that every non-linear extension will reintroduce a pathological sixth degree of freedom, the *Boulware-Deser* (BD) ghost [7, 8, 9].

A promising ansatz to derive a consistent non-linear theory is to extend the unique ghost-free linear massive gravity. For this, one can introduce a covariant tensor H through rewriting the metric as [8]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = H_{\mu\nu} + \eta_{ab}\, \partial_\mu \phi^a\, \partial_\nu \phi^b, \tag{3.14}$$

where ϕ are *Stückelberg fields* that can be introduced to restore diffeomorphism invariance, which is originally broken due to the presence of a mass term. Additionally, these scalar fields can be used to decompose all graviton degrees of freedom, for instance the helicity-0 mode π through

$$\varphi^a = x^a - \eta^{a\mu}\,\partial_\mu \pi, \tag{3.15}$$

which yields an explicit expression for $H_{\mu\nu}$ in terms of π [35],

$$H_{\mu\nu} = h_{\mu\nu} + 2\Pi_{\mu\nu} - \eta^{\alpha\beta}\,\Pi_{\mu\alpha}\Pi_{\beta\nu}, \quad \text{with } \Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi. \tag{3.16}$$

Note that the combination $[\Pi]^2 - [\Pi^2]$ is just a total derivative which ensures the ghost-freedom at this stage. A non-linear generalization to the potential is given by

$$\mathcal{L}_{\text{pot}} \propto \mathcal{U}(g,H) \equiv -4\left(\left(g^{\mu\nu}\,\mathcal{K}_{\mu\nu}\right)^2 - g^{\alpha\beta}\,g^{\mu\nu}\,\mathcal{K}_{\alpha_\mu}\,\mathcal{K}_{\beta\nu}\right), \tag{3.17}$$

where $\mathcal{K}$ is defined such that it reduces to $\Pi_{\mu\nu}$ in the limit $h_{\mu\nu} \to 0$,

$$\mathcal{K}_{\mu\nu} \equiv g_{\mu\alpha}\left(\delta^\alpha_\nu - \sqrt{\delta^\alpha_\nu - H^\alpha_\nu}\right). \tag{3.18}$$

An expansion of the potential (3.17) leads to an infinite series in H and a subsequent resummation in which only total derivatives are added provides a recursivly defined mass term [35]

$$\mathcal{L}_{\text{pot}} \propto \sum_{n\geq 2} a_n\,\mathcal{L}^{(n)}(\mathcal{K}), \tag{3.19}$$

with

$$\mathcal{L}^{(n)}(\mathcal{K}) = -\sum_{m=1}^{n} (-1)^m\,\frac{(n-1)!}{(n-m)!}\,[K^m]\,\mathcal{L}^{(n-m)}(\mathcal{K}). \tag{3.20}$$

This de Rham-Gabadadze-Tolley (dRGT) mass term has been shown to be free of ghosts up to quartic order in the decoupling limit [35, 36], which corresponds to

$$M_{\text{P}} \to \infty \quad \text{and} \quad m \to 0 \quad \text{while} \quad \Lambda_3 \equiv \left(m^2 M_{\text{P}}\right)^{1/3} = \text{fixed}, \tag{3.21}$$

and effectively decouples the helicity-0, the usually most dangerous degree of freedom, from all other modes. Almost at the same time, Hassan and Rosen discovered that the infinite series in the mass term indeed terminates and have presented a proof of the equivalence of this mass term with [37]

$$\mathcal{L} \propto \sum_{n=0}^{4} \beta_n\,e_n\left(\sqrt{g^{-1}f}\right), \tag{3.22}$$

where $f_{\mu\nu}$ is an arbitrary fixed tensor field, the parameters β_n denote dimension-free real coefficients and $e_n(X)$ represents the elementary symmetric polynomials of the eigenvalues λ_i of a matrix X, i.e.,

$$e_0(X) = 1, \tag{3.23}$$

$$e_1(X) = \sum_{i=1}^{4} \lambda_i, \tag{3.24}$$

$$e_2(X) = \sum_{1\le i<j}^{4} \lambda_i \lambda_j, \tag{3.25}$$

$$e_3(X) = \sum_{1\le i<j<k}^{4} \lambda_i \lambda_j \lambda_k, \tag{3.26}$$

$$e_4(X) = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = \det X. \tag{3.27}$$

A full, but tedious, Hamiltonian analysis showed that the linear combination of these terms indeed provides an additional constraint, that removes one degree of freedom, and with this the absence of the BD ghost in the full non-linear theory [38, 39, 40, 41, 42]. The complete non-linear and ghost-free massive gravity theory with a minimally coupled matter Lagrangian $\mathcal{L}_m$ is dubbed *dRGT massive gravity* and reads

$$S_{\mathrm{dRGT}} = -M_\mathrm{P}^2 \int \mathrm{d}^4x\, \sqrt{-g} \left(R - 2m^2 \sum_{n=0}^{4} \beta_n\, e_n\left(\sqrt{g^{-1}f}\right)\right) + \int \mathrm{d}^4x\, \sqrt{-g}\, \mathcal{L}_m. \tag{3.28}$$

Note that even though an explicit CC has been omitted, the parameter β_0 indeed takes over its role as $e_0 = 1$.

From now one, we will assume the *reference metric* to be Minkowskian, $f_{\mu\nu} = \eta_{\mu\nu}$, which will not only significantly simplify the analysis, but, in addition, this restriction also ensures the absence of gradient or even ghost instabilities [43].[1] Furthermore, the square root $\sqrt{g^{-1}f}$ is not uniquely defined for matrices. In order to ensure differentiability, we will define it as the positive solution, i.e., the one that obeys $\sqrt{X}\sqrt{X} = X$. Finally, a variation with respect to $g_{\mu\nu}$ and the assumption of diagonality of g yields the EoM [37],

$$G^{\mu\nu} + m^2 \sum_{n=0}^{3} (-1)^n \beta_n\, g^{\mu\alpha}\, Y^{\nu}_{(n)\alpha}\left(\sqrt{g^{-1}f}\right) = M_\mathrm{P}^{-2}\, T^{\mu\nu}. \tag{3.29}$$

Here, the matrices $Y^{\nu}_{(n)\alpha}(X)$ follow from the variation of $e_n(X)$ and read

$$Y_{(0)}(X) = \mathbb{1}, \tag{3.30}$$

$$Y_{(1)}(X) = X - \mathbb{1}\,[X]\,, \tag{3.31}$$

$$Y_{(2)}(X) = X^2 - X\,[X] + \frac{1}{2}\mathbb{1}\left([X]^2 - \left[X^2\right]\right), \tag{3.32}$$

$$Y_{(3)}(X) = X^3 - X^2\,[X] + \frac{1}{2}X\left([X]^2 - \left[X^2\right]\right) - \frac{1}{6}\mathbb{1}\left([X]^3 - 3\,[X]\left[X^2\right] + 2\left[X^3\right]\right). \tag{3.33}$$

3.3 Cosmology with Massive Gravitons

Contrary to the original expectation that an additional massive term in the action will weaken gravitational interactions on large scales, the modified Einstein eq. (3.29) with the freedom in the parameters β_n is able to influence gravity at all scales and might even

[1] The claim that dRGT massive gravity is ghost-free is only restricted to the absence of an additional sixth ghost degree of freedom. All other five helicity modes crucially depend on the background and might also carry a wrong sign in their kinetic term (see also section 4.5 for a more detailed discussion).

strengthen gravity. The cosmological phenomenology can, similar to the previously discussed Einstein case, be studied by assuming an FLRW background, with one exception: The loss of the gauge freedom due to a breaking of diffeomorphism invariance forbids us to choose the lapse arbitrarily. In GR, the combination of both Friedmann equations (2.21, 2.22) is redundant with the energy-meomentum conservation, ${T^{\mu\nu}}_{;\mu} = 0$. It is different in massive gravity where a conserved $T^{\mu\nu}$ just enters as an, though very well motivated, assumption. In addition to the Friedmann equations, the combination of this assumption together with a reformulation of the *Bianchi identities*,

$$\nabla_\mu G^{\mu\nu} = 0, \tag{3.34}$$

enforces the term that modifies the EoM to be covariantly conserved:

$$m^2 \nabla_\mu \left(\sum_{n=0}^{3} (-1)^n \beta_n \, g^{\mu\alpha} \, Y^{\nu}_{(n)\alpha} \right) = 0. \tag{3.35}$$

In an FLRW universe, this *Bianchi constraint* yields [43]

$$m^2 a^2 \left(\beta_1 + 2\beta_2 \, a^{-1} + \beta_3 \, a^{-2} \right) = 0. \tag{3.36}$$

Remarkably, this equation does not constrain the lapse function, as one would expect, but fixes the value of the scale factor. It is not only the inability of dRGT to describe the accelerating epoch of our Universe, it is, in fact, not even compatible with a dynamical universe.

If the Bianchi constraint were slightly different, it would indeed serve as a constraint for the lapse and allow for a dynamical universe. The specific form of the dRGT potential, which ensures the ghost freedom, manifests itself in the lack of dynamics. If the potential stays untouched, then additional dynamics should be added by hand to, for instance, the matter sector. Tiny, even unobservable, anisotropies, that are larger than the horizon, could allow for dynamical FLRW solutions [43] (see also refs. [44, 45] for a general FLRW metric with inhomogeneous Stückelberg fields). But such a modification would surely render a massive gravity less appealing.

3.4 Generalized Matter Couplings

Without explicitly utilizing a new freedom in the matter sector by modifying the EM tensor, the dRGT Lagrangian still makes a strong assumption on how the matter content is coupled. In GR, the *minimal coupling* of the matter Lagrangian, i.e., to just the volume element $\sqrt{-g}$, is a direct consequence of General Covariance. If this has been broken by a mass term, then there is no fundamental reason why the coupling should stay minimally. Especially in the presence of two tensor fields a generalization of the coupling has to be taken into account [46, 47, 48]. But such a modification is expected to reintroduce the BD ghost and was therefore thought to destabilize the theory [38, 49, 50].

Recently, a coupling to a certain combination of both metrics was suggested [49], shown to be free of ghosts below the strong coupling scale Λ_3 [49, 51, 17], and can therefore evade the ghost problem. In this scenario, all matter is coupled to the effective metric

$$g^{\text{eff}}_{\mu\nu} \equiv \alpha^2 g_{\mu\nu} + 2\alpha\beta\, g_{\mu a} X^a_\nu + \beta^2\, \eta_{\mu\nu}, \qquad \alpha,\beta \in \mathbb{R}, \tag{3.37}$$

where $X^\mu_\nu \equiv \left(\sqrt{g^{-1}\eta}\right)^\mu_\nu$. The modification will then become perceivable on the right hand side of eq. (3.29),

$$G^{\mu\nu} + m^2 \sum_{n=0}^{3} (-1)^n \beta_n\, g^{\mu a}\, Y^\nu_{(n)a}\left(\sqrt{g^{-1}f}\right) = M_\mathrm{P}^{-2}\, \alpha \det(\alpha + \beta X)\left(\alpha\, T^{\mu\nu} + \beta X^\mu_a\, T^{\nu a}\right). \tag{3.38}$$

Its consequences for the cosmological evolution have been analyzed in publication 4 and will be summarized in the following.

If, again, an FLRW ansatz for g is chosen, then the line element of the effective metric can be written has

$$g^{\text{eff}}_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -N^2_{\text{eff}}\, \mathrm{d}t^2 + a^2_{\text{eff}}\, \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \tag{3.39}$$

and only depends on an effective lapse function N_{eff} and scale factor a_{eff}:

$$N_{\text{eff}} \equiv \alpha\, N + \beta, \qquad a_{\text{eff}} \equiv \alpha\, a + \beta. \tag{3.40}$$

Let us assume a scenario in which the matter fluid is still conserved with respect to the effective metric, i.e., $\nabla^{\text{eff}}_\mu T^{\mu\nu} = 0$, but, contrary to what has been assumed in section 2.3, is allowed to have a non-vanishing pressure

$$p = \frac{1}{3} - g^{\text{eff}}_{00}\, T^{00}. \tag{3.41}$$

Per construction, the new dynamics in the matter sector propagate into the Bianchi constraint,

$$m^2\, M_\mathrm{P}^2\, a^2 \left(\beta_1 + 2\beta_2\, a^{-1} + \beta_3\, a^{-2}\right) = \alpha\,\beta\, a^2_{\text{eff}}\, p\, \dot{a}, \tag{3.42}$$

If the pressure is parameterized by an *equation-of-state* (EoS) parameter w with $p = w\rho$, then the energy conservation,

$$\frac{\mathrm{d}\log\rho}{\mathrm{d}\log a_{\text{eff}}} + 3\left(1 + w(a_{\text{eff}})\right) = 0, \tag{3.43}$$

can be integrated to obtain an expression for $\rho(a_{\text{eff}})$ which, in fact, conflicts with eq. (3.42).

However, the situation sounds worse than it actually is. In order to use the standard methods to describe cosmology, the pressure requires a lapse dependence, which is automatically the case for any fundamental field with a kinetic term. Consider for instance an EM tensor from a canonical scalar field χ with potential $V(\chi)$,

$$T^{\mu\nu} = \nabla^\mu_{\text{eff}}\chi\nabla^\nu_{\text{eff}}\chi - \left(\frac{1}{2}\nabla_{\text{eff}\,a}\,\chi\,\nabla^a_{\text{eff}}\chi + V(\chi)\right) g^{\mu\nu}_{\text{eff}}. \tag{3.44}$$

Although the scalar field is now explicitly allowed to depend on the lapse, the Klein-Gordon equation [52]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\chi}^2}{2N_{\mathrm{eff}}^2}+V(\chi)\right)+3\frac{\dot{a}_{\mathrm{eff}}}{a_{\mathrm{eff}}}\frac{\dot{\chi}^2}{N_{\mathrm{eff}}^2}=0 \tag{3.45}$$

clearly shows that if the potential is independent of N_{eff} then neither can $\dot{\chi}^2/2N_{\mathrm{eff}}^2$. Since the pressure is just

$$p=\frac{\dot{\chi}^2}{2N_{\mathrm{eff}}^2}-V(\chi), \tag{3.46}$$

the no-go theorem of dRGT massive gravity for dynamical FLRW solutions can be extended to doubly-coupled theories, in which either the pressure of an effective fluid or, in case there is just one scalar field present, the potential does not depend on the lapse.

The situation changes in the presence of a perfect fluid with an additional canonical scalar field. The new freedom evades the no-go theorem but it will lead to a highly nonstandard cosmological evolution: Both χ^2 and H_{eff}^2 will become negative for positive-definite potentials, or, otherwise, H_{eff}^2 will blow up as $a_{\mathrm{eff}} \to \infty$, indicating a late-time background instability [52].

A viable theory of doubly-coupled massive gravity seems to be hardly constructable, even in the presence of additional fields, and, if possible, it requires the usage of techniques beyond those, which are used in standard cosmology.

3.5 Haunted Massive Gravity

Since the time when the linear FP theory was originally formulated and even before, ghosts have always been seen as an unacceptable pathological behavior. They seem to destabilize the vacuum as well as produce potentially dangerous classical instabilities. The latter is still acceptable as long as no contradictions with observations appear. A vacuum instability, however, is often thought to be a disastrous behavior. The only hope to tame a ghost seems to be a modification at UV scales.

3.5.1 Lorentz Breaking UV Operators

The motivation to even think about a UV modification are twofold. Theories of massive gravity do not just add a mass to the graviton, their entire foundation is based on the breaking of General Covariance. One possible consequence, a generalization of the coupling to matter, has been discussed in the previous section. However, this symmetry under diffeomorphisms is directly related to an LI. How justified would it be to give up one symmetry and still enforce to other? Furthermore, all theories of modified gravity, regardless of whether they introduce a mass to the graviton or break other Lovelock assumptions, still live inside the framework of the EFT. A UV completion will surely introduce new operators above the cutoff of the theory, or already above the strong-coupling scale in theories of massive gravity. As some of these operators are expected to break Lorentz invariance, any conclusion about the stability of the vacuum state necessarily has to take a possible LB into account.

Before summarizing the consequences of an LB for both the classical and quantum stability for one specific model that was discussed in publication 8, we assume an arbitrary scattering between a ghost mode and other fields. The decay rate Γ is then obtained by integrating the scattering amplitude $\mathcal{M}$ over the entire phase space. More specifically,

$$\Gamma = \frac{1}{2m_g} \int \prod_f \mathrm{d}^3 p_f \, \frac{\pi}{E_f} \left|\mathcal{M}\right|^2 \delta^{(4)} \left(p_g - \sum_f p_f \right), \tag{3.47}$$

where p_g and m_g denote the ghost momentum and its mass, respectively, and the subscript f indicates all final particles. In a Lorentz invariant theory, the phase-space is infinitely large leading to a divergent decay rate. Note that an EFT cutoff is not able to simply cut the integral. However, an LV operator will automatically induce an LB scale that renders the decay rate finite. The decay rate can then be determined by the most dominant scattering process. This possibility was already proposed in refs. [15, 16] and simple scalar field models have been studied [15, 53, 54]. But their results should be taken with care, when applying them to theories of modified gravity with a minimal coupling, where derivative interactions play the major role.

3.5.2 Cosmological Viability with Ghosts

Classical Instability

To tackle the question of how the decay rate generally scales with the LB cutoff in minimally coupled theories of modified gravity, we have discussed a specific model of massive gravity that differs from dRGT and, thus, introduces a BD ghost. Doing this, one can kill two birds with one stone: Besides understanding the vacuum decay in modified gravity, at the same time the question, whether a theory of massive gravity can be made cosmologically viable, without explicitly introducing new degrees of freedom, can finally be answered positively! To see this, the model

$$\begin{aligned} S_{\mathrm{HMG}} = M_{\mathrm{P}}^2 \int \mathrm{d}^4 x \sqrt{-g} \Big[R + 2m^2 \Big(& (1 - a_1(g,f)) \left[\sqrt{g^{-1} f} \right] \\ & - \frac{1}{2} (1 - a_2(g,f)) \left(\left[\sqrt{g^{-1} f} \right]^2 - \left[g^{-1} f \right] \right) \Big) \Big] \end{aligned} \tag{3.48}$$

with

$$a_i(g,f) \equiv \bar{a}_i \, g^{\alpha\beta} f_{\beta\alpha}, \qquad \bar{a}_i \in \mathbb{R} \tag{3.49}$$

has been studied and was dubbed *Haunted Massive Gravity* (HMG). This action contains the first and second order interactions of the dRGT potential and explicitly violates its ghost-free structure by a detuning with $a_i(g,f)$.

The no-go result for FLRW backgrounds in dRGT has signalized that additional dynamics at background level are required in order obtain a viable theory. In HMG, the ghost will take over this part. This is certainly a dangerous endeavor, even at classical level one has to expect an instability due to the pathological BD ghost, that is usually associated with growing modes. For instance, the scale factor might grow exponentially – a ghost as the origin of the cosmic acceleration?

If one assumes a dark matter fluid, then the combination of the Friedmann equation

$$3H^2 = \rho + \frac{m^2}{a^4 N_g}\left[a^3\left(-\left(a\bar{\alpha}_1 + 6\bar{\alpha}_2\right)\right) - 3a^2 N_g\left(2a\bar{\alpha}_1 + \bar{\alpha}_2\right) + 3N_g^3\left(a\left((a-1)a - 3\bar{\alpha}_1\right) + 3\bar{\alpha}_2\right)\right], \tag{3.50}$$

together with the Bianchi constraint

$$\begin{aligned}&\left(1 + 3a^{-2}N_g^2\right)\left[N_g'\left(a\left(a\bar{\alpha}_1 + 6\bar{\alpha}_2\right) + 2N_g\left(2a\bar{\alpha}_1 + \bar{\alpha}_2\right)\right)\right.\\ &\left.+HN_g\left(-6a\bar{\alpha}_2 + N_g^2\left(4\bar{\alpha}_1 - (a-2)a\right) - 2N_g\left(a\bar{\alpha}_1 + \bar{\alpha}_2\right) + 9\bar{\alpha}_1 a^{-1}N_g^3\right)\right] = 0,\end{aligned} \tag{3.51}$$

provides a simply relation between the lapse and the scale factor in the limit $a \ll 1$ [55],

$$N_g = \pm\frac{1}{3}\sqrt{\frac{\bar{\alpha}_2}{\bar{\alpha}_1}a}, \tag{3.52}$$

and implies $H^2 \propto a^{-3}$. This result is indeed consistent with the early-time evolution in ΛCDM. The late time behavior in HMG can easily computed numerically and, for $\bar{\alpha}_i = \mathcal{O}(1)$, predicts an effective EoS parameter $w_{\text{eff}} < -1/3$ [55], indicating an acceleration. It is exactly the ghost instability that was expected to be visible at classical level and could potentially solve the DE problem.

Once again, it needs to be emphasized that HMG should not be seen as a new candidate of modified gravity that was intended to compete with ΛCDM, but it serves as a perfect counter-example for the conjecture that every massive gravity without an extra freedom does not produce a viable cosmological evolution.

Still, the dangerous quantum instability is not cured yet. But as the scattering processes at tree-level do indeed correspond to the classical background instability, we expect a maximization of the timescale of the background instability to likewise slow down the vacuum decay. For this, the time t_c can be computed at which the lapse N_g crosses zero denoting a Big Bang singularity. A maximization of the timescale then corresponds to $t_c = 0$, which is achieved if the parameters of the theory approximately obey the linear relation [55]

$$\bar{\alpha}_2 \simeq \frac{1}{6}\bar{\alpha}_1 - \frac{2}{45}. \tag{3.53}$$

Quantum Instability

As mentioned before, the instability at quantum level can be cured by the influence of an LB operator, but it is neither obvious which interaction is dominant, nor at which scale this operator has to set in to preserve viability. Let us first discuss the linear HMG mass term:

$$S^{(2)}_{\text{mass}} = M_{\text{P}}^2 m^2 \int \mathrm{d}^4 x \left(\frac{1}{4} + \bar{a}_1 - 2\bar{a}_2\right)[\eta(\delta g)]^2 - \left(\frac{1}{4} + 3\bar{a}_1 - 8\bar{a}_2\right)[\eta(\delta g)\,\eta(\delta g)]\,, \tag{3.54}$$

which, as constructed, violates the FP tuning for non-vanishing parameters $\bar{a}_i$. In addition, a canonical non-linear matter scalar field φ minimally coupled to gravity is considered. Since the vacuum decay is expected (and for HMG also explicitly shown in ref. [55]) to occur at small scales at which the metric is very well approximated by fluctuations around a flat Minkowski metric, the ansatz (2.24) with $a(t) = 1$ can be used to obtain the second-order action,

$$\begin{aligned} S^{(2)}_{\text{HMG}} = \int \mathrm{d}^4 x \Big[& 4M_{\text{P}}^2 \left(\Phi_i^2 - 2\Delta\Phi\Psi - 2\Phi'\Delta E' - 3\Phi'^2 - 2B_i\Phi_i'\right) \\ & + \frac{1}{2} m^2 M_{\text{P}}^2 \left(c_1 B_i^2 + c_2\left(\Psi^2 + (\Delta E)^2\right) + 8c_3\Phi\Delta E + 12c_3\Phi^2 + 4c_4\Psi(\Delta E + 3\Phi)\right) \\ & - \left(1 + B_i^2 - (\Delta E)^2 + 3\Phi^2 + 6\Phi\Psi - \Psi^2 + 2\Delta E(\Phi + \Psi)\right) X_\varphi \Big]\,, \end{aligned} \tag{3.55}$$

where we have defined

$$X_\varphi \equiv -\varphi'^2 + \varphi_i^2 + m_\varphi^2\,\varphi^2 \tag{3.56}$$

together with the parameters c_i

$$c_1 \equiv 1 + 12\bar{a}_1 - 32\bar{a}_2\,, \tag{3.57}$$

$$c_2 \equiv -16\bar{a}_1 + 12\bar{a}_2\,, \tag{3.58}$$

$$c_3 \equiv 1 + 4\bar{a}_2\,, \tag{3.59}$$

$$c_4 \equiv 1 + 4\bar{a}_1 - 8\bar{a}_2\,. \tag{3.60}$$

The combination of all five scalars in the action (3.55) should describe one helicity-0 mode, a BD ghost, and an external matter field. Surprisingly, not two but three of them, i.e., Ψ, B_i, and ΔE, are, in fact, auxiliary. However, integrating them out introduces fourth-order derivatives, which indicates the presence of an Ostrogradsky ghost in addition to the two dynamical degrees of freedom and, therefore, matches up the counting of the total number of degrees of freedom.

To properly discuss the ghost instability, all degrees of freedom have to be decoupled from each other. A recipe, how to separate all modes in a general, not even necessarily covariant, theory of both two and three interacting scalar fields, was first presented in publication 8.[2] These results were used to find that HMG can equivalently be described by an interaction of two tachyonic fields π and ξ, respectively, and one ghost $\Phi_{👻}$. The tachyonic instability should, however, not be taken too seriously as this is just an effective reformulation to compute the scattering amplitudes and does not imply that the physical

[2]During the analysis, a remarkable side product was found: While Ostrogradsky's theorem does not make any statement about the number of ghosts that will show up in higher-derivative theories, we have found that, in fact, the property of a theory being covariant is crucially related to the number of propagating degrees of freedom.

helicity-0 or the matter field is really tachyonic.

Once all modes are decoupled, the most dominant interactions can be read off from the Lagrangian and were found to be

$$\mathcal{L}^{\text{dom}} \propto \frac{m^6}{m_\varphi^6\, M_\text{P}^2}\, \Phi_{\text{ghost}}^2\, \xi^3\, \partial_\mu\, \partial^\mu \xi\,. \tag{3.61}$$

The timescale of the vacuum instability is, therefore, determined by the derivative interactions between two ghost fields and four matter fields, which leads to a vacuum decay rate [55]

$$\Gamma_{\Phi_{\text{ghost}}} = \frac{3A^2 m_\xi^4\, \Lambda_\text{LB}^6}{2\,(2\pi)^{10}\, m_{\Phi_{\text{ghost}}}} + \mathcal{O}\left(\Lambda_\text{LB}^5\right), \tag{3.62}$$

where A comprises the prefactor of the Lagrangian (3.61). Cosmological viability requires the decay time to be larger than the age of our Universe, i.e., $\Gamma^{-1} \gtrsim H_0^{-1}$. Assuming the graviton mass m to be of order H_0, which is necessary to ensure that modifications of gravity appear at cosmological scales, shows that the scale at which LI has to be broken,

$$\Lambda_\text{LB} \lesssim \Lambda_\text{LB}^{(\text{max})} \equiv \mathcal{O}\left(\left(\frac{m_\varphi^{12}\, M_\text{P}^8}{m^{14}}\right)^{1/6}\right), \tag{3.63}$$

can easily be much larger than not only the strong coupling scale of the theory $\Lambda_5 = (m^4 M_\text{P})^{1/5}$ but even the Planck mass! Only for incredibly tiny masses of φ, i.e., $m_\varphi \ll H_0 \simeq 2 \times 10^{-33}$ eV, the decay of the vacuum might occur too fast, but a massive scalar field with such a small mass has never been observed.

To summarize, a model of massive gravity has been found that disproves two old conjectures. Firstly, a massive graviton alone is indeed able to provide a viable cosmological evolution and, secondly, a ghost in theories of modified gravity can be harmless. Even more, the interaction that dominates the vacuum decay is expected to be the same for most minimally coupled theories of modified gravity and, thus, will also render many theories viable again that had been discarded due to the existence of a ghost.

Chapter 4

BIMETRIC GRAVITY

OR: THE UNIQUE THEORY OF TWO INTERACTING SPIN-2 FIELDS

Maximizing the symmetry of an action plays a crucial role in field theory and is often regarded as a step towards a more fundamental description of gravity. Certainly the biggest disadvantage in a theory of a massive gravity is the breaking of diffeomorphism invariance. Quite naturally, one will ask whether this symmetry can be restored. The answer turns out to lead to a theory of two metrics, a bimetric theory.

4.1 Generalizing Massive Gravity to a Bimetric Theory

A massive gravity action that exhibits a diffeomorphism invariance requires additional kinetic terms to compensate the transformation behavior of the mass term. While any modification in the Einstein-Hilbert action is likely to introduce new degrees of freedom, the kinetic term for the metric g should be kept and the reference metric has to obtain its own kinetics which will cause the second metric to become dynamical. In other words, restoring General Covariance in massive gravity seems to lead to a theory of two interacting spin-2 fields. In order to construct the corresponding action it is reasonable to just use two copies of GR, one for each tensor field, and add a suitable interaction term. The ghost-free proof in dRGT massive gravity can be generalized to a dynamical reference metric and will not affect the form of the potential [38, 40, 41]. Therefore, the action can finally be written as

$$S = -\frac{1}{2}M_g^2 \int \mathrm{d}^4x\,\sqrt{-g}\left(R(g) - 2m^2\sum_{n=0}^{4}\beta_n\, e_n(X)\right) - \frac{1}{2}M_f^2 \int \mathrm{d}^4x\,\sqrt{-f}\,R(f) + \int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L}_m, \tag{4.1}$$

where a second Planck scale M_f in addition to $M_g \equiv M_\mathrm{P}$ is introduced. Because each single Einstein-Hilbert term is invariant under diffeomorphisms, the invariance of the mass term, and therefore the whole action, possesses this symmetry, too, if both g and f are transformed with the same diffeomorphism. Thus, giving dynamics to f has indeed restored this gauge symmetry.

Even though the second tensor field f gets its own Einstein-Hilbert action with the volume element $\sqrt{-f}$ and looks like a second metric, there is still only one metric spacetime over which all integrations are performed. However, this language of "two metrics" is often used and explains its name *bimetric gravity*, or in short, *bigravity*. Right after this action was proposed by Hassan and Rosen, it was explicitly shown to be free of the BD ghost [38, 40].

At first glace, the metric g seems to play a special role. This is, in fact, not wrong since the entire matter sector couples to g only. But apart from that, the mass term carries a symmetry that ensures that both metrics are equally footed. To see this explicitly, one can utilize the properties of the elementary symmetric polynomials [38],

$$e_k\left(\sqrt{g^{-1}f}\right) = \frac{e_{4-k}\left(\sqrt{f^{-1}g}\right)}{e_4\left(\sqrt{f^{-1}g}\right)}, \tag{4.2}$$

and finds

$$\sqrt{-g}\sum_{n=0}^{4}\beta_n\, e_n\left(\sqrt{g^{-1}f}\right) = \sqrt{-f}\; e_4\left(\sqrt{f^{-1}g}\right)\sum_{n=0}^{4}\beta_n\, e_n\left(\sqrt{g^{-1}f}\right) \tag{4.3}$$

$$= \sqrt{-f}\sum_{n=0}^{4}\beta_n\, e_{4-n}\left(\sqrt{f^{-1}g}\right). \tag{4.4}$$

The bimetric action is then indeed symmetric under the exchanges $f \leftrightarrow g$, $\beta_n \to \beta_{4-n}$, and $M_g \leftrightarrow M_f$ together with a simultaneous rescaling $m^2 \to m^2\, M_g^2/M_f^2$ [56].

Clearly, with the new dynamics for the reference metric $f_{\mu\nu}$, we have entered a new field with more than just a massive graviton. Generally, analyzing the spectrum to find all propagating modes is quite complicated, if possible at all, and crucially depends on the background. In a simple case where both metrics just describe small fluctuations around the same background $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + M_g^{-1}\delta g_{\mu\nu} \qquad \text{and} \qquad f_{\mu\nu} = \bar{g}_{\mu\nu} + M_f^{-1}\delta f_{\mu\nu}, \tag{4.5}$$

and all higher-order interactions are switched-off by using $\beta_i = (-3, -1, 0, 0, 1)$ [56], the action (4.1) at second-order becomes [38]

$$S = \int d^4x \left(\delta g_{\mu\nu}\, \hat{\mathcal{E}}^{\mu\nu\alpha\beta}\, \delta g_{\alpha\beta} + \delta f_{\mu\nu}\, \hat{\mathcal{E}}^{\mu\nu\alpha\beta}\, \delta f_{\alpha\beta}\right) - \frac{m^2}{4} M_{\mathit{eff}}^2 \int d^4x \left[\left(\frac{\delta g^{\mu}_{\ \nu}}{M_g} - \frac{\delta f^{\mu}_{\ \nu}}{M_f}\right)^2 - \left(\frac{\delta g^{\mu}_{\ \mu}}{M_g} - \frac{\delta f^{\mu}_{\ \mu}}{M_f}\right)^2\right], \tag{4.6}$$

where $\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$ denotes the Lichnerowicz operator, defined such that $\delta g_{\mu\nu}\, \hat{\mathcal{E}}^{\mu\nu\alpha\beta}\, \delta g_{\alpha\beta}$ describes the linearized Einstein-Hilbert action, and the effective Planck mass M_{eff} is defined as

$\left(M_g^{-2} + M_f^{-2}\right)^{-2}$. To decouple all modes, one introduces

$$\tilde{G}_{\mu\nu} \equiv \frac{M_{\text{eff}}}{M_f}\delta g_{\mu\nu} + \frac{M_{\text{eff}}}{M_g}\delta f_{\mu\nu}, \tag{4.7}$$

$$\tilde{M}_{\mu\nu} \equiv \frac{M_{\text{eff}}}{M_g}\delta g_{\mu\nu} + \frac{M_{\text{eff}}}{M_f}\delta f_{\mu\nu} \tag{4.8}$$

to obtain [38]

$$S = \int d^4x \left(\tilde{G}_{\mu\nu}\,\hat{\mathcal{E}}^{\mu\nu\alpha\beta}\,\tilde{G}_{\alpha\beta} + \tilde{M}_{\mu\nu}\,\hat{\mathcal{E}}^{\mu\nu\alpha\beta}\,\tilde{M}_{\alpha\beta}\right) - \frac{m^2}{4}M_{\text{eff}}^2 \int d^4x \left[\tilde{M}^{\mu\nu}\tilde{M}_{\mu\nu} - \tilde{M}^2\right]. \tag{4.9}$$

The mass term that appears in (4.9) depends only on $\tilde{M}_{\mu\nu}$ and precisely obeys the FP structure. With this, both a massless $\tilde{G}_{\mu\nu}$ and massive $\tilde{M}_{\mu\nu}$ mode, respectively, are decoupled, pointing to the presence of two spin-2 fields of which one is massive. Bimetric gravity does not just generalize massive gravity by giving the reference metric dynamics, it is rather a completely different theory of two interacting gravitons.

4.2 Cosmological Background Solutions

With the development of bigravity, a new hope arose to finally tackle the DE problem successfully. Suddenly, many different classes of solutions describing completely different phenomenologies were available. Most of them are able to explain the accelerated expansion [56, 57, 58, 59, 60], others contain an inflationary epoch or could even describe bouncing solutions [55]. However, a comprehensive analysis of their phenomenology and a comparison with observational data in publication 1 demonstrated that quite a number of models are not cosmologically viable. To see this explicitly, we choose an FLRW ansatz for both metrics,

$$\mathrm{d}s_g^2 = a^2\left(-\mathcal{H}^{-2}\,\mathrm{d}t^2 + \mathrm{d}x_i\,\mathrm{d}x^i\right), \tag{4.10}$$

$$\mathrm{d}s_f^2 = b^2\left(-\mathcal{H}^{-2}\,N_f^2\,\mathrm{d}t^2 + \mathrm{d}x_i\,\mathrm{d}x^i\right). \tag{4.11}$$

Note that the combined diffeomorphism invariance allows us to choose one time parametrization, in this case the e-folding time $t \equiv \log a$. Before analyzing the set of background equations, it is useful to introduce the ratio between both scale factors,

$$r \equiv \frac{b}{a}, \tag{4.12}$$

which will be assumed to be positive. Furthermore, we can fix two redundancies in the set of free parameters of the theory. Under the transformation $f_{\mu\nu} \to M_f^{-2} f_{\mu\nu}$ the elementary symmetric polynomials transform as

$$\sum_{n=0}^{4} \beta_n\, e_n\left(\sqrt{g^{-1}f}\right) \to \sum_{n=0}^{4} M_f^{-n}\,\beta_n\, e_n\left(\sqrt{g^{-1}f}\right), \tag{4.13}$$

demonstrating that the Planck scale M_f can be set to M_g by rescaling $\beta_n \to M_f^n \beta_n$. Finally, the graviton mass scale m will be absorbed into the β_n and all masses will be expressed in units of Planck masses.

Because the bimetric action should be varied with respect to all dynamical fields, we obtain two EoM in addition to the EM conservation. The set of independent equations becomes

$$3\mathcal{H}^2 = a^2\left(\rho + \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3\right), \tag{4.14}$$

$$3\mathcal{H}^2 = \frac{a^2 r N_f^2}{(r' + r)^2}\left(\beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3\right), \tag{4.15}$$

$$\rho' = -3\rho\,(1 + w_{tot})\,. \tag{4.16}$$

As already mentioned earlier, the parameter β_0 appears as a CC. Because the entire matter sector is minimally coupled to g, it is only this parameter that will receive quantum corrections [61, 62, 63, 64] leading to the CC problem. All other coupling parameter β_n are protected against loops and, therefore, preserve technical naturalness. This motivates the search for models in which β_0 is set to zero in order to obtain self-accelerating models.[1]

All background equations (4.14) - (4.16) can now be used to solve for the lapse, which leads to

$$N_f = 1 + \frac{r'}{r}. \tag{4.17}$$

The same constraint would also directly follow from the Bianchi constraint. In addition, the background equations also allow to directly solve for ρ in terms of r only:

$$\rho = \beta_1 r^{-1} - \beta_0 + 3\beta_2 + 3\,(\beta_3 - \beta_1)\,r + (\beta_4 - 3\beta_2)\,r^2 - \beta_3 r^3. \tag{4.18}$$

Therefore, to analyze the phenomenology of possible cosmological solutions, it is only important to know how the ratio of the scale factors evolves. One way to analyze the evolution of r is the usage of

$$r' = \frac{\rho'}{\rho_{,r}} = -3\,(1 + w_{\rm tot})\,\frac{\rho}{\rho_{,r}}, \tag{4.19}$$

and employing eq. (4.18) to obtain a differential equation for r. Instead of solving this equation explicitly, it turns out to be very convenient to discuss its phase space. Even beyond the background level, the evolution in the phase space will allow us to immediately draw conclusions about the existence of different types of instabilities, as discussed in sections 4.3.1 and 4.5. One representative phase space diagram is illustrated in Fig. 4.1 [28].

All viable cosmological solutions, i.e., those in which the density is positive, the Hubble expansion real, and a matter dominated period exists, fall into one of the following three

[1]By assuming $\beta_0 = 0$, one has, of course, not obtained a solution for the CC problem. It is rather assumed that a symmetry, possibly predicted by a more fundamental theory, exists that enforces the CC to vanish and, consequently, requires an explanation for the late-time acceleration of the Universe.

qualitatively different types of branches [60, 28]:

Finite Branches

During the entire cosmic evolution the ratio of the scale factors stays finite, $r \in (0, r_c)$, and increases with time, i.e., $r' \geq 0$. In the asymptotic past, $r \to 0$ causes the density (4.18) to diverge and indicates a Big Bang singularity. While the universe expands, r increases and approaches a root at r_c: $r'|_{r_c} = 0$, which corresponds to the asymptotic future. Because of eq. (4.19), a constant r implies a vanishing density and the Hubble function will approach a constant; we have entered a *de Sitter* epoch in which the universe is dominated by DE only.

A model that deserves special attention is the *Minimal Bimetric Model* (MBM) and has been analyzed in publication 2. It is the only viable one-parameter model that is not just equivalent to ΛCDM. Here, all β-parameters except β_1 are zero and, therefore, the model contains the same number of free parameters as in ΛCDM. Furthermore, it is particulary simple as the entire evolution can be solved analytically and agrees with observational data [60]. Interestingly, the EoS always evolves from -2 to -1; a phantom behavior that distinguishes all finite branch models in bigravity from the standard cosmological evolution.

Infinite Branches

If the scale factor b dominates over a at early times, then eq. (4.19) shows that r' is negative. Such a model evolves from the limit $r \to \infty$ in the asymptotic past towards a root at r_c, which, again, indicates a de Sitter point. Even though no viable one-parameter model exists in this branch, it can reproduce the success of ΛCDM if at least $\beta_1, \beta_4 > 0$, the so-called *Infinite Branch Bigravity* (IBB) model [60, 65].

Exotic Branches

All cosmological solutions discussed so far start with a Big Bang singularity and reach a de Sitter state at late times. However, there are branches in which r is always finite and non-zero, even in the early- and late-time limits. In the phase-space diagram the asymptotic points are then described by either a pole or a root. While the latter has already been discussed in the other branches and corresponds to a vanishing density, the existence of a pole[2] indicates a point in time at which $\mathcal{H} = 0$: A bounce! One possible scenario is a contraction from an infinitely large universe until a non-singular bounce occurs, followed by an (accelerated) expansion. Such a model could potentially explain not only the recent accelerated expansion of our Universe but, at the same time, does not require an inflationary epoch and comes out without any Big Bang singularity.

[2]Note that a pole in r' is not an unphysical behavior because the prime denotes the derivative with respect to e-folding time, i.e., $r' = a\mathcal{H}^{-1}\dot{r}$, and leads to a divergence if $\mathcal{H}$ crosses zero.

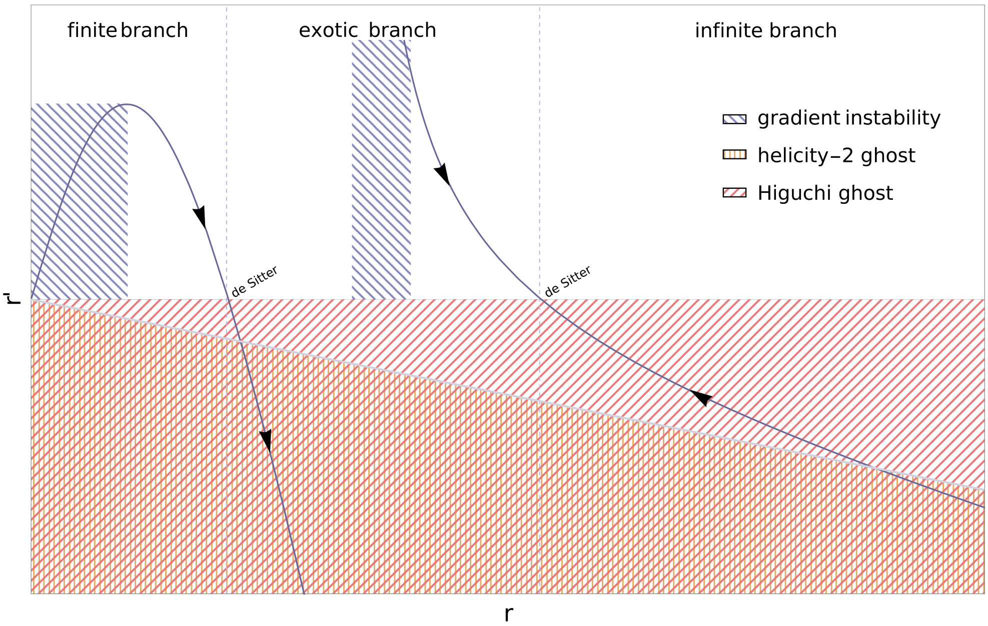

Figure 4.1: *Representative phase space diagram of a model that contains three different cosmological solutions: Two of them, those on the finite and infinite branch, describe an expanding universe towards a de Sitter state, while the third solution on the exotic branch corresponds to a bouncing model. The colored regiones indicate the existence of gradient instabilities (diagonal blue stripes) and ghost instabilities (vertical orange stripes signal that the helicity-2 modes are ghosts, whereas diagonal red stripes denote a helicity-0 ghost), respectively.*

4.3 Linear Scalar Perturbations

At background level, bimetric gravity provides a huge number of viable models that are not only consistent with current data but also predicts new phenomenologies that might be testable in the near future. Consequently, studying their behavior at linear level received an increasing interest [66, 67, 68, 69, 70, 65, 71, 55, 72, 73, 74].

4.3.1 Gradient Instabilities

It has not last long until the first simple models were found to develop dangerous instabilities [68, 69, 65, 71]. To properly understand their origin and consequences, we have studied the particular MBM in detail in Publication 2. By finding stability conditions for general models in publication 3 we could then identify all possible models that do not suffer from gradient instabilities.

Let us first focus on scalar perturbations around an FLRW background that can be described by the line elements

$$\mathrm{d}s^2_{\delta g} = 2a^2\left[-\Psi\,\mathrm{d}t^2 + \left(\Phi\delta_{ij} + k_i\,k_j\,E\right)\mathrm{d}x^i\,\mathrm{d}x^j\right]\exp\left(i\,\vec{k}\,\vec{r}\right), \tag{4.20}$$

$$\mathrm{d}s^2_{\delta f} = 2b^2\left[-N_f^2\,\Psi_f\,\mathrm{d}t^2 + \left(\Phi_f\,\delta_{ij} + k_i\,k_j\,E_f\right)\mathrm{d}x^i\,\mathrm{d}x^j\right]\exp\left(i\,\vec{k}\,\vec{r}\right) \tag{4.21}$$

in Fourier space. Because solving the set of perturbation equations turns out to be quite cumbersome, a suitable gauge transformation can be used to simplify the analysis by, e.g., rendering some variables auxiliary. A convenient choice is the transformation to gauge-invariant variables [68] (see also refs. [75, 71] on how to choose a useful gauge by using the Noether identities),

$$\Phi \longrightarrow \Phi - \mathcal{H}^2 E', \tag{4.22}$$

$$\Psi \longrightarrow \Psi - \mathcal{H}\left(\mathcal{H}' E' + \mathcal{H}\left(E'' + E'\right)\right), \tag{4.23}$$

$$\Phi_f \longrightarrow \Phi_f - \frac{\mathcal{H}^2 r\, E_f'}{r' + r}, \tag{4.24}$$

$$\Psi_f \longrightarrow \Psi_f - \frac{\mathcal{H} r^2 \mathcal{H}'\left(r' + r\right) E_f' + \mathcal{H}^2 r\left(r\left(r' + r\right) E_f'' + E_f'\left(2r'^2 + r\left(2r' - r''\right) + r^2\right)\right)}{\left(r' + r\right)^3}. \tag{4.25}$$

In total, one obtains ten independent perturbation equations of which eight equations follow from the (0,0)-, (0,i)-, (i,i)-, and (i,j)-EoM for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively; the remaining two arise from the energy-momentum conservation of the perturbed EM tensor. Since we have just one propagating helicity-0 mode and a scalar matter fluctuation, we should expect the set of equations to be reducible to only two second-order differential equations. After integrating out all auxiliary variables one, indeed, obtains [76, 65, 71]

$$X_i'' + F_{ij} X_j' + S_{ij} X_j = 0, \tag{4.26}$$

where $X_i \equiv (\Phi, \Psi)$ and F and S are matrices that contain the model dependency. To dicsuss the stability of X it is sufficient to use the ansatz $X_i \propto e^{\omega t}$ in combination with the assumption that w does not depend on t.[3] Furthermore, a possible instability will lead to a growth at especially sub-horizon scales. In this régime, the eigenfrequencies are [65, 28]

$$\omega^2 = \left(\frac{k}{\mathcal{H}}\right)^2 \left[\frac{r'\left(\frac{(r^2+1)(\beta_1-\beta_3 r^2) r'}{\rho(w+1)} - \frac{r^2(\beta_1+4\beta_2 r+3\beta_3 r^2)}{\beta_1+2\beta_2 r+\beta_3 r^2}\right)}{3r^3} - 1\right]. \tag{4.27}$$

For models in which the highest-order interactions are switched-off, i.e., $\beta_2 = \beta_3 = 0$, it reduces to the remarkably simple expression

$$\omega^2_{\beta_0\beta_1\beta_4} = \left(\frac{k}{\mathcal{H}}\right)^2 \frac{r''}{3r'}, \tag{4.28}$$

which is very convenient to study the stability as just the sign is of importance. A negative one implies imaginary eigenfrequencies and, thus, stable, oscillating modes. Otherwise, if the sign os positive, the scalar perturbations undergo an exponentially fast growth. Especially the scale dependence signalizes a dramatic behavior at small scales which does not seem to withstand any comparison with observations.

In the finite branch model MBM, the early-time evolution satisfies $r', r'' > 0$ and,

[3] Neglecting the time dependence is allowed if the WKB approximation, i.e., $\left|\omega'/\omega^2\right| \ll 1$, holds. In the sub-horizon approximation, this was indeed found to be valid [65].

therefore, indicates unstable linear perturbations! In fact, the presence of gradient instabilities at early times can be generalized to all finite branch models that produce a viable background [65]. All models living in the exotic branch are also plagued by instabilities [55]. Only one candidate has survived the instability check: the models on the infinite branch.

4.3.2 Quasi-Static Approximation

In order to compare the evolution of scalar perturbations in IBB, the only model that has a viable background evolution and is free of gradient instabilities, with observations of large-scale structure, it is sufficient to use both the sub-horizon limit together with the quasi-static approximation, since these experiments especially probe modes within the horizon [65]. While the sub-horizon approximation only considers small modes that satisfy $k/\mathcal{H} \gg 1$, the quasi-static limit restricts to slowly oscillating modes. In this limit, we can introduce the anisotropic stress

$$\eta \equiv -\frac{\varphi}{\psi} \tag{4.29}$$

and the effective gravitational coupling

$$Y \equiv -\frac{2k^2\,\psi}{3\mathcal{H}^2\Omega_m\delta}, \tag{4.30}$$

where Ω_m denotes the ratio between the matter density and the critical density, i.e., for which the universe would be flat. Both modified gravity parameters are defined to be unity if the evolution is equivalent to the one in ΛCDM. We have shown that IBB predicts [65]

$$\lim_{t\to-\infty} \eta = \frac{1}{2} \qquad \text{and} \qquad \lim_{t\to-\infty} Y = \frac{4}{3}, \tag{4.31}$$

and therefore deviates significantly from the standard model in the asymptotic past. So far, both ΛCDM as well as IBB agree with all current observed growth data [65], but near-future experiments will soon be able to discriminate between IBB and ΛCDM [77].

4.4 Growing Tensor Modes

Cosmological perturbations comprise small fluctuations in the scalar as well as in the vector and tensor sector. The gradient instabilities that have been discussed in the previous section only affect scalar perturbations. In publication 5 we have focused on the tensor sector to extend the picture of cosmological viability.

Consider transverse gravitational waves propagating in the z direction for both metrics, g and f,

$$h_{f/g\,(ij)} = \begin{pmatrix} h_{f/g\,(+)} & h_{f/g\,(\times)} & 0 \\ h_{f/g\,(\times)} & -h_{f/g\,(+)} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.32}$$

The EoM for both metrics provide [68, 71, 73]

$$h_g'' + \gamma_g\, h_g' + \left(m_g^2 + c_g^2\, \mathcal{H}^{-2}\, k^2\right) h_g = q_g\, h_f, \tag{4.33}$$

$$h_f'' + \gamma_f\, h_f' + \left(m_f^2 + c_f^2\, \mathcal{H}^{-2}\, k^2\right) h_f = q_f\, h_g, \tag{4.34}$$

where the following coefficients have been defined:

$$\gamma_g = 2 + \frac{\mathcal{H}'}{\mathcal{H}}, \qquad \gamma_f = \frac{2r^2 + 3r'^2 + r\,(4r' - r'')}{r\,(r' + r)} + \frac{\mathcal{H}'}{\mathcal{H}}\ , \tag{4.35}$$

$$m_g^2 = \mathcal{H}^{-2} a^2 B\, r, \qquad m_f^2 = \frac{(r' + r)}{\mathcal{H}^2 r^2} a^2 B\ , \tag{4.36}$$

$$c_g^2 = 1, \qquad c_f^2 = \frac{(r' + r)^2}{r^2}\ , \tag{4.37}$$

$$q_g = \mathcal{H}^{-2} a^2 B\, r, \qquad q_f = \frac{(r' + r)}{\mathcal{H}^2 r^2} a^2 B\ , \tag{4.38}$$

with

$$B \equiv \beta_1 + \beta_3 r^2 + r\,(2\beta_2 + \beta_3 r') + \beta_2 r'. \tag{4.39}$$

Contrary to the scalar case, the phenomenological behavior of the gravitational waves is much easier to understand and can easily be computed numerically. The tensor fluctuations h_g are, as in GR, damped by $\gamma_g > 0$. However, the friction term for h_f, γ_f, is negative if $b' < 0$, which is always fulfilled at early times in models on the infinite branch, and signalizes a fast grow. Due to the coupling of both modes, this growth propagates into h_g.

Because growing modes would influence the CMB, we have compared the IBB model to present CMB data in publication 5. Whether the gravitational waves h_g grow too fast or not does not only depend on the choice of parameters, but also crucially on the initial conditions set by an inflationary epoch. The list of models for inflation is long. Often the end of inflation occurs at energies around some MeV up to 10^{15} GeV [27].

In addition, fast growing tensor modes will rapidly reach large values that are not consistent anymore with the linear perturbation approximation and non-linearities should be taken into account. One should, therefore, only trust the solutions of the perturbation equations when $h_{f/g} \lesssim 1$.

However, for the IBB model a very optimistic assumption of a low-energy inflation, that stops at an energy scale of $\mathcal{O}$(GeV), with an additional cutoff is required to evade conflicts with recent observations of the CMB [73]. Albeit less appealing, another possible solution is a tuning of the model parameters to achieve a very small coupling q_g with an additional CC to ensure the viability at background level.

4.5 Reopening the Ghost Hunt

Bimetric gravity contains many viable cosmological background solutions that easily pass all observational tests. All their linear perturbations are, however, plagued by either scalar gradient instabilities or growing tensor modes. To properly understand their origin it is useful to directly check the behavior of the single helicity modes of both gravitons, as it has been done in publication 6.

4.5.1 Higuchi Ghost in Massive Gravity

The absence of ghosts was one, and perhaps the most important, requirement that a theory of a massive bimetric gravity has to fulfill. All ghost-free proofs have concentrated on finding only an additional Hamiltonian constraint, though, which ensures that no BD ghost will appear. This does, however, not automatically imply that all graviton degrees of freedom are well behaved. Before discussing the bimetric case, let us focus on a linear massive gravity around a de Sitter background first (for a generalization to an FLRW background, see ref. [78]), i.e.,

$$g_{\mu\nu} = g^{(\mathrm{dS})}_{\mu\nu} + M_{\mathrm{P}}^{-1} h_{\mu\nu}. \tag{4.40}$$

After a decomposition into the two tensor ($\bar{h}_{\mu\nu}$) and one scalar (π) degrees of freedom by replacing $h_{\mu\nu} = \bar{h}_{\mu\nu} + \pi\, g^{(\mathrm{dS})}_{\mu\nu}$, the Lagrangian for the helicity-0 can be then read off from the FP action and becomes [79]

$$\mathcal{L}^{\text{hel-0}}_{\mathrm{dS}} = -\frac{3}{4}\left[1 - 2\left(\frac{H}{m}\right)^2\right]\left((\partial\pi)^2 - m^2\,\bar{h}\,\pi - 2m^2\pi^2\right). \tag{4.41}$$

Because the scalar mode of the graviton is coupled to the trace T of the EM tensor (which is, in fact, the origin of the vDVZ discontinuity [31]), the matter Lagrangian reads

$$\mathcal{L}^{\text{hel-0}}_{\text{matter}} = \frac{m^2}{M_{\mathrm{P}}\sqrt{m^2 - 2H^2}}\,\varphi\, T, \tag{4.42}$$

where φ is the normalized helicity-0 mode,

$$\varphi = \sqrt{1 - \frac{2H^2}{m^2}}\;\pi. \tag{4.43}$$

This expression is quite useful as the following cases can easily be distinguished:

- $m^2 = 0$: This corresponds to a vanishing helicity-0 mode, as expected for massless gravitons. When taking the limit $m \to 0$, the Lagrangian (4.42) indicates a strong coupling and the validity of the linear FP theory breaks down.
- $m^2 < 0$: Even though we cannot directly read off the properties of this theory from the Lagrangian, one finds that this yields an unhealthy helicity-1 mode [79].
- $0 < m^2 < 2H^2$: A forbidden mass range due to the change of the overall sign. In this case, the helicity-0 of the graviton becomes a ghost, which is named *Higuchi ghost*, after Higuchi who has observed this condition first [80, 81].
- $m^2 > 2H^2$: This bound preserves the overall sign and avoids the Higuchi ghost.

- $m^2 = 2H^2$: Such a tuning of the graviton mass implies a precise cancellation of the first bracket in the Lagrangian (4.41) indicating a vanishing helicity-0 mode. In fact, in this case we observe an additional gauge symmetry [82]

$$h_{\mu\nu} \to h_{\mu\nu} + \left(\nabla_\mu \nabla_\nu + H^2\, g_{\mu\nu}^{(dS)}\right) \xi(x) \tag{4.44}$$

where $\xi(x)$ plays the role of an arbitrary gauge parameter. A theory that possesses this symmetry is called *partially massless* (PM).

4.5.2 Higuchi Ghost in Bimetric Gravity

In a bimetric theory, the condition to ensure a healthy helicity-0 mode was derived in a minisuperspace approximation around an FLRW background by Fasiello and Tolley who found [4] [83]

$$\frac{3}{2}\left(\beta_1 + 2\beta_2 r + \beta_3 r^2\right)\left(1 + r^2\right) \geq \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 = 3r\left(\frac{\mathcal{H}}{a}\right)^2 . \tag{4.45}$$

Remarkably, this condition is similar to [28]

$$r' \geq 0, \tag{4.46}$$

and immediately shows that all models on the infinite branch suffer from the Higuchi ghost. Therefore, the absence of gradient instabilities in these models does not automatically imply a healthy theory. The ghost will cause a fast growth of scalar perturbations and, even worse, a quantum instability will arise.

4.5.3 Tensor Ghosts

To understand the growing modes in the tensor sector for IBB, the authors in ref. [72] pointed out that the relative factor between the kinetic tensor modes for $g_{\mu\nu}$ and $f_{\mu\nu}$ is the lapse N_f. Because of eq. (4.17) and the early time limit $r \to \infty$ and, thus, $r' \simeq -3/2(1 + w_{\rm tot})r$, the lapse is negative leading to a sign difference in both kinetic terms [28]: The helicity-2 modes of the second metric are ghosts, too.

All infinite branch models are therefore plagued by in total three ghost degrees of freedom which, without any modifications in the UV, immediately rules the models out. Additionally, all models on the exotic branch suffer from either gradient instabilities or ghosts, too [28]; likewise all finite branch solutions that seem to be nonviable at early times due to (only) unstable scalar perturbations.

[4] Note that the authors have used a slightly different convention in the mass term that was compensated by a rescaling of the β-parameters.

4.6 Evading Gradient Instabilities

Let us now focus on the finite branch solutions, the only one being free of ghosts. Gradient instabilities which affect the scalar sector can only hardly be compatible with the observed structure in our Universe that is, in fact, very well described by linear fluctuations. Since the enhanced growth only occurs at early times, one can hope to find a suitable parameter region to push this instability to the far past in order to minimize observable consequences.

4.6.1 Pushing the Instability Away

In Publication 2, we have noted that the possibility of adding a CC to the MBM can stop the instability to evolve at $r = \beta_1/2\beta_0$, which can, if $\beta_1 \ll \beta_0$, happen at arbitrarily early times [69]. The growth could stop very early and might not be observable today. One can even go further and push the instability beyond the strong coupling scale of the theory. At these energy scales, the theory is not trustable anymore and new operators become important.

A non-vanishing CC can indeed solve the instability problem in the scalar sector, but at the same time it introduces new, though less dangerous, problems. The limit $\beta_1 \to 0$ implies the limit of a vanishing graviton mass, $m \to 0$, which is associated with the vDVZ discontinuity and, thus, non-linear effects become important. Furthermore, a non-zero CC lets the theory again run into the CC problem that was originally intended to get solved.

4.6.2 Planck Mass Scaling

Besides taking the limit $\beta_1 \to 0$, there is the possibility to assume extremely large values for the β-parameters. This approach might look highly unnatural but can be translated to a much more meaningful limit in which the Planck mass M_f for the second metric goes to zero. The redundancy in the parameter space of the Lagrangian (4.1) has been used to fix $M_f = M_g$ just for convenience but it encumbers the real massless limit. In publication 7 we have revealed this limit again and reintroduced both Planck masses and the graviton mass scale. To explicitly see the massive and massless limits, one can write the field equations as

$$G_{\mu\nu}(g) + m^2\, V^g_{\mu\nu} = \frac{1}{M_g^2} T_{\mu\nu}, \tag{4.47}$$

$$\alpha^2 G_{\mu\nu}(f) + m^2\, V^f_{\mu\nu} = 0, \tag{4.48}$$

where $\alpha \equiv M_f/M_g$ and $V^{g/f}$ denotes the variation of the potential with respect to g and f, respectively. For $\alpha \to 0$, the f-equation (4.48) becomes a constraint for $V^f_{\mu\nu}$ that can be used together with the identity [84]

$$\sqrt{-g}\, g^{\mu\alpha}\, V^{g}_{\alpha\nu} + \sqrt{-f}\, f^{\mu\alpha}\, V^{f}_{\alpha\nu} = \sqrt{-g} \sum_{n=1}^{4} \beta_n\, e_n\left(\sqrt{g^{-1}f}\right) \delta^{\mu}_{\nu}, \tag{4.49}$$

to solve for $f_{\mu\nu}$.[5] Plugging this solution into eq. (4.47) yields an ordinary Einstein equation with an effective CC. Let us make this explicit for the $\beta_1\beta_2$ model with $\beta_0 = \beta_3 = \beta_4 = 0$. In the limit $\alpha \to 0$, the combination of both EoM leads to

$$r \to -\frac{1}{3}\frac{\beta_1}{\beta_2} \tag{4.50}$$

and the g-equation (4.47) becomes

$$3H^2 = \frac{\rho}{M_g^2} - \frac{2}{3}\frac{\beta_1^2}{\beta_2} m^2. \tag{4.51}$$

Because the former parameter choice $M_f = 1$ has implicitly rescaled the coupling parameters by $\beta_n \to \Omega^{-n}\beta_n$ when $M_f \to \Omega\, M_f$ for $\Omega \in \mathbb{R}^+$, the GR limit $\alpha \to 0$ corresponds to the limit where all parameters β_n go to infinity in a parametrically different way.

Since ΛCDM is free of any type of instability at linear level, the limit of a small Planck scale M_f to approach GR has reanimated the hope to solve the gradient instabilities in bigravity. And indeed, the eigenfrequencies (4.27) can be solved for the transition time t_* at which all scalar modes stabilize and yields for the $\beta_1\beta_2$ model [85]

$$H_*^2 = \pm\frac{\beta_2 m^2}{\sqrt{3}\alpha^2} + \mathcal{O}(\alpha'), \tag{4.52}$$

where $H_* \equiv H(t_*)$. This directly shows that the transition time can be pushed to arbitrarily early times by suitably lowering α.

It seems that the only way to evade unstable scalar modes is the use of the limit $\alpha \to 0$. Even though the complete non-linear bimetric theory will almost exactly look like GR, it keeps the advantage of solving the DE problem with a technically-natural effective CC.

The entire theory of two interacting spin-2 fields that generalizes massive gravity and GR, that contains so many free parameters, that allows the choice between various solutions on different branches – all this freedom seems to collapse to just one class of models: Those that look exactly like ΛCDM.

[5] During this procedure, one has to assume that at least two of β_i-parameters do not vanish, where $i \neq 0$.

Part II

Publications

Publication 1

Viable cosmological solutions in massive bimetric gravity

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JCAP 1403 (2014) 029, arXiv:1312.3208

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Viable cosmological solutions in massive bimetric gravity

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Received December 19, 2013
Accepted March 12, 2014
Published March 14, 2014

Abstract. We find the general conditions for viable cosmological solution at the background level in bigravity models. Furthermore, we constrain the parameters by comparing to the Union 2.1 supernovae catalog and identify, in some cases analytically, the best fit parameter or the degeneracy curve among pairs of parameters. We point out that a bimetric model with a single free parameter predicts a simple relation between the equation of state and the density parameter, fits well the supernovae data and is a valid and testable alternative to ΛCDM. Additionally, we identify the conditions for a phantom behavior and show that viable bimetric cosmologies cannot cross the phantom divide.

Keywords: modified gravity, cosmology of theories beyond the SM, dark energy theory

ArXiv ePrint: 1312.3208

JCAP03(2014)029

doi:10.1088/1475-7516/2014/03/029

Contents

1 Introduction

The discovery of cosmic acceleration has sparked a renewed interest in theories that go beyond standard gravity. Beside the possibility of explaining dark energy, the main motivation is to find new observationally testable features of gravity that allow one to test it beyond the narrow limits of the solar system.

It is possible to identify three main classes of models of modified gravity: based on additional scalar fields, vectors or tensors, respectively. The first one is perhaps the most studied one, owing to the similarity with inflation and to its simpli city. Even restricting oneself to single scalar fields with second order equation of motion, the class of possible Lagrangians, represented by the so-called Horndeski Lagrangian [8, 16], is however huge. In this paper we concern us with the third class, namely models that modify Einstein's gravity by introducing a massive term in the equations of motion.

The history of massive gravity is an old one, dating back to 1939, when the linear model of Fierz and Pauli was published. We refer to the review [15] for a reconstruction of the steps leading to the modern approach. The key point of these new forms of massive gravity is the introduction of a second tensor field, beside the metric. Such a theory of massive gravity was studied in [7] and was later shown to be free of ghosts [11]. Furthermore, the interaction of the two tensor fields creates a mixture of massless and massive gravitons that apparently avoid the appearance of ghosts [10].

In ref. [10, 12] the authors proposed to render the second tensor field dynamical, just as the standard metric, although only the latter is coupled to matter (for a generalization, see [1]). This approach, denoted bimetric gravity, keeps the theory ghosts-free and has the advantage of allowing cosmologically viable solutions. The cosmology of bimetric gravity has been studied in several papers, e.g. in refs. [2, 4–6, 19, 20]. The main conclusion is that bimetric gravity allows for a cosmological evolution that can approximate the ΛCDM universe and can therefore be a candidate for dark energy. For a criticism of these theories see e.g. ref. [9], whose conclusions are however apparently contradicted by the results in [14].

Bimetric gravity has been compared to background data, in particular supernovae Ia, in [2, 20], where confidence regions have been obtained for various cases. We will recover indeed several results already presented in [2]. We feel however that several interesting questions concerning the possibility of obtaining a viable cosmological evolution in bimetric

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models have not been fully addressed yet. Some of the questions that this paper addresses are: 1) for which values of the parameters and of the initial conditions does bimetric gravity allow for viable cosmologies? 2) For which values of the parameter there appear an effective phantom (i.e. an equation of state less than -1) behavior? 3) Can one find simple expressions for the parameters for which the supernovae data can be fitted?

We will find that in several cases these questions can be answered in a simple analytical way, providing a number of alternatives to ΛCDM. Interestingly, these alternative models do not reduce to ΛCDM for some values of the parameters (unless of course a cosmological constant is added as an additional parameter) and can therefore be ruled out by precise cosmological observations (if they are not yet ruled out!). In particular, we point out that a minimal bimetric model with a single free parameter predicts a simple relation between the equation of state and the density parameter, fits well the supernovae data and is a valid and testable alternative to ΛCDM.

The results of this paper provide a preliminary choice of well-behaved cosmological evolutions that can be further analyzed at the perturbation level. This task will be carried out in a companion paper.

2 Background equations

We start with the action of the form [10]

$$S = -\frac{M_g^2}{2}\int d^4x\,\sqrt{-\det g}\,R(g) - \frac{M_f^2}{2}\int d^4x\,\sqrt{-\det f}\,R(f) + m^2 M_g^2\int d^4x\,\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) + \int d^4x\,\sqrt{-\det g}\,L_m(g,\Phi) \tag{2.1}$$

where e_n are suitable polynomials and β_n arbitrary constants. Here $g_{\mu\nu}$ is the standard metric coupled to matter fields in the L_m Lagrangian, while $f_{\mu\nu}$ is a new dynamical tensor field. In the following we express masses in units of M_g^2 and the mass parameters m^2 will be absorbed into the parameters β_n. The action then becomes

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-\det g}\,R(g) - \frac{M_f^2}{2}\int d^4x\,\sqrt{-\det f}\,R(f) + \int d^4x\,\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) + \int d^4x\,\sqrt{-\det g}\,L_m(g,\Phi)\,. \tag{2.2}$$

Varying the action with respect to $g_{\mu\nu}$, one obtains the following equations of motion,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}\sum_{n=0}^{3}(-1)^n\beta_n\left[g_{\mu\lambda}Y^{\lambda}_{(n)\nu}\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) + g_{\nu\lambda}Y^{\lambda}_{(n)\mu}\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right)\right] = T_{\mu\nu} \tag{2.3}$$

where the expressions $Y^{\lambda}_{(n)\nu}\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right)$ are defined as, putting $X = \left(\sqrt{g^{-1}f}\right)$,

$$Y_{(0)}(X) = I, \tag{2.4}$$

$$Y_{(1)}(X) = X - I[X], \tag{2.5}$$

$$Y_{(2)}(X) = X^2 - X[X] + \frac{1}{2}I\left([X]^2 - [X^2]\right) \tag{2.6}$$

$$Y_{(3)}(X) = X^3 - X^2[X] + \frac{1}{2}X\left([X]^2 - [X^2]\right) - \frac{1}{6}I\left([X]^3 - 3[X][X^2] + 2[X^3]\right) \tag{2.7}$$

where I is the identity matrix and $[\ldots]$ is the trace operator.

Varying the action with respect to $f_{\mu\nu}$ we get

$$\bar{R}_{\mu\nu} - \frac{1}{2} f_{\mu\nu} \bar{R} + \frac{1}{2M_f^2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[f_{\mu\lambda} Y^{\lambda}_{(n)\nu} \left(\sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) + f_{\nu\lambda} Y^{\lambda}_{(n)\mu} \left(\sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) \right] = 0 \tag{2.8}$$

where the overbar indicates $f_{\mu\nu}$ curvatures. Under the rescaling $f \to M_f^{-2} f$, the Ricci scalar transforms as $\bar{R}(f) \to M_f^2 \bar{R}(f)$ which results in

$$\sqrt{-\det f}\bar{R}(f) \to M_f^{-2} \sqrt{-\det f} \bar{R}(f) \,. \tag{2.9}$$

Next to the Einstein-Hilbert term for $f_{\mu\nu}$, there is another term in the action that depends on $f_{\mu\nu}$ which transforms as

$$\sum_{n=0}^{4} \beta_n e_n \left(\sqrt{g^{-1} f} \right) \to \sum_{n=0}^{4} \beta_n e_n \left(M_f^{-1} \sqrt{g^{-1} f} \right) \,. \tag{2.10}$$

Since the elementary symmetric polynomials $e_n(X)$ are of order X^n, the rescaling of $f_{\mu\nu}$ by a constant factor M_f^{-2} translates into a redefinition of the couplings $\beta_n \to M_f^n \beta_n$ which allows us to assume $M_f = 1$ in the following.

We assume now a cosmological spatially flat FRW metric

$$ds^2 = a^2(t) \left(-dt^2 + dx_i dx^i \right) \tag{2.11}$$

where t represents the conformal time and a dot will represent the derivative with respect to it. The second metric is chosen as

$$f_{\mu\nu} = \begin{bmatrix} -\dot{b}(t)^2/\mathcal{H}^2(t) & 0 & 0 & 0 \\ & b(t)^2 & & \\ 0 & 0 & b(t)^2 & 0 \\ 0 & 0 & 0 & b(t)^2 \end{bmatrix} \tag{2.12}$$

where $\mathcal{H} \equiv \dot{a}/a$ is the conformal Hubble function. This form of the metric $f_{\mu\nu}$ ensures that the equations satisfy the Bianchi identities (see e.g. [12]).

Inserting $g_{\mu\nu}$ in eq. (2.3) we get

$$3\mathcal{H}^2 = a^2 \left(\rho_m + \rho_{\rm mg} \right) \tag{2.13}$$

where the massive gravity energy density is

$$\rho_{\rm mg} = B_0 \equiv \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \tag{2.14}$$

with

$$r = \frac{b}{a} \tag{2.15}$$

The matter energy density follows the usual conservation law

$$\dot{\rho}_m + 3\mathcal{H}\rho_m = 0 \,. \tag{2.16}$$

Notice that although we do not consider explicitly a radiation epoch (since we confine ourselves to observations at low redshifts), a radiation component could be easily added to the pressureless matter and would not change qualitatively any of the conclusions below. We can also define

$$\Omega_{\rm mg} = \frac{\rho_{\rm mg}}{\rho_m + \rho_{\rm mg}} = 1 - \Omega_m \tag{2.17}$$

where $\Omega_m = \rho_m/(\rho_m + \rho_{\rm mg})$.

Similarly, the background equation for the f metric is

$$\mathcal{H}^2 = \frac{a^2}{3r} B_1 \tag{2.18}$$

if $B_1 \neq 0$ (and $\dot{b} = 0$ if $B_1 = 0$) where

$$B_1 = \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 \,. \tag{2.19}$$

Combining (2.13) and (2.18), differentiating and inserting (2.16) we obtain the constraint

$$\dot{b} = -\frac{\left(4\beta_0 + 9\beta_1 r + 6\beta_2 r + \beta_3 r^2\right)\dot{a}}{3B_2} \tag{2.20}$$

where

$$B_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2 \,. \tag{2.21}$$

The background equations can be conveniently written as a first order system for r and $\mathcal{H}$, where the prime denotes the derivative with respect to $N = \log a$:

$$2\mathcal{H}'\mathcal{H} + \mathcal{H}^2 = a^2\left(B_0 + B_2 r'\right) \,, \tag{2.22}$$

$$r' = \frac{3rB_1\Omega_m}{\beta_1 - 3\beta_3 r^2 - 2\beta_4 r^3 + 3B_2 r^2} \,, \tag{2.23}$$

$$\Omega_m = 1 - \frac{B_0}{B_1} r \tag{2.24}$$

(the r' equation has been first obtained in ref. [2]). We can define the effective equation of state

$$w_{\rm eff} \equiv \Omega_{\rm mg} w_{\rm mg} = -\frac{1}{3}\left(1 + 2\frac{\mathcal{H}'}{\mathcal{H}}\right) = -\frac{r\left(B_0 + B_2 r'\right)}{B_1} \tag{2.25}$$

$$= -1 + \Omega_m - \frac{B_2 r r'}{B_1} \,. \tag{2.26}$$

Eq. (2.23) is particularly useful for our discussion below. Notice that it can be written also as

$$r' = -\frac{3\rho_m}{\rho_{m,r}} \tag{2.27}$$

where $\rho_{m,r}$ denotes differentiation with respect to r of the function

$$\rho_m(r) = \frac{B_1}{r} - B_0 \tag{2.28}$$

obtained by combining eqs. (2.13) and (2.18).

It is convenient from now on to express the β parameters in units of H_0^2 and H in units of H_0.

3 Conditions for cosmological viability

Several possible branches of the solution of eq. (2.23) are possible, depending on the initial condition for r. We distinguish in the following between *finite* branches, that are confined within two successive roots or poles of r', and *infinite* branches, which can extend to infinite values of r. We define now a viable cosmological solution one in which the following conditions are satisfied: a) $\rho_m > 0$ and $\rho_{\rm mg}$ not identically zero, b) a monotonic expansion, i.e. $\rho_m + \rho_{\rm mg} > 0$, c) the evolution in the asymptotic past is dominated by ρ_m, i.e. $\rho_m(N \to -\infty) \to \infty$, $\Omega_m(N \to -\infty) = 1$ (and therefore $w_{\rm eff}(N \to -\infty) = 0$), d) no singularities in r' at finite times and e) $r \geq 0$ at all times. Violations of these conditions do not necessarily imply contradiction with observable data if they occur outside the observable range and could in principle be lifted or relaxed. However, when they are satisfied the cosmological evolution is much safer, simpler and requires no special tuning. Most of what follows is devoted to determining the conditions under which cosmological viable solutions take place.

Combining these conditions and analyzing eq. (2.23) yields the following properties of viable models:

1. All viable models except $\beta_i = 0 \,\forall\, i > 0$, i.e. the ΛCDM case, must fulfill $r' \to -\infty$ as $r \to \infty$. To see this, we use eq. (2.23) to find that models in which we can not observe this limit need to satisfy $\beta_1 = \beta_3 = 0$ and $\beta_2 = \frac{1}{3}\beta_4$. With this choice, the combination of eq. (2.22) and the background equation (2.18) together with its derivative yields

$$\sqrt{\beta_2\,(1 + r^2)}\,(\beta_0 - 3\beta_2) = 0 \tag{3.1}$$

   which provides the constraint $\beta_0 = 3\beta_2$. But this corresponds to a vanishing matter density ρ_m which is not viable. Note that the choice of parameters $\beta_1 = \beta_3 = 0$ and $\beta_0 = 3\beta_2 = \beta_4$ matches with those of the partially massless bimetric theory which was studied in [13]. However, in those theories the authors assumed the reference metric to be proportional to $g_{\mu\nu}$ which is explicitly avoided in this work due our choice of the Bianchi constraint.

2. If a viable range in r is infinite then, as just shown, r decreases with time since the limit $r' \to -\infty$ as $r \to \infty$ must hold. Then $r \to \infty$ corresponds to the infinite past and therefore, if this branch is viable, then it needs to satisfy $\lim_{r\to\infty} \Omega_m = 1$. With eq. (2.24) one finds that a viable solution with an infinite range in r requires $\beta_2 = \beta_3 = 0 \neq \beta_4$. Moreover, β_4 is enforced to be positive in order to produce a positive expansion rate at early times.

3. A non-vanishing massive gravity part, i.e. $B_0 \neq 0$, always implies that if there is a root $r = 0$, then for this root, and only for this one, $\Omega_m = 1$. For all other roots we need $\Omega_m = 0$ in order to fulfill eq. (2.23).

4. Let $r \in (r_1, r_2)$ be a branch with $r'|_{r_1} = r'|_{r_2} = 0$ for r_1, r_2 strictly positive. As seen before, a root at $r > 0$ corresponds to $\rho_m = 0$. For a non-constant evolution of the matter density, the mean value theorem always provides a $\bar{r} \in (r_1, r_2)$ with $\rho_{m,r} = 0$ causing a singularity in r'. Since eq. (2.28) shows that the matter density can not become divergent at a finite and non-zero r, a viable model always evolves from either $r = 0$ or $r = \infty$ to a root of ρ.

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5. We will find that $r = 0$ always corresponds to the asymptotic past. If it would instead describe a final state, then a vanishing ρ_m as $N \to \infty$ (which has to hold since the matter density follows the usual conservation rule) needs $\beta_1 = 0$ and $\beta_0 = 3\beta_2$. Additionally, this requires $\beta_3 > 0$, otherwise we have either a negative β_3 which means that the density is not positive or $\beta_3 = 0$ in which the branch would be infinite with $\rho_{\rm mg} = 0$, i.e. $\Omega_m = 1$, at all times. However, we then obtain a finite branch between two roots of $\rho(r)$ at $r = 0$ and $r_c > 0$ but we already concluded in point 4 that $r = 0$ must then correspond to the asymptotic past.

6. The previous conclusions imply for all viable cases an evolution from $\Omega_m = 1$ to the final state $\Omega_m = 0$, just like ΛCDM.

7. We can use eq. (2.23) to find that there is always a root at $r = 0$ for non-vanishing β_1. All models without a root at $r = 0$ need to satisfy

$$\lim_{r\to 0} \Omega_m\big|_{\beta_1=0} = 1 - \lim_{r\to 0} \frac{\beta_0 + 3\beta_2 r^2 + \beta_3 r^3}{3\left(\beta_2 + \beta_3 r\right) + \beta_4 r^2} = 1 \tag{3.2}$$

In this case, viability enforces $\beta_0 = 0$. Models with a pole at $r = 0$ need to satisfy $\beta_1 = \beta_3 = 0$ with $\beta_2 \neq \frac{1}{3}\beta_0$ and must fulfill

$$\lim_{r\to 0} w_{\rm eff}\big|_{\beta_0=\beta_1=\beta_3=0} = -\frac{3\beta_2}{3\beta_2 - \beta_4} = 0\ . \tag{3.3}$$

This contradicts the condition $\beta_2 \neq 0$. If $r = 0$ is neither a root nor a pole, then from eq. (2.23) we see that this corresponds to $\beta_3 \neq 0$ and $\beta_0 \neq 3\beta_2$ (note that this implies $\beta_2 \neq 0$) instead. However, the resulting matter density

$$\rho_m\Big|_{\beta_0=\beta_1=0} = 3\beta_2 + 3\beta_3 r + \left(\beta_4 - 3\beta_2\right) r^2 - \beta_3 r^3 \tag{3.4}$$

violates the requirement of a divergent density for $r \to 0$. Therefore, every viable branch that evolves from $r = 0$ must satisfy $r'|_{r=0} = 0$.

8. If r evolves from $r = 0$, then a positive H^2 at early times implies $\beta_k > 0$ where β_k denotes the non-vanishing β-parameter with the smallest index $k \neq 0$.

9. A model which produces two viable branches has to satisfy $\beta_1 \geq 0$ and $\beta_4 > 0$, in order to produce positive Hubble functions in both branches.

10. From eq. (2.26) we find that the equation of state always evolves from $w_{\rm eff} = 0$, as required from the conditions of viability, to $w_{\rm eff} = -1$ on a viable solution. Notice that $w_{\rm eff} = -1$ even for a vanishing explicit cosmological constant $\beta_0 = 0$.

    Depending on the number of non-negative roots, we therefore find that several cases can not be viable:

11. The number of non-negative roots can be zero only if $\beta_1 = \beta_2 = \beta_3 = 0$, which leads to

$$r' = \frac{3\left(\beta_0 - \beta_4 r^2\right)}{2\beta_4 r}\ . \tag{3.5}$$

As already remarked, a viable model must therefore evolve from $r = \infty$ to $r = 0$ since $r' < 0$ for $r \to \infty$ and this requires a positive and non-zero β_4. However, this produces a singular r' at $r = 0$ (unless $\beta_0 = 0$ but we are now only interested in models with no positive roots) which was already shown to be non-viable.

12. A model that has at least one positive root and does not have a root at $r = 0$ may only be able to produce a viable infinite branch. A finite but non-zero r' at $r = 0$ can not be achieved with a vanishing β_3 but this is enforced by the criteria of viable infinite branches (see point 2). Thus, all models with only non-zero roots must fulfill $\beta_1 = \beta_2 = \beta_3 = 0 < \beta_4, \beta_0$. With e.g. Descartes' rule of sign we see that we can not expect more than one positive root. Whenever there is a model with at least two positive roots producing a viable branch, there must be one root at $r = 0$.

13. If there is only one root $r = 0$, then this root is reached in the asymptotic future, i.e. for $N = \infty$, since the range must be infinite. This contradicts the previous conclusion that $r = 0$ has to correspond to the asymptotic past. Therefore, no viable cosmologies exist if there is only one root at $r = 0$.

14. If there are $n \geq 2$ positive roots at $r_{c_1}, \ldots, r_{c_n}$, where $r_{c_i} < r_{c_j}$ for $i < j$, then only the two branches $r \in (0, r_{c_1})$ and $r \in (r_{c_n}, \infty)$ may be viable.

15. Models with two viable branches require $\beta_2 = \beta_3 = 0$ and $\beta_1, \beta_4 > 0$. Descartes' rule of sign then shows that those models must have exactly two positive roots.

16. With, again, Descartes' rule of sign we find that there is no model with $\beta_2 = \beta_3 = 0$ that produce three positive roots. For this reason, we can not expect any viable infinite branch in models with three positive roots.

Finally, we can employ these results to show that several simple models do not produce viable solutions:

- Consider models in which only one β-parameter does not vanish. Let's call them β_i models. Then only β_0 or β_1 models may produce viable solutions. This first one is not surprising since it is equivalent to a ΛCDM universe. For all the other β_i models, we find

$$r'\Big|_{\beta_i=0, i\neq 2} = -\frac{3\left(r^2-1\right)}{2r}, \qquad r'\Big|_{\beta_i=0, i\neq 3} = -\frac{r\left(r^2-3\right)}{r^2-1}, \qquad r'\Big|_{\beta_i=0, i\neq 4} = -\frac{3}{2}r\,. \tag{3.6}$$

  The infinite branch in β_2 or β_3 models can not be viable. In addition, their finite branches suffer from a pole in r'. Therefore, we can not expect any viable solutions. These arguments do not hold for the β_4 model. However, we already concluded (see point 13) that a model with only one root at $r = 0$ is not viable.

- In a more general case, in which two free β-parameters are allowed to vary (let's denote them $\beta_i\beta_j$ models), we will find that only the combination involving β_0 or β_1 are generally able to produce viable solutions. To see that the models $\beta_2\beta_3$, $\beta_2\beta_4$ and $\beta_3\beta_4$ can not be viable, we first assume that both couplings in all three combinations do not vanish, otherwise we would obtain non-viable minimal models. This also rejects the possibility of viable models with an infinite branch in these cases. In the $\beta_2\beta_3$ model, the matter density evolves like

$$\rho_m = 3\left(\beta_2 + \beta_3 r - \beta_2 r^2\right) - \beta_3 r^3\,, \tag{3.7}$$

  and is therefore finite at $r = 0$, which contradicts condition c). In fact this solution can be continued to negative r, which implies that $|b|$ reaches zero and increases again.

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This is therefore a bouncing cosmology which is interesting on its own but violates our viability condition and we leave its study to future work. For the $\beta_2\beta_4$ model we have already shown that only a finite branch $(0, r_c)$ could be viable. Simplifying eq. (2.23) yields

$$r'\Big|_{\beta_i=0, i\neq 2,4} = -\frac{3}{2}r + \frac{3\beta_2}{2r\,(3\beta_2 - \beta_4)} \,. \tag{3.8}$$

This exhibits a pole at $r = 0$ which indicates non-viability. To analyze the $\beta_3\beta_4$ models, we again use eq. (2.23) which directly shows that we need to have $\beta_3 \neq 0$ in order to get a positive root. In this case, the only positive root is given by

$$r_c = \frac{\beta_4 + \sqrt{12 + \beta_3^2 + \beta_4^2}}{2\beta_3} \,. \tag{3.9}$$

In addition, we will find that r' is singular at

$$r_s = \frac{\beta_4 + \sqrt{9\beta_3^2 + \beta_4^2}}{3\beta_3} \,. \tag{3.10}$$

Since $\beta_3 \neq 0$, only the branch $(0{,}r_c)$ could be viable and therefore either $r_s < 0$ or $r_s > r_c$ must hold. Notice that $r_s = 0$ is not viable. Both relations require $\beta_3 < 0$. However, a positive Hubble function enforces $\beta_3 > 0$ which shows that the branch $(0, r_c)$ always contains a singularity in r'. We therefore conclude that models with $\beta_0 = \beta_1 = \beta_2 = 0$ are not able to produce viable solutions.

- The subset of cosmological solutions with an infinite range in r and without an explicit cosmological constant is described by the relation $\beta_0 = \beta_2 = \beta_3 = 0 < \beta_4$ together with $\beta_1 \neq 0$. For these models, we obtain

$$\Omega_{m,r} = \frac{3\beta_1 r\left(-2\beta_1 + \beta_4 r^3\right)}{\left(\beta_1 + \beta_4 r^3\right)^2} \tag{3.11}$$

from which we see that Ω_m increases with time when the following condition holds:

$$\Omega_{m,r} < 0 \iff \left(\beta_1 < 0 \;\wedge\; \beta_1 + \beta_4 r^3 \neq 0\right) \;\vee\; \left(\beta_1 > 0 \;\wedge\; r < \left(\frac{2\beta_1}{\beta_4}\right)^{\frac{1}{3}}\right) \,. \tag{3.12}$$

Viable models are therefore only possible if $\beta_1 > 0$. In addition, the solution r_c of the equation

$$\Omega_m = 1 - \frac{3\beta_1 r_c^2}{\beta_1 + \beta_4 r_c^3} = 0 \tag{3.13}$$

is negative (or zero but this, as already discussed, does not correspond to a viable solution) if $\beta_4 > 2\beta_1$. This shows that only models with $\beta_0 = 0$ satisfying $\beta_0 = \beta_2 = \beta_3 = 0 < \frac{1}{2}\beta_4 \leq \beta_1$ are able to produce viable branches (r_c, ∞).

- A simple model with all identical couplings, i.e. $\beta_0 = \beta_i = \hat{\beta}$, needs $\hat{\beta} > 0$ in order to produce a positive expansion rate. The matter density

$$\rho_m = -\hat{\beta}\frac{(r-1)(r+1)^3}{r} \tag{3.14}$$

then shows that only the finite branch $(0,r_c)$ with $r_c = 1$ could be viable. Additionally, the Hubble function at present time

$$\frac{1}{3}\hat{\beta}\,(1+r_0)^3\,r_0^{-1} = 1 \tag{3.15}$$

is only solved by a purely real and positive present value r_0 if $\hat{\beta} \leq \frac{4}{9}$.

In practice, to see if a viable solution exists, one first has to find all positive solutions r_0 that fulfill both Friedmann equations (2.13) and (2.18) at present time. One then needs to check whether the branches $r \in (0, r_{c_1})$ and $r \in (r_{c_n}, \infty)$, where r_{c_1} and r_{c_n} denote the smallest and largest strictly positive root of $\rho_m(r)$, respectively, contain r_0 and, finally, ensure that those branches do not contradict the criteria of viability. In general, one can show that a finite branch between two roots $(0, r_c)$ with $0 < r_0 < r_c$ in which r' is positive and does not have any pole is always viable if the matter density is positive in this range. This provides a very simple recipe to find viable cosmologies without solving the evolution equations.

It is also interesting to provide the general conditions for a phantom ($w_{\rm mg} < -1$) solution to appear. From $w_{\rm eff}$ we see that

$$w_{\rm mg} = -1 - \frac{B_2 r r'}{\Omega_{\rm mg} B_1}\;. \tag{3.16}$$

Combining with eq. (2.24) we obtain

$$w_{\rm mg} = -1 - \frac{B_2}{B_0} r'\;. \tag{3.17}$$

Near the de Sitter final state we can assume $\Omega_m \to 0$ and therefore $B_0 = B_1/r$ from eq. (2.24). This implies

$$w_{\rm mg} \approx -1 - \frac{B_2}{B_1} r r'\;. \tag{3.18}$$

In a viable branch with a finite range in r, both r and r' are positive. If the range is infinite, then r' is negative. In addition, B_1 is always positive due to eq. (2.18). We conclude that a necessary and sufficient condition for a phantom equation of state is $B_2 > 0$ for a finite branch $(0, r_{c_1})$. If the branch is infinite, then a phantom requires $B_2 < 0$ which results in $\beta_1 < 0$ since viable models in infinite branches need to fulfill $\beta_2 = \beta_3 = 0$. From eq. (2.20) we notice that B_2 cannot be zero in a viable region of r and therefore $w_{\rm mg}$ cannot cross the -1 line. This shows that every viable bigravity cosmology is either phantom or non-phantom throughout its evolution. Conversely, finding a phantom crossing would rule out the entire class of viable bimetric cosmologies.

We chose two representative models to sketch a possible viable evolution of a bimetric gravity in figures 1 and 2. The model A, described by $\beta_i = \left(1, \frac{1}{5}, 0, 0, 1\right)$, produces two viable branches. Although Ω_m and $w_{\rm eff}$ evolve similarly in both branches, we find a phantom equation of state only in the finite one. An one-parameter model $\beta_0 = \beta_i = \hat{\beta}$, such as model B with $\hat{\beta} = \frac{4}{9}$, is only able to produce a viable finite branch. Those models always produce a phantom since a positive expansion rate requires $\hat{\beta} > 0$.

4 Comparing to supernovae Ia Hubble diagram

To compare the background evolution of bimetric gravity models with observed SNe Ia, we use the SCP Union 2.1 Compilation [18] containing 580 SNe Ia. For each observed SN Ia we

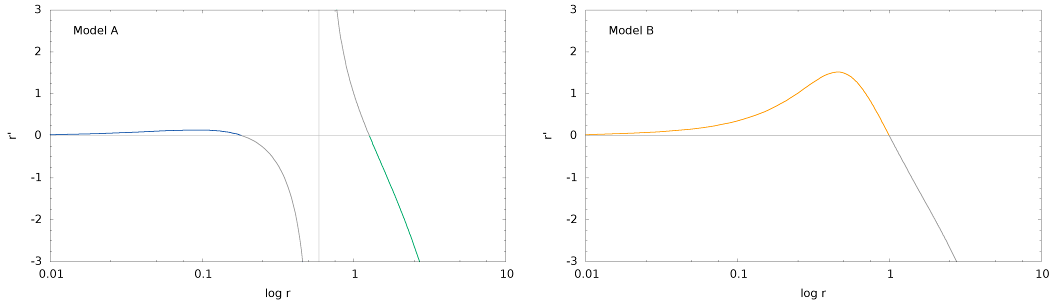

Figure 1. The evolution of $r'(r)$ corresponding to the models A and B with $\beta_i = \left(1, \frac{1}{5}, 0, 0, 1\right)$ (left) and $\beta_i = \frac{4}{9}$ (right), respectively, visualizing all possible branches. The first model contains two finite ($\sim (0, 0.2)$ and $\sim (0.2, 1.30)$) and one infinite branch ($\sim (1.3, \infty)$). However, only the first and third branch may be viable which, indeed, turns out to be the case. On the contrary, the one-parameter model B only produces one viable branch $(0, 1)$ with $r_0 = \frac{1}{2}$, though r' seems to evolve viable even in the infinite branch.

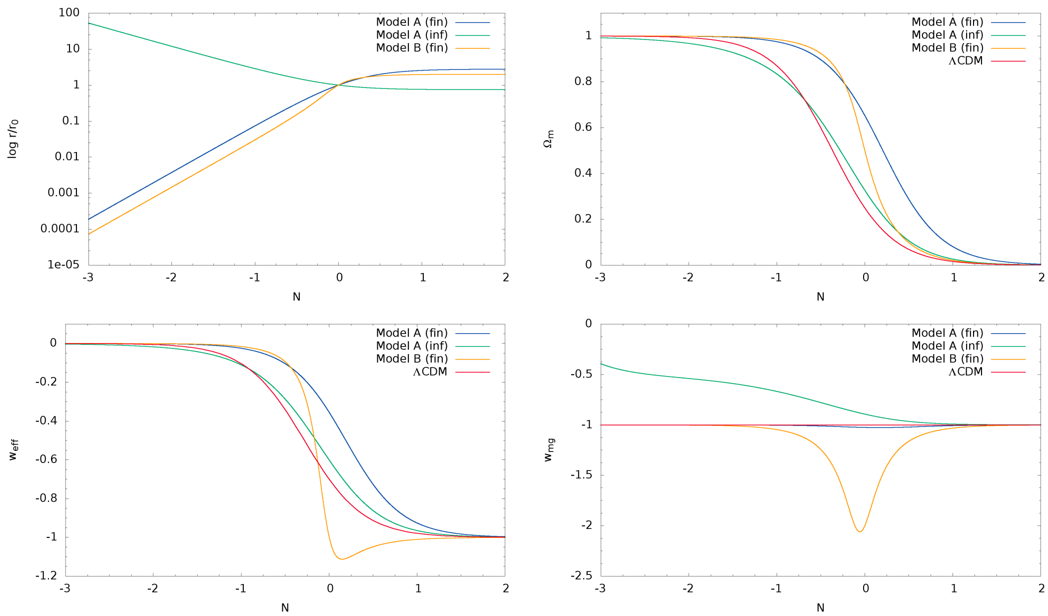

Figure 2. A comparison of r (top left), Ω_m (top right), $w_{\rm eff}$ (bottom left) and $w_{\rm mg}$ (bottom right) of all viable branches in the models A (blue and green) and B (yellow) whose evolution of r' were already discussed in figure 1. Additionally, the latter three plots contain the ΛCDM expectation for $\Omega_m = 0.3$.

can use the measured maximum magnitude in the B-band $m_B^{\max}$ together with the stretch correction s and the color correction c to compute the likelihood for a bimetric model θ with

$$L(\theta) \propto \int \exp\left(-\sum_{i=1}^{N} \frac{(\mu_i - \mu_{\rm theo})^2}{2\sigma_i^2}\right) \mathrm{d}M\, \mathrm{d}\alpha\, \mathrm{d}\beta\ , \tag{4.1}$$

where α and β are nuisance parameters which weight the stretch- and color correction and M denotes the absolute magnitude,

$$\mu_i = m_{B_i}^{\max} - M + \alpha\,(s_i + 1) - \beta c_i\ . \tag{4.2}$$

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The marginalization over M can be performed analytically, whereas we simplify the computation by using the values for α and β that minimize $\chi^2_{\rm red}$ instead of computing the marginalization numerically. In addition, we add an intrinsic dispersion which we assume to be $\sigma_{\rm int} = 0.1345\,\text{mag}$ in order to obtain a $\chi^2_{\rm red} = 1$ for the best fit in a ΛCDM cosmology.

We decided not to use other cosmological datasets like baryon acoustic oscillations because they are at the moment far weaker than SN Ia and CMB peak positions because their analysis depends on various assumptions which are not warranted in a non-standard model as the one we explore here. Nevertheless, our results are in agreement with ref. [2], where these additional datasets have been employed.

5 Minimal models: 1-parameter models

It is very instructive to study in detail some simple subset among all the possible viable cosmologies. During our analysis we will mostly assume a vanishing explicit cosmological constant, i.e. $\beta_0 = 0$ (the only model with a non-vanishing cosmological constant that is studied in this work will be the 1-parameter model $\beta_0 = \beta_i = \hat{\beta}$). This subset of models is a very interesting one since those models may fit observational data without the need of a cosmological constant. In this section we assume moreover that all of the other β_i vanish, except one, i.e. we restrict ourselves to β_i models. In this case, as already shown, only one possibility, the β_1 model, turns out to be viable. In terms of simplicity, this is the minimal bimetric model, so it can help us gaining intuition on the behavior of this class of models. This model was already studied and compared to the SN Ia data in [2]; the same paper excludes the other β models on the ground of their poor fit to data. The β_1 model is interesting also because r can be easily solved analytically. Its evolution follows from eq. (2.23),

$$r' = \frac{3r\left(1-3r^2\right)}{1+3r^2} \,. \tag{5.1}$$

Note that the evolution of r does not depend on β_1. In terms of the scale factor, the solution reads

$$r(a) = \frac{1}{6}a^{-3}\left(-A \pm \sqrt{12a^6 + A^2}\right) \,. \tag{5.2}$$

To determine the constant A, we use the background equation (2.13) at current time which provides $r_0 = \frac{1-\Omega_{m0}}{\beta_1}$ and therefore

$$A = \frac{3(\Omega_{m0}-1)^2 - \beta_1^2}{\beta_1\left(\Omega_{m0}-1\right)} \,. \tag{5.3}$$

Depending on β_1 and Ω_{m0}, both a negative and positive A is possible. To satisfy $r(a\to 0) = 0$, we need to choose the positive sign in eq. (5.2) if A is positive, or the negative sign in case of a negative A. The comparison with the SNIa Hubble diagram shows that A has to be positive (see below).

With this result, the equation of state and Ω_m are fully described through

$$\Omega_m(a) = -\frac{1}{6}Aa^{-6}\left(A \mp \sqrt{12a^6 + A^2}\right) \,, \tag{5.4}$$

$$w_{\rm eff}(a) = \pm\frac{A}{\sqrt{12a^6 + A^2}} - 1 \,, \tag{5.5}$$

$$w_{\rm mg}(a) = \mp\frac{A}{\sqrt{12a^6 + A^2}} - 1 \,. \tag{5.6}$$

Thus, in the β_1 viable minimal model, the equation of state always evolves from -2 to -1. These equations imply a simple and testable relation between $w_{\rm mg}$ and Ω_m valid at all times during matter domination:

$$w_{\rm mg} = \frac{2}{\Omega_m - 2} \, . \tag{5.7}$$

In general, denoting with a subscript 0 the present time, the following conditions must be satisfied by any model:

$$\Omega_{m0} = 1 - \frac{B_0(r_0)}{B_1(r_0)} r_0 \, , \tag{5.8}$$

$$1 = \frac{B_1(r_0)}{3r_0} \tag{5.9}$$

(the last one is obtained from eq. (2.18) after expressing the βs in units of H_0^2). In particular, for the β_1 minimal model we obtain then a direct relation to the present value of the matter fractional density, $\beta_1 = \sqrt{3(1-\Omega_{m0})}$ which yields

$$A = \frac{\sqrt{3}\Omega_{m0}}{\sqrt{1-\Omega_{m0}}} \, . \tag{5.10}$$

We fitted the β_1 model to the SN Union 2.1 catalog (see figure 3) and obtained $A \approx 0.8$ for the best fit. The most likely values for β_1 and Ω_{m0} are summarized in table 1. We list also the present value of the equation of state expressed using the simple CPL parametrization [3, 17]

$$w(a) = w_0 + w_a(1-a) \tag{5.11}$$

in order to provide a quick comparison to present and future cosmological data.

The β_1 model is then a valid alternative to ΛCDM in terms of simplicity, and although it does not reduce to ΛCDM in any limit, it gives a good fit to the background data.

A second type of minimal models is described by identical couplings $\beta_0 = \beta_i = \hat{\beta}$. As noted earlier, only those models with $0 < \hat{\beta} \leq \frac{4}{9}$ produce one viable finite branch. The evolution of r, described by

$$r' = \frac{3r\left(1-r^2\right)}{1-2r+3r^2} \, , \tag{5.12}$$

has an analytical solution, though it is much more complicated than in a minimal model with only one non-vanishing coupling. However, the matter density parameter follows the simple relation

$$\Omega_m = 1 - r \tag{5.13}$$

which, just like r', is independent of $\hat{\beta}$. Of course, the present value r_0 is a function of $\hat{\beta}$. Again, we can use the set of equations (5.9) to obtain a relation between $\hat{\beta}$ and Ω_{m0},

$$\hat{\beta} = \frac{3\left(\Omega_{m0}-1\right)}{\left(\Omega_{m0}-2\right)^3} \, . \tag{5.14}$$

We found that both types of minimal models are only able to produce viable branches if the coupling parameters are positive and r_0 is located in a finite branch. Then eq. (3.17) directly implies that all these minimal models are described by a phantom equation of state at any time. A comparison of both minimal models with observed SNe Ia yields the likelihoods in figure 3 which provide the best fits listed in table 1. Their equation of state is plotted in figure 5.

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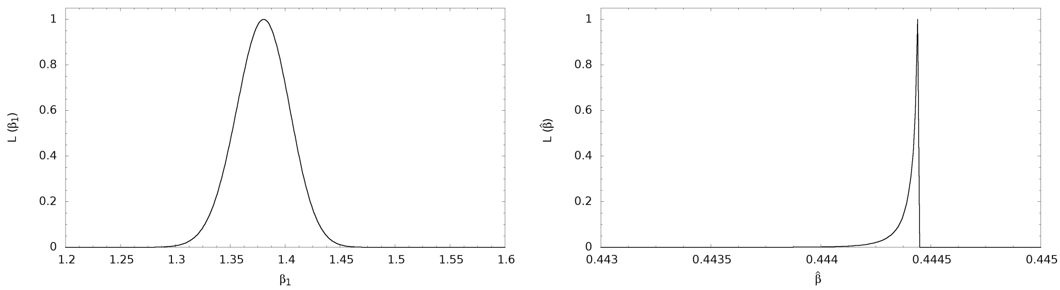

Figure 3. Likelihood for the coupling parameter in the minimal β_1 (left) and $\hat{\beta}$ model (right). The maxima of the likelihoods were rescaled to unity. Note that the $\hat{\beta}$ model produces non-viable solutions for $\hat{\beta} > \frac{4}{9}$.

	$\beta(\Omega_{m0})$	$\chi^2_{\rm min}$	β_1 or $\hat{\beta}$	Ω_{m0}	w_0	w_a
β_1	$\sqrt{3(1-\Omega_{m0})}$	578.3	$1.38^{+0.03}_{-0.03}$	$0.37^{+0.02}_{-0.02}$	$-1.22^{+0.02}_{-0.02}$	$-0.64^{+0.05}_{-0.04}$
$\hat{\beta}$	$\frac{3(\Omega_{m0}-1)}{(\Omega_{m0}-2)^3}$	606.3	$0.44^{+0.00}_{-0.01}$	$0.50^{+0.01}_{-0.00}$	$-2.00^{+0.00}_{-0.01}$	$-1.97^{+0.07}_{-0.00}$

Table 1. Best fit values for the two minimal models. The column $\beta(\Omega_{m0})$ lists the relation between the value of the coupling parameter β_1 and $\hat{\beta}$, respectively, and the present matter density parameter. The parameters w_0 and w_a describe the CPL parametrization at present time.

6 Two-parameter models

We move now to models in which all β_i vanish except two, taken in turn to be all possible combinations (we keep $\beta_0 = 0$). As already shown, we need to exclude all cases in which $\beta_1 = 0$ since we do not expect any viable models.

To compute the likelihood for $\Omega_{m,0}$, we divide the range in $\Omega_{m,0}$ in bins B_k of constant width and marginalize the likelihood over both β-parameters with the restriction $\Omega_{m,0} \in B_k$. Our results are summarized in figure 4 where the left plots show the 68%, 95% and 99.7% confidence regions in the $\beta_i - \beta_j$ plane, the corresponding likelihoods for $\Omega_{m,0}$ are illustrated in the right column. In all cases that are shown in figure 4, we found bimetric gravity models which are consistent with observed SNe Ia. We always observe a strong degeneracy between the two free parameters, as already remarked in ref. [2].

As in the minimal cases, the system (5.9) gives a relation between pairs of β and Ω_{m0}:

1. For $\beta_1\beta_2$:

$$\beta_2 = \frac{\beta_1^2 + \sqrt{\beta_1^4 - 9\beta_1^2\Omega_{m0} + 9\beta_1^2}}{9\left(\Omega_{m0} - 1\right)} + 1\,. \tag{6.1}$$

2. For $\beta_1\beta_3$:

$$\beta_3 = \frac{-32\beta_1^3 \pm \sqrt{\left(8\beta_1^2 + 27\left(\Omega_{m0}-1\right)\right)^2\left(16\beta_1^2 - 27\left(\Omega_{m0}-1\right)\right)} - 81\beta_1\left(\Omega_{m0}-1\right)}{243\left(\Omega_{m0}-1\right)^2}\,, \tag{6.2}$$

where the positive sign should be taken if $\beta_1 < \frac{3}{2}\sqrt{\frac{3}{2}}\sqrt{1-\Omega_{m0}}$ and the negative one otherwise.

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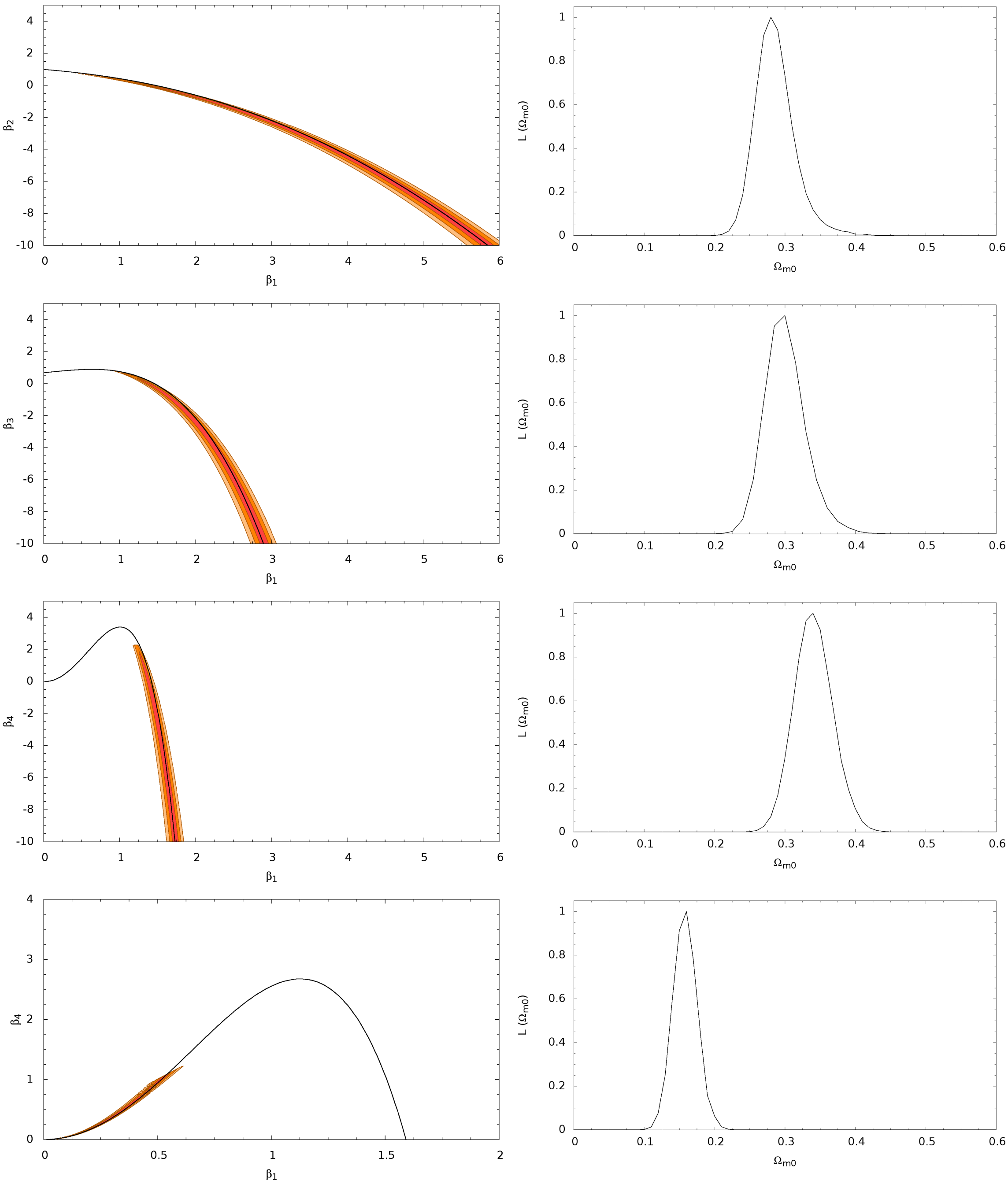

Figure 4. Left: likelihoods from observed SNe Ia with only two β-parameter varying while all other β_i vanish. In $\beta_1\beta_4$ models we distinguish between finite (plots in third row) and infinite (last row plots) branches. The filled regions correspond to the 68% (red), 95% (orange) and 99.7% (yellow) confidence level. In each two-dimensional likelihood, the analytic result $\beta_j(\beta_i, \Omega_{m0})$ is illustrated by a black solid line and corresponds to the most likely value Ω_{m0}. Right: likelihood for $\Omega_{m,0}$ obtained after a marginalization over the β parameters corresponding to the likelihoods on the left side.

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In all cases Ω_{m0} should be taken as the best fit value. The $\beta_1\beta_4$ model does not have a simple analytic solution but the relation is easily solved numerically. These relation are plotted in the same figure 4; as one can see, they fit very well the degeneracy curves.

At 1σ, the relative error Δ on the fitted $\beta_j(\beta_i, \Omega_{m0})$, with $(\beta_i < \beta_j)$, are given in table 4, where we determined the error by fitting the 68% contour with $\beta'_j = \beta_j(1+\Delta)$. For the best fit in all analyzed combinations, we show the evolution of the equation of state $w_{\rm mg}$ in figure 5 and the distance moduli $\mu(z)$ in comparison with the measured SNe Ia of the Union 2.1 Compilation in figure 6.

Note that the analytic fit does not always need to correspond to a viable solution since it ignores the condition $0 < r_0 < r_c$ and $r_c < r_0$ in the finite and infinite branch, respectively. We therefore need to exclude some parameter regions. As an example, we analyze all $\beta_1\beta_3$ models with positive β_3 and obtain

$$r_0 = \frac{3 \pm \sqrt{9 - 12\beta_1\beta_3}}{6\beta_3} \quad \text{and} \quad r_c = \pm\sqrt{\frac{-3\left(\beta_1 - \beta_3\right) \pm \sqrt{9\left(\beta_1^2 + \beta_3^2\right) - 14\beta_1\beta_3}}{2\beta_3}}\,. \tag{6.3}$$

A necessary condition to satisfy the relation $0 < r_0 < r_c$ is

$$\beta_3 < \frac{1}{243}\left(81\beta_1 - 32\beta_1^3 + \sqrt{\left(27 + 16\beta_1^2\right)\left(-27 + 8\beta_1^2\right)^2}\right) \tag{6.4}$$

which excludes most of the models with $\beta_3 > 0$. Similar boundaries of the coupling parameter corresponding to the highest order interaction exist in the $\beta_1\beta_2$ and $\beta_1\beta_3$ models, too.

Only the model $\beta_1\beta_4$ is able to produces infinite branches. The likelihoods in figure 4 for finite and infinite branches show that there is no parameter region in which the contours of both likelihoods overlap. If there is a $\beta_1\beta_4$ model in which two viable branches co-exist, then at least one branch is strongly disfavored by SNe Ia observations.

7 Conclusions

In this paper we studied a class of bimetric gravitational models that have been shown to be ghost-free and to induce cosmological acceleration. We define a viable cosmology as one in which the cosmic evolution broadly resembles the standard one, without bounces, singularities at finite time, and with a matter (or radiation) dominated past. Adopting spatially flat metrics we find that the system becomes effectively unidimensional and in some cases even analytical. This allows us to find a number of simple rules for viability which selects a subset of models and initial conditions. We show that if a branch is viable, then its final state is always deSitter. We also find the analytical condition for the occurrence of a phantom phase and we remark that observing a phantom crossing would rule out the entire class of viable bimetric models.

Then we show that among the models with only a single non-zero parameter, only one gives a viable cosmology, which well reproduces the SN data and can be taken as a simple, distinguishable alternative to ΛCDM. The relation (5.7) provides a stringent test for this minimal model. For models with two coupling constants and without a cosmological constant, only three cases produce a viable cosmology. In several cases we find also an analytic expression for the background best fit which very closely approximates our numerical likelihood results.

These results allow to pre-select a number of cases for which a detailed study, including perturbation growth, can be performed. This task is carried out in a companion paper.

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Model	χ^2_{min}	Ω_{m0}	Δ
ΛCDM	578.00	$0.27^{+0.02}_{-0.02}$	
β_1, β_2	577.99	$0.28^{+0.04}_{-0.03}$	~ 0.03
β_1, β_3	578.02	$0.30^{+0.02}_{-0.04}$	~ 0.08
β_1, β_4	578.04	$0.34^{+0.03}_{-0.04}$	~ 0.20
β_1, β_4 (inf. branch $r \in (r_c, \infty)$)	578.60	$0.16^{+0.02}_{-0.03}$	~ 0.03

Table 2. Numerical results of the best fit to SNe Ia data for different models with only two free β-parameter. The relative error on the fit $\beta_j(\beta_i, \Omega_{m0})$ $(i < j)$ corresponding to the most likely value for Ω_{m0} is denoted by Δ.

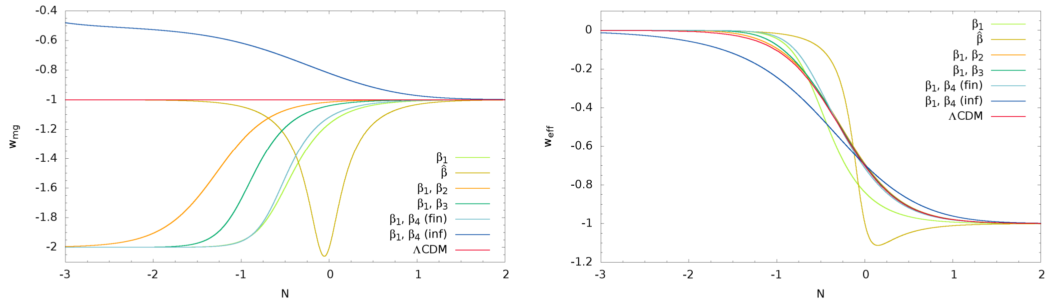

Figure 5. Evolution of the equation of state in the best fits in the minimal β_1 and $\hat{\beta}$ models and the two-parameter models $\beta_1\beta_2$, $\beta_1\beta_3$ and $\beta_1\beta_4$. Here, we distinguish between finite (light blue) and infinite (dark blue) branches in $\beta_1\beta_4$ models.

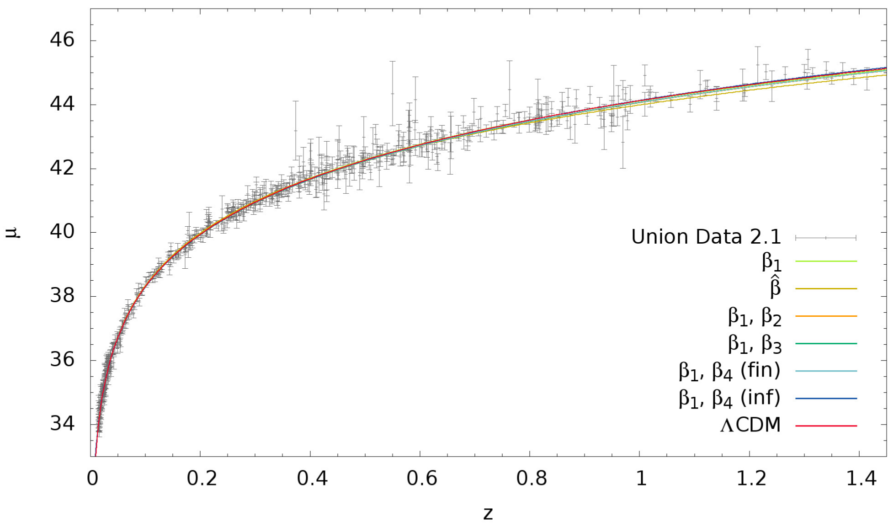

Figure 6. Hubble diagram with the best fit in the minimal one-parameter models and two-parameter models compared to all measured SNe Ia from the Union Data 2.1. As already indicated by the numerical values of the χ^2 (see tables 1 and 2), the best fit in the β_1 and in all analyzed two parameter models are close to the ΛCDM result (red).

Acknowledgments

L.A. acknowledges support from DFG through the TRR33 project "The Dark Universe". We are grateful to Y. Akrami, S.F. Hassan, S. Hoffmann, T. Koivisto, D. Mota, M. Sandstand, A. Schmidt-May, M. von Strauss for useful discussions. Special thanks to Valerio Marra for help with the supernovae data and to Mariele Motta for checking various equations. A.P. thanks the DAAD-WISE Fellowship for support.

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JCAP03(2014)029

Publication 2

Instability in a minimal bimetric gravity model

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Physical Review D 90, 044030 (2014), arXiv:1402.1988

PHYSICAL REVIEW D **90,** 044030 (2014)

Instability in a minimal bimetric gravity model

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(Received 28 February 2014; published 13 August 2014)

We discuss in detail a particularly simple example of a bimetric massive gravity model which seems to offer an alternative to the standard cosmological model at background level. For small redshifts, its equation of state is $w(z) \approx -1.22^{+0.02}_{-0.02} - 0.64^{+0.05}_{-0.04} z/(1+z)$. Just like ΛCDM, it depends on a single parameter, has an analytical background expansion law and fits the expansion cosmological data well. However, confirming previous results, we find that the model is unstable at early times at small scales and speculate over possible ways to cure the instability. In the regime in which the model is stable, we find that it fits the linear perturbation observations well and has a growth index of approximately $\gamma = 0.47$.

DOI: 10.1103/PhysRevD.90.044030 PACS numbers: 04.50.Kd

I. INTRODUCTION

The history of massive gravity dates back to 1939, when the linear model of Fierz and Pauli was published (see e.g. Refs. [1] and [2] for a review). Massive gravity requires the introduction of a second tensor field in addition to the metric (or some form of nonlocality in the action; see Ref. [3]). The interaction of the two tensor fields creates a mixture of massless and massive gravitons that apparently avoids the appearance of ghosts [4–7].

In the model introduced in Refs. [8,9], the second tensor field becomes dynamical, just like the standard metric, although only the latter is coupled to matter (for a generalization, see Ref. [10]). This approach, denoted bimetric gravity, keeps the theory ghost free and has the advantage of allowing cosmologically viable solutions. The cosmology of bimetric gravity has been studied in several papers, e.g. in Refs. [11–17].

In this paper we select among the class of bimetric models a particularly simple case, which we dub the minimal bimetric model (MBM). Just like ΛCDM, this model depends on a single parameter and has an analytical background behavior that is at all times distinguishable from ΛCDM. In a previous paper we have shown that the MBM is the only one-parameter version of bimetric gravity (beside the trivial case in which only a cosmological constant is left) that is cosmologically well behaved at the background level and fits the supernovae Hubble diagram well [18] (see also Refs. [11,12]).

Unfortunately, considering the full set of equations beyond the quasistatic limit, we find that the model is unstable at large wave numbers k in the past and up to a redshift of order unity. This instability has been discussed previously by other authors for bimetric models in general [13,19] and, if taken at face value, would rule out the model. Nevertheless, we believe it is worth analytically identifying the epoch in which the instability takes place and discussing possible ways to overcome it. This could help to find other cases, within the class of bimetric models, that do not suffer from the same problem.

In the regime in which the model is stable we derive its scalar cosmological perturbation equations in the subhorizon limit and integrate them numerically. We then compare the results with a recent compilation of growth data [20]. We find that the MBM fits both supernovae and growth rate data, while remaining well distinguishable from ΛCDM. If a variant of the model is found that cures the instability in the past, the model could be an interesting competitor to ΛCDM.

II. BACKGROUND EQUATIONS

We start with the action of the form [8]

$$S = -\frac{M_g^2}{2}\int d^4x\sqrt{-g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-f}R(f) + m^2M_g^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(X) + \int d^4x\sqrt{-g}L_m, \tag{1}$$

where $X^\alpha_\gamma \equiv \sqrt{g^{\alpha\beta}f_{\beta\gamma}}$, e_n are elementary symmetric polynomials, β_n are arbitrary constants and $L_m = L_m(g,\psi)$ is a matter Lagrangian. Here $g_{\mu\nu}$ is the standard metric coupled to matter fields in the L_m Lagrangian, while $f_{\mu\nu}$ is an additional dynamical tensor field. In the following we express masses in units of the Planck mass M_g and the mass parameter m^2 will be absorbed into the parameters β_n. Varying the action with respect to $g_{\mu\nu}$, one obtains the following equations of motion:

$$G_{\mu\nu} + \frac{1}{2}\sum_{n=0}^{3}(-1)^n\beta_n[g_{\mu\lambda}Y^\lambda_{(n)\nu}(X) + g_{\nu\lambda}Y^\lambda_{(n)\mu}(X)] = T_{\mu\nu}, \tag{2}$$

where $G_{\mu\nu}$ is Einstein's tensor, and the expressions $Y^{\lambda}_{(n)\nu}(X)$ are defined as

$$Y_{(0)} = I, \tag{3}$$

$$Y_{(1)} = X - I[X], \tag{4}$$

$$Y_{(2)} = X^2 - X[X] + \frac{1}{2}I([X]^2 - [X^2]), \tag{5}$$

$$Y_{(3)} = X^3 - X^2[X] + \frac{1}{2}X([X]^2 - [X^2]) - \frac{1}{6}I([X]^3 - 3[X][X^2] + 2[X^3]), \tag{6}$$

where I is the identity matrix and $[\ldots]$ is the trace operator. Varying the action with respect to $f_{\mu\nu}$ we get

$$\bar{G}_{\mu\nu} + \sum_{n=0}^{3}\frac{(-1)^n\beta_{4-n}}{2M_f^2}[f_{\mu\lambda}Y^{\lambda}_{(n)\nu}(X^{-1}) + f_{\nu\lambda}Y^{\lambda}_{(n)\mu}(X^{-1})] = 0, \tag{7}$$

where the overbar indicates $f_{\mu\nu}$ curvatures. Notice that β_0 acts as a pure cosmological constant. Finally, the rescaling $f \to M_f^{-2}f, \beta_n \to M_f^n\beta_n$ allows us to assume $M_f = 1$ in the following (see Ref. [17]).

We assume now a cosmological spatially flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = a^2(t)(-dt^2 + dx_i dx^i), \tag{8}$$

where t represents the conformal time and a dot will represent the derivative with respect to it. The second metric is chosen also in a FRW form,

$$ds_f^2 = -[\dot{b}(t)^2/\mathcal{H}^2(t)]dt^2 + b(t)^2 dx_i dx^i, \tag{9}$$

where $\mathcal{H} \equiv \dot{a}/a$ is the conformal Hubble function. This form of the metric $f_{\mu\nu}$ ensures that the equations satisfy the Bianchi identities (see e.g. Ref. [9]).

Defining $r = b/a$, the background equations can be conveniently written as a first-order system for $r, \mathcal{H}$, using $N = \log a$ as the time variable and denoting d/dN with a prime [18] (see also Ref. [12]):

$$2E'E + E^2 = a^2(B_0 + B_2 r'), \tag{10}$$

$$r' = \frac{3rB_1\Omega_m}{\beta_1 - 3\beta_3 r^2 - 2\beta_4 r^3 + 3B_2 r^2}, \tag{11}$$

where $\Omega_m = 1 - \frac{B_0}{B_1}r$, $E \equiv \mathcal{H}/H_0$ and the couplings β_i are measured in units of H_0^2 and finally

$$B_0 = \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3, \tag{12}$$

$$B_1 = \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3, \tag{13}$$

$$B_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2. \tag{14}$$

III. MINIMAL BIMETRIC MODEL

In Ref. [18] we identified the conditions for standard cosmological viability, i.e. for a matter epoch followed by a stable acceleration, without bounces or singularities beside the big bang. We found that among the models with a single nonvanishing parameter only two cases give a viable cosmology, namely, the cases with only β_0 or only β_1. The former one is indeed the ΛCDM model, while the β_1 case is what we call the minimal bimetric model. One has then for the MBM

$$r' = \frac{3r(1 - 3r^2)}{1 + 3r^2}, \tag{15}$$

independent of β_1. This equation has two branches for $r > 0$, but only the one that starts at $r = 0$ and ends at $r = 1/\sqrt{3}$ is cosmologically viable. In terms of the scale factor, this solution reads [18,21]

$$r(a) = \frac{1}{6}a^{-3}(-A \pm \sqrt{12a^6 + A^2}), \tag{16}$$

where $A = -\beta_1 + 3/\beta_1$. These equations imply a remarkably simple and testable relation between the equation of state w and Ω_m valid at all times during matter domination:

$$w = \frac{2}{\Omega_m - 2}, \tag{17}$$

where the density parameter is given by

$$\Omega_m = 1 - 3r(a)^2. \tag{18}$$

Since the Friedmann equation of the second metric provides $r_0 = \beta_1/3$, the present value of the matter density parameter is therefore simply related to single parameter value of the model. Together with Eq. (17) this shows that all viable parameter values for β_1 lead to a phantom equation of state at present time. Another useful relation for the MBM that we will use below is $\mathcal{H}^2 = \beta_1 a^2/3r$.

In Ref. [18] we found that the MBM fits the supernovae data well if $\beta_1 = 1.38 \pm 0.03$, corresponding to $\Omega_{m0} = 1 - \beta_0^2/3 = 0.37 \pm 0.02$. The equation of state turns out to be approximated at small redshifts by $w(z) \approx -1.22^{+0.02}_{-0.02} - 0.64^{+0.05}_{-0.04}z/(1+z)$. However this parametrization is not adequate at $z \geq 0.5$ and the analytic expressions (16)–(18) should be employed instead.

IV. PERTURBATION EQUATIONS

We now find the perturbation equations for the MBM. For the perturbed part of the metrics we adopt the gauge defined in Fourier space as

$$ds_f^2 = 2Fb^2\left[-\frac{\dot{b}(t)^2\Psi_f}{b(t)^2\mathcal{H}^2(t)}dt^2 + (\Phi_f\delta_{ij} + k_ik_jE_f)dx^idx^j\right],$$
$$ds^2 = 2Fa^2[-\Psi dt^2 + (\Phi\delta_{ij} + k_ik_jE)dx^idx^j], \quad (19)$$

where $F = e^{i\mathbf{k}\cdot\mathbf{r}}$. After a transformation to the gauge-invariant variables [13]

$$\tilde{\Phi} = \Phi - \mathcal{H}^2E', \quad (20)$$

$$\tilde{\Psi} = \Psi - (\mathcal{H}^2 + \mathcal{H}\mathcal{H}')E' - \mathcal{H}^2E'', \quad (21)$$

$$\tilde{\Phi}_f = \Phi_f - \frac{r\mathcal{H}^2E_f{}'}{(r' + r)}, \quad (22)$$

$$\tilde{\Psi}_f = \Psi_f - \frac{\mathcal{H}r^2(\mathcal{H}E_f{}')'}{(r' + r)^2} - \frac{\mathcal{H}^2E_f{}'r(r^2 + 2r'^2 + 2rr' - rr'')}{(r' + r)^3}, \quad (23)$$

we obtain from the Einstein equations a set of perturbation equations in $\Xi = \{\tilde{\Phi}, \tilde{\Psi}, \tilde{\Phi}_f, \tilde{\Psi}_f, E, \Delta E \equiv E - E_f\}$,

$$[00]\Phi\left(1 + \frac{2k^2}{3a^2r\beta_1}\right) - \Phi_f + \frac{a^2(1-3r^2)\beta_1}{-4r+6r^3}E' + \frac{\mathcal{H}^2(1+3r^2)}{-4+6r^2}\Delta E' + \frac{1}{3}k^2\Delta E - \frac{a(-1+3r^2)\sqrt{\beta_1}}{\sqrt{3}k^2r^{5/2}}\theta - \frac{\delta\rho}{3B_2r} = 0, \quad (24)$$

$$[0i]\Phi' - \Psi + \frac{a^2\rho}{2\mathcal{H}k^2}\theta + (\mathcal{H}^2 - \mathcal{H}\mathcal{H}')E' = 0, \quad (25)$$

$$[ij]\Phi + \Psi + a^2r\beta_1\Delta E = 0, \quad (26)$$

$$[ii] - \left(2 + \frac{2k^2}{3a^2r\beta_1}\right)\Phi + 2\Phi_f - \Psi\left(1 + \frac{2k^2}{3a^2r\beta_1}\right) + \frac{6(2-3r^2)}{3+9r^2}\Psi_f + \frac{\mathcal{H}^3r(3+9r^2)(\mathcal{H}-\mathcal{H}')}{a^2(2-3r^2)\beta_1}E'' + \frac{3a^2(2+9r^2)(1-3r^2)^2\beta_1}{4r(2-3r^2)^2(1+3r^2)}E'$$
$$-\frac{2k^2}{3}\Delta E + \frac{a^2(1+3r^2)\beta_1}{6r(2-3r^2)}\Delta E'' + \frac{a^2(22-9r^2(-19+42r^2+15r^4))\beta_1}{12r(4-27r^4+27r^6)}\Delta E' = 0, \quad (27)$$

$$[00]\Phi + \left(-1 - \frac{2k^2r}{3a^2\beta_1}\right)\Phi_f + \frac{k^2}{3}\Delta E + \frac{a^2(-1+3r^2)\beta_1}{4r-6r^3}E' - \frac{a^2(1+3r^2)\beta_1}{6r(2-3r^2)}\Delta E' = 0, \quad (28)$$

$$[0i]\Phi_f{}' + \frac{(-4+6r^2)}{1+3r^2}\Psi_f + \frac{3a^2(-1+3r^2)\beta_1}{4r(-2+3r^2)}E' + \frac{3a^2(1-3r^2)\beta_1}{4r(-2+3r^2)}\Delta E' = 0, \quad (29)$$

$$[ij]\Psi_f + \Phi_f + \frac{a^2(1+3r^2)\beta_1}{-4r+6r^3}\Delta E = 0, \quad (30)$$

$$[ii]\Phi + \left(-1 + \frac{2k^2r(-2+3r^2)}{3a^2(1+3r^2)\beta_1}\right)\Phi_f + \frac{\Psi}{2} + \left(1 - \frac{3}{1+3r^2} + \frac{2k^2r(-2+3r^2)}{3a^2(1+3r^2)\beta_1}\right)\Psi_f + \frac{a^2(1-3r^2)\beta_1}{4r(-2+3r^2)}E'' + \frac{a^2(1+3r^2)\beta_1}{12r(-2+3r^2)}\Delta E''$$
$$+\frac{1}{3}k^2\Delta E + \frac{a^2(-22+9r^2(-19+42r^2+15r^4))\beta_1}{24r(4-27r^4+27r^6)}\Delta E' - \frac{3(2+9r^2)(a-3ar^2)^2\beta_1}{8(2-3r^2)^2(r+3r^3)}E' = 0, \quad (31)$$

and from the conservation of matter we get two more equations for the matter density contrast δ and the velocity divergence θ,

$$\delta' + \theta\mathcal{H}^{-1} + 3\Phi' - 3\mathcal{H}^2E'' - 6\mathcal{H}\mathcal{H}'E' + k^2E' = 0, \quad (32)$$

$$\theta' + \theta + k^2E'\mathcal{H}' - k^2\Psi\mathcal{H}^{-1} + k^2\mathcal{H}(E'' + E') = 0. \quad (33)$$

V. INSTABILITY

Recently some authors [13,19] found an instability at small scales in massive bimetric theories. Here we revisit this issue in the MBM. Starting from the set of general perturbation equations (24)–(31), one can replace all $\Psi_f, \Phi_f, \Delta E$ and their derivatives by using g_{00}, g_{ii}, and g_{ij}. This also shows that g_{ij} and f_{ij} are linearly dependent.

Then we can replace δ, θ with the help of g_{0i} and f_{00}. Finally, one can find a linear combination of f_{0i}, and g_{ii} which allows one to express E' as a function of Ψ, Φ and their derivatives. In this way, we can express our original ten equations as just two second-order equations for $X \equiv \{\Psi, \Phi\}$ which can be written as $(i, j = 1, 2)$

$$X_i'' + M_{ij}X_j' + N_{ij}X_j = 0, \quad (34)$$

where M_{ij} and N_{ij} are two matrices that depend only on k, β_1 and r. For the explicit expressions of their elements see Appendix A. The eigenfrequencies of this equation can be found by substituting $X = X_0 e^{i\omega N}$, assuming that the dependence of ω on time is negligibly small. In the limit of large k we find

$$\omega_{\mp} = \pm \frac{k}{\mathcal{H}} \frac{\sqrt{-1 + 12r^2 + 9r^4}}{1 + 3r^2} \quad (35)$$

(here k is in the same units as $\mathcal{H}$) plus two other solutions, one of which is zero while the other is independent of k and therefore subdominant. One can then see that real solutions (needed to obtain an oscillating, rather than a growing, solution for X) are found only for $r > 0.28$, which occurs for $N \approx -0.4$, i.e. $z \approx 0.5$. This is exactly the same instant at which r'' crosses zero. At any epoch before this, the perturbation equations are unstable for large k, i.e. they grow as a^{ω_+}. Notice that $\omega_{\mp}$ are independent of β_1; this means that the instability remains even in the limit of zero mass, which is similar to the van Dam-Veltman-Zakharov discontinuity [22,23]. Similar to that case, one might speculate that when nonlinear order effects start being important they might cure the instability. Notice also that the large-k limit we have taken is valid only for $k/\mathcal{H} \gg 1$, i.e. for $r > r_H$, where $r_H(k)$ is the solution of the equation $a(r)^2 = 3rk^2/\beta_1$ and $a(r)$ is obtained by inverting Eq. (16).

This explosively large growth is in obvious contrast with what we know about the growth of linear perturbations in our Universe, for instance, with the smoothness of the microwave cosmic background and the linearity of present fluctuations on scales larger than a few megaparsecs. However, one might imagine that by adjusting for instance the initial conditions or by playing with other assumptions, the model could be saved. Therefore, in order to quantify the real impact of the instability, we estimate a directly observable quantity that is independent of initial conditions: the growth rate of the linear perturbations as measured with redshift distortions. Since all the perturbation variables can be written as a linear combination of Φ and Ψ, their dominant behavior will have the same growth $\sim e^{i\omega_+ N}$. This means that during the instability epoch the matter density contrast grows as $\delta \sim a^{\omega}$ where $\omega = |\omega_+|$. This allows us to estimate the growth rate $f \equiv d \log \delta / dN$ and to obtain the observable combination $f(z)\sigma_8(z) = \sigma_8 f \delta/\delta_0$ as

$$f(z)\sigma_8(z) = Aa^{\omega}(\omega + N\omega'), \quad (36)$$

where A is a normalization constant. The combination $f\sigma_8(z)$ has been estimated through redshift distortions at various redshifts up to unity (see for instance Ref. [20]), and it has been found to be practically constant in the range from $z = 0.8$ to $z = 0.3$ for scales around $k = 0.1\ h/\mathrm{Mpc}$, corresponding to $k/H_0 \approx 50$. In stark contrast, using the expression (36), we estimate an extremely fast growth during the instability epoch; for instance, between $z = 0.8$ and $z = 0.6$ the growth of $f\sigma_8(z)$ is found to be around 180 000 times.

Adding the cosmological constant β_0, one obtains

$$\omega_{\mp} = \pm \frac{k}{\mathcal{H}} \frac{\sqrt{-1 + 2(\beta_0/\beta_1)r + 12r^2 + 9r^4}}{1 + 3r^2}. \quad (37)$$

In this case the instability region occurs for any $r < \beta_1/2\beta_0$; if $\beta_1/\beta_0 \ll 1$ this unstable epoch can be pushed arbitrarily back into the past but then the model would effectively behave like ΛCDM.

It is possible that a different choice of parameters β_i leads to an evolution which is free from instabilities, or a value of $r_H(k)$ such that (at least for the scales that are today in the linear regime) the subhorizon evolution occurs during the stable phase. Finally, one could also assume that β_1 is actually a time-dependent variable [e.g., it could be a function of a scalar field, $\beta_1(\phi)$], so that its value is very small in the past—therefore recovering a standard evolution—and comparable to H_0 near the present epoch.

VI. QUASISTATIC LIMIT

Taken at face value, the instability rules out the MBM, unless nonlinear effects are able to rescue it. However, we think it is still worthwhile to consider some of its cosmological effects for two reasons. First, one of the mentioned mechanisms or some variants thereof might be able to cure the past instability while leaving unaltered the recent epoch. Second, the methods we investigate below can be applied to other choices of parameters in the bimetric class that allow for a stable evolution.

In the regime in which the model is stable, i.e. for $z \leq 0.5$, one can simplify the perturbation equations by taking the quasistatic limit. In this regime and at subhorizon scales, i.e. $k/\mathcal{H} \gg 1$, we can in fact assume that $\Xi_i(k/\mathcal{H})^2$ is much larger than Ξ_i and its derivatives Ξ_i', Ξ_i'' for any $\Xi_i = \{\Phi, \Psi, \Phi_f, \Psi_f, \Delta E, E\}$ and also $\delta(k/\mathcal{H})^2$, $\delta'(k/\mathcal{H})^2 \gg \theta/\mathcal{H}$; then the set of differential equations becomes algebraic (except for the matter conservation equations) and we obtain the Poisson-like relations

$$\Psi = -\frac{\mathcal{H}^2\Omega_m\delta(2k^2r^3(11 + 6r^2) + 3\beta_1 a^2(1 + 7r^2 - 6r^4))}{2k^2(\beta_1 a^2(1 + r^2)^2(1 + 3r^2) + k^2r^3(7 + 3r^2))}, \quad (38)$$

$$\Phi = \frac{\mathcal{H}^2\Omega_m\delta(2k^2r^3(10+3r^2)+3\beta_1 a^2(1+4r^2+3r^4))}{2k^2(\beta_1 a^2(1+r^2)^2(1+3r^2)+k^2r^3(7+3r^2))}, \tag{39}$$

$$\Psi_f = -\frac{\mathcal{H}^2\Omega_m\delta(3r^2+1)(3\beta_1 a^2(6r^4-7r^2-1)+2k^2r(6r^2-1))}{4k^2(3r^2-2)(\beta_1 a^2(r^2+1)^2(3r^2+1)+k^2(3r^2+7)r^3)}, \tag{40}$$

$$\Phi_f = \frac{\mathcal{H}^2\Omega_m\delta(3r^2+1)(3\beta_1 a^2(r^2+1)+k^2r)}{2k^2(\beta_1 a^2(r^2+1)^2(3r^2+1)+k^2(3r^2+7)r^3)}, \tag{41}$$

$$\Delta E = \frac{\mathcal{H}^2\Omega_m\delta r(9\beta_1 a^2(1-3r^2)+2k^2r(3r^2+1))}{2a^2\beta_1 k^2(\beta_1 a^2(r^2+1)^2(3r^2+1)+k^2(3r^2+7)r^3)}, \tag{42}$$

which reduce to the standard ones during the matter epoch, i.e. for $r\to 0$. In the quasistatic limit the set of equations does not contain the $(0,i)g_{\mu\nu}$ and $(0,i)f_{\mu\nu}$ equations. Since both equations were used to simplify the remaining ones, we have checked the consistency of the solutions with both $(0,i)$ equations. We then obtain the two modified gravity parameters

$$\eta \equiv -\frac{\Phi}{\Psi} = H_2\frac{1+H_4(k/\mathcal{H})^2}{1+H_3(k/\mathcal{H})^2}, \tag{43}$$

$$Y \equiv -\frac{2k^2\Psi}{3\mathcal{H}^2\Omega_m\delta_m} = H_1\frac{1+H_3(k/\mathcal{H})^2}{1+H_5(k/\mathcal{H})^2}, \tag{44}$$

where

$$H_1 \equiv \frac{1+7r^2-6r^4}{(1+r^2)^2(1+3r^2)}, \tag{45}$$

$$H_2 \equiv \frac{1+4r^2+3r^4}{1+7r^2-6r^4}, \tag{46}$$

$$H_3 \equiv \frac{2\mathcal{H}^2r^3(11+6r^2)}{3\beta_1 a^2(1+7r^2-6r^4)}, \tag{47}$$

$$H_4 \equiv \frac{2\mathcal{H}^2r^3(10+3r^2)}{3\beta_1 a^2(1+4r^2+3r^4)}, \tag{48}$$

$$H_5 \equiv \frac{\mathcal{H}^2r^3(7+3r^2)}{\beta_1 a^2(1+r^2)^2(1+3r^2)}. \tag{49}$$

For $\beta_1\to 0$ the only consistent background solution is $r\to 0$; in this limit the model reduces to pure CDM and consequently $H_{1,2}=1$ and $H_{3,4,5}=0$. The expressions (43) and (44) have the same structure as the Horndeski Lagrangian [24–26] since both Lagrangians produce second-order equations of motion. The matter evolution equations can now be written as a single equation:

$$\delta_m'' + \delta_m'\left(1+\frac{\mathcal{H}'}{\mathcal{H}}\right) - \frac{3}{2}Y(k)\Omega_m\delta_m = 0. \tag{50}$$

Integrating numerically this equation along the background solution (16), we find that near $k=0.1\ h/\mathrm{Mpc}$ and $\beta_1 = 1.39$ we can approximate $f\equiv\delta'/\delta\approx\Omega_m^\gamma$ [27] with $\gamma\approx 0.47$ in the range $z\in(0,5)$ (see Fig. 1). Near $\beta_1=1.39$ the dependence on β_1 at $k=0.1\ h/\mathrm{Mpc}$ can be linearly approximated as $\gamma = 0.26+0.15\,\beta_1$, while the weak dependence on k is approximately

$$\gamma(k) = 0.47 + 0.001\left(\frac{k}{0.1\ h/\mathrm{Mpc}}\right)^{-1/2}. \tag{51}$$

Future experiments, like the Euclid satellite [28], plan to measure γ to within 0.02; this will amply allow one to distinguish the MBM from ΛCDM and standard quintessence, which predict $\gamma\approx 0.54$.

Let us remark, however, that the growth rate is significantly larger than 1 for redshifts $z\gtrsim 1$ and cannot be well approximated with the standard Ω_m^γ fit. We find that an additional correction

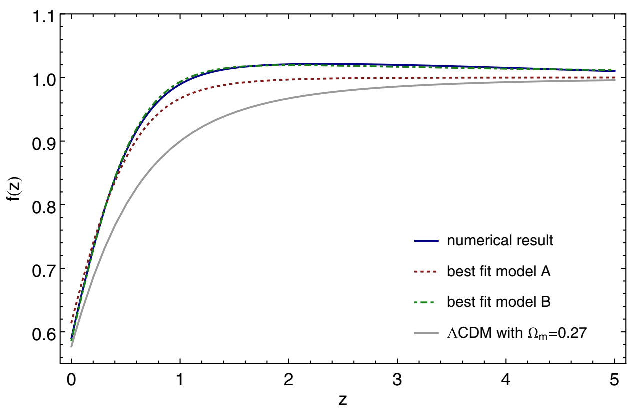

FIG. 1 (color online). Growth rate $f=\delta'/\delta$ in the quasistatic limit for $\beta_1=1.39$ and $k=0.1\ h/\mathrm{Mpc}$. The numerical result (in blue) is approximated by the fitting model $f=\Omega_m^\gamma$ (model A, red dotted curve) and $f=\Omega_m^{\gamma_0}(1+\frac{\gamma_1}{z+1})$ (model B, green dash-dotted curve). For a comparison we plot the ΛCDM result (gray dashed line) while using $\Omega_{m0}=0.37$ which corresponds to the present matter density in our analyzed MBM.

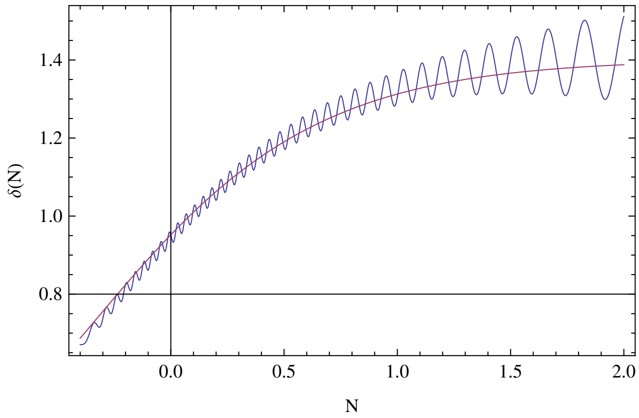

FIG. 2 (color online). The oscillating blue line represents the numerical integration of the full set of perturbation equations for $k/H_0 = 100, \beta_1 = 1.4$. The red smooth line is the solution of the growth equation (50) in the quasistatic limit for the same parameters.

$$f \approx \Omega_m^{\gamma_0}\left(1 + \frac{\gamma_1}{z+1}\right) \tag{52}$$

with $\gamma_0 = 0.58$ and $\gamma_1 = 0.07$ is better able to reproduce our numerical result.

The quasistatic limit is an excellent approximation to the full behavior, provided one considers only the stable epoch $z < 0.5$, as shown in Fig. 2.

VII. COMPARISON TO THE GROWTH RATE

The quasistatic limit can be compared to measurements of $f(z)\sigma_8(z)$ where $\sigma_8(z) = \sigma_8 G(z)$, with $G(z)$ being the growth rate normalized to unity today. Most of the present measurements actually extend to redshifts higher than 0.5, which is outside the stability regime. Nevertheless, as a way to demonstrate the feasibility of constraining this model with growth data, we include these measurements as well. The likelihood is given by

$$\chi^2_{f\sigma_8} = \sum_{ij}(d_i - \sigma_8 t_i)C^{-1}_{ij}(d_j - \sigma_8 t_j), \tag{53}$$

in which d_i and t_i are vectors containing the measured and theoretically expected data, respectively, and C_{ij} denotes the covariance matrix. Since the current constraints on σ_8 depend on the theory of gravity, for generality we marginalize analytically the likelihood over $\sigma_8 > 0$, obtaining

$$\chi^2_{f\sigma_8} = S_{20} - \frac{S_{11}^2}{S_{02}} + \log S_{02} - 2\log\left(1 + \mathrm{Erf}\left(\frac{S_{11}}{\sqrt{2S_{02}}}\right)\right), \tag{54}$$

where

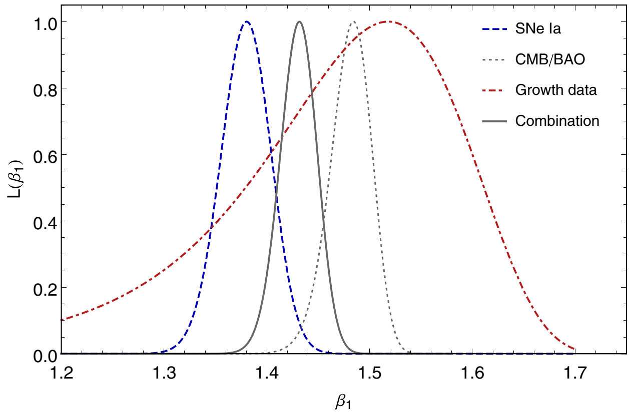

FIG. 3 (color online). Likelihood for β_1 obtained from observed SNe Ia (blue dashed), measured growth data (red dot-dashed) and the combination of CMB and BAO measurements (dotted gray). The full combined likelihood is indicated by a gray solid line. All likelihoods are rescaled to unity at their maximum. For the most likely values we obtain $\beta_1 = 1.38^{+0.03}_{-0.03}$ ($\chi^2_{\min} = 578.3$) and $\beta_1 = 1.52^{+0.09}_{-0.13}$ ($\chi^2_{\min} = 10.48$) for the comparison with SNe Ia and growth data, respectively. Due to the broad width of the growth likelihood, its combination with the other probes does not sensibly change the results.

$$S_{11} = d_i C^{-1}_{ij} t_j, \tag{55}$$

$$S_{20} = d_i C^{-1}_{ij} d_j, \tag{56}$$

$$S_{02} = t_i C^{-1}_{ij} t_j \tag{57}$$

(see also e.g. Ref. [29]). Since current data are not binned in k space, we choose an average value $k = 0.1\ h/\mathrm{Mpc}$ in Eq. (50).

We compute the likelihood from the data set compiled in Ref. [20] which contains measured growth histories from the 6dFGS [30], LRG_{200}, LRG_{60} [31], BOSS [32], WiggleZ [33] and VIPERS [34] surveys. Our results are shown in Fig. 3. The growth data constraints appear much broader than, but consistent with, the supernova type Ia (SN Ia) data. The combined result from SNe and growth data is $\beta_1 = 1.39 \pm 0.03$, practically identical to the best fit from SN Ia alone. We also plot in Fig. 3 the likelihood from cosmic microwave background (CMB) and baryon acoustic oscillation (BAO) measurements where we use the results from the first peak angular size WMAP 7.2 data [35] and the SDSS DR7 sample including the LRG and 2dFGRS data set [36]. The combined result from all data, SN + CMB + BAO + growth turns out to be $\beta_1 = 1.43 \pm 0.02$. However, one should keep in mind that the CMB data analysis assumes a pure ΛCDM so it is not obvious that it can be applied here without corrections. Note that including the CMB/BAO data does not change the best-fit parameters for $w(z)$ and γ significantly.

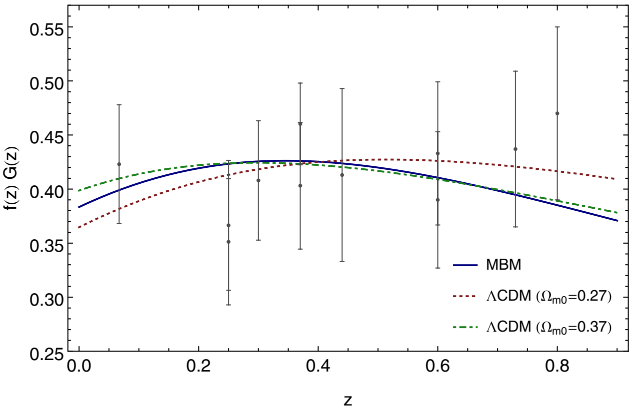

FIG. 4 (color online). Comparison of growth histories in the MBM with $\beta_1 = 1.39$ (blue) and ΛCDM (dotted red: $\Omega_{m0} = 0.27$; dot-dashed green: $\Omega_{m0} = 0.37$ which corresponds to the present matter density in the best-fit MBM). The data points are taken from Ref. [20]. Note that the normalization of the curves is immaterial due to the marginalization over σ_8.

Finally, in Fig. 4 we compare the growth history corresponding to the most likely MBM with the measured growth data and the ΛCDM expectation.

VIII. CONCLUSIONS

We have shown that a minimal bimetric model exists which closely reproduces the success and the simplicity of ΛCDM at the background level. We fixed its single parameter, β_1, to percent accuracy by fitting to supernovae and growth data. The MBM has several unique signatures, like the $w - \Omega_m$ relation (17), the phantom equation of state, the k dependence of the growth factor [Eq. (51)] and the values of f above unity [Eq. (52)], all of which will make it easily distinguishable from ΛCDM with future experiments.

We have shown however that the model suffers from a perturbation instability at large k at epochs before $z \approx 0.5$, confirming previous results [13,19] but also identifying the exact epoch of transition. Taken at face value, such an instability seems to rule out this particular form of bimetric model. A possible way to save the model is to assume that when the perturbations become nonlinear the instability becomes under control. This conjecture can be confirmed only by going to higher order in perturbation theory. Of course the instability might also disappear by choosing a different set of parameters. We leave a complete analysis of other models to future work.

ACKNOWLEDGMENTS

We acknowledge support from DFG through the project TRR33 "The Dark Universe." We thank Mariele Motta for discussions and cross-checks on the perturbation equations and Valerio Marra for help with the supernovae catalog, and Y. Akrami, T. Koivisto, and A. Solomon for discussions on the perturbation equations.

APPENDIX A: EXPLICIT EXPRESSION FOR THE MATRICES M_{IJ} AND N_{IJ}

In this appendix we will present the elements of the matrices M_{ij} and N_{ij} appearing in the second-order differential equation (34). We start by defining the functions

$$K_1 \equiv K_2 \equiv \frac{4r - 6r^3}{a^2(3\beta_1 r^2 + \beta_1)}, \qquad K_3 \equiv K_4 \equiv \frac{1}{a^2 r \beta_1}, \tag{A1}$$

$$K_5 \equiv K_6 \equiv -\frac{24k^2(2-3r^2)^2}{a^2 r(3r^2+1)^2\beta_1}, \qquad K_7 \equiv -\frac{12(2-3r^2)^2(3r^2-1)}{(3r^3+r)^2}, \tag{A2}$$

$$K_8 \equiv -\frac{12\sqrt{3}a(2-3r^2)^2(3r^2-1)\sqrt{\beta_1}}{r^{5/2}(3kr^2+k)^2}, \qquad K_9 \equiv -\frac{a(3r^2-1)\sqrt{\beta_1}}{k\sqrt{r}}, \tag{A3}$$

$$K_{10} \equiv -\frac{a^2 k(3r^2-1)\beta_1}{\sqrt{3}(3r^3+r)}, \qquad K_{11} \equiv \frac{4k(3r^2-2)}{\sqrt{3}(3r^2+1)}, \tag{A4}$$

$$K_{12} \equiv K_{13} \equiv \frac{\sqrt{3}a^2 k(3r^2-1)\beta_1}{4r-6r^3}, \qquad K_{14} \equiv K_{15} \equiv \frac{1}{a^2 r\beta_1}, \tag{A5}$$

$$K_{16} \equiv \frac{9r^2-6}{k^2(3r^2+1)}, \qquad K_{17} \equiv K_{18} \equiv \frac{1}{a^2 r\beta_1} + \frac{3}{2k^2}, \tag{A6}$$

$$K_{19} \equiv \frac{a^2(135r^6+378r^4-171r^2-22)\beta_1}{8k^2(2-3r^2)^2(3r^3+r)}, \qquad K_{20} \equiv -\frac{9a^2(1-3r^2)^2(9r^2+2)\beta_1}{8k^2(2-3r^2)^2(3r^3+r)}, \tag{A7}$$

$$K_{21} \equiv \frac{a^2(3r^2+1)\beta_1}{4k^2 r(3r^2-2)}, \qquad K_{22} \equiv -\frac{3a^2(3r^2-1)\beta_1}{4k^2 r(3r^2-2)}, \tag{A8}$$

$$K_{23} \equiv \frac{1}{\mathcal{H}}, \qquad K_{24} \equiv k^2 - 6\mathcal{H}\mathcal{H}', \tag{A9}$$

$$K_{25} \equiv -3\mathcal{H}^2, \qquad K_{26} \equiv -\frac{k^2}{\mathcal{H}}, \tag{A10}$$

$$K_{27} \equiv k^2(\mathcal{H}+\mathcal{H}'), \qquad K_{28} \equiv k^2\mathcal{H}, \tag{A11}$$

$$K_{29} \equiv -\frac{12k^2(2-3r^2)^2}{(3r^2+1)^2}, \qquad K_{30} \equiv -\frac{36(2-3r^2)^2}{(3r^2+1)^2}, \tag{A12}$$

$$K_{31} \equiv \frac{12(2-3r^2)^2(3\beta_1 a^2+2k^2 r)}{a^2(3r^2+1)^2\beta_1},$$

$$K_{32} \equiv -\frac{6a^2(3r^2-2)\beta_1}{3r^3+r}, \tag{A13}$$

$$K_{33} \equiv \frac{18a^2(3r^2-2)(3r^2-1)\beta_1}{r(3r^2+1)^2},$$

$$K_{34} \equiv \frac{4r-6r^3}{a^2(3\beta_1 r^2+\beta_1)}, \tag{A14}$$

$$K_{35} \equiv \frac{4r-6r^3}{a^2(3\beta_1 r^2+\beta_1)},$$

$$K_{36} \equiv K_{37} \equiv \frac{1}{a^2 r\beta_1}, \tag{A15}$$

which only depend on the background quantities β_1, r, $\mathcal{H}$ and the wave number k. Although we introduced several redundant functions, the definitions of those functions turn out to be useful since in every bimetric model the dependencies of both M_{ij} and N_{ij} on the K_i functions are the same. We proceed by defining

$$L_1 \equiv K_{13} + (3K_2K_{31}^2)^{-1}(3K_1K_{11}K_{31}K_{33} + 2\sqrt{3}kK_2(K_{33}K'_{31} - K_{31}K'_{33})), \tag{A16}$$

$$L_2 \equiv (3K_2K_{31}^2)^{-1}[2\sqrt{3}kK_2(K_{14}(K_{31}(K_{29}+K'_{32}) - K_{32}K'_{31}) - K_{31}(K_{30} - 2K_{32}K'_{14})) - 3K_{14}K_{31}(K_{12}K_2K_{31} + K_1K_{11}K_{32})], \tag{A17}$$

$$L_3 \equiv (3K_2K_{31}^2)^{-1}[2\sqrt{3}kK_2(2K_{31}K_{32}K'_{15} + K_{15}(K_{31}(K_{29}+K'_{32}) - K_{32}K'_{31})) - 3K_{15}K_{31}(K_{12}K_2K_{31} + K_1K_{11}K_{32})], \tag{A18}$$

$$L_4 \equiv -(3K_2K_{31}^2)^{-1}[3(K_{11}K_3 + K_{12}K_2K'_{14})K_{31}^2 + 3K_1K_{11}(K_{14}K_{29} - K_{30} + K_{32}K'_{14})K_{31} - 2\sqrt{3}kK_2(K_{14}K'_{29} - K'_{30} + K'_{14}(K_{29}+K'_{32}) + K_{32}K''_{14})K_{31} + 2\sqrt{3}kK_2(K_{14}K_{29} - K_{30} + K_{32}K'_{14})K'_{31}], \tag{A19}$$

$$L_5 \equiv (3K_2K_{31}^2)^{-1}[-3K_{11}K_4K_{31}^2 - 3K_1K_{11}(K_{15}K_{29} + K_{32}K'_{15})K_{31} + K_2(2\sqrt{3}k(-K_{32}K'_{15}K'_{31} + K_{29}(K_{31}K'_{15} - K_{15}K'_{31}) + K_{31}(K_{15}K'_{29} + K'_{15}K'_{32} + K_{32}K''_{15})) - 3K_{12}K_{31}^2K'_{15})], \tag{A20}$$

$$L_6 \equiv -\frac{2kK_{33}}{\sqrt{3}K_{31}}, \tag{A21}$$

$$L_7 \equiv \frac{2kK_{14}K_{32}}{\sqrt{3}K_{31}}, \tag{A22}$$

$$L_8 \equiv \frac{2kK_{15}K_{32}}{\sqrt{3}K_{31}}, \tag{A23}$$

$$L_9 \equiv K_{20} + \frac{(K_1K_{16}k^2 + 3K_2)K_{33}}{k^2K_2K_{31}}, \tag{A24}$$

$$L_{10} \equiv K_{14}\left(-K_{19} - \frac{(K_1K_{16}k^2+3K_2)K_{32}}{k^2K_2K_{31}}\right) - 2K_{21}K'_{14}, \tag{A25}$$

$$L_{11} \equiv K_{15}\left(-K_{19} - \frac{(K_1K_{16}k^2+3K_2)K_{32}}{k^2K_2K_{31}}\right) - 2K_{21}K'_{15}, \tag{A26}$$

$$L_{12} \equiv -(k^2K_2K_{31})^{-1}[-K_{18}K_2K_{31}k^2 + K_{16}K_3K_{31}k^2 + K_1K_{16}(K_{14}K_{29} - K_{30} + K_{32}K'_{14})k^2 + K_{19}K_2K_{31}K'_{14}k^2 + K_2K_{21}K_{31}K''_{14}k^2 - 3K_2K_{30} + K_{14}K_2(K_{31}k^2 + 3K_{29}) + 3K_2K_{32}K'_{14}], \tag{A27}$$

$$L_{13} \equiv K_{17} - K_{19}K'_{15} - K_{21}K''_{15} - (k^2K_2K_{31})^{-1}[K_{16}K_{31}K_4k^2 + K_1K_{16}(K_{15}K_{29} + K_{32}K'_{15})k^2 + K_{15}K_2(K_{31}k^2 + 3K_{29}) + 3K_2K_{32}K'_{15}], \tag{A28}$$

$$L_{14} \equiv K_{22}, \tag{A29}$$

$$L_{15} \equiv -K_{14}K_{21}, \tag{A30}$$

$$L_{16} \equiv -K_{15}K_{21}, \tag{A31}$$

$$L_{17} \equiv [K_9(K_7(K_{24}K_7K_9K_{31}^2 + (K_{31}K_8K'_{10} + K_6K_9K'_{33} + K_{33}K_9K'_6)K_{31} - K_{33}K_6K_9K'_{31}) - K_{31}K_{33}K_6K_9K'_7) - K_{10}K_{31}^2(K_9(K_{23}K_7^2 - K'_8K_7 + K_8K'_7) + K_7K_8K'_9)](K_{31}K_7K_9)^{-2}, \tag{A32}$$

$$L_{18} \equiv (\sqrt{3}K_{31}K_7K_9)^{-2}[-2\sqrt{3}kK_{31}^2(K_9(K_{23}K_7^2 - K'_8K_7 + K_8K'_7) + K_7K_8K'_9) + 3K_9^2(K_{14}K_{31}K_{32}K_6K'_7 - K_7((K_5 - 3K_7)K_{31}^2 + (K_6(-K_{30} + 2K_{32}K'_{14} + K_{14}(K_{29} + K'_{32})) + K_{14}K_{32}K'_6)K_{31} - K_{14}K_{32}K_6K'_{31}))], \tag{A33}$$

$$L_{19} \equiv (3K_{31}^2K_7^2K_9)^{-1}[-2\sqrt{3}kK_7K_8K_{31}^2 + 3K_{15}K_{32}K_6K_7K_9K'_{31} + 3K_9(K_{15}K_{32}K_6K'_7 - K_7(2K_{32}K_6K'_{15} + K_{15}(K_6(K_{29} + K'_{32}) + K_{32}K'_6)))K_{31}], \tag{A34}$$

$$L_{20} \equiv -(K_{31}K_7)^{-2}[K_7K'_5K_{31}^2 - K_5K'_7K_{31}^2 + K_{14}K_6K_7K'_{29}K_{31} - K_6K_7K'_{30}K_{31} + K_6K_7K'_{14}K'_{32}K_{31} - K_{30}K_7K'_6K_{31} + K_{32}K_7K'_{14}K'_6K_{31} + K_{30}K_6K'_7K_{31} - K_{32}K_6K'_{14}K'_7K_{31} + K_{32}K_6K_7K''_{14}K_{31} + K_{30}K_6K_7K'_{31} - K_{32}K_6K_7K'_{14}K'_{31} + K_{29}(K_{31}(K_6K_7K'_{14} + K_{14}K_7K'_6 - K_{14}K_6K'_7) - K_{14}K_6K_7K'_{31})], \tag{A35}$$

$$L_{21} \equiv (\sqrt{3}K_{31}K_7K_9)^{-2}[2\sqrt{3}kK_{31}^2(K_9(K_{23}K_7^2 - K'_8K_7 + K_8K'_7) + K_7K_8K'_9) - 3K_9^2(K_{15}K_{31}K_6K_7K'_{29} + K_{29}(K_{31}(K_6K_7K'_{15} + K_{15}K_7K'_6 - K_{15}K_6K'_7) - K_{15}K_6K_7K'_{31}) + K'_{15}(K_{31}K_6K_7K'_{32} + K_{32}(K_{31}K_7K'_6 - K_6(K_7K'_{31} + K_{31}K'_7))) + K_{31}K_{32}K_6K_7K''_{15})], \tag{A36}$$

$$L_{22} \equiv K_{25} + K_7^{-1}\left(\frac{K_{33}K_6}{K_{31}} + \frac{K_{10}K_8}{K_9}\right), \tag{A37}$$

$$L_{23} \equiv (3K_7)^{-1}\left(\frac{2\sqrt{3}kK_8}{K_9} - \frac{3K_{14}K_{32}K_6}{K_{31}}\right), \tag{A38}$$

$$L_{24} \equiv -\frac{K_{15}K_{32}K_6}{K_{31}K_7}, \tag{A39}$$

$$L_{25} \equiv K_{26} + \frac{2k(K_9 - K'_9)}{\sqrt{3}K_9^2}, \tag{A40}$$

$$L_{26} \equiv \frac{K_{10}K'_9 - K_9(K_{10} - K_{27}K_9 + K'_{10})}{K_9^2}, \tag{A41}$$

$$L_{27} \equiv \frac{2k(K'_9 - K_9)}{\sqrt{3}K_9^2}, \tag{A42}$$

$$L_{28} \equiv \frac{2k}{\sqrt{3}K_9}, \tag{A43}$$

$$L_{29} \equiv K_{28} - \frac{K_{10}}{K_9}, \tag{A44}$$

$$L_{30} \equiv -\frac{2k}{\sqrt{3}K_9}, \tag{A45}$$

and

$$G_1 \equiv \frac{-L_{12}L_{23}L_8 + L_{12}L_{24}L_7 + L_{15}L_{20}L_8 - L_{15}L_{24}L_4 - L_{16}L_{20}L_7 + L_{16}L_{23}L_4}{-L_1L_{15}L_{24} + L_1L_{16}L_{23} + L_{15}L_{17}L_8 - L_{16}L_{17}L_7 - L_{23}L_8L_9 + L_{24}L_7L_9}, \tag{A46}$$

$$G_2 \equiv \frac{-L_{13}L_{23}L_8 + L_{13}L_{24}L_7 + L_{15}L_{21}L_8 - L_{15}L_{24}L_5 - L_{16}L_{21}L_7 + L_{16}L_{23}L_5}{-L_1L_{15}L_{24} + L_1L_{16}L_{23} + L_{15}L_{17}L_8 - L_{16}L_{17}L_7 - L_{23}L_8L_9 + L_{24}L_7L_9}, \tag{A47}$$

$$G_3 \equiv \frac{-L_{10}L_{23}L_8 + L_{10}L_{24}L_7 + L_{15}L_{18}L_8 - L_{15}L_2L_{24} - L_{16}L_{18}L_7 + L_{16}L_2L_{23}}{-L_1L_{15}L_{24} + L_1L_{16}L_{23} + L_{15}L_{17}L_8 - L_{16}L_{17}L_7 - L_{23}L_8L_9 + L_{24}L_7L_9}, \tag{A48}$$

$$G_4 \equiv \frac{-L_{11}L_{23}L_8 + L_{11}L_{24}L_7 + L_{15}L_{19}L_8 - L_{15}L_{24}L_3 - L_{16}L_{19}L_7 + L_{16}L_{23}L_3}{-L_1L_{15}L_{24} + L_1L_{16}L_{23} + L_{15}L_{17}L_8 - L_{16}L_{17}L_7 - L_{23}L_8L_9 + L_{24}L_7L_9}. \tag{A49}$$

The elements of the matrices M_{ij} and N_{ij} are then given by

$$M_{11} = \frac{L_{11} - G_4L_9 - L_{14}(G_2 + G_4') + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(-G_4L_{17} + L_{19} - L_{22}(G_2 + G_4'))}{-G_4L_{14} + L_{16} + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(L_{24} - G_4L_{22})}, \tag{A50}$$

$$M_{12} = \frac{L_{10} - G_3L_9 - L_{14}(G_1 + G_3') + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(-G_3L_{17} + L_{18} - L_{22}(G_1 + G_3'))}{-G_4L_{14} + L_{16} + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(L_{24} - G_4L_{22})}, \tag{A51}$$

$$\begin{aligned} M_{21} = {} & (L_{16}(L_{23} - G_3L_{22}) + L_{15}(G_4L_{22} - L_{24}) + L_{14}(G_3L_{24} - G_4L_{23}))^{-1}[L_{14}L_{17}G_4^2 - L_{22}L_9G_4^2 - L_{14}L_{19}G_4 \\ & + L_{24}L_9G_4 + L_{11}(G_4L_{22} - L_{24}) + G_2L_{14}L_{24} + L_{14}L_{24}G_4' + L_{16}(-G_4L_{17} + L_{19} - L_{22}(G_2 + G_4'))], \end{aligned} \tag{A52}$$

$$\begin{aligned} M_{22} = {} & (L_{16}(G_3L_{22} - L_{23}) + L_{15}(L_{24} - G_4L_{22}) + L_{14}(G_4L_{23} - G_3L_{24}))^{-1}[-G_3G_4L_{14}L_{17} + G_4L_{14}L_{18} \\ & - G_1L_{14}L_{24} + L_{10}(L_{24} - G_4L_{22}) + G_3G_4L_{22}L_9 - G_3L_{24}L_9 - L_{14}L_{24}G_3' + L_{16}(G_3L_{17} - L_{18} + L_{22}(G_1 + G_3'))], \end{aligned} \tag{A53}$$

and

$$N_{11} = \frac{L_{13} - G_2L_9 - L_{14}G_2' + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(-G_2L_{17} + L_{21} - L_{22}G_2')}{-G_4L_{14} + L_{16} + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(L_{24} - G_4L_{22})}, \tag{A54}$$

$$N_{12} = \frac{L_{12} - G_1L_9 - L_{14}G_1' + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(-G_1L_{17} + L_{20} - L_{22}G_1')}{-G_4L_{14} + L_{16} + (L_{23} - G_3L_{22})^{-1}(G_3L_{14} - L_{15})(L_{24} - G_4L_{22})}, \tag{A55}$$

$$\begin{aligned} N_{21} = {} & (L_{16}(L_{23} - G_3L_{22}) + L_{15}(G_4L_{22} - L_{24}) + L_{14}(G_3L_{24} - G_4L_{23}))^{-1}[G_2G_4L_{14}L_{17} - G_4L_{14}L_{21} \\ & + L_{13}(G_4L_{22} - L_{24}) - G_2G_4L_{22}L_9 + G_2L_{24}L_9 + L_{14}L_{24}G_2' + L_{16}(-G_2L_{17} + L_{21} - L_{22}G_2')], \end{aligned} \tag{A56}$$

$$\begin{aligned} N_{22} = {} & (L_{16}(L_{23} - G_3L_{22}) + L_{15}(G_4L_{22} - L_{24}) + L_{14}(G_3L_{24} - G_4L_{23}))^{-1}[G_1G_4L_{14}L_{17} - G_4L_{14}L_{20} \\ & + L_{12}(G_4L_{22} - L_{24}) - G_1G_4L_{22}L_9 + G_1L_{24}L_9 + L_{14}L_{24}G_1' + L_{16}(-G_1L_{17} + L_{20} - L_{22}{G_1}')]. \end{aligned} \tag{A57}$$

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Publication 3

Stable and unstable cosmological models in bimetric massive gravity

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and Adam R. Solomon

Physical Review D 90, 124014 (2014), arXiv:1407.4331

PHYSICAL REVIEW D **90,** 124014 (2014)

Stable and unstable cosmological models in bimetric massive gravity

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(Received 7 August 2014; published 3 December 2014)

Nonlinear, ghost-free massive gravity has two tensor fields; when both are dynamical, the mass of the graviton can lead to cosmic acceleration that agrees with background data, even in the absence of a cosmological constant. Here the question of the stability of linear perturbations in this bimetric theory is examined. Instabilities are presented for several classes of models, and simple criteria for the cosmological stability of massive bigravity are derived. In this way, we identify a particular self-accelerating bigravity model, infinite-branch bigravity (IBB), which exhibits both viable background evolution and stable linear perturbations. We discuss the modified gravity parameters for IBB, which do not reduce to the standard ΛCDM result at early times, and compute the combined likelihood from measured growth data and type Ia supernovae. IBB predicts a present matter density $\Omega_{m0} = 0.18$ and an equation of state $w(z) = -0.79 + 0.21z/(1+z)$. The growth rate of structure is well approximated at late times by $f(z) \approx \Omega_m^{0.47}[1 + 0.21z/(1+z)]$. The implications of the linear instability for other bigravity models are discussed: the instability does not necessarily rule these models out, but rather presents interesting questions about how to extract observables from them when linear perturbation theory does not hold.

DOI: 10.1103/PhysRevD.90.124014 PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k

I. INTRODUCTION

Testing gravity beyond the limits of the Solar System is an important task of present and future cosmology. The detection of any modification of Einstein's gravity at large scales or in past epochs would be an extraordinary revolution and change our view of the evolution of the Universe.

A theory of a massless spin-2 field is either described by general relativity [1–6] or unimodular gravity [7,8].

Consequently, most modifications of gravity proposed so far introduce one or more new dynamical fields, in addition to the massless metric tensor of standard gravity. This new field is usually a scalar field, typically through the so-called Horndeski Lagrangian [9,10], or a vector field, such as in Einstein-aether models (see Refs. [11,12] and references therein). A complementary approach which has gained significant attention in recent years is, rather than adding a new dynamical field, to promote the massless spin-2 graviton of general relativity to a massive one.

The history of massive gravity is an old one, dating back to 1939, when the linear theory of Fierz and Pauli was published [13]. We refer the reader to the reviews [14,15] for a reconstruction of the steps leading to the modern approach, which has resulted in a ghost-free, fully nonlinear theory of massive gravity [16] (see also Refs. [17–21]). A key element of these new forms of massive gravity is the introduction of a second tensor field, or "reference metric," in addition to the standard metric describing the curvature of spacetime. When this reference metric is fixed (e.g., Minkowski), this theory propagates the five degrees of freedom of a ghost-free massive graviton.

However, the reference metric can also be made dynamical, as proposed in Refs. [22,23]. This promotes massive gravity to a theory of bimetric gravity. This theory is still ghost free and has the advantage of allowing cosmologically viable solutions. The cosmology of bimetric gravity has been studied in several papers, e.g., in Refs. [24–30]. The main conclusion is that bimetric gravity allows for a cosmological evolution that can approximate the ΛCDM universe and can therefore be a candidate for dark energy without invoking a cosmological constant. Crucially, the parameters and the potential structure leading to the accelerated expansion are thought to be stable under quantum corrections [31], in stark contrast to a cosmological constant, which would need to be fine-tuned against the energy of the vacuum [32,33].

Bimetric gravity has been successfully compared to background data [cosmic microwave background, baryon acoustic oscillations, and type Ia supernovae (SNe)] in

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Refs. [24,25], and to linear perturbation data in Refs. [34,35]. The comparison with linear perturbations has been undertaken on subhorizon scales assuming a quasistatic (QS) approximation, in which the potentials are assumed to be slowly varying. This assumption makes it feasible to derive the modification to the Poisson equation and the anisotropic stress, two functions of scale and time which completely determine observational effects at the linear level.

The quasistatic equations are, however, a valid subhorizon approximation only if the full system is stable for large wave numbers. Previous work [27,36,37] has identified a region of instability in the past.[1] Here we investigate this problem in detail. We reduce the linearized Einstein equations to two equations for the scalar modes, and analytically determine the epochs of stability and instability for all the models with up to two free parameters which have been shown to produce viable cosmological background evolution. The behavior of more complicated models can be reduced to these simpler ones at early and late times.

We find that several models which yield sensible background cosmologies in close agreement with the data are in fact plagued by an instability that only turns off at recent times. This does not necessarily rule these regions of the bimetric parameter space out, but rather presents a question of how to interpret and test these models, as linear perturbation theory is quickly invalidated. Remarkably, we find that only a particular bimetric model—the one in which only the β_1 and β_4 parameters are nonzero (that is, the linear interaction and the cosmological constant for the reference metric are turned on)—is stable and has a cosmologically viable background at all times when the evolution is within a particular branch. This shows that a cosmologically viable bimetric model without an explicit cosmological constant (by which we mean the constant term appearing in the Friedmann equation) does indeed exist, and raises the question of how to nonlinearly probe the viability of other bimetric models.

This paper is part of a series dedicated to the cosmological perturbations of bimetric gravity and their properties, following Ref. [35].

II. BACKGROUND EQUATIONS

We start with the action of the form [23]

$$S = -\frac{M_g^2}{2}\int d^4x\sqrt{-\det g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R(f) \tag{1}$$

$$+ m^2M_g^2\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) + \int d^4x\sqrt{-\det g}\mathcal{L}_m(g,\Phi), \tag{2}$$

where e_n are elementary symmetric polynomials and β_n are free parameters. Here $g_{\mu\nu}$ is the standard metric coupled to the matter fields Φ in the matter Lagrangian, $\mathcal{L}_m$, while $f_{\mu\nu}$ is a new dynamical tensor field with metric properties. In the following we express masses in units of M_g and absorb the mass parameter m^2 into the parameters β_n. The graviton mass is generally of order $m^2\beta_n$. The action then becomes

$$S = -\frac{1}{2}\int d^4x\sqrt{-\det g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R(f) \tag{3}$$

$$+ \int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) + \int d^4x\sqrt{-\det g}\mathcal{L}_m(g,\Phi). \tag{4}$$

There has been some discussion in the literature over how to correctly take square roots. We will find solutions in which $\det\sqrt{g^{-1}f}$ becomes zero at a finite point in time (and only at that time), and so it is important to determine whether to choose square roots to always be positive, or to change sign on either side of the $\det = 0$ point. This was discussed in some detail in Ref. [39] (see also Ref. [40]), where continuity of the vielbein corresponding to $\sqrt{g^{-1}f}$ demanded that the square root not be positive definite. We will take a similar stance here, and make the only choice that renders the action differentiable at all times, i.e., such that the derivative of $\sqrt{g^{-1}f}$ with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ is continuous everywhere. In particular, using a cosmological background with $f_{\mu\nu} \equiv \mathrm{diag}(-X^2, b^2, b^2, b^2)$, this choice implies that we assume $\sqrt{-\det f} = Xb^3$, where $X = \dot{b}/\mathcal{H}$ with $\mathcal{H}$ is the g-metric Hubble rate. This is important because, as we will see later on, it turns out that in the cosmologically stable model, the f metric bounces, so X changes sign during cosmic evolution. Consequently the square roots will change sign as well, rather than develop cusps. Note that sufficiently small perturbations around the background will not lead to a different sign of this square root.

Varying the action with respect to $g_{\mu\nu}$, one obtains the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\lambda}Y^{\lambda}_{(n)\nu}\left(\sqrt{g^{\alpha\beta}f_{\beta\gamma}}\right) = T_{\mu\nu}. \tag{5}$$

[1] This should not be confused with the Higuchi ghost instability, which affects most massive gravity cosmologies and some in bigravity, but is, however, absent from the simplest bimetric models which produce ΛCDM-like backgrounds [38].

Here the matrices $Y^{\lambda}_{(n)\nu}(\sqrt{g^{\alpha\beta}f_{\beta\gamma}})$ are defined as, setting $\mathbb{X} = (\sqrt{g^{-1}f})$,

$$Y_{(0)}(\mathbb{X}) = \mathbb{I}, \quad (6)$$

$$Y_{(1)}(\mathbb{X}) = \mathbb{X} - \mathbb{I}[\mathbb{X}], \quad (7)$$

$$Y_{(2)}(\mathbb{X}) = \mathbb{X}^2 - \mathbb{X}[\mathbb{X}] + \frac{1}{2}\mathbb{I}([\mathbb{X}]^2 - [\mathbb{X}^2]), \quad (8)$$

$$Y_{(3)}(\mathbb{X}) = \mathbb{X}^3 - \mathbb{X}^2[\mathbb{X}] + \frac{1}{2}\mathbb{X}([\mathbb{X}]^2 - [\mathbb{X}^2]) - \frac{1}{6}\mathbb{I}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]), \quad (9)$$

where $\mathbb{I}$ is the identity matrix and $[\ldots]$ is the trace operator. Varying the action with respect to $f_{\mu\nu}$ we find

$$\bar{R}_{\mu\nu} - \frac{1}{2}f_{\mu\nu}\bar{R} + \frac{1}{M_f^2}\sum_{n=0}^{3}(-1)^n\beta_{4-n}f_{\mu\lambda}Y^{\lambda}_{(n)\nu}(\sqrt{f^{\alpha\beta}g_{\beta\gamma}}) = 0, \quad (10)$$

where the overbar indicates the curvature of the $f_{\mu\nu}$ metric.

The f-metric Planck mass, M_f, is a redundant parameter and can be freely set to unity [41]. To see this, consider the rescaling $f_{\mu\nu} \to M_f^{-2}f_{\mu\nu}$. The Ricci scalar transforms as $\bar{R}(f) \to M_f^2\bar{R}(f)$, so the full Einstein-Hilbert term in the action becomes

$$\frac{M_f^2}{2}\sqrt{-\det f}\bar{R}(f) \to \frac{1}{2}\sqrt{-\det f}\bar{R}(f). \quad (11)$$

The other term in the action that depends on $f_{\mu\nu}$ is the mass term, which transforms as

$$\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}) \to \sum_{n=0}^{4}\beta_n e_n(M_f^{-1}\sqrt{g^{-1}f}) = \sum_{n=0}^{4}\beta_n M_f^{-n} e_n(\sqrt{g^{-1}f}), \quad (12)$$

where in the last equality we used the fact that the elementary symmetric polynomials $e_n(\mathbb{X})$ are of order $\mathbb{X}^n$. Therefore, by additionally redefining the interaction couplings as $\beta_n \to M_f^n\beta_n$, we end up with the original bigravity action but with $M_f = 1$.[2] Consequently we set $M_f = 1$ in the following.

Let us now consider the background cosmology of bimetric gravity. We assume a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds_g^2 = a^2(\tau)(-d\tau^2 + dx_i dx^i), \quad (13)$$

where τ is conformal time and an overdot represents the derivative with respect to it. The second metric is chosen as

$$ds_f^2 = -[\dot{b}(\tau)^2/\mathcal{H}^2(\tau)]d\tau^2 + b(\tau)^2 dx_i dx^i, \quad (14)$$

where $\mathcal{H} \equiv \dot{a}/a$ is the conformal-time Hubble parameter associated with the physical metric, $g_{\mu\nu}$. The particular choice for the f-metric lapse, f_{00}, ensures that the Bianchi identity is satisfied (see, e.g., Ref. [22]).

Inserting the FLRW ansatz for $g_{\mu\nu}$ into Eq. (5) we get

$$3\mathcal{H}^2 = a^2(\rho_{\rm tot} + \rho_{\rm mg}), \quad (15)$$

where we define an effective massive-gravity energy density as

$$\rho_{\rm mg} = B_0 \equiv \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \quad (16)$$

with

$$r \equiv \frac{b}{a}, \quad (17)$$

while $\rho_{\rm tot}$ is the density of all other matter components (e.g., dust and radiation). The total energy density follows the usual conservation law,

$$\dot{\rho}_{\rm tot} + 3\mathcal{H}\rho_{\rm tot} = 0. \quad (18)$$

It is useful to define the density parameter for the mass term (which will be the effective dark energy density):

$$\Omega_{\rm mg} \equiv \frac{\rho_{\rm mg}}{\rho_{\rm tot} + \rho_{\rm mg}} = 1 - \Omega_m - \Omega_r, \quad (19)$$

where $\Omega_i = \rho_i/(\rho_{\rm tot} + \rho_{\rm mg})$ for matter and radiation.

The background dynamics depend entirely on the g-metric Hubble rate, $\mathcal{H}$, and the ratio of the two scale factors, $r = b/a$ [25]. Moreover, by using $N = \log a$ as time variable, with $'$ denoting derivatives with respect to N, the background equations can be conveniently reformulated as a first-order autonomous system [42]:

$$2\mathcal{H}'\mathcal{H} + \mathcal{H}^2 = a^2(B_0 + B_2 r' - w_{\rm tot}\rho_{\rm tot}), \quad (20)$$

$$r' = \frac{3(1 + w_{\rm tot})B_1\Omega_{\rm tot} r}{\beta_1 - 3\beta_3 r^2 - 2\beta_4 r^3 + 3B_2 r^2}, \quad (21)$$

$$\Omega_{\rm tot} = 1 - \frac{B_0}{B_1}r, \quad (22)$$

[2] Recall that we are expressing masses in units of the Planck mass, M_g. In more general units, the redundant parameter is M_f/M_g.

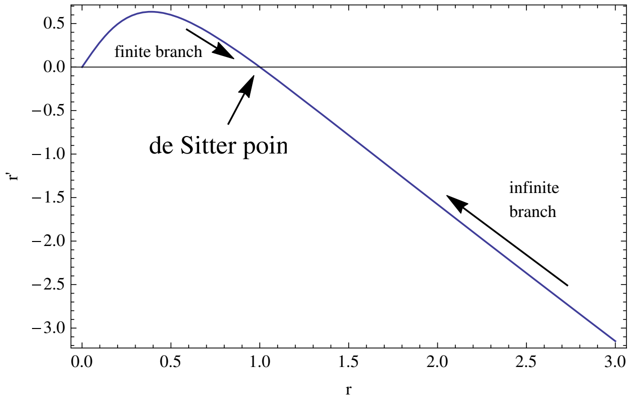

FIG. 1 (color online). Plot of the function $r'(r)$ for the $\beta_1\beta_4$ model for $\beta_1 = 0.5$, $\beta_4 = 1$. For both the finite and infinite branches, the final state is the de Sitter point. The arrows show the direction of movement of r.

where

$$B_1 \equiv \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3, \tag{23}$$

$$B_2 \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2, \tag{24}$$

and w_{tot} denotes the equation of state (EOS) corresponding to the sum of matter and radiation density parameter Ω_{tot}. We can define the effective equation of state

$$w_{\mathrm{eff}} \equiv \Omega_{\mathrm{mg}} w_{\mathrm{mg}} + \Omega_{\mathrm{tot}} w_{\mathrm{tot}} = -\frac{1}{3}\left(1 + 2\frac{\mathcal{H}'}{\mathcal{H}}\right) = -\frac{r(B_0 + B_2 r')}{B_1} \tag{25}$$

$$= -1 + \Omega_{\mathrm{tot}} - \frac{B_2 r r'}{B_1}, \tag{26}$$

from which we obtain

$$w_{\mathrm{mg}} = -1 - \frac{B_2 r r'}{\Omega_{\mathrm{mg}} B_1} = -1 - \frac{B_2}{B_0} r'. \tag{27}$$

Another useful relation gives the Hubble rate in terms of r without an explicit ρ dependence,

$$\mathcal{H}^2 = \frac{a^2 B_1}{3r}. \tag{28}$$

The background evolution of r will follow Eq. (21) from an initial value of r until $r' = 0$, unless r hits a singularity. In Ref. [42] it was shown that cosmologically viable evolution can take place in two distinct ways, depending on initial conditions: when r evolves from 0 to a finite value (we call this a finite branch) and when r evolves from infinity to a finite value (infinite branch). In all viable cases, the past asymptotic value of r corresponds to $\Omega_m = 1$ while the final point corresponds to a de Sitter stage with $\Omega_m = 0$ (see Fig. 1 for an illustrative example).

In the following, we consider only pressureless matter, or dust, with $w_{\mathrm{tot}} = 0$. The reason is that we are interested only in the late-time behavior of bigravity when the Universe is dominated by dust. We also assume $r \geq 0$, although in principle nothing prevents a negative value of b.

We will find it convenient to express all the β_i parameters in units of H_0^2 and $\mathcal{H}$ in units of H_0.[3] In this way all the quantities that enter the equations are dimensionless.

III. PERTURBATION EQUATIONS

In this section we study linear cosmological perturbations. We define our perturbed metrics in Fourier space by

$$g_{\alpha\beta} = g_{0,\alpha\beta} + h_{\alpha\beta}, \tag{29}$$

$$f_{\alpha\beta} = f_{0,\alpha\beta} + h_{f,\alpha\beta}, \tag{30}$$

where $g_{0,\alpha\beta}$ and $f_{0,\alpha\beta}$ are the background metrics with line elements

$$ds_g^2 = a^2(t)(-dt^2 + dx_i dx^i), \tag{31}$$

$$ds_f^2 = -[\dot{b}(t)^2/\mathcal{H}^2(t)]dt^2 + b(t)^2 dx_i dx^i, \tag{32}$$

while $h_{\alpha\beta}$ and $h_{f,\alpha\beta}$ are perturbations around the backgrounds $g_{0,\alpha\beta}$ and $f_{0,\alpha\beta}$, respectively, whose line elements are

$$ds_h^2 = 2a^2[-\Psi dt^2 + (\Phi\delta_{ij} + k_i k_j E)dx^i dx^j]\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{33}$$

$$ds_{h_f}^2 = 2b^2\left[-\frac{\dot{b}^2\Psi_f}{b^2\mathcal{H}^2}dt^2 + (\Phi_f\delta_{ij} + k_i k_j E_f)dx^i dx^j\right] \times \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{34}$$

After transforming to gauge-invariant variables [27],

$$\Phi \longrightarrow \Phi - \mathcal{H}^2 E', \tag{35}$$

$$\Psi \longrightarrow \Psi - \mathcal{H}(\mathcal{H}'E' + \mathcal{H}(E'' + E')), \tag{36}$$

$$\Phi_f \longrightarrow \Phi_f - \frac{\mathcal{H}^2 r E_f'}{r' + r}, \tag{37}$$

[3]With this convention, our β_i parameters are equivalent to the $B_i \equiv m^2\beta_i/H_0^2$ used in Refs. [25,26,35].

$$\Psi_f \longrightarrow \Psi_f - \frac{\mathcal{H}r^2\mathcal{H}'(r'+r)E'_f + \mathcal{H}^2 r(r(r'+r)E''_f + E'_f(2r'^2 + r(2r'-r'') + r^2))}{(r'+r)^3}, \tag{38}$$

and using $N = \log a$ as the time variable, the perturbation equations for the $g_{\mu\nu}$ metric read

$$[00]\left(\frac{2k^2}{3B_2a^2r}+1\right)\Phi - \Phi_f + \frac{1}{3}k^2\Delta E + \frac{2H^3r(-\mathcal{H}+\mathcal{H}')}{\mathcal{A}_2}E' - \frac{\mathcal{H}^2\mathcal{A}_1}{\mathcal{A}_2}\Delta E' - \frac{2\mathcal{H}^2(\mathcal{A}_1 + a^2r^2B_2)(\mathcal{H}-\mathcal{H}')}{a^2k^2r\mathcal{A}_1B_2}\theta - \frac{\delta\rho}{3B_2r} = 0, \tag{39}$$

$$[0i]\Phi' - \Psi + \frac{a^2\rho}{2\mathcal{H}k^2}\theta + (\mathcal{H}^2 - \mathcal{H}\mathcal{H}')E' = 0, \tag{40}$$

$$[ij]\Phi + \Psi + \frac{1}{2}a^2r\mathcal{A}_3\Delta E = 0, \tag{41}$$

$$[ii]\left(\frac{2k^2}{3B_2a^2r}+\frac{\mathcal{A}_3}{B_2}\right)\Phi + \left(\frac{2k^2}{3B_2a^2r}+1\right)\Psi - \frac{\mathcal{A}_3}{B_2}\Phi_f - \frac{\mathcal{A}_2}{\mathcal{A}_1}\Psi_f + \frac{k^2\mathcal{A}_3}{3B_2}\Delta E - \frac{2\mathcal{H}^3r(\mathcal{H}-\mathcal{H}')}{\mathcal{A}_2}E'' - \frac{\mathcal{H}^2\mathcal{A}_1}{\mathcal{A}_2}\Delta E'' + \mathcal{A}_4E' + \mathcal{A}_5\Delta E' = 0, \tag{42}$$

while the corresponding equations for $f_{\mu\nu}$ are

$$[00]\Phi - \left(1 + \frac{2k^2r}{3a^2B_2}\right)\Phi_f + \frac{k^2}{3}\Delta E - \frac{\mathcal{A}_1\mathcal{H}^2}{\mathcal{A}_2}\Delta E' - \frac{2\mathcal{H}^3r(\mathcal{H}-\mathcal{H}')}{\mathcal{A}_2}E' = 0, \tag{43}$$

$$[0i]\Phi'_f - \frac{\mathcal{A}_2}{\mathcal{A}_1}\Psi_f + \frac{a^2\mathcal{H}B_2(\mathcal{H}'-\mathcal{H})}{\mathcal{A}_2}\Delta E' - \frac{a^2\mathcal{H}B_2(\mathcal{H}'-\mathcal{H})}{\mathcal{A}_2}E' = 0, \tag{44}$$

$$[ij]\Phi_f + \Psi_f - \frac{a^2\mathcal{A}_1\mathcal{A}_3}{2r\mathcal{A}_2}\Delta E = 0, \tag{45}$$

$$[ii]\left(\frac{2rk^2\mathcal{A}_2}{3a^2B_2\mathcal{A}_1}+\frac{\mathcal{A}_3}{B_2}\right)\Phi_f + \left(\frac{2k^2r\mathcal{A}_2}{3a^2B_2\mathcal{A}_1}+\frac{\mathcal{A}_2}{\mathcal{A}_1}\right)\Psi_f - \frac{\mathcal{A}_3}{B_2}\Phi - \Psi - \frac{k^2\mathcal{A}_3}{3B_2}\Delta E + \frac{2\mathcal{H}^3r(\mathcal{H}'-\mathcal{H})}{\mathcal{A}_2}E'' + \frac{\mathcal{H}^2\mathcal{A}_1}{\mathcal{A}_2}\Delta E'' - \mathcal{A}_4E' - \mathcal{A}_5\Delta E' = 0, \tag{46}$$

where $\Delta E \equiv E - E_f$ and the $\mathcal{A}_i$ coefficients are defined as

$$\mathcal{A}_1 = a^2B_2 - 2\mathcal{H}^2r, \tag{47}$$

$$\mathcal{A}_2 = a^2B_2 - 2\mathcal{H}r\mathcal{H}', \tag{48}$$

$$\mathcal{A}_3 = 2B_2 + B'_2, \tag{49}$$

$$\mathcal{A}_4 = -\frac{(\mathcal{A}_1-\mathcal{A}_2)^2(-a^4(1+2r^2)B_2^2 + \mathcal{A}_1(\mathcal{A}_1+\mathcal{A}_2) + a^2r^2B_2(2\mathcal{A}_1+\mathcal{A}_2))}{2r(a^2r^2B_2+\mathcal{A}_1)\mathcal{A}_2^2} + \frac{(-a^2B_2+\mathcal{A}_1)(\mathcal{A}_1-\mathcal{A}_2)(\mathcal{A}_1\mathcal{A}_2 - a^2B_2((1+r^2)\mathcal{A}_1 - r^2\mathcal{A}_2))B'_2}{2rB_2(a^2r^2B_2+\mathcal{A}_1)\mathcal{A}_2^2}, \tag{50}$$

$$\mathcal{A}_5 = \frac{\mathcal{A}_1^2(\mathcal{A}_1^2 - \mathcal{A}_1\mathcal{A}_2 - 4\mathcal{A}_2^2) + a^2B_2\mathcal{A}_1(2r^2\mathcal{A}_1^2 - 3r^2\mathcal{A}_1\mathcal{A}_2 + (4-3r^2)\mathcal{A}_2^2)}{2r(a^2r^2B_2+\mathcal{A}_1)\mathcal{A}_2^2} - \frac{a^4B_2^2((1+2r^2)\mathcal{A}_1^2 - 2(1+2r^2)\mathcal{A}_1\mathcal{A}_2 + (1-2r^2)\mathcal{A}_2^2)}{2r(a^2r^2B_2+\mathcal{A}_1)\mathcal{A}_2^2} + \frac{\mathcal{A}_1(-a^2B_2+\mathcal{A}_1)(-\mathcal{A}_1\mathcal{A}_2 + a^2B_2((1+r^2)\mathcal{A}_1 - (1+2r^2)\mathcal{A}_2))B'_2}{2rB_2(a^2r^2B_2+\mathcal{A}_1)\mathcal{A}_2^2}. \tag{51}$$

These equations are in agreement with those presented in Refs. [27,35,41] (for a more detailed derivation see, e.g., Ref. [43]).

The matter equations are

$$\delta' + \theta\mathcal{H}^{-1} + 3\Phi' - 3\mathcal{H}^2E'' - 6\mathcal{H}\mathcal{H}'E' + k^2E' = 0, \quad (52)$$

$$\theta' + \theta + k^2E'\mathcal{H}' - k^2\Psi\mathcal{H}^{-1} + k^2\mathcal{H}(E'' + E') = 0, \quad (53)$$

where δ and θ are the matter density contrast and peculiar velocity divergence, respectively. Differentiating and combining Eqs. (52) and (53) we obtain

$$\delta'' + \left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta' + \frac{k^2\Psi}{\mathcal{H}^2} - 6E'(2\mathcal{H}'^2 + \mathcal{H}(\mathcal{H}'' + \mathcal{H}')) - 3\mathcal{H}E''(5\mathcal{H}' + \mathcal{H}) - 3E^{(3)}\mathcal{H}^2 + 3\left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\Phi' + 3\Phi'' = 0. \quad (54)$$

Note that E enters the equations only with derivatives; one could then define a new variable $Z = E'$ to lower the degree of the equations.[4] One could also adopt the gauge-invariant variables

$$\delta \to \delta + 3\mathcal{H}^2E', \quad (55)$$

$$\theta \to \theta - k^2\mathcal{H}E' \quad (56)$$

to bring the matter conservation equations into the standard form of a longitudinal gauge but since this renders the other equations somewhat more complicated we will not employ them.

IV. QUASISTATIC LIMIT

Large-scale structure experiments predominantly probe modes within the horizon. Conveniently, in the subhorizon and quasistatic limit, the cosmological perturbation equations simplify dramatically. In this section we consider this QS limit of subhorizon structures in bimetric gravity.

The subhorizon limit is defined by assuming $k \gg \mathcal{H}$, while the QS limit assumes that modes oscillate on a Hubble timescale: $\Xi' \sim \Xi$ for any variable Ξ.[5] Concretely, this means that we consider the regime where $(k^2/\mathcal{H}^2)\Xi_i \gg \Xi_i \sim \Xi_i' \sim \Xi_i''$ for each field $\Xi_i = \{\Psi, \Phi, \Psi_f, \Phi_f, \Delta E, E\}$. We additionally take $\delta(k/\mathcal{H})^2, \delta'(k/\mathcal{H})^2 \gg \theta/\mathcal{H}$. In this limit we obtain the system of equations

$$3k^2\Delta E + \left(9 + \frac{6k^2}{B_2a^2r}\right)\Phi - 9\Phi_f - \frac{3\delta\rho}{B_2r} = 0, \quad (57)$$

$$\frac{1}{2}a^2r\mathcal{A}_3\Delta E + \Phi + \Psi = 0, \quad (58)$$

$$3\frac{k^2\mathcal{A}_3}{B_2}\Delta E + \left(9\frac{\mathcal{A}_3}{B_2} + \frac{6k^2}{B_2a^2r}\right)\Phi + \left(9 + \frac{6k^2}{B_2a^2r}\right)\Psi - 9\frac{\mathcal{A}_3}{B_2}\Phi_f - 9\frac{\mathcal{A}_2}{\mathcal{A}_1}\Psi_f = 0, \quad (59)$$

$$3k^2\Delta E - \left(9 + \frac{6k^2r}{a^2B_2}\right)\Phi_f + 9\Phi = 0, \quad (60)$$

$$-\frac{a^2\mathcal{A}_1\mathcal{A}_3}{2r\mathcal{A}_2}\Delta E + \Phi_f + \Psi_f = 0, \quad (61)$$

$$\frac{3k^2\mathcal{A}_3}{B_2}\Delta E + \frac{9\mathcal{A}_3}{B_2}\Phi + 9\Psi - \left(\frac{6rk^2\mathcal{A}_2}{a^2B_2\mathcal{A}_1} + \frac{9\mathcal{A}_3}{B_2}\right)\Phi_f - \left(\frac{6k^2r\mathcal{A}_2}{a^2B_2\mathcal{A}_1} + \frac{9\mathcal{A}_2}{\mathcal{A}_1}\right)\Psi_f = 0, \quad (62)$$

where we have used the momentum constraints, Eqs. (40) and (44), to replace time derivatives of Φ and Φ_f. The above set of equations can be solved for Ψ, Φ, Ψ_f, Φ_f, and ΔE in terms of δ (see also Ref. [35]):

$$\Psi = \frac{3(3a^2\mathcal{A}_1\mathcal{A}_3B_2^2 + 3a^2\mathcal{A}_2\mathcal{A}_3B_2^2r^2 + k^2(2\mathcal{A}_1\mathcal{A}_3^2r^3 - 2B_2r(\mathcal{A}_2B_2 - 2\mathcal{A}_1\mathcal{A}_3)))\Omega_m\mathcal{H}^2}{k^4(B_2^2(4\mathcal{A}_1r^3 + 4\mathcal{A}_2r) - 8\mathcal{A}_1\mathcal{A}_3B_2r(r^2+1)) - 6k^2(r^2+1)^2a^2\mathcal{A}_1\mathcal{A}_3B_2^2}\delta, \quad (63)$$

$$\Phi = -\frac{3(3a^2\mathcal{A}_1\mathcal{A}_3B_2 + 3a^2\mathcal{A}_1\mathcal{A}_3B_2r^2 + k^2(r(4\mathcal{A}_1\mathcal{A}_3 - 2\mathcal{A}_2B_2) + 2\mathcal{A}_1\mathcal{A}_3r^3))\Omega_m\mathcal{H}^2}{k^4(B_2(4\mathcal{A}_1r^3 + 4\mathcal{A}_2r) - 8\mathcal{A}_1\mathcal{A}_3r(r^2+1)) - 6k^2(r^2+1)^2a^2\mathcal{A}_1\mathcal{A}_3B_2}\delta, \quad (64)$$

$$\Psi_f = -\frac{3(-3a^4\mathcal{A}_1^2\mathcal{A}_3B_2^2 - 3a^4\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3B_2^2r^2 + 2\mathcal{A}_1k^2r(a^2\mathcal{A}_1\mathcal{A}_3^2 - a^2(\mathcal{A}_1 + \mathcal{A}_2)\mathcal{A}_3B_2 + \mathcal{A}_2B_2^2))\Omega_m\mathcal{H}^2}{k^4(B_2^2(4\mathcal{A}_1\mathcal{A}_2r^3 + 4\mathcal{A}_2^2r) - 8\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3B_2r(r^2+1)) - 6k^2(r^2+1)^2a^2\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3B_2^2}\delta, \quad (65)$$

$$\Phi_f = -\frac{3(3a^2\mathcal{A}_1\mathcal{A}_3B_2 + 3a^2\mathcal{A}_1\mathcal{A}_3B_2r^2 + 2\mathcal{A}_1k^2r(\mathcal{A}_3 - B_2))\Omega_m\mathcal{H}^2}{k^4(B_2(4\mathcal{A}_1r^3 + 4\mathcal{A}_2r) - 8\mathcal{A}_1\mathcal{A}_3r(r^2+1)) - 6k^2(r^2+1)^2a^2\mathcal{A}_1\mathcal{A}_3B_2}\delta, \quad (66)$$

[4] E only appears without derivatives in the mass terms, specifically in differences with E_f, and so all appearances of E are accounted for by the separate gauge-invariant variable ΔE.

[5] Recall that we are using the dimensionless $N = \log a$ as our time variable.

$$\Delta E = \frac{3r(3a^2(\mathcal{A}_1 - \mathcal{A}_2)B_2^2 + 2\mathcal{A}_1 k^2 r(B_2 - \mathcal{A}_3))\Omega_m \mathcal{H}^2}{k^4 a^2(B_2^2(2\mathcal{A}_1 r^3 + 2\mathcal{A}_2 r) - 4\mathcal{A}_1\mathcal{A}_3 B_2 r(r^2+1)) - 3a^2k^2(r^2+1)^2 a^2 \mathcal{A}_1\mathcal{A}_3 B_2^2}\delta. \tag{67}$$

The QS limit is, however, only a good approximation if the full set of equations produces a stable solution for large k. In fact, if the solutions are not stable, the derivative terms we have neglected are no longer small (as their mean values vary on a faster timescale than Hubble), and the QS limit is never reached. We therefore need to analyze the stability of the full theory.

V. INSTABILITIES

Let us go back to the full linear equations, presented in Sec. III. While we have ten equations for ten variables, there are only two independent degrees of freedom, corresponding to the scalar modes of the two gravitons. The degrees-of-freedom counting goes as follows (see Ref. [44] for an in-depth discussion of most of these points): four of the metric perturbations (δg_{00}, δg_{0i}, δf_{00}, and δf_{0i}) and θ are nondynamical, as their derivatives do not appear in the second-order action. These can be integrated out in terms of the dynamical variables and their derivatives. We can further gauge fix two of the dynamical variables. Finally, after the auxiliary variables are integrated out, one of the initially dynamical variables becomes auxiliary (its derivatives drop out of the action) and can itself be integrated out.[6]

This leaves us with two independent dynamical degrees of freedom. The aim of this section is to reduce the ten linearized Einstein equations to two coupled second-order equations, and then ask whether the solutions to that system are stable. We will choose to work with Φ and Ψ as our independent variables, eliminating all of the other perturbations in their favor.

We can begin by eliminating Ψ_f, Φ_f, ΔE, and their derivatives using the $0-0$, $i-i$, and $i-j$ components of the g-metric perturbation equations. We will herein refer to these equations as g_{00}, g_{ii}, and so on for the sake of conciseness. Doing this we see also that the g_{ij} and f_{ij} equations are linearly related. Then we can replace δ and θ with the help of the g_{0i} and f_{00} equations. Finally, one can find a linear combination of the f_{0i} and g_{ii} equations which allows one to express E' as a function of Φ, Ψ, and their derivatives. In this way, we can write our original ten equations as just two second-order equations for $X_i \equiv \{\Phi, \Psi\}$ with the following structure:

$$X_i'' + F_{ij}X_j' + S_{ij}X_j = 0, \tag{68}$$

where F_{ij} and S_{ij} are complicated expressions that depend only on background quantities and on k. The eigenfrequencies of these equations can easily be found by substituting $X = X_0 e^{i\omega N}$, assuming that the dependence of ω on time is negligibly small.[7] For instance, assuming that only β_1 is nonzero, in the limit of large k we find [34]

$$\omega_{\beta_1} = \pm\frac{k}{\mathcal{H}}\frac{\sqrt{-1+12r^2+9r^4}}{1+3r^2}, \tag{69}$$

plus two other solutions that are independent of k and are therefore subdominant. One can see then that real solutions (needed to obtain oscillating, rather than growing and decaying, solutions for X) are found only for $r > 0.28$, which occurs for $N = -0.4$, i.e., $z \approx 0.5$. At any epoch before this, the perturbation equations are unstable for large k. In other words, we find an imaginary sound speed. This behavior invalidates linear perturbation theory on subhorizon scales and may rule out the model, if the instability is not cured at higher orders, for instance by a phenomenology related to the Vainshtein mechanism [45,46].

Now let us move on to more general models. Although the other one-parameter models are not viable in the background[8] (i.e., none of them have a matter dominated epoch in the asymptotic past and produce a positive Hubble rate) [42], it is worthwhile to study the eigenfreqencies in these cases too, particularly because they will tell us the early time behavior of the viable multiple-parameter models. For simplicity, from now on we refer to a model in which, e.g., only β_1 and β_2 are nonzero as the $\beta_1\beta_2$ model, and so on.

At early times, every viable, finite-branch, multiple-parameter model reduces to the single-parameter model with the lowest-order interaction. For instance, the $\beta_1\beta_2$, $\beta_1\beta_3$, and $\beta_1\beta_2\beta_3$ models all reduce to β_1, the $\beta_2\beta_3$ model reduces to β_2, and so on. Similarly, in the early Universe, the viable, infinite-branch models reduce to single-parameter models with the highest-order interaction. Therefore, in order to determine the early time stability, we need to only look at the eigenfrequencies of single-parameter models, for which we find

$$\omega_{\beta_2} = \pm\frac{k}{\mathcal{H}r}, \tag{70}$$

$$\omega_{\beta_3} = \pm\frac{ik\sqrt{r^4 - 8r^2 + 3}}{\sqrt{3}\mathcal{H}(r^2-1)}, \tag{71}$$

[6]We thank Macarena Lagos and Pedro Ferreira for discussions on this point.

[7]The criterion for this WKB approximation to hold is $|\omega'/\omega^2| \ll 1$. We find that for large k this approximation is almost always valid.

[8]With the exception of the β_0 model, which is simply ΛCDM.

$$\omega_{\beta_4} = \pm \frac{k}{\sqrt{2}\mathcal{H}}. \tag{72}$$

Therefore, the only single-parameter models without instabilities at early times are the β_2 and β_4 models. Using the rules discussed above, we can now extend these results to the rest of the bigravity parameter space.

Since much of the power of bigravity lies in its potential to address the dark energy problem in a technically-natural way, let us first consider models without an explicit g-metric cosmological constant, i.e., $\beta_0 = 0$. On the finite branch, all such models with $\beta_1 \neq 0$ reduce, at early times, to the β_1 model, which has an imaginary eigenfrequency for large k (69) and is therefore unstable in the early Universe. Hence the finite-branch $\beta_1\beta_2\beta_3\beta_4$ model and its subsets with $\beta_1 \neq 0$ are all plagued by instabilities. All of these models have viable background evolution [42]. This leaves the $\beta_2\beta_3\beta_4$ model; this is stable on the finite branch as long as $\beta_2 \neq 0$, but its background is not viable. We conclude that there are no models with $\beta_0 = 0$ which live on a finite branch, have a viable background evolution, and predict stable linear perturbations at all times.

This conclusion has two obvious loopholes: either including a cosmological constant, β_0, or turning to an infinite-branch model. We first consider including a nonzero cosmological constant, although this may not be as interesting theoretically as the models which self accelerate. Adding a cosmological constant can change the stability properties, although it turns out not to do so in the finite-branch models with viable backgrounds. In the $\beta_0\beta_1$ model, the eigenfrequencies,

$$\omega_{\beta_0\beta_1} = \pm \frac{k\sqrt{9r^4 + 2(\beta_0/\beta_1)r + 12r^2 - 1}}{\mathcal{H}(3r^2 + 1)}, \tag{73}$$

are unaffected by β_0 at early times and therefore still imply unstable modes in the asymptotic past. This result extends (at early times) to the rest of the bigravity parameter space with $\beta_0, \beta_1 \neq 0$. No other finite-branch models yield viable backgrounds. Therefore, all of the solutions on a finite branch, for any combination of parameters, are either unviable (in the background) or linearly unstable in the past.

Let us now turn to the infinite-branch models. In this case, it turns out that there exists a small class of viable models which has stable cosmological evolution: models where the only nonvanishing parameters are β_0, β_1, and β_4, as well as the self-accelerating $\beta_1\beta_4$ model. Here, r evolves from infinity in the past and asymptotes to a finite de Sitter value in the future. As mentioned in Ref. [42], a nonvanishing β_2 or β_3 would not be compatible with the requirement $\lim_{t\to-\infty}\Omega_{\mathrm{tot}} = 1$. This can be seen directly from Eq. (22) in the limit of large r. For these $\beta_0\beta_1\beta_4$ models we perform a similar eigenfrequency analysis and obtain

$$\omega_{\beta_0\beta_1\beta_4} = \pm \frac{k\sqrt{(9 + 2\beta_0\beta_4/\beta_1^2)r^4 + 2(\beta_0/\beta_1)r + 12r^2 - 1 + (\beta_4/\beta_1)[2(\beta_4/\beta_1)r^6 - 6r^5 - 8r^3]}}{\mathcal{H}(3r^2 + 1 - 2(\beta_4/\beta_1)r^3)}. \tag{74}$$

Restricting ourselves to the self-accelerating models (i.e., $\beta_0 = 0$), we obtain

$$\omega_{\beta_1\beta_4} = \pm \frac{k\sqrt{9r^4 + 12r^2 - 1 + (\beta_4/\beta_1)[2(\beta_4/\beta_1)r^6 - 6r^5 - 8r^3]}}{\mathcal{H}(3r^2 + 1 - 2(\beta_4/\beta_1)r^3)}. \tag{75}$$

Notice that, for large r, the eigenvalues (74)–(75) reduce to the expression (72) for ω_{β_4}. This frequency is real, and therefore the $\beta_1\beta_4$ model, as well as its generalization to include a cosmological constant, is stable on the infinite branch at early times.

It is interesting to note that the eigenfrequencies can also be written as

$$\omega_{\beta_0\beta_1\beta_4} = \pm \frac{ik}{\mathcal{H}}\sqrt{\frac{r''}{3r'}}. \tag{76}$$

Therefore, the condition for the stability of this model in the infinite branch, where $r' < 0$, is simply $r'' > 0$. One might wonder whether this expression for ω is general or model specific. While it does not hold for the β_2 and β_3 models, Eqs. (70)–(71), it is valid for all of the submodels of $\beta_0\beta_1\beta_4$, including Eqs. (69) and (72). We can see from this, for example, that the finite-branch ($r' > 0$) β_1 model is unstable at early times because initially r'' is positive. In Fig. 1 we show schematically the evolution of the $\beta_1\beta_4$ model on the finite and infinite branches. The stability condition on either branch is $r''/r' = dr'/dr < 0$. For the parameters plotted, $\beta_1 = 0.5$ and $\beta_4 = 1$, one can see graphically that this condition is met, and hence the model is stable, only at late times on the finite branch but for all times on the infinite branch. Our remaining task is to extend this to other parameters.

Let us now prove that the infinite-branch $\beta_1\beta_4$ model is stable at all times for all viable choices of the parameters. In a previous work we showed that background viability and the condition that we live on the infinite branch restrict us to the parameter range $0 < \beta_4 < 2\beta_1$ [35,42]. We have already seen that at early times, $r \to \infty$, and the eigenfrequencies match those in the β_4 model (72) which are purely real. What about later times? The discriminant is positive and hence the model is stable whenever $r > 1$. The question then is the following: do the infinite-branch models in this region of the parameter space always have $r > 1$?

The answer is yes. To see this, consider the algebraic equation for r, which can be determined by combining the g- and f-metric Friedmann equations [see Eq. (2.17) of Ref. [35]], and focus on the asymptotic future by taking $\rho \to 0$. This gives

$$\beta_4 r_c^3 - 3\beta_1 r_c^2 + \beta_1 = 0, \tag{77}$$

where r_c is the far-future value of r. When $\beta_4 = 2\beta_1$ exactly, this is solved by $r_c = 1$. We must then ask whether for $0 < \beta_4 < 2\beta_1$, r_c remains greater than 1. Writing $p \equiv r_c - 1$, using Descartes' rule of signs, and restricting ourselves to $0 < \beta_4 < 2\beta_1$, we can see that p has one positive root, i.e., there is always exactly one solution with $r_c > 1$ in that parameter range. Therefore, in all infinite-branch solutions with $0 < \beta_4 < 2\beta_1$, r evolves to some $r_c > 1$ in the asymptotic future. We conclude that all of the infinite-branch $\beta_1\beta_4$ cosmologies which are viable at the background level are also linearly stable at all times, providing a clear example of a bimetric cosmology which is a viable competitor to ΛCDM.

The models without quadratic- and cubic-order interactions were also discussed in Ref. [47]. Interestingly, for those models, as well as other models where only one of the three parameters β_1, β_2, and β_3 is nonvanishing, the authors found that if one metric is an Einstein metric, i.e., $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, then the other metric is proportional to it. This automatically avoids pathologic solutions when choosing the nondynamical constraint in the Bianchi constraint [47] (which are, however, explicitly avoided in the present work by imposing the dynamical constraint in order to find cosmological solutions that differ from ΛCDM).

VI. QUASISTATIC LIMIT OF INFINITE-BRANCH BIGRAVITY

In the previous section we found that most bigravity models which are viable at the background level suffer from a linear instability at early times. A prominent exception was the model with the β_1 and β_4 interactions turned on (i.e., the first-order interaction between the two metrics and the f-metric cosmological constant) in the case of solutions on the infinite branch, where r evolves from infinity at early times to a finite value in the far future. This means that we can safely use the QS approximation for the subhorizon modes in the infinite-branch $\beta_1\beta_4$ model, hereafter referred to (interchangeably) as infinite-branch bigravity (IBB); in this section, we compare the QS limit of this model to observations.

The background cosmology of IBB was studied in Refs. [35,42]. Reference [35] further studied the linear perturbations and quasistatic limit, finding results in agreement with those presented in the following two sections. Using the Friedmann equations, it has been shown that the background cosmology only selects a curve in the parameter space, given by

$$\beta_4 = \frac{3\Omega_{\mathrm{mg},0}\beta_1^2 - \beta_1^4}{\Omega_{\mathrm{mg},0}^3}, \tag{78}$$

where we recall that $\Omega_{\mathrm{mg},0} \equiv \beta_1 r_0$ is the present-day effective density of dark energy that appears in the Friedmann equation (15). This does not need to coincide with the value of Ω_Λ derived in the context of ΛCDM models; indeed, the best-fit value to the background data is $\Omega_{\mathrm{mg},0} = 0.84^{+0.03}_{-0.02}$ [42]. Furthermore, as discussed in the previous subsection, to ensure that we are on the infinite branch we impose the condition $0 < \beta_4 < 2\beta_1$.

The QS-limit equations in terms of δ now read [recall $B_1 = \beta_1 + \beta_4 r^3$; see Eq. (23)]

$$k^2\Psi = \frac{(\frac{3}{2}a^2\beta_1(9\beta_1(r^2-1)r^2 + (r^2-2)\mathcal{B}) - \frac{1}{2}k^2 r(9\beta_1(r^2-1) + (8r^2+9)\mathcal{B}))\Omega_m\mathcal{H}^2}{3a^2\beta_1(r^2+1)^2\mathcal{B} + k^2(2r^3\mathcal{B} + 3\beta_1(r^2-1)r + 3r\mathcal{B})}\delta, \tag{79}$$

$$k^2\Phi = \frac{(3a^2\beta_1(r^2+1)\mathcal{B} + \frac{1}{2}k^2 r(9\beta_1(r^2-1) + (4r^2+9)\mathcal{B}))\Omega_m\mathcal{H}^2}{2a^2\beta_1(r^2+1)^2\mathcal{B} + k^2(2r^3\mathcal{B} + 3\beta_1(r^2-1)r + 3r\mathcal{B})}\delta, \tag{80}$$

$$k^2\Psi_f = \frac{(-3a^2\beta_1\mathcal{B}(9\beta_1(r^2-1)r^2 + (r^2-2)\mathcal{B}) - k^2 r\mathcal{B}(9\beta_1(r^2-1) + 5\mathcal{B}))\Omega_m\mathcal{H}^2}{2a^2\beta_1(r^2+1)^2\mathcal{B}(9\beta_1(r^2-1) + \mathcal{B}) + k^2 r(3\beta_1(r^2-1) + (2r^2+3)\mathcal{B})(9\beta_1(r^2-1) + \mathcal{B})}\delta, \tag{81}$$

$$k^2\Phi_f = \frac{(3a^2\beta_1(r^2+1)\mathcal{B} + k^2 r\mathcal{B})\Omega_m\mathcal{H}^2}{2a^2\beta_1(r^2+1)^2\mathcal{B} + k^2(2r^3\mathcal{B} + 3\beta_1(r^2-1)r + 3r\mathcal{B})}\delta, \tag{82}$$

$$k^2\Delta E = \frac{(2k^2r^2\mathcal{B} - \frac{9}{2}a^2\beta_1 r(3\beta_1(r^2-1) + \mathcal{B}))\Omega_m\mathcal{H}^2}{2a^4\beta_1^2(r^2+1)^2\mathcal{B} + \beta_1 a^2k^2 r(3\beta_1(r^2-1) + (2r^2+3)\mathcal{B})}\delta, \tag{83}$$

where we have used the combination $\mathcal{B} \equiv 3\beta_1(r^2+1) - 2B_1$ to further simplify the expressions.

In order to compare with observations, we calculate two common modified gravity parameters: the anisotropic stress, $\eta \equiv -\Phi/\Psi$, and the effective gravitational coupling for the growth of structures, $Y \equiv -2k^2\Psi/(3\mathcal{H}^2\Omega_m\delta_m)$. In general relativity with ΛCDM, $\eta = Y = 1$, while in $\beta_1\beta_4$ IBB they possess the following structure,

$$\eta = H_2 \frac{1 + H_4(k/\mathcal{H})^2}{1 + H_3(k/\mathcal{H})^2}, \qquad (84)$$

$$Y = H_1 \frac{1 + H_3(k/\mathcal{H})^2}{1 + H_5(k/\mathcal{H})^2}, \qquad (85)$$

with coefficients

$$H_1 = -\frac{9\beta_1(r^2-1)r^2 + (r^2-2)\mathcal{B}}{2(r^2+1)^2\mathcal{B}}, \qquad (86)$$

$$H_2 = -\frac{2(r^2+1)\mathcal{B}}{9\beta_1(r^2-1)r^2 + (r^2-2)\mathcal{B}}, \qquad (87)$$

$$H_3 = -\frac{\mathcal{H}^2 r(9\beta_1(r^2-1) + (8r^2+9)\mathcal{B})}{3a^2\beta_1(9\beta_1(r^2-1)r^2 + (r^2-2)\mathcal{B})}, \qquad (88)$$

$$H_4 = \frac{\mathcal{H}^2 r(9\beta_1(r^2-1) + (4r^2+9)\mathcal{B})}{6a^2\beta_1(r^2+1)\mathcal{B}}, \qquad (89)$$

$$H_5 = \frac{\mathcal{H}^2 r(6r^2\mathcal{B} + 9\beta_1(r^2-1) + 9\mathcal{B})}{6a^2\beta_1(r^2+1)^2\mathcal{B}}. \qquad (90)$$

As a side remark, we note that in this model the asymptotic past corresponds to the limit $r \to \infty$ and $r' \to -\frac{3}{2}r$, i.e., $r \to a^{-3/2}$. This implies that $b \sim a^{-1/2}$, i.e., the second metric initially collapses while our metric expands. On the approach to the final de Sitter stage, r approaches a constant r_c, so the scale factors a and b both expand exponentially. The f-metric scale factor, b, therefore undergoes a bounce in this model.

This bounce has an unusual consequence. Recall from Eq. (14) that, after imposing the Bianchi identity, we have $f_{00} = -\dot{b}^2/\mathcal{H}^2$. Therefore, when b bounces, f_{00} becomes zero: at that one point, the lapse function of the f metric vanishes.[9] We believe, however, that this does not render the solution unphysical, for the following reasons. First, the f metric does not couple to matter and so, unlike the g metric, it does not have a geometric interpretation. A singularity in the f metric therefore does not necessarily imply a singularity in observable quantities. In fact, we find

[9]Moreover, the square root of this, $\dot{b}/\mathcal{H}$, appears in the mass terms. This quantity starts off negative at early times and then becomes positive.

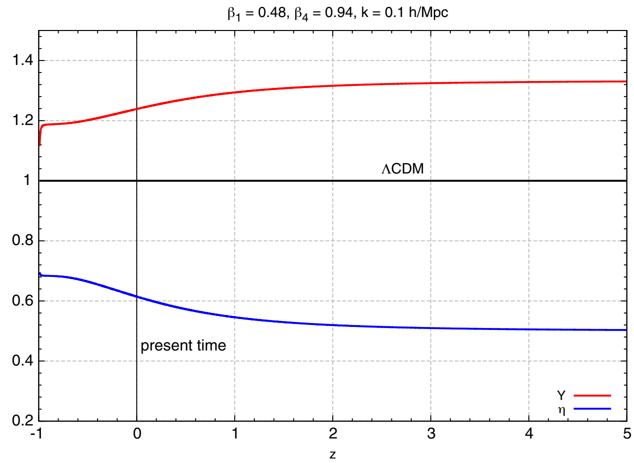

FIG. 2 (color online). The modified-gravity parameters, Y and η, for the $\beta_1\beta_4$ IBB model, from $z = 5$ until the asymptotic (de Sitter) future. Notice that the parameters approach a constant late-time value until a late era of horizon exit, when the $k = 0.1h/\mathrm{Mpc}$ mode becomes superhorizon and the QS limit breaks down. The horizontal line corresponds to the ΛCDM prediction for Y and η, and the vertical line is the present day. These curves are very weakly dependent on k. For concreteness, we use the best-fit values $\beta_1 = 0.48$ and $\beta_4 = 0.94$, calculated in Sec. VII.

no singularity in any of our background or perturbed variables. Second, although the Riemann tensor for the f metric is singular when $f_{00} = 0$, the Lagrangian density $\sqrt{-\det f}R_f$ remains finite and nonzero at all times, so the equations of motion can be derived at any points in time.

In the asymptotic past, every infinite-branch $\beta_1\beta_4$ model satisfies

$$\lim_{N\to-\infty} \eta = \frac{1}{2} \quad \text{and} \quad \lim_{N\to-\infty} Y = \frac{4}{3} \qquad (91)$$

and therefore does not reduce to the standard ΛCDM. In the future one finds $\eta \to 1$ if k is kept finite, but this is somewhat fictitious: for any finite k there will be an epoch of horizon exit in the future after which the subhorizon QS approximation breaks down. We can see both this asymptotic past and future behavior in Fig. 2, although the late-time approach of η to unity is not easily visible.

VII. COMPARISON TO MEASURED GROWTH DATA

In this section we compare the predictions in the quasistatic approximation to the measured growth rate. In Ref. [35], we discussed the numerical results of the modified-gravity parameters, Eqs. (84)–(85), for $\beta_1\beta_4$ infinite-branch bigravity and their early time limits,[10] and compared to the data. Although we found strong deviations

[10]Note that Ref. [35] uses a slightly different effective gravitational constant, $Q \equiv \eta Y$.

from the ΛCDM values, the model is at present still in agreement with the observed growth data. However, as we mentioned, future experiments will be able to distinguish between the predictions of the ΛCDM and bimetric gravity for η and Y.

We use the data set compiled by Ref. [48] containing the current measurements of the quantity

$$f(z)\sigma_8(z) = f(z)G(z)\sigma_8, \qquad (92)$$

where $f(z) \equiv \delta'/\delta$ and $G(z)$ is the growth factor normalized to the present. The data come from the 6dFGS [49], LRG_{200}, LRG_{60} [50], BOSS [51], WiggleZ [52], and VIPERS [53] surveys. These measurements can be compared to the theoretical growth rate which follows from integrating Eq. (54) in the QS limit:

$$\delta_m'' + \delta_m'\left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right) - \frac{3}{2}Y(k)\Omega_m\delta_m = 0. \qquad (93)$$

The theoretically expected and observed data, t_i and d_i, respectively, can be compared to compute

$$\chi^2_{f\sigma_8} = \sum_{ij}(d_i - \sigma_8 t_i)C^{-1}_{ij}(d_j - \sigma_8 t_j), \qquad (94)$$

where C_{ij} denotes the covariance matrix. Since no model-free constraints on σ_8 exist, one can remove this dependency with a marginalization over positive values which can be performed analytically:

$$\chi^2_{f\sigma_8} = S_{20} - \frac{S_{11}^2}{S_{02}} + \log S_{02} - 2\log\left(1 + \mathrm{Erf}\left(\frac{S_{11}}{\sqrt{2S_{02}}}\right)\right). \qquad (95)$$

Here, $S_{11} = d_i C^{-1}_{ij} t_j$, $S_{20} = d_i C^{-1}_{ij} d_j$, and $S_{02} = t_i C^{-1}_{ij} t_j$. Note that Y is (weakly) scale dependent but the current observational data are averaged over a range of scales. For the computation of the likelihood, we assume an average scale $k = 0.1h/\mathrm{Mpc}$.

As shown in Fig. 3, the confidence region obtained from the growth data is in agreement with type Ia SNe data (see Ref. [42] for the likelihood from the SCP Union 2.1 Compilation of SNe Ia data [54]). The growth data alone provides $\beta_1 = 0.40^{+0.14}_{-0.15}$ and $\beta_4 = 0.67^{+0.31}_{-0.38}$ with a $\chi^2_{\mathrm{min}} = 9.72$ (with nine degrees of freedom) for the best-fit value and is in agreement with the SNe Ia likelihood. The likelihood from growth data is, however, a much weaker constraint than the likelihood from background observations. Thus, the combination of both likelihoods, providing $\beta_1 = 0.48^{+0.05}_{-0.16}$ and $\beta_4 = 0.94^{+0.11}_{-0.51}$, is similar to the SNe Ia result alone.

Note that those favored parameter regions were obtained by integrating the two-dimensional likelihood and are not Gaussian distributed due to the degeneracy in the

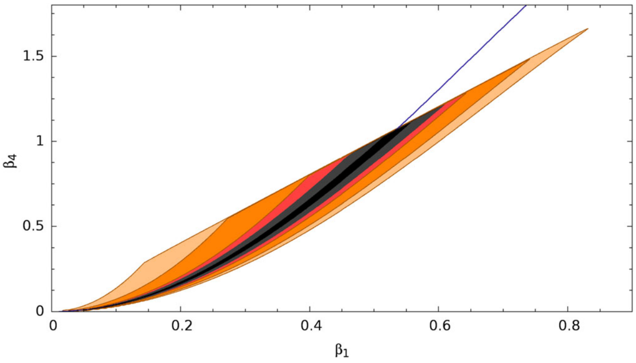

FIG. 3 (color online). Likelihood from measured growth rates, where the gray, light gray and lighter gray filled regions correspond to 68%, 95%, and 99.7% confidence levels. Both black (68%) and dark gray (99.7%) regions illustrate the combination of the likelihoods from measured growth data and type Ia supernovae. The blue line indicates the degeneracy curve corresponding to the background best-fit points. Note that the viability condition enforces the likelihood to vanish when $\beta_4 > 2\beta_2$.

parameters β_1 and β_4 [see Eq. (78)]. This degeneracy curve is unaffected by additional growth data and is still parametrized by the SNe Ia result $\Omega_{m0} = 1 - \Omega_{\mathrm{mg}0} = 0.16^{+0.02}_{-0.03}$ (note that the combination of the most likely parameters predicts, however, $\Omega_{m0} = 0.18$). According to Eq. (27), the EOS of modified gravity, w_{mg}, is best fit by $w_0 = -0.79$ and $w_a = 0.21$, where we use the Chevallier-Polarski-Linder (CPL) parametrization [55,56],

$$w(z) = w_0 + w_a z/(1+z). \qquad (96)$$

However, since we approximated the EOS near the present time, we cannot expect Eq. (96) to fit the real EOS well at early times or in the future. As shown in Fig. 4, the fit is in

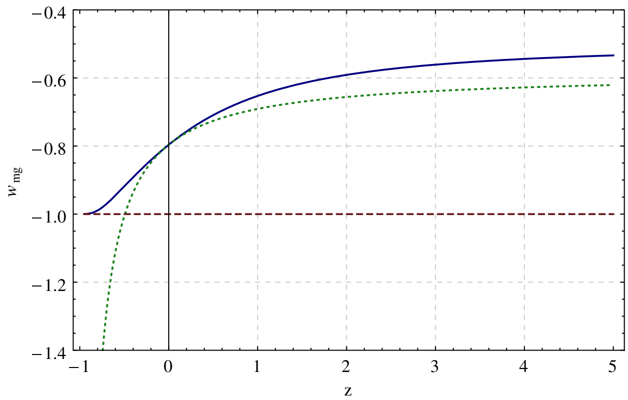

FIG. 4 (color online). The equation of state (EOS, solid blue) in the IBB model with $\beta_1 = 0.48$, $\beta_4 = 0.94$, along with the CPL approximation $w(a) \approx w_0 + w_a z/(1+z)$ (dotted green) where w_a corresponds to the slope at present time. In the asymptotic future, w_{mg} tends to −1, i.e., the EOS of a cosmological constant (dashed red).

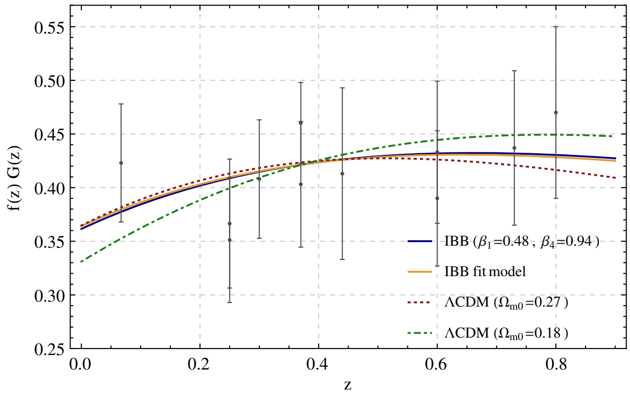

FIG. 5 (color online). Growth history for the best-fit IBB model (solid blue) with $\beta_1 = 0.48$ and $\beta_4 = 0.94$ compared to the result obtained from the best fit (97) (solid orange) with $\gamma_0 = 0.47$ and $\alpha = 0.21$, and the ΛCDM predictions for $\Omega_{m0} = 0.27$ (dotted red) and $\Omega_{m0} = 0.18$ (dotted-dashed green). The latter value for the matter density is similar to that corresponding to IBB. Note that a vertical shift of each single curve is possible due to the marginalization over σ_8. Here, we choose that value for σ_8 for each curve individually such that it fits the data best. The growth histories are compared to observed data compiled by Ref. [48].

fact valid in the past only up to $z \approx 0.5$, while in the future the limit $w_{\rm mg} \to -1$ is lost.

For one specific choice of parameters, corresponding to the best-fit values, we compared the quantity $f(z)G(z)$ with the measured growth data and fits from ΛCDM in Fig. 5. Although the modified-gravity parameters differ significantly from the ΛCDM result $Y = \eta = 1$, the prediction for $f(z)G(z)$ is in good agreement with measurements and is close to the ΛCDM result.

The difference between the growth rate $f(z)$ in the best-fit model and ΛCDM is, however, quite large. Therefore, the common approximation $f \approx \Omega_m^\gamma$ fits the growth rate very badly, even if the range in the redshift is small [where $f(z)$ is still smaller than unity] [35]. We have found a two-parameter scheme,

$$f \approx \Omega_m^{\gamma_0}\left(1 + \alpha \frac{z}{1+z}\right), \qquad (97)$$

which is able to provide a much better fit (see Fig. 5). Using this approximation, we obtain $\gamma_0 = 0.47$ and $\alpha = 0.21$ as best-fit values.

VIII. CONCLUSIONS AND OUTLOOK

We have investigated the stability of linear cosmological perturbations in bimetric gravity. Many models with viable background cosmologies exhibit an instability on small scales until fairly recently in cosmic history. However, we also found a class of viable models which are stable at all times: IBB with the interaction parameters β_1 and β_4 turned on. In these models, the ratio $r = b/a$ of the two scale factors decreases from infinity to a finite late-time value. IBB is able to fit observations at the level of both the background (type Ia supernovae) and linear, subhorizon perturbations (growth histories) without requiring an explicit cosmological constant for the physical metric, although the region of likely parameters is small. The combination of both likelihoods yields the parameter constraints $\beta_1 = 0.48^{+0.05}_{-0.16}$ and $\beta_4 = 0.94^{+0.11}_{-0.51}$. IBB with these best-fit parameters predicts $\Omega_{m0} = 0.18$ and an equation of state $w(z) \approx -0.79 + 0.21z/(1+z)$. The growth rate, $f \equiv d\ln\delta/d\ln a$, is approximated very well by the two-parameter fit $f(z) \approx \Omega_m^{0.47}[1 + 0.21z/(1+z)]$. Additionally, the two main modified-gravity parameters, the anisotropic stress η and modification to Newton's constant Y, tend to $\eta = \frac{1}{2}$ and $Y = \frac{4}{3}$ for early times and therefore do not reduce to the standard ΛCDM result. The predictions of this two-parameter model will be testable by near-future experiments [57].

On the surface, our results would seem to place in jeopardy a large swath of bigravity's parameter space, such as the minimal β_1-only model which is the only single-parameter model that is viable at the background level [42]. It is important to emphasize that the existence of such an instability does not automatically rule these models out. It merely impedes our ability to use linear theory on deep subhorizon scales (recall that the instability is problematic specifically for large k). Models that are not linearly stable can still be realistic if only the gravitational potentials become nonlinear, or even if the matter fluctuations also become nonlinear but in such a way that their properties do not contradict observations. The theory can be saved if, for instance, the instability is softened or vanishes entirely when nonlinear effects are taken into account. We might even expect such behavior: bigravity models exhibit a Vainshtein mechanism [45,46] which restores general relativity in environments where the new degrees of freedom are highly nonlinear.

Consequently there are two very important questions for future work: can these unstable models still accurately describe the real Universe, and if so, how can we perform calculations for structure formation?

Until these questions are answered, the $\beta_1\beta_4$ infinite-branch model seems to be the most promising target at the moment for studying bigravity. Because this instability appears to be absent in the superhorizon limit, it may also be feasible to test the unstable models using large-scale modes.

What other escape routes are there? Throughout this analysis we have assumed that only one of the metrics couples to matter. A possible way to cure bimetric gravity from instabilities while only allowing one nonvanishing β parameter could be to allow matter to couple to both metrics [26,58]. In such a theory, the finite-branch solutions asymptote to a nonzero value for r in the far past, so these theories may avoid the instability. This would introduce a new coupling parameter, so if only one β parameter is turned on, there are two free parameters and such a model is arguably as predictive as the $\beta_1\beta_4$ model. Unfortunately, this way of double-coupling would introduce a ghost [59–61]. However, the authors in Ref. [59] proposed a coupling to matter using a new composite metric which is free of the ghost in the decoupling limit. The cosmological background solutions in bigravity with this type of coupling together with a comparison to observations were studied in [62] (see also Ref [63] for the case of massive gravity). The consequences for linear perturbations will be discussed in a future work (in preparation).

ACKNOWLEDGMENTS

We thank Antonio de Felice, Jonas Enander, Pedro Ferreira, Emir Gumrukcuoglu, Tomi Koivisto, Martin Kunz, Macarena Lagos, Michele Maggiore, Edvard Mörtsell, Malin Renneby, Jeremy Sakstein, Ippocratis Saltas, Ignacy Sawicki, and Takahiro Tanaka for many useful discussions. L. A. and F. K. acknowledge support from Deutsche Forschungsgemeinschaft (DFG) through the TR33 project "The Dark Universe." Y. A. is supported by the European Research Council (ERC) Starting Grant No. StG2010-257080. M. M. acknowledges funding from the Swiss National Science Foundation. A. R. S. is supported by the David Gledhill Research Studentship, Sidney Sussex College, University of Cambridge; and by the Isaac Newton Fund and Studentships, University of Cambridge. We acknowledge the use of resources from the Norwegian national supercomputing facilities, NOTUR.

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Publication 4

Cosmological viability of massive gravity with generalized matter coupling

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JCAP 1504 (2015) 027, arXiv:1409.8300

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Cosmological viability of massive gravity with generalized matter coupling

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Received December 13, 2014
Revised February 26, 2015
Accepted March 27, 2015
Published April 20, 2015

Abstract. There is a no-go theorem forbidding flat and closed FLRW solutions in massive gravity on a flat reference metric, while open solutions are unstable. Recently it was shown that this no-go theorem can be overcome if at least some matter couples to a hybrid metric composed of both the dynamical and the fixed reference metric. We show that this is not compatible with the standard description of cosmological sources in terms of effective perfect fluids, and the predictions of the theory become sensitive either to the detailed field-theoretical modelling of the matter content or to the presence of additional dark degrees of freedom. This is a serious practical complication. Furthermore, we demonstrate that viable cosmological background evolution with a perfect fluid appears to require the presence of fields with highly contrived properties. This could be improved if the equivalence principle is broken by coupling only some of the fields to the composite metric, but viable self-accelerating solutions due only to the massive graviton are difficult to obtain.

Keywords: modified gravity, dark energy theory

ArXiv ePrint: 1409.8300

JCAP04(2015)027

 doi:10.1088/1475-7516/2015/04/027

Contents

1 Introduction

Massive gravity has a long history [1–5], but only recently has the fully nonlinear, consistent theory of a massive graviton been constructed by de Rham, Gabadadze, and Tolley (dRGT) [6–11] (see ref. [12] for a comprehensive review). However, this theory does not possess flat or closed Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological solutions with a flat reference metric [13], and the solutions which do exist, by choosing open curvature or a different reference metric, are unstable to the Higuchi ghost [14] or other linear and nonlinear instabilities [15–20].

The search for viable cosmologies with a massive graviton has involved two routes. One is to extend dRGT by adding extra degrees of freedom. For example, these problems are at least partially cured in bigravity, where the second metric is given dynamics [21–33]. Other extensions of massive gravity, such as quasidilaton [34], varying-mass [13, 35], nonlocal [36–38], and Lorentz-violating [39, 40] massive gravity, also seem to possess improved cosmological behavior. The other approach is to give up on homogeneity and isotropy. While FLRW solutions are important for their mathematical simplicity, which renders them easy both to compute and to compare to observations, the Universe could in principle have anisotropies which have such low amplitude, are so much larger than our horizon, or both, that we cannot readily observe them. Remarkably, these cosmologies not only exist in massive gravity but are locally (i.e., within the horizon) arbitrarily close to the standard FLRW case [13]. The general scenario of an FLRW metric with inhomogeneous Stückelberg fields has been derived in refs. [41, 42]. This includes, but is not limited to, the case in which the reference metric is still Minkowski space, but only has the canonical form $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ in coordinates where $g_{\mu\nu}$ is not of the FLRW form [43]. The inhomogeneous and anisotropic solutions are reviewed thoroughly in ref. [12]. See ref. [44] for a review of cosmology in massive gravity and some of its extensions.

Recently a workaround that allows consistent flat FLRW solutions in the context of dRGT massive gravity — i.e., the theory with only a single, massive graviton — was discovered in ref. [45]. This solution is based on the fact that massive gravity contains a fixed

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reference metric, and matter can in principle couple to both metrics [46–48], although care must be taken to ensure this coupling does not reintroduce the exorcised ghost [45, 49]. In this scenario, matter is coupled to an effective or Jordan-frame metric given by

$$g^{\text{eff}}_{\mu\nu} \equiv \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\alpha} X^{\alpha}_{\nu} + \beta^2 \eta_{\mu\nu}, \tag{1.1}$$

where $g_{\mu\nu}$ is the dynamical metric, $\eta_{\mu\nu}$ is the Minkowski reference metric, and $X^{\mu}_{\nu} \equiv (\sqrt{g^{-1}\eta})^{\mu}_{\nu}$. This effective metric was arrived at in the vielbein formulation by a complementary derivation in ref. [50], and is claimed to be ghost-free at least within the effective theory's régime of validity [45, 51, 52]. In ref. [45], it was shown that flat FLRW solutions exist when $\alpha, \beta \neq 0$, and a worked example was presented in which one or more scalar fields couples to $g^{\text{eff}}_{\mu\nu}$. The matter coupling has since been studied in the context of bigravity in refs. [53, 54], where it was shown to be consistent with observational data of the cosmic expansion.

In this paper we explore the basic properties of these newly-allowed massive cosmologies from an observationally-oriented standpoint. Unusually, the proof that FLRW cosmologies exist leans heavily on the choice of a fundamental field as the matter source coupled to the effective metric. In a standard late-Universe setup where matter is described by a perfect fluid with constant equation of state w (or even more generally when w only depends on the scale factor), this result does not hold, and FLRW solutions are constrained to be nondynamical, just as in standard dRGT. More generally, the pressure of at least one of the matter components coupled to $g^{\text{eff}}_{\mu\nu}$ must depend on something besides the scale factor — such as the lapse or the time derivative of the scale factor — for massive-gravity cosmologies to be consistent. This is why fields, which have kinetic terms where the lapse necessarily appears, are good candidates to obtain sensible cosmological solutions. Consequently the standard techniques of late-time cosmology cannot be applied to this theory. We emphasize this does not necessarily imply that cosmological solutions do not exist, but rather that we must either employ a more sophisticated description of the matter sector or include new degrees of freedom in order to obtain realistic models which can be reliably confronted with data.

Our focus here is on models with an extra, "dark" scalar degree of freedom coupled to $g^{\text{eff}}_{\mu\nu}$. While we do not aim to rule these out, we show that these solutions exhibit pathologies in the early- and late-time limits if all matter couples to the effective metric, and the scalar-field physics would need to be highly contrived to avoid these issues, although these pathologies are largely avoided if the equivalence principle is broken and only the new dark sector couples to $g^{\text{eff}}_{\mu\nu}$. Moreover, the reliance on a dark sector which may well be gravitationally subdominant and high-energy implies a violation of the decoupling principle, in which the low-energy expansion of the Universe should not be overly sensitive to high-energy physics.

During the completion of this paper, ref. [55] appeared which studied the background cosmology of this theory with a scalar field coupled to the effective metric, and demonstrated its perturbative stability. We agree with their results wherever we overlap. Our emphasis differs, however, as we focus on the effects of the perfect fluids, particularly dust and radiation, expected to be gravitationally dominant in the late Universe.

The rest of this paper is organized as follows. In section 2 we derive and discuss the cosmological evolution equations in this theory. In section 3 we elucidate the conditions under which the no-go theorem is violated and dynamical cosmological solutions exist. We discuss in section 4 some of the nonintuitive features of the Einstein-frame formulation of the theory, and how these are resolved in the Jordan-frame description. In section 5 we study cosmologies containing only a scalar field, and generalize this to include a perfect fluid

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coupled to the effective metric in section 6. In section 7 we consider an alternative setup in which the scalar field couples to the effective metric while the perfect fluid couples to the dynamical metric. We discuss our results and conclude in section 8.

2 Cosmological backgrounds

If all matter fields couple to $g^{\rm eff}_{\mu\nu}$, the theory is defined by the action

$$S = -\frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}R + m^2 M_{\rm Pl}^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(X\right) + \int d^4x\sqrt{-g_{\rm eff}}\mathcal{L}_m\left(g_{\rm eff},\Phi\right), \tag{2.1}$$

where e_n are the elementary symmetric polynomials of the eigenvalues of X, β_n are dimensionless free parameters characterizing the strength of the different graviton interactions, and Φ represents the matter fields. The Einstein equation for this theory was derived in ref. [54] and can be written in the form[1]

$$(X^{-1})^{(\mu}{}_{\alpha}G^{\nu)\alpha}+m^2\sum_{n=0}^{3}(-1)^n\beta_n g^{\alpha\beta}(X^{-1})^{(\mu}{}_{\alpha}Y^{\nu)}_{(n)\beta}=\frac{\alpha}{M_{\rm Pl}^2}\det(\alpha+\beta X)(\alpha(X^{-1})^{(\mu}{}_{\alpha}T^{\nu)\alpha}+\beta T^{\mu\nu}), \tag{2.2}$$

where the stress-energy tensor is defined in the usual way with respect to the effective metric,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g_{\rm eff}}}\frac{\delta\left[\sqrt{-g_{\rm eff}}\mathcal{L}_m\left(g^{\rm eff}_{\mu\nu},\Phi\right)\right]}{\delta g^{\rm eff}_{\mu\nu}}, \tag{2.3}$$

and the matrices $Y_{(n)}$ are given by

$$\begin{aligned}
Y_{(0)} &\equiv \mathbb{1},\\
Y_{(1)} &\equiv X - \mathbb{1}\left[X\right],\\
Y_{(2)} &\equiv X^2 - X\left[X\right] + \frac{1}{2}\mathbb{1}\left(\left[X\right]^2 - \left[X^2\right]\right),\\
Y_{(3)} &\equiv X^3 - X^2\left[X\right] + \frac{1}{2}X\left(\left[X\right]^2 - \left[X^2\right]\right)\\
&\quad -\frac{1}{6}\mathbb{1}\left(\left[X\right]^3 - 3\left[X\right]\left[X^2\right] + 2\left[X^3\right]\right).
\end{aligned} \tag{2.4}$$

Notice that for diagonal metrics, including the FLRW metric, the symmetrization in the Einstein equation can be dropped and we can obtain a simpler version,

$$G^{\mu\nu} + m^2\sum_{n=0}^{3}(-1)^n\beta_n g^{\mu\alpha}Y^{\nu}_{(n)\alpha} = \frac{\alpha}{M_{\rm Pl}^2}\det(\alpha+\beta X)\left(\alpha T^{\mu\nu} + \beta X^{\mu}_{\alpha}T^{\nu\alpha}\right). \tag{2.5}$$

Let us assume a flat FLRW ansatz for $g_{\mu\nu}$ of the form

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{2.6}$$

[1] Our convention is that indices on the Einstein tensor $G^{\mu\nu}$ are raised with $g^{\mu\nu}$.

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and choose unitary gauge for the Stückelberg fields, $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, so the effective metric is given by

$$g^{\rm eff}_{\mu\nu}dx^\mu dx^\nu = -N^2_{\rm eff}(t)dt^2 + a^2_{\rm eff}(t)\delta_{ij}dx^i dx^j, \tag{2.7}$$

where the effective lapse and scale factor are related to N and a by

$$N_{\rm eff} = \alpha N + \beta, \qquad a_{\rm eff} = \alpha a + \beta. \tag{2.8}$$

We will define the Hubble rates for $g_{\mu\nu}$ and $g^{\rm eff}_{\mu\nu}$ by

$$H \equiv \frac{\dot{a}}{aN}, \qquad H_{\rm eff} \equiv \frac{\dot{a}_{\rm eff}}{a_{\rm eff}N_{\rm eff}}. \tag{2.9}$$

Notice that these are defined slightly differently than usual because of the inclusion of the lapse. This is because in diffeomorphism-invariant theories, such as general relativity, the lapse can be fixed by a gauge transformation and so corresponds to a choice of time coordinate. Indeed, in such a theory $\dot{a}/aN$ would simply be the Hubble rate defined in cosmic time (i.e., $\dot{a}/a$ with $N=1$). Because we do not have this freedom in massive gravity (once we have fixed the Stückelberg fields), we cannot freely choose a time coordinate in this way, and neither the lapse nor the time coordinate, t, has any physical meaning on its own. Instead these quantities will only appear through the combinations Ndt and $N_{\rm eff}dt$. This motivates the Hubble rates we have defined in eq. (2.9), which are simply $d\ln a/Ndt$ and $d\ln a_{\rm eff}/N_{\rm eff}dt$.

Now let us derive the cosmological equations of motion. The time component of eq. (2.2) yields the Friedmann equation,

$$3H^2 = \frac{\alpha\rho}{M^2_{\rm Pl}}\frac{a^3_{\rm eff}}{a^3} + m^2\left(\beta_0 + \frac{3\beta_1}{a} + \frac{3\beta_2}{a^2} + \frac{\beta_3}{a^3}\right), \tag{2.10}$$

where $\rho \equiv -g^{\rm eff}_{00}T^{00}$ is the density of the matter source coupled to $g^{\rm eff}_{\mu\nu}$.[2] The spatial component of eq. (2.2) gives us the acceleration equation,

$$3H^2 + \frac{2\dot{H}}{N} + \frac{\alpha p}{M^2_{\rm Pl}}\frac{N_{\rm eff}a^2_{\rm eff}}{Na^2} = m^2\left[\beta_0 + \beta_1\left(\frac{1}{N} + \frac{2}{a}\right) + \beta_2\left(\frac{2}{aN} + \frac{1}{a^2}\right) + \frac{\beta_3}{Na^2}\right], \tag{2.11}$$

where $p \equiv (1/3)g^{\rm eff}_{ij}T^{ij}$ is the pressure. Notice that the double coupling leads to a time-dependent coefficient multiplying the density and pressure terms in eqs. (2.10) and (2.11) and hence a varying gravitational constant for cosmological solutions. The Friedmann equation for the effective Hubble rate, $H_{\rm eff}$, can be determined from eq. (2.10) by the relation

$$H_{\rm eff} = \alpha\frac{Na}{N_{\rm eff}a_{\rm eff}}H, \tag{2.12}$$

which follows from eq. (2.8). Note that for practical purposes one could freely set $\alpha = 1$ here by rescaling $g_{\mu\nu}$, $M_{\rm Pl}$, and β_n; only the ratio β/α is physical [53].

Matter is covariantly conserved with respect to $g^{\rm eff}_{\mu\nu}$,

$$\nabla^{\rm eff}_\mu T^{\mu\nu} = 0, \tag{2.13}$$

[2]If we have additional matter coupled to $g_{\mu\nu}$, its density will enter the Friedmann equation (2.10) in the standard way.

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from which we can obtain the usual energy conservation equation written in terms of the effective metric,

$$\dot{\rho} + 3\frac{\dot{a}_{\rm eff}}{a_{\rm eff}}(\rho + p) = 0. \tag{2.14}$$

As in general relativity, this holds independently for each species of matter as long as we assume that interactions between species are negligible. Finally, we can take the divergence of the Einstein equation (2.2) with respect to $g_{\mu\nu}$ and specialize to the FLRW background to find, after imposing stress-energy conservation, the "Bianchi constraint,"

$$m^2 M_{\rm Pl}^2 a^2 P(a)\dot{a} = \alpha\beta a_{\rm eff}^2 p\dot{a}, \tag{2.15}$$

where we have defined

$$P(a) \equiv \beta_1 + \frac{2\beta_2}{a} + \frac{\beta_3}{a^2}. \tag{2.16}$$

This can equivalently be derived using eqs. (2.10), (2.11), and (2.14), as well as by leaving the Stückelberg fields unfixed (recall that we have been working in unitary gauge from the start) and taking their equation of motion [13, 45]. The pressure, p, appearing in eq. (2.15) is the total pressure of the Universe, or, if different species couple to different metrics, the total pressure of all matter coupled to $g^{\rm eff}_{\mu\nu}$.

3 When do dynamical solutions exist?

In the original, singly-coupled formulation of massive gravity, $\beta = 0$ and so the right-hand side of eq. (2.15) vanishes, with the result that a is constrained to be constant. This is nothing other than the no-go theorem on flat FLRW solutions in massive gravity. A nondynamical cosmology is, of course, still a solution when α and β are nonzero, in which case the values of a and N are determined from eqs. (2.10) and (2.11). The question is now under which circumstances the theory also allows for dynamical a.

To begin with, let us follow the standard techniques of cosmology by modeling the matter as a perfect fluid with $p = w\rho$, where w is either a constant or depends only on $a_{\rm eff}$. Assuming that $\dot{a} \neq 0$, eq. (2.15) becomes

$$m^2 M_{\rm Pl}^2 a^2 P(a) = \alpha\beta w a_{\rm eff}^2 \rho. \tag{3.1}$$

Notice that due to our equation of state, ρ is a function only of a (or, equivalently, $a_{\rm eff}$). To see this, consider eq. (2.14) in the form

$$\frac{d\ln\rho}{d\ln a_{\rm eff}} + 3\left[1 + w(a_{\rm eff})\right] = 0. \tag{3.2}$$

Integrating this will clearly yield $\rho = \rho(a_{\rm eff})$. Unless the left-hand side of eq. (3.1) has exactly the same functional form for $a_{\rm eff}$ as the right hand side (which is, e.g., the case when $w = -1/3$ and $\beta_2 = \beta_3 = 0$), this equation is not consistent with a time-varying a. The theory does therefore not give viable cosmologies where all matter coupled to $g^{\rm eff}_{\mu\nu}$ is described with an equation of state $p = w\rho$ if w is constant or depends only on the scale factor, as is the case with, e.g., a standard perfect fluid.

This conclusion is avoided if the pressure also depends on the lapse. In this case, eq. (2.15) becomes a constraint on the lapse, unlocking dynamical solutions.[3] The most obvious way to

[3]Another possibility is that the pressure depends on $\dot{a}$. Given the functional form of this dependence, the effective Hubble parameter in terms of $a_{\rm eff}$ can then be determined by combining eqs. (2.12), (2.15), and (4.3). We do not discuss this case any further.

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obtain a lapse-dependent pressure is to source the Einstein equations with a fundamental field rather than an effective fluid. This was exploited by ref. [45] to find dynamical cosmologies with a scalar field coupled to $g^{\rm eff}_{\mu\nu}$. We discuss this case in more detail below. Therefore, while physical dust-dominated solutions may exist, we must either include additional degrees of freedom or treat the dust in terms of fundamental fields.[4] The standard methods of late-time cosmology cannot be applied to doubly-coupled massive gravity.

4 Einstein frame vs. Jordan frame

Before examining the cosmological solutions in detail, it behooves us to further clarify the somewhat unusual differences between this theory's Einstein and Jordan frames. If all matter couples to the effective metric, then, as we show below, the Friedmann equation in the Einstein frame is completely independent of the matter content of the Universe (up to an integration constant which behaves like pressureless dust). In the Einstein-frame description, matter components with nonzero pressure affect the cosmological dynamics not through the Hubble rate but rather through the lapse, N. Because the lapse is involved in the transformation from the Einstein-frame H to the Jordan-frame $H_{\rm eff}$, cf. eq. (2.12), the Jordan-frame Friedmann equation (corresponding to the observable Hubble rate) does depend on matter.

We proceed to demonstrate this explicitly. Regardless of the functional form of p, and whether or not it depends on the lapse, as long as $\dot{a} \neq 0$ the pressure is constrained by eq. (2.15) to have an implicit dependence on a given by

$$p(a) = \frac{m^2 M_{\rm Pl}^2 a^2 P(a)}{\alpha\beta a_{\rm eff}^2}. \tag{4.1}$$

The continuity equation (2.14) can then be integrated to obtain

$$\rho(a) = \frac{C}{a_{\rm eff}^3} - \frac{3m^2 M_{\rm Pl}^2}{\beta a_{\rm eff}^3}\left(\frac{\beta_1}{3}a^3 + \beta_2 a^2 + \beta_3 a\right), \tag{4.2}$$

where C is a constant of integration. Inserting this into eq. (2.10), we find a generic form for the Einstein-frame Friedmann equation,

$$3H^2 = m^2\left(c_0 + 3\frac{c_1}{a} + 3\frac{c_2}{a^2} + \frac{c_3}{a^3}\right), \tag{4.3}$$

where we have defined the coefficients

$$\begin{aligned} c_0 &\equiv \beta_0 - \frac{\alpha}{\beta}\beta_1, \\ c_1 &\equiv \beta_1 - \frac{\alpha}{\beta}\beta_2, \\ c_2 &\equiv \beta_2 - \frac{\alpha}{\beta}\beta_3, \\ c_3 &\equiv \beta_3 + \frac{\alpha C}{m^2 M_{\rm Pl}^2}. \end{aligned} \tag{4.4}$$

[4] We note, however, that if the pressure of the dust is truly taken to be vanishing on large scales, then it would seem from the Bianchi constraint (2.15) that the no-go theorem is still a problem.

Notice that the functional forms of $p(a)$, $\rho(a)$, and $H^2(a)$ are completely independent of the energy content of the Universe, except for an integration constant scaling like pressureless matter. It is interesting to note that in the vacuum energy case (see ref. [53]) with $\beta_n = (\alpha/\beta)\beta_{n+1}$, all of the c_i coefficients apart from c_3 vanish. Therefore if the metric interactions took the form of a cosmological constant for $g^{\rm eff}_{\mu\nu}$, then the Einstein-frame Friedmann equation would scale as a^{-3}.

We emphasize that the dependence of the Einstein-frame quantities solely on a and the mass terms is interesting and is certainly unusual, but it does not mean that matter does not affect the cosmological dynamics; as discussed above, if all matter couples to $g^{\rm eff}_{\mu\nu}$, then the observable Hubble rate is $H_{\rm eff}$, and this *does* depend on the matter content. If not all matter were coupled to $g^{\rm eff}_{\mu\nu}$ — for example, if the standard model fields were coupled to $g_{\mu\nu}$ (as it is argued they should in ref. [45]) — then the expression (4.2) for $\rho(a)$ would only apply to the total density of the matter coupling to the effective metric, while the density of the fields coupled to the dynamical metric would appear in the Friedmann equation for H in the usual way.

5 Massive cosmologies with a scalar field

Let us turn to the properties of cosmological solutions. Recall that if we include matter whose pressure does not only depend on the scale factor, $a_{\rm eff}$, then the Bianchi constraint (2.15) may not rule out dynamical cosmological solutions. For a pressure that also depends on the lapse, eqs. (2.15) and (4.3) determine N and H, respectively. These can be used in turn to derive the Jordan-frame Friedmann equation. Because the lapse enters into the frame transformation (2.12), the Jordan frame can be sensitive to matter even though the Einstein frame is not. The lapse thus plays an important and novel role in massive gravity compared to general relativity.

As discussed above, lapse-dependent pressures are not difficult to obtain: they enter whenever we consider a fundamental field with a kinetic term. Consider a universe dominated by a scalar field, χ, with a canonical kinetic term and an arbitrary potential.[5] Its stress-energy tensor is given by

$$T^{\mu\nu} = \nabla^\mu_{\rm eff}\chi\nabla^\nu_{\rm eff}\chi - \left(\frac{1}{2}\nabla_\alpha\chi\nabla^\alpha_{\rm eff}\chi + V(\chi)\right)g^{\mu\nu}_{\rm eff}, \tag{5.1}$$

where $\nabla^\mu_{\rm eff} \equiv g^{\mu\nu}_{\rm eff}\nabla^{\rm eff}_\nu$ and $V(\chi)$ is the potential. The density and pressure associated to χ are

$$\rho_\chi = \frac{\dot\chi^2}{2N^2_{\rm eff}} + V(\chi), \qquad p_\chi = \frac{\dot\chi^2}{2N^2_{\rm eff}} - V(\chi). \tag{5.2}$$

The constraint (2.15) now has a new ingredient; the lapse, $N_{\rm eff}$, which appears through the scalar field pressure.[6]

One can then use the Bianchi identity to solve for the lapse and substitute it into the Friedmann equation to obtain an equation for the cosmological dynamics that does not

[5] We note here that, for illustrative purposes, all of our discussions of a scalar field will assume that it is canonical. The more general $P(X)$ case is discussed in some detail in ref. [45].

[6] The α_2 theory studied in ref. [45] can be obtained by setting $\beta_0 = 3$, $\beta_1 = -3/2$, $\beta_2 = 1/2$, and $\beta_3 = 0$ [10]. With this parameter choice, the Bianchi constraint (2.15) reproduces eq. (5.8) of ref. [45].

involve the lapse [45]. A simple way to substitute out the lapse is to use the relation, following straightforwardly from eq. (2.15),

$$\frac{\dot{\chi}^2}{2N_{\rm eff}^2} = V(\chi) + \frac{m^2 M_{\rm Pl}^2 a^2 P(a)}{\alpha\beta a_{\rm eff}^2}, \tag{5.3}$$

as the lapse only appears in the Einstein-frame Friedmann equation through $\dot{\chi}^2/2N_{\rm eff}^2$. Note however that we can also use eq. (5.3) to solve for the potential, $V(\chi)$, and write the Einstein-frame Friedmann equation in a form that does not involve the potential. Of course, if we were to additionally integrate the continuity equation as discussed above, then the Einstein-frame Friedmann equation would take the form of eq. (4.3) which contains neither the kinetic nor the potential term.

Using eqs. (4.1) and (4.2) we can find expressions for the kinetic and potential energies purely in terms of a,

$$K(a) = \frac{m^2 M_{\rm Pl}^2 a^3}{2\alpha a_{\rm eff}^3}\left(\frac{c_1}{a} + 2\frac{c_2}{a^2} + \frac{c_3}{a^3}\right), \tag{5.4}$$

$$V(a) = -\frac{m^2 M_{\rm Pl}^2 a^3}{2\alpha a_{\rm eff}^3}\left(2d_0 + \frac{d_1}{a} + 2\frac{d_2}{a^2} + \frac{d_3}{a^3}\right), \tag{5.5}$$

where $K \equiv \dot{\chi}^2/2N_{\rm eff}^2$, the c_i are defined in eq. (4.4), and we have further defined

$$\begin{aligned} d_0 &\equiv \frac{\alpha}{\beta}\beta_1, \\ d_1 &\equiv \beta_1 + 5\frac{\alpha}{\beta}\beta_2, \\ d_2 &\equiv \beta_2 + 2\frac{\alpha}{\beta}\beta_3, \\ d_3 &\equiv \beta_3 - \frac{\alpha C}{m^2 M_{\rm Pl}^2}. \end{aligned} \tag{5.6}$$

The integration constant, C, appears when solving the continuity equation (2.14). The Friedmann equation is given by the generic eq. (4.3). That is, we are left with the peculiar situation that the pressure, energy density, and Einstein-frame Friedmann equation are completely insensitive to the form of the scalar field potential. As discussed above, this lack of dependence on the details of the scalar field physics is illusory; the lapse does depend on $V(\chi)$ and $\dot{\chi}$, cf. eq. (5.3), and in turn the Jordan-frame expansion history depends on the lapse, cf. eq. (2.12).

Let us briefly remark on a pair of important exceptions. The no-go theorem forbidding dynamical a still applies when there is a scalar field present if either the potential does not depend on the lapse (such as a flat potential) or the field is not rolling. Let us rewrite eq. (2.14) (which is equivalent to the Klein-Gordon equation) as

$$\frac{d}{dt}\left(\frac{\dot{\chi}^2}{2N_{\rm eff}^2} + V(\chi)\right) + 3\frac{\dot{a}_{\rm eff}}{a_{\rm eff}}\frac{\dot{\chi}^2}{N_{\rm eff}^2} = 0. \tag{5.7}$$

If $V(\chi)$ is independent of $N_{\rm eff}$ then $\dot{\chi}^2/N_{\rm eff}^2$ cannot depend on $N_{\rm eff}$ and, by extension, neither can $p = \dot{\chi}^2/2N_{\rm eff}^2 - V(\chi)$. In the specific case of $V(\chi) =$ const. this is clearly true, and we find $\dot{\chi}^2/N_{\rm eff}^2 \propto a_{\rm eff}^{-6}$, so $p = p(a)$. Similarly, if the field is not rolling, $\dot{\chi} = 0$, then it is clear from eq. (5.2) that p loses its dependence on the lapse.

To conclude this section, when a scalar field is coupled to the effective metric, we avoid the no-go theorem and it is possible to have dynamical a, unless the potential does not depend on the lapse (including a constant potential) or the field is not rolling. This result agrees with and slightly generalizes that presented in refs. [45, 55]. In a realistic scenario, however, we will have not only a scalar field but also matter components present. We now turn to that scenario.

6 Adding a perfect fluid

We have seen that the no-go theorem on FLRW solutions in dRGT massive gravity continues to hold in the doubly-coupled theory if the only matter coupled to the effective metric is a perfect fluid whose energy density and pressure depend only on the scale factor. This complicates the question of computing dust-dominated or radiation-dominated solutions in massive gravity. One solution might be to treat the dust in terms of fundamental fields. Another would be to add an extra degree of freedom such as a scalar field. Its role is to introduce a lapse-dependent term into the Bianchi constraint (2.15) and thereby avoid the no-go theorem.

It is this possibility which we study in this section. In section 5 we examined the scalar-only case. Let us now include other matter components, such as dust or radiation, also coupled minimally to $g^{\rm eff}_{\mu\nu}$. We assume that the density and pressure of these matter components, $\rho_{\rm m}$ and $p_{\rm m}$, only depend on $a_{\rm eff}$.[7] We can then write the total density and pressure as

$$\begin{aligned} \rho &= K + V + \rho_{\rm m}, \\ p &= K - V + p_{\rm m}, \end{aligned} \tag{6.1}$$

so that

$$\begin{aligned} K &= \frac{\rho + p - (\rho_{\rm m} + p_{\rm m})}{2}, \\ V &= \frac{\rho - p - (\rho_{\rm m} - p_{\rm m})}{2}. \end{aligned} \tag{6.2}$$

Note that eqs. (5.4) and (5.5) no longer hold, as they were derived in the absence of other matter, but eqs. (4.1) and (4.2) are still valid and are crucial.

We would like to investigate the cosmological dynamics of this model. Rather than explicitly solving for the lapse and substituting it into the Friedmann equation for $H_{\rm eff}$, which leads to a very complicated result, we will take advantage of the known forms of $K(a_{\rm eff})$ and $V(a_{\rm eff})$, as well as the fact that $N_{\rm eff}$ only appears in $H_{\rm eff}$ and K through the operator

$$\frac{d}{d\tau} = \frac{1}{N_{\rm eff}}\frac{d}{dt}. \tag{6.3}$$

The physical Hubble rate is given by

$$H_{\rm eff} \equiv \frac{\dot{a}_{\rm eff}}{a_{\rm eff} N_{\rm eff}} = \frac{\alpha \dot{a}}{a_{\rm eff} N_{\rm eff}}. \tag{6.4}$$

[7] As discussed above and in ref. [45], in principle any dust or radiation is made of fundamental particles for which the stress-energy tensor does depend on the lapse. We introduce this effective-fluid description because it is the standard method of deriving cosmological solutions in nearly any gravitational theory and is thus an important tool for comparing to observations.

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Using the chain rule, we can write

$$\dot{a} = \frac{da}{dt} = \frac{da}{dV}\frac{dV}{d\chi}\frac{d\chi}{dt} = \frac{V'\dot{\chi}}{(dV/da)}, \tag{6.5}$$

where a prime denotes a derivative with respect to χ. We also know that $\dot{\chi} = N_{\rm eff}\sqrt{2K}$, giving

$$\dot{a} = \frac{V' N_{\rm eff}\sqrt{2K}}{(dV/da)}, \tag{6.6}$$

which we can plug into eq. (6.4) to obtain

$$H_{\rm eff}^2 = \frac{(V')^2 2K}{a_{\rm eff}^2 (dV/da_{\rm eff})^2}. \tag{6.7}$$

This is the Friedmann equation for any universe with a scalar field rolling along a nonconstant potential. Every term in eq. (6.7) can be written purely in terms of $a_{\rm eff}$, allowing the full cosmological dynamics to be solved in principle. K and $dV/da_{\rm eff}$ are given in terms of $a_{\rm eff}$ by eq. (6.2) [using eqs. (4.1) and (4.2)]. V' as a function of $a_{\rm eff}$ can be determined from the same equations once the form of $V(\chi)$ is specified. Note that while the lapse is not physically observable, its evolution in terms of a can then be fixed by using eq. (2.12) to find

$$\frac{N^2}{N_{\rm eff}^2} = 2K\left(\frac{V'}{\alpha a H (dV/da_{\rm eff})}\right)^2, \tag{6.8}$$

where $H(a)$ is given by eq. (4.3).

Assuming that the matter has a constant equation of state, we can use the known forms of $K(a)$ and $V(a)$ to find a relatively simple expression for the Friedmann equation up to V',

$$\left(\frac{H_{\rm eff}}{V'}\right)^2 = \frac{4\alpha^3\beta a_{\rm eff}^3\left(\mathcal{C}_0 + \mathcal{C}_1 a_{\rm eff} + \mathcal{C}_2 a_{\rm eff}^2 + \mathcal{C}_\rho a_{\rm eff}^3\right)}{\left[3\mathcal{C}_0 + 4\mathcal{C}_1 a_{\rm eff} + 5\mathcal{C}_2 a_{\rm eff}^2 + 3(1-w)\mathcal{C}_\rho a_{\rm eff}^3\right]^2}, \tag{6.9}$$

where for brevity we have defined

$$\begin{aligned}
\mathcal{C}_0 &\equiv \beta\left[\alpha^3 C + \beta^2\beta_1 + m^2 M_{\rm Pl}^2\left(3\alpha\left(\alpha\beta_3 - \beta\beta_2\right)\right)\right], \\
\mathcal{C}_1 &\equiv -2m^2 M_{\rm Pl}^2\left[\alpha\left(\alpha\beta_3 - 2\beta\beta_2\right) + \beta^2\beta_1\right], \\
\mathcal{C}_2 &\equiv m^2 M_{\rm Pl}^2\left(\beta\beta_1 - \alpha\beta_2\right), \\
\mathcal{C}_\rho &\equiv -\alpha^3\beta(1+w)\rho_{\rm m}.
\end{aligned} \tag{6.10}$$

Notice that the right-hand side is a function of a only.

The Friedmann equation (6.9) cannot be straightforwardly solved for generic choices of the potential, so we will make progress by examining past and future asymptotics, taking into account radiation ($w = 1/3$) in the former and dust ($w = 0$) in the latter. Before we do this, it is important to note that taking these asymptotics is not always simple, as we cannot necessarily assume that $a_{\rm eff} \to 0$ at the beginning of the Universe or that $a_{\rm eff} \to \infty$ as $t \to \infty$. This means that, for example, our late-time analysis (in which $a_{\rm eff}$ is taken to infinity) will only be applicable for cosmologies in which the Universe expands *ad infinitum*. Depending on the choice of scalar field potential, the Universe might end up, for example, recollapsing or approaching an asymptotic maximum value of $a_{\rm eff}$. A major aim of this section is to show

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the difficulties in obtaining standard cosmologies with a scalar field and perfect fluid both coupled to $g^{\rm eff}_{\mu\nu}$; since a Universe which does not expand to infinity is highly nonstandard, we will find it sufficient to take $a_{\rm eff} \to \infty$ as the late-time limit in our search for viable cosmologies.[8] We will see an example of when this limit may not be applicable.

Taking $a_{\rm eff} \to \infty$ in eq. (6.2), we find

$$\frac{\dot{\chi}^2}{2N_{\rm eff}^2} \xrightarrow{a_{\rm eff}\to\infty} \frac{m^2 M_{\rm Pl}^2(\beta\beta_1 - \alpha\beta_2)}{2\alpha^3\beta a_{\rm eff}}, \tag{6.11}$$

$$V(\chi) \xrightarrow{a_{\rm eff}\to\infty} -\frac{\beta_1 m^2 M_{\rm Pl}^2}{\alpha^3\beta}. \tag{6.12}$$

We see that the scalar field slows to a halt: $V(\chi)$ approaches a constant, while $d\chi/d\tau$, where $d\tau = N_{\rm eff}dt$ is the proper time, approaches zero. Notice that $V(\chi)$ is forced by the dynamics to approach a specific value, $V \to -\beta_1 m^2 M_{\rm Pl}^2/\alpha^3\beta$. *A priori* there is no guarantee this value is within the range of $V(\chi)$, assuming the scalar field potential is not somehow set by gravitational physics. For example, a positive-definite potential like $V \sim \chi^2$ or $V \sim \chi^4$ would never be able to reach such a value, assuming α, β, and β_1 are positive. Indeed, one can solve eq. (6.2) explicitly for $\chi(a_{\rm eff})$ in such a case and find that, for large values of $a_{\rm eff}$, χ and $H_{\rm eff}$ are imaginary: there is a maximum value of $a_{\rm eff}$ at which χ^2 and $H^2_{\rm eff}$ cross zero and become negative. Because such cosmologies are highly nonstandard and are unlikely to agree with data, we leave their study for future work.

Taking the large-$a_{\rm eff}$ limit of the Friedmann equation (6.9), we obtain

$$\left(\frac{H_{\rm eff}}{V'}\right)^2 \xrightarrow{a_{\rm eff}\to\infty} \frac{4\alpha^3\beta}{25\mathcal{C}_2} a_{\rm eff}. \tag{6.13}$$

Because $V(\chi)$ approaches a late-time value given by eq. (6.12), then assuming $V(\chi)$ is invertible, χ must also approach a constant χ_c. This means that $V' = (dV/d\chi)|_{\chi=\chi_c}$ contributes a constant to eq. (6.13). This is counter-intuitive; while the scalar field approaches a constant, $\dot{\chi} \to 0$, V' can and generically will approach a nonzero constant, which is just the slope of the potential evaluated at the asymptotic-future value of χ, χ_c. The Klein-Gordon equation (5.7) is still satisfied because, as long as V' does not go to zero, we can see from eq. (6.13) that $H_{\rm eff} \to \infty$ at late times. Therefore, the reason the scalar field slows down, in terms of the Klein-Gordon equation, is that the Hubble friction grows arbitrarily large, bringing the field to a halt even on a potential with a nonzero slope.[9] Unless the potential is contrived such that $V' \to 0$ as $V \to -\beta_1 m^2 M_{\rm Pl}^2/\alpha^3\beta$, we see from eq. (6.13) that $H_{\rm eff}$ generically blows up, which is potentially disastrous behavior. This implies a violation of the null energy condition.

As we discuss below, if V' goes to 0 then, depending on the speed at which it does so, $H_{\rm eff}$ may be better behaved.

At early times, demanding the existence of a sensible radiation era leads to further problems. Assuming radiation couples to $g^{\rm eff}_{\mu\nu}$, then $\rho_{\rm m} \sim a_{\rm eff}^{-4}$ with $p_{\rm m} = \rho_{\rm m}/3$. We have, cf. eq. (6.2), that $2K = \rho + p - (\rho_{\rm m} + p_{\rm m})$, but, cf. eq. (4.2), ρ and p do not have any terms scaling as steeply as $a_{\rm eff}^{-4}$. Therefore, in the presence of radiation, ρ_χ and p_χ pick up a *negative* term going as $a_{\rm eff}^{-4}$ to exactly cancel out $\rho_{\rm m}$ and $p_{\rm m}$, leading to $K < 0$ at sufficiently early times. From eq. (6.7) we see that this would lead to a negative $H^2_{\rm eff}$, and hence to an

[8] Of course, observations do not necessarily rule out the possibility of the scale factor not evolving to infinity, but it seems likely that making such a model agree with the data would require some serious contrivances.

[9] We thank the referee for helpful discussions on this point.

imaginary Hubble rate. Equivalently, we can take the early-time limit of eq. (6.9) to show, setting $\rho_{\rm m} = \rho_0 a_{\rm eff}^{-4}$,

$$\left(\frac{H_{\rm eff}}{V'}\right)^2 \xrightarrow{a_{\rm eff}\to 0} -\frac{3}{4\rho_0} a_{\rm eff}^4, \tag{6.14}$$

so that again we see (for a real potential) $H_{\rm eff}$ becoming imaginary.

How could these conclusions be avoided? We can reproduce sensible behavior, but only if the potential is extremely contrived. At early times, we would need to arrange the scalar's dynamics so that $V' \to \infty$ "before" (i.e., at a later $a_{\rm eff}$ than) K crosses zero.[10] We would then reach the initial singularity, $H_{\rm eff} \to \infty$, before the kinetic term turns negative.[11] Moreover, we would need to tune the parameters of the theory so that $K = 0$ happens at extremely early times, specifically before radiation domination. At intermediate times, V' would need to scale in a particular way to [through eq. (6.9)] reproduce $H_{\rm eff}^2 \sim a_{\rm eff}^{-4}$ and $H_{\rm eff}^2 \sim a_{\rm eff}^{-3}$ during the radiation- and matter-dominated eras, respectively. Finally, in order to have $H_{\rm eff} \to$ const. at late times, we see from eq. (6.13) that we would require V' to decay as $a_{\rm eff}^{-1/2}$. We can construct such a potential going backwards by setting $H_{\rm eff} = H_{\Lambda{\rm CDM}}$ in eq. (6.9), but there is no reason to expect such an artificial structure to arise from any fundamental theory. Even then we may still get pathological behavior: we can see from eq. (2.12) that $N_{\rm eff}$ diverges if, at some point during the cosmic evolution, $H_{\rm eff} a_{\rm eff} = Ha$.

7 Mixed matter couplings

Before concluding, we briefly discuss a slightly different formulation which avoids some of these problems. If we consider a scalar field and a perfect fluid, the avoidance of the no-go theorem on FLRW solutions only requires that the scalar field couple to $g_{\mu\nu}^{\rm eff}$. In principle, all other matter could still couple to $g_{\mu\nu}$. In fact, this is the theory that was studied in ref. [45], where it was argued more generally that only a new dark sector should couple to $g_{\mu\nu}^{\rm eff}$, while the standard model, as well as dark matter and dark energy, should couple to $g_{\mu\nu}$. This theory violates the equivalence principle in the scalar sector, but is not *a priori* excluded, and will turn out to have somewhat better cosmological behavior. Moreover, there is a compelling theoretical reason to consider such "mixed" couplings: matter loops would only generate a cosmological constant and would not destabilize the rest of the potential. This is because the vacuum energy associated to $g_{\mu\nu}^{\rm eff}$ takes the form of the dRGT potential with all β_n parameters nonzero, while the vacuum energy of matter coupled to $g_{\mu\nu}$ only contributes to β_0 [56]. We note that this problem may nevertheless persist with the dark fields that couple to $g_{\mu\nu}^{\rm eff}$, unless their vacuum energy is somehow protected from loop corrections. We have seen in section 5 that one simple possibility, using a massless field, does not seem to work because after integrating the Klein-Gordon equation, the pressure loses its dependence on the lapse.

Because the perfect fluid couples to $g_{\mu\nu}$ and we derived the Bianchi constraint (2.15) by taking the g-metric divergence of the Einstein equation, the constraint will now only contain p_χ rather than the total pressure, i.e.,

$$m^2 M_{\rm Pl}^2 a^2 P(a)\dot{a} = \alpha\beta a_{\rm eff}^2 p_\chi \dot{a}. \tag{7.1}$$

[10] The other obvious possibility, having $dV/da_{\rm eff}$ reach 0 before K does, is impossible given the forms of $K(a)$ and $V(a)$.

[11] This proposal has an interesting unexpected advantage: the Universe would begin at finite $a_{\rm eff}$, so a UV completion of gravity might not be needed to describe the Big Bang in the matter sector.

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This is the same constraint as in the scalar-only case discussed in section 5, so the scalar's kinetic and potential energies have the same forms, $K(a)$ and $V(a)$, as in eqs. (5.4) and (5.5). The physical Hubble rate is now H, which after solving for the lapse is determined by the equation[12]

$$3H^2 = \frac{\rho_{\rm m}}{M_{\rm Pl}^2} + m^2\left(c_0 + 3\frac{c_1}{a} + 3\frac{c_2}{a^2} + \frac{c_3}{a^3}\right), \tag{7.2}$$

where the c_i coefficients are defined in eq. (4.4). We emphasize that eq. (7.2) is completely generic when some matter couples to $g_{\mu\nu}$ and some, possibly in a dark sector, couples to $g^{\rm eff}_{\mu\nu}$. We have not assumed anything about the structure of the fields coupling to the effective metric, as we can derive eq. (4.2) for $\rho(a)$ and hence eq. (7.2) simply by using the Bianchi constraint to integrate the stress-energy conservation equation.

The cosmological behavior in this theory is fine. Because the scalar field does not have to respond to matter to maintain a particular form of $\rho(a)$ and $p(a)$, we no longer have pathological behavior in the early Universe, where there will be a standard a^{-4} evolution. Moreover, as was pointed out in ref. [55], there is late-time acceleration: as $\rho_{\rm m} \to 0$, $3H^2 \to m^2(\beta_0 - (\alpha/\beta)\beta_1)$, which, if positive, leads to an accelerating expansion.

However, these are not always *self*-accelerating solutions. We will demand two conditions for self-acceleration: that the late-time acceleration not be driven by a cosmological constant, and that it not be driven by $V(\chi)$, as both of these could easily be accomplished without modifying gravity. In other words, we would like the effective cosmological constant at late times to arise predominantly from the massive graviton.

Let us start with the first criterion, the absence of a cosmological constant. One can write the dRGT interaction potential in terms of elementary symmetric polynomials of the eigenvalues of either $X \equiv \sqrt{g^{-1}f}$ or $\mathbb{K} \equiv \mathbb{I} - X$, with the strengths of the interaction terms denoted by β_n in the first case and by α_n in the latter [10, 12]. What is notable is that $\alpha_0 \neq \beta_0$: the cosmological constant is not the same in these two parametrizations. Terms proportional to $\sqrt{-g}$ arise from the other interaction terms when transforming from one basis to the other. We have worked in terms of β_n as it is mathematically simpler, but in massive gravity with a Minkowski reference metric, the presence of a Poincaré-invariant preferred metric allows for a more concrete definition of the cosmological constant in terms of α_n.[13] Consider expanding the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu} + h_{\mu\alpha}h_{\nu\beta}\eta^{\mu\nu}. \tag{7.3}$$

This expansion is useful because the metric is quadratic in $h_{\mu\nu}$ but is fully nonlinear, i.e., we have not assumed that $h_{\mu\nu}$ is small [12]. In this language, the cosmological constant term, proportional to $\sqrt{-g}$, can be eliminated by setting $\alpha_0 = \alpha_1 = 0$. Making this choice of parameter, and using the fact that α_n and β_n are related by [10]

$$\beta_n = (4-n)!\sum_{i=n}^{4}\frac{(-1)^{i+n}}{(4-i)!(i-n)!}\alpha_i, \tag{7.4}$$

[12] Using the transformations to the α_2 theory in footnote 6, we recover eq. (5.9) of ref. [45].
[13] We thank Claudia de Rham for helpful discussions on this point.

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we find the effective cosmological constant can be expressed in terms of $\alpha_{2,3,4}$ by

$$\begin{aligned}\Lambda_{\text{eff}} &= \frac{m^2}{3}\left(\beta_0 - \frac{\alpha}{\beta}\beta_1\right) \\ &= \frac{m^2}{3}\left[3\alpha_2\left(2+\frac{\alpha}{\beta}\right) - \alpha_3\left(4+3\frac{\alpha}{\beta}\right) + \alpha_4\left(1+\frac{\alpha}{\beta}\right)\right].\end{aligned} \tag{7.5}$$

Part of this constant comes from the fixed behavior of the scalar field potential.[14] This piece is not difficult to single out: it consists exactly of the terms in eq. (7.5) proportional to α/β. Taking the late-time limit of eq. (5.5), we can see that $V(\chi)$ asymptotes to

$$V(\chi) \xrightarrow{a_{\text{eff}}\to\infty} -\frac{m^2 M_{\text{Pl}}^2 \beta_1}{\alpha^3 \beta}. \tag{7.6}$$

Now consider the Friedmann equation in the form (2.10) with, at late times, $\rho \to 0$. We can define a cosmological-constant-like piece solely due to the late-time behavior of V given by

$$\Lambda_\chi \equiv \frac{\alpha V}{3M_{\text{Pl}}^2}\left(\frac{a_{\text{eff}}}{a}\right)^3 \xrightarrow{a_{\text{eff}}\to\infty} \frac{m^2}{3}\frac{\alpha}{\beta}\left(3\alpha_2 - 3\alpha_3 + \alpha_4\right). \tag{7.7}$$

Then eq. (7.5) can simply be written in the form

$$\Lambda_{\text{eff}} = \frac{m^2}{3}\left(6\alpha_2 - 4\alpha_3 + \alpha_4\right) + \Lambda_\chi = \frac{m^2}{3}\beta_0 + \Lambda_\chi, \tag{7.8}$$

where in the last equality we mention that the residual term is nothing other than $m^2\beta_0/3$, which is simply a consistency check.

The modifications to gravity induced by the graviton mass therefore lead to a constant contribution to the Friedmann equations at late times, encapsulated in $m^2\beta_0/3$ (with $\alpha_0 = \alpha_1 = 0$, so we do not identify this term with a cosmological constant). In a truly self-accelerating universe, this term should dominate Λ_χ. If it did not, the acceleration would be partly caused by the scalar field's potential, and one could get the same end result in a much simpler way with, e.g., quintessence. For generic values of α_n and for $\beta \sim \mathcal{O}(1)$, both of these contributions are of a similar size and will usually have the same sign. To ensure self-accelerating solutions, one could, for example, tune the coefficients so that $3\alpha_2 - 3\alpha_3 + \alpha_4 = 0$ (the scalar field contributes nothing to Λ_{eff}) or $3\alpha_2 - 3\alpha_3 + \alpha_4 < 0$ (the scalar field contributes negatively to Λ_{eff}), or take $\beta \ll 1$ (the scalar field contributes negligibly to Λ_{eff}).

We end this section by briefly discussing the link between theory and observation in this particular model. One might worry that the predictivity of the theory is hurt by demanding that there be a new dark sector coupled to $g^{\text{eff}}_{\mu\nu}$. It is then natural to suspect that the task of confronting doubly-coupled massive gravity with observations is hopelessly dependent on the nature of this new dark sector, and the theory's parameters will consequently be more difficult to constrain. Yet we have seen in this section that that is not true: the Friedmann equation (7.2) makes no reference to any details of the dark field or fields.[15]

[14] Notice from eq. (5.4) that, as in section 6, the scalar field slows down to a halt at late times, so there is no contribution from the kinetic energy.

[15] This is not the case when all matter couples to the effective metric, as observations would trace $g^{\text{eff}}_{\mu\nu}$, which *is* sensitive to the nature of the dark sector, rather than $g_{\mu\nu}$. We have, however, seen that the case where the standard model couples to $g_{\mu\nu}$ is by far the best-behaved version.

Recall from section 4 that this is a consequence of the Bianchi constraint on the dark sector. We thus have the unusual result that the expansion history in the theory with a new dark sector, and nothing else, coupled to $g^{\rm eff}_{\mu\nu}$ is completely insensitive to the nature of the dark fields.[16] There could be one scalar field or more, with any assortment of potentials and kinetic terms, and as long as they exist, and are subject to the technical conditions discussed above (such as having a nontrivial potential, if the kinetic term is canonical), then their contribution to the cosmological dynamics is given by the mass term in eq. (7.2). This is good news for observers looking to perform geometrical tests of this theory. However, we are not aware of any reason that this lack of dependence on the details of the dark sector should extend beyond the simple background FLRW case. Even linear cosmological perturbations might be sensitive to the dark physics [55], which would present a challenge in comparing this theory to structure formation.

8 Discussion and conclusions

One can extend dRGT massive gravity by allowing matter to couple to an effective metric constructed out of both the dynamical and the reference metrics. The no-go theorem ruling out flat or closed homogeneous and isotropic cosmologies in massive gravity [13] can be overcome when matter is "doubly coupled" in such a way [45, 55]. We have shown that this result is, unusually, dependent on coupling the effective metric to a fundamental field, as the no-go theorem is specifically avoided because the pressure of such matter depends on the lapse function. This lapse dependence is not present in the perfect-fluid description typically employed in late-time cosmological setups, such as radiation ($p \sim a_{\rm eff}^{-4}$) and dust ($p = 0$), and therefore a universe containing *only* such matter will still run afoul of the no-go theorem. While this may not be a strong physical criterion — cosmological matter is still built out of fundamental fields — it presents a sharp practical problem in relating the theory to cosmological observations. If we assume that matter is described by perfectly pressureless dust, which is sensible on very large scales, then even the field description might not be sufficient, as the absence of pressure would set the right-hand side of eq. (2.15) to zero. Furthermore, if one uses a scalar field to avoid the no-go theorem, it cannot live on a flat potential and must be rolling. The latter consideration would seem to rule out the use of the Higgs field to unlock massive cosmologies, as we expect it to reside in its minimum cosmologically.

Overall, in principle one can obtain observationally-sensible cosmologies in doubly-coupled massive gravity, but either a new degree of freedom must be included, such as a new dark field or some other matter source with a nontrivial pressure, or we must treat cosmological matter in terms of their constituent fields. Thus we cannot apply the standard techniques of late-time cosmology to this theory.

We have further shown that if dust and radiation are doubly coupled as well — which is necessary if we demand the new scalar matter obey the equivalence principle — then the cosmologies generically are unable to reproduce a viable radiation-dominated era, and in the far future the Hubble rate diverges, rather than settling to a constant and producing a late-time accelerated expansion. These pathologies can only be avoided if the scalar field potential is highly contrived with tuned theory parameters, or dust and radiation do not doubly couple. In the latter case, there is generically late-time acceleration, but for much of the parameter space this is driven in large part by the potential of the scalar field. In

[16] See ref. [45] for a complementary derivation of this result.

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those cases the modification to general relativity may not be especially well motivated by cosmological concerns, as the scalar field would play the role of dark energy and not provide much benefit over simple quintessence. Otherwise, the parameters of the theory need to be tuned to ensure that the theory truly self-accelerates.

It seems that the dRGT massive gravity only has viable FLRW cosmological solutions — i.e., that evade the no-go theorems on existence [13] and stability [20] — if one either includes a scalar field or some other "exotic" matter with a lapse-dependent pressure (or possibly a pressure depending on $\dot{a}$) and couples it to the effective metric proposed in ref. [45] or goes beyond the perfect-fluid description of matter. Even if one includes a new scalar degree of freedom, significant pathologies arise if normal matter couples to the same effective metric. In all setups, the need for descriptions beyond a simple perfect fluid makes this theory problematic from an observational standpoint. Indeed, one might compare this to the situation with the original dRGT theory, in which all matter couples to $g_{\mu\nu}$. While FLRW solutions do not exist in this case, it is possible by mildly breaking the assumption of isotropy and homogeneity to evade the no-go theorem [13]. The real problem is that by dropping the highly-symmetric FLRW ansatz, we lose a great deal of predictability and it becomes significantly more difficult to unambiguously compare the theory to observations.

We end with three small caveats. Notice that we have assumed that in unitary gauge for the Stückelberg fields, i.e., choosing coordinates such that $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the metric has the usual FLRW form (2.6). However, that form is arrived at by taking coordinate transformations of a more general homogeneous and isotropic metric, so that assumption may be overly restrictive.[17] Equivalently, one could consider a more general, inhomogeneous and/or anisotropic gauge for the Stückelbergs.

We also note that if this theory does possess a ghost, even with a mass above the strong-coupling scale, solutions to the nonlinear equations of motion could contain the ghost mode and therefore not be physical.[18] In other words, the ghost-free effective theory below the strong coupling scale and the theory we have been studying may not have coinciding solutions. However, a Hamiltonian analysis showed that the ghost does not appear around FLRW backgrounds [45], suggesting that we have studied the correct cosmological solutions to any underlying ghost-free theory.

Finally, if one simply gives dynamics to the reference metric, we end up with a theory of doubly-coupled bigravity which treats the two metrics on completely equal footing and has been shown to produce observationally viable cosmologies [53], although some of the issues with doubly-coupled massive gravity, such as the potential ghost problem, will still remain.

Acknowledgments

We are grateful to Luca Amendola, Claudia de Rham, Matteo Fasiello, Fawad Hassan, Lavinia Heisenberg, Mikica Kocic, Johannes Noller, Raquel Ribeiro, Angnis Schmidt-May, and Andrew Tolley for highly enlightening discussions. A.R.S. has been supported by the David Gledhill Research Studentship, Sidney Sussex College, University of Cambridge; the Isaac Newton Fund and Studentships, University of Cambridge; the Cambridge Philosophical Society; and the STFC. Y.A. is supported by the European Research Council (ERC) Starting Grant StG2010-257080. Y.A. also acknowledges support from DFG through the project TRR33 "The Dark Universe." F.K. acknowledges support from DFG Graduiertenkolleg

[17] We thank Fawad Hassan for pointing this out to us.

[18] We thank Angnis Schmidt-May for discussions on this point.

GRK 1940 "Particle Physics Beyond the Standard Model." E.M. acknowledges support for this study from the Swedish Research Council.

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Publication 5

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JCAP 1605 (2015) 052, arXiv:1503.02490

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Surfing gravitational waves: can bigravity survive growing tensor modes?

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Received March 19, 2015
Accepted May 6, 2015
Published May 26, 2015

Abstract. The theory of bigravity offers one of the simplest possibilities to describe a massive graviton while having self-accelerating cosmological solutions without a cosmological constant. However, it has been shown recently that bigravity is affected by early-time fast growing modes on the tensor sector. Here we argue that we can only trust the linear analysis up to when perturbations are in the linear regime and use a cut-off to stop the growing of the metric perturbations. This analysis, although more consistent, still leads to growing tensor modes that are unacceptably large for the theory to be compatible with measurements of the cosmic microwave background (CMB), both in temperature and polarization spectra. In order to suppress the growing modes and make the model compatible with CMB spectra, we find it necessary to either fine-tune the initial conditions, modify the theory or set the cut-off for the tensor perturbations of the second metric much lower than unity. Initial conditions such that the growing mode is sufficiently suppresed can be achieved in scenarios in which inflation ends at the GeV scale.

Keywords: modified gravity, CMBR polarisation, gravitational waves and CMBR polarization, dark energy theory

ArXiv ePrint: 1503.02490

doi:10.1088/1475-7516/2015/05/052

JCAP05(2015)052

Contents

1 Introduction

Evidence from an increasing number of cosmological observables favours an accelerating universe at late times [1–9]. This era of accelerated expansion may be due to novel gravitational physics, which will be tested by ongoing and future experiments [10]. This possibility has triggered vigorous interest in alternative theories of gravity [8, 11, 12]. Any modification of gravity requires new degrees of freedom (dof). Since the theory of a massless graviton is unique, new dofs are often gained by adding new fields. The simplest possibility is the addition of a scalar field, typically resulting in theories belonging to the Horndeski class [13, 14] or beyond [15–17].

Formulating a theory of massive gravity has been a long standing problem in theoretical physics due to the difficulties to incorporate the right degrees of freedom. The linear Fierz-Pauli theory had been developed long time ago [18], but until recently all non-linear completions introduced the so called Bouleware-Deser (BD) ghost [19], an extra dof that makes the theory not viable. Despite the difficulties, a class of healthy theories has been recently identified [20] in which a specific choice of the potential terms makes the theory ghost-free [21]. All these theories of massive gravity describe an interaction of two tensor fields in which the second one, the so called *reference metric*, is fixed. While massive gravity only allows static solutions on homogeneous backgrounds [22], a bimetric theory with a dynamical reference metric does not introduce the BD ghost and describes dynamical cosmologies [23–25] (see also the reviews [26, 27]). Cosmological solutions in these bimetric theories often allow for self-acceleration without the introduction of a cosmological constant [28] and were successfully compared to observations at background level [28–30].

Many bigravity theories are however affected by gradient instabilities in their scalar sector, as has been shown by studies of the linear perturbations [31–33] (see refs. [32, 34] for derivations of the equations refs. [35–39] for discussion of their dynamics). Stable evolution can be achieved only in a two parameters class of models known as Infinite-Branch Bigravity (IBB) [36]. In IBB, the reference metric (in keeping with common usage, we keep referring to the second metric as reference metric even if in reality is dynamical; we also use the notation f-metric) is contracting during the radiation and most of the matter era, until it undergoes a bounce at low redshift and begins to expand, coinciding with the onset of accelerated expansion in the physical metric without the need for a cosmological constant. The early time contraction of the reference metric makes tensor perturbations grow with time in IBB theories, as it was first shown in refs. [37, 40] (see also [32, 41] for modified tensor perturbation equations). This growing mode couples to the physical metric and severely modifies its dynamics, leading to observable consequences.

In this paper we will investigate the effects of these large tensor perturbations on the Cosmic Microwave Background (CMB) and possible mechanisms to make the theory compatible with current observations. The perturbations in the reference metric grow very fast and rapidly become non-linear. At this point we will assume that tensor perturbations stabilize, modeling this effect by introducing a cut-off in the perturbations of the reference metric. Despite this treatment, the tensor growing mode significantly affects the evolution of the physical metric, and the consequences can be seen as an enhancement of both temperature and polarization spectra on low multipoles. These effects cannot be sufficiently reduced by varying the bigravity or other cosmological parameters: making the theory viable requires either fine tuning of the initial conditions, lowering the cut-off or modifying the theory. As it will be shown below, sufficient suppression of the growing mode can be achieved by an inflationary mechanism that produces Hubble-scale tensor perturbations at an energy scale of order few GeV.

2 Bigravity

We start with the action of the form [23]

$$S = -\frac{M_g^2}{2}\int d^4x\sqrt{-g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-f}R(f) \tag{2.1}$$
$$+m^2M_g^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(X) + \int d^4x\sqrt{-g}\mathcal{L}_m$$

where $e_n(X)$ are the elementary symmetric polynomials of the eigenvalues of the matrices $X^\alpha_\gamma \equiv \sqrt{g^{\alpha\beta}f_{\beta\gamma}}$, M_g and M_f are the Planck masses for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, m is the mass scale of the graviton, β_n are arbitrary constants and $\mathcal{L}_m = \mathcal{L}_m(g,\psi)$ is the matter Lagrangian. Throughout the paper we will use a mostly plus metric signature convention and natural units in which the speed of light c is set to one.

Here $g_{\mu\nu}$ is the standard metric coupled to matter fields in the $\mathcal{L}_m$ Lagrangian, while $f_{\mu\nu}$ is an additional dynamical tensor field. In the following we express masses in units of the Planck mass M_g and the mass parameter m^2 will be absorbed into the parameters β_n. Varying the action with respect to $g_{\mu\nu}$, one obtains the following equations of motion:

$$G_{\mu\nu} + \frac{1}{2}\sum_{n=0}^{3}(-1)^n m^2\beta_n\left[g_{\mu\lambda}Y^\lambda_{(n)\nu}(X) + g_{\nu\lambda}Y^\lambda_{(n)\mu}(X)\right] = T_{\mu\nu} \tag{2.2}$$

where $G_{\mu\nu}$ is Einstein's tensor, and the expressions $Y^{\lambda}_{(n)\nu}(X)$ are defined as

$$Y_{(0)} = I, \tag{2.3}$$

$$Y_{(1)} = X - I[X], \tag{2.4}$$

$$Y_{(2)} = X^2 - X[X] + \frac{1}{2}I\left([X]^2 - [X^2]\right) \tag{2.5}$$

$$Y_{(3)} = X^3 - X^2[X] + \frac{1}{2}X\left([X]^2 - [X^2]\right) - \frac{1}{6}I\left([X]^3 - 3[X][X^2] + 2[X^3]\right) \tag{2.6}$$

where I is the identity matrix and [...] is the trace operator.

Varying the action with respect to $f_{\mu\nu}$ we get

$$\bar{G}_{\mu\nu} + \sum_{n=0}^{3} \frac{(-1)^n m^2 \beta_{4-n}}{2M_f^2}\left[f_{\mu\lambda}Y^{\lambda}_{(n)\nu}(X^{-1}) + f_{\nu\lambda}Y^{\lambda}_{(n)\mu}(X^{-1})\right] = 0 \tag{2.7}$$

where the overbar indicates $f_{\mu\nu}$ curvatures. Notice that β_0 acts as a pure cosmological constant, which is however not required to satisfy current observations. Finally, the rescaling $f \to M_f^{-2} f$, $\beta_n \to M_f^n \beta_n$ allows us to assume $M_f = 1$ in the following (see [42]). Additionally, from now on we absorb the graviton mass m into the constants β_i.

We assume now a cosmological spatially flat FRW metric:

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx_i dx^i\right) \tag{2.8}$$

where τ represents the conformal time and a dot will represent the derivative with respect to it. The second metric is chosen also in a spatially FRW form

$$ds_f^2 = -\left[\dot{b}(\tau)^2/\mathcal{H}^2(t)\right]d\tau^2 + b(\tau)^2 dx_i dx^i \tag{2.9}$$

where $\mathcal{H} \equiv \dot{a}/a$ is the conformal Hubble function and $b(\tau)$ is the 'scale' factor associated with the second metric f. This form of the metric $f_{\mu\nu}$ ensures that the equations satisfy the Bianchi constraints (see e.g. [25]).

The background equations for the two metrics have been obtained and discussed at length in several papers [28, 29, 36, 43]. Here we summarize the main properties in the notation of [36]. Defining $r(\tau) \equiv b(\tau)/a(\tau)$ as the ratio of the two scale factors, the background equations can be conveniently written as a first order system of two equations for $r(t)$ and $\mathcal{H}$:

$$2\mathcal{H}'\mathcal{H} + \mathcal{H}^2 = a^2(B_0 + B_2 r' - w_{\rm tot}\rho_{\rm tot}), \tag{2.10}$$

$$r' = \frac{3rB_1\Omega_{\rm tot}(1 + w_{\rm tot})}{\beta_1 - 3\beta_3 r^2 - 2\beta_4 r^3 + 3B_2 r^2}, \tag{2.11}$$

where the prime denotes derivative with respect to $N \equiv \log a$ [29, 30], $w_{\rm tot}$ denotes the equation of state (EOS) corresponding to the total density parameter $\Omega_{\rm tot}$ and the functions $B_0(\tau), B_1(\tau), B_2(\tau)$ are related to the β_i and $r(\tau)$ as follows:

$$B_0(\tau) = \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3, \tag{2.12}$$

$$B_1(\tau) = \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3, \tag{2.13}$$

$$B_2(\tau) = \beta_1 + 2\beta_2 r + \beta_3 r^2. \tag{2.14}$$

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For simplicity the time dependence of $B_{0,1,2}$ will be understood from now on. The Friedmann equation (i.e. the $(0,0)$ component of eq. (2.2)) gives

$$3\mathcal{H}^2 = a^2(\rho_{\rm tot} + B_0)\,, \tag{2.15}$$

and by combining with the $(0,0)$ component of eq. (2.7) we obtain a useful relation between $\mathcal{H}$ and the ratio $r(\tau)$:

$$\mathcal{H}^2 = \frac{a^2 B_1}{3r}. \tag{2.16}$$

Finally, the combination of the last two expressions for $\mathcal{H}$ provide $\Omega_{\rm tot}(\tau) = 1 - \frac{B_0}{B_1} r(\tau)$ which can be inserted in eq. (2.11) to produce a closed differential equation for $r(\tau)$ alone.

The behavior of the background solutions depends on the choice of the β_i constants and on the initial value of r. We denote solutions with the same β_i but different initial conditions as *branches* of the same theory. In ref. [30] it was shown that for each choice of β_i only two branches exist that agree with a standard cosmological early time evolution (like a matter dominated era at early times) and allow for physical solutions (e.g. ρ, $\mathcal{H} > 0$). In the first branch, r evolves from $r = 0$ to a de Sitter point at a finite value $r_c > 0$. These branches, however, suffer from scalar gradient instabilities [31]. Only choosing $\beta_2 = \beta_3 = 0$ and the second type of branches in which r evolves from $r \to \infty$ in the asymptotic past towards a de Sitter point at a constant $r_c > 0$, leads to the absence of these gradient instabilities in the scalar sector and is compatible with background data [36]: we dubbed this case infinite-branch bigravity (IBB). Note that even though an additional non-vanishing effective cosmological constant β_0 is viable, we assume $\beta_0 = 0$ since it would not affect the early-time evolution and is not required in order to fit observational data (see [29, 30]). From now on we restrict ourselves to IBB, in which only β_1 and β_4 are non-zero. This choice avoids introducing an explicit cosmological constant, which would make the entire bigravity model somewhat less appealing.

As shown in [36], IBB models have to satisfy $0 < \beta_4 < 2\beta_1$ in order to get an initial value of r on the infinite branch. In particular, it was found that the best fit model occurs for $\beta_1 = 0.48$ and $\beta_4 = 0.94$: from now on we refer to this choice as the reference IBB model. We then have

$$\mathcal{H}^2 = \frac{a^2(\beta_1 + \beta_4 r^3)}{3r}. \tag{2.17}$$

Here we derive the early time behaviour of the background evolution for later use. (corresponding to early time in IBB), eq. (2.11) for IBB reduces to

$$r' \sim -\frac{3}{2}(1 + w_{\rm tot})r \tag{2.18}$$

so that for $w_{\rm tot} = \text{const}$ (i.e., in radiation or matter dominated epochs) one has

$$r \sim a^{-3(1+w_{\rm tot})/2}\,, \tag{2.19}$$

and the $f_{\mu\nu}$ scale factor $b(\tau) = r(\tau)a(\tau)$ goes as:

$$b \sim a^{-(1+3w_{\rm tot})/2}\,. \tag{2.20}$$

The scale factor $b(\tau)$ therefore contracts instead of expanding as long as $w_{\rm tot} > -1/3$. Moreover, in the same approximation,

$$\mathcal{H}^2 \approx \frac{a^2 \beta_4 r^2}{3}. \tag{2.21}$$

It is useful to derive an approximated estimate for the $b(\tau)$ bounce epoch. The bounce occurs $b' = 0 \;\Leftrightarrow\; r'\big|_{r_b} = -r_b$. If we assume a bounce after the radiation epoch, then the ratio at the bounce has to satisfy

$$4\beta_1 - 6\beta_1 r_b^2 + \beta_4 r_b^3 = 0. \tag{2.22}$$

From comparisons with observational data, we know that the best fit is close to $\beta_4 \approx 2\beta_1$ which leads to

$$r_b \simeq 1 + \sqrt{3}\,. \tag{2.23}$$

It is also useful, for future purposes, to take note of the approximate observational relation between β_1 and β_4:

$$\beta_4 = \beta_1^2 \frac{3\left(1-\Omega_{\text{tot0}}\right) - \beta_1^2}{\left(1-\Omega_{\text{tot0}}\right)^3} = \frac{3\beta_1^2}{\left(\Omega_{\text{tot0}}-1\right)^2} + \mathcal{O}(\beta_1^4)\ ; \tag{2.24}$$

This relation approximately corresponds to the degenerate line between β_1 and β_4 for a flat universe, when fitting data sets such as supernovae, CMB and baryonic acoustic oscillation data [36]. Moreover, we require $\beta_1 \lesssim 0.5$, to ensure that the solution of eq. (2.16) at present time lies on the infinite branch.

3 Tensor perturbations

As we are interested on the effect of bigravity on gravitational waves, we now proceed with writing the tensor perturbation equations [32, 37, 40]. For the perturbed part of the metrics we adopt the transverse-traceless (TT) gauge, i.e. we select a transverse wave propagating along the z direction. Then, the tensor metric perturbations are given by:

$$h_{g(ij)} = \begin{pmatrix} h_{g(+)} & h_{g(\times)} & 0 \\ h_{g(\times)} & -h_{g(+)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

and similarly for the tensor modes of the f metric. We then obtain the following equations for both components (suppressing the subscripts $+, \times$)

$$h_n'' + \gamma_n h_n' + \left(m_n^2 + c_n^2 \mathcal{H}^{-2} k^2\right) h_n = q_n h_m\,, \tag{3.2}$$

where the indices $n \neq m$ refer to g-metric and f-metric, respectively; we have then

$$\gamma_g = 2 + \frac{\mathcal{H}'}{\mathcal{H}}, \qquad \gamma_f = \frac{2r^2 + 3r'^2 + r\left(4r' - r''\right)}{r\left(r' + r\right)} + \frac{\mathcal{H}'}{\mathcal{H}}; \tag{3.3}$$

$$m_g^2 = q_g = \mathcal{H}^{-2} a^2 B r, \qquad m_f^2 = q_f = \frac{\left(r' + r\right)}{\mathcal{H}^2 r^2} a^2 B; \tag{3.4}$$

$$c_g^2 = 1, \qquad c_f^2 = \frac{\left(r' + r\right)^2}{r^2}, \tag{3.5}$$

and where:

$$B \equiv \beta_1 + \beta_3 r^2 + r\left(2\beta_2 + \beta_3 r'\right) + \beta_2 r'. \tag{3.6}$$

These equations are equivalent to the ones in refs. [32, 37, 40]. In IBB (i.e. for $\beta_0 = \beta_2 = \beta_3 = 0$), B is simply given by β_1. The coefficients (3.3)–(3.5) for the two tensor equations

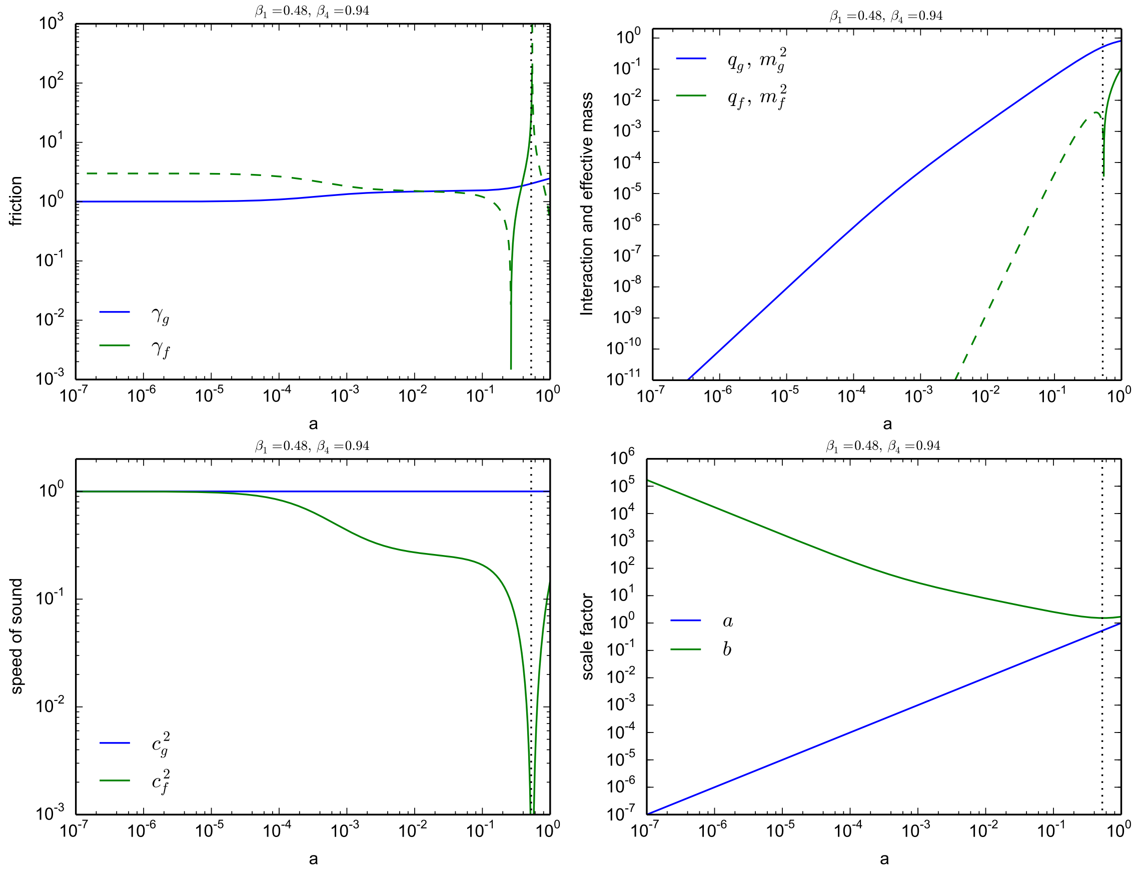

Figure 1. Coefficients of the tensor equations for IBB (3.3)–(3.5). Solid/dashed lines indicate positive/negative values and the vertical dotted line marks the bounce of the reference metric. Note that the friction term in the f-metric is negative at early times, when the f-metric is contracting. Note also that the coupling and effective mass terms are equal, and the ones corresponding to the f metric are very suppressed at early times. The bottom-right panel shows the evolution of the two scale factors.

are plotted in figure 1 as a function of redshift, for the choice $\beta_1 = 0.48$ and $\beta_2 = 0.94$. For this reference model, when considering both matter and radiation, the bounce happens at a redshift $z_b \simeq 0.9$, with a corresponding $r_b \simeq 2.8$.

Let us anticipate here an important feature of these equations. As it will be shown below (see also ref. [40]), the equation for h_f is unstable at early times since its friction term is negative as long as the scale factor $b(t)$ is collapsing instead of expanding. The fast growth of h_f will then drive a fast growth of h_g as well, through the coupling term. However, in the limit $r \to \infty$, the coupling coefficient q_g becomes

$$q_g = \frac{a^2 r \beta_1}{\mathcal{H}^2} \xrightarrow{r\to\infty} 3\frac{\beta_1}{\beta_4 r} \ , \tag{3.7}$$

and is therefore relatively small for large r. For the reference IBB model we have $\beta_1/\beta_4 \approx 0.5$; more in general, according to eq. (2.24), $\beta_1/\beta_4 \approx 1/3\beta_1$ to within factors of order unity, and therefore since $\beta_1 < 0.5$, we have the lower bound

$$q_g \approx \frac{1}{\beta_1 r} \geq \frac{1}{r} \ . \tag{3.8}$$

At recombination, for instance, we have $r \approx 10^4$ in IBB so that one needs a h_f roughly 10^4 times bigger than h_g before the coupling term $q_g h_f$ becomes comparable to the h_g terms in eq. (3.2) and it starts driving the evolution of h_g. This means that, in principle, a growing mode in h_f will take some time before affecting h_g. Whether this is enough to spoil the physical metric, is what we are going to test below.

In the following subsections we discuss more in detail the time behavior of h_g, h_f during the inflationary, radiation and matter eras.

3.1 Inflation

During a de Sitter epoch in which $H = const$, one has $\mathcal{H}' \sim \mathcal{H}$ and from eq. (2.11):

$$r \sim const \; . \tag{3.9}$$

The tensor equations (3.2) then reduce to:

$$h_g'' + 3h_g' + h_g \left(\frac{k^2}{\mathcal{H}^2} + \frac{a^2 \beta_1 r}{\mathcal{H}^2} \right) = \frac{a^2 \beta_1 r}{\mathcal{H}^2} h_f \; , \tag{3.10}$$

$$h_f'' + 3h_f' + h_f \left(\frac{k^2}{\mathcal{H}^2} + \frac{a^2 \beta_1}{r \mathcal{H}^2} \right) = \frac{a^2 \beta_1}{r \mathcal{H}^2} h_g \tag{3.11}$$

We can now assume $a^2 \beta_1 r \ll \mathcal{H}^2$ during inflation (ie $\rho_{inf} \gg \rho_{mg}$) so h_g behaves as in GR. The same is true for h_f since $a^2\beta_1 / r\mathcal{H}^2 \sim (\beta_1/\beta_4) r^{-3} \ll 1$. Since the inflationary equations are the standard ones, we expect the initial conditions to be unchanged and to apply equally well to h_g and h_f.

3.2 Radiation and matter dominated era

In the early time, we can approximate the ratio of scale factors as $r' = -\frac{3}{2}\left(1 + w_{\text{tot}}\right) r$ which is solved by

$$r = A a^{-\frac{3}{2}(1+w_{\text{tot}})}, \tag{3.12}$$

where A is a suitable normalization constant of order unity. Furthermore we approximate $\mathcal{H}^2 \simeq \frac{1}{3}\beta_4 a^2 r^2$. If the initial conditions for h_g and h_f are similar, then the source terms (3.5) are negligible at early times, i.e. small a, and the equations decouple. Furthermore, we find

$$\gamma_g \simeq \frac{3}{2}\left(1 - w_{\text{tot}}\right), \qquad \gamma_f \simeq -\frac{3}{2}\left(3w_{\text{tot}} + 1\right), \tag{3.13}$$

$$m_g^2 \simeq \frac{3\beta_1}{A\beta_4} a^{\frac{3}{2}(w_{\text{tot}}+1)}, \qquad m_f^2 \simeq -\frac{3\beta_1(3w_{\text{tot}}+1)}{2A^3\beta_4} a^{\frac{9}{2}(w_{\text{tot}}+1)}, \tag{3.14}$$

$$c_g^2 \simeq 1, \qquad c_f^2 \simeq \frac{(3w_{\text{tot}}+1)^2}{4}. \tag{3.15}$$

Neglecting the mass term m_f^2 at early times, the tensor evolution for $f_{\mu\nu}$ is described by

$$h_f'' - \frac{3}{2}(3w_{\text{tot}}+1) h_f' + \frac{h_f k^2 (3w_{\text{tot}}+1)^2}{4\mathcal{H}^2} = 0. \tag{3.16}$$

At large scales the last term is negligible and one finds a growth of h_f as $a^{3(3w_{\text{tot}}+1)/2}$. Thus, when radiation dominates, h_f increases very fast as a^3. Clearly, if one starts with $h_f' = 0$ then this growing mode is initially absent and it takes some time before it becomes visible.

The evolution of h_g has instead a constant mode $h \sim const$ until h_f is large enough to source the growth of h_g, cf eq. (3.7).

The early time approximation that leads to eq. (3.16) turns out to be a very good approximation also in the matter domination. In this regime h_f increases as $a^{3/2}$ for super-horizon modes. When the coupling term becomes important, h_g is driven by h_f and acquires the same trend. Finally, when MDE ends and the system approaches a de Sitter behavior, the perturbations begin to decay.

For sub-horizon scales the behavior is influenced by the h_f time-dependent sound speed. An asymptotic form for large k can however be found. In this regime we can neglect the mass and the coupling terms, and the h_g, h_f equations during either RDE or MDE have the general form

$$h_n'' + \gamma_n h_n' + \beta_n k^2 a^\eta h_n = 0 \, , \tag{3.17}$$

where the index n stands for g, f and $\eta = 1 + 3w_{\rm tot}$ and β_n is an irrelevant constant. The general solution can be easily written in terms of the Bessel functions but here we need only the asymptotic behavior for large k or late times, which is

$$h_n \sim a^{-\left(\frac{\gamma_n}{2}+\frac{\eta}{4}\right)} \tag{3.18}$$

times fast oscillations. We see then that for sub-horizon modes h_f grows as a^1 in RDE, as $a^{1/2}$ in MDE and a final decay as a^{-1} when approaching the future deSitter phase, while h_g decays as in the standard case as a^{-1} in all eras (before being driven to growth by the coupling to h_f). Both the super-horizon and the sub-horizon behaviors found here analytically are confirmed by our numerical findings and agree, and generalize, the analytic results of [40]. For very large wavenumbers the coupling and the mass terms are ineffective at all times and the h_g equation reduces to the standard case. This implies that there is no large effect to be expected for the directly detectable range of gravitational waves, which is around 0.1Hz or $k \approx 10^{14}\,\mathrm{Mpc}^{-1}$ (see e.g. [44]), although a precise calculation is beyond the scope of this paper.

3.3 Inflationary initial conditions

We can now use the a^3 growth mode during radiation to estimate the order of magnitude effect of the tensor modes at recombination (a more precise estimation will be obtained numerically in sections 4, 5). Inflation ends at some energy scale that can vary from 10^{15} GeV to few MeV depending on the model. The upper limit comes from the bounds on the amplitude of tensor perturbations, indicating that the energy scale of inflation is at most that of Grand Unified Theories when observable modes are produced. The lower bound is inferred from the need of a radiation dominated universe in thermal equilibrium during big bang nucleosynthesis. These values are reached when the scale factor was $a_{inf} \approx 10^{-9}$ at the latest. Since super-horizon tensor modes grow as a^3 during radiation domination, in the most favourable case of inflation ending just before big bang nucleosynthesis one would obtain an enhancement until recombination $a_{\rm rec} \approx 10^{-3}$ of $h_{f(\rm rec)} \sim 10^{18} h_{f(e)}$ roughly, where the subscript e denotes the end of inflation. If $h_{f(e)}$ has the standard value approximately equal to $H_e/T_P \approx T_e^2/T_P^2$ during inflation, where $T_P \approx 2.4 \cdot 10^{18}\,\mathrm{GeV}$ is the reduced Planck temperature/energy and T_e is the inflationary energy scale (here for simplicity assumed to be similar to the energy at the end of inflation), then the value at recombination of h_f for a wave that reenters horizon at recombination or larger is roughly

$$h_{f(\rm rec)} \approx \left(\frac{a_{\rm rec}}{a_e}\right)^3 h_{f(e)} = \left(\frac{T_e}{T_{\rm rec}}\right)^3 \left(\frac{T_e}{T_P}\right)^2 \tag{3.19}$$

Shorter waves reenter before and therefore grow less. If this value has to be compatible with the level of fluctuations in the CMB polarization spectra, then it should be lower than about one tenth of the temperature fluctuations; we take conservatively the level 10^{-7}. The same value should be taken for h'_f since inflation excites both the tensor mode and its momentum conjugate. However we do not detect directly h_f but rather the g-metric mode h_g which is coupled to matter, so as already noticed one can have a value of h_f larger than h_g by a factor of $q_g^{-1} \approx 10^4$ at recombination. Putting therefore $h_{f(\text{rec})} \leq 10^{-3}$, we obtain an upper limit to the temperature at the end of inflation $T_e \approx 10\,\text{GeV}$. Since tensor modes impact CMB also at reionization, when the coupling term q_g is closer to unity, this limit should be lowered to roughly

$$T_e \approx 1\,\text{GeV} \tag{3.20}$$

(a similar limit has been obtained also in [40]). It might be interesting to remark that the superhorizon growing mode breaks the standard link between tensor modes and inflationary scale due to the presence of the coupling: now in principle one can have observable tensor modes even in low scale inflation.

Any inflationary model with higher energy scale will generate excessive power on the tensor modes unless the inflationary initial conditions are suppressed with respect to the standard value or their growth is reduced. Taken at face value, this shows that the a^3 growing mode can be reconciled with observations only in the rather extreme scenarios of very low-energy inflation, as e.g. in the models discussed in ref. ([45, 46]). Fixing the initial conditions to a more conservative era for the end of inflation, e.g. $T \sim 10^3\,\text{GeV}$, would produce the huge value $h_{f(\text{rec})} \sim 10^{39} h_{f(e)} \sim 10^7$.

Barring the case of very low-energy inflation, then, the IBB model is at odds with CMB observations. In the rest of the paper we will explore more or less contrived ways to overcome this difficulty.

3.4 The non-linear cut-off

The tensor perturbations in the reference metric grow so fast that they will eventually become non-linear. At this point, the perturbative treatment followed so far breaks down and one has to take into account higher order corrections, or even the full equations of motion in order to correctly reproduce the dynamics. A natural question is then what happens to the two metrics after non-linearity is reached: the evolution would then need to be calculated self-consistently in a non-linear theory for bigravity, which is beyond the scope of this paper. This problem is not new to Dark Energy models. There are cases such as growing neutrino cosmologies [47] in which the effect of non-linearities becomes important and needs to be taken into account also when dealing with the CMB predictions [48]. In that scenario, the fast growth actually leads to stable non-linear structures (which are a way to test the model rather than an argument to exclude it based on linear theory).

Following the idea of ref. [48], we then stop the evolution of perturbations at some cut off amplitude value $h^{\text{cut}} \approx 1$, when non-linearity is approximately reached. This prescription is applied to both h_g and h_f and has the effect to partially stop the 'dragging' of the second metric h_f over the standard one h_g. Such assumption is adopted here for simplicity, as a toy model, our interest being to give a consistent estimate of how big is the impact of the growing mode on CMB spectra and tensor perturbations when non-linearity is reached.

In practice, since the growth of h_f is very rapid in the early Universe, this is equivalent to fixing $h_f = h^{\text{cut}} = 1$ from the very beginning, with the consequence that the perturbations

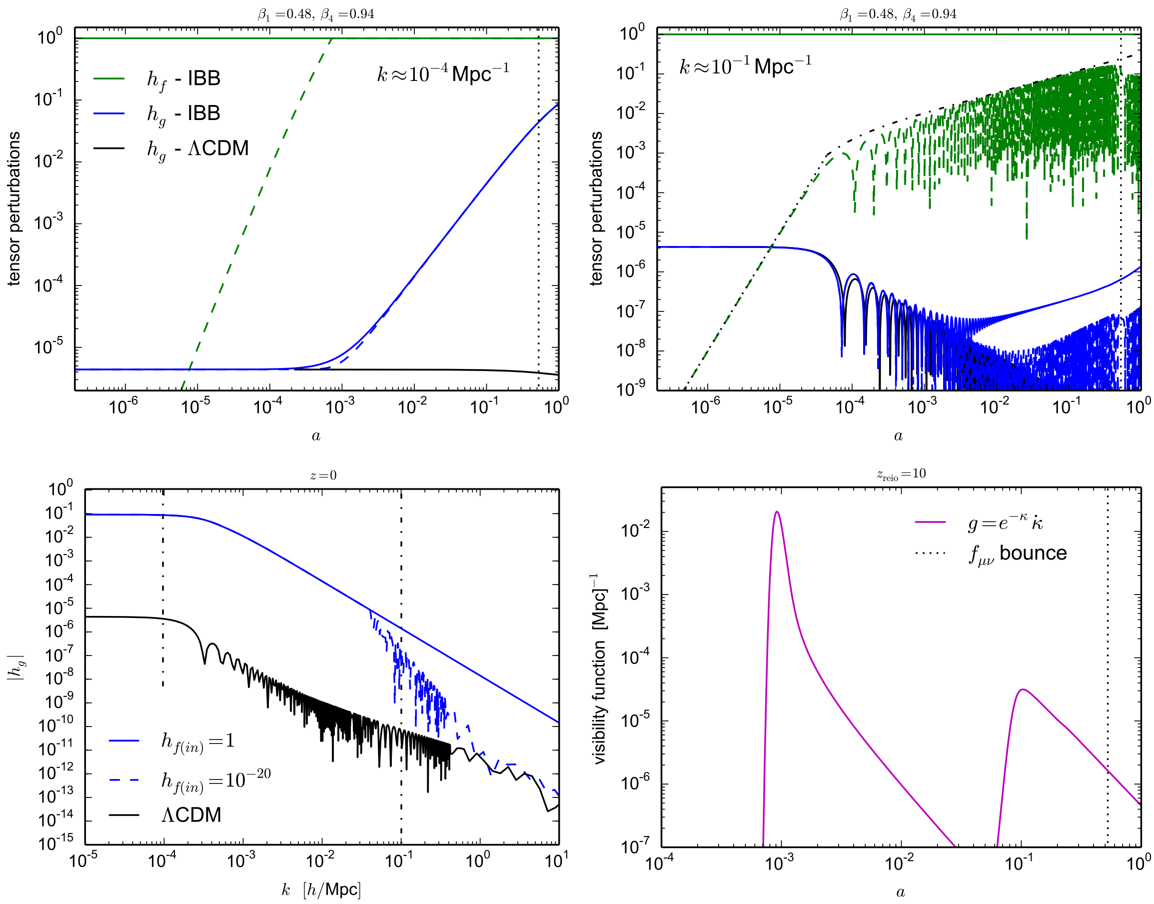

Figure 2. Evolution of tensor perturbations at different scales (top panels) and scale dependence of tensor perturbations at $z = 0$ (bottom left panel). All scales include a cut-off when non-linearity is reached; h_g does not reach the cut-off for any scale and redshift. The initial conditions have been chosen so that $h_f = 1$ is initially non-linear (solid lines) or starts at a small value (dashed lines) and becomes non-linear at later times. The latter choice corresponds to $h'_{f(in)} \approx h_{f(in)} = 10^{-20}$ at $a = 10^{-10}$ (see section 5.1). The top right panel overlays the analytic approximaton for super-horizon evolution and sub-horizon envelope (black dotted/dash-dotted lines) described in section 3.2. The two modes shown in the top panel correspond to the vertical dash-dotted lines in the bottom left panel. For reference, we recall the standard CMB photon visibility function (bottom right panel), whose peaks correspond to recombination and reionization epochs (see section 4). The bounce of the reference metric has been indicated with a dotted vertical line.

of the reference metric are not dynamical anymore. Nevertheless they still affect the tensor modes of the physical metric due to the coupling (3.7). The overall evolution as a function of the scale factor for two values of k and two different choices of the initial conditions is plotted in figure 2 for both h_g and h_f, once the bound has been applied. Figure 2 also shows a similar behavior between a model in which h_f starts saturated and one in which the cutoff value is reached during the evolution. It is also shown how, even though the cutoff is applied to both metrics, h_g never reaches it during its evolution. Note that the full non-linear dynamics might produce other effects. For example, in the limit of scales smaller than the horizon, one

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finds the oscillating behavior in e-folding time $h_g, h_f \sim e^{imN}$ with eigenfrequencies:

$$m = \pm \frac{r + r'}{\mathcal{H} r}. \tag{3.21}$$

This oscillating behavior is then always present in the sub-horizon solutions, overimposed to an amplitude modulation, as shown for the smallest scale in figure 2 (top right panel). In this case, setting h_f to the non-linear cut-off value leads to a growing behavior plus a damping of the initial oscillations. On the other hand, the model with fine tuned initial conditions displays the oscillatory behavior expected from linear theory (3.21), and in this case the non-linear value $h_f \sim 1$ is not reached, at least for Fourier modes corresponding to small scales. In this case the negative friction of the reference metric gets compensated by the positive friction from the physical metric. Nonetheless we expect our method to give a consistent qualitative estimate of the observable effects of the growing mode on the CMB, focusing on the tensor contribution and neglecting scalar or vector perturbations.

4 Cosmic Microwave Background anisotropies in bigravity

The Cosmic Microwave Background (CMB) was shown to be a powerful probe to test not only early time cosmology but also Dark Energy and Modified Gravity models [49]. In particular, in this paper we are interested in the effect that tensor perturbations in bigravity have on the CMB power spectra. At recombination, when photons are not anymore tightly coupled to baryons but decoupling has not occurred yet, electrons can be scattered simultaneously by photons coming from cold and hot spots. In presence of a quadrupole temperature anisotropy, the scattered photons will be linearly polarized and the CMB radiation will be characterized not only by its intensity, but also by its polarization. CMB polarization can be expressed in a tensor normal basis in Fourier space, in terms of E and B modes. While scalar perturbations can only produce an E mode (primordial) polarization pattern, tensor perturbations can feed both E and B primordial modes. Therefore any change in the evolution of tensor perturbations predicted in bigravity will affect the polarization spectra.

At later times, polarization and temperature anisotropies are further modified during reionization. Reionization occurs at a much lower redshift, when the universe becomes partially ionized due to the formation of the first stars, allowing CMB photons to partially rescatter. The recombination and reionization eras correspond to peaks in the visibility function shown in figure 2. The visibility function $g(t) = \exp(-\kappa)\dot{\kappa}$ (where κ is the optical depth and $\dot{\kappa}$ is its derivative with respect to conformal time τ) gives the probability that a photon last scattered in the conformal time interval $[\tau, \tau + d\tau]$. Due to the importance of the coupling at relatively low redshift, the most important effects of tensor modes on the CMB are imprinted during the reionization epoch.

In the following, we have only modified tensor perturbations, assuming that the contribution of scalar perturbation is small enough to be neglected, as scalar modes affect B mode polarization only indirectly, via lensing of E modes, at scales $\ell \gtrsim 150$. Of course, if polarization is large enough, it might also feed back the scalar spectra. However, this seems a good enough first approximation to test the specific effect of the growing mode on the BB spectra. We implemented the tensor evolution equations in two publicly available Boltzmann codes, CAMB [50] and CLASS [51], and compared the results obtained in the various cases to verify their mutual consistency.

As discussed in the previous section, our aim is to check the effect on the CMB spectra consistently, i.e. taking into account that, by definition, we cannot trust any result derived

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assuming linear perturbation theory when perturbations become non-linear. We then fix $h_f = h^{\rm cut} = 1$ from the beginning and evolve only the g-metric tensor h_g, for which we will assume standard initial conditions:

$$A_t = r_{T/S} A_s \left(\frac{k}{k_p}\right)^{n_T} , \tag{4.1}$$

with a fiducial tensor-to-scalar ratio $r_{T/S} = 0.05$, a scalar amplitude $A_s = 2.21 \times 10^{-9}$ and a tensor spectral index given by the self-consistency condition of single field slow-roll inflation $n_T = -(2 - r_{T/S}/8 - n_s)r_{T/S}/8$, where the fiducial scalar spectral index is $n_s = 0.9645$ (in section 5.1 we explore the effects of changing the IC on the tensor modes.). For bigravity we choose the best fit model $\beta_1 = 0.48$, $\beta_4 = 0.94$ with $\Omega_{cdm} = 0.13$, $\Omega_b = 0.05$, while for the reference ΛCDM model we choose *Planck* 2015 TT, EE, TE+lowP marginalized values [52], i.e. $\Omega_{cdm} = 0.26$, $\Omega_b = 0.05$. In both bigravity and ΛCDM cases the fiducial optical depth is $\tau = 0.079$, corresponding to $z_{\rm reio} = 10$. In order to test the IBB model, we compare the achieved spectra with up to date CMB observations, using Planck 2015 data [53] for TT, TE and EE spectra, while for the BB spectrum we rely on WMAP [54] and BICEP1 [55] together with the joint BICEP2, Keck, Planck analysis [56]. Figure 3 shows the tensor contribution to the CMB temperature, polarization and cross spectra for the fiducial bigravity model described above. Large angular scales are the most sensitive to the growing modes in the f-metric, because the coupling to the physical metric is only important at low redshifts, after recombination. On these scales the tensor perturbations give contributions to the power spectra that are too large to be compatible with CMB data. Even for the T and E polarization spectra, the tensor contribution in IBB bigravity overshoots the observed values by several orders of magnitude for $\ell \lesssim 60$. Since both the scalar and tensor contributions to TT and EE spectra are positive definite, it is impossible that a reduced scalar contribution compensates the (large) tensor part in order to fit the data. The conclusion is that the cut-off on the growing mode is not enough to save the model.

5 Possible solutions to the growing mode problem

A first attempt to overcome the growing mode problem is to verify how much the effect on the spectra depends on the choice of the fiducial model. We investigated the effect of changing the bigravity parameters β_1 and β_4, choosing them close to the degeneracy curve (2.24), the redshift of reionization $z_{\rm reio}$, the tensor to scalar ratio $r_{T/S}$ and the tensor spectral index n_T characterizing the shape of the primordial power spectrum for tensor perturbations. These effects are shown in figure 4, showing that simple variations of these cosmological parameters do not offer a sufficient improvement.

The observable impact on the CMB is produced in the reionization era because the coupling between the physical and reference metric's tensor perturbations is only relevant at low redshift. Shifting $z_{\rm reio}$ in the range $(5, 15)$ has only a small effect on the BB spectrum. Further changes would spoil the predictions for EE and TE spectra and enter in contradiction with the Gunn-Peterson limit [57]. Higher reionization redshifts reduce the tension slightly, mainly because the perturbations in h_g are smaller at earlier times and on smaller scales. Another attempt that does not work is to modify the initial conditions for perturbations of the physical metric. Varying the spectral index and the tensor to scalar ratio only has an impact on relatively high multipoles $\ell > 30$, on which the tensor perturbations are predominantly

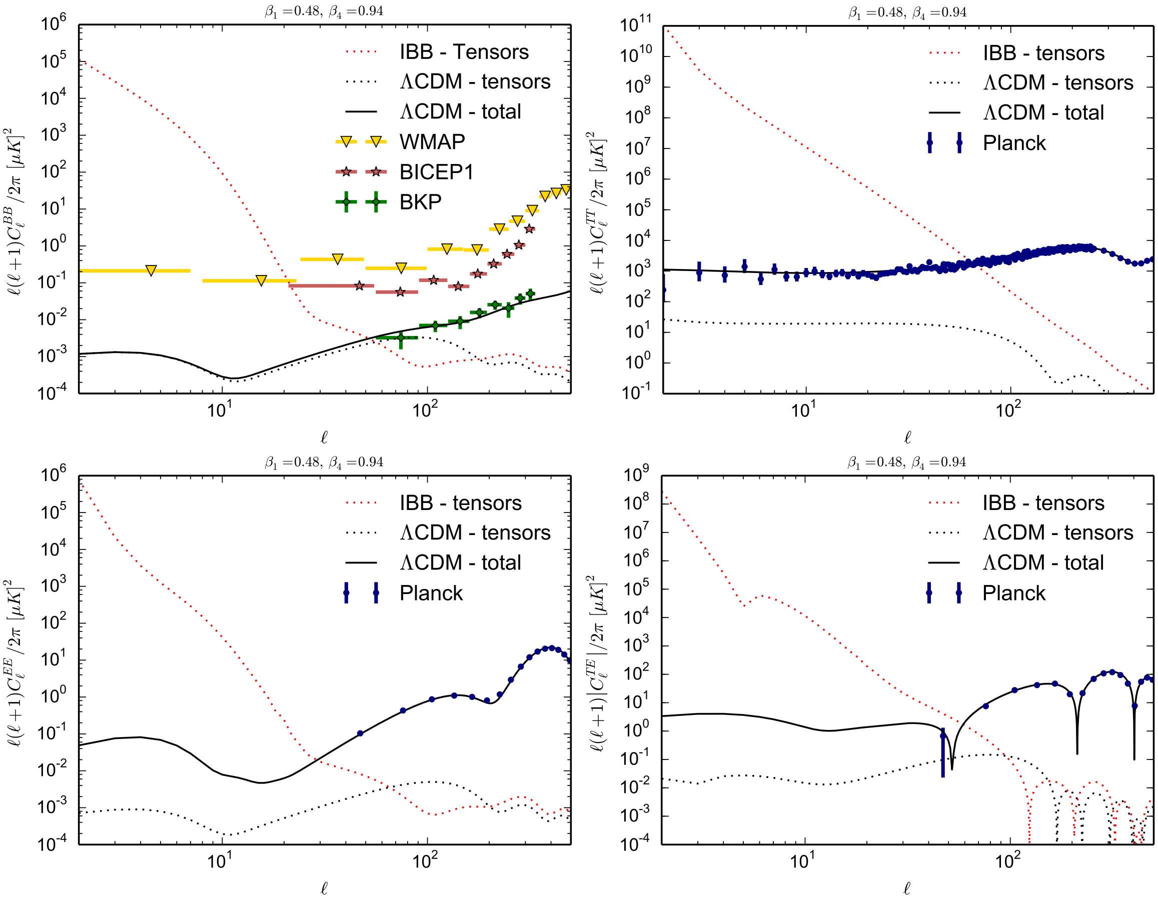

Figure 3. Effects of the growing mode in the f-tensors on the CMB. All the plots assume IBB with $\beta_1 = 0.48$, $\beta_4 = 0.94$, $\Omega_m = 0.18$ with $\Omega_b = 0.05$, $h = 0.67$, $r_{T/S} = 0.05$ and the spectral index determined by inflationary self-consistency conditions. The tensor perturbations have been assumed to start at non-linear cutoff value $h_f = 1$.

imprinted during recombination. For lower multipoles the evolution of h_g is dominated by the coupling to h_f, which overshoots the initial conditions on large scales due to its large value.

Concordance with observations can neither be achieved by varying the IBB parameters β_1, β_4. The fast growth of the h_f tensor perturbations leads to growing h_g tensors due to the coupling in (3.2) which is proportional to $\beta_1 r/\mathcal{H}^2$. One might expect that a change in the betas could lead to higher values of r and, thus, a suppression of the coupling at early times (note that $\mathcal{H}^2 \propto r^2$ in RDE). Higher values of r are possible when lowering β_1 since $r_0 = (1 - \Omega_{\text{tot}0})/\beta_1$, where r_0 denotes the present value of r and one has $r > r_0$ during the entire evolution. However, in order to fit observations we have to choose parameters being close to the degeneracy curve (2.24), i.e. $\beta_4/\beta_1^2 \simeq$ const. As already mentioned, at early times the coupling term is then proportional to $q_g \approx 1/\beta_1 r$ (see eq. (3.8)). A smaller coupling would of course help in delaying or reducing the effect of the h_f growing mode on h_g. Since the evolution for large r is nearly independent of the coupling parameters (see eq. (2.18)), and since $\beta_1 \lesssim 0.5$, the ratio $1/(\beta_1 r)$ cannot decrease much below the reference case. Figure 4 shows clearly that the effect of varying β's is negligible, at least when the coefficients remain along the degeneracy line.

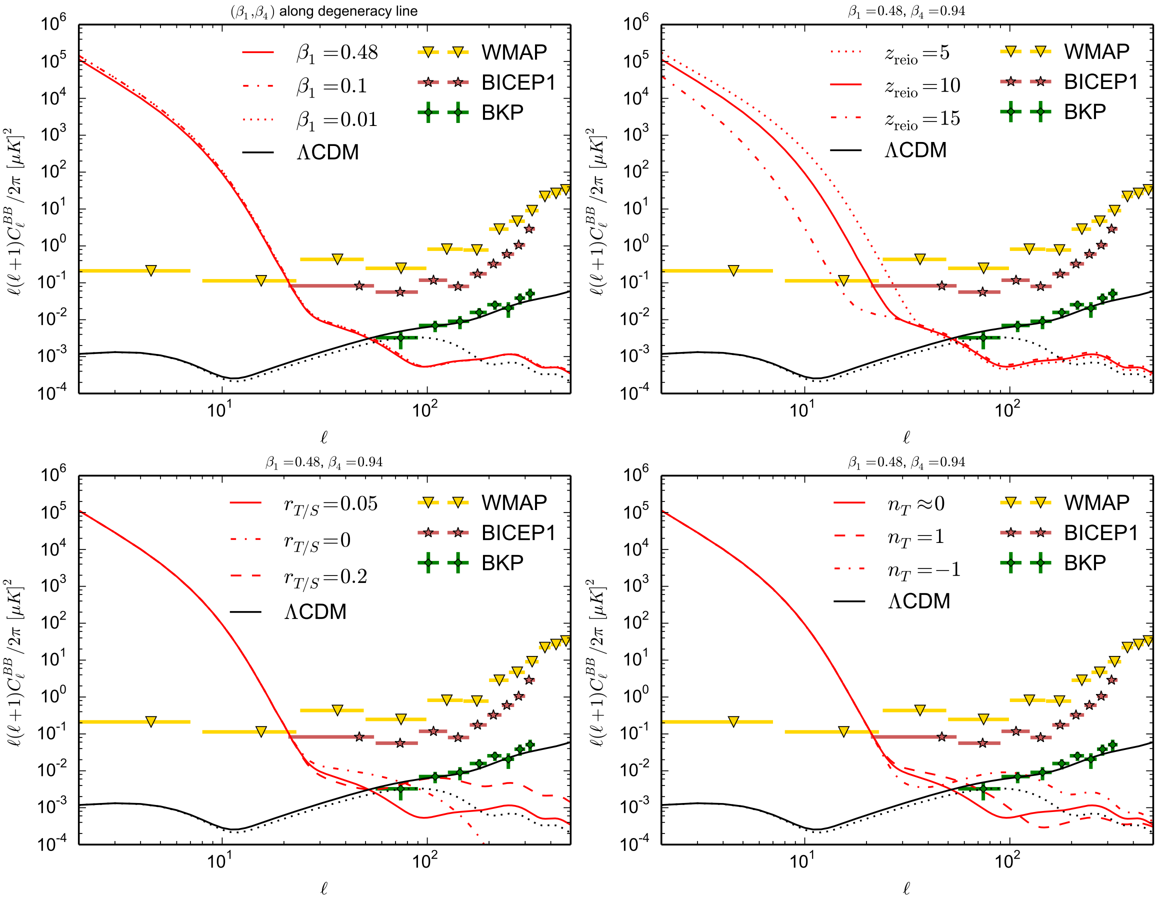

Figure 4. Non-solutions to the problem of growing modes in the reference metric: varying the IBB parameters across the degeneracy line (top left panel), the reionization redshift (top right) and the initial conditions for tensor perturbations in the physical metric (bottom panels). The remaining model details are the same as in figure 3. In particular, the reference metric perturbations are set initially to the non-linear cutoff value. All solid lines correspond to the standard values described in section 4.

One loophole in the line of argument above is to leave the observational degeneracy curve (2.24) and add an explicit cosmological constant β_0. The coupling coefficient q_g does not explicitly depend on β_0 and reduces to:

$$q_g = \frac{3\beta_1 r^2}{\beta_1 + \beta_4 r^3}. \tag{5.1}$$

From the $f_{\mu\nu}$-Friedmann eq. (2.16) we find that r scales like $\beta_4^{-1/2}$ for $\beta_1 \ll \beta_4 r^3$ which leads to

$$q_g \propto \sqrt{3}\frac{\beta_1}{\sqrt{\beta_4}}. \tag{5.2}$$

Thus, one is able to get an arbitrarily small coupling when choosing values of β_1 that are sufficiently smaller than $\sqrt{\beta_4}$. In this regime, the massless graviton dominates over the massive one and the cosmological evolution tends to that of ΛCDM. It is clear that such a model is not particularly interesting from a cosmological point of view because hardly distinguishable from the standard model and therefore here we will not investigate it further.

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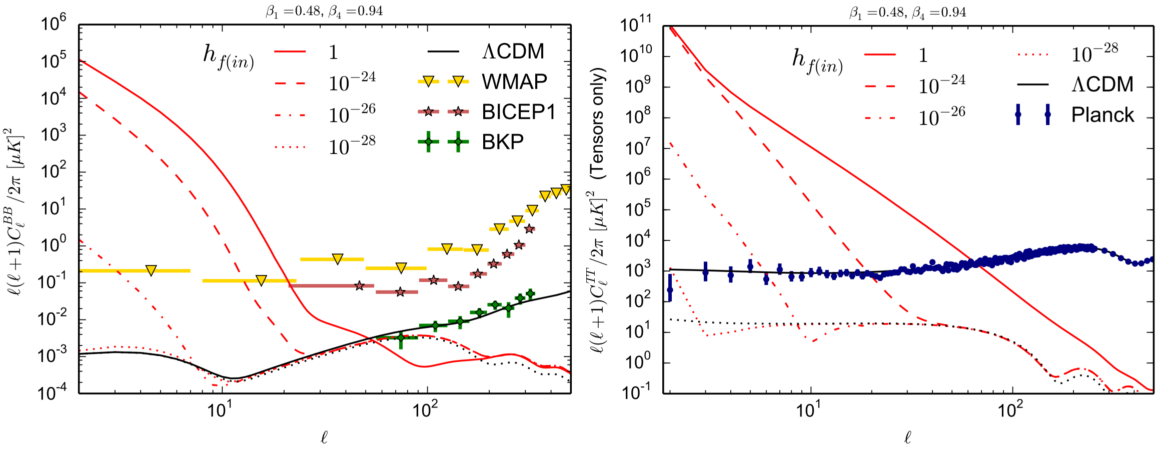

Figure 5. BB (left panel) and TT (right panel) spectra using fine-tuned initial conditions as described in section 5.1. The initial amplitude and time derivative have been specified according to the growing solution (5.3) at a fiducial scale factor $a_{\rm in} = 10^{-10}$. Bigravity BB spectra (red lines) only contain the primordial tensor contribution, while ΛCDM spectra (black solid lines) includes the contribution of both scalar and tensor perturbations. The evolution of tensor perturbations has been stopped whenever $h_{g,f} = 1$ is reached (cf. section 3.4).

In order to render the model viable without adding a cosmological constant, it is necessary to adopt a more radical solution. In the following subsections we explore how IBB can be reconciled with observations by fine-tuning the initial conditions in the reference metric perturbations, lowering the non-linear cut-off or modifying the theory.

5.1 Changing the initial conditions for h_f

Our next attempt consists in checking whether fine tuning the initial conditions can compensate the effect of the growing mode on the CMB spectra, as illustrated in figure 5. We specify the initial conditions in terms of the growing solution in the radiation era

$$h_f(a) = h_{f({\rm in})}\,(a/a_{\rm in})^3\,, \quad h_f' = 3h_f(a)\,, \tag{5.3}$$

found in section 3.2.[1] We find that the initial conditions have to be fine tuned to zero to at least the level of one part in 10^{26} at $z_{\rm in} = 10^{10}$ in order to fit current limits on the BB spectrum. This choice of the initial epoch corresponds to an era before Big Bang Nucleosynthesis, which as already mentioned is a hard lower bound for the end of inflation. One can easily relate to earlier times in order to specify the IC at the reheating epoch, when inflation ends and tensor perturbations start growing. Table 1 extrapolates the result to the range of energies in which inflation might have ended.

As already noticed, the only way to generate naturally a very low level of tensor modes compatible with CMB without any cut-off is to assume that inflation ends at an energy scale

[1]One can in general fix h_f and h_f' independently for each wavenumber, but we restrict to the simpler choice (5.3) here. If more general IC are considered, a necessary condition for the growing mode to be sufficiently suppressed is that the time derivative is small. This condition is sufficient as long as $h_{f({\rm in})}$ is well below the cutoff value (see next subsection).

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	BBN	fiducial	1 GeV	GUT
Bounds on $h_{f(\mathrm{in})}$	$\lesssim 10^{-19}$	$\lesssim 10^{-25}$	$\lesssim 10^{-31}$	$\lesssim 10^{-82}$
Scale factor a	$\sim 10^{-8}$	$\sim 10^{-10}$	$\sim 10^{-12}$	$\sim 10^{-29}$
Temperature T	$\sim 0.1\,\mathrm{MeV}$	$\sim 10\,\mathrm{MeV}$	$1\,\mathrm{GeV}$	$\sim 10^{16}\,\mathrm{Gev}$

Table 1. Upper bounds on h_f extrapolated to different epochs using the growing mode, eq. (5.3). The limits shown are based on the results for BB spectra, which only depend on the tensor sector at low multipoles where the theory enters in tension with the data. Considering TT spectra would tighten these bounds, as can be inferred from the left panel of figure 5.

not larger than 1 GeV.[2] If however one assumes the non-linear cut-off, then the inflation energy scale bound can be relaxed. From table 1 we see empirically that the initial condition for h_f is related to the end of inflation energy scale T_e (expressed in GeV) as $h_{f(\mathrm{in})} \approx 10^{-31} T_e^{-3}$. Then one has

$$h_{f(\mathrm{in})} \approx \left(\frac{T_e}{T_P}\right)^2 \approx 10^{-31}\left(\frac{1\,\mathrm{GeV}}{T_e}\right)^3 , \tag{5.4}$$

or $T_e \approx 25\,\mathrm{GeV}$, a more realistic scale range for low-energy inflation.

A sufficiently small value of h'_f might also be provided by a more exotic inflationary mechanism. During inflation no growing modes occur on the perturbations of the f metric. Some solutions, such as increasing the mass of the graviton at very early times, might naturally generate the low values needed to reconcile the model with observations (the problem of growing classical perturbations is common to ekpyrotic scenarios alternative to inflation [59]). Outside of these rather unconventional, although not impossible, cases, the conclusion we draw is that only very fine tuned initial conditions allow to reconcile bigravity with CMB observations.

Even if inflation ends at a sufficiently low scale or a mechanism to suppress $h'_{f(\mathrm{in})}$ exists, it has been argued by Cusin et al. [40] that non-linear corrections would spoil the small value of h'_f. Although we will not investigate this issue further here, we note that the nature of the theory might protect the tensor modes against such terms. This is precisely what happens in the linear equation (3.2), in which the source term is highly suppressed. If a similar suppression occurs also on the non-linear source terms, the fine tuned initial conditions can render the model compatible with CMB observations (assuming that there are no additional complications in the scalar sector).

5.2 Lowering the cut-off

Another possible solution to mitigate the effect of IBB on CMB spectra is to assume that non linear effects begin to be not negligible before h_f reaches unity; this can be thought as an effective way to treat the impact of non linear effects which, even if dominant when h_f is above unity, can start to affect the evolution of the perturbations even for lower values. Therefore, in figure 6 we show the behavior of TT and BB spectra for different values of h^{cut}

[2] After this manuscript had been submitted, a work studying inflation in bigravity appeared in which the authors argue that highly suppressed reference metric tensor initial conditions are naturally generated if inflation leads to a cosmology of the IBB type [58]. This is because of the large hierarchy between the physical and reference metric, that makes the tensor modes in f suppressed by a factor $1/r$ with respect to those in g. The results of ref. [58] make it unnecessary to lower the inflationary scale to meet the bounds of table 1.

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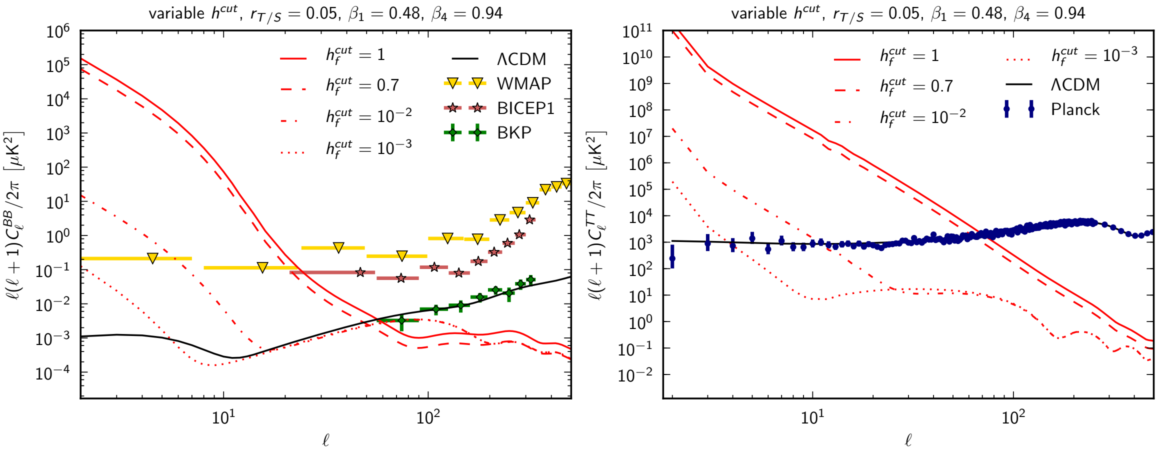

Figure 6. BB (left panel) and TT (right panel) using a varying cutoff $h^{\rm cut}$. The evolution of h_f and h_g is frozen when those reach, respectively, the cutoff value $h^{\rm cut}$. Bigravity BB spectra (red lines) only contain the primordial tensor contribution, while ΛCDM spectra (black solid lines) and observational data points contains the contribution of both scalar and tensor perturbation.

at which we freeze the evolution of the metric perturbations. One would need to suppress the cutoff scale by at least three orders of magnitude to reconcile theoretical BB spectra with currently available limits and possibly an even lower value to make the TT spectrum acceptable for current data, once the scalar contribution is taken into account.

It seems contrived that non-linear effects might play a role at such small values of the cutoff. However, theories of massive gravity are known for having strong non-linear effects in certain limits, such as the Vainshtein mechanism [60, 61]. As long as numerical results for the non-linear evolution of tensor perturbations are not available, we must contemplate the possibility that the cut-off could be lower than unity and even significantly lower.

5.3 Modifying the theory

The growing modes in the reference metric could possibly be reconciled with CMB data by a suitable modification of the theory. Here we explore a phenomenological modification in which a redshift dependence of the β_i parameters is assumed; this kind of behavior might be achieved in generalized massive gravity models [62], where a time dependence of the mass parameters is introduced without the addition of any new dynamical degree of freedom. Our modification consists on setting $\beta_1 = \beta_4 = 0$ until a certain switch redshift z_s is reached. Then bigravity becomes active and the evolution described in section 3 is switched on. The considered background evolution instead is the one produced by bigravity at all redshift, as IBB well approximates the standard background, which should take place at $z > z_s$, at early times.

The results in figure 7 show how switching on bigravity at approximately the redshift of matter-radiation equality ($z_s \approx 10^3$) can produce an acceptable BB spectra when comparing with current data. In the TT case instead, the contribution of scalar modes to the spectrum can possibly lead to the necessity of an even lower value of z_s.

There are additional ways in which the theory might be modified while retaining the original field content of bigravity. In the following we will describe these possibilities, although addressing them in detail will be left for future work. A possible modification is to allow for branches different than IBB. So far, only branches in which r evolves from $r = 0$

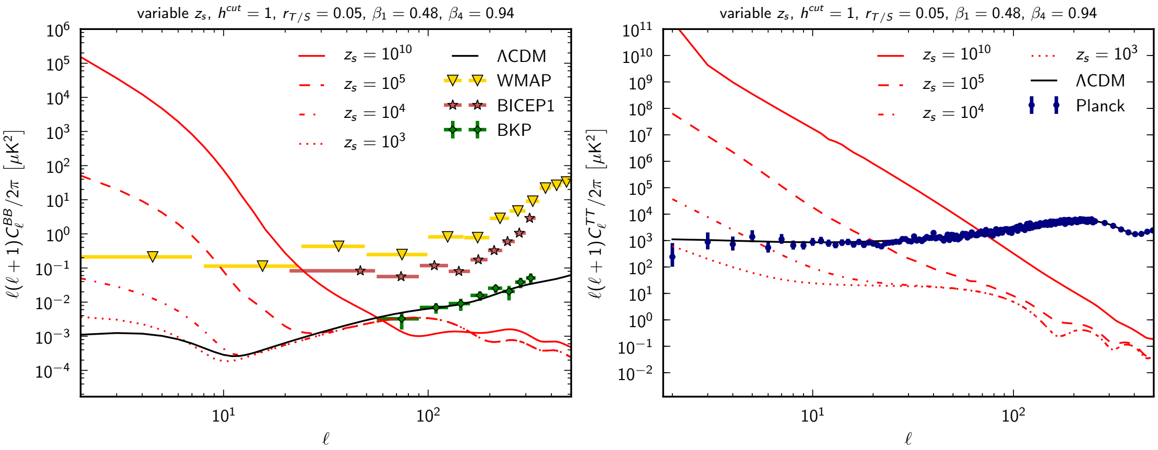

Figure 7. BB (left panel) and TT (right panel) using a varying initial redshift z_s. If $z > z_s$ h_f follows the same standard equation as h_g, while for $z \leq z_s$ the evolution described in section 3 is switched on. Bigravity BB spectra (red lines) only contain the primordial tensor contribution, while ΛCDM spectra (black solid lines) and observational data points contains the contribution of both scalar and tensor perturbation.

or $r \to \infty$ were considered. In [30] all remaining cases due to a non-viable behavior were excluded. However, some of those conditions based on expectations of a standard cosmological evolution, like an expansion at all times and the existence of a matter/radiation dominated era in the asymptotic past. It might be interesting to study these disregarded branches. Additionally, both metrics are usually assumed to be FLRW at background level, even though this is assumed only for simplicity. In ref. [63] the authors considered more general types of metrics which might lead to interesting evolutions at background and linear level and could also have an impact on the tensor evolution.

An additional possibility is to modify the coupling to matter. Even though both metrics are a priori equally footed, we let only one of them couple to matter while the remaining metric stays unobservable. An additional coupling of the reference metric to matter would influence the tensor perturbation and might be able to tame the fast growth (see e.g. refs. [64, 65] for further discussion on bi-metric couplings) . An additional coupling of the same matter Lagrangian to $f_{\mu\nu}$ is not possible as it will generally reintroduce the BD ghost [66–68]. Even though this will not happen if a different matter Lagrangian (an unobservable dark sector) is coupled to $f_{\mu\nu}$, we will usually meet a new fine tuning [69] that would make the theory less appealing. One way out would be a coupling through a new composite metric that is constructed such that it avoids the BD ghost [66]. This choice would lead to viable, self-accelerating backgrounds [70] but still does not yield a realistic cosmological evolution at the linear level [71, 72].

6 Conclusions

We have analyzed the behavior of tensor perturbations in the infinite-branch bigravity (IBB) model and the signatures they produce on the CMB. In this model the reference metric contracts at early times, causing tensor perturbations in this metric to grow rapidly. These modes have ample time to grow since the end of inflation. However, a consistent analysis of linear perturbations and CMB spectra cannot include the regime in which perturbations become non-linear. We then assume that this growth stops when perturbations become non-linear, with an amplitude saturated at a value of order unity. The coupling between the two metrics produces in turn a growth of the tensor perturbations in g. If the coupling is weak enough at early times, the growing mode will in principle propagate to the physical metric only after some time. Our first objective has been then to check whether this effect is late enough to keep the spectra of tensor perturbations compatible with present CMB data. Our conclusion is that, even when perturbations remain below order unity, they are still large enough to have a twofold impact on CMB spectra: first, the tensor modes provide a large contribution to the TT, EE and TE spectra which is orders of magnitude larger than the scalar contribution; second, the tensor modes induce a strong B-mode polarization on the CMB. Both effects dominate on the largest angular scales and are incompatible with observations from CMB experiments.

Varying the IBB parameter (β_1, with β_4 being derived from it via eq. (2.24)) or other cosmological parameters offers little help in reconciling IBB bigravity theory with observations. We further explore five scenarios in which the theory might be rendered viable:

1. Lowering the energy scale of inflation. If the energy scale of inflation is very small, around 1 GeV, the tensor modes are naturally suppressed *and* the growing mode has less time to grow until recombination or reionization: the combined effect makes the IBB model acceptable without any change. If one invokes the freezing of the growing mode when it reaches non-linearity, then the inflationary energy scale can be increased up to 25 GeV roughly. Such a scale is much lower than the one predicted in simple slow roll inflation but could in principle be achieved in alternative scenarios [59].

2. Fine-tuning the initial conditions. If h_f' is very small at early times, the perturbations will not have reached the non-linear value at late times, when the coupling to g becomes important. This requires a fabulous degree of fine-tuning, of one part in 10^{26} at $z = 10^{10}$ (10^{28} when TT modes are considered). For this solution to work beyond the linear approximation, it is necessary that non-linear sources in the equation for h_f are suppressed at early times. Even if stable agains non-linear corrections, fine tuning the initial conditions for h_f seems a highly ad-hoc requirement for the theory in the absence of a mechanism, perhaps generalizing inflation, able to naturally produce such small initial conditions. As mentioned in the previous point, this fine-tuning occurs naturally only in the case of very low-energy inflation.

3. Lowering the non-linear cutoff. IBB becomes safe if h_f stabilizes at a value smaller than unity due to non-linear effects. This requires the non-linear effects to act at most when $h_f \lesssim 10^{-3}$ (and $\lesssim 10^{-4}$ when TT modes are considered).

4. Adding a cosmological constant. In that case the theory still describes a massive graviton, although it would not be responsible for the acceleration of the universe. From a purely cosmological point of view, the model will be very similar to ΛCDM.

5. Modifying the theory. One possibility is to allow for a time dependence to the theory parameters β_1, β_4 in the tensor perturbation equations. This phenomenological parametrization of modified IBB allows to satisfy CMB data if the parameters become non-zero only after $z = 1000$.

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While our assumption stops the growth of perturbations when they become non-linear, a full analysis would require an understanding of the actual non-linear behaviour of perturbations, which could be used to exclude or validate such scenarios in a fully consistent way.

Finding a modification of bigravity that overcomes the difficulties of the growing modes could be possible within the framework of generalized massive gravity [62]. In this class of theories, the interaction terms are given an additional dependence in the Stückelberg fields, which allows the couplings to vary over time without introducing additional degrees of freedom. Another possibility in this direction is allowing a composite coupling that involves both tensor, possibly on an equal footing. Theories with additional degrees of freedom, such as scalar-bitensor or multigravity, might as well prove useful to solve the problem of growing modes. More exotic modifications of the theory remain to be explored.

Constructing a viable theory of massive gravity has proven to be a challenge. Only after eight decades could the linear Fierz-Pauli theory be generalized to a ghost-free non-linear completion, albeit one that forbids any interesting cosmological solution. This difficulty could be overcome by giving a kinetic term to the reference metric, allowing the existence of accelerating cosmologies at the price of two additional degrees of freedom, corresponding to a massless tensor. Of all the five-parameter set of bigravity theories, only the two-parameter IBB family is able to accelerate the universe with neither a cosmological constant nor scalar instabilities. Yet, such a theory is affected by growing modes that generically spoil the predictions of the cosmic microwave background. The results presented here represent a setback for the simple and appealing self-accelerating bigravity paradigm, a paradigm that, unless saved by non-linear effects or a tiny amplitude of the initial conditions, will have to be abandoned.

Acknowledgments

We thank Yashar Akrami, Ruth Durrer, Pedro Ferreira, Daniel G. Figueroa, Fawad Hassan, Macarena Lagos, Gerasimos Rigopoulos, Angnis Schmidt-May, Adam Solomon and Christof Wetterich for useful discussions. This research has been supported by DFG through the grant TRR33 “The Dark Universe”. F.K. acknowledges support from the Graduate College “Astrophysics of fundamental probes of gravity”.

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JCAP05(2015)052

Publication 6

Higuchi ghosts and gradient instabilities in bimetric gravity

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Physical Review D 91, 104019 (2015), arXiv:1503.07436

PHYSICAL REVIEW D **91,** 104019 (2015)

Higuchi ghosts and gradient instabilities in bimetric gravity

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(Received 30 March 2015; published 19 May 2015)

Bimetric gravity theories allow for many different types of cosmological solutions, but not all of them are theoretically allowed. In this work we discuss the conditions to satisfy the Higuchi bound and to avoid gradient instabilities in the scalar sector at the linear level. We find that in expanding universes the ratio of the scale factors of the reference and observable metric has to increase at all times. This automatically implies a ghost-free helicity-2 and helicity-0 sector and enforces a phantom dark energy. Furthermore, the condition for the absence of gradient instabilities in the scalar sector will be analyzed. Finally, we discuss whether cosmological solutions can exist, including exotic evolutions like bouncing cosmologies, in which both the Higuchi ghost and scalar instabilities are absent at all times.

DOI: 10.1103/PhysRevD.91.104019 PACS numbers: 04.50.Kd, 98.80.-k, 95.36.+x, 95.30.Sf

I. INTRODUCTION

The question whether the graviton can have a mass has been asked for a long time, and its answer has always been accompanied by uncertainties. The linear theory of a massive gravity was first analyzed by Fierz and Pauli [1]. Since then the van Dam-Veltman-Zakharov discontinuity [2,3] and the appearance of the Boulware-Deser (BD) ghost [4] has been challenging the theory. Recently, a theory of a massive spin-2 field was presented in which the coupling between an additional fixed tensor field and the metric has a specific structure and is free of the BD ghost [5–12] (see Refs. [13,14] for recent reviews on massive gravity). To promote this theory of a massive gravity to a bimetric theory, Hassan and Rosen considered a dynamical tensor field $f_{\mu\nu}$ where its kinetic term has the same Einstein-Hilbert structure as $g_{\mu\nu}$ and does not introduce the BD ghost [11,15]. This bimetric theory is described by the action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}R(g) - \frac{1}{2}\int d^4x\sqrt{-f}R(f) + \int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(X) + \int d^4x\sqrt{-g}\mathcal{L}_m, \quad (1)$$

where we already set the Planck mass for $f_{\mu\nu}$ to M_g (see Refs. [16,17] for further explanation), absorbed m, the mass scale of the graviton, into β_n and expressed masses in units of M_g^2. The interaction between both tensor fields is determined by the elementary symmetric polynomials e_n of the eigenvalues of the matrices $X^\alpha_\gamma \equiv \sqrt{g^{\alpha\beta}f_{\beta\gamma}}$, multiplied by arbitrary real coupling constants β_n. It is convenient to express these free parameters in units of the present Hubble expansion rate, H_0^2.

A remarkable property of bimetric gravity theories is the possibility of nonstandard, self-accelerating cosmological solutions and the ability of making predictions that are different from ΛCDM. Some of these might be useful for future measurements in order to distinguish standard ΛCDM from bigravity. To benefit from that, one has to pay the price and needs to disentangle all the nonviable models from the viable ones.

Even though this theory has five free parameters, it is not clear whether viable models exist (except for $\beta_1 = \ldots = \beta_4 = 0$ which is simply ΛCDM) and, if they do, what they look like. In Ref. [18] simple criteria of viability were considered and viable background solutions were presented (see also Ref. [19]). One choice of the coupling parameters, in the following simple "*model*," will usually lead to several different cosmological solutions [16,18–22]. In the following, every possible solution will be called a "*branch*." We distinguish between different types of branches, depending on how the ratio of the scale factors r of the metrics $f_{\mu\nu}$ and $g_{\mu\nu}$ evolves. In solutions on "*finite branches*," the ratio evolves from zero towards a finite asymptotic value, whereas on "*infinite branches*" r becomes infinitely large at early times and decreases with time. We call all other branches "*exotic branches*," and these usually describe bouncing cosmologies or a static universe in the asymptotic past or future.

So far, only finite and infinite branches were studied in the literature. While many of these are in good agreement with observational data at the background level [18,20,23], most of them suffer from scalar instabilities [24–27]. It seems that only one specific class of models, the infinite branch bigravity (IBB), is free of scalar instabilities [24].

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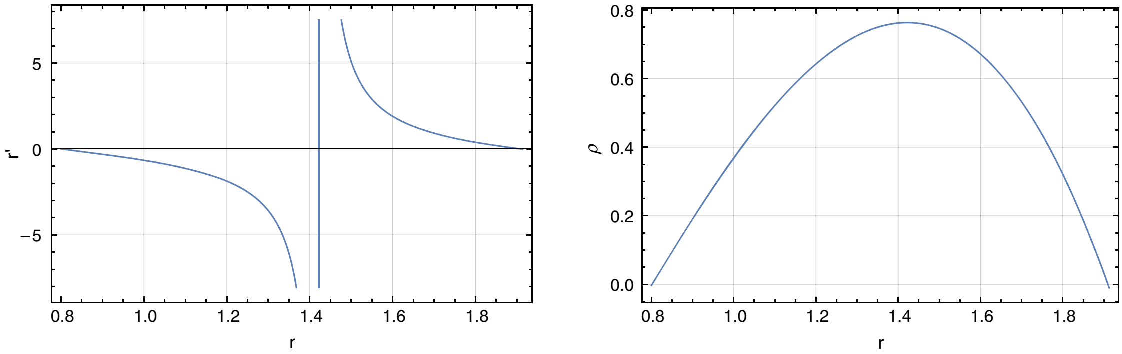

FIG. 1 (color online). Example of a model [$\beta_i = (0, 0.3, -0.8, 1, -1)$] that describes a bouncing universe. Here, the asymptotic past of this universe is described by a root at $r \simeq 0.8$.It then contracts, i.e.dt<0, until r reaches the pole and, finally, expands towards a root at $r \simeq 1.9$, which describes a de Sitter point.

These models are specific infinite branch solutions in which β_2 and β_3 vanish. Moreover, IBB agrees very well with observations at the background and linear level [24,28]. Unfortunately, the authors in Ref. [29] noted that the Higuchi bound is generally violated in the early time limit. This bound, first derived in Ref. [30], ensures a healthy helicity-0 mode of the graviton. A violation leads to the appearance of the Higuchi ghost, named after Higuchi who found that a spin-2 particle with mass m and $0 < m^2 < 2H^2$ in a de Sitter space leads to a negative norm [31,32] (see also Ref. [33] in which the Higuchi bound was derived for arbitrary spatially flat FLRW metrics in massive gravity). Note that even though IBB seems to be well behaved at the linear level, the appearance of the Higuchi ghost may only be visible at higher orders or maybe even only in the full solution [34]. Furthermore, it was found that cosmological solutions on this infinite branch suffer from a ghost in the helicity-2 sector at early times [35].

The analysis of viable backgrounds in Ref. [18], that leaded e.g. to the exclusion of solutions on the exotic branch or a vanishing β_2 and β_3 in the infinite branch, are, however, based on assumptions like the existence of a matter dominated past or the absence of poles in r', where the prime indicates the derivative with respect to the e-folding time t. Even though it would be probably difficult to get exotic solutions in agreement with observations, they are *a priori* not excluded. Moreover, poles in $r' = \frac{d}{dt} r$ could have a very physical meaning: If r' reaches a pole, then dt becomes zero and the Universe undergoes a bounce. Such an example is shown in Fig. 1.[1]

In the following, we will first briefly discuss the background evolution in Sec. II, before we then analyze conditions for the absence of the Higuchi ghost and scalar instabilities (Secs. III–IV) to draw conclusions about the viability of all theoretically possible solutions (Sec. V).

[1]Note that this specific model is not viable due to a negative $\mathcal{H}^2$ and is only shown for motivation purposes.

II. EQUATIONS OF MOTION AT BACKGROUND LEVEL

To find the cosmological background evolution, we vary the action (1) with respect to both metrics and find the equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^{\lambda}_{(n)\nu} \left(\sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) + g_{\nu\lambda} Y^{\lambda}_{(n)\mu} \left(\sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) \right] = T_{\mu\nu}, \tag{2}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} f_{\mu\nu} \bar{R} + \frac{1}{2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} [f_{\mu\lambda} Y^{\lambda}_{(n)\nu} \left(\sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) + f_{\nu\lambda} Y^{\lambda}_{(n)\mu} \left(\sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) \Big] = 0, \tag{3}$$

where the overbar denotes curvature of $f_{\mu\nu}$ and $Y^{\lambda}_{(n)\nu}$ are suitable polynomials (see Ref. [20] for their definitions). At the background level, we will use a Friedmann-Lemaître-Robertson-Walker (FLRW) ansatz for both metrics with two different scale factors, a and b, together with two different time parametrizations t and $\tilde{t} \equiv Xt$. Throughout this work, t represents the e-folding time and a prime denotes the derivative to it. With this ansatz for the metrics,

$$g_{\mu\nu} dx^\mu dx^\nu = a^2 (-\mathcal{H}^{-2} dt^2 + d\vec{x}^2), \tag{4}$$

$$f_{\mu\nu} dx^\mu dx^\nu = b^2 (-X^2 \mathcal{H}^{-2} dt^2 + d\vec{x}^2), \tag{5}$$

where $\mathcal{H}$ is the dimensionless conformal Hubble function, we obtain the g_{00} and f_{00} equations,

$$3\mathcal{H}^2 = a^2 (\rho + \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3), \tag{6}$$

$$3\mathcal{H}^2 = \frac{a^2 r X^2}{(r' + r)^2} (\beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3). \tag{7}$$

Here we introduced the ratio of the scale factors $r \equiv b/a$. As usual, both the Friedmann and acceleration equations for $g_{\mu\nu}$ are degenerated with the conservation of the energy,

$$\rho' = -3\rho(1 + w_{\rm tot}), \tag{8}$$

where $w_{\rm tot}$ denotes the equation of state (EOS) parameter of the total energy density, while there is no extra constraint from the acceleration equation for $f_{\mu\nu}$ due to the missing coupling to the energy-momentum tensor. The combination of this set of equations leads to

$$X = 1 + \frac{r'}{r}. \tag{9}$$

Replacing this constraint in the equations of motion yields

$$3\mathcal{H}^2 = a^2(\rho + \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3), \tag{10}$$

$$3\mathcal{H}^2 = \frac{a^2}{r}(\beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3). \tag{11}$$

The second alternative Friedmann equation is particularly interesting since it directly determines the evolution of the scale factor if the evolution of r is known.

The sign of b is *a priori* unknown and, therefore, r could be negative. However, odd powers of r are always proportional to either β_1 or β_3. All cosmological solutions with negative r due to a negative scale factor for $f_{\mu\nu}$ are therefore equivalent to those with positive r after the redefinition $\beta_{2n+1} \to -\beta_{2n+1}$. From now on, we will assume $r \geq 0$.[2] The combination of both Friedmann equations leads to an equation for the density as a function of r only,

$$\rho = \beta_1 r^{-1} - \beta_0 + 3\beta_2 + 3(\beta_3 - \beta_1) r + (\beta_4 - 3\beta_2) r^2 - \beta_3 r^3. \tag{12}$$

It will be useful to study r', which can be written as

$$r' = \frac{\rho'}{\rho_{,r}} = -3(1 + w_{\rm tot})\frac{\rho}{\rho_{,r}}, \tag{13}$$

where we used Eq. (8) in the last step.

III. HIGUCHI GHOSTS

Bimetric theories are called ghost-free since the specific structure of the potential term in the Lagrangian avoids an additional degree of freedom (d.o.f.), which usually would be the BD ghost. This, however, does not imply that all d.o.f. of the massless and massive graviton are not ghosts.

A. Higuchi bound

Bimetric gravity theories describe a mixture of a massless and massive spin-2 field. The latter carries five dofs, including one helicity-0 mode. In pure massive gravity around a de Sitter spacetime, Higuchi derived a bound for the graviton mass to ensure positive norm states [31,32]. A negative norm would imply a ghost helicity-0 mode and is usually dubbed an Higuchi ghost. The condition for its absence in bimetric gravity theories around a FLRW background was derived in Ref. [30].[3] In our notations, the bound is

$$\frac{3}{2}(\beta_1 + 2\beta_2 r + \beta_3 r^2)(1 + r^2) \geq \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 = 3r\left(\frac{\mathcal{H}}{a}\right)^2, \tag{14}$$

which is equivalent to

$$\beta_1 + 3r^2(\beta_1 - \beta_3) + 2r^3(3\beta_2 - \beta_4) + 3r^4\beta_3 \geq 0. \tag{15}$$

Interestingly, using Eqs. (12)–(13) leads to the simple bound

$$\rho_{,r} \leq 0. \tag{16}$$

This condition for the absence of the Higuchi ghost was already derived in Ref. [36] (see also Ref. [27]). Since

$$\rho_{,r} = -3(1 + w_{\rm tot})\frac{\rho}{r'} \tag{17}$$

and $\rho > 0$ together with $1 + w_{\rm tot} > 0$ (we are usually considering a combination of pressureless and relativistic matter), the bound is equivalent to

$$r' \geq 0. \tag{18}$$

Note that this holds even for negative values of r. Therefore, in an expanding universe the ratio of the scale factors b and a has to increase at all times in order to satisfy the Higuchi bound. Since r' is negative on all infinite branches [18], this directly shows that these branches suffer from the Higuchi ghost at all times and confirms the findings in Ref. [29] that the bound is violated at least at early times on infinite branches, i.e. large r. On the other hand, all finite branches that produce viable backgrounds are free from the Higuchi ghost since viability in these branches enforces $r' \geq 0$ [18]. This especially includes the

[2] This assumption might only be unjustified if both positive and negative values of r are reached at some time. In the later discussion we will find that this requires finite, nonzero values of r' at $r = 0$ in order to produce viable branches. It turns out that these specific models will not be able to produce viable cosmologies.

[3] Note that the authors in Ref. [30] used an overall factor of $\frac{1}{2}$ in front of the potential term in the Lagrangian, which can be compensated for by a redefinition of the β couplings.

finite branch in the β_1 model, i.e. only $\beta_1 \neq 0$, which was already shown to be free of the ghost in Ref. [30].

The rhs of the bound (14) has to be non-negative at all times. Since we already concluded that $r \geq 0$ is a valid assumption without loss of generality, the Higuchi bound enforces

$$B_2 \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2 \geq 0, \tag{19}$$

where B_2 is simply the derivative of ρ_{mg}, the modified part in the Friedmann equation (10), with respect to r. Therefore, the Higuchi bound is related to the change of the amount of dark energy in our Universe with time.

B. Phantom dark energy

It is often useful to study the equation of state parameter (EOS), w_{mg}, i.e. the ratio between the pressure and the density, from contributions of the modification of gravity. If we know how the matter density in our Universe evolves, then the knowledge of w_{mg} enables us to draw conclusions about the acceleration and even the future of our Universe.

In Ref. [18] we showed that Eq. (19) is directly related to the EOS via

$$w_{mg} = -1 - \frac{B_2}{\rho_{mg}} r'. \tag{20}$$

If $\rho_{mg} > 0$ (which, as observations indicate, should hold at least around present time), then the Higuchi bound enforces a phantom dark energy. Every cosmological solution in bimetric gravity should therefore have either a Higuchi ghost or a phantom dark energy.

The property of being a phantom is usually thought to come along with a future instability, the "big rip" [37]. Note, however, that the EOS is highly time dependent and tends to -1 in the asymptotic future if it described by a root in r', e.g. in most of the finite branch models. A sufficiently fast increase of w_{mg} could then avoid this instability and guarantee a better behaved future. A phantom in bimetric gravity is, therefore, not as frightening as in ΛCDM. Thus, a model implying a phantom dark energy should not automatically be related with a problematic future, much less be rejected.

C. Tensor ghosts

Interestingly, the only factor in the lapse of $f_{\mu\nu}$ that is not strictly positive is $r + r'$. Thus, the only way to get a negative lapse is a negative r'. Therefore, fulfilling the Higuchi bound implies a nonvanishing and especially positive lapse at all times.

It was mentioned in Ref. [35] that the relative factor between the kinetic tensor modes for $g_{\mu\nu}$ and $f_{\mu\nu}$ is the lapse function of $f_{\mu\nu}$ and, therefore, a negative lapse is responsible for a ghost in the helicity-2 sector. We conclude that the absence of the Higuchi ghost automatically implies the absence of a ghost in the helicity-2 sector.

As shown in Ref. [38], the lapse of $f_{\mu\nu}$ directly enters in the friction and mass term of the $f_{\mu\nu}$-tensor perturbation equation leading to negative values at early times, which is responsible for a fast grow of the tensor modes [35,38]. This is already a signal of the existence of a ghost. To get such a fast growth in the tensor evolution in accordance with observations is a challenging but not undoable task [38]. The main problem, however, is the existence of the ghost itself.

D. Consequences of the existence of ghosts

A ghost helicity-0 or helicity-2 will have a dramatic impact on the viability of a theory. It will lead to an unbounded Hamiltonian from below and allow the existence of particles with positive and negative energies. As expounded in Ref. [34], the vacuum state will immediately decay into positive and negative energy particles. This behavior is enough to rule out the underlying theory.[4] It is, therefore, not a question of how problematic the evolution is of a field described by the equation of motion. A ghost might influence its evolution in a (more or less) unacceptable way, e.g. through a negative friction. However, it is not the possibly ill-behaved solution of the perturbation equations that renders the theory unphysical, but rather the absence of a stable vacuum state and interactions with negative energy particles. It is even possible that such a system could seem to be completely well behaved at all orders in perturbation theory, but the perturbative solution still not converge to the exact solution. An example where perturbation theory is even able to hide the negative energy solutions, which are present in the full theory, is discussed in Ref. [34].

Since bimetric gravity is only an effective field theory, one might wonder whether a ghost could be harmless in this setup or whether a ghost is necessarily excited. This is, unfortunately, not the case. As explained in Ref. [39], only modes with positive energy are able to decouple, but not a ghost state since there is no positive energy necessary to excite a ghost (see also Ref. [40]). Even in effective field theories (and even if the mass of the ghost lies above the cutoff) one has to avoid ghosts at all costs.

IV. EIGENFREQUENCIES OF SCALAR PERTURBATIONS

After reducing the number of possible cosmological solutions with the demand of the absence of ghosts, we will analyze the behavior of scalar perturbations at the linear level. Even though there are already quite a number of

[4]Note that there are "good ghosts," e.g. the Faddeev-Popov ghost, which are not related to physical degrees of freedom and are, therefore, harmless.

works in which similar properties were studied, all these investigations were based on strong assumptions, mostly a restriction in the parameter space, fixing the EOS of the matter fluid, or focusing on a specific type of branch. In the majority of cases, this is a consequence of the complexity of the perturbation equations. Since the aim of this work is to draw conclusions about the viability of the most general cosmological solutions in bimetric theories with a FLRW background, we will now work out conditions for the absence of gradient instabilities without resigning from generality regarding the parameter space, type of branch, and nature, i.e., EOS, of the fluid.

The set of scalar perturbation equations at the linear level can be reduced to a system of two second-order differential equations for two potentials Ξ_i describing the two propagating scalar degrees of freedom [26] (see also Refs. [16,22,25,29,35,41]),

$$\Xi_i'' + A_{ij}\Xi_j' + B_{ij}\Xi_j = 0, \tag{21}$$

where A_{ij} and B_{ij} are matrices which depend on the background quantities r, $\mathcal{H}$ and the parameters of the models. The complexity of this system depends crucially on the choice of the gauge. A very convenient one was used in Ref. [29], leaned on Ref. [42]. In this work, we take advantage of the relatively simple[5] perturbation equations that the authors found in this gauge (see Ref. [29] for the derivation and printed equations) and analyze them by using the ansatz $\Xi_i \propto e^{\omega t}$. For simplicity, we assume that the eigenfrequencies ω do not depend on time. This is a valid assumption as long as $|\omega'/\omega^2| \ll 1$ holds and was confirmed for all models studied in Ref. [24]. In the subhorizon limit, we obtain a surprisingly simple expression for the eigenfrequencies,

$$\omega^2 = \left(\frac{k}{\mathcal{H}}\right)^2 \left[\frac{r'\left(\frac{(r^2+1)(\beta_1-\beta_3 r^2) r'}{\rho(w+1)} - \frac{r^2(\beta_1+4\beta_2 r+3\beta_3 r^2)}{\beta_1+2\beta_2 r+\beta_3 r^2}\right)}{3r^3} - 1\right] \tag{22}$$

$$= \left(\frac{k}{\mathcal{H}}\right)^2 \times \left[\frac{r'\left(-\rho_{,r}^{-1}(r^2+1)(\beta_1-\beta_3 r^2) - \frac{r^2(\beta_1+4\beta_2 r+3\beta_3 r^2)}{\beta_1+2\beta_2 r+\beta_3 r^2}\right)}{3r^3} - 1\right], \tag{23}$$

which agrees with all previous, but much more complicated, results for one- and two-parameter models that were studied in [24]. As already mentioned in Ref [24], if we assume dark matter only, then for models in which $\beta_2 = \beta_3 = 0$ this reduces to

$$\omega^2_{\beta_0\beta_1\beta_4} = \left(\frac{k}{\mathcal{H}}\right)^2 \frac{r''}{3r'}. \tag{24}$$

In order to discuss stability, we only need to analyze the sign of ω^2: A negative value would imply oscillating and, therefore, stable potentials Ξ_i. If, however, ω^2 is positive, then Ξ_i grows quickly with time and even faster as the scales become smaller. Such an instability is not compatible with the structure in our Universe and needs to be avoided in a viable model.

Let us now introduce $B_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2$ to obtain

$$\omega^2 = \frac{k^2}{3r\rho_{,r}\mathcal{H}^2}\left[r'\left(3(r^2+1)\left(\frac{B_2}{r}\right)_{,r} - \rho_{,r}\left(r\frac{B_{2,r}}{B_2}+1\right)\right) - 3r\rho_{,r}\right] \tag{25}$$

Interestingly, the condition for stability depends on how dark energy (and the density of the cosmic fluid) changes but not explicitly on how large it is. We observed a similar property during the analysis of the Higuchi bound. Note that B_2 is related to the change of the energy density, $\rho_{,r}$, and the Hubble expansion via

$$B_2 = -\frac{r}{1+r^2}\left(\frac{1}{3}r\rho_{,r} - 2\left(\frac{\mathcal{H}}{a}\right)^2\right). \tag{26}$$

Together with

$$\left(\frac{B_2}{r}\right)_{,r} = r^{-2}B_2\left(r\frac{B_{2,r}}{B_2} - 1\right), \tag{27}$$

we finally arrive at

$$\omega^2 = \left(\frac{k}{\mathcal{H}}\right)^2\left(\frac{2r'(r(r^2+1)B_{2,r}\rho_{,r} - (\frac{\mathcal{H}}{a})^2(3(r^2+1)B_{2,r} + r\rho_{,r}) + 6(\frac{\mathcal{H}}{a})^4)}{r^2\rho_{,r}(r\rho_{,r} - 6(\frac{\mathcal{H}}{a})^2)} - 1\right). \tag{28}$$

[5]Where "simple" means that printing these equations would fill only a couple of pages.

As we will see later, this expression for the eigenfrequencies will become very convenient when analyzing the stability around poles in r', which e.g. always appear in exotic branches.

It might be useful to study an expression for ω^2 which does not explicitly depend on the β parameters but on r and its derivatives, like Eq. (24). Finding such an expression is always possible when using a set of five independent equations to eliminate all coupling parameters. One possibility is the set of equations for r', r'', r''', $\mathcal{H}^2$ and ρ (note that the result will not depend on r''') which yields

$$\omega^2 = \left(\frac{k}{\mathcal{H}}\right)^2 \frac{a^2\rho r^2(w+1)[2(w+1)r'' + r'(6w^2 - 2w' + 9w + 3)] - 2\mathcal{H}^2 r'[r'((w+1)(r' - 3rw) + rw') - r(w+1)r'']}{3r(w+1)r'(a^2\rho r(w+1) + 2\mathcal{H}^2 r')} \tag{29}$$

Here, and in all the following equations, we dropped the subscript in w_{tot} for simplicity. If we are interested in analyzing the eigenfrequencies at specific epochs, e.g. radiation-dominated era (RDE) and matter-dominated era, we can assume $w \simeq$ const and obtain

$$\omega^2 = \left(\frac{k}{\mathcal{H}}\right)^2 \frac{a^2\rho r^2(w+1)[2r'' + 3r'(2w+1)] + 2\mathcal{H}^2 r'[r(r'' + 3wr') - r'^2]}{3rr'(a^2\rho r(w+1) + 2\mathcal{H}^2 r')}. \tag{30}$$

This leads to the condition

$$r'[a^2\rho r(w+1) + 2\mathcal{H}^2 r'][a^2\rho r^2(w+1)(2r'' + (6w+3)r') + 2\mathcal{H}^2 r'(r(r'' + 3wr') - r'^2)] < 0 \tag{31}$$

in order to get stable scalar perturbations, i.e. $\omega^2 < 0$. When using the Higuchi bound, $r' > 0$, the first bracket term is always positive and, thus, the second one has to be negative. This is equivalent to

$$r'' < \frac{r'}{2r} \frac{2\mathcal{H}^2 r'(r' - 3rw) - 3a^2\rho r^2(w+1)(2w+1)}{a^2\rho r(w+1) + \mathcal{H}^2 r'}, \tag{32}$$

where we also used $r' > 0$. Note that the denominator is always positive. If the numerator would be negative, then the bound would especially imply $r'' < 0$. However, this is not generally the case and, thus, the condition for stable scalar modes is not automatically equivalent to $r'' < 0$ in contrast to the case for $\beta_0\beta_1\beta_4$ models during matter domination [see Eq. (24)].

A. Radiation-dominated era

Even though we will not aim to exclude models which are theoretically allowed but do very likely not reproduce observational data (an example would be a nearly static universe that did not have a radiation-dominated epoch), it is worthwhile to analyze the conditions when the Universe is filled with either relativistic particles or pressureless matter only.

When radiation dominates, i.e. $w \simeq 1/3$, the eigenfrequencies simplify to

$$\omega^2 = \frac{k^2}{3\mathcal{H}^2 rr'} \frac{\frac{4}{3}a^2\rho r^2(2r'' + 5r') + 2\mathcal{H}^2 r'(rr'' - r'^2 + rr')}{\frac{4}{3}a^2\rho r + 2\mathcal{H}^2 r'}. \tag{33}$$

In the early Universe, the Hubble expansion is usually driven by radiation, i.e. $3\mathcal{H}^2 \simeq a^2\rho$. With this approximation, the condition for stability in the scalar sector becomes

$$r'' > -\frac{r'(r' + 10r)}{r' + 4r}. \tag{34}$$

For large absolute values of r', which is the case e.g. near a pole, we simply obtain $r'' > -r'$.

In a previous work [24], we studied the eigenfrequencies for IBB and confined ourselves to a universe filled with dark matter only. According to Eq. (24), we concluded stable scalar modes because r' increases with time but stays negative until reaching the final de Sitter point. Since r' is always negative in IBB, the condition (34) is not necessarily valid anymore. However, we can still use condition (31). Here, the product of the first two terms is always positive since

$$r'(a^2\rho r(w+1) + 2\mathcal{H}^2 r')|_{\mathrm{IBB}} = 9a^2\beta_1 r(r^2+1)\left(\frac{(w+1)(\beta_1 + \beta_4 r^3 - 3\beta_1 r^2)}{\beta_1 - 2\beta_4 r^3 + 3\beta_1 r^2}\right)^2 > 0. \tag{35}$$

Therefore, we can analyze the third factor and, assuming $w \in (-1, 1)$ for simplicity, find that stable modes are guaranteed if

$$3\beta_1 r^2 < \beta_1 + \beta_4 r^3, \tag{36}$$

which is not only satisfied in the RDE, i.e. large r (note that both β_1 and β_4 have to be positive in order to get a viable cosmological background), but, in fact, is equivalent to the condition $\rho > 0$ on that branch and, therefore, trivially satisfied at all times.

B. Matter-dominated era

Let us study the regime when matter dominates the Universe. Now the EOS vanishes and the scalar modes are described through

$$\omega^2 = \frac{k^2}{3\mathcal{H}^2 r r'} \frac{a^2\rho r^2(2r'' + 3r') + 2\mathcal{H}^2 r'(rr'' - r'^2)}{a^2\rho r + 2\mathcal{H}^2 r'}. \tag{37}$$

For stability, we need to satisfy the condition

$$r'' < \frac{2\mathcal{H}^2 r'^3 - 3a^2\rho r^2 r'}{2a^2\rho r^2 + 2r\mathcal{H}^2 r'}. \tag{38}$$

If we assume that $\frac{a^2\rho}{\mathcal{H}^2} \to 0$ for late times, which should be true when dark energy starts to dominate, then the condition of stability reduces to

$$r'' \lesssim \frac{r'^2}{r}. \tag{39}$$

V. FINDING VIABLE BRANCHES

We will now raise the question whether branches exist that satisfy both the Higuchi bound and the condition for scalar stability. Here we will only focus on cosmological solutions that are not equivalent to ΛCDM, which of course satisfy both conditions. We therefore assume that at least one of the couplings $\beta_1, \ldots, \beta_4$ is nonzero. Together with conditions of physicality, $a, \rho, \mathcal{H}^2 > 0$, we define these as criteria of viability. Note that we allow for solutions that have a very nonstandard past, e.g. no matter- or radiation-dominated epoch, or even contracting backgrounds, even though these might be hard to compare with observational data. This extends the more restrictive background analysis of [18]. Therefore, not only the finite branch with small r or the infinite one could be viable but also many solutions on exotic branches. Many different types of branches exist: some of them start from a root $r' = 0$, while others may evolve from a pole or even pass a pole at some finite time. In many cases, it is not directly clear whether such branches solve the equations of motion. In particular, every branch always needs to contain a solution of Eq. (11) at present time, i.e. when $\mathcal{H} = a = 1$.

We start with focusing on finite branches with a root at $r = 0$. Let us first concentrate on models with $\beta_1 \neq 0$, which always have a root at $r = 0$ [see Eq. (13)]. In Ref. [24] we generally found scalar instabilities in these type of branches. Even though this is based on the assumption of a universe filled with dark matter only, this conclusion does not change when considering arbitrary but reasonable EOS parameters. We take the same line of argument and study the simple β_1-models, i.e. models with only nonvanishing β_1, since all other models will reduce to these in the limit when r gets close to $r = 0$. The eigenfrequencies in β_1 models are given by

$$\omega^2_{\beta_1} = \frac{1 + 2w - 6r^2(w+2) - 9r^4}{(3r^2+1)^2}\left(\frac{k}{\mathcal{H}}\right)^2$$
$$\simeq \frac{1+2w}{(3r^2+1)^2}\left(\frac{k}{\mathcal{H}}\right)^2 \tag{40}$$

and, therefore, indicate unstable modes for small values of r as long $w > -1/2$. Let us consider the previously excluded models with $\beta_1 = 0$ and find

$$r'|_{r=0} = \frac{\beta_0 - 3\beta_2}{\beta_3}(w+1). \tag{41}$$

Even though the combination $\beta_0 = 3\beta_2$ is able to produce a root at $r = 0$, it will not lead to viable solutions since in this case $r' = -3(1+w) + \mathcal{O}(r^2)$ indicates a violation of the Higuchi bound. From this we conclude that

(1) Finite branches with a root at $r = 0$ always lead to either unstable modes (if $\beta_1 \neq 0$) or violate the Higuchi bound (if $\beta_1 = 0$) for small r.

On the other hand, r' could be nonzero but still finite at $r = 0$. In this case, one of the asymptotic points is either a pole or the whole branch evolves between two roots at negative and positive r. In the first case, we can assume that at least one of the poles is reached at $r > 0$, otherwise we are able to analyze viability in the "mirrored" model corresponding to $\beta_{2n+1} \to -\beta_{2n+1}$. If the branch does not contain any pole, then $\rho_{,r}$ has to vanish at $r = 0$ (roots at $r \neq 0$ always indicate a vanishing density whereas a maximum of the density at $r \neq 0$ leads to a pole). The position of the maximum of ρ at $r = 0$ requires $\beta_3 = 0$ and leads to $\rho_{,r} \propto r$ which cannot be negative for both regions, $r > 0$ and $r < 0$. We can summarize that

(2) All finite branches with a nonzero and finite r' at $r = 0$ have to have a pole either in the asymptotic past or future.

Roots, except for those at $r = 0$, always correspond to a vanishing density. Due to Eq. (13), we will always find a pole between two roots r_1 and r_2, if both r_1 and r_2 are nonzero. Therefore, poles could be interesting starting or final points of a branch. Whenever such a pole describes the asymptotic future, then $\dot{r}$ has to go to zero, otherwise the pole would not be a stable asymptotic point. Since $r' = \mathcal{H}^{-1}\frac{d}{d\tau}r = \mathcal{H}^{-1}a\dot{r}$ diverges, we find that $\mathcal{H}$ needs to vanish at this point. On the other hand, if a pole describes the asymptotic past, then we can use the fact that the density starts from a finite value. For nonzero values of r, this is clear from Eq. (12). It also holds if $r = 0$ is a pole, since this would require $\beta_1 = \beta_3 = 0$ due to Eq. (13) [18] and, therefore, implies $\rho|_{r=0} = 3\beta_2 - \beta_0$. If the density is finite at early times, the scale factor a has to have a finite but nonzero value. In this case, $\mathcal{H}$ needs to be zero at early times, too, otherwise one could go backwards in time and we would not have an asymptotic past. Thus, we conclude

(3) $\mathcal{H}$ has to become zero on a pole, if it describes an asymptotic point.

Let us assume a pole at $r = 0$, which, as we already noted, requires $\beta_1 = \beta_3 = 0$ and leads to $\mathcal{H}^2|_{r=0} = \beta_2 a^2$. From the previous conclusion, we need a vanishing $\mathcal{H}^2$ at $r = 0$. Note that $a > 0$, otherwise this would contradict a finite density. Therefore, we need $\beta_2 = 0$ and, thus, obtain $B_2 = 0$ for all r, which means that

(4) A pole at $r = 0$ violates the Higuchi bound.

For simplicity, we will from now on assume that if there is a pole at r_p, then $r_p > 0$. Additionally, we can exclude $r = 0$ from being an asymptotic point due to the previous conclusions.Furthermore, Eqs. (13) and (12) provide the limit $r' \propto -r$ when taking $r \to \infty$ as long as the density does not vanish (see Ref [18] more detailed explanations). This excludes infinite branches, i.e. branches in which r evolves from or to $r \to \infty$, from being viable due to the violation of the Higuchi bound and we find that

(5) The limits $r \to 0$ and $r \to \infty$ are no viable asymptotic points.

We will now consider a root at $r \neq 0$ as the asymptotic past. Due to Eq. (13), the density vanishes on a root. To fulfill the conservation of energy, those models require a contracting universe at early times. If this universe evolves to another root (on which again $\rho = 0$), then it has to undergo a bounce at $\rho_{,r} = 0$ leading to a pole at which $\mathcal{H} = 0$. Employing the previous conclusions, we find the general statement

(6) Every viable branch needs to contain at least one pole on which $\mathcal{H}$ vanishes.

This result is particularly interesting as it will allow us to draw conclusions when connecting this with the requirement of stable scalar perturbations and the absence of the Higuchi ghost. The necessary condition for a pole is $\rho_{,r} \to 0$. Then, the eigenfrequencies of scalar perturbations around the pole (28) reduce to

$$\omega^2 \to \left(\frac{k}{\mathcal{H}}\right)^2 \left(2B_{2,r}\frac{r'(1+r^2)}{r^2\rho_{,r}}\frac{r\rho_{,r} - 3(\frac{\mathcal{H}}{a})^2}{r\rho_{,r} - 6(\frac{\mathcal{H}}{a})^2} - 1\right) \quad (42)$$

$$\simeq \left(\frac{k}{\mathcal{H}}\right)^2 \left(2B_{2,r}\frac{r'}{\rho_{,r}}\frac{(1+r^2)}{r^2} - 1\right), \quad (43)$$

where we used $(\frac{\mathcal{H}}{a})^4 \ll (\frac{\mathcal{H}}{a})^2$, $(\frac{\mathcal{H}}{a})^2\rho_{,r} \ll (\frac{\mathcal{H}}{a})^2$ (and, additionally, $B_{2,r} \neq 0$ which, as we will see later, is justified), together with

$$\frac{r\rho_{,r} - 3(\frac{\mathcal{H}}{a})^2}{r\rho_{,r} - 6(\frac{\mathcal{H}}{a})^2} = \frac{B_2(1+r^2) - (\frac{\mathcal{H}}{a})^2 r}{B_2(1+r^2)} \simeq 1, \quad (44)$$

which follows from Eq. (26) and $B_2(1+r^2) - (\frac{\mathcal{H}}{a})^2 r \simeq B_2(1+r^2)$ (note that the Higuchi bound (14) implies $B_2(1+r^2) > 2r(\frac{\mathcal{H}}{a})^2 > 0$). Since $r' \to \infty$ and $\rho_{,r} \to 0$ (but still $\rho_{,r} < 0$ and $r' > 0$), the first term in the bracket of Eq. (43) dominates unless $B_{2,r} = 0$.

Let us first assume that $B_{2,r} = 0$ at the pole r_p. We then find

$$B_{2,r}|_{r=r_p} = 0 \Rightarrow \beta_2 = -\beta_3 r_p, \quad (45)$$

$$\left(\frac{\mathcal{H}}{a}\right)^2\Bigg|_{r=r_p} = 0 \Rightarrow \beta_1 = -\beta_4 r_p^3, \quad (46)$$

$$\rho_{,r}|_{r=r_p} = 0 \Rightarrow \beta_3 = -\beta_4 r_p, \quad (47)$$

which leads to

$$3\mathcal{H}^2 = \frac{a^2}{r}\beta_4(r - r_p)^3, \quad (48)$$

as well as

$$B_2 = -\beta_4 r_p (r_p - r)^2. \quad (49)$$

If r increases with time (which implies that $dt > 0$ since $r' > 0$), then β_4 has to be positive in order to get a positive $\mathcal{H}^2$. On the other hand, this would imply a negative B_2 and, thus, would violate the Higuchi bound. We therefore have to have a decrease of r with time (which implies contraction, $dt < 0$). Now, this is only compatible with negative values for β_4. Of course, such a model would be hard to believe in since it would contract at all times. But there is a more solid argument for ruling out these models: The contraction would lead to an increasing density. Since a root corresponds to a vanishing density, there must be a point of maximum density which always indicates a pole [see Eq. (13)]. Note, that we already excluded both $r = 0$ and $r \to \infty$ as asymptotic states, which are the only ones that would be able to describe an infinitely large density. However, on this one, $\mathcal{H}$ cannot vanish. Even though this second pole does not necessarily need to be an asymptotic point, $\mathcal{H} = 0$ is required due to the bounce. Neither the positive nor the negative values for β_4 lead to viable solutions, and we conclude that

(7) Viability enforces a nonzero value for $B_{2,r}$ around a pole.

We are now allowed to assume $B_{2,r} \neq 0$. Then the term -1 in Eq. (43) is negligible and, thus, $B_{2,r}$ has to be positive in order to get stability, i.e. $\omega^2 < 0$.

We will now study the expansion rate around the pole r_p and check whether $\mathcal{H}^2$ is positive. Note that $(\mathcal{H}^2)_{,r}|_{r_p}$ does not automatically vanish since $\mathcal{H}^2$ could become negative, too (which, however, would not correspond to physical solutions). However, the conditions for scalar stability ($B_{2,r}|_{r_p} > 0$), the existence of a pole ($\rho_{,r}|_{r_p} = 0$ with $\mathcal{H}^2|_{r_p} = 0$) and physicality ($\rho|_{r_p} \geq 0$) together with the assumption that $(\mathcal{H}^2)_{,r}|_{r_p} \neq 0$ lead to a contradiction. Therefore, let us assume $(\mathcal{H}^2)_{,r}|_{r_p} = 0$ which, together with $\mathcal{H}^2|_{r_p} = 0$, implies

$$\beta_2 = -\frac{1}{3} r_p^{-1}(2\beta_1 - \beta_4 r_p^3), \tag{50}$$

$$\beta_3 = \frac{1}{3} r_p^{-2}(\beta_1 - 2\beta_4 r_p^3). \tag{51}$$

If we now assume that $\mathcal{H}^2$ is positive and nonzero at second order, then we need to have

$$3\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{(r_p - r)^2}{r r_p^2}(\beta_1 + \beta_4 r r_p^2) = r_p^{-3}(\beta_1 + \beta_4 r_p^3)(r - r_p)^2 + \mathcal{O}((r - r_p)^3) > 0. \tag{52}$$

However, this would imply that

$$\rho_{,r} = 2\frac{1 + r_p^2}{r_p^3}(\beta_1 + \beta_4 r_p^3)(r - r_p) + \mathcal{O}((r - r_p)^2) \tag{53}$$

only becomes negative when leaving the pole, if r decreases with time, i.e. in a contracting universe. We can now use the same argument that we used before and conclude that we need to reach a second pole which will either describe an asymptotic point or a bounce. Equation (52) provides the possibility of another point $r_{p_2} = -\beta_1/(\beta_4 r_p^2)$ at which the expansion stops, but this cannot be a pole since then we would find

$$\rho_{,r}|_{r_{p_2}} = -\frac{(\beta_1 + \beta_4 r_p^3)^2(\beta_1^2 + \beta_4^2 r_p^4)}{\beta_1 \beta_4^2 r_p^6} \neq 0. \tag{54}$$

Our last chance are models in which $\mathcal{H}^2$ vanishes up to second order, implying that

$$\beta_1 = -\beta_4 r_p^3, \tag{55}$$

$$\beta_2 = \beta_4 r_p^2, \tag{56}$$

$$\beta_3 = -\beta_4 r_p. \tag{57}$$

These solutions lead to $B_{2,r} = 0$, which we already excluded earlier. Therefore,

(8) A negative $B_{2,r}$ around a pole leads to gradient instabilities whereas a positive value violates either the Higuchi bound or leads to unstable scalar perturbations.

In combination with the requirement of a positive $B_{2,r}$, in order to get stable scalar perturbations this shows that every branch is plagued by either the Higuchi ghost or scalar gradient instabilities.

VI. CONCLUSIONS AND OUTLOOK

We analyzed general models in singly coupled bimetric gravity around a FLRW background and found that all physical cosmological solutions that are not equivalent to ΛCDM have a period in time in which either linear scalar perturbations undergo a gradient instability or the Higuchi ghost appears. The condition for the absence of ghosts is surprisingly equivalent to $r' > 0$, which means that the ratio of the scale factors b and a has to increase as long as the Universe expands. Moreover, satisfying this bound ensures a positive lapse of $f_{\mu\nu}$ which is related to the absence of a helicity-2 ghost.

In fact, all infinite branches suffer from the Higuchi ghost at all times and a ghost in the helicity-2 sector at early times, whereas in all finite branches, and even exotic branches that do not contain the limit $r \to 0$, there exists at least one epoch in which there is either a gradient instability in the scalar sector or a ghost appears. A schematic illustration of a typical phase space diagram with the forbidden regions is presented in Fig. 2.

While the existence of a ghost already renders the model unphysical and forces us to discard this type of model, unstable scalar modes will not necessarily rule out the theory. A Vainshtein screening may be able to prevent the scalar sector from getting unstable. Furthermore, this gradient instability is not present at all times. Every finite branch has a point in time at which the instability stops and the scalar perturbations begin to oscillate. As shown in [43], a small, but natural, Planck mass for $f_{\mu\nu}$[6] would shift this gradient instability to very early times or even to energy scales above the cutoff of the effective field theory. In the latter case, the cosmological evolution would be very close to ΛCDM. On the other hand, if the instability ended between inflation and big bang nucleosynthesis, only very small scales would be affected [43]. These could, in principle, lead to a creation of many seeds for black holes.[7]

All models which we do not have to exclude due to the presence of a ghost will describe a phantom dark energy. Such a property would cause an anxious future in a ΛCDM model but not necessarily in bimetric theories due the time dependence of EOS corresponding to dark energy. In fact, it could cause welcome signatures that might allow observations to distinguish bimetric gravity from general relativity.

[6] Note that in this and many previous works, the Planck mass M_f was set to M_g, which is allowed due to a redundancy in the parameters but is, however, not the most natural choice.

[7] Since the cosmological evolution at the time where the instability would end is not yet close to ΛCDM, the fast change in ρ_{mg} might have an even stronger influence on the evolution of primordial black holes compared to standard ΛCDM [44].

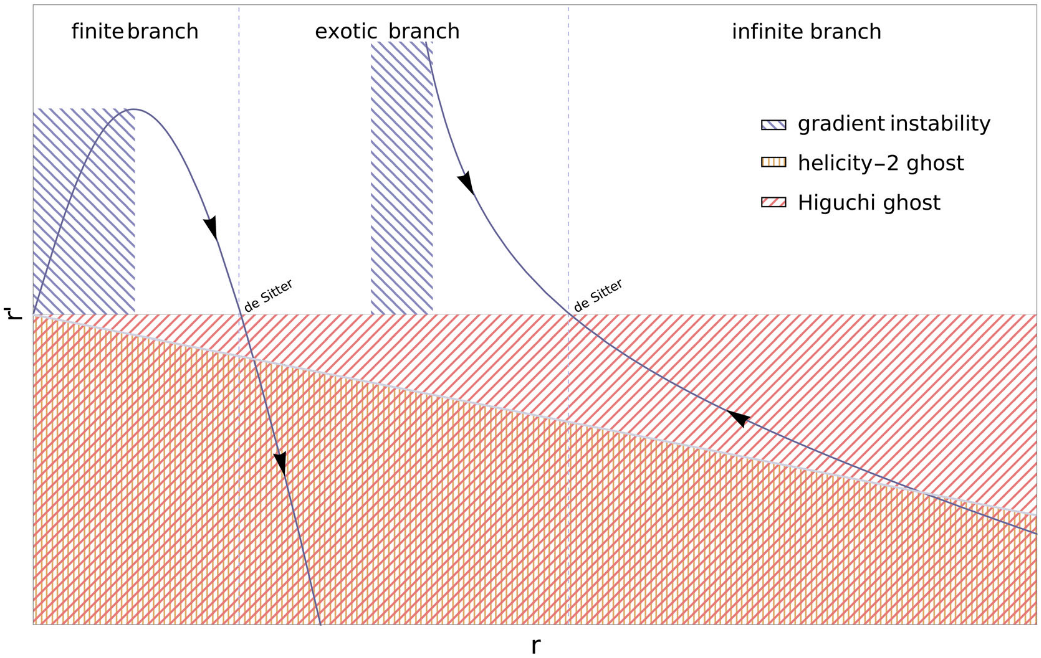

FIG. 2 (color online). Illustration of a phase space diagram of a typical model together with colored regions corresponding to different types of instabilities. While finite branches are plagued from gradient instabilities in the scalar sector (diagonal blue stripes from top-left to bottom-right) at early times, the infinite branches suffer from the Higuchi ghost (diagonal red stripes from bottom-left to top-right) at all times and a ghost in the helicity-2 sector (vertical orange stripes) at early times. Finally, all exotic branches, including e.g. bouncings, have always a pathological behavior at least around the pole in r'.

Throughout this work we assumed a very simple, but well-motivated, type of bigravity. We considered a fluid that is only singly coupled to an observable metric and where both metrics are of FLRW type. Several extensions exist in the literature. One example would be the coupling of matter to both metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ simultaneously [27,45–49], which, however, would introduce the BD ghost if the same matter sector is coupled to both metrics [45,50,51]. Ghost-free (but not always with well-behaved cosmological solutions) scenarios exist if one assumes a coupling through a composite metric [40,50,52–59]. But even the bimetric gravity with a standard matter coupling could allow for cosmological solutions without any gradient or ghost instabilities at the cost of giving up a FLRW background [60,61].

ACKNOWLEDGMENTS

I am grateful to Yashar Akrami, Luca Amendola, Jonas Enander, Matteo Fasiello, Fawad Hassan, Macarena Lagos, Edvard Mörtsell, Angnis Schmidt-May, and Adam Solomon for fruitful discussions and suggestions. Additionally, I am thankful to NORDITA (Stockholm) for organizing the "Extended Theories of Gravity" workshop that inspired discussions during the completion of this work and to both NORDITA and the Oskar Klein Centre for the warm hospitality during my subsequent visit. I acknowledge support from the Graduate College "Astrophysics of Fundamental Probes of Gravity" and from the DFG through the Grant No. TRR33, "The Dark Universe".

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Publication 7

Bimetric gravity is cosmologically viable

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Phys. Letters B B748 (2015), arXiv:1503.07521

Physics Letters B 748 (2015) 37–44

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Bimetric gravity is cosmologically viable

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ARTICLE INFO

Article history:
Received 4 May 2015
Received in revised form 25 June 2015
Accepted 25 June 2015
Available online 29 June 2015
Editor: M. Trodden

Keywords:
Modified gravity
Massive gravity
Background cosmology
Cosmic acceleration
Dark energy
Bimetric gravity
Bigravity

ABSTRACT

Bimetric theory describes gravitational interactions in the presence of an extra spin-2 field. Previous work has suggested that its cosmological solutions are generically plagued by instabilities. We show that by taking the Planck mass for the second metric, M_f, to be small, these instabilities can be pushed back to unobservably early times. In this limit, the theory approaches general relativity with an effective cosmological constant which is, remarkably, determined by the spin-2 interaction scale. This provides a late-time expansion history which is extremely close to ΛCDM, but with a technically-natural value for the cosmological constant. We find M_f should be no larger than the electroweak scale in order for cosmological perturbations to be stable by big-bang nucleosynthesis. We further show that in this limit the helicity-0 mode is no longer strongly-coupled at low energy scales.

"The reports of my death have been greatly exaggerated."
—*Metrics Twain*

1. Introduction

The Standard Model of particle physics contains fields with spins 0, 1/2, and 1, describing matter as well as the strong and electroweak forces. General relativity (GR) extends this to the gravitational interactions by introducing a massless spin-2 field. There is theoretical and observational motivation to seek physics beyond the Standard Model and GR. In particular, GR is nonrenormalizable and is associated with the cosmological constant, dark energy, and dark matter problems. To compound the puzzle, the GR-based Λ-cold dark matter (ΛCDM) model provides a very good fit to observational data, despite its theoretical problems. In order to be observationally viable, any modified theory of gravity must be able to mimic GR over a wide range of distances.

A natural possibility for extending the set of known classical field theories is to include additional spin-2 fields and interactions. While "massive" and "bimetric" theories of gravity have a long history [1,2], nonlinear theories of interacting spin-2 fields were found, in general, to suffer from the Boulware–Deser (BD) ghost instability [3]. Recently a particular bimetric theory (or bigravity) has been shown to avoid this ghost instability [4,5]. This theory describes nonlinear interactions of the gravitational metric with an additional spin-2 field. It is an extension of an earlier ghost-free theory of massive gravity (a massive spin-2 field on a nondynamical flat background) [6–8] for which the absence of the BD ghost at the nonlinear level was established in Refs. [5,9–11].

Including spin-2 interactions modifies GR, *inter alia*, at large distances. Bimetric theory is therefore a candidate to explain the accelerated expansion of the Universe [12,13]. Indeed, bigravity has been shown to possess Friedmann–Lemaître–Robertson–Walker (FLRW) solutions which can match observations of the cosmic

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http://dx.doi.org/10.1016/j.physletb.2015.06.062

expansion history, even in the absence of vacuum energy [14–20].[1] Linear perturbations around these cosmological backgrounds have also been studied extensively [28–41]. The epoch of acceleration is set by the mass scale m of the spin-2 interactions. Unlike a small vacuum energy, m is protected from large quantum corrections due to an extra diffeomorphism symmetry that is recovered in the limit $m \to 0$, just as fermion masses are protected by chiral symmetry in the Standard Model (see Ref. [42] for an explicit analysis in the massive gravity setup). This makes interacting spin-2 fields especially attractive from a theoretical point of view.

Cosmological solutions lie on one of two branches, called the finite and infinite branches.[2] The infinite-branch models can have sensible backgrounds [19,32], but the perturbations have been found to contain ghosts in both the scalar and tensor sectors [33,34,41]. Most viable background solutions lie on the finite branch [16–19]. While these avoid the aforementioned ghosts, they contain a scalar instability at early times [29,32,33] that invalidates the use of linear perturbation theory and could potentially rule these models out. For parameter values thought to be favored by data, this instability was found to be present until recent times (i.e., a similar time to the onset of cosmic acceleration) and thus seemed to spoil the predictivity of bimetric cosmology.

In this Letter we study a physically well-motivated region in the parameter space of bimetric theory that has been missed in earlier work due to a ubiquitous choice of parameter rescaling. We demonstrate how in this region the instability problem in the finite branch can be resolved while the model still provides late-time acceleration in agreement with observations.

Our search for viable bimetric cosmologies will be guided by the precise agreement of GR with data on all scales, which motivates us to study models of modified gravity which are close to their GR limit. Often this limit is dismayingly trivial; if a theory of modified gravity is meant to produce late-time self-acceleration in the absence of a cosmological constant degenerate with vacuum energy, then we would expect that self-acceleration to disappear as the theory approaches GR. We will see, however, that there exists a GR limit of bigravity which retains its self-acceleration, leading to a GR-like universe with an effective cosmological constant produced purely by the spin-2 interactions.

2. Bimetric gravity

The ghost-free action for bigravity containing metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ is given by [4,43]

$$S = \int d^4x \Big[-\frac{M_{\rm Pl}^2}{2}\sqrt{g}R(g) - \frac{M_f^2}{2}\sqrt{f}R(f) + m^2 M_{\rm Pl}^2 \sqrt{g}\, V(\mathbb{X}) + \sqrt{g}\mathcal{L}_m(g,\Phi_i)\Big]. \tag{1}$$

Here $M_{\rm Pl}$ and M_f are the Planck masses for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, and we will frequently refer to their ratio,

$$\alpha \equiv \frac{M_f}{M_{\rm Pl}}. \tag{2}$$

The potential $V(\mathbb{X})$ is constructed from the elementary symmetric polynomials $e_n(\mathbb{X})$ of the eigenvalues of the matrix $\mathbb{X} \equiv \sqrt{g^{-1}f}$, defined by

$$\mathbb{X}^{\mu}{}_{\alpha}\mathbb{X}^{\alpha}{}_{\nu} \equiv g^{\mu\alpha} f_{\alpha\nu}, \tag{3}$$

and has the form [8,43],[3]

$$\sqrt{g}\, V(\mathbb{X}) = \sqrt{g}\beta_0 + \sqrt{g}\sum_{n=1}^{3}\beta_n e_n(\mathbb{X}) + \sqrt{f}\beta_4. \tag{4}$$

In the above, m is a mass scale and β_n are dimensionless interaction parameters. β_0 and β_4 parameterize the vacuum energies in the two sectors. Guided by the absence of ghosts and the weak equivalence principle, we take the matter sector to be coupled to $g_{\mu\nu}$.[4] Then the vacuum-energy contributions from the matter sector $\mathcal{L}_{\rm m}$ are captured in β_0. We can interpret $g_{\mu\nu}$ as the spacetime metric used for measuring distance and time, while $f_{\mu\nu}$ is an additional symmetric tensor that mixes nontrivially with gravity. As we discuss further below, the two metrics do not correspond to the spin-2 mass eigenstates but each contain both massive and massless components. Even before fitting to observational data, the parameters in the bimetric action are subject to several theoretical constraints. For instance, the squared mass of the massive spin-2 field needs to be positive, it must not violate the Higuchi bound [59,60], and ghost modes should be absent.

In terms of the Einstein tensor, $G_{\mu\nu}$, the equations of motion for the two metrics take the form

$$G_{\mu\nu}(g) + m^2 V^g_{\mu\nu} = \frac{1}{M_{\rm Pl}^2}T_{\mu\nu}, \tag{5}$$

$$\alpha^2 G_{\mu\nu}(f) + m^2 V^f_{\mu\nu} = 0, \tag{6}$$

where $V^{(g,f)}_{\mu\nu}$ are determined by varying the interaction potential, V. Taking the divergence of eq. (5) and using the Bianchi identity leads to the *Bianchi constraint*,

$$\nabla^{\mu}_{(g)} V^g_{\mu\nu} = 0. \tag{7}$$

The analogous equation for $f_{\mu\nu}$ carries no additional information due to the general covariance of the action.

Finally, note that the action (1) has a status similar to Proca theory on curved backgrounds. It is therefore expected to require an analogue of the Higgs mechanism, with new degrees of freedom, in order to have improved quantum behavior. The search for a ghost-free Higgs mechanism for gravity is still in progress [61].

3. The GR limit

When bigravity is linearized around proportional backgrounds $\bar{f}_{\mu\nu} = c^2\bar{g}_{\mu\nu}$ with constant c,[5]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_{\rm Pl}}\delta g_{\mu\nu}, \tag{8}$$

$$f_{\mu\nu} = c^2\bar{g}_{\mu\nu} + \frac{c}{M_f}\delta f_{\mu\nu}, \tag{9}$$

the canonically-normalized perturbations can be diagonalized into massless modes $\delta G_{\mu\nu}$ and massive modes $\delta M_{\mu\nu}$ as [4,62]

$$\delta G_{\mu\nu} \propto \left(\delta g_{\mu\nu} + c\alpha\, \delta f_{\mu\nu}\right), \tag{10}$$

$$\delta M_{\mu\nu} \propto \left(\delta f_{\mu\nu} - c\alpha\, \delta g_{\mu\nu}\right). \tag{11}$$

[1] Stable FLRW solutions do not exist in massive gravity [21–27].

[2] There is a third branch containing bouncing solutions, but these tend to have pathologies [41].

[3] This is a generalization of the massive-gravity potential [8] (to which it reduces for $f_{\mu\nu} = \eta_{\mu\nu}$ and a restricted set of β_n) given in Ref. [43].

[4] More general matter couplings not constrained by these requirements have been studied in Refs. [20,44–58].

[5] These correspond to Einstein spaces and, for nonvanishing α, solve the field equations only in vacuum. A quartic equation determines $c = c(\beta_n, \alpha)$.

Notice that when $\alpha \to 0$ (or $M_{\rm Pl} \gg M_f$), the massless state aligns with $\delta g_{\mu\nu}$, i.e., up to normalization,

$$\delta G_{\mu\nu} \to \delta g_{\mu\nu} + \mathcal{O}(\alpha^2). \tag{12}$$

Because $g_{\mu\nu}$ is the physical metric, this suggests that $\alpha \to 0$ is the general-relativity limit of bigravity.[6] We will see below that the nonlinear field equations indeed reduce to Einstein's equations for $\alpha = 0$ and that the limit is continuous. Thus $g_{\mu\nu}$ is close to a GR solution for sufficiently small values of α. We therefore identify $M_{\rm Pl}$ with the measured physical Planck mass whenever $\alpha \ll 1$, holding it fixed while making M_f smaller. Interestingly, in the bimetric setup a large physical Planck mass is correlated with the fact that gravity is approximated well by a massless field. In other words, when bimetric theory is close to GR, the gravitational force is naturally weak.

The GR limit can be directly realized at the nonlinear level [64,65]. The metric potentials satisfy the identity

$$\sqrt{g} g^{\mu\alpha} V^g_{\alpha\nu} + \sqrt{f} f^{\mu\alpha} V^f_{\alpha\nu} = \sqrt{g} V \delta^\mu{}_\nu, \tag{13}$$

where V is the potential in the action (1). For $M_f = 0$, the $f_{\mu\nu}$ equation (6) gives $V^f_{\mu\nu} = 0$, an algebraic constraint on $f_{\mu\nu}$. Then, using the above identity, the $g_{\mu\nu}$ equation (5) becomes

$$G_{\mu\nu}(g) + m^2 V g_{\mu\nu} = \frac{1}{M^2_{\rm Pl}} T_{\mu\nu}. \tag{14}$$

Since $T_{\mu\nu}$ is conserved, taking the divergence gives

$$\partial_\mu V = 0. \tag{15}$$

We see that eq. (14) is the Einstein equation for $g_{\mu\nu}$ with cosmological constant $m^2 V$. Remarkably, because V depends on $f_{\mu\nu}$ and all the β_n, this effective cosmological constant is generically *not* simply the vacuum energy from matter loops (which is parameterized by β_0). Even in the GR limit, the impact of the spin-2 interactions remains and bigravity's self-acceleration survives.

It is straightforward to see that, unlike the $m \to 0$ limit, the $\alpha \to 0$ limit is not affected by the van Dam–Veltman–Zakharov (vDVZ) discontinuity [66,67]. The cause of this discontinuity is the Bianchi constraint (7) which constrains the solutions even when $m = 0$. On the contrary, when $\alpha \to 0$, the Bianchi constraint simply reduces to eq. (15) and is automatically satisfied.

The conditions $V^f_{\mu\nu} = 0$ and $\partial_\mu V = 0$ determine $f_{\mu\nu}$ algebraically in terms of $g_{\mu\nu}$, generically as $f_{\mu\nu} = c^2 g_{\mu\nu}$. In the limit $M_f = 0$, the f sector is infinitely strongly coupled.[7] Due to the nontrivial potential, this causes the f metric to exactly follow the g metric (both at the background and perturbative levels), while the g sector remains weakly coupled.

4. Strong-coupling scales

We now argue that at energy scales relevant to cosmology, this model avoids known strong-coupling issues, sometimes contrary to intuition gained from massive gravity.

There are several strong-coupling scales one might expect to arise. At an energy scale k, the f sector has an effective coupling k/M_f, as can be seen from expanding the Einstein–Hilbert action in $\delta f_{\mu\nu}/M_f$, just as in GR. Then, for small but nonzero α, which is the case of interest here, one might worry that perturbations of $f_{\mu\nu}$ with momentum k become strongly coupled at low scales $k \sim M_f$. However, we have seen that in the limit of infinite strong coupling, $M_f = 0$, $f_{\mu\nu}$ becomes nondynamical and is entirely determined in terms of $g_{\mu\nu}$, while the $g_{\mu\nu}$ equation is degenerate with GR and its perturbations remain weakly coupled. Due to the continuity of the limit, we expect that, for small enough α, strong-coupling effects will continue to not affect the g sector, even when perturbations of $f_{\mu\nu}$ are strongly coupled at relatively small energy scales. In practice, however, since the measured value of $M_{\rm Pl}$ is very large, even reasonably high values of M_f can still lead to small α. In cosmological applications, all observable perturbations satisfy $k/M_f \ll 1$ for $M_f \gg 100 H_0 \sim 10^{-31}$ eV, roughly the scale at which linear cosmological perturbation theory breaks down at recent times, so that perturbations of $f_{\mu\nu}$ remain weakly coupled in any case.

Another potentially-problematic scale is associated with the helicity-0 mode of the massive graviton. In massive gravity, this mode becomes strongly coupled at the scale [72,73]

$$\Lambda_3 \equiv \left(m^2 M_{\rm Pl}\right)^{1/3}, \tag{16}$$

where m is defined to coincide with the Fierz–Pauli mass [1] on flat backgrounds. This scale is rather small, $\Lambda_3 \sim 10^{-13}$ eV $\sim$ $(1000 \text{ km})^{-1}$ for $m \sim H_0 \sim 10^{-33}$ eV, and severely restricts the applicability of massive gravity [74]. The same scale also appears in the decoupling-limit analysis of bimetric theory [37], where m is now the parameter in front of the potential in the action (1). In the limit $\alpha \to 0$, the f sector approaches massive gravity [65] and one might worry that the strong-coupling problem persists or becomes worse with the emergence of an even lower scale $(m^2 M_f)^{1/3}$. This is not the case. In the bimetric context, the scale defined in eq. (16) is not physical, since m^2 is degenerate with the β_n. The physically relevant strong-coupling scale must be defined with respect to the bimetric Fierz–Pauli mass [62],

$$m^2_{\rm FP} = m^2 \left(\frac{1}{c^2\alpha^2} + 1\right)\left(c\beta_1 + 2c^2\beta_2 + c^3\beta_3\right), \tag{17}$$

which is only defined around proportional backgrounds, $f_{\mu\nu} = c^2 g_{\mu\nu}$. In the massive-gravity limit, $\alpha \to \infty$, the helicity-0 mode is mostly contained in g with a strong-coupling scale

$$\Lambda_3 \equiv \left(m^2_{\rm FP} M_{\rm Pl}\right)^{1/3}, \tag{18}$$

consistent with eq. (16) for appropriately restricted parameters. However, in the GR limit, $\alpha \to 0$, the helicity-0 mode resides mostly in f, where the strong-coupling scale is

$$\tilde{\Lambda}_3 \equiv \left(m^2_{\rm FP} M_f\right)^{1/3} \to \left(\frac{m^2 M_{\rm Pl}}{\alpha}\mathcal{O}(\beta_n)\right)^{1/3}, \tag{19}$$

which is no longer small. Note that for solutions that admit this limit, c becomes independent of α. We can also consider the $\alpha \to 0$ limit of eq. (18), to verify that the small part of the helicity-0 mode in g is not strongly coupled,

$$\Lambda_3 \to \left(\frac{m^2 M_{\rm Pl}}{\alpha^2}\mathcal{O}(\beta_n)\right)^{1/3}. \tag{20}$$

This is even higher than $\tilde{\Lambda}_3$. Therefore the strong-coupling issues with the helicity-0 mode are alleviated, rather than exacerbated, when $\alpha \to 0$.

[6] See Ref. [63] for an early discussion of such a limit.

[7] Strongly-coupled gravity in the context of GR has been studied, for instance, in Refs. [68–71] and has been argued to allow for a simplified quantum-mechanical treatment.

5. Cosmology

We now proceed to apply the above arguments to the particular example of a homogeneous and isotropic universe. We will take both metrics to be of the diagonal FLRW form [14–16],[8]

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{21}$$

$$f_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -X^2(t)\mathrm{d}t^2 + Y^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{22}$$

where we can freely choose the cosmic-time coordinate for $g_{\mu\nu}$ ($g_{00} = -1$) because of general covariance. Because matter couples minimally to $g_{\mu\nu}$, this choice is physical, and $a(t)$ corresponds to the scale factor inferred from observations. We furthermore take the matter source to be a perfect fluid, $T^\mu_{\ \nu} = \mathrm{diag}(-\rho, p, p, p)$. The g-metric equation (5) leads to the Friedmann equation,

$$3H^2 = \frac{\rho}{M_{\mathrm{Pl}}^2} + m^2\left(\beta_0 + 3\beta_1 y + 3\beta_2 y^2 + \beta_3 y^3\right), \tag{23}$$

where the Hubble rate is defined as $H \equiv \dot{a}/a$ and the ratio of the scale factors is

$$y \equiv \frac{Y}{a}. \tag{24}$$

The analogous equation for the f metric is

$$3K^2 = \frac{m^2}{\alpha^2}X^2\left(\frac{\beta_1}{y^3} + 3\frac{\beta_2}{y^2} + 3\frac{\beta_3}{y} + \beta_4\right), \tag{25}$$

with $K \equiv \dot{Y}/Y$. The final ingredient is the Bianchi constraint (7), which yields

$$(HX - Ky)\left(\beta_1 + 2\beta_2 y + \beta_3 y^2\right) = 0. \tag{26}$$

Taking the first or second term of eq. (26) to vanish selects the so-called dynamical or algebraic branches, respectively. Perturbations in the algebraic branch are pathological [29], so we will consider the dynamical branch in which the f-metric lapse is fixed,

$$X = \frac{Ky}{H}. \tag{27}$$

Inserting this into the $f_{\mu\nu}$ equation (25) transforms it into an "alternate" Friedmann equation,

$$3\alpha^2 H^2 = m^2\left(\frac{\beta_1}{y} + 3\beta_2 + 3\beta_3 y + \beta_4 y^2\right). \tag{28}$$

We take at least two of the β_n for $n \geq 1$ to be nonzero in order to ensure the existence of interesting solutions in the GR limit $\alpha \to 0$. The solutions to eq. (28) in the GR limit are always on the "finite" branch, i.e., y evolves from 0 to a finite late-time value. The perturbations on this branch are healthy *except* for a scalar instability, which we discuss below.

Equation (28) has two features which are useful for our purposes. First, in the limit $\alpha \to 0$ it tends to a polynomial constraint that leads to a constant solution for y, so that the potential term in the Friedmann equation (23) becomes a cosmological constant. This provides an explicit example of the statement above that as $\alpha \to 0$, the theory approaches general relativity with an effective cosmological constant (even with $\beta_0 = 0$). Recall that even though the theory approaches GR in this limit, the bigravity interactions survive in the form of this constant. The other useful feature is that, because eq. (28) does not involve ρ, it can be used to rephrase the potential term in eq. (23) in terms of the Hubble rate. This will allow us to determine the time-dependence of the potential term order by order in α.[9]

6. The effective cosmological constant

Let us illustrate the new viable bimetric cosmologies qualitatively by selecting the model with $\beta_0 = \beta_3 = \beta_4 = 0$,[10] which we will refer to as the $\beta_1\beta_2$ model. The Friedmann and "alternate" Friedmann equations (23) and (28) are

$$3H^2 = \frac{\rho}{M_{\mathrm{Pl}}^2} + 3m^2\left(\beta_1 y + \beta_2 y^2\right), \tag{29}$$

$$3\alpha^2 H^2 = m^2\left(\frac{\beta_1}{y} + 3\beta_2\right). \tag{30}$$

We can use eq. (30) to eliminate y in eq. (29). It is instructive to work in the GR limit where eq. (30) gives

$$y \xrightarrow{\alpha\to 0} -\frac{1}{3}\frac{\beta_1}{\beta_2}. \tag{31}$$

The $\alpha \to 0$ limit is nonsingular only if both β_1 and β_2 are nonzero. Plugging this into eq. (29) we obtain

$$3H^2 = \frac{\rho}{M_{\mathrm{Pl}}^2} - \frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2. \tag{32}$$

The effective cosmological constant is

$$\Lambda_{\mathrm{eff}} = -\frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2. \tag{33}$$

Late-time acceleration requires $\beta_2 < 0$.

When we are not exactly in the GR limit, we should consider corrections to eq. (32),

$$\begin{aligned} 3H^2 &= \frac{\rho}{M_{\mathrm{Pl}}^2} + \frac{\beta_1^2 m^4}{3\left(H^2\alpha^2 - \beta_2 m^2\right)^2}\left(3\alpha^2 H^2 - 2\beta_2 m^2\right) \\ &= \frac{\rho}{M_{\mathrm{Pl}}^2} - \frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2 - \frac{\alpha^2\beta_1^2}{3\beta_2^2}H^2 + \mathcal{O}(\alpha^4). \end{aligned} \tag{34}$$

This expansion is valid as long as

$$H^2 \lesssim \frac{\beta_2 m^2}{\alpha^2}. \tag{35}$$

Rearranging and again keeping terms up to $\mathcal{O}(\alpha^2)$, we find a standard Friedmann equation with a time-varying effective cosmological constant given by

$$\Lambda_{\mathrm{eff}} = -\frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2 - \frac{2}{9}\frac{\beta_1^2}{\beta_2^2}\alpha^2\left(\frac{\rho}{2M_{\mathrm{Pl}}^2} - \frac{\beta_1^2}{3\beta_2}m^2\right) + \mathcal{O}(\alpha^4). \tag{36}$$

Because matter is coupled minimally to $g_{\mu\nu}$, it will have the standard behavior $\rho \sim a^{-3(1+w)}$, where $w = p/\rho$ is the equation-of-state parameter, allowing ρ to stand in for time. This captures the first hint of the dynamical dark energy that is typical of bigravity [16–20].

These results generalize easily to other parameter combinations. We list the effective cosmological constant up to $\mathcal{O}(\alpha^2)$ for all the

[8] See Ref. [75] and the references therein for other possible metrics in bimetric cosmology.

[9] One can also combine eqs. (23) and (28) to obtain a quartic equation for y involving ρ [14–17,31], but this is more cumbersome as it involves higher powers of y than eq. (28) does.

[10] Since we are interested in finding self-accelerating solutions in the absence of vacuum energy, we will set $\beta_0 = 0$ herein, but emphasize that this is not necessary.

Table 1
The effective cosmological constant and lowest-order corrections (which are time-dependent through ρ) for a variety of two-parameter models. We have chosen solution branches which lead to positive $\Lambda_{\rm eff}$ for appropriate signs of the β_n, and generally take $\beta_1 \geq 0$ based on viability conditions [19]. The $\beta_3, \beta_4 \neq 0$ model does not possess a finite-branch solution [19].

Model	$\Lambda_{\rm eff}$ $(\alpha \to 0)$	$\mathcal{O}(\alpha^2)$ correction
$\beta_1, \beta_2 \neq 0$	$-\frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2$	$-\frac{2}{9}\frac{\beta_1^2}{\beta_2^2}\alpha^2\left(\frac{\rho}{2M_{\rm Pl}^2} - \frac{\beta_1^2}{3\beta_2}m^2\right)$
$\beta_1, \beta_3 \neq 0$	$\frac{8}{3\sqrt{3}}\frac{\beta_1^{3/2}}{\sqrt{-\beta_3}}m^2$	$\frac{\beta_1}{\beta_3}\alpha^2\left(\frac{\rho}{3M_{\rm Pl}^2} - \frac{8\beta_1^{3/2}}{9\sqrt{-3\beta_3}}m^2\right)$
$\beta_1, \beta_4 \neq 0$	$3\frac{\beta_1^{4/3}}{\sqrt[3]{-\beta_4}}m^2$	$-\left(-\frac{\beta_1}{\beta_4}\right)^{\frac{2}{3}}\alpha^2\left(\frac{\rho}{M_{\rm Pl}^2} + 3\frac{\beta_1^{4/3}}{\sqrt[3]{-\beta_4}}m^2\right)$
$\beta_2, \beta_3 \neq 0$	$2\frac{\beta_2^3}{\beta_3^2}m^2$	$-\frac{\beta_2^2}{\beta_3^2}\alpha^2\left(\frac{\rho}{M_{\rm Pl}^2} + \frac{2\beta_2^3}{\beta_3^2}m^2\right)$
$\beta_2, \beta_4 \neq 0$	$-9\frac{\beta_2^2}{\beta_4}m^2$	$3\frac{\beta_2}{\beta_4}\alpha^2\left(\frac{\rho}{M_{\rm Pl}^2} - \frac{9\beta_2^2}{\beta_4}m^2\right)$

two-parameter models (setting $\beta_0 = 0$) in Table 1. We remind the reader that, in order for the $\alpha \to 0$ limit to be well-behaved, at least two of the β_n parameters (excluding the vacuum energy contribution, β_0) must be nonzero.

7. Exorcising the instability

The stability of cosmological perturbations in bigravity was investigated in Ref. [32] by determining the full solutions to the linearized Einstein equations in the subhorizon régime. The perturbations were shown to obey a WKB solution given by

$$\Phi \sim e^{i\omega N}, \tag{37}$$

where Φ represents any of the scalar perturbation variables, $N \equiv \ln a$, and we have taken the limit $k \gg aH$ where k is the comoving wavenumber. The eigenfrequencies ω were presented for particular models in Ref. [32], where it was found that all models with viable backgrounds have $\omega^2 < 0$ at early times, revealing a gradient instability that only ends at a very low redshift. Using the formulation of the linearized equations of motion presented in Ref. [33], we can write the eigenfrequencies for general β_n and α in the compact form [41]

$$\left(\frac{aH}{k}\right)^2\omega^2 = 1 + \frac{\left(\beta_1 + 4\beta_2 y + 3\beta_3 y^2\right)y'}{3y\left(\beta_1 + 2\beta_2 y + \beta_3 y^2\right)} - \frac{\left(1+\alpha^2 y^2\right)\left(\beta_1 - \beta_3 y^2\right)y'^2}{3\alpha^2 y^3 \tilde{\rho}(1+w)}, \tag{38}$$

where $\tilde{\rho} \equiv \rho/m^2 M_{\rm Pl}^2$ and primes denote $\mathrm{d}/\mathrm{d}\ln a$.

We apply this to the $\beta_1\beta_2$ model. Assuming a universe dominated by dust ($w = 0$), ω^2 crosses zero when[11]

$$18\alpha^2\beta_2\left(\alpha^2\beta_1^2 + 4\beta_2^2\right)y^5 + 9\alpha^2\beta_1\left(\alpha^2\beta_1^2 + 10\beta_2^2\right)y^4 + 48\alpha^2\beta_1^2\beta_2 y^3 + 6\beta_2\left(2\alpha^2\beta_1^2 - \beta_2^2\right)y^2 - 6\beta_1^2\beta_2 y - \beta_1^3 = 0. \tag{39}$$

Solving this for y, we can then use eq. (30) to determine the value of Hubble rate at the *transition era*, before which the gradient instability is present and after which it vanishes. While this solution is too complicated to write down explicitly, in the limit $\alpha \to 0$ the leading-order term is remarkably simple,[12]

$$H_\star^2 = \pm\frac{\beta_2 m^2}{\sqrt{3}\alpha^2} + \mathcal{O}(\alpha^0), \tag{40}$$

where $H_\star$ is defined as the Hubble rate at the time when $\omega^2 = 0$, i.e., after which the gradient instability is absent. We pick the negative branch of eq. (40) for physical reasons, i.e., so that $H_\star^2 > 0$ given that $\beta_2 < 0$. We have checked explicitly that by solving for y with this value of H and plugging it into ω^2, all terms up to $\mathcal{O}(\alpha^2)$ vanish.

Interestingly, eq. (40) is the same as the condition (35) for the small-α expansion of the background solution to be valid. Therefore, simply by pushing the instability back to early times, one gets late-time bimetric dynamics that can be described as perturbative corrections to GR, except for the effective cosmological constant which remains nonperturbative. This is nontrivial; while we expect everything to reduce to GR at late times when we can expand in $\alpha H/\sqrt{\beta_n}m$, there could in principle have been earlier times during which perturbations were stable but still fundamentally different than in GR.

We can rewrite eq. (40) in more physical terms as

$$H_\star^2 = -\frac{3\sqrt{3}}{2\alpha^2}\left(\frac{\beta_2}{\beta_1}\right)^2 H_\Lambda^2, \tag{41}$$

where H_Λ is the far-future value of H and should be comparable to the present Hubble rate, H_0. For $|\beta_1| \sim |\beta_2|$, this implies simply

$$H_\star \sim \frac{H_0}{\alpha}. \tag{42}$$

We see that as we approach the GR limit, the smaller one takes the f-metric Planck mass, the earlier in time bigravity's gradient instability is cured. Our goal is to make this era so early as to be effectively unobservable. One has a variety of choices for the scale where the instability sets in; the values of α and M_f for various choices are summarized in Table 2.

Table 2
The values of α and M_f for a few choices of the era at which perturbations become stable.

Era of transition to stability	$H_\star$	α	M_f
BBN	10^{-16} eV	10^{-17}	100 GeV
$\tilde{\Lambda}_3 = (m^2 M_{\rm Pl}/\alpha)^{1/3}$	10^{-3} eV	10^{-31}	10^{-3} eV
GUT-scale inflation	10^{13} GeV	10^{-55}	10^{-27} eV
$M_{\rm Pl}$	10^{19} GeV	10^{-61}	10^{-33} eV

A natural requirement would be to push the instability outside the range of the effective field theory, i.e., above either the cut-off scale where new physics must enter, or the strong-coupling scale where tree-level unitary breaks down.[13] The cut-off scale in massive and bimetric gravity is not known. The strong-coupling scale, to the extent it is understood, was discussed above. Here we focus on observational constraints. It is natural to demand that the instability lie beyond some important cosmic era which we can indirectly probe, such as big-bang nucleosynthesis (BBN) or inflation. Both of these possibilities are then likely to be observationally safe as long as the Universe is decelerating (e.g., is

[11] We have used eqs. (29) and (30) and their derivatives to solve for y' and ρ in eq. (38) in terms of β_n and y [31]. Note that $\omega^2 = 0$ does not imply strong coupling because, while the gradient terms vanish, the kinetic terms remain nonzero.

[12] While eq. (40) only holds exactly in the presence of dust, $w = 0$, for other reasonable equations of state, such as radiation ($w = 1/3$), it will only be modified by an $\mathcal{O}(1)$ factor. Since we will be using this analysis only to make order-of-magnitude estimates, the exact factors are unimportant.

[13] These two are not always the same, and may not be in massive and bimetric gravity [76,77].

radiation-dominated) after inflation, because the instability is only a problem for subhorizon modes with large k/aH, and during a decelerating epoch modes with fixed comoving wavelength always become smaller with respect to the horizon. Consider, as an example, that the transition to stability occurs between inflation and BBN. During that period, modes will grow rapidly on small scales, but those will be far, far smaller than the modes relevant for the cosmic microwave background or large-scale structure. One might worry that inflation's ability to set initial conditions is spoiled in this scenario (assuming that the linear theory is even valid during inflation, which is not guaranteed due to the arguments above). However, the instability should be absent during inflation; notice from eq. (38) that ω^2 generically becomes large and positive for w close to -1.[14] Therefore the instability would not affect the generation of primordial perturbations during inflation. If the instability later appears with the onset of radiation domination, it would only affect small scales which are irrelevant for present-day cosmology.

If the instability ends at the time of BBN, M_f can be as high as about 100 GeV, far larger than the wavenumbers probed by cosmological observations. We remind the reader that for such a "large" M_f, perturbations in the Einstein–Hilbert term for $f_{\mu\nu}$ remain weakly-coupled for all observationally-relevant k.

While analytic results like eq. (40) cannot be obtained for most of the other two-parameter models, we have checked that in each case the relevant behavior, $H_\star \sim H_\Lambda/\alpha$, holds.[15] The values given in Table 2 are therefore fairly model-independent.

The other pathology that is typical of massive and bimetric gravity, the Higuchi ghost, is not present in these models. There is a simple condition for the absence of this ghost, $\mathrm{d}\rho/\mathrm{d}y < 0$ [35,36] (see also Refs. [33,37]). Because for normal matter ρ is always decreasing with time, this amounts to demanding that y be increasing. In the "finite-branch" solutions which we are considering, y evolves monotonically from 0 at early times to a fixed positive value at late times, and so the Higuchi bound is always satisfied [41].

8. Parameter rescalings

We have presented a physically well-motivated region of bimetric parameter space, near the GR limit, in which observable cosmological perturbations are stable and yet self-acceleration remains. One is naturally led to ask how this has been missed by the many previous studies of bimetric cosmology. The issue lies in a rescaling which leaves the action (1) invariant [28,62],

$$f_{\mu\nu} \to \Omega^2 f_{\mu\nu}, \qquad \beta_n \to \frac{1}{\Omega^n}\beta_n, \qquad M_f \to \Omega M_f, \tag{43}$$

and hence gives rise to a redundant parameter. It has become common to let α play this role and perform the rescaling $\Omega = 1/\alpha$ such that α is set to unity. While our results do not invalidate this rescaling, they do show that it picks out a particular region of parameter space which may not capture all physically-meaningful situations. In particular, the $\alpha \to 0$ limit, in which the theory approaches GR—the behavior at the heart of our removing the gradient instability—would look extremely odd after this rescaling: the β_n would not only be very large, but each β_{n+1} would be *parametrically* larger than β_n.[16] Therefore, studies which set α to unity could in principle have found the GR-like solutions which we study here, but only by looking at what would have appeared to be a highly unnatural and tuned set of parameters, even though they have a simple and sensible physical explanation. Without performing this rescaling, we can simply take the nonzero β_n to be $\mathcal{O}(1)$ and consider that we are in the small-M_f régime.

It is clear that in phenomenological studies of bigravity, α must not automatically be set to unity. When working with a two-β_n model, perhaps a more sensible rescaling would be one such that the two β_n are equal to each other (up to a possible sign). They can further be absorbed into m^2. In this case, the free parameters are effectively the spin-2 interaction scale, m^2, and the f-metric Planck mass, M_f. Their effects decouple nicely: M_f controls the earliness of the instability, while m sets the acceleration scale. Alternatively, one may consider that the rescaling (43) simply tells us that rather different regions of parameter space happen to have the same solutions, and therefore not perform any rescaling *a priori* at all.

9. Summary and discussion

We have shown that a well-motivated but heretofore underexplored region of parameter space in bimetric gravity can lead to cosmological solutions which are observationally viable and close to general relativity, with an effective cosmological constant that is set by the spin-2 interaction scale m. In this limit, obtained by taking a small f-metric Planck mass, the gradient instability that seems to generically plague bimetric models at late times is relegated to the very early Universe, where it can be either made unobservable or pushed outside the régime of validity of the effective theory. This instability had been considered in previous work to make bimetric cosmologies nonpredictive even at late times. Furthermore, in this limit the theory avoids the usual low-scale strong-coupling issue that affects the helicity-0 sector in the massive-gravity limit.

What is encouraging is that the one property of bigravity which survives in the small-α limit is its cosmologically most useful feature, the technically-natural dark energy scale. In other words, the effective cosmological constant of bigravity in a region close to GR is not just the vacuum-energy contribution and can give rise to self-acceleration in its absence.

The model we have presented is expected to be extremely close to GR at all but very high energy scales. In particular the Newtonian limit is well-behaved; unlike $m^2 \to 0$, which suffers from the vDVZ discontinuity, the GR limit $\alpha \to 0$ is completely smooth because all the helicity states of the massive spin-2 mode decouple from matter. Note also that massive gravity does not possess such a continuous GR limit.

It is worth emphasizing that the $\alpha \to 0$ limit brings bimetric theory arbitrarily close to GR even for a large value of the spin-2 mass scale, $m \gg H_0$. The presence of heavy spin-2 fields in the Universe is therefore not excluded as long as their self-interaction scale (set by M_f) is sufficiently small compared to M_{Pl}. In this case, however, the β_n parameters need to be highly tuned for the effective cosmological constant small enough to be compatible with observations.[17] Note however that, since the β_n are

[14] This depends on the exact β_n parameters and the evolution of y. Background viability requires $\beta_1 > 0$ [19], so as long as $\beta_3 \le 0$, at the very least the last term in eq. (38) is large and manifestly positive.

[15] Specifically, this holds in the models with $\beta_1 \neq 0$. The gradient instability is absent from the $\beta_2\beta_3$ and $\beta_2\beta_4$ models at early times [32]. These were shown in Ref. [19] to have problematic background behavior at early times, but these again can be made unobservably early in the GR limit.

[16] We can recast this as a large m^2, but there would remain a specific tuning among the β_n of the form $\beta_n/\beta_{n+1} \sim \epsilon$, where ϵ is the value of α before the rescaling.

[17] Indeed, without this tuning of the β_n, the interaction term would lead to acceleration at an unacceptably early epoch. This scenario is related to the findings of Ref. [36], where it was shown that the instability becomes negligible for large values of m.

protected against loop corrections [42,78,79], this tuning does not violate *technical* naturalness.

Finally we comment on the potential observable signatures of this theory. While at low energies, corresponding to recent cosmological epochs, this limit of bigravity is extremely close to GR, there may be observable effects at early times when the effects of strong coupling become important. In this case, given by $H > H_\star$, the small-α approximation breaks down and modified-gravity effects must be taken into account. This may be particularly important for inflation, which will see such effects unless M_f is extraordinarily small. A better understanding of strong coupling in the $f_{\mu\nu}$ sector will therefore point the way towards tests of this important region of bimetric parameter space, since at this point it is not clear how to perform computations in the strong-coupling regime. There may also be effects related to the Vainshtein mechanism [80,81]. We conclude that the closeness of this theory to GR is both a blessing and a curse: while it is behind the exorcism of the gradient instability and brings the theory in excellent agreement with experiments, it presents a serious observational challenge if it is to be compared *against* GR. It is nevertheless encouraging that this "GR-adjacent" bigravity naturally explains cosmic acceleration while avoiding the instabilities that plague other bimetric models, and therefore merits serious consideration.

Acknowledgments

We are grateful to Luca Amendola, Jonas Enander, Matteo Fasiello, Emir Gümrükçüoğlu, Nima Khosravi, Edvard Mörtsell, Luigi Pilo, Marit Sandstad, and Sergey Sibiryakov for useful discussions. We thank the "Extended Theories of Gravity" workshop at Nordita for providing a stimulating atmosphere during the completion of this work. Y.A. and F.K. acknowledge support from DFG through the project TRR33 "The Dark Universe." F.K. is also supported by the Graduate College "Astrophysics of Fundamental Probes of Gravity." The work of A.S.M. is supported by ERC grant No. 615203 under the FP7 and the Swiss National Science Foundation through the NCCR SwissMAP. A.R.S. acknowledges support from the STFC.

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Publication 8

A spectre is haunting the cosmos: Quantum stability of massive gravity with ghosts

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JHEP 11 (2016) 118, arXiv:1605.08757

JHEP

Published for SISSA by Springer

Received: June 13, 2016
Revised: October 21, 2016
Accepted: November 17, 2016
Published: November 21, 2016

A spectre is haunting the cosmos: quantum stability of massive gravity with ghosts

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Abstract: Many theories of modified gravity with higher order derivatives are usually ignored because of serious problems that appear due to an additional ghost degree of freedom. Most dangerously, it causes an immediate decay of the vacuum. However, breaking Lorentz invariance can cure such abominable behavior. By analyzing a model that describes a massive graviton together with a remaining Boulware-Deser ghost mode we show that even ghostly theories of modified gravity can yield models that are viable at both classical and quantum levels and, therefore, they should not generally be ruled out. Furthermore, we identify the most dangerous quantum scattering process that has the main impact on the decay time and find differences to simple theories that only describe an ordinary scalar field and a ghost. Additionally, constraints on the parameters of the theory including some upper bounds on the Lorentz-breaking cutoff scale are presented. In particular, for a simple theory of massive gravity we find that a breaking of Lorentz invariance is allowed to happen even at scales above the Planck mass. Finally, we discuss the relevance to other theories of modified gravity.

Keywords: Classical Theories of Gravity, Cosmology of Theories beyond the SM

ArXiv ePrint: 1605.08757

This work is dedicated to our friend, Tham.

doi:10.1007/JHEP11(2016)118

Contents

1 Introduction

Despite the fact that the standard theory of gravity, general relativity (GR), is already around one century old, there has always been interest in finding viable modifications to it. In particular, the discovery of the late-time acceleration of our Universe [1, 2] driven by some dark energy has led to an additional motivation, as GR requires a technically unnatural cosmological constant (CC) in order to be compatible with observations. The list of problems with the standard theory goes much further (see for example ref. [3] for a recent review): GR is not renormalizable and can only be regarded as an effective field theory (EFT). Furthermore, the formation of structure at early times needs an additional inflationary epoch, and even the requirement of some additional dark matter might be the consequence of the inability of GR to properly describe the evolution of the cosmic structure. It is however not only these problems that make a search for modifications of GR attractive; there is also the more fundamental question of which classes of theories are allowed and consistent.

Under certain assumptions, Vermeil and Cartan independently proved that Einstein equations are the only allowed field equations to describe gravity [4, 5]. In particular, if a rank-2 tensor K is naturally constructed from only a pseudo-Riemannian metric g, and is symmetric, divergence-free, only second order in the derivatives of g, and linear in these derivatives, then K has to be a linear combination of the Einstein tensor and the metric itself. Later, Lovelock showed that the requirements of symmetry and linearity are redundant in four dimensions [6]. A generalization of this theorem has recently been suggested by Navarro and Sancho [7], who replaced the assumptions of the number of dimensions and absence of higher order derivatives by a simpler requirement that K is homogeneous, i.e., $K(\lambda^2 g) = \lambda^w K(g)$ $\forall g$, $\forall \lambda > 0$, and of weight $w > -2$.

Thinking about modifying GR can be translated into relaxing these so-called Lovelock assumptions. Higher dimensional spacetimes, as well as unnaturalness (understood in the mathematical sense, i.e., either breaking of locality or generally non-$\mathcal{C}^\infty$), enable a richer phenomenology and do not necessarily require a CC in order to fit current observations. Additionally, a pseudo-Riemannian geometry is quite restrictive as it both implies a vanishing torsion and enforces the connection to be metric-compatible. Certainly the strongest assumption, however, is the dependence on the metric only. Consequently, most theories of modified gravity assume additional fields that can be either scalar, vector, or tensor.

There is, however, one assumption that usually stays untouched: the absence of higher order derivatives. An old theorem from Ostrogradski states that non-degenerate Lagrangians that lead to third or higher order derivatives in the equations of motion (EoM) always house an additional ghost, i.e., a degree of freedom with the wrong kinetic sign. But even degenerate Lagrangians producing third order derivatives are affected by ghosts [8]. The consequences that come along with a ghost are usually believed to be fatal (see, e.g., refs. [9, 10]). Such a negative energy mode could drive the classical theory into an instability. Even though this might still be acceptable as long as the theory is in agreement with observations, it indeed limits the number of viable theories drastically. The real catastrophe appears, however, at the quantum level: ghost fields can decay into ordinary matter fields by reaching arbitrarily large negative energy states. And, even worse, this decay will practically happen instantaneously (see refs. [10, 11] for more details). Such a theory cannot describe a stable vacuum and therefore has to be ruled out. The origin of the fast decay lies in an integration over the entire phase space when computing scattering amplitudes which diverge in the ultraviolet (UV) region. Therefore, the only way to tame the ghost is to modify the integration in the UV. In ref. [12], the authors suggested that new operators beyond the EFT would allow us to cut this integral and, therefore, a theory with ghosts could theoretically be cured. In fact, it has been shown that the vacuum in simple theories with two oscillators, of which one is a ghost, can indeed have a decay time that is larger than the Hubble time [12, 13] (see also refs. [14, 15] for discssions of ghosts in Chern-Simons and Hořava-Lifshitz theories, respectively). The energy scale at which new physics might enter and break Lorentz invariance (LI) can be low enough to slow down the vacuum decay sufficiently and circumvent any violation of experimental constraints, but at the same time be high enough to be above the cutoff of the EFT. In fact, a Lorentz breaking (LB) does not render the theory unappealing as long as it occurs above the EFT cutoff scale.

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In this work, we discuss a theory of modified gravity that automatically introduces a ghost instead of adding a simple ghost field by hand to a well behaved theory. In fact, as will be shown, many properties of such a theory, like the decay time of the vacuum, may be significantly different, and therefore the conclusions from simpler toy models should not be adopted blindly. In order to modify GR suitably, we assume a massive graviton. Even though the idea of studying massive gravity is very old [16], ghost-free non-linear theories were discovered only recently [17–23]. Here we use the so-called ghost-free de Rham-Gabadadze-Tolley (dRGT) theory to construct a theory of a massive graviton with an additional Boulware-Deser (BD) ghost [24], which we then dub *haunted massive gravity* (HMG). We first study the classical behavior of the theory for a Friedmann-Lemaître-Robertson-Walker (FLRW) background in order to identify the models that do not introduce potentially dangerous instabilities already at this level. With HMG we find the first theory of a canonical non-linear massive gravity which possesses models that are free of any background pathologies, and allows for dynamical, even self-accelerating, FLRW solutions. We finally discuss the quantum stability of the viable models by computing the ghost and vacuum decay rates in HMG.

Although the theory that is discussed in this work has some nice features, e.g., it can provide a solution to the dark energy problem, it is certainly not the most promising contender of GR. For example, the construction of the mass term is mainly based on keeping simplicity, and the strong coupling scale of the theory is very low, losing predictivity on small scales. This work, however, does not intend to present a new theory of modified gravity in order to address the problems with GR, but rather to examine the behavior of ghosts appearing in more realistic theories of gravity. We find that the scattering processes that dominate in the ghost and vacuum decay rates do not coincide with those that appear in theories with two coupled canonical scalar fields where one of the fields is a ghost. While in such a simple scenario the decay time has been found to scale only quadratically with the cutoff at which Lorentz violation (LV) occurs [12] (see also refs. [13, 25] for a discussion on decay rates for other setups), we find a completely different scaling for HMG. Furthermore, we expect the type of interaction that we find in this work as the most important one in HMG to in fact determine the decay time also in many other theories of gravity with a present ghost mode.

2 HMG with dRGT limit

Since we are interested in a theory of a massive graviton with an additional BD ghost, we could simply study any non-linear theory that does not coincide with the dRGT theory. However, we would like to keep a ghost-free limit, and therefore, we start with dRGT and modify the tuning between the coefficients of interaction terms.

The dRGT massive gravity can be written as [22, 26]

$$S_{\rm dRGT} = M_{\rm P}^2 \int {\rm d}^4x \sqrt{-g}\left[R + 2m^2 U(\mathbb{K})\right], \tag{2.1}$$

where $U(\mathbb{K})$ is the mass term, which depends on the eigenvalues of $\mathbb{K} \equiv \sqrt{g^{-1}f} - \mathbb{1} \equiv \mathbb{X} - \mathbb{1}$, and is equivalent to $\sum_{n=0}^{3} \beta_n U_n$ with $U_n \equiv e_n(\mathbb{X})$, e_n being the elementary symmetric polynomials of the eigenvalues of $\mathbb{X}$. Additionally, f is a non-dynamical symmetric rank-2 tensor field, and β_n depend on the two free dimensionless parameters of the theory, α_3 and α_4 [26]:

$$\beta_0 = 6 - 4\alpha_3 + \alpha_4, \tag{2.2}$$
$$\beta_1 = -3 + 3\alpha_3 - \alpha_4, \tag{2.3}$$
$$\beta_2 = 1 - 2\alpha_3 + \alpha_4, \tag{2.4}$$
$$\beta_3 = \alpha_3 - \alpha_4. \tag{2.5}$$

Let us first discuss the simplest model, the so-called minimal model [26], and choose α_3 and α_4 such that we can switch off the highest order interactions, i.e. $\beta_2 = \beta_3 = 0$, and obtain

$$\alpha_3 = \alpha_4 = 1 \quad \Leftrightarrow \quad \beta_0 = 3,\ \beta_1 = -1. \tag{2.6}$$

The action in this case becomes

$$S_{\rm min} = M_{\rm P}^2 \int \mathrm{d}^4x\sqrt{-g}\left[R + 2m^2\left(3 - [\mathbb{X}]\right)\right], \tag{2.7}$$

where $[\mathbb{X}]$ denotes the trace of $\mathbb{X}$. If we change the prefactors of the mass term, we then change either the CC or the graviton mass (or make it tachyonic). Thus, in order to introduce a ghost, we should switch on higher order interactions. One possibility would be to remove the CC (which would make the model very appealing especially for cosmology) and allow for β_1 and β_2 to be non-vanishing. We then find $\beta_0 = \beta_3 = 0$ together with

$$\alpha_3 = \alpha_4 = 2 \quad \Leftrightarrow \quad \beta_1 = 1 = -\beta_2, \tag{2.8}$$

which results in the action

$$S = M_{\rm P}^2 \int \mathrm{d}^4x\sqrt{-g}\left[R + 2m^2\left(\beta_1[\mathbb{X}] + \frac{1}{2}\beta_2\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right)\right)\right] \tag{2.9}$$
$$= M_{\rm P}^2 \int \mathrm{d}^4x\sqrt{-g}\left[R + 2m^2\left(\left[\sqrt{g^{-1}f}\right] - \frac{1}{2}\left(\left[\sqrt{g^{-1}f}\right]^2 - \left[g^{-1}f\right]\right)\right)\right]. \tag{2.10}$$

Note that this choice, like all other combinations that satisfy $\alpha_3 + \alpha_4 > 0$, does not lead to a Higuchi ghost, at least around an FLRW background for large H^2 [27]. This is important because we will use this theory as a ghost-free limit which should ensure not only the absence of the BD ghost but also the presence of five healthy graviton degrees of freedom.

We now modify the theory to introduce a ghost. The simplest way would be to modify the prefactor in one of the interaction terms which in the linear theory corresponds to a violation of the Fierz-Pauli (FP) tuning. However, we do not expect this modification to enable dynamical FLRW solutions for a flat reference metric since the combination of the Bianchi identities and the conservation of energy-momentum tensor will still be a constraint for the scale factor, as it has been shown for dRGT in ref. [28]. One way out could be to

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introduce a metric-dependent (and, thus, lapse-dependent) prefactor like $\alpha\left[g^{-1}f\right]$, with $\alpha\in\mathbb{R}$, and study the action

$$S = M_{\mathrm{P}}^2\int \mathrm{d}^4x\sqrt{-g}\left[R+2m^2\left(\left[\sqrt{g^{-1}f}\right]+\frac{1}{2}\left(\alpha\left[g^{-1}f\right]-1\right)\left(\left[\sqrt{g^{-1}f}\right]^2-\left[g^{-1}f\right]\right)\right)\right]. \tag{2.11}$$

Although this theory would certainly allow for dynamical FLRW backgrounds, we found only unviable solutions for which the scale factor would become imaginary or the lapse would cross zero, indicating an instability. Therefore, we consider a slightly more complicated theory in which both interaction terms are modified, and dub this theory haunted massive gravity (HMG):

$$\begin{aligned} S_{\mathrm{HMG}} = M_{\mathrm{P}}^2\int \mathrm{d}^4x\sqrt{-g}\Bigg[R+2m^2\Bigg(&\left(1-\alpha_1\left(g,f\right)\right)\left[\sqrt{g^{-1}f}\right] \\ &-\frac{1}{2}\left(1-\alpha_2\left(g,f\right)\right)\left(\left[\sqrt{g^{-1}f}\right]^2-\left[g^{-1}f\right]\right)\Bigg)\Bigg], \end{aligned} \tag{2.12}$$

with

$$\alpha_1\left(g,f\right)\equiv\bar{\alpha}_1\mathbb{X}^2=\bar{\alpha}_1 g^{\alpha\beta}f_{\beta\alpha}, \tag{2.13}$$

$$\alpha_2\left(g,f\right)\equiv\bar{\alpha}_2\mathbb{X}^2=\bar{\alpha}_2 g^{\alpha\beta}f_{\beta\alpha}, \tag{2.14}$$

where $\bar{\alpha}_i$ are two dimensionless parameters.

This theory has some interesting properties. Firstly, the limit $\bar{\alpha}_i\to 0$ corresponds to the ghost-free dRGT theory, whereas any other values should introduce a new ghost degree of freedom as it does not coincide with dRGT, the unique non-linear ghost-free theory of a massive graviton. Secondly, the additional dynamical factors will enable us to have dynamical FLRW solutions by modifying the Bianchi constraint, and, finally, we expect the vacuum to decay more slowly at late times since $\left[g^{-1}f\right]\propto a^{-2}$ for FLRW backgrounds.[1]

3 Background cosmology

From now on, let us fix the reference metric to a flat Minkowski background, i.e., $f_{\mu\nu}=\eta_{\mu\nu}$. Since massive gravity with $f_{\mu\nu}=\eta_{\mu\nu}$ breaks diffeomorphism invariance, the lapse of $g_{\mu\nu}$ must not be chosen arbitrarily. For an FLRW background we therefore choose

$$\mathrm{d}s^2=-N_g^2\mathrm{d}t^2+a^2\mathrm{d}\vec{x}^2, \tag{3.1}$$

with N_g and a denoting the lapse and the scale factor, respectively, and t being cosmic time. Varying the action (2.12) with respect to $g_{\mu\nu}$ yields

$$-2m^2M_{\mathrm{P}}^2\delta\left(\sqrt{-g}U_1\left(X\right)\right)=-\sqrt{-g}\beta_1 m^2M_{\mathrm{P}}^2 g^{\mu\alpha}Y^{\nu}_{(1)\,\alpha}\left(\sqrt{g^{-1}f}\right)\delta g_{\mu\nu}, \tag{3.2}$$

$$-2m^2M_{\mathrm{P}}^2\delta\left(\sqrt{-g}U_2\left(X\right)\right)=\sqrt{-g}\beta_2 m^2M_{\mathrm{P}}^2 g^{\mu\alpha}Y^{\nu}_{(2)\,\alpha}\left(\sqrt{g^{-1}f}\right)\delta g_{\mu\nu}, \tag{3.3}$$

[1]This could have an interesting impact on the phenomenology at early times and might lead to an enhanced creation of particles. The relevant time period would, however, presumably lie above the cutoff scale of the theory.

with

$$Y_{(1)}(\mathbb{X}) \equiv \mathbb{X} - \mathbb{1}[\mathbb{X}], \tag{3.4}$$

$$Y_{(2)}(\mathbb{X}) \equiv \mathbb{X}^2 - \mathbb{X}[\mathbb{X}] + \frac{1}{2}\mathbb{1}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right). \tag{3.5}$$

Furthermore, we need the variation of α_i:

$$\delta\alpha_i(g) = -\bar{\alpha}_i g^{\alpha\mu} g^{\nu\beta} \eta_{\beta\alpha} \delta g_{\mu\nu}. \tag{3.6}$$

With this, the variation of the mass term yields

$$\sqrt{-g} M_{\mathrm{P}}^2 V^{\mu\nu} \equiv -m^2 \sqrt{-g} M_{\mathrm{P}}^2 \Bigg[(\alpha_1(g) - 1) g^{\mu\alpha} Y^{\nu}_{(1)\,\alpha} + (\alpha_2(g) - 1) g^{\mu\alpha} Y^{\nu}_{(2)\,\alpha} - 2U_1(\mathbb{X}) \frac{\delta\alpha_1(g)}{\delta g_{\mu\nu}} - 2U_2(\mathbb{X}) \frac{\delta\alpha_2(g)}{\delta g_{\mu\nu}} \Bigg]. \tag{3.7}$$

Combining the Bianchi identities with the assumption of a conserved energy-momentum tensor leads to the Bianchi constraint

$$\nabla_\mu V^{\mu\nu} = 0, \tag{3.8}$$

which implies

$$\left(1 + 3a^{-2}N_g^2\right)\left[N_g'\left(a\left(a\bar{\alpha}_1 + 6\bar{\alpha}_2\right) + 2N_g\left(2a\bar{\alpha}_1 + \bar{\alpha}_2\right)\right) + HN_g\left(-6a\bar{\alpha}_2 + N_g^2\left(4\bar{\alpha}_1 - (a-2)a\right) - 2N_g\left(a\bar{\alpha}_1 + \bar{\alpha}_2\right) + 9\bar{\alpha}_1 a^{-1} N_g^3\right)\right] = 0. \tag{3.9}$$

Here, $H \equiv a'/a$ is the Hubble rate, and a prime denotes a derivative with respect to t. In the limit $\bar{\alpha}_i \to 0$, eq. (3.9) fixes the scale factor confirming the no-go theorem for FLRW solutions in dRGT massive gravity. In our case, however, we are able to switch on the dynamics since the Bianchi constraint now depends on both the scale factor and the lapse. This constraint together with the Friedmann equation

$$3H^2 = \rho + V_{00} \tag{3.10}$$

$$= \rho + \frac{m^2}{a^4 N_g}\left[a^3\left(-\left(a\bar{\alpha}_1 + 6\bar{\alpha}_2\right)\right) - 3a^2 N_g\left(2a\bar{\alpha}_1 + \bar{\alpha}_2\right) + 3N_g^3\left(a\left((a-1)a - 3\bar{\alpha}_1\right) + 3\bar{\alpha}_2\right)\right] \tag{3.11}$$

and assuming a universe filled with dark matter only,

$$\rho \equiv \rho_0 a^{-3}, \tag{3.12}$$

can be solved numerically. In the limit $a \ll 1$, the combination of the Bianchi constraint and the Friedmann equation provides

$$N_g = \pm\frac{1}{3}\sqrt{\frac{\bar{\alpha}_2}{\bar{\alpha}_1}}\,a, \tag{3.13}$$

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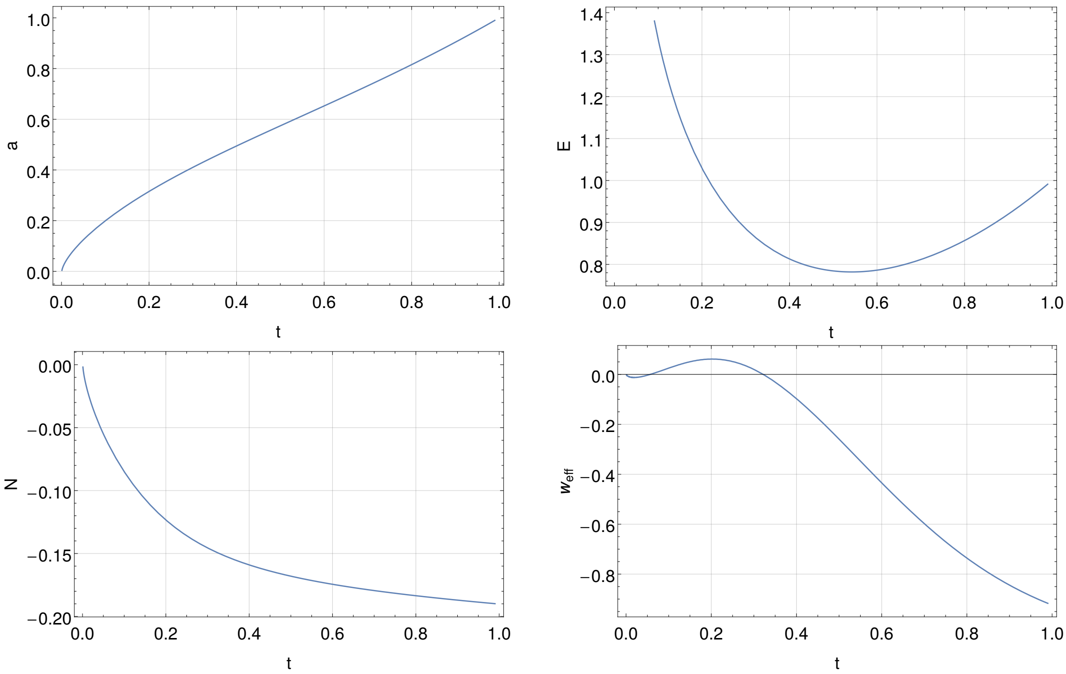

Figure 1. Numerical solution of the FLRW background evolution in HMG, corresponding to the model with $\bar{\alpha}_1 = 0.9$ and $\bar{\alpha}_2 \simeq 0.1025$. The plots show the scale factor (upper left), the expansion rate (upper right), the lapse (lower left), and the effective equation of state (lower right). All the quantities are plotted versus the cosmic time, t, scaled in such a way that $a(t = 1) = 1$.

and implies $H^2 \propto a^{-3}$. Therefore, we find a singularity for $a \to 0$. The time at which it occurs will be denoted by t_c, i.e., $N_g(t_c) = 0$.

Interestingly, for a given $\bar{\alpha}_1$ one can find a value for the parameter $\bar{\alpha}_2$ that maximizes the timescale of the background evolution by reaching $t_c \to 0$. One example for such a model is

$$\bar{\alpha}_1 = 0.9 \;\Rightarrow\; \bar{\alpha}_2 \simeq 0.1025. \tag{3.14}$$

By solving the background equations numerically, we can search for the parameter region that leads to $t_c = 0$ and find that it can be fitted very well linearly with

$$\bar{\alpha}_2 \simeq \frac{1}{6}\bar{\alpha}_1 - \frac{2}{45}. \tag{3.15}$$

If we promote the maximization of the classical timescale to a constraint, then this model will effectively lose one free parameter. Furthermore, eq. (3.15) indicates that both $\bar{\alpha}_1$ and $\bar{\alpha}_2$ can be of $\mathcal{O}(1)$.

Surprisingly, after solving the background equations for the model (3.14) numerically, we find that at late times the effective equation of state parameter $w_{\rm eff} < -1/3$ indicates a period of self-acceleration and we thus have found a candidate for a model that could be able to provide a solution to the dark energy problem. See figure 1 for a numerical solution of the background evolution, corresponding to model (3.14).

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4 Second order action for HMG around Minkowski

In order to compute the vacuum decay, we have to first identify the ghost mode. For this, we perturb the background, expand the action to second order, and finally integrate out[2] all auxiliary fields.

4.1 Gravity sector

We now choose to work with perturbations around a Minkowski background. Note that we expect a generalization to an FLRW background to merely modify the decay rate insignificantly. In fact, corrections from an FLRW background are proportional to H/k and become negligible in the high-momentum limit, on which we will focus later. Furthermore, ignoring the cosmological expansion and, therefore, a smaller volume at early times, should then correspond to maximizing the decay rate and computing an upper bound for a more realistic scenario.

In this case, the Bianchi constraint (3.9) enforces the lapse to be a constant; here we set $N_g = 1$. Furthermore, in order to excite the scalar BD ghost we consider the following scalar perturbations δg around the background $\bar{g} \equiv \eta$:

$$\mathrm{d}s^2_{\delta g} = 2\left[-\Psi \mathrm{d}t^2 + B_{,i}\mathrm{d}x^i\mathrm{d}t + \left(\Phi\delta_{ij} + E_{,ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right]. \tag{4.1}$$

The Einstein-Hilbert action at second order therefore reads

$$S^{(2)}_{\mathrm{EH}} = 4M_{\mathrm{P}}^2 \int \mathrm{d}^4x \left(\Phi_i^2 + 2\Phi_i\Psi_i - 2\Phi'\Delta E' - 3\Phi'^2 - 2B_i\Phi_i'\right), \tag{4.2}$$

where we have used the notation

$$X_iY_i \equiv \sum_j X_{,j}Y_{,j}\,, \qquad \Delta X \equiv \sum_j X_{,jj}. \tag{4.3}$$

We now expand the mass term of action (2.12) to second order and use

$$\sqrt{g^{-1}\eta} \simeq \sqrt{\bar{g}^{-1}\eta}\left[1 - \frac{1}{2}\bar{g}^{-1}(\delta g) + \frac{3}{8}\bar{g}^{-1}(\delta g)\,\bar{g}^{-1}(\delta g)\right] \tag{4.4}$$

$$= \mathbb{1} - \frac{1}{2}\eta\,(\delta g) + \frac{3}{8}\eta\,(\delta g)\,\eta\,(\delta g)\,, \tag{4.5}$$

to obtain

$$\begin{aligned}
&\left(1 - \bar{\alpha}_1\left[g^{-1}\eta\right]\right)\left[\sqrt{g^{-1}\eta}\right] \\
&\simeq \left[\mathbb{1} - \frac{1}{2}\eta\,(\delta g) + \frac{3}{8}\eta\,(\delta g)\,\eta\,(\delta g)\right] \\
&\quad - \bar{\alpha}_1\left([\mathbb{1}] - [(\delta g)\,\eta] + [\eta\,(\delta g)\,\eta\,(\delta g)]\right)\left([\mathbb{1}] - \left[\frac{1}{2}\eta\,(\delta g)\right] + \left[\frac{3}{8}\eta\,(\delta g)\,\eta\,(\delta g)\right]\right) &(4.6)\\
&\simeq 4\left(1 - 4\bar{\alpha}_1\right) - \left(\frac{1}{2} - 6\bar{\alpha}_1\right)[\eta\,(\delta g)] - \frac{1}{2}\bar{\alpha}_1\,[\eta\,(\delta g)]^2 + \left(\frac{3}{8} - \frac{11}{2}\bar{\alpha}_1\right)[\eta\,(\delta g)\,\eta\,(\delta g)] &(4.7)
\end{aligned}$$

[2]Since we are interested in computing only tree-level diagrams in order to discuss the quantum behavior of the theory, the elimination of all auxiliary fields in the action by using their EoM is equivalent to properly integrating out these fields.

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and

$$
\begin{aligned}
&(1-\bar{\alpha}_2\,[g^{-1}\eta])\left(\left[\sqrt{g^{-1}\eta}\right]^2-[g^{-1}\eta]\right)\\
&\quad\simeq(1-\bar{\alpha}_2\,(4-[(\delta g)\,\eta]+[\eta\,(\delta g)\,\eta\,(\delta g)]))\times\\
&\quad\times\left(\left[\mathbb{1}-\frac{1}{2}\eta\,(\delta g)+\frac{3}{8}\eta\,(\delta g)\,\eta\,(\delta g)\right]^2-4+[(\delta g)\,\eta]-[\eta\,(\delta g)\,\eta\,(\delta g)]\right) \qquad (4.8)\\
&\quad\simeq 12\,(1-4\bar{\alpha}_2)-3\,(1-8\bar{\alpha}_2)\,[(\delta g)\,\eta]+\left(\frac{1}{4}-4\bar{\alpha}_2\right)[\eta\,(\delta g)]^2\\
&\qquad+2\,(1-10\bar{\alpha}_2)\,[\eta\,(\delta g)\,\eta\,(\delta g)]\,. \qquad (4.9)
\end{aligned}
$$

Finally, with

$$
\begin{aligned}
\sqrt{-g}&\simeq\frac{1}{4}\sqrt{-\bar{g}}\left(4+2\,(\delta g)^{\mu}_{\ \mu}-(\delta g)_{\mu\nu}\,(\delta g)^{\mu\nu}+\frac{1}{2}\left((\delta g)^{\mu}_{\ \mu}\right)^2\right) \qquad (4.10)\\
&=\frac{1}{4}\left(4+2\,[\eta\,(\delta g)]-[\eta\,(\delta g)\,\eta\,(\delta g)]+\frac{1}{2}\,[\eta\,(\delta g)]^2\right) \qquad (4.11)
\end{aligned}
$$

we find the second order action of the mass term as

$$
S^{(2)}_{\rm mass}=M_{\rm P}^2m^2\int \mathrm{d}^4x\left(\frac{1}{4}+\bar{\alpha}_1-2\bar{\alpha}_2\right)[\eta\,(\delta g)]^2-\left(\frac{1}{4}+3\bar{\alpha}_1-8\bar{\alpha}_2\right)[\eta\,(\delta g)\,\eta\,(\delta g)]\,. \qquad (4.12)
$$

For the ansatz (4.1), this becomes

$$
S^{(2)}_{\rm mass}=\frac{1}{2}M_{\rm P}^2m^2\!\int\!\mathrm{d}^4x\Big[-c_1B\Delta B+c_2\left(\Psi^2+(\Delta E)^2\right)+8c_3\Phi\Delta E+12c_3\Phi^2+4c_4\Psi\,(\Delta E+3\Phi)\Big], \qquad (4.13)
$$

where we have defined the parameters c_i as

$$
\begin{aligned}
c_1&\equiv 1+12\bar{\alpha}_1-32\bar{\alpha}_2, &(4.14)\\
c_2&\equiv -16\bar{\alpha}_1+12\bar{\alpha}_2, &(4.15)\\
c_3&\equiv 1+4\bar{\alpha}_2, &(4.16)\\
c_4&\equiv 1+4\bar{\alpha}_1-8\bar{\alpha}_2. &(4.17)
\end{aligned}
$$

Therefore, the (minimal) ghost-free massive gravity corresponds to the limit $c_1, c_3, c_4 \to 1$ and $c_2 \to 0$. Note that we should not necessarily expect a smooth limit due to the change in the number of degrees of freedom in HMG compared to dRGT.

4.2 Full action including matter

For simplicity, we assume the matter sector to contain only a single, minimally-coupled, scalar field φ with mass m_φ, i.e.,

$$
S_{\rm matter}=-\int \mathrm{d}^4x\sqrt{-g}\left(\partial^\mu\varphi\partial_\mu\varphi+m_\varphi^2\varphi^2\right), \qquad (4.18)
$$

and consider the weak-field limit where only the gravitational sector is expanded to second order and matter is kept unperturbed. The kinetic and mass terms of φ are then given by its coupling to the background metric.

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By combining the actions (4.2), (4.13), and (4.18), we obtain the final leading order action of HMG, containing a Boulware-Deser ghost and a matter field, around Minkowski (modulo total derivatives),

$$
\begin{aligned}
S^{(2)}_{\rm HMG} = \int {\rm d}^4x \Big[& 4M_{\rm P}^2\left(\Phi_i^2 - 2\Delta\Phi\Psi - 2\Phi'\Delta E' - 3\Phi'^2 - 2B_i\Phi_i'\right) \\
& + \frac{1}{2}m^2M_{\rm P}^2\left(c_1B_i^2 + c_2\left(\Psi^2 + (\Delta E)^2\right) + 8c_3\Phi\Delta E + 12c_3\Phi^2 + 4c_4\Psi\left(\Delta E + 3\Phi\right)\right) \\
& - \left(1 + B_i^2 - (\Delta E)^2 + 3\Phi^2 + 6\Phi\Psi - \Psi^2 + 2\Delta E\left(\Phi + \Psi\right)\right)X_\varphi\Big] ,
\end{aligned} \tag{4.19}
$$

where we have defined

$$
X_\varphi \equiv -\varphi'^2 + \varphi_i^2 + m_\varphi^2\varphi^2. \tag{4.20}
$$

In total, all the scalar potentials Φ, Ψ, B, E, and the matter field φ should describe at most three propagating scalar degrees of freedom: one helicity-0 mode of the graviton, one scalar field from the matter sector, and one additional Boulware-Deser ghost. All these scalar degrees of freedom are, however, not always excited around all backgrounds. Since we are interested in the interaction of the ghost with the matter field, we need to ensure that the BD ghost is indeed a propagating mode in eq. (4.19). To see this, we first integrate out all the auxiliary fields by using their EoM

$$
\frac{\partial\mathcal{L}}{\partial X} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial X'}\right) - \partial_i\left(\frac{\partial\mathcal{L}}{\partial X_i}\right) + \partial_t^2\left(\frac{\partial\mathcal{L}}{\partial X''}\right) + \partial_i^2\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_i^2X\right)}\right) = 0. \tag{4.21}
$$

For $X \in \{\Psi,\, B,\, \Delta E\}$ this leads to

$$
\Psi = \frac{8M_{\rm P}^2\Delta\Phi - (\Delta E + 3\Phi)\left(2c_4m^2M_{\rm P}^2 + X_\varphi\right)}{c_2m^2M_{\rm P}^2 - X_\varphi}, \tag{4.22}
$$

$$
B_i = \frac{8M_{\rm P}^2\Phi_i'}{c_1m^2M_{\rm P}^2 + X_\varphi}, \tag{4.23}
$$

$$
\Delta E = -\frac{\Phi\left(4c_3m^2M_{\rm P}^2 + X_\varphi\right) + \Psi\left(2c_4m^2M_{\rm P}^2 + X_\varphi\right) + 8M_{\rm P}^2\Phi''}{c_2m^2M_{\rm P}^2 - X_\varphi}. \tag{4.24}
$$

Solving this set of equations for Ψ, B, and ΔE as functions of Φ, X_φ, and their derivatives, yields

$$
\Psi = \frac{8M_{\rm P}^2\Delta\Phi\left(c_2m^2M_{\rm P}^2 - X_\varphi\right) - \left(2c_4m^2M_{\rm P}^2 + X_\varphi\right)\left[\Phi\left(\left(3c_2 - 4c_3\right)m^2M_{\rm P}^2 - 4X_\varphi\right) - 8M_{\rm P}^2\Phi''\right]}{\left(c_2^2 - 4c_4^2\right)m^4M_{\rm P}^4 - 2\left(c_2 + 2c_4\right)m^2M_{\rm P}^2X_\varphi}, \tag{4.25}
$$

$$
B_i = \frac{8M_{\rm P}^2\Phi_i'}{c_1m^2M_{\rm P}^2 + X_\varphi}, \tag{4.26}
$$

$$
\Delta E = \frac{8\Delta\Phi\left(2c_4m^2M_{\rm P}^2 + X_\varphi\right) + \Phi\left[4\left(c_2c_3 - 3c_4^2\right)m^4M_{\rm P}^4 + \left(c_2 - 4c_3 - 12c_4\right)m^2M_{\rm P}^2X_\varphi - 4X_\varphi^2\right] + 8M_{\rm P}^2\Phi''\left(c_2m^2M_{\rm P}^2 - X_\varphi\right)}{-\left(c_2^2 - 4c_4^2\right)m^4M_{\rm P}^4 + 2\left(c_2 + 2c_4\right)m^2M_{\rm P}^2X_\varphi}. \tag{4.27}
$$

Note that the determinant of the mixing matrix for Ψ and ΔE in eqs. (4.22) and (4.24) is proportional to the factors m^2 and $c_2 + 2c_4 = 2 - 8\bar{\alpha}_1 + 32\bar{\alpha}_2$. If one of these terms vanishes, the mixing matrix becomes singular, which is equivalent to it having a zero

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eigenvalue. These eigenvalues correspond to the kinetic terms of the diagonalized degrees of freedom (given by the eigenvectors). Therefore, the singular situation corresponds to a combination of the auxiliary fields losing its kinetic term, and leads to a strong coupling and a breakdown of perturbativity.

Finally, the full second order HMG action

$$S^{(2)}_{\mathrm{HMG}} = S^{(2)}_{\mathrm{EH}} + S^{(2)}_{\mathrm{mass}} + S_{\mathrm{matter}} \tag{4.28}$$

can be written as

$$S^{(2)}_{\mathrm{HMG}}\left[\Phi,\varphi\right] = S^{(2)}_{\mathrm{m}}\left[\Phi,\varphi\right] + S^{(2)}_{\mathrm{kin}}\left[\Phi,\varphi\right] + S^{(2)}_{\mathrm{int}}\left[\Phi,\varphi\right]. \tag{4.29}$$

The action now depends only on two remaining interacting massive scalar fields φ and Φ described by the mass term

$$S^{(2)}_{\mathrm{m}} = -\int \mathrm{d}^4x \left(-6c_3 m^2 M_{\mathrm{P}}^2 \Phi^2 + m_\varphi^2 \varphi^2\right), \tag{4.30}$$

and rather complicated kinetic and interaction terms, $S^{(2)}_{\mathrm{kin}}$ and $S^{(2)}_{\mathrm{int}}$, respectively. Because the action (4.19) contains a term that is proportional to $\Phi'\Delta E'$, we find, after integrating out ΔE, terms that include $(\Phi'')^2$. The occurrence of fourth-order derivatives in Φ signals that the theory is inevitably plagued by an Ostrogradsky ghost. In total, we expect a composition of three scalar degrees of freedom consisting of a helicity-0 mode from the graviton, a matter field, and a ghost. In order to analyze the interactions between the ghost and the other degrees of freedom, we need to decouple all of them.

4.3 Decoupling of the ghost

4.3.1 Decoupling in vacuum

Before analyzing the UV limit, i.e., $X_\varphi \gg m^2 M_{\mathrm{P}}^2$, we study the simpler case first in which the matter field is absent, i.e., $X_\varphi = 0$. The action can then be written as

$$S^{(2)}_{\mathrm{HMG}} = \int \mathrm{d}^4x \left[C_1 \left(\Delta\Phi\right)^2 + C_2 \Phi\Delta\Phi + C_3 \Phi''\Delta\Phi + C_4 \Phi\Phi'' + C_5 \left(\Phi''\right)^2 + C_6 \Phi^2\right], \tag{4.31}$$

where

$$C_1 \equiv -\frac{32 c_2 M_{\mathrm{P}}^2}{m^2\left(c_2^2 - 4c_4^2\right)}, \qquad C_2 \equiv -\frac{4M_{\mathrm{P}}^2\left(c_2^2 - 12c_2c_4 + 4c_4(4c_3 - c_4)\right)}{c_2^2 - 4c_4^2}, \tag{4.32}$$

$$C_3 \equiv -\frac{32M_{\mathrm{P}}^2\left(4c_4(c_1 - c_4) + c_2^2\right)}{c_1 m^2\left(c_2^2 - 4c_4^2\right)}, \quad C_4 \equiv \frac{4M_{\mathrm{P}}^2\left(3c_2^2 - 8c_2c_3 + 12c_4^2\right)}{c_2^2 - 4c_4^2}, \tag{4.33}$$

$$C_5 \equiv -\frac{32 c_2 M_{\mathrm{P}}^2}{m^2\left(c_2^2 - 4c_4^2\right)}, \qquad C_6 \equiv \frac{2m^2 M_{\mathrm{P}}^2 (3c_2 - 4c_3)\left(c_2c_3 - 3c_4^2\right)}{c_2^2 - 4c_4^2}. \tag{4.34}$$

In order to make the additional scalar degree of freedom manifest we can try to find an equivalent action that descibes two fields with at most second derivatives instead of one field having fourth order derivatives. A special case where the interaction term is just $(\Box\Phi)^2$ has already been presented in ref. [18].

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For this, we introduce an auxiliary field χ together with seven unknown constants D_i, and consider a general action of two scalars Φ and χ that contains at most second-order derivatives,

$$S' = \int \mathrm{d}^4x \left[D_1\Phi\Phi'' + D_2\Phi\Delta\Phi + D_3\chi\Phi'' + D_4\chi\Delta\Phi + D_5\chi^2 + D_6\chi\Phi + D_7\Phi^2\right]. \tag{4.35}$$

The coupling to χ is constructed such that this auxiliary field can easily be integrated out by using its equation of motion,

$$\chi = -\frac{1}{2D_5}\left(D_6\Phi + D_3\Phi'' + D_4\Delta\Phi\right). \tag{4.36}$$

We would then obtain an action that looks similar to the action (4.31) with which we started, except for the coefficients that will now depend on the constants D_i. Since we are interested in finding an equivalent action with two degrees of freedom, we equate these coefficients and solve for the unknown constants D_i. Interestingly, a solution does exist only if

$$C_1 = \frac{C_3^2}{4C_5} \quad \Leftrightarrow \quad C_1\left(\Delta\Phi\right)^2 + C_3\Phi''\Delta\Phi + C_5\left(\Phi''\right)^2 = \left(\sqrt{C}_1\Delta\Phi + \sqrt{C}_5\Phi''\right)^2, \tag{4.37}$$

which, as one can easily check, is also satisfied for HMG.[3] After fixing the redundancy due to a free rescaling of the actions by choosing $D_5 = -m^2M_\mathrm{P}^2$,[4] we find

$$D_1 = C_4 \pm \sqrt{C}_5 D_6, \qquad D_2 = C_2 \pm \frac{C_3D_6}{2\sqrt{C_5}}, \qquad D_3 = \mp 2\sqrt{C}_5, \tag{4.38}$$

$$D_4 = \mp\frac{C_3}{\sqrt{C_5}}, \qquad D_7 = C_6 - \frac{1}{4}D_6^2. \tag{4.39}$$

We can now introduce two scalar fields π and $\Phi_{\text{👻}}$ (to be pronounced "phi spectre"), described by a superposition of Φ and χ, which can finally be decoupled with the transformations

$$\Phi \longrightarrow A_1\pi - A_2\Phi_{\text{👻}} \quad \text{and} \quad \chi \longrightarrow \Phi_{\text{👻}}, \tag{4.40}$$

where A_1 and A_2 are free coefficients. They can be used to diagonalize the mass terms,

$$S'_m = \int \mathrm{d}^4x \left[D_5\Phi_{\text{👻}}^2 + D_6\Phi_{\text{👻}}\left(A_1\pi - A_2\Phi_{\text{👻}}\right) + D_7\left(A_1\pi - A_2\Phi_{\text{👻}}\right)^2\right], \tag{4.41}$$

to obtain the physical degrees of freedom which can be achieved by setting

$$A_1D_6 = 2A_1A_2D_7. \tag{4.42}$$

[3]This condition enforces the theory to be covariant. Since we have started with a covariant theory and then performed a time-space splitting, it is indeed expected that this condition is satisfied. For theories that violate this constraint due to terms that break covariance, this does, however, not imply that there is no ghost but rather that our ansatz is not sufficient. This could indicate that the theory does not propagate only one but more degrees of freedom.

[4]This choice does also ensure real coefficients for parameter values that we will focus on later. Otherwise, if $c_2^2 - 4c_4^2$ is negative, the coefficient D_5 should be positive such that $\sqrt{C}_5$ is real.

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For the choice $A_1 = 1$ and $D_6 = m^2 M_\mathrm{P}^2$, and using the solution corresponding to the upper signs in eqs. (4.38) and (4.39), we see that if $c_2^2 - 4c_4^2 > 0$ (the parameter region that we are interested in) then the prefactor in the kinetic terms for $\Phi_{\text{👻}}$ is negative and, therefore, describes a (BD) ghost, whereas the one for π is positive and, thus, corresponds to the healthy helicity-0 mode. From eq. (4.41) we can read off the diagonalized mass terms for both scalars and find

$$m^2_{\Phi_{\text{👻}}} M_\mathrm{P}^2 = \frac{4D_5 D_7 - D_6^2}{4D_7} = \frac{-4m^2 M_\mathrm{P}^2 D_7 - m^4 M_\mathrm{P}^4}{4D_7}, \tag{4.43}$$

$$m_\pi^2 M_\mathrm{P}^2 = D_7. \tag{4.44}$$

Since D_7 is positive if eq. (3.15) is satisfied, this indicates that the ghost is indeed a tachyon.

4.3.2 Decoupling in the presence of matter

So far, we have been able to decouple the ghost and the helicity-0 in the absence of an additional matter field. Due to integrating out all auxiliary fields in the full action (4.19), the coupling between matter and both the ghost and the helicity-0 mode is, however, not trivial and requires a proper decoupling of all present degrees of freedom. Fortunately, the procedure is conceptually similar to what has been done in the vacuum case. Furthermore, we can simplify the calculations by considering the small scale limit $X_\varphi \gg m^2 M_\mathrm{P}^2$. Because the analysis nevertheless becomes a bit lenghtier, we present some itermediate steps in appendix A.

In the presence of a matter field φ and assuming small scales, the action can be decomposed as

$$\begin{aligned} S^{(2)}_\mathrm{HMG} = \int \mathrm{d}^4x \Big[&(\Phi'')^2 \left(C_1\Phi^2\varphi^2 + C_2\Phi\varphi + C_3\right) + (\varphi'')^2\left(C_4\Phi^2\varphi^2 + C_5\Phi\varphi + C_6\right) \\ &+ (\Delta\Phi)^2\left(C_7\Phi^2\varphi^2 + C_8\Phi\varphi + C_9\right) + (\Delta\varphi)^2\left(C_{10}\Phi^2\varphi^2 + C_{11}\Phi\varphi + C_{12}\right) + C_{13}\Delta\Phi\Delta\varphi\Phi\varphi \\ &+ \Phi''\big(C_{14}\Delta\varphi\Phi\varphi + C_{15}\varphi''\Phi\varphi + C_{16}\Delta\Phi\varphi^2\Phi^2 + C_{17}\Delta\Phi\varphi\Phi \\ &\qquad + C_{18}\Delta\Phi + C_{19}\Delta\varphi\varphi^2\Phi^2 + C_{20}\Delta\Phi\varphi\Phi^2\big) \\ &+ \Phi''\left(C_{21}\Delta\varphi\varphi^2\Phi^2 + C_{22}\Delta\varphi + C_{23}\varphi''\varphi^2\Phi^2 + C_{24}\varphi''\right) \\ &+ \Delta\Phi\left(C_{25}\varphi''\varphi^2\Phi^2 + C_{26}\varphi'' + C_{27}\Delta\varphi\varphi^2\Phi^2 + C_{28}\Delta\varphi\right) + C_{29}\Delta\varphi\varphi''\Phi\varphi + C_{30}\Delta\varphi\varphi'' \\ &+ \Phi''\left(C_{31}\Phi\varphi^2 + C_{32}\Phi\right) + \varphi''\left(C_{33}\Phi^2\varphi^3 + C_{34}\Phi^2\varphi + C_{35}\varphi\right) + \varphi\Phi\left(C_{36}\Phi\varphi^2 + C_{37}\Phi\right) \\ &+ \Delta\varphi\left(C_{38}\Phi^2\varphi^3 + C_{39}\Phi^2\varphi + C_{40}\varphi\right) + C_{41}\Phi^2\varphi^4 + C_{42}\Phi^2\varphi^2 + C_{43}\varphi^2 + C_{44}\Phi^2\Big]. \end{aligned} \tag{4.45}$$

Note that for HMG, some of the constants C_i do indeed vanish. All of them are explicitly listed in eq. (A.1). Again, we start with an ansatz for an action that explicitly describes three degrees of freedom with at most second-order derivatives,

$$\begin{aligned} S'^{(2)}_\mathrm{HMG} = \int \mathrm{d}^4x \Big[&\Phi''\left(D_1\Phi\varphi^2 + D_2\Phi\right) + \varphi''\left(D_3\varphi^3\Phi^2 + D_4\varphi\Phi^2 + D_5\varphi\right) \\ &+ \Delta\Phi\left(D_6\Phi\varphi^2 + D_7\Phi\right) + \Delta\varphi\left(D_8\varphi^3\Phi^2 + D_9\varphi\Phi^2 + D_{10}\varphi\right) \\ &+ D_{11}\Phi^2\varphi^4 + D_{12}\Phi^2\varphi^2 + D_{13}\varphi^2 + D_{14}\Phi^2 \\ &+ \chi\left(D_{15}\Phi'' + D_{16}\Phi''\varphi\Phi + D_{17}\Delta\Phi + D_{18}\Delta\Phi\varphi\Phi + D_{19}\varphi''\right. \\ &\left. + D_{20}\varphi''\varphi\Phi + D_{21}\Delta\varphi + D_{22}\Delta\varphi\varphi\Phi\right) + D_{23}\chi^2\Big]. \end{aligned} \tag{4.46}$$

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After solving for all coefficients D_i, applying the field transformations

$$\Phi \longrightarrow A_1\pi + \Phi_{\text{ghost}} + M_{\mathrm{P}}^{-1}\xi, \tag{4.47}$$

$$\varphi \longrightarrow M_{\mathrm{P}}\pi - M_{\mathrm{P}}\Phi_{\text{ghost}} + A_2\xi, \tag{4.48}$$

$$\chi \longrightarrow A_3\pi - \Phi_{\text{ghost}} + M_{\mathrm{P}}^{-1}\xi, \tag{4.49}$$

and setting $D_{23} = -m^2 M_{\mathrm{P}}^2$, we find by checking the relative signs of all kinetic terms that Φ_{ghost} is the ghost mode with the mass

$$m^2_{\Phi_{\text{ghost}}} M_{\mathrm{P}}^2 = C_{44} - m_\varphi^2 M_{\mathrm{P}}^2 - m^2 M_{\mathrm{P}}^2, \tag{4.50}$$

and π and ξ describe the helicity-0 mode and the matter field, respectively, with masses

$$m_\pi^2 M_{\mathrm{P}}^2 = C_{43} M_{\mathrm{P}}^2 + \frac{\left(M_{\mathrm{P}}^2\left(m^2 - C_{43}\right) + C_{44}\right)^2}{4C_{44}} - \frac{\left(M_{\mathrm{P}}^2\left(C_{43} + m^2\right) + C_{44}\right)^2}{4m^2 M_{\mathrm{P}}^2} \tag{4.51}$$

$$= m^2_{\Phi_{\text{ghost}}} M_{\mathrm{P}}^2 \left(-1 + \frac{1}{4} m^2_{\Phi_{\text{ghost}}}\left(C_{44}^{-1} M_{\mathrm{P}}^2 - m^{-2}\right)\right), \tag{4.52}$$

$$m_\xi^2 = M_{\mathrm{P}}^{-2}\left(C_{44} - m^2 M_{\mathrm{P}}^2\right) + \frac{1}{C_{43} M_{\mathrm{P}}^4}\left(C_{44} + m^2 M_{\mathrm{P}}^2\right)^2 \tag{4.53}$$

$$= \frac{m^2_{\Phi_{\text{ghost}}} m^2 M_{\mathrm{P}}^2 - C_{44}\left(m^2_{\Phi_{\text{ghost}}} + 4m^2\right)}{C_{44} - \left(m^2_{\Phi_{\text{ghost}}} + m^2\right) M_{\mathrm{P}}^2}. \tag{4.54}$$

We observe that all masses are mainly determined by the coefficient C_{44}. In our favored parameter region that satisfies eq. (3.15) and $\alpha_1 = \mathcal{O}(1)$ we find $C_{44} \gg m^2 M_{\mathrm{P}}^2$. Hence, if m_φ^2 is small (which we will also assume later for the analysis of the vacuum decay) then eq. (4.50) indicates a positive $m^2_{\Phi_{\text{ghost}}}$ but tachyonic scalars π and ξ. This is not surprising as we have already seen the existence of a tachyon in the vacuum case. With an additional coupling to a new, even non-tachyonic, scalar field the tachyonic instability can leak into all other mass terms. However, this does not render the theory more dangerous and rather tells us that the decay processes in our theory of massive gravity with an additional scalar field can be described by the equivalent setting of one ghost and two tachyonic fields.

4.4 Strong coupling scale of the theory

The constraint that removes the BD ghost in a non-linear theory of a massive graviton automatically removes all interactions that are suppressed by scales $\Lambda < \Lambda_3$ with

$$\Lambda_\lambda \equiv \left(M_{\mathrm{P}} m^{\lambda-1}\right)^{1/\lambda}. \tag{4.55}$$

All other non-linear theories that reduce to FP at the linear level contain terms suppressed by Λ_5. However, this does not necessarily hold for theories that do not reduce to the FP theory at the linear level.

We can find the cutoff scale of HMG by using the expansion of the mass term (4.12) and introducing the Stückelberg fields

$$\delta g_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu, \tag{4.56}$$

and, subsequently,

$$A_\mu \longrightarrow A_\mu + \partial_\mu \phi. \tag{4.57}$$

This decomposition into the three helicity modes allows us to read off the energy scales with which all single interactions are suppressed (see, e.g., refs. [29, 30]). For this we need to canonically normalize all the modes through the rescaling

$$h_{\mu\nu} \longrightarrow \frac{2}{M_{\mathrm{P}}} h_{\mu\nu}, \tag{4.58}$$

$$A_\mu \longrightarrow \frac{2}{m M_{\mathrm{P}}} A_\mu, \tag{4.59}$$

$$\phi \longrightarrow \frac{2}{m^2 M_{\mathrm{P}}} \phi. \tag{4.60}$$

One now finds that the interactions in HMG that are suppressed by the smallest scale are of the type

$$\propto \frac{\bar{\alpha}_i}{M_{\mathrm{P}} m^4} \left(\partial\partial\phi\right)^3 = \frac{\bar{\alpha}_i}{\Lambda_5^5} \left(\partial\partial\phi\right)^3, \tag{4.61}$$

and correspond to the cutoff scale Λ_5.

5 Quantum instability

5.1 Most dominant interaction terms

The quantum stability depends on the scattering between the ghost $\Phi_{\ghost}$ and the matter field ξ. In order to compute the scattering amplitude we move to Fourier space and introduce $k_{\Phi_{\ghost}}$ and k_ξ for the momenta of $\Phi_{\ghost}$ and ξ, respectively. The final action of the interaction between these two fields is rather complicated. In general, the interaction terms contain derivatives of both fields that describe the so-called derivative interactions, which, thus, have momentum-dependent vertices. It is exactly this type of interaction that might be dangerous since the scattering amplitude requires an integration over the entire phase space of the initial and final states of the fields. Therefore, all derivative interactions lead to UV divergent terms $\propto k^\alpha$ with $\alpha \in \mathbb{R}^+$. Even though such derivative interactions exist in the Standard Model (SM), this problem is usually solved by introducing counter terms which regularize the divergent parts. In our case, we require a Lorentz violation to cut the integral over the phase space.

If the integral of the phase space is cut at some energy level due to some new Lorentz breaking operators, then the decay rate might not necessarily be dominated by the UV behavior anymore. As seen in section 4.4, the cutoff of the EFT is much below the Planck scale. Depending on the mass of the graviton, terms with a lower number of derivatives could then become dominant. At the end of this section we will, however, find that these types of interactions are indeed less important.

From now on, we will need to only focus on the interactions with the highest number of derivatives where we are allowed to assume that both $k_{\Phi_{\ghost}}$ and k_ξ are of the same order since the cutoff scales above which Lorentz invariance is broken are equivalent for both momenta. Even though one can directly see from the action (4.46) that there are many different types of derivative interations, most of them are suppressed by powers of M_{P}^{-1} or m_{φ}. We find the Lagrangian corresponding to the most dangerous process to be

$$\mathcal{L}^{\mathrm{dom}} \simeq \frac{\left(c_2^2+4c_2(4c_3+3c_4+1)-8\left(2c_3^2+2c_3c_4+c_4(2c_4-1)\right)\right)^4 m^6}{32m_{\varphi}^6 M_{\mathrm{P}}^2(c_2+2c_4)^5}\, \Phi_{\text{👻}}^2\, \xi^3\, \partial_\mu \partial^\mu \xi. \tag{5.1}$$

For the analysis of the vacuum decay it will be useful to apply the transformation $\Phi_{\text{👻}} \longrightarrow M_{\mathrm{P}}^{-1}\Phi_{\text{👻}}$ to obtain the same dimensions for both $\Phi_{\text{👻}}$ and ξ. Finally, the interaction in Fourier-space becomes

$$\mathcal{L}^{\mathrm{dom}} \simeq \frac{\left(c_2^2+4c_2(4c_3+3c_4+1)-8\left(2c_3^2+2c_3c_4+c_4(2c_4-1)\right)\right)^4 m^6}{32m_{\varphi}^6 M_{\mathrm{P}}^4(c_2+2c_4)^5} k_\xi^2\, \Phi_{\text{👻}}^2\, \xi^4. \tag{5.2}$$

Note that this interaction arises from a matter sector, which, as in many other theories of modified gravity, couples minimally to gravity. Thus, this type of derivative interaction is not only a property of HMG but rather occurs in a much broader class of theories, even beyond massive gravity. Since we are studying the Lagrangian on-shell, the exact term describing the most dominant interaction is, of course, still model-dependent. Especially the occurrence of derivatives in the potential term of the theory might lead to different results. However, we expect that the qualitative results for HMG will still be valid for a huge class of theories of modified gravity that introduce a ghost and have a matter sector minimally coupled to gravity.

5.2 Ghost decay

The total decay rate of the ghost particle is the sum of the decay rates from all possible decay channels.[5] As already discussed, the dominant contribution to the total decay rate comes from the process shown in figure 2 (left) and, therefore, the rate can be very well approximated by

$$\Gamma_{\Phi_{\text{👻}}} = \frac{1}{2m_{\Phi_{\text{👻}}}} \int \prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3\, 2E_f}\, |\mathcal{M}|^2\, (2\pi)^4\, \delta^{(4)}\left(p_{\Phi_{\text{👻}}} - \sum_f p_f\right). \tag{5.3}$$

Here, $\mathcal{M}$ is the scattering amplitude, $m_{\Phi_{\text{👻}}}$ and $p_{\Phi_{\text{👻}}}$ are the mass and four-momentum of the ghost particle, respectively,[6] and E_f is the energy of a particle appearing in a final decay product. The dominance of high momenta in the decay rate further justifies the high energy limit leading to eq. (5.1).

It is important to note that we have different dispersion relations for a ghost and a standard field. While for a ghost we have $E_{\Phi_{\text{👻}}} = -\sqrt{m_{\Phi_{\text{👻}}}^2 + \vec{p}_{\Phi_{\text{👻}}}^{\,2}}$, the dispersion relation for standard matter fields is $E_{\mathrm{sm}} = \sqrt{m_{\mathrm{sm}}^2 + \vec{p}_{\mathrm{sm}}^{\,2}}$. Here $\vec{p}_{\Phi_{\text{👻}}}$ and $\vec{p}_{\mathrm{sm}}$ are the spatial momenta for the ghost and matter fields, respectively.

[5] Since the ghost is a boson, due to spin-statistics its production rate will be enhanced by a factor $1 + n_{\vec{p}}$ depending on the occupation number of the final state. However, we are interested in the case in which the phase-space density of ghosts is negligible.

[6] Even though a (non-tachyonic) ghost is usually recognized as a field with negative kinetic energy, its mass term does also carry an additional minus sign [11].

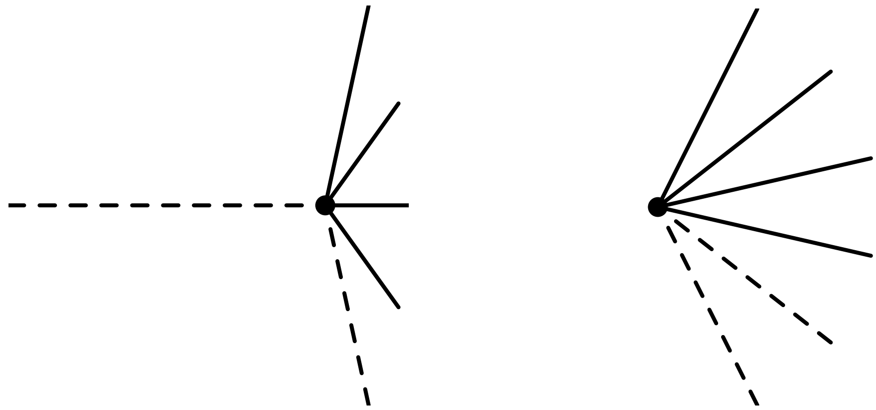

Figure 2. Left: Feynman diagram for the most dominant decay of a Boulware-Deser ghost particle (left dashed leg) into another ghost and four minimally coupled matter particles (solid legs). Right: Feynman diagram for the most dominant vacuum decay into ghost and matter paricles.

Since eq. (5.3) contains an integral over the entire phase space of all decay products, the decay rate is usually expected to be infinitely large. As mentioned earlier, a LB allows us to cut the integral at Λ_{LB}, which is, in fact, the energy scale that determines the decay time.

We now need to find the $\mathcal{M}$ matrix that corresponds to the derivative interaction between $\Phi_{\clubsuit}$ and ξ as described by eq. (5.1). The derivatives yield two powers from the vertex of the interaction. Furthermore, we multiply the vertex by a factor of 3! as we can freely swap all lines that correspond to ξ. Thus, the scattering amplitude from the Feynman diagram shown in figure 2 (left) becomes

$$\mathcal{M} = 3!A\,(ip_3)\,(ip_3) = -3!A\eta^{\mu\nu}\,(p_3)_\mu\,(p_3)_\nu\,, \tag{5.4}$$

where we have introduced

$$A \equiv \frac{\left(c_2^2 + 4c_2(4c_3 + 3c_4 + 1) - 8\left(2c_3^2 + 2c_3c_4 + c_4(2c_4 - 1)\right)\right)^4 m^6}{32 m_\varphi^6 M_{\mathrm{P}}^4 (c_2 + 2c_4)^5}. \tag{5.5}$$

In order to find an upper bound on the decay rate or, equivalently, a lower bound on the decay time, we consider the worst-case scenario in which the matter field is almost massless. Even though this will generally lead to higher decay rates, it is still a good approximation as the decay will be dominant at energies near the LI-violating cutoff scale. Assuming isotropy in the decay process, i.e., $\mathrm{d}^3p_f = 4\pi p_f^2 \mathrm{d}p_f$, fixing the angles between different vectors, and using the momentum conservation, we finally obtain the differential decay rate,

$$\mathrm{d}\Gamma_{\Phi_{\clubsuit}} \simeq -\frac{18A^2 p_2p_3p_4p_5p_6 m_\xi^4 (2\pi)^4 \delta^{(4)}\left(p_{\Phi_{\clubsuit}} - \sum_{f=1}^5 p_f\right)}{(2\pi)^{10}\, m_{\Phi_{\clubsuit}}}. \tag{5.6}$$

We are now able to perform the phase-space integral in eq. (5.3) up to the cutoff scale Λ_{LB}, at which Lorentz breaking occurs, and obtain

$$\Gamma_{\Phi_{\clubsuit}} \simeq \frac{3A^2 m_\xi^4 \Lambda_{\mathrm{LB}}^6}{2\,(2\pi)^{10}\, m_{\Phi_{\clubsuit}}} + \mathcal{O}\left(\Lambda_{\mathrm{LB}}^5\right). \tag{5.7}$$

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Note that A contains the scale with which the tree-level interaction term (5.2) is suppressed. If one would consider contributions from loops then their vertices that are suppressed by Λ_5 might lower the scale with which the decay rate is suppressed down to Λ_5.[7]

As mentioned before, the decay rate (5.7) corresponds only to the scattering process that dominates in the UV. The validity of this assumption is not obvious for low cutoff scales. From eqs. (4.25) and (4.26) we find that interactions with less derivatives of φ introduce additional factors of $m^2 M_{\mathrm{P}}^2$ in A. From a power counting we find that the corresponding decay rate $\tilde{\Gamma}_{\Phi_{\text{ghost}}}$ behaves like

$$\tilde{\Gamma}_{\Phi_{\text{ghost}}} \propto m^4 M_{\mathrm{P}}^4 \Lambda_{\mathrm{LB}}^{-8} \Gamma_{\Phi_{\text{ghost}}}. \tag{5.8}$$

Therefore, for all cutoff scales that satisfy $\Lambda_{\mathrm{LB}} \gtrsim \sqrt{m M_{\mathrm{P}}} \simeq 10^{-2}\,\mathrm{eV}$ (for $m = \mathcal{O}(H_0)$) we do not expect higher decay rates. As we will see in the next section, this condition is always satisfied for cutoff scales $\Lambda_{\mathrm{LB}}^{(\mathrm{max})}$ with which the decay would happen on a timescale of the Hubble time.

5.3 Vacuum decay

Besides the decay of a ghost, the vacuum itself can also decay into two ghosts and additional matter particles. The Feynman diagram for the most dominant vacuum decay is shown in figure 2 (right). The main contribution to the decay rate of the vacuum comes from the same vertex that we found for the ghost decay and, thus, we find

$$\Gamma_{\mathrm{vac}} = \int \prod_{f_{\Phi_{\text{ghost}}}} \frac{\mathrm{d}^3 p_{f_{\Phi_{\text{ghost}}}}}{(2\pi)^3\, 2E_{f_{\Phi_{\text{ghost}}}}} \prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3\, 2E_f} \left|\mathcal{M}\right|^2 (2\pi)^4\, \delta^{(4)}\left(\sum_{f_{\Phi_{\text{ghost}}}} p_{f_{\Phi_{\text{ghost}}}} + \sum_f p_f\right). \tag{5.9}$$

After choosing the rest-frame of the ghost particle f_1, the $\mathcal{M}$ matrix is, up to a symmetry factor 2!, similar to eq. (5.4) and, thus, eq. (5.9) reduces to

$$\Gamma_{\mathrm{vac}} = 2\Gamma_{\Phi_{\text{ghost}}}. \tag{5.10}$$

The total decay rate of the interaction described by eq. (5.2) is, however, not simply the sum of all decay rates Γ_i. The vacuum $|0\rangle$ is defined as the state without any excitations, which is not a stable state if the theory contains ghost fields. The particle production rate from the vacuum decay is, therefore, only a good approximation for an initial vacuum state and might become less trustable as the vacuum decays. For this reason and since the decay of the vacuum is not more dangerous than the decay of the ghost, we now focus on the ghost decay only.

5.4 Numerical calculations

The decay rate possesses some model dependencies. A priori the graviton mass scale m is a free parameter. However, if HMG is to be regarded as a theory of modified gravity that is supposed to solve the dark energy problem by providing self-accelerating solutions, then

[7] We thank Claudia de Rham for discussions on this aspect.

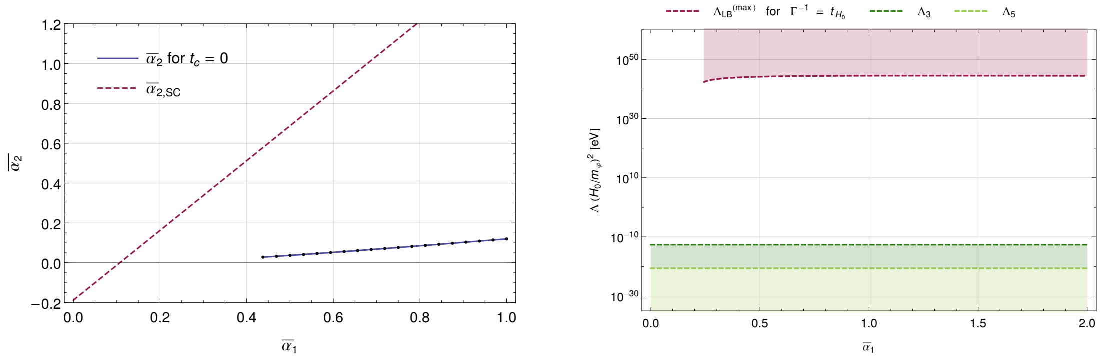

Figure 3. Left: constraints on the parameter space of HMG. The blue solid line indicates the region in which eq. (3.15) is satisfied, i.e., the timescale of the instability at the classical level to develop is maximized. Models on the red dashed line satisfy $c_2 + 2c_4 = 0$, indicating the strong coupling regime. Right: numerical results for the upper bound on the Lorentz-breaking cutoff scale, $\Lambda_{\mathrm{LB}}^{(\mathrm{max})}$ (red dashed line) corresponding to a decay time of the order of the Hubble time H_0^{-1}. As indicated by the lowest and second-lowest dashed lines, denoting Λ_5 and Λ_3, respectively, the LB cutoff scale can be much larger than the strong coupling scale of the EFT.

m should not be chosen arbitrarily. The mass parameter m determines the scale at which modifications to GR become important and is therefore expected to be $\sim H_0$.

There is however an additional model dependency. By using the fit (3.15), we get

$$A \propto 1/\left(c_2 + 2c_4\right)^5 \simeq \left(\frac{45}{26 - 120\bar{\alpha}_1}\right)^5 , \tag{5.11}$$

which, by tuning $\bar{\alpha}_1$, might diverge, leading to an infinite decay rate. As discussed previously, this limit corresponds to a strong coupling of the matter and ghost mode, and thus, the perturbative approach breaks down.

Since the classical background should also be unstable if the vacuum decays at tree level, we do not expect to find stable classical backgrounds for $\bar{\alpha}_1 \simeq 13/60$. For a cross-check, we determine the parameter region that maximizes the timescale of the classical instability to develop. As shown in figure 3 (left panel), we find that the results of the background analysis indeed agree with this constraint.

The viability of the theory depends on the decay time of the most dominant scattering process, and requires $\Gamma^{-1} \gtrsim H_0^{-1}$, which sets an upper bound on the scale Λ_{LB}. For a graviton mass $m = \mathcal{O}\left(H_0\right)$ and $\bar{\alpha}_1 = \mathcal{O}\left(1\right)$, where we approximately find $m_\xi \simeq m_{\Phi_\clubsuit} \simeq m$, we can estimate the order of magnitude for the upper bound $\Lambda_{\mathrm{LB}}^{(\mathrm{max})}$,

$$\Lambda_{\mathrm{LB}} \lesssim \Lambda_{\mathrm{LB}}^{(\mathrm{max})} \equiv \left(\frac{2\left(2\pi\right)^{10} m m_{\Phi_\clubsuit}}{3A^2 m_\xi^4}\right)^{1/6} = \mathcal{O}\left(\left(\frac{m_\varphi^{12} M_{\mathrm{P}}^8}{m^{14}}\right)^{1/6}\right) . \tag{5.12}$$

This gives us an upper limit that is much above the cutoff scale of the theory. For more accurate numerical results see figure 3 (right panel). In the limit $m_\varphi \to 0$, the amplitude A diverges and indicates an infinitely large decay rate. However, in this limit the Lagrangian

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of the interaction (5.2) would enter a strongly coupled regime and our perturbative ansatz would not be trustable anymore. Nevertheless, even considering extremely small masses $m_\varphi \simeq m = \mathcal{O}(H_0)$ would lead to $\Lambda_{\mathrm{LB}}^{(\max)} > M_{\mathrm{P}}$.

Even though this LB cutoff scale $\Lambda_{\mathrm{LB}}^{(\max)}$ is much above the strong coupling scale that was found in section 4.4, all of our results are still trustable and should be taken seriously. In fact, the decay products can reach energies near $\Lambda_{\mathrm{LB}}^{(\max)}$. In addition, we should expect the decay processes to occur even above Λ_{EFT} (which is Λ_5 for our massive gravity theory). We should however note that it could indeed be possible that energies above Λ_{EFT} would lead to new interactions that dominate and result in much larger decay rates, depending on the underlying new physics at energies above the EFT cutoff scale.

5.5 Comparison to observations

To date, no high-energy physics experiments have found any signals for the violation of Lorentz invariance, which may seem to indicate that Lorentz-violating operators, if exist, play a role only at very high energies, perhaps even above the Planck scale. Even though our results are compatible with this conclusion, a LB at much smaller energy scales but above Λ_3 would nevertheless be allowed.

Even though the arguments above require some speculation about the UV-completed physics, there is a more profound reason why it is not surprising to find no LV at higher energies. As recently pointed out in ref. [31], most of the operators that break LI lead to a strong coupling already above energies of $\mathcal{O}\,(\mathrm{meV})$. It has been conjectured that the strongly coupled degree of freedom (in our case the one that leads to a LV) effectively decouples from the high-energy theory and can therefore not be observed yet [31]. Similar problems appear in QCD (confinement) and massive gravity (Vainshtein screening).

Fortunately, there are possibilities to indirectly detect a breaking of LI that stabilizes the vacuum decay. For the decay products it is most likely to have energies of order Λ_{LB}, even if $\Lambda_{\mathrm{LB}} \gg \Lambda_{\mathrm{EFT}}$. As long as they do not scatter at these energy levels, they can still consistently be described by our EFT. A direct observation of these decay products could then hint towards a breaking of LI. If one assumes a LB above $\sim 1\mathrm{MeV}$ then one could search for observable effects such as peaks in the gamma-ray background, along the lines of the studies in ref. [25]. However, the background flux is not well constrained yet for all (especially higher) energies.

6 Summary and conclusions

In this work, we have discussed the influence of a ghost on the viability of an EFT by considering the violation of Lorentz invariance above certain energy scales in a particular theory of modified gravity describing a massive graviton with an additional Boulware-Deser ghost, which we called haunted massive gravity (HMG). Even though we do not believe that our HMG model is able to play a major role in the class of theories of modified gravity attempting to explain, e.g., the late-time acceleration of our Universe, we do expect that its quantum properties can be mapped onto a huge class of other theories of gravity that also introduce an Ostrogradski ghost.

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In contrast to simple toy models with a canonical scalar field interacting with a ghost, we have found a decay rate that does not scale as $\Lambda_{\rm LB}^2$, where $\Lambda_{\rm LB}$ denotes the energy scale above which Lorentz invariance is broken, or $\Lambda_{\rm LB}^8$ if one assumes the simplest interaction with a graviton; the decay rate scales, instead, as $\Lambda_{\rm LB}^6$. The origin of this difference lies in the different dominating scattering processes involved. If the ghost mass is of the order of the Hubble parameter H_0, which is expected for theories of modified gravity that provide solutions to the dark energy problem, then the upper bound on the cutoff scale at which LB has to occur is allowed to be extremely high and could even be above the Planck scale.

Finally, with HMG we have found an example of a massive gravity theory which allows for dynamical, and even self-accelerating, FLRW solutions with a flat reference metric, contrary to the ghost-free dRGT theory. Furthermore, we obtained a parameter region in which both free parameters of the theory are of $\mathcal{O}(1)$ and maximizes the timescale on which the classical instability is suppressed to obtain a viable cosmological solution. This is indeed surprising as one might expect that a ghost that is present at the background level (which is required in order to obtain dynamical FLRW solutions) will automatically destabilize the theory. We have however studied only the background solutions, and one should therefore note that it is very likely that the cosmological perturbations would be classically unstable, although it is not obvious with which timescale this instability is suppressed. Furthermore, it might also be possible that quantum loops would render the theory unviable due to interactions that could theoretically be much more dangerous than the tree-level interactions which we have studied in this work; we leave the investigation of these questions for future work.

In general, ghosts are potentially dangerous and can rule out a theory if the quantum behavior is not under control. However, if one accepts the possibility of Lorentz-violating physics above the cutoff of the theory, then all these theories should be studied carefully and might be acceptable and well behaved.

Acknowledgments

We are grateful to Arthur Hebecker for initial discussions. We would also like to thank Claudia de Rham, Florian Führer, Johannes Noller, Sabir Ramazanov, Javier Rubio, Angnis Schmidt-May, Adam R. Solomon, and Mikael von Strauss for useful comments and suggestions. We are also grateful to the anonymous referee for important and helpful comments on a previous version of the manuscript. Y.A., L.A., F.K., and H.N. acknowledge support from DFG through the TRR33 project "The Dark Universe." F.K. is also supported by the Landesgraduiertenförderung (LGFG) through the Graduate College "Astrophysics of Fundamental Probes of Gravity." H.N. also acknowledges financial support from DAAD through the program "Forschungsstipendium für Doktoranden und Nachwuchswissenschaftler." Significant parts of the tensor algebra benefited from the package xAct for Mathematica [32].

A Detailed expressions for decoupling of the helicity-0 mode, the ghost and the matter field

The coefficients C_i corresponding to the action (4.45) read

$$
\begin{aligned}
&C_1 = 0, && C_2 = 0, && C_3 = -\frac{16M_P^2}{(c_2+2c_4)\,m^2},\\
&C_4 = -\frac{4}{(c_2+2c_4)\,m^2M_P^2}, && C_5 = 0, && C_6 = 0,\\
&C_7 = 0, && C_8 = 0, && C_9 = -\frac{16M_P^2}{(c_2+2c_4)\,m^2},\\
&C_{10} = -\frac{4}{(c_2+2c_4)\,m^2M_P^2}, && C_{11} = 0, && C_{12} = 0,\\
&C_{13} = -\frac{16}{(c_2+2c_4)\,m^2}, && C_{14} = \frac{16}{(c_2+2c_4)\,m^2}, && C_{15} = -\frac{16}{(c_2+2c_4)\,m^2},\\
&C_{16} = 0, && C_{17} = 0, && C_{18} = \frac{32M_P^2}{(c_2+2c_4)\,m^2},\\
&C_{19} = \frac{8}{(c_2+2c_4)\,m^2M_P^2}, && C_{20} = \frac{16}{(c_2+2c_4)\,m^2}, && C_{21} = 0,\\
&C_{22} = 0, && C_{23} = 0, && C_{24} = 0,\\
&C_{25} = 0, && C_{26} = 0, && C_{27} = 0,\\
&C_{28} = 0, && C_{29} = 0, && C_{30} = 0,\\
&C_{31} = -\frac{16m_\varphi^2}{(c_2+2c_4)\,m^2}, && C_{32} = \frac{8M_P^2\,(c_2-2c_3-c_4)}{c_2+2c_4}, && C_{33} = -\frac{8m_\varphi^2}{(c_2+2c_4)^2\,M_P^2},\\
&C_{34} = \frac{2\,(c_2-4c_3-4c_4)}{c_2+2c_4}, && C_{35} = -1, && C_{36} = \frac{16m_\varphi^2}{(c_2+2c_4)\,m^2},\\
&C_{37} = -\frac{8\,(c_2-2c_3-c_4)}{c_2+2c_4}, && C_{38} = \frac{8m_\varphi^2}{(c_2+2c_4)\,m^2M_P^2}, && C_{39} = -\frac{2\,(c_2-4c_3-4c_4)}{c_2+2c_4},\\
&C_{40} = 1, && C_{41} = -\frac{4m_\varphi^4}{(c_2+2c_4)\,m^2M_P^2}, && C_{42} = \frac{2\,(c_2-4c_3-4c_4)\,m_\varphi^2}{c_2+2c_4},\\
&C_{43} = -m_\varphi^2, && C_{44} = \frac{\left(c_2^2+4c_2\,(4c_3+3c_4)-16\,(c_3^2+c_3c_4+c_4^2)\right)m^2M_P^2}{4\,(c_2+2c_4)}. &&
\end{aligned}
\tag{A.1}
$$

After integrating out the auxiliary field χ in eq. (4.46), the comparison of the resulting action with the original one (4.45) provides a set of equations that can be solved with

$$D_i = C_i \quad \text{for} \quad 1 \le i \le 14, \tag{A.2}$$

$$D_{15} = \mp 2\sqrt{-C_3D_{23}}, \tag{A.3}$$

$$D_{16} = D_{18} = D_{19} = D_{21} = 0, \tag{A.4}$$

$$D_{17} = \mp\frac{2C_{13}\sqrt{-C_3D_{23}}}{C_{14}}, \tag{A.5}$$

$$D_{20} = \mp C_{15}\sqrt{\frac{D_{23}}{C_3}}, \tag{A.6}$$

$$D_{22} = \mp C_{14}\sqrt{\frac{D_{23}}{C_3}}, \tag{A.7}$$

if the following contraints are fulfilled:

$$C_{15}^2 = 4C_3C_4, \tag{A.8}$$
$$C_{13}^2C_3 = D_{14}^2C_9, \tag{A.9}$$
$$4C_3C_{10} = C_{14}^2, \tag{A.10}$$
$$C_{14}C_{18} = 2C_3C_{13}, \tag{A.11}$$
$$2C_3C_{19} = C_{14}C_{15}, \tag{A.12}$$
$$C_{13}C_{15} = C_{14}C_{20}. \tag{A.13}$$

All of them are indeed satisfied for HMG.

For the transformations given in eq. (4.47) and the choice $D_{23} = -m^2M_P^2$ we find that the mass matrix is diagonalized if

$$2A_1C_{44} + 2A_3m^2M_P^2 - 2C_{43}M_P^2 = 0, \tag{A.14}$$
$$-2A_2C_{43}M_P + \frac{2C_{44}}{M_P} + 2m^2M_P = 0, \tag{A.15}$$
$$\frac{2A_1C_{44}}{M_P} + 2A_2C_{43}M_P - 2A_3m^2M_P = 0. \tag{A.16}$$

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Part III

Summary and Discussion

Summary and Discussion

The cosmological standard model based on Einstein gravity with an additional CC and DM is neither plagued by theoretical inconsistencies nor stays in contradiction with any observation so far. It rather introduces an unsatisfying fine-tuning problem which is, from the cosmological point of view, the main reason to search for modifications of GR with self-accelerating solutions, or even theories that are able to solve the CC problem. In addition, understanding the gravitational interaction from the field theory perspective without the restriction to GR might be an important step towards a more fundamental theory. In this thesis we have combined both approaches by analyzing theories in which gravity is mediated by a massive graviton (massive gravity) and a combination of two gravitons where one of them is massive (bimetric gravity), respectively. While both massive and bimetric gravity require an additional tensor field, it crucially depends on the presence of its dynamics whether the theory describes one or two propagating gravitons.

dRGT Massive Gravity

Giving a mass to the graviton has turned out to be a very cumbersome challenge. Almost all potentials will introduce an additional BD ghost mode [7] which renders the theory inconsistent due to a destabilization of the vacuum state. Only one exception was found, the dRGT massive gravity [35, 38],

$$S_{\mathrm{dRGT}} = -M_{\mathrm{P}}^2 \int \mathrm{d}^4x \sqrt{-g} \left(R - 2m^2 \sum_{n=0}^{4} \beta_n \, e_n \left(\sqrt{g^{-1}f} \right) \right) + \int \mathrm{d}^4x \sqrt{-g} \, \mathcal{L}_m, \tag{4.53}$$

that has been discussed in section 3.2. It considers an additional fixed tensor field $f_{\mu\nu}$ and ensures the absence of the BD ghost [38] by the precise structure of the potential term, which is built from the linear combination of the elementary symmetric polynomials of the eigenvalues of the matrix $\sqrt{g^{-1}f}$.

Unfortunately, when using standard techniques of cosmology, it does not serve as an alternative to ΛCDM. The Hamilonian constraint, that was found to ensure the absence of the BD ghost, indirectly affects the background evolution and propagates into the Bianchi constraint (3.35), which enforces the scale factor to be a constant. However, this does not immediately rule out massive gravity. For instance, the existence of large anisotropies in the matter sector would introduce an additional freedom that can be used to obtain a viable cosmological model [43].

Doubly-coupled Massive Gravity

In this thesis, a second way to enable dynamical FLRW solutions in dRGT massive gravity has been discussed. In publication 4 we have studied the case when the entire matter content is coupled to an effective metric, described by the line-element

$$g^{\text{eff}}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N_{\text{eff}}^2\,\mathrm{d}t^2 + a_{\text{eff}}^2\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{4.54}$$

which introduces an effective lapse function N_{eff} and scale factor a_{eff}. If, in addition, a fundamental field is introduced whose pressure depends on the lapse function, then the original no-go theorem that forbids dynamical FLRW backgrounds can be evaded [52, 86]. In this case, the Bianchi constraint gets additional lapse-dependent modifications and becomes a constraint for the lapse rather the scale factor.

However, if both dust and radiation are similarly doubly-coupled, then the absence of a viable radiation dominated era at early times as well as a divergent Hubble rate indicates the inability of the model to reproduce the comological evolution of our Universe [52]. This pathological behavior can be avoided if only one scalar field is coupled to the effective metric while all other oberservable fields are coupled to the physical metric g [49], at the cost of giving up the equivalence principle.

The examples that have been discussed in section 3.4 demonstrate that doubly-coupled dRGT massive gravity might be cosmologically viable if suitable modifications and additional fields are assumed. However, it should be emphasized that a coupling to an effective metric is still not proven to be quantum mechanically viable. Although it was claimed in refs. [51, 49] that the BD ghost that will show up is harmless, because its mass will be above the strong coupling scale Λ_3, a proof that the ghost is indeed only an artifact of a theory that is valid above Λ_3 is still missing. As we have argued in section 2.2.2, if an EFT, and therefore any theory of massive gravity, contains a ghost with mass above the cutoff then an explicit demonstration that the ghost is indeed unphysical is necessary. As this proof has not been presented yet, one should assume that the ghost is harmful [16, 17]. Thus, as a precaution, any theory of massive or bimetric gravity, in which the matter sector is coupled to an effective metric, has to be regarded as theoretically inconsistent.

Haunted Massive Gravity

Combining all results, that have been obtained in the framework of dRGT massive gravity, suggests that a theory of a massive graviton cannot be viable if no additional fields are considered. Our aim in section 3.5 was to tackle this question and present one counter example, the so-called HMG. Our results have been presented in publication 8. We have shown that the ghost field, which automatically will appear if the dRGT structure is modified, is sufficient to obtain viable cosmological solutions. With this, we have presented the first theory of massive gravity that allows for cosmologically viable and self-accelerating solutions, and does not require any additional degrees of freedom in the matter sector. Furthermore, we could argue that the ghost is harmless at classical and quantum level, even though it is indeed a propagating, physical degree of freedom.

The structure of HMG contains several interesting particularities. Its potential term was constructed such that the limit in which the two free parameters vanish, i.e., $\bar{a}_i, \bar{a}_2 \to 0$, corresponds to the original dRGT mass term. However, not only models with very tiny parameter values are viable, contrary to what one might expect, but the choice $\bar{a}_i = \mathcal{O}(1)$ does indeed provide physical solutions. However, taking this limit is only allowed at the level of the (off-shell) action. Similar to the vDVZ discontinuity in the FP theory, i.e., the break-down of the linear theory when taking the massless limit, the BD ghost will get strongly coupled and the linear approximation is not valid. In fact, HMG exhibits up to three vDVZ discontinuities. Besides the one corresponding the formerly mentioned limit $\bar{a}_i \to 0$, the perturbations get strongly coupled if

$$m^2\left(c_2 + 2c_4\right) = m^2\left(2 - 8\bar{a}_1 + 32\bar{a}_2\right) \to 0. \tag{4.55}$$

For $m \to 0$, this indicates a strongly coupled helicity-0 mode, similar to the vDVZ discontinuity in dRGT massive gravity. In addition, if $c_2 + 2c_4 \to 0$, then the mixing matrix for two auxiliary potentials becomes singular, indicating the loss of their kinetic terms. Because these auxiliary fields will introduce the Ostrogradsky ghost (see also section 3.5.2 for details on the counting of propagating degrees of freedom), which is, in fact, the BD ghost in HMG, the limit corresponds to a strongly coupled ghost. This is a particularly interesting observation because such a tuning of the parameters might be able to cure the theory from the ghost.[6] However, such a setting requires the consideration of non-linear corrections as the linear theory breaks down.

We have shown that the ghost instability in HMG at classical and quantum level is harmless if an LB operator exists, which is even allowed to set in above the Planck scale. However, during the analysis the contributions of loops were neglected. Since classical linear perturbations correspond to scattering processes at tree-level, the consistency of our result depends on whether the assumption, that interactions without loops dominate the vacuum decay, is justified.[7] Because loop contributions would include vertices that are suppressed by the strong-coupling scale, we should indeed expect that the same LB operator, which cures the vacuum decay at tree-level, would simultaneously stabilize the interactions at higher-orders.

While the energy scale at which the invariance under Lorentz transformations has to get broken was found to be above M_{P} [55], the result might change if interactions with other particles than just a scalar field are considered. However, even if such an analysis requires an LB to occur at lower energies, it would be unlikely to detect signatures of an LB at, e.g., the Large Hadron Collider. Since a violation of LI has to occur in the

[6] It should be noted that quantum corrections might be able to destabilize this tuning and could reintroduce the ghost.

[7] Note that this line of argument is only valid because the BD ghost already appears at background level. This is not generally true for all theories of modified gravity that excite an Ostrogradsky ghost. Depending on the background, the ghost might not get excited at background level and, therefore, contributions from loops should be taken into account.

gravity sector and most of the operators that could be considered are strongly coupled above energies of $\mathcal{O}(\text{meV})$ [87], the degrees of freedom being responsible for an LB will effectively decouple and cannot be observable at higher energies. This indicates that a search for Lorentz violating physics should include indirect observations. If the vacuum, even slowly, decays, then one expects emitted particles to have an energy which is around the scale at which LI breaks [54]. An unexplained peak in the spectrum of the observed cosmic background radiation could then signal an LB.

With HMG we have constructed a viable theory of modified gravity that contains an Ostrogradsky ghost and serves as an example that a theory with a ghost, not necessarily inside the framework of massive gravity, must not immediately be ruled out. It rather shows that modifications above the Planck scale might be able to cure quantum instabilities.

However, even if a theory of massive gravity containing a BD ghost exists that is theoretically consistent and passes all observational tests, the cosmological solution will not be as predictive as the one in a ghost-free massive gravity or even GR: The region where the linear FP theory is invalid, i.e., inside the Vainshtein radius (3.13), corresponds to the strong coupling scale of the theory [8], $\Lambda_5 = \left(m^4 M_\text{P}\right)^{1/5}$.[8] Interestingly, this cut-off corresponds to the same scale with which the operators are suppressed that introduce the BD ghost [8]. Hence, the precise dRGT structure, that ensures the absence of a sixth ghost-degree of freedom, raises the cut-off of the theory to $\Lambda_3 = \left(m^2 M_\text{P}\right)^{1/3}$ and, thus, allows the usage of linear perturbation theory for the analysis of sub-horizon scales in our Universe. On the other hand, it indicates that every theory of massive gravity with a ghost is not predictive above Λ_5. While this statement has only been proven for extensions of the linear FP theory [8], we have analyzed the strong coupling scale in HMG and found that all interactions suppressed by the smallest scales are [55]

$$\propto \frac{\bar{a}_i}{M_\text{P} m^4} \left(\partial\partial\varphi\right)^3 = \frac{\bar{a}_i}{\Lambda_5^5} \left(\partial\partial\varphi\right)^3 , \tag{4.56}$$

where φ denotes the canonically normalized helicity-0 mode, and, indeed, shows that the cut-off of HMG corresponds to Λ_5.

Bimetric Gravity

Fortunately, a generalization of dRGT massive gravity was suggested that does not suffer from a ghost. Instead of adding new fundamental matter fields to dRGT massive gravity or modifying the coupling to matter, one can simply consider a dynamical reference metric instead of a fixed second tensor field. This bimetric theory reintroduces a gauge symmetry, a combined diffeomorphism invariance, and treats both metrics on equal footing [38] by possessing an invariance under the interchange of both metrics.

[8]For a graviton mass of order the Hubble scale, $H_0 \simeq 10^{-33}$ eV, the Vainshtein radius around the Sun would already be as large as the size of the Milky Way [31].

Furthermore, if the graviton mass scale is of order of the Hubble parameter[9], many comological solutions contain a self-accelerating late-time epoch and agree with observational data at background level without the need of an additional CC [56, 58, 60]. Especially two models, the MBM and IBB, were extensively studied in publication 2 and 3, respectively, and were found to be compatible with current observational data [60, 69, 65] but predict a significant different evolution of the growth of structure compared to ΛCDM, which will be testable soon with future probes [77]. However, many other models with different possible phenomenological consequences have been found [60, 28]. In section 4.2, we have classified all cosmological solutions into finite-, infinite-, and exotic branch solutions, discussed their viability in publication 1 and 3, and concluded in publications 6 that every model contains a period in time at which it develops either a gradient instability of scalar perturbations or ghost instabilities.

Gradient instabilities in the scalar sector alone do not render a model unphysical [65]. Although the scalar perturbations would grow exponentially and eventually form many black holes, the theory could still be viable if the instability has not had enough time to develop. Such a scenario is indeed the case for models that were discussed in publication 7. We have found a class of models in which the perturbations get stabilized at energy scales that can be arbitrarily large[10] and only the smallest scales were affected by the fast growth. This could possibly lead to an enhancement of the number of primordial black holes. Such a signature might be the only possible way to discriminate these models from GR with a CC since, as discussed in section 4.6.2, a model, that is only affected by the instability at very early times, behaves equivalently to ΛCDM for energy scales at which linear perturbations are stable.

However, all results that have led to the discovery of unstable scalar modes are based on the assumption that the fluctuations are well describable by linear perturbations. An exponential growth will eventually cause the potentials to exceed unity, especially on deep sub-horizon scales, indicating that non-linear effects are not negligible anymore. It could be possible that higher-order perturbations are then able to counteract the linear growth and stabilize the modes. A similar property was already discusses in massive gravity; the break-down of the FP massive gravity at Solar System scales is a consequence of the theory being linear. In the non-linear régime, a Vainshtein screening ensures the viability by taking higher-order perturbations into account and restores GR [34, 89].

[9]Note that the mass scale m which enters in the potential of the bimetric action (4.1) does not correspond to the mass of the (massive) graviton mode. Due to the mixing of the massive and massless graviton modes, the physical mass depends on the background. In the de Sitter state described by proportional metrics, i.e., $f_{\mu\nu} = c^2 g_{\mu\nu}$, the bimetric FP mass reads [88]

$$m_{\mathrm{FP}}^2 = m^2 \left(\frac{1}{c^2\alpha^2} + 1\right)\left(c\beta_1 + 2c^2\beta_2 + c^3\beta_3\right), \tag{4.57}$$

and, therefore, depends on the mass scale m as well as on the Planck mass ratio $\alpha = M_f/M_g$ and the model parameters.

[10]In order to agree with all observations, it could be sufficient to consider a stabilization at energies before Big Bang nucleosynthesis sets in, which requires $M_f = \mathcal{O}(100\ \mathrm{GeV})$. Surprisingly, to push the gradient instability above the Planck mass M_g, the Planck scale for the second metric should be of order H_0, i.e., $M_f = \mathcal{O}(10^{-33}\ \mathrm{eV})$ [85].

It is still unknown whether bimetric theories exhibit a similar screening mechanism, too. However, if a Vainshtein screening exists then it will very likely able to reanimate many finite-branch models, including the simplest MBM, that has been analyzed in publication 2. In fact, by analyzing static, spherically symmetric matter distributions, the authors in ref. [90] have shown that a screening in models, in which the perturbations are unstable until redshift $z \simeq 0.5$, is sufficient to avoid contradictions with observations.

While solutions on the finite (and exotic) branch can potentially be cured by a classical screening mechanism, the pathologies on the infinite branch are of quantum mechanical nature and much harder to tackle. Initially, the fast growing tensor modes in these models [72, 73] were thought to be curable [73] by, e.g., considering non-linearities or with the help of an additional CC, as discussed in publication 5. But any of these modifications would just deal with the classical symptoms of a much more dangerous disease: In all solutions on the infinite branch both helicity-2 modes of the massive graviton are ghosts at early times and, additionally, the Higuchi bound, which ensures the helicity-0 mode of the graviton to not be a ghost, is violated at all times, indicating a pathological quantum instability [28]. At classical level, all three ghosts appear as growing modes in the tensor as well as in the scalar sector. However, if the quantum interactions, that cause a decay of the vacuum, are not suppressed by, e.g., LB physics at higher energies, then all solutions on the infinite branch have to be ruled out.

It seems that the only viable models in bimetric theories, that are free of any type of instability, are those on the finite branch that satisfy $M_f \ll M_g$ [85]. As already discussed, the gradient instability is then pushed back to very early times such that the entire observable phenomenology is equivalent to the standard picture of cosmology. Even though the Planck masses for both metrics M_f and M_g, respectively, are required to differ by many orders of magnitude, there is a priori no reason why both values should be similar. It is rather justified by the already existing hierarchy problem, i.e., the huge deviation between the coupling constants of gravity and the weak force.

Modifications of Theories with Massive Spin-2 Fields

In this thesis we have studied the cosmological viability of massive and bimetric gravity and seen that the way towards a solution to the Dark Energy problem with the use of massive spin-2 fields is quite cumbersome. However, many modifications, and extensions of theories, in which the graviton is allowed to carry a mass, exist. For instance, a generalization of bimetric gravity to a theory that contains a doubly-coupled matter sector [91, 92, 93, 94, 95], the promotion of the graviton mass to be dynamical in time [96], and an extension towards a multi-metric theory [46, 97, 98, 99].

While all these approaches add a new freedom to theory, one can also hope to find a massive (bimetric) gravity that possesses an additional gauge symmetry, which removes the helicity-0 mode of the massive graviton. Because this mode is often responsible for ghost- and gradient pathologies, such a theory might be able to provide an attractive explanation for the origin of Dark Energy. Furthermore, the additional symmetry seems to provide a link between the value of the CC and the graviton mass [100] and, therefore, could provide a solution to the CC problem. While a non-linear version of such a PM theory with an additional vector field has recently been found [101, 102], the search for this additional gauge symmetry in the original massive and bimetric setting is still ongoing [103, 104, 100, 105, 106, 107, 108].

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Abbreviations

BD	Boulware-Deser (additional ghost mode in many theories of massive gravity)
CC	Cosmological Constant (Λ or β_0)
CDM	Cold Dark Matter
CMB	Cosmic Microwave Background
DE	Dark Energy
DM	Dark Matter
dRGT	de Rham-Gabadadze-Tolley (unique ghost-free non-linear massive gravity)
EFT	Effective Field Theory
EM	Energy-Momentum
EoM	Equations of Motion
EoS	Equation of State ($p = w\rho$)
FLRW	Friedmann-Lemaître-Robertson-Walker (homogeneous and isotropic background)
FP	Fierz-Pauli (action of unique ghost-free linear massive gravity)
GR	General Relativity
HMG	Haunted Massive Gravity (toy model of massive gravity with a ghost)
IBB	Infinite Branch Bigravity ($\beta_1\beta_4$ models)
IR	Infrared (large scales)
LB	Lorentz Breaking
LI	Lorentz Invariance
MBM	Minimal Bimetric Model (β_1 model)
PM	Partially Massless (theory possessing a symmetry that removes the helicity-0 mode)
QFT	Quantum Field Theory
SNe Ia	Supernovae of Type Ia
UV	Ultraviolet (small scales)
vDVZ	van Dam-Veltman-Zakharov (discontinuity in the massless limit)
ΛCDM	Λ Cold Dark Matter (the cosmological standard model)

Colophon

This thesis was typeset with LaTeX. Its style is inspired by the one developed by John Liaperdos and is based on the teipel-thesis pdfLaTeX class.

Declaration

I hereby declare that I have completed this work solely and only with the help of the references mentioned.

Heidelberg, November 28th, 2016

FRANK KÖNNIG