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Rabinowitz Floer homology, leafwise intersections and topological entropy

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ABSTRACT. We study dynamical properties of contact manifolds using methods from Floer theory.

In the first part of this thesis we exhibit examples of contact structures on spheres of dimensions greater than 5 having positive topological entropy. We give two different types of constructions, each requiring a different approach, each leading to positive entropy.

The first approach uses the algebraic growth of wrapped Floer homology and its invariance properties under some class of contact surgeries. By carrying out a suitable series of those surgeries we then obtain contact spheres (S^{2n-1}, ξ) of dimensions $2n - 1 > 5$ such that the topological entropy of every Reeb flow on (S^{2n-1}, ξ) is positive. Those spheres admit an exact filling by a domain that is homotopy equivalent to a bouquet of spheres. In dimension 5 this approach leads also to the construction of a contact structure on $S^3 \times S^2$ such that all its Reeb flows have positive topological entropy. The results of this part already appeared in [AM17].

The second approach uses the Floer homology of perturbations of the Rabinowitz action functional. This allows us in particular to show that there exist contact spheres in dimensions greater then 5 that are exactly fillable by a domain diffeomorphic to a ball and such that the topological entropy of every Reeb flow on it is positive.

In the second part of the thesis we define a version of Rabinowitz Floer homology for hypertight contact manifolds in symplectizations and prove versions of conjectures by Sandon and Mazzucchelli on the existence of translated points and invariant Reeb orbits. Furthermore we give a proof of the existence of non-contractible Reeb orbits on hypertight contact manifolds that admit positive loops of contactomorphisms. The results in the second part already appeared in [MN17].

ZUSAMMENFASSUNG. Die vorliegende Dissertation behandelt dynamische Eigenschaften von Kontaktmannigfaltigkeiten unter Verwendung der Floer-Theorie.

Im ersten Teil dieser Arbeit werden wir Beispiele von Kontaktstrukturen positiver Entropie auf Sphären in den Dimensionen größer als 5 konstruieren. Wir werden zwei unterschiedliche Konstruktionen entwickeln, und dabei jeweils andere Methoden benutzen, die in beiden Fällen zu positiver Entropie führen.

Der erste Ansatz benutzt das algebraische Wachstum der "wrapped Floer homology" und dessen Invarianzeigenschaften unter einer gewissen Klasse von Kontakt-Chirurgien. Indem wir ein geeignete Folge dieser Chirurgien durchführen, erhalten wir Kontaktsphären (S^{2n-1}, ξ) der Dimensionen $2n - 1 > 5$, so dass die topologische Entropie eines jeden Reeb-Flusses auf (S^{2n-1}, ξ) positiv ist. Diese Sphären lassen eine exakte Füllung durch ein Gebiet zu, das homotopieäquivalent zu einem Bouquet von Sphären ist. In Dimension 5 führt dieser Ansatz auch zur Konstruktion einer Kontaktstruktur auf $S^3 \times S^2$, so dass all ihre Reeb-Flüsse positive topologische Entropy besitzen. Die Resultate in diesem Teil erschienen bereits in [AM17].

Die zweite Methode stützt sich auf die Floer-Homologie von Störungen des Rabinowitz-Wirkungsfunktionals. Sie erlaubt uns insbesondere zu zeigen, dass eine Kontaktsphäre in Dimensionen größer als 5 existiert, die exakt füllbar durch ein Gebiet ist, welches diffeomorph zu einem Ball ist, und deren Reeb-Flüsse alle positive topologische Entropie besitzen.

Im zweiten Teil definieren wir eine Version der Rabinowitz-Floer-Homologie für hyperstraffe Kontaktmannigfaltigkeiten in Symplektisierungen und beweisen Versionen von Vermutungen von Sandon und Mazzucchelli über die Existenz von "translated points" und invarianten Reeb-Bahnen. Darüber hinaus geben wir einen Beweis für die Existenz von nicht-zusammenziehbaren Reeb-Bahnen auf hyperstraffen Kontaktstrukturen, die eine Schleife aus positiven Kontaktomorphismen besitzen, an. Die Resultate im zweiten Teil sind bereits in [MN17] enthalten.

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Contents

CHAPTER 1

Introduction

1. General geometric setup

Weinstein, in the course of the celebrated existence results for periodic orbits for certain classes of Hamiltonian systems of Rabinowitz [**Rab78**] and of himself [**Wei78**], subsumed in his influential note [**Wei79**] some assumptions on the energy levels, in different situations where existence of periodic orbits have been observed, under the contact type condition. A Hamiltonian function $H : M^{2n} \to \mathbb{R}$ on a symplectic manifold (M, ω) induces a Hamiltonian flow ϕ_H^t on M, which leaves the "energy levels" $H^{-1}(c)$, $c \in \mathbb{R}$, invariant. The Hamiltonian flow restricted to a regular energy level of contact type Σ can be defined through a single 1-form α on Σ that satisfies the non-degenerate condition $\alpha \wedge d\alpha^{n-1} \neq 0$; the restriction of ϕ_H^t to Σ is then also called the Reeb flow of α. This allows one to shift the viewpoint away from some ambient symplectic manifold, and study directly properties of a dynamical system given by such a pair (Σ, α). It turns out that many qualitative properties of the system do only depend on the kernel of α, the contact structure.

A *(co-oriented) contact manifold* (Σ, ξ) is a compact odd-dimensional manifold Σ^{2n-1} equipped with a contact structure ξ, that is, a (co-oriented) hyperplane distribution on Σ which is given by $\xi = \ker \alpha$ for a 1-form α with $\alpha \wedge (d\alpha)^{n-1} \neq 0$. Such an α is called a *contact form* on (Σ, ξ). We can associate to it the *Reeb vector field* R_α defined by $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(X_\alpha) = 1$. Denote the flow of R_α, the *Reeb flow* of α, by $\theta_\alpha = (\theta_\alpha^t)_{t \in \mathbb{R}}$. We denote the set of supporting contact forms for a given contact structure ξ by $\mathcal{C}(\xi)$. It is clear from the definition that for two supporting contact forms $\alpha_1, \alpha_2 \in \mathcal{C}(\xi)$ there is a positive function $\mathsf{f} : \Sigma \to \mathbb{R}_{>0}$ with $\alpha_1 = \mathsf{f}\alpha_0$.

EXAMPLE 1.1. Every starshaped hypersurface $i : S^{2n-1} \to \mathbb{R}^{2n} = \{(q_1, \cdots q_n, p_1, \cdots p_n) \mid p_i, q_i \in \mathbb{R}\}$ has a contact form given by $\alpha = i^*(pdq)$. The contact forms arising this way differ by multiplication with a positive function and the resulting contact structure $\ker \alpha$ is called the *standard contact structure* on S^{2n-1}. As usual, we often use the suggestive notation $\alpha = \lambda|_{i(S^{2n-1})}$.

EXAMPLE 1.2. More generally, let Q be closed manifold and λ_{geo} be the canonical Liouville form on T^*Q, which can be written as pdq in local coordinates. Let $S^*Q = (T^*Q \setminus O)/\mathbb{R}_{>0}$ be the spherization of T^*Q. Here O denotes the 0-section and the action by $\mathbb{R}_{>0}$ the radial multiplication. Let $i : S^*Q \to T^*Q$ be an embedding and bundle map such that $i(\Sigma) \cap T^*_qQ$ is starshaped. Then $(S^*Q, \xi = \ker \alpha)$ with $\alpha := i^*\lambda_{geo}$ is a contact manifold. Again, ξ does not depend on the choice of embedding i. In particular, if $i(S^*Q) = (S^*Q)_g = \{(q,p) \mid |p|_g = 1\}$, with respect to some Riemannian metric g on Q then the Reeb flow with respect to $\alpha = i^*\lambda_{geo}$ corresponds to the (co-)geodesic flow on $(S^*Q)_g$. More generally, the geodesic flow of any Finsler norm on Q is the Reeb flow on (S^*Q, ξ) for some $\alpha \in \mathcal{C}(\xi)$.

EXAMPLE 1.3. Let $\mathbb{T}^3 = \{(r, s, t) \,|\, r, s, t \in \mathbb{R}/\mathbb{Z}\}$ be the three torus. A family of contact structures on $\mathbb{T}^3$ is given by $\xi_k = \ker \alpha_k$, $k > 0$, with $\alpha_k = \cos(2\pi kr)\, ds + \sin(2\pi kr)\, dt$.[1]

An important class of contact manifolds including example 1.1 and 1.2 are contact manifolds that arise as boundaries of Liouville domains, the exactly fillable contact manifolds. In the first part of the thesis we exclusively deal with exactly fillable contact manifolds.

A *Liouville domain* (M, λ) is a compact manifold M with boundary $\Sigma = \partial M$, equipped with a 1-form λ, the *Liouville form*, such that $\omega = d\lambda$ is nondegenerate, i.e. $\omega^n \neq 0$, and such that the *Liouville vector field Y* given by $i_Y\omega = \lambda$ points strictly outwards along Σ. It follows that $\alpha_{(M,\lambda)} = \lambda|_{\partial M}$ is

[1]The family of ξ_k are actually pairwise non-diffeomorphic [**Gir94, Kan97**].

a contact form on $\Sigma = \partial M$. We denote the induced contact structure on the boundary Σ by $\xi_{(M,\lambda)} := \ker \alpha_{(M,\lambda)}$. If the choice of λ is clear from the context we often just write M, α_M ξ_M, etc. A contact manifold (Σ, ξ) is called *exactly fillable* if there is a Liouville domain (M, λ) with $\Sigma = \partial M$ and a contact form $\alpha \in \mathcal{C}(\xi)$ with $\alpha = \alpha_{(M,\lambda)}$. Whenever we regard Liouville domains (M, λ) as symplectic manifolds we implicitly take the symplectic form $\omega = d\lambda$.

All examples of Liouville domains that we will consider below can be equipped, after a small perturbation of λ, with a Morse function $\psi : M \to \mathbb{R}$ such that the Liouville vector field X is gradient-like with respect to ψ. These so-called *Weinstein domains* can be regarded as a symplectic analogue of the important complex geometric notion of Stein domains. A good reference here is [**CE12**].

Using the flow of Y, one can attach an infinite cone to M along Σ that gives us the *completion* $(\widehat{M}, \widehat{\lambda})$ of M with $\widehat{M} = M \cup_\Sigma ([1, \infty) \times \Sigma)$, $\widehat{\lambda}|_M = \lambda$, $\widehat{\lambda}|_{[1,\infty)\times\Sigma} = r\alpha_M$. Note that $\widehat{\omega} := d\widehat{\lambda}$ is still a symplectic form. If the choice of λ, $\widehat{\lambda}$ is clear we usually write M, $\widehat{M}$ respectively.

2. Contact manifolds with positive entropy

2.1. Main results. An important measure of the complexity of a dynamical system on a manifold N is the topological entropy h_{top}, which quantifies in a single number the exponential complexity of the system. Given a flow $\phi = (\phi^t)_{t\in\mathbb{R}}$ on a compact manifold N, we define for an (auxiliary) metric d on N the metric d_t on N by $d_t(x, y) := \sup_{0\leq\tau\leq t} d(\phi^t(x), \phi^t(y))$. Furthermore, call a set S (ϵ, t)-separated if $d_t(x, y) > \epsilon$ for all $x, y \in S$, $x \neq y$. Let $s(\epsilon, t)$ the maximal cardinality of an (ϵ, t)-separated set. The *topological entropy of* ϕ is then defined by $h_{\text{top}}(\phi) := \lim_{\epsilon\searrow 0} \limsup_{t\to\infty} \frac{\log s(\epsilon,t)}{t}$. So, roughly speaking, the topological entropy measures the exponential rate of divergence of orbits of the flow. We refer the reader to [**HK95**] for basic properties of h_{top}. By a deep result of Yomdin [**Yom87**] $h_{\text{top}}(\phi)$ is bounded from below by the exponential volume growth rate of any submanifold in M, i.e.

$$h_{\mathrm{top}}(\phi) \geq v(\phi, N) := \limsup_{t\to\infty} \frac{\log \mathrm{Vol}_g^n(\phi^t(N))}{t} \tag{1}$$

for any submanifold N in M. Here, $n = \dim N$ and Vol_g^n is the n-dimensional volume with respect to some Riemannian metric g on M.

A well-known and also well-studied class of examples of flows for which the topological entropy is positive are geodesic flows. It is due to the work of Dinaburg [Din71], see also the work of Manning [Man79], that if the fundamental group of a Riemannian manifold (Q, g) grows exponentially, the topological entropy of the induced geodesic flow on $(S^*Q)_g$ is positive. In particular this property does not depend on the chosen metric g. Also, due to the work of Gromov, Paternain and Petean [Gro78, PP06] simply connected manifolds Q for which $\mathrm{rkH}_i(\Omega Q)$ grows exponentially for some field of coefficients, have the same property. Here ΩQ denotes the based loop space of Q. See also [Pat99] for more details on geodesic flows and entropy.

The more general problem for Reeb flows on (S^*Q, ξ), see example 2 above, was studied by Macarini and Schlenk [MS11], and they showed, based on the geometric ideas by Frauenfelder and Schlenk [FS06], that also the topological entropy of all Reeb flows on S^*Q is positive, where Q is a manifold of the same type as considered above.

In view of those results we say that a contact manfiold (Σ, ξ) has *positive entropy* if the topological entropy of the Reeb flow of any supporting contact form of ξ is positive.

In a series of papers [Alv16a, Alv16b, Alv17] Alves exhibited many new examples of contact manfolds of positive entropy in dimension 3, all having a fundamental group of exponential growth. More examples in 3 dimensions were obtained in [ACH17].

The first part of the thesis is devoted to the study of contact manifolds having positive entropy in dimensions greater than 3. It turns out that this phenomenon in higher dimensions is quite flexible from the topological point of view. In particular, we will see that on any exactly fillable contact manifold of dimension ≥ 7 the given contact structure can be locally modified to obtain a contact structure of positive entropy.

Let us state the main results.

THEOREM 2.1.

A) Let S^{2n-1} be the $(2n-1)$-dimensional sphere with its standard smooth structure. For $n \geq 4$ there exists a contact structure on S^{2n-1} with positive entropy.

B) There exists a contact structure on $S^3 \times S^2$ with positive entropy.

From Theorem 2.1 and the methods developed below we obtain the following more general result.

THEOREM 2.2.

♣ *If V is a manifold of dimension $2n-1 \geq 7$ that admits an exactly fillable contact structure, then V admits a contact structure with positive entropy.*

♢ *If V is a 5-manifold that admits an exactly fillable contact structure, then the connected sum $V\#(S^3 \times S^2)$ admits a contact structure with positive entropy.*

Note that the standard contact structure on spheres as well as the canonical contact structure on $S^*S^3 \cong S^3 \times S^2$ have a contact form with periodic Reeb flow. In particular these are not diffeomorphic to the contact structures in Theorem 2.1. Other exotic contact spheres have been constructed by several authors, see [**Eli91**, **Ust99**, **DG04**, **McL11**]. The contact spheres constructed here are, from our perspective, the "most exotic" ones. From the dynamical point of view they are the most remote from the standard contact spheres since they admit Legendrian submanifolds that have exponential volume growth under every Reeb flow.

All examples of contact manifolds that we consider in the theorems above arise as boundaries of Liouville domains. In view of that it is therefore a natural question to ask if there is a Liouville form on a ball such that the induced contact structure on the boundary sphere has positive entropy. The Liouville domains that we construct to obtain the examples of contact spheres in Part A) of Theorem 2.1 are not of that form, they have non-trivial homology in the middle degree and are homotopy equivalent to a wedge sum of spheres. By carrying out a completely different construction we will prove the following improvement of Part A) of Theorem 2.1.

THEOREM 2.3. *For every $n \geq 4$ there is Liouville form λ on B^{2n} such that the boundary sphere $(S^{2n-1}, \xi_{(B^{2n},\lambda)})$ equipped with the induced contact structure $\xi_{(B^{2n},\lambda)}$ has positive entropy.*

Again, from Theorem 2.3 and the methods developed below we obtain the following

THEOREM 2.4. *Let $n \geq 4$ and let M^{2n} be a compact manifold with boundary that admits some Liouville form λ_0. Then there exists a (possibly non-diffeomorphic) Liouville form λ on M^{2n} such that $(\partial M, \xi_{(M,\lambda)})$ has positive entropy.*

2.2. Methods to prove Theorems 2.1 and 2.2. To prove Theorems 2.1 and 2.2 we introduce the notion of algebraic growth of wrapped Floer homology. This notion is useful because, on one hand, it gives a lower bound for the growth rate of wrapped Floer homology defined using its action filtration and, on the other hand, it is stable under several geometric modifications of Liouville domains.

We consider the wrapped Floer homology $\mathrm{HW}(M, L_0 \to L_1)$ of a triple (M, L_0, L_1) where $M = (M, \lambda)$ is a Liouville domain and L_0 and L_1 are two exact Lagrangians that are asymptotically conical, i.e. conical near ∂M with Legendrian boundaries Λ_0 and Λ_1 in $(\partial M, \xi_\lambda)$. We write $\mathrm{HW}(M, L)$ for $\mathrm{HW}(M, L \to L)$. See Section 5.0.2. Results on positive entropy can be obtained from the exponential *symplectic growth* of HW, which is defined as follows. By considering only critical points below an action value a, one obtains the filtered Floer homology $\mathrm{HW}^a(M, L_0 \to L_1)$. The homologies

$\mathrm{HW}^a(M, L_0 \to L_1)$ form a filtered directed system $\widetilde{\mathrm{HW}}(M, L_0 \to L_1)$ with direct limit $\mathrm{HW}(M, L_0 \to L_1)$. The *exponential symplectic growth rate* of HW is defined as the exponential growth rate Γ of this filtered directed system, cf. 5.2, which in case of $\widetilde{\mathrm{HW}}$ is given by

$$\Gamma(\widetilde{\mathrm{HW}}(M, L_0 \to L_1)) = \limsup_{a \to \infty} \frac{\log(\dim \mathrm{Im}\ \iota_a)}{a}; \tag{2}$$

where $\iota_a : \mathrm{HW}^a(M, L_0 \to L_1) \to \mathrm{HW}(M, L_0 \to L_1)$ is the natural map to the direct limit. Since the generators of $\mathrm{HW}(M, L_0 \to L_1)$ correspond essentially to Reeb chords from Λ_0 to Λ_1, the symplectic growth gives a lower bound on the growth of Reeb chords with respect to their action. Assuming that Λ_1 is a sphere, we adapt the ideas in [**Alv17**] to get lower bounds for the volume growth $v(\phi_\alpha, \Lambda_0)$ in terms of the exponential symplectic growth rate of $\mathrm{HW}(M, L_0, L_1)$ for every contact form α on ξ_M.

A *topological operation* on a Liouville domain M is a recipe for producing a new Liouville domain N from M. To obtain examples of contact manifolds with positive entropy we perform certain topological operations on Liouville domains. The operations we consider are: attaching symplectic handles on M and, in the case M is the unit disk bundle of a manifold, plumbing M with the unit disk bundle of another manifold. Although one can understand the change or invariance of the (unfiltered) wrapped Floer homology under these operations, it is often much harder or not even possible to understand the effect of these operations on the symplectic growth.

To overcome this difficulty we look at a notion of growth that is defined purely in terms the algebraic structure on wrapped Floer homology, the *algebraic growth*. Let us explain this briefly. Let A be a (not necessarily unital) K-algebra with multiplication $\star$ and $S \subset A$ a finite set of elements of A. Given $j \geq 0$, let $N_S(j) = \{a \in A \mid a = s_1 \star s_2 \star \cdots \star s_j;\ s_1, \ldots, s_j \in S\}$; i.e. $N_S(j)$ is the set of elements of A that can be written as a product of j not necessarily distinct elements of S. We define $W_S(n) \subset A$ to be the smallest

K-vector space that contains the union $\bigcup_{j=1}^{n} N_S(j)$. The *exponential algebraic growth rate* of the pair (A, S) is defined as

$$\Gamma_S^{\mathrm{alg}}(A) = \limsup_{n\to\infty} \frac{1}{n} \log \dim_K W_S(n) \in [0, \infty).$$

In case $A = K\langle G\rangle$ is the group algebra over a finitely generated group $(G = \langle S\rangle, \star)$, it is elementary to see that $\Gamma_S^{\mathrm{alg}}(A)$ coincides with the exponential algebraic growth of G in the usual geometric group theoretical sense. Now, induced by the triangle product in Floer homology, $\mathrm{HW}(M, L)$ is equipped with a ring structure $\star$ turning it into a $\mathbb{Z}_2$-algebra. Given a finite set S of $\mathrm{HW}(M, L)$ we define (cf. Definition 5.6) $\Gamma_S^{\mathrm{alg}}(M, L) := \Gamma_S^{\mathrm{alg}}(\mathrm{HW}(M, L))$. We say that $\mathrm{HW}(M, L)$ has *exponential algebraic growth* if there exists a finite subset S of $\mathrm{HW}(M, L)$ such that $\Gamma_S^{\mathrm{alg}}(M, L) > 0$.

Our main motivation for studying the exponential algebraic growth of HW is the following

PROPOSITION 2.5. *Let M be a Liouville domain and L be an asymptotically conical exact Lagrangian in it, and assume that* $\mathrm{HW}(M, L)$ *has exponential algebraic growth. Then we have:*

A) The Liouville domain M' obtained by attaching subcritical handles to M has exponential algebraic growth of HW*. More precisely, if the attachments are made away from L (so that L survives as an asymptotically conical exact Lagrangian submanifold of M') then* $\mathrm{HW}(M', L)$ *has exponential algebraic growth.*

B) If M is the unit disk bundle of a closed orientable manifold Q^n whose fundamental group grows exponentially, and M' is obtained by a plumbing whose graph is a tree and one of the vertices is M, then M' has exponential algebraic growth of HW*. More precisely, if L_q is a unit disk fibre in M and the plumbing is done away from L_q then* $\mathrm{HW}(M', L_q)$ *has exponential algebraic growth.*

This result essentially says that plumbing and subcritical surgeries are topological operations that preserve exponential algebraic growth of HW, and will allow us to construct many examples of Liouville domains which admit asymptotically conical exact Lagrangian disks with exponential algebraic growth of HW.

The exponential algebraic growth of our examples stems from the algebraic growth of the homology of the based loop space $H_*(\Omega Q)$ equipped with the Pontrjagin product, where Q is a compact manifold. In fact, we will only use the degree 0 part whose algebraic growth is that of $\pi_1(Q)$.

In order to obtain our main results we will bound the topological entropy of Reeb flows from below in terms of the algebraic growth of $\mathrm{HW}(M,L)$. For that we will use the crucial fact that the spectral number $c : \mathrm{HW}(M,L) \to \mathbb{R}_+$ defined by $c(x) = \inf\{a \in \mathbb{R} \,|\, x \in \mathrm{Im}\, i_a\}$ is subadditive, i.e. we have $c(x \star y) \leq c(x) + c(y)$ for all $x, y \in \mathrm{HW}(M,L)$. It follows, cf. Proposition 5.7, that for any finite $S \subset \mathrm{HW}(M,L)$ we have

$$\Gamma(\widetilde{\mathrm{HW}}(M,L)) \geq \frac{1}{\rho(S)} \Gamma_S^{\mathrm{alg}}(M,L),$$

where $\rho(S) = \max_{s \in S} c(s)$. By using that $\mathrm{HW}(M, L \to L_1)$ is a module over $(\mathrm{HW}(M,L), \star)$, this lower bound can be extended to $\Gamma(\widetilde{\mathrm{HW}}(M, L \to L_1))$ for all L_1 that are exact Lagrangian isotopic to L, cf. Lemma 7.3. In other words, exponential algebraic growth of $\mathrm{HW}(M,L)$ implies positive symplectic growth of $\mathrm{HW}(M, L \to L_1)$. This, combined with ideas from [Alv17], leads to

THEOREM 2.6. *Let L be an asymptotically conical exact Lagrangian on a Liouville domain (M, λ), $\Sigma := \partial M$ and $\alpha_0 := \lambda|_\Sigma$. Assume that there is a finite set $S \subset \mathrm{HW}(M,L)$ such that $\Gamma_S^{\mathrm{alg}}(M,L) > 0$ and that $\Lambda = \partial L$ is a sphere. Then, for every contact form α on (Σ, ξ_λ) the topological entropy of the Reeb flow θ_α is positive. Moreover, if f_α is the function such that $\mathsf{f}_\alpha \alpha_0 = \alpha$ then*

$$h_{\mathrm{top}}(\phi_\alpha) \geq \frac{\Gamma_S^{\mathrm{alg}}(M,L)}{\rho(S) \max(\mathsf{f}_\alpha)}.$$

2.3. Methods to prove Theorems 2.3 and 2.4. The construction to obtain the examples in Theorem 2.3 is a variation of that of McLean in [McL11]. He obtained examples of higher dimensional Liouville domains that are diffeomorphic to balls and have positive polynomial growth of symplectic homology. Symplectic homology of a Liouville domain M, $\mathrm{SH}(M)$, is the open string analogue of $\mathrm{HW}(M, L)$. It is the homology of a chain complex that is essentially generated by closed Reeb orbits on ∂M. It can also be viewed naturally as a direct limit of a filtered directed system $\widetilde{\mathrm{SH}}(M)$ and we can consider its exponential growth $\Gamma(\widetilde{\mathrm{SH}}(M))$.

Also the examples of Theorem 2.3 are obtained by applying suitable topological operations on Liouville domains. The main idea here is to take cross products $M \times T$ of some Liouville domain M with $\Gamma(\widetilde{\mathrm{SH}}(M)) > 0$ and a contractible 4-dimensional Liouville domain T with $\mathrm{SH}(T) \neq 0$. Examples of such T were exhibited in [SS05] and [McL08, Theorem 3.1]. This allows one to get Liouville domains with positive exponential growth of SH and for which the degree where its singular homology is non-trivial lies in the subcritical range. By attaching suitable subcritical handles and by using a result of McLean in [McL11] on the invariance of the filtration of SH under subcritical surgery, see Theorem 9.2, we obtain Liouville domains (B^{2n}, λ) with $\Gamma(\widetilde{\mathrm{SH}}(B^{2n})) > 0$.[2]

That the boundary spheres have positive entropy follows from the following statement, an analogue of Theorem 2.6

PROPOSITION 2.7. *Assume that* $\Gamma(\widetilde{\mathrm{SH}}(M)) > 0$. *Then* (Σ, ξ_M) *has positive topological entropy. Moreover, let* $\alpha = \mathsf{f}\alpha_M$ *be any supporting contact form on* (Σ, ξ_M)*, then* $h_{\mathrm{top}}(\theta_\alpha) \geq \frac{\Gamma(\widetilde{\mathrm{SH}}(M))}{\max_\Sigma \mathsf{f}}$.

The proof of Proposition 2.7 makes use of Rabinowitz Floer homology (RFH), a variant of SH. RFH was first defined by Cieliebak and Frauenfelder

[2]Note that, although SH has also an algebraic structure, its corresponding exponential algebraic growth vanishes since this product is commutative. The proof of the result of McLean needs several deep geometric ideas and is more involved than a proof of only the invariance of SH under subcritical surgery.

in [**CF09**]. The basic idea is to apply Floer theory to a certain Lagrange multiplier functional that was introduced by Rabinowitz to prove existence of periodic orbits in starshaped hypersurfaces, see [**Rab78**]. A variant, the positive Rabinowitz Floer homology $\mathrm{RFH}^{\geq 0}$ has a filtered version $\widetilde{\mathrm{RFH}}^{\geq 0}$ and one has $\Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M)) = \Gamma(\widetilde{\mathrm{SH}}(M))$ for any Liouville domain M.

Although RFH is closely related to SH, there are some major conceptual advantages of RFH. For dynamical applications, one main advantage is the possibility to treat Hamiltonian perturbations of the underlying action functional quite naturally. This goes back to Albers and Frauenfelder who introduced in [**AF10a**] RFH of the perturbed action functional. Let M be a Liouville domain and consider its completion $\widehat{M}$. Let $\mathrm{Ham}_c(\widehat{M})$ denote the group of compactly supported Hamiltonian diffeomorphisms on $\widehat{M}$, cf. Definition 10.1. For any $\phi \in \mathrm{Ham}_c(\widehat{M})$ one can define the homology $\mathrm{RFH}(M;\phi)$. The generators of the underlying chain complex correspond to well-known dynamical objects, the *leafwise intersections* of ϕ.[3] These are points $x \in \Sigma = \partial M$ such that $\phi(x) \in \Sigma$ and $\phi(x) = \theta^{\eta}_{\alpha_M}(x)$, see the figure on page 12. Leafwise intersection correspond to intersection points of $(id \times \theta^{\eta}_{\alpha_M})(\Delta_\Sigma)$ and $(id \times \phi)(\Delta_\Sigma)$, where Δ_Σ denotes the diagonal in $\Sigma \times \Sigma$. $\Gamma(\mathrm{RFH}^{\geq 0}(M)) > 0$ yields a positive exponential growth rate of leafwise intersections for non-degenerate $\phi \in \mathrm{Ham}_c(\widehat{M})$, and the main point of the proof of Proposition 2.7 is to deduce the positivity of the volume growth rate $v(\Delta_\Sigma, \theta^t_{\alpha_M})$ from the exponential growth of leafwise intersections.

The main difference between the situation here and that of the Legendrian setting of Theorem 2.6 is that here a neighbourhood of the diagonal Δ_Σ is not foliated by submanifolds of the form $id \times \phi(\Delta_\Sigma)$, whereas in the proof of 2.6 we can make use of a Legendrian foliation of a neighbourhood of a Legendrian sphere.

[3]The study of leafwise intersections was initiated by Moser [**Mos78**] and since then many authors have proved existence and multiplicity results, e.g. [**Ban78, EH89, Hof90, Gin07, Zil10, AF10b, San13**].

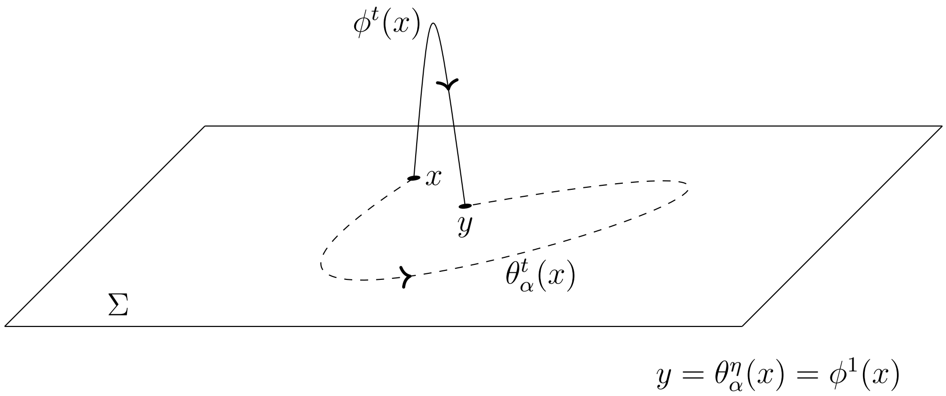

FIGURE 1. Leafwise intersection

To prove Proposition 2.7 we will consider a large parameter family of diffeomorphisms in $\mathrm{Ham}_c(\widehat{M})$ and a pigeon hole type principle to pick for any time T a ball B and a suitable subfamily of Hamiltonian diffeomorphisms $\mathcal{H}_T$ such that $id \times \phi(\Delta_B)$, $\phi \in \mathcal{H}_T$, foliate a neighbourhood of Δ_B and such that the number of the leafwise intersections for ϕ in a "sufficiently large" subset $\widetilde{\mathcal{H}_T}$ of $\mathcal{H}_T$ is of order $\exp(T)$. Then, with similar ideas that were used to prove Theorem 2.6 we can obtain Proposition 2.7.

Let me finish this section with a few additional remarks on Proposition 2.7. First of all it is, to my knowledge, the first time that any closed string version of Floer homology or contact homology is used to obtain the positivity of topological entropy of Reeb flows where one does not have to impose the positivity of exponential growth of certain free homotopy classes of loops in ∂M, cf. [**Alv16a**]. The advantage of obtaining positive h_{top} from a closed string version of Floer homology is that it is in many situations easier to compute. It is not necessary to find a suitable Legendrian Λ with Lagrangian filling L, such that $\mathrm{HW}(M, L)$ has exponential growth; the definition of SH only involves the Liouville domain M. Besides its application to prove Theorems 2.3 and 2.4, I expect in particular that Proposition 2.7 can be used to exhibit more examples in dimension 3. In particular, one can hope to show that all contact 3-manifolds of exponential growth type of contact homology exhibited in [**CH13**] have positive entropy and treat the remaining cases additionally to those that were handled recently in [**ACH17**] by using Legendrian contact homology. Another problem that I expect to be naturally approachable by this method, is the question of estimating the topological entropy of more general flows on manifolds having positive entropy. E.g. by using a variant of RFH that was defined by Albers and Frauenfelder in [**AF12a**] I expect that one can treat flows that are generated by time-dependent Reeb vector fields. For spherizations of cotangent bundles results in this direction were obtained by Dahinden in [**Dah17**].

3. Translated points on hypertight contact manifolds

As explained above, Rabinowitz Floer homology is well suited to treat existence and multiplicity problems of leafwise intersections. Let now (Σ, ξ) be a contact manifold. We denote by $\mathrm{Cont}_0(\Sigma, \xi)$ the space of all contactomorphisms on Σ which are contact-isotopic to the identity and similarly $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ the space of all paths $\hat{\varphi} = \{\varphi_t\}_{t \in [0,1]}$ of contactomorphisms which start at the identity. A variant of the notion of leafwise intersection is that of a translated point which was introduced by Sandon, see [**San13**].

Let $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$ and $\alpha \in \mathcal{C}(\xi)$. A point $x \in \Sigma$ is a *translated point of φ with respect to α* if there exists $\eta \in \mathbb{R}$ such that

$$\varphi(x) = \theta_\alpha^\eta(x) \quad \text{and} \quad \varphi^*\alpha|_x = \alpha|_x\,,$$

where again θ_α^t denotes the Reeb flow of α. We stress that the definition of a translated point depends on the contact form.

Translated points can be viewed as a special type of leafwise intersections: Every $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$ lifts to an element $\phi \in \mathrm{Ham}(S\Sigma)$ where $S\Sigma$ is the symplectisation of Σ with respect to α. The translated points of φ correspond to leafwise intersections of ϕ on $\Sigma \subset S\Sigma$, see 15.2.

Sandon in [San13] raised the question about the multiplicity of translated points and conjectured, motivated by the Arnold conjecture for periodic orbits, that the number of translated points is always bounded from below by the least number of critical points a function on Σ may have. An adaptation of Rabinowitz Floer homology that is well suited to treat existence and multiplicity problems of translated points was introduced by Albers and Merry [AM13], and they obtained various results on the existence and multiplicity of translated points. For many classes of contact manifolds, the problem is that Rabinowitz Floer homology is not yet defined or it is not known what it computes. In particular difficulties appear if one would like to define RFH in the symplectisation of Σ since in general Floer trajectories could escape to the negative end, which leads to a lack of compactness of moduli-spaces of Floer trajectories and the Floer differential can not be properly defined. In certain situation one can avoid the "bubbling-off" phenomenon at the negative end. Albers, Fuchs and Merry [AFM13] constructed Rabinowitz Floer homology associated to a contact form that does not admit any contractible periodic Reeb orbits. A contact manifold (Σ, ξ) is called *hypertight* if it admits such a supporting contact form, i.e. $\alpha_0 \in \mathcal{C}(X)$ without any contractible Reeb orbits. This includes Reeb orbits which are iterates of simple closed Reeb orbits. For instance, $(\mathbb{T}^3, \xi_k)$ in Example 1.3 is a family of hypertight contact manifolds. See Example 3.4 below for infinitely many contact structures on more general 3-dimensional manifolds. Certain prequantisation spaces are hypertight as well, see Example 3.3 below. Note that for some hypertight (Σ, ξ) there are always $\alpha \in \mathcal{C}(\xi)$ that admit contractible Reeb orbits. We extend the construction from [AFM13] and define Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, \alpha)$ for *any* contact form α supporting a hypertight contact structure, including those which do have contractible Reeb orbits. Moreover, this Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, \alpha)$ does not depend on the choice of supporting contact form. More precisely, denoting by $\mathcal{C}(\xi)$ the set of all supporting contact forms of ξ, we construct an isomorphism between $\mathrm{RFH}_*(\Sigma, \alpha_1)$ and $\mathrm{RFH}_*(\Sigma, \alpha_2)$ for any two $\alpha_1, \alpha_2 \in \mathcal{C}(\xi)$.

We denote by $\mathrm{RFH}_*\left(\Sigma, \alpha_1; \hat{\varphi}\right)$ the Rabinowitz Floer homology of the pair $(\hat{\varphi}; \alpha)$ whose chain complex has as generators orbits where first one flows along a Reeb orbit and then along the path $\hat{\varphi}$ until one hits the Reeb orbit again. In particular, if $\hat{\varphi} = id$, the generators are closed Reeb orbits, and (by definition) $\mathrm{RFH}_*(\Sigma, \alpha; id) = \mathrm{RFH}_*(\Sigma, \alpha)$.

The main technical result is the following.

THEOREM 3.1. *Let* (Σ, ξ) *be a hypertight contact manifold and let* $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0\left(\Sigma, \xi\right)$. *For any two* $\alpha_1,\ \alpha_2 \in \mathcal{C}(\xi)$*, the Rabinowitz Floer homology groups* $\mathrm{RFH}_*(\Sigma, \alpha_i; \hat{\varphi})$ *are well defined, and there are isomorphisms*

$$\mathrm{RFH}_*\left(\Sigma, \alpha_1; \hat{\varphi}\right) \cong \mathrm{RFH}_*\left(\Sigma, \alpha_2; \hat{\varphi}\right) \cong \mathrm{H}_{*+n-1}\left(\Sigma; \mathbb{Z}_2\right).$$

One application of Theorem 3.1 is the following theorem which proves versions of a conjecture by Sandon [**San13**, Conjecture 1.2] for hypertight contact manifolds.

THEOREM 3.2. *Let* (Σ, ξ) *be a hypertight contact manifold. Then:*

i) For any $\alpha \in \mathcal{C}(\xi)$ *and for any* $\varphi \in \mathrm{Cont}_0\left(\Sigma, \xi\right)$ *there exists a translated point of* φ *with respect to* α.

ii) Let $\alpha \in \mathcal{C}(\xi)$ *be nondegenerate. Then for a residual set of* $\varphi \in \mathrm{Cont}_0\left(\Sigma, \xi\right)$ *the number of translated points of* φ *with respect to* α *is bounded from below by* $\sum_{i=0}^{\dim(\Sigma)} \dim \mathrm{H}_i(\Sigma; \mathbb{Z}_2)$.

Moreover, the statement ii) can be improved as follows:

ii') For any α *and generic* φ *one of the following holds: (a) there is a translated point on a closed contractible Reeb orbit or (b) there are at least* $\sum_{i=0}^{\dim(\Sigma)} \dim \mathrm{H}_i(\Sigma; \mathbb{Z}_2)$ *many translated points.*

If the oscillation norm of the associated contact Hamiltonian (cf. Definition 13.1) of φ *is smaller than the smallest contractible Reeb period of the contact form, option (b) above is always true for nondegenerate* φ.

The above result is already known for supporting contact forms α without contractible Reeb orbits, [**AFM13**]. Theorem 3.2 extends this to *all* supporting contact forms. In [**San13**] similar results were proved for the specific contact manifolds S^{2n-1} and $\mathbb{R}\mathrm{P}^{2n-1}$ equipped with their standard contact forms.

EXAMPLE 3.3. An important class of examples of hypertight manifolds comes from certain *prequantisation spaces*. Let (M, ω) be a closed symplectic manifold and assume that the de Rham cohomology class $[\omega]$ has a primitive integral lift in $\mathrm{H}^2(M;\mathbb{Z})$. Now, we look at the circle bundle $p:\ \Sigma_k \to M$ with corresponding Euler class $k[\omega],\ 0 \neq k \in \mathbb{Z}$ and connection 1-form α with $p^*(k\omega) = -d\alpha$. Then, (Σ_k, α) is a contact manifold with periodic Reeb flow. The closed Reeb orbits are the fibres of the bundle. Moreover, the long exact sequence of the fibration

$$\pi_2(M) \xrightarrow{q_k} \pi_1\left(S^1\right) \to \pi_1(\Sigma_k) \to \pi_1(M) \to 0$$

shows that the map q_k is non-trivial if and only if the homotopy class of the fibre is torsion. Note that if q_k is non-trivial, then q_{nk} is non-trivial for each $n \neq 0$. It follows from [**AFM13**, Theorem 1.5] that a prequantisation space is hypertight if the fibre is not torsion.

A class of examples in dimension three was given by [**CH05**], see also [**Alv16a**]:

EXAMPLE 3.4. Let M be a closed connected oriented 3-manifold which can be cut along a nonempty family of incompressible tori into a family of irreducible manifolds with boundary, then M can be given infinitely many non-diffeomorphic hypertight contact structures ξ_k.

Another application of Theorem 3.1 concerns the existence of φ-invariant Reeb orbits. We define $\mathcal{S}\mathrm{Cont}_0(\Sigma, \alpha) := \{\varphi \in \mathrm{Cont}_0(\Sigma, \xi) \mid \varphi^*\alpha = \alpha\}$ to be the set of strict contactomorphisms in $\mathrm{Cont}_0(\Sigma, \xi)$ with respect to the supporting contact form α. For $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma, \xi)$ a Reeb orbit $x:\ \mathbb{R} \to \Sigma$ is called φ-*invariant* if $\varphi(x(t)) = x(t+\tau)$ for some $\tau \in \mathbb{R} \setminus \{0\}$. In [**Maz15**, Conjecture 1.2] Mazzucchelli conjectures that for $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma, \xi)$ there always is a φ-invariant Reeb orbit. Specialising Theorem 3.2 i) to strict contactomorphims, we prove a result closely related to Mazzucchelli's conjecture for hypertight contact manifolds:

COROLLARY 3.5. *Let (Σ, ξ) be a hypertight contact manifold. Let $\alpha \in \mathcal{C}(\xi)$ and fix $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma, \alpha)$. Then either there exists a φ-invariant Reeb orbit or an entire Reeb orbit is left fixed by φ.*

Another application of Theorem 3.1 is the study of the existence of *non-contractible* closed Reeb orbits. Given a loop $\hat{\varphi} = \{\varphi_t\}_{t\in[0,1]} \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ let us denote by $u_{\hat{\varphi}} \in [S^1, \Sigma]$ the free homotopy class of the loop $t \mapsto \varphi_t(x)$. A loop of contactomorphisms is called *positive* if the associated contact Hamiltonian is positive, see Definition 13.1. In [**AFM13**] it was shown that on hypertight contact manifolds there do not exist any *contractible* positive loops of contactomorphisms.

THEOREM 3.6. *Let (Σ, ξ) be a hypertight contact manifold. Assume there exists a positive loop $\hat{\varphi}$ of contactomorphisms. Then the class $u_{\hat{\varphi}}$ is a non-trivial element of $[S^1, \Sigma]$, and for any $\alpha \in \mathcal{C}(\xi)$ there exists a closed Reeb orbit of α in the free homotopy class of $-u_{\hat{\varphi}}$ (which is thus necessarily non-contractible).*

Note that again this result holds for any supporting contact form.

EXAMPLE 3.7. If a contact manifold (Σ, ξ) admits a supporting contact form with periodic Reeb flow, then the Reeb flow itself constitutes a positive loop. Thus prequantisation spaces always admit a positive loop. If in addition the fibre is not torsion, then they are examples of hypertight contact manifolds with a positive loop of contactomorphisms.

CHAPTER 2

Reeb flows and topological entropy

In the present chapter we prove the results introduced in section 2 above. Section 4 contains growth properties of filtered directed systems in general. Then, in section 5 we give the definition of wrapped Floer homology HW on Liouville domains and its product structure. We give the definition of a version of the important Viterbo transfer map for HW in section 6 and derive some of its properties. Section 7 establishes implications of the growth properties of HW to topological entropy. In section 8 we give a proof of the invariance of HW under subcritical handle attachment, recollect a result on HW of plumbings and prove Proposition 2.5. Briefly we recall the definition of symplectic homology in section 9 as well as Rabinowitz Floer homology RFH in section 10.1, before in section 10.2 we prove the implication of the growth of RFH on topological entropy. Section 11 recalls the important isomorphisms between the Floer homology of cotangent bundles and the homology of the based and free loop space of the base manifold. Finally, in section 12 we prove Theorems 2.1, 2.2, 2.3 and 2.4.

4. Filtered directed systems, symplectic growth, algebraic growth

The homology theories that we consider naturally come with an $\mathbb{R}_+$- filtration. In order to unify the treatment we consider in this section general filtered directed systems and their exponential growth rate. In the case of wrapped Floer homology the direct limit has a structure of an algebra and we will be interested in the algebraic growth properties. We will include important relations between the algebraic growth and the growth as a filtered directed system in this general framework here.

4.1. Algebraic growth and growth of filtered directed systems. Fix a field K. We use the convention that $\log(0) := 0$.

4.1.1. *Filtered directed systems and growth.*

DEFINITION 4.1. A *filtered directed system* over $\mathbb{R}_+ = [0,\infty)$ or for short *f.d.s.* is a pair (V, π) where

- V_t, $t \in [0,\infty)$, are finite dimensional K-vector spaces.
- $\pi_{s\to t} : V_s \to V_t$, for $s \leq t$ are homomorphisms (*persistence homomorphisms*), such that $\pi_{s\to t} \circ \pi_{r\to s} = \pi_{r\to t}$ for $r \leq s \leq t$, and $\pi_{t\to t} = id_{V_t}$ for all $t \in \mathbb{R}_+$.

Let $\mathfrak{J}$ be the smallest vector space of $\bigoplus_{t\in\mathbb{R}_+} V_t$ containing the set $\bigcup_{s\leq t}\{\pi_{s\to t}(x_s) - x_s\}$. The *direct limit* $\varinjlim V$ of V is defined by the quotient $\varinjlim V := \bigoplus_{t\in\mathbb{R}_+} V_t/\mathfrak{J}$. The inclusions $V_t \hookrightarrow \bigoplus_{t\in\mathbb{R}_+} V_t$ induce maps to $\varinjlim V$ which we denote by i_t. The *spectral number* c_V, or just c if the context is clear, of an element $x \in \varinjlim V$ is

$$c_V(x) := \inf\{t \in [0,\infty) \,|\, \exists x_t \in V_t \text{ such that } i_t(x_t) = x\}.$$

It is clear from the definition of c_V that if $x_1, .., x_n \in V$ and $k_1, ..., k_n \in K$ we have

$$c_V\left(\sum_{i=1}^{n} k_i x_i\right) \leq \max_{1\leq i\leq n} c_V(x_i). \tag{3}$$

DEFINITION 4.2. Let $d_t^V := \dim\{x \,|\, c_V(x) \leq t\}$. The *exponential growth rate* of the f.d.s. V is

$$\Gamma(V) := \limsup_{t\to\infty} \frac{1}{t} \log d_t^V.$$

We say that V has exponential growth if $0 < \Gamma(V) < \infty$.

REMARK 4.3. We have that $\Gamma(V) > \mu$ if and only if there is a sequence $t_k \to +\infty$ such that $d_{t_k}^V > e^{\mu t_k}$.

DEFINITION 4.4. A *morphism* between f.d.s. (V, π) and (V', π') is a collection of homomorphisms $f = (f_t)_{t\in[0,\infty)}$, $f_t : V_t \to V_t'$, that are compatible with respect to the persistence homomorphisms:

$$f_t \circ \pi_{s\to t} = \pi'_{s\to t} \circ f_s. \tag{4}$$

An *asymptotic morphism* is a collection of homomorphisms $f_t : V_t \to V_t'$, $t \in (K, \infty)$, for some $K > 0$ such that (4) holds for $K < s \leq t$.

The f.d.s together with the morphisms form a category and we can speak about isomorphisms of f.d.s.. One can carry over many vector space constructions to f.d.s., and define quotients, exact sequences , direct sums of f.d.s. etc.

Let (V, π) be a f.d.s. and $\eta \geq 1$. We can dilate V by η to a filtered directed system $(V(\eta), \pi(\eta))$ given by $V(\eta)_t = V_{\eta t}$, $\pi(\eta)_{s \to t} = \pi_{\eta s \to \eta t}$. It follows that π gives rise to a canonical morphism $\pi[\eta] : V \to V(\eta)$ by $\pi[\eta]_t = \pi_{t \to \eta t}$. For a morphism $f : V \to W$ we get a dilated morphism $f(\eta) : V(\eta) \to W(\eta)$ by setting $f(\eta)_t = f_{\eta t}$.

DEFINITION 4.5. Let (V, π_V) and (W, π_W) be f.d.s.. We call them (η_1, η_2)-*interleaved*, if there are asymptotic morphisms $f : V \to W(\eta_1)$ and $g : W \to V(\eta_2)$ for two real numbers $\eta_1, \eta_2 \geq 1$ such that

$$f(\eta_2) \circ g = \pi_W[\eta_1\eta_2] \text{ and } g(\eta_1) \circ f = \pi_V[\eta_1\eta_2].$$

We say that V and W are *interleaved* if they are (η_1, η_2)-interleaved for some $\eta_1, \eta_2 \geq 1$.

$$\begin{tikzcd}
V(\eta_1\eta_2)_a & \\
& W(\eta_1)_a \arrow[ul, "g(\eta_1)_a=g_{\eta_1 a}"'] \\
V_a \arrow[uu, "\pi_V[\eta_1\eta_2]_a=(\pi_V)_{a\to\eta_1\eta_2 a}"] \arrow[ur, "f_a"] &
\end{tikzcd}$$

The direct limits of interleaved f.d.s. are isomorphic. It is also easy to see the following

LEMMA 4.6. *Let V and W be (η_1, η_2)-interleaved for some $\eta_1, \eta_2 \geq 1$. Then*

$$\Gamma(V) \leq \eta_1\Gamma(W) \textit{ and } \Gamma(W) \leq \eta_2\Gamma(V).$$

REMARK 4.7. Although the definition of filtered directed system is that one in [**McL11**], we have a stronger notion of isomorphism type here. The category of filtered directed systems considered here fits more with that of persistence modules, see [**PS16**] for beautiful applications of persistence modules in symplectic geometry. We decided to call the objects here filtered directed systems, since the definition here is not exactly that of [**PS16**] and we are only interested in the growth properties, while further features of persistence modules that usually play a crucial role are not important in our discussion.

4.1.2. *Algebras and their algebraic growth.* We recall from the introduction the definition of the algebraic growth of a K-algebra A and a finite subset $S \subset A$. For a given $j \geq 0$ let

$$N_S(j) = \{a \in A \mid a = s_1 \star s_2 \star \cdots \star s_j;\ s_1, \ldots, s_j \in S\};$$

i.e. $N_S(j)$ is the set of elements of A that can be written as a product of j, not necessarily distinct, elements of S. We define $W_S(n) \subset A$ to be the smallest K-vector space that contains the union $\bigcup_{j=1}^{n} N_S(j)$. The exponential algebraic growth rate of the pair (A, S) is defined as

$$\Gamma_S^{\mathrm{alg}}(A) = \limsup_{n \to \infty} \frac{1}{n} \log \dim_K W(n) \in [0, \infty).$$

We will need the following definition.

DEFINITION 4.8. Let M be a module over an algebra A with scalar multiplication denoted by $*$. The module M is called *stretched* if there exists an element $m_0 \in M$ such that for all elements $a \neq 0 \in A$ we have $a * m_0 \neq 0$. An element $m_0 \in M$ satisfying this condition is called a *stretching* element.

In the following let V be a filtered directed system and assume that the vector space $A = \varinjlim V$ has a K-algebra structure with multiplication $\star$. We do not assume that A is finitely generated. Furthermore, let W be a filtered directed system, such that $M = \varinjlim W$ is a module over A with multiplication

$*$, i.e. a module over $(A, \star)$ with scalar multiplication $*$ which is compatible with the K-vector space structure of A and M.

Furthermore assume that the spectral numbers c_V and c_W are *subadditive* with respect to $\star$ and $*$, i.e.

$$c_V(a \star b) \leq c_V(a) + c_V(b), \text{ for all } a, b \in A, \tag{5}$$

and

$$c_W(a * m) \leq c_V(a) + c_W(m), \text{ for all } a \in A \text{ and } m \in M. \tag{6}$$

LEMMA 4.9. *Let V be a f.d.s. such that $A = \varinjlim V$ has a K-algebra structure with multiplication $\star$, and assume that c_V is subadditive with respect to $\star$. Then for every finite subset $S \subset A$ we have*

$$\Gamma(V) \geq \frac{1}{\rho(S)} \Gamma_S^{\mathrm{alg}}(A),$$

where $\rho(S) = \max_{x \in S} c_V(x)$.

PROOF. From the subadditivity of c_V with respect to $\star$ it follows that if $a = s_1 \star s_2 \star \cdots \star s_n$, $s_i \in S$, we have

$$c_V(a) = c_V(s_1 \star \cdots \star s_n) \leq c_V(s_1) + \cdots + c_V(s_n) \leq \rho(S) n.$$

It then follows from (3) that $W(n) \subset \{x \in A \mid c(x) \leq \rho(S)n\}$. We thus conclude that

$$\begin{aligned} \Gamma_S^{\mathrm{alg}}(A) &= \limsup_{n \to \infty} \frac{1}{n} \log \dim W(n) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \, \dim\{x \mid c(x) \leq \rho(S)n\} \leq \rho(S)\Gamma(V). \end{aligned}$$

□

LEMMA 4.10. *Let V and W be f.d.s. and assume that the vector space $A = \varinjlim V$ has an K-algebra structure with multiplication $\star$, and that $M := \varinjlim W$ has the structure of a module over A with multiplication $*$. Assume that c_V and c_W are subadditive with respect to $\star$ and $*$, respectively,* *and that $M \neq 0$ is a stretched module over the algebra A. Then*

$$\Gamma(W) \geq \Gamma(V). \tag{7}$$

Moreover, for every finite set $S \subset A$ we have

$$\Gamma(W) \geq \frac{1}{\rho(S)} \Gamma_S^{\mathrm{alg}}(A). \tag{8}$$

PROOF. Take a stretching element $m_0 \neq 0$ in M. We have $a * m_0 \neq b * m_0$ for $a \neq b$, $a, b \in A$. In particular $a \mapsto a * m_0$ is an injective homomorphism from A to M. Therefore, by (6) we have for all $t > 0$

$$\begin{aligned} d_t^V &= \dim\{a \in A \,|\, c_V(a) \leq t\} \\ &\leq \dim\{m \in M \,| c_W(m) \leq t + c_W(m_0)\} = d^W_{t+c_W(m_0)}. \end{aligned}$$

We then get

$$\begin{aligned} \Gamma(V) &= \limsup_{t\to\infty} \frac{\log\, d_t^V}{t} \leq \limsup_{t\to\infty} \frac{\log\, d^W_{t+c_W(m_0)}}{t} \\ &= \limsup_{t\to\infty} \frac{\log\, d^W_{t+c_W(m_0)}}{t + c_W(m_0)} \frac{t + c_W(m_0)}{t} = \Gamma(W). \end{aligned}$$

This proves (7). Inequality (8) is obtained by combining (7) with Lemma 4.9. □

In order to get results on entropy, we will need the following notions.

DEFINITION 4.11. Let $\mathcal{W} = W(i)_{i\in I}$ be a family of f.d.s. with direct limits $M(i)$ that are modules over $A := \varinjlim V$. We say that the family $M(i)_{i\in I}$ is *uniformly stretched* if there exists a constant $B \geq 0$ such that for every $i \in I$ there exists a stretching element $m_i \in M(i)$ with $c_{M(i)}(m_i) \leq B$.

DEFINITION 4.12. Let $\mathcal{W} = W(i)_{i\in I}$ be a family of filtered directed systems. The *uniform exponential growth rate* of $\mathcal{W}$ is

$$\Gamma_{i\in I}(\mathcal{W}) := \limsup_{t\to\infty} \frac{1}{t} \log\left(\inf_I d_t^{W(i)}\right).$$

REMARK 4.13. $\Gamma_{i\in I}(\mathcal{W}) > \mu$ if and only if there is a sequence $t_k \to +\infty$ with $d_{t_k}^{W(i)} > e^{\mu t_k}$ for all $i \in I$.

LEMMA 4.14. *Let V be a f.d.s. such that $A = \varinjlim V$ has a K-algebra structure with multiplication $\star$. Let $\mathcal{W} = W(i)_{i\in I}$ be a family of f.d.s. such that for every $i \in I$ the direct limit $M(i) = \varinjlim W(i)$ is a module over A with multiplication $*(i)$. Assume that c_V is subadditive with respect to $\star$, that $c_{W(i)}$ is subadditive with respect to $*(i)$ for every $i \in I$, and that the family $M(i)_{i\in I}$ is uniformly stretched over the algebra A. Then*

$$\Gamma_{i\in I}(\mathcal{W}) \geq \Gamma(V). \tag{9}$$

PROOF. Since $M(i)_{i\in I}$ is uniformly stretched there exists $B > 0$ such that for every $i \in I$, we can find a stretching element $m_i \in M(i)$ with $c_{M(i)}(m_i) \leq B$. Hence we have by (6) that $d_t^V \leq \inf_I d_{t+B}^{W(i)}$ and the result is obtained as in the proof of Lemma 4.10. □

4.2. Filtered directed system and homology. We finish this section with an technical observation that will be used in Sections 7 and 10. In those situations we need to get uniform lower bounds on critical points of certain functionals that have critical values above a given number. This amounts in estimating the dimension of Floer homology restricted to certain action intervals.

If $0 \to U \to V \to W \to 0$ is a short exact sequence of f.d.s., it is easy to see that

$$d_t^V = d_t^U + d_t^W. \tag{10}$$

Let now (V, π^V) be a filtered directed system. For any $a > 0$ let $({}^a\!U, \pi^{{}^a\!U})$ be the f.d.s. given by

$${}^a\!U_t = \begin{cases} \operatorname{Im} \pi_{a\to t}^V, \text{ for } t \geq a \\ \{0\} \text{ if } t < a, \end{cases}$$

and with the persistence maps given by restriction of $\pi_{s\to t}^V$ to ${}^a\!U_s$ if $s \geq a$ and 0 otherwise. Let ${}^a\!W = V/{}^a\!U$ be the quotient of V by ${}^a\!U$. Then we have a short exact sequence

$$0 \to {}^{a}U \to V \to {}^{a}W \to 0$$

and equation (10) in this situation becomes

$$d_t^V = d_a^V + d_t^{{}^{a}W}. \tag{11}$$

Now, assume there is a f.d.s. Z and a commuting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & V_a \arrow[r, "(\pi_V)_{a\to t}"] \arrow[d, "="'] & V_t \arrow[r] \arrow[d, "(\pi_V)_{t\to t'}"'] & Z_t \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & V_a \arrow[r, "(\pi_V)_{a\to t'}"] \arrow[d, "="'] & V_{t'} \arrow[r] \arrow[d] & Z_{t'} \arrow[r] \arrow[d] & \cdots \\
& \cdots & \cdots & \cdots &
\end{tikzcd} \tag{12}$$

where the rows are exact and the homomorphisms in the columns are persistence homomorphisms.

From (12) we obtain an injective morphism ${}^{a}W = V/\operatorname{Im}\pi^V_{a\to t} \to Z$ of filtered directed systems, hence $d_t^Z \geq d_t^{{}^{a}W}$. Combining this with (11) we get

$$d_t^Z \geq d_t^V - d_a^V. \tag{13}$$

5. Wrapped Floer homology HW, product and module structure

In the following we give the definition and conventions for wrapped Floer homology used here. This Floer type homology theory is a version of Lagrangian Floer homology for open manifolds. The latter goes back to [**Flo88**]. Wrapped Floer homology appeared in [**AS06**] for contangent bundles, and the case of general Liouville domains can be found in [**AS10b**]. We refer to these papers and [**Rit13**, Section 4] for more details and references.

5.0.1. *Asymptotically conical Lagrangians.* In order to introduce wrapped Floer homology, we give here the definition of a special class of Lagrangians in Liouville domains (M, λ).

We consider Lagrangians $(L, \partial L)$ in (M, Σ) that are exact, i.e. $\lambda|_L = df$, and that satisfy

$$\begin{aligned}&\Lambda = \partial L \text{ is a Legendrian submanifold in } (\Sigma, \xi_M),\\ &L \cap [1-\epsilon, 1] \times \Sigma = [1-\epsilon, 1] \times \Lambda \text{ for a sufficiently small } \epsilon > 0.\end{aligned} \tag{14}$$

We will call a Lagrangian that satisfies (14) *asymptotically conical*. We can extend it naturally to an exact Lagrangian $\widehat{L} = L \cup_\Lambda ([1, \infty) \times \Lambda)$ in $\widehat{M}$. We will refer to a Lagrangian in $\widehat{M}$ of this form also as *asymptotically conical (with respect to M)*. More generally, given a subset $U \subset \widehat{M}$ we say that L is *conical in U* if the Liouville vector field is tangent to $L \cap \mathrm{int}(U)$ in the interior $\mathrm{int}(U)$ of U.

5.0.2. *Wrapped Floer homology.* Let (M, λ) be a Liouville domain with vanishing first chern class $c_1(M) \subset \mathrm{H}^2(M; \mathbb{Z})$. For two asymptotically conical exact Lagrangians L_0 and L_1 in M denote the space of (smooth) paths from $\widehat{L}_0$ to $\widehat{L}_1$ by $\mathcal{P}_{L_0 \to L_1} = \{\gamma : [0,1] \to \widehat{M} \,|\, \gamma(0) \in \widehat{L}_0,\, \gamma(1) \in \widehat{L}_1\}$.

Denote by R_{α_M} the Reeb vector field on the boundary $(\Sigma, \xi_M = \ker \alpha_M)$. A *Reeb chord of length T* of α_M from $\Lambda_0 = \partial L_0$ to $\Lambda_1 = \partial L_1$ is a map $\gamma : [0, T] \to \Sigma$ with $\dot{\gamma}(t) = R_{\alpha_M}(\gamma(t))$ with $\gamma(0) \in \Lambda_0$ and $\gamma(T) \in \Lambda_1$. Denote the set of Reeb chords of length $< T$ by $\mathcal{T}^T_{\Lambda_0 \to \Lambda_1}(\alpha_M)$, and the set of all Reeb chords by $\mathcal{T}_{\Lambda_0 \to \Lambda_1}(\alpha_M)$. The Reeb chord γ of length T of α_M from Λ_0 to Λ_1 is said to be *transverse* if the subspaces $T_{\gamma(1)}(\phi^T_{R_{\alpha_M}}(\Lambda_0))$ and $T_{\gamma(1)}\Lambda_1$ of $T_{\gamma(1)}\Sigma$ intersect at only one point. The *spectrum* of the triple $(M, L_0 \to L_1)$, denoted by $\mathcal{S}(M, L_0 \to L_1)$, is the set of lengths of Reeb chords from Λ_0 to Λ_1 in Σ. It is a nowhere dense set in $[0, \infty)$.

Given a contact form α on (Σ, ξ_M) and a pair of Legendrian submanifolds (Λ_0, Λ_1) on (Σ, ξ_M), we say that the triple $(\alpha, \Lambda_0 \to \Lambda_1)$ is *regular* if all Reeb chords of α from Λ_0 to Λ_1 are transverse. We say that $(M, L_0 \to L_1)$ is *regular* if $(\lambda_\Sigma, \Lambda_0 \to \Lambda_1)$ is regular and L_0 and L_1 intersect transversely.

From now on, we assume that for the contact form α_M induced by M on (Σ, ξ_M) the triple $(\alpha_M, \Lambda_0 \to \Lambda_1)$ is regular.

An autonomous Hamiltonians $H : \widehat{M} \to \mathbb{R}$ is called *admissible* if

- $H < 0$ on M, and
- there exist $\mu > 0$ and $b \leq -\mu$ such that $H(x,r) = h(r) = \mu r + b$ on $[1,\infty) \times \partial M$.

If $H : \widehat{M} \to \mathbb{R}$ is admissible and satisfies $H(x,r) = \mu r + b$ on $[1,\infty) \times \partial M$ we say that H is admissible with *slope* μ *(at infinity)*.

Define the action functional $\mathcal{A}_H^{L_0 \to L_1} = \mathcal{A}_H : \mathcal{P}_{L_0 \to L_1} \to \mathbb{R}$ by

$$\mathcal{A}_H(\gamma) = f_0(x(0)) - f_1(x(1)) + \int_0^1 \gamma^*\lambda - \int_0^1 H(\gamma(t))dt,$$

where f_0 and f_1 are functions on L_0 and L_1 respectively with $df_i = \lambda|_{\widehat{L}_i}$ for $i = 0,1$. The critical points of $\mathcal{A}_H$ are Hamiltonian chords from $\widehat{L}_0$ to $\widehat{L}_1$ that reach $\widehat{L}_1$ at time 1. We define

$$\mathcal{T}_{L_0 \to L_1}(H) := \operatorname{Crit} \mathcal{A}_H = \{\gamma \in \mathcal{P}_{L_0 \to L_1} \mid \dot{\gamma}(t) = X_H(\gamma(t))\},$$

and write $\mathcal{T}_L(H)$ instead of $\mathcal{T}_{L \to L}(H)$. Here X_H is the Hamiltonian vector field defined by $\omega(X_H, \cdot) = -dH$. We call an admissible Hamiltonian *non-degenerate* for $L_0 \to L_1$ if all elements in $\mathcal{T}_{L_0 \to L_1}(H)$ are non-degenerate, i.e. $\phi^1_{X_H}(\widehat{L_0})$ is transverse to $\widehat{L_1}$. Such a Hamiltonian H must have slope $\mu \notin \mathcal{S}(M, L_0 \to L_1)$. Note that every admissible Hamiltonian can be made non-degenerate for $L_0 \to L_1$ after a generic perturbation ([**AS10b**, Lemma 8.1]). We denote by

$$\mathcal{H}_{\text{reg}}(M, L_0 \to L_1) \tag{15}$$

the set of admissible Hamiltonians which are non-degenerate for $L_0 \to L_1$. For a Hamiltonian $H \in \mathcal{H}_{\text{reg}}(M, L_0 \to L_1)$ all elements in $\mathcal{T}_{L_0 \to L_1}(H)$ have their image contained in M.

For admissible Hamiltonians H with slope $\mu \notin \mathcal{S}$ that are constant in M away from the boundary, depend on r and increase sharply near ∂M, $\mathcal{T}_{L_0 \to L_1}(H)$ corresponds to $\mathcal{T}^{\mu}_{\Lambda_0 \to \Lambda_1}(\alpha_M)$ and intersection points of L_0 and L_1 in M. If $(M, L_0 \to L_1)$ is regular, such Hamiltonians belong to the set $\mathcal{H}_{\text{reg}}(M, L_0 \to L_1)$.

An almost complex structure J on $((0,\infty)\times\partial M,\lambda=r\alpha_M)$ is called *cylindrical* if it preserves $\xi_M=\ker\alpha_M$, if $J|_{\xi_M}$ is independent of r and compatible with $d(r\alpha_M)|_{\xi_M}$, and if $JX_{\alpha_M}=r\partial_r$. In the following we take almost complex structures J on $\widehat{M}$ that are *asymptotically cylindrical*, i.e. cylindrical on $[r,\infty)\times\partial M$ for some $r>1$. The L^2-gradient of the action functional with respect to the Riemannian metric given by $d\lambda(J\cdot,\cdot)=g(\cdot,\cdot)$ is given by

$$\nabla\mathcal{A}_H(\gamma)=-J(\gamma)\left(\partial_t\gamma-X_H(\gamma)\right),$$

and we interpret the negative gradient flow lines as Floer strips

$$\begin{aligned}&u:\mathbb{R}\times[0,1]\to\widehat{M},\\&\overline{\partial}_{J,H}(u)=\partial_s u+J(u)(\partial_t u-X_H(u))=0,\\&u(\cdot,0)\in\widehat{L}_0,\ \text{and } u(\cdot,1)\in\widehat{L}_1.\end{aligned}\tag{16}$$

We define the moduli space of parametrized Floer strips connecting two critical points x and y of $\mathcal{A}_H$

$$\begin{aligned}\widetilde{\mathcal{M}}(x,y,H,J)=\Big\{&u:\mathbb{R}\times[0,1]\to\widehat{M}\,|\,u\text{ satisfies (16)},\\&\lim_{s\to-\infty}=x\text{ and }\lim_{s\to+\infty}=y\Big\}.\end{aligned}$$

There is a natural $\mathbb{R}$-action on $\mathcal{M}(x,y,H,J)$ coming from the translations in the domain. Letting $\widetilde{\mathcal{M}}^1(x,y,H,J)$ be the set of elements of $\widetilde{\mathcal{M}}(x,y,H,J)$ that have Fredholm index 1 we write

$$\mathcal{M}^0(x,y,H,J):=\widetilde{\mathcal{M}}^1(x,y,H,J)/\mathbb{R},$$

where the quotient is taken with respect to the $\mathbb{R}$-action mentioned above. The *energy* of an element u is

$$E(u):=\int_{-\infty}^{\infty}|\nabla\mathcal{A}_H|_{L^2}^2\,ds=\mathcal{A}_H(x)-\mathcal{A}_H(y).$$

For a generic J and non-degenerate admissible H define the wrapped Floer chain complex

$$\mathrm{CW}(H, L_0 \to L_1) = \bigoplus_{x \in \mathrm{Crit}\,(\mathcal{A}_H)} \mathbb{Z}_2 \cdot x,$$

with differential $\partial : \mathrm{CW}(H, L_0 \to L_1) \to \mathrm{CW}(H, L_0 \to L_1)$ given by

$$\partial(x) = \sum_{y \in \mathrm{Crit}\,(\mathcal{A}_H)} \#_{\mathbb{Z}_2} \mathcal{M}^0(x, y, H, J) \cdot y.$$

For generic J the differential is well-defined and moreover $\partial^2 = 0$. For simplicity we will write $\mathrm{CW}(H)$ instead of $\mathrm{CW}(H, L_0 \to L_1)$ when there is no possibility of confusion. We will not be concerned with gradings in CW. The homology of $(\mathrm{CW}(H, L_0 \to L_1), \partial)$ is called the wrapped Floer homology of $(H, L_0 \to L_1)$ and is denoted by $\mathrm{HW}(H; L_0 \to L_1)$, or in short $\mathrm{HW}(H)$.

Next we consider continuation maps. Let H_- and H_+ be non-degenerate admissible Hamiltonians with $H_+(x) \geq H_-(x)$ for all $x \in \widehat{M}$, in short $H_+ \succ H_-$. Take a non-decreasing homotopy through admissible Hamiltonians $(H_s)_{s \in \mathbb{R}}$, $\partial_s H_s \geq 0$, with $H_s = H_\pm$ near $\pm\infty$. For elements in $\mathcal{M}^0(x_-, x_+, H_s, J)$, i.e. Floer strips

$$\begin{aligned}
&u : \mathbb{R} \times [0,1] \to \widehat{M},\\
&\overline{\partial}_{J,H_s}(u) := \partial_s u + J(\partial_t u - X_{H_s}(u)) = 0,\\
&\lim_{s \to \pm\infty} u(s,t) = x_\pm,\\
&u(\cdot, 0) \in \widehat{L}_0, \text{ and } u(\cdot, 1) \in \widehat{L}_1,
\end{aligned} \tag{17}$$

with Fredholm index 0 connecting $x_- \in \mathrm{Crit}\,(\mathcal{A}_{H_-})$ and $x_+ \in \mathrm{Crit}\,(\mathcal{A}_{H_+})$, the action difference is

$$\mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+) = E(u) + \int_{\mathbb{R} \times [0,1]} \partial_s H_s(u).$$

Hence the action decreases under the continuation maps

$$\hat{\chi}^{H_- \to H_+} : \mathrm{CW}(H_-) \to \mathrm{CW}(H_+),$$

given by

$$\hat{\chi}^{H_- \to H_+}(x_-) = \sum_{x^+ \mathrm{Crit}\,(\mathcal{A}_{H_+})} \#_{\mathbb{Z}_2} \mathcal{M}^0(x_-, x_+, H_s, J) \cdot x_+.$$

The induced maps in homology

$$\chi^{H_- \to H_+} : \mathrm{HW}(H_-) \to \mathrm{HW}(H_+)$$

are independent of the choice of homotopy H_s. The *wrapped Floer homology of* (M, L_0, L_1) is

$$\mathrm{HW}(M, L_0 \to L_1) := \varinjlim \mathrm{HW}(H, L_0 \to L_1),$$

where the direct limit is taken over $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ with respect to $\prec$.

Since the differential decreases action, we have for any open, closed, or half-open interval I well-defined chain complexes $\mathrm{CW}^I(H)$, where one only considers generators with action in I. Let $\mathrm{HW}^I(H)$ by their homology groups. We also write $\mathrm{HW}^a(H)$ instead of $\mathrm{HW}^{(-\infty,a)}(H)$. Furthermore there are well-defined maps $\mathrm{HW}^{[\delta,a)}(H) \to \mathrm{HW}^{[\delta,b)}(H)$ for $\delta < a < b$, induced by inclusion of chain complexes, and $\mathrm{HW}^I(H) \to \mathrm{HW}^I(H')$ if $H \prec H'$, given by continuation as above. For I of the form $(-\infty, a)$ we denote the maps by

$$\iota^H_{a\to b} : \mathrm{HW}^a(H) \to \mathrm{HW}^b(H),\ \iota^H_{a\to} : \mathrm{HW}^a(H) \to \mathrm{HW}(H) \text{ and}$$
$$\chi^{H\to H'}_a : \mathrm{HW}^a(H) \to \mathrm{HW}^a(H'),\ \chi^{H\to H'} : \mathrm{HW}(H) \to \mathrm{HW}(H').$$

We have a filtered directed system $\left(\widetilde{\mathrm{HW}}(M, L_0 \to L_1), \iota\right)$, given by

$$\left(\widetilde{\mathrm{HW}}(M, L_0 \to L_1)\right)_a = \mathrm{HW}^a(M, L_0 \to L_1) := \varinjlim \mathrm{HW}^a(H, L_0 \to L_1)$$

and $\iota_{a\to b} = \varinjlim \iota^H_{a\to b}$, where the direct limits are taken over $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ with respect to $\prec$. The map to the direct limit of the filtered directed system is denoted by $\iota_a : \mathrm{HW}^a(M, L_0 \to L_1) \to \varinjlim \widetilde{\mathrm{HW}}(M, L_0 \to L_1)$. Additionally we get for any interval $I \subset \mathbb{R}$ well-defined vector spaces

$$\mathrm{HW}^I(M, L_0 \to L_1) := \varinjlim \mathrm{HW}^I(H, L_0 \to L_1).$$

We have a commuting diagram

$$\begin{array}{ccc} \mathrm{HW}^a(H) & \xrightarrow{\chi_a^{H \to H'}} & \mathrm{HW}^a(H') \\ {\scriptstyle \iota^H_{a\to b}}\big\downarrow & & {\scriptstyle \iota^{H'}_{a\to b}}\big\downarrow \\ \mathrm{HW}^b(H) & \xrightarrow{\chi_b^{H \to H'}} & \mathrm{HW}^b(H') \\ {\scriptstyle \iota^H_{b\to}}\big\downarrow & & {\scriptstyle \iota^{H'}_{b\to}}\big\downarrow \\ \mathrm{HW}(H) & \xrightarrow{\chi^{H \to H'}} & \mathrm{HW}(H') \end{array} \tag{18}$$

which induces a commutative diagram in the direct limit

$$\begin{array}{ccc} \mathrm{HW}^a(H) & \xrightarrow{\chi_a^{H \to}} & \mathrm{HW}^a(M) \\ {\scriptstyle \iota^H_{a\to b}}\big\downarrow & & {\scriptstyle \iota_{a\to b}}\big\downarrow \\ \mathrm{HW}^b(H) & \xrightarrow{\chi_b^{H \to}} & \mathrm{HW}^b(M) \\ {\scriptstyle \iota^H_{b\to}}\big\downarrow & & {\scriptstyle \iota_{b\to}=\iota_b}\big\downarrow \\ \mathrm{HW}(H) & \xrightarrow{\chi^{H \to}} & \mathrm{HW}(M) \end{array} \tag{19}$$

with maps $\chi_a^{H\to}$ and $\chi^{H\to}$ induced by $\chi_a^{H\to H'}$ and $\chi^{H\to H'}$, respectively, and $\iota_{b\to} := \varinjlim \iota^H_{b\to}$ which is identical to the map $\iota_b : \mathrm{HW}^b(M) \to \varinjlim \widetilde{\mathrm{HW}}(M)$ and in particular the wrapped Floer homology as defined above is identical to the direct limit of $\widetilde{\mathrm{HW}}$:

$$\mathrm{HW}(M, L_0 \to L_1) = \varinjlim \widetilde{\mathrm{HW}}(M, L_0 \to L_1).$$

DEFINITION 5.1. As $\mathrm{HW}(M, L_0 \to L_1)$ is the direct limit of the the filtered directed system $\widetilde{\mathrm{HW}}(M, L_0 \to L_1)$, we define the spectral number c of elements of $\mathrm{HW}(M, L_0 \to L_1)$ via the definition given in Section 4.1.2.

We will now define the symplectic growth rate of HW.

DEFINITION 5.2. The exponential symplectic growth rate is defined by

$$\Gamma^{\mathrm{symp}}(M, L_0 \to L_1) := \Gamma(\widetilde{\mathrm{HW}}(M, L_0 \to L_1)) = \limsup_{a\to\infty} \frac{\log(\dim \mathrm{Im}\ \iota_a)}{a}.$$

Analogously, for a family $(L_i)_{i\in I}$ of asymptotically conical exact Lagrangians in M we define $\Gamma^{\mathrm{symp}}_{i\in I}(M, L_0 \to L_i) := \Gamma_{i\in I}(\widetilde{\mathrm{HW}}(M, L_0 \to L_i)_{i\in I})$, where $\Gamma_{i\in I}(\widetilde{\mathrm{HW}}(M, L_0 \to L_i)_{i\in I})$ is defined as in Definition 4.12.

REMARK 5.3. Note that for any interval $I \subset \mathbb{R}$ we get well-defined vector spaces $\mathrm{HW}^I(M, L_0 \to L_1) := \varinjlim \mathrm{HW}^I(H, L_0 \to L_1)$. Analogously as above we define for any $\delta > 0$ the f.d.s. $\widetilde{\mathrm{HW}}^{\geq\delta}(M, L_0 \to L_1)$ by

$$\left(\widetilde{\mathrm{HW}}^{\geq\delta}(M, L_0 \to L_1)\right)_a = \begin{cases} \mathrm{HW}^{[\delta,a]}(M, L_0 \to L_1),\ \text{if } a \geq \delta \\ \{0\},\ \text{if } 0 \leq a < \delta, \end{cases}$$

and with the obvious persistence maps. Consider the short exact sequences of chain complexes

$$0 \to \mathrm{CW}^{(-\infty,\delta)}(H) \to \mathrm{CW}^{(-\infty,a)}(H) \to \mathrm{CW}^{[\delta,a)}(H) \to 0,$$

for admissible Hamiltonians H, $a \geq \delta$. Taking the direct limit of the associated long exact sequences in homology gives a commuting diagram of the form (12) with $V = \widetilde{\mathrm{HW}}(M, L_0 \to L_1)$ and $Z = \widetilde{\mathrm{HW}}^{\geq\delta}(M, L_0 \to L_1)$. So we are in the situation of 4.2 and by (13) we conclude that

$$\begin{aligned} \dim \mathrm{HW}^{[\delta,a)}(M, L_0 \to L_1) \geq{}& \dim\{x \in \mathrm{HW}(M, L_0 \to L_1) \mid c(x) \leq a\} \\ &- \dim\{x \in \mathrm{HW}(M, L_0 \to L_1) \mid c(x) \leq \delta\}. \end{aligned} \tag{20}$$

By the direct limit construction above HW and its filtration becomes independent of a choice of Hamiltonian. The definition has the practical problem that in order to prove properties of HW one has to go through the algebraic construction and one looses geometric insight. We will now first describe an alternative definition of spectral numbers and then show what kind of dynamical information $\mathrm{HW}^a(M, L_0 \to L_1)$ contains by considering a special kind of admissible Hamiltonians.

5.0.3. *Spectral numbers in* HW. We present an equivalent definition of c which is more geometrical. Given $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$, and any cycle $w \in \mathrm{CW}(H, L_0 \to L_1)$ we denote by $[w] \in \mathrm{HW}(H, L_0 \to L_1)$ the homology class of w in $\mathrm{HW}(H, L_0 \to L_1)$. The cycle w can be expressed in a unique way as a sum of elements of $\mathcal{T}_{L_0 \to L_1}(H)$ and we denote by $\mathcal{A}(w)$ the maximum of the actions of these elements.

If $w' \in \mathrm{CW}^a(H, L_0 \to L_1)$, then it can be expressed in a unique way as a sum of elements in $\mathcal{T}^a_{L_0 \to L_1}(H)$. This expression is identical to the one of $\hat{\iota}^H_{a\to}(w')$, where $\hat{\iota}^H_{a\to} : \mathrm{CW}^a(H) \to \mathrm{CW}(H)$ is the inclusion. from what we conclude

$$\mathcal{A}(\hat{\iota}^H_{a\to}(w')) < a \text{ for all } w' \in \mathrm{CW}^a(H, L_0 \to L_1).$$

The right hand side, $d(\mathfrak{h})$, in the following identity is often taken as the definition of the spectral number $c(\mathfrak{h})$.

LEMMA 5.4. *For a homology class $\mathfrak{h} \in \mathrm{HW}(M, L_0 \to L_1)$ we have*

$$c(\mathfrak{h}) = \inf_{H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)} \inf\{\mathcal{A}(w) \mid w \in \mathrm{CW}(H, L_0 \to L_1) \textit{ is a cycle with } \chi^{H\to}([w]) = \mathfrak{h}\} =: d(\mathfrak{h}). \tag{21}$$

PROOF. Let $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ and $w \in \mathrm{CW}(H, L_0 \to L_1)$ be a cycle with $\chi^{H\to}([w]) = \mathfrak{h}$. For each $a > \mathcal{A}(w)$ we know that there exists a cycle $w' \in \mathrm{CW}^a(H, L_0 \to L_1)$ such that $\hat{\iota}^{H\to}_a(w') = w$. We obtain

$$\chi^{H\to} \circ \iota^H_{a\to}([w']) = \chi^{H\to}([w]) = \mathfrak{h}.$$

By (19) $\mathfrak{h}$ is in the image of ι_a, from what we get $c(\mathfrak{h}) \leq a$. Since this is valid for each $a > \mathcal{A}(w)$ we obtain that $c(\mathfrak{h}) \leq \mathcal{A}(w)$, and it follows that

$$c(\mathfrak{h}) \leq d(\mathfrak{h}). \tag{22}$$

To obtain the reverse inequality let $a > c(\mathfrak{h})$. Then there exists some $\beta \in \mathrm{HW}^a(M)$ such that $\iota_a(\beta) = \mathfrak{h}$. By construction of $\mathrm{HW}^a(M)$ we know that there is $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ and a cycle $w' \in \mathrm{CW}^a(H, L_0 \to L_1)$ such that $\chi_a^{H\to}([w']) = \beta$. Let $w := \hat{\iota}^H_{a\to}(w')$. By the observation we made before the lemma we have $\mathcal{A}(w) < a$. Using (19) we obtain

$$\chi^{H\to}([w]) = \chi^{H\to}(\iota^H_{a\to}([w']) = \iota_a \circ \chi_a^{H\to}([w']) = \iota_a(\beta) = \mathfrak{h}.$$

We have shown that for each $a > c(\mathfrak{h})$ there exists $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ and a cycle $w \in \mathrm{CW}(H, L_0 \to L_1)$ such that $\mathcal{A}(w) < a$ and $\chi^{H\to}([w]) = \mathfrak{h}$. It follows that

$$c(\mathfrak{h}) \geq d(\mathfrak{h}).$$

$\square$

5.0.4. *A special type of Hamiltonians.* First of all note that for any $H \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$ and any $b > \max_{x \in \mathcal{T}_{L_0 \to L_1}(H)}\{\mathcal{A}(x)\}$ we have that $\mathrm{CW}(H, L_0 \to L_1) = \mathrm{CW}^b(H, L_0 \to L_1)$ and the continuation map $\chi_b^{H\to}$ gives us a map

$$\chi^{H\to}_{\to b} : \mathrm{HW}(H, L_0 \to L_1) \to \mathrm{HW}^b(M, L_0 \to L_1).$$

Given an admissible Hamiltonian H in M and a number $a > 0$ we write $H \prec a$ if the slope of H is $< a$. We define

$$\mathsf{K}(M, L_0 \to L_1) := \max\{\max\{f_0(x) - f_1(x) \,|\, x \in L_0 \cap L_1\},\, 0\}. \tag{23}$$

Take a collar neighbourhood $\mathfrak{V} = ([1-\delta, 1] \times \Sigma) \subset M$ of ∂M on which L_0 and L_1 are conical, and λ is given by $r\alpha_M$. If $a > \mathsf{K}(M, L_0 \to L_1)$ we can choose $\mathsf{K}(M, L_0 \to L_1) < \mu < a$, such that there is no element in $\mathcal{T}_{\Lambda_0 \to \Lambda_1}(\alpha_M)$ with length in the interval $[\mu, a)$, since $(\alpha_M, \Lambda_0 \to \Lambda_1)$ is regular. We now choose an admissible Hamiltonian H^μ in $\widehat{M}$ with slope μ such that

- H^μ is a negative constant $-\mathrm{k}$ in $M \setminus \mathfrak{V}$, with k small,
- H^μ depends only on r in $\mathfrak{V}$, and is a convex function of r that increases sharply close to ∂M.

If k is small enough, and H^μ increases sharply enough close to ∂M then we have

■ the action of all elements of $\mathcal{T}_{L_0 \to L_1}(H^\mu)$ have action $< a$;

see for example [Rit13, Lemma 9.8].

If $(M, L_0 \to L_1)$ is regular then $H^\mu \in \mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$. In this case we have that the set $\mathcal{T}_{L_0 \to L_1}(H^\mu)$ is in bijective correspondence with $\mathcal{T}^a_{\Lambda_0 \to \Lambda_1}(\alpha_M) \cup (L_0 \cap L_1)$.

In case $(\alpha_M, \Lambda_0 \to L_1)$ is regular but $(M, L_0 \to L_1)$ is not, we can make a C^∞-small perturbation of H^μ inside M that still satisfies ■ and is in $\mathcal{H}_{\mathrm{reg}}(M, L_0 \to L_1)$; for simplicity we still denote this perturbation by H^μ.

As an important observation we note that

$$\chi^{H^\mu \to}_{\to a} : \mathrm{HW}(H^\mu, L_0 \to L_1) \to \mathrm{HW}^a(M, L_0 \to L_1) \tag{24}$$

is an isomorphism. This can be seen as follows. Up to a slight shift of H^μ, it is possible for each $b > \mu$ to choose an admissible Hamiltonian H^b with slope b that is identical to H^μ in $M \setminus \widetilde{\mathcal{U}}$, where $\widetilde{\mathcal{U}}$ is a tiny neighbourhood of the boundary ∂M, and such that $\mathrm{Crit}\,\mathcal{A}_{H^b}$ coincides with $\mathrm{Crit}\,\mathcal{A}_{H^\mu}$ plus some additional elements with action $\geq a$, see figure 2. A non-decreasing homotopy from H^μ to H^b that is constant on $M \setminus \widetilde{\mathcal{V}}$ induce an isomorphism

$$\hat{\chi}^{H^\mu \to H^b}_a : \mathrm{CW}^a(H^\mu, L_0 \to L_1) \to \mathrm{CW}^a(H^b, L_0 \to L_1)$$

on the chain level. Namely, since continuation maps decrease action and the generators of the chain complex with action $< a$ coincide, $\hat{\chi}^{H^\mu \to H^b}_a$ is a lower triangular matrix with 1 on the diagonal, hence an isomorphism. With this

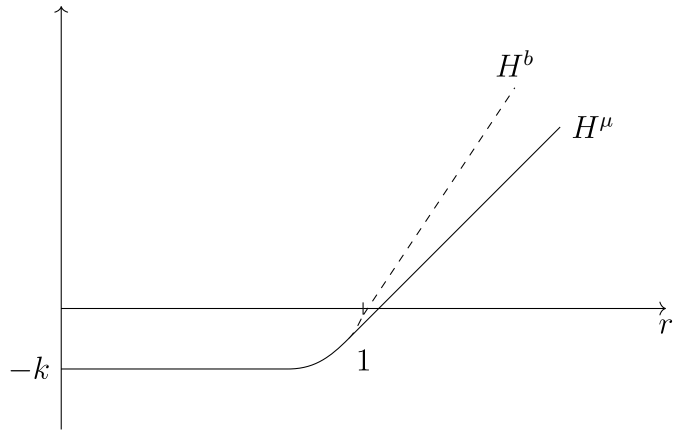

FIGURE 2.

and by carfully choosing a cofinal family of Hamiltonians on can see that the map $\chi_{\to a}^{H^\mu \to}$ of (24) is an isomorphism.

In particular it follows that for $a > \mathsf{K}(M, L_0 \to L_1)$ we have

$$\mathrm{HW}^a(M, L_0 \to L_1) \cong \varinjlim_{H \prec a} \mathrm{HW}(H; L_0 \to L_1). \tag{25}$$

Analogously there are, for any $\delta > 0$ and a suitable choice of H^μ, isomorphisms

$${}^{\delta}\chi_{\to a}^{H^\mu \to} : \mathrm{HW}^{[\delta, +\infty)}(H^\mu, L_0 \to L_1) \to \mathrm{HW}^{[\delta, a)}(M, L_0 \to L_1).$$

5.1. Algebra and module structures on wrapped Floer homology.

5.1.1. *Algebra structure in* HW. Let L be an exact asymptotically conical Lagrangian on a Liouville domain M. We endow M with an asymptotically cylindrical almost complex structure as in Section 5.0.2. We recall the definition of the triangle product in the wrapped Floer homology $\mathrm{HW}(M, L)$, and follow the conventions of [**AS10a**].

We first define the triangle Δ. One first takes the disjoint union $\mathbb{R} \times [-1, 0] \cup \mathbb{R} \times [0, 1]$. We identify the points $(s, 0^-) \in \mathbb{R} \times [-1, 0]$ and $(s, 0^+) \in \mathbb{R} \times [0, 1]$ for all $s \geq 0$, and denote the resulting space by Δ. Let

p_{sing} be the point in Δ which comes from the points $(0, 0^-) \in \mathbb{R} \times [-1, 0]$ and $(0, 0^+) \in \mathbb{R} \times [0, 1]$.

The interior of Δ coincides with $(\mathbb{R} \times (-1, 1)) \setminus ((-\infty, 0] \times \{0\})$. As $(\mathbb{R} \times (-1, 1)) \setminus ((-\infty, 0] \times \{0\})$ is a subset of $\mathbb{C}$ we can restrict the complex structure of $\mathbb{C}$ to $(\mathbb{R} \times (-1, 1)) \setminus ((-\infty, 0] \times \{0\})$. We get a complex structure j in the interior of Δ. j extends to a complex structure on $\Delta \setminus p_{\text{sing}}$. We can define global coordinates (s, t) on $\Delta \setminus p_{\text{sing}}$, using again that the interior of Δ coincides with $(\mathbb{R} \times (-1, 1)) \setminus ((-\infty, 0] \times \{0\})$.

For an admissible Hamiltonian H on $\widehat{M}$, the solutions of the Floer equation on Δ are maps $u : \Delta \to \widehat{M}$ that satisfy

$$\overline{\partial}_{J,H}(u) := \partial_s u + J(u)(\partial_t u - X_H(t, u)) = 0.$$

We write $\widehat{H} = 2H \in C^\infty(M)$.

Given $x_1, x_2 \in \mathcal{T}_L(H)$ and $y \in \mathcal{T}_L(\widehat{H})$ we let $\mathcal{M}(x_1, x_2; y, L, J)$ be the space of maps $u : \Delta \to \widehat{M}$ that satisfy $\overline{\partial}_{J,H}(u) = 0$ and such that $u(z) \in L$ for all $z \in \partial(\Delta)$, $\lim_{s\to-\infty} u(s, t-1) = x_1(t)$ for $t \in [0, 1]$, $\lim_{s\to-\infty} u(s, t) = x_2(t)$ for $t \in [0, 1]$, and $\lim_{s\to+\infty} u(s, 2t-1) = y(t)$ for $t \in [0, 1]$. Define $n(x_1, x_2; y)$ as the number of elements of $\mathcal{M}(x_1, x_2; y, L, J)$ which have Fredholm index 0. If the moduli spaces $\mathcal{M}(x_1, x_2; y, L, J)$ are transversely cut out, something that can be achieved by perturbing H and J, the numbers $n(x_1, x_2; y)$ are always finite.

Define $\Upsilon_L : \mathrm{CW}(H, L) \otimes \mathrm{CW}(H, L) \to \mathrm{CW}(\widehat{H}, L)$ by

$$\Upsilon_L(x_1, x_2) = \sum_{y \in \mathcal{T}_L(\widehat{H})} (n(x_1, x_2; y) \mod 2) y$$

for $x_1, x_2 \in \mathcal{T}_L(H)$, and extending it linearly to $\mathrm{CW}(H, L) \otimes \mathrm{CW}(H, L)$. It is proved in [AS10a] that the map Υ_L descends to a map $H\Upsilon_L : \mathrm{HW}(H, L) \otimes \mathrm{HW}(H, L) \to \mathrm{HW}(\widehat{H}, L)$, that endows $\mathrm{HW}(H, L)$ with a product which we denote by $\star$. It is compatible with the continuation maps, as follows by the results in [Sch, Chapter 5], and passing to the direct limit $H\Upsilon_L$ endows $\mathrm{HW}(M, L)$ with a product. For homology classes

$\mathfrak{h}, \mathfrak{h}' \in \mathrm{HW}(M, L)$ we will also denote their product by $\mathfrak{h} \star \mathfrak{h}'$. The product $\star$ is associative: the proof is identical to the proof in [**Sch**] that the pair of pants product in Floer homology is associative. As $\star$ is distributive with respect to the vector space structure of $\mathrm{HW}(M, L)$ it gives $\mathrm{HW}(M, L)$ the structure of a ring. Since we defined $\mathrm{HW}(M, L)$ with coefficients in $\mathbb{Z}_2$ the product $\star$ actually endows $\mathrm{HW}(M, L)$ with the structure of an algebra.

It was proved in [**AS10a**] that in the case where $M = T^*Q$ of a compact manifold Q and $L = T_qQ$ for some point $q \in Q$, the triangle product coincides with the Pontrjagin product.

An important property of the triangle product is given by

LEMMA 5.5. *The spectral numbers c of $\mathrm{HW}(M, L)$ are subadditive with respect to $\star$.*

PROOF. We will need the triangle inequality

$$\mathcal{A}_{\widehat{H}}(y) \leq \mathcal{A}_H(x_1) + \mathcal{A}_H(x_2), \tag{26}$$

that must be satisfied by the actions of $x_1, x_2 \in \mathcal{T}_L(H)$ and $y \in \mathcal{T}_L(\widehat{H})$ if the moduli space $\mathcal{M}(x_1, x_2; y, L, J) \neq \emptyset$ (see [**AS10a**, Formula 3.18]).

Let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{HW}(M, L)$. Given $\delta > 0$, we know from Lemma 5.4 that there exits Hamiltonians $H_1, H_2 \in \mathcal{H}_{\mathrm{reg}}(M, L)$ and cycles $w_i' \in \mathrm{CW}(H_i, L)$ such that

$$\chi^{H_i \to}([w_i']) = \mathfrak{h}_i \;\text{ and }\; \mathcal{A}(w_i') < c(\mathfrak{h}_i) + \frac{\delta}{2}$$

for $i = 1, 2$. Let now $H \in \mathcal{H}_{\mathrm{reg}}(M, L)$ such that $H \geq H_1$ and $H \geq H_2$. We define $w_i := \chi^{H_i \to H}(w_i')$ for $i = 1, 2$. Since the action decreases under the continuation maps $\chi^{H_i \to H}$ we have $\mathcal{A}(w_i) < c(\mathfrak{h}_i) + \frac{\delta}{2}$, and we obtain

$$\chi^{H \to}([w_i]) = \chi^{H \to}(\chi^{H_i \to H}([w_i'])) = \chi^{H_i \to}([w_i']) = \mathfrak{h}_i,$$

for $i = 1, 2$. By (26) we have $\mathcal{A}(\Upsilon_L(w_1 \otimes w_2)) \leq c(\mathfrak{h}_1) + c(\mathfrak{h}_2) + \delta$. By definition $[\Upsilon_L(w_1 \otimes w_2)] = [w_1] \star [w_2]$, and by our construction of $\star$ in $\mathrm{HW}(M, L)$

we have

$$\chi^{\widehat{H}\to}([w_1]\star[w_2]) = \chi^{H\to}([w_1])\star\chi^{H\to}([w_2]) = \mathfrak{h}_1\star\mathfrak{h}_2.$$

By Lemma 5.4 we conclude $c(\mathfrak{h}_1\star\mathfrak{h}_2) \leq \mathcal{A}(\Upsilon_L(w_1\otimes w_2)) \leq c(\mathfrak{h}_1)+c(\mathfrak{h}_2)+\delta$.

Summing up, we have shown that $c(\mathfrak{h}_1\star\mathfrak{h}_2) < c(\mathfrak{h}_1)+c(\mathfrak{h}_2)+\delta$ for any $\delta > 0$, which implies

$$c(\mathfrak{h}_1\star\mathfrak{h}_2) \leq c(\mathfrak{h}_1)+c(\mathfrak{h}_2).$$

□

We are ready to define the algebraic growth of HW.

DEFINITION 5.6. Let S be a finite set of elements of $\mathrm{HW}(M,L)$. We define

$$\Gamma_S^{\mathrm{alg}}(M,L) := \Gamma_S^{\mathrm{alg}}(\mathrm{HW}(M,L)). \tag{27}$$

Combining Lemma 4.9 and Lemma 5.5 we obtain:

PROPOSITION 5.7. *For every finite set S of $\mathrm{HW}(M,L)$ we have*

$$\Gamma^{\mathrm{symp}}(M,L) \geq \frac{\Gamma_S^{\mathrm{alg}}(M,L)}{\rho(S)}. \tag{28}$$

5.1.2. $\mathrm{HW}(M, L\to L')$ *as a module over* $\mathrm{HW}(M,L)$. We start by picking two exact asymptotically conical Lagrangians L and L' on (M,ω,λ). The boundary $\partial(\Delta)$ contains three connected components: the component $\mathcal{D}_{left}$ which is equal to $\mathbb{R}\times\{-1\}$, the component $\mathcal{D}_{mid}$ which contains the singular point, and the component $\mathcal{D}_{right}$ which is equal to $\mathbb{R}\times\{1\}$.

Let $x\in\mathcal{T}_L(H)$, $z\in\mathcal{T}_{L\to L'}(H)$ and $\widetilde{z}\in\mathcal{T}_{L\to L'}(H)$. Let $\mathcal{M}(x;z,\widetilde{z},J,H)$ be the moduli space of maps $u:\Delta\to\widehat{M}$ which satisfy (16) and such that $u(\mathcal{D}_{left})\subset L$, $u(\mathcal{D}_{mid})\subset L$, $u(\mathcal{D}_{right})\subset L'$, and $\lim_{s\to-\infty}u(s,t-1)=x(t)$ for $t\in[0,1]$, $\lim_{s\to-\infty}u(s,t) = z(t)$ for $t\in[0,1]$, and $\lim_{s\to+\infty}u(s,2t-1)=\widetilde{z}(t)$ for $t\in[0,1]$. Let $n(x;z,\widetilde{z})$ be the number of elements in $\mathcal{M}(x;z,\widetilde{z},J,H)$ that have Fredholm index 0. For non-degenerate

H and a generic choice of J, all the spaces $\mathcal{M}(x; z, \widetilde{z}, J, H)$ are transversely cut out, and therefore the numbers $n(x; z, \widetilde{z})$ are all finite.

We then define a map

$$\Theta_{L,L'} : \mathrm{CW}(H, L) \otimes \mathrm{CW}(H, L \to L') \to \mathrm{CW}(\widehat{H}, L \to L')$$

by letting

$$\Theta_{L,L'}(x \otimes z) = \sum_{\widetilde{z} \in \mathcal{T}_{L \to L'}(H)} (n(x; z, \widetilde{z}) \mod 2)\, \widetilde{z},$$

for $x \in \mathcal{T}_L(H)$, $z \in \mathcal{T}_{L \to L'}(H)$, and by extending it linearly.

The map $\Theta_{L,L'}$ descends to a map

$$H\Theta_{L,L'} : \mathrm{HW}(H, L) \otimes \mathrm{HW}(H, L \to L') \to \mathrm{HW}(\widehat{H}, L \to L').$$

The proof is again identical to the one used in [**Sch**] to show that the pair of pants product descends to the Floer homology. Taking direct limits we obtain a product $H\Theta_{L,L'} : \mathrm{HW}(M, L) \otimes \mathrm{HW}(M, L \to L') \to \mathrm{HW}(M, L \to L')$. We will use the notation $H\Theta_{L,L'}(\mathfrak{h}, \mathfrak{m}) = \mathfrak{h} * \mathfrak{m}$.

In order to conclude that $\mathrm{HW}(M; L \to L')$ is a module over the algebra $\mathrm{HW}(M; L)$ we must prove that:

$$\mathfrak{h} * (\mathfrak{m}_1 + \mathfrak{m}_2) = \mathfrak{h} * \mathfrak{m}_1 + \mathfrak{h} * \mathfrak{m}_2$$
$$(\mathfrak{h}_1 + \mathfrak{h}_2) * \mathfrak{m} = \mathfrak{h}_1 * \mathfrak{m} + \mathfrak{h}_2 * \mathfrak{m}$$
$$(\mathfrak{h}_1 \star \mathfrak{h}_2) * \mathfrak{m} = \mathfrak{h}_1 * (\mathfrak{h}_1 * \mathfrak{m})$$

for all $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{HW}(H; L)$ and $\mathfrak{m}, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{HW}(H; L \to L')$.

The first two properties follow from the linearity of $H\Theta_{L,L'}$. The proof of the third one is a cobordism argument identical to the of [**Sch**, Chapter 5] that proves the associativity of the triangle product $\star$. An argument identical to one used to prove Lemma 5.5 gives

LEMMA 5.8. *The spectral numbers c are subadditive with respect to $*$.*

6. Viterbo functoriality

The Viterbo transfer map on HW will be described. As first applications we then deduce invariance properties under a graphical change of the boundary of the Liouville domain in the completion.

6.1. The Viterbo transfer map on HW**.** The Viterbo transfer map was first introduced as a map for symplectic homology in [Vit99], see also [Cie02, McL09]. The analogous map in wrapped Floer homology was studied by [AS10b], see also [Rit13]. Our focus lies on its compatibility with the action filtration.

Let (M, λ_M) be a Liouville domain and let $j : W \to M$ be a codimension 0 exact embedding of a Liouville domain (W, λ_W) into (M, λ_M), i.e. $j^*\lambda_M = \lambda_W$. Let L_0 and L_1 be asymptotically conical exact Lagrangians in M, and assume $L_0^{'} := L_0 \cap W$ and $L_1^{'} := L_1 \cap W$ are asymptotically conical in W. Additionally, assume that

$$L_0 \text{ is also conical on } M \setminus W$$

and L_1 satisfies the property

$$\begin{aligned} &\lambda|_{L\setminus L'} \text{ vanishes on the boundary } \partial(L \setminus L^{'}) = \partial L \cup \partial L^{'}, \text{ and} \\ &\text{one can write } \lambda|_{L\setminus L'} = df, \text{ where } f \text{ vanishes near } \partial L \cup \partial L^{'}. \end{aligned} \tag{29}$$

We will call a Lagrangian with this property *transfer admissible* for the pair (M, W). See [AS10b] for a discussion of that condition and why the transfer map can in general not be defined if one removes this condition.

We give the construction of the *Viterbo transfer map* as an asymptotical morphism of filtered directed systems

$$j_!(L_0, L_1) : \widetilde{\mathrm{HW}}(M, L_0 \to L_1) \to \widetilde{\mathrm{HW}}(W, L_0^{'} \to L_1^{'}).$$

More precisely we get for $a > K = \mathsf{K}(M, L_0 \to L_1)$, defined in (23), homomorphisms

$$j_!(L_0, L_1)_a : \mathrm{HW}^a(M, L_0 \to L_1) \to \mathrm{HW}^a(W, L_0^{'} \to L_1^{'})$$

that are compatible with the persistence morphisms $\iota_{a\to b}$, for $K < a < b$. Moreover, the homomorphisms are functorial with respect to a composition of embeddings $W_1 \subset W_2 \subset M$ and the induced maps in the direct limit

$$\bar{j}_!(L_0) = \bar{j}_!(L_0, L_0) : \mathrm{HW}(M, L_0) \to \mathrm{HW}(W, L_0^{'}), \text{ and}$$

$$\bar{j}_!(L_0, L_1) : \mathrm{HW}(M, L_0 \to L_1) \to \mathrm{HW}(W, L_0^{'} \to L_1^{'})$$

are compatible with the algebra and module structure, i.e.

$$\bar{j}_!(L_0)(x \star y) = \bar{j}_!(L_0)(x) \star \bar{j}_!(L_0)(y) \tag{30}$$

and

$$\bar{j}_!(L_0, L_1)(x * z) = \bar{j}_!(L_0)(x) * \bar{j}_!(L_0, L_1)(z) \tag{31}$$

for all $x, y \in \mathrm{HW}(M, L_0)$ and $z \in \mathrm{HW}(M, L_0, L_1)$.

We first give the definition of $j_!(L_0, L_1)$. We may assume that the triples $(M, L_0 \to L_1)$ and $(W, L_0^{'} \to L_1^{'})$ are regular. Otherwise we can perform the construction considering suitable compactly supported Hamiltonian perturbations of L_0 and L_1. Let $\mathcal{S} := \mathcal{S}(M, L_0 \to L_1) \cup \mathcal{S}(W, L_0^{'} \to L_1^{'})$. We furthermore assume that actually $W \subset M_{\tau^2}$ for some $\tau < 1$, sufficiently close to 1. One can get the maps for general $W \subset M$ by an inverse limit.

First of all, for every $R > 1$ one can construct a compactly supported Hamiltonian isotopy $(\psi_t^R)_{t\in[0,1]}$ on $\widehat{M}$, $(\psi_0^R = id,\ \psi := \psi_1^R)$ that leaves $\widehat{L}_0$ invariant and maps $\widehat{L}_1$ to a Lagrangian $\widehat{L}_1^R$ that is conical on $(\widehat{M} \setminus M_R)$ and $(W_R \setminus W)$, and that is transfer admissible for the pair (M_R, W_R) as follows. Map $L_1 \setminus W$ by the Liouville flow $(\phi_{\log t})_{t\in[1,R]}$ into $A_R = M_R \setminus W_R$. Since L_1 is conical near ∂W, we can extend $\big(L_1^{'} \cup \phi_{\log t}(L_1 \setminus W)\big)_{t\in[1,R]}$ to a 1-parameter family of exact Lagrangians interpolating between $\widehat{L}_1$ and a Lagrangian $\widehat{L}_1^R$. Therefore we can choose a Hamiltonian isotopy $(\psi_t^R)_{t\in[0,1]}$ in $\widehat{M}$ that realizes this Lagrangian isotopy and is supported in $M_{\frac{1}{\tau}R} \setminus W_\tau$. Since $\widehat{L}_0$ is conical outside W, we can choose the isotopy to leave $\widehat{L}_0$ invariant. We can choose the isotopy such that $(\psi \circ \zeta)^*\lambda = R\zeta^*\lambda$, where $\zeta : L_1 \setminus W \hookrightarrow \widehat{M}$

is the embedding of L_1 restricted to $L_1 \setminus W$. The function

$$f_R : \widehat{L}_1^R \to \mathbb{R}, \text{ with } f_R(x) = \begin{cases} f_1(x), & \text{if } x \in L_1 = \widehat{L}_1^R \cap W, \\ Rf_1(\psi^{-1}x), & \text{elsewhere} \end{cases}$$

is a primitive of $\lambda|_{\widehat{L}_1^R}$.

We now carefully choose for every $\mu \notin \mathcal{S}$ sufficiently large a step-shaped Hamiltonian H_μ^{step} on $\widehat{M}$. Let $k_W := \min\{f_0(x) - f_1(x) \,|\, x \in L_0 \cap L_1 \cap W\}$ where f_i are the primitives of $\lambda|_{L_i}$, $i = 0, 1$. Let $\widetilde{k} = \max\{-k_W, 0\}$. Let

$$\begin{aligned} \widetilde{K} &= \mathsf{K}(M, W, L_0 \to L_1) \\ &= \max\{\max\{f_0(x) - f_1(x) \,|\, x \in L_0 \cap L_1 \cap M \setminus W\}, 0\}. \end{aligned}$$

Choose some small $\epsilon > 0$. Let $\mu > \widetilde{K}$, $\mu \notin \mathcal{S}$, and define $\delta_\mu := \min\{\text{dist}\,(\mu, \mathcal{S}), \mu - \widetilde{K}\}$. Choose $R > \frac{\widetilde{k}+\mu+4\epsilon}{\delta_\mu}$. We choose a smooth function $H_\mu^{step} : \widehat{M} \to \mathbb{R}$ (see the figure on page 46) that only depends on the radial coordinate $r = r_W$ in $(0, R) \times \partial W$ and only on the radial coordinate $r = r_M$ in $(\tau R, \infty) \times \partial M$, and such that

$$H_\mu^{step}(x) = \begin{cases} -\epsilon, & \text{if } x \in W_\tau \\ \frac{\partial^2 H}{\partial r} \geq 0, & \text{if } x = (r, y) \in W \setminus W_\tau \\ \mu r - \mu, & \text{if } x = (r, y) \in W_{\tau R} \setminus W \\ \frac{\partial^2 H}{\partial r} \leq 0, & \text{if } x = (r, y) \in W_R \setminus W_{\tau R} \\ (R-1)\mu - \epsilon, & \text{if } x \in M_{\tau R} \setminus W_R \\ \frac{\partial^2 H}{\partial r} \geq 0, & \text{if } x = (r, y) \in M_R \setminus M_{\tau R} \\ \mu r - \mu, & \text{if } x = (r, y) \in \widehat{M} \setminus M_R. \end{cases} \tag{32}$$

We divide the critical points of the action functional $\mathcal{A} := \mathcal{A}_{H_\mu^{step}}^{\widehat{L}_0 \to \widehat{L}_1^R}$ of H_μ^{step} with respect to $\widehat{L}_0$ and $\widehat{L}_1^R$ into four classes: Intersections of L_0 and L_1 in W_τ denoted by $\mathfrak{A}^*$, Hamiltonian chords close to ∂W denoted by $\mathfrak{A}^{**}$, intersections of $\widehat{L}_0$ and $\widehat{L}_1^R$ in $M_R \setminus W_R$ denoted by $\mathfrak{B}^*$, and chords close to ∂W_R and ∂M_R

denoted by $\mathfrak{B}^{**}$. We can estimate the action values as follows.

$$\mathcal{A}(x) \geq k_w - \epsilon \geq -\widetilde{k} - \epsilon, \text{ if } x \in \mathfrak{A}^*, \tag{33}$$

$$\mathcal{A}(x) > -\epsilon \geq -\widetilde{k} - \epsilon, \text{ if } x \in \mathfrak{A}^{**}, \tag{34}$$

$$\mathcal{A}(x) \leq R\widetilde{K} - ((R-1)\mu - \epsilon) < -\widetilde{k} - 3\epsilon, \text{ if } x \in \mathfrak{B}^*, \text{ and} \tag{35}$$

$$\mathcal{A}(x) < (\mu - \operatorname{dist}(\mu, \mathcal{S}))R - ((R-1)\mu - \epsilon) < -\widetilde{k} - 3\epsilon, \text{ if } x \in \mathfrak{B}^{**}. \tag{36}$$

In (35) we use that $f_0(x) - f_R(x) \leq \widetilde{K}R$ for every $x \in \mathfrak{B}^*$. Altogether we get that $\mathcal{A}(x) \geq -\widetilde{k} - \epsilon$, if $x \in \mathfrak{A} = \mathfrak{A}^* \cup \mathfrak{A}^{**}$ and $\mathcal{A}(x) < -\widetilde{k} - 3\epsilon$, if $x \in \mathfrak{B} = \mathfrak{B}^* \cup \mathfrak{B}^{**}$. Hence there are no Floer trajectories from $\mathfrak{B}$ to $\mathfrak{A}$. So $\mathrm{CW}_*^{(-\widetilde{k}-2\epsilon,+\infty)}(H_\mu^{step}; \widehat{L}_0 \to \widehat{L}_1^R) = \mathrm{CW}_*(H_\mu^{step})/\mathrm{CW}_*^{(-\infty,-\widetilde{k}-2\epsilon)}(H_\mu^{step})$ is generated by elements of action larger then $-\widetilde{k} - 2\epsilon$ is a chain complex, and the projection $\mathrm{CW}(H_\mu^{step}) \to \mathrm{CW}^{(-\widetilde{k}-2\epsilon,+\infty)}(H_\mu^{step})$ induces a map

$$\mathrm{HW}(H_\mu^{step}, \widehat{L}_0 \to \widehat{L}_1^R) \to \mathrm{HW}^{(-\widetilde{k}-2\epsilon,+\infty)}(H_\mu^{step}, \widehat{L}_0 \to \widehat{L}_1^R) \tag{37}$$

on homology.

Let now H_μ^M be a non-degenerate admissible Hamiltonian with respect to M on $\widehat{M}$ with slope μ, and H_μ^W a non-degenerate admissible Hamiltonian with respect to W on $\widehat{W}$ with slope μ. We have the following isomorphisms:

$$\begin{aligned} \mathrm{HW}(H_\mu^M; L_0 \to L_1) &\xrightarrow{\cong} \mathrm{HW}((\psi^{-1})^* H_\mu^M; \widehat{L}_0 \to \widehat{L}_1^R) \\ &\xrightarrow{\cong} \mathrm{HW}(H_\mu^{step}; \widehat{L}_0 \to \widehat{L}_1^R), \text{ and} \end{aligned} \tag{38}$$

$$\mathrm{HW}^{(-\widetilde{k}-2\epsilon,+\infty)}(H_\mu^{step}; \widehat{L}_0 \to \widehat{L}_1^R) \xrightarrow{\cong} \mathrm{HW}(H_\mu^W; L_0' \to L_1'). \tag{39}$$

Here, the second isomorphism in (38) holds, since $(\psi^{-1})^* H_\mu^M$ and H_μ^{step} can be connected by a compactly supported homotopy of Hamiltonians. To get the isomorphism in (39) we choose a conical almost complex structure near ∂W. By [**AS10b**, Lemma 7.2], see also [**Rit13**, Appendix D] there are

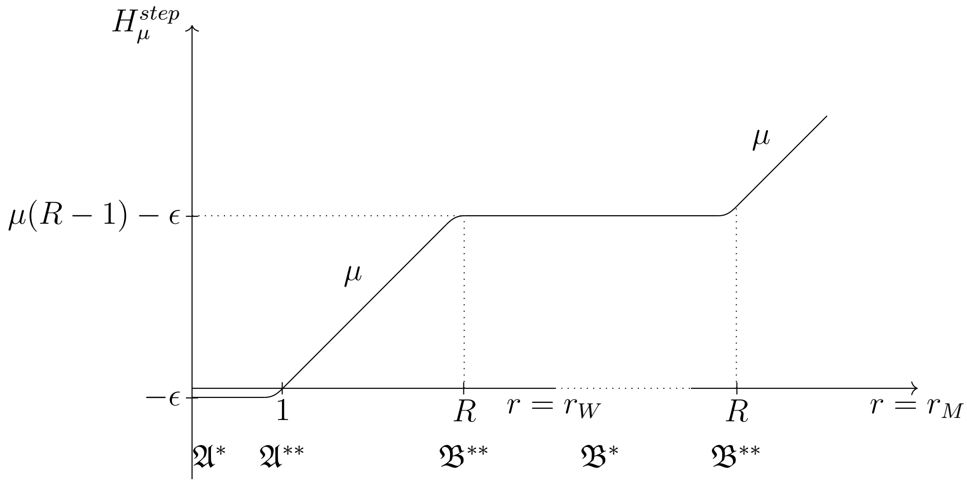

FIGURE 3.

no Floer trajectories with asymptotics in W that leave W and hence the differential of $\mathrm{CW}^{(-\widetilde{k}-2\epsilon,+\infty)}(H_\mu^{step}, \widehat{L}_0 \to \widehat{L}_1^R)$ only counts Floer trajectories that map into W.

Combining (37), (38), and (39) gives maps

$$j_\mu : \mathrm{HW}(H_\mu^M; L_0 \to L_1) \to \mathrm{HW}(H_\mu^W; L_0' \to L_1') \tag{40}$$

for any $\mu > \widetilde{K}$, $\mu \notin \mathcal{S}$. The isomorphisms (37), (38), and (39) are all compatible with Floer continuation maps induced by non-decreasing homotopies of the corresponding Hamiltonians. We do not give the details here and refer the reader to [Rit13, Theorem 9.8]. We thus get commutative diagrams

$$\begin{tikzcd}
\mathrm{HW}(H_\mu^M, L_0 \to L_1) \arrow[r, "j_\mu"] \arrow[d, "\chi^{H_\mu^M, H_\eta^M}"'] & \mathrm{HW}(H_\mu^W, L_0' \to L_1') \arrow[d, "\chi^{H_\mu^W, H_\eta^W}"'] \\
\mathrm{HW}(H_\eta^M, L_0 \to L_1) \arrow[r, "j_\eta"] & \mathrm{HW}(H_\eta^W, L_0' \to L_1')
\end{tikzcd}$$

for any $\eta > \mu > \widetilde{K}$, $\mu, \eta \notin \mathcal{S}$.

Hence, for any $a > K = \mathsf{K}(M, L_0 \to L_1) \geq \widetilde{K}$ one obtains, because of the construction in Section 5.0.4, a map

$$j_!(L_0, L_1)_a : \mathrm{HW}^a(M, L_0 \to L_1) \to \mathrm{HW}^a(W, L_0^{'} \to L_1^{'})$$

induced in the direct limit taken over all non-degenerate admissible Hamiltonians with slope μ, $K < \mu < a$. By the construction these maps are compatible with the persistence morphisms $\iota_{a\to b}$, for $K < a < b$.

By a standard compactness-cobordism argument, and by using once again the non-escaping result [**AS10b**, Lemma 7.2] one can show the compatibility of the algebra and module structure with the Viterbo transfer maps (30) and (31); for this see [**Rit13**].

6.2. Graphical change of the contact hypersurface ∂M. Using Viterbo transfer maps one can deduce invariance properties of HW under a graphical change of ∂M in $\widehat{M}$. This will be used to bound the growth rate of Reeb chords for different choices of contact forms on $(\partial M, \xi_M)$.

Let us introduce some notation. Let $\mathsf{f} : \partial M \to (0, +\infty)$ be a smooth function. Let $M_{\mathsf{f}} = \widehat{M} \setminus \{(r, x)\,|\, r > \mathsf{f}(x), x \in \partial M\}$. It is easy to see that $M_{\mathsf{f}} = (M_{\mathsf{f}}, \widehat{\lambda}|_{M_{\mathsf{f}}})$ is a Liouville domain. For example, given $\delta > 0$ we denote by $M_{1+\delta}$ the Liouville domain $(M_{1+\delta}, \lambda_{1+\delta})$ embedded in $\widehat{M}$ defined by $M_{1+\delta} = M \cup_\Sigma ([1, 1 + \delta] \times \Sigma)$, $\lambda_{1+\delta} = \widehat{\lambda}|_{M_{1+\delta}}$.

Let us describe the boundary of M as an embedding $i : \Sigma \to M$ with $i(\Sigma) = \partial M$. We have embeddings $i_{\mathsf{f}} : \Sigma \to \widehat{M}$ given by $i_{\mathsf{f}}(x) = (\mathsf{f}(x), x) \in (0, \infty) \times \partial M \subset \widehat{M}$; their images coincide with the boundaries ∂M_{f}. Moreover, let $\alpha_M = i^*\lambda$, and β any supporting contact form on $(\Sigma, \xi_{(M,\lambda)})$, i.e. $\beta = \mathsf{f}\alpha$ for a smooth function $\mathsf{f} : \Sigma \to (0, +\infty)$. Then $\beta = i_{\mathsf{f}}^*\widehat{\lambda}$. In other words, we have a bijection between the supporting contact forms on $(\Sigma, \xi_{(M,\lambda)})$ and the forms induced by $\widehat{\lambda}$ on the boundary of Liouville domains of the form M_{f}.

Let now M be a Liouville domain with asymptotically conical exact Lagrangians L_0 and L_1 as above, let $0 < \epsilon < 1$.

LEMMA 6.1. *Assume that L_i, $i = 0, 1$, are conical on $M \setminus M_\epsilon$. Then, for $a > K = \mathsf{K}(M, L_0 \to L_1)$, we have an isomorphism*

$$\Xi_a : \mathrm{HW}^a(M_\epsilon, L_0 \cap M_\epsilon \to L_1 \cap M_\epsilon) \xrightarrow{\cong} \mathrm{HW}^{\frac{1}{\epsilon}a}(M, L_0 \to L_1), \qquad (41)$$

where the maps Ξ_a are compatible with persistence maps.

Furthermore, the Viterbo map

$$\mathrm{HW}^a(M, L_0 \to L_1) \to \mathrm{HW}^a(M_\epsilon, L_0 \cap M_\epsilon \to L_1 \cap M_\epsilon)$$

composed with Ξ_a is the persistence homomorphism

$$\mathrm{HW}^a(M, L_0 \to L_1) \to \mathrm{HW}^{\frac{1}{\epsilon}a}(M, L_0 \to L_1).$$

PROOF. Note, that adding a constant to any Hamiltonian H or applying a compactly supported deformation to H does not change its Floer homology. Let H be an admissible Hamiltonian with slope μ with respect to M_ϵ. Then $H - \mu(\frac{1}{\epsilon} - 1)$ is an admissible Hamiltonian with slope $\frac{1}{\epsilon}\mu$ with respect to M. Moreover, if one chooses a cofinal sequence of Hamiltonians H_k of the first kind with slopes $K < \mu_k < a$, $\mu_k \to a$, there are compactly supported homotopies of the shifted Hamiltonians to a cofinal sequence of admissible Hamiltonians H'_k with respect to $(M, L_0 \to L_1)$ with slopes $\frac{1}{\epsilon}\mu_k$. This gives isomorphisms

$$\Xi_k : \mathrm{HW}(H_k, L_0 \cap M_\epsilon \to L_1 \cap M_\epsilon) \to \mathrm{HW}(H'_k, L_0 \to L_1),$$

that do not depend on the choice of homotopy and lead to the isomorphism $\Xi_a = \varinjlim \Xi_k$ in the direct limit. Observe, that both the Viterbo transfer map in the present situation and the persistence morphisms are given by a continuation map induced by a monotone homotopy. One can apply a usual chain homotopy argument in Floer homology to see the second statement. $\square$

Let $\mathsf{f} : \partial M \to [1, \infty)$ and $\zeta = \max_{\partial M} \mathsf{f}$.

LEMMA 6.2. *The filtered directed systems $\widetilde{\mathrm{HW}}(M, L_0 \to L_1)$ and $\widetilde{\mathrm{HW}}(M_{\mathsf{f}}, \widehat{L}_0 \cap M_{\mathsf{f}} \to \widehat{L}_0 \cap M_{\mathsf{f}})$ are $(\zeta, 1)$-interleaved.*

PROOF. Let $a > \mathsf{K}(M, L_0 \to L_1)$. Via Viterbo transfer maps and Lemma 6.1 we get asymptotic morphisms of filtered directed systems f and g, with

$$f_a : \mathrm{HW}^a(M, L_0 \to L_1) \cong \mathrm{HW}^{\zeta a}(M_\zeta, L_0 \to L_1) \to \\ \mathrm{HW}^{\zeta a}(M_{\mathsf{f}}, \widehat{L}_0 \cap M_{\mathsf{f}}, \widehat{L}_1 \cap M_{\mathsf{f}}) \quad \text{and}$$

$$g_a : \mathrm{HW}^a(M_{\mathsf{f}}, \widehat{L}_0 \cap M_{\mathsf{f}}, \widehat{L}_1 \cap M_{\mathsf{f}}) \to \mathrm{HW}^a(M, L_0 \to L_1).$$

By the functoriality of Viterbo maps and Lemma 6.1 one can easily see that $f_a \circ g_a$ and $g_{\zeta a} \circ f_a$ are equal to the corresponding persistence homomorphisms. This gives the claimed $(\zeta, 1)$-interleaving of $\widetilde{\mathrm{HW}}(M, L_0 \to L_1)$ and $\widetilde{\mathrm{HW}}(M_{\mathsf{f}}, \widehat{L}_0 \cap M_{\mathsf{f}} \to \widehat{L}_0 \cap M_{\mathsf{f}})$.

$$\begin{tikzcd}
 & & \mathrm{HW}^{\zeta a}(M_{\mathsf{f}}) \\
\mathrm{HW}^{\zeta a}(M_\zeta) \arrow[urr] & \mathrm{HW}^a(M) \arrow[l, "\cong"'] \arrow[ur, dashed, "f_a"'] & \mathrm{HW}^a(M_{\mathsf{f}}) \arrow[l, "g_a"] \arrow[u]
\end{tikzcd}$$

□

7. HW, algebraic growth and entropy

This section we give a proof of Theorem 2.6. The proof here and the proof in [AM17, Section 4] differ slightly. We make here additionally use of Lemma 11.

7.1. Legendrian isotopies, transfer admissible Lagrangians and growth. We start by introducing some notation. Let (M, λ) be a Liouville domain and L be an asymptotically conical exact Lagrangian disk in M. We denote by Λ the Legendrian sphere ∂L. Letting $\Sigma := \partial M$ and $\alpha_M := \lambda|_\Sigma$ be the contact form induced by M on Σ we assume that $(\alpha_M, \Lambda \to \Lambda)$ is regular. As usually, we denote by ξ_M the contact structure $\ker \alpha_M$.

Our approach to prove invariance of the exponential symplectic growth of HW differs from the ones developed by [MS11, McL12]. It makes extensive use of the module and algebra structures that exist on HW. We will need the following

DEFINITION 7.1. Let $\mu > 0$ and Λ_0 be a Legendrian sphere in (Σ, ξ_M). Assume that Λ_1 is Legendrian isotopic to Λ_0. We say that Λ_1 is *μ-close to Λ_0 in the C^3-sense* if there exists a Legendrian isotopy $\theta : [-1,1] \times S^{n-1} \to (\Sigma, \xi_M)$ from Λ_0 to Λ_1 whose C^3-norm is $< \mu$, and which is stationary in the first coordinate outside a compact subset of $(-1, 1)$.

Recall that the symplectisation of a contact form α on (Σ, ξ_M) is the exact symplectic manifold $((0, +\infty) \times \Sigma, d r\alpha, r\alpha)$ where r denotes the first coordinate in $(0, +\infty) \times \Sigma$. The following lemma is essentially due to Chantraine [**Cha10**] and is proved in [**AM17**, Appendix].

LEMMA 7.2. *Fix a constant $\epsilon > 0$, a contact form α on (Σ, ξ), a Legendrian Λ_0 in (Σ, ξ), and a tubular neighbourhood $U(\Lambda_0)$ of Λ_0 in Σ. Then there exists $\delta > 0$ such that if Λ_1 is δ-close to Λ_0 in the C^3-sense, then there exist exact Lagrangian cobordisms $\mathcal{L}^-$ from Λ_1 to Λ_0 and $\mathcal{L}^+$ from Λ_0 to Λ_1 in the symplectization of α satisfying:*

a) $\mathcal{L}^-$ is conical outside $[1 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{4}] \times \Sigma$,
b) $\mathcal{L}^+$ is conical outside $[1 + \frac{\epsilon}{4}, 1 + \frac{\epsilon}{2}] \times \Sigma$,
c) the projections of $\mathcal{L}^+$ and $\mathcal{L}^-$ to Σ are completely contained in $U(\Lambda_0)$,
d) the primitives $f^\pm$ of $(r\alpha)|_{\mathcal{L}^\pm}$ have support in $[1 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{4}] \times \Sigma$ and $[1 + \frac{\epsilon}{4}, 1 + \frac{\epsilon}{2}] \times \Sigma$, respectively, and $|f^\pm|_{C^0} < \epsilon$.

Moreover if $\mathcal{L}$ is the exact Lagrangian cylinder obtained by gluing $\mathcal{L}^+ \cap [1, +\infty) \times \Sigma)$ on top of $\mathcal{L}^- \cap ((0, 1] \times \Sigma)$ we have that

e) $\mathcal{L}$ is Hamiltonian isotopic to $\mathbb{R} \times \Lambda_0$ in the symplectization of α, and the Hamiltonian producing the isotopy can be taken to have support in $[1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}] \times \Sigma$.

We now fix $\epsilon > 0$ such that L is conical on $M \setminus M_{1-2\epsilon}$. We choose a Legendrian tubular neighbourhood $\mathcal{U}(\Lambda)$ of Λ on (Σ, ξ_M). For these choices of $\epsilon > 0$ and $\mathcal{U}(\Lambda)$, we choose $\delta_1 > 0$ given by Lemma 7.2.

Let $\mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$ be the set of Legendrian sphere Λ_1 which are δ_1-close to Λ in the C^3 sense, are disjoint from Λ, and satisfy that $(\alpha_M, \Lambda \to \Lambda_1)$ is regular. Choose $L_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$.

It follows from Lemma 7.2 that there exists an exact Lagrangian cobordism $\mathcal{L}^-$ from Λ_1 to Λ in the symplectization of α_M which is conical outside $[1 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{4}] \times \Sigma$. We can then glue $\mathcal{L}^- \cap [1 - \frac{\epsilon}{2}, 1] \times \Sigma$ to $L \cap M_{1-\frac{\epsilon}{2}}$ to obtain an exact Lagrangian submanifold L_1 in M. The Lagrangian L_1 is an exact filling of Λ_1. Let f_L be the primitive of $\lambda|_L$ which vanishes in Λ. Using Lemma 7.2 we can glue f^- to the restriction of f_L to $L \cap M_{1-\frac{\epsilon}{2}}$ to obtain primitive of f_{L_1} of $\lambda|_{L_1}$ which vanishes in Λ_1.

Because of the control given by Lemma 7.2 on the function $|f^-|_{C^0}$ on $\mathcal{L}^-$, and the facts that L and L_1 coincide on $M_{1-\frac{\epsilon}{2}}$ and f_L vanishes on the collar $L \cap (M \setminus M_{1-\frac{\epsilon}{2}})$ we have

$$\mathsf{K}(M, L \to L_1) < \epsilon. \tag{42}$$

By Lemma 7.2 d) the Lagrangian L_1 is transfer admissible for the pair $(M, M_{1-\epsilon})$. Combining this with (42) we obtain for $a > \epsilon \geq \mathsf{K}(M, L \to L_1)$ a Viterbo map $\Psi^a_{\mathcal{L}^-} : \mathrm{HW}^a(M, L \to L_1) \to \mathrm{HW}^a(M_{1-\epsilon}, L)$, where to simplify notation we keep denoting by L and L_1 the restrictions of L and L_1 to $M_{1-\epsilon}$. Passing to the direct limit we obtain a map $\Psi_{\mathcal{L}^-} : \mathrm{HW}(M, L \to L_1) \to \mathrm{HW}(M_{1-\epsilon}, L)$.

By Lemma 7.2 we also have an exact Lagrangian cobordism $\mathcal{L}^+$ from Λ to Λ_1, which is diffeomorphic to $\mathbb{R} \times S^{n-1}$, and is conical over Λ for $r \geq 1 + \frac{\epsilon}{2}$ and conical over Λ_1 for $r \leq 1 + \frac{\epsilon}{4}$. By gluing $\mathcal{L}^+ \cap ([1, 1+\epsilon] \times \Sigma)$ to L_1 we obtain an exact Lagrangian $\overline{L}$ in $M_{1+\epsilon}$. By Lemma 7.2 d) the Lagrangian $\overline{L}$ is transfer admissible for the pair $(M_{1+\epsilon}, M)$. By gluing f^+ to f_{L_1} we obtain a primitive $f_{\overline{L}}$ of $\lambda|_{\overline{L}}$. Reasoning as in the proof of (42) one obtains

$$\mathsf{K}(M_{1+\epsilon}, L \to \overline{L}) < \epsilon. \tag{43}$$

We thus obtain for each $a > \epsilon$ a Viterbo map $\Psi^a_{\mathcal{L}^+} : \mathrm{HW}^a(M_{1+\epsilon}, L \to \overline{L}) \to \mathrm{HW}^a(M, L \to L_1)$, where by abuse of notation we denote by L the conical extension of L to $M_{1+\epsilon}$. Passing to the direct limit we then obtain a map $\Psi_{\mathcal{L}^+} : \mathrm{HW}(M_{1+\epsilon}, L \to \overline{L}) \to \mathrm{HW}(M, L \to L_1)$.

By Lemma 7.2, $\overline{L}$ is Hamiltonian isotopic to the conical extension of L to $M_{1+\epsilon}$, which we will still denote by L, for a Hamiltonian function which vanishes outside $M_{1+\frac{\epsilon}{2}} \setminus M_{1-\frac{\epsilon}{2}}$. A continuation argument then implies that for each admissible Hamiltonian H that is regular for both $(M_{1+\epsilon}, L)$ and $(M_{1+\epsilon}, L \to \overline{L})$, and has slope $> \epsilon$ we have continuation isomorphisms $\mathrm{HW}(H, L \to \overline{L}) \to \mathrm{HW}(H, L)$. By Section 5.0.4 we conclude that for each $a > \epsilon$ that we have isomorphisms

$$\Phi_a : \mathrm{HW}^a(M_{1+\epsilon}, L \to \overline{L}) \text{ and } \mathrm{HW}^a(M_{1+\epsilon}, L). \tag{44}$$

This induces an isomorphism $\Phi : \mathrm{HW}(M_{1+\epsilon}, L) \to \mathrm{HW}(M_{1+\epsilon}, L \to \overline{L})$.

Let us consider, see Lemma 6.1, the isomorphisms

$$\begin{aligned} \Xi_a^- &: \mathrm{HW}^a(M_{1-\epsilon}, L) \to \mathrm{HW}^{\frac{1}{1-\epsilon}a}(M, L) \text{ and} \\ \Xi_a^+ &: \mathrm{HW}^a(M, L) \to \mathrm{HW}^{(1+\epsilon)a}(M_{1+\epsilon}, L). \end{aligned}$$

Then the family of homomorphisms $f_a = \Xi_a^- \circ \Psi_{\mathcal{L}^-}^a$ and $g_a = \Psi_{\mathcal{L}^+}^{(1+\epsilon)a} \circ \Phi^a \circ \Xi_a^+$, $a > \mathsf{K}(M, L \to L_1)$ are asymptotic morphisms and we have, by Lemma 6.1 and by the functoriality of Viterbo transfer maps, the commutative diagram

$$\begin{tikzcd}
\mathrm{HW}^{\frac{1+\epsilon}{1-\epsilon}a}(M, L \to L_1) & \mathrm{HW}^{\frac{1+\epsilon}{1-\epsilon}a}(M_{1+\epsilon}, L) \arrow[l, dotted] \\
 & \mathrm{HW}^{\frac{1}{1-\epsilon}a}(M, L) \arrow[u, dotted] \arrow[ul, "g_{(\frac{1}{1-\epsilon})a}"'] \\
\mathrm{HW}^a(M, L \to L_1) \arrow[uu] \arrow[ur, "f_a"] \arrow[r, dotted] & \mathrm{HW}^a(M_{1-\epsilon}, L) \arrow[u, dotted]
\end{tikzcd} \tag{45}$$

where the vertical arrow on the left is the persistence homomorphism.[1].

[1]Actually $\widetilde{\mathrm{HW}}(M, L \to L_1)$ and $\widetilde{\mathrm{HW}}(M, L)$ are $(\frac{1}{1-\epsilon}, 1+\epsilon)$-interleaved

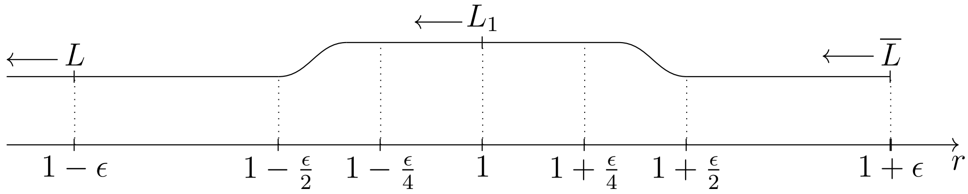

FIGURE 4.

Since L is conical on $M_{1+\epsilon} \setminus M_{1-\epsilon}$, $M \setminus M_{1-\epsilon}$ and $M_{1+\epsilon} \setminus M$, we have transfer maps

$$\begin{aligned}
&\Psi_L^{\pm} : \mathrm{HW}(M_{1+\epsilon}, L) \to \mathrm{HW}(M_{1-\epsilon}, L), \\
&\Psi_L^{-} : \mathrm{HW}(M, L) \to \mathrm{HW}(M_{1-\epsilon}, L), \\
&\Psi_L^{+} : \mathrm{HW}(M_{1+\epsilon}, L) \to \mathrm{HW}(M, L).
\end{aligned}$$

that are algebra isomorphisms by Lemma 6.1.

We denote the algebra $\mathrm{HW}(M, L)$ by A_L. The homologies $\mathrm{HW}(M_{1-\epsilon}, L)$, $\mathrm{HW}(M, L \to L_1)$ and $\mathrm{HW}(M_{1+\epsilon}, L \to \overline{L})$ are modules over the algebras $\mathrm{HW}(M, L)$, $\mathrm{HW}(M_{1+\epsilon}, L)$ and $\mathrm{HW}(M_{1-\epsilon}, L)$, respectively: they are therefore A_L-modules. By this discussion and (31) in section 6 the maps Φ, $\Psi_{\mathcal{L}^-}$ and $\Psi_{\mathcal{L}^+}$ are A_L-module homomorphisms.

Since the Viterbo transfer map is functorial and is invariant under a Hamiltonian deformation in the cobordism, the diagram

$$\begin{tikzcd}
\mathrm{HW}(M_{1-\epsilon}, L) & \\
\mathrm{HW}(M, L \to L_1) \arrow[u, "\Psi_{\mathcal{L}^-}"] & \\
\mathrm{HW}(M_{1+\epsilon}, L \to \overline{L}) \arrow[u, "\Psi_{\mathcal{L}^+}"] & \mathrm{HW}(M_{1+\epsilon}, L) \arrow[l, "\Phi"] \arrow[uul, "\Psi_L^{\pm}"']
\end{tikzcd}$$

is commutative. It thus follows that the map $\Psi_{\mathcal{L}^-} \circ \Psi_{\mathcal{L}^+}$ is an A_L-module isomorphism. We thus conclude that $\Psi_{\mathcal{L}^+}$ is injective. Let $\mathbf{1}_L$ be the unit in

$\mathrm{HW}(M_{1+\epsilon}, L)$. As Φ is an A_L-module isomorphism and $\Psi_{\mathcal{L}^+}$ is an injective A_L-module homomorphism we know that the element $m_{L_1} := \Psi_{\mathcal{L}^+} \circ \Phi(\mathbf{1}_L)$ in $\mathrm{HW}(M, L \to L_1)$ is a stretching element. We have thus proved the following:

LEMMA 7.3. $\mathrm{HW}(M, L \to L_1)$ *is a stretched module over* $\mathrm{HW}(M, L)$. *It follows from Lemma 4.10, Lemma 5.5, and Lemma 5.8 that*

$$\Gamma^{\mathrm{symp}}(M, L \to L_1) \geq \Gamma^{\mathrm{symp}}(M, L).$$

Let us estimate $c(m_{L_1})$. We have that $c(\mathbf{1}_L) = 0$, hence $c(\Phi(\mathbf{1}_L)) \leq \epsilon$ by (44). Therefore also

$$c(m_{L_1}) \leq \epsilon \tag{46}$$

By the discussion above this holds independently of a choice of $\Lambda_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$.

PROPOSITION 7.4. *The family* $(\mathrm{HW}(M, L \to L_1))_{\Lambda_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)}$ *of* A_L-*modules is uniformly stretched. It follows from Lemma 4.14, Lemma 5.5, and Lemma 5.8 that*

$$\Gamma^{\mathrm{symp}}_{\Lambda_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)}(M, L \to L_1) \geq \Gamma^{\mathrm{symp}}(M, L).$$

Now, let $\Lambda_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$ and $\mathsf{N}^a_{\alpha_M}(\Lambda \to \Lambda_1) = \#\mathcal{T}^a_{\Lambda \to \Lambda_1}(\alpha_M)$. We define

$$\mathsf{N}^a_{\alpha_M}(\Lambda \to \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)) := \inf_{\Lambda_q \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)} \{\mathsf{N}^a_{\alpha_M}(\Lambda \to \Lambda_q)\}.$$

We can now prove the following proposition, that is crucial for the estimate of the topological entropy.

PROPOSITION 7.5. *The sequence of numbers* $\mathsf{N}^a_{\alpha_M}(\Lambda \to \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda))$ *satisfies*

$$\limsup_{a \to +\infty} \frac{\log \mathsf{N}^a_{\alpha_M}(\Lambda \to \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda))}{a} \geq \Gamma^{\mathrm{symp}}(M, L). \tag{47}$$

PROOF. Let $a > \epsilon$. Let $L_1 \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$. By the results of Section 5.0.4 there exists a Hamiltonian $H^a \in \mathcal{H}_{\mathrm{reg}}(M, L \to L_1)$ with slope $< a$ such that

- all elements in $\mathcal{T}_{L \to L_1}(H^a)$ have action $< a$,
- the map ${}^{\epsilon}\chi^{H^a \to}_{\to a} : \mathrm{HW}^{[\epsilon, +\infty)}(H^a, L \to L_1) \to \mathrm{HW}^a(M, L \to L_1)$ is an isomorphism.

By the discussion before all intersection points $L \cap L_1$ have action $< \epsilon$, hence we have that

$$\begin{aligned} \mathsf{N}^a_{\alpha_M}(\Lambda \to \Lambda_1) &\geq \dim \mathrm{CW}^{[\epsilon,+\infty)}(H^a, L \to L_1) \\ &\geq \mathrm{HW}^{[\epsilon,a)}(M, L \to L_1). \end{aligned} \tag{48}$$

Using (20) we conclude that

$$\begin{aligned} \mathsf{N}^a_{\alpha_M}(\Lambda \to \Lambda_1) \geq \dim\{x \in \mathrm{HW}(M, L \to L_1) \,|\, c(x) \leq a\} \\ - \dim\{x \in \mathrm{HW}(M, L \to L_1) \,|\, c(x) \leq \epsilon\}. \end{aligned} \tag{49}$$

It follows that

$$\begin{aligned} \mathsf{N}^a_{\alpha_M}(\Lambda \to \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)) \geq \inf_{\Lambda_q \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)} \dim\{x \in \mathrm{HW}(M, L \to L_q) \,|\, c(x) \leq a\} \\ - \sup_{\Lambda_q \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)} \dim\{x \in \mathrm{HW}(M, L \to L_q) \,|\, c(x) \leq \epsilon\}. \end{aligned}$$

It follows from diagram (45) that there is a uniform upper bound of $\dim\{x \in \mathrm{HW}(M, L \to L_q) | \, c(x) \leq \epsilon\}$ for $\Lambda_q \in \mathcal{V}^{\mathrm{reg}}_{\alpha_M}(\Lambda)$, hence the statement follows with Proposition 7.4. $\square$

7.2. From the growth of Reeb chords to topological entropy. Let α be a contact form on a contact manifold (Σ, ξ), and X_α be its Reeb vector field. Recall that a Riemannian metric g on X is said to be compatible with α if $g(X_\alpha, X_\alpha) = 1$ and X_α is orthogonal to ξ with respect to g.

We proceed by fixing some more notation. We denote by $\mathbb{D}^n(\rho)$ the n-dimensional disk of radius $\rho > 0$ around the origin. We endow $\mathbb{D}^n(\rho)$ with the Euclidean metric, and consider on $T^*_1\mathbb{D}^n(\rho) = \mathbb{D}^n(\rho) \times S^{n-1}$ the contact form α_{euc} associated to the Euclidean metric. For each $z \in \mathbb{D}^n(\rho)$ the sphere $S^{n-1}_z := \{z\} \times S^{n-1}$ is Legendrian in $(\mathbb{D}^n(\rho) \times S^{n-1}, \ker \alpha_{\mathrm{euc}})$. Let g_{round} be the metric with constant curvature 1 on S^{n-1} and g_{euc} be the Euclidean metric on $\mathbb{D}^n(\rho)$. The metric $\widetilde{g} = g_{\mathrm{euc}} \oplus g_{\mathrm{round}}$ on $\mathbb{D}^n(\rho) \times S^{n-1}$ is compatible with the contact form α_{euc}; see [Cal05].

PROPOSITION 7.6. *Let $\alpha = \alpha_M$ be as above the contact form on (Σ, ξ_M) induced by λ on Σ and assume that we have $\Gamma^{\mathrm{alg}}_S(M, L) > 0$. Then there* *exists a Riemannian metric g on (Σ, ξ_M) adapted to α_M, such that*

$$\limsup_{t\to+\infty} \frac{\log \mathrm{Vol}_g^{n-1}(\phi_\alpha^t(\Lambda))}{t} \geq \Gamma^{\mathrm{symp}}(M, L) > 0,$$

where Vol_g^{n-1} is the $(n-1)$-dimensional volume with respect to g.

PROOF. The proof will consist of several steps.

Step 1. Construction of the metric g. It follows from the Legendrian neighbourhood theorem (see [**KM97**, Proposition 43.18]) that there exists a tubular neighbourhood $\mathcal{V}(\Lambda)$ of Λ and a contactomorphism $\Upsilon : (\mathcal{V}(\Lambda), \xi_M) \to (\mathbb{D}^n(\rho) \times S^{n-1}, \ker \alpha_{\mathrm{euc}})$ that satisfies

$$\Upsilon^* \alpha_{\mathrm{euc}} = \alpha, \tag{50}$$

$$\Upsilon(\Lambda_1) = \{0\} \times S^{n-1}. \tag{51}$$

We extend the Riemannian metric $\Upsilon^* \widetilde{g}$, which is compatible with α on $\mathcal{V}(\Lambda)$, to a metric g on Σ which is compatible with the contact form α.

After shrinking the neighbourhood $\mathcal{V}(\Lambda)$ and $\rho > 0$, we can assume that for every $z \in \mathbb{D}^n(\rho)$ the Legendrian $\Lambda^z := \Upsilon^{-1}(\{z\} \times S^{n-1})$ is in the neighbourhood $\mathcal{V}_{\alpha_M}(\Lambda)$ constructed in Section 7.1.

Step 2. For each $a > 0$ we define the map $F_\Lambda^a : \Lambda \times [0, a] \to \Sigma$ by

$$F_\Lambda^a(q, t) = \phi_\alpha^t(q).$$

Let $\mathrm{Cyl}_\alpha^a(\Lambda)$ be the image $F_\Lambda^a(\Lambda \times [0, a])$. We want to estimate from below the n-dimensional volume $\mathrm{Vol}_g^n(\mathrm{Cyl}_\alpha^a(\Lambda))$ of $\mathrm{Cyl}_\alpha^a(\Lambda)$ with respect to the Riemannian metric g. For this we define $\mathfrak{B}_\alpha^a(\Lambda) := \Upsilon(\mathrm{Cyl}_\alpha^a(\Lambda) \cap \mathcal{V}(\Lambda_1))$. We have

$$\mathrm{Vol}_g^n(\mathrm{Cyl}_\alpha^a(\Lambda)) \geq \mathrm{Vol}_g^n(\mathrm{Cyl}_\alpha^a(\Lambda) \cap \mathcal{V}(\Lambda)) = \mathrm{Vol}_{\widetilde{g}}^n(\mathfrak{B}_\alpha^a(\Lambda)). \tag{52}$$

Let $\Pi : \mathbb{D}^n(\rho) \times S^{n-1} \to \mathbb{D}^n(\rho)$ be the projection to the first coordinate. Applying Sard's Theorem to the map $\Pi \circ \Upsilon \circ F_\Lambda^a : (\{a\} \times \Lambda) \cap (F_\Lambda^a)^{-1}(\mathcal{V}(\Lambda)) \to \mathbb{D}^n(\rho)$ we conclude that the set $\mathbb{D}^n(\rho) \setminus \Pi \circ \Upsilon(\phi_\alpha^a(\Lambda))$ is an open set of full

Lebesgue measure in $\mathbb{D}^n(\rho)$. We define the set $\mathfrak{U}^a_\alpha(\Lambda) \subset \mathbb{D}^n(\rho) \setminus \Pi \circ \Upsilon(\phi^a_\alpha(\Lambda))$ by the property

- $z \in \mathfrak{U}^a_\alpha(\Lambda)$ if all α-Reeb chords from Λ to Λ^z with length $< a$ are transverse.

The proof of the next lemma is identical to the one of [Alv17, Lemma 3].

LEMMA 7.7. *The set $\mathfrak{U}^a_\alpha(\Lambda)$ is an open subset of $\mathbb{D}^n(\rho)$ of full Lebesgue measure. The set $\widetilde{\mathfrak{U}}^a_\alpha(\Lambda) \subset \mathfrak{U}^a_\alpha(\Lambda)$ of those elements $z \in \mathfrak{U}^a_\alpha(\Lambda)$ such that $\Lambda^z \in \mathcal{V}^{\alpha-\mathrm{reg}}_{\alpha_M}(\Lambda_1)$ is a dense subset of full Lebesgue measure in $\mathfrak{U}^a_\alpha(\Lambda)$.*

Step 3. A volume estimate. The function $h^a : \mathfrak{U}^a_\alpha(\Lambda) \to [0, +\infty)$ defined by $h^a(z) := \#(\mathcal{T}^a_{\Lambda \to \Lambda^z}(\alpha))$ is locally constant on $\mathfrak{U}^a_\alpha(\Lambda)$ since it is continuous and takes only integer values.

We define $\mathfrak{R}^a_\alpha(\Lambda) := \Pi^{-1}(\mathfrak{U}^a_\alpha(\Lambda)) \cap \mathfrak{B}^a_\alpha(\Lambda)$. Since $\mathfrak{R}^a_\alpha(\Lambda) \subset \mathfrak{B}^a_\alpha(\Lambda)$ we have $\mathrm{Vol}^n_{\tilde{g}}(\mathfrak{B}^a_\alpha(\Lambda)) \geq \mathrm{Vol}^n_{\tilde{g}}(\mathfrak{R}^a_\alpha(\Lambda))$. As the map $\Pi : \mathbb{D}^n(\rho) \times S^{n-1} \to \mathbb{D}^n(\rho)$ is a Riemannian submersion we have that $\mathrm{Vol}^n_{\tilde{g}}(\mathfrak{R}^a_\alpha(\Lambda)) \geq \mathrm{Vol}^n_{g_{\mathrm{euc}}}(\Pi(\mathfrak{R}^a_\alpha(\Lambda)))$, where $\mathrm{Vol}^n_{g_{\mathrm{euc}}}(\Pi(\mathfrak{B}^a_\alpha(\Lambda)))$ is computed with multiplicities. If an open set is covered k-times by $\Pi : \mathfrak{R}^a_\alpha(\Lambda) \to \mathfrak{U}^a_\alpha(\Lambda)$, then its volume contributes k-times to $\mathrm{Vol}^n_{g_{\mathrm{euc}}}(\Pi(\mathfrak{R}^a_\alpha(\Lambda)))$.

For each $z \in \mathfrak{U}^a_\alpha(\Lambda)$ the number of times $\Pi : \mathfrak{R}^a_\alpha(\Lambda) \to \mathfrak{U}^a_\alpha(\Lambda)$ covers z is $h^a(z) = \#(\mathcal{T}^a_{\Lambda \to \Lambda^z}(\alpha))$. We thus obtain

$$\mathrm{Vol}^n_{g_{\mathrm{euc}}}(\Pi(\mathfrak{R}^a_\alpha(\Lambda))) = \int_{\mathfrak{U}^a_\alpha(\Lambda)} h^a(z)\, d\mathsf{vol}_{g_{\mathrm{euc}}},$$

where $d\mathsf{vol}_{g_{\mathrm{euc}}}$ is the volume form generated by g_{euc} on $\mathbb{D}^n(\rho)$.

Since $\Gamma^{\mathrm{symp}}(M, L) > 0$, we can fix $0 < \eta < \Gamma^{\mathrm{symp}}(M, L)$. It follows from Proposition 7.5 that there exists a sequence $a_j \to +\infty$ such that $h^{a_j}(z) \geq e^{\eta a_j}$ for all $z \in \widetilde{\mathfrak{U}}^{a_j}_\alpha(\Lambda)$. Since $\widetilde{\mathfrak{U}}^{a_j}_\alpha(\Lambda)$ is dense in $\mathfrak{U}^{a_j}_\alpha(\Lambda)$ and h^{a_j} is locally constant on $\mathfrak{U}^{a_j}_\alpha(\Lambda)$ we obtain $h^{a_j}(z) \geq e^{\eta a_j}$ for all $z \in \mathfrak{U}^{a_j}_\alpha(\Lambda)$ and all a_j. With (52) it follows that

$$\mathrm{Vol}^n_g(\mathrm{Cyl}^{a_j}_\alpha(\Lambda)) \geq \int_{\mathfrak{U}^{a_j}_\alpha(\Lambda)} h^{a_j}(z)\, d\mathsf{vol}_{g_{\mathrm{euc}}} \geq e^{\eta a_j} \pi \rho^2 \tag{53}$$

for every a_j.

Step 4. A Fubini type equality. We define $\widehat{g} := (F^a_\Lambda)^* g$. Then

$$\mathrm{Vol}^n_g(\mathrm{Cyl}^a_\alpha(\Lambda)) = \int_{\Lambda \times [0,a]} d\mathsf{vol}_{\widehat{g}}, \tag{54}$$

where $d\mathsf{vol}_{\widehat{g}}$ is the volume form associated to $\widehat{g}$. Since the metric g is adapted to the contact form α the Reeb vector field has length 1 and is orthogonal to the Legendrian spheres $F^a_\Lambda(t,\Lambda) = \phi^t_\alpha(\Lambda)$ for every $t \in [0,a]$. Letting ∂_t be the tangent vector field on $[0,a] \times \Lambda$ associated to the first coordinate $t \in [0,a]$, and using the definition of F^a_Λ, it follows that $D(F^a_\Lambda)\partial_t = X_\alpha$. Therefore ∂_t has $\widehat{g}$-norm equal to 1 at every point in $[0,a] \times \Lambda$, and is orthogonal to the spheres $\{t\} \times \Lambda$. We thus conclude that

$$\begin{aligned} \mathrm{Vol}^n_g(\mathrm{Cyl}^a_\alpha(\Lambda)) &= \int_{\Lambda \times [0,a]} d\mathsf{vol}_{\widehat{g}} = \int_0^a \mathrm{Vol}^{n-1}_{\widehat{g}}(\{t\} \times \Lambda) dt \\ &= \int_0^a \mathrm{Vol}^{n-1}_g(\phi^t_\alpha(\Lambda)) dt, \end{aligned} \tag{55}$$

where $\mathrm{Vol}^{n-1}_{\widehat{g}}$ is the $(n-1)$-dimensional volume associated to $\widehat{g}$.

Step 5. To finish the proof we argue by contradiction and assume that $\limsup_{t\to+\infty} \frac{\log \mathrm{Vol}^{n-1}_g(\phi^t_\alpha(\Lambda))}{t} < \eta$. In this case, there exist $a_0 > 0$ and $\varepsilon > 0$ such that for all $t \geq a_0$ we have $\mathrm{Vol}^{n-1}_g(\phi^t_\alpha(\Lambda)) \leq e^{t(\eta-\varepsilon)}$. Integrating both sides of this inequality from 0 to $a \geq a_0$ and invoking (55) we obtain

$$\mathrm{Vol}^n_g(\mathrm{Cyl}^a_\alpha(\Lambda)) \leq \frac{e^{a(\eta-\varepsilon)} - e^{a_0(\eta-\varepsilon)}}{\eta - \varepsilon} + \int_0^{a_0} \mathrm{Vol}^{n-1}_g(\phi^t_\alpha(\Lambda)) dt. \tag{56}$$

For a large enough the right hand side of (56) is smaller than $e^{\eta a}\pi\rho^2$, contradicting (53). We thus conclude that

$$\limsup_{t\to+\infty} \frac{\log \mathrm{Vol}^{n-1}_g(\phi^t_\alpha(\Lambda))}{t} \geq \eta.$$

Since this is valid for any $\eta < \Gamma^{\mathrm{symp}}(M,L)$, the proof of the proposition is completed. $\square$

PROOF OF THEOREM 2.6. Let α be any supporting contact form of ξ_M, and f_α the positive function on Σ with $\alpha = \mathsf{f}_\alpha \alpha_M$. Then by Lemma 6.2 we get that

$$\Gamma^{\mathrm{symp}}(M_{\mathsf{f}_\alpha}) \geq \frac{\Gamma^{\mathrm{symp}}(M, L)}{\max(\mathsf{f}_\alpha)}$$

From Proposition 7.6 applied to the Liouville domain M_{f_α} and Yomdin's inequality (see (1)) it follows that

$$h_{\mathrm{top}}(\phi_\alpha) \geq \frac{\Gamma^{\mathrm{symp}}(M, L)}{\max(\mathsf{f}_\alpha)}. \tag{57}$$

We then obtain Theorem 2.6 by combining (57) with the inequality

$$\Gamma^{\mathrm{symp}}(M, L) \geq \frac{\Gamma^{\mathrm{alg}}_S(M, L)}{\rho(S)}.$$

from Lemma 5.7. □

8. Modifications of Liouville domains and HW

8.1. Contact surgery. One main method of constructing contact manifolds is to apply contact surgery. The classical surgery construction on an n-dimensional differential manifold Q is the procedure of constructing an n-dimensional manifold Q' by replacing a neighbourhood of a sphere S^k in Q, provided that it is diffeomorphic to $S^k \times D^{n-k}$, by $D^{k+1} \times S^{n-k-1}$, which means that we glue the latter space along its boundary to the boundary of $Q \setminus (S^k \times D^{n-k})$, see for example [Mil61]. In [Eli90] Eliashberg showed that this construction can as well be applied in the contact category. He actually proved a more general statement about extendability of Stein structures of fillings of the contact manifold on which surgery is applied. We here shortly describe the contact surgery construction and follow Weinstein's paper [Wei91], where a proof of some parts of Eliashbergs construction was given in a simplified form. Good references are also [Cie02], [Gei97] or [Gei08, Chapter 6] and we refer to them for more details.

Let $W = (W, \lambda_W)$ be a Liouville domain, $\Sigma = \partial W$, $\lambda_W|_\Sigma = \alpha$ and $\xi = \ker \alpha$. We recall some notions using the terminology of [**Gei08**, Section 2.5.2]. The form $d\alpha$ endows ξ with a natural conformal symplectic bundle structure. Let S be an isotropic submanifold of (Σ, ξ). We write $TS^\perp$ for the sub-bundle of ξ that is $d\alpha$-orthogonal to TS. We have $TS \subset TS^\perp$ since S is isotropic. We can therefore write the normal bundle of S in Σ as

$$T\Sigma/TS = T\Sigma/\xi \oplus \xi/TS^\perp \oplus TS^\perp/TS.$$

The *conformal symplectic normal bundle* $\mathrm{CSN}(\Sigma, \mathrm{S}) = TS^\perp/TS$ has a natural conformal symplectic structure via $d\alpha$. If S is a sphere, $T\Sigma/\xi \oplus \xi/TS^\perp$ has a trivialization. The following theorem is due to Weinstein.

THEOREM 8.1. [**Wei91**] *Let S^n be an isotropic sphere in Σ with a trivialization of* $\mathrm{CSN}(\Sigma, S)$. *Then there is a Liouville domain M with an exact embedding $(W, \lambda_W) \subset (M, \lambda_M)$, such that $\Sigma' = \partial M$ is obtained from Σ by surgery on S using the natural framing. In particular Σ' also carries a contact structure.*

REMARK 8.2. (1) In fact, if W is equipped with a Weinstein structure, one can extend it to M.
(2) The cobordism $M \setminus W$ is essentially given by attaching a handle along $\partial W \times \{1\}$ to $\partial W \times [0, 1]$. On the handle the Weinstein/Liouville structure can be described explicitly, see [**Wei91**], we refer to it as a *Weinstein handle*, and say that M is obtained by attaching an $(n+1)$-dimensional Weinstein handle along $S^n \subset \partial W$.
(3) The Liouville vector field X can be chosen such that there is exactly one point $p \in M \setminus W$ where X vanishes. The integral lines of X that are asymptotic to p intersect Σ in S^n and ∂M in the *co-core sphere* $B \subset \partial M$.

That usefulness of 8.1 for constructions is the following h- principle which allows one to apply contact surgery in a prescribed isotopy class of embeddings $S^k \to \Sigma$, see [**Gei08**, Prop. 6.3.5/6].

THEOREM 8.3. *Let Σ be a $2n-1$-dimensional contact manifold. Let $i : N \to \partial W, k < n$ be an embedding of a closed manifold N. Assume that i is covered by a fibrewise injective bundle map $TN \otimes \mathbb{C} \to \xi$. Then there is an isotropic embedding $i_0 : N \to \partial M$ that is isotopic to i.*

When applying contact surgery on a contact $2n-1$ manifold Σ we distinguish between *critical* and *subcritical* surgery, depending on whether the dimension of the isotropic sphere S^k along which the surgery takes place has critical dimension $k = n-1$ or subcritical dimension $k < n-1$.

8.2. Subcritical surgery. A particular feature of the subcritical case is the following phenomenon. If a Liouville domain M is obtained by a subcritical Weinstein handle attachment to a Liouville domain W, the symplectic homologies $\mathrm{SH}(M)$ and $\mathrm{SH}(W)$ are isomorphic. This is in contrast to the effect such a handle attachment has on the topology of W. That result was proved by Cieliebak in [**Cie02**]. For a detailed account, clarifying different parts of the proof, see also [**McL09**, Appendix C] or [**Fau17**]. We will use later a more general statement on the effect of SH under subcriticial surgery due to McLean, see Theorem 9.2. In this section we prove an analogous result for wrapped Floer homology.

8.2.1. *Subcritical surgery and* HW. We give a proof of the invariance of HW in a situation that is sufficient for our purpose, that is we assume that the Lagragians do not intersect the handle. Here the situation is a bit simpler as in the case of SH. For special situations and where also the Lagrangians cross the handle the invariance of HW was proved in [**Iri13**].

Let M be a Liouville domain obtained by attaching an $(n+1)$-handle to W. The Liouville vector field X can by chosen such that there is exactly one point $p \in M \setminus W$ where X vanishes in $M \setminus W$. The integral lines of X that are asymptotic to p intersect Σ in S and ∂M in the co-core sphere $B \subset \partial M$. (See [**Wei91, Cie02**] or [**Gei08**, Chapter 6] for details.)

Let now L_0', L_1' be two asymptotically conical exact Lagrangians in W whose boundaries Λ_0' and Λ_1' in Σ do not intersect S. Outside S the integral

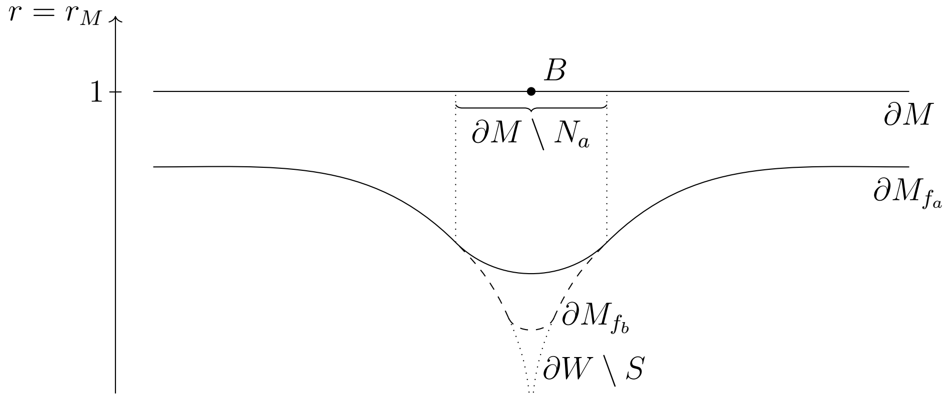

FIGURE 5.

lines of the Liouville vector field starting at ∂W intersect ∂M and so the completed Lagrangians $\widehat{L_i'} \subset \widehat{M}$ intersect ∂M. Moreover, $L_i = \widehat{L_i'} \cap M \subset M$ for $i = 0, 1$ are exact and conical in the complement of W. We say that (M, L_0, L_1) is *obtained by surgery* from (W, L_0', L_1').

As described in section 6 we get a Viterbo transfer map

$$j_!(L_0, L_1) : \widetilde{\mathrm{HW}}(M, L_0 \to L_1) \to \widetilde{\mathrm{HW}}(W, L_0' \to L_1').$$

Assume that the isotropic sphere S has the property that there is no Reeb chord from Λ_0' to S. If S is subcritical, i.e. $\dim(S) < n - 1$, this can be achieved by a generic perturbation of S, or equivalently, a generic perturbation of λ.

The following proposition is an analogous result for HW of Cieliebak's result [**Cie02**, Theorem 1.11]. Its proof for HW is even simpler.

PROPOSITION 8.4. *The Viterbo transfer map in the direct limit,*

$$\bar{j}_!(L_0, L_1) : \mathrm{HW}(M, L_0, L_1) \to \mathrm{HW}(W, L_0', L_1'),$$

is an isomorphism.

For the proof of Proposition 8.4 it is convenient to introduce the following weaker form of interleaving of f.d.s. Let $\sigma : [0, \infty) \to [0, \infty)$ be a monotone increasing function and V a filtered directed system. Analogously to the notation in 4.1.1 let $(V(\sigma), \pi(\sigma))$ be given by $V(\sigma)_t = V_{\sigma(t)t}$, $\pi(\sigma)_{s\to t} = \pi_{\sigma(s)s\to\sigma(t)t}$ and $\pi[\sigma]_t = \pi_{\sigma(t)t}$. If f is a morphism from (V, π) to another f.d.s. we write $f(\sigma)_t = f_{\sigma(t)t}$ for the induced morphism with domain $(V(\sigma), \pi(\sigma))$. A *weakly interleaving* of two f.d.s. (V, π_V) and (W, π_W) is a pair (f, g) of morphisms, $f : V \to W(\sigma_1)$ and $g : W \to V(\sigma_2)$, where $\sigma_1, \sigma_2 \geq 1$ are monotonically increasing functions, such that

$$f(\sigma_2) \circ g = \pi_W[\widetilde{\sigma}_1] \text{ and } g(\sigma_1) \circ f = \pi_V[\widetilde{\sigma}_2],$$

where $\widetilde{\sigma}_1$, and $\widetilde{\sigma}_2$ are suitably chosen. We also call then V and W weakly interleaved. The fact that the map $\bar{j}_!(L_0, L_1)$ in Proposition 8.4 is an isomorphism will follow from a weak interleaving of the corresponding f.d.s., which is in general not an interleaving. This is the reason why we cannot directly prove lower bounds for $\Gamma^{\mathrm{symp}}(M, L_0 \to L_1)$ in terms of $\Gamma^{\mathrm{symp}}(W, L_0' \to L_1')$ and this was originally our motivation to introduce the algebraic growth of wrapped Floer homology.

PROOF OF PROPOSITION 8.4. Let $U := \widetilde{\mathrm{HW}}(M, L_0 \to L_1)$ and let $V := \widetilde{\mathrm{HW}}(W, L_0' \to L_1')$. We will construct a filtered directed system Q that is isomorphic to V and weakly interleaved with U.

For convenience we may assume $\mathsf{K}(M, L_0 \to L_1) = 0$. Let $S \subset \partial W$ be the attaching sphere and $B \subset \partial M$ be the co-core sphere. For $a > 0$, choose a tubular neighbourhood $U_a \subset \partial W$ of S such that there is no Reeb trajectory starting at Λ_0' that intersects U_a at a time less than a, and such that $U_b \subset U_a$ if $a < b$. Denote the Liouville flow on $\widehat{M}$ by φ_t and let $g : \partial M \setminus B \to (0, 1]$ be given by $g(x) = t$ where $t \in (0, 1]$ is the unique number such that $\varphi_{\log t}(x) \in \partial W$. Note that g tends to 0 as x tends to B. Define the set $N_a \subset \partial M$ by $N_a := \{x \in \partial M \mid \varphi_{\log g(x)}(x) \in \partial W \setminus U_a\}$. Choose a family of smooth functions $f_a : \partial M \to (0, 1]$, $a \in (0, \infty)$, with the property

- $f_a|_{N_a} = g$,
- $\forall x \in \partial M,\ f_a(x)$ is monotonically decreasing in a.

Note that $W \subset M_{f_b} \subset M_{f_a}$, for $b \geq a$ and $\partial W \setminus U_a \subset \partial M_{f_a}$. See also the figure on page 62.

Define $\sigma(a) = \frac{1}{\min_{\partial M} f_a}$. Define $Q_a := \mathrm{HW}^a(M_{f_a}, L_0 \to L_1)$, where by abuse of notation we write L_i instead of $L_i \cap M_{f_a}$ for $i = 0, 1$. For $a < b$ define the map $\pi_{a \to b} : Q_a \to Q_b$ as the composition of the Viterbo transfer map $\mathrm{HW}^a(M_{f_a}, L_0 \to L_1) \to \mathrm{HW}^a(M_{f_b}, L_0 \to L_1)$ and the persistence map $\mathrm{HW}^a(M_{f_b}, L_0 \to L_1) \to \mathrm{HW}^b(M_{f_b}, L_0 \to L_1)$. By the commutativity of the Viterbo map with persistence maps and by functoriality of the Viterbo map it follows that $\pi_{a \to c} = \pi_{b \to c} \circ \pi_{a \to b}$, for $a \leq b \leq c$, and hence (Q, π) is a filtered directed system. Furthermore, via the family of homomorphisms $\phi_a : \mathrm{HW}^a(M) \to \mathrm{HW}^a(M_{f_a})$ we obtain a morphism of f.d.s. $\phi : U \to Q$. We define $\psi : Q \to U(\sigma)$ by the Viterbo transfer $\mathrm{HW}^a(M_{f_a}) \to \mathrm{HW}^a(M_{\min f_a}) = \mathrm{HW}^{\sigma(a) \cdot a}(M)$. It is clear that (ϕ, ψ) is a weak interleaving of U and Q.

It remains to show that Q and V are isomorphic. Let $a > 0$. We may assume that L_0 and L_1 are conical in the complement of $W_{\frac{1}{2}}$. Let H_μ be an admissible Hamiltonian with slope μ with respect to $W_{\frac{1}{2}}$. Consider a Hamiltonian K_μ such that

$$K_\mu(x) = H_\mu(x), \text{ if } x \in W_{\frac{3}{4}}. \tag{58}$$

$$K_\mu(x) = H_\mu(x), \text{ if } x \in (0, +\infty) \times \partial W \setminus U_a. \tag{59}$$

It follows that on $(1, +\infty)\partial W \setminus U_a$, $K_\mu(x) = 2\mu r_W + b$, for some $b \in \mathbb{R}$, where x is written in the coordinates $(r_W, y) \in (1, +\infty) \times \partial W \setminus U_a$. Hence we can assume additionally that

$$K_\mu(x) = 2\mu r_{M_{f_a}} + b, \text{ where } x = (r_{M_{f_a}}, y) \in (1, +\infty) \times \partial M_{f_a}. \tag{60}$$

By definition of U_a, $\mathcal{A}_{H_\mu}$ and $\mathcal{A}_{K_\mu}$ have the same critical points, and so it follows from [**AS10b**, Lemma 7.2] that we actually have that $\mathrm{HW}(K_\mu) = \mathrm{HW}(H_\mu)$. On the other hand the shift $K_\mu - \frac{1}{2}\mu$ is admissible with respect to M_{f_a} with slope 2μ. One concludes, reasoning as in Lemma

6.1, that $Q_a = \mathrm{HW}^a(M_{f_a}, L_0 \to L_1) \cong \mathrm{HW}^{\frac{1}{2}a}(W_{\frac{1}{2}}, L_0' \to L_1')$ which is, by Lemma 6.1, isomorphic to $\mathrm{HW}^a(W, L_0' \to L_1') = V_a$. That this identification respects the persistence morphisms of Q and V is again deduced from the functoriality of the Viterbo maps and the fact that the Viterbo maps are themselves morphisms of filtered directed systems. Denote the isomorphism from Q to V by τ. We have obtained a weak interleaving $(\tau \circ \phi, \psi \circ \tau^{-1})$. Moreover $\tau \circ \phi = j_!$ by construction. It follows that $\bar{j}_! = \varinjlim j_!$ is an isomorphism. $\square$

8.3. Plumbing. Let Q_1 and Q_2 be closed orientable n-dimensional manifolds. We let D^*Q_i be the unit cotangent bundle of Q_i. We choose balls $B_i \subset Q_i$ in each Q_i. The plumbing N of D^*Q_1 and D^*Q_2 is obtained by identifying D^*B_1 and D^*B_2 via a symplectomorphism that swaps the momentum and position coordinates of these manifolds; see [**AS12**, **Gei08**] for the details. There are obvious embeddings of $D^*(Q_1 \setminus B_1)$ and $D^*(Q_2 \setminus B_2)$ into N. It is shown in [**AS12**, Section 4] that N admits a Liouville structure which coincides with those of $D^*(Q_i \setminus B_i)$ on the image of these embeddings. This implies that for points $q_1 \in Q_1 \setminus B_1$ the cotangent disc fibre L_{q_1} over q_1 survives as a conical exact Lagrangian in the Liouville domain N.

This construction can be generalised in the following way. Let Q_i, $1 \leq i \leq k$ be a finite collection of orientable n-dimensional manifolds. Let T be a tree with k vertices and use a bijection to associate to each vertex a manifold Q_i. For each edge η leaving the "vertex" Q_i we choose an embedded open ball $B_i(\eta)$ in Q_i. We assume that these balls are chosen to be disjoint and do not cover Q_i. For all $i \neq j$ and every edge η connecting Q_i and Q_j (there can be at most one such edge as T is a tree) we identify $D^*(B_i(\eta))$ and $D^*(B_j(\eta))$ by the recipe explained in the previous paragraph. The resulting manifold N can be given a Liouville structure as explained in [**AS12**, Section 4] and [**Gei08**]. Let $\dot{Q}_1$ be the complement of the "edge balls" in Q_1, and $q_1 \in \dot{Q}_1$. In [**AS12**, Section 4] the following result is proved.

THEOREM 8.5. [**AS12**] *There exists an injective algebra homomorphism from the group algebra* $\mathbb{Z}_2[\pi_1(Q_1)]$ *to* $\mathrm{HW}(N, L_{q_1})$.

In fact the injective algebra homomorphism obtained in [AS12] is for the respective homologies with $\mathbb{Z}$ coefficients, and applying the Universal Coefficient Theorem one obtains the homomorphism mentioned above. Thus if $\pi_1(Q_1)$ grows exponentially then $\mathrm{HW}(N, L_{q_1})$ has exponential algebraic growth; see Section 2.

PROOF OF PROPOSITION 2.5. Part A) follows from Proposition 8.4 and Part B) follows from Theorem 8.5. $\square$

9. Symplectic homology

Symplectic homology (SH) of a Liouville domain M is the closed string analogue of wrapped Floer homology. Its construction goes back to [CFH95] and many properties go back to [Vit99]. We give here very briefly the necessary definitions, mainly stressing the differences to the setting of HW.

The action functional is defined on the free loop space of $\widehat{M}, \Lambda\widehat{M}$, and is given by

$$\mathcal{A}_H(\gamma) = \int_0^1 \gamma^*\lambda - \int_0^1 H(\gamma(t))\, dt$$

for any admissible $H : \widehat{M} \to \mathbb{R}$. Critical points $x \in \mathrm{Crit}\,(\mathcal{A}_H)$ are 1-periodic orbits of the Hamiltonian flow of H.

The Floer homology of $\mathcal{A}_H$ will be defined as above. Different to the situation of HW one has to allow time-dependent perturbations of H to guarantee that $\mathcal{A}_H$ is Morse. The negative gradient flow lines are interpreted as solution of

$$\begin{aligned} &u : \mathbb{R} \times S^1 \to \widehat{M}, \\ &\overline{\partial}_{J,H}(u) = \partial_s u + J(u)(\partial_t u - X_H(u)) = 0. \end{aligned} \tag{61}$$

By counting elements of 0-dimensional moduli-spaces analogously as for HW, one gets a well-defined chain complex (FC, ∂),

$$\mathrm{FC}(H) = \bigoplus_{x \in \mathrm{Crit}\,(\mathcal{A}_H)} \mathbb{Z}_2 \cdot x,$$

We denote the homology of $(\mathrm{FC}(H), \partial)$ by $\mathrm{FH}(H)$. As before there is a filtration of the chain complex by action and we obtain for any $a > 0$ homologies $\mathrm{FH}^a(H) := \mathrm{H}(\mathrm{FC}^{(-\infty,a)})$.

One can define as before for any two admissible Hamiltonians H_- and H_+ with $H_+ \succ H_-$ and a monotonically increasing homotopy between H_- and H_+ chain maps $\mathrm{FC}(H_-) \to \mathrm{FC}(H_+)$. And as before we obtain a filtered directed system $\widetilde{SH}(M) = (\mathrm{SH}^a(M))_{a\in(0,+\infty)}$ given by

$$\mathrm{SH}^a(M) := \varinjlim{}_H \mathrm{FH}^a(H),$$

with respect to the ordering $\succ$. The direct limit of $\widetilde{SH}(M)$ will be also denoted by $\mathrm{SH}(M)$ and is isomorphic to $\varinjlim{}_H \mathrm{FH}(H)$. By abuse of language we call $\mathrm{SH}(M)$ and $\widetilde{\mathrm{SH}}(M)$ the symplectic homology of M.

Also transfer maps are defined analogously to the situation of HW in 6. For an exact codimension-0 embedding of Liouville domains W into M one defines the transfer map as a homomorphism of filtered directed systems

$$j : \widetilde{\mathrm{SH}}(M) \to \widetilde{\mathrm{SH}}(W).$$

The construction is analogous to the construction of $j(L_0, L_1)$ in section 6.

In analogy to Lemma 6.2 one can prove the following lemma. For that let $\mathsf{f} : \partial M \to [1, \infty)$ be a smooth function. Recall that M_f is the Liouville domain given by $M_\mathsf{f} = \widehat{M} \setminus \{(r, x) \,|\, r > \mathsf{f}(x), x \in \partial M\}$.

LEMMA 9.1. *The filtered directed systems* $\widetilde{\mathrm{SH}}(M)$ *and* $\widetilde{\mathrm{SH}}(M_\mathsf{f})$ *are* $(\zeta, 1)$*-interleaved, where* $\zeta = \max_{\partial M} \mathsf{f}$.

We recall now some results that we will use for the construction in section 12.

9.0.1. *Subcritical surgery and* SH. As mentioned before, if M is a Liouville domain that is obtained by attaching a subcritical Weinstein handle to a Liouville domain W, the symplectic homologies $\mathrm{SH}(M)$ and $\mathrm{SH}(W)$ are isomorphic, see [**Cie02**]. We now state a more general statement statement due to McLean.

THEOREM 9.2 (McLean [McL09]). *Let W be a Liouville domain, and M a Liouville domain obtained from W by attaching a subcritical handle on M. Then the filtered directed systems $\widetilde{\mathrm{SH}}(M)$ and $\widetilde{\mathrm{SH}}(W)$ are interleaved.*

The proof is geometrically more involved then the prove of Cieliebak on the isomorphism of $\mathrm{SH}(M)$ and $\mathrm{SH}(W)$ and it uses the result of Giroux that every contact manifold admits an open book decomposition, see [Gir02, Theorem 10].

As an immediate corollary we get, see Lemma 4.6

COROLLARY 9.3. *Let M and W be as above. Then $\Gamma(\widetilde{\mathrm{SH}}(M)) > 0$ if and only if $\Gamma(\widetilde{\mathrm{SH}}(W)) > 0$.*

9.0.2. *Künneth formula for* SH. Let (M_1, λ_1) and (M_2, λ_2) be two Liouville domains. By smoothing the corners of $M_1 \times M_2$ we get a Liouville domain M with $\widehat{M} = \widehat{M}_1 \times \widehat{M}_2$. We have the following Künneth formula,

THEOREM 9.4. [Oan, Theorem A] *Let M_1, M_2, and M be as above. Then there is an isomorphism*

$$\gamma : \bigoplus_{k+l=n} \mathrm{SH}_k(M_1) \otimes \mathrm{SH}_l(M_2) \to \mathrm{SH}_n(M).$$

Moreover, we have the triangle inequality $c_M(\gamma(a \otimes b)) \leq c_{M_1}(a) + c_{M_2}(b)$, where c_M, c_{M_1}, and c_{M_2} are the spectral numbers of $\widetilde{\mathrm{SH}}(M)$, $\widetilde{\mathrm{SH}}(M_1)$, and $\widetilde{\mathrm{SH}}(M_2)$, respectively.

For a proof see [Oan]. The statement on triangle inequality is not explicitly stated in [Oan] but follows directly from the proof, see also [McL09].

9.0.3. *A special Weinstein domain of dimension* 4. Finally we recall an example of a special Liouville domain in dimension 4. While the symplectic homology of the standard ball vanishes, there are contractible Liouville domains that have non-vanishing symplectic homology already in dimension 4. Examples beeing the Ramanujam's surface, see [SS05] and the tom-Dieck Petrie surface, see [McL08, Theorem 3.1].

THEOREM 9.5. *There is a contractible* 4*-dimensional Weinstein domain* T *such that* $\mathrm{SH}(T) \neq 0$.

10. Rabinowitz Floer homology RFH and entropy

10.1. RFH on Liouville domains and leafwise intersections. In this subsection we recall the definition of the perturbed Rabinowitz Floer action functional of [**AF10a**] for Liouville domains, introduce the notion of leafwise intersections, and state some basic properties, mainly from [**AF10a**]. First, let us define a few important notions and let (N, ω) be any symplectic manifold.

DEFINITION 10.1. A diffeomorphism ϕ on N is *generated* by a Hamiltonian function $H : N \times S^1 \to \mathbb{R}$ if it is the time-one map of the flow of the Hamiltonian vector field X_{H_t} given by $\omega(X_{H_t}, \cdot) = -dH_t$. Denote by $\mathrm{Ham}_c(N)$ the group of *compactly supported (c.s.)* Hamiltonian diffeomorphisms, i.e. all diffeomorphisms ϕ that are generated by Hamiltonians $H : N \times S^1 \to \mathbb{R}$ that are constant outside a compact set.

DEFINITION 10.2. For a compactly supported $H : N \times S^1$ define $\|H\|_{\text{Hofer}} = \int_0^1 \max_{x \in N} H(x,t) - \min_{x \in N} H(x,t)\, dt$.

Let $\phi \in \mathrm{Ham}_c(N)$. The *Hofer norm* of ϕ is

$$\|\phi\|_{\text{Hofer}} := \inf_{H \text{c.s., gener. } \phi} \|H\|_{\text{Hofer}}.$$

We also consider, for $i = 0, 1$, the following norms on $\mathrm{Ham}_c(N)$:

$$\|\phi\|_i := \inf_{H \text{c.s., gener. } \phi} \|H\|_{C^i}.$$

Now, consider a Liouville domain (M, λ). Regard the boundary $\Sigma = \partial M$ as the hypersurface $\{1\} \times \partial M \subset \widehat{M}$. Let $\alpha_M = \alpha_{(M,\lambda)}$ be the induced contact form on Σ, and let ϱ_{α_M} be the smallest period of all periodic Reeb orbits with respect to α_M.

Let $\phi \in \mathrm{Ham}_c(\widehat{M})$. Recall from the introduction, that a *leafwise intersection (point)* of ϕ (on Σ) with (time-)shift η is a point $x \in \Sigma$ such that $\phi(x) \in \Sigma$

and $\phi(x) = \theta^{\eta}_{\alpha_M}(x)$, see Figure 1 on page 12. By abuse of language we sometimes also say that the pair (x, η) is a leafwise intersection. We now introduce the perturbed Rabinowitz action functional. It has leafwise intersections as critical points.

10.1.1. *Perturbed Rabinowitz Floer action functional.* Let $1 > \delta > 0$, and let $G : \widehat{M} \to \mathbb{R}$ be a smooth function such that $G(r, y) = r$ for all $(r, y) \in (1-\delta, 1+\delta) \times \partial M$, and such that it is constant outside some compact subset of $(0, +\infty) \times \partial M \subset \widehat{M}$.

Fix a smooth function $\chi : [0, 1] \to \mathbb{R}$ with support $\operatorname{supp} \chi \subset [0, \frac{1}{2}]$ and such that $\int_0^1 \chi(t)\, dt = 1$. Define

$$F(x, t) := \chi(t)G(x).$$

Let $H : \widehat{M} \times S^1 \to \mathbb{R}$ be a compactly supported Hamiltonian function and let $K \subset \widehat{M}$ be a compact set such that H vanishes outside K for all t. Furthermore, we assume that $H(x, t) = 0$ for $t \in [0, \frac{1}{2}]$. Let $\phi \in \mathrm{Ham}_c(\widehat{M})$ be the diffeomorphism generated by H. Note that every $\phi \in \mathrm{Ham}_c(\widehat{M})$ has a generating Hamiltonian of that form.

Define the *perturbed Rabinowitz Floer action functional (of* (H, F)*)*

$$\mathcal{A}^F_H : C^\infty(S^1, \widehat{M}) \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}^F_H(u, \eta) = \int_0^1 u^*\lambda - \eta \int F(u, t)\, dt - \int H(u, t)\, dt.$$

A simple calculation shows that critical points $(u, \eta) \in \operatorname{Crit} \mathcal{A}^F_H$ satisfy

$$\left.\begin{aligned} \partial_t u &= \eta X_F(u, t) + X_H(u, t) \\ 0 &= \int_0^1 F(u, t). \end{aligned}\right\} \tag{62}$$

LEMMA 10.3. *Let* $(u,\eta) \in \operatorname{Crit} \mathcal{A}_H^F$*. Then* $u(0)$ *is a leafwise intersection of* ϕ^{-1} *with shift* η*. Moreover, there is a constant* C_λ *such that*

$$|\mathcal{A}_H^F(u,\eta) - \eta| \leq C_\lambda \|H\|_{C^1}.$$

PROOF. Let $(u,\eta) \in \operatorname{Crit} \mathcal{A}_H^F$. Since $X_H(\cdot,t) = 0$ for $t \in [0, \frac{1}{2}]$, we have for $t \in [0, \frac{1}{2}]$

$$\begin{aligned}\frac{d}{dt} G(u(t)) &= dG(u(t))[\partial_t u(t)] \\ &= dG(\eta X_F(u(t))) \\ &= \chi(t)\eta dG(X_G(u(t))) = 0\end{aligned}$$

and hence G is constant along $u(t)$ for $0 \leq t \leq \frac{1}{2}$. By the second equation of (62) we get therefore $G(u(t)) \equiv 0$, i.e. $u(t) \in \Sigma$ for $0 \leq t \leq \frac{1}{2}$. Furthermore, since $X_G|_\Sigma = R$, we get $u(\frac{1}{2}) = \theta^\eta_{\alpha_M}(u(0))$. On the other hand we have $\partial_t u(t) = X_F(u,t)$ for $\frac{1}{2} \leq t \leq 1$, and so $u(1) = \phi(u(\frac{1}{2}))$. We obtain the equation $\phi^{-1}(x) = u(\frac{1}{2}) = \theta^\eta(x)$, where $x := u(0) = u(1)$.

To see the second statement we calculate

$$\begin{aligned}\left|\mathcal{A}_H^F(u,\eta) - \eta\right| &= \left| \int_0^{\frac{1}{2}} \chi(t)\eta\lambda(R(u(t)))\,dt + \int_{\frac{1}{2}}^1 u^*\lambda - \int_{\frac{1}{2}}^1 H(u,t)\,dt - \eta \right| \\ &= \left| \int_{\frac{1}{2}}^1 u^*\lambda - \int_{\frac{1}{2}}^1 H(u,t) \right| \\ &\leq \frac{1}{2} \|\lambda|_K\|_\infty \max_{t\in S^1} \|X_{H_t}\|_\infty + \|H\|_{C^0} \\ &\leq C_\lambda \|H\|_{C^1}\end{aligned}$$

for a suitable constant $C_\lambda > 0$. □

If $\mathcal{A}_H^F$ is Morse, i.e. if all critical points are non-degenerate, we can define, cf. [**AF10a**], the Floer homology of $\mathcal{A}_H^F$ with $\mathbb{Z}_2$-coefficients which is independent of the choice of F and we will denote it by $\mathrm{RFH}(M;H)$. It is the homology of a chain complex $\mathrm{RFC}(M;H)$ that is generated by the critical points of $\mathcal{A}_H^F$ and with a differential that is given by counting rigid Rabinowitz

Floer gradient trajectories connecting critical points. We will not repeat the details here and refer the reader to [**CF09, AF10a**] and to chapter 3, where a version of Rabinowitz Floer homology is defined for hypertight contact manifolds and where we provide more details. We will call H *non-degenerate* if $\mathcal{A}_H^F$ is Morse, and $\phi \in \mathrm{Ham}_c(\widehat{M})$ *non-degenerate* if there is a non-degenerate c.s. H that generates ϕ.

Let us collect a few properties for $\mathrm{RFH}(M; H)$. In the special case $H = 0$ the functional $\mathcal{A}_0^F$ is not Morse, but in general Morse-Bott. In this situation one can still define the Floer homology of the functional which we then denote also by $\mathrm{RFH}(M)$, see [**CF09**].

Since the differential decreases the action, one can filter the Rabinowitz Floer chain complex $\mathrm{RFC}(M; F, H)$ by action, which gives chain complexes $\mathrm{RFC}^I(M; F, H)$ for any open, closed or half-open interval $I \subset \mathbb{R}$. And hence one can define the vector spaces $\mathrm{RFH}^I(M; F, H) := \mathrm{H}(\mathrm{RFC}^I(M; F, H))$. $\mathrm{RFH}^I(M; F; H)$ turns out to be independent of F and we write $\mathrm{RFH}^I(M; H)$ instead. This follows by energy estimates for s-dependent Rabinowitz Floer trajectories, see e.g. [**CF09**, Prop. 3.4], and the fact that the action value of a critical point is independent of the choice of F. Note that in our situation the same is not true with respect to the choice of the Hamiltonian H that generate a given $\phi \in \mathrm{Ham}_c(\widehat{M})$.

$\widetilde{\mathrm{RFH}}^{\geq 0}(M) := (\mathrm{RFH}^{[0,a)}(M))_{a \geq 0}$ is naturally a filtered directed system, where the persistence maps are induced by inclusion of chain complexes. We call the direct limit $\mathrm{RFH}^{\geq 0}(M)$, which is isomorphic to $\mathrm{RFH}^{[0,+\infty)}(M)$ the *positive Rabinowitz Floer homology* of M.

Note also, that by definition, for any $a > 0$, $\mathrm{RFH}^{[0,a)}(M) \cong \mathrm{RFH}^{(-\delta,a)}(M)$ for $0 < \delta < \varrho_{\alpha_M}$.

For two compactly supported Hamiltonains H_1, H_2 there are isomorphisms

$$i_{H_1,H_2} : \mathrm{RFH}(M; H_1) \to \mathrm{RFH}(M; H_2)$$

that are induced by chain maps given through Floer continuation, see [**AF10a**]. Moreover, one can easily deduce from energy estimates of s-dependend Floer gradient trajectories, see e.g. [AF10a, Lemma 2.7] , that for all compactly supported H_1, H_2 there is $\kappa > 0$ that depends on $\|H_1 - H_2\|_{\mathrm{Hofer}}$ such that Floer continuation actually induce homomorphisms

$$\mathrm{RFH}^{(b,c)}(M, H_1) \to \mathrm{RFH}^{(b-\kappa,c+\kappa)}(M, H_2)$$

for all $b < c$. As a special case, which will be important later, we have the following: For a fixed positive $\delta < \frac{1}{2}\varrho_{\alpha_M}$, there is a small $D > 0$ such that for all compactly supported H with $\|H\|_{\mathrm{Hofer}} < D$ we have homomorphisms

$$\begin{aligned} \mathrm{RFH}^{[0,a)}(M) \to \mathrm{RFH}^{(-\delta,a+\delta)}(M;H) \text{ and} \\ \mathrm{RFH}^{(-\delta,a)}(M;H) \to \mathrm{RFH}^{[0,a+\delta)}(M), \end{aligned} \tag{63}$$

for all $a > 0$. It is then easy to see by the property of continuation maps that these maps are compatible with the persistence maps of the filtered directed systems $\widetilde{\mathrm{RFH}}^{\geq 0}(M)$ and $(\mathrm{RFH}^{(-\delta,a)}(M;H))_{a\geq 0}$, and that we have isomorphisms

$$i^+_{0,H} : \mathrm{RFH}^{\geq 0}(M) \cong \mathrm{RFH}^{(-\delta,+\infty)}(M;H). \tag{64}$$

We finish this section by citing a theorem that provides a strong relationship between symplectic homology and positive Rabinowitz Floer homology.

THEOREM 10.4. [CFO10, Proposition 1.4] *There is a long exact sequence*[2]

$$\cdots \to \mathrm{H}_{*+n}(M, \mathbb{Z}_2) \to \mathrm{SH}_*(M) \to \mathrm{RFH}^{\geq 0}_*(M) \to \mathrm{H}_{(*+n)-1}(M, \mathbb{Z}_2) \to \cdots$$

Moreover, for any $a > 0$ there is a long exact sequence

$$\cdots \to \mathrm{H}_{*+n}(M) \to \mathrm{SH}^a_*(M) \to \mathrm{RFH}^{[0,a)}_*(M) \to \mathrm{H}_{(*+n)-1}(M) \to \cdots$$

The sequences are compatible with the persistence maps $\mathrm{SH}^a_*(M) \to \mathrm{SH}^b_*(M)$ and $\mathrm{RFH}^{[0,a)}_*(M) \to \mathrm{RFH}^{[0,b)}_*(M)$, for $a < b$.

REMARK 10.5. The second statement on the filtered version is not explicitly mentioned in [CFO10] but follows directly from their proof of the first statement.

[2]For gradings see [CFO10].

Since $\dim \mathrm{H}(M)$ is finite, we conclude

COROLLARY 10.6. $\Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M)) = \Gamma(\widetilde{\mathrm{SH}}(M))$.

10.2. Growth of RFH **and entropy.** As before, let $M = (M, \lambda)$ be a Liouville domain with boundary Σ and induced contact form $\alpha_M = \lambda|_\Sigma$, $\xi_M = \xi_{(M,\lambda)} = \ker \alpha_M$ and denote the Reeb flow with respect to an arbitrary supporting contact form α on (Σ, ξ_M) by $\theta_\alpha = (\theta^t_\alpha)_{t\in\mathbb{R}}$. We consider the filtered directed system $\widetilde{\mathrm{RFH}}^{\geq 0}(M) = (\mathrm{RFH}^{[0,a)})_{a>0}$. In this section we prove the following

THEOREM 10.7. $h_{\mathrm{top}}(\theta_{\alpha_M}) \geq \Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M))$.

This yields Proposition 2.7 stated in the introduction.

PROPOSITION 2.7. *Assume that* $\Gamma(\widetilde{\mathrm{SH}}(M)) > 0$. *Then* (Σ, ξ_M) *has positive topological entropy. Moreover, let* $\alpha = \mathsf{f}\alpha_M$ *be any supporting contact form on* (Σ, ξ_M), *then* $h_{\mathrm{top}}(\theta_\alpha) \geq \frac{\Gamma(\widetilde{\mathrm{SH}}(M))}{\max_\Sigma \mathsf{f}}$.

PROOF. Let $\alpha = \mathsf{f}\alpha_M$, $\mathsf{f} : \Sigma \to \mathbb{R}_{>0}$, be any contact form supporting ξ_M. Consider the Liouville domain $M_{\mathsf{f}} \subset \widehat{M}$. By Lemma 9.1 we have that $\Gamma(\widetilde{\mathrm{SH}}(M_{\mathsf{f}})) > \frac{\Gamma(\widetilde{\mathrm{SH}}(M))}{\max_\Sigma \mathsf{f}}$. By Corollary 10.6, $\Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M_{\mathsf{f}})) = \Gamma(\widetilde{\mathrm{SH}}(M_{\mathsf{f}}))$. Since $\alpha_{M_{\mathsf{f}}} = \alpha$, the statement follows from Theorem 10.7 applied to the Liouville domain M_{f}. $\square$

We start with an observation about the growth of leafwise intersection for Hamiltonian diffeomorphism with small Hofer norm.

Denote by $N_T(\phi, \Sigma) = N_T(\phi, \Sigma, \alpha_M)$ the number of leafwise intersections (x, η) of $\phi \in \mathrm{Ham}_c(M)$ with shift $0 \leq \eta \leq T$ with respect to the Reeb flow $\theta^t_{\alpha_M}$ on Σ.

LEMMA 10.8. *Let* $\Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M)) > \mu > 0$. *Then there is a small* $D > 0$ *and a sequence* $T_l \to +\infty$ *such that for all non-degenerate* $\phi \in \mathrm{Ham}_c(\widehat{M})$ *with* $\|\phi\|_1 < D$

$$N_{T_l}(\phi, \Sigma) \geq e^{\mu T_l}.$$

PROOF. Choose $\delta < \frac{1}{2}\varrho_{\alpha_M}$. By (64) there is a small $D > 0$ such that, for all non-degenerate compactly supported H as above with $\|H\|_{\mathrm{Hofer}} < D$, there are isomorphisms

$$i^+_{0,H} : \mathrm{RFH}^{\geq 0}(M) \cong \mathrm{RFH}^{(-\delta,+\infty)}(M; H).$$

We have a filtered directed system

$$V^H = \mathrm{RFH}^{(-\delta,a)}(M, H),\ a \geq 0.$$

Furthermore, let $\epsilon > 0$ and consider the filtered directed system $Z^{\epsilon,H}$ given by

$$Z^{\epsilon,H}_a := \begin{cases} \mathrm{RFH}^{[\epsilon,a)}(M; H) \text{ for } a \geq \epsilon \\ \{0\} \text{ if } 0 \leq a < \epsilon. \end{cases}$$

Then we are in the situation of section 4.2, i.e. (12) holds with $V = V^H$ and $Z = Z^{\epsilon,H}$.

Consider also the constant C_λ from Lemma 10.3. Denote by $c_H(x)$ the spectral number of $x \in \mathrm{RFH}^{(-\delta,+\infty)}(M; H)$. Since $\Gamma(\widetilde{\mathrm{RFH}}^{\geq 0}(M)) > \mu$, we can find in particular a sequence $T_l \to +\infty$ such that

$$\begin{aligned} &\dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M) \mid c_0(x) \leq T_l - C_\lambda D - \delta\} \\ &- \dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M) \mid c_0(x) \leq C_\lambda D + \delta\} \\ &\geq e^{\mu T_l}. \end{aligned} \tag{65}$$

Let now $\phi \in \mathrm{Ham}_c(M)$ be non-degenerate with $\|\phi\|_1 < D$, hence there is H as described above that generates ϕ^{-1} with $\|H\|_{C^1} < D$ and such that $\mathcal{A}^F_H$ is non-degenerate for F as above.

We can now estimate

$$\begin{aligned}
N_{T_l}(\phi,\Sigma) &= \#\{(p,\eta) \text{ l. int. of } \phi \,|\, 0 \le \eta \le T_l\}\\
&\overset{(*)}{\ge} \#\{(p,\eta) \text{ l. int. of } \phi \,|\, C_\lambda D \le \mathcal{A}_H^F(p,\eta) < T_l - C_\lambda D\}\\
&\ge \dim \mathrm{RFH}^{[C_\lambda D, T_l - C_\lambda D)}(M;H)\\
&\overset{(**)}{\ge} \dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M;H) \,|\, c_H(x) < T_l - C_\lambda D\}\\
&\quad - \dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M;H) \,|\, c_H(x) \le C_\lambda D\}\\
&\overset{(***)}{\ge} \dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M) \,|\, c_0(x) < T_l - C_\lambda D - \delta\}\\
&\quad - \dim\{x \in \mathrm{RFH}^{(-\delta,\infty)}(M) \,|\, c_0(x) \le C_\lambda D + \delta\}\\
&\overset{(65)}{\ge} e^{\mu T_l}.
\end{aligned}$$

In $(*)$ we have used Lemma 10.3, in $(**)$ we have used the inequality (13) with $V = V^H$ and $Z = Z^{C_\lambda D, H}$, and to obtain inequality $(***)$ we have used that $|c_0(x) - c_H(i^+_{0,H}(x))| < \delta$, see (63). □

Let now in general (Y,ω) be a symplectic manifold, and (Σ, α_0) any *contact type* hypersurface in Y, i.e. α_0 is a contact form on Σ with $d\alpha_0 = \omega|_\Sigma$. Let $\theta^t = \theta^t_{\alpha_0}$ the Reeb flow on Σ. Note, that ∂M is a contact type hypersurface in $\widehat{M}$. The following proposition together with Lemma 10.8 gives Theorem 10.7.

PROPOSITION 10.9. *Assume that there is $\mu > 0$, $D > 0$ and a sequence $T_l \to +\infty$ such that for any non-degenerate $\phi \in \mathrm{Ham}_c(Y)$ with $\|\phi\|_1 < D$ we have that $N_{T_l}(\phi, \Sigma, \alpha_0) \ge e^{\mu T_l}$ for all $l \in \mathbb{N}$. Then $h_{\mathrm{top}}(\theta^t) \ge \mu$.*

The proof of the proposition, as the results on h_{top} in section 7, rests on Yomdins inequality (1). We apply it to the diagonal $\Lambda_\Sigma \in \Sigma \times \Sigma$ and the flow $id \times \theta^t$. To detect the volume growth we will use a large parameter family of small Hamiltonian perturbations of the identity and then for any time T_l we suitably choose a ball $B \in \Sigma$ and a $2n$-dimensional subfamily of Hamiltonian diffeomorphisms such that the number of leafwise intersections with time shift between 0 and T_l is of order $\exp(T_l)$. This allows us to bound the volume growth of the diagonal from below.

10.2.1. *A family of Hamiltonian diffeomorphisms.* We are going to construct a large parameter family of Hamiltonian diffeomorphisms on Y with suitable properties.

Choose a sufficiently big $K \in \mathbb{N}$ and a collection of open Darboux balls U_k, $k \in \{1, \cdots, K\}$ in Y, such that $\overline{U_k}$ lies in an open ball in Y, with open subsets $W_k \subset V_k \subset U_k$ such that

- $\overline{W_k} \subset V_k$ and $\overline{V_k} \subset U_k$, for all $k \in \{1, \cdots, K\}$
- $\Sigma \subset \bigcup_{k=1}^N W_k$.

For each $k \in \{1, \cdots, K\}$ we choose Darboux coordinates $(u_1, \cdots, u_{2n}) = (x_1, y_1, \cdots, x_n, y_n)$, i.e. $\omega|_{U_k} = \sum_{i=1}^n dx_i \wedge dy_i$.

Choose $2nK$ Hamiltonian functions $H_i^k,\ k \in \{1, \cdots, K\},\ i \in \{1, \cdots, 2n\}$ with the properties that

- On U_k,
$$H_i^k(u_1, u_2, \cdots, u_{2n}) = \begin{cases} -u_{i+1}, & \text{if } i \text{ odd} \\ u_{i-1}, & \text{if } i \text{ even} \end{cases}$$
- H_i^k vanish outside a compact subset of Y.

We rescale the Hamiltonian functions by some small constants ρ_k, that we will choose later, and obtain functions

$$G_i^k = \rho_k H_i^k. \tag{66}$$

Denote the Hamiltonian vector fields of G_i^k on Y by X_i^k and the flows by $(\chi_i^k)_t, t \in [0, \infty)$. For each $2n$-tuple $\tau_k = (t_1, t_2, \cdots, t_{2n}) \in [0,1]^{2n}$ we obtain a Hamiltonian diffeomorphism

$$(\psi^k)_{\tau_k} := (\chi_{2n}^k)_{t_{2n}} \circ (\chi_{2n-1}^k)_{t_{2n-1}} \circ \cdots \circ (\chi_1^k)_{t_1}.$$

And for each K-tuple of $2n$-tuples $\tau = (\tau_1, \tau_2, \cdots, \tau_K) \in ([0,1]^{2n})^K$ we obtain a Hamiltonian diffeomorphism

$$\varphi_\tau := (\psi^K)_{\tau_K} \circ \cdots \circ (\psi^1)_{\tau_1}.$$

LEMMA 10.10. *We can choose the constants ρ_k in* (66)*, $k \in \{1, \cdots, K\}$, sufficiently small such that for every $k \in \{1, \cdots, K\}$ and all $\tau_k \in [0,1]^{2n}$,*

$$(\psi^k)_{\tau_k}(V_k) \subset U_k, \tag{67}$$

and, for every $k \in \{1, \cdots, K-1\}$ and all $\tau_1, \tau_2, \cdots, \tau_k \in [0,1]^{2n}$,

$$(\psi^k)_{\tau_k} \circ \cdots \circ (\psi^1)_{\tau_1}(W_{k+1}) \subset V_{k+1}. \tag{68}$$

PROOF. First of all, it is clear that (67) is satisfied if $\rho_1, \cdots, \rho_K < \epsilon$ for some sufficiently small $\epsilon > 0$.

Choose for every $k \in \{1, \cdots, K\}$ open subsets W_k^j, $1 \le j \le k$, such that $W_k = W_k^1, \overline{W_k^1} \subset W_k^2, \overline{W_k^2} \subset W_k^3, \cdots, \overline{W_k^k} \subset V_k$.

Let $\rho_K < \epsilon$ and choose, for $j \in \{1, \cdots, K-1\}$, $\rho_j < \epsilon$ sufficiently small such that for all $\tau_j \in [0,1]^{2n}$ and for every $k \in \{j, \cdots, K\}$,

$$(\psi^j)_{\tau_j}(W_{k+1}^j) \subset W_{k+1}^{j+1}.$$

Therefore, for every $j \in \{1, \cdots, K-1\}$ and for all $k \in \{j, \cdots, K\}$, $\tau_1, \cdots, \tau_j \in [0,1]^{2n}$,

$$(\psi^j)_{\tau_j} \circ \cdots \circ (\psi^1)_{\tau_1}(W_{k+1}) \subset W_{k+1}^{j+1}.$$

In particular this holds for $k = j$ and hence (68) is satisfied for this choice of $\rho_1, \cdots, \rho_K$. $\square$

From now on we fix a choice of the ρ_k such that (67) and (68) hold. We make the following observations

LEMMA 10.11. *For every $k \in \{1, \cdots, K\}$,*

(i) *the vector fields $X_i^k, 1 \le i \le 2n$, are linearly independent on U_k,*
(ii) *the flows $(\chi_i^k)_t, 1 \le i \le 2n$, pairwise commute on V_k for $0 \le t \le 1$,*

(iii) $(\psi^k)_{\tau_k}$, $\tau_k \in [0,1]^{2n} \setminus (0,\cdots,0)$, *have no fixed points on* V_k.

PROOF. By the prescribed form of the Hamiltonian functions G_i^k on U_k, we have that for $x = (u_1, u_1, \cdots, u_{2n}) \in U_k$

$$X_i^k(u_1, u_2, \cdots, u_{2n}) = \rho_k \partial_{u_i}. \tag{69}$$

Hence (i) holds. By (67) and (69) we get on V_k an explicit formula for the diffeomorphisms $(\psi^k)_{\tau_k}$ for all $\tau_k = (t_1^k, \cdots, t_{2n}^k) \in [0,1]^{2n}$. Namely, for $x = (u_1, \cdots, u_{2n}) \in V_k$ we have

$$(\psi^k)_{\tau_k}(u_1, u_2, \cdots, u_{2n}) = (u_1 + \rho_k t_1^k, u_2 + \rho_k t_2^k, \cdots, u_{2n} + \rho_k t_{2n}^k), \tag{70}$$

in coordinates on U_k. In particular, (ii) and (iii) hold. $\square$

We set $B_k := W_k \cap \Sigma$ for all $k \in \{0, \cdots, K\}$. The B_k form an open covering of Σ. For simplicity we may assume that the B_k are diffeomorphic to balls.

Let $k \in \{1, \cdots, K\}$. Define $I_k : V_k \times [0,1]^{2n} \to Y \times Y$ by

$$I_k(x, \sigma) := (x, (\psi^k)_\sigma(x)). \tag{71}$$

LEMMA 10.12. *I_k is a (codimension-0) embedding.*

PROOF. The map I_k is injective. Namely, let $I_k(x,\sigma) = I_k(y, \widetilde{\sigma})$ then $x = y$. And we hence also have $(\psi^k)_\sigma(x) = (\psi^k)_{\widetilde{\sigma}}(x)$. By Lemma 10.11(ii) we get $(\psi^k)_{\sigma - \widetilde{\sigma}}(x) = x$ and by Lemma 10.11(iii) it follows that $\sigma = \widetilde{\sigma}$.

It remains to show that $\partial_{t_1} I_k, \cdots, \partial_{t_{2n}} I_k, \partial_{u_1} I_k, \cdots, \partial_{u_{2n}} I_k$ are linearly independent.

One computes:

$$\begin{aligned}\partial_{t_i} I_k &= (0, D\left((\chi_{2n}^k)_{t_{2n}} \circ \cdots \circ (\chi_{i+1}^k)_{t_{i+1}}\right) X_i^k \left((\chi_i^k)_{t_i} \circ \cdots \circ (\chi_1^k)_{t_1}(x)\right) \\ &= (0, X_i^k((\psi^k)_\sigma(x))).\end{aligned} \tag{72}$$

For the second equality we have used 10.11(ii).

Hence, by 10.11(i), $\partial_{t_1} I_k, \cdots, \partial_{t_{2n}} I_k$ are linearly independent. Obviously $\partial_{u_1} I_k, \cdots, \partial_{u_{2n}} I_k$ are linearly independent, and since they span a transverse subspace to the span of $\partial_{t_1} I_k, \cdots, \partial_{t_{2n}} I_k$, the statement follows. □

For every $k \in \{1, \cdots, K\}$ we choose a metric g_k on Y such that restricted to U_k it is the euclidean metric with respect to the coordinates $(u_1, \cdots, u_{2n})$. Denote the product metric $g_k \times g_k$ on $Y \times Y$ by $\overline{g}_k$.

On $V_k \times [0,1]^{2n}$, I_k induces a metric

$$\mathfrak{g}_k = \mathfrak{g} := (I_k^{-1})^* \overline{g}_k. \tag{73}$$

We observe the following.

LEMMA 10.13. *With respect to the coordinates $(u_1, \cdots, u_{2n}, t_1, \cdots t_{2n})$ on $V_k \times [0,1]^{2n} \subset U_k \times [0,1]^{2n}$ the components of $\mathfrak{g}$ are*

$$\mathfrak{g}_{i,j} = \begin{cases} 2\delta_{i,j}, \text{ if } 1 \le i, j \le 2n, \\ \rho_k \delta_{i,j-2n}, \text{ if } 1 \le i \le 2n, 2n+1 \le j \le 4n, \\ \rho_k \delta_{i-2n,j}, \text{ if } 2n+1 \le i \le 4n, 1 \le j \le 2n, \\ (\rho_k)^2 \delta_{i,j}, \text{ if } 2n+1 \le i, j \le 4n. \end{cases} \tag{74}$$

In particular, one can estimate norms on $V_k \times [0,1]^{2n}$ with respect to $\mathfrak{g}$ only in terms of ρ_k.

PROOF. In terms of coordinates $(u, \hat{u}) \in U \times U$, it follows by (72) that $\frac{\partial}{\partial t_i} I_k(x, \sigma) = X_i^k(z) = \rho_k \partial_{\hat{u}_i}(z)$, where $z := (\psi^k)_\sigma(x)$. Furthermore, by (70) we have that $\frac{\partial}{\partial u_i} I_k(x, \sigma) = \partial_{u_i}(x) + \frac{\partial}{\partial u_i}(\psi^k)_\sigma(x) = \partial_{u_i}(x) + \partial_{\hat{u}_i}(z)$. The statement follows. □

10.2.2. *Leafwise intersections, non-degenericity and growth.* We will deduce the lower bound on the entropy in Proposition 10.9 by proving a lower bound on the volume growth of the "graph" of the Reeb flow in $\Sigma \times \Sigma$, i.e. the growth of the function $t \mapsto \mathrm{Vol}^{2n-1}_{\overline{g}_k}((id \times \theta^t)(\Delta_\Sigma))$ for some $k \in \{1, \cdots, K\}$, cf. Claim 10.18. For that we consider for each $T > 0$

the maps $\Theta_T : \Sigma \times [0,T] \to \Sigma \times \Sigma$, $(x,t) \mapsto (x, \theta^t(x))$, and we will give a lower bound on the exponential growth rate of $T \mapsto \mathrm{Vol}^{2n}_{g_k}(\mathrm{Im}\,\Theta_T)$, cf. (78).

We use now the suggestive notation $\tau^{<k>}$ for some $(K-1)$-tuple of $2n$-tuples $(\tau_1, \cdots \tau_{k-1}, \tau_{k+1}, \cdots \tau_K) \in ([0,1]^{2n})^{K-1}$, and write $\tau = \tau^{<k>}[\sigma]$ for the K-tuple of $2n$-tuples $\tau = (\tau_1, \cdots, \tau_{k-1}, \sigma, \tau_{k+1}, \cdots, \tau_K) \in ([0,1]^{2n})^K$ that one obtains by inserting $\sigma \in [0,1]^{2n}$ at the k-th entry.

Fix now any k in $\{1, \cdots, K\}$ and $\tau^{<k>} = (\tau_1, \tau_{k-1}, \tau_{k+1}, \cdots \tau_K)$ in $([0,1]^{2n})^{K-1}$ as above. Let $\zeta = \zeta^{\tau^{<k>}} := (\psi^K)_{\tau_K} \circ \cdots (\psi^{k+1})_{\tau_{k+1}}$ and $\xi = \xi^{\tau^{<k>}} := (\psi^{k-1})_{\tau_{k-1}} \circ \cdots \circ (\psi^1)_{\tau_1}$. Hence, in our notation we may write $\varphi_{\tau^{<k>}[\sigma]} = \zeta \circ (\psi^k)_\sigma \circ \xi$.

We consider the restriction of Θ_T to $B_k \times [0,T]$ and "perturb" it into a function

$$F^{\tau^{<k>}}_{T,k} : B_k \times [0,T] \to Y \times Y,$$

given by

$$F^{\tau^{<k>}}_{T,k} := (\xi \times \zeta^{-1}) \circ \Theta_T|_{B_k \times [0,T]},$$

i.e. $F^{\tau^{<k>}}_{T,k}(x,t) = (\xi(x), \zeta^{-1} \circ \theta^t(x))$. Let $I_k : V_k \times [0,1]^{2n} \to Y \times Y$ be defined as in (71) above. Let $\mathcal{U} = \mathcal{U}^{\tau^{<k>}}_{T,k} := (F^{\tau^{<k>}}_{T,k})^{-1}(\mathrm{Im}\, I_k) \subset B_k \times [0,T]$ and let

$$\Phi^{\tau^{<k>}}_{T,k} := (I_k)^{-1} \circ F^{\tau^{<k>}}_{T,k}|_{\mathcal{U}} : \mathcal{U} \to V_k \times [0,1]^{2n}.$$

Denote by $\Pi_k : V_k \times [0,1]^{2n} \to [0,1]^{2n}$ the projection to $[0,1]^{2n}$, and let

$$\Psi^{\tau^{<k>}}_{T,k} := \Pi_k \circ \Phi^{\tau^{<k>}}_{T,k} : \mathcal{U} \to [0,1]^{2n}.$$

We will write also F, Φ, Π, Ψ, etc. if the choices of $T, k, \tau^{<k>}$ are clear.

LEMMA 10.14. *The points $(x,t) \in (\Psi^{\tau^{<k>}}_{T,k})^{-1}(\sigma) \subset B_k \times [0,T]$ correspond exactly to the leafwise intersections (x,t) of $\varphi_{\tau^{<k>}[\sigma]}$ with $x \in B_k$ and time-shift t between 0 and T. Furthermore, (x,t) is non-degenerate if and only if it is a regular point of $\Psi^{\tau^{<k>}}_{T,k}$.*

PROOF. Let $(x,t) \in (\Psi^{\tau^{<k>}}_{T,k})^{-1}(\sigma)$. Hence there exists $y \in V_k$ with $I_k(y,\sigma) = (y, (\psi^k)_\sigma(y)) = (\xi(x), \zeta^{-1}\theta^t(x))$. Hence we have $y = \xi(x)$ and $\theta^t(x) = \zeta \circ (\psi^k)_\sigma \circ \xi(x) = \varphi_{\tau^{<k>}[\sigma]}(x)$.

Conversely, assume that for some $(x,t) \in B_k \times [0,T]$ we have that $\theta_t(x) = \varphi_{\tau^{<k>}[\sigma]}(x)$. Then, since $\xi(x) \in V_k$ by (68) we observe that $(\xi(x), \zeta^{-1} \circ \theta^t(x)) = (\xi(x), (\psi^k)_\sigma(\xi(x))) \subset \operatorname{Im} I_k$, and therefore conclude that $(x,t) \in \Phi^{-1}(\Pi^{-1}(\sigma)) = \Psi^{-1}(\sigma)$.

Consider now a leafwise intersection $(x,t) \in B_k \times [0,T]$ of $\varphi_{\tau^{<k>}[\sigma]}$ that is non-degenerate, i.e. $D\theta^t(x)[v] \neq D\varphi_{\tau^{<k>}[\sigma]}(x)[v] (= D(\zeta \circ (\psi^k)_\sigma \circ \xi)(x)[v])$ for all $v \in T_xY$, $v \neq 0$. We can write this with $y := \xi(x)$ as

$$D(\zeta^{-1} \circ \theta^t)(x)[v] \neq D(\psi^k)_\sigma(y)[D\xi v], \text{ for all } v \in T_yY, v \neq 0.$$

Therefore, $(\xi \times (\zeta^{-1} \circ \theta^t)(\Delta_Y)$ is transverse to $(id \times (\psi^k)_\sigma)(\Delta_Y)$ at the point $(\xi \times (\zeta^{-1} \circ \theta^t)(x,x)$, where $\Delta_Y \subset Y \times Y$ is the diagonal. By considering the image under I^{-1} we follow that $(I_k)^{-1} \circ F(B_k \times [0,T]) = \Phi(B_k \times [0,T])$ is transverse to $\Pi^{-1}(\sigma)$ at the point $\Phi(x,t)$. But this means that actually $\sigma = \Pi \circ \Phi(x,t) = \Psi(x,t)$ is a regular value of Ψ. The reverse direction is analogous. □

Before we prove Proposition 10.9 let us observe the following.

LEMMA 10.15. *The set*

$$G := \{\tau \in [0,1]^{2n \cdot K} \mid \textit{all leafwise intersections of } \varphi_\tau \textit{ are non-degenerate}\}$$

has full Lebesgue-measure.

PROOF. Let $G_{k,T} \subset [0,1]^{2nK}$ be the set of $\tau \in [0,1]^{2nK}$ such that all leafwise intersections (x,t) of φ_τ with $x \in B_k$ and action shift t between 0 and T are non-degenerate.

Let $\tau^{<k>} \in ([0,1]^{2n})^{K-1}$ as defined above. Define $\Lambda^{\tau^{<k>}}_{k,T} \subset [0,1]^{2n}$ by $\Lambda^{\tau^{<k>}}_{k,T} := \{\sigma \in [0,1]^{2n} \mid \tau^{<k>}[\sigma] \in G_{k,T}\}$, i.e. $\Lambda^{\tau^{<k>}}_{k,T}$ is the set of $\sigma \in [0,1]^{2n}$ such that all leafwise intersections (x,t) of $\varphi_{\tau^{<k>}[\sigma]}$ with $x \in B_k$ and with $0 \leq t \leq T$ are non-degenerate.

By Lemma 10.14 and Sard's theorem $\mathcal{L}^{2n}(\Lambda_{k,T}^{\tau^{<k>}}) = 1$. Furthermore,

$$\mathcal{L}^{2nK}(G_{k,T}) = \int_{[0,1]^{2n(K-1)}} \mathcal{L}^{2n}(\Lambda_{k,T}^{\tau^{<k>}}) d\mathcal{L}^{2n(K-1)} = 1,$$

and therefore $\mathcal{L}^{2nK}(G) = \mathcal{L}^{2nK}(\bigcap_{T=1}^{\infty} \bigcap_{k=1}^{K} G_{k,T}) = 1$. $\square$

PROOF OF PROPOSITION 10.9. Let $D, \mu > 0$ and $T_l \to +\infty$ a sequence such that the assumptions of the proposition hold. We may assume, by choosing sufficiently small ρ_k in (66), that for all $\tau \in [0,1]^{2nK}$, $\|\varphi_\tau\|_1 < D$.

Define for each $l \in \mathbb{N}$ and $k \in \{1, \cdots, K\}$, $A_{l,k} \subset [0,1]^{2nK}$ by

$$A_{l,k} := \{\tau \in G \mid N_{T_l}(\varphi_\tau, B_k) \geq \frac{1}{K} e^{\eta T_l}\}.$$

The sets $A_{l,k}$ are Lebesgue-measurable. Since $\bigcup_{k=1}^{K} B_k = \Sigma$, we have for all $T > 0$ $\sum_{k=1}^{K} N_T(\varphi_\tau, B_k) \geq N_T(\varphi_\tau, \Sigma)$ and hence for all $l \in \mathbb{N}$ $\bigcup_{k=1}^{K} A_{l,k} = G$. By Lemma 10.15 we can therefore for each $l \in \mathbb{N}$ choose $k_l \in \{1, \cdots, K\}$ such that

$$\mathcal{L}^{2nK}(A_{l,k_l}) \geq \frac{1}{K}. \tag{75}$$

CLAIM 10.16. For every $l \in \mathbb{N}$ there is $\tau^{<k_l>} \in ([0,1]^{2n})^{K-1}$ as above such that $\daleth_l := \{\sigma \in [0,1]^{2n} \mid \tau^{<k_l>}[\sigma] \in A_{l,k_l}\}$ has Lebesgue measure

$$\mathcal{L}^{2n}(\daleth_l) \geq \frac{1}{K}. \tag{76}$$

Proof. This follows directly from

$$\mathcal{L}^{2nK}\left(A_{l,k_l}\right) = \int_{[0,1]^{2n(K-1)}} \mathcal{L}^{2n}\left(A_{l,k_l} \cap \left(\pi^{<k_l>}\right)^{-1}(x)\right) d\mathcal{L}^{2n(K-1)},$$

where we denote by $\pi^{<k>} : [0,1]^{2nK} \to [0,1]^{2n(K-1)}$ the projection given by $\pi^{<k>}(\tau_1, \cdots, \tau_K) = (\tau_1, \cdots, \tau_{k-1}, \tau_{k+1}, \cdots \tau_K)$. $\blacksquare$

For every $l \in \mathbb{N}$ we fix now k_l and $\tau^{<k_l>}$ as above, i.e. such that (75) and (76) hold.

Let $l \in \mathbb{N}$. We write $\widehat{F}^l$ instead of $F^{\tau^{<k_l>}}_{T_l,k_l} : B_{k_l} \times [0,T_l] \to Y \times Y$. Furthermore, we write $\mathcal{U}^l := \mathcal{U}^{\tau^{<k_l>}}_{T_l,k_l}$, $\Phi^l := \Phi^{\tau^{<k_l>}}_{T_l,k_l}$ and $\Psi^l := \Psi^{\tau^{<k_l>}}_{T_l,k_l}$.

We define a constant $\widehat{c}$ as

$$\widehat{c} := \max_{k \in \{1,\cdots K\}} \max_{x \in V_k} \|D\Pi_k(x)\|_k,$$

where $\|\cdot\|_k$ denotes here the operator norm with respect to $\mathfrak{g}_k$ on $V_k \times [0,1]^{2n}$, see (73), and the euclidean metric on $[0,1]^{2n}$.

CLAIM 10.17. For all $l \in \mathbb{N}$

$$Vol_{\overline{g}_{k_l}}\left(\operatorname{Im}\widehat{F}^l\right) \geq \frac{e^{\mu T_l}}{\widehat{c}^{2n} K^2},$$

We prove the claim below. Since $\widehat{F}^l = (\xi^{\tau^{<k_l>}}, (\zeta^{\tau^{<k_l>}})^{-1}) \circ \Theta_{T_l}|_{B_{k_l} \times [0,T_l]}$, we have

$$\begin{aligned}\mathrm{Vol}\left(\operatorname{Im}\widehat{F}^l\right) &\leq \max_{x \in M} \|D\xi^{\tau^{K_l}}(x)\|^{2n}_{g_{k_l}} \max_{x \in M} \|D\left(\zeta^{\tau^{<k_l>}}\right)^{-1}(x)\|^{2n}_{g_{k_l}} \mathrm{Vol}_{\overline{g}_{k_l}}(\operatorname{Im}\Theta_{T_l})\\ &\leq \mathcal{M}\,\mathrm{Vol}_{\overline{g}_{k_l}}(\operatorname{Im}\Theta_{T_l}),\end{aligned} \tag{77}$$

where $\mathcal{M} = \prod_{j=1}^K \mathcal{M}_j$ and

$$\mathcal{M}_j = \sup_{\sigma \in [0,1]^{2n}} \max_{k \in \{1,\cdots,K\}} \max_x \|D(\psi^j)_\sigma(x)\|_{g_k} < \infty.$$

Note that $\mathcal{M}_j$ are finite since we consider a compact family of diffeomorphisms.

By Claim 10.17 and (77), we get that

$$\max_{k \in \{1,\cdots,K\}} \mathrm{Vol}_{\overline{g}_k}(\operatorname{Im}\Theta_{T_l}) \geq \frac{e^{\mu T_l}}{\mathcal{M}\widehat{c}^{2n} K^2},$$

hence there is $k \in \{1,\cdots,K\}$ such that

$$\limsup_{T \to \infty} \frac{\log(\mathrm{Vol}_{\overline{g}_k}(\operatorname{Im}\Theta_T))}{T} \geq \mu. \tag{78}$$

From (78) one can deduce that also

CLAIM 10.18.
$$\limsup_{t\to\infty} \frac{\log \mathrm{Vol}_{\overline{g}_k}((id\times\theta^t)(\Delta_\Sigma))}{t} \geq \mu.$$

Proof. This follows by a Fubini-type estimate. Let J be an ω-compatible almost complex structure on Y, cylindrical near Σ, and let the metric g on Y by defined by $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and let $h := g|_\Sigma$. We consider the product metric $\overline{h} = h \times h$ on $\Sigma\times\Sigma$. Note that $h(R,R) = 1$ for the Reeb vector field R on Σ. Furthermore, to simplify the calculations, we can consider a ball $B \in \Sigma$ such that
$$\limsup_{T\to\infty} \frac{\log(\mathrm{Vol}_{\overline{g}_k}(\operatorname{Im}\Theta_T|_{B\times[0,T]}))}{T} \geq \mu. \tag{79}$$
It is sufficent to proof that
$$\limsup_{t\to\infty} \frac{\log \mathrm{Vol}_{\overline{h}}((id\times\theta^t)(\Delta_B))}{t} \geq \mu. \tag{80}$$
We write $\overline{\Theta}_T = \Theta_T|_{B\times[0,T]}$.

Let ν be a volume form on B. For a given T, the volume form on $B\times[0,T]$ induced by the metric $(\overline{\Theta}_T)^*\overline{h}$ can be written as $\Omega((\overline{\Theta}_T)^*\overline{h}) = f\,\nu\wedge dt$ for a positive function $f : B\times[0,T] \to \mathbb{R}_{>0}$. We also write for a fixed $0 \leq t \leq T$ the volume form $\Omega((\overline{\Theta}_T(\cdot,t))^*\overline{h})$ on B as $\hat{f}_t\,\nu$. We claim that $f(\cdot,t) \leq \hat{f}_t$ on B.

Namely, given some local coordinates $(v_1,\cdots,v_{2n-1})$ on B such that $\nu = dv_1\wedge\cdots\wedge dv_{2n-1}$, we can write f and $\hat{f}_t$ in terms of pull-backs of $\overline{h}$ as $f = \sqrt{\det[\overline{\Theta}_T^*\overline{h}]}$ and $\hat{f}_t = \sqrt{\det[\overline{\Theta}_T(\cdot,t)^*\overline{h}]}$, where by $[\overline{\Theta}_T^*\overline{h}]$ and $[\overline{\Theta}_T(\cdot,t)^*\overline{h})]$ we denote the $(2n\times 2n)$- and $(2n-1\times 2n-1)$-matrices, respectively, given by the components of the pullbacks of $\overline{h}$.

We have $\det[(\overline{\Theta}_T)^*\overline{h}] \leq \|\partial_t\overline{\Theta}_T\|_{\overline{h}}\det[\overline{\Theta}_T(\cdot,t)^*\overline{h}]$. Since by definition of $\overline{h}$ $\|\partial_t\overline{\Theta}_T\|_{\overline{h}} = h(R,R) = 1$, we have $f(\cdot,t) \leq \hat{f}_t(\cdot)$ on B.

We now conclude

$$\begin{aligned}
\mathrm{Vol}_{\overline{h}}(\mathrm{Im}\,\overline{\Theta}_T) &= \int_{B\times[0,T]} \Omega(\overline{\Theta}_T^*\overline{h}) \\
&= \int_{B\times[0,T]} f\nu \wedge\, dt \\
&= \int_0^T \int_B f(\cdot,t)\nu\, dt \\
&\le \int_0^T \int_B \hat{f}_t\, \nu\, dt \\
&= \int_0^T \int_B \Omega(\overline{\Theta}_T(\cdot,t)^*\overline{h})\, dt \\
&= \int_0^T \mathrm{Vol}_{\overline{h}}((id\times\theta^t)(\Delta_B))dt.
\end{aligned}$$

Assume that (80) does not hold. In this case there is t_0 and $\epsilon > 0$ such that $\mathrm{Vol}_{\overline{h}}((id\times\theta^t)(\Delta_B)) < e^{(\mu-\epsilon)t}$, for all $t \ge t_0$. By the above estimate we would then get for all $T \ge t_0$

$$\mathrm{Vol}_{\overline{h}}(\mathrm{Im}\,\overline{\Theta}_T) < M + (T-t_0)e^{(\mu-\epsilon)T},$$

where $M = \int_0^{t_0} \mathrm{Vol}_{\overline{h}}((id\times\theta^t)(\Delta_B))\, dt$. This contradicts (79). ■

By Claim 10.18 the volume of the immage of the immersions (id,θ^t) of Σ into $\Sigma\times\Sigma$, i.e. $\mathrm{Vol}\{(x,\theta^t(x) \mid x\in\Sigma\}$ grow exponentially in t with growth rate $\ge\mu$. By the inequality of Yomdin [**Yom87**], cf. (1), it follows that the topological entropy of (id,θ^t) is bounded from below by μ. From the definition of h_{top} applied to some product metric on $\Sigma\times\Sigma$ it follows easily that the topological entropy of the Reeb flow θ^t and that of (id,θ^t) coincide, and hence the proposition follows.

It now only remains to prove Claim 10.17.

Proof of Claim 10.17. Let $l\in\mathbb{N}$. By definition

$$\mathrm{Vol}_{\overline{g}_{k_l}}(\mathrm{Im}\,\widehat{F}^l) = \int_{\mathcal{U}^l} (\widehat{F}^l)^*\overline{g}_{k_l} = \int_{\mathcal{U}^l} \Omega((\Phi^l)^*\mathfrak{g}).$$

Equip $[0,1]^{2n}$ with the euclidean metric. If we choose a local orthonormal frame $(\epsilon_1, \cdots, \epsilon_{2n})$ for the metric $(\Phi^l)^*\mathfrak{g}_{k_l}$ around some regular point $p \in \mathcal{U}^l$ of Φ^l, we have that $\|D\Phi^l(\epsilon_i)\|_{\mathfrak{g}_{k_l}} = 1$, for $i = 1, \cdots, 2n$, and we compute that the volume form

$$\begin{aligned}
\Omega\left((\Phi^l)^*\mathfrak{g}_{k_l}\right) &= \epsilon_1^* \wedge \cdots \wedge \epsilon_{2n}^* \\
&\geq \frac{1}{\hat{c}^{2n}} |D\Pi_{k_l}(D\Phi^l(\epsilon_1))| \cdots |D\Pi_{k_l}(D\Phi^l(\epsilon_{2n}))| \epsilon_1^* \wedge \cdots \wedge \epsilon_{2n}^* \\
&\geq \frac{1}{\hat{c}^{2n}} |\det D\Psi^l| \epsilon_1^* \wedge \cdots \wedge \epsilon_{2n}^* \\
&= \frac{1}{\hat{c}^{2n}} \Omega((\Psi^l)^* g_{eucl}),
\end{aligned} \tag{81}$$

where the matrix $D\Psi^l$ is represented here with respect to $(\epsilon_1, \cdots, \epsilon_{2n})$ and standard coordinates in $[0,1]^{2n}$. Since the right hand side vanishes for singular points, (81) holds for all $p \in \mathcal{U}^l$.

Choose local coordinates $(v_1, \cdots, v_{2n-1})$ in B and standard coordinates $(t_1, \cdots, t_{2n})$ in $[0,1]^{2n}$.

We conclude that

$$\begin{aligned}
\int_{\mathcal{U}^l} \Omega\left((\Phi^l)^*\mathfrak{g}\right) &\overset{(81)}{\geq} \int_{\mathcal{U}^l} \frac{1}{\hat{c}^{2n}} \Omega\left((\Psi^l)^* g_{eucl}\right) \\
&= \frac{1}{\hat{c}^{2n}} \int_{\mathcal{U}^l} |\det D\Psi^l| dv_1 \wedge \cdots dv_{2n-1} \wedge dt \\
&\overset{(*)}{=} \frac{1}{\hat{c}^{2n}} \int_{[0,1]^{2n}} \#\left((\Psi^l)^{-1}(\sigma)\right) dt_1 \wedge \cdots \wedge dt_{2n} \\
&\geq \frac{1}{\hat{c}^{2n}} \int_{\daleth_l} \#\left((\Psi^l)^{-1}(\sigma)\right) d\mathcal{L}^{2n}(\sigma) \\
&\overset{(**)}{\geq} \frac{1}{\hat{c}^{2n}} \mathcal{L}^{2n}\left(\daleth_l\right) \min_{\sigma \in \daleth_l} N_{T_l}\left(\varphi_{\tau^{< k_l >}[\sigma]}\right) \\
&\geq \frac{1}{\hat{c}^{2n}} \frac{1}{K} \left(\frac{1}{K} e^{\nu T_l}\right).
\end{aligned}$$

In $(*)$ we have applied the coarea formula [**BZ88**, p. 103], and for $(**)$ we have used Lemma 10.14 and the definition of $\gimel_l$.

Hence the claim holds. ■

This finishes the proof of the proposition. □

11. The Floer homology of cotangent bundles

Let (Q, g) be a Riemannian manifold. We recall here important and fundamental isomorphisms between the homology of the free loop space and based loop space of Q, and SH and HW of the cotangent bundle of Q, respectively. We also state some additional properties of the isomorphisms that we will use later. The isomorphism between the homology of the free loop space and symplectic homology of the cotangent bundle goes back to Viterbo [**Vit99**]. Different approaches and further properties of that isomorphism can be found in Salamon-Weber [**SW06**] and Abbondandolo-Schwarz [**AS06**]. The analogous isomorphism for the based loop space appeared in [**AS10a**].

11.1. Based loop space and wrapped Floer homology. Let Q be a compact manifold and fix a point $q \in Q$. We denote by $\Omega_q(Q)$ the based loop space of Q with basepoint in q, which is the space of continuous maps from $[0, 1]$ to V that map 0 and 1 to q.

The concatenation of based loops gives $\Omega_q(Q)$ the structure of an H-space (see [**Hat02**]). More precisely, the concatenation induces the so-called Pontrjagin product on the singular homology $\mathrm{H}_*(\Omega_q(Q))$ of $\Omega_q(Q)$ with $\mathbb{Z}_2$ coefficients. The Pontrjagin product $[a_1] \cdot [a_2]$ of two homology classes $[a_1], [a_2] \in \mathrm{H}_*(\Omega_q(Q))$ is well-known to be associative. As it is distributive with respect to the vector space structure of $\mathrm{H}_*(\Omega_q(Q))$, it makes $\mathrm{H}_*(\Omega_q(Q))$ into a ring. Because the homology $\mathrm{H}_*(\Omega_q(Q))$ is considered with coefficients in $\mathbb{Z}_2$ it actually has the structure of an algebra.

Abbondandolo-Schwarz in [**AS10a**] construct an algebra isomorphism

$$\mathrm{H}_*(\Omega_q(V)) \to \mathrm{HW}(H, L_q), \tag{82}$$

where H is a function on T^*Q quadratic in the fibres, L_q is the cotangent fibre over q and T^*Q is equipped with the canonical Liouville form λ_{geo}.

Let $(D^*Q, \lambda_{geo}) \subset (T^*Q, \lambda_{geo})$ be the unit disk bundle with respect to some Riemannian metric on Q. It can be shown, see [**Rit13**] that $\mathrm{HW}(H)$ is isomorphic as an algebra to the wrapped Floer homology $\mathrm{HW}(D^*Q, L_q)$. Altogether we get the following

THEOREM 11.1 (Abbondandolo-Schwarz [**AS10a**]). *There is an algebra isomorphism*

$$\Psi_{AS,q} : \mathrm{H}_*(\Omega_q(V)) \to \mathrm{HW}(D^*Q, L_q). \tag{83}$$

11.2. Free loop space and symplectic homology. Consider now the free loop space $\Lambda(Q)$ of Q. Let g be a Riemannian metric on Q and let $l_g(\gamma)$ be the length of a path γ with respect to g. Let $\Lambda^a(Q) := \{\gamma \in \Lambda(Q) \,|\, l_g(\gamma) \leq a\}$. $H_0(\Lambda^a(Q))$ is naturally a filtered directed system.

Abbondandolo-Schwarz in [**AS06**] construct an isomorphism between $\mathrm{H}_*(\Lambda(Q))$ and $\mathrm{SH}(M)$. It is induced by a chain isomorphism between the Morse complex of an energy functional on $\Lambda(Q)$ and the Floer homology of a Hamiltonian on T^*Q that is quadratic in the fibres.

Let S on $\Lambda(Q)$ be defined as $S(\gamma) := \frac{1}{2}\int_0^1 |\dot\gamma|_g^2\, dt$ and let $\mathcal{E}^t := \{\gamma \in \Lambda(Q) \,|\, S(\gamma) \leq t\}$. Let $H_g(q,p) = \frac{1}{2}|p|_g^2$ the Legendre transform of S. As it follows from [**AS06**], there are isomorphisms $H(\mathcal{E}^{\frac{1}{2}m^2}) \to \mathrm{HF}^{\frac{1}{2}m^2}(H_g)$ that are compatible with the filtration, in other words $(H(\mathcal{E}^{\frac{1}{2}m^2}))_{m\geq 0}$ and $(\mathrm{HF}^{\frac{1}{2}m^2}(H_g))_{m\geq 0}$ are isomorphic as filtered directed systems. Strictly speaking, we have to perturb here H_g to guarantee that HF is well-defined.

Let $m \geq 0$. The space $\Lambda^m_{reg}(Q)$ of loops in $\Lambda^m(Q)$ that are parametrized with constant speed is homotopy equivalent to $\Lambda^m(Q)$. Furthermore, see [**Ano80**], $\Lambda^m_{reg}(Q)$ is homotopy equivalent to $\mathcal{E}^{\frac{1}{2}m^2}$. Hence $(\mathrm{H}(\Lambda^m(Q)))_{m\geq 0}$ is isomorphic to $(\mathrm{HF}^{\frac{1}{2}m^2}(H_g))_{m\geq 0}$ as filtered directed system.

Finally it can be shown, see e.g. [**McL12**, Lemma 4.18], that $\mathrm{HF}^{\frac{1}{2}m^2}(H_g)$ is isomorphic to $\mathrm{SH}^m((D^*Q)_g)$, the symplectic homology defined via linear

Hamiltonians as in section 9 of the unit disc bundle $(D^*Q)_g$ of T^*Q with respect to g. Altogether we get

THEOREM 11.2 (Abbondandolo-Schwarz [**AS06**]). *There is an isomorphism of filtered directed systems*

$$\Phi_{AS} : (\mathrm{H}(\Lambda^m(Q)))_{m\geq 0} \to \widetilde{\mathrm{SH}}(T^*Q). \tag{84}$$

12. Contact manifolds with positive entropy

In this section we will exhibit various examples of contact structures with positive entropy. In a preliminary subsection 12.1 we first consider cotangent bundles T^*Q of Riemannian manifolds such that HW or SH of D^*Q have positive exponential growth. With the methods developed above this provides alternative proofs of results in [**MS11**]. In subsection 12 we will carry out some topological constructions and prove Theorems 2.1, 2.2, 2.3 and 2.4.

12.1. Entropy on spherizations S^*Q.

12.1.1. HW *and positive entropy on* S^*Q. Let Q be a closed connected manifold and g a Riemannian metric on Q. Again, let $(D^*Q)_g, \lambda_{geo})$ in (T^*Q, λ_{geo}) be the unit disk bundle with respect to the Riemannian metric g. By Theorem 11.1 of Abbondandolo and Schwarz the map

$$\Psi_{AS,q_1} : \mathrm{H}_*(\Omega_{q_1}(Q)) \to \mathrm{HW}((D^*Q)_g, L_{q_1})$$

is an algebra isomorphism. It is well-known that there is an algebra isomorphism

$$\Phi : \mathbb{Z}_2[\pi_1(Q, q_1)] \to \mathrm{H}_0(\Omega_{q_1}(Q)).$$

Composing these two maps we obtain an injective algebra homomorphism

$$\widetilde{\Phi} : \mathbb{Z}_2[\pi_1(Q, q_1)] \to \mathrm{HW}((D^*Q)_g, L_{q_1}).$$

For a finitely generated group G and a finite set σ of generators of G, let $\widehat{\Gamma}_\sigma(G)$ be the usual exponential growth of the group G with respect to the set σ; see [**dlH00**, Section VI.C]. To a finite set σ of generators of $\pi_1(Q, q_1)$, we associate the finite set $S \subset \mathbb{Z}_2[\pi_1(Q, q_1)]$ that is formed by the elements of σ and its inverses. It is immediate to see that

$$\widehat{\Gamma}_\sigma(\pi_1(Q, q_1)) = \Gamma_S^{\mathrm{alg}}(\mathbb{Z}_2[\pi_1(Q, q_1)]).$$

Using that $\widetilde{\Phi}$ is injective we obtain

$$\widehat{\Gamma}_\sigma(\pi_1(Q, q_1)) = \Gamma_S^{\mathrm{alg}}(\mathbb{Z}_2[\pi_1(Q, q_1)]) \leq \Gamma_{\widetilde{\Phi}(S)}^{\mathrm{alg}}(\mathrm{HW}((D^*Q)_g, L_{q_1})).$$

We have shown the following

LEMMA 12.1. *It $\pi_1(Q, q_1)$ has exponential growth then there exists a finite set $S \subset \mathrm{HW}((D^*Q)_g, L_{q_1})$ such that $\Gamma_S^{\mathrm{alg}}((D^*Q)_g, L_{q_1}) > 0$.*

With Theorem 2.6 we get

COROLLARY 12.2. *It $\pi_1(Q, q_1)$ has exponential growth then (S^*Q, ξ) has positive topological entropy.*

REMARK 12.3. The statement of the corollary is also proved in [**MS11**]. Their proof uses that Ψ_{AS,q_1} respects the action resp. energy filtration. Our proof does not need this fact. Another interesting class of manifolds S^*Q with positive entropy is given by Q that are rationally hyperbolic, see [**MS11**]. Since for most rationally hyperbolic manifolds Q few is known about the algebraic growth properties of $\mathrm{H}(\Omega_q(Q))$, our approach is limited here. But conjecturally, see [**FHT01**, p.517], rationally hyperbolic manifolds Q give also rise to exponential algebraic growth of $\mathrm{H}(\Omega_q(Q))$.

12.1.2. RFH *and positive entropy of S^*Q.* Now let us consider the free loop space $\Lambda(Q)$ of Q. By Theorem 11.2, we have

$$\Gamma(\widetilde{SH}(T^*Q)) \geq \Gamma(\mathrm{H}(\Lambda^m(Q))_{m\geq 0}),$$

hence, by 2.7 we get

COROLLARY 12.4. *Assume that homology of the free loop space grows exponentially with respect to length, i.e. $\Gamma(\mathrm{H}(\Lambda^m(Q))_{m\geq 0}) > 0$, then (S^*Q, ξ) has positive entropy.*

Let us consider a class of manifolds Q that satisfy the assumption of Corollary 12.4. For that we recall the definition of the growth of conjugacy classes of a group.

DEFINITION 12.5. Let G be a finitely generated group. Fix a finite generating set $S \subset G$. For every conjugacy class $\mathfrak{c}$ of G we define the length L of $\mathfrak{c}$ by $L(\mathfrak{c}) = \inf\{n \in \mathbb{N} \mid h = s_1 \cdot s_2 \cdot \dots \cdot s_n$ and $\langle h \rangle = \mathfrak{c}\}$, where $\langle h \rangle$ denotes the conjugacy class of $h \in G$. We define for each $t \geq 0$,

$$N_t^{\mathrm{conj}}(G) := \#\{\mathfrak{c} \text{ conjugacy class of } G \mid L(\mathfrak{c}) \leq t\}.$$

Define the *exponential conjugacy growth rate* by

$$\Gamma_S^{\mathrm{conj}}(G) = \limsup_{t \to \infty} \frac{\log N_t^{\mathrm{conj}}(G)}{t}.$$

The growth of the number of conjugacy classes of groups was studied by many authors and the motivation was originally to find lower bounds on the growth of the number of closed geodesics on Riemannian manifolds. Note that there is a one-to-one correspondence between free homotopy classes of loops in a manifold Q and conjugacy classes of $\pi_1(Q)$. First and fundamental results were obtained by Margulis [**Mar69**]. He obtained good estimates on the growth of the number of conjugacy classes of $\pi_1(Q)$ for manifolds Q of negative sectional curvature. Further examples are provided also by the following statement, for a proof see [**McL12**, Lemma 4.21].

LEMMA 12.6. *Let $G = \langle S \rangle$ be a finitely generated group that is a free product of at least three non-trivial groups, then $\Gamma_S^{\mathrm{conj}}(G) > 0$.*

The exponential conjugacy growth of the fundamental group of a manifold Q implies the exponential growth of the homology of free loop space $\Lambda(Q)$, more precisely.

PROPOSITION 12.7. *Let $\pi_1(Q)$ be finitely generated by S and let g be a Riemannian metric on Q. Let l_g denote the length of paths in Q with respect* *to g and let $\rho := \max_{s\in S}\min\{l_g(e)\,|\,[e]=s\}$. Then*

$$\Gamma((\mathrm{H}_0(\Lambda^a))_{a\geq 0}) \geq \frac{1}{\rho}\Gamma^{\mathrm{conj}}(\pi_1(Q)).$$

We obtain in particular

$$\Gamma(\widetilde{SH}(T^*Q)) \geq \frac{1}{\rho}\Gamma^{\mathrm{conj}}(\pi_1(Q)). \tag{85}$$

PROOF. See also [**McL12**, Section 4.2]. To avoid confusion we denote for any loop γ, $[[\gamma]] \in [S^1, Q]$ its associated free homotopy class in $[S^1, Q]$ and as before for a based loop e, $[e] \in \pi_1(Q)$ its associated element in $\pi_1(Q)$. Let as before $\Lambda^a(Q) = \{\gamma \in \Lambda(Q)\,|\,l_g(\gamma) \leq a\}$ and consider the filtered directed system $(\mathrm{H}_0(\Lambda^a(Q)))_{a\geq 0}$. The elements in $\mathrm{Im}\,(\mathrm{H}_0(\Lambda^a(Q)) \to \mathrm{H}_0(\Lambda(Q)))$ are in one-to-one correspondence to the free homotopy classes $\xi \in [S^1, Q]$ for which there is $\gamma \in \Lambda^a(Q)$ with $[[\gamma]] = \xi$. Let now $t > 0$ and take any conjugacy class $\mathfrak{c}$ of $\pi_1(Q)$ with $L(\mathfrak{c}) \leq t$, so we have a representative $h \in \pi_1(Q)$ such that $h = s_1 \cdots s_n,\ n \leq t$ and $s_i \in S,\ 1 \leq i \leq n$. Now by the triangle inequality for l_g there is a (based) loop e with $[e] = h$ with $l_g(e) \leq n\rho$. But via $[[c]] \in [S^1, Q]$ it gives us an element in $\mathrm{Im}\,(\mathrm{H}_0(\Lambda^{n\rho}) \to \mathrm{H}_0(\Lambda(Q)))$. And it is immediate that this assignment, $\mathfrak{c} \mapsto \xi$, of conjugacy classes of $\pi_1(Q)$ with $L(\mathfrak{c}) \leq t$ to such elements ξ, is injective. Hence, by a standard estimate

$$\Gamma(\mathrm{H}_0(\Lambda^a)) \geq \frac{1}{\rho}\Gamma^{\mathrm{conj}}(\pi_1(Q)).$$

□

REMARK 12.8. There are many more examples of manifolds Q that satisfy the assumption of Corollary 12.4, see e.g. [**PP05**]. To my knowledge it is not known if for a general (non-simply connected) manifold Q with exponential growth of $\mathrm{H}(\Lambda(Q))$ also $\mathrm{H}(\Omega(Q))$ growths exponentially. If the latter is not true, new manifold of the form S^*Q with positive entropy will be covered with Corollary 12.4. This would then in particular give new examples of manifolds such that the topological entropy of all geodesic flows is positive.

12.2. Contact spheres with positive entropy.

12.2.1. *Some subcritical Handle attachments.* We first start with the following Lemma. This will allow us to simplify the topology of a Weinstein (or Liouville domain) by applying Weinstein 2- and 3-handle attachments.

LEMMA 12.9. *Let W be a Weinstein domain of dimension $2n \geq 6$ such that $c_1(W) = 0$, and assume that the $\mathbb{Z}$-homology of ∂W is freely generated. Then, by only attaching successively Weinstein 2- and and 3-handles to W, it is possible to obtain a simply connected Weinstein domain $\widetilde{W}$ such that $\mathrm{H}_1(\partial\widetilde{W}) = 0$, $\mathrm{H}_2(\partial\widetilde{W}) = 0$, and, for $i \geq 3$, $\mathrm{H}_i(\partial\widetilde{W}) \cong \mathrm{H}_i(\partial W)$ and $\mathrm{H}_i(\widetilde{W}) \cong \mathrm{H}_i(W)$.*

PROOF. We describe handle attachments in several steps.

Step 1. If $\mathrm{H}_1(\partial W) = 0$, we directly start with step 2. If not, by assumption $\mathrm{H}_1(\partial W) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k \text{ times}}$, $k > 0$, and we take a generator e of one $\mathbb{Z}$-factor. Consider a (based) loop c that represents a class in $\pi_1(\partial W)$ whose image under the Hurewicz homomorphism $\pi_1(\partial W) \to \mathrm{H}_1(\partial W)$ is e. By the h-principle for isotropic submanifolds, cf. Theorem 8.3, we can find an isotropic embedded loop γ in the same free homotopy class as c. In our situation the assumptions of Theorem8.3 are satisfied since every oriented bundle over S^1 is trivial. And by the same reason we can find a trivialization of $\mathrm{CSN}(\partial W, \gamma)$ and can attach a Weinstein 2-handle to W along γ, see Theorem 8.1, and obtain a new Weinstein domain W'.

Topologically, $\partial W'$ is obtained by gluing the handle $D^2 \times S^{2n-2}$ and $\partial W \setminus (S^1 \times D^{2n-2})$ along their boundaries. So in order to see the effect on homology we consider the long exact sequences of the pairs $(\partial W, S^1 \times D^{2n-2})$ and $(\partial W', D^2 \times S^{2n-3})$,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_2(\partial W) \arrow[r] & \mathrm{H}_2(\partial W, S^1 \times D^{2n-2}) \arrow[r] & \mathrm{H}_1(S^1) \arrow[r] & \mathrm{H}_1(\partial W) \\ 0 \arrow[r] & \mathrm{H}_2(\partial W') \arrow[r] & \mathrm{H}_2(\partial W', D^2 \times S^{2n-3}) \arrow[r] \arrow[u, "\text{exc}"', "\cong"] & 0 & \end{tikzcd} \tag{86}$$

where the isomorphism linking the two sequences is given by excision. Since the map $\mathrm{H}_1(S^1) \to \mathrm{H}_1(\partial W)$ is injective, we get that $\mathrm{H}_2(\partial W') \cong \mathrm{H}_2(\partial W)$. Similarly, by looking at the same two sequences at other degrees, we obtain that $\mathrm{H}_1(\partial W') \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{(k-1)\text{ times}}$, and $\mathrm{H}_i(\partial W') \cong \mathrm{H}_i(\partial W)$, for $i \geq 3$.

We repeat the above another $(k-1)$ times and obtain a Weinstein domain W_1 with $\mathrm{H}_i(\partial W_1) \cong \mathrm{H}_i(\partial W)$, for $i \neq 1$, and $\mathrm{H}_1(\partial W_1) = 0$. Since $W \subset W_1$ induces an isomorphism on H^2, the first chern class c_1 of W_1 still vanishes.

Step 2. The next step is to attach handles to W_1 to obtain a simply connected Weinstein domain W_2. Now we choose an isotropic γ in ∂W_1 that represents a non-zero element g in $\pi_1(\partial W_1)$. As above we can attach a Weinstein 2-handle to γ and obtain a Weinstein domain W_1'. We have that $\pi_1(\partial W_1') \cong \pi_1(\partial W_1)/G$, where G is a normal subgroup that contains the element g, see e.g. [**Mil61**, p.44]. Consider now analogously to Step 1 the diagram (86) obtained by the long exact sequences of the pairs $(\partial W_1, S^1 \times D^{2n-2})$ and $(\partial W_1', D^2 \times S^{2n-3})$. The upper row in (86) is now short exact and, since $\mathrm{H}_1(S^1) \cong \mathbb{Z}$, it splits. Hence $\mathrm{H}_2(\partial W_1') \cong \mathrm{H}_2(\partial W_1) \oplus \mathbb{Z}$. Let σ be a 2-cycle that represents the generator of the new $\mathbb{Z}$-factor of $\mathrm{H}_2(W_1')$. Before the attachment of the 2-handle we had a freedom in the framing of the attaching sphere γ in W_1 by a choice of an element in $\pi_1(U(n-2)) \cong \mathbb{Z}$. From this we can guarantee that the chern class c_1 of $TW_1'|_\sigma$ vanishes. By assumption, c_1 evaluated on elements in $\mathrm{H}_2(\partial W_1)$ vanishes and hence altogether $c_1(W_1') = 0$.

After attaching sufficiently many such handles we obtain a simply connected Weinstein domain W_2. Moreover, $\mathrm{H}_1(\partial W_2) = 0$, $\mathrm{H}_2(\partial W_2)$ is freely generated, for $i \geq 3$ $\mathrm{H}_i(\partial W_2) \cong \mathrm{H}_i(\partial W_1)$, and $c_1(W_2) = 0$.

Step 3. Since ∂W_2 is simply connected, we can, by Hurewicz' Theorem, represent the generators of $\mathrm{H}_2(\partial W_2)$ by embedded spheres. Let now $j : S^2 \to \partial W_2$ be any embedded sphere. An almost complex structure J compatible with $d\lambda$ and cylindrical near ∂W_2 induces a complex structure on the bundles j^*TW_2, $j^*\xi$ and $j^*(\underline{R} \oplus \underline{Y})$ over S^2, where $\underline{R}$ and $\underline{Y}$ are the bundles over ∂W_2 induced by the Reeb vector field and Liouville vector field, respectively. We have $j^*TW_2 = j^*\xi \oplus j^*(\underline{R} \oplus \underline{Y})$, and since $j^*(\underline{R} \oplus \underline{Y})$ is trivial, it follows from $c_1(W_2) = 0$ that $c_1(j^*\xi) = 0$, which means that $j^*\xi$ is trivial. Therefore the trivial bundle $TS^2 \otimes \mathbb{C}$ admits a fibrewise injective injection into $j^*\xi$ and by Theorem 8.3 there is an isotropic embedding $j_0 : S^2 \to \partial W$ that is homotopic to j. Again, since $c_1(j_0^*TW_2) = 0$ we obtain by the Whitney sum formula that $c_1(\mathrm{CSN}(\partial W_2, j_0(S^2))) = 0$. This implies that $\mathrm{CSN}(\partial W_2, j_0(S^2))$ admits a trivialization and we can attach a Weinstein 3-handle to $j_0(S^2)$. Similar as in Steps 1 and 2 above we observe that the new Weinstein domain W_2' satisfies $\mathrm{H}_2(\partial W_2') \oplus \mathbb{Z} \cong \mathrm{H}_2(\partial W_2)$ and $\mathrm{H}_i(\partial W_2') \cong \mathrm{H}_i(\partial W_2)$ for $i \neq 2$.

By attaching sufficiently many 3-handles we obtain a Weinstein domain $\widetilde{W}$ that is simply connected, has $\mathrm{H}_1(\partial \widetilde{W}) = \mathrm{H}_2(\partial \widetilde{W}) = 0$, and for $i \geq 3$ $\mathrm{H}_i(\partial \widetilde{W}) \cong \mathrm{H}_i(\partial W)$. It is also easy to see that the 2- and 3-handle attachments don't effect the homology groups H_i of the domain for $i \geq 3$. Hence $\widetilde{W}$ satisfies all the properties that are stated in the lemma. $\square$

Let us also state the following useful statement, that hold for Weinstein domains, see also [**McL11**, Lemma 2.9].

LEMMA 12.10. *Let W be Weinstein domain of dimension $2n \geq 6$. Then the maps $\pi_1(\partial W) \to \pi_1(W)$ and $\mathrm{H}_i(\partial W) \to \mathrm{H}_i(W)$, $i < n$ induced by the inclusion $\partial W \to W$ are isomorphisms.*

PROOF. W admits a Morse function whose critical points have index not exceeding n, see [**CE12**]. Therefore W is homotopy equivalent to ∂W with cells of dimension $\geq n$ attached. Attaching cells of dimension $\geq n > 2$ has no effect on π_1 or H_i, for $i \leq n-1$. $\square$

12.2.2. *Proof of statements (A) of Theorem 2.1 and ♣ of Theorem 2.2.*

PROOF OF STATEMENT (A) OF THEOREM 2.1. Let G be a finitely presented group such that

- $\mathrm{H}_1(G) = \mathrm{H}_2(G) = 0$,
- G has exponential growth,

- G admits a presentation on which the number of relations does not exceed the number of generators.

For example the group $G(2, 3, 7)$ considered below satisfies these properties. Then, it follows from [**Ker69**], that for every $n \geq 4$ there exists a manifold Q^n which is an integral homology sphere and which satisfies $\pi_1(Q^n) \cong G$. We denote by $\varrho(G)$ the minimal number of generators of G.

We denote by D^*Q^n the unit disk bundle of Q^n, with respect to a Riemannian metric g in Q^n, endowed with the canonical symplectic and Liouville forms. We choose a point $q \in Q^n$ and g generically so that q is not conjugate to itself. Let $S^*Q^n = \partial D^*Q^n$ be the unit cotangent bundle of Q^n. In order to prove our result we consider two distinct cases.

Case 1: n is odd and ≥ 5.
In this case the Euler characteristic of Q^n vanishes. Because G grows exponentially, we know that $\mathrm{HW}_0(D^*Q^n, L_q)$ has exponential algebraic growth. Let N^1 be the plumbing of D^*Q^n and D^*S^n performed far from L_q. By Proposition 2.5, $\mathrm{HW}_0(N^1, L_q)$ has exponential algebraic growth.

It is a result of Milnor that the boundary of the plumbing of the unit disk bundles of two odd-dimensional homology spheres of dimension ≥ 3 is a homology sphere; see [**Bre93**, Chapter VI - Section 18]. Applying this to the pair D^*Q^n and D^*S^n we conclude that ∂N^1 is a homology sphere. Since N^1 retracts to the one point union of Q and S^n we know that the homology of N^1 is zero in every degree different from 0 and n, where we have $\mathrm{H}_0(N^1) \cong \mathbb{Z}$ and $\mathrm{H}_n(N^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Case 2: n is even and ≥ 4.
In this case the Euler characteristic of Q^n is 2. We consider the plumbing associated to the E8 tree; see [**Bre93**, Chapter VI - Section 18]. To each vertex of the E8 tree we associate a disk bundle in the following way:

- to the leftmost vertex we associate D^*Q^n,
- to every other vertex we associate D^*S^n.

We let N^1 be the plumbing associated to the E8 tree determined by this choice of disk bundles at each vertex, and assume that the plumbing is done away from a cotangent fibre $L_q \subset D^*Q^n$. It was shown by Milnor (see [**Bre93**, Chapter VI - Section 18]) that ∂N^1 is a homology sphere. Since N^1 retracts to the wedge sum of Q and seven copies of S^n determined by the E8 tree, we know that the homology of N^1 is zero in every degree different from 0 and n, where we have $\mathrm{H}_0(N^1) \cong \mathbb{Z}$ and $\mathrm{H}_n(N^1) \cong \oplus_{i=1}^{8}\mathbb{Z}$. By Proposition 2.5, $\mathrm{HW}_0(N^1, L_q)$ has exponential algebraic growth.

We now treat both cases simultaneously. By Lemma 12.9 we can attach Weinstein 2- and 3-handles to N^1 and obtain a Weinstein domain N^2 that is simply connected and is a homology sphere. It follows from Whitehead's Theorem for homology [**Hat02**, Corollary 4.33] that ∂N^2 also has the homotopy groups of a sphere. Since the dimension of ∂N^2 is greater than 5 the h-cobordism theorem tells us that ∂N^2 is homeomorphic to a sphere. And since the smooth spheres under connected sum form a finite group, we can take the (contact) boundary connected sum of finitely many copies of N^2, i.e. connecting the domains by Weinstein 1-handles, to get a domain N^3 such that the sphere ∂N^3 has the standard smooth structure. We conically extend the Lagrangian L_q of one summand to N_3. By Proposition 2.5 we have that $\mathrm{HW}(N^3, L_q)$ has exponential algebraic growth. By applying Theorem 2.6, this proves statement (A) of Theorem 2.1. □

PROOFS OF STATEMENT ♣ OF THEOREM 2.2. Let $n \geq 4$, let V be a $(2n-1)$-dimensional manifold, and assume that there exists an exactly fillable contact structure ξ on V. Denote by M_V a Liouville domain whose boundary is (V, ξ). Let N^4 be the Liouville domain constructed in the proof of statement (A) of Theorem 2.1. By Proposition 2.5, the Liouville domain $N^5 = N^4 \# M_V$ has an asymptotically conical exact Lagrangian L such that $\mathrm{HW}(N^5, L)$ has exponential algebraic growth. By Theorem 2.6, the statement follows. □

12.2.3. *Proof of statements (B) of Theorem 2.1 and ♢ of Theorem 2.2.*

PROOF OF STATEMENT (B) OF THEOREM 2.1. We will consider a carefully chosen 3-manifold Q. Consider the Brieskorn manifolds of dimension 3, $M(p,q,r) = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \,|\, {z_1}^p + {z_2}^q + {z_3}^r = 0\} \cap S^5$. $M(p,q,r)$ is a $\mathbb{Z}$-homology sphere if p,q,r are relatively prime (see for example [Sav02]). It was shown by Milnor [Mil75] that its fundamental group $\pi_1(M(p,q,r))$ is the commutator subgroup of the group

$$G = G(p,q,r) := \langle \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3 \rangle,$$

see also [Sea06]. The groups $\Sigma(p,q,r) = G(p,q,r)/Z(G(p,q,r))$ are called the triangle groups, where $Z(G)$ is the center of G. Consider the case $p = 2$, $q = 3$, $r = 7$. A short computation shows that $G(2,3,7)$ coincides with its commutator subgroup $[G(2,3,7), G(2,3,7)]$. It is well-known that the exponential growth of $\Sigma(2,3,7)$ is $\log(x)$, where $x \approx 1.17628$ is equal to Lehmer's Salem number (see [Hir03] or [Bre14]). Hence we have that $\widehat{\Gamma}(G(2,3,7)) \geq \widehat{\Gamma}(\Sigma(2,3,7)) > 0$. We take $Q = M(2,3,7)$. The integral homology of D^*Q is the same as that of Q, which is $\mathbb{Z}$ in degrees 0 and 3 and vanishes in all other degrees. Moreover it is clear that $\pi_1(S^*Q) = \pi_1(Q \times S^2) = \pi_1(Q)$ is generated by the elements γ_1 and γ_2.

Let N^1 be the Liouville domain obtained by plumbing D^*Q with the unit disk bundle D^*S^3 of S^3. We assume that the plumbing is performed away from the cotangent fibre L_q over a point $q \in Q$. Therefore L_q survives as a conical exact Lagrangian in N^1. By Proposition 2.5 we know that $\mathrm{HW}_*(N^1, L_q)$ has exponential algebraic growth.

Since N^1 is the plumbing of D^*Q and D^*S^3, and Q and S^3 are both homology spheres we obtain that ∂N^1 is a homology sphere; see [Bre93, Chapter VI - Section 18].

Combining this with the fact that N^1 retracts to the one point union of S^3 and Q we conclude that

- $\mathrm{H}_0(N_1) \cong \mathbb{Z}$, $\mathrm{H}_3(N_1) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $\mathrm{H}_i(N_1) = 0$ for $i \neq 0, 3$,
- $\mathrm{H}_0(\partial N_1) \cong \mathbb{Z}$, $\mathrm{H}_5(\partial N_1) \cong \mathbb{Z}$, and $\mathrm{H}_i(\partial N_1) = 0$ for $i \neq 0, 5$.

Let now $\{\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3\}$ be generators of $\pi_1(\partial N_1) \cong \pi_1(Q)$ corresponding to γ_1, γ_2 and γ_3 respectively. As in Step 2 of the proof of Lemma 12.9 we apply the Weinstein handle attachment and attach a 2-handle to N^1 over σ_3, obtaining a Weinstein domain N^2. From the presentation of $\pi_1(Q)$ that we used, it is clear that ∂N^2 is simply connected, and so is N^2 by Lemma 12.10. We choose the framing of the handle attachment so that ∂N^2 is spin. We obtain that $\mathrm{H}_0(\partial N^2) \cong \mathbb{Z}$, $\mathrm{H}_2(\partial N_2) \cong \mathbb{Z}$, and $\mathrm{H}_1(\partial N_2) = 0$. Since we can choose the framing of the symplectic normal bundle such that the first chern class of N_2 is 0, in particular the Stiefel Whitney class of ∂N^2 must vanish. By Smale's classification of spin simply-connected five manifolds [**Sma62**] it follows that ∂N^2 is diffeomorphic to $S^3 \times S^2$.

Since N^2 is obtained from N^1 via a subcritical handle attachment and the Lagrangian L_q is far from the attaching locus of this handles, we know that L_q survives as a conical exact Lagrangian in N^2. Moreover, Proposition 2.5 implies that $\mathrm{HW}_*(N^2, L_q)$ has exponential algebraic growth, and it follows from Theorem 2.6 that the contact manifold ∂N^2 has positive entropy. □

PROOF OF STATEMENT $\diamondsuit$ OF THEOREM 2.2. The statement is proved by a connected sum argument identical to the one in the proof of statement $\clubsuit$. □

12.2.4. *Weinstein domains diffeomorphic to balls with boundary of positive entropy.* We prove Theorems 2.3 and 2.4.

PROOF OF THEOREM 2.3. By Theorem 2.7 it is sufficient to construct Liouville domains M diffeomorphic to a ball with $\Gamma(\widetilde{\mathrm{SH}}(M)) > 0$. We distinguish the cases of even and odd $n \geq 4$.

Let first consider even n, $n = 2k$ with $k = 2, 3 \cdots$. Let Σ_g be a surface of genus $g \geq 2$. Let $D^*\Sigma_g$ be the unit disk bundle of Σ_g with respect to a given metric on Σ_g. Let T be a contractible 4-dimensional Liouville domain with $\mathrm{SH}(T) \neq 0$, cf. Theorem 9.5, and consider $M_n = \underbrace{T \times \cdots \times T}_{(k-1) \text{ times}} \times D^*\Sigma_g$.

Since T is contractible, M_n is homotopy equivalent to Σ_g. Hence $\mathrm{H}(M_n)$ is freely generated, and moreover $\mathrm{H}_2(M_n) \cong \mathbb{Z}$ and $\mathrm{H}_i(M_n) = 0$ for $i \geq 3$. By Lemmata 12.9 and 12.10 we can add Weinstein 2- and 3-handles to ∂M_n and obtain a Weinstein domain $\widetilde{M}_n$ such that $\widetilde{M}_n$ is simply connected and $\mathrm{H}_i(\widetilde{M}_n) = 0$ for $i \neq 0$. Hence $\widetilde{M}_n$ is contractible and by the h-cobordism theorem $\widetilde{M}_n$ is diffeomorphic to a ball. Note that $\Gamma^{\mathrm{conj}}(\pi_1(\Sigma_g)) > 0$, hence by Proposition 12.7 $\Gamma(\widetilde{\mathrm{SH}}(D^*\Sigma_g)) > 0$. By the Künneth formula for SH (Theorem 9.4) $\Gamma(\widetilde{\mathrm{SH}}(M_n)) \geq \Gamma(\widetilde{\mathrm{SH}}(D^*\Sigma_g)) > 0$. Finally, by Theorem 9.3 it follows that $\Gamma(\widetilde{\mathrm{SH}}(\widetilde{M}_n)) > 0$.

Now we proceed to construct the examples for odd $n \geq 5$, i.e. $2n - 1 = 4k + 1$ for $k = 2, 3, \cdots$. Let P now be a 3-dimensional homology sphere with $\Gamma^{\mathrm{conj}}(\pi_1(P)) > 0$; for example a connected sum of three homology spheres that are not simply connected, see Lemma 12.6. We consider D^*P with respect to a given metric and define $M := T \times D^*P$. Since T is contractible $\mathrm{H}(M) \cong \mathrm{H}(Q) \cong \mathrm{H}(S^3)$, and $\pi_1(M) \cong \pi_1(Q)$. By Lemma 12.9 and Lemma 12.10 we can attach 2-and 3-handles to obtain a domain $\widehat{M}$ that is simply connected with $\mathrm{H}_i(\widehat{M}) = 0$ for $i \neq 0, 3$ and $\mathrm{H}_3(\widehat{M}) \cong \mathrm{H}_3(\partial\widehat{M}) \cong \mathbb{Z}$. Furthermore, $\mathrm{H}_1(\partial\widehat{M}) = \mathrm{H}_2(\partial\widehat{M}) = 0$. Hence we can find by Hurewicz' Theorem an embedding $i : S^3 \to \partial\widehat{M}$ that represents a generator in $\mathrm{H}_3(\partial\widehat{M})$. The assumptions of Theorem 8.3 are satisfied; namely $TS^3 \otimes \mathbb{C}$ is trivial, and since every 4-dimensional complex bundle over S^3 is trivial due to $\pi_2(U(4)) = 0$, also $\xi|_{S^3}$ is trivial. Therefore, we can find an isotropic embedding $i_0 : S^3 \to \partial\widehat{M}$ representing a generator of $\mathrm{H}_3(\partial\widehat{M})$. Since also every complex line bundle over S^3 vanishes, $\mathrm{CSN}(\partial\widehat{M}, i_0(S^3))$ admits a trivialization and we can attach a Weinstein 4-handle along $i_0(S^3)$. We obtain a Liouville domain $\widetilde{M}_5$. The same reasoning as in the case of 2- and 3-handle attachments shows that the attachment of this 4-handle only affects the homology of $\widehat{M}$ in degree 3 and we get that $\mathrm{H}_i(\widetilde{M}_5) = 0$ for $i \neq 0$. We define then for odd $n = 4k + 1 \geq 7$, $\widetilde{M}_n = \underbrace{T \times \cdots \times T}_{(k-2)\text{ times}} \times \widetilde{M}_5$. The domains $\widetilde{M}_n$ for all odd $n \geq 5$ are simply connected, have the homology of a point, hence are contractible and by the h-cobordism theorem are diffeomorphic to balls B^{2n}. Using here again Theorems 11.2, 9.4 and 9.3 we obtain $\Gamma(\widetilde{\mathrm{SH}}(\widetilde{M_n})) > 0$. $\square$

PROOF OF THEOREM 2.4. Let $n \geq 4$. In the proof of Theorem 2.3 we find a Liouville form λ_0 on B^{2n} such $\Gamma(\widetilde{\mathrm{SH}}(B^{2n}), \lambda_0) > 0$. Let now (M^{2n}, λ_1) be any Liouville domain. Denote by $\widetilde{\lambda}$ the induced Liouville form on the disjoint union $B \cup M$. Attach a Weinstein 1-handle to $B^{2n} \cup M^{2n}$ that connects B and M. The resulting Liouville domain M_1, which is diffeomorphic to the boundary connected sum of the two domains, is hence diffeomorphic to M, see [**Kos93**, p.97]. By this construction we get a new Liouville form λ on M. By Corollary 9.3 $\Gamma(\widetilde{\mathrm{SH}}(M, \lambda)) = \Gamma(\widetilde{\mathrm{SH}}(M \cup B, \widetilde{\lambda}))$. By the definition of SH we have $\widetilde{\mathrm{SH}}(B \cup M, \widetilde{\lambda}) \cong \widetilde{\mathrm{SH}}(B, \lambda_0) \oplus \widetilde{\mathrm{SH}}(M, \lambda_1)$ as filtered directed systems. Hence $\Gamma(\widetilde{\mathrm{SH}}(B \cup M)) \geq \Gamma(\widetilde{\mathrm{SH}}(B, \lambda_0)) > 0$. Again by Theorem 2.7 the boundary $(\partial M, \xi_{(M,\lambda)})$ has positive entropy. $\square$

CHAPTER 3

RFH on hypertight contact manifolds and translated points

In this chapter we prove the results introduced in section 3. First we define the Rabinowitz action functional in the symplectisation of any hypertight contact manifold, section 13.1, and prove compactness properties of moduli space of solutions of Floer equations, section 13.2. This allows us to define the Rabinowitz Floer homology in our setting, section 13.3. In section 14 we define continuation maps, prove the relevant compactness properties of s-dependent moduli spaces which leads to the invariance of RFH under the change of the contact form. In the last section, 15, we give dynamical applications and prove Theorems 3.2 and 3.6 from the Introduction.

13. Definition of RFH

13.1. The Rabinowitz action functional. Let (Σ, ξ) be a closed co-orientable contact manifold with ξ a hypertight contact structure. In this paper, we always assume that the contact manifold is hypertight. Let $\alpha_0 \in \mathcal{C}(\xi)$ be a supporting contact form without contractible Reeb orbits. Denote by R_0 the Reeb vector field of α_0 and by $\theta^t_{\alpha_0} : \Sigma \to \Sigma$ its Reeb flow. Let $\alpha_1 \in \mathcal{C}(\xi)$ be any other supporting contact form, which possibly has contractible Reeb orbits. Then, there is a function $g : \Sigma \to (0, +\infty)$ such that $\alpha_1 = g \cdot \alpha_0$.

Let $M := (0, +\infty) \times \Sigma$. We want to equip M with a symplectic form with primitive λ such that λ equals $r\alpha_1$ near $\{1\} \times \Sigma$ and λ equals $r\alpha_0$ near $\{0\} \times \Sigma$. For this contact form to be symplectic it is crucial that we can homotope $r\alpha_1$ to $r\alpha_0$ in an increasing way along r. Let $0 < \epsilon < \inf_{x\in\Sigma} g(x)$ and let $\nu > 0$. Define

$$\lambda := f(r, x)\alpha_0,$$

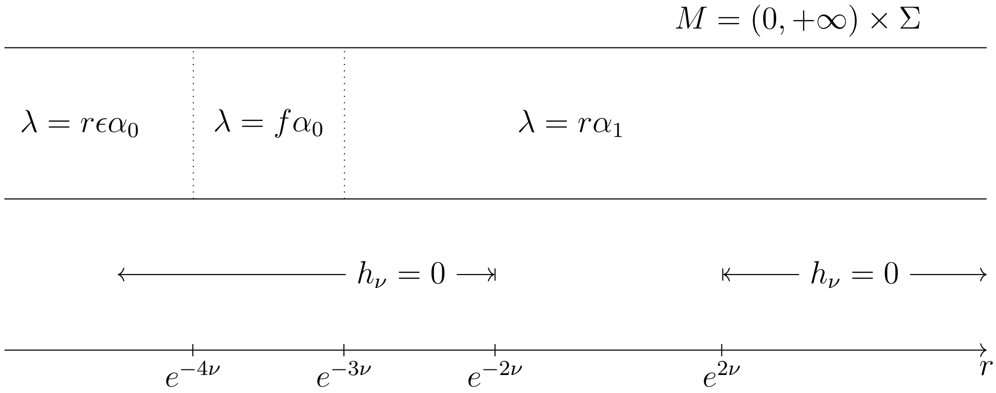

FIGURE 6.

where $f : M \to (0, +\infty)$ is defined as

$$f(r, x) = \begin{cases} rg(x), & \text{for } r > e^{-3\nu} \\ r\epsilon, & \text{for } r < e^{-4\nu}, \end{cases} \tag{87}$$

and such that $\frac{\partial f}{\partial r} > 0$. A direct computation shows that $d\lambda$ is nondegenerate if and only if $\frac{\partial f}{\partial r} > 0$. We set

$$\Omega_\nu(\alpha_0, \alpha_1) = \Big\{ \lambda \in \Omega^1(M) \mid \lambda = f\alpha_0,\ f \text{ satisfies (87) for some } \epsilon > 0 \\ \text{and } \frac{\partial f}{\partial r} > 0 \Big\}.$$

In the construction below we will fix $\lambda \in \Omega_\nu(\alpha_0, \alpha_1)$ after choosing a suitable $\nu > 0$.

Suppose $\varphi : \Sigma \to \Sigma$ is a contactomorphism. Then there is a smooth positive function $\rho : \Sigma \to (0, +\infty)$ such that $\varphi^*\alpha_1 = \rho\alpha_1$. In the following, we always consider a contactomorphism which is contact-isotopic to the identity.

Let $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$, then there is a path $\hat{\varphi} = \{\varphi_t\}_{t\in[0,1]}$ with $\varphi_t = \mathbb{1}$, for $t \in \left[0, \frac{1}{2}\right]$ and $\varphi_1 = \varphi$. We call such a path *admissible*. Also, there exists a smooth family of positive functions $\rho_t : \Sigma \to (0, +\infty)$ such that $\varphi_t^* \alpha_1 = \rho_t \alpha_1$.

DEFINITION 13.1. The *contact Hamiltonian* of $\hat{\varphi}$ with respect to α_1 is the function $l : \Sigma \times [0,1] \to \mathbb{R}$ defined by

$$l_t \circ \varphi_t = \alpha_1 \left(\frac{d}{dt} \varphi_t \right). \tag{88}$$

We extend l to a Hamiltonian function $L : M \times [0,1] \to \mathbb{R}$ by

$$L_t(r,x) := r l_t(x).$$

The Hamiltonian diffeomorphism $\phi_L^t : M \to M$ associated to L and the symplectic form $d\lambda$ is given by

$$\phi_L^t(r,x) = \big(r\rho_t(x)^{-1}, \varphi_t(x) \big) \tag{89}$$

on the interior of $\{ (r,x) \in M |\ \lambda = r\alpha_1 \}$ and so in particular on $(e^{-\nu}, e^{\nu}) \times \Sigma$.

Moreover, let

$$H(r) = \begin{cases} c, & \text{for } r \in (e^{2\nu}, +\infty) \\ r - 1, & \text{for } r \in (e^{-\nu}, e^{\nu}) \\ -c, & \text{for } r \in (0, e^{-2\nu}) \end{cases}$$

for some constant $c \geq \max\{1 - e^{-\nu}, e^{\nu} - 1\}$ such that $H'(r) \geq 0$. We will also use H for the function $H(r,x) := H(r)$ on M. Note that

$$X_H(r,x) = \frac{\partial H}{\partial r}(r,x) R_1(x)$$

since $\frac{\partial H}{\partial r} = 0$ on the region where $\lambda \neq r\alpha_1$. Here, R_1 denotes the Reeb vector field of α_1.

Now, let $\kappa : S^1 \to \mathbb{R}$ be a smooth function with

$$\kappa(t) = 0, \quad \text{for all} \quad t \in \left[\frac{1}{2}, 1\right] \quad \text{and} \quad \int_0^1 \kappa(t)\, dt = 1.$$

We use κ to modify the Hamiltonian H to $\kappa(t) H(r,x)$.

In the following, we want to cut off the Hamiltonian L. In order to do so, we have to take care that we cut off outside of the region where our perturbed functional will have its periodic orbits. Define a smooth function $\beta_\nu \in C^\infty((0,\infty),[0,1])$ such that

$$\beta_\nu(r) = \begin{cases} 1, & r \in (e^{-\nu}, e^{\nu}) \\ 0, & r \in (0, e^{-2\nu}] \cup [e^{2\nu}, +\infty)\,. \end{cases}$$

We use β_ν to cutoff the Hamiltonian L via $\beta_\nu L$.

Let $\mathcal{L}M$ denote the set of contractible smooth loops $u = (r,x) : S^1 \to M$. Finally, we are ready to define the *perturbed Rabinowitz action functional* associated to $\hat{\varphi}$,

$$\mathcal{A}^{(\hat{\varphi},\nu)}_{(\alpha_0,\alpha_1)} : \mathcal{L}M \times \mathbb{R} \to \mathbb{R},$$

$$\mathcal{A}^{(\hat{\varphi},\nu)}_{(\alpha_0,\alpha_1)}(r,x,\eta) := \int_{S^1} (r,x)^*\lambda - \eta \int_{S^1} \kappa(t) H(r) dt - \int_{S^1} \beta_\nu(r) L_t(r,x)\, dt. \tag{90}$$

A point $(u(t),\eta) \in M$ (where $u(t) = (r(t),x(t))$) is a critical point of $\mathcal{A}^{(\hat{\varphi},\nu)}_{(\alpha_0,\alpha_1)}$ if

$$\begin{cases} \dot{u}(t) = \eta\kappa(t) X_H(u(t)) + \beta_\nu(r(t)) X_L(u(t)) \\ \int_0^1 \kappa(t) H(u(t))\, dt = 0. \end{cases}$$

LEMMA 13.2. *Assume that* $\hat{\varphi} = \{\varphi_t\}_{t\in[0,1]} \in \widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ *is an admissible path of contactomorphisms and define*

$$C(\hat{\varphi};\alpha_1) := \max_{t\in[0,1]} \int_0^t \max_{x\in\Sigma} \left| \frac{\dot{\rho}_s(x)}{\rho_s(x)^2} \right| ds. \tag{91}$$

Let $(r,x,\eta) \in \mathrm{Crit}\left(\mathcal{A}^{(\hat{\varphi},\nu)}_{(\alpha_0,\alpha_1)}\right)$*. If* $\nu > C(\hat{\varphi};\alpha_1)$*, then every critical point of* $\mathcal{A}^{(\hat{\varphi},\nu)}_{(\alpha_0,\alpha_1)}$ *has image contained in* $(e^{-\nu}, e^{\nu}) \times \Sigma \times \mathbb{R}$*, i.e.* $r(S^1) \subset (e^{-\nu}, e^{\nu})$*.*

The proof of the lemma is analogous to the proof of [**AFM13**, Lemma 3.5] after the observation that for $\nu > C(\hat{\varphi};\alpha_1)$ we have $r(S^1) \subset (e^{-\nu}, e^{\nu})$

and the fact that $\lambda = r\alpha_1$ in a neighborhood of all the critical points of the action functional $\mathcal{A}^{(\hat\varphi,\nu)}_{(\alpha_0,\alpha_1)}$.

Note that it follows that for a critical point $H(u(t)) = 0$ for $t \in [0, \frac{1}{2}]$, and thus $r(t) = 1$ for $t \in [0, \frac{1}{2}]$.

Later, we will also need the oscillation norm of contact Hamiltonian. We define the *oscillation norm* of the contact Hamiltonian l to be

$$e^{C(\hat\varphi;\alpha_1)}\left(\int_0^1 \max_{x\in\Sigma} l_t(x)dt - \int_0^1 \min_{x\in\Sigma} l_t(x)dt\right). \tag{92}$$

From now on, we fix $\nu > C(\hat\varphi;\alpha_1)$ and $\lambda \in \Omega_\nu(\alpha_0,\alpha_1)$ and write $\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)} := \mathcal{A}^{(\hat\varphi,\nu)}_{(\alpha_0,\alpha_1)}$.

REMARK 13.3. Note that after the above choice of ν at a critical point (u,η) it holds that

$$\begin{aligned}\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)}(u,\eta) &= \int_{S^1}\lambda(\dot u(t))\,dt - \int_{\frac{1}{2}}^1 L_t(u(t))\,dt\\ &= \eta + \int_{S^1}\left(\lambda(X_L(u)) - L_t(u)\right)\,dt\\ &= \eta\end{aligned}$$

since $L_t = rl_t$, where l_t is the contact Hamiltonian of the path $\hat\varphi$ with respect to α_1.

DEFINITION 13.4. A path $\hat\varphi$ is *nondegenerate* if $\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)} : \mathcal{L}M\times\mathbb{R}\to\mathbb{R}$ is a Morse-Bott function which means $\mathrm{Crit}\left(\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)}\right) \subset \mathcal{L}M$ is a submanifold and for each $(u,\eta)\in \mathrm{Crit}\left(\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)}\right)$ we have

$$T_{(u,\eta)}\mathrm{Crit}\left(\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)}\right) = \ker \mathrm{Hess}\left(\mathcal{A}^{\hat\varphi}_{(\alpha_0,\alpha_1)}\right)(u,\eta).$$

See [Fra04] for more details.

It is standard to show that nondegeneracy is a generic property for paths of contactomorphisms.

We want to choose J an almost complex structure on M in such a way that it satisfies the following properties.

DEFINITION 13.5. Let $\alpha \in \mathcal{C}(\xi)$. We say that an almost complex structure J is *SFT-like with respect to* α, if

- it is invariant under translations $(r, x) \mapsto (e^c r, x)$ for $c \in \mathbb{R}$,
- it preserves ξ and
- satisfies $JR_\alpha = r\partial_r$,

where R_α denotes the Reeb vector field with respect to α.

Let $J := \{J_t\}_{t\in S^1}$ be a family of almost complex structures compatible with $d\lambda$. With the sign conventions that we use this means that $d\lambda(J\cdot, \cdot)$ defines a family of Riemannian metrics on M. In the following, we always assume that J is independent of t outside a compact set and

$$\begin{gathered} J \text{ is SFT-like with respect to } \alpha_0 \text{ on } (0, e^{-4\nu}] \times \Sigma \text{ and} \\ \text{SFT-like with respect to } \alpha_1 \text{ on } [e^{2\nu}, +\infty) \times \Sigma. \end{gathered} \tag{93}$$

Note that the set of almost complex structures of the form (93) with respect to some α is connected.

For $(u, \eta) \in \mathcal{L}M \times \mathbb{R}$, let $\langle\!\langle \cdot, \cdot \rangle\!\rangle_J$ on $T_{(u,\eta)}(\mathcal{L}M \times \mathbb{R})$ denote the L^2-inner product defined by

$$\langle\!\langle (\hat{u}, \hat{\eta}), (\hat{v}, \hat{\tau}) \rangle\!\rangle_J := \int_{S^1} d\lambda\,(J_t \hat{u}, \hat{v})\,dt + \hat{\eta}\hat{\tau},$$

$(\hat{u}, \hat{\eta}), (\hat{v}, \hat{\tau}) \in T_{(u,\eta)}(\mathcal{L}M \times \mathbb{R})$.

The gradient $\nabla_J \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u, \eta)$ with respect to the above inner product is given by

$$\begin{aligned} &\nabla_J \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u, \eta) = \\ &\quad \left(J_t(u) \left(\partial_t u - \eta\kappa X_H(u) - \frac{\partial \beta_\nu(r)}{\partial r} X_L(u) \right), -\int_{S^1} \kappa H(u)dt \right). \end{aligned}$$

We look at negative gradient flow lines of $\nabla_J \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u,\eta)$, i.e. maps $(u,\eta) \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty(\mathbb{R},\mathbb{R})$ satisfying

$$\partial_s(u,\eta) + \nabla_J \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u,\eta) = 0.$$

Thus the Floer equations of $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$ are given by

$$\partial_s u + J_t(u)\left(\partial_t u - \eta\kappa X_H(u) - \frac{\partial \beta_\nu(r)}{\partial r} X_L(u)\right) = 0$$

$$\partial_s \eta - \int_{S^1} \kappa H(u) dt = 0.$$

The energy of a solution is

$$E(u,\eta) = \int_{\mathbb{R}} \int_{S^1} |\partial_s(u,\eta)|_J^2 \, dt \, ds.$$

Let $a_-, a_+ \in \mathbb{R}$. The moduli space $\mathcal{M}^{a_+}_{a_-}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, J\right)$ is the set of all solutions $(u(s),\eta(s))$ of the Floer equations with

$$a_- \geq \lim_{s\to-\infty} \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u(s),\eta(s)) \quad \text{and} \quad \lim_{s\to+\infty} \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u(s),\eta(s)) \geq a_+.$$

Note that in this case the energy is precisely given by the difference of the action values. Thus, it actually holds that $a_- \geq a_+$ since solutions of the Floer equations with nonnegative energy must be decreasing.

13.2. Compactness of the moduli spaces.

THEOREM 13.6. *Let J be a family of almost complex structures compatible with $d\lambda$ that are both independent of t and SFT-like outside of $[e^{-4\nu}, e^{2\nu}] \times \Sigma$. Then the moduli spaces $\mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, J\right)$ are compact in the C^∞_{loc}-topology.*

The crucial property to achieve compactness is the fundamental lemma.

LEMMA 13.7. [CF09, Proposition 3.2] *There exist constants $C_0, C_1 > 0$ such that for any $(u,\eta) \in \mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, J\right)$ we have that*

$$\left\|\nabla_J \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u,\eta)\right\|_J \leq C_0 \quad \Rightarrow \quad |\eta| \leq C_1\left(1 + |\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u,\eta)|\right). \tag{94}$$

The proof given in [CF09] still goes through. The proof uses the behaviour of flow lines in a neighborhood of the hypersurface. In our setting, a neighborhood of the hypersurface still looks the same apart from rescaling of the contact form.

We will show the following proposition from which Theorem 13.6 follows immediately.

PROPOSITION 13.8. *In the setting of Theorem 13.6 there exist $k, l > 0$ such that*

$$Im(u) \subset [k,l] \times \Sigma \quad \text{for any } (u,\eta) \in \mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, J\right).$$

We need the notion of a trivial cylinder.

DEFINITION 13.9. A mapping $u : \mathbb{R} \times S^1 \to \mathbb{R}^+ \times \Sigma$ of the form $(ce^{\pm Ps}, \gamma(\pm tP))$ for some $c \in \mathbb{R}^+$ and $j\partial_t = \partial_s$ for the complex structure j on $\mathbb{R} \times S^1$ is called a *trivial cylinder* over a P-periodic Reeb orbit γ. Note that such a cylinder is a J-holomorphic map for any SFT-like J.

Moreover, we need the definition of the Hofer energy.

DEFINITION 13.10. Let (Z, j) be a compact Riemann surface (possibly disconnected and with boundary). Let $u = (r, x) : Z \to M$ be a (j, J)-holomorphic map. The *Hofer energy* of a flow line u is given by

$$\begin{aligned} E_H(u) &= \sup_{m \in \mathcal{S}} \int_Z u^* d(m\alpha) \\ &= \sup_{m \in \mathcal{S}} \left(\int_Z u^*(m d\alpha) + \int_Z u^*(m'(s)\, ds \wedge \alpha) \right) \in [0, +\infty], \end{aligned}$$

where $\mathcal{S} := \{m \in C^\infty(\mathbb{R}, [0,1]) \mid m' \geq 0\}$.

To prove Proposition 13.8 we use the following theorem from [AFM13] which is the special case of the SFT-compactness result that we need.

THEOREM 13.11. [AFM13, Theorem 5.3] *Let (M, λ) be as before. Suppose (Z_k, j_k) is a family of compact (possibly disconnected) Riemann surfaces with boundary and uniformly bounded genus. Assume that*

$$u_k = (a_k, y_k) : Z_k \to \mathbb{R}^+ \times \Sigma = M$$

is a sequence of (j_k, J)-holomorphic maps with $E_H(u_k) < K$ for some $K > 0$ and which are nonconstant on each connected component of Z_k and satisfy $a_k(\partial_k Z_k) \subset [D, +\infty)$, where $D < e^{-4\nu}$. Also, assume that $\inf_k \inf_{Z_k} a_k = 0$. Then there exists a subsequence k_n and cylinders $C_n \subset Z_{k_n}$ biholomorphically equivalent to standard cylinders $[-L_n, +L_n] \times S^1$ such that $L_n \to +\infty$ and such that $u_{k_n}|_{C_n}$ converges (up to an $\mathbb{R}$-shift) in $C^\infty_{loc}(\mathbb{R} \times S^1, \mathbb{R}^+ \times \Sigma)$ to a trivial cylinder over a Reeb orbit of α_0 with period $\leq K$.

PROOF OF PROPOSITION 13.8. First, we note that on $(e^{2\nu}, +\infty) \times \Sigma$ the Hamiltonian vector fields of H and L vanish and thus solutions of the Floer equations are in fact J-holomorphic curves. Hence, we can apply the maximum principle to keep Floer trajectories from escaping to $+\infty$ and consequently there is $l > 0$ such that $Im(u) \subset (0, l] \times \Sigma$ for all $(u, \eta) \in \mathcal{M}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0, \alpha_1)}, J\right)$.

Moreover, we observe that for any $(u,\eta) \in \mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, J\right)$ the restriction $u|_{u^{-1}((0,e^{-4\nu}]\times\Sigma)}$ is a J-holomorphic map.

Claim: For $(u,\eta) \in \mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}; J\right)$ the Hofer energy $E_H(u_k)$ is uniformly bounded by $e^{4\nu}(a_- - a_+)$.

The proof of this claim can be found in [**AFM13**, Proof of Theorem 3.9]. For the convenience of the reader, we include it here. Indeed, we can estimate

$$\begin{aligned} a_- - a_+ \geq E(u,\eta) &= \int_{-\infty}^{+\infty} \left\|\nabla \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u,\eta)\right\|_J^2 ds \\ &= \int_{-\infty}^{+\infty} \|\partial_s(u,\eta)\|_J^2 \, ds \\ &\geq \int_{-\infty}^{+\infty}\int_{S^1} d\lambda(J\partial_s u, \partial_s u)\, dt\, ds \\ &\geq \int_{u^{-1}((0,e^{-4\nu})\times\Sigma)} d\lambda(J\partial_s u, \partial_s u)\, dt\, ds \\ &= \int_{u^{-1}((0,e^{-4\nu})\times\Sigma)} u^* d\lambda, \end{aligned}$$

where on this domain $d\lambda = d(r\epsilon\alpha_0)$. Here we used that u restricted to $u^{-1}((0,e^{-4\nu})\times\Sigma)$ is J-holomorphic.

On the other hand we can estimate for $m \in S$

$$\begin{aligned} \int_{u^{-1}((0,e^{-4\nu})\times\Sigma)} u^* d(m\epsilon\alpha_0) &\leq \int_{u^{-1}((0,e^{-4\nu})\times\Sigma)} u^* d(\epsilon\alpha_0) \\ &\overset{\text{Stokes}}{=} e^{4\nu}\int_{u^{-1}((0,e^{-4\nu})\times\Sigma)} u^* d\lambda, \end{aligned}$$

which concludes the proof of the claim.

Assume by contradiction that there exists no $k > 0$ such that $Im(u) \subset [k,l]\times\Sigma$ and thus there is a sequence $u_k = (a_k, y_k)$ such that $\lim_k \inf_{Z_k} a_k = 0$. Choose $T < e^{-4\nu}$ such that T is a regular value for all a_k's. Let $Z_k := (u_k)^{-1}((0,T]\times\Sigma)$ and consider the J-holomorphic curves $v_k := u_k|_{Z_k}$. Since for each k the v_k is a gradient flow line of $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$ and its asymptotes are critical points (u, η) of $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$, where u is contained in $(e^{-\nu}, e^{\nu}) \times \Sigma$, the Z_k's are compact possibly disconnected Riemann surfaces of genus 0. Since we choose $T < e^{-4\nu}$ and so that T is a regular value, no Floer cylinder is constant and thus v_k has no constant components. By choice of T, v_k also satisfies $a_k(\partial Z_k) \subset [T, +\infty)$. Thus v_k satisfies all the assumptions of Theorem 13.11 and hence there exists a subsequence $v_{k_n} \to v_{k_0}$ whose restriction to cylinders converges to a trivial cylinder over a Reeb orbit of α_0 since we have $a_k \to 0$.

Thus there is an embedded circle S in the domain $\mathbb{R} \times S^1$ of v_{k_0} such that the restriction of y_{k_0} to S is homotopic to a Reeb orbit γ of α_0. The domain of y_{k_0} is $\mathbb{R} \times S^1$ and thus either S is a circle bounding a disk or S is a circle of the form $s \times S^1$. In the first case it is clear that S and thus y_{k_0} restricted to S is contractible. In the latter case it follows that the image of S under y_{k_0} is contractible since it is homotopic to the asymptotic end of the cylinder. These asymptotic ends are contractible since they lie in $\mathcal{L}M$.

Finally, we have shown that y_k converges to a contractible Reeb orbit of α_0, which is a contradiction since α_0 is without contractible Reeb orbits.

$\square$

13.3. Definition of Rabinowitz Floer homology. Here we give a sketch of the definition of Rabinowitz Floer homology in our setting. For details see for example [**CF09**] or [**AF10a**]. By an analogous argument as in [**AF10a**, Appendix A] for all $\alpha_1 \in \mathcal{C}(\xi)$ the functional $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$ is Morse for a generic choice of φ. Note that since every critical point has image contained in $(e^{-\nu}, e^{\nu}) \times \Sigma \times \mathbb{R}$, the property of being Morse does in fact not depend on $\lambda \in \Omega_\nu(\alpha_0, \alpha_1)$. One can extend the definition of the Rabinowitz Floer homology groups to all functionals $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$ as follows: Any path of contactomorphisms can be written as the limit of nondegenerate paths since those form a residual set. We use this sequence to define the Rabinowitz Floer homology of a degenerate path as the limit of the homology of the functionals associated to a sequence of nondegenerate paths. Invariance of Rabinowitz Floer homology shows that the homology does not depend on the chosen sequence.

Note that the choice $\hat{\varphi} = id$ is not generic. Whilst the functional $\mathcal{A}^{id}_{(\alpha_0,\alpha_1)}$ is not Morse it satisfies the Morse-Bott conditions. Thus we will explain here how to construct the Rabinowitz Floer complex under the assumption that $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}$ is Morse-Bott.

Assume that the functional is Morse-Bott. Choose a Morse function f and a Riemannian metric g on the critical submanifold $\mathrm{Crit}\,\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$. Denote by $\mathrm{Crit}\,(f)$ the set of critical points of f. Let $\mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$ be the $\mathbb{Z}_2$-vector space consisting of formal sums $\sum_{w\in \mathrm{Crit}\,(f)} n_w\, w$, where the coefficients n_w in $\mathbb{Z}_2$ satisfy

$$\#\left\{w \in \mathrm{Crit}\,(f)|\ n_w \neq 0,\, \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(w) \leq \kappa\right\} < \infty,$$

for every $\kappa \in \mathbb{R}$.

Assume that (f, g) is a Morse-Smale pair, where f is a smooth function on $\mathrm{Crit}\,\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$ and g is a Riemannian metric on $\mathrm{Crit}\,\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$. Then, for any $z, w \in \mathrm{Crit}\,(f)$ we denote by $\mathcal{M}\left(z, w; \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f, J, g\right)$ the moduli space of gradient trajectories with cascades from z to w with respect to the Riemannian metric g on $\mathrm{Crit}\,\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$, and we denote by $\mathcal{M}^0\left(z, w; \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f, J, g\right)$ its zero-dimensional part. The compactness result, Theorem 13.6, shows that if $|z|-|w| = 1$, then $\mathcal{M}\left(z, w; \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f, J, g\right)$ is compact and thus its zero-dimensional part is a finite set.

We define the boundary operator

$$\partial :\ \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f; J, g\right) \to \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f; J, g\right)$$

as the linear extension of

$$\partial(z) = \sum_{w\in \mathrm{Crit}\,(f)} n(z, w)\, w,$$

where $z \in \mathrm{Crit}\,(f)$ and $n(z,w) = \#_{\mathbb{Z}_2}\mathcal{M}^0\left(z, w; \mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f, J, g\right)$. Standard arguments in Floer theory also yield that $\partial^2 = 0$. Finally, we can define the Floer homology groups of the above constructed chain complex

$$\mathrm{HF}\left(\mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f; J, g\right), \partial\right) := \mathrm{H}\left(\mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f; J, g\right), \partial\right).$$

It is also standard to show that these do not depend on the choice of f, g and J and thus we define the Rabinowitz Floer homology of (M, λ) as

$$\mathrm{RFH}\left((M,\lambda);\hat{\varphi}\right) := \mathrm{HF}\left(\mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}, f; J, g\right), \partial\right).$$

See [**CF09**] for more details on the definition of Rabinowitz Floer homology.

14. Continuation maps

In the next step we show that the above defined groups are independent of α_1. We construct an isomorphism

$$\widetilde{\Phi} : \mathrm{RFH}\left((M,\lambda_1);\hat{\varphi}\right) \to \mathrm{RFH}\left((M,\lambda_2);\hat{\varphi}\right)$$

for any $\alpha_1, \alpha_2 \in \mathcal{C}(\xi)$, and $\lambda_1 \in \Omega_\nu(\alpha_0,\alpha_1), \lambda_2 \in \Omega_\nu(\alpha_0,\alpha_2)$, where $\nu > \max\left\{C(\hat{\varphi};\alpha_1), C(\hat{\varphi};\alpha_2)\right\}$.

In order to do so, we define a homotopy λ_s between λ_1 and λ_2. Choose a function $\zeta \in C^\infty\left(\mathbb{R},[0,1]\right)$ with $0 \leq \dot{\zeta}(s) \leq 2$ and such that

$$\zeta(s) = \begin{cases} 1, & \text{for } s \geq 1 \\ 0, & \text{for } s \leq 0. \end{cases}$$

Define

$$\lambda_s := \lambda_1 + \zeta(s)\left(\lambda_2 - \lambda_1\right). \tag{95}$$

Note that since $\partial_r f_i > 0$, where $\lambda_i = f_i\alpha_0$ for $i = 1, 2$, we still have that $d\lambda_s$ is a nondegenerate symplectic form. Note that this symplectic form $d\lambda_s$ is independent of s for $s \notin [0,1]$ and that $\alpha_s := \lambda_s|_{\Sigma\times\{1\}}$ is a supporting contact form for ξ.

Let J_1 and J_2 denote the $d\lambda_1$-, resp. $d\lambda_2$-, compatible families of SFT-like almost complex structures defined as in (93). Moreover, let J_s be such that

$$J_s = \begin{cases} J_1, & \text{for } s \geq 1 \\ J_2, & \text{for } s \leq 0, \end{cases}$$

where J_s is a $d\lambda_s$-compatible almost complex structure which is independent of t outside a compact set and with J_1, J_2 SFT-like as defined in (93). That is J_1 is SFT-like with respect to α_0 on $(0, e^{-4\nu}] \times \Sigma$ and SFT-like with respect to α_1 on $[e^{2\nu}, +\infty) \times \Sigma$ and J_2 analogously for α_0 and α_2.

Look at the s-dependent functional

$$\mathcal{A}_{\lambda_s}(u, \eta) := \int_{S^1} u^* \lambda_s - \eta \int_{S^1} \kappa(t) H(r) dt - \int_{S^1} \beta_\nu(r) L_t(u)\, dt,$$

where $u = (r, x) : S^1 \to M$. The corresponding Floer equations are

$$\partial_s u + J_{s,t} \left(\partial_t u - \eta \kappa X_H^{\lambda_s}(u) - \beta_\nu(r) X_L^{\lambda_s} \right) = 0$$
$$\partial_s \eta + \int_{S^1} \kappa H(u) dt = 0,$$

where $u = u(s,t) : \mathbb{R} \times S^1 \to M$ and $\eta = \eta(s) : \mathbb{R} \to \mathbb{R}$. Note that now, the Hamiltonian vector fields of H and L with respect to $\lambda_s|_{\{1\}\times\Sigma}$ depend on s since $\lambda_s|_{\{1\}\times\Sigma} = \alpha_s$ is an s-dependent contact form.

From now on, we will denote $u(s) := u(s, \cdot) : \mathbb{R} \times S^1 \to M$. Let $a_-, a_+ \in \mathbb{R}$. Denote as before by $\mathcal{M}_{a_-}^{a_+}(\mathcal{A}_{\lambda_s}, J_s)$ the solutions of the corresponding Floer equations of $\mathcal{A}_{\lambda_s}(u(s), \eta(s))$ whose asymptotes have action bounded by $a_\pm$, i.e.

$$a_- \geq \lim_{s \to -\infty} \mathcal{A}_{\lambda_s}(u(s), \eta(s)) \quad \text{and} \quad \lim_{s \to +\infty} \mathcal{A}_{\lambda_s}(u(s), \eta(s)) \geq a_+.$$

14.1. Compactness for the s-dependent moduli spaces.

THEOREM 14.1. *Assume λ_s is a homotopy of 1-forms as defined in* (95). *There exists $\epsilon > 0$ such that if $\sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s \lambda_s\|_\infty < \epsilon$, then the moduli space $\mathcal{M}_{a_-}^{a_+}(\mathcal{A}_{\lambda_s}, J_s)$ is relatively compact in the C^∞_{loc}-topology.*

Recall that $\|\partial_s\lambda_s\|_\infty$ is only nonzero for $s \in [0,1]$ since outside of $[0,1]$ λ_s is independent of s.

The main task to achieve compactness is to bound the energy. We start with the observation that Lemma 13.7 remains true in the s-dependent case.

LEMMA 14.2. *There exist constants $C_0, C_1 > 0$ such that for any element $(u,\eta) \in \mathcal{M}_{a_-}^{a_+}(\mathcal{A}_{\lambda_s}, J_s)$ we have that*

$$\|\nabla_{J_s}\mathcal{A}_{\lambda_s}(u,\eta)\|_{J_s} \le C_0 \quad \Rightarrow \quad |\eta| \le C_1\left(1 + |\mathcal{A}_{\lambda_s}(u,\eta)|\right). \tag{96}$$

PROOF. We can apply Lemma 13.7 for any of the functionals $\mathcal{A}_{\lambda_s}$. For a fixed functional $\mathcal{A}_{\lambda_s}$ the constants $C_{0,s}$ and $C_{1,s}$ in (94) depend continuously on s. Take $C_0 := \min\{C_{0,s} \,|\, s \in [0,1]\}$ and $C_1 := \max\{C_{1,s} \,|\, s \in [0,1]\}$. □

The following lemma is the main content of this paper and uses ideas of Bae-Frauenfelder, see [**BF11**, Lemma 2.9; Theorem 2.10]. It establishes uniform bounds on $\|\eta(s)\|_\infty$ and the energy of Floer trajectories in the s-dependent case provided that the homotopy is sufficiently slow.

LEMMA 14.3. *Let C_0 and C_1 be chosen as required in Lemma 14.2 and fix $a_-, a_+ \in \mathbb{R}$. Then, there is a constant $\rho = \rho_{\{\lambda_s\}} > 0$ such that if $\sup_{s\in[0,1]}\sup_{(0,e^{2\nu}]\times\Sigma}\|\partial_s\lambda_s\|_{J_s} < \rho$ the following holds: There exist constants $C_\eta, C_E > 0$ such that for any $(u(s),\eta(s)) \in \mathcal{M}_{a_-}^{a_+}(\mathcal{A}_{\lambda_s}, J_s)$*

$$\|\eta(s)\|_\infty < C_\eta \quad \textit{and} \quad E(u(s),\eta(s)) < C_E.$$

Using the above estimates we can prove Theorem 14.1.

PROOF OF THEOREM 14.1. The uniform energy bound of Lemma 14.3 and the fact that X_H and X_L have support in $(e^{-2\nu}, e^{2\nu}) \times \Sigma$ as well as the fact that λ_s restricted to $(0, e^{-4\nu}] \times \Sigma$ is just some multiple of $r\alpha_0$, respectively $r\alpha_1$ and $r\alpha_2$ on $(e^{+3\nu}, +\infty) \times \Sigma$, shows that outside of $(e^{-4\nu}, e^{2\nu}] \times \Sigma$ solutions of the Floer equations are exactly J-holomorphic curves. This allows us to apply the maximum principle to keep Floer trajectories from escaping to $+\infty$. As in the proof of Theorem 13.6 we use Theorem 13.11 to argue that neither do Floer trajectories escape to the negative end.

This shows that $u(s)$ stays in a compact subset of $(0,+\infty)\times\Sigma$, say $[k,l]\times\Sigma$. With the uniform L^∞-bound for $\eta(s)$ from Lemma 14.3 the usual bubbling-off arguments apply, thus yielding L^∞-bounds for the derivatives of $u(s)$. Thus we can use Arzelà-Ascoli for families of Rabinowitz Floer trajectories whose asymptotes have action bounded by $a_\pm$. We finally get a $C^\infty_{\text{loc}}\left(\mathbb{R}\times S^1, M\right)\times C^\infty_{\text{loc}}\left(\mathbb{R},\mathbb{R}\right)$ convergence of a subsequence of trajectories. The result then follows by the usual arguments. $\square$

It remains to prove Lemma 14.3.

PROOF OF LEMMA 14.3. In the following we will denote

$$\sup_{s\in[0,1]}\ \sup_{(0,e^{2\nu}]\times\Sigma}\left\|\partial_s\lambda_s\right\|_{J_s}=:\widetilde{C}.$$

We want to estimate the energy

$$\begin{aligned}
E(u(s),\eta(s)) &= \int_{\mathbb{R}}\left\|\partial_s(u,\eta)\right\|^2_{J_s}\,ds=\int_{-\infty}^{+\infty}\left\|-\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right\|^2_{J_s}\,ds\\
&=\int_{-\infty}^{+\infty}\langle\!\langle-\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s)),-\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s))\rangle\!\rangle_{J_s}\,ds\\
&=\int_{-\infty}^{+\infty}-d\mathcal{A}_{\lambda_s}(u(s),\eta(s))\left(-\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right)\,ds\\
&=\int_{-\infty}^{+\infty}-d\mathcal{A}_{\lambda_s}(u(s),\eta(s))(\partial_s(u(s),\eta(s)))\,ds\\
&=\int_{-\infty}^{+\infty}\left[-\frac{d}{d_s}\left(\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right)+\left(\frac{\partial}{\partial_s}\mathcal{A}_{\lambda_s}\right)(u(s),\eta(s))\right]\,ds\\
&=\mathcal{A}_{\lambda_s}\left(u_-,\eta_-\right)-\mathcal{A}_{\lambda_s}\left(u_+,\eta_+\right)+\int_0^1\int_{S^1}(u(s))^*(\partial_s\lambda_s)\,ds\\
&\le a_- - a_+ +\int_0^1\int_{S^1}(u(s))^*(\partial_s\lambda_s)\,ds.
\end{aligned}$$

Recall that by the maximum principle $u(s) \subset (0, e^{2\nu}] \times \Sigma$. Using the Floer equation for $\partial_s u(s)$ we have

$$
\begin{aligned}
&\left| \int_0^1 \int_{S^1} (u(s))^*(\partial_s \lambda_s)\, ds \right| \\
&\leq \int_0^1 \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s \lambda_s\|_{J_s} \cdot \int_{S^1} |\partial_t u(s)|\; dt\, ds \\
&= \widetilde{C} \int_0^1 \|\partial_t u(s)\|_{J_s}\; ds \\
&= \widetilde{C} \int_0^1 \left\| J_s \partial_s u(s) + \eta(s)\kappa X_H^{\lambda_s} + \beta_\nu X_L^{\lambda_s} \right\|_{J_s}\; ds \\
&\leq \widetilde{C} \int_0^1 \|\partial_s u(s)\|_{J_s}\; ds + \widetilde{C} \int_0^1 \left\| \eta(s)\kappa X_H^{\lambda_s} + \beta_\nu X_L^{\lambda_s} \right\|_{J_s}\; ds \\
&\overset{(*)}{\leq} \widetilde{C} \left[\left(\int_0^1 \|\partial_s u(s)\|_{J_s}\; ds \right)^2 + 1 \right] + \widetilde{C} \int_0^1 \left\| \eta(s)\kappa X_H^{\lambda_s} + \beta_\nu X_L^{\lambda_s} \right\|_{J_s}\; ds \\
&\leq \widetilde{C} \int_0^1 \|\partial_s u(s)\|_{J_s}^2\; ds + \widetilde{C} + \widetilde{C} \int_0^1 \left\| \eta(s)\kappa X_H^{\lambda_s} + \beta_\nu X_L^{\lambda_s} \right\|_{J_s}\; ds \\
&= \widetilde{C} \left(E(u(s), \eta(s)) + 1 + \|\eta\|_\infty \left\| \kappa X_H^{\lambda_s} \right\|_\infty + \left\| \beta_\nu X_L^{\lambda_s} \right\|_\infty \right) \\
&\leq \widetilde{C} \left(E(u(s), \eta(s)) + 1 + \|\eta\|_\infty \left\| X_H^{\lambda_s} \right\|_\infty + \left\| X_L^{\lambda_s} \right\|_\infty \right),
\end{aligned}
$$

where we wrote $\|X_H^{\lambda_s}\|_\infty := \sup_{s\in[0,1]} \sup_M \|X_{H_s}\|_{J_s}$ (and analogously for $\|X_L^{\lambda_s}\|_\infty$) and where $(*)$ used the fact that for $x \geq 0$ it holds that $x \leq x^2 + 1$.

Putting everything together yields

$$
\begin{aligned}
&E(u(s), \eta(s)) \\
&\qquad \leq a_- - a_+ + \widetilde{C} \left(E(u(s), \eta(s)) + 1 + \|\eta\|_\infty \left\| X_H^{\lambda_s} \right\|_\infty + \left\| X_L^{\lambda_s} \right\|_\infty \right)
\end{aligned}
$$

and thus

$$
\left(1 - \widetilde{C}\right) E(u(s), \eta(s)) \leq a_- - a_+ + \widetilde{C} \left(1 + \|\eta\|_\infty \left\| X_H^{\lambda_s} \right\|_\infty + \left\| X_L^{\lambda_s} \right\|_\infty \right).
$$

So for $\widetilde{C} < 1$ we have that

$$E(u(s),\eta(s)) \leq \frac{a_- - a_+}{1-\widetilde{C}} + \frac{\widetilde{C}}{1-\widetilde{C}}\left(1 + \|\eta\|_\infty \left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right)$$

and thus, if $\widetilde{C} < \frac{1}{2}$, also

$$E(u(s),\eta(s)) < 2\left|a_- - a_+\right| + 2\widetilde{C}\left(1 + \|\eta\|_\infty \left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right). \quad (97)$$

Note also that

$$\begin{aligned} \left|\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right| &\leq \max\left\{|a_+|,|a_-|\right\} + \int_{-\infty}^{+\infty}\left|\frac{\partial}{\partial s}\Big(\mathcal{A}_{\lambda_s}(u(s),\eta(s))\Big)\right| ds \\ &= \max\left\{|a_+|,|a_-|\right\} + \int_0^1\left|\int_{S^1}(u(s))^*(\partial_s\lambda_s)\right| ds \qquad (98) \\ &\leq \max\left\{|a_+|,|a_-|\right\} \\ &\quad + \widetilde{C}\left(E(u(s),\eta(s)) + 1 + \|\eta\|_\infty \left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right) \\ &\qquad (99) \end{aligned}$$

Indeed, to show (98), we note that

$$\mathcal{A}_{\lambda_s}(u(s),\eta(s)) = \underbrace{\mathcal{A}_{\lambda_-}(u_-,\eta_-)}_{\leq a_-} + \int_{-\infty}^{s}\frac{\partial}{\partial s}\left(\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right) ds \quad \text{and}$$

$$\mathcal{A}_{\lambda_s}(u(s),\eta(s)) = \underbrace{\mathcal{A}_{\lambda_+}(u_+,\eta_+)}_{\geq a_+} - \int_{s}^{+\infty}\frac{\partial}{\partial s}\left(\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right) ds,$$

where

$$\frac{\partial}{\partial s}\left(\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right) = \left(\frac{\partial}{\partial s}\mathcal{A}_{\lambda_s}\right)(u(s),\eta(s)) + \underbrace{d\mathcal{A}_{\lambda_s}(\partial_s u(s),\partial_s\eta(s))}_{=-\left\|\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s))\right\|^2_{J_s}}.$$

But in fact the homotopy λ_s only changes on $[0,1]$ and so we have that

$$\begin{aligned}\int_{-\infty}^{s}\left(\frac{\partial}{\partial s}\mathcal{A}_{\lambda_s}\right)(u(s),\eta(s))\,ds &= \int_{0}^{s}\left(\frac{\partial}{\partial s}\mathcal{A}_{s}\right)(u(s),\eta(s))\,ds\\ &\le \int_0^1\left|\left(\frac{\partial}{\partial_s}\mathcal{A}_s\right)(u(s),\eta(s))\right|\,ds\\ &= \int_0^1\left|\int_{S^1}(u(s))^*(\partial_s\lambda_s)\right|\,ds\end{aligned}$$

and analogously for $\int_s^{+\infty}$. Finally, we have

$$\begin{aligned}\mathcal{A}_{\lambda_s}(u(s),\eta(s)) &\le a_- + \int_{-\infty}^{s}\left(\frac{\partial}{\partial s}\mathcal{A}_{\lambda_s}\right)(u(s),\eta(s)) - \|\nabla\mathcal{A}_{\lambda_s}(u(s),\eta(s))\|^2\\ &\le a_- + \int_0^1\left|\int_{S^1}(u(s))^*(\partial_s\lambda_s)\right|\,ds\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_{\lambda_s}(u(s),\eta(s)) &\ge a_+ - \int_{s}^{+\infty}\left(\frac{\partial}{\partial s}\mathcal{A}_{\lambda_s}\right)(u(s),\eta(s)) + \|\nabla\mathcal{A}_{\lambda_s}(u(s),\eta(s))\|^2\\ &\ge a_+ - \int_0^1\left|\int_{S^1}(u(s))^*(\partial_s\lambda_s)\,dt\right|\,ds.\end{aligned}$$

This gives (98)

$$|\mathcal{A}_{\lambda_s}(u(s),\eta(s))| \le \max\{|a_-|,|a_+|\} + \int_0^1\left|\int_{S^1}(u(s))^*(\partial_s\lambda_s)\right|\,ds.$$

Now define for $s\in\mathbb{R}$

$$\tau(s) := \inf\left\{\tau\ge 0:\quad \left\|\nabla_{J_s}\mathcal{A}_{\lambda_s}\Big(u(s+\tau),\eta(s+\tau)\Big)\right\|_{J_s} < C_0\right\},$$

where C_0 is the constant from Lemma 14.2. Then $\tau(s)$ satisfies

$$\tau(s) \le \frac{E(u(s),\eta(s))}{C_0^2}. \tag{100}$$

Indeed,

$$E(u(s),\eta(s)) \ge \int_s^{s+\tau(s)} \|\nabla_{J_s}\mathcal{A}_{\lambda_s}(u(s),\eta(s))\|_{J_s}^2\,ds \ge \tau(s)C_0^2.$$

Hence, using the estimate (96) from Lemma 14.2 and Hölder's inequality we have

$$
\begin{aligned}
|\eta(s)| &= \left|\eta(s+\tau(s)) - \int_s^{s+\tau(s)} \partial_s\eta(s)\,ds\right| \\
&\leq |\eta(s+\tau(s))| + \int_s^{s+\tau(s)} |\partial_s\eta(s)\,ds|\;ds \\
&\leq C_1\left(|\mathcal{A}_{\lambda_s}(u(s),\eta(s))|+1\right) + |\tau(s)|^{\frac{1}{2}}\left(\int_s^{s+\tau(s)} |\partial_s\eta(s)|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

We can estimate

$$
\begin{aligned}
\left(\int_s^{s+\tau(s)} |\partial_s\eta(s)|^2\right)^{\frac{1}{2}} &\leq \left(\int_s^{s+\tau(s)} \|\partial_s(u(s),\eta(s))\|_{J_s}^2\;ds\right)^{\frac{1}{2}} \\
&\leq \left(E(u(s),\eta(s))\right)^{\frac{1}{2}}
\end{aligned}
$$

and use equations (99) and (100) to deduce that

$$
\begin{aligned}
|\eta(s)| \overset{(99),(100)}{\leq} C_1\Big[&\max\{|a_+|,|a_-|\} \\
&+ \widetilde{C}\left(E(u(s),\eta(s)) + 1 + \|\eta\|_\infty \left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right)\Big] \\
&+ \frac{E(u(s),\eta(s))}{C_0}.
\end{aligned}
$$

Inserting the above in (97) yields

$$
\begin{aligned}
\|\eta\|_\infty &\leq C_1\left(1+\max\{|a_+|,|a_-|\}\right) \\
&\quad + \left(2\,|a_- - a_+| + 2\widetilde{C}\left(1+\|\eta\|_\infty\left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right)\right)\left(C_1\widetilde{C} + \frac{1}{C_0}\right) \\
&\qquad + C_1\widetilde{C}\left(1+\|\eta\|_\infty\left\|X_H^{\lambda_s}\right\|_\infty + \left\|X_L^{\lambda_s}\right\|_\infty\right).
\end{aligned}
$$

Hence we have that

$$\begin{aligned}\left(1-\widetilde{C}\left\|X_H^{\lambda_s}\right\|_\infty\left(2C_1\widetilde{C}+\frac{2}{C_0}+C_1\right)\right)&\|\eta\|_\infty\\ &\leq C_1\left(1+\max\left\{|a_+|,|a_-|\right\}\right)\\ &\quad+\left(2\left|a_--a_+\right|+2\widetilde{C}\left(1+\left\|X_L^{\lambda_s}\right\|_\infty\right)\right)\left(C_1\widetilde{C}+\frac{1}{C_0}\right)+C_1\widetilde{C}\\ &=: B.\end{aligned}$$

To finally get a bound for η we need to have

$$\underbrace{\widetilde{C}\left\|X_H^{\lambda_s}\right\|_\infty\left(2C_1\widetilde{C}+\frac{2}{C_0}+C_1\right)}_{=:A}<1 \quad\text{and}\quad \widetilde{C}<\frac{1}{2}. \tag{101}$$

Let $\rho>0$ denote a value so that (101) is satisfied for $\widetilde{C}=\rho$.

Now, for $\widetilde{C}<\rho$ we get that

$$\|\eta\|_\infty \leq \frac{1}{1-A}B =: C_\eta$$

and thus the first assertion of the lemma. Combining this with (97) gives the energy bound and thus completes the proof

$$E(u(s),\eta(s))<2\left|a_--a_+\right|+2\widetilde{C}\left(1+2C_\eta\left\|X_H^{\lambda_s}\right\|_\infty+\left\|X_L^{\lambda_s}\right\|_\infty\right)=: C_E.$$

□

14.2. Invariance. With the help of Theorem 14.1 we are now able to define a quasi-isomorphism

$$\widetilde{\Phi}:\mathrm{CF}(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}) \to \mathrm{CF}(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_2)}).$$

First, we remark that $\lambda_s=f(s,x,r)\alpha_0$ and J_s is SFT-like for λ_s apart from a small neighborhood of the hypersurface $\{1\}\times\Sigma$. Outside of this neighborhood we have $\|\lambda_s\|_{J_s}=\sqrt{r}$. Recall that by the maximum principle Floer trajectories remain in $(0,e^{2\nu})\times\Sigma$, thus we can estimate

$$\begin{aligned}
\widetilde{C} &= \sup_{s\in[0,1]} \sup_{(0,+\infty)\times\Sigma} \|\partial_s \lambda_s(r,x)\|_{J_s} = \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s f\alpha_0\|_{J_s} \\
&= \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \left\|\frac{\partial_s f}{f} f\alpha_0\right\|_{J_s} = \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \left\|\frac{\partial_s f}{f}\lambda_s\right\|_{J_s} \\
&\leq \sqrt{e^{2\nu}} \left\|\frac{\partial_s f}{f}\right\|_\infty + \epsilon < K
\end{aligned}$$

for a constant $K \in \mathbb{R}$. Here, the ϵ-Error comes from the fact that J_s is not SFT-like on a small neighborhood.

Now, use an adiabatic argument to subdivide the homotopy into a family of homotopies $\{(\lambda_s^j, J_s^j)\}_j$, each of which satisfies the smallness assumption on $\sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s \lambda_s^j\|_{J_s^j}$. Note that since the norms of the vector fields $X_H^{\lambda_s}$ and $X_L^{\lambda_s}$ are bounded, this still holds true after a subdivision of the one form and the almost complex structure. Therefore, we can apply Theorem 14.1 to get the desired compactness properties for $\mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda_s^j}, J_s\right)$, and we can define maps

$$\widetilde{\Phi_j} : \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda^j}\right) \to \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda^{j+1}}\right)$$

by

$$\widetilde{\Phi}_j(z) = \sum_{w\in \mathrm{Crit}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda^{j+1}}\right)} \#\mathcal{M}^0\left(z, w; \mathcal{A}^{\hat{\varphi}}_{\lambda_s^j}, J_s\right) w$$

for $z \in \mathrm{Crit}\,(\mathcal{A}^{\hat{\varphi}}_{\lambda^j})$. Here $\mathcal{M}^0\left(z, w; \mathcal{A}^{\hat{\varphi}}_{\lambda_s^j}, J_s\right)$ denotes the zero-dimensional part of the space of Floer trajectories from z to w.

Furthermore, the $\widetilde{\Phi_j}$ descend to define continuation homomorphisms

$$\Phi_j : \mathrm{RFH}((M, \lambda^j); \hat{\varphi}) \to \mathrm{RFH}((M, \lambda^{j+1}); \hat{\varphi}).$$

With the inverse homotopy $\widetilde{\lambda}_s^j = \lambda^{j+1} + \beta(s)(\lambda^{j+1} - \lambda^j)$ we analogously get maps

$$\widetilde{\Psi}_j : \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda^{j+1}}\right) \to \mathrm{CF}\left(\mathcal{A}^{\hat{\varphi}}_{\lambda^j}\right).$$

A homotopy of homotopies argument shows that Φ_j is an isomorphism. This holds for every j with $0 \leq j \leq n$, and we conclude:

PROPOSITION 14.4. *Let* (Σ, ξ) *be a hypertight contact manifold and let* $\alpha_1, \alpha_2 \in \mathcal{C}(\xi)$. *Let* $\lambda_1 \in \Omega_\nu(\alpha_0, \alpha_1)$ *and* $\lambda_2 \in \Omega_\nu(\Sigma, \xi)$. *Then the homology groups* $\mathrm{RFH}((M, \lambda_i); \hat{\varphi})$ *are well-defined for* $i = 1, 2$, *and we have an isomorphism*

$$\mathrm{RFH}((M, \lambda_1); \hat{\varphi}) \cong \mathrm{RFH}((M, \lambda_2); \hat{\varphi}).$$

Choosing $\alpha_2 = \epsilon\alpha_0$ shows that $\mathrm{RFH}\left((M, \lambda_1); \hat{\varphi}\right) \cong \mathrm{RFH}\left((M, r\alpha_0); \hat{\varphi}\right)$, which is the Rabinowitz Floer homology of the actual symplectisation $((0, +\infty) \times \Sigma, r\epsilon\alpha_0)$ and thus we have proved Theorem 3.1, using that the Rabinowitz Floer homology is invariant under rescaling of a contact form. Thus it is justifiable to write $\mathrm{RFH}(\Sigma, \xi; \hat{\varphi})$ for a hypertight contact manifold (Σ, ξ).

15. Applications

15.1. A few properties of RFH**.** In the previous section, we have elaborated that for a hypertight contact manifold, the Rabinowitz Floer homology is independent of the supporting contact form. This allows us to deduce results for hypertight manifolds by using the knowledge of the Rabinowitz Floer homology groups for a convenient choice of supporting contact form, namely one that has no contractible Reeb orbits.

For that we first state some properties of Rabinowitz Floer homology that we need, for more details see [**AFM13**]. If we write $\mathrm{RFH}\left(\Sigma, \alpha_1; \hat{\varphi}\right)$, we mean the Rabinowitz Floer homology of the perturbed action functional on (M, λ), where $\lambda \in \Omega_\nu(\alpha_0, \alpha_1)$.

(1) For α_0 a supporting contact form without any contractible Reeb orbits and $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0\left(\Sigma, \xi\right)$ the Rabinowitz Floer homology is canonically isomorphic to the singular homology

$$\mathrm{RFH}_*\left(\Sigma, \alpha_0; \hat{\varphi}\right) \cong \mathrm{H}_{*+n-1}\left(\Sigma; \mathbb{Z}_2\right).$$

Rabinowitz Floer homology is independent of $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ in the following sense: There are canonical isomorphisms

$$\Phi_{\hat{\varphi}} : \mathrm{RFH}(\Sigma, \alpha) \to \mathrm{RFH}(\Sigma, \alpha; \hat{\varphi}),$$

where $\mathrm{RFH}(\Sigma, \alpha) = \mathrm{RFH}(\Sigma, \alpha; \mathrm{id})$.

For $\hat{\varphi},\ \hat{\psi} \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ there is a canonical map

$$\Phi_{\hat{\varphi},\hat{\psi}} : \mathrm{RFH}(\Sigma, \alpha; \hat{\varphi}) \to \mathrm{RFH}(\Sigma, \alpha; \hat{\psi})$$

such that $\Phi_{\hat{\psi}} = \Phi_{\hat{\varphi},\hat{\psi}} \circ \Phi_{\hat{\varphi}}$.

In particular, $\mathrm{RFH}_n(\Sigma, \alpha; \hat{\varphi})$ contains a class $[\Sigma_{\hat{\varphi}}] \neq 0$ which is defined by

$$\Phi_{\hat{\varphi},\hat{\psi}}([\Sigma_{\hat{\varphi}}]) = [\Sigma_{\hat{\psi}}] \quad \text{and}$$

$$[\Sigma_{\mathrm{id}}] = [\Sigma] \in \mathrm{RFH}_n(\Sigma, \alpha) \cong \mathrm{H}_{2n-1}(\Sigma; \mathbb{Z}_2).$$

(2) We denote by $\mathrm{RFH}^c(\Sigma, \alpha; \hat{\varphi})$ the Rabinowitz Floer homology generated by the subcomplex of $(u, \eta) \in \mathrm{Crit}\left(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}\right)$ with $\eta \leq c$. At a critical point (u, η) the action value is exactly $\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}(u, \eta) = \eta$, see Remark 13.3, and thus the critical points with $\eta \leq c$ form indeed a subcomplex. Then the inclusion of critical points induces a map

$$\iota^c_{\hat{\varphi}} : \mathrm{RFH}^c(\Sigma, \alpha; \hat{\varphi}) \to \mathrm{RFH}(\Sigma, \alpha; \hat{\varphi}).$$

In particular, for two paths $\hat{\varphi}, \hat{\psi}$ there is a constant $K(\hat{\varphi}, \hat{\psi})$ such that the map $\Phi_{\hat{\varphi},\hat{\psi}}$ from property (1) defines a map

$$\Phi_{\hat{\varphi},\hat{\psi}} : \mathrm{RFH}^c_*(\Sigma, \alpha_0, \hat{\varphi}) \to \mathrm{RFH}^{c+K(\hat{\varphi},\hat{\psi})}_*\left(\Sigma, \alpha_0, \hat{\psi}\right)$$

for any $c \in \mathbb{R}$.

REMARK 15.1.

1. For property (1) and (2) we use the fact that one can also define the s-dependent moduli spaces $\mathcal{M}_{a_-}^{a_+}\left(\mathcal{A}_\lambda^{\hat{\varphi}_s}, J_s\right)$ for a family of s-dependent contactomorphisms and show that they are compact. For a family of compactly supported contactomorphisms the estimates for the s-dependent functional are immediate.
2. We can estimate the constant K via the two contact Hamiltonians of $\hat{\varphi}, \hat{\psi}$. This estimate will be the content of Proposition 15.13.

15.2. Translated points for hypertight contact manifolds. Let φ be a contactomorphism which is contact-isotopic to the identity and let ρ be the smooth function $\rho : \Sigma \to (0, \infty)$ such that $\varphi^*\alpha = \rho\alpha$. Recall that a point $x \in \Sigma$ is a *translated point of* φ if there exists $\eta \in \mathbb{R}$ such that

$$\varphi(x) = \theta_\alpha^\eta(x), \quad \rho(x) = 1.$$

REMARK 15.2. We identify the hypersurface $\Sigma = \{1\} \times \Sigma$. We recall that a *leafwise intersection point* for ϕ_L^1 is a point $(1, x)$ such that

$$\phi_L^1(1, x) = \left(1, \theta_\alpha^{-\eta}(x)\right),$$

where ϕ_L is the Hamiltonian diffeomorphism associated to L defined in (89).

A point $x \in \Sigma$ is a translated point of φ if and only if $(1, x) \in M$ is a leafwise intersection point for ϕ_L^1.

To detect translated points with Rabinowitz Floer homology, we use the following analogue of [**AFM13**, Lemma 3.5].

LEMMA 15.3. *(r, x, η) is a critical point of $\mathcal{A}_{(\alpha_0,\alpha_1)}^{\hat{\varphi}}$ if and only if the point $p := x\left(\frac{1}{2}\right)$ is a translated point for φ with respect to α_1 with time-shift $-\eta$. In this case, we have*

$$\mathcal{A}_{(\alpha_0,\alpha_1)}^{\hat{\varphi}}(r, x, \eta) = \eta.$$

As an immediate consequence of Theorem 3.1 and property (1) about Rabinowitz Floer homology, we get

COROLLARY 15.4. $\mathrm{RFH}_*((M,\lambda_1);\hat{\varphi}) \overset{3.1}{\cong} \mathrm{RFH}_*(\Sigma,\alpha_0;\hat{\varphi}) \overset{(1)}{\cong} \mathrm{H}_{*+n-1}(\Sigma)$.

REMARK 15.5. Note that $\mathrm{RFH}_*\left(\Sigma,\alpha_0;\mathrm{id}\right)$ is the Rabinowitz Floer homology of the actual symplectisation of (Σ,α_0), namely $((0,+\infty)\times\Sigma,\omega)$ equipped with the symplectic form $\omega = d(r\alpha_0)$ as defined in [**AFM13**].

In order to prove the second statement of Theorem 3.2, we need the following proposition.

PROPOSITION 15.6. *Assume* $\dim(\Sigma)\geq 3$. *Let* $\mathcal{R}\subset\Sigma$ *be the union of the images of all contractible closed Reeb orbits on* (Σ,α). *Then for generic* α *the set* $\mathcal{C} = \{\varphi\in\mathrm{Cont}_0(\Sigma)\,|\,\forall x\in\mathrm{Crit}\,(\mathcal{A}^{\hat{\varphi}}_{(\alpha_0,\alpha_1)}),\, x(\frac{1}{2})\cap\mathcal{R}=\emptyset\}$ *is a residual subset in* $\mathrm{Cont}_0(\Sigma)$.

The proof of the above Proposition goes as [**AF12b**, Theorem 3.3].

PROOF OF THEOREM 3.2. Let us first take any contact form $\alpha\in\mathcal{C}(\xi)$ and any contactomorphism $\varphi\in\mathrm{Cont}_0\left(\Sigma,\xi\right)$ and assume there is no translated point. This means that there are no critical points of the action functional which a posteriori implies that the action functional is trivially Morse. Thus, the Rabinowitz Floer homology is defined and equal to $\mathrm{RFH}(\Sigma,\alpha;\varphi)=0$, which contradicts Corollary 15.4. Hence, i) follows.

If Σ is one-dimensional, statement ii) is easy to prove and is left as an exercise to the reader. For nondegenerate φ the perturbed Rabinowitz action functional is Morse-Bott, and thus for generic φ either the lower bound in ii) holds or there is a translated point on a closed contractible Reeb orbit. Proposition 15.6 above asserts that there is a generic set of contact forms $\alpha\in\mathcal{C}(\xi)$ such that the second alternative can be avoided by a generic $\varphi\in\mathrm{Cont}_0(\Sigma,\xi)$. Altogether ii) follows.

Let ϵ be the smallest period of a contractible Reeb orbit of $\alpha\in\mathcal{C}(\xi)$. The proof of Proposition 15.16 shows that $\mathrm{RFH}^{(-\epsilon,+\epsilon)}\left(\Sigma,\alpha;id\right)$ is isomorphic to $\mathrm{RFH}\left(\Sigma,\alpha_0;id\right)$. Moreover, by a standard action estimate, if the oscillation norm of the contact Hamiltonian of φ, as given by equation (92), is less than

ϵ, $\mathrm{RFH}^{(-\epsilon,+\epsilon)}(\Sigma,\alpha;id)$ is isomorphic to $\mathrm{RFH}^{(-\epsilon,+\epsilon)}(\Sigma,\alpha;\hat{\varphi})$. Statement ii') follows. $\square$

REMARK 15.7. If φ has a translated point on a closed contractible Reeb orbit, then the loop which is given by first going along the contactomorphism φ and then close up along the Reeb orbit is contractible. This is true because critical points of the perturbed Rabinowitz action functional are contractible.

REMARK 15.8. We can also consider any Hamiltonian diffeomorphism $\phi^t : M \to M$ which is supported in a neighborhood of Σ and get the analogous result to Theorem 3.2 for leafwise intersections of ϕ^1.

15.3. Existence of invariant Reeb orbits. We can also use the nonvanishing of Rabinowitz Floer homology to deduce the existence of a so called φ-invariant Reeb orbit for $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma,\alpha)$. The details are below.

Recall that $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma,\alpha)$ is the set of strict contactomorphisms with respect to α, i.e. $\varphi^*\alpha = \alpha$. Note that if $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma,\xi)$, then φ commutes with the Reeb flow.

Moreover, recall that a Reeb orbit $x : \mathbb{R} \to \Sigma$ is called φ-invariant if $\varphi(x(s)) = x(s+\tau)$ for some $\tau \in \mathbb{R}\setminus\{0\}$ and all $s \in \mathbb{R}$. In particular, if φ is strict, then a translated point $x \in \Sigma$ gives rise to a φ-invariant orbit. Indeed, if x is a translated point, then every point on the Reeb orbit $\{\theta^s_\alpha(x) |\ s \in \mathbb{R}\}$ is also a translated point

$$\varphi\left(\theta^s_\alpha(x)\right) = \theta^s_\alpha\left(\varphi(x)\right) = \theta^s_\alpha\left(\theta^\eta_\alpha(x)\right) = \theta^{\eta+s}_\alpha(x) = \theta^\eta_\alpha\left(\theta^s_\alpha(x)\right).$$

Thus the Reeb orbit $\{\theta^s_\alpha(x) |\ s \in \mathbb{R}\}$ is φ-invariant.

After the above consideration we have the following corollary of Theorem 3.2.

COROLLARY 15.9. *If $\varphi \in \mathcal{S}\mathrm{Cont}_0(\Sigma,\xi)$ is a strict contactomorphism, then either there is a φ-invariant Reeb orbit (if $\tau \neq 0$) or there is a fixed point for φ (if $\tau = 0$).*

15.4. Existence of non-contractible Reeb orbits. In the following, we will explain how Theorem 3.2 implies the existence of non-contractible Reeb orbits provided there exists a positive loop of contactomorphisms of Σ.

We call a *loop* of contactomorphisms a *positive loop* if the contact Hamiltonian associated to it is everywhere positive, see Definition 13.1.

REMARK 15.10. The notion of a positive loop does not depend on the choice of contact form but only on the contact structure. This is true because the positivity of a loop $\hat{\varphi}$ is equivalent to the property that the vector field $\frac{d}{dt}\varphi_t$ should define the given coorientation of the contact structure ξ, and this does not depend on the choice of contact form.

Most of this subsection is analogous to [**AFM13**, Section 3 and 4]. The results in this section are extensions of the results [**AFM13**, Proposition 4.6] in the sense that we use an arbitrary contact form.

First, we introduce the notion of spectral numbers. Recall the map from property (2) about Rabinowitz Floer homology

$$\iota^c_{\hat{\varphi}} : \mathrm{RFH}^c(\Sigma, \alpha; \hat{\varphi}) \to \mathrm{RFH}(\Sigma, \alpha; \hat{\varphi}).$$

DEFINITION 15.11. Let $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ be nondegenerate. Then its *spectral number* $c(\hat{\varphi}; \alpha) \in \mathbb{R}$ is defined as

$$c(\hat{\varphi}; \alpha) := \inf \left\{ c \in \mathbb{R} | \ [\Sigma_{\hat{\varphi}}] \in \iota^c_{\hat{\varphi}}\left(\mathrm{RFH}^c_*(\Sigma, \alpha; \hat{\varphi})\right) \right\}.$$

REMARK 15.12. This number is not always $-\infty$. Take $\hat{\varphi} = \mathrm{id}$ and $\alpha = \alpha_0$ the contact form which does not possess any contractible Reeb orbits. We know that $[\Sigma_{\mathrm{id}}] = [\Sigma] \in \mathrm{RFH}_n(\Sigma, \alpha_0; \mathrm{id}) \cong \mathrm{H}_{2n-1}(\Sigma; \mathbb{Z}_2)$ and because there are no contractible Reeb orbits this class cannot be represented by a sequence of Reeb orbits. In particular, it cannot be represented by a sequence of Reeb orbits with arbitrarily long negative period and thus $c(\mathrm{id}; \alpha_0) = 0$.

Actually, after the observation that the spectral number is zero for $(\mathrm{id}; \alpha_0)$ we can deduce that $c(\mathrm{id}; \alpha) = 0$ for any supporting contact form. This will be the content of Corollary 15.17.

Recall that by property (1) of Rabinowitz Floer homology for two paths of contactomorphisms $\hat{\varphi}, \hat{\psi} \in \widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi)$ we can define a map

$$\Phi_{\hat{\varphi},\hat{\psi}}:\ \mathrm{RFH}^c\,(\Sigma,\alpha;\hat{\varphi}) \to \mathrm{RFH}^{c+K(\hat{\varphi},\hat{\psi})}\left(\Sigma,\alpha;\hat{\psi}\right).$$

For $K(\hat{\varphi},\hat{\psi})$ we have the following estimate.

PROPOSITION 15.13. [**AFM13**] *Let l_t and k_t denote the contact Hamiltonians of $\hat{\varphi}$ and $\hat{\psi}$ and $C(\hat{\varphi};\alpha)$ and $C(\hat{\psi};\alpha)$ the values defined in Lemma 13.2 equation* (91)*. Then,*

$$K(\hat{\varphi},\hat{\psi}) \leq e^{\max\left\{C(\hat{\varphi};\alpha),C(\hat{\psi};\alpha)\right\}} \int_0^1 \max\left\{\max_{x\in\Sigma}\left(l_t(x)-k_t(x)\right),0\right\} dt. \tag{102}$$

We can estimate the difference between spectral numbers of two nondegenerate paths via this number K.

PROPOSITION 15.14. *Let $\hat{\varphi}$, $\hat{\psi} \in \widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi)$ be two nondegenerate paths of contactomorphisms and denote again by l_t and k_t their contact Hamiltonians. Then we can estimate*

$$c(\hat{\psi};\alpha) \leq c\,(\hat{\varphi};\alpha) + K(\hat{\varphi},\hat{\psi}).$$

Furthermore, we have

$$l_t(x) \leq k_t(x), \quad \forall x\in\Sigma,\ 0\leq t\leq 1 \quad \Rightarrow \quad c(\hat{\varphi};\alpha) \geq c(\hat{\psi};\alpha).$$

The proof of Proposition 15.14 is analogous to the proof in [**AFM13**, Proposition 4.2].

REMARK 15.15. It is possible to extend c to all of $\widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi)$ via the limit of nondegenerate paths in a unique manner and such that all the previously mentioned properties are still satisfied.

We equip $\widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi)$ with the C^2-topology. For fixed $\alpha \in \mathcal{C}(\xi)$ the continuity of the map

$$\begin{aligned} c:\ \widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi) &\to \mathbb{R} \\ \hat{\varphi} &\mapsto c(\hat{\varphi};\alpha) \end{aligned}$$

was proved in [**AFM13**, Lemma 4.3]. We prove the following extension.

PROPOSITION 15.16. *We consider $\alpha \in \mathcal{C}(\xi)$. For any fixed path $\hat{\varphi}$ of contactomorphisms the map*

$$(\hat{\varphi};\alpha) \mapsto c(\hat{\varphi};\alpha)$$

is continuous, where $\mathcal{C}(\xi)$ is equipped with the natural C^0-topology.

As a corollary of Proposition 15.16 we have

COROLLARY 15.17. *For $\alpha \in \mathcal{C}(\xi)$, where ξ is a hypertight contact structure, we have*

$$c(\mathrm{id},\alpha) = 0.$$

Proposition 15.14 together with Corollary 15.17 shows that $c(\hat{\varphi};\alpha)$ is finite for every $\hat{\varphi}$ and α.

The full proof of Proposition 15.16 can be found in the [**MN17**, Appendix A]. Here, we only give a proof of Corollary 15.17 for which we need the following lemma, which gives a lower bound on the periods of all non-constant Reeb orbits appearing during a homotopy of supporting contact forms.

LEMMA 15.18. *Let $\alpha_1,\alpha_2 \in \mathcal{C}(\xi)$ and let α_s, $s \in [0,1]$, be a homotopy of supporting contact forms between α_0 and α_1. For $s \in [0,1]$, denote by $\mathcal{P}_s \subset \mathbb{R}\setminus\{0\}$ the set of periods of closed orbits of the Reeb vector field R_{α_s}, and set $T_s := \inf\{|\eta| \,|\, \eta \in \mathcal{P}_s\}$. Then $T := \inf\{T_s \,|\, s \in [0,1]\} > 0$.*

PROOF. Let V be a non-vanishing vector field on a compact manifold M and let $p \in M$. One can choose $\tau_p > 0$ and a neighborhood U_p of p, such that the flow ϕ_V^t of V satisfies $\phi_V^{\tau_p}(U_p) \cap U_p = \emptyset$. Now let the vector field V_s

depend continuously on a parameter s. Fix s_0 and choose as before τ_p and U_p for the vector field V_{s_0}. Then for a sufficiently small interval I_p containing s_0 it still holds that $\phi^{\tau_p}_{V_s}(U_p) \cap U_p = \emptyset$ for all $s \in I_p$. Choose a finite covering of M by the sets U_p. We have a corresponding finite intersection $I_{s_0} = \bigcap I_p \neq \emptyset$ and a corresponding infimum $\tau_{s_0} = \inf \tau_p > 0$. Then the smallest period of a closed orbit of every $V_s,\ s \in I$, is bounded from below by τ_{s_0}.

Applying this to the family of Reeb vector fields R_{α_s}, one finds for every $s_0 \in [0,1]$ a small interval I_{s_0} containing s_0 such that $T_{s_0} = \inf\{T_s \,|\, s \in I_{s_0}\} > 0$. Since $[0,1] = \bigcup_{k=1}^n I_{s_k}$ for some $s_k \in [0,1]$, $n \in \mathbb{N}$, we conclude $T = \inf_{1\leq k\leq n} T_{s_k} > 0$. $\square$

PROOF OF COROLLARY 15.17. Let α_0 be as above, and $\lambda_0 = r\alpha_0$. Let $\lambda \in \Omega_\nu(\alpha_0, \alpha_1)$ for some suitable $\nu > 0$. Let λ_s be a homotopy between λ_0 and λ of the type considered before. As in Lemma 15.18 denote by $T > 0$ the smallest period of a closed Reeb orbit for the homotopy $\alpha_s = \lambda_s|_{\Sigma\times\{1\}}$.

As explained in Section 14, for fixed $s_0, s_1 \in [0,1]$ we can choose a reparametrised homotopy $\widetilde{\lambda}_s$ between λ_{s_0} and λ_{s_1} such that

$$\widetilde{C} := \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s \widetilde{\lambda}_s\|_{J_s} < 2|s_1 - s_0| \sup_{s\in[0,1]} \sup_{(0,e^{2\nu}]\times\Sigma} \|\partial_s \lambda_s\|_{J_s}.$$

We can apply an adiabatic argument for this homotopy to get uniform bounds on η, as in Lemma 14.3, by taking a small enough partition of $[0,1]$. An analogous argument applies here.

We first show the following.

CLAIM 15.19. Let C_0 and C_1 as in Lemma 14.2. There is a constant c_T depending on the original homotopy λ_s, C_0, C_1 and T such that if $\widetilde{C} < c_T$, then

$$\mathcal{M}_0^{a+}(\mathcal{A}^{\mathrm{id}}_{\widetilde{\lambda}_s}) = \emptyset, \quad \text{if} \quad a_+ \geq T \tag{103}$$

and

$$\mathcal{M}^0_{a_-}(\mathcal{A}^{\mathrm{id}}_{\widetilde{\lambda}_s}) = \emptyset, \quad \text{if} \quad a_- \leq -T. \tag{104}$$

To prove the claim we assume $a_+ \geq a_-$ and consider a trajectory $(u(s), \eta(s))$ in $\mathcal{M}_{a_-}^{a_+}(\mathcal{A}_{\widetilde{\lambda}_s}^{\mathrm{id}})$. Assume $\widetilde{C} < 1$. Then, see the proof of Lemma 14.3,

$$\begin{aligned} a_+ - a_- &\leq -E(u(s), \eta(s)) + \left| \int_0^1 \int_{S^1} (u(s))^* (\partial_s \lambda_s)\, ds \right| \\ &\leq -E(u(s), \eta(s)) \\ &\quad + \widetilde{C} \left(E(u(s), \eta(s)) + 1 + \|\eta\|_\infty \sup_{s \in [s_0, s_1]} \|X_H^{\lambda_s}\|_{J_s} \right) \\ &\leq \widetilde{C} \left(1 + \|\eta\|_\infty \sup_{s \in [0,1]} \|X_H^{\lambda_s}\|_{J_s} \right) \\ &\leq \widetilde{C} \left(1 + \frac{1}{1-A} B \left\| X_H^{\lambda_s} \right\|_\infty \right). \end{aligned}$$

Note that $\sup_{s \in [s_0, s_1]} \left\| X_H^{\lambda_s} \right\|_{J_s} \leq \sup_{s \in [0,1]} \left\| X_H^{\lambda_s} \right\|_{J_s} = \left\| X_H^{\lambda_s} \right\|_\infty$ is finite, where by $\left\{ X_H^{\lambda_s} \right\}_{s \in [s_0, s_1]}$ we mean a possibly reparametrised homotopy.

Here, we can estimate A from the proof of Lemma 14.3:

$$A = \widetilde{C} \sup_{s \in [s_0, s_1]} \|X_H\|_{J_s} \left(2C_1 \widetilde{C} + \frac{2}{C_0} + C_1 \right) \leq \widetilde{C}\, \underbrace{\left\| X_H^{\lambda_s} \right\|_\infty (3C_1 + \frac{2}{C_0})}_{=:K} < \frac{1}{2}$$

if we assume $\widetilde{C} < \min\{1, \frac{1}{2K}\}$.

Also

$$\begin{aligned}
B &= C_1\left(1+\max\{|a_+|,|a_-|\}\right)+\left(2\,|a_- - a_+|+2\widetilde{C}\right)\left(C_1\widetilde{C}+\frac{1}{C_0}\right)\\
&\qquad\qquad + C_1\widetilde{C}\\
&\overset{\widetilde{C}<1}{\leq} \underbrace{\left(C_1+2\left(C_1+\frac{1}{C_0}\right)+C_1\right)}_{=:K_0}+C_1\max\{|a_+|,|a_-|\}\\
&\qquad\qquad + \underbrace{2\left(C_1+\frac{1}{C_0}\right)}_{=:K_1}|a_- - a_+|\\
&\leq K_0+C_1\max\{|a_+|,|a_-|\}+K_1\,|a_- - a_+|\,.
\end{aligned}$$

Hence

$$\begin{aligned}
a_+ - a_- &\leq \widetilde{C}\big(1+2K_0\left\|X_H^{\lambda_s}\right\|_\infty\\
&\qquad + 2C_1\left\|X_H^{\lambda_s}\right\|_\infty \max\{|a_+|,|a_-|\}+K_1\left\|X_H^{\lambda_s}\right\|_\infty|a_- - a_+|\big)\\
&= \widetilde{C}\underbrace{\left(1+2K_0\left\|X_H^{\lambda_s}\right\|_\infty\right)}_{=:\widetilde{K}_0}+\widetilde{C}\underbrace{\left\|X_H^{\lambda_s}\right\|_\infty\cdot 2C_1}_{=:\widetilde{K}_1}\max\{|a_+|,|a_-|\}\\
&\qquad + \widetilde{C}\underbrace{\left\|X_H^{\lambda_s}\right\|_\infty K_1}_{=:\widetilde{K}_2}|a_- - a_+|.
\end{aligned}$$

Now, choose $c_T > 0$ so that

$$c_T < \min\left\{1,\frac{1}{2K},\frac{T}{2\widetilde{K}_0},\frac{1}{4\widetilde{K}_1},\frac{1}{4\widetilde{K}_2}\right\}.$$

If $\widetilde{C} < c_T$, then

$$a_+ - a_- < \frac{1}{2}T+\frac{1}{4}\max(|a_+|,|a_-|)+\frac{1}{4}|a_- - a_+|,$$

and thus

$$\frac{3}{4}(a_+ - a_-) < \frac{1}{2}T + \frac{1}{4}\max(|a_+|, |a_-|).$$

From this, the desired upper bounds follow. Indeed, if $a_- = 0$, then $a_+ < T$, and if $a_+ = 0$, then $a_- > -T$. This concludes the proof of the claim.

Recall that the original homotopy λ_s goes from $r\alpha_0$ to λ. Assume now s_0 and s_1 have been chosen such that $\widetilde{C} < c_T$. Let $[X]$ be a nonzero class in $\mathrm{RFH}(\Sigma, \lambda_{s_0}; \mathrm{id})$ with

$$[X] \in \iota^0_{\mathrm{id}}(\mathrm{RFH}^0(\Sigma, \lambda_{s_0}; \mathrm{id})), \quad [X] \notin \iota^{-T}_{\mathrm{id}}(\mathrm{RFH}^{-T}(\Sigma, \lambda_{s_0}; \mathrm{id})).$$

Consider the non-zero class $\Phi_{s_0,s_1}([X]) \in \mathrm{RFH}(\Sigma, \lambda_{s_1}; \mathrm{id})$. Result (103) from the previous claim shows that it can be represented by chains consisting of elements in $\mathrm{Crit}\,(\mathcal{A}^{\mathrm{id}}_{\lambda_{s_1}})$ with nonpositive period, so

$$\Phi_{s_0,s_1}([X]) \in \iota^0_{\mathrm{id}}(\mathrm{RFH}^0(\Sigma, \lambda_{s_1}; \mathrm{id})).$$

We can also conclude that

$$\Phi_{s_0,s_1}([X]) \notin \iota^{-T}_{\mathrm{id}}(\mathrm{RFH}^{-T}(\Sigma, \lambda_{s_1}; \mathrm{id})).$$

Namely assume the contrary. So $\Phi_{s_0,s_1}([X])$ can be represented by chains consisting of critical points all of whose periods are $< -T$. Equation (104) from the claim for the inverse homotopy shows

$$\Psi_{s_1,s_0}(\Phi_{s_0,s_1}([X])) \in \iota^{-T}_{\mathrm{id}}(\mathrm{RFH}^{-T}(\Sigma, \alpha_{s_0}; \mathrm{id})).$$

But since $\Psi_{s_1,s_0} \circ \Phi_{s_0,s_1} = \mathrm{id}$, it follows that

$$[X] \in \iota^{-T}_{\mathrm{id}}\mathrm{RFH}^{-T}\left(\Sigma, \alpha_{s_0}; \mathrm{id}\right),$$

which contradicts the assumption about $[X]$.

To conclude the proof, note that $[\Sigma] \in \mathrm{RFH}(\Sigma, \alpha_0; \mathrm{id})$ satisfies exactly the assumption about $[X]$ above. In fact $[\Sigma] \in \mathrm{RFH}^{(-\epsilon,+\epsilon)}(\Sigma, \alpha_0; \mathrm{id})$, because there are no contractible Reeb orbits for α_0. Hence the same follows for

$\Phi([\Sigma]) \in \mathrm{RFH}(\Sigma, \alpha, \mathrm{id})$. We conclude that

$$c(\mathrm{id}; \alpha) = \inf \{ c \in \mathbb{R} | \ [\Sigma_\alpha] \in \iota_{id}^c (\mathrm{RFH}_*^c (\Sigma, \alpha; id)) \} = 0.$$

□

REMARK 15.20. For $\hat{\varphi} = \{ t \mapsto \theta_\alpha^{tT} \}$ or $\hat{\varphi}$ a loop, analogous arguments as in the proof of Corollary 15.17 show that if we assume $\|\alpha_s\|_{J_s}$ to be sufficiently small, then the corresponding spaces of Floer trajectories $\mathcal{M}_{-T}^{-T+a}(\mathcal{A}_{\lambda_s}^{\hat{\varphi}_s})$ and $\mathcal{M}_{-T+a}^{-T}(\mathcal{A}_{\lambda_s}^{\hat{\varphi}_s})$ are empty for $a \notin (-\epsilon, \epsilon)$ (see Property (a) below). The main point about this argument is that there is a spectral gap around the value $0 \in \mathrm{Spec}\,(\hat{\varphi}_s; \alpha_s)$.

Meanwhile for $\hat{\varphi}$ any path of contactomorphisms (not of the type mentioned above), we have

$$\mathrm{Spec}\,(\hat{\varphi}; \alpha) = \{ \eta \,|\, \hat{\varphi} \text{ has a translated point with time shift } -\eta \text{ w.r.t. } \alpha \} .$$

Let α_0 be without contractible Reeb orbits and $\eta \in \mathrm{Spec}\,(\hat{\varphi}_0; \alpha_0)$. Let λ_s be a homotopy as usual between $r\alpha_0$ and $\lambda \in \Omega(\alpha_0, \alpha_1)$ where $\alpha_1 \in \mathcal{C}(\xi)$ and look at flow lines in $\mathcal{M}_\eta^\tau \left(\mathcal{A}_{\lambda_s}^{\hat{\varphi}_s} \right)$. Define

$$\delta_s := \inf \{ \tau_s - \eta \,|\, \tau_s \in \mathrm{Spec}\,(\hat{\varphi}; \alpha_s) \,, \tau_s \neq \eta \} .$$

Then for $\hat{\varphi} = \mathrm{id}$ (or $\hat{\varphi}$ a loop, or a path along the Reeb flow) we can bound δ_s away from 0. For general $\hat{\varphi}$, however, it may happen that $\inf_s \delta_s = 0$ and thus there is no spectral gap around 0.

Furthermore, one can show that the spectral number satisfies the following properties.

(a) For any $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0\,(\Sigma, \xi)$ the spectral number is a critical value of $\mathcal{A}_{(\alpha_0, \alpha)}^{\hat{\varphi}}$, that is $c(\hat{\varphi}; \alpha) \in \mathrm{Spec}(\hat{\varphi})$. In particular, for the Reeb flow $t \mapsto \theta_\alpha^{tT}$ we have that $c(t \mapsto \theta_\alpha^{tT}; \alpha) = -T$. Indeed, let $\hat{\varphi}$ be the path $t \mapsto \theta_\alpha^{tT}$. The critical points are closed Reeb orbits and then one moves along the Reeb orbit for time $-T$. Thus,

$$\mathrm{Spec}\left(\theta_\alpha^{tT}; \alpha \right) = \{ -T + \{ \text{contract. Reeb periods of } \alpha \} \} .$$

Compare this also to Remark 15.2. From this it is clear that $c(\theta^{tT}_{\alpha_0};\alpha_0) = -T$.

For $(\hat{\theta}_{\alpha_s};\alpha_s)$ a family it is still true that

$$\mathrm{Spec}\left(\theta^{tT}_{\alpha_s};\alpha_s\right) = \{-T\} + \{\text{contract. Reeb periods of } \alpha_s\}\,.$$

By Lemma 15.18 there is a constant $\epsilon > 0$ so that the family of Reeb vector fields R_{α_s} has no nonconstant Reeb orbit with period in $(-\epsilon,+\epsilon)$ and hence

$$\mathrm{Spec}\,(id;\alpha_s) \cap (-\epsilon,+\epsilon) = \{0\}\,.$$

Hence, $\{-T\}$ is an isolated component of the set $\mathrm{Spec}\left(\theta^{tT}_{\alpha_s};\alpha_s\right)$. By continuity of the map $(\hat{\varphi};\alpha) \mapsto c(\hat{\varphi};\alpha)$, as proved in Proposition 15.16, and the fact that $c(\theta^{tT}_{\alpha_0};\alpha_0) = -T$, we have

$$c\left(\theta^{tT}_{\alpha_s};\alpha_s\right) = -T, \quad \forall s.$$

(b) The map c descends to a well defined map

$$c:\ \widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi) \to \mathbb{R},$$

where $\widetilde{\mathrm{Cont}}_0\,(\Sigma,\xi)$ denotes the the universal cover of $\mathrm{Cont}_0\,(\Sigma,\xi)$.

In particular, using Proposition 15.14 and property (a) one can show the following.

COROLLARY 15.21. *Let $\hat{\varphi} \in \widetilde{\mathrm{Cont}}_0\,(\Sigma,\alpha)$ with contact Hamiltonian l_t. Then, if*

$$\begin{aligned} l_t < 0,\ \forall t \in [0,1] \quad &\Rightarrow \quad c(\hat{\varphi};\alpha) > 0 \quad \textit{and if} \\ l_t > 0,\ \forall t \in [0,1] \quad &\Rightarrow \quad c(\hat{\varphi};\alpha) < 0. \end{aligned}$$

The following argument is from [**AFM13**, Proof of Corlollary 4.4].

PROOF OF COROLLARY 15.21. This corollary follows now from Proposition 15.14 and property (a).

For $l_t < 0$ there is $\epsilon > 0$ so that $l_t \leq -\epsilon < 0$. The constant function ϵ generates the path $\{t \mapsto \varphi_{R_\alpha}^{t\epsilon}\}$ which by property (a) has spectral number $-\epsilon$. Proposition 15.14 implies that for $l_t \leq -\epsilon$,

$$c(\hat{\varphi}; \alpha) \geq c(t \mapsto \theta_\alpha^{-t\epsilon}; \alpha) = \epsilon > 0.$$

Analogously, we have for $l_t \geq \epsilon > 0$ that

$$c(\hat{\varphi}; \alpha) \leq c(t \mapsto \theta_\alpha^{t\epsilon}; \alpha) = -\epsilon < 0.$$

□

Now, let $\hat{\varphi} = \{\varphi_t\}_{t \in S^1}$ be a loop of contactomorphisms. Then one can show a more specific result about the spectral number.

PROPOSITION 15.22. *Let $\hat{\varphi}$ be a loop of contactomorphisms. Then, if $c(\hat{\varphi}; \alpha) \neq 0$ there exists a Reeb orbit of period $-c(\hat{\varphi}; \alpha)$ which belongs to the free homotopy class $-u_{\hat{\varphi}}$.*

Also, $c(\hat{\varphi}; \alpha) = 0$ if and only if $-u_{\hat{\varphi}}$ is the class of contractible loops.

The proof goes analogously to the proof of [**AFM13**, Lemma 4.3].

PROOF OF PROPOSITION 15.22. In the situation where $\hat{\varphi}$ is in fact a loop, a critical point (x, r, η) of $\mathcal{A}^{\hat{\varphi}}$ is precisely the concatenation of a closed Reeb orbit of period η with the loop $t \mapsto \varphi_t(x)$ for some $x \in \Sigma$.

Assume that $c(\hat{\varphi}; \alpha) \neq 0$. Then, by property (a), there is a Reeb orbit with period $\eta = c(\hat{\varphi}; \alpha)$. Again the loop (x, r, η) is contractible, and since it consists of the concatenation of a Reeb orbit with the orbit $x \mapsto \varphi_t(x)$, this means that the Reeb orbit lies in the free homotopy class $-u_{\hat{\varphi}}$.

Now, assume that $c(\hat{\varphi}; \alpha) = 0$. Then, by property (a) of the spectral number, there must be a Reeb orbit with $\eta = 0$ and thus there is a critical point which is of the form $(x, r, 0)$, where $x(t) = \varphi_t(x)$. But since $\mathcal{A}^{\hat{\varphi}}$ is defined on the set of contractible orbits, this implies that $x(t)$ is contractible, and thus $u_{\hat{\varphi}}$ is trivial.

To prove the converse, note that the same argument as in the proof of Proposition 15.16 shows that for a loop $\hat{\varphi}$, $c(\hat{\varphi}, \alpha_0) = 0$ implies $c(\hat{\varphi}, \alpha) = 0$.

So if $c(\hat{\varphi}, \alpha) \neq 0$, we have that $c(\hat{\varphi}, \alpha_0) \neq 0$, and hence there is a Reeb orbit with respect to α_0 in $-u_{\hat{\varphi}}$. The assumptions on α_0 then imply that $-u_{\hat{\varphi}}$ is a non-trivial class. $\square$

Note that by Corollary 15.17 and Corollary 15.21 we have for each positive loop

$$c(\hat{\varphi}; \alpha) < c(\mathrm{id}; \alpha) = 0, \ \forall \alpha \in \mathcal{C}(\xi).$$

Therefore, in this setting a positive loop is never contractible.

As a corollary of Proposition 15.22 we have the following result which proves Theorem 3.6.

COROLLARY 15.23. *Let (Σ, ξ) be a hypertight contact manifold. If there exists a positive loop $\hat{\varphi} \in \mathrm{Cont}_0\,(\Sigma, \xi)$, then for any supporting contact form there exists a closed Reeb orbit in the non-trivial free homotopy class $-u_{\varphi}$, i.e. there always exists a* ***non-contractible*** *Reeb orbit.*

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