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# The Geometry and Physics of F-theory Compactifications

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**The Geometry and Physics of F-Theory Compactifications**

**Abstract:** In this PhD thesis we study the structure of gauge and gravitational anomalies in effective theories obtained by compactfication of F-theory on Calabi-Yau manifolds. In particular, we study the continuous local anomalies in 2D $\mathcal{N} = (0,2)$ effective theories from elliptically fibered Calabi-Yau five-fold compactifications and discrete gauge anomalies in 6D $\mathcal{N} = (1,0)$ theories from F-theory on genus-one fibrations of Calabi-Yau three-folds.
Certain anomalies associated with these symmetries, induced at 1-loop in perturbative theories, can be cancelled by a corresponding generalized Green-Schwarz mechanism operating at the level of chiral fields in the effective theories. We derive closed expressions for types Green-Schwarz mechanisms in F-theory compactifications, as well as the gravitational and gauge anomalies. These expressions in both cases involve topological invariants of the underlying fibrations of Calabi-Yau manifolds. Cancellation of these anomalies in the effective theories predicts intricate topological identities which must hold on every corresponding Calabi-Yau manifold. Some of the identities we find on elliptic 5-folds are related in an intriguing way to previously studied topological identities governing the structure of local anomalies for continuous symmetry in 6D $\mathcal{N} = (1,0)$ and 4D $N = 1$ theories obtained from F-theory.

**Zusammenfassung:** In dieser Doktorarbeit untersuchen wir die Strukturen von Eich- und Gravitationsanomalien in effektiven Theorien, welche durch Kompaktifizierung von F-Theorie auf Calabi-Yau-Mannigfaltigkeiten erhalten wurden. Insbesondere betrachten wir lokale Anomalien in 2D $\mathcal{N} = (0,2)$ effektiven Theorien von Kompaktifizierungen elliptisch gefaserter Calabi-Yau 5-faltigkeiten und diskrete Eich-Anomalien in 6D $\mathcal{N} = (1,0)$ Theorien von F-Theorie auf Genus-1 Faserungen von Calabi-Yau 3-faltigkeiten.
Bestimmte mit diesen Symmetrien assoziierte Anomalien, in perturbativen Theorien in 1-Schleifen-Ordnung induziert, können durch verallgemeinerte Green-Schwarz-Mechanismen gekürzt werden, welche auf der Ebene der chiralen Felder der effektiven Theorie wirken. Wir leiten einen geschlossenen Ausdruck sowohl für die Green-Schwarz-Mechanismen in F-Theorie-Kompaktifizierungen als auch für die Eich- und Gravitationsanomalien her. In beiden Fällen beinhalten diese Ausdrücke topologische Invarianten der zugrundeliegenden Faserung der Calabi-Yau-Mannigfaltigkeiten. Die Kürzung dieser Anomalien in den effektiven Theorien impliziert komplizierte topologische Identitaetäten, welche auf jeder dazugehörigen Calabi-Yau-Mannigfaltigkeit gelten müssen. Einige dieser Identitäten auf elliptischen 5-faltigkeiten hängen in verblüffender Weise mit den zuvor betrachteten Identitäten zusammen, die die Struktur der lokalen Anomalien von kontinuierlichen Symmetrien in von F-Theorie erhaltenen 6D $\mathcal{N} = (1,0)$ und 4D $N = 1$ Theorien bestimmen.

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*X: Why is the Math so ...?*
*F: Because it's real.*

# Contents

# Part I.

# Introduction

## 0.1. Physical Motivations for String Theory

### 0.1.1. The Standard Model and the Beyonds

On the 4th of July 2012, the discovery of the last missing keystone in the Standard Model (SM) of Particle Physics - the Higgs boson - was announced at CERN at a mass scale around 126 GeV, fitting into the range of theoretical expectations. The discovery of the Higgs boson completes the Standard Model, which describes three of the four fundamental interactions in nature that have been observed - the Strong, the Electromagnetic and the Weak interactions. It also has a fairly simple structure, partially unifying the electromagnetic and the weak interaction into the electroweak theory, and can be characterized as a gauge theory with gauge group $G = SU(3) \times SU(2) \times U(1)$ together with 19 real parameters.[1] The Standard Model is renormalizable and with the Higgs mass around $\sim 126$ GeV, it could remain consistent (at least conceptually) even if we extrapolate it all the way to the Planck scale $\ell_P \sim 10^{19}$ GeV [2], where the fourth interaction, gravity, becomes too strong to be neglected and the gauge theory should be replaced by a more fundamental theory (see the reasons below). The Standard Model can be considered as one of the biggest triumphs in the history of mankind and subsumes our understanding of elementary particle physics to the extent of what has been tested to date in experiments. In other words, we can explain (almost [3]) all phenomena at the sub-atomic scale which we have observed so far by such a (fairly) simple underlying theory.

However, despite such impressive successes of the Standard Model, it suffers from some (conceptual and mathematical) shortcomings. Among others[4], we are going to list several caveats which are particularly relevant from the perspective of this thesis:

**The electroweak hierarchy problem** The most questionable one is the electroweak hierarchy problem, i.e. the question of why the electroweak scale at 250 GeV is so much smaller than the Planck scale ($\sim 10^{19}$ GeV). Such a problem would typically lead to the fine-tuning problem which says that an incredible amount of fine-tuning of several parameters (estimated around 30 orders of magnitude ) is required to stabilize the Higgs mass at around 126 GeV in order to remove the radiative one-loop corrections to the Higgs mass from fermion coupling in the Standard Model. Although such an fine-turning does not lead to fundamental inconsistencies, it is not perceived to be "natural". Several ideas have been put forward for attacking this issue such as Large Extra Dimensions (see e.g. [1]). Let us focus on another option: supersymmetry. Supersymmetry (SUSY) is a spacetime symmetry which maps bosonic particles and fields (of integer spin) into fermionic ones (of half integer spin) and vice versa. So that every boson in a supersymmetric theory has a corresponding fermion with the same mass, as well as other quantum numbers, and vice verse. The reason why SUSY can (partially) solve the fine-tuning problem, roughly speaking, is that supersymmetry, as a symmetry, protects the Higgs mass from quantum corrections, due to a cancellation of bosonic and fermionic loop contributions. In global supersymmetry, realistic models are only possible for 4D $N = 1$ supersymmetry (with

[1] These include 6 quark masses, 3 charged lepton masses, 3 gauge couplings, 3 mixing angles and 1 CP violation phase for the quarks, 2 parameters in the Higgs potential and 1 QCD vacuum angle.

[2] The Landau pole associated with the $U(1)$ in the Standard Model is expected to occur far above the Planck scale based on the currently known particle content and is believed to be solved by a complete theory including quantum gravity.

[3] There are few observations that are not incorporated in the Standard Model in its transitional form, clearly the non-vanishing neutrino mass is one of them.

[4] The SM also fails to convincingly explain several puzzles such as the origin of Dark Matter, baryon asymmetry or the strong CP problem.

four real supercharges) as only these are compatible with a chiral spectrum. As the amount supersymmetry is enhanced, all helicity $\frac{1}{2}$ fermions are accompanied by helicity $-\frac{1}{2}$ fermions in the same gauge representation, and hence such theories with $N > 1$ supercharges (for which the number of supercharges exceeds four) are non-chiral.

**Arbitrariness** Another conceptual shortcoming of the Standard Model is the arbitrariness in the structure of the model. For example, the 19 parameters in the model are determined by empirical data, rather than by the theory itself. Likewise, there is no guiding principle to explain the gauge structure $SU(3) \times SU(2) \times U(1)$ of the Standard Model based on its characteristics such as freedom of anomalies or renormalisability.

**Simple structure** Related to the two issues above, the three fundamental interactions, in terms of the gauge couplings, are treated separately in the Standard Model even though their mathematical formulation is essentially the same. Surprisingly, given the current particle content of the Standard Model and taking into account the running of the gauge couplings, the three gauge couplings become equally strong pairwise at three points scattered between $10^{13}$ and $10^{17}$ GeV. This leads to a proposal that there is some Grant Unified Theory (GUT) based on a simple group such as $SU(5)$ [2] that includes all the particle contents of the SM, which is spontaneously broken at the energy $M_{GUT} \sim 10^{16}$ GeV (see e.g. [3]). Such a unified description is extremely appealing and natural to physicists as it fits with the lessons we learned from Newton and Maxwell. In order for the three gauge couplings to meet at one and the same point, one promising way is argue to that there exists one more intermediate threshold for new physics between the GUT scale and the electroweak scale. One option is to combine the GUT idea with supersymmetry and to introduce the minimal supersymmetric extension of the Standard Model (MSSM). In the MSSM, indeed the three gauge couplings almost perfectly unify (at one-loop level in perturbation theory) under the condition that the SUSY breaking scale lies around the TeV range. However, this model in the recent years has been under severe tension as the recent data from the LHC at CERN have not shown any significant signs of SUSY around the TeV scale.

String theory, as we are going to introduce shortly, can in principle solve the above caveats and encompass all the above ideas we mentioned.

### 0.1.2. Gravity and Quantum theory

However, even if we were to ignore the above (and other) shortcomings of the SM and believe there is no any new physics between the electroweak scale and the Planck scale, there is still one big elephant in the room–Gravity. In the last century, we learned that classical gravity is described by Einstein's General Relativity, which states that gravity arises from the geometry of spacetime and the dynamics of gravity is entirely governed by the Einstein equation in terms of the geometry of the spacetime. Remarkably, Einstein's theory successfully describes physics at large macroscopic scales (corresponding a low-energy scale) and has been well tested (at least up to the scale of the solar system) by lots of experiments, from Eddington's early tests of light deflection by the gravitational field of the sun to the recent LIGO/VIRGO networks of gravitational wave detectors. However, Einstein gravity is a classical theory and the current viewpoint is that this classical theory should be replaced by a certain theory of quantum gravity at a high energy scale. There are several well-posed arguments from black hole physics in need of such a quantum theory of gravity such as the singularity problem and the information-loss problem (they are related). Here we would like to give another heuristic argument based on the conflict between the non-linear nature of Einstein's classical gravity theory and the superposition principle of a general quantum theory. To see that, we write down the four-dimensional Einstein-Hilbert action coupled to the Standard Model describing a microscopic system

$$S = \frac{1}{16\pi G_N} \int \sqrt{-g}(R - 2\Lambda_c + \mathcal{L}_{SM}) \,, \tag{0.1}$$

where the matter action $\mathcal{L}_{SM}$) denotes the Lagrangian for the Standard Model. Here we are working in units where $c = 1$. The Einstein equations state that

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - \Lambda_c g_{\mu\nu} \,. \tag{0.2}$$

The energy-momentum $T_{\mu\nu}$ on the right-hand shall receive contributions from the matter sector, which is quantum in nature. Thereby, technically, it should be written as $\langle \widehat{T}_{\mu\nu} \rangle$, meaning that it should be replaced by its expectation value evaluated in the Hilbert space. Hence the semi-classical Einstein equations yield

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \langle \widehat{T}_{\mu\nu} \rangle \,, \tag{0.3}$$

where we have omitted the vacuum energy from the cosmological constant term $\Lambda_c g_{\mu\nu}$. The form of the equation suggests that the non-linearity of Einstein gravity (on the left of the equation) conflicts with the linearity of the quantum theory (on the right), i.e. the superposition principle. In other words, Einstein's theory should be described by a quantum version subject to the superposition principle. This heuristic argument dates back to Richard Feynman, who was the first to devise a thought experiment to argue that classical gravity theory spoils the superposition principle for a certain quantum system. Though it is not a strict logical necessity to draw this conclusion, the argument indicates that the gravity theory should somehow change its classical appearance at a certain energy scale. For more details we refer to [4], which also reviews other arguments suggesting that the gravitational interaction shall be quantized.

However, when we try to quantize Einstein's gravity directly, one can readily realize that it is non-renormalizable simply because that the gravitational coupling has negative mass dimension and every computation of physical quantities diverges from some loop order onwards in a way that cannot be remedied by adding a finite number of counterterms. These divergences are not even cured by adding supersymmetry, dubbed as a supergravity. Hence one needs to introduce a different quantum theory of gravity whose classical limit should correspond to Einstein's gravity. However, searching for such theories is hard. This partially relates to the fact that the relevant scale, where quantum effects of gravity are strong and relevant, is far beyond what can directly be explored experimentally. The scale is believed to at the Planck scale $\ell_P$, given by

$$\ell_p := \sqrt{\hbar G/c^3} \approx 10^{-33} cm \,. \tag{0.4}$$

Nevertheless, there are several candidate proposals. Among them, string theory is the most promising one, which provides a consistent, elegant and powerful framework to unify gravity and quantum theory, and further it is believed to be UV finite (at least at up to four-loop corrections).

## 0.2. String Theory

The idea of string theory is very simple, it replaces point-like particles as the building blocks of a theory by one-dimensional objects called strings with a string length $\ell_s$ and describes how these strings propagate through spacetime and interact with each other. Further, there is no single free dimensionless parameter in string theory and every parameter characterizing its effective theory can be in principle derived from the string theory itself. Since we are going to elaborate in more detail on string theory in chapter 1, here let us give a narrative way to provide a bird's-eye view on string theory.

### 0.2.1. A Nod on the history of string theory

In the summer of 1968 [5], Veneziano wrote the paper [5] while visiting CERN, in which he postulated a formula, known as "Veneziano formula", to characterize some duality properties in the soft behavior of high-energy hadronic resonances. This paper marked the beginning of string theory, as later people realized that such a formula can naturally explain the scattering amplitudes of a vibrating string. At that time, string theory was developed in a bid to understand the strong interaction. However, in order to explain the observed spectrum of hadrons and their interactions, there are certain issues, for example string theory could not explain the point-like structure of hadrons in deep inelastic scattering experiments, and more importantly, the spectrum of (closed) string theory contains a massless spin-2 state (as will be introduced in 1) which does not fit with the hadronic spectrum. Further investigations revealed that such massless spin-2 states interact more like gravitons. Motivated by this, Scherk and Schwarz in 1974 proposed that string theory should be viewed as a theory of quantum gravity [6], which necessarily means that the string length $\ell_s$ should be identified roughly with the Planck scale $\ell_p$. With such appreciation, string theory recaptured some (but still not many) people's attention. Later, it was found that for consistency and other reasons, one needs to introduce supersymmetry, leading to the construction of three different superstring theories: Type IIA, IIB and Type I strings, where the last two theories are chiral and it was believed they might be anomalous. It was until 1983 that Alvarez-Gaumé and Witten calculated 10D gravitational anomalies [7] and revealed that Type IIB anomalies are automatically cancelled. But Type I string (whose low-energy effective theory is a 10D $\mathcal{N} = 1$ supergravity), the only known one including gauge sector at that time, was believed to have local anomalies, which seems to indicate the inconsistency of the theory. However, Green and Schwarz discovered that the anomalies of the 10D Type I string theory could also be cancelled by introducing what became known as the Green-Schwarz mechanism [8]. Remarkably, the Green-Schwarz mechanism shows that both gauge groups $E_8 \times E_8$ and $SO(32)$ in a 10D $\mathcal{N} = (1,0)$ supergravity satisfy the anomalies cancellation. The UV completion of supergravity with $SO(32)$ can be realized by Type I string theory, but the one with $E_8 \times E_8$ did not have a known UV completion. Inspired by this, it was the "Princeton string quartet" of Gross, Harvey, Martinec and Rohm in 1985 who constructed two heterotic string theories embedding both of the consistent gauge groups [9].

Hence by 1985, people knew there are five different (perturbative) consistent superstring theories: Type IIA and Type IIB, Type I, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$ theory. This, however, does not fit with the aesthetic criterion of aiming for a unified theory. Remarkably, in 1995, Witten claimed that all these five theories should be understood as a different limits in the moduli space of one theory [10], known as M-theory later, and these five string theories could be related to each other by so-called dualities (cf. figure 0.1). The idea of the duality

[5]This was even before the Standard Model came to life.

originally did not arise from string theory as such but instead roots in quantum field theories. It states that two seemingly different theories could describe the physics of the same system within different regions in its moduli space. The celebrated example is electromagnetic duality. This fundamentally changed the appreciation of string theory and today we still rely on such an understanding, though we have not made much significant progress in describing the unified theory - M-theory. Another strikingly important discovery before the end of 1995 was made by Polchinski and his collaborators [11, 12] stating that by its very nature as a quantum gravity theory, superstring theory also includes dynamical extended brane objects[6], known as D-brane. This led to a revolution in the history of string theory. With such appreciation of D-branes, Strominger and Vafa in 1996 presented the first microscopic calculation of black hole entropy [13], which was a remarkable victory for string theory.

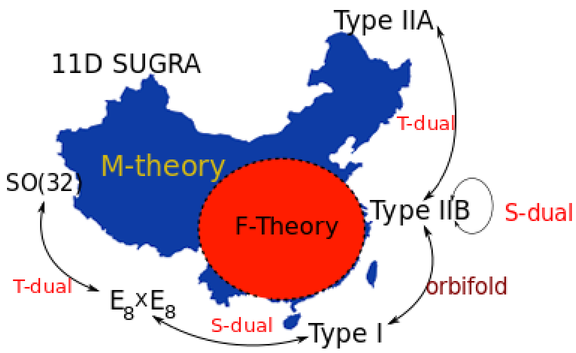


**Figure 0.1.:** *A schematic depiction of the relations amongst M-theory and its various perturbative limits. The blue area represents the (putative) underlying fundamental M-theory. The five different superstring theories at certain corners represent certain limits in the moduli space of M-theory. We also denote some (not all) of the dualities amongst the five superstring theories. In addition, we refer to the red area as F-theory, as an indication that F-theory provides a huge number of vacua if not all. There might be some string vacua not being realized by F-theory, indicated by that the read area does not cover the whole map.*

### 0.2.2. Superstring theory

One of the features of superstring theory is that, roughly speaking, its consistency conditions enforce the total dimension of the defining spacetime to be 10 (or 11 for M-theory). In order to connect to our four dimensions, one typical choice is that we require the extra 6-dimensional space, dubbed 'internal space', to be compact and extremely tiny[7] so that it cannot be probed in current experiments. This is the paradigm of string compactifications. The idea of compactifications, however, dates back to much earlier attempts to unify gravity and electromagnetism - the

[6]The fundamental objects in M-theory are not strings anymore, but are M2/M5-branes.

[7]According to general relativity spacetime is dynamical, in agreement with our current cosmological observation that our universe is expanding. In other words, our observed 3D space was much smaller at early in the history of the universe. Likewise, it is well conceivable that there are extra spacial dimensions that remain small even today.

Kaluza-Klein (KK) compactifications. In KK compactifications (with 5d spacetime $\mathbb{R}^{1,3} \times S^1$), the geometry of the extra dimension, i.e. the one-circle $S^1$ representing the "internal space", determines the physics in the 4D spacetime $R^{1,3}$. The relevant geometric data which (partially) control the 4D physics is the size of the $S^1$, i.e. the radius $R$, called a "modulus". Each modulus gives rise to a massless scalar field which freely propagates in 4D spacetime. To avoid constraints from fifth-force experiments, its mass is required to be stabilized at a certain non-zero value, the process of which is called moduli stabilization. In string compactifications, the internal space is 6-dimensional and its geometry is more involved (typically coming with lots of moduli) than the $S^1$ in the KK compactification. In the simplest configurations, the internal 6-dimensional spaces are required to be Calabi-Yau manifolds in order to preserve supersymmetry (for many good reasons such as solving the electroweak hierarchy problem we mentioned), which are complex spaces admitting a Ricci-flat metric.[8] Such Calabi-Yau manifolds provide solutions to Einstein gravity and define spacetime vacua for superstrings.

Prior to the discovery of D-branes, the old endeavor to connect the superstrings to our observed physics (i.e. SM+general relativity) was via heterotic $E_8 \times E_8$ string Calabi-Yau three-fold compactifications, which give rise to 4D $N = 1$ supergravity theories coupled to supersymmetric gauge theories. And at that time, people thought one only needs to find a typical Calabi-Yau manifold which gives rise to the same context of particles as Standard Model. Indeed, along this idea, Candelas, Horowitz, Strominger, and Witten in 1985 found that one particular Calabi-Yau space realized a 4D $\mathcal{N} = 1$ theory with promising number of chiral spectra. This can be viewed as one of the first endeavors, if not the first, to build a standard-like model, which is the main concern in string phenomenology. From such endeavors, one learned that the gauge couplings, parameters in the Standard Model, gauge structures in principle can be tuned by adjusting certain parameters characterizing internal spaces, i.e. Calabi-Yau spaces. Although Calabi-Yau spaces have special properties, it turns out one can construct lots of them and further each of them carry abundant moduli, and consequently, the variety of 4D possible physical theories[9] from string compactifications are huge. We call such enormous sets of possible models the "string landscape". However, a clear solution describing exactly our nature has not been obtained, especially there is no known reasons or a first principle to pick up such standard model-like from the string landscape. Furthermore, it turns out that in Heterotic strings, there is no completely satisfactory was to deal with the issue associated with moduli stabilization, especially for the bundle moduli (arise from the vector bundles for gauge groups). The stabilization of these bundle moduli turns out to involve a non-perturbative superpotential from world-sheet instantons [15], whose explicit form is generically impossible to write down unless a Calabi-Yau metric is explicitly provided (for recent developments, see e.g. [16]).

With the discovery of D-branes in type II strings, numerous additional types of string vacua were revealed. A Dp-brane can be understood as a $(p + 1)$-dimensional subspace where open strings end. From the viewpoint of the closed string sector, a D-brane is defined as a non-trivial higher-dimensional soliton solution of the effective 10/11D supergravity theory. Furthermore, the massless degrees of freedom of a D-brane define themselves a gauge theory on the world-volume of the D-brane (for more details, see section 1.5). Hence one can utilize this fact to build standard-like gauge theories (typically supersymmetric ones) from Type II strings. Further, the gauge sector is confined on the $(p+1)$-dimensional brane while the gravity sector is propagating along the whole 10 dimensions, which is exactly the idea of Large Extra Dimensions (for example

[8]Throughout this thesis, we focus on Calabi-Yau spaces as Kähler spaces.

[9]Note that not all of them are standard-like models. However, the number of standard-like model in string landscape is still huge, the recent result from F-theory compactification on the number of models with exact chiral spectra and gauge groups with standard-model is around $10^{15}$ [14].

the ADD model [17]) in a bid to solve the electroweak hierarch problem. For certain parameters (or rather values of the dynamical moduli fields) of the internal space the hierarchy problem can in principle be solved along these lines.

As we mentioned at the beginning, the preferred choice is a 4D $\mathcal{N} = 1$ theory with a chiral spectrum. One of the central properties of D-branes is that a vacuum state with a single D-brane is not annihilated by all the supercharges but only by half of them. If one wants to construct 4D $N = 1$ theories from type II strings, one option is to first reduce to a 4D $N = 2$ theort by Calabi-Yau compactification and then to add (spacetime filling) D-branes under certain conditions. However, adding D-branes on a compact manifold will in general render the theory inconsistent. Typically additional extended objects are required with charges opposite to those of the Dp-brane, known as Op-planes. Such Op=planes are the result of an orientifold projection on the Calabi-Yau spaces. We call them orientifold compactifications. One popular choice are Type IIB O3/O7 orientifold compactifications, which will be discussed in more detail in chapter 1.

Compared with Heterotic strings, Type II strings in principle have good prospects for the moduli stabilization via for example the KKLT mechanism [18] or the Large Volume String Scenario [19]. These benefits extend to Type IIB O3/O7 orientifold compactifications. However, there are certain caveats in such orientifold compactifications. On theoretical grounds, the main drawback of such compactifications is that they are treated at the level of a probe approximation to the extent that the important back-reaction of the branes on the geometry is neglected. As far as model building is concerned, especially for constructing GUTs (as a major paradigm for model building), the main drawback is that it takes certain non-perturbative effects to generate exceptional gauge degrees of freedom (while in the old endeavor through Heterotic string, exceptional gauge groups are provided in the first place), which are not well-controlled and typically requiring fine-tunings. Circumventing these two drawbacks properly leads us to F-theory, which we will introduced shortly.

Before we move to F-theory, we would like to mention other motivations for studying string theory. Apart from the fact that string theory provides a great opportunity to unify gravity and quantum theory, with an elegant structure that fits with our aesthetic criteria, lots of deep and intriguing ideas have emerged from string theory such as duality, emergence, geometrisation. These ideas not only help us deepen our understanding of string theory, but also affect other branches of modern physics such as condensed matter. And even beyond physics, string theory, as well as (supersymmetric) field theories inspired from string theory, provide great intuition and many guiding principles for numerous interesting areas in mathematics such as (holomorphic) mirror symmetry, knots, Langland correspondence, Donaldson invariants, quantum geometry, etc. And of course, in Calabi-Yau compactifications, as well as compactifications on manifolds with special holonomies, string theory provides lots of results enriching even mathematician's appreciation of these geometries.

## 0.3. F-theory

In 1996, Vafa introduced F-theory as a geometric formulation of Type IIB string theory which automatically incorporates the non-trivial profile of the axio-dilaton $\tau := C_0 + ig_s^{-1}$ in the presence of seven-branes [20]. Such a formulation involves necessarily strongly coupled type IIB theory, as the back-reaction of seven-branes generates a holomorphically varying profile of $\tau$ which inevitably attains large $g_s$ in certain regions. By noting that the axio-dilaton $\tau$ in the presence of seven-branes exhibits the same transformation as the complex structure moduli of a torus $T^2$ (or more technically, an elliptic curve), the main idea of such geometric formulation is to introduce an extra torus $T^2$ attached to each point in the 10D spacetime of type IIB. Its complex structure moduli $\tau$ encodes the axio-dilaton of Type IIB theory. Thus in F-theory, we have, formally, a 12-dimensional spacetime.[10] By allowing the axio-dilaton $\tau$ to vary over the type IIB spacetime, such a 12-dimensional spacetime attains the structure of an elliptic fibration, where the torus (or the elliptic curve) plays the role of the fiber. In order to obtain a low-dimensional supersymmetric theory from an F-theory compactification, the elliptic fibration should be an elliptically fibered Calabi-Yau manifold.

However, in some sense, a more accurate description of F-theory should involve M-theory through a T-duality over the fiber $T^2$ together with certain limits. Such duality naturally passes to F/M-theory compactifications. And more importantly, by going to the M-theory side, one can use well-studied tools such as geometric engineering to analyze various aspects of the effective theories from the compactifications, such as gauge structures, matter spectra and couplings even though F-theory typically involves a regime non-perturbative in the string coupling $g_s$. Remarkably, such information is almost entirely encoded in the geometry of elliptic fibrations and one can read off this crucial physical information by studying and analyzing these geometries with well-studied tools in the algebraic geometry. As summarized in the table 1.1 in [21], there is a clear dictionary between the physics of F-theory compactifications and the geometries of elliptic fibrations. By further studies, the lists in this dictionary would be expected to be refined and extended. Concerning model building, F-theory naturally incorporates exceptional gauge groups in typical regions of its moduli space, while at the same time it inherits beneficial properties of Type IIB theory, including its ingredients to address moduli stabilization, at least in principle. In some sense, F-theory thus inherits both the attractive properties of Heterotic string compactifications and Type II string compactifications. One can also relate F-theory to Heterotic $E_8 \times E_8$ theory by applying duality in certain cases, which we will not cover in this thesis. We will explain more details on the geometry and physics of F-theory compactification in chapter 2.

## 0.4. Anomalies in F-theory Compactifications

We have mentioned several aspects of anomalies. Here let us briefly introduce some basic facts. The term 'anomaly' describes the breakdown of a classical symmetry at the quantum level. To be precise, it refers to the phenomenon that a classical conserved current associated with a (continuous) symmetry of a classical theory may fail to be conserved when the theory is quantized. Depending on the type of symmetries, we speak of global symmetry anomalies and gauge symmetry anomalies. The famous example of a global symmetry anomaly is the chiral anomaly. The concept of the chiral anomaly dates back to the study of the triangle diagram for the decay $\pi^0 \to 2\gamma$, where it was found that gauge invariance of the amplitude is incompatible with the conservation of the axial current. Contrary to global anomalies, gauge anomalies have to be cancelled for consistency of theories. The study of anomalies has played an important role in field theories such as the Standard Model and GUTs, as it turns out that anomaly cancellation poses strong constraints on the content of the spectrum and is an efficient guiding principle for the construction of consistent models. The power and beauty of anomalies is that they have has both infrared (IR) and ultraviolet (UV) implications. The UV implications come from the fact that an anomaly appears in the process of regularizing UV quantities and represents the failure to be able to find consistent regulators in the UV. On the other hand, as t 'Hooft pointed out,

[10] As we will explain in chapter 3, this is not a fundamental 12-dimensional theory with signature $(1, 11)$.

an anomaly should most appropriately be interpreted as an infrared effect as it turns out that all the contributions to an anomaly come from the massless part of the spectrum of the theory, which only needs IR descriptions. This means that once an anomaly is present in a fundamental theory, it has to leave some imprints in the effective theory and vice versa. This led t 'Hooft to propose the idea of the t' Hooft anomaly.

From a mathematical viewpoint, chiral anomalies can be given a topological interpretation in terms of an index theorem. This relation to topology in fact pertains to all types of local anomalies: The reason is that all gauge anomalies - including gravitational anomalies - in $d = 2n$ dimensions can be related to chiral anomalies in $d = 2n + 2$ through the Stora-Zumino descent formalism.

In string theory, the study of anomalies provides even more valuable insights. We have briefly mentioned that 10D (supersymmetric) chiral theories are restricted as there are 3 possibilities: the $\mathcal{N} = (2, 0)$ type IIB supergravity and the $\mathcal{N} = (1, 0)$ supergravity with two gauge groups $E_8 \times E_8$ and $SO(32)$. Indeed, as we are going to discuss in more detail in the thesis, such strict restrictions on the possible form of the theory can even pass to six-dimensional effective theories. Furthermore, studying the anomaly structure in certain theories will shed more light on some crucial properties of the theories or their dual theories, especially for those strongly coupled systems which do not admit a weakly coupled Lagrangian description such as higher-dimensional SCFTs and their AdS duals. Indeed, recent studies of 6D SCFTs from F-theory compactifications (see e.g. [22]) and their further compactifications gained lots of information from corresponding anomaly polynomials, and leads to a lot of fruitful results.

In F-theory, anomaly cancellation bears additional benefits. As we mentioned above, many essential data of the effective theory of F-theory compactifications, such as gauge structures, the massless spectra and holomorphic couplings, are encoded in the F-theory geometry. Hence one would expect that the constraints from (local) anomaly cancellation can be written in terms of topological invariants or geometric relations on the underlying F-theory geometries. Such relations provide a great deal of intuition and serve as guidelines for mathematicians. Indeed, as we will show in chapter 4, some of the anomaly equations in F-theory compactifications on Calabi-Yau spaces $X_n, n = 3, 4, 5$ of various complex dimension $n$ manifest themselves as relations in the cohomology ring of the underlying Calabi-Yau $X_n$. These relations exhibit striking similarities for different dimension $n$.

## 0.5. A Short Summary of the Results

Motivated by the above, in this thesis, we investigate the consistency conditions, especially anomaly cancellations in F-theory compactifications.

In chapter 4, we provide closed expressions for the gravitational and gauge anomalies in 2D $N = (0, 2)$ compactifications of F-theory on elliptically fibered Calabi-Yau 5-folds. In particular, we have derived the Green-Schwarz counterterms for the cancellation of abelian gauge anomalies. The Green-Schwarz mechanism operates in a manner very similar to its 6D $N = (1, 0)$ cousin: Dimensional reduction of the self-dual Type IIB 4-form results in real chiral scalar fields whose axionic shift symmetries are gauged and whose Chern-Simons type couplings hence become anomalous. We have uplifted our results for the gauging and the couplings to an expression valid in the most general context of F-theory on elliptically fibered Calabi-Yau 5-folds. Anomaly cancellation in the 2D $(0, 2)$ supergravity is then equivalent to (4.61) for the gauge and (4.78) for the gravitational part. Each equation splits into a purely geometric and a flux dependent identity. These must hold separately on every elliptic Calabi-Yau 5-fold and for every consistent background of $G_4$ fluxes. We have verified this explicitly in a family of fibrations and for all vertical gauge fluxes thereon. As a spin-off of a systematical analysis of the 2D chiral spectrum, we also determine the anomalies from 3-7 sectors. Such anomalies can be reformulated as the t' Hooft anomalies from the perspective of strings from wrapping D3-branes when they intersect with 7-branes and play an important role also in understanding the role of (tensionless) strings in 6D N = (1,0) superconformal field theories when extrapolating the result to the 6D.

In chapter 5, after presenting the 6D $\mathcal{N} = (1,0)$ anomaly cancellations associated with continuous symmetry and their F-theory embeddings in the chapter 3, we study anomaly cancellation in the context of discrete gauge symmetries in 6D $\mathcal{N} = (1,0)$ theories from F-theory compactifications. As we discussed in 2.10, the type of fibrations giving rise to such discrete symmetries shall be the genus-one fibrations, which have not been studied very well compared to elliptically fibered Calabi-Yau. By applying similar ideas in chapter 3, we identify the correspondence between the geometry of the genus-one fibrations and the anomaly coefficients in the anomaly equations of discrete symmetries. Such results can shed more lights on our understandings of genus-one fibrations.

## 0.6. Outline of the Thesis

This thesis is organized as follows: Part II provides some of the foundations of Type IIB orientifold compactifications and F-theory which are needed to explain our new results in a self-contained manner. Our own work is then presented in Part III.

In Part II, we will first give an overview of string theory in chapter 1, focusing in particular on the massless spectrum. After introducing the massless spectra, we turn to 10D supergravity theories and the cancellation of their anomalies by introducing the 10D Green-Schwarz mechanism. The latter serves as the prototype for anomaly cancellation also in lower dimensions. We will then provide a detailed review of D-branes and their associated physics. Later we will shift to Type II string compactifications, focusing on various aspects of Type IIB orientifold compactifications as needed in the context of chapters 4 and 5.

In chapter 2, we discuss in detail F-theory compactifications and their most important physical aspects such as their low-energy limit, the appearance of non-abelian gauge groups, matter spectra, abelian gauge groups, discrete symmetry and the description of the flux sector, together with the corresponding geometric description of elliptically fibered Calabi-Yau spaces. These ingredients will be heavily relied on in the following chapters.

In chapter 3, we are going to present the basic features of 6D $\mathcal{N} = (1,0)$ supergravity and the cancellation of its anomalies, as a basis for the developments both in section 4 and 5. We will argue that anomaly cancellation imposes strong constraints on the massless spectra and gauge structures of the theories. Finally we will embed such theories into F-theory compactifications and find the corresponding geometric terms for the cancellation of anomalies.

In chapter 4, we present our own work on elliptically fibered Calabi-Yau five-manifold compactifications and establish a closed expression for the complete gauge and gravitational anomalies of a 2D $\mathcal{N} = (0,2)$ theory obtained from F-theory compactification. In order to derive the Green-Schwarz term, we will use the Type IIB orientifold picture and extrapolate these results to general F-theory compactifications. This chapter closely follows our publication [23].

In chapter 5, after presenting anomaly cancellation associated with continuous symmetries in 6D $\mathcal{N} = (1,0)$ theories and their F-theory correspondence, we will focus on discrete symmetries and their anomaly cancellation. Such discrete symmetries are viewed as part of continuous $U(1)$ symmetries in the UV which is broken at a certain scale with a remaining discrete part in the effective theory. The anomalies of the latter hence determine the anomalies of the remaining discrete symmetry at low energies, depending on the specifics of how the symmetry is broken. We find proper anomaly equations for the discrete symmetry and relate the anomaly coefficient to the geometry of genus-one fibrations on which F-theory is compactified.

In the appendix, we list the necessary notations and collect some mathematical facts relevant for our discussion in the main text.

# Part II.

# Preliminaries

# Chapter 1.

# Type II String Theory and Compactifications

In this chapter, we review the relevant aspects of superstring theory and its compactifications. In particular, the section 1.1 gives a short overview on perturbative string theory and its massless spectrum. Then we introduce supergravity, as the low-energy description of superstring theory. Some 10D supergravity theories are chiral and exist possible anomalies, thereby we analyze the relevant anomalies structures and introduced the 10D Green-Schwarz mechanism to cancel the gauge and gravitational anomaly for 10D $N = 1$ supergravity. In particular, we discuss string universality which holds in ten dimension. In section 1.5, we introduce D-branes and gauge theories in the closed Type IIB superstring theory. Calabi-Yau compactification and orientifold compactification are then followed, particularly, we focus on the Type IIB orientifold compactification with O7/O3-planes. To that end, we analyze the massless spectrum of the intersecting space-filling D7-branes and relevant consistent conditions.

## 1.1. Introduction to Perturbative String Theory

To make the context self-contained, we are starting with a lightning review of some basic aspects of perturbative bosonic string and superstring theory, focusing on the massless spectrum of the strings. The material of this section can be found in any standard textbook on string theory.

### 1.1.1. From the point particle to extended objects

The Lagrangian of (perturbative) string theory is very simple, it replaces the length of the world-line of a particle by the area of a string world-sheet. However, it turns out this simple action has far-reaching consequences, as we are going to elaborate.

To be more precise, the propagation of a string in $D$-dimensional spacetime $\mathcal{M}$ sweeps out a 2-dimensional world-sheet (WS) $\Sigma(\tau, \sigma)$ with $\tau$ being the time-like coordinate, and $\sigma$ the space-like coordinate, which topologically speaking, could either be an infinitely strip corresponding to an open string or an infinite cylinder for an closed string. For closed string, we then have $\sigma \in [0, 2\pi)$, while for the open strings, we have $\sigma \in (0, \pi)$. For convenience, we will package them together as $\xi^a = (\tau, \sigma), a = 0, 1$ in the following.

Further, we can define a map from the world-sheet $\xi^a$ to spacetime $\mathcal{M}$ as

$$\Sigma : (\tau, \sigma) \to X^\mu(\tau, \sigma) \in \mathcal{M} \tag{1.1}$$

where $X^\mu$, as a field on the world-sheet $\Sigma$, defines an embedding of $\Sigma$ into the D-dimensional spacetime $\mathcal{M}$.

The string action, dubbed Nambu-Goto action, reads

$$S_{NG} = -T_s \int_\Sigma dA \tag{1.2}$$

where the area element $dA$ of the WS $\Sigma$ is

$$dA = \sqrt{-det(\gamma) d^2\xi}, \qquad \gamma_{ab} := \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu}. \tag{1.3}$$

The induced metric $\gamma_{ab}$ is the pull-back of the spacetime $\mathcal{M}$ metric $g_{\mu\nu}$ onto $\Sigma$. Here $T_s$ denotes the string tension, meaning the mass/energy per unit length, which has mass dimension $[\text{mass}]^2$. For history reason, we define the string tension $T_s$ as

$$\boxed{T_s = \frac{1}{2\pi\alpha'}, \qquad \alpha' = \ell_s^2} \tag{1.4}$$

where $\ell_s$ refers to the string's intrinsic length and is the only parameter in string theory. And $\alpha'$ is dubbed universal Regge slope. In order to render string theory capable of describing quantum gravity, then one has $l_s \leqslant [10^{-33}]$cm [1].

**The Polyakov Action** The square-root in the Nambu-Goto action renders the theory difficult to quantize via path integral techniques. However, it turns out there is an equivalent (at the classical level) string action, which eliminates the square root at the expense of introducing the auxiliary field $h_{ab}$. This was first introduced by Polyakov, and hence dubbed Polyakov action

$$S_P = -\frac{T_s}{2}\int_\Sigma d^2\xi \sqrt{-h} h^{ab}\partial_a X^\mu \partial_b X_\nu g_{\mu\nu} \tag{1.5}$$

where $h = \det h_{ab}$. The new field $h_{ab}$ is now treated as a dynamic metric on the WS $\Sigma$ [2]. The equation of motion (E.O.M) of the metric $h^{ab}$ gives rise to the vanishing energy-momentum tensor $T_{ab} := \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} = 0$. One can use this constraint to eliminate the auxiliary metric $h_{ab}$ to recover the Nambu-Goto action. The Polyakov action, is nothing other than the non-linear sigma model and hence we will call the spacetime $\mathcal{M}$ the target space.

The Polyakov action, as a two-dimensional gravity theory, enjoys two important world-sheet local symmetries:

- local diffeomorphisms: For any reparametrization $\xi^a \to \widetilde{\xi}^a$, the theory is invariant under

$$h_{ab} \to \widetilde{h}_{ab} = \frac{\partial \xi^c}{\partial \widetilde{\xi}^a}\frac{\partial \xi^d}{\partial \widetilde{\xi}^b} h_{cd}, \qquad X^\mu(\xi) \to \widetilde{X^\mu}(\xi). \tag{1.8}$$

- Weyl rescaling:

$$X^\mu \to X^\mu, \qquad h_{ab} \to \widetilde{h}_{ab} = e^{2\Lambda} h_{ab}. \tag{1.9}$$

[1] The exact value of the string length also depends on many other aspects, especially the string coupling $g_s$ and the volume of the extra dimensions. But since we view string theory as a fundamental theory of quantum gravity, the string length should be super, super tiny, around the order of the 4D Planck scale hence $l_s \leqslant [10^{-33}]$cm

[2] As a side remark, the above can also be naturally generalized to any $p$-dimensional spatially extended object, with the Nambu-Gotto action in a flat spacetime given by

$$S_p = -T_p \int \sqrt{h} d^{p+1}\xi \tag{1.6}$$

And the equivalent covariant Polyakov action yields

$$S_p = -\frac{T_s}{2}\int d^{p+1}\xi \sqrt{h}\{h^{ab}\partial_a X^\mu \partial_b X_\nu g_{\mu\nu} - (p-1)\}. \tag{1.7}$$

Using these two gauge freedoms, one can fix the metric $h_{\alpha\beta}$ completely and it reduces into the flat form

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.10}$$

In this conformal gauge fixing, the Polyakov action reduces to

$$S = -\frac{T_s}{2} \int d^2\xi \eta^{ab} \partial_a X^\mu \partial_b X_\nu. \tag{1.11}$$

From now on, we focus on the cases that the spacetime/target space $\mathcal{M}$ is $D$ dimensional Minkowski space $\mathbb{R}^{1,D-1}$ [3] . Introducing the light-cone coordinates in the world-sheet as $(\xi^\pm = \tau \pm \sigma, \partial_\pm = \frac{\partial}{\partial \xi^\pm})$, the classical E.O.M for $X^\mu$ can be solved as

$$\partial_+ \partial_- X^\mu = 0 \tag{1.12}$$

where further $X$ can be decomposes into left- and right-movers:

$$X^\mu(\xi_\pm) = X_L^\mu(\xi^+) + X_R^\mu(\xi^-). \tag{1.13}$$

**Boundary conditions** The open string has two boundary conditions, known as Neumann and Dirichlet conditions:

$$\begin{aligned} \text{Neumann} &: \partial_\sigma X^\mu|_{\sigma=0,\pi} = 0; \\ \text{Dirichlet} &: \partial_\tau X^\mu|_{\sigma=0,\pi} = 0; \end{aligned} \tag{1.14}$$

As a trained physicist, it is not hard to figure out that the Dirichlet condition, fixing the endpoints of the string, leads to breaking of Poincaré invariance which made people ignore the Dirichlet condition at the early days. We will see later that this boundary condition will lead to the appreciation of D-branes, which are the main topic in this thesis. For the time being, however, let us merely focus on the Neumann boundary condition and its solution.

Hence for the Neumann boundary condition, the general solution of the **open string** can be obtained by modes expansion, which yields

$$X^\mu(\tau, \sigma) = X_0^\mu + 2\alpha' P_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} a_n^\mu e^{-in\tau} cos(n\sigma) \tag{1.15}$$

where the fist two terms describe the motion of the open string center of mass, while the remaining terms correspond to string oscillations/vabriations. Reality of the $X^\mu$ implies $(a_n^\mu)^* = a_{-n}^\mu$.

For **closed strings**, which have the periodicity condition

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi), \tag{1.16}$$

[3] The first step to quantize a (field) theory, one should choose a "vacuum configuration". The Minkowski space $\mathbb{R}^{1,D-1}$ is the simplest "vacuum configuration," from the target spacetime perspective, as it is Ricci-flat. Indeed, as it turns out, one of the classical equations of motion from the spacetime theory is the Ricci-flatness condition on the spacetime metric $g_{\mu\nu}$.

one has the similar modes expansions

$$\begin{aligned} X^\mu(\tau,\sigma) =& X_L^\mu(\xi^+) + X_R^\mu(\xi^-), \\ X_L^\mu(\xi^+) =& \frac{1}{2}X_0^\mu + P_0^\mu\tau + i\sqrt{\alpha'/2}\sum_{n\neq 0}\frac{1}{n}\widetilde{a}_n^\mu e^{-2in\xi^+}, \\ X_R^\mu(\xi^-) =& \frac{1}{2}X_0^\mu + P_0^\mu\tau + i\sqrt{\alpha'/2}\sum_{n\neq 0}\frac{1}{n}a_n^\mu e^{-2in\xi^-}. \end{aligned} \tag{1.17}$$

Similarly, the reality of the $X^\mu$ implies $(a_n^\mu)^* = a_{-n}^\mu$ and $(\widetilde{a}_n^\mu)^* = \widetilde{a}_{-n}^\mu$. Note that the right- and left-mover circulate the string in opposite directions.

**Mass-shell constraints** The above classical solutions should satisfy the physical constraints of a vanishing energy-momentum tensor $T_{ab} = 0$. In light-cone coordinates, $T_{ab}$ has two non-trivial components $T_{++}, T_{--}$, which in the closed string are given by

$$T_{++}(\xi^+) = \sum_{n=-\infty}^{+\infty} \widetilde{L}_n e^{2in\xi^+}, \qquad T_{--}(\xi^+) = \sum_{n=-\infty}^{+\infty} L_n e^{2in\xi^-} \tag{1.18}$$

where the coefficients are the so-called $Virasoro$ generators

$$\boxed{\widetilde{L}_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty}\widetilde{a}_{m-n}\cdot\widetilde{a}_n, \qquad L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} a_{m-n}\cdot a_n} \tag{1.19}$$

which we further define $\widetilde{a}_0 = a_0 = \sqrt{\frac{\alpha'}{2}}P_0^\mu$.

Hence the vanishing of the $T_{ab}$ translates into the infinite number of constraints

$$\widetilde{L}_n = L_n = 0, \qquad \forall n \in \mathbb{Z}. \tag{1.20}$$

This fact is related to that the Polyakov action enjoying conformal symmetry (see discussion below), which in 2D gives infinite constraints. One can further check the $Virasoro$ generators satisfy the Possion bracket algebra classically

$$[L_m, L_n]|_{P.B} = i(m-n)L_{m+n}, \tag{1.21}$$

together with their tilded parts.

For the open string, we can have the $Virasoro$ generators in the same way as

$$\boxed{L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} a_{m-n}\cdot a_n} \tag{1.22}$$

where now $a_0 = \sqrt{2\alpha'}P_0^\mu$.

The important relevant fact for us is that the mass-shell condition can be derived from one of the above constraints, specially $L_0 = 0$. Namely, for open string we have

$$L_0 = \sum_{n=1}^{\infty} a_{-n}\cdot a_n + \frac{1}{2}a_0^2 = \sum_{n=1}^{\infty}\alpha_{-n}\cdot a_n + \alpha' P_0^2 = 0. \tag{1.23}$$

Hence we have

$$\boxed{M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} a_{-n}\cdot a_n} \tag{1.24}$$

In the same way, we have for the closed string

$$\boxed{M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty} a_{-n}\cdot a_n + \widetilde{a}_{-n}\cdot \widetilde{a}_n} \tag{1.25}$$

**Free string spectrum** Now we are at the position to deduce the string spectrum. Applying the canonical quantization procedure

$$[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = 2\pi i\delta(\sigma-\sigma')\eta^{\mu\nu}, \tag{1.26}$$

the non-vanishing commutations relation for the open string can be obtained as

$$[a^\mu_m, a^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [X^\mu_0, P^\nu] = i\eta^{\mu\nu}. \tag{1.27}$$

In the same way, we can obtain similar commutations for each oscillations $a$ and $\widetilde{a}$ of left- and right-movers of the closed string.

With the commutations (1.27), one can identify $a^\mu_n$ with $n>0$ as the annihilation operators and $a^\mu_{-n}$s as creation operators. Without lose of generality, let us first focus on the open string cases. As the zero mode operator $X^\mu_0$ and $P^\mu_0$ satisfy the standard Heisenberg relation, one can build the Hilbert space spanned by the usual plane wave basis $|p\rangle = e^{ip\cdot x}$ of eigenstates of $P^\mu_0$. The vacuum thereby can be defined as $|p,0\rangle$ such that

$$a^\mu_n|p,0\rangle = 0, \qquad P^\mu_0|p,o\rangle = p^\mu|p\rangle, \forall n>0. \tag{1.28}$$

We define the "physical states" $|\text{phys}\rangle$ of the full Hilbert space such that they satisfy the Virasoro constraints $T_{ab}=0$:

$$[(L_0-c)|\text{phys}\rangle = 0]\cup[L_n|\text{phys}\rangle = 0], \qquad n>0. \tag{1.29}$$

where now at the quantum level, the Virasoro algebra should be defined with normal ordering as

$$\begin{aligned} L_m &:= \frac{1}{2}\sum_{n\in\mathbb{Z}} a_{m-n}\cdot a_n = 0, \qquad \forall m>0 \\ L_0 &:= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_{-n}\cdot a_n. \end{aligned} \tag{1.30}$$

The appearance of constant $c$ denotes the effect of normal ordering, which is essential due to the Casimir effect and is given by $c = \frac{D-2}{2}\sum_{n=1}^{\infty} n = \frac{D-2}{24}$. Without giving further discussions, we would also like to point out that the value of $c = \frac{D-2}{24}$ can also be obtained by Weyl anomaly cancellation.

One obtains the mass-shell condition from the constraint $L_0 = c$ as

$$m^2 = -P_0^2 = \frac{1}{\alpha'}(N-c). \tag{1.31}$$

where $N := \sum_{n=1}^{\infty} a_{-n} \cdot a_n$ denotes the level number.

One more physical constraint should be mentioned before employing the standard procedure of constructing the Fock space by the raising operator $a^{\mu}_{-n}$. There are potential states of negative norm due to the wrong sign $\eta_{00} = -1$ associated with commutant algebra along the time direction, i.e.$[a^0_m, (a^0_m)^{\dagger}] = -1$, which can lead to a non-unitary theory. In order to eliminant these negative norm states, one should invoke the light-cone gauge. As a result, we are only left with the transverse physical raising operator $a^i_{-n}, i = 1, ..., D-2$.

With this, one can obtain the spectrum at each exicted level $N$ by acting the physical raising operators $a^i_{-n}, i = 1, ..., D-2$ on the vacuum as

$$\begin{aligned}
&N = 0, |p\rangle, \qquad -\frac{1}{4}p^2 = -1, \qquad \text{tachyon},\\
&N = 1, a^i_{-1}|p\rangle, \qquad -\frac{1}{4}p^2 = 1 - c, \qquad \text{vector},\\
&N = 2, a^i_{-1}a^j_{-1}|p\rangle \cup a^i_{-2}|p\rangle, \qquad -\frac{1}{4}p^2 = 2 - c, \qquad \text{spin 2},
\end{aligned} \tag{1.32}$$

where the ground state gives rise to a tachyon, which signals that the bosonic string is not a consistent quantum theory as the vacuum is not stable. This problem will be remedied by the superstring later, so let us just carry on without worrying too much about it.

The first excited level $N = 1$ gives rise to a vector in the spacetime, which would carry a polarization $\zeta_\mu$. It turns out that the polarization and the momentum must obey $p \cdot \zeta = 0$ in order to preserve Lorentz invariance of the spacetime. In other words, this vector fields should be massless. This sets $c = 1$. Combining with the condition for the absence of Weyl anomaly $c = \frac{D-2}{24}$, the dimension of space time hence is fixed to be

$$\boxed{D = 26}. \tag{1.33}$$

The second excited level $N = 2$ has a total of $324 = 24 + 24 \cdot 25/2$ degrees of freedoms, which together can be packaged into the symmetric, traceless second-rank tensor representation of $SO(24)$. It is the massive spin 2 state.

The appearance of a tachyon in the spectrum makes string theory less appealing, which needs to be solved by other means. On the other hand, the massless vectors give some hopes for describing gauge theories. However, a massless vector is not automatically a gauge field, unless it transforms under certain gauge group. Indeed, it turns out there is, roughly speaking, one extra freedom to introduce additional quantum numbers associated to the ends of open strings and hence the vacuum can be denoted as $\lambda^a_{mn}|p, mn\rangle$, where the indices $\lambda^a_{mn}$ are so-called Chan-Paton labels living on the two ends of a open string and $m, n = 1, ..., N$ represents the $N$ choices for each endpoint. The $\lambda^a_{mn}$ can be represented by matrices that satisfy a Lie algebra as a symmetry group of string interactions. Due to this, it can define a gauge group and the massless states are now gauge fields $A^i := a^i_{-1}\lambda^a_{mn}|p, mn\rangle$, which can form antisymmetric, symmetric or complex representations corresponding to orthogonal $SO(N)$, symplectic $Sp(N)$ or unitary gauge groups $U(N)$, respectively.

The most fascinating aspects of bosonic strings comes from the spectrum of the closed strings. The spectrum of closed strings can be obtained by the direct product of states from left- and right-movers with the condition $M_L^2 = M_R^2$. Apart from the annoying tachyons, at the massless level, one thus has

$$a^i_{-1}\widetilde{a}^j_{-1}|p>, \qquad (i, j) \in 1, ..., 24 \tag{1.34}$$

It can be decomposed as

$$\boxed{\begin{aligned}\mathbf{24_v}\otimes\mathbf{24_v}&=\mathbf{1}\otimes\mathbf{276}_A\otimes\mathbf{300}_S\\&=\phi\otimes B_{\mu\nu}\otimes g_{\mu\nu}\end{aligned}} \tag{1.35}$$

where the indices $A$ and $S$ denote the antisymmetric and symmetric 2-form. This contains a spin 2 graviton $g_{\mu\nu}$, a scalar dilaton $\phi$, and a 2-form antisymmetric tensor $B_{\mu\nu}$.

The presence of spin 2 graviton in the closed string makes string theory a candidate for the theory of gravity. However, for the open strings, there are no gravitons. But it turns out in the presence of interactions, an open string theory will automatically require the presence of closed strings by unitarity and hence embody the gravitons, which further make string theory as a candidate for unification of gauge and gravity theory.

**2D Conformal field theory** So far we have determined the spectrum of the bosonic string in flat spacetime $R^{1,25}$. How about the cases in generic background? How to determine them? Well it turns out that the 2D Polyakov action for the bosonic strings describes a 2D conformal theory. To see this, one should notice this gauge-fixing does not eliminate the complete local symmetry, there are still residual symmetries, which is a conformal symmetry. To see this, after fixing the $h_{ab}$ to the flat one, one can always act with a reparametrization $\xi^a \to \widetilde{\xi}^a$ such that

$$\frac{\partial\xi^c}{\partial\widetilde{\xi}^a}\frac{\partial\xi^d}{\partial\widetilde{\xi}^b}\eta_{cd} := \Lambda(\xi)\eta_{ab} \tag{1.36}$$

and come back to the fixed flat metric by a corresponding Weyl rescaling. The above reparametrization is exactly the conformal transformation! Hence the 2D Polyakov action reduces to a conformally invariant theory when background metric $g_{\alpha\beta}$ is fixed. Similarly, any conformal theory can give rise to a classical theory which enjoys both diffeomorphism and Weyl invariance when it couples to 4D gravity.

## 1.2. Superstring Theory

As we discussed, there are two big problems associated with the bosonic string if it is meant to describe our world. The first one is the presence of tachyons in the spectrum, both for closed string and open strings. The other is that there is no spacetime fermions. This is where superstring theory is motivated and cames to the rescue which is obtained by adding a fermionic part on the world-sheet.

**The RNS formalism of superstrings** The world-sheet action in the conformal gauge $h_{ab} = \eta_{ab}$ takes the form

$$S = -\frac{T_s}{2}\int d^2\xi(\partial X^\mu\bar{\partial}X_\mu - i\bar{\psi}^\mu\rho^a\partial_a\psi_\mu) \tag{1.37}$$

where $\rho^a$ is the 2D Dirac matrices defined as

$$\rho^0 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \rho^1 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{1.38}$$

and hence $\psi^\mu$, the 2D world-sheet Majorana fermion with a two-component spinor, is given by

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} \tag{1.39}$$

with the reality conditions $\psi_\pm^* = \psi_\pm$. These two components are also Weyl spinors satisfying the massless Dirac equation

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0. \tag{1.40}$$

Here we are still focusing on the flat Minkowski spacetime $R^{1,D-1}$. The bosonic parts are same to the one discussed in the bosonic strings. Note that the world-sheet Majorana-Weyl spinor $\psi_\pm^\mu$ are vector fields in the spacetime just as the scalar fields $X^\mu$. Further, with this action, they enjoy the world-sheet 2D $\mathcal{N} = (2,2)$ supersymmetry for closed superstrings. For details on the supersymmetric transformations we refer to the standard textbooks on string theory.

The relevant informations we would like to obtain are the massless spectra. To this end, we shall perform the same mode decomposition as for the bosonic field $X^\mu$ before. To do that, we first notice there are also two boundary conditions: Ramond(R) and Neveu-Schwarz(NS) for the WS fermions $\psi_\pm^\mu$ in the open superstrings, which are given by

$$\begin{aligned} R: \quad & \psi_+^\mu(\tau, \sigma = 0) = \psi_-^\mu(\tau, \sigma = 0), \qquad \psi_+^\mu(\tau, \sigma = \pi) = \psi_-^\mu(\tau, \sigma = \pi), \\ NS: \quad & \psi_+^\mu(\tau, \sigma = 0) = \psi_-^\mu(\tau, \sigma = 0), \qquad \psi_+^\mu(\tau, \sigma = \pi) = -\psi_-^\mu(\tau, \sigma = \pi). \end{aligned} \tag{1.41}$$

Given these boundary conditions, following the same procedures shown in the bosonic strings, we first obtain the general solutions for the 2D fermions in the open superstrings by mode decomposition as

$$\psi_\pm^\mu = \frac{1}{\sqrt{2}} \sum_r b_r^\mu e^{-ir(\tau \pm \sigma)}, \qquad (b_r^\mu)* = b_{-r}^\mu \tag{1.42}$$

where $r$ is half-integer for NS sector and integer for R sector.

In the same way, we can obtain similar mode expansions in the closed string sector, but now with independent boundary conditions for the left-mover $\psi_+^\mu$ and the right-mover$\psi_-^\mu$ as

$$\psi_+^\mu = \sum_r \widetilde{b}_r^\mu e^{-2ir(\tau+\sigma)}; \qquad \psi_-^\mu = \sum_r b_r^\mu e^{-2ir(\tau-\sigma)} \tag{1.43}$$

where for each of $\widetilde{b}_r, b_r$ one can assign half-integers and integers for $r$ corresponding to NS-sectors and R-sectors. As we shall see later, the R-sector will give rise to spacetime fermions whereas the NS-sector yields spacetime bosons. Depending on different pairing for $\psi_+, \psi_-$ we can group four closed string sectors depending on their spacetime states as:

- Bosons: NS-NS and R-R
- Fermions: NS-R and R-NS

The $Virasoro$ algebras in the superstring have been extended to $super-Virasoro$ as

$$\begin{aligned} L_n =& \frac{1}{2} \sum_{m=-\infty}^{\infty} a_{n-m} \cdot a_m + \frac{1}{4} \sum_r (2r - n) b_{n-r} \cdot b_r \\ G_r =& \sum_{m=-\infty}^{\infty} a_m \cdot b_{r-m} \end{aligned} \tag{1.44}$$

where $G_r$'s are the modes of the supercurrent. Together with the tilded part for the closed superstrings. The vanishing of the energy-momentum tensor and the supercurrent impose the constraint $L_n = G_r = 0, \forall(n,r)$, which again is infinite number. This actually reflects that the 2D RNS action of the superstring is a 2D $\mathcal{N} = (2,2)$ superconformal theory.

**The superstring spectrum** Let us first focus on the case of the open superstrings. Applying the standard canonical quantization, the canonical anti-commutator relations for the WS fermion generators are given by

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}. \tag{1.45}$$

Given this, we can interpret the $b_r^\mu$ with $r < 0$ as the raising operators and ones with $r > 0$ as lowering operators. Note that only R-sector gives a zero mode $b_0^\mu$. The full Hilbert space is the free tensor product of the bosonic and the fermionic spaces.

By the same token in the bosonic strings, one can obtain the mass formula from the super Virasor algebra constraints, i.e. $(L_0 - c_s|\text{phys}\rangle = 0)$, which reads

$$M^2 = \frac{1}{\alpha'}(N + N_\psi) - c_s \tag{1.46}$$

where $N_\psi = \sum_{r>0} r b_{-r} \cdot b_r$. The normal ordering constant $c_s$ turns out to be 1/2 for the NS sector and $c_s = 0$ for the R sector as the fermionic modes $b_r$ contribute. The critical dimension $D$ of the superstring can be shown to be

$$\boxed{D = 10} \tag{1.47}$$

as a consequence of being a consistent theory.

In the sequel, we only focus on analyzing the spectrum of the fermionic part, as the bosonic part essentially is as the one in the pervious section.

**NS sector**: In the NS sector, the ground state is still a **tachyon** with the mass given by $-\frac{1}{4}p^2 = -\frac{1}{2}$ based on (1.46) with $N = N_\phi = 0$. At the first excited level, there is a massless vector

$$|\mu, p\rangle = b_{-1/2}^i |p\rangle, i = 1, ..., 8. \tag{1.48}$$

Here in order to manifest the unitarity, we have employed the light-cone gauge, where only the transverse components of raising operators $b_{-1/2}^i, i = 1, .., 8$ are physical. Thereby $b_{-1/2}^i|p\rangle$ transforms as the vector representation of $SO(8)$ denoted as $\mathbf{8_v}$.

**R sector**: In the R sector, the ground state in the R sector is massless and given by the solution

$$a_n^i |p; 0\rangle_R = b_r^i |p; 0\rangle_R = 0, \qquad (n,r) > 0, \tag{1.49}$$

together with the massless Dirac equation. Further, there is a zero mode $b_0^\mu$ which satisfies the aniti-commutantor algebra, generating the 10D Clifford algebra

$$\{b_0^\mu, b_0^\nu\} \sim \eta^{\mu\nu}. \tag{1.50}$$

Thus the $b_0^\mu$'s shall be viewed as 11D Dirac matrices. Applying this zero mode on the ground state $|p; 0\rangle_R$, one can see that the ground state gives rise to a massless $Spin(1,9)$ spinor. It turns out all states in the R sector are spacetime fermions. In 10D, one can impose both the Majorana and Weyl conditions at the same time for massless spinor, so that the ground state $|p; 0\rangle_R$ can be chosen to have a definite spacetime chirality. It turns out that there are two possible chiralities associated with little group $SO(8)$, labeled as $\mathbf{8_s}$ and $\mathbf{8_c}$.

It seems that the tachyons still remain in the spectrum of NS sectors in the superstring theories. However, it turns out that the above spectrum is not consistent with world-sheet modular invariance, which is required for eliminating the absence of global anomalies under 2D large diffeomorphisms disconnected from the identity in some topologically non-trivial Riemann surfaces. This consistency, on the other hand, implies that one should impose the GSO projection

$$P_{GSO} = \frac{1}{2}(1 - (-1)^F) \tag{1.51}$$

where $F = \sum_{r>0} b_{-r} \cdot b_r$ is the fermion number operator, satisfying $\{(-1)^F, \psi^\mu\} = 0$. The GSO projection turns out to eliminate the tachyons from the NS sectors while in the R sector it acts as spacetime chirality. It follows that at the massless level of open superstring there is one massless vector and one Weyl Majorana spinor, both having 8 degrees of freedom. This is a consequence of a resulting spacetime supersymmetry.

Now we are at the position to talk about the massless spectrum of the closed superstring, known as Type II superstrings [4]. The spectrum can be obtained by taking tensor products of left- and right-movers, each of which can be viewed as the open superstring described as above and hence there are four possible sectors: R-R, NS-NS and R-NS, NS-R. It turns out there are two different theories of closed superstrings, dubbed as Type IIA and Type IIB, depending on chiralities of left- and right- movers. Type IIA have opposite chralities for the R-sectors of the two independent movers wheras type IIB has same chirality.

### 1.2.1. Massless spectrum of Type IIA superstring

The massless spectrum of type IIA superstring is obtained by the tensor product

$$(\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}), \tag{1.52}$$

where $\mathbf{8_v}$ denotes the NS sector and $\mathbf{8_s}$ and $\mathbf{8_c}$ are the two representations of Majorana-Weyl spinors for the R sector with different chiralities.

In the NS-NS sector, the massless spectra are given by

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28}_A \oplus \mathbf{35}_S = \Phi \oplus B_{\mu\nu} \oplus g_{\mu\nu}, \tag{1.53}$$

which contains the spacetime metric $g_{\mu\nu}$, the dilaton $\phi$ and a 2-form Kalb-Ramond potential $B_2$. In the RR sector, the Type IIA spectra are given by

$$\mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8} \oplus \mathbf{56} = C_1 \oplus C_3, \tag{1.54}$$

which gives rise to the generalized $p$-form potentials $C_i$ for $i = 1, 3$.

The Fermions, on the other hand, are given by two NS-R sectors

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8}_R \oplus \mathbf{56}_L, \\ \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8}_L \oplus \mathbf{56}_R. \end{aligned} \tag{1.55}$$

Indeed, the fermions contains the two 10D Majorana-Weyl spinors $\zeta_\alpha$ and Weyl gravitino $\psi_{\mu\alpha}$ of opposite chirality.

[4] By closed strings, we really means that only the closed string can propagates on the bulk spacetime. In Type II strings, open strings do exist, but they only propagate on the defects of the bulk, i.e. D-branes.

The effective theories describing the massless spectrum of superstrings are supergravities. The type IIA supergravity has $N = 2$ supersymmetry, and the two supersymmetries are of opposite chirality generated by $Q^1_\alpha \in 16$ and $Q^2_\alpha \in 16'$.

### 1.2.2. Massless of the Type IIB superstrings

The massless spectrum of type IIB superstring is encoded in the tensor product of various representations of $SO(8)$:

$$(\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}). \tag{1.56}$$

Similarly, the bosonic spectrum is given by the NS-NS sector and the R-R sector. In the NS-NS sector, the massless spectrum is same as in Type IIA, given by

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \phi \oplus B_{\mu\nu} \oplus g_{\mu\nu}, \tag{1.57}$$

which contains the spacetime metric $g_{\mu\nu}$, the dilaton $\phi$ and a 2-form Kalb-Ramond potential $B_2$. In the RR sector, the Type IIB spectrum is

$$\mathbf{8_s} \otimes \mathbf{8_s} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ = C_0 \oplus C_2 \oplus C_4^+, \tag{1.58}$$

which gives rise to the generalized $p$-form potential fields $C_i$ for $i = 0, 2, 4$. The " + " on $C_4$ denotes that $C_4$ is self-dual.

The fermions are given by two NS-R sectors

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8}_R \oplus \mathbf{56}_L, \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8}_R \oplus \mathbf{56}_L, \end{aligned} \tag{1.59}$$

which contains the two 10D Left-handed Majorana-Weyl spinors $\zeta_\alpha$ and right-handed Weyl gravitino $\psi_{\mu\alpha}$.

Type IIB string theory can be well described by the ten dimensional type IIB supergravity at low energy, which also has $N = 2$ supersymmetries, and the two supersymmetries are of same chirality generated by $Q^1_\alpha \in 16$ and $Q^2_\alpha \in 16$.

Before closing this subsection, we would like to mention that upon compactification to 9 dimension on a circle of radius $R$, the two type II superstrings are equivalent under T-duality:

$$R \to \frac{1}{R}, \qquad IIA \leftrightarrow IIB. \tag{1.60}$$

**Insert 2.1: Some facts on the $p+1$-form potential fields $A_p$ in superstring theory**

A $p + 1$ form field $A_{p+1}$ can be viewed as the higher dimensional generalization of the gauge vector potential field $A_1$. They enjoy similar gauge transformation as

$$A_{p+1} \to A_{p+1} + d\lambda_p, \tag{1.61}$$

where $\lambda_p$ is an arbitrary closed $p$ form and $d$ denotes the Exterior derivative. In superstring theories, there are two types of higher $p + 1$ potential forms: $B_2$ and $C_{p+1}$. Like the vector gauge potential $A_1$ coupling to a (electrically) charged point particle with charge $e$ as

$$e \int A_1 \tag{1.62}$$

, the $p+1$ form $A_{p+1}$ could also couple to a $p$ spatial dimensional extended object with charge $\mu_p$ as

$$\mu_p \int A_{p+1}, \tag{1.63}$$

where we are using standard differential form calculus. One can also, like the 4D Maxwell theory, defines a magnetically charged under $A_{p+1}$ objects with spatial dimension $d-4-p$ by defining it electrically charged under the hodge dual form $d\widetilde{A}_{d-3-p} = *dA_{p+1}$ in d dimension.

The two-form Kalb-Ramond $B_{\mu\nu}$ could couple directly to the string world-sheet, as one can see from the non-linear sigma model action for strings in the curved spacetime with non-trivial backgrounds $B_{\mu\nu}$ and dilaton $\phi$

$$S = -\frac{T_s}{2}\int d^2\xi\sqrt{h}(g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu h^{\alpha\beta} + iB_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon_{\mu\nu} h^{\alpha\beta} + \alpha' \phi R), \tag{1.64}$$

where $R$ is the 2D world-sheet Ricci scalar. The reason why they appear together in the non-linear sigma model is that $B_{\mu\nu}, \phi, g_{\mu\nu}$ are generated at the same excited level. In terms of conformal theory, generated by the same vertex operator. Hence the string carries (electric) charge with respect to $B_{\mu\nu}$.

Now let's turn to the RR potential fields $C_p$ for the type II superstrings. The main difference to the $B_{\mu\nu}$ is that the vertex operators for the R-R fields $C_p$ involves only their field strengths $F_{p+1} = dC_p$ and thereby only the field strengths couple to the string, rather than the potential fields $C_p$ themselves. In physical lingo, it means that elementary, perturbative string states are neutral under the RR gauge symmetries and cannot carry any charges with respect to the RR gauge fields $C_p$. This fact leads to (rather) poor understanding of the world-sheet theory, as a conformal theory, with non-trivial background RR fields $C_p$. As we will see later, D-branes/O-planes, both as non-perturbative solitons states of the corresponding supergravity, couple to the RR $p$-form fields $C_p$.

## 1.3. Supergravity of String Theory

After we have introduced massless spectra for Type II strings, we are going to describe the relevant supergravities (including the 11D supergravity) describing the dynamics of these massless spectra, and thus viewed as low-energy limit of superstrings. For our purpose on Green-Schwarz mechanism, we will also introduce 10D $\mathcal{N}=1$ supergravity coupled by a super Yang-Mills theory. And we will show how the 10/11D chiral supergravities are anomaly free.

### 1.3.1. 11D supergravity

A higher spacetime dimension leads to larger possible representations of the Clifford algebra. Spacetime dimensions of more than 11 with the corresponding spin group of dimension higher than 64 would lead to supersymmetric massless particles of spin greater than 2 [5].. It is thought that such theories are ill defined. In this sense 11 is the largest allowed number of spacetime dimensions for supergravity. Due to the high constraints from supersymmetry, supergravity in 11 dimensions is unique.

[5] Here we are only consider the signature $(1, D-1)$ for the theory of gravity, for other choices, higher dimensional supergravity is still possible. In fact, naively speaking, F-theory, the main theme of this thesis, has 12 dimensions and we view the signature as $(2, 10)$ in certain senses.

Remarkably, the 11D Supergravity is viewed as the low-energy limit of M-theory, whose precise formulation and dynamics are still not yet well-known. By construction, the 11D supergravity has 32 supercharges $Q_\alpha$ in the 32 dimensional spinor representation of $SO(1,10)$. All massless fields reside in one supermultiplet, the gravity supermultiplet, which contains:
• $g_{\mu\nu}$: the graviton with $(9\times 10)/2-1=44$ degrees of freedom, as it is symmetric and traceless.
• $C_{\mu\nu\lambda}$: an antisymmetric 3-form field with $(9\times 8\times 7)/6=84$ degrees of freedom.
• $\psi_{\mu\alpha}$ the gravitino, with 128 degrees of freedom.

### 1.3.2. 10D $\mathcal{N}=2$ supergravity

The 11D supergravity gives rise to a 10D $N=2$ supergravity when performing a Kaluza-Klein compactification on $S^1$. More precisely, wrapping the 11D theory on a circle of radius $R$, when $R$ becomes small the Kaluza-Klein modes on the $S^1$ become very massive as the masses of KK modes are proportional to $1/R$, so only the zero modes of the 11D theory make up the effective theory in 10D. The correspondence between the 11D and 10D massless fields is given by

$$(g^{11}_{\mu\nu}, g^{11}_{\mu 11}, g^{11}_{1111}, C^{11}_{\mu\nu\lambda}, C^{11}_{\mu\nu 11}) \to (g_{\mu\nu}, A_\mu, \phi, C_{\mu\nu\lambda}, B_{\mu\nu}). \tag{1.65}$$

One can see that the spectrum here exactly coincidents with the massless spectrum of Type IIA superstring. Further, the resulting 10D $\mathcal{N}=2$ supergravity has two opposite chirialties, and hence it is non-chiral.

There exists a second 10D $\mathcal{N}=2$ supergravity whose two supersymmetries are of same chirality and thereby it is chiral. It describes the low-energy behavior of the classical effective theory of Type IIB superstring. The bosonic part of 10D Type IIB supergravity pseudo-action in its democratic formulation, where each RR gauge potential $C_{p+1}$ is accompanied by its magnetic dual, is given by [24]

$$\begin{aligned} S_{\rm IIB} = \frac{1}{2\kappa_{10}^2}\Bigg(&\int d^{10}x\, e^{-2\phi}(\sqrt{-g}R + 4\partial_M\phi\,\partial^M\phi) - \frac{1}{2}\int e^{-2\phi} H_3\wedge *H_3 \\ &-\frac{1}{4}\sum_{p=0}^{4}\int F_{2p+1}\wedge *F_{2p+1} - \frac{1}{2}\int C_4\wedge H_3\wedge F_3\Bigg). \end{aligned} \tag{1.66}$$

where the gravitational coupling $\kappa_{10}$ is given by $\kappa_{10}^2 = \frac{1}{4\pi}(4\pi^2\alpha')^4 = \frac{\ell_s^8}{4\pi}$ and the field strengths are defined as

$$\begin{aligned} &H_3 = dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2 - C_0\, dB_2, \\ &F_5 = dC_4 - \frac{1}{2}C_2\wedge dB_2 + \frac{1}{2}B_2\wedge dC_2, \end{aligned} \tag{1.67}$$

together with the duality relations $F_9 = *F_1$, $F_7 = -*F_3$, $F_5 = *F_5$, which hold at the level of equations of motion. Note that in order to give rise to a complete equation of motion, one should also impose the self-dual constraints on $F_5$ as $*F_5 = F_5$ as the action (A.1) is not the standard manifestly covariant action.

### 1.3.3. 10D $\mathcal{N}=1$ supergravity

In 10D $\mathcal{N}=1$ supergravity, there is a vector multiplet containing a vector field and a gaugino in the vector representation of the little group $SO(8)$, in addition to a gravity multiplet.

The gravity multiplet contains a spacetime metric $g_{\mu\nu}$, the 2-form $B_{\mu\nu}$ and a scalar dilaton $\phi$, as well as their superpartner fermions: 10D Weyl-Majonara $\zeta_\alpha$ and gravitino $\psi_{\mu\alpha}$.

The vector multiplet contains a gauge field $A_\mu$ and the gaugino $\lambda_\alpha$.

The low energy limit of 10D $\mathcal{N}=1$ supergravity has the following action

$$S \sim \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g}[e^{-2\phi}(R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}|H|^2)-\frac{e^{-\phi}}{g_{YM}^2}F^\alpha_{\mu\nu}F^{\mu\nu\alpha}] \tag{1.68}$$

where the three-form field strength $H$ is given by

$$H = dB-\omega_Y, \qquad d\omega_Y=\sum_a F^a\wedge F^a. \tag{1.69}$$

The gauge group realized by the vector multiplet seems arbitrary. However, as we are going to show in next subsection, there are only two choices, $E_8\times E_8$ and $SO(32)$, as is required by some consistency conditions. Remarkablely, there are also two types of superstrings described by these 10D $\mathcal{N}=1$ supergravities in the low energy limit, known as the Heterotic string and Type I string. In terms of the gauge algebra, the Heterotic strings allow two possibilities, $E_8\times E_8$ and $SO(32)$. Type I string carries the gauge group $SO(32)$.

## 1.4. Anomaly Cancellation in 10D Supergravity

In this section, we are going to check anomaly cancellations in 10D chiral supergravities. As we review anomalies in the appendix A.2, the anomaly structure of a theory in $D$ dimension can be encoded in a $D+2$-form anomaly polynomial $I_{(D+2)}$ with the degree $(D+2)/2$ in $F, R$. The 10D anomaly polynomial $I_{12}$ contains $I_{1/2}$ contributed from $n$ Weyl spinor field $\zeta_\alpha$, $I_{3/2}$ from the gravitino field $\psi_{\mu\alpha}$ and a self-dual $I_{\text{s.d.}}$ 4-form field $C_4$, and is given by [7] (see also summary in [25])

$$\begin{aligned} I_{1/2}=&-\frac{\mathrm{Tr}(F^6)}{1440}+\frac{\mathrm{Tr}(F^4)\mathrm{tr}(R^2)}{2304}-\frac{\mathrm{Tr}(F^2)\mathrm{tr}(R^4)}{23040}-\frac{\mathrm{Tr}(F^2)(\mathrm{tr}(R^2))^2}{18432}\\ &\frac{n\mathrm{tr}(R^6)}{725760}+\frac{n\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960}+\frac{n(\mathrm{tr}(R^2))^3}{1327104};\\ I_{3/2}=&-495\frac{\mathrm{tr}(R^6)}{725760}+225\frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960}-63\frac{(\mathrm{tr}(R^2))^3}{1327104};\\ I_{\text{s.d.}}=&992\frac{\mathrm{tr}(R^6)}{725760}-448\frac{\mathrm{tr}(R^4)tr(R^2)}{552960}+128\frac{(\mathrm{tr}(R^2))^3}{1327104}. \end{aligned} \tag{1.70}$$

Here the Tr denotes traces calculated in the adjoint representation of the gauge groups (For the abelian U(1) groups, they simply vanish because the adjoint of them are trivial.)

There are no gauge fields in Type IIB supergravity, hence there is no gauge anomaly per se. However, there could be a gravitational anomaly in Type IIB supergravity as it is chiral. With inspection of its massless spectrum, one can verify that the total anomaly for the Type IIB theory vanishes, i.e.

$$-2I_{1/2}(F\to 0, n\to 1)+2I_{3/2}+I_{\text{s.d.}}=0. \tag{1.71}$$

Here we have taken into account that there are two gravitinos $\psi_{\mu\alpha}$ and two Majorana-Weyl spinor $\zeta_\alpha$, and the minus signals the opposite chirality of gravitinos and Weyl spinors.

However, this seeming "coincidence" does not apply for the other choices of supergravity, especially the $\mathcal{N}=1$ theories. It is easy to spot, as pointed out by [25], that the cancellations of the two terms $\mathrm{trR}^6$ and $[\mathrm{tr}(\mathrm{R}^2)]^3$ in $I_{1/2}$ and $I_{2/3}$ requires conflicting conditions for the rank of gauge groups. Indeed, assuming the dimension of the gauge group being $n$, then the total anomaly of 10D $\mathcal{N}=1$ supergravities, in terms of the anomaly polynomial $I_{12}$, yields

$$\begin{aligned} I_{12} =& \frac{1}{1440}\left(-\mathrm{Tr}(F^6) + \frac{\mathrm{Tr}(F^4)\mathrm{Tr}F^2}{48} - \frac{[\mathrm{Tr}(F^2)]^3}{14400}\right) \\ &+ (n-496)\left(\frac{\mathrm{tr}(R^6)}{725760} + \frac{\mathrm{Tr}(R^4)\mathrm{Tr}R^2}{552960} + \frac{[\mathrm{Tr}(R^2)]^3}{1327104}\right) \\ &- \frac{Y_4 X_8}{768}, \end{aligned} \tag{1.72}$$

where

$$X_8 = \mathrm{tr}(R^4) + \frac{[\mathrm{tr}(R^2)]^2}{4} - \frac{\mathrm{Tr}(F^2)(\mathrm{tr}R^2)}{30} + \frac{\mathrm{Tr}F^4}{3} - \frac{(\mathrm{Tr}F^2)^2}{900}, \qquad Y_4 = \mathrm{tr}R^2 - \frac{1}{30}\mathrm{Tr}F^2. \tag{1.73}$$

So it seems that all 10D $\mathcal{N}=1$ supergravities are doomed to fail due to anomalies. What could be the savior here?

### 1.4.1. The 10D Green-Schwarz mechanism

The key feature associated with the Green-Schwarz mechanism is that the anomalous gauge variation from 1-loop contributions for the effective theories can be non-vanishing but it should factorize and hence can be cancelled by adding a tree-level term to the action by hand, which is called Green-Schwarz term. Before delving into more details on the celebrated Green-Schwarz mechanism, let us illustrate why that works. To this end, recall that the low-energy effective theory we are working with refers to the **Wilsonian** effective theory, which is obtained by integrating out the massive modes of the parent theory, rather than truncating the massive modes as one typically does in the 1-PI effective theory. The main difference between these two effective theories is that the Wilsonian effective theory typically would generate additional terms involving irrelevant higher-derivative terms by integrating out the massive modes. These additional terms have no a *priori* obligation to respect the symmetries of the theory, especially in a non-renormalizable effective theory, and hence can generally have anomalous variations. This is exactly the role the Green-Schwarz terms play.

In 10D cases, the Green-Schwarz term reads as

$$S_{GS} \sim -\int B \wedge X_8(F,R), \tag{1.74}$$

where $X_8$ is given by (1.73). To implement this mechanism, the three-form field strength $H$ must be modified by adding higher orders such that

$$H = dB - \omega_Y + k\omega_R, \qquad d\omega_R = \mathrm{tr}R^2. \tag{1.75}$$

Now the two-form potential field $B_2$, gauge field $A$ and spin connection $\widehat{\omega}$ transform as

$$\delta B = Tr(\lambda F) - tr(\widehat{l}R), \qquad \delta A = d\Lambda, \qquad \delta\widehat{\omega} = d\widehat{l}. \tag{1.76}$$

Accompanying with these gauge transformation, the $S_{GS}$ picks up a gauge variation of the form

$$\delta S_{GS} = \int [\mathrm{Tr}(\lambda F) - tr(\widehat{l}R)] \wedge X_8 =: 2\pi \int_{\mathbb{R}^{1,1}} I_{10}^{(1),\mathrm{GS}}(\lambda)\,, \tag{1.77}$$

with $I_{10}^{(1),\mathrm{GS}}$ a gauge invariant 10-form. By the standard descent procedure, it defines an anomaly-polynomial $I_{12}^{\mathrm{GS}}$ encoding the contribution to the total anomaly from the Green-Schwarz sector. Concretely, the descent equations

$$I_{12}^{\mathrm{GS}} = dI_{11}^{\mathrm{GS}}, \qquad \delta_\lambda I_{11}^{\mathrm{GS}} = dI_{10}^{(1),\mathrm{GS}}(\lambda) \tag{1.78}$$

imply

$$2\pi I_{12}^{\mathrm{GS}} = Y_4 X_8. \tag{1.79}$$

Consistency of the theory then requires that

$$I_{12} + I_{12}^{\mathrm{GS}} = 0\,. \tag{1.80}$$

together with

$$\mathrm{Tr}(F^6) = \frac{\mathrm{Tr}(F^4)\mathrm{Tr}F^2}{48} - \frac{[\mathrm{Tr}(F^2)]^3}{14400}, \tag{1.81}$$

which enforces that $k = 30$ and the rank of gauge algebra $n = 496$ under the condition (1.81), which essentially says that the adjoint representation of the gauge group does not have a sixth order Casimir.

It turns out that the gauge algebras which are compatible with the Green-Schwarz mechanism and (1.81) are only [26]

$$\{\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{so}(32), \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{248}, \mathfrak{u}(1)^{248} \oplus \mathfrak{u}(1)^{248}\}, \tag{1.82}$$

so that the total anomaly can be cancelled.

However, as it turns out [27], the last two possibilities with abelian factors are inconsistent with the two requirements of gauge invariance and gravitational anomaly cancellation. And hence anomaly cancellation enforces the 10D $\mathcal{N} = 1$ supergravities to have gauge algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ or $\mathfrak{so}(32)$. And as we have said, they can both be realized in the string theory.

This remarkable fact indicates the string universality of the supergravity in 10 dimensions, for more details see the section 2.6 of [25].

## 1.5. Branes and Gauge Theory

We finally come to the crux of one of the main topics of this thesis- D-branes. As we have said, Type IIA superstring theory is proved to be T-dual to the Type IIB superstring theory. From the perspective of boundary conditions of the strings, T-duality essentially exchanges the Dirichlet condition and the Neumann condition and vice versa. Hence one should take the two boundary conditions on the same footing. But how to solve the issue that the Dirichlet condition breaks Poincaré invariance? Well, it turns out that the object that the open string ends on itself carries energy and hence is dynamical [6]. In this way, the end of an open string can evade the

[6] As a side remark, this appreciation of boundary conditions as dynamical objects in their own rights is not only for D-branes in string theory but also plays important roles in field theories and leads to many fruitful

momentum conservation. In other words, the dynamics of this object can be described by the excitations of the open strings who ends on them. With such "new" appreciation, this object, known as D-brane, has proven to be important for the understanding of lots of physics, including string theory itself, and the discovery of the D-branes [11, 12], led to the second revolution in the development of string theory. There are many excellent reviews on this broad topic such as [28–30]. In this section, we are going to present two main aspects of the nature of D-branes in Type II strings. They either can be viewed in microscopic way as certain boundaries of the fundamental open strings, or on the other hand, viewed in macroscopic way effectively as the solitonic solutions to equation of motion of type II supergravity and can relate to macroscopic solitonic objects such as black holes, domain walls, cosmic strings, monopoles or instantons depending on the dimension in the effective theory.

### 1.5.1. Descriptions in world-sheets of open strings

In the weakly coupled Type II string theories, a (static) D-brane can be viewed as a hyperplane in the spacetime $\mathcal{M}$ where the open strings end. If the open string with Neumann boundary conditions for coordinates $(X^0, X^1, ..., X^p)$ in the 10D spacetime and Dirichlet boundary condition for $(X^{p+1}, ..., X^9)$, then D-branes are defined as a hyperplane in $(X^0, .., X^p)$ that hosts the (open) fundamental strings. Depending on their spatial dimension, we sometimes denote them as Dp-branes. Before exploring the D-brane physics, let us first set up some conventions. Similar to the description of the string using the world-sheet, one can use the world-volume to embed the Dp-branes into the 10D spacetime $\mathcal{M}$. Denoting the $p+1$ dimensional world-volume as $\mathcal{W}$ swept out by the Dp-brane propagating through the target space $\mathcal{M}$, parametrizing by the coordinates $\xi^a, a = 0, ..., p$, one can define the embedding map as $\imath : \mathcal{W} \to \mathcal{M}$, $\xi^a \to X^\mu(\xi^a)$.

| Sector | State | $SO(p-1)$ | $(p+1)$ fields |
|---|---|---|---|
| NS | $b^a_{-1/2}\lvert p\rangle$ | Vector | $A^a(\xi^a)$ |
| NS | $b^m_{-1/2}\lvert p\rangle$ | Scalar | $X^m(\xi^a)$ |
| R | $\mathbf{B}_C$ | Spinor | fermions $\lambda_\alpha$ |

**Table 1.1.:** *The spectra assoicated with the Dp-brane in type II strings.*

Now let us start with the analysis of perturbative open string spectrum in flat spacetime. The massless bosonic spectra of open strings ending on a Dp-brane with world-volume coordinates contain a massless gauge field $A_a, a = 0, 1, .., p$ and scalar fields $X^m(\xi^a), m = p+1, .., 9$, which we listed in **??**. The scalar fields $X^m$ actually describe the transverse **positions** of the Dp-brane embedded in the spacetime $\mathcal{M}$. The gauge fields $A^a(\xi^a)$, on the other hand, can be viewed as tangent space of the hyperplane of Dp-branes, which describe **shapes** of the D-branes as a "soliton" background, i.e. as a fixed topological defect in spacetime. As we said above, D-branes are viewed as dynamical extended objects by themselves, hence the hyperplanes are not rigid and instead, the shapes and positions can fluctuate. The low-energy theories of world-volume of a Dp-brane (or more general $(p, q)$-branes) should capture the dynamics of the above massless spectra generated by the open strings, and it turns out it reduce to super Yang Mills (SYM) theory [7].

results, see, for example, Witten's work on topological field theories.

[7] The case with M5-branes in M-theory is subtle, as the fundamental objects would be M2-branes in M-theory. Hence the low-energy limit of world-volume of M5-brane should be describing the dynamics of strings, which

To be more precise, the kinetic term for the low energy effective theory of D-brane, describing the above open string massless spectrum, is given by the Dirac-Born-Infeld (DBI) action [32]

$$S_p = -\mu_p \int d^{p+1}\xi e^{-\phi(\xi)} \sqrt{-\det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{1.83}$$

where $g_{ab} := \imath^* g_{\mu\nu} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, B_{ab} := \imath^* B_{\mu\nu} = B_{\mu\nu}\partial_a X^\mu \partial X^\nu_b$ refer to the pullbacks of the corresponding 10D bulk ones, $F$ is the field strength of the world-volume $U(1)$ gauge field $A_a$. The prefactor of the dilaton $e^{-\phi(\xi)}$ arises from the open string tree level, i.e. the disk. Turning down the field strength $F$ and Kalb-Ramond 2-form field $B$, one can find that the DBI action is the $p+1$ dimensional generalization of the 2D Polyakov action and can be justified by checking that the equation of motion of DBI satisfies conformal invariance for the open string in the D-brane background. Thereby the tension $T_p$ of a Dp-brane is given by [8]

$$T_p = \frac{2\pi}{g_s \ell_s^{p+1}} = \frac{|\mu_p|}{g_s}, \tag{1.84}$$

where the coefficient $\mu_p$ will be discussed later. The D-brane tension $T_p$ scales with the string coupling as $T_p \simeq 1/g_s$, reflecting the non-perturbative nature of these states, and we will see it leads to a lot of peculiar properties of D-branes [9] . At the weak coupling limit, $T_P$ tends to $T_P \to \infty$ and it can be viewed as the rigid hyperplane in the weak coupling regime. In some senses, one can view the tension $T_p$ as the NS-NS charge.

As we said, the low-energy limit of D-brane should be captured by a super Yang Mills (SYM) theory. How can DBI action be linked to the SYM theory? Well, based on two derivative form of SYM, one should extract the two derivative leading order of DBI action. To do that, one should invoke the formula

$$\det(1+M) = 1 + tr(M) - \frac{1}{2}tr(M^2) + ..., \tag{1.85}$$

and expand the DBI action (1.83) at the almost flat space $g_{\mu\nu} = \eta_{\mu\nu}$ with slowly-varying fields to order $F^4, (\partial X)^4$ and then we have

$$g_{ab} \sim \eta_{ab} + \partial_a X^\mu \partial_b X_\mu + \mathcal{O}((\partial X)^4), \qquad B_{\mu\nu} = 0. \tag{1.86}$$

If we further assume that $2\pi\alpha' F_{\alpha\beta}$ and $\partial X$ are small and of the same order, then we can easily see that the DBI action reduces to

$$S_p = T_p V_p - \frac{1}{4g_{YM}^2} \int d^{p+1}x (F_{ab}F^{ab} + (\frac{2}{2\pi\alpha'})^2 \partial_a X^m \partial^a X_m) + \mathcal{O}(F^4), \tag{1.87}$$

arise from the boundary of M2-branes on M5-branes, this necessarily involves higher structures of gauge theories, the proper mathematics would be gerbes rather than the vector bundles/sheaves. For details on this aspect, we refer to [31] and references therein.

[8] Here we already evaluated the $e^{-\phi(\xi)}$ at the asymptotic value $\phi_0$ which gives rise to string coupling $g_s^{-1}$.

[9] One may wonder what would be the back-reaction on the geometry since D-brane carries tension. To see this, one can easily check that $N$ coincident branes in a $D$ dimensional spacetime give rise to a gravitational potential (classically) $V \sim G_N N T_p / r^{D-p-3}$ with Newton's constant $G_N$ given by $G_N \sim \ell_s^8 g_s^2$, hence for $N$ coincident Dp-branes the strength of the gravitational potential is $g_s N$. This implies that the back-reaction of the D-branes on the spacetime geometry can be neglected at distances larger than the string scale if $g_s N \ll 1$, which corresponds to the limit of the weakly coupled string theory. Hence this suggests that it is reasonable to view the D-branes at the weakly coupled string theory as a hyperplane in the flat Minkowski space where open fundamental string ends.

where the $V_p$ is the Dp-brane volume and the coupling $g_{YM}^2$ is given by

$$g_{YM}^2 = g_s \ell_s^{p-3}. \tag{1.88}$$

Note that for D3-branes, the string coupling $g_s$ coincidents with the square of the SYM coupling. The first term contributes to the vacuum energy and can be omitted, thereby the DBI action reduces to the SYM action! Note that the above limit we've taken turns out to be equivalent to sending the string length $\ell_s \to 0$, namely we have decoupled the gravity and also massive string modes as their mass scales as $m \sim \frac{1}{\ell_s}$. In such a limit, in order to keep $g_{YM}$ fixed, one needs to impose $g_s \to 0$ for $p < 3$ and $g_s \to \infty$ for $p > 3$.

Furthermore, a Dp-brane (with suitable dimension, which will be explained shortly) can be described as a Bogomol'ny-Prasad-Sommerfeld (BPS) saturated state preserving half of the 32 supercharges of type II string theory, as the open string boundary conditions are invariant under only half of the transformations. In other words, the Type II vacua without D-branes preserve all 32 supersymmetries, but the states containing a D-brane are invariant under only half the supersymmetries. Being BPS states implies that parallel Dp-branes do not exert forces on each other, since roughly speaking, the repulsion forces from the charge exactly cancel their gravitational(and dilaton) attractive force. As a result of this, a stack of Dp-branes can be placed on top of each other without any repulsive or attractive forces and the individual Dp-branes are indistinguishable. This essentially promotes the Chan-Paton labels to be a non-abelian $U(N_c)$ matrix. Correspondingly, the low-energy world-volume of a stack of $N_c$ Dp-branes turns out to be a SYM with the non-abelian gauge group $U(N_C)$ [10] and 16 supercharges. The massless spectra $X^m(\xi^a), A^a$ then transform under the adjoint representation of $U(N_c)$ gauge group, where the diagonal components of $X^m$ and Cartan generators of $A^a$ arise from the states generated by open strings whose endpoints are on the same Dp-brane. On the other hand, the off-diagonal components of $X^m$ and charged gauge bosons come from open strings ending on different Dp-branes. The Lagrangian for the $U(N_c)$ SYM reads

$$\mathcal{L} = -\frac{1}{g_{SYM}^2}\mathrm{Tr}(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{\ell_s^4}\mathcal{D}_a X^m \mathcal{D}^a X + V), \tag{1.89}$$

where the covariant derivative acts as $\mathcal{D}_a X^m = \partial_a X^m - i[A_a, X^m]$. Flat directions of the potential V, defined as

$$V \propto \sum \frac{1}{g_{YM}^2 \ell_s^8}\mathrm{Tr}[X^m, X^n]^2, \tag{1.90}$$

are parametrized by the diagonal $X^m$, which describes the Coulomb branch of the $U(N_c)$ theory. The vev of $X^m$ then represents the transverse position of these $N_c$ Dp-branes. As an aside, it turns out that one can also describe the low energy dynamics of a stack $N_c$ Dp-branes in **flat** space in static gauge by the dimensional reduction to $p+1$ dimensions of 10D $\mathcal{N} = 1$ SYM.

The above analysis can also carried over to the cases that D-branes have a curved world-volume. In such cases, the gauge groups would not necessarily be $U(N)$. It turns out, in perturbative Type II string theories, the gauge groups can be realized by D-branes are only three options: SU(N), SO(N) and Sp(N), which we will mention in 1.9. For other options, especially the exceptional ones such as $E_n$ gauge groups, they necessarily invoke non-perturbative degrees of freedom, while certain $(p, q)$ strings become light and build up the necessary components for exceptional groups.

[10] Note there are other choices such as $SO(N)$ or $Sp(N)$, but that depends on further configurations, especially the O-planes, which will be introduced momentarily.

Having said the Dp-branes are BPS states of the Type II strings, one would wonder which conserved charges, determined entirely by their mass/tension in the corresponding supersymmetric algebra, are carried by the Dp-branes. Well, it turns out there is only one set of charges for Dp-branes with the correct Lorentz transformation properties- Ramond-Ramond charges. Indeed, the Dp-branes carry charges with respect to the RR gauge symmetries by coupling to the R-R $p+1$ potential fields $C_{p+1}$ through the integral $\mu_p \int_{\mathcal{W}} \imath^* C_{p+1}$ [11]. This fact is consistent with the statement we listed in Insert 2.1 that the perturbative string states cannot see the charges of the RR fields, as the Dp-brane, whose tension is proportional to $1/g_s$, is genuinely non-perturbative. Since they carry the conserved charge, they are hence stable.

As we said earlier, $C_{p+1}$, as a $p+1$-form potential field, is hodge dual to $C_{7-p}$ by $dC_{7-p} = *dC_{p+1}$. Based on the discussion in Insert 2.1, we then say that $p+1$ form $C_{p+1}$ is electrically coupled to a Dp-brane and magnetically coupled to a D(6-p)-brane in Type II strings. We see that in type IIA theory with $p$ in RR potentials $C_{p+1}$ being even, we thus have stable Dp-branes with $p = 0, 2, 4, 6, 8$. In type IIB string theory, we can have stable Dp-branes with $p = -1, 1, 3, 5, 7, (9)$ [12]. Dp-branes with wrong dimension in type II strings, for example a Dp-brane with even value of $p$ in Type IIB, can also host the open strings. However, they do not carry the conserved charge as there are no suitable RR fields. As a consequence, they are not stable and break all spacetime supersymmetry. They are essentially the same objects as the ones in the bosonic strings, and generate tachyons in their spectrum.

In terms of the charges, as a generalization of 6D version $eg = 2\pi n$, they similarly are subject to the Dirac quantization

$$\mu_p \mu_{6-p} \in 2\pi\mathbb{Z}, \tag{1.91}$$

where the charge $\mu_p$ is measured by the generalized Gauss law

$$\int_{S^{8-p}} *F_{p+2} = \mu_p. \tag{1.92}$$

However, it turns out that Dp-branes not only carry the charge of $C_{p+1}$, but also of lower form $C_n$ fields with $n < p+1$. There are various ways to see that. For example one can use T-duality to see that Dp-branes also couple to $C_{p-1}$ RR fields. The full action of RR fields coupling had been derived in [34] by invoking the anomaly inflow mechanism [13], which is given by the Chern-Simons action, sometime also known as Wess-Zumino action.

The Chern-Simons action for the Dp-branes takes the form

$$S^{Dp} = -\frac{\mu_p}{2} \int_{\mathcal{W}} \sum_{2p} \imath^* C_{2p} \wedge \mathrm{Tr}(e^{i\mathcal{F}}) \wedge \sqrt{\frac{\widehat{A}(\mathrm{T}D_p)}{\widehat{A}(\mathrm{N}D_p)}}, \tag{1.93}$$

[11] This point can be also reflected by the fact that the Dp-branes can be viewed as the soliton solutions of classical equation of motion in the supergravity [33], as we will discuss shortly

[12] As we have stressed, there are only closed string propagating on the bulk of type II strings. Including D9-branes in the Type IIB then implies that open strings can also propagate on the bulk, which would leads to a contradiction. Indeed, this is only consistent in Type I theory, which can be viewed as type IIB with 32 D9-branes and an orientifold (to be discussed later). D8-branes is also subtle, It is actually a domain-wall coupling to a non-dynamical 9-form field $C_9$.

[13] The anomaly inflow mechanism, firstly introduced in the context of gauge theory [35], could usually be applied to the anomalous theory such that it can be coupled/embedded in a higher dimensional theory whose anomalous variation of the classical action localizes at ("flow" to) the world-volume of the original anomalous theory and cancels its anomaly.

where the charge is given by $\mu_p = \frac{2\pi}{\ell_s^{p+1}}\alpha_p$ [14]. T$D_p$ and N$D_p$ denote the tangent and normal space to the Dp-brane along $\mathcal{W}$ and the conventions of Chern Character and the $A$-roof genus can be found in the appendix B.1.1. Note furthermore that we are writing the brane action in terms of $\mathrm{Tr} = \frac{1}{\lambda}\mathrm{tr}_{\mathbf{fund}}$, where the Dynkin index $\lambda$ is given in Table 4.1. $\sum_{2p} \imath^* C_{2p}$ denotes the formal sum of RR fields $\imath^* C_{2p}$, which are pull-backs of the bulk RR fields[15]. Since we are working in the democratic formulation, where each RR gauge potential $C_{p+1}$ is accompanied by its magnetic dual, the Chern-Simons action has to include a factor of $\frac{1}{2}$ [40], which we are making manifest in (A.3). Note that the Chern-Simons action does not involve the metric and is thus of topological nature.

The gauge invariant field strength $\mathcal{F}$ above is defined as

$$\mathcal{F} = i(\ell_s^2 \mathbf{F} + 2\pi \imath^* B_2 \mathbf{I})\,. \tag{1.94}$$

Compared to expressions oftentimes used in the literature we have absorbed a factor of $\frac{-1}{2\pi}$ in the definition of $\mathcal{F}$. The minus "-" here is in order to be consistent with the other conventions for anomaly in A.2.

### 1.5.2. D-branes as soliton solutions of supergravity

We have deduced from the tension $T_p$ of Dp-branes that D-branes should be viewed as non-perturbative states in string theory. In other words, such states are not the oscillation states of the string from a world-volume perspective, but rather should be thought as analogs of solitons in field theories. Recall that the solitons, such as the monopoles, can be described as collective excitations of spacetime fields, describing the classical solutions of the spacetime equation of motions. Inspired by this, one would wonder whether D-branes have similar interpretations, though in string theory. However, we do not have a complete description of the spacetime action of string theory yet, but only a classic description of the massless fields known as supergravity.

The interaction of D-branes with closed strings, typically via open strings, would create non-trivial backgrounds for the metric $g_{\mu\nu}$ and RR fields, which can be described as a solution of the supergravity equation of motion.

The BPS solution for a stack of $N$ $p$-branes along directions $a = 0, 1 \ldots, p$, $p < 7$, takes the form (see e.g. [41])

$$ds^2 = H_p^{-1/2} \eta_{ab} dx^a dx^b + H_p^{1/2} \sum_m dx^m dx^m, \tag{1.95}$$

$$e^{2\phi} = e^{2\phi_0} H_p^{\frac{3-p}{2}}, \quad C_{p+1} = \frac{H_p^{-1} - 1}{e^{\phi_0}} dx^0 \wedge \ldots \wedge dx^p, \tag{1.96}$$

[14] Here $|\alpha_p| = 1$ fits the BPS algebra condition $M = |Z|$ for the case of Dp-branes. We are working in the convention that $\alpha_7 = 1 = -\alpha_3$. Note that the sign difference between the D7-brane and the D3-brane is crucial for obtaining the correct matching between the D3-brane tadpole conditions [36,37]. Together with the minus sign, this convention also makes sure that for the supergravity fields the D7-brane couples magnetically to the axio-dilaton $\tau = C_0 + ie^{-\phi}$.

[15] Note that it does not necessarily mean that for Dp-branes, higher p-form RR fields such as $C_{p+3}$ in the bulk does not affect the Dp-branes action at all. For example, $C_{p+3}$ can be pulled back to $C_{p+1}$ on the world-volume of Dp-branes through "interior derivative" contraction. In fact, this is crucial for the Myers effects [38] (or [39] for a review), where $N$ Dp-branes get "polarized" by background fluxes and then puff up into a higher dimensional non-commutative world-volume geometry. In the large $N$ limit, it will end up as a $D_{p+2}$ with extra two dimensional space being a fuzzy sphere $S^2$. However, we can safely ignore this effect in this thesis and hence use the same notations for the RR-fields in the bulk of type II strings and in the world-volume of Dp-branes.

$$H_p = 1 + \left(\frac{r_p}{r}\right)^{(7-p)}, \qquad r_p^{(7-p)} = e^{\phi_0}\, N\alpha'^{(7-p)/2}(4\pi)^{(5-p)/2}\Gamma(\frac{7-p}{2}) \tag{1.97}$$

$$\widetilde{F}_{ia_1\ldots.a_{p+1}} = -\epsilon_{a_1\ldots a_{p+1}}\partial^i H^{-1}, \tag{1.98}$$

where $\widetilde{F}_{p+2} := dC_{p+1}$ the RR field strength, and $r = \sum_m |x^m|^2$ denotes the radial coordinate in the transverse space of D-branes. The above solution has Poincaré invariance along the coordinate $x^a$, but only rotational invariance in $x^m$, thus is consistent with a $p+1$ dimensional Dp-brane configuration. Further we have $\int_{S^{8-p}} *\widetilde{F} = N$, where $S^{8-p}$ is an $8-p$ dimensional sphere at the infinity of the transverse space, thus it measure the charge of the Dp-brane. The solution asymptotes to a flat 10D spacetime $\mathbb{R}^{1,9}$ at large $r$, but when tending to small $r$, it looks like a throat of characteristic size $r_p \sim g_s N$. For $p = 7, 8$, we can still have similar solution but such solutions do not have the above asymptotically flat spacetime $\mathbb{R}^{1,9}$.

So through above solutions, we can see that in the presence of Dp-branes, the type II SUGRA spectra such as the metric $g_{\mu\nu}$, the dilaton $\phi$ and $B_2$ unavoidably receive the back-reactions. This is not surprise given that the D-branes carry tensions (energy). However, as we described in the footnote 9, for the weakly coupled type IIB when $g_s \to 0$, such back-reactions can be safely ignored.

Up to now, we have described two natures of D-branes: one is from the open string perspective and the dynamics at low energy can be described by the gauge theory, and the other is from the closed string perspective and the dynamics at low energy can be described by soliton-like solutions of supergravity. These two natures of D-branes, heuristically speaking, can be viewed more broadly as the gauge/gravity correspondence, among which is the celebrated AdS/CFT correspondence initiated by Maldacena in 1998. However, for our purpose, we will not going to that the fascinating directions.

## 1.6. Calabi-Yau Compactifications and Flux Compactifications

In this section we are going to review the Calabi-Yau compactifications of Type II theories. For definiteness, we will focus on Calabi-Yau three-folds but the arguments are readily generalized the arguments to other dimensions.

### 1.6.1. Kaluza-Klein Reduction and Calabi-Yau compactifications

Let us briefly review Kaluza-Klein compactification of 10D supergravity to lower dimensional effective theory. For definiteness, we will focus on the 4D dimensional effective theory in this subsection, similar ideas can also apply to other lower dimensional theories.

As stated, the consistency of superstrings requires the critical dimension of the spacetime $\mathcal{M}$ should be **ten** [16]. In order to make contact with our observed flat 4D Minkowski space

[16] To be more precise, the superstring strings and more fundamental, M-theory, in certain cases, would not always stick to the 10/11 dimension as some informations cannot only be captured by the 10D or 11D theories. The best idea is to view superstring theory as a 2D $\mathcal{N} = (2, 2)$ superconformal theory with the central charges equal to 15. One can then decompose the central charge $c = 15$ as the product of two theories with central charge $c = 6$ and $c = 9$, where the one with $c = 6$ can be easily realized by the free theory in the Minkowski space $\mathbb{R}^{1,3}$. The $c = 9$ one, in terms of geometric compactification, typically refers to the non-linear sigma model with the target space being the internal space $\mathcal{X}$. However, one can also realize a 2D $\mathcal{N} = (2, 2)$ superconformal theory with the central charge $c = 9$ alternatively such as the Landau-Ginzburg model and Minimal models. In such examples, one losts the geometric pictures. Thanks to the seminal work [42] by Witten, one can view the geometric compactifications and the non-geometric compactifications describe different phases of string theory and such two phases typically could be connected by varying suitable

$\mathbb{R}^{1,3}$, the typical approach is to consider the geometric compactifications, i.e. considering the other 6 spatial dimensions $\mathcal{X}$ as an extremely tiny space as a hidden space [17], namely we have $\mathcal{M}_{10} = \mathcal{X} \times \mathbb{R}^{1,3}$ with the metric being

$$ds^2 = e^{A(y)} g_{ab} dx^a dx^b + e^{-4A(y)} g_{ij} dy^i dy^j, \tag{1.99}$$

where $x_a, a = 0, 1, 2, 3$ are coordinates on 4D $\mathbb{R}^{1,3}$ and $y_i, i = 4, ..., 9$ are coordinates on the space $\mathcal{X}$, dubbed as the internal space. The warp factor $e^A$ depends on the internal space and can be determined by solving the equation of motion of the corresponding supergravity. A non-trivial warp factor typically indicates there are non-trivial local sources. In the type II strings, when there are no D-branes or fluxes involved, then the warp factor $A$ can be dropped.

The idea of the Kaluza-Klein (KK) reduction is to expand all the 10D fields into modes of the internal manifold $\mathcal{X}$. Taking 10D massless scalar fields $\Phi$ as the examples, they should satisfy Laplace's equation:

$$\Delta^{10} \Phi(x_a, y_i) = 0. \tag{1.100}$$

Splitting the Laplacian operator $\Delta^{10}$ into a 4D d'Alembert operator $\Delta^4$ and into 6D Laplace one $\Delta^6$ as $\Delta^{10} = \Delta^4 + \Delta^6$, together with the ansatz $\Phi(x, y) = \sum_k \phi_k(x)\psi_k(y)$, we see that

$$(\Delta^4 + m_k^2)\phi_k(x) = 0, \tag{1.101}$$

where $\psi_k$ is an eigenfunction of the Laplacian $\Delta^6$ on the internal space with eigenvalue $m_k^2$- i.e., from the 4D viewpoint we have a particle with mass $m_k$. The hodge theorem implies that the Laplacian operator $\Delta^6$ on a compact internal space $\mathcal{X}$ are always discrete and non-degenerate. Hence from the 4D perspective, we have a tower of KK states with discrete masses $m_k$s. Since the eigenvalues of the Laplacian operator on the internal space $\mathcal{X}$ scale with its size as $1/R^2$ ( viewed as the "radius"of the internal space $\mathcal{X}$), this implies $m_k^2 \sim R^{-2}$. If the internal space is very small, i.e. $R \to 0$, it means that most of the Fourier modes living on it will be very massive and can be integrated out

The above arguments for the scalars can be generalized to other 10D massless fields including the graviton $g_{\mu\nu}$. In each case the mass, as seen from 4 dimensions, are determined by eigenvalues of a suitable operator on $\mathcal{X}$. And hence one see that the 4D effective theory, describing the massless or light degrees of freedoms, are determined by certain operators in $\mathcal{X}$ and as we will show shortly, these zero modes are typically determined by the topology and geometry of $\mathcal{X}$. For those bosonic zero modes, we will show there are one-to-one correspondence with the harmonic forms of $\mathcal{X}$.

After discussing the general features of KK reduction, one would ask what kinds of the internal space $\mathcal{X}$ shall be. Well, although we are free to choose arbitrary internal space, it turns out if one would keep certain supersymmetries for the 4D effective theories, there are strong constraints imposing on the internal space $\mathcal{X}$.

To see that, recall the Lorentz group of $\mathcal{M}$ under the ansatz $\mathcal{M} = \mathbb{R}^{1,3} \times \mathcal{X}$ decompose as $SO(1,9) \to SO(1,3) \times SO(6)$. Then a supercharge, as a spinor representation **16** in 10D, generically decomposes as

$$\mathbf{16} \to (\mathbf{2}, \bar{\mathbf{4}}) \otimes (\bar{\mathbf{2}}, \mathbf{4}). \tag{1.102}$$

parameters.

[17]According to general relativity that space is flexible which does not conflict with our current cosmological observation that our universe is expanding. In other words, our observed 3d space was much smaller. Likewise, it is not unacceptable to imagine there is an extra space that remains small today.

As one can see there is no singlet spinor **1** in the internal space, which is crucial for the presence of supercharges in the 4D effective theories. In order to that, it typically requires that the internal space $\mathcal{X}$ has a reduced structure group $SU(3)$, which is a topological condition, then one has

$$\mathbf{4} \to \mathbf{1} \otimes \mathbf{3}, \tag{1.103}$$

which indicates a singlet of $SU(3)$, meaning that this spinor is independent of the tangent bundle of the internal space $\mathcal{X}$ and hence the singlets, dubbed as $\eta$, is well-defined and non-vanishing. However, further constraint from the supersymmetry requires that the globally defined spinor $\eta$ should be covariantly constant with respect to the metric of $\mathcal{X}$. This in turn requires that the internal space whose holonomy group should be $SU(3)$, which is a differential condition on the metric, or rather on its connection. The requirement for a six dimensional space with holonomy $SU(3)$ turns out to be very strong, which requires if and only if it is a Kähler manifold and further Ricci flat. These types of internal spaces $X_n$ are known as the Calabi-Yau manifolds [18].

One can further show, by using the same tricks, there are also non-vanishing globally well-defined real 2-form and complex 3-form in the Calabi-Yau three-manifolds. To see this, recall that a two-form and a three-form in $SO(6)$ are in the **15** and **20** representations. One can then reduce them similarly in representations of $SU(3)$, together with a vector **6**, as

$$\begin{aligned} \mathbf{6} &\to \bar{\mathbf{3}} \otimes \mathbf{3}; \\ \mathbf{15} &\to \mathbf{8} \otimes \bar{\mathbf{3}} \otimes \mathbf{3} \otimes \mathbf{1}; \\ \mathbf{20} &\to \bar{\mathbf{6}} \otimes \mathbf{6} \otimes \bar{\mathbf{3}} \otimes \mathbf{3} \otimes \mathbf{1} \otimes \mathbf{1}. \end{aligned} \tag{1.104}$$

So indeed there are non-vanishing globally well-defined real 2-form and complex 3-form in the Calabi-Yau three-manifolds, which denotes as $J$ and $\Omega_3$ respectively. In addition, we can see there is no invariant vector (equivalent to five-form in Calabi-Yau three-folds), which further infers $\Omega_3 \wedge J = 0$. Similarly, one can also see the six-form is a singlet and also is unique up to a constant prefactor, which infers that $J \wedge J \propto \Omega_3 \wedge \bar{\Omega}_3$. As we'll elaborate more details on the main properties for Calabi-Yau manifold in the appendix B.1.2, $J$ and $\Omega_3$ in Calabi-Yau three manifolds are known as the Kähler form and the holomorphic three-form, which are characterized the Kähler moduli and complex structure moduli of $X_3$.

In summary, 1/4 of the supercharges can be preserved in a Calabi-Yau compactification, thus hence the Calabi-Yau compactification of Type II strings (In total 32 real supercharges) give rise to an 4D $\mathcal{N} = 2$ (8 real supercharges) ungauged supergravity. One can generalize this to other dimensional Calabi-Yau compactification [19], and we list the corresponding the effective supergravity in 1.2, together with those from the M-theory Calabi-Yau compactifications (which will be employed in chapter 2 ).

### 1.6.2. Calabi-Yau compactification of Type IIB string theory

In this subsection, we will take a Calabi-Yau three-fold $X_3$ compactification of Type IIB as an example to give an analysis of the massless spectrum, which has been well studied. We will follow the notations in [43].

The parameter space of a Calabi-Yau manifold $X_3$ is that of Ricci-flat Kähler metrices $g_{i\bar{j}}$

[18] This can also be justified that the non-linear sigma model with the target space being the Calabi-Yau manifolds can attain the conformal point, hence in total viewed as the 2D $\mathcal{N} = (2, 2)$ superconformal theory with central charge $c = 9$.

[19] Here we assume all of them to be compact Calabi-Yau manifolds

| Calabi-Yau | Effective Supergravity from Type II string Theory | Effective theories from M-Theory |
|---|---|---|
| $X_1(T^2)$ | 8D $\mathcal{N}=4$ (32) on $\mathbb{R}^{1,7}$ | 9D $\mathcal{N}=2$ (32) on $\mathbb{R}^{1,8}$ |
| $X_2$(K3) | 6D $\mathcal{N}=(2,0)$ (16) on $\mathbb{R}^{1,5}$ | 7D $\mathcal{N}=1$ (16) on $\mathbb{R}^{1,6}$ |
| $X_3$ | 4D $\mathcal{N}=2$ (8) on $\mathbb{R}^{1,3}$ | 5D $\mathcal{N}=1$ (8) on $\mathbb{R}^{1,4}$ |
| $X_4$ | 2D $\mathcal{N}=(2,2)$ (4) on $\mathbb{R}^{1,1}$ | 3D $\mathcal{N}=2$ (4) on $\mathbb{R}^{1,2}$ |

**Table 1.2.:** *Type II and M-theory in various dimensions. The number in the brackets denote the number of real supercharges.*

[20]. Now given such a $g$, can we continuously deform it to a new metric $g+\delta g$ such that the Ricci-tensor still vanishes $R_{i\bar{j}}(g+\delta g)=0$? To see this, we firstly notice that there are two basic types of perturbations $\delta g$: those with pure and those with mixed type indices:

$$\delta g = \delta g_{ij} dy^i dy^j + \delta g_{i\bar{j}} dy^i d\bar{y}^{\bar{j}} + c.c.. \tag{1.105}$$

The defomations of the mixed type turns out simply correspond to those of the Kähler form $J = i g_{i\bar{j}} dy^i \wedge d\bar{y}^{\bar{j}}$, which give rise to $h^{1,1}(X_3)$ **real** scalar field $v^A(x)$ from the expansion

$$J = v^A \omega_A, \qquad A = 1, ..., h^{1,1}(X_3), \tag{1.106}$$

where the set of harmonic forms $\omega_A$s is a basis of the cohomology group $H^{1,1}(X_3)$. Such $h^{1,1}(X_3)$ real scalars in type IIB are complexified by the other $h^{1,1}(X_3)$ real scalars $b^A$, sometime dubbed theta angles, arising from the expansion of the NSNS two-form field $B_2$ and hence we have the complex fields [21] $t^A = b^A + iv^A$, parameterizing the $h^{1,1}$ dimensional complexified Kähler cone [45].

On the other hand, the deformations of the pure type is a bit subtle, it corresponds to the deformation of complex structure, which are parametrized by **complex** scalars $z^K(x)$ and are in one-to-one correspondence with harmonic $(1,2)$-forms in $X_3$

$$\delta g_{ij} = \frac{i}{||\Omega_3||^2} \bar{z}^K (\bar{\chi}_K)_{i\bar{i}\bar{j}} \Omega^{\bar{i}\bar{j}}_{j}, \qquad K = 1, ..., h^{1,2}, \tag{1.107}$$

where $\bar{\chi}_K$ denotes a basis of $H^{1,2}(X_3)$ and $||\Omega_3||^2$ stands for $\frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$.

Some basis of various cohomology groups on $X_3$ are denoted as follows. We denote the hodge dual forms to $(1,1)$-froms $\omega_A$ as $\widetilde{\omega}^A \in H^{2,2}(X_3)$. And the real, symplectic 3-forms on $H^3(X_3)$ are denoted as $(\alpha_{\widehat{K}}, \beta^{\widehat{K}})$ such that they subject to

$$\int_{X_3} \alpha_{\widehat{K}} \wedge \beta^{\widehat{L}} = \delta^{\widehat{L}}_{\widehat{K}}, \qquad \int_{X_3} \alpha_{\widehat{K}} \wedge \alpha_{\widehat{L}} = 0 = \int_{X_3} \beta^{\widehat{K}} \wedge \beta^{\widehat{L}}, \qquad \widehat{K} = 0, ..., h^{1,2}(X_3). \tag{1.108}$$

We can now expand other various type IIB supergravity bosonic fields $(\phi, B_2, C_0, C_2, C_4)$ in

[20] Here we have introduced the complex coordinates $y^i, \bar{y}^{\bar{i}}, (i,\bar{i}) = 1, ..., 3$ to parametrize the Calabi-Yau threefolds. By abuse of notation, we use the same notation for coordinates in a real internal space and a complex internal space.

[21] As a side remark, such complexifying Kähler moduli does not happen in M-theory compactification, simply because there is no $B_2$ fields. This fact leads to some interesting differences, see [44] for example.

terms of harmonic forms on $X_3$, which yields as

$$\begin{aligned} \phi =&\widehat{\phi}(x), \qquad C_0 = \widehat{c}_0(X), \\ B_2 =&\widehat{B}_2(x) + b^A\omega_A, \qquad C_2 = \widehat{C}_2 + c^A\omega_A, \\ C_4 =&\widehat{D}_2^A \wedge \omega_A + V^{\widehat{K}} \wedge \alpha_{\widehat{K}} - U_{\widehat{K}} \wedge \beta^{\widehat{K}} + \rho_A, \widetilde{\omega}^A \end{aligned} \tag{1.109}$$

where $\widehat{\phi}, \widehat{c}_0, b^A, c^A, \rho_A$ are scalars, $V^{\widehat{K}}, U_{\widehat{K}}$ are one-form fields, and $\widehat{B}_2, \widehat{C}_2, \widehat{D}_2$ are two-forms in the 4D $\mathcal{N} = 2$ effective theories. Altogether they build the 4D $\mathcal{N} = 2$ massless multiplets as the table 1.3 shows. Note that RR 4-form field $C_4$ only has half of degrees of freedoms surviving due to the self-duality condition, and hence we drop out $(\widehat{D}_2^A, U_{\widehat{K}})$ in favor of $(\rho_A, V^{\widehat{K}})$.

| Multiplets | number | components |
|---|---|---|
| Gravity | 1 | $(g_{\mu\nu}, V^0)$ |
| Vector | $h^{2,1}(X_3)$ | $(V^K, z^K)$ |
| Hypermultiplet | $h^{1,1}(X_3)$ | $(v^A, b^A, c^A, \rho^A)$ |
| Double-tensor | 1 | $(\widehat{B}_2, \widehat{C}_2, \widehat{\phi}, \widehat{c}_0)$ |

**Table 1.3.:** *4D $\mathcal{N} = 2$ multiplets from type IIB Calabi-Yau compactification*

One can also dualize the two-forms fields $\widehat{B}_2, \widehat{C}_2$ into scalars in 4 dimension, and thus the double-tensor multiplet turns to an extra hypermultiplet. The 4D effective action then can be entirely expressed in terms of vector- and hyper multiplets and collectively reads as

$$S_{IIB}^{(4)} = \int -\frac{1}{2}R{*}\mathbf{1} + \frac{1}{4}\text{Re}\mathcal{M}_{\widehat{K}\widehat{L}}F^{\widehat{K}} \wedge {*}F^{\widehat{L}} + \frac{1}{4}\text{Im}\mathcal{M}_{\widehat{K}\widehat{L}}F^{\widehat{K}} \wedge F^{\widehat{L}} - G_{KL}dz^K \wedge {*}d\bar{z}^L - h_{\widehat{A}\widehat{B}}dq^{\widehat{A}} \wedge {*}dq^{\widehat{B}}. \tag{1.110}$$

Here $q^{\widehat{A}}$ represents all $h^{1,1}+1$ hypermultiplets containing the extra dual double-tensor multiplets parametrizing a quaternionic scalar manifold $\mathcal{M}^Q$ , and $h_{\widehat{A}\widehat{B}}$ is the quaternionic metric. And $z^K$ represents all $h^{2,1}$ vector multiplets parametrizing a special-Kähler manifold $\mathcal{M}^{SK}$ with the metric $G_{KL}$. The total moduli space $\mathcal{M}$ of a 4D $\mathcal{N} = 2$ theory locally is a product of these two manifolds:

$$\mathcal{M} = \mathcal{M}^Q \times \mathcal{M}^{SK}. \tag{1.111}$$

Let us briefly discuss the special Kähler manifold $\mathcal{M}^{SK}$, which has a very elegant structure. It turns out that the geometry of such special Kähler manifold $\mathcal{M}^{SK}$ is completely determined by a holomorphic function-known as prepotential $\mathcal{F}(z)$. It determines the Kähler potential $K$, as well as its metric $G_{KL}$ as follows

$$G_{KL} = \frac{\partial}{\partial z^K}\frac{\partial}{\partial \bar{z}^L}K, \qquad K = -\ln[-i\int \Omega_3 \wedge \bar{\Omega}_3] = -\ln i[\bar{X}^{\widehat{K}}\mathcal{F}_{\widehat{K}} - X^{\widehat{K}}\bar{\mathcal{F}}_{\widehat{K}}], \tag{1.112}$$

where

$$\mathcal{F}_{\widehat{K}} = \frac{\partial \mathcal{F}}{\partial X^{\widehat{K}}} = \int_{\alpha_{\widehat{K}}} \Omega_3, \qquad X^{\widehat{K}} = \int_{\beta^{\widehat{K}}} \Omega_3. \tag{1.113}$$

Before we close this section, let us mention one fascinating fact pointed out by Aspinwall in [46]. Recall that the above special Kähler moduli space $\mathcal{M}^{SK}$, known as the moduli space of vector multiplets of a 4D $\mathcal{N} = 2$, is consistent with the complex structure moduli of a Calabi-Yau manifold $X_3$. In this section this is not surprise as the complex structure of $X_3$ in Type IIB compactifications gives rise to vector multiplets in 4D $\mathcal{N}=2$. However, there are other choices to generate a 4D $\mathcal{N}=2$ supergravity. But such equivalence suggests that the 4D $\mathcal{N}=2$ theories of supergravity "knew" that they are related in some way to the Calabi-Yau manifolds. Put in other words, string theory provides a natural way to realize this equivalence! One may wonder how about the vector multiplets in type IIA compactification on $X_3$, whose moduli space $\mathcal{M}^{SK}$ should not be the one for $X_3$ itself simply because we have wrong dimension for RR fields, but according to above arguments it should corresponds to the complex structure moduli space of a Calabi-Yau $\widetilde{X}_3$. What is the relation between $X_3$ and $\widetilde{X}_3$? It turns out this question would lead to one of the most fascinating topic in mathematics and physics- Mirror symmetry, which we will not cover in this thesis. Simply put, $\widetilde{X}_3$ is the mirror dual to $X_3$, where Type IIB compactifying on $\widetilde{X}_3$ gives rise to a same effective 4D $\mathcal{N}=2$ theory as Type IIA on $X_3$. There are abundant references on such topics, for example see [47].

## 1.7. Fluxes in Type IIB String Theory

In the previous section, we have discussed Calabi-Yau compactifications of Type II strings which give rise to 4D $N=2$ theories. Though 4D $\mathcal{N}=2$ theories exhibit a fascinating structures, it is, however, not appealing to our realistic world as eight real supercharges would necessarily break the chiralities. The only supersymmetric theories in 4D exhibiting possible chirality are those $\mathcal{N}=1$ with four real supercharges. And since type IIB compactification only gives rise to a low dimensional supergravity (with only abelian vector multiplets), in order to realize a standard model-like, which is a non-abelian gauge theory, one needs to introduce D-branes. Furthermore, recall that we have mentioned in 1.5 one of the center properties of D-branes is that it breaks half of supersymmetry, namely vacuum states with a single D-brane are not annihilated by all the supercharges but only half of them. Hence adding spacetime-filling D-branes [22] in the Calabi-Yau compactification could serve both purposes. However, before we start to discuss D-branes, let us shortly discuss Type IIB background fluxes in such compactifications, which are typically required from consistency conditions(we will introduce in 1.10) when introducing D-branes in Calabi-Yau compactifications.

As we know from Maxwell's electric-magnetism, fluxes refer to non-trivial vacuum expectation values of the field strength $F=dA$ of a gauge potential $A$, and typically they requires topological non-trivial cycles. For example, in abelian $U(1)$ gauge theory with magnetic monopoles, one has

$$\text{Flux} \approx \int_{\Sigma} F, \tag{1.114}$$

where $\Sigma$ refers to certain topologically non-trivial cycles that enclose the magnetic monopoles. It can be naturally extended to higher dimensional theories like string theories, while the topological defects are branes. In type IIB, one could also turn on so-called bulk three-forms fluxes [23] $G_3 := F_3 - \tau H_3 = dC_2 - ie^{\phi} dB_2$. Turning on such background fluxes in Type IIB Calabi-Yau threefold compactification [24] would break the 4D $\mathcal{N}=2$ supergravity to 4D $\mathcal{N}=1$

[22] "Spacetime-filling "means that the D-branes span the flat Minkowski space, as $\mathbb{R}^{1,3}$ here.

[23] Here we are only allowed to turn on the fluxes whose legs are either entirely on the internal space or in the macroscopic spacetime, in order to preserve the Lorentz invariance of the effective theory after the compactification. Throughout this thesis, we only focus on internal fluxes, as the macroscopic spacetime in this thesis are always assuming to being the flat Minkowski space hence no non-trivial cycles to support the fluxes.

[24] One should note that there is a no-go theorem stating that if the internal space is compact and non-singular,

supergravity by mass deformations as showed in [48]. More generally, in a conformally flat three dimensional internal space, one can turn on fluxes in a controlled and stable way to break 4D $\mathcal{N}=4$ to 4D $\mathcal{N}=3,2,1,0$ as showed in [49–52].

### 1.7.1. Gauge fluxes on D7-branes

Besides the above bulk three-form fluxes, one can also turn on the gauge fluxes $F$ originating from the world-volume gauge field $A$ of the D-branes in compactifications.

**D-branes with curved world-volume** Here we would like to introduce relevant aspects for D-branes with curved world-volumes, which is a typical situation in compactification. As we know from DBI action, in absence of fluxes $\mathcal{F}$, a static space-filling Dp-brane with the world-volume $\mathbb{R}^{1,3}\times\Sigma$ has energy $E$ from the 4D perspective with

$$E(\Sigma)\propto \mathrm{Vol}(\Sigma), \tag{1.115}$$

where Vol(Σ) denotes the volume of the cycles wrapped by Dp-brane in the internal space. Hence if one would like to preserve (partial) supersymmetry in the 4D, the Dp-brane should satisfy the BPS condition, which especially implies that Dp-brane shall stay at the lowest energy level. From the above, it means that they have to wrap along the submanifolds whose volumes are the minimums among the same dimensional submanifolds, which are the supersymmetric cycles, or in mathematic terms, calibrated submanifolds. The calibrated submanifolds $\Sigma_{cali}$ can be defined with respect to a calibration $\Phi_c$, which is $r$-form with two conditions:

$$\begin{aligned}\text{Algebraic condition}:&\Phi_c|_{\Sigma_{cali}}\leqslant vol(\Sigma_{cali}):=\sqrt{g|_{\Sigma_{cali}}}d\sigma\\ \text{Differential condition}:&d\Phi_c=0.\end{aligned} \tag{1.116}$$

In the case of Calabi-Yau three-manifolds, the top-form is unique $\Omega_3$ and the (product of) Kähler form $J_2$ can be calibrations. The corresponding calibrated manifolds with the top-form $\Omega_3$ are special Lagrangian submanifolds, and holomorphic cycles for the Kähler form $J_2$. In type IIA, the supersymmetric cycles wrapped by space-filling D-branes are only special Lagrangian cycles as it has odd dimensions and wrapped by D6-branes and O6-planes. Whereas in Type IIB, holomorphic curves can be wrapped by D3, D5, D7, D9-branes, as well as Op-plane, and preserve the (partial) supersymmetry. To be more precise, let's take the Type IIB strings three-fold Calabi-Yau compactification as an example. The calibration with respect to the Kähler form $J_2$ for the BPS space-filling Dp-branes are (see e.g. [53])

$$d^{p-3}\xi\sqrt{\det g}=\frac{1}{\frac{(p-3)}{2}!}J^{\frac{(p-3)}{2}}. \tag{1.117}$$

One should also need to take into account the $\mathcal{F}$ when the non-trivial background flux $F$ and $B_2$ are turned on. For example, the modified calibration condition for a D7-brane which wraps a four-cycle $S$ in $X_3$ yields

$$\int_S d^4\xi\sqrt{\det(g-i\mathcal{F})}=\int_S\frac{1}{2}e^{-i\theta}(J+\mathcal{F})(J+\mathcal{F}), \tag{1.118}$$

the necessary condition for turning on such non-trivial fluxes is that there are branes sources.

where $\theta$ is a real parameter that characterizing the unbroken supersymmetry in terms of linear combination of two supercharges. Noting that

$$\int_S d^4\xi \sqrt{\det(g - i\mathcal{F})} := \int_S \frac{1}{2} J \wedge J - \frac{1}{2}\mathcal{F} \wedge \mathcal{F}, \tag{1.119}$$

combining with the fact that in the presence of orientifold plane, then one can obtain that the condition that the D7-branes to be BPS, and necessarily being calibrated as

$$\int_S J \wedge \mathcal{F} = 0. \tag{1.120}$$

This should be reflected by the D-term or F-term in the 4D effective world-volume gauge theory of the D7-brane. It turns out the D-term take the job, that is we expect that D-term should read as [53]

$$D \sim \int_S J \wedge \mathcal{F}. \tag{1.121}$$

As for the effective gauge theory, one can still use the dimensional reduction of 10D $\mathcal{N} = 1$ SYM to obtain the low energy SYM theory, as we mentioned in 1.5. However, such dimensional reduction typically needs to perform topological twists for some supercharges in order to persevere certain supersymmetries. In a curved space, there is no guarantee for the existences of covariantly constant spinor, as we have seen from the conditions leading to Calabi-Yau spaces. A topological twist typically changes the spin structures and thus certain supercharges could survive in a curved space [54].

For later purpose, let us focus on D7-branes. The consistent configuration for internal gauge fluxes can be described by a stable holomorphic vector bundles [25] with the identification of the curvature with the gauge field strength. For all the concrete applications in this thesis we will restrict ourselves further to using a line bundles $L_a$ to characterize gauge informations for D7-branes. Typically, the gauge information on a D7-brane wrapping a divisor $D_i$ shall be encoded in the Picard group of the line bundle $L_i$, which is isomorphic to the first sheaf cohomology group of $\mathcal{O}^*_{D_i}$: $H^1(\mathcal{O}^*_{D_i})$. The Picard group enjoys a short exact sequence as

$$0 \to \mathcal{J}^1 H^1(D_i) \to H^1(\mathcal{O}^*_{D_i}) \to H^{1,1}_Z(D_i) \to 0, \tag{1.122}$$

where the third term is the gauge flux $F = dA$, given by the first Chern class $c_1(L_i) \in H^{1,1}_Z(D_i)$ and the second term $\mathcal{J}^1 H^1(D_i) := H^1_{\mathbf{C}}(D_i)/F^1 H^1_{\mathbf{C}}(D_i) + H^1_{\mathbf{Z}}(D_i)$ is a Jacobian, topology of a tours, which, in the absence of the gauge flux, parametrizes the Wilson line moduli of the gauge field $A$, i.e. the holonomy of the gauge field $A$ over a non-trivial 1-cycle. Note that if $D_i$ is simply connected, i.e. $\pi(D_i) = 0$ and no non-trivial one-cylces, then the Jacobian $\mathcal{J}^1 H^1(D_i)$ is trivial and the gauge dates are uniquely specified by the first Chern class $c_1(L_i) \in H^{1,1}_Z(D_i)$. In the sequel, we omit the degrees of freedoms of Wilson lines and focus on the gauge fluxes.

[25] A finer and more reasonable description shall involve the (coherent) sheaf, which was first suggested by J. Harvey and G. Moore in [55] for modeling D-branes on large-radius Calabi-Yau manifolds and since then it has been under vigorous development and become a common weapon for string physicists to attack the physics of D-branes, see more details in [56]. Simply put, a sheaf is the mathematical machinery need to make sense when a bundle is no longer a sensible concept such as a vector bundle living only over a submanifold or certain singular space. For our purpose, the holomorphic vector (line) bundles are good enough. Throughout this thesis, unless they are mentioned, we only talk about vector bundles instead of sheaves for characterizing the non-trivial gauge backgrounds of D-branes.

Recall that in the DBI action and Chern-Simons action, the field strength $F_a$ appears together with the pull-backs of $B_2$ fields as a gauge invariant $\mathcal{F}_a = i(\ell_s^2 F_a + i2\pi\imath^* B)$. From now one let us set $\ell_s^2 = 1$. Then given a stack of $N_a$ D-branes, which typically hosts the $U(N)$ gauge theory, we decompose the background value of the field strength $\mathcal{F}_a$ as

$$\mathcal{F}_a = T_0(F_a^0 + i\imath^* B) + \sum_i T_i F_a^{(i)}, \tag{1.123}$$

where $T_0$ denotes the unity element $\mathbb{1}$ in $U(N_a)$ representing the diagonal subgroup $U(1)_a$ and $T_i$ are the traceless abelian elements of $SU(N_a)$. In terms of the line bundles $L_a^i$, one has

$$c_1(L_a^0) = \frac{-1}{2\pi}(F_a^0 + i\imath^* B) \in H^2(D_a), \qquad c_1(L_a^i) = \frac{-1}{2\pi} F_a^i \in H^2(D_a). \tag{1.124}$$

Turning on the gauge flux will break the gauge group $U(N_a)$ to its subgroup which commutes to the $U(1)_i$. For example turning on the $F_a^{(0)}$ will breaks $U(N_a)$ to $SU(N_a)$ while turning on the $F_a^i$ breaks $SU(N_a)$ to its commuting ones, For example, if the $F_a^i$ coincides with the hypercharge generator in $SU(5)$, then one has the breaking $SU(5) \to SU(3) \times Su(2) \times U(1)_i$.

As an a side remark, following [57], we should notice that there is an important fact for constructing the gauge flux $c_1(L_a)$ from the relative cohomology group, namely a non-trivial $c_1(L_a)$ can be trivial in $H^2(X_3)$. To see this, recall that the D7-branes divisor $D_a$ defines a inclusion $\imath : D_a \to X_3$ which can further induce the pushforward map $\imath_*$and the pullback map $\imath^*$ in the corresponding homology group and its Poincaré dual cohomology group respectively, as

$$\imath_* : H_2(D_a) \to H_2(X_3); \qquad \imath^* : H^2(X_3) \to H^2(D_a). \tag{1.125}$$

Note that the pullback map $\imath^*$ induces a long sequence on the cohomology group involving the relative cohomology group $H^2(X_3, D_a)$. The relevant fact for us is that there are non-trivial part of $H^2(D_a)$ which is trivial in $H^2(X_3)$. In other words, one can split

$$L_a = \imath^*(\mathbb{L}_a) \otimes L_a^{non}, \tag{1.126}$$

where the part $\imath^*(\mathbb{L}_a)$ denote the pullback of the line bundle $\mathbb{L}_a$ defined in the $X_3$. As a consequence, the two parts, as divisors, have a vanishing intersection on $D_a$, namely

$$\int_{D_a} c_1(\imath^*(\mathbb{L}_a)) \wedge c_1(L_a^{non}) = 0. \tag{1.127}$$

## 1.8. Orientifold Compactifications of Type II Strings

In this section, we will mainly discuss consequences when introducing spacetime-filling D-brane in Type II Calabi-Yau compactifications.

As we learned from the Gauss law in electrodynamics, the total charges in a compact manifold have to be vanishing. D-branes, as higher dimensional extended objects, also carry the RR charges. Based on the same reason, such RR charges in a compact manifold such as the Calabi-Yau manifolds have to vanish. The similar argument can also applied to the D-branes tension (As we mentioned, D-branes tensions can be viewed as NSNS charges). Such considerations are dubbed as tadpole cancellation, which we will discuss more together with other consistency conditions in 1.10.2. Hence in a compact Calabi-Yau manifold, one needs to add certain sources with negative RR charges and negative tension in order to be consistent [26]. Fortunately, in string theory, such objects exist, known as orientifold planes or simply O-planes.

### 1.8.1. Orientifold planes

The orientifold planes typically arise when gauging a discrete $\mathbb{Z}_2$ symmetry, where such $\mathbb{Z}_2$ symmetry $\mathcal{O}$ in type II string theories typically contains a world-sheet parity $\Omega$ and a $\mathbb{Z}_2$ involution symmetry $\sigma$ acting on 10D spacetime $\mathcal{M}$ coordinates [27] $X^\mu$ and reverses the orientation of the strings [58]. Then an *Op*-plane can be introduced as the fix-point locus of the involution symmetry $\sigma$ from the spacetime perspective. More precisely, the involution $\sigma$ transforms [28]

$$\sigma : X^m(y,\bar{y}) \leftrightarrow -X^m(y,\bar{y}); \qquad m = p+1, ...., 9 \tag{1.128}$$

and an Op-plane resides at the fixed plane of this $\sigma$ and extends in the $(X^0, X^1, ...., X^p)$ directions.

In perturbative type II string theories, orientifold planes, like D-branes, also couple to massless closed string modes and carry RR charges, as well as breaking half of bulk supersymmetries. However, unlike D-branes, they have fixed **negative** tensions and hence are not dynamical (at least in the perturbative strings) in a sense that they cannot fluctuate and therefore cannot carry degrees of freedom [29]. Based on their dimension of spatial space, we distinguish them as *Op*-planes in the same sense as *Dp*-branes. Similarly, Op-planes are stable in type IIA with $p$s being even and in Type IIB with $p$s being odd. As mentioned above, a Op-plane carries charge under the RR fields $C_{p+1}$, and this couplings are also captured by the Chern-Simons action, which is given :

$$S^{Op} = -\frac{Q_p \mu_p}{2} \int_{O_p} \sum_{2p} \imath^* C_{2p} \sqrt{\frac{L(\frac{1}{4}\mathrm{T}O_7)}{L(\frac{1}{4}\mathrm{N}O_7)}}\,. \tag{1.129}$$

where $|Q_p| = 2^{p-4}$ [30] and $L$ denotes the Hirzebruch L-genus. The coupling to $C_{p+1}$ is then

$$-\frac{Q_p \mu_p}{2} \int_{O_p} C_{p+1}. \tag{1.130}$$

Note that there is opposite sign comparing with the Dp-branes, which is exactly the reason we are introducing them. As they are not dynamical and no open strings attaching them, thereby

[26] Of course, such two conditions can be satisfied separately. For example, one can add anti D-brane to cancel the RR tadpole and invoking other mechanisms to tackle with the issue with the NSNS tadpoles. However, the anti Dp-branes need to be fixed at certain loci otherwise they would approach to Dp-branes by attraction and eventually will annihilate, inspired by the interaction between a particle and its anti-particle, and would be not wanted. One famous example with such stabilized anti-Dp-branes is the anti D3-branes, sitting in the deeply warped region on CY, in the KKLT setting with an effort to construct dS spaces from String theory compactification. However, such cases needs extra and maybe subtle conditions, which makes this option not so economic.

[27] Note in Calabi-Yau compactification, it shall act trivially on Minkowski spaces and thus the O-planes are space-time filling.

[28] Here we rewrite the 10D coordinates in terms real ones $X$s.

[29] Since a fluctuating negative object necessarily has negative-norm states

[30] Note that Op-planes do not always carry negative RR charge. We usually denote the Op-planes with negative planes as $\mathrm{O}^-$p-planes and $\mathrm{O}^+$p-planes with positive RR charges. But they do always carry negative tensions. Unless specified, we will omit the superscript $\pm$ and assume they are with negative RR charges in this chapter.

no massless gauge fields lives on their world-volume. The DBI-like action for Op-planes reads

$$S_o = 2^{p-4}T_o \int d^{p+1}\xi e^{-\phi}(\sqrt{-\det(g)}), \tag{1.131}$$

where $g := \det(g_{ab})$ denotes the determinant of the inducing metric $g_{ab}$ from the spacetime metric $g_{\mu\nu}$ to the Op-plane world-volume and the tension $T_o = Q_p T_p$. Such action is essentially the higher dimensional generalization of Nambu-Goto action.

### 1.8.2. Type IIB O3/O7 orientifold compactifications

Now we are in the position to introduce Calabi-Yau orientifold compactifications. Before we go to the specific Type IIB O3/O7 orientifold compactifications, let us say a bit more on general aspects of the above $\mathbb{Z}_2$ action $\mathcal{O}$ on Calabi-Yau compactifications, dubbed orientifold action. In order for such action to preserve supersymmetries, there are certain constraints. In particular, the involution $\sigma$ must be isometric and (anti) holomorphic [59]. To be more precise, the involution [31] $\sigma^*$ must be anti-holomorphic ($\sigma^* J_n^m = -J_n^m$) in type IIA and holomorphic $\sigma J_n^m = J_n^m$ in type IIB, respectively. Further the unique holomorphic $(n, 0)$-from $\Omega_3$ must also be an eigenform of $\sigma^*$ in type IIB, namely $\sigma^*\Omega_n = \pm\Omega_n$. In type IIA, it acts on the holomorphic n-form as $\sigma^*\Omega_n = e^{2\pi\theta}\bar{\Omega}_n$, where $\theta$ is some phase, typically set to 0 by redefining $\Omega_n$.

In order to preserve same supercharges as the one preserved by spacetime-filling D-branes, it turns out that in the presence of D3/D7-brane systems in type IIB Calabi-Yau compactifications, the orientifold action $\mathcal{O}$ should be taken as

$$\Omega(-1)^{F_L}\sigma, \qquad \Omega_n = -\sigma^*\Omega_n, \tag{1.133}$$

while in the presence of D5/D9-brane systems, it is taken as

$$\Omega\sigma, \qquad \Omega_n = \sigma^*\Omega_n. \tag{1.134}$$

And in the presence of D6-brane systems in Type IIA Calabi-Yau compactifications, it's taken as

$$\Omega\sigma, \qquad \Omega_n = \sigma^*\bar{\Omega}_n, \tag{1.135}$$

where $F_L$ is the spacetime left-mover fermion number. Now having said only odd-valued $p$ for Op-planes stay stable in Type IIB and even-valued for Type IIA, one might wonder can all allowed values for $p$ coexist in the same Calabi-Yau compactification. To see that one needs to look at the involution action on $\Omega_n$ in the vicinity of O-planes. For definiteness, let us work on a Calabi-Yau threefold $X_3$ . Assuming nearby the Op-planes, the holomorphic top form $\Omega_n$ locally takes the form as $\Omega_n = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^3$. Now the action reads

$$\sigma^*(dy^1 \wedge dy^2 \wedge \cdots \wedge dy^3)|_{Op} = \pm(dy^1 \wedge dy^2 \wedge \cdots \wedge dy^3) \tag{1.136}$$

as at the fixed locus $Op$ either an even or an odd number of $y^i$ have to change the sign. For an even number, i.e. $\sigma^*(\Omega_n) = \Omega_n$, the internal fixed locus is 2 or 6 dimensions. And an

[31] Here $\sigma^*$ denotes the pullback of $\sigma$ for the Dolbeault cohomology of $X_n$. The involution $\sigma$ induces an eigenspace splitting of the Dolbeault cohomology groups

$$H^{p,q}(X_n) = H_+^{p,q}(X_n) \oplus H_-^{p,q}(X_n) \tag{1.132}$$

corresponding to the eigenvalues $\pm 1$.

odd number, the internal fixed locus is 0 or 4 dimensions. This is exactly the dimension of the D-branes systems. In other words, the orientifold action $\mathcal{O}$ generating the Op-planes is coincident with the one for Dp-branes. Hence in orientifold compactification, with the same spacetime-filling dimensional Op-planes and Dp-branes, one can in principle cancel the tadpole issue and preserve certain supersymmetries in low-dimensional effective theories. Otherwise, the supersymmetries will be totally broken if one use the Op-planes to solve the RR tadpole issue. In the rest, we will always stick to the first situation.

From here we will focus on orientifold compactification of Type IIB string theory with O7/O3-planes in Calabi-Yau compactifications with generic dimensions. For our purposes, we take a look on how the the 10D Type IIB supergravity fields transform under the orientifold projection $\mathcal{O} := \Omega_p(-1)^F\sigma$. In order to enter the low dimensional effective action, the Type IIB various fields should survive, i.e. they should transform even under the orientifold projection. We listed the type IIB supergravity fields under the $\Omega_p(-1)^F$ in 1.4, those who transform even (odd) under $\Omega_p(-1)^F$ should be required to transform even (odd) under the the involution $\sigma$.

| | even | odd |
|---|---|---|
| $\Omega_p$ | $\phi, g_{\mu\nu}, C_2$ | $C_0, B_2, C_4$ |
| $(-1)^F$ | $\phi, g_{\mu\nu}, B_2$ | $C_0, C_2, C_4$ |

**Table 1.4.:** *The various Type IIB supergravity fields transformations under the world-sheet parity $\Omega_p$ and fermion number sign $(-1)^F$.*

| | even | odd |
|---|---|---|
| $\sigma$ | $\phi, g_{\mu\nu}, C_0, C_4$ | $C_2, B_2$ |

**Table 1.5.:** *The various Type IIB supergravity fields transformations under the involution $\sigma$.*

For more details on the explicit dimensional reductions and low-energy effective theory action, we refer to references, for examples [53,60,61] with Calabi-Yau three-folds. Before moving to next step, we would like to make a crucial comment. Recall we have said that D-branes and O-planes, as dynamic objects with tensions, should typically have back-reactions on supergravity bulk fields (together with the background fluxes), if there are not on top of each others. Let us first talk about back-reactions on metric $g_{\mu\nu}$. Thus it implies that the geometry of internal spaces would be changed after introducing D-branes and O-planes, and thus the internal spaces might not be Calabi-Yau manifolds. Indeed, it turns out in our cases that after the involution action $\sigma$, the geometry $B_n := X_n/\sigma$, dubbed downstairs geometry, with respect to the upstair geometry: $X_n$, is not a Calabi-Yau space anymore, but a Kähler manifold. The reason is that in such compactification, the background fluxes typically need to be turned on for consistency conditions and such fluxes will be obstructions for the internal space being a Calabi-Yau, i.e. contributing some torsions and obstructing $J$ and $\Omega_n$ to be integrable. However, one of specialties of Type IIB O3/O7 orientifold compactifications is that the Kähler manifold $B_n$ is a so-called conformal Calabi-Yau, which means that the metric has related a Calabi-Yau one by a conformal factor, namely:

$$ds^2(B_n) = e^{-2A} ds^2(X_n), \tag{1.137}$$

where $A$ is the warp factor, and typically small near certain limits in the moduli space of $B_n$.

This fact make the type IIB O3/O7 compactifications the most popular choices among other orientifold compactifications for model building, which we will discuss shortly. As one can still use the well-studied mathematics for Calabi-Yau to study. A systematically descriptions for such flux compactification would involve the G-structures for complex manifolds, we refer to the review [62] for more details.

The other important back-reactions for the discussions laid out in chapter is on the axio-dilaton $\tau$. As we have learned that D7-branes and O7-planes are under magnetically charged by $C_0$ and hence by the axio-dilaton $\tau$, the presence of D7-branes and O7-planes generically would have back-reactions on the axion-dilaton $\tau$. Indeed, in order to characterize such physics, one necessarily need to invoke F-theory, which is the main topic in the next chapter 2. In the contexts of Type IIB orientifold compactifications, we assume that such back-reactions are small and only at the perturbative limit, i.e. $g_s \to 0$.

## 1.9. Open String Sectors in Type IIB O3/O7 Orientifold Compactifications

From the previous section, we mainly answered the question what the consequences would be after adding spacetime-filling D-brane into Calabi-Yau compactification of Type IIB. In short, one typically needs to introduce the O-planes to solve the RR tadpoles issue and hence is necessarily leading to the orientifold compactification.

Recall that the involution $\sigma$ in the O3/O7 orientifold projection 1.133 acts non-trivially on Calabi-Yau manifolds $X_n$, hence it also acts non-trivially on the D7-brane (and probably D3-brane in Calabi-Yau four-fold compactifications) as extended objects in the spacetime. Without loss of generality, Let us focus on D7-branes. We assume a stack of $N_i$ spacetime-filling D7-branes wrapping holomorphic divisor $D_i$ in the $X_n$. The involution $\sigma$ then maps $D_i$ to its orientifold image $D_i' = \sigma^* D_i$, which we viewed as wrapped by a stack of $N_i$ image D7-brane. Thus, in the upstairs geometry, each brane is accompanied by its image brane. Three qualitatively different classes of D7-branes configurations need to be distinguished:

$$\begin{aligned} &1.[D_i] \neq [D_i'] \equiv [\sigma^* D_i];\\ &2.[D_i] = [D_i'] \text{ but } D_i \neq D_i' \text{ point-wise;}\\ &3.D_i = D_i' \text{ point-wise.} \end{aligned} \tag{1.138}$$

Here the class $[D_i] \in H^2(X_n)$ is Poincaré dual to the divisor class $D_i$, $i = 1, ..., n$ the number of D7-brane stacks. In the third case, the D7-branes sit on top of O7-plane, whereas the first two cases they may or may not intersect the O7-plane. Without providing any proofs [32], we state that if there are no any gauge fluxes on the D7-branes, the first case carries unitary gauge groups $U(n)$ while the other two yield symplectic $Sp(n)$ or orthogonal gauge groups $SO(n)$, respectively. Note that the exceptional gauge groups would not arise from orientifold compactifications, such groups would inevitably require non-perturbative effects.

Let us merely give a brief look on the orientifold action on the gauge invariant fluxes $\mathcal{F}_i$ to their image $\mathcal{F}_i'$. The involution acts on the D7-branes divisors as $\sigma : D_i \to D_i'$ which induces a map $\sigma^*$ on the cohomology group of $D_a$ as $\sigma^* : H^2(D_i) \to H^2(D_i')$. Further the world-sheet parity map $\Omega$ also acts on the gauge fluxes. In terms of the line bundle, it acts as

$$L_i \to L_i^\vee, \tag{1.139}$$

[32] One needs to analyze the spectrum of open strings in these setting, see any standard textbook on string theory.

where $L_a^\vee$ represent the dual bundle of $L_a$. In particular, for the gauge invariant $\mathcal{F}_a$, it yields

$$\mathcal{F}_i \to \mathcal{F}'_i = -\sigma^* \mathcal{F}_i, \tag{1.140}$$

where the $-1$ sign is due to the world-sheet parity $\Omega$. In general, the Chern character of the image bundle $L'_i$ reads

$$\mathrm{ch}_k(L'_i) = (-1)^k \sigma^* \mathrm{ch}_k(L_i) = \sigma^* \mathrm{ch}_k(L_i^\vee). \tag{1.141}$$

For future reference, let us present an important fact relating the above divisors in the upstairs geometry $X_n$ to the corresponding one in downstairs geometry $B_n$. To start we define a combination

$$D_i^\pm = D_i \cup (\pm D'_i), \tag{1.142}$$

with its Poincaré dual $[w_i^\pm] \in H^2(X_n)^\pm$. From above, it is easy to see that the divisor class $[w_\alpha^+], \alpha \in h_+^{1,1}(X_n)$ survives under the involution $\sigma$ and remains as 2-form in the downstairs geometry, which we denote $[\omega^\alpha]$ in $H^2(B_n)$. Denoting the map $\pi_o : X_n \to B_n$, then we have a pullback

$$\pi_o^*(\omega^\alpha) = w_\alpha^+. \tag{1.143}$$

Note that the intersection numbers in $B_n$ are related to ones in $X_n$ as

$$\int_{X_n} [\pi_o^*(\omega^\alpha)] \wedge [\pi_o^*(\omega^\beta)] = \int_{\pi_0(X_n)} [\omega^\alpha] \wedge [\omega^\beta] = 2 \int_{B_n} \omega^\alpha \wedge \omega^\beta, \tag{1.144}$$

where the last equality is due to $\pi_0(X_n) = 2B_n$ as a double covering.

### 1.9.1. Massless spectra in intersecting D7-brane models

Now we are in the position to discuss (charged) massless spectra in the Type IIB O3/O7 orientifold compactification. As we have implied, one of the primary motivation to consider the orientifold compactification is for the favor of model building. Namely we would like to study whether string theory can provide a standard model-like or their supersymmetric correspondences. The main characteristics for such models is that they have chiral matter in (bi) fundamental representations of certain gauge groups. Up to now, we have implied that the Type IIB O3/O7 orientifold compactification can possible provide chiralities, for example with three-dimensional upstair geometry $X_3$ we can have 4D $\mathcal{N} = 1$ non-abelian gauge theories from D-brane sector coupled to supergravity. But we have seen that in the absence of gauge fluxes, the fermions in the gauge theories on a stacks of D-branes are in the adjoint representation. Note in the 4D cases, the adjoint representation is real and does not lead to non-trivial chiral spectrum. So we need to get some other representations in such compactification.

Such other representations do exist and arise from the intersecting D-branes [33]. Let us focus on D7-branes. Note when two stacks of D7-branes intersects, say wrapping divisors $D_i$ and $D_j$, then at the intersecting loci $D_i \cap D_j$ the open strings stretching these two D7-branes will generate the massless spectrum and contribute additional representations. To see that, a heuristic argument goes like this: suppose we start with a stack $N + M$ D7-branes with no gauge fluxes, and then rotate the $N$ copies away so that they intersects with the other $M$ copies.

[33] In parts of literature, they may refer the method we are discussing as magnetized D-branes as we need to turn on gauge fluxes later to generate chiral spectra.

From the gauge theory perspective, such process corresponds to higgsing $U(M+N)$ gauge groups to $U(M) \times U(N)$ groups. In terms of matter representation, it yields

$$\mathbf{Adj}(U(M+N)) \quad \rightarrow \mathbf{Adj}(U(M)) \oplus \mathbf{Adj}(U(M)) \oplus [(M_i, \bar{N}_j) \oplus c.c.] \tag{1.145}$$

and hence we can realize the bi-fundamental representations than the adjoint representations. As we will see later, turning on gauge fluxes even on a single stack of D7-branes can also decompose adjoint representaitons to other representations. So in principle one can realize the purpose. Let us go to the details. In the upstairs geometry $X_n$, Depending on where open strings end, there are four possibilities of strings in $(ii')$ sector, $(ij')$-sector and $(jj')$ sector

| Sector | $U(N_a)$ | $U(N_b)$ | Chirality |
|---|---|---|---|
| $(ij)$ | $\overline{\Box}_{-1}$ | $\Box_1$ | $I_{ij}$ |
| $(i'j)$ | $\Box_1$ | $\Box_1$ | $I_{i'j}$ |
| $(i'i)$ | $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2$ | 1 | $\frac{1}{2}(I_{i'i} + 2I_{\text{O7i}})$ |
| $(i'i)$ | $\Box\Box_2$ | 1 | $\frac{1}{2}(I_{i'i} - 2I_{\text{O7i}})$ |

**Table 1.6.:** *Localized matter spectra for intersecting D7-branes, the subscripts denote abelian gauge $U(1)$ group charge. Taken from [57]. The third line with two index antisymmetric representation refers to the case 3 in 1.138, i.e. the D7-branes are on top of a O7-plane. The fourth line with two index symmetric representation refers to the case that D7-branes are away from O7-branes*

We are particularly interesting in chiral spectra and their chirality. It turns out there are two types of chiral spectra in this setting, namely the so-called bulk matter and the localized matter if the dimension of the upstairs geometry $X_n$ allows. In the following, we will present the chirality from these matter for Calabi-Yau three-folds and Calabi-Yau four-folds.

**Bulk matter** The bulk matter refers to the massless states that propagate along the whole divisors $D_a$. To be more precise, one can imagin that the matter is generated by the open string residing on two stacks of D7-branes with number $N_i$ and $N_j$ which wrap along a same divisor, namely $D_i = D_j = D$ but in principle can carry different line bundles $L_i$ and $L_j$. As we have said, turning on the gauge flux $L_i$ and $L_j$ breaks to the corresponding non-abelian groups associated with the 7-branes $(G)$ to its commutant subalgebras $(H_i, H_j)$ as

$$G \quad \rightarrow \quad (H_i, H_j) \tag{1.146}$$

$$(\mathbf{Adj}(G) \quad \rightarrow \quad (\mathbf{Adj}(H_i), 1) \oplus (1, \mathbf{Adj}(H_j)) \oplus \bigoplus_{i,j} [(N_i, \bar{N}_j) \oplus (c.c)]. \tag{1.147}$$

According to the analysis in [63], open strings connecting two stacks of D-branes can generate bifundamental matter in representation $N_i, \bar{N}_j$ and its associated zero modes are counted by the extension group $Ext^n(\imath_* \mathcal{L}_i, \imath_* \mathcal{L}_j), n = 0, .., \dim(X_n)$[34]. These extension groups can equivalently be expressed in terms of certain Dolbeault cohomology groups.

On a Calabi-Yau three-fold $X_3$, the chiral index for the bulk modes in the representation

[34]Here we focus on the pushforward line bundle $\imath_* \mathcal{L}_i$. For the line bundle $L_a^{non}$ which is trivial on the $X_n$, this chiral index would not change, as the index eventually is evaluated on $X_n$ hence it would not get any contributions from $L_a^{non}$

$(N_i, \bar{N}_j)$ can be calculated by

$$\begin{aligned} I_{ij}^{bulk} =& \sum_{n=0}^{3} (-1)^n \mathrm{dimExt}^n(\imath_*\mathcal{L}_i, \imath_*\mathcal{L}_j) \\ =& \int_{D_i \bigcap D_j} (c_1(\imath_*\mathcal{L}_i) - c_1(\imath_*\mathcal{L}_j)) = \int_{X_3} (c_1(\mathcal{L}_i) - c_1(\mathcal{L}_j)) \wedge [D] \wedge [D]. \end{aligned} \tag{1.148}$$

According to [57,63], the groups with $i = 0, 3$ are trivial (at least in all situations which allow for a solution to the supersymmetry equations). The states counted by the groups with $i = 1$ refer to the anti-chiral multiplets in 4D $\mathcal{N} = 1$ theories which count Wilson line moduli while for $i = 2$, the states refer to the D7-brane deformation moduli and form chiral multiplets. Note that they both are accompanied by their CPT conjugate partners, which are, respectively, chiral and anti-chiral multiplets in the complex conjugate representations. For Calabi-Yau three-folds, this fact implies that the multiplets in the adjoint representation can not have non-vanishing chiral index, as the adjoint representation in the 4D is self-conjugate.

On a Calabi-Yau four-fold $X_4$, the sheaf extension group $Ext^i$ in principle counts the massless spectra of the 2D $\mathcal{N} = (0, 2)$ theories, but we phrase the discussion directly in terms of the related Dolbeault cohomology groups. It has been argued in [23, 64, 65] that the Dolbeault cohomology $H^p_{\bar{\partial}}(D, L_{\mathbf{R}})$ counts the massless matter in representation $\mathbf{R} = (N_i, \bar{N}_j)$ for some vector bundle $L_{\mathbf{R}}$ on the divisor $D$. More precisely, $H^p_{\bar{\partial}}(D, L_{\mathbf{R}})$ counts the vector multiplets, chiral multiplets, Fermi multiplets and chiral multiplets of 2D $\mathcal{N} = (0, 2)$ theories for $n = 0, 1, 2, 3$ respectively. The chiral index for the bulk matter in the representation $\mathbf{R}$ reads

$$\begin{aligned} I^{bulk} =& \sum_{n=0}^{3} (-1)^n H^n_{\bar{\partial}}(D, L_{\mathbf{R}}) \\ =& 2 \int_D (c_1(D)(\frac{1}{12} rk(L_{\mathbf{R}}) c_2(D) + ch_2(L_{\mathbf{R}})). \end{aligned} \tag{1.149}$$

Applying above to our cases, i.e. $L_{\mathbf{R}} = L_i - L_j$ and $\mathbf{R} = (N_i, \bar{N}_j)$, we have $ch_2(L_{\mathbf{R}}) = \frac{1}{2}(c_1^2(\mathcal{L}_i) - c_1^2(\mathcal{L}_j))$, and we have

$$I_{ij}^{bulk} = \int_{X_4} (c_1^2(\mathcal{L}_i) - c_1^2(\mathcal{L}_j)) \wedge [D] \wedge [D]. \tag{1.150}$$

Here we use that $c_1(D) = -D$ by applying the adjunction formulas B.2.4 in the Calabi-Yau $X_4$.

Note that, in the Calabi-Yau four-folds, the effective theory from the Type IIB compactification is two dimensional. In two dimension, the CPT conjugate of a chiral fermion is still chiral but in the complex conjugate representation. Hence the adjoint representation does contribute to the chiral index, and in this case the chiral index for the adjoint representation $H_i$ reads

$$I_{adj}^{bulk} = \frac{1}{12} \int_D c_1(D) \wedge c_2(D). \tag{1.151}$$

Here we used the fact that for a real representation, this expression is to be multiplied by $\frac{1}{2}$.

Note that for Calabi-Yau four-fold, the counting differs by a factor of 1/2 from the similar one in 2.12.1 for F-theory compactification on a Calabi-Yau five-manifold $\widehat{X}_5$, which can be viewed as the Type IIB orientifold compactification on $B_4$. The 1/2 arises from 1.144 with the fact that $X_4$ can be viewed as the double-covering space of $B_4$.

**Localized matter** The localized zero modes refer to massless matter which are trapped at the intersection loci $C_{ij}$ between two different D7-branes wrapping the divisor $D_i$ and $D_j$.

By the same token, the chiral index for the localized matter in a Calabi-Yau three-fold $X_3$ reads

$$\begin{aligned} I_{ij}^{loc} &= \sum_{n=0}^{3} (-1)^n \text{dimExt}^n(\imath_*\mathcal{L}_i, \imath_*\mathcal{L}_j)) \\ &= \int_{D_i \bigcap D_j} (c_1(\imath_*\mathcal{L}_i) - c_1(\imath_*\mathcal{L}_j)) = \int_{X_3} (c_1(\mathcal{L}_i) - c_1(\mathcal{L}_j)) \wedge [D_i] \wedge [D_j]. \end{aligned} \tag{1.152}$$

The above $(c_1(\mathcal{L}_i) - c_1(\mathcal{L}_j))$ can be viewed as the line bundle $L_{ij}$ induced by the $\mathcal{L}_i$. Namely, given a gauge line bundle $L_i$ along the 7-brane $D_i$, it induces a corresponding gauge line bundle $L_{ij}$ along the intersection $C_{ij}$ with another D7-brane $D_j$ as

$$L_{ij} := L_i|_{C_{ij}} \otimes L_j^\vee|_{C_{ij}}, \tag{1.153}$$

where $L_j$ represents the gauge bundle over the other D7-brane $D_j$. Simply, in terms of the first Chern class, one has

$$c_1(L_{ij}) = c_1(L_i) + c_1(L_j^\vee) = c_1(L_i) - c_1(L_j). \tag{1.154}$$

While on a Calabi-Yau four-fold $X_4$, we turn to the Dolbeault cohomology group

$$H^n(C_{\mathbf{R}}, L_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}), n = 0, 1, 2 \tag{1.155}$$

which counts 2D chiral multiplets, Femi multiplets and chiral multiplets, respectively. Then the chirality is given by

$$\begin{aligned} I_{ij}^{loc} &= \sum_{n=0}^{3} (-1)^n h^n(C_{\mathbf{R}}, L_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}) \\ &= -\frac{1}{12} \int_{C_{\mathbf{R}}} (-c_1^2(C_{\mathbf{R}}) + \frac{1}{2} c_2(C_{\mathbf{R}}) + 12 ch_2(L_{\mathbf{R}})), \end{aligned} \tag{1.156}$$

where the last term $ch_2(L_{\mathbf{R}})$ is determined by vector bundle $L_{\mathbf{R}}$. Note that in the Calabi-Yau four-fold $X_4$, there are localized matter whose chirality are not totally depending on the gauge fluxes. In terms of our cases with $L_{\mathbf{R}} = L_i - L_j$ and $\mathbf{R} = (N_i, \bar{N}_j)$, the last term is given by

$$ch_2(L_{\mathbf{R}}) = \int_{D_i \bigcap D_j} (ch_2(\imath_*\mathcal{L}_i) - ch_2(\imath_*\mathcal{L}_j)) = \int_{X_4} (c_1^2(\mathcal{L}_i) - c_1^2(\mathcal{L}_j)) \wedge [D_i] \wedge [D_j]. \tag{1.157}$$

Finally, one can also have non-trivial chiral index localized at the intersection between the O7-plane and D7-brane. On a Calabi-Yau 3-fold $X_3$, the chiral index can be obtained by similar reason as

$$I_{o7i} = \int_{X_3} c_1(\mathcal{L}_i) \wedge [D_i] \wedge [O7], \tag{1.158}$$

which measures the chiral index for the anti-symmetric representation.

On a Calabi-Yau four-fold $X_4$, the chirality is total depending on the gauge fluxes, and in our cases, is given by

$$I_{o7i} = \int_{X_4} c_1^2(\mathcal{L}_i) \wedge [D_i] \wedge [O7]. \tag{1.159}$$

However, one novel aspect of the chiral spectra in a Calabi-Yau four-fold $X_4$ is that strings sitting in the intersecting loci between D3-brane and D7-branes can also generate chiral spectrum. Due to the time reasons, we are not going to present the results here, but refer to the section 4.7 and the corresponding F-theory lifts in 2.12.1 for more details.

## 1.10. Consistency Conditions

### 1.10.1. Freed-Witten anomaly

The gauge flux on D7-branes should satisfy certain quantisation and consistency condition. The consistency condition arise from the Freed-Witten anomaly [66], which read as

$$c_1(L_i) + i i^* B + \frac{1}{2} c_1(K_{D_i}) \in H^{1,1}_{\mathbb{Z}}(D_i), \tag{1.160}$$

where $L_i$ is the line bundle describing the gauge flux on D7-branes and $D_i$ denotes the divisor wrapped by D7-branes. Such condition grantees that single valuedness of the world-sheet path integral of an open string wrapping the two-surface $\Sigma$ whose boundary $\partial\Sigma$ along the $D_i$. For more details, we refer to [66](see also the short review in [57]).

### 1.10.2. Tadpole cancellations

Recall that in Type II string theories Dp-branes carry an RR charges and couples to the background RR $C_{p+1}$ fields indicated by the Chern-Simons action (A.3). For the spacetime-filling D-branes, such coupling could rise the issue RR tadpoles. To be more precise, the RR charges carried with the space-filling D-branes could be sources in a compact internal space. However, as we learned from the Gauss law in the electrodynamics, the total charges in a compact space have to vanish otherwise the flux lines associated with them have nowhere to escape. As for the RR charges, which can be viewed as the high dimensional generalization of the Gauss law, these conditions are dubbed as the RR tadpole conditions. The appearance of the RR tadpoles make the theory inconsistent, and typically the 1-loop open string amplitude would be divergent. By the same token, one should also consider the NS-NS tadpoles cancellation. The NS-NS tadpoles arise from the DBI action of the D-branes coupling to the NS-NS fields such as the graviton $g_{\mu\nu}$ and the dilaton $\phi$ and the tension $T_p$ in this sense can be viewed as the NS-NS fields charge. In a consistent compactification, RR tadpoles and NS-NS tadpoles have to be cancelled. However, the presence of NS-NS tadpoles typically would give rise potential for the NS-NS background fields which can be cured by the Fischer-Susskind mechanism [67] involving certain background fields, and hence the NS-NS tadpole indicate that the backgrounds are not stable. Let us focus on RR tadpole cancellations in the sequel.

For definiteness, let us mainly focus on Calabi-Yau three-folds as the main example to illustrate the details and similar results can be generalized to other dimensional Calabi-Yau spaces. In order to cancel the RR tadpoles, we have mentioned one can introduce Op-branes. From the CS actions of D7-branes and O7-planes, apart from D7-brane charges, one can see that they can also induce D5-branes and D3-branes charges in the presence of non-trival gauge fluxes $F$ and the internal curvature terms.

As for the $C_8$ coupling/D7-brane tadpole conditions, we have to impose

$$\sum_i n_i([D_i] + [D_{i'}]) = 8[D_{O7}], \tag{1.161}$$

where $D_{O7}$ denotes the divisors wrapped by the O7-planes. Note here we are working in the upstairs geometry. In the downstairs geometry, instead we have

$$\sum_a n_a([D_a]) = 4[D_{O7}]. \tag{1.162}$$

Let us stick to the upstair geometry for the rest. As for the $C_6$ coupling, we have

$$\sum_a \int_{X_3} \omega_a \wedge ([D_a] \wedge \mathrm{tr}\mathcal{F}_a + [D'_a] \wedge \mathrm{tr}\mathcal{F}'_a) = 0. \tag{1.163}$$

Note that for the conditions with $[D_a] = [D'_a]$ and $\mathrm{tr}\mathcal{F}_a = -\mathrm{tr}\mathcal{F}'_a$, this D5-brane tadpole can be automatically cancelled.

As for the $C_4$ coupling, By summing over all the induced $C_4$ coupling we have

$$(N_{D3} + N_{D3'}) + N_{Flux} - \sum_a (Q_{D7} + Q'_{D7}) = \frac{N_{O3}}{2} + Q_{O7}, \tag{1.164}$$

where $N_{D3}, N_{O3}$ counts the number of spacetime-filling D3-branes and O3-planes, respectively. And $Q_{O7}$ is given by

$$Q_{O7} = \chi(O7)/6 = \int_{X_3} c_2(O7) \wedge [D_{O7}] = \int_{X_3} [D^3_{O7}] + c_2(X_3) \wedge [D_{O7}]. \tag{1.165}$$

However, $Q_{D7}$ is a bit subtle for calculation. It depends on the D7-brane and image D7-branes configuration in 1.138. Namely, in the case (1) and (3), one has

$$Q^a_{D7} = N_a \chi(D_a)/24 + \frac{1}{8\pi^2} \int_{D_a} \mathrm{tr}\mathcal{F}^2_a. \tag{1.166}$$

While in the case (2), the D7-brane divisor would develop singularities. Taking this into account, one needs to modify the Euler characteristic of $D_a$ to $\chi_o(D_a)$ as

$$\chi_0(D_a) = \chi(D_a) - n_{pp}, \tag{1.167}$$

where $n_{pp}$ denotes the compensating number whose precise value can be referred to [68].

# Chapter 2.

# F-theory

In this section we are going to embark on an analysis of F-theory compactifications, which is the main theme in this thesis. We will discuss in detail F-theory compactifications and their most important physical aspects such as their low-energy limit, the appearance of non-abelian gauge groups, matter spectra, abelian gauge groups, discrete symmetry and the description of the flux sector, together with the corresponding geometric description of elliptically fibered Calabi-Yau spaces. These ingredients will be heavily relied on in the following chapters. As we will show, the geometry and physics are heavily intertwined in F-theory compactifications. There are a plethora of good introductory reviews on F-theory, for examples [25, 69, 70], among others, of which we will rely on. Some parts of disuccsion in this chapter also follow closely with the comprehensive 2018 TASI lecture by T. Weigand [21].

## 2.1. $SL(2,\mathbb{Z})$ Invariance of Type IIB and Monodromies of Seven-branes

In order to appreciate that the Type IIB superstring enjoys $SL(2,\mathbb{Z})$ duality symmetry, we rewrite the low-energy Lagrangian (1.66) of Type IIB string in the Einstein frame, which is given by [24]

$$S_{\text{IIB}} = 2\pi \left( \int d^{10}x\sqrt{-g}(R - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2\tau_2^2} - \mathcal{M}_{IJ} F_3^I \cdot F_3^J - \frac{1}{4}|F_5^2|) - \frac{\epsilon_{IJ}}{4} \int C_4 \wedge F_3^I \wedge F_3^J \right). \tag{2.1}$$

Here we rewrite $\tau \coloneqq C_0 + ie^{-\phi} = \tau_1 + i\tau_2$ being the axio-dilaton and $F_3^I$ being the three-form field strength doublet:

$$\begin{pmatrix} F_3^1 \\ F_3^2 \end{pmatrix} = \begin{pmatrix} dB_2 \\ dC_2 \end{pmatrix}. \tag{2.2}$$

Further, the matrix $\mathcal{M}_{IJ}$ is given by

$$\mathcal{M}_{IJ} = \frac{1}{\text{Im}(\tau)} \begin{pmatrix} |\tau|^2 & -\text{Re}(\tau) \\ -\text{Re}(\tau) & 1 \end{pmatrix}. \tag{2.3}$$

As one can check, the action (2.1) is manifestly invariant under the $SL(2,\mathbb{Z})$ duality as follows[1].

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad F_3^I \to M^I_J F_3^J, \qquad F_5 \to F_5, \qquad g_{MN} \to g_{MN}, \qquad M^I_J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.4}$$

[1] More precisely, the (2.1) is the type IIB classical supergravity whose duality group can be extended to $SL(2,\mathbb{R})$. However, the whole type IIB superstring theory is believed to enjoy the duality group of $SL(2,\mathbb{Z})$ as one expects the quantum effects such as D(-1)-instantons would break the continuous one to the discrete subgroup. It has a role somewhat like that of $\mathbb{Z}$ inside of $\mathbb{R}$.

Note that the RR $C_4$ potential field is invariant under the $SL(2,\mathbb{Z})$, hence the charged object-D3-branes- are also invariant in F-theory, which leads them to be good probes. D3-branes, by themselves, enjoys lots of novel properties in the context of F-theory. In particular, the complexified gauge coupling of the gauge theories on the D3-branes world-volume coincident with the axio-dilaton $\tau$, which makes them, among other properties, intriguing topics and lead a lot of new understanding of the gauge theories and gravity. However, D3-brane is not the main part of the discussion in this chapter and we will not give many details unless the need rises.

The identity element $\mathbf{1}$ in $SL(2,\mathbb{Z})$ means that $\tau, (B_2, C_2)$ are identical in all regions of space-time and it necessarily implies that there are no any non-trivial sources for them. The sources for $\tau$ in type IIB are D7-branes (and also O7-planes, with 4 multiple of D7-branes charges of opposite sign), as we have seen that 7-branes carry magnetic charges under $C_0$: the real part of $\tau$ [2]. In order to define a non-trivial type IIB background which is patched together by $SL(2,\mathbb{Z})$ symmetry, it is inevitable to include D7-branes and O7-planes. To be more precise, we assume that a D7-brane extends along spacetime with coordinates $x^0, x^1, ..., x^7$ and define a complex coordinate $z \in \mathbb{C}$ parametrizing the transverse space $z = x^8 + ix^9$ of the D7-brane where D7-brane is point-like source. The equation of motion (2D Possion eqaution) for $C_8$ in presence of a 7-brane at $z = z_0$ then takes the form (in the normalized unit)

$$d * F_9 = \delta^2(z - z_0). \tag{2.5}$$

Gauss Law tells us the integrated form should be

$$1 = \int_C d * F_9 = \oint_{S^1} F_1 = \oint_{S^1} dC_0. \tag{2.6}$$

Taking into account constraints from supersymmetry, which implies that the axio-dilaton $\tau$ must be a holomorphic function in $z$, one can obtain the simple solution:

$$\tau(z) = \tau_0 + \frac{1}{2\pi i}\log(z - z_0) + \text{regular at } z_0 \tag{2.7}$$

in the vicinity of a D7-brane. Note that at $z = z_0$, where the D7-brane locates, the value of $\tau$ diverges. Hence we can view the degenerations of $\tau$ as a "detection" to signal the presence of D7-branes. Further, note that the logarithmic branch cut induces a monodromy $T : \tau \to \tau + 1$ with $SL(2,\mathbb{Z})$ matrix $M_{[1,0]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ when we move around $z_0$ in a circle. This suggests that one can identify D7-branes by their monodromy effect $M_{[1,0]}$ on the axio-dilaton profile $\tau$.

We have mentioned in last chapter that in general there are dyonic-like $(p, q)$ strings in the string theory, which are BPS bound states of $p$ fundamental strings (donated as $(1, 0)$ strings) and $q$ D1-branes (donated as $(0, 1)$ strings) when $(p, q)$ are coprime [71]. By analogy, one can also expect there are $(p, q)$-branes, which can be defined as where $(p, q)$-strings end at. One should note that the existence of $(p, q)$ strings, as well as $(p, q)$-branes are non-perturbatively objects and necessary beyond the perturbative string regime.

Given that, as we presented above, one can identify the $SL(2,\mathbb{Z})$ monodromy $M_{[1,0]}$ as D7-branes, what about the corresponding monodromy associated with general $(p, q)$7-branes? It

[2] the complexities of $\tau$ in this way is due to type IIB supersymmetries

turns out that the monodromy matrix takes the form (For details, please refer to [3] [21])

$$M_{p,q} = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix}. \tag{2.8}$$

It is well-known to mathematicians that the $SL(2,\mathbb{Z})$ is precisely the group of transformation on the upper half-plane corresponding to modular transformation on a two-torus $T^2$ with a complex structure modulus $\tau$, which we give a brief review on the tori $T^2$ in Insert 3.1. This coincidence leads to the proposal of F-theory [20], which interpretes the axio-dilaton $\tau$ as a complex structure modulus of an extra torus $T^2$ fibered over the physical 10D spacetime. Note that the torus $T^2$ is not strictly a physical spacetime in this setting [4], but rather gives a geometrization of type IIB backgrounds with varying axio-dilaton $\tau$. In other words, the $SL(2,\mathbb{Z})$ gauge symmetry of the Type IIB string theory hence is interpreted as the geometrical $SL(2,\mathbb{Z})$ reparametrization of the $T^2$, as a monodromy group acting on

**Insert 3.1: $SL(2,\mathbb{Z})$ and Torus**

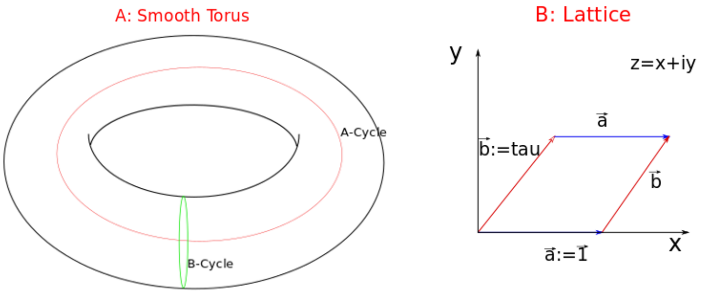


**Figure 2.1.:** *Torus. The left side represents a generic smooth one with two non-trivial one-cycles: A/B-cycles. The right side denotes the lattice representation with the vector $\widehat{a}, \widehat{b}$ corresponds to the 1-cycles $A, B$.*

A torus $T^2$ can be understood as the quotient $T^2 = \mathbb{C}/(a\mathbb{Z}\oplus b\mathbb{Z})$ of the complex plane $\mathbb{C}$ by a lattice $\Lambda_{a,b} = (a\mathbb{Z}\oplus b\mathbb{Z})$ such that the torus $T^2$ can be given by identifying $\widehat{z} \sim \widehat{z}+\widehat{a} \sim \widehat{z}+\widehat{b}$. The ration $\frac{b}{a}$ describes the shape of the Torus and is dubbed complex structure moduli (or more precisely Teichmüller parameter) describing points in Teichmüller space, which is the universal covering space of the moduli space of the $T^2$. With the above identification, one can always normalize the lattice $\Lambda_{a,b}$ by setting $a = 1, b = \tau, \tau \in H_\perp$ where $H_\perp$ stands for the upper half complex plane, and thus is the Teichmüller space for torus. In this form, one can easily show that the following transformation $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$ leaves the lattice $\Lambda_{1,\tau}$ invariant. Recall that $S$ and $T$ generate the group $SL(2,\mathbb{Z})$, and $SL(2,\mathbb{Z})$ hence is the symmetry group of $\Lambda_{1,\tau}$, i.e. the torus $T^2$ with the complex structure modulus $\tau$, whose image part $Im(\tau) \geqslant 0$.

The full family of inequivalent tori is given by the so-called fundamental domain $\mathcal{F} = H_\perp/PSL(2,\mathbb{Z})$, and $P$ denotes the extra $\mathbb{Z}_2$, which swaps the sign on the $SL(2,\mathbb{Z})$ matrixes,

[3] One should note that every $(p,q)$-branes could be transformed into a $(1,0)$-branes, i.e. D-branes once we fixed a $SL(2,\mathbb{Z})$ frame. However, for multiple different $(p,q)$-branes, it is not always so. In thses cases, we dubbed them as non-local, we will come back this point again later.

[4] By that, we mean there are no physical observables propagating on the torus $T^2$ in F-theory. However, as we are going to discuss in the next section, the torus $T^2$ does become a part of the physical spacetime in the M-theory.

which clearly does not give a new $T^2$. More precisely, it is given by

$$\mathcal{F} = \{\tau \in H_{\perp}| \qquad |\tau| \geqslant 1, \qquad -\frac{1}{2} < Re(\tau) \leqslant \frac{1}{2}\}. \tag{2.9}$$

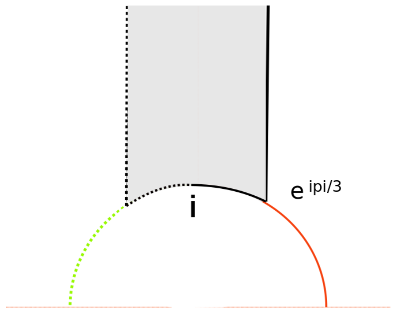


**Figure 2.2.:** *The Fundamental domain of $T^2$ displayed as the above gray region*

Note there are three orbifold-like singularities associated with the fundamental domain of $\tau$, which are listed as follows:

$$\begin{aligned} \tau = e^{i\pi/3}, \qquad & j(\tau) = 0, \qquad & ST; \\ \tau = i, \qquad & j(\tau) = 24^3, \qquad & S; \\ \tau = i\infty, \qquad & j(\tau) = e^{-2\pi\tau}, \qquad & T. \end{aligned} \tag{2.10}$$

Here $j(\tau)$ function is the Klein $j-$function, which will be introduced later, the third column denotes the invariant monodromy. The lattice vectors $1, \tau$ correspond to two non-trivial 1-cycles $A/B$-cycles in Figure.(2.4).

the 1-cycles of the torus $T^2$ (will be elaborated in 2.2 ). The tori can be patched together via the $SL(2,\mathbb{Z})$ symmetry and can be mapped to one another. However, there is one issue with this setting. Notice that the 2-form doublet also transforms under the $SL(2,\mathbb{Z})$ symmetry. This suggests that it should have an interpretation in terms of a three-form potential $\widehat{C}_3$ (with four-form field strength $\widehat{F}_4$) which, once integrated over the one-cycles of the torus, reproduces the doublet of two-forms in Type IIB. However, there does not exist such a three-form gauge potential in Type IIB string theory. We will circumvent this issue with an alternative geometrical interpretation in M-theory later, where we can see this identification of $SL(2,\mathbb{Z})$ is deeply rooted in duality with M-theory. The punchline is that the volume of this extra $T^2$ bears no physical significance.

### 2.1.1. Seven-branes and non-abelian gauge theories on their world volume

In this subsection, we first detour to the world-volume of $(p,q)$ 7-branes. And in the intersecting D-branes models, we have mentioned they can in principle generate all Lie groups except the exceptional gauge groups.

We have said that $(p,q)$-branes are BPS bound states of $p$ $(1,0)$-branes and $q$ $(0,1)$ branes. And we also know that one can place a stacks of $N$ D-branes on top of each other in flat space and regard them as BPS bound states of type II supergravity, which further generate a $U(N)$ gauge theory. How about the situation with two different $(p,q)$-branes and $(p',q')$-branes? Can they form BPS bound states? It is not hard to realize that it is possible as we can view that a stacks of D-branes as the special cases, namely $(p,q)$ and $(p',q')$ can both simultaneously transformed into $[1,0]$ branes, i.e. D-branes, in a local $SL(2,\mathbb{Z})$ frame. It turns out this happens when their monodromies (read off from (2.8)) commute. Otherwise, we call the $(p, q)$ brane and $(p', q')$ are mutually non-local[5]. In general cases, two such mutually non-local $(p, q)$ branes cannot be brought on top of each other in a supersymmetric way and hence form BPS bound states. However, for certain cases they do form bound states and further they similarly realize the simply laced A-D-E Lie groups in flat space. Let us now focus on 7-branes. To proceed, we first denote the three cases of $(p, q)$ 7-branes as

$$A : [1, 0], \qquad B : [3, 1], \qquad C : [1, 1]. \tag{2.11}$$

The A-D-E gauge theories in a 8D flat spacetime are then obtained from the following world-volume of $(p, q)$ 7-branes

$$SU(N) : A^N \qquad SO(2N) : A^n BC, \qquad E_k : A^{k-1}BC^2, k = 6, 7, 8. \tag{2.12}$$

As a side remark, one can easily notice that the $BC$ bound states are perturbative $O7$-planes [6] by checking the monodromy of BC system

$$M_{BC} = M_{[3,1]}M_{[1,1]} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \tag{2.13}$$

as its action on $\tau \to \tau - 4$ since it carries $-4$ charge in the units where D7-brane charges is 1. We have then $M_{O7} = -M_{D7}^{-4}$. The minus sign here is crucial, though it acts trivially on $\tau$, but it changes the sign of the three-form doublet $F_3^I$, as expected from the orientifold action. This also fits with an experience in the intersecting branes model, which we mentioned in the third case in1.9, that a stack of $N$ Dp-branes on top of an $O^-$ p-plane give rise to a gauge group $SO(2N)$ [7].

Further, when 4 D7-branes together are placed on top of an O7-plane, i.e. $SO(8)$ system with $A^4BC$, there is a cancellation of net charge and tension locally and hence there is no back-reaction on $\tau$, with a residual monodromy

$$M_{SO(8)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.14}$$

which $\tau$ is globally constant and render the three-form doublet $F_3^I$ changing the sign. This is nothing but the $\mathbb{Z}_2$ involution of a perturbative type IIB orientifold.

We also know that the perturbative type II strings cannot generate exceptional gauge group $E_k, k = 6, 7, 8$, hence the above exceptional gauge group must be generated at strongly coupled regime. Indeed, the axio-dilaton $\tau$ associated with the above $E_k$ cases are all of order 1 and

[5] Locality here means that say if one becomes massless, the other one does not get affected. The typical examples are dyons in the Seiberg-Witten theory. And from geometric engineering viewpoint, the corresponding cycles typically have vanishing intersections, which encoded in the Picard-Lefshetz formula as we will show momentarily.

[6] More precisely, here $O7$-plane denotes the $O7^-$-plane. In the sequel, we will skip the sup-script $-$ in $O7^-$-plane order to avoid the clutter without any specifics. However, One should note that in F-theory, there are configurations with $O7^+$-planes, especially in the frozen singularities [72, 73] (See a brief discussion in the appendix C.5).

[7] It would generate $SP(2N)$ gauge group if placing on top of an $O^+$ p-plane.

further they are all globally constant. We list the details as following:

$$\begin{aligned} SO(8): M &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{Z}_2, \qquad \tau = arbitrary; \\ E_6: M &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathbb{Z}_3, \qquad \tau = e^{i\pi/3}; \\ E_7: M &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbb{Z}_4, \qquad \tau = e^{i2\pi/4} = i; \\ E_8: M &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbb{Z}_6, \qquad \tau = e^{i\pi/3}. \end{aligned} \tag{2.15}$$

We will come back to this point later in F-theory settings later.

The features of constant axio-dilaton $\tau$ near the singularities could also be verified by the probe D3-branes. When a D3-brane probe the above cases, the world-volume of the D3-brane should be a 4D $N=2$ SCFT with the corresponding flavor symmetry as $SO(8), E_n$.

### 2.1.2. Closed strings description of F-theory via M-theory

In the previous subsections, we gave some heuristic explanations why one can view the axio-dilaton $\tau$ as a complex structure modulus $\tau$ of an extra torus $T^2$, mainly based on the monodromies in the presence of 7-branes. Now we are going to give a rigorous proof in this subsection. In order to do that, we need to switch gear to the M-theory dual side. We will mainly follows the discussion that had been laid down lucidly in section 3.1 of [69].

The low energy effective theory action of M-theory is

$$S_{\mathrm{M}} = 2\pi[\int d^{10}x\sqrt{-g}R - \frac{1}{2}\int G_4 \wedge *G_4 - \frac{1}{6}C_3\wedge G_4\wedge G_4 + \int \quad C_3 \wedge I_8(R) + ...], \tag{2.16}$$

where $I_8(R)$ is a polynomial of degree 4 in the curvature which is generated by higher derivative at one-loop [74].

Now let us look how it can dualize to F-theory. First considering the theory compactified on $T^2$ fibration over a generic complex manifold $X_9$, with metric being

$$ds_M^2 = \frac{v}{\tau_2}((dx+\tau_1 dy)^2 + \tau_2^2 dy^2) + ds_9^2, \tag{2.17}$$

where $x, y$ are coordinates on a torus $T^2$ and are periodic, with periodicity 1 for example, and $v$ is the area of $T^2$. The followed complex structure of this torus $T^2$ reads $\tau = \tau_1 + i\tau_2$. In general, the $v$ and $\tau$ are varied at different points on $X_9$. Without losing generality, we dubbed the 1-cycle along the $x$-direction as A-cycle $S^1_A$, and the one along $y$-direction as B-cycle $S^1_B$. Then as we discussed in the last chapter, one can reduce to the IIA theory by reduction of M-theory along the A-cycle in the limit of zero size of A-cycle $R_A \to 0$, with the space-time being $\mathbb{R}^{1,8}\times S^1_B$. Further with the help of T-duality, we can further go to the type IIB theory on $\mathbb{R}^{1,8}\times \widetilde{S}^1_B$with the radius $\widetilde{R}_B = \frac{\ell_s^2}{R_B}$. In particular, if we take the decompactification limit $\widetilde{R}_B \to +\infty$( equivalently $R_B \to 0$), the circle $\widetilde{S}^1_B$ goes to infinity flat line and it would be naturally expect it recovers the 10D Lorentz invariance of type IIB on $\mathbb{R}^{1,9}$. From M-theory perspective, we can perform this limit by keeping the complex structure modulus of torus $T^2$ $\tau \sim R_A/R_B$ fixed and also shrinking its area to zero size $v_{T^2} \sim R_A R_B \to 0$. In diagrammatically words, we want to see the following duality/equivalence is valid

$$\text{M-theory on } \mathbb{R}^{1,8} \times (S^1_A \times S^1_B)|_{v_{T^2}\to 0, \tau=\text{fixed}} \simeq \text{Type IIB on } \mathbb{R}^{1,9}. \tag{2.18}$$

In order to present in a rigorous way, one should discuss it at the level of metric. For this, one need to recall the general relation between the $S^1$ circle compactified M-theory and the Type IIA metric, which is given by

$$ds^2_M = L^2 e^{4\chi/3}(dx + C_1)^2 + e^{-2\chi/3} ds^2_{IIA}, \tag{2.19}$$

where $x$ is the coordinate on the $S^1_A$ cycle with the normalization $2\pi R_A = 1$ and $L$ is a conventional length which sets the scale of the M-theory, meaning that given a string length $\ell_s$, one can define the M-theory length $l_M$ by $\ell_s L = \ell^3_M$. Then comparing to the $T^2$ fibration above, we can immediately obtain

$$C_1 = \tau_1 dy, \qquad L^2 e^{4\chi/3} = \frac{v}{\tau_2}, \qquad ds^2_{IIA} = \frac{\sqrt{v}}{L\sqrt{\tau_2}}(v\tau_2 dy^2 + ds^2_9). \tag{2.20}$$

Noting the facts that compactifying the M2-branes on $S^1$ give rise to the fundamental strings (F1-strings), otherwise if $S^1$ is normal to the world-volume of M2-branes then it gives rise to the D2-branes, we then have $g_{IIA}\ell^3_s = e^{4\chi/3}\ell^3_M$.

Then T-duality maps type IIA to Type IIB with the circle length $R_A$ changing to $R_B = \frac{\ell_s^2}{R_A}$, as well as $C_0 = (C_1)_y$, With these, one can express $\ell_s$ and $g_{IIA}$ as a function of $v, \tau_2, R$ and $\ell_M$, as well as the IIB metric and couplings. Indeed, One finally have

$$C_0 + \frac{i}{g_{IIB}} = \tau_1 + i\tau_2 = \tau, \qquad ds^2_{IIB,S} = \frac{\sqrt{v g_{IIB}}}{L}(\frac{\ell^6_M}{v^2_{T^2}} dy^2 + ds^2_9). \tag{2.21}$$

Transforming it to Einstein frame, it reads

$$ds^2_{IIB,E} = \frac{\sqrt{v}}{L}(\frac{L^2\ell^4_s}{v^2} dy^2 + ds^2_9). \tag{2.22}$$

In order to preserve the supersymmetry at $\mathbb{R}^{1,d-1}$ dimensional effective theory, we consider $X_9 = \mathbb{R}^{1,d-1} \times X_{9-d}$ with $X_{9-d}$ being a Kälher manifold, and further assuming $T^2$ varies holomorphically on $B_{9-d}$, namely the total space $X_{11-d}$ must be Calabi-Yau of complex dimensional $(11-d)/2$. Note that the area $v_{T^2}$ remains constant along the base $B_{9-d}$, then we can simply take our conventional scale $L = \sqrt{v}$, and the metric reduces to

$$ds^2_{IIB,E} = -dx^2_0 + dx^2_1 + ... + dx^{d-1} + \frac{\ell^4_s}{v} dy^2 + ds^2_{B_{9-d}}. \tag{2.23}$$

If we send now $v_{T^2} \to 0$, with fixing $\ell_s$ at finite value, we can see that this decompactified to flat $d+1$ dimensional Minkowski space times $B_{9-d}$, with a non-trivial profile for the dilaton-axion $\tau(u), u \in B_{9-d}$. This shows us that the one cycle in the fiber of Calabi-Yau space in M-theory side, compensates part of the visible non-compact Minkowski space in Type IIB, without breaking the full Lorentz invariance in the limit $v_{T^2} \to 0$.

To give a summary, we have the following map for the duality

$$\text{M-theory on } \mathbb{R}^{1,10-2n} \times X_n \xrightarrow{v\to 0} \text{type IIB on } \mathbb{R}^{1,10-2n} \times B_{n-1}\ , \tag{2.24}$$

where $X_n$ is the $T^2$ fibration of Calabi-Yau manifolds, and $B_{n-1}$ is its base.

Thus far, we have given a rigorous way to encode the axio-dilaton $\tau := C_0 + ie^{-\phi}$ of type IIB with the geometric interpretation, namely as the complex structure $\tau$ of $T^2$ from the duality to M-theory. And it crucially requires the volume $v$ of the $T^2$ to zero size $v_{T^2} \to 0$ in order to restore Lorentz invariance of type IIB spacetime, which explains why there is no four-form field strength $\widehat{F}_4$ gives rise to the $SL(2,\mathbb{Z})$ doublet $F_3^I$ simply because the volume $v$ is zero.

We are now in the position to introduce one way of defining F-theory:

F-theory on a torus fibration $X_n$ is defined to be the type IIB compactification on $B_{n-1}$, which is dual to the M-theory compactification on $X_n$ with the size of fiber being zero.

To give a summary, we have the following map for the duality

$$\text{M-theory on } \mathbb{R}^{1,10-2n} \times X_n \xrightarrow{v_{T^2}\to 0} \text{F-theory on } \mathbb{R}^{1,10-2n} \times X_n \times S^1 \ , \tag{2.25}$$

where $X_n$ is the $T^2$ fibration of Calabi-Yau manifolds, and $B_{n-1}$ is its base.

From the type IIB picture, F-theory is a consistent description of strongly coupled type IIB with 7-branes and varying axio-dilaton, whose space-time now is a torus fibration. The extra torus $T^2$ is not a physical space-time (as its volume goes to zero limit), but as a book-keeping device that accounts for the variation of axio-dilatons as the consequence of back-reactions of 7-branes. The base $B_{n-1}$, on the other hand, is "visible" part of the space for Type IIB. Hence the variation of axio-dilaton $\tau$ in presence of a set of 7-branes is therefore modelled as the variation of complex structure of an torus $T^2$ transverse the positions of the 7-branes. Such a structure defines an elliptic fibration.

Given by the amount of supersymmetries in M-theory Calabi-Yau compactifications 1.2, we can hence obtain the amounts of supersymmetries for F-theory compactifications. For future reference, we list every dimensional Calabi-Yau compactfications and their persevered supersymmetries for F/M-theory in 2.1.

| Calabi-Yau | Effective theories from F-Theory | Effective theories from M-Theory |
|---|---|---|
| $X_2$(K3) | 8D $\mathcal{N}=1$ (16) on $\mathbb{R}^{1,7}$ | 7D $\mathcal{N}=1$ (16) on $\mathbb{R}^{1,6}$ |
| $X_3$ | 6D $\mathcal{N}=(1,0)$ (8) on $\mathbb{R}^{1,5}$ | 5D $\mathcal{N}=1$ (8) on $\mathbb{R}^{1,4}$ |
| $X_4$ | 4D $\mathcal{N}=1$ (4) on $\mathbb{R}^{1,3}$ | 3D $\mathcal{N}=2$ (4) on $\mathbb{R}^{1,2}$ |
| $X_5$ | 2D $\mathcal{N}=(2,0)$ (2) on $\mathbb{R}^{1,1}$ | 1D $\mathcal{N}=2$ (2) Quantum Mechanism |

**Table 2.1.:** *F- and M-theory compactifications in various dimensions. The number in the brackets denotes the number of real supercharges.*

## 2.2. The Geometry of Elliptic Curves and Elliptic Fibrations

So in the previous sections we have stressed that a torus fibration is a proper setting for F-theory. In this section, we first give a review on the geometry of fibrations.

A torus fibration is essentially a tuple $(X_n, B_{n-1}, T^2), \pi : X_n \to B_{n-1}$ (the subscript $n$ represents the complex dimensions) with a projection map $\pi : X_n \to B_{n-1}$ such that for each

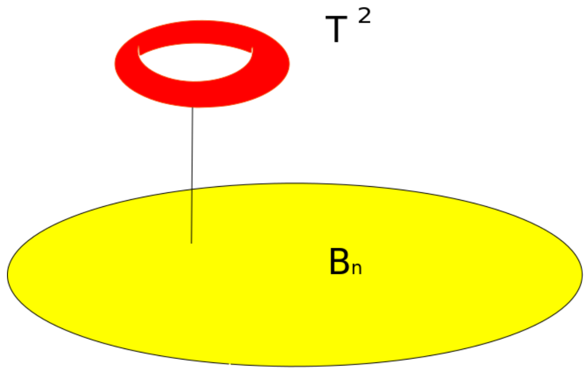


**Figure 2.3.:** *Torus fibration*

point $p$ in the base $B_{n-1}$, the preimage $\pi^{-1}(b)$ is homotopy equivalent to the fiber: $T^2$[8],

$$\begin{array}{ccc} T^2 & \longrightarrow & X_n \\ & & \big\downarrow \pi \\ & & B_{n-1}. \end{array} \tag{2.26}$$

For a schematic visualization, we refer to the Figure 2.3.

Tours fibrations may have a rational section $\sigma_0$, which is a map from the base $B_{n-1}$ into the total space $X_n$ such that $\sigma(B_n)$ intersects each fibre once [9]. In this case, the tours fibration are dubbed an elliptic fibration, whose fibre now is an elliptic curve. An elliptic curve $\mathbb{E}_\tau$ with a complex structure $\tau$ is a one dimensional complex manifold of genus one, which is isomorphic to a torus $T^2$ with a marked point called the origin $\mathcal{O}$. Then an elliptic fibration can be shown as

$$\begin{array}{ccc} \mathbb{E}_\tau & \longrightarrow & X_n \\ & & \pi \big\downarrow \big\uparrow s_A \\ & & B_{n-1}. \end{array} \tag{2.27}$$

Now the pre-image $\pi^{-1}(p)$ of a generic point $p$ in the base $B_{n-1}$ is an elliptic curve, and as the point $p$ moves along the base, the origin $\mathcal{O}$ in the pre-image $\pi^{-1}(p)$ varies as a meromorphic function through $X_n$, thereby defines the so-called rational section $s_A : B_{n-1} \to X_n$ of the elliptic fibration. In mathematical term, it defines a divisor $S_A$ in the $X_n$. Particularly, if the rational section further is also a holomorphic section, namely the marked point $\mathcal{O}$ in the pre-image $\pi^{-1}(p)$ varies holomorphic through $X_n$, then this elliptic fibration is called the Weierstrass model [10].

[8] A fibration is a generalization of the notion of a fiber bundle, except that the fibers at each point $p$ in the base $B_{n-1}$ need not be isomorphic; rather, they are just homotopy equivalent.

[9] The torus fibration with no rational sections may also have multi-sections/$p$-sections, in which case the $p$-section means it locally intersects each fiber at $p$-point, i.e. $p$-times but globally these $p$ points are connected by a monodromy along the loci in the base, which dubbed genus-one fibration and we will come back to this point with more details in 2.10.

[10] Every elliptic fibration is birational equivalent (isomorphic up to higher codimensional loci) to the Weierstrass model. More precisely, a rational section has no essential difference with the holomorphic section up to

### 2.2.1. Elliptic curves

There are many mathematical reviews covering the details on elliptic curves, for instance in [75] and references therein. In this subsection we are going to pick up some relevant basics for our purpose.

Having said that an elliptic curve is a projective curve of genus one with a marked point $\mathcal{O}$, we are giving algebraic descriptions of elliptic curves. Algebraic speaking, the elliptic curve can be described as a hypersurface, or more generally, a complete intersection in some weighted projective ambient spaces. A typical example is a smooth cubic in the $\mathbb{P}^2$. And it is naturally expect the choices of the ambient space should not be unique. However, a typical choice for our purpose would be the projective weighted space $\mathbb{P}_{231}$ and this is so-called Weierstrass model:

$$F = -y^2 + x^3 + fxz^4 + gz^6 = 0. \tag{2.28}$$

Here $[x, y, z]$ are homogeneous coordinates in the ambient space $\mathbb{P}_{231}$, which by definition enjoys the following equivalences

$$(x, y, z) \approx (\lambda^2 x, \lambda^3 y, \lambda z), \qquad \lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}. \tag{2.29}$$

Indeed, one can easily check that its genus is 1 by using the equivalence (2.29) to set $z = 1$ so that the equation in this local patch can be defined by

$$F = -y^2 + x^3 + fx + g = 0. \tag{2.30}$$

Thinking of $x$ now as a local coordinate chart on $\mathbb{P}^1$, this equation generically defines a genus one $g = 1$ smooth curve as $g = \frac{(d-2)(d-1)}{2} = 1$ with the degree $d = 3$.

One should notice that there could be singularity associated with an elliptic curve. A singular loci on the Weierstrass model can be obtained by solving the defining equation:

$$F = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial x} = -2y = x^3 + fx + g = 3x^2 + f = 0 \to x = \frac{-3g}{2f}. \tag{2.31}$$

Rewriting the above equation, we would say the curve is singular when the Discriminant $\Delta$ vanishes

$$\Delta = 4f^3 + 27g^2 = 0. \tag{2.32}$$

Note that there are two set of solutions of the Discriminant $\Delta$ leading to two different singularities for the elliptic curve $\mathbb{E}_\tau$, one of them, known as **nodal** curve, is when $f \sim g^{\frac{2}{3}} \neq 0$ rendering the curve possesses a self-intersecting point, the other one, known as **cuspidal** curve, is for $f = g = 0$.

One can relate the above algebraic description of the elliptic curve to the standard representation $T^2 = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ (see Insert 3.1) by identifying the holomorphic coordinates in both pictures. From the perspective of $T^2 = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, the holomorphic coordinate is $z = x + \tau y$, which for any point $P$ can be written as

$$z(P) = \int_0^P \Omega_1, \qquad \Omega_1 = dz := \frac{cdx}{y}, \tag{2.33}$$

where $c$ is some normalization constant. Choosing a basis of 1-cycles $(A, B)$ the algebraic $T^2$,

codimension-one loci. When going to higher codimensional loci, the meromorphic functions defining the rational section may hit the poles.

the modulus is then given by

$$\tau = \frac{\oint_B \Omega_1}{\oint_A \Omega_1} \tag{2.34}$$

in a specific $SL(2,\mathbb{Z})$ S-daulity frame. Now, one want to find out the relation between $\tau$ and $f, g$ by computing period integral. It turns out that

$$j(\tau) = \frac{4(24f)^3}{\Delta}, \qquad \Delta = 27g^2 + 4f^3, \tag{2.35}$$

where $j(\tau)$ is a modular invariant $j$-function, $j(\tau) = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \mathcal{O}(e^{2\pi i\tau})$ [11].

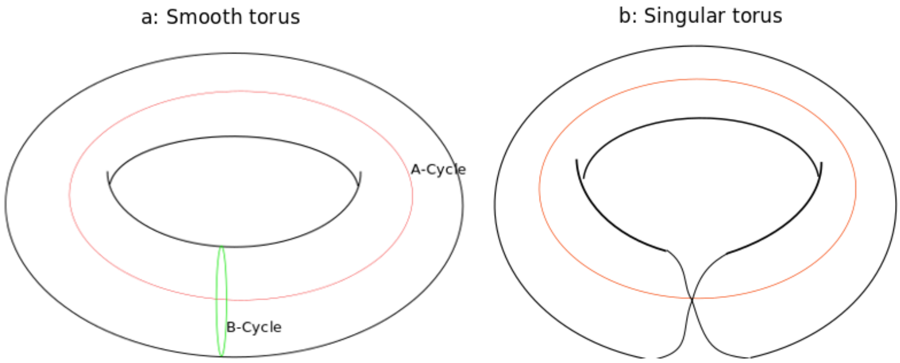


**Figure 2.4.:** *Torus. The left side represents a generic smooth one with two non-trivial one-cycles: A/B-cycles. The right side denotes a singular one, whose B-cycle degenerates hence the torus pinches, and it has the topology of a two-sphere $\mathbb{P}^1$.*

In terms of the torus, one of the one-cycle is pitched as shown in figure 2.4

### 2.2.2. Elliptic fibrations

The above analysis could be easily generalized to the elliptic fibration. And the Weierstrass model turns to be

$$P_W = -y^2 + x^3 + f(u_i)zx^4 + g(u_i)z^6 = 0, \tag{2.36}$$

where $x, y, z$ still are weighted coordinates in projective space $\mathbb{P}^2_{231}$ describing the fiber $\mathbb{E}_\tau$ which again enjoy the equivalence (2.29). The constants $(f, g)$ are now promoting as holomorphic functions in the base $B_{n-1}$ with the coordinates $u_i$. Globally speaking, $X_n$ is a hypersurface $P_W = 0$ in a $\mathcal{P}_{231}$-bundle [12] over the base $B_{n-1}$ given by

$$\mathbb{P}_{231}(\mathcal{E}) = \mathbb{P}_{231}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}). \tag{2.37}$$

[11] As a side note, the coefficents in $j(\tau)$ are related to the dimensions of irreducible representation of the Monster group, which was firstly observed by John McKay in 1978 with the following remark $196884 = 196883 + 1$. Later, this relation was called Monstrous Moonshine correspondence.

[12] Of course one could embed the fibre $\mathbb{E}_\tau$ into other ambient space, for detail construction we refer to [76]. Note that for other choices of fibrations there, they do not have the full $SL(2,\mathbb{Z})$ monodromy group [77], which would typically lead to moduli spaces describing Calabi-Yau manifold with frozen moduli [72, 73], we will come back to this point shortly.

In terms of bundles, globally speaking, the fiber coordinates $(x, y, z)$ now are viewed as sections of line bundles $L_x, L_y, L_z$ over $B_{n-1}$, respectively, where

$$L_x := \mathcal{L}^2 \otimes L_z, \qquad L_y = \mathcal{L}^3 \otimes L_z^3. \tag{2.38}$$

Further the $f, g$ should satisfy the following global conditions in order to be Calabi-Yau manifold

$$f \in \Gamma(B_{n-1}, \mathcal{L}^4), \qquad g \in \Gamma(B_{n-1}, \mathcal{L}^6). \tag{2.39}$$

As we stressed, the whole fibration need to be Calabi-Yau if the low-dimensional effective theories enjoy the supersymmetry. Here and in the sequel, we will denote the fibration $X_n$ as a n-dimensional Calabi-Yau manifold.

Here the line bundle $\mathcal{L}$ is a $SL(2, \mathbb{Z})$ holomorphic line bundle, which can be taken as anti-canonical bundle of the base $B_{n-1}$: $\mathcal{L} = \bar{K}_{B_{n-1}}$ in order to render the whole fibration $X_n$ being Calabi-Yau. To see this, note if we denote the divisor associate the bundle $\mathcal{L}$ by $D_{\mathcal{L}} = [\mathcal{L}]$ and the hyperplane divisor of $\mathbb{P}_{2,3,1}$ by $D_z = [z = 0]$, thanks to the adjunction formula, the Chern class of $X_{n+1}$ reads

$$\begin{aligned} c(X_n) =& \frac{c(B_{n-1})(1 + 3D_z + 3D_{\mathcal{L}})(1 + 2D_z + 2D_{\mathcal{L}})(1 + D_z)}{1 + 6D_z + 6D_{\mathcal{L}}} \\ \Longrightarrow & c_1(X_n) = c_1(B) - D_{\mathcal{L}} \end{aligned} \tag{2.40}$$

in order to make the whole elliptic fibration $X_n$ being Calabi-Yau, i.e. $c_1(X_n) = 0$, we have to take $D_{\mathcal{L}} = [-K_{B_{n-1}}]$.

One can also verify this from the following viewpoint. We firstly allow the $x, y$ vary holomorphically over $B_{n-1}$. More precisely, we can take $(x, y)$ to be local coordinates on suitable line bundle $\mathcal{L}$ over $B_{n-1}$, the weighted degree of $(x, y)$ tells us if $y$ is a section of a line bundle $3\mathcal{L}$, then $x$ is a section of a line bundle $2\mathcal{L}$. In order to let $X_n$ being the Calabi-Yau manifold, this means that there should be a non-zero holomorphic n-form $w_n$ on $X_n$ yielding

$$w_X = \frac{dx}{y} \wedge w_{B_{n-1}}, \tag{2.41}$$

where $w_{B_{n-1}}$ is the canonical form which is section of the canonical bundle $K_{B_{n-1}}$. Since $w_X$ transforms trivially, it follows that $\mathcal{L} = -K_{B_{n-1}}$.

The origin point $\mathcal{O}$ in the elliptic curve $\mathbb{E}_\tau$ now promote to be the zero (holomorphic) section $s_0 : [x : y : z] \to [1 : 1 : 0]$ which maps any point $b \in B_{n-1}$ to the single point $z = 0$ in the fiber and define a divisor $S_0 := [z = 0]$.

Now let's quickly look at the base $B_{n-1}$. Since $X_n$ is assumed to be an elliptically fibered Calabi-Yau manifold, this necessarily requires that the base should be subject to

$$h^{i,0}(B_{n-1}) = 0, \qquad i = 1, ..., n-1. \tag{2.42}$$

Throughout this thesis, we also assume the base $B_{n-1}$ to be smooth Kähler manifold [13] without any further specifics as we have stressed that the base $B_{n-1}$ shall be viewed as the physical space-time of the type IIB orientifold compactification.

[13]Note that a smooth manifolds does not exclude that its submanifolds can develop singular by themselves. This is an important fact that many crucial discussions in this thesis build on.

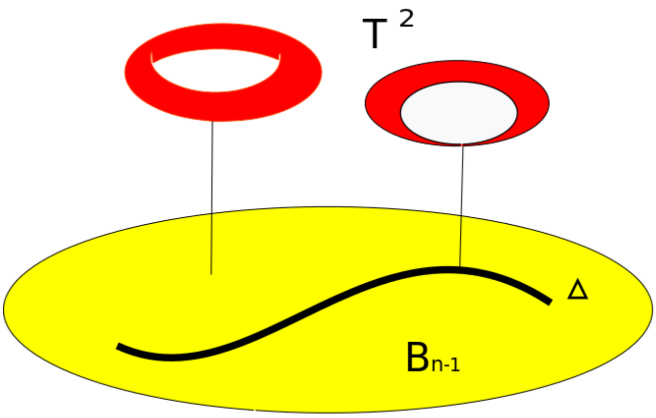


**Figure 2.5.:** *Elliptic fibration, where the tours $T^2$ pinches at the loci $\Delta$ of the base $B_{n-1}$, which is a divisor.*

Similar to the elliptic curve $\mathbb{E}_\tau$, the Weierstrass model (2.48) also has the discriminant

$$\Delta = 4f^3 + 27g^2, \tag{2.43}$$

which now the vanishing loci $\Delta = 0$ defines a divisor $W$ in the base $B_{n-1}$, encodes the singularities of the fiber $\mathbb{E}_\tau$, see 2.5. If we associate the discriminant $\Delta$ with a curvature class of a certain bundle, then the condition for the total space $X_n$ as an elliptic fibration being a Calabi-Yau is given as [14]

$$-12K_{B_{n-1}} = [W]. \tag{2.44}$$

This is known as the "Kodaira condition". As we have mentioned, the loci of the Discriminant $\Delta = 0$ is where the elliptic fiber $\mathbb{E}_\tau$ develops singular, generically with 1-cycle collapsing to zero size. More importantly, it tells the location of the seven-branes in F-theory. To see this, recall that one can identify the $SL(2,\mathbb{Z})$ monodromy with the $(p,q)$ seven-branes. How does the monodromy relate to these 1-cycles? To this end, one should invoke the Picard-Lefschetz monodromy formula

$$\eta \to \eta - (\eta \cdot \gamma)\gamma, \tag{2.45}$$

which characterizes the monodromic behaviors of $\eta$ cycle around the point in the complex structure moduli space where the $\gamma$ cycle vanishes. Now in the case of the elliptic fiber $\mathbb{E}_\tau$, of the topology of $T^2$, with the convention with the orientation $A \cdot B = 1$ for the $T^2$ if the cycle $pA + qB$ vanishes, an arbitrary cycle $aA + bB$ undergoes a monodromy accordingly is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a - (aq - bp)p \\ b - (aq - bp)q \end{pmatrix} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.46}$$

This is exactly the same as the monodromy of $(p,q)$ branes (2.8). This signals that one can view the degeneration of $pA + qB$ cycles of the $\mathbb{E}_\tau$ as the signals of the presence of the $(p,q)$ 7-branes.

[14] Through this thesis, given a divisor $W$, we both denote its cohomology class and homology class (mod out torsion) as $[W]$.

Note from the Picard-Lefshetz formula above, one can readily obtain the condition for two $(p,q)$ and $(a,b)$ 7-brane being mutually local:

$$aq - bp = 0. \tag{2.47}$$

Essentially, it represents that the two corresponding 1-cycles in the fiber have vanishing intersection. In physics lingo, if one becomes massless, the other does not get affected and hence they are mutually local. If the intersecting number is not zero, we call them mutually non-local.

It's not hard to see that one can bring a $(p,q)$ 1-cycles back into $(1,0)$ 1-cycle locally by the $SL(2,\mathbb{Z})$ transformation. However, one cannot do this globally as the obstruction comes from these mutually non-local 1-cycles. In general, it is very hard to figure out exactly which $(p,q)$ 1-cycles degenerates at a given points in the base; worse even, this in fact depends on the path we take through the base. Reflecting on the physics, it means that one cannot globally transform all the $(p,q)$ 7-branes back to D7-branes in a bid to do a perturbative string analysis, which we will see some more details on this aspect from a golden example in the next subsection. This fact, from other perspective, suggests that F-theory is intrinsically **strongly coupled** type IIB theory.

In the following, we would like to start with a simple example–K3, to illustrate the various aspects of the elliptically fibered Calabi-Yau spaces and its implications on physics.

## 2.3. A Golden Example: F-theory on Elliptically Fibred K3s

An elliptically fibred K3 is the simplest example of the elliptic fibration, which requires the base being $B_1 = \mathbb{P}^1$. The corresponding Weierstrass model yields to

$$P_W = -y^2 + x^3 + f(u,v)zx^4 + g(u,v)z^6 = 0, \tag{2.48}$$

where $x, y, z$ are coordinates in projective space $\mathbb{P}^2_{231}$ describing the fiber and $u, v$ are the homogenous coordinates in the base $B_1 := \mathbb{P}^1$. As a Calabi-Yau, the coordinates enjoy the following equivalences

$$\begin{aligned}(u,v,x,y,z) \approx& (\mu u, \mu v, \mu^4 x, \mu^6 y, z) \\ \approx& (\lambda^2 x, \lambda^3 y, \lambda z, 0, 0), \qquad (\mu,\lambda) \in \mathbb{C}^* := \mathbb{C}\backslash\{0\},\end{aligned} \tag{2.49}$$

and $f(u,v), g(u,v)$ are homogeneous polynomials of degree $8, 12$ in $u, v$, respectively, as the request (2.39) and $c_1(\mathbb{P}^1) = 2H$ with $H$ being the hyperplane divisor, whose section is given by a homogeneous polynomials of degree 2. This is consistent with the rule to determine whether such a hypersurface is a Calabi-Yau space is simply that the weighted degree of the defining polynomials equals to the sum of the weights. Note that for fixed $(u,v)$, (2.48) describes a Calabi-Yau one-fold: elliptic curve, i.e. $T^2$ with a marked point.

In these cases, the discriminant $\Delta$ describe points in the base manifold where the elliptic fiber degenerates, i.e. one cycle in the $T^2$ shrinks to zero size so the fiber pinch at that point. Since $\Delta = 4f^3 + 27g^2$ is a degree 24 polynomial in the coordinate $(u,v)$ on the base $\mathbb{P}^1$, we expect that there are generic 24 points on the $\mathbb{P}^1$ as discussed above. For simplicity, we choose a local patch in the $\mathbb{P}^1$ by setting $v = 1$, then we denote these 24 points as $u_i, i = 1, ..., 24$. One should note that although the fiber is singular at the loci $u_i, i = 1, ..., 24$ of the discriminant $\Delta = 0$, the whole manifold $K3$ does not generally develop singularities as $\partial P_W / \partial w = (f'x + g')|_{\Delta(u_i)=0}, i = 1, ..., 24$ is generally non-zero. Furthermore, near a generic zero $u_i, i = 1, ..., 24$, it has been worked out that the $\tau$ can always yield as

$$\tau(u) \approx \frac{1}{2\pi i}\ln(u-u_i) \tag{2.50}$$

up to $SL(2,\mathbb{Z})$ transformations. Note that when $u \to u_i$, $\tau \to i\infty$. This corresponds to weak coupling in type IIB theory $g_{IIB} \to 0$, as $\tau = C_0 + \frac{i}{g_{IIB}}$. Moreover, when circling once around $u = u_i$, i.e. $u = |u-u_i|e^{2\pi i}$, we see that $\tau$ undergoes *monodromy*:

$$T : \tau \to \tau + 1. \tag{2.51}$$

This indicates that there is a D7-brane at $u = u_i$ as $C_0 \to C_0 + 1 \leftrightarrow \oint_{u_i} F_1 = \oint_{u_i} dC_0 = 1$.

However, one should not expect that all the solutions at the 24 degenerating points $u_i$ should take the form as (2.50) and hence can be interpreted as the locations of D7-branes. To see this, one simply recall the Gauss theorem that there is no way that one can put objects with non-vanishing net charges in a compact space, otherwise the fluxes lines have nowhere to go. Nevertheless, it should be generic 24 $(p,q)$ 7-branes residing there and hence there could possible to reach a vanishing net charge in $\mathbb{CP}^1$. More directly, although we can alway go to an $SL(2,\mathbb{Z})$ frame locally where $\tau(u)$ lies in the fundamental domain and $(p,q)$ 7-brane be a D7-brane, there is no way to extend it *globally*. In other words, one can always pick a reference point $u_*$ with $\text{Im}\tau(u_*) \to \infty$ in the fundamental domain $\mathcal{F}$, however, once $u$ walking around in the $\mathbb{P}^1$, $\tau(u)$ might move off to some other region of the $H_\perp$ which requires a $SL(2,\mathbb{Z})$ matrix $M$ to bring to form in (2.50). Correspondingly, the $(1,0)$ 7-brane, i.e. D7-brane, should transform as $MTM^{-1}$ and usually ends up with a $(p,q)$ 7-brane. From this, we can see F-theory is intrinsically strongly coupled, as $(p,q)$ 7-branes are non-perturbative objects.

It turns out that the total rank of gauge groups from F-theory compactifications on $K3$ has to be 20[15], which can also be verified by the dual Heterotic string on $T^2$.

### 2.3.1. The weak coupling limit-Type IIB intepretation

The weak coupling limit requires that $\tau(u)$ is constant and have large imaginary part over everywhere on the whole base space $\mathbb{P}^1$. How to achieve this? While the condition indicates that the $j$ function should be $\infty$ and hence requires that

$$\frac{f^3}{g^2} = \alpha, \qquad \alpha = \text{constant}. \tag{2.52}$$

This clever limit was firstly put out by Sen [78, 79], hence dubbed Sen limit. Here we are going to reproduce the limit following [78, 79]. The starting point to achieve (2.52) is to parametrise

$$\begin{aligned} f =& \alpha p^2, \\ g =& p^3, \end{aligned} \tag{2.53}$$

[15] It is not saying that the rank of gauge groups for a 8d $N = 1$ supergravity from string theory has to be 20, there are alternative constructions of 8d $N = 1$ supergravities. Even in F-theory, there are some non-geometric backgrounds such as frozen singularities that could possible give other choices of rank such as $12, 4$ with simply replacing $8D7 + O7^-$-plane by $O7^+$-plane. The crucial message is that the choices of gauge groups from string theory constructions are limited. From the field theory perspective, there is no known obstructions for a 8d $N = 1$ supergravity coupled to arbitrary number of vector multiplets. But quantum gravity is believed to impose further (putative) constraints, which is the sprit of the recent swampland conjectures.

with $p$ being a polynomial of degree 4 in terms of $(u, v)$. For simplicity, we choose the patch with $v = 1$, then $p$ generically has the form

$$p(u) = \prod_{i=1}^{4}(u - u_i), \tag{2.54}$$

where $u_i$ are constant over $\mathcal{P}^1$, and also we also set the coefficient $u^4$ to be 1 via the equivalence (2.49). Plugging into the Klein modular function $j(\tau)$ in (2.35) we have

$$\Delta = (27 + 4\alpha^3)\prod_{i=1}^{4}(u - u_i)^6, \qquad j(\tau) = \frac{4(24\alpha^3}{27 + 4\alpha^3}, \tag{2.55}$$

hence we can tune $\alpha = -3/4^{1/3}$ and arrive the type IIB weak coupling everywhere on the base $\mathbb{P}^1$.

Now since the axio-dilaton is constant and perturbatively small everywhere, it means that there are no sources for the axio-dilaton $\tau$. Does it imply that D-branes configurations are trivial, i.e. no 7-branes configurations at all? One can easily see the answer is **NO**, otherwise the supersymmetry would be enhanced. Indeed, we have encountered the situation with a constant axio-dilaton in 2.1.1, namely the 7-brane configurations are the $A^4BC$ systems. More precisely, the 24 (p,q) 7-branes are grouped into 4 stacks of 6 7-branes situated at the points $u_i, i = 1, ..., 4$, each stack corresponding to a $A^4BC$ system.

Therefore, we could expect the monodromy of the whole system yields to

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.56}$$

which acting on the two one-cycles $(A, B)$ in the fiber $T^2$ by a $\mathbb{Z}_2$ involution

$$(A, B) \to (-A, -B). \tag{2.57}$$

Note that in the type IIB picture, this monodromy implies that the $SL(2, \mathbb{Z})$ doublet $(B_2, C_2)$ are doubled valued on the base $B_1 : \mathbb{P}^1$. The standard way to describe this effect is to construct a double cover $X_1$ branched over the loci of D7-branes and describe the F-theory space as a $\mathbb{Z}_2$-quotient $B_1 = X_1/\sigma$ with $\sigma$ being the involution. The new space $X_1$ [16] is defined by

$$X_1 : \xi^2 = p(u, v). \tag{2.58}$$

Indeed, it is a torus $X_1 = T_b^2$ [17], whose coordinates satisfies

$$(u, v, \xi) \sim (\lambda u, \lambda v, \lambda^2 \xi). \tag{2.59}$$

In other words, the base $\mathbb{P}^1$ could be viewed as quotient $T_b^2/\mathbb{Z}_2$ by

$$\mathbb{Z}_2 : \xi \to -\xi, \tag{2.60}$$

[16] As we stressed in the previous chapters, a double-covering space $X_{n-1}$ of a base $B_{n-1}$ is not necessarily a elliptically fibered, though we use the same notation for them.

[17] The subscript $b$ indicates that the torus $T_b^2$ is a different one than the fiber $T^2$.

together the whole K3 surface could be viewed as

$$K3 = (T^2 \times T^2)/\mathbb{Z}_2. \tag{2.61}$$

Is this the whole story for the Sen limit? Or in other words, all the weak coupling limits of F-theory corresponds to $A^4BC$ system of type IIB theory? The answer is **No**! One can also expect that the D7-branes can be pulled away from the O7-planes and keep $\mathrm{Im}(\tau)$ large enough. Indeed, by ignoring the higher term, the solution (2.7) can be rewritten as

$$\tau(z) = \frac{1}{2i\pi}\ln\frac{z - z_0}{\lambda}, \tag{2.62}$$

if we assume that $z - z_0 \ll \lambda$ then the geometry is approximately flat, which we may argue that the probe language of D7-branes are valid since the region for the weak coupling limit is large enough to trust the description by a effective supergravity. However when we approach a O7-plane, the naive supergravity solution breaks down as we can see the $\mathrm{Im}(\tau)$ becomes negative! $\tau \sim -\frac{4}{2\pi i}\ln(z - z_{07})$. In other words, one finds nasty singularities at finite distance from the O7-plane! The reason is simply: O7-planes carry negative tension. In F-theory, the pathology is cured as the O7-planes at non-zero string coupling ($g_s \neq 0$) splits in two $(p, q)$ branes: $B$ and $C$, with the distance of the separation of the order $\exp(-\pi/2\mathrm{Im}(\tau))$. This splitting of O7-planes is truly non-perturbative effects as the string coupling now is non-perturbatively small. In other words, it is in principle to obtain certain weak coupling limits from a F-theory configuration and trust the analysis in effective supergravities away from vicinity of O7-planes. In order to extract the limits, the starting point is to parameterize

$$\begin{aligned} f =& - 3h^2 + \epsilon\eta, \\ g =& - 2h^3 + \epsilon h\eta - \epsilon^2\chi/12, \end{aligned} \tag{2.63}$$

where $h, \eta$ and $\chi$ are a homogeneous polynomials of degree 4, 8 and 12 in the $(u, v)$, respectively, and $\epsilon$ is a constant. The sen limit is taken by sending $\epsilon \to 0$ while keeping everything else fixed, one then finds that for the discriminant $\Delta$ and $j(\tau)$

$$\Delta \approx -9\epsilon^2 h^2(\eta^2 - h\chi) + \mathcal{O}(\epsilon^3), \qquad j(\tau) \approx \frac{24^4 h^4}{2\epsilon^2(\eta^2 - h\chi)}. \tag{2.64}$$

Thus, in this limit, we have

$$g_{IIB} \sim -\frac{1}{\log|\epsilon|} \to 0 \tag{2.65}$$

everywhere except near $h = 0$ and we may expect a weakly coupled type IIB vacuum. In order to relate this to IIB data, one needs to consider the $SL(2, \mathbb{Z})$ monodromy around the discriminant $\Delta = 0$. Note at the above sen limit with $\epsilon \to 0$ all the roots of the discriminant $\Delta$ are located at $h = 0$ and $\eta^2 = h\chi$. Thanks to Sen [78, 79], a monodromy anaylze shows the results of these two components of $\Delta = 0$ yield

$$\begin{aligned} h = 0: &\qquad M = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \\ \eta^2 = \chi: &\qquad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{2.66}$$

which we know are generated by O7-plane and D7-brane, respectively. Hence the two roots should be identified with the O7-plane divisor and the D7-brane divisor $W$ in the base $B_1 = \mathbb{P}^1$ as follows

$$O7 : h(u,v) = 0, \qquad W : \eta(u,v)^2 = h\chi(u,v). \tag{2.67}$$

And the Calabi-Yau one-fold, which is viewed as the double cover over $B_1$ is given by the equation

$$X_1 : \xi^2 = h(u,v) \tag{2.68}$$

with the orientifold involution

$$\sigma : \xi \to -\xi. \tag{2.69}$$

Given that, one should note that in the covering Calabi-Yau one-fold $X_1$, the D7-brane divisor $\widetilde{W}$ is then given by

$$\eta^2 = \xi^2\chi(u,v) \tag{2.70}$$

reflecting the double cover over $W$.

Away from the sen limit $\epsilon \to 0$, the O7-plane would split into two $(p,q)$ 7-branes, i.e. B and C, which necessarily involves the non-perturbative effects. Including the higher order of $\epsilon$, the factorization of D7-branes and O7-planes is lost, which could be thought as the recombination of D7-branes and O7-planes.

As aside, there are additional two ways to make axio-dilaton constant along the base $B_1$: taking $f \equiv 0$ or $g \equiv 0$. However, in these cases, $\tau$ generally takes a large values and necessarily correspond to strongly coupled type IIB strings. It turns out it generalizes the $\mathbb{Z}_2$ orientifolds to $\mathbb{Z}_n$ actions [80]. Note that the $n = 3,4,6$ cases have been already covered in the $(p,q)$ 7-brane description 2.1.1, which corresponds to the constant axio-dilaton.

The procedure with the above sen limit can formally extend to any higher dimensional elliptic fibered Calabi-Yau $X_n, n > 2$ and accordingly, the base $B_{n-1}$ can be viewed as the involution of a double cover space $X_{n-1}$ which is a $n-1$ dimensional Calabi-Yau manifold (again but not necessarily elliptic fibered)!.

The Sen limit has drawn significant attentions in the past decades, for examples [68, 81–85]. For our purpose, let's pick up some properties discussed in [68, 82]. Backing to the loci of the D7-branes in (2.67), one can easily find that it is generically [18] singular along the locus $\eta = \xi = 0$ of double points, further degenerates at the pinch locus $\eta = \xi = \chi = 0$. Locally around these singular locus, the shape of D7-brane $\widetilde{W}$ looks like a Whitney umbrella, hence dubbed this singular D7-brane as a *Whitney* D7-brane [68]. Note that the D7-branes appearing in this Sen limit always have a double intersection with an O7-plane, which can be seen as a consequence of Dirac quantization in the case of $O7^-$-plane [68]. The singular of the D7-brane divisor $\widetilde{W}$ would render the definition of certain topological number such as Euler characteristic of $\widetilde{W}$ open to interpretation. In [68, 82], they use the condition of the D3-brane tadpole cancellation to give a consistent definition of the Euler characteristic of singular surface $\widetilde{W}$ in Calabi-Yau four-folds, which we have mentioned in (1.167).

Later, in [81], they argued that the above Sen limit, as a way to connect F-theory and IIB has many subtle problems, mainly because of lacking a proper way to track the relation between

[18] By "generically" we mean the local functions $(\eta, \xi, \chi)$ are generic on the base. In other words, one can obtain a smooth D7-brane configuration with specific forms of $(\eta, \xi, \chi)$. For example, we will see soon in next subsection, if one takes $\chi$ to be a perfect square, say $\chi = \psi^2$, then $\widetilde{W}$ split as $\widetilde{W} = W_+ + W_-$ with $W_\pm = \eta \pm \xi\psi$, known as brane-image-brane pair. It turns out that these two divisors are homologous to each other in the double-covering space $X_{n-1}$. Later, we will show these configurations will uplift to the abelian $U(1)$ symmetries in F-theory compactifications.

two sides physical quantities such as the superpotential. The technique reason is that the degenerating limit (weak coupling limit) $\epsilon = 0$ in this approach is too severe to extract all the relevant information. Instead, they proposal a new way to understand the Sen limit, dubbed the stable degenerating limit of F-theory. The basic idea is that introducing a new parameter $\epsilon$ and consider the Sen's family of Weierstrass model $X_n$ as a singular $n+1$ fold Calabi-Yau $Y_{n+1}$ in its own right and blow up the singularity to obtain a new smooth $\widehat{Y}_{n+1}$. The key part is that the double-covering Calabi-Yau $X_{n-1}$ can arise from the intersection of two components of $\widehat{Y}_{n+1}$ at the weak coupling limit $\epsilon = 0$. And in this limit, one can in principle have a good track of the degenerating limit. For details, we refer to the original work [81].

Before we close this subsection, we would like to emphasis a very important fact concerning of the sen limits of F-theory. It seems that every elliptic fibered Calabi-Yau $X_n$ has the sen limit, and physically speaking, it seems that every F-theory vacua has a type IIB weak coupled limit. However, this is **not** true. The problematic point is that the double covered Calabi-Yau space $X_{n-1}$ typically has conifold singularities, which would render the track of $\tau(u)$ unclear rather than $g_{IIB} \to 0$. We will come back to this point in section 5.3.1 and similar situation we found in conifold $I_1$ model. As a matter of fact, It is believed that most of F-theory vacua do not have a weakly coupled type IIB limit. This point can be also **inferred** from the geometric viewpoint that the base $B_{n-1}$ of a given elliptically fibered Calabi-Yau $X_n$ typically is not a conformal Calabi-Yau space, while we have mentioned at the end of section 1.8.2 that the downstairs geometries of Type IIB orientifold compactifications are conformal Calabi-Yau.

## 2.4. Codimension-one Singularities and Non-abelian Gauge Algebras

From a physics perspective, the most essential data of an elliptic fibration are the loci and also the type of fiber degenerations as they encode the information on nature of the seven-branes and further determines the gauge algebra, matter spectra and holomorphic interactions, etc. In this section, we are going to review the classification of codimension-one singularities on elliptically fibered Calabi-Yau spaces. We first start with the two dimensional K3, which has been classified by associating with the A-D-E Lie algebra by Kodaira and Néron. On higher-dimensional elliptic fibrations, there are certain monodromies effects affecting the global structures of the codimension-one fibers and hence could lead to non-simply laced Lie algebras, which we will discussed in the next section.

We denote the loci of the discriminant as the divisor on the base $B_{n-1}$

$$W := \{\Delta = 0\} \in B_{n-1}. \tag{2.71}$$

If $f, g$ are maximally generic functions, then the discriminant divisor $W$ is an irreducible divisor on the base $B_{n-1}$, i.e. it can be described by a single meromorphic functions over $B_{n-1}$. At a generic point $t \in \Delta = 0$, the $(f, g)$ do not simultaneously vanish, however the fiber $\mathbb{E}_\tau$ over the point $t$ would degenerate at $[x : y : z = * : 0 : 1]$ and turns out to be a **nodal** curve, as we discussed in the subsection 2.2.1. This is so-called Kodaira-type $I_1$ in the Kodaira classification of elliptic surfaces. This can be read off from the vanishing order of the zeros of $(f, g, \Delta)$, namely we have

$$\text{type} - I_1 : \qquad \text{ord}(f, g, \Delta) = (0, 0, 1). \tag{2.72}$$

Throughout this thesis, the discriminant for the Kodaira type $I_1$ will be denoted as $\Delta_0$ as well as its divisor $W_0$, which is a single irreducible divisor. It turns out that F-theory on this

Weierstrass model has exactly the sen limit we have discussed in 2.3.1. Namely it has a type IIB interpretation: Orientifold compactification on double covering space $X_{n-1}$ of the base $B_{n-1}$ with a single *Whitney* D7-brane along the divisor $[\eta^2 - \xi^2\chi = 0]$ on $X_{n-1}$ together with an O7-plane along the divisor $[\xi^2 = 0]$. The novel feature of this Whitney D7-brane is that it carries a trivial gauge group $SO(1) \cong \{1\}$ as the brane and image brane are now coincident, on top of the O7-plane.

Similar to the elliptic curves, there are also some non-generic points $p \in \{\Delta = 0\}$ such that $(f, g)$ both vanish hence the fibre $\mathbb{E}_\tau$ over $p$ develops to **cuspidal** curve t $[x : y : z = 0 : 0 : 1]$. In this cases, it is Kodaira Type II with the following feature:

$$\text{type} - II : \text{ord}(f, g, \Delta) = (\geqslant 1, 1, 2). \tag{2.73}$$

Comparing with the above Type $I_1$ singularity, this case has a subtle type IIB interpretation. We refer to [82] for more details.

Note that the whole elliptic fibration Calabi-Yau manifolds $X_n$ is **smooth** for both of the above two types. However, if the vanishing order $(f, g, \Delta)$ over certain discriminant $\Delta_I$ exceed the above two cases, the whole fibration $X_n$ would also be singular over the divisor $[\Delta_I = 0]$ in the base. For the rest of this thesis, we will assume that the discriminant $\Delta$ generically decomposes as

$$\Delta = \Delta_0 \prod_{I=1}^{N} (\Delta_I)^{p_I}, \tag{2.74}$$

where we single out the 0 component $\Delta_0$ whose fiber is Kodaira type $I_1$ and $\Delta_I$ describe irreducible polynomials and $p_I$s are the multiplicities for each $I$(i.e. the degree of W along $W_I$), e.g. $p_I = n$ for $A_{n-1}$ and $p_I = 10$ for $E_8$. The discriminant divisor $W = \{\Delta = 0\}$ hence has

$$W = W_0 \cup_I W_I. \tag{2.75}$$

And the whole fibration $X_n$, not only the fibers, are singular at $W_I$. Each divisor $W_I$ are wrapped by the seven-branes in the base $B_{n-1}$. However, for the (accepted) singular Calabi-Yau, there are typically no well-defined topological invariants and hence we would lose the controls in terms of descriptions. Hence we have to resolve these singularities in order to study them. We are going to introduce one of the blow-up method. We firstly start with the simplest elliptic fibered Calabi-Yau two-fold, i.e. K3 surfaces and later give a systematically description for higher dimensional elliptic fibrations with certain subtleties.

### 2.4.1. Starting with the golden example: K3

**A-D-E Classification** We have already said, an elliptically fibered K3 surface is generic smooth even though the fiber degenerates at various points $u_i, i = 1, ..., 24$ in the base $\mathbb{P}^1$. However, when some $u_i$ collide, it would expect that the whole elliptic fibration would render singular. As for the Weierstrass model, one would expect it should be reflected by the $(f, g, \Delta)$. Indeed, these questions was firstly analyzed in the seminal work by Kodaira [86, 87] and classified the singularities by the A-D-E types and later by Néron [88] for codimension-one fibers for general dimensional Weierstrass models. For mathematicians at that time, it was hard to image how the classification has anything to do with the simple-laced Lie algebra A-D-E. In order to appreciate how does the A-D-E classification arise, It is necessary to discuss it in the context of the crepant resolution $\widehat{X}_2$ of the singular $X_2$. We will give a precise definition of crepant resolution of general algebraic variety in 2.8 later, here we focus on K3 and also give an example to illustrate various properties in the Insertion 3.2.

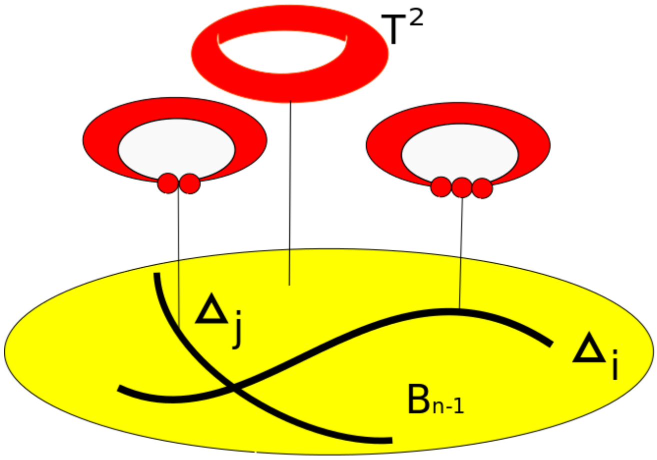


**Figure 2.6.:** *Elliptic fibration of $\widehat{X}_n$, where the tours $T^2$ pinches at the loci $\Delta$ of the base $B_{n-1}$ are blowing up into collections of $\mathbb{P}^1$s*

The idea of a crepant resolution $\widehat{X}_2$ of $X_2$, shortly speaking, is a local operation which replaces the singular fibers $\mathbb{E}_s$ over $W_I$ in $X_2$ by a collection of rational curves $\mathbb{P}^1$ while keep the smooth parts of $X_2$ unchanged, and importantly the canonical bundle stay the same thereby the resolved $\widehat{X}_2$ is also Calabi-Yau two-manifold, i.e. K3. After this local operator, the new fiber on the resolved $\widehat{X}_2$ over each point in $W_I$ topologically looks like [19]

$$[\mathbb{E}_\tau] = \sum_{i=0}^{\mathrm{rk}(\mathfrak{g})} a_i[\mathbb{P}^1_i], \tag{2.76}$$

where $a_0 = 1$ and $a_i$ represents the multiplicities of each of $\mathbb{P}^1$s and $\mathfrak{g}$ is the corresponding A-D-E Lie algebra, such that the set of these rational curves

$$\mathbb{P}^1_i, \qquad i =, ..., \mathrm{rk}(\mathfrak{g}) \tag{2.77}$$

have a matrix of intersection numbers among themselves. In the K3 cases, Kodaira [86, 87] showed that this intersecting matrix is exactly the Cartan matrix of the corresponding A-D-E lie algebra $\mathfrak{g}$. More precisely, these rational curves intersect as [20]

$$\int_{\mathbb{P}^1_i} w_i = [\mathbb{P}^1_j] \cdot [\mathbb{P}^1_i] = -C_{ij}, \tag{2.78}$$

where the $C_{ij}$s are the Cartan matrixes of the A-D-E algebra $\mathfrak{g}$ and $w_i$ denotes the Poincarë dual to the $\mathbb{P}^1_i$. One can represent them diagrammatically by associating a node with each degenerating cycle $\mathbb{P}^1_I, I = 1, ..., \mathrm{rk}(\mathfrak{g})$ and by connecting the nodes by a line for each intersection

[19] Here for our purpose, we only assume there are only one Kodaira singularity corresponding to the algebra $\mathfrak{g}$.
[20] Throughout this chapter, given a divisor $D$, we will denote both the cohomology class and homology class (mode out torsion) as $[D]$. And the intersection product for both of them are also denoted as "$\cdot$". See more details on our notations in the appendix B.

of the two cycles, see 2.7. Note that we have singled out $\mathbb{P}^1_0$, which represents the original singular fiber (whose topology is $\mathbb{P}^1$), and this can be viewed as the affine node for the Dynkin diagram. And the multiplicity $a_i$ coincides with the dual Kac label for the corresponding node in the Dynkin diagram.

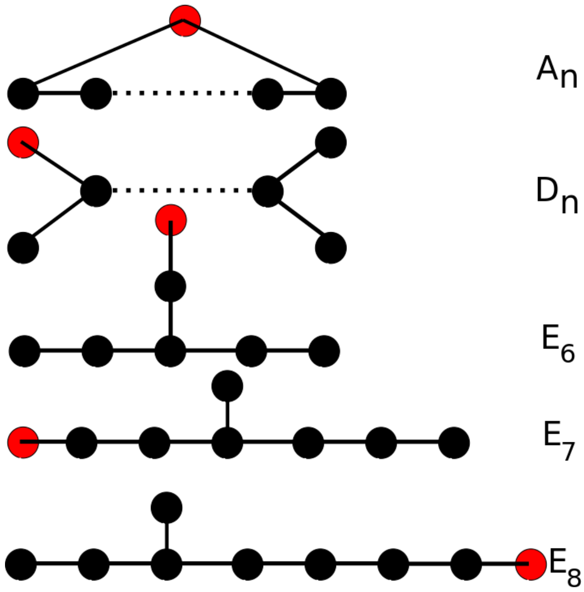


**Figure 2.7.:** *The affine Dynkin diagram of the simple-laced A-D-E Lie algebras. Each node represents a rational curve $\mathbb{P}^1_i, i = 0, 1..., rk(\mathfrak{g})$. The red one represents the extended/affine one $\mathbb{P}^1_0$, which is the original pitched torus in the singular limit, intersected once by the zero section.*

In summary, replaces the singular fibers over $W_I$ by a chain of rational curves $\mathbb{P}^1$ hence exceptional divisor $E_{i_I}$ in a new total space $\widehat{X}_n$, which under the projection $\pi : \widehat{X}_n \to B_{n-1}$ maps to the divisor $W_I$. the blowing-up resolution replaces the singular fibers over $W_I$ by a collection of rational curves $\mathbb{P}^1$ and one can introduce a set of exceptional divisor $E_{i_I}$ which are themselves fibration over $W_i$ whose fibers is

$$\begin{array}{ccc} \mathbb{P}^1_{i_I} & \longrightarrow & E_{i_I} \\ & & \downarrow \\ & & W_I, \end{array} \tag{2.79}$$

where $i_I = 0, .., \mathrm{rk}(\mathfrak{g}_I)$s are the dual Kac label of the corresponding node in Dynkin diagram and $a_0 = 1$, $\mathbb{P}^1_I$ are the resolved rational curves.

Contracting all fiber $\mathbb{P}^0_i$ to a point corresponds to blow down from $\widehat{X}_2$ to the original singular fibration $\widehat{X}_2$. Kodaira and Néron have showed that the various types of the Kodaira fibers for all minimal [21] elliptic surfaces can all be classified as the vanishing order of $(f, g, \Delta)$, where we list in (2.2), as well as the local geometry around the singular points associated with them.

[21]**minimality** refers that a complex manifold $X$ contains no curves $C$ with self-intersecting number $C \cdot C = -1$. On a complex manifold $X$, such $(-1)$ curves are the only types curves that can be blow down to a smooth point without changing the canonical bundle of that manifold $X$. Note that in the smooth K3 surface $\widehat{X}_2$,

| ord$(f)$ | ord$(g)$ | ord$(\Delta)$ | fiber-type | local geometry in $\mathbb{C}^3$ | singularity-type | monodromy |
|---|---|---|---|---|---|---|
| $\geq 0$ | $\geq 0$ | 0 | $I_0$ | | smooth | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ |
| $\geq 0$ | $\geq 0$ | 1 | $I_1$ | $y^2 = x^2 + z$ | smooth with nodal curve | $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ |
| 0 | 0 | $n$ | $I_n$ | $y^2 + x^2 + z^n = 0$ | $A_{n-1}$ | $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ |
| $\geq 1$ | 1 | 2 | $II$ | | smooth with cuspidal curve | $\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$ |
| 1 | $\geq 2$ | 3 | $III$ | $y^2 + x^2 + z^2 = 0$ | $A_1$ | $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ |
| $\geq 2$ | 2 | 4 | $IV$ | $y^2 + x^2 + z^3 = 0$ | $A_2$ | $\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$ |
| 2 | 3 | 6 | $I_0^*$ | $y^2 + x^2 z + z^3 = 0$ | $D_4$ | $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ |
| 2 | $\geq 3$ | $n+6$ | $I_n^*$ | $y^2 + x^2 z + z^{n+3} = 0$ | $D_{n+4}$ | $\begin{pmatrix}-1 & n\\ 0 & -1\end{pmatrix}$ |
| $\geq 2$ | 3 | $n+6$ | $I_n^*$ | $y^2 + x^2 z + z^{n+3} = 0$ | $D_{n+4}$ | $\begin{pmatrix}-1 & n\\ 0 & -1\end{pmatrix}$ |
| $\geq 3$ | 4 | 8 | $IV^*$ | $y^2 + x^3 + z^4 = 0$ | $E_6$ | $\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$ |
| 3 | $\geq 5$ | 9 | $III^*$ | $y^2 + x^3 + xz^3 = 0$ | $E_7$ | $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ |
| $\geq 4$ | 5 | 10 | $II^*$ | $y^2 + x^3 + z^5 = 0$ | $E_8$ | $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$ |

**Table 2.2.:** *Kodaira classification of elliptic fiber based on truplet $f, g, \Delta$. The monodromy listed above of course up to an $SL(2,\mathbb{Z})$ transformation. Note that Blow-up is a local operations, hence we also include the local geometry of the A-D-E surface singularities, note that here $(x, y, z)$ are coordinates in the $\mathbb{C}^3$, rather than in the weighted projective space $\mathbb{P}^2_{2,3,1}$ as the same notations in the Weierstrass model.*

Note that If ord$(f, g, \Delta) \geqslant (4, 6, 12)$ then the singularity of the total space is so severe that it destroys the triviality of the canonical bundle on any resolved spaces, and hence no **minimal** smooth elliptic K3 $\widehat{X}_2$ exists. That means, there is no crepant resolution of a Weierstrass model with this property. As a consequence, the effective theory from F-theory compactifications does not have supersymmetry [22]. Throughout this thesis, we constraint us with the discussions on the minimal singularities at the codimension-one. At higher codimensional loci, the vanishing orders of singularities of the fibers can be allowed to exceeds the $(4, 6)$ but still not too severe, which we will explain a bit later in 2.7.

The interpretation of the Kodaira table is as following: assuming that the codimension-one divisor on the base $B_1$ locally can be described by the vanishing of a local coordinate $w = 0$,

there are no $-1$ curves as one has

$$C \cdot (C + K) = 2(g-1) \rightarrow C \cdot C = 2(g-1) \neq -1 \tag{2.80}$$

where $C$ represents any curves in $K3$ with the canonical bundle $K = 0$.

[22]Note that the breaking of supersymmetry in these cases happens already at the compactification scale, which is different from the desired supersymmetric breaking in the favorable phenomenological applications.

then except for all Kodaira type fibers $I_n$ series, $f, g, \Delta$ must factorize as (in the given patch)

$$f = w^m \widetilde{f}, \qquad g = w^n \widetilde{g}, \Delta = w^k \widetilde{\Delta}, \qquad (m, n, k) = \mathrm{ord}(f, g, \Delta), \tag{2.81}$$

where $\widetilde{f}, \widetilde{g}$ are sufficiently generic such that the discriminant $\Delta$ has the prescribed vanishing order $k$. As for the special cases with $I_n$ Kodaira fibers, apart from the above cases with$f, g$, they can also obtain the presecribed vanishing order $k$ for $\Delta$ even though that $(f, g)$ have no zeros at the divisor $[w = 0]$. In more words, starting from a general ansatz

$$f = \sum_i f_i w^i, \qquad g = \sum_i g_i w^i, \tag{2.82}$$

one can tune the coefficients $f_i, g_i$ such that the discriminant

$$\Delta = (4f_0^3 + 27g_0^2) + (12 f_1 f_0^2 + 54 g_0 g_1) w + \mathcal{O}(w^2) \tag{2.83}$$

can be vanish to the higher prescribed order! The above interpretations in general can be carried over to higher dimensional Calabi-Yau $X_n$ with the base $B_{n-1}$.

Before we move to the next subsection for higher dimensional Calabi-Yau, we would like to mention an aside remark. According to the language of (geometric) [23] "non-Higgable gauge groups" in [89], gauge groups from $I_n$ fiber are not (geometric) non-Higgable. Essentially speaking, the (geometric) non-Higgable conditions requires that the $\Delta$ vanishes to the lowest order on any divisor $W$ over the base $B_{n-1}$ while the $(f, g)$ are generic sections of the corresponding line bundles. As a consequence, the gauge groups are "minimal" for any possible fibration over $B_{n-1}$ and as one tunes them, it should worsen the singularities and hence enhance the gauge algebras. Thereby, in the sense, the counter example are those with $I_N$ singularities, the $f, g$ are non-generic polynomials in order to render the $\Delta$ vanishes at the higher order. Based on the table 4.1 in citeWeigand:2018cod for higher dimensional Calabi-Yau $X_n$, we can see $I_0, I_1, II, III, IV, I_0^*, IV^*, III^*$ and $II^*$ allows the generic fibrations and accordingly, the possible (geometric) non-Higgsable gauge algebras are $\mathfrak{so}(2), \mathfrak{so}(7), \mathfrak{so}(8), \mathfrak{su}(3), \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_i, i = 6, 7, 8$.

**Insertion 3.2: An example with $A_2$ singularities**

In this insertation, we give a brief introduction on how to resolve the singularities of elliptic surface. i.e. K3 surface. We focus on the example of $\mathbb{C}^2/\mathbb{Z}_2$, which is given by $z_1, z_2 \in \mathbb{C}^2$ modding out $\mathbb{Z}_2$ action as $(z_1, z_2) \sim (-z_1, -z_2)$. For more details on general singular $K3$s and how to resolve its A-D-E singularities, we refer to the Aspinwall's nice review [90]. We can choose three $\mathbb{Z}_2$ invariant coordinates as

$$x_1 = z_1 z_2, x_2 = z_1^2, x_3 = z_2^2. \tag{2.84}$$

It is easy to see they hold the equality as

$$x_1^2 = x_2 x_3. \tag{2.85}$$

This is so-called conifold singularities in $\mathbb{C}^3$. This form could be transformed to the canonical

[23] By "geometric", we mean that in the non-Higgsable structure, one cannot higgsing the gauge group by tuning the coefficients $(f, g)$, which is a geometric operation. Nevertheless, One can also higgsing the gauge group by turning on the $G_4$ flux, typically at the higher dimensional Calabi-Yau manifolds compactification.

form of an $A_1$ singularities in Kodaira classification

$$z^2 + y^2 + x^2 = 0 \tag{2.86}$$

by changing the coordinate as

$$z = ix_1, y = x_2 + ix_3, \qquad x = x_2 = ix_3. \tag{2.87}$$

Now in order to blow up the singularities at $x = y = z = 0$ we introduce a new $\mathbb{P}^2$ with coordinates $\xi_i, i = 1, 2, 3$ which are subject to the relations as

$$z\xi_2 = y\xi_1, \qquad z\xi_3 = x\xi_1, \qquad y\xi_3 = x\xi_2. \tag{2.88}$$

One can check that the $\xi_i$ are uniquely determined at a given point $P$ away from the singular point $O : x = y = z = 0$. However, at the singular point $x = y = z = 0$, they are freely unfixed, namely at the singular point in the $\mathbb{C}^3$, we have the entire $\mathbb{P}^2$. The space (2.88) is thus referred to as a blow-up of $\mathbb{C}^3$ at the origin $x = y = z = 0$. A space $X$ blown-up at a point, denoted as $\widehat{X}$ is birationally equivalent to $X$

$$\rho : \widehat{X} \to X, \tag{2.89}$$

here $\rho$ represents the blow-down of $\widehat{X}$. The $\mathbb{P}^2$ which has grown out of the origin $O$ is dubbed the exceptional divisor in $\mathbb{C}^3$.

In terms of the the K3 surface $A$ defined in (2.86), we can see the exceptional divisor is $\mathbb{P}^1$. In order to see it, we firstly introduce the **proper transfom** $\widehat{A} \subset \widehat{X}$, which is defined as the closure of the point set $\rho^{-1}(A \setminus O)$ in $\widehat{X}$. Then considering a path in $A$ towards the origin $O$, which approach the origin $O$ with an angle, say following $xt, yt, zt, t \in \mathbb{C}^1$, then it will land on the point in $\mathbb{P}^2$ where again $\xi_1^2 + \xi_2^2 + \xi^3 = 0$,. Thus the point set that provides the closure away from $O$ is a quadratic $\xi_1^2 + \xi_2^2 + \xi^3 = 0$ in $\mathbb{P}^2$, which is easy to show that this curve has genus $g = 0$, i.e. $\mathbb{P}^1$. We hence show that the proper transform $\widehat{A}$ replace the origin $O$, at which is singular, by a $\mathbb{P}^1$ and further the resolved space $\widehat{A}$ is smooth. Again this $\mathbb{P}^1$ is the exceptional divisor $E$ in $\widehat{A}$.

As the exceptional divisor $E$ in $\widehat{A}$ is rational curve, i.e. its genus equals to 0, one can calculate that their self-intersection number is $-2$ as in smooth K3 one has

$$E \cdot (E + K) = 2(g - 1) = E \cdot E, \tag{2.90}$$

where the first equality we employ the adjunction formula for the curve and the second one is due to $K = 0$ for K3.

Similarly, the above blow-up can also apply to the generalized orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ as singular $K3$ spaces. In such cases, a set of exceptional divisors $E_i, i = 1, .., n-1$, of all topological $\mathbb{P}^1$, shall be introduced to resolve the orbifold singularity. Further, these exceptional divisors has the $A_{n-1}$ intersection matrix, with the diagonal items being $-2$ and fits with the Kodaria result. For $D$ and $E$ cases, the discrete group $G$ in the orbifold $\mathbb{C}^2/G$ would not be simple cyclic $\mathbb{Z}_n$, but other discrete subgroups $G$ of the $SU(2)$. For more details on these resolution, we refer to section 2.6 in [90].

We call this resolution of singularities as blow-up, as the singular point has blow up to a new space $\mathbb{P}^2$. Note that the blow-up resolution can also operates at a generic smooth points. The single blow-up has increased the rank of Picard group by one coming from the exceptional divisor $E$.

This operation can be similarly generalized to any dimensional complex manifolds. To blow up a d-dimensional complex manifold $X$ at a point $p$, we can introduce a $\mathbb{P}^d$ with coordinates $\xi_i, i = 1, ..., d+1$ subjecting to the equations:

$$z_i\xi_j = z_j\xi_i. \tag{2.91}$$

### 2.4.2. General description of codimension-one fibers in higher dimensional elliptical fibrations

In the previous sections, we have discussed the Kodaira classification of the elliptic surfaces. The same results of Kodaira and Néron carry over to higher dimensional elliptical fibrations. However, there are new features for elliptic fibration Calabi-Yau $X_n$ for $n > 2$. Note that when $n > 2$, the discriminant divisor $W$ is not zero-dimensional but a higher dimensional spaces hence the monodromy effects on the fiber components over the discriminant divisor $W$ should be taken into account, , which are equivalent to outer automorphisms on the simply-laced Lie algebra A-D-E. More precisely, if one transport one $\mathbb{P}^1$ component of a generic fiber $\mathbb{E}_\tau$ along a non-trivial loop within $W$, then the $\mathbb{P}^1$ would map to another $\mathbb{P}^1$ and thus can be effectively identified as the fibre of the same exceptional divisor $E_{i_I}$. From the standpoint of Dynkin diagram, these effects correspond to the folding of Dynkin diagrams of A-D-E type to other one, hence produce the Dynkin diagrams associated with non-simply laced Lie algebras $B_n, C_n, G_2$ and $F_4$. For the details we refer to the work [91] (and also the review in [21]).

In order to fully distinguish the Lie algebras after taking into account these effects, we are going to replace the Weierstrass model by the Tate's algorithm. In the vincinity of a codimension-one singularity a Weierstrass model can locally viewed as the Tate form by methods of a general algorithm [92] placing the singularity in the fiber at the points $[x : y : z = 0 : 0 : 1]$. By local coordinate redefinition, the Weierstrass model (2.48) can be brought into the Tate form

$$P_T : y^2 = x^3 + xyza_1 + x_2z^2a_2 + yz^3a_3 + xz^4a_4 + z^6a_6, \tag{2.92}$$

where the coefficients $a_i$ is the section of the line bundle $K^{-i}_{B_{n-1}}$ over the base, i.e. $a_i \in \Gamma(B_{n-1}, K^{-i}_{B_{n-1}})$ and $[x : y : z]$ are the same coordinates as the Weierstrass model in the weighted projective space $\mathbb{P}^2_{2,3,1}$. To recover the Weierstrass model from the Tate form, one can first introduce the combinations

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = a_1a_3 + 2a_4, \qquad b_6 = a_3^2 + 4a_6. \tag{2.93}$$

Then the $f, g, \Delta$ in Weierstrass model in the terms of $a_i$ follows

$$f = -\frac{1}{48}(b_2^2-24b_4), \quad g = -\frac{1}{864}(-b_2^3+36b_2b_4-216b_6), \quad \Delta = -8b_4^3+9b_2b_6b_4-27b_6^2+\frac{1}{4}b_2^2\left(b_2b_6 - b_4^2\right). \tag{2.94}$$

With these identities, the Kodaira classification of the type of singularities can be labeled by the $a_i$, which listed in the (2.3). One can argue for a generic coefficient $a_i$ the Tate form is equivalent to a generic Weierstrass model globally. However, for many cases with specific $f, g$, especially those leading to singularities and interesting physics, it may in general not be possible to write the Weierstrass model globally in Tate form. In other words, The Tate form can only be obtained locally, where the particular examples are those with $I_n, n = 6, 7, 8, 9$ or $I_3^*$. For details we refer to [93].

| sing. type | discr. deg(Δ) | gauge enhancement type | group | coefficient vanishing degrees $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ |
|---|---|---|---|---|---|---|---|---|
| $\mathrm{I}_0$ | 0 | | — | 0 | 0 | 0 | 0 | 0 |
| $\mathrm{I}_1$ | 1 | | — | 0 | 0 | 1 | 1 | 1 |
| $\mathrm{I}_2$ | 2 | $A_1$ | $SU(2)$ | 0 | 0 | 1 | 1 | 2 |
| $\mathrm{I}_{2k}^{\mathrm{ns}}$ | $2k$ | $C_{2k}$ | $SP(2k)$ | 0 | 0 | $k$ | $k$ | $2k$ |
| $\mathrm{I}_{2k}^{\mathrm{s}}$ | $2k$ | $A_{2k-1}$ | $SU(2k)$ | 0 | 1 | $k$ | $k$ | $2k$ |
| $\mathrm{I}_{2k+1}^{\mathrm{ns}}$ | $2k+1$ | | [unconv.] | 0 | 0 | $k+1$ | $k+1$ | $2k+1$ |
| $\mathrm{I}_{2k+1}^{\mathrm{s}}$ | $2k+1$ | $A_{2k}$ | $SU(2k+1)$ | 0 | 1 | $k$ | $k+1$ | $2k+1$ |
| II | 2 | | — | 1 | 1 | 1 | 1 | 1 |
| III | 3 | $A_1$ | $SU(2)$ | 1 | 1 | 1 | 1 | 2 |
| $\mathrm{IV}^{\mathrm{ns}}$ | 4 | | [unconv.] | 1 | 1 | 1 | 2 | 2 |
| $\mathrm{IV}^{\mathrm{s}}$ | 4 | $A_2$ | $SU(3)$ | 1 | 1 | 1 | 2 | 3 |
| $\mathrm{I}_0^{*\,\mathrm{ns}}$ | 6 | $G_2$ | $G_2$ | 1 | 1 | 2 | 2 | 3 |
| $\mathrm{I}_0^{*\,\mathrm{ss}}$ | 6 | $B_3$ | $SO(7)$ | 1 | 1 | 2 | 2 | 4 |
| $\mathrm{I}_0^{*\,\mathrm{s}}$ | 6 | $D_4$ | $SO(8)$ | 1 | 1 | 2 | 2 | 4 |
| $\mathrm{I}_1^{*\,\mathrm{ns}}$ | 7 | $B_4$ | $SO(9)$ | 1 | 1 | 2 | 3 | 4 |
| $\mathrm{I}_1^{*\,\mathrm{s}}$ | 7 | $D_5$ | $SO(10)$ | 1 | 1 | 2 | 3 | 5 |
| $\mathrm{I}_2^{*\,\mathrm{ns}}$ | 8 | $B_5$ | $SO(11)$ | 1 | 1 | 3 | 3 | 5 |
| $\mathrm{I}_2^{*\,\mathrm{s}}$ | 8 | $D_6$ | $SO(12)$ | 1 | 1 | 3 | 3 | 5 |
| $\mathrm{I}_{2k-3}^{*\,\mathrm{ns}}$ | $2k+3$ | $B_{2k}$ | $SO(4k+1)$ | 1 | 1 | $k$ | $k+1$ | $2k$ |
| $\mathrm{I}_{2k-3}^{*\,\mathrm{s}}$ | $2k+3$ | $D_{2k+1}$ | $SO(4k+2)$ | 1 | 1 | $k$ | $k+1$ | $2k+1$ |
| $\mathrm{I}_{2k-2}^{*\,\mathrm{ns}}$ | $2k+4$ | $B_{2k+1}$ | $SO(4k+3)$ | 1 | 1 | $k+1$ | $k+1$ | $2k+1$ |
| $\mathrm{I}_{2k-2}^{*\,\mathrm{s}}$ | $2k+4$ | $D_{2k+2}$ | $SO(4k+4)$ | 1 | 1 | $k+1$ | $k+1$ | $2k+1$ |
| $\mathrm{IV}^{*\,\mathrm{ns}}$ | 8 | $F_4$ | $F_4$ | 1 | 2 | 2 | 3 | 4 |
| $\mathrm{IV}^{*\,\mathrm{s}}$ | 8 | $E_6$ | $E_6$ | 1 | 2 | 2 | 3 | 5 |
| $\mathrm{III}^*$ | 9 | $E_7$ | $E_7$ | 1 | 2 | 3 | 3 | 5 |
| $\mathrm{II}^*$ | 10 | $E_8$ | $E_8$ | 1 | 2 | 3 | 4 | 5 |
| non-min | 12 | | — | 1 | 2 | 3 | 4 | 6 |

**Table 2.3.:** *Refined Kodaira classification resulting from Tate's algorithm, taken from [70].*

As stated in 2.4 and here we repeat, the location of 7-branes are determined by where the discriminant vanishes

$$\Delta = \Delta_0 \prod_{I=1}^{N} (\Delta_I)^{p_I} = 0, \tag{2.95}$$

where we have factorized the discriminant $\Delta$ into several irreducible components and single out the 0 component $\Delta_0$ whose fiber is Kodaira type $I_1$, which has no such monodromy effects and $\Delta_I$ describe irreducible polynomials and $p_I$s are the multiplicities for each $I$ (i.e. the degree of W along $W_I$. e.g. $p_I = n$ for $A_{n-1}$ and $p_I = 10$ for $E_8$. ). The discriminant divisor $W = \{\Delta = 0\}$ hence has

$$W = W_0 \cup_I W_I. \tag{2.96}$$

And the whole fibration are singular at $W_I$, not only the fibers. Each divisor $W_I$ are wrapped by the seven-branes in the base $B_{n-1}$ and can be associated with a non-abelian gauge algebra $\mathfrak{g}_I$. After taking into account monodromy effects which appear for non-simply laced groups, one can systematically define the exceptional divisor $E_{i_I}$ as fibering the invariant orbits $C_{i_I}$ of fiber components $\mathbb{P}^1_{i_I}$ over the irreducible divisor $W_I$. The orbits $C_{i_I}$ is the resolved rational curves $\mathbb{P}^1_{i_I}$ itself for the simply-laced cases. On the other hand, the orbits $C_{i_I}$ are the images of $\mathbb{P}^1_{i_I}$ under monodromies in the non-simply laced case, which can be seen as a union of several rational curves which are related by monodromies.

For our purposes in part III, we list several important properties associated with these exceptional divisors $E_{i_I}$ as following:

$$\begin{aligned} &[E_{i_I}]\cdot[\mathbb{P}^1_{j_J}] = -\delta_{IJ}\, C_{i_I j_J}\,. \\ &\pi_*([E_{i_I}]\cdot[E_{j_J}]) = -\delta_{IJ}\,\mathfrak{C}_{i_I j_I}\,[W_I] = -\mathrm{Tr}\,\mathcal{T}_{i_I}\mathcal{T}_{j_J}\,[W_I]\,, \\ &[S_0]\cdot[\mathbb{P}^1_{i_I}] = \delta_{i0}, \qquad i_I = 0,1,...,\mathrm{rk}(\mathfrak{g}_I). \end{aligned} \tag{2.97}$$

Here $[E_{i_I}]$ denotes the homology class of the divisor $E_{i_I}$ and unless noted otherwise, all intersection products are taken on the resolved elliptic fibered Calabi-Yau $\widehat{X}_n$. The matrix $C_{i_I j_I}$ is the Cartan matrix of $\mathfrak{g}_I$ (in conventions where the entries on its diagonal are +2).

### 2.4.3. Determining the non-abelian gauges group via geometric engineering

Sofar we have reviewed that the codimension-one singularities of the fibers in the elliptic fibered K3 can be classified by the simply-laced A-D-E Lie algebra as the intersecting pattens of the degenerating 2-cycles in elliptic surfaces according to the A-D-E Dynkin diagrams. By same token, it can be extended to as their outer automorphism versions $B-C-F-G$ Lie algebras for higher dimensional elliptic fibrations. These facts was well-known to mathematicians, however, it was completely mysterious what, if anything, this has to do with the A-D-E algebras. In this subsection, we are going to explain the connection from the standpoint of string theory, namely how the A-D-E classification of singularities in K3 surface relates to the corresponding A-D-E Lie algebras.

We have already mentioned several times that the low-energy theories of world-volume of D-branes (or more general $(p,q)$-branes) reduce to the gauge theory, describing the dynamics of the massless spectra generated by the open strings connecting the branes [24]. So it is naturally to think of the corresponding Lie algebra $\mathfrak{g}_I$ in the Kodaira-Tate table living on the stacks of 7-branes wrapped along the divisors $W_I$.

We would like to start with backing to the above $A^4BC$ system in the limit of 2.52. From the viewpoint of type II intersecting branes model, we have an $SO(8)$ gauge symmetry associated with each $A^4BC$ system, as 4 D7-branes are on top of $O7^-$-plane. Thus we get an $SO(8)$ gauge group at each point $u_i, i = 1,...,4$ and $(SO(8))^4$ in total in that limit, which is a special point on moduli spaces of F-theory compactifications. And we have stressed, that the 7-branes in F-theory correspond to a singular fibration of Calabi-Yau. In this example, the Calabi-Yau manifold is K3 surface. Furthermore, the singularities of K3 surface (or more general elliptic surfaces) are classified by the A-D-E types. Would the K3 surface with the $A^4BC$ systems coincidently be the $D_4$ type of singularities? To see this, note that after a suitable rescaling of

[24] The cases with M5-branes are subtle, as the fundamental objects would be M2-branes in M-theory. Hence the low-energy limit of world-volume of M5-branes should be describing the dynamics of strings, which is the boundary of M2-branes on M5-branes, this necessarily involves higher structure of gauge theory, the proper mathematics would be gerbes rather than the vector bundles/sheaves.

the coordinates and denote the new ones as $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \mathbb{C}^3$, the Weierstrass model in (2.48) takes the form

$$\widetilde{y}^2 = \widetilde{x}^2\widetilde{z} + \widetilde{z}^3 + ..., \tag{2.98}$$

this is exactly the $D_4$ type of the K3 surface. Hence this example strongly verify the idea is solid. In principle, one can use the language of $(p,q)$ strings connecting the $(p,q)$ 7-branes to obtain the massless spectra and hence can lead to the gauge algebras $\mathfrak{g}$. However, this is technically difficult to apply in the generic form as the $SL(2,\mathbb{Z})$ monodromies exert the global effects. For more details on the $(p,q)$ strings junction techniques and recent F-theory application, we refer to [94] and [95].

Fortunately, one can take advantage of the dual M-theory side to make the gauge algebras alive. But one should notice that we are still lack of a proper way to study the M-theory compactification on singular manifolds. Hence the only way one can do is to firstly resolved the singular Calabi-Yau manifolds, and then study the effective theory on $\mathbb{R}^{1,10-2n}$ by compactifying the M-theory on the smooth Weierstrass model $\widehat{X}_n$. As we stated, there are two proper ways for resolution of singularities: Deformation and Blow-up. The former resolution typically changes the complex structure moduli and the later one change the Käher moduli. One will mainly discuss the blow-up resolution, as we did in the previous sections. From the effective field theory language, the blow-up resolution corresponds to entering the Coulomb branch. Let us focus on the cases that the codimension-one singularities in the elliptically fibered Calabi-Yau $X_n$ admits a crepant(flat) blow-up resolution $\widehat{X}_n$. As we stressed, the resolution requires the introduction of independent exceptional divisors $E_{i_I}, i_I = 1, ..., \mathrm{rk}\mathfrak{g}_I$, which are rationally fibred over irreducible discriminant divisor $W_I$ within $\widehat{X}_n$. By the Shioda-Tate-Wazir theorem, the NS groups $NS(\widehat{X}_n)$, which counts the holomorphic cycles, can be splited into three categories: the zero section $S_0$, the vertical divisors $\pi^*(D^b_\alpha)$ from pulling divisors $D^b_\alpha$ in the base $B_{n-1}$ back to the resolved Calabi-Yau $\widehat{X}_n$ and the resolution divisor $E_{i_I}$, as well as the divisor classes associated with the generators of the free part of the Mordell-Weil group of rational sections of $\widehat{X}_n$. For our purpose in this subsection, we leave behind the Model-Weil generators for a moment and will be discussed in 2.9.1. Hence we have

$$h^{1,1}(\widehat{X}_n) = 1 + h^{1,1}(B_{n-1}) + \sum_I \mathrm{rk}(\mathfrak{g}_I). \tag{2.99}$$

In M-theory, the abelian gauge symmetries can be obtained from 3-form gauge potential $C_3$ reduction along a basis of harmonic 2-form on $h^{1,1}(\widehat{X}_n)$. More precisely, we have [25]

$$C_3 = \widetilde{A}^0 \wedge [\widetilde{S}_0] + \sum_\alpha A^\alpha \wedge \pi^*[D^b_\alpha] + \sum_{i_I} A^{i_I} \wedge [E_{i_I}], \tag{2.101}$$

which results in a $U(1)^{h^{1,1}(\widehat{X}_n)+1}$ effective theory in $\mathbb{R}^{1,8-2n}$. However, note that $\widetilde{A}^0, A^\alpha$ does not uplift into the vector multiplets in the dual F-theory. The $\widetilde{A}^0$ should be interpreted as the Kaluza-Klein $U(1)$ gauge field reduction along $S^1$ of the metric $g_{\mu\nu}$ in the dual F-theory side. While the $A^\alpha$ should uplift to the 2-form fields $b^\alpha_2$, which from Type IIB picture are reducted

[25] Here we have shifted the zero section to

$$[\widetilde{S}_0] := [S_0] - \frac{1}{2}[K_{B_{n-1}}] \tag{2.100}$$

from RR-fields $C_4$

$$C_4 = \sum_{\alpha} b_2^{\alpha} \wedge D_{\alpha}^b. \tag{2.102}$$

Hence $A^{\alpha}, \alpha \in h^{1,1}(B_{n-1})$ uplifts to tensor fields $b_2^{\alpha}$ in F-theory effective theories. More details, for elliptically fibered Calabi-Yau two-folds, i.e. K3s, the tensor from the base $B_1 = \mathbb{P}^1$ belongs to the 8D gravitational multiplet. For the three-folds $X_3$, $h^{1,1}(B_2) - 1$ of these tensors $b_2$s sit in the tensor multiplet with anti-self duality and the remaining one belongs to the self-dual tensors in the gravitational multiplet. In the cases of $X_4$, these tensors $b_2$ are dual to the axionic scalar fields in $\mathbb{R}^{1,3}$ which complexify the $h^{1,1}(B_3)$ Kähler moduli.

In general, given a divisor $D_A$ as well as their cohomological class $[D_A]$ in $\widehat{X}_n$, there are certain conditions imposing on $[D_A]$ such that the massless $U(1)$ in M-theory arising from reduction of $C_3 = A^A \wedge [D_A]$ can uplift to F-theory gauge fields. These conditions are known as the transversality conditions

$$\begin{aligned} [D_A] \cdot \pi^*(w_{2n-2}^b), \qquad & \forall w_{2n-2}^b \in H^{2n-4}(B_{n-1}); \\ [D_A] \cdot [S_0] \cdot \pi^*(w_{2n-4}^b) = 0, \qquad & \forall w_{2n-4}^b \in H^{2n-4}(B_{n-1}). \end{aligned} \tag{2.103}$$

Intuitively speaking, it means that the divisor class has "one leg along the fibre and the other leg along the base". From physical viewpoint, the first condition, saying that the divisor class $D_A$ has vanishing intersections with any curve classes in the base, ensures that there are no axionic gaugings which would lead to massive $U(1)$. While the second condition indicates its intersection with the generic fiber $\mathbb{E}_{\tau}$ vanishes, ensuring that all the KK tower states that originate from the same state in F-theory side have the same charge under the abelian $U(1)$ fields $A^A$.

Hence only the parts $A^{i_I}$ reduction from exceptional divisor $E_{i_I}$ (as well as the rational section divisor, which will be introduced in 2.9.1 ) uplift to the gauge fields in the effective theoryieson $\mathbb{R}^{1,11-2n}$ from F-theory compactifications. Given the dimension, we can identify these gauge fields $A^{i_I}$ as the Cartan generators of non-abelian gauge algebra $\mathfrak{g}_I$. What are the other degrees of freedom associated with the non-Cartan generators, i.e. W- bosons, in the non-abelian gauge algebra $\mathfrak{g}_I$? Given the fact that the origin of $A_{i_I}$ is $C_3$ fields in M-theory, it is not hard to see that the gauge bosons should comes from M2-branes. Indeed, recall that M2-branes are electrically charged under $C_3$, with the coupling form being $\mu_{M2} \int_{\Sigma} C_3$. Now as the M2-branes wraps the holomorphic curve $C$ in the fiber, i.e. the world-sheet $\Sigma$ takes the form $C \times \mathbb{R}^1$,this, together with (2.101), leads to the charge $q_{i_I}$ of the wrapped M2-brane states under $A_{i_I}$ as (in the units of $\mu_{M2} = 1$)

$$q_{i_I} = \int_C E_{i_I} = [C] \cdot [E_{i_I}]. \tag{2.104}$$

If we take $C$ as one of the components $\mathbb{P}^1_{j_I}$

$$\int_{\mathbb{P}^1_{j_I}} E_{i_I} = \mathbb{P}^1_{j_I} \cdot [E_{i_I}] = -C_{i_I j_I}, \tag{2.105}$$

where $C_{i_I j_I}$ again is the Cartan matrix associated with the Lie algebra $\mathfrak{g}_I$. It is known that the columns/rows of the Cartan matrix represent the Cartan charges of the *simple* roots of $\mathfrak{g}_I$, which forms a basis of weights in the *adjoint* representations of $\mathfrak{g}_I$. Taking into accounts of the linear combinations and the contributions from the anti-M2-branes [26], it turns out one can reproduce the whole states in the full adjoint representation of the A-D-E algebra $\mathfrak{g}_I$ by

[26] Anti M2-branes here can be interpreted as the M2-branes wrap the $\mathbb{P}^1_{i_I}$ in opposite way.

wrapping the M2- and anti-M2-branes along the exceptional curves $C$, i.e. the gauge bosons [27]. By the same token, the mass associated with the state arising from the M2-brane wrapping $C$ yields

$$m(C) \simeq \mathrm{vol}(C) = \int_C J, \tag{2.106}$$

with $J$ being the Kähler form of $\widehat{X}_n$. Hence in the resolved space $\widehat{X}_n$, these non-diagonal gauge bosons are generically massive, as the volume of fibers generically do not vanish. From the standpoint of the effective gauge field theories from the M-theory compactification on $\widehat{X}_n$, by assuming the existence of the resolution of $\widehat{X}_n$, this means that the gauge algebra $\mathfrak{g}_I$ is broken to their subalgebras by turning on vev for the scalars $\xi_i$ in the vector multiplets $V_I$, with the vev given by

$$\langle \xi_{i_I} \rangle = \mathrm{vol}_J(\mathbb{P}^1_{i_I}) = \int_{\mathbb{P}^1_{i_I}} J. \tag{2.107}$$

And typically the effective gauge theories completely breaks as

$$\mathfrak{g}_I \xrightarrow{X_n \to \widehat{X}_n} \mathfrak{u}(1)^{r_I}, \tag{2.108}$$

where $r_I$ denotes the rank of $\mathfrak{g}_I$. The F-theory limit, i.e. blowing down all the resolved fiber in $\widehat{X}_n$ and backing to $X_n$, should correspond to the origin of the Coulomb branch. This also fits with a fact in the 4D $\mathcal{N} = 1$ field theory, namely the 4D effective gauge theories from F-theory compactifications does not have Coulomb branch as there are no scalars in the vector multiplets. When compactified the $\mathbb{R}^{1,11-2n}$ F-theory effective theories further on the circle $S^1$, they identify with the $\mathbb{R}^{1,11-2n}$ effective theories from dual M-theory compactifications, with the condition that the radius $R$ of the $S^1$ being the inverse of the volume of the generic fiber $E_f$: $R = 1/\mathrm{vol}(\mathbb{E}_f)$. In terms of these scalar fields, they correspond to the Wilson lines of the gauge fields $A_I$ in F-theory along the circle $S^1$. And F-theory effective actions are recovered at the limit of $R \to \infty$.

As for the non simply-lace Lie algebras, there are certain subtle aspects due to the monodromy effects. In shorts words, the actual Lie algebra $\mathfrak{g}_I$ lives on the divisor $W_I$ has some discrepancy with the one $\widetilde{\mathfrak{g}}_I$ directly reading off from the structures of the local fiber types. In fact, $\mathfrak{g}_I$ is a subalgebra of $\widetilde{\mathfrak{g}}_I$ and hence there exists some representations that are not in the adjoints of $\mathfrak{g}_I$ as

$$\mathbf{adj}(\widetilde{\mathfrak{g}}_I) = \mathbf{adj}(\mathfrak{g}_I) \oplus \widetilde{\rho}_0. \tag{2.109}$$

For our purposes, we will not cover the relevant details but refer to the Sec. 3.3 in the TASI lecture [21].

Via duality with M-theory, M2-branes wrapping the fibral curves $\mathbb{P}^1_{j_J}$ give rise to states associated with the simple roots $-\alpha_{i_I}$, and the Cartan $U(1)_{i_I}$ gauge fields arise from KK reduction of the M-theory 3-form, which follows as

$$C_3 = A_{i_I} \wedge [E_{i_I}] + ..... \tag{2.110}$$

In this sense the resolution divisors $[E_{i_I}]$ can be identified with the generators $\mathcal{T}_{i_I}$ of the Cartan subgroup of $G_I$ in the so-called co-root basis, whose trace over the fundamental representation

[27] In principle, a M2-brane can wrap an arbitrary linear combination of the resolved curves $\mathbb{P}^1_{i_I}$ corresponding to the (-) simple roots, and forms a infinite number of 1-particle states. It is still not clear which mechanism choose such the combinations that the gauge bosons remain at the effective theory.

of $G_I$ is normalized such that

$$\mathrm{tr}_{\mathrm{fund}} \mathcal{T}_{i_I} \mathcal{T}_{j_J} = \delta_{IJ}\, \lambda_I\, \mathfrak{C}_{i_I j_I} \qquad \text{with} \qquad \mathfrak{C}_{i_I j_I} = \frac{2}{\lambda_I} \frac{1}{\langle \alpha_{j_I}, \alpha_{j_I} \rangle} C_{i_I j_I} \,. \tag{2.111}$$

The quantity $\lambda_I$ denotes the Dynkin index in the fundamental representation and is tabulated in Table 4.1. Note that for simply-laced groups $\mathfrak{C}_{i_I j_I} = C_{i_I j_I}$. The geometric manifestation of this identification is the important relation

$$\pi_*([E_{i_I}] \cdot [E_{j_J}]) = -\delta_{IJ}\, \mathfrak{C}_{i_I j_I} [W_I] = -\mathrm{Tr}\, \mathcal{T}_{i_I} \mathcal{T}_{j_J} [W_I] \,, \tag{2.112}$$

where Tr is related to the trace in the fundamental representation via

$$\mathrm{Tr} = \frac{1}{\lambda_I} \mathrm{tr}_{\mathbf{fund}} \,. \tag{2.113}$$

And one can identify $\mathbb{P}^1_{i_I} \simeq -\alpha_{i_I}$.

## 2.5. A Nod to the Resolutions

Up to now, we have analyzed the gauge symmetries of compactification of F-theory on elliptically fibered Calabi-Yau manifolds $X_n$ by blow-up resolutions of the codimension-one singularities. The Blow-ups resolutions of the singularities, seen from the dual M-theory compactifications, correspond to moving to the Coulomb branches of the gauge theories on the 7-branes. The M2-branes wrapped the resolved fibers correspond to the massive non-abelian gauge bosons in the gauge symmetry breaking process. One should note such breaking process essentially happens at the M-theory side, as the F-theory limit corresponds to the blow-down of all resolved fibers and the origin of the Coulomb branches of the M-theory effective theories.

However, before we go to the next steps, we would like to make a few remarks on the resolutions.

**1. Multiple choices** For certain types of singularities, especially those with $I_N$ singularities, there are multiple ways to resolve the fiber singularities at various codimensional loci. These resolved manifolds should be birationally equivalent to each other but the physics should not be different with each other. In addition, one can also perform deformations to resolve these singularities, which necessarily changes the complex structures moduli. From the field theory, this corresponds to enter the Higgs branches of the effective gauge theories.

**2. Drawback.** It turns out by resolutions of the singularities, one cannot access all non-abelian gauge degrees of freedom and would possible rule out a significant parts of possible supersymmetric backgrounds of the effective theories. One particular example is involving certain types of bound states of 7-branes, dubbed "gluing branes" [96,97] or "T-branes" [98–100]. In terms of field theory, the reason behind that is that the singular limits of the Calabi-Yau $X_n$ is located at the intersections of several branches of moduli spaces, and by performing certain resolutions, some other branches may be obstructed and hence lost that part of information. For details on their phenomenological implications, we refer to [101] and the references thereof.

**3. Existences.** We would like to stress here that for a typical Calabi-Yau space $X_n$, there is no guarantees for smoothing the singularities. On opposite, it is believed that for a typical Calabi-Yau manifold, it has both non-Higgsable clusters [102] and terminal singularities, which obstruct resolving the singularities through complex structure deformation and Kähler resolution, respectively. However, the typical favorable example $SU(5)$ is neither non-Higgable cluster nor generic with terminal singularities.

The last two points suggests a proper framework for studying F-theory on singular space by itself are in needs. Several recent attempts along this line have been developed, for examples in [103] and [104].

## 2.6. Chiral Matter and Codimension-two Singularities of Calabi-Yau

In the previous section, we have seen that the structures of the gauge groups of the $\mathbb{R}^{1,11-2n}$ low-energy theories from F-theory compactifications on elliptically fibered Calabi-Yau spaces $X_n$ are primarily encoded in the codimension-one singularities of $X_n$. Such codimension-one singularities were systematically analyzed by Kodaira on $K3$ and corresponding A-D-E gauge algebras in the effective theories, known as A-D-E classification. When the base $B_{n-1}$ is of high dimension, monodromies around these codimension-one loci can lead to non-simply laced groups.

When going to higher codimensional loci, typically in the intersection of two irreducible divisors $W_I$, one expect that the vanishing orders $(f, g, \Delta)$ would increases and hence the singular types enhance. Indeed, as we will show momentarily, the matter would be generated at higher codimensional loci, and this has a counterpart in the effective fields on the 7-branes. Namely, these matter can be thought as arising from Higgsing of a higher rank gauge group located at the higher codimensional loci, known as the Vafa-Katz picture [105].

To be more precise, let's first assume that two stacks of 7-branes wrapping divisor $W_I$ and $W_J$ intersect on $2n-6$-cycles $\Sigma_{IJ}$ in the base $B_{n-1}$:

$$C_{IJ} = W_I \bigcap W_J, \tag{2.114}$$

with the vanishing order of $(a_i, \Delta)$ at $W_I$ and $W_J$ being $(n_{i_I}, m_I)$ and $(n_{i_J}, m_J)$, respectively. One can then read off the gauge algebra carried by the 7-brane wrapped on $W_I$ and $W_J$ from the Kodaira-Tate table 2.3, denotes as $\mathfrak{g}_I$ and $\mathfrak{g}_J$, respectively. Then at the codimension-two loci $\Sigma_{IJ}$ the vanishing order typically would enhance to

$$(a_i, \Delta)|_{C_{IJ}} = (n_{i_I} + n_{i_J}, m_I + m_J). \tag{2.115}$$

One again can read off the gauge algebra $\mathfrak{g}_C$ from the Kodaira-Tate table 2.3, which contains the two $\mathfrak{g}_{I,J}$ as subalgebras.

However, this does not necessarily mean there are new gauge algebras in the effective theories but rather indicates new degrees of freedoms such as charged matter, which is more consistent with the picture of type IIB intersecting D-branes. How to show it? One can view the new charged matter arises from the decomposing of the adjoint representation of higher gauge algebra $\mathfrak{g}_C$ to its two subalgebras. In terms of branching, it reads

$$\begin{aligned} \mathfrak{g}_C &\to \mathfrak{g}_I \times \mathfrak{g}_J \\ \mathrm{Ad}(\mathfrak{g}_C) &\to (\mathrm{Ad}(\mathfrak{g}_I), 1_J) \oplus (1_I, \mathrm{Ad}(\mathfrak{g}_J)) \oplus_r [(\mathbf{R}_I^r, \mathbf{R}_J^r) \oplus c.c], \end{aligned} \tag{2.116}$$

where $(\mathbf{R}_I^r, \mathbf{R}_J^r)$ denotes the charged matter. Similar to the discriminant $\Delta$, the codimension-two loci $C_{IJ}$ also split into several irreducible components based on the above branching, and we denote as

$$C_{IJ} = \cup_r C_{IJ}^r. \tag{2.117}$$

We will omit the superscript $IJ$ occasionally if not confused, and denote the irreducible components $C^r_{IJ}$ as $C_{\mathbf{R}_r}$ as to each of component one can associate a certain representation $\mathbf{R}^r_I$.

Namely, one can view that the gauge theory on $W_I$ with corresponding gauge algebra $\mathfrak{g}_I$ as the Higgsing of a parent gauge theory with $\mathfrak{g}_C$ by vevs $< \phi_I >$s of the adjoint Higgs $\phi$, where at the $C_{IJ}$ the vevs are zero and only at these loci the charged matter could be reflected at the effective theory. In other words, if one move out from the codimension-two loci $C_{IJ}$ to the codimension-one $W_I$ for example, only the adjoint representation $\mathfrak{g}_I$ appears in the effective theory of the 7-brane wrapping $W_I$.

Now we would ask how are these new degrees of freedoms reflected from the geometry of $X_n$? Recall that we have said that $\mathrm{Ad}(\mathfrak{g}_I)$ arise from the M2-branes wrapping the resolved fiber components over $W_I$, together with the $C_3$ decomposition along the resolved fiber. Hence one naturally expect that these new degrees of freedom come from the the fiber at the $C_{IJ}$. Let's focus on the fibre $\mathbb{E}_I = \sum_i a_i \mathbb{P}^1_{i_I}$ over any generic point $p \in W_I$. To proceed, let us drop the superscript $I$ for conveniences. It turns out that the fiber typically becomes $\widetilde{\mathbb{E}} = \sum_k \widetilde{a}_k \mathbb{P}^1_k$ when one move the point $p$ inside $C_{IJ}$, with more components $\mathbb{P}^1$ arising from the splitting one (or more) component(s) $\mathbb{P}^1_i$ as $\mathbb{P}^1_i \to \bigcup_k \mathbb{P}^1_k$. In the language of representation theory, the $\mathbb{P}^1_i$ correspond to a weight $w^{adj}_i$, then the splitting of $\mathbb{P}^1_i \to \bigcup_k \mathbb{P}^1_k$ amounts to a decomposition of the adjoint weight $w^{adj}_i \to \sum_k w^k$, where $w^k$s are charges of the states with respect to the Cartan subalgebra $\mathfrak{g}$. In other words, M2-branes wrapped on these new components $\mathbb{P}^1_k$ give rise to states with weight $\beta^a_k$ in representation $\mathbf{R}^a$ of a corresponding gauge algebra $\mathfrak{g}$. Note that the representations $\mathbf{R}^a$ of $\mathfrak{g}$ are typically not the adjoint representation $\mathrm{ad}(\mathfrak{g})$ anymore. Indeed, [106, 107] showed that these new states generated by wrapped M2-branes and anti M2-branes lift to 4D $\mathcal{N} = 1$ chiral multiplet and anti-chiral multiplets in F-theory compactification on a Calabi-Yau four-fold $X_4$. And similar interpretation also applys to other dimensional Calabi-Yau manifolds.

Note that the above analysis also apply for the situations where the 7-brane divisor $W_I$ is singular, typically the single divisor $W_I$ has self-intersection, namely $I = J$ in (2.114). However, such cases are subtle in many contexts. Nevertheless, the codimension-two loci arising from singular divisors will typically leads to exotic representation such as symmetric tensor representations or even more exotic representations [108, 109]. While for the smooth divisors only leads to the fundamental, two-index antisymmetric ones for $\mathfrak{su}(n)$ cases, as well as three-index antisymmetric ones for $\mathfrak{su}(n), n = 6, 7, 8$. Similar phenomenons also happen in the abelian gauge symmetries, which we will discuss in 2.9.2.

One should note that the possible singular types at codimension-two and higher loci, unlike the ones at codimension-one, are not completely classified yet. In [91], they classified cases with rank one enhancement with the assumption of the smooth 7-branes.

For our purposes in chapter 4, we would also introduce matter cycles for the localized charged matter in the Calabi-Yau five manifolds compactification. Namely, a fibration can be introduced by fibering the new fibers $\mathbb{P}^1_k$, corresponding the weights $\beta^a_k, k = 1, ..., \dim(\mathbf{R}^a)$, over the codimension-two irreducible loci $C_{\mathbf{R}_I}$ as

$$\begin{array}{ccc} \mathbb{P}^1_k & \longrightarrow & S^k_{\mathbf{R}} \\ & & \downarrow \\ & & C_{\mathbf{R}}. \end{array} \tag{2.118}$$

Here the fibration (complex ) 3-cycle $S^k_{\mathbf{R}}$ is called the matter 3-cycle [28].

### 2.6.1. Holomorphic couplings and higher codimensional singularities of Calabi-Yau manifolds

The parts of Yukawa couplings and higher codimensional singularities and their physical implications in F-theory are beyond the concrete applications in this thesis, thereby we will not give many details instead we will give a short summary and refer to the review [21] for more discussions.

From the codimension-two singularities, we have learned accompanying the enhancement of the singularities, there are new states arising from these singularities. Now as one approaches the codimension-three loci (apparently it needs $n > 3$ for $X_n$), Do we expect that are there any new states? It was argued in [106, 107] though that there are no more new matter, instead codimension-three singulairities are responsible for hosting holomorphic Yukawa couplings at the perturbative level. For Calabi-Yau five-manifolds, there are also possible codimension-four singuliarites, and leads to $E$ and $J$ quartic interactions in 2D $\mathcal{N} = (0, 2)$ effective theories [64].

## 2.7. Non-flat Resolutions at High Codimensional Loci

So far we have sticked to the assumption that the singularities over the various codimensional loci in the base $B_{n-1}$ admit a flat, Calabi-Yau resolution. This typically requires that the vanishing order of $(f, g)$ are not at the same time exceed $(4, 6)$ at the various codimensional loci. We call these Kadiaro singularities as the **minimal** singularities. However, even the minimal condition is satisfied, there are nevertheless no guarantees that a flat, Calabi-Yau (crepant) resolution of the singularities exists. Such types of singularities, typically occur in elliptic fibrations in higher codimensional loci than one, are so-called $\mathbb{Q}$-factorial terminal singularities and will have some physical implications in the F/M-theory compactifications, which we will discuss in 2.8. In this section, let's concentrate the situations with non-minimal singularities of Weierstrass models.

If the non-minimal singularities of $X_n$ exist in codimension-one loci, one is still capable of carrying resolutions but such resolutions are not Calabi-Yau (crepant) resolution anymore and hence such a manifold cannot yield to a supersymmetric F-theory vaccum [29], as the dual M-theory compactification is not. Similar stories apply to the cases that codimension-two loci with the vanishing order $(f, g, \Delta) \geqslant (8, 12, 24)$ and codimesnion-three loci with $(f, g, \Delta) \geqslant (12, 18, 36)$. However, if there are Calabi-Yau $X_n$ with minimal codimension-one singularities but whose higher codimensional singularities are non-minimal but wild, i.e., for those whose vanishing orders at codimension-two loci and codimension-three loci are $(4, 6, 12) \leqslant (f, g, \Delta)|_{\text{codim-2}} < (8, 12, 24)$ and $(8, 12, 24) < (f, g, \Delta)|_{\text{codim-3}} < (12, 18, 36)$ respectively, some novel interesting physics comes to play. Such situations have been worked out extensively and nicely in the elliptically fibered (local) Calabi-Yau three-folds $X_3$, whose $(f, g)$ vanishes to order $(4, 6, 12) \leqslant (f, g, \Delta)|_{\text{codim-2}} < (8, 12, 24)$ at some points $p$ on the base $B_2$ and have been dubbed conformal matter, which is a strongly coupled sector [110]. To see this, let's focus on an example with two colliding $E_8$ 7-branes. The Weierstrass model can be written as

$$y^2 = x^3 + \alpha u^2 v^2 x + \beta u^3 v^3, \qquad (u, v) \in \mathbb{C}^2, \tag{2.119}$$

[28] It is known as the matter **surface** in the four-folds $\widehat{X}_4$.

[29] The supersymmetries are broken already at the compactification scale.

where we work on the local patch with $z = 1$ and $\alpha, \beta$ are some constants on the base $B_2 = \mathbb{C}^2$. Following the Kodaira table, one can read there are two 7-branes sitting at the divisors $\Sigma_1 : (u = 0)$ and $\Sigma_2 : (v = 0)$ on the base $\mathbb{C}^2$ with $(f, g, \Delta)|_{\Sigma_i} = (2, 3, 6), i = 1, 2$ , respectively, each of them is $D_4$ singularities at the loci $\Sigma_i$ and hence carries an $SO(8)$ gauge algebra. Then along the locus $u = v = 0$, one can see the vanishing orders $(f, g, \Delta)|_{\Sigma_1 \cap \Sigma_2} = (4, 6, 12)$, satisfying the above interval. Such singularities of fibers though have two choices of crepant resolutions. The first choice is to carry a non-flat resolution without modifying the base $B_2$, i.e. the resolved fibres $\mathbb{E}_f$ over some points $p$ in the base have complex dimension more than one, i.e. $\dim(\mathbb{E}_f) > 1$, and necessarily carries new degrees of freedoms. Several discussions for these non-flat fibrations have been discussed in various contexts, for example from SCFT viewpoint [111] and also from the phenomenogical applications [112, 113].

In order to see that these new degrees of freedom lead to a strongly coupled system, it is better to resolve it in alternative way–performing a finite number of blow-ups over the points $p$ in the base $B_2$ without modifying the dimensions of all the resolved fibers $\mathbb{E}_f$. The blow-ups in the base $B_2$ introduces new exceptional divisors $\Sigma_i^{ex}$ in the $\widetilde{B}_2$ and D3-branes can wrap on these new exceptional divisors, which appear as dynamical string in the 6D $\mathcal{N} = (1, 0)$ effective theories. It is also equivalent to say that in the original base $B_2$, these exceptional divisors have zero volume. Then there are tensionless strings propagating from these D3-brane. From the 6D $\mathcal{N} = (1, 0)$ effective theory standpoint, the volumes of these exceptional divisors parameter the scalar fields on the tensor multiplets and blow-up these exceptional divisors corresponds to moving along the flat directions in the Coulomb branches [30] of the 6D $\mathcal{N} = (1, 0)$ effective theory. The appearance of the 6D tensionless strings, which from the 6D SCFT experience [31], is a strong evidence to indicate this tensionless limits carry strongly coupled systems. One can also justify this idea from the gauging coupling of the 7-brane wrapping on these exceptional divisors $\Sigma_i^{ex}$, which reads $\frac{1}{g_i^2} \simeq \mathrm{vol}(\Sigma_i^{ex}) = 0$. From the mass dimension of the gauge coupling in the 6 dimension, this limit is the strong coupling limit. In some cases when the gravity is decoupled, this actually leads to a superconformal theory, and hence the matter locating on such loci dubbed conformal matter, in contrast to the ordinary perturbative matter located at the codimension-two loci which satisfy the miminal bound $(8, 12, 24)$. For example, in the above example, one can write the blow-up in the base $\mathbb{C}^2$ locally as a birational transformation as

$$u \to u, v \to uv, \tag{2.120}$$

and the corresponding proper transformation for $(f, g, \Delta)$ to the new ones in the new base $\widetilde{B}_2$, as sections of $(\bar{K}^4_{\mathbb{C}^2}, \bar{K}^6_{\mathbb{C}^2}, \bar{K}^{12}_{\mathbb{C}^2})$, goes as

$$\begin{aligned} &f(u, v) \to f(u, uv)/u^4 = \alpha v^2, \; g(u, v) \to g(u, uv)/u^6 = \beta v^3, \\ &\Delta(u, v) \to \Delta(u, uv)/u^{12} = 4\alpha^2 v^6 + 27\beta^2 v^6. \end{aligned} \tag{2.121}$$

In this new base, the fiber configuration is denoted as

$$[D_4], 1, [D_4] \tag{2.122}$$

[30]Note that in 6D $\mathcal{N} = (1, 0)$ theories, there are no scalars in the vector multiplets as we will show in 3.2, hence we, among others, refer the parameters of the Coulomb branches to the scalars in the tensor multiplets. Note that the tensor multiplets give rise to vector multiplets in the $S^1$ and $T^2$ compactifications.

[31]Namely, if the blow-ups happens at the codimension-one loci in the base, then by the same token, it also signals the existence of a 6D SCFT. In this thesis, we are not going to introduce such an interesting topic, we refer to [114] for more details.

which essentially is the tensor branch of $D_4 \times D_4$ conformal matter. For details, we refer to the review [114].

## 2.8. $\mathbb{Q}$-factorial Terminal Singularities

We have already spoiled that even though the vanishing order $f, g, \Delta$ in higher codimensional loci $\Gamma$ are minimal in certain Calabi-Yau $X_n$, i.e, $(f, g, \Delta)|_\Gamma < (4, 6, 12)$, the singularities of the fibers over $\Gamma$ are not guaranteed to admit a Calabi-Yau resolution $\widehat{X}_n$. One of the obstructions comes from the $\mathbb{Q}$-factorial terminal singularities.

In this section we will present some facts on $\mathbb{Q}$-factorial terminal singularities, of which we will massive employ in chapter 5. The presentation here follows closely with the section 5.6 in [21]. We refer more details on the relevant physics implication in [115, 116] and the references thereof.

Let's first set up the stage: Assuming $X$ be a complex algebraic variety with possible singularities, a resolution of $X$ is a **birational** map $\rho : \widetilde{X} \to X$ to a smooth variety $\widetilde{X}$ such that the smooth sets $\mathcal{V} \subset X$ are in isomorphism with $\mathcal{U} \subset \widetilde{X}$: $\rho^{-1}(\mathcal{U}) \cong \mathcal{V}$ and the two only differ in the so-called exceptional set $E$ on $\widehat{X}$. A big resolution introduces divisors, as codimension-one loci in the $E$, whereas the small resolution introduces higher codimensiional loci in $E$. In this sense, the blow-up resolutions on the elliptically fibered Calabi-Yau $\rho : \widehat{X}_n \to X_n$ are big resolutions we have discussed in the previous sections. A typical example of the small resolution is the blow-up in the conifold singularity, which we laid out in the appendix B.4. Further, the singularities of a complex algebra variety $X$ is $\mathbb{Q}$-Gorenstein if the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier (Namely $rK_X$ is a Cartier divisor for some integers $r \in \mathbb{Z}$). Particularly, for the $r = 1$ cases, the singularities of $X$ are Gorenstein types. For a $\mathbb{Q}$-Gorenstein variety $X$, the resolution can be described by

$$rK_{\widetilde{X}} = rK_X + \sum_I c_i r E_i, \tag{2.123}$$

where $E_i$ denotes the exceptional divisor in $\widetilde{X}$. If $\forall i, c_i > 0$, then $X$ is said to have at worst terminal singularities. The coefficients $c_i \in \mathbb{Q}$ are called discrepancies, measuring the types of the singularities. The *crepant* resolution refers to the resolution does not change the canonical divisor, i.e. $c_i = 0, \forall i$. For example, the Calabi-Yau resolution, which we have refereed many times, is a crepant resolution. The other typical types of singularities, which are interesting for F/M-theory compactification, are **terminal** if $c_i > 0, \forall i$, **canonical** if $c_i \geqslant 0, \forall i$.

Let's now focus on $X$ being an elliptically fibered Calabi-Yau $X_n$. Then for $X_n$, $r = 1$ as $K_{X_n} = \mathcal{O}_{X_n}$. The question we are interested in, is that given a $X_n$, does a Calabi-Yau resolution $\widehat{X}_n$ exist[32]? Noting that the small resolution does not change the canonical divisor $K_X$, so the positive answer for the question depends on the two choices: either a small resolution exists, or a big crepant resolution exsits, i.e. $c_i = 0, \forall i$.

Let's first discuss the existence condition for a small resolution. To this end, we recall a definition:

**Definition:** An algebraic variety $X$ is $\mathbb{Q}$-factorial when every Weil divisor $D$ in $X$ is also $\mathbb{Q}$-Cartier, namely, $kD$ can be locally expressed as the vanishing locus of a single function on $X$ for some $k \in \mathbb{Z}$.

[32] Logically, the question should be answered firstly before starting the discussions in this chapter. However, in the previously sections, the discussions are under the assumption that they exist.

**Lemma:** the conditions for the existence of a small resolution of a canonical singularity is iff $X$ is not $\mathbb{Q}$-factorial.

Now one may wonder why we care about a Calabi-Yau manifold $X_n$ with $\mathbb{Q}$ terminal singularities, whose singularities do not admit crepant resolutions either by small resolutions or by big resolutions [33]. To answer this question, we first recall that the main reason why we distance ourselves away from the (acceptable) singularities in compactified manifolds and alway try to resolve them, especially in M-theory compactification, is that we lack proper controls and lots of physical quantities are not computable, which involves the poor understanding of the topological prosperities of compactified spaces such as Euler characteristic, intersecting numbers, Poincaré duality. However, a Calabi-Yau manifold $X_n$ with $\mathbb{Q}$ terminal singularities is not in this category. Instead, it still has a well-defined rational Poincaré invariance and hence admit a non-degenerate intersection pairing for Calabi-Yau three-folds $X_3$ [116], though it affects a pure Hodge structure and Hodge duality. Recall that in the previous sections, we heavily use the intersecting pair (2.105) to determine the gauge algebras and the representations of matter and hence we can still have a green light for this operation with the presence of $\mathbb{Q}$ terminal singularities.

However, $\mathbb{Q}$ terminal singularities do lead to some novel physical phenomenons. From mathematical viewpoint, the $\mathbb{Q}$ terminal singularities, like other singularities, change the counting of the complex structure moduli of $X_n$. Recall in a smooth Calabi-Yau $\widehat{X}_n$, this number is given by hodge number $h^{1,n-1}(\widehat{X}_n)$ and in F-theory compactifications, these number count the bulk matter that uncharged under any continuous gauge groups. Indeed, in [115, 116] they discussed the $\mathbb{Q}$ terminal singularities in the Calabi-Yau three-folds $X_3$ cases, and proposed the right guy for counting the complex structure deformations is $\mathrm{CxDef}(X_3)$, given by

$$\mathrm{CxDef}(X_3) := \frac{1}{2} b_3(\widetilde{X}_3) = \frac{1}{2}(b_3(X_3) + \sum_P m_P), \tag{2.124}$$

where $\widetilde{X}_3$ denotes a smooth deformation of $X_3$, which always exists for the singularities and $\mathrm{CxDef}(X_3)$ is defined by one of its smooth cousin $\widetilde{X}_3$. And here the $m_P$, known as Milnor number, counts the number of the new $S^3$ in the process of the deformation, which can be easily obtained for a hypersurface singularity. From physics viewpoint, the presence of $\mathbb{Q}$ terminal singularities at codimension-two loci in $X_3$ indicates the presence of uncharged *localized* hypermultiplets under any continuous gauge groups, which compensate the difference between the $\mathrm{CxDef}(X_3)$ and the number of unlocalized uncharged hypermultiplets. Indeed, in [115, 116] they used the gravitational anomaly cancellation (Note that in a 6D chiral theory, there are possible pure gravitational anomalies ) to justify the above counting.

From the field theory standpoint, the fact that a codimension-two terminal singularities indicates that the presence of uncharged localized matter is fair reasonable. This would indicate that one cannot give masses to these states in M-theory on $X_3$ by going to the Coulomb branch in a supersymmetric way. Recall that in the previous sections, the mass of localized matter is proportional to the volume of the corresponding resolved fiber. By resolving the singularities

[33] One should note that the higher codimensional $\mathbb{Q}$ terminal singularities are totally different with the non-minimal singularities in codimension-one loci in Calabi-Yau manifolds $X_n$. The last types do not provide any supersymmetric theories from the dual M-theory compactifications and hence should not be considered as supersymmetric vacua, while the former ones do. Furthermore, the non-minimal singularities, are non-canonical and typically sit at infinite distances in moduli spaces of $X_n$. The terminal singularities, however, sit at finite distances in moduli space. In physical lingo, singularities at the infinite distances in moduli spaces will render the effective theories unphysical, as one believes that infinite tower of massless states are be generated

of the fiber and denoting the resolved space as $X_3'$, the effective theory essentially enters a Coulomb branch of the gauge theory from M-theory compactifying on $X_3'$. Now if the resolution cannot be crepant, as the $\mathbb{Q}$ terminal singularities do, then $X_3'$ is not Calabi-Yau, hence breaks the supersymmetry of the M-theory effective theory.

Note that we keep stressing that the localized matter arises form the codimension-two $\mathbb{Q}$ terminal singularities are uncharged under any *continuous* gauge groups, it, however, can be charged under the discrete symmetries. In 5, we discuss the $\mathbb{Z}_p$ abelian discrete gauge symmetry, viewed as a massive $U(1)$ gauge symmetry, and will continues to discuss this with other aspects.

Terminal singularities can also appear in the codimension-three loci and the similar interpretations with the codimension-two ones are still not clear. However, there are some work on studying the supersymmetric theories concerning with codimension-three terminal singularities in Calabi-Yau. In [117], they used the D3-branes to probe certain types of these singularities and proposed that it gives rise to 4D $\mathcal{N} = 3$ supersymmetric field theories on the world-volume of the D3-branes.

## 2.9. Abelian $U(1)$ Gauge Symmetries in F-theory Compactifications

In the previous sections, we have discussed the non-abelian gauge algebras in F-theory compactifications, which are determined by local data, in a sense that the information about the types of Lie algebra, their representations and holomorphic interactions are encoded in the structures of singular fibers over strata of codimension-one, two and three loci in the base $B_{n-1}$, respectively. Gauge theories with abelian $U(1)$ symmetries in F-theory compactifications, on the other hand, requires the information of global properties of the elliptically fibered Calabi-Yau manifolds. One might be surprise of this fact, as the gauge symmetries in type IIB, regardless of non-abelian and abelian, should arise from the seven-branes sectors and naively expects there are no many differences. However, we would find that this is not the case for F-theory. In fact, we have already encountered in the Sen limit of the $I_1$ Kodaira type singularities, where the $U(1)$ gauge algebra in the type IIB orientifold is projected out by orientifolding, prior to uplifting to F-theory. The reason behind that, as we will give more details shortly, is that the geometric Stückelberg mechanism always plays a role in the Kodaira type $I_1$ cases. The only way to avoid this type of Stückelberg mechanism and thereby manage to uplift to F-theory requires the divisors of D7-brane and its image ones are in the same homologous class, i.e. case (2) in (1.9). This necessarily would involve the rational sections, as we are going to talk them now.

### 2.9.1. The Mordell-Weil Group of Rational Sections

The rational section $s_A$ of an elliptic fibration is a rational map $s_A : B_{n-1} \to X_n$ such that the combination of the projection map in (2.146) and $s_A$ are identical map in the base $B_{n-1}$: $\pi \circ s_A = id_B$. In the Weierstrass models, the rational map $s_A$ assign a rational/meromorphic function on the base to each of Weierstrass coordinates $[x : y : z]$. Note that the rationality ensures that the rational section intersects/marks a single point with each fiber over all points in the base. Hence It defines a copy of the base $B_{n-1}$ insider of $X_n$ and hence a divisor, denoted by $S_A$.

We have already met one type of the rational section, the zero section $s_0 : [x : y : z] = [1 : 1 : 0]$ of the Weierstrass model. In fact, for the generic Weierstrass model, i.e. the coefficients $(f, g)$

attains generic values over a generic base $B_{n-1}$ [34], the Weierstrass model only admits one rational section-zero section, which further is a holomorphic section [35]. However, if $(f, g)$ taken a specific value, then the Weierstrass model might have more rational sections, which we will give an example, dubbed the restricted $U(1)$ model in 2.9.4.

The set of all rational sections $s_A$ on an elliptic fibration carries an abelian group structure, i.e. the Mordell-Weil (MW) group, which can be traced back to the abelian group structure of rational points on the fibres-elliptic curves. To see this, one can go to a local patch, say $z = 1$. Then the elliptic curve $\mathbb{E}_\tau$ in the Weierstrass model is a curve in the $x - y$ plane, with the infinity being zero/marked point $\mathcal{O}$. Now given two points $P_1, P_2 \in \mathbb{E}_\tau$, then one can find a third point $P_3$ arising from the intersection of the elliptic curve $\mathbb{E}_\tau$ with a line who passes through the two points $P_1, P_2$. Indeed, one can check this does give rise to a commutative, associated map $\odot$ such that $P_1 \odot P_2 \odot P_3 = \mathcal{O}$. The zero element in this abelian group is then the zero point $\mathcal{O}$. Note that when the line is a vertical line, the intersecting point $P_3$ then is at infinity $\mathcal{O}$, i.e. it means that $P_1 = -P_2$. In the terms of a lattice $\mathbb{C}/\lambda_{1,\tau}$, the map $\odot$ is exactly the additional map $+$ in $\mathbb{C}$ modulo the lattice equivalence. As one expects, this group structure can be thus fibre-wise generalized to elliptic fibrations as additive structure on rational sections $s_A$, defined by the additive of the points on each fibres marked by the rational sections $s_A$. The zero-elements in this abelian group, dubbed Mordell-Weil group, is then the zero section $S_0$, as it intersects each fibre $\mathbb{E}_\tau$ at the zero points $\mathcal{O}$.

According to the Mordell-Weil theorem, the Mordell-Weil group is finitely generated and hence has following decomposition:

$$MW(\pi) \sim \mathbb{Z}^r \oplus \mathbb{Z}_{k_1} \oplus ... \oplus \mathbb{Z}_{k_n}, \tag{2.125}$$

where $\mathbb{Z}$s are the free generators and the subscript $r$ denotes the copies of $\mathbb{Z}$, i.e. the rank of MW group. The other discrete parts represent the torsional generators $s_{k_i}$ which satisfy $k_i s_{k_i} := s_{k_i}^{\odot k_i} = s_0$ for some finite $k_i \in \mathbb{Z}$. Both of free and torsional parts of the Modell-Weil group turn out to bear physical meanings in F-theory compactifcations. The torsional part is related to the global structure of the gauge groups [119, 120], which we will not discuss it in the reset but refer to the reviews [21, 121] for more details, especially the section 3.3 in [121]. From now on, we focus on the free parts.

### 2.9.2. The Shioda's map

Having said that a rational section $s_A$ defines a divisor $S_A$ on the whole fibration $X_n$. Throughout the whole thesis, we take the $X_n$s as simply-connected projective variety. After the resolution of singularities [36], the resolved ones $\widehat{X}_n$s should be simply-connected and smooth projective variety, whose divisor class groups coincide with the Néron-Severi groups $NS(\widehat{X}_n)$. In this sense, the rational section defines a divisor $S_A$ in the $NS(\widehat{X}_n)$, whose Poincaré dual cohomology class,

[34] As a side remark, one should note that for some types of base, even the coefficients $f, g$ takes the generic values, the Weierstrass model still could admits extra rational section. For example, in 3d cases, [118] showed that for a small number of "semi-toric" (a single $\mathbb{C}^*$ action) bases, the theory admits a rational section, dubbed the non-Higgable abelian gauge structure.

[35] At codimensional one, there are no essential difference between the zero section and the rational section. In other words, the meromorphic functions defining section hits the poles only at higher codimensional space. That is the reason why every elliptic fibration, which admits at least a rational section is birational equivalent (isomorphic up to higher codimensional loci) to the Weierstrass model

[36] We have not discussed the singularities associated with the abelian $U(1)$ in F-theory. As we will stress later, the singularities of the fibre essentially arise from the codimension-two loci, where a charged singlet under the $U(1)$ resides, typically an $I_2$ singularities.

denoted as $[S_0]$, is characterized by the cohomology group $H^{1,1}(\widehat{X}_n)$ [37]. As we have mentioned in the previous section, the Shioda-Tate-Wazir theorem states that $H^{1,1}(\widehat{X}_n), n > 2$ is generated by four parts

$$H^{1,1}(\widehat{X}_n) = \langle [S_0], [S_A], [E_{i_I}], [\pi^{-1}(D_\alpha^b)] \rangle, \alpha = 1, ..., h^{1,1}(B_{n-1}), \tag{2.126}$$

and hence

$$h^{1,1}(\widehat{X}_n) = h^{1,1}(B_{n-1}) + \mathrm{rk}(\mathfrak{g}) + 1 + r. \tag{2.127}$$

All of these give rise to abelian $U(1)$s in the M-theory compactification by the decomposition of $C_3$ along these divisors. As we have already showed, the divisor classes have to satisfy the transversality conditions listed in (2.103) in order to guarantee the abelian gauge fields can be uplifted to F-theory. And we have already mentioned that the $U(1)$s generated by exceptional divisors $E_{i_I}$ in the M-theory can be uplifted to **Cartan** $U(1)$s in F-theory and further under the F-theory limit, all these Cartan $U(1)$s resemble into non-abelian gauge symmetries, together with the other massless gauge bosons from wrapped M2-branes. Now, one might expect that the rational divisors $S_A$, as the only left type in above four types, should contribute to the abelian $U(1)$ symmetries in F-theory compactifications. Indeed, rational sections are the ones for geometric engineering of abelian $U(1)$ symmetries in F-theory. However, to properly specify the massless $U(1)$ gauge fields in the F-theory dual side, the divisors which the $C_3$ fields decompose should be modified, as it turns out that by decompose $C_3$ along this divisors

$$C_3 = A^A \wedge [S_A], \tag{2.128}$$

the resulting gauge fields $A^A$s do not corresponde to the gauge fields $A$ in F-theory compactifications.

The mathematical reason behind this is that the map from the divisor groups to the Mordell-Weil group is not a group homomorphism. More precisely, the additive map of a NS group is not compatible with the one in a MW group, namely, $S_A + S_B$ of two divisors $S_A, S_B$ is not the divisor class associated with the additive of the two corresponding sections $s_A \odot s_B$. In order to find such group homomorphism, one should introduce the Shioda homomorphism $\phi(s_A)$ [122] which maps a section $s_A$ to a divisor class $U_A$ on $\widehat{X}_n$ with rational coefficients, i.e. $[U_A := \phi(s_A)] \in NS(\widehat{X}_n) \otimes \mathbb{Q}$ as

$$U_A := \phi(s_A) = S_A - S_0 - \pi^{-1}(\pi_*((S - S_0) \cdot S_0)) + \sum_I \sum_{i_I} l_A^{i_I} E_{i_I}, \tag{2.129}$$

where $k_A^{i_I} := l_{Aj_I} (C^{-1})^{j_I i_I} \in \mathbb{Q}$ with the $(C^{-1})^{j_I i_I}$ being the $ji$ components of the inverse Cartan matrix associated with non-abelian gauge algebra $\mathfrak{g}_I$, as we have introduced. Further, $l_{Aj_I}$ is the intersecting number between the rational section divisor $S_A$ and the fibre curves $\mathbb{P}_i^1$ at the codimension-one loci. As it shall be, the Shioda homomorphism is group homomorphism, i.e. it is compatible with the structure of MW group as

$$\phi(s_A \odot s_B) = \phi(s_A) + \phi(s_B) := U_A + U_B. \tag{2.130}$$

In particular, by the above construction (2.129) and especially with the choices of the fractional

[37] Note that the Poincaré dual cohomology of NS group by definition is $H^{1,1}_{\mathbb{Z}}(\widehat{X}_n) = H^{1,1}(\widehat{X}_n) \cap H^2(\widehat{X}_n, \mathbb{Z})$. However, for the simply-connected complex algebraic variety there is no difference between $H^{1,1}_{\mathbb{Z}}(\widehat{X}_n)$ and $H^2(\widehat{X}_n, \mathbb{Z})$.

coefficients $k^A_{i_I}$, the Shioda homomorphism associates each rational section $s_A$ to a divisor class $[U_A]$ and satisfies the transversality conditions, which in Calabi-Yau five-manifolds [38] reads

$$[U_A] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] \cdot_{\widehat{X}_5} [D_\delta] = 0, \qquad [U_A] \cdot_{\widehat{X}_5} [S_0] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] = 0$$
$$[U_A] \cdot_{\widehat{X}_5} [E_{i_I}] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] = 0\,, \tag{2.131}$$

where $D_\alpha = \pi^* D^{\mathrm{b}}_\alpha$ are the vertical divisors. Here the first two conditions $(a)$ and $(b)$ echo with the transversality conditions in (2.103). The third condition, however, is a normalization such that the non-gauge bosons from M2-branes wrapping the resolved fibre curves $\mathbb{P}^1_i$ at the codimension-one loci do not carry the charge under the $U(1)$s from these extra rational sections. However, the third condition does not extend to the high codimensional loci as the matter in certain representation of a non-abelian gauge algebra $\mathfrak{g}_I$ could also carry the charge under the $U(1)$ from these extra sections. The above construction and relevant discussion in abelian symmetry in F-theory have been extensively studied in the past years, for examples in [119, 123–126, 126, 127]

In summary, the set of

$$r = h^{1,1}(\widehat{X}_n) - h^{1,1}(B_{n-1}) - 1 \tag{2.132}$$

U(1) vector multiplets reduction from M-theory compactifications can uplift to the vector multiplets in F-theory compactifications, which include the Cartan $U(1)$s associated within the non-abelian algebras and the non-Cartan $U(1)$s. The non-Cartan ones are in one-to-one correspondence with the independent rational sections in the Mordell-Weil groups, as will be introduced shortly.

Now by the same token, one can determine the charge $q_A$ for states from M2-brane wrapping on a fibre curve $\mathcal{C}$ under the $A^A$ from

$$C_3 = A^A \wedge [U_A)] + ..., \tag{2.133}$$

by evaluating the intersecting number

$$q_A = \int_{\mathcal{C}} [U_A] = [U_A] \cdot [\mathcal{C}]. \tag{2.134}$$

Realizing higher $U(1)_A$ charges $q_A$ in string theory in general would be a very interesting but also difficult question. From the perturbative Type IIB picture, charges of a state under the $U(1)$ are either $\pm 1$ or 0. This essentially arises from the Chan-Paton factors and a single open string ending on a single D7-brane only has at most one end. With certain combinations of various $U(1)$, it possible to increase the absolute values for the charge to be 2. In the context of F-theory, due to its non-perturbative nature, higher charge matter are possible as in the non-perturbative limit, the essential objects are light $(p, q)$-strings and one might use the string junctions to analyze them. The string junctions, intuitively speaking, can be viewed as strings with multiple ends and can end on branes at multiple points and hence increase charges. Up to the time of this writing, the highest charge under the abelian $U(1)$ is 4, which has been worked out in [128]. This essentially due to the fact that the section are not smooth, similar to the sprit that the 7-branes divisor are singular at the end of the discussion of 2.6, which leads to new representations. However, a systematically description has not yet given.

[38] For generic dimensional Calabi-Yau $\widehat{X}_n$, they can be readily generalized.

### 2.9.3. The height pairing and gauge couplings

An important geometrical quantity in studying the abelian $U(1)$ gauge theories is the height pairing of two sections $s_A$ and $s_B$, which is given by [39]

$$b_{AB} := -\pi_*(\sigma(s_A)\cdot\sigma(s_B)), \tag{2.135}$$

and it gives rise to an effective divisor in the base $B_{n-1}$. From the physical viewpoint, the Kähler volume of $b_{AB}$ is dictated by the inverse of square $U(1)$ gauging coupling $g^2_{U(1)}$ in the $\mathbb{R}^{1,11-2n}$ effective theory. Indeed, it is easy to check that the $\mathbb{R}^{1,11-2n}$ effective theory contains the terms

$$S = \frac{M_{pl}^2}{2}\int\sqrt{-g}R + \frac{1}{4g^2_{U(1)}}\int dA^A\wedge *dA^B, \tag{2.136}$$

where $M_{pl}^2 = 4\pi\mathrm{vol}(B_{n-1})$ and $\frac{1}{g^2_{U(1)}} = \frac{1}{2\pi}\mathrm{vol}_J(b_{AB}) = \frac{1}{2\pi}\int J\wedge[b_{AB}[-\pi_*(\sigma(s_A)\cdot\sigma(s_B))]$.

Further, the height pairing for same rational section $S_A$ is given by

$$b_{AA} = 2\bar{K}_{B_{n-1}} + 2\pi_*(S_A\cdot S_0) - \sum_I \pi_{Ak_I}(C^{-1})^{k_I i_I}\pi_{Ai_I}\Sigma_I. \tag{2.137}$$

Note that the first term suggests that the gauge coupling of abelian symmetries is controlled partially by the volume of the anti-canonical divisor of the base $\bar{K}_B$. This fact leads to an important statement that the abelian symmetries cannot be consistently decoupled from the gravitational sector. Indeed, In the Calabi-Yau three-folds compactification of F-theory, it has been proven rigorously in [129] that, due to this part, the hight pairing $b_{AB}$ is not contractible and hence the $U(1)$ abelian gauge symmetry of the 6D effective theories stays as a global symmetry when the gravity is decoupled. Furthermore, as we will show in chapter 4 and the chapter 3, the height pairings also play important roles in the Green-Schwarz mechanism of anomaly cancellations.

One particular point we would like to mention is that the height pairing, as a divisor in the base $B_{n-1}$, is not typically the components of a discriminant $\Delta$ in a Weierstrass model. As we mentioned at the beginning of this section, the abelian $U(1)$ gauge symmetry associated with D7-branes in Type IIB orientifold would typically become massive by the so-called geometric Stückelberg mechanism [40] [126, 130] when uplifted to F-theory, as the mass of the gauge field generated in this way is proportional to the string coupling $g_s$. The geometric Stückelbergberg mechanism typically involves of the setting of the brane-image-brane with different homology group in Type IIB, i.e. $[D]\neq\sigma^*[D']$ and the only way to avoid this mechanism is the cases that the $U(1)_A$ is associated with some linear combination of 7-brane divisors $\sum a_I[D]_I$ so that their image one under the involution $\sigma$ is at the same homologous class in the double-covering space $X_{n-1}$ of the base $B_{n-1}$. In the base $B_{n-1}$, this divisor associated with the $U(1)_A$ is then the height pairing $b_{AB}$. And hence typically it can not be attributed to a single 7-brane divisor $W_I$ and hence to the discriminant $\Delta$. From the F/M-theory picture, the geometric Stückelberg mechansim typically invokes the non-harmonic 2-forms to get the massive $U(1)$ from the $C_3$ decomposition [131], as we will show momentarily in 2.10.2.

[39] As aside, the height pairing is the essential part for the construction of the Shioda map [122], of which we refer to a nice summary at the section 3.2.1 in [121].

[40] The mechanism involves only geometric configurations and independent of the gauge fluxes, hence with the name *geometric*. The geometric Stückerbreg mechanism is the center theme in chapter 5 for discrete symmetries, viewed as the massive $U(1)$ gauge symmetries

### 2.9.4. Example: $U(1)$ restricted Tate model

In this subsection, we will give an example to illustrate various aspects of abelian gauge symmetry in F-theory compactifications. The example we are going to introduce is so-called $U(1)$ restricted Tate model which was first studied in [126] and given by the Tate model with $a_6 = 0$

$$y^2 + a_1 xyz + a_3 yz^3 = x^3 + a_2 x^2 z^2 + a_4 xz^4. \tag{2.138}$$

As a result, the fibration adimits an extra section

$$s_1 : b \to s_1(b) = [x : y : z] = [0 : 0 : 1]. \tag{2.139}$$

In terms of Weierstrass model, the coefficients $f, g$ are specified by

$$f = \frac{a_1 a_3}{2} + a_4 - \frac{1}{48}(a_1^2 + 4a_2)^2. \tag{2.140}$$

For our purpose, it is convenient to rewrite our restricted $U(1)$ Weierstrass form as

$$AB = CD \tag{2.141}$$

with

$$A = y, \qquad B = y + a_1 xz + a_3 z^3, \qquad C = x, \qquad D = a_2 xz^2 + a_4 z^4. \tag{2.142}$$

This is exactly a conifold form and the singular locus at $A = B = C = D = 0$ is a codimension-two locus $C_{I_2} = \{a_3 = 0\} \cup \{a_4 = 0\}$ at the base $B_{n-1}$ with the fibre singular at the point $x = y = 0$. This also explains that the rational section $s_1$ is not holomorphic but a rational section, as it passes through the singular point in the fibre. However, this kind of conifold singularity admits the small, Kähler resolution and hence the whole Calabi-Yau manifolds can be resolved crepantly, which excludes it be a terminal singularity. This codimension-two singularities (codimension-three on the whole Calabi-Yau ) actually manifest the $U(1)$ gauge symmetries at F-theory compactificaitons. To see this, after the small resolution, one can introduce two new Cartier divisors $D_\pm$ whose ancestors are the ideals $(A, C)$ and $(B, D)$, as wll as their Poincaré dual two forms $\omega_\pm \in H^{1,1}(X_n)$. Then one can use this two forms, more precisely the difference $\omega_+ - \omega_-$ to obtain the gauge field $A^-$ by decomposing $C_3$ as $C_3 = A^- \wedge (\omega_+ - \omega_-)$ such $A^-$ can further lift to the abelian gauge field in the F-theory dual side.

One can also combine this abelian $U(1)$ with other non-abelian gauge groups by invoking corresponding geometric engineering. For example, let us briefly mention the restricted $SU(5) \times U(1)$ model. Namely, we extend the $U(1)$ restricted model and combine it with further enhancement to non-abelian gauge algebra $\mathfrak{su}(5)$ over a divisior $W : w = 0$ on the base $B_{n-1}$. One can apply the standard procedure that we laid out previously to generate an $SU(5)$ singularity over a divisor $W$ by setting the vanishing orders of the coefficients $a_i$ of the Tate polynomials, and in addition enforcing $a_6 = 0$ for the $U(1)$. To be precisely, The Tate section are fixed as

$$a_1 = a_1, \qquad a_2 = a_{21} w, \qquad a_2 = a_{21} w, \qquad a_3 = a_{32} w^2, \qquad a_4 = a_{43} w^3, \qquad a_6 = 0. \tag{2.143}$$

The discriminant $\Delta$ hence yields

$$\Delta = w^5 (P + Qw + Rw^2 + Sw^3 Tw^4), \tag{2.144}$$

with corresponding coefficients of $(P, Q, R, S)$. We can further analyze the pattern of singularity's enhancements in the codimension-two loci hence leads to the localized matter. We will work on this example for a Calabi-Yau five-manifold $X_5$ in chapter 4.

## 2.10. Discrete Symmetries in F-theory Compactifications

From the field theory perspective, a discrete gauge symmetry [41] can be constructed by the Higgsing of an continuous gauge theory, say abelian $U(1)$ gauge symmetry. Namely, given a theory with $U(1)$ and several charged state, including a scalar field $\phi$ with the charge $p$, then one can give a vev for a scalar $\phi$ and then after the Higgsing, the remaining effective theory enjoys a $\mathbb{Z}_p$ symmetry. Having said that the abelian $U(1)$ symmetry manifests itself in F-theory compactfications through the rational sections, one would expect the abelian discrete symmetry would manifest itself, if possible, through F-theory compactified manifolds without any rational sections. Indeed, it turns out the proper fibrations for studying the discrete symmetries are the genus-one fibrations, which do not admit rational sections. And in the past years, we have seen lots of progresses in this direction, and interestingly, the above Higgsing can be realized geometrically as a conifold transition in a Calabi-Yau manifolds (see e.g. [21]).

### 2.10.1. Genus-one fibrations

F-theory compactifications do not require the presence of rational sections of the fibrations *per se*. Instead, the fiber could in principle be any algebraic curve $\mathcal{C}$ of genus one, as long as it carries the $SL(2, \mathbb{Z})$ modular invariance. As we stressed many time, if the genus-one curve $\mathcal{C}$ has a marked point $\mathcal{O}$, i.e. the elliptic curve $\mathbb{E}$, and hence the fibration is the elliptic fibration, which has (at least) a rational section which intersects a single point with the fiber over each point in the base $B_{n-1}$ and is birational equivalent to the Weierstrass model where the rational section is further a holomorphic section-zero section $s_0$. On the other hand, if the $\mathcal{C}$ does not have any marked points, then the corresponding fibration $\mathcal{X}_n$, defined as

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{X}_n \\ & & \Big\downarrow \pi \\ & & B_{n-1}, \end{array} \tag{2.145}$$

has no rational sections and will be referred as genus-one fibration. However, given a genus-one curve $\mathcal{C}$, one can construct an associated elliptic curve $J(\mathcal{C})$, which is the Jacobian of the curve. Correspondingly, one can also construct an elliptic fibration $\mathbb{X}_n$ with the same base $B_{n-1}$:

$$\begin{array}{ccc} J(\mathcal{C}) & \longrightarrow & \mathbb{X}_n \\ & & \Big\downarrow \pi \Big\uparrow s_0 \\ & & B_{n-1}, \end{array} \tag{2.146}$$

with the Jacobian $J(\mathcal{C})$ as the elliptic fiber. The key point is that the $\tau$ and the discriminant of both fibration $\mathcal{X}_n$ and $\mathbb{X}_n$ are identical.

[41] Through this thesis, we only focus on abelian discrete symmetries

Although the genus-one fibrations do not have rational section, they typically have $p-$sections (also known as muti-sections [132]) $s^{(p)} : B_{n-1} \to X_n$, which locally maps each point $t \in B_{n-1}$ to $p$ points in the fiber $\mathcal{C}$ over that point $t$. Globally, however, these $p$ points are exchanged by a monodromy action when moving around the branch loci in the base $B_{n-1}$, so that they together defines a $p$-section $s^{(p)}$ of the fibration and therefore a single divisor $S^{(p)}$. Consequencely, the divisor $S^{(p)}$ should be thought of an $p$-fold cover of the base $B_{n-1}$. Note that in general, there can be several $p$-sections in a genus-one fibrations $\mathcal{X}_n$ and these independent multi-sections are encoded in the so-called Tate-Schafarevich group $Ш(\mathbb{X}_n)$ associated with the so-called Tate-Schafarevich group $Ш(\mathbb{X}_n)$ which can be thought as all genus-one fibrations with the same Jacobian fibraion. And the zero-element is the Jacobian $\mathbb{X}_n$ itself while the other $p-1$ elements can be thought as differernt $p$-section fibrations.

And how does the discrete symmetries arise from F-theory genus-one fibration compactifca-tions?

Insert 3.1: **$\mathbb{Z}_p$ symmetry and Stückelberg mechanims** Given a discrete $\mathbb{Z}_p$ gauge symmetry in a field theory, one would wonder how to describe them? Is there a Lagrangian description? Is there a corresponding gauge field? well there are no satisfying answer to these questions so far, however, there exists some equivalent ways to describe a $\mathbb{Z}_p$ gauge symmetry, for example, the Stückelberg mechanism. To this end, one could start with the following Lagrangian

$$\mathcal{L} \supset t^2(d\phi - pA) \wedge *(d\phi - pA), \tag{2.147}$$

which describe the higgsing of the Abelian Higgs model as

$$\mathcal{L} \supset D\varphi\bar{D}\varphi = (d\varphi + ipA\varphi)(d\bar{\varphi} - ipA\bar{\varphi}) \tag{2.148}$$

by giving a vev to $\varphi$ as $\langle\varphi\rangle = (t+h)e^{i\phi}$. Here $\phi$, as a phase factor, subjects to the shift symmetry as $\phi \sim \phi + 2\pi$ and hence the $U(1)$ is Higgsed to $\mathbb{Z}_p$. In the low-energy limit $t \to \infty$, one has $A = d\phi/p$ and therefore they are no local degrees of freedoms. It is actually a topological field theory and is equivalent to the famous $BF$-theory in the four dimension [133]. In this section, we nevertheless take the $t$ as finite value. To see that it is $\mathbb{Z}_p$ discrete symmetry, one notes that (2.147) is invariant under the following transformation [a]

$$A \to A + d\chi, \qquad \phi \to \phi - p\chi, \qquad \chi \text{ arbitraty scalar fields} \tag{2.149}$$

And only the discrete part $\mathbb{Z}_p$ of the $U(1)$ survive under the shift symmetry, namely only $\chi = 2\pi n/p, \forall n > 1$ subset of the transfomation survive under the shift symmetry $\phi \to \phi + 2\pi$.

So in this sense, a $\mathbb{Z}_p$ gauge symmetry can be interpreted as massive $U(1)$ gauge symmetry.

[a] This is essentially 0-form version of the typical gaugings of the $n$-form global symmetry: introduce a $n+1$-form gauge potential, where the 0-form global symmetry is essential the shift symmetry.

## 2.10.2. Torsional cohomology and M-theory picture

In order to describe that the effective theory form F-theory genus-one fibration $\mathcal{X}_n$ compactifica-tion admits a discrete $\mathbb{Z}_p$ symmetry, it is better to go to the picture of M-theory Calabi-Yau $X_n$ compactification with discrete $\mathbb{Z}_p$ symmetry. Note that at this subsection, we do not require that the M-theory and F-theory compactify on the same Calabi-Yau manifolds.

In the M-theory Calabi-Yau $X_n$ compactification (let's focus on the Calabi-Yau three-folds

$X_3$ in this section), we have applied many times that a massless $U(1)$ gauge potential $A^A$ can be obtained by the decomposition of $C_3$ along a harmonic two-form as

$$C_3 = A^A \wedge \omega_A + ...., \omega \in H^{1,1}(X_3, \mathbb{Z}). \tag{2.150}$$

However, one would ask what is the implication of decomposing the $C_3$ fields along a non-harmonic, i.e. not-closed, 2-form? To this end, firstly notice that A non-closed 2-form $\mathrm{w}_2$ can be related to a torsional 3-form $\alpha_3$ as follows:

$$d\mathrm{w}_2 = p\,\alpha_3 \qquad \forall p \in \mathbb{Z}_+ \neq 0, 1. \tag{2.151}$$

Then we have the decomposition as

$$C_3 = A \wedge \mathrm{w}_2 + \phi\,\alpha_3 + ..., \tag{2.152}$$

where $A$ is not closed 1-form anymore and should be treated as massive gauge potential rather than the massless one. However, one can do the derivative as

$$dC_3 = dA \wedge \mathrm{w}_2 + (pA + d\phi) \wedge \alpha_3. \tag{2.153}$$

Here the $pA$ in the second term arise from (2.151) and insert the above into the relevant term in the 11D Supergravity, i.e.,

$$\mathcal{L}_{11d} \supset \int dC_3 \wedge *dC_3. \tag{2.154}$$

One obtains a kinetic term in the 5D effective theory as

$$S \simeq \int (d\phi - pA) \wedge *(d\phi - pA) + \dots . \tag{2.155}$$

This is exactly the Stückelberg term (see Insert 3.1 above ) in the 5D. The Stückelberg term is invariant under a simultaneous gauge transformation

$$A \to A + d\chi \qquad \phi \to \phi + p\,\chi\,, \tag{2.156}$$

which allows us to go to a gauge with only a massive vector field and it is a consequence of a gauging of the axionic shift symmetry and the $U(1)$ symmetry breaks to the its discrete part $\mathbb{Z}_p$.

The above torsional 3-forms $\alpha_3$ exists if the torsional cohomology $\mathrm{Tor}(H^3(X_d, \mathbb{Z}))$ does not vanish. In fact, if a Calabi-Yau three-fold $X_3$ [42] with such non-trivial torsional cohomology, i.e. if $\mathrm{Tor}(H^3(X_d, , \mathbb{Z})) = \mathbb{Z}_p = \mathrm{Tor}(H_2(X_d, , \mathbb{Z}))$ [43], then one would expect that the 4D effective theories from Type II $X_3$ compactification carries the discrete $\mathbb{Z}_p$ symmetry. The reason is that, say in Type IIA, the D2-branes and D4-branes can then wrap along these $p$-torsional 2-cycles and 3-cycles and give rise to the $\mathbb{Z}_p$ charged particles and strings, i.e., $p$-copies of them are hence

[42] $X_3$ can be any Calabi-Yau manifolds, not necessarily being elliptic fibrations. Here we denote the same notation with the Calabi-Yau manifolds and elliptic fibred Calabi-Yau manifolds $X_3$.

[43] Here we have used the universal coefficient theorem. Shortly speaking, For a smooth complex dimension $d$ complex variety $Y_d$, we have two relations between the torsional cohomology

$$\mathrm{Tor}(H_k(Y_d, , \mathbb{Z})) \simeq \mathrm{Tor}(H^{k+1}(Y_d, \mathbb{Z})) = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_n}, \qquad \mathrm{Tor}(H^k(Y_d, \mathbb{Z})) \simeq \mathrm{Tor}(H^{2d-k}(Y_d, \mathbb{Z})), \tag{2.157}$$

where the first relation is the so-called universal coefficient theorem and the second is Poincare duality. Here $\mathrm{Tor}(H_k(Y_d, , \mathbb{Z}))$ denotes the $p$-torsional cycles, meaning that $p$ copies of them are homological trivial, i.e. as a boundary of a $k+1$-chain. For a detail review we refer to [134].

uncharged and can thus decay, which is the smoking gun for the presence of the $\mathbb{Z}_p$ discrete symmetry in the four-dimensional theory with gravity, according to [133].

### 2.10.3. Discrete symmetry in genus-one fibrations

So according to the above discussion, in M-theory picture, the proper geometry for generating a $\mathbb{Z}_p$ discrete symmetry is a Calabi-Yau $X_n$ with non-trivial torsional cohomology $\mathrm{Tor}(H^3(X_n,\mathbb{Z})) = \mathbb{Z}_p$. However, the genus-one fibrations $\mathcal{X}_n$ are in general smooth and have no non-trivial torsional cohomology groups and M-theory compactification on such smooth geometry with the a divisor $S^{(k)}$ would have a massless $U(1)$ symmetry, so how to interpret the appearance of $\mathbb{Z}_p$ in F-theory?

To that end, we need to recall in the circle compactification of F-theory effective theories to M-theory effective theory. For definiteness, let us focus on Calabi-Yau three-folds for the discussion. As we have said in 2.4.3, in such circle compactification, one can turn on the Coulomb parameter $\xi$, which from the field theory perspective, is a holonomy associated with the gauge field $A$ in $\mathbb{R}^{1,5}$ from F-theory compactifications. However, if gauge symmetry is discrete symmetry $\mathbb{Z}_p$, which is interpreted as a massive $U(1)$, such holonomies are discrete and have $p$ copies. Then the resulting theories after circle compactification with such holonomies would not connect continuously as do the massless $U(1)$ cases. In other words, there exist $p$ supersymmetric isolated M-theory vacua along the Coulomb branch which are identified modulo $p$ and all these $p$ different M-theory vacua uplifts to the same vacua in F-theory compactifications. Then it is natural to beg the question which geometry can provide these $p$ different vacua in the M-theory compactification. It turns out that it is the Jacobian fibration $\mathbb{X}_3$ associated with the genus-one fibration $\mathcal{X}_3$. Indeed, as we said the Jacobian fibration has the Tate-Shafarevich group $\text{Ш}(\mathbb{X}_3)$ which contains $p-1$ $p$-section fibrations together with Jacobian fibration itself.

Further, thanks to the hard-working mathematicians, we have known that for elliptically fibered Calabi-Yau three-fold $X_3$ with no reducible fibers in comdimension-one that the torsional cohomology is encoded in the Tate-Schafarevich group

$$\mathrm{Tor}(H^3(\mathbb{X}_3,\mathbb{Z})) \cong \text{Ш}(\mathbb{X}_3). \tag{2.158}$$

Hence if the holonomy $\xi$ is trivial, then M-theory compactifies on the zero element in $\text{Ш}(\mathbb{X}_3)$, i.e. the Jacobian fibration $\mathbb{X}_3$ which, according to (2.158) has non-trivial torsion, hence the $\mathbb{Z}_p$ discrete symmetry in $\mathbb{R}^{1,5}$ passes to the effective theory in $\mathbb{R}^{1,4}$ from M-theory. If the holonomy is non-trivial, then it gives rise to a kinetic mixing term between the $U(1)$ from the KK reduction and the massive $U(1)$ so that a massless $U(1)$ can be obtained through such combinations, and this $U(1)$ is exactly the $U(1)$ from the the divisor $S^{(p)}$ in the genus-one fibration $\mathcal{X}_3$ on which M-theory compactified [135, 136]. For more details we refer to the review [21, 121].

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### 2.10.4. The geometrical description of the Higgsing

Here we would like to see how does the higgsing $U(1) \to \mathbb{Z}_p$ reflects on the geometry of the fibration. To this end, one typically start from the elliptic fibration $X_n$ with only one abelian $U(1)$, i.e. the rank $r$ of the Mordell-Weil group is 1, and have a codimension-two singularity (conifold) $I_2$ singularity. Instead of doing the small resolution which leads to the M-theory Coulomb branch, one can perform the deformation of this conifold singularities, which essentially changes the complex structure moduli of the whole fibration. In physical terms, this means that we enter the Higgs branch of the gauge theory by giving a vev to a charge $p$ state. After the deformation, the singular elliptic fibration $X_n$ acquires a $\mathbb{Q}$-factorial terminal singularities at the codimension-two loci on the base $B_{n-1}$ and the states trapped in the singular fiber now are charged under a discrete symmetry $\mathbb{Z}_p$.

## 2.11. Gauge Backgrounds

So far we have concentrated the pure geometric background of F-theory compactifications. The component group $\pi_0(G)$ of the gauge group $G$ in F-theory compactifications is dictated by the Tate-Shafarevich group $\text{Ш}(X_n)$ of the fibration $X_n$ and fundamental group $\pi_1(G)$ is dictated by the Mordell-Weil group $MW(X_n)$. Particularly, we have stressed that given a fibration $X_n$, the Discriminant $\Delta$ typically splits into irreducible components $W_I$, each of them is wrapped by 7-branes and carries a gauge group $G_I$ determined by the singularities of the fibers over it. And the charged matter are localized in the codimensional loci $\Sigma_{IJ}$, where typically are the intersection between two 7-branes $W_I$ and $W_J$. Further, the Yukawa coupling, as well as other types of couplings, are encoded in the higher codimensional loci. And all of this information can be (in principle) determined once a fibration $X_n$ is given. Now one would ask does this information is enough for determining a F-theory vacuum? Well, as alluded in the previous chapter of orientifold compactification, in order to define a type IIB vacuum, further information on the gauge background should also need to be provided, which typically contains the D7-brane gauge background, i.e. the Picard group of the line bundle $L$ whose discrete part contains the gauge fluxes $F$, as well as the bulk 3-form fluxes $H_3$. Hence one expect that F-theory compactifications, which alternatively viewed as the strongly coupled Type IIB orientifold compactification, should also require these gauge background to characterize a proper vaccum.

Indeed, in the dual M-theory compactification, additional information on the $C_3$ fields should be included. Such information in supersymmetric vacua should be recapitulated by *Deligne cohomology* $H^4_D(\widehat{X}_n, \mathcal{D}(2))$ [44], and can be parametrized by equivalence classes of rational complex codimension-2-cycles [137, 138], which form the second Chow group $\text{CH}^2(\widehat{X}_5)$. The *Deligne cohomology* also fits into the short exact sequence [45]

$$0 \to J^2(\widehat{X}_n) \hookrightarrow H^4_D(\widehat{X}_4, \mathbb{Z}(2)) \xrightarrow{\widehat{c}_2} H^{2,2}_{\mathbb{Z}}(\widehat{X}_n) \to 0\,, \tag{2.159}$$

where the continuous part $J^2(\widehat{X}_n)$, the intermediate Jacobian,

$$J^2(\widehat{X}_n) = H^3(\widehat{X}_n, \mathbb{C})/(H^{2,1}(\widehat{X}_n, \mathbb{C}) + H^3(\widehat{X}_n, \mathbb{C}) \tag{2.160}$$

counts, in the absence of $G_4 = dC_3$ fluxes [46], the Wilson line of $C_3$, i.e. the holonomies of $C_3$ around non-trivial 3-cycles. And $G_4$ fluxes are encoded in the discrete part $H^{2,2}_{\mathbb{Z}}(\widehat{X}_n)$. For more details on the Deligne cohomology and its F-theory implication we refer to [137, 138], as well as the PhD thesis [139].

In this thesis, we will only focus on the its discrete part: $G_4$ fluxes, which is sufficient for the purpose to determine the chiral index of charged massless states. In a supersymmetric vacua, the $G_4$ in the M-theory compactification should satisfy the self-duality condition $G_4 = *G_4$ and

[44] Here $\mathcal{D}(2)$ represents the Deligne complex, for details please refer to [137].

[45] Intuitively, one can view the Deligne cohomology as the higher dimensional generalization of the Picard group listed in (1.122).

[46] This fits with the above exact sequence: the kernel of the surjective map $\widehat{c}_2$ is given by the intermdiate Jacobian $J^2$.

also the primitivity condition $J \wedge G_4 = 0$ [140, 141]. More precisely, it should subject to

$$G_4^{1,3} = G_4^{3,1} = 0, \qquad G_4^{2,2} \wedge J = 0 \tag{2.161}$$

Further, similar to the gauge flux $F$ in the pervious chapter, the $G_4$ fluxes also need to subject to the Freed-Witten quantization condition [142] as

$$G_4 + \frac{1}{2} c_2(\widehat{X}_n) \in H^4(\widehat{X}_n, \mathbb{Z})\,. \tag{2.162}$$

In summary, the $G_4$ fluxes in a supersymmetric F/M-theory vacuum should sit in the following cohomology group as

$$G_4 + \frac{1}{2} c_2(\widehat{X}_n) \in H^{2,2}(\widehat{X}_n) \bigcap H^4(\widehat{X}_n, \mathbb{Z}), \tag{2.163}$$

together with the primitive condition $G_4^{2,2} \wedge J = 0$.

The above conditions should be imposed on the $G_4$ fluxes in order to give rise a supersymmetric M-theory vacuum. If the vacuum is also given by F-theory compactifications, then addition constraints on the $G_4$ fluxes should be imposed in order to preserving the Lorentz invariance, which was firstly put forward in [143]. Particularly, the harmonic 4-form for $G_4$ should only have one leg from the fiber as 1-cycle of the fiber $\mathbb{E}_\tau$ turns out to be the non-compact space-time of F-theory vacua. These conditions in Calabi-Yau five manifolds $\widehat{X}_5$, which is known as the transversality conditions for $G_4$ fluxes, read as

$$\int_{\widehat{X}_5} G_4 \wedge S_0 \wedge \pi^* \omega_4 = 0 \quad \text{and} \qquad \int_{\widehat{X}_5} G_4 \wedge \pi^* \omega_6 = 0\,, \qquad \forall\, \omega_4 \in H^4(B_4),\ \omega_6 \in H^6(B_4)\,, \tag{2.164}$$

where $[D_\alpha^b] \in H^{1,1}(B_4)$.

One should note the above story on $G_4$ fluxes makes sense only in the F/M-theory compactification of Calabi-Yau manifolds $\widehat{X}_n, n = 4, 5$. For lower dimensional Calabi-Yau manifolds, a $G_4$ flux background is incompatible with Lorentz invariance and supersymmetry. Note that in (2.16), the $G_4$ fluxes in the Chern-Simons coupling contribute to the Bianchi identity of the $C_3$ as

$$d * G = \frac{1}{2} G_4 \wedge G_4 - I_8(R) + \sum_i \delta_{M2_i}, \tag{2.165}$$

where $\delta_{M2_i}$ denotes 8-form current sourced by the spacetime-filling M2-brane, which further can be defined as

$$\delta_{M2_i} := \delta(f^1) df^1 \wedge \cdots \wedge \delta(f^8) df^8. \tag{2.166}$$

Here the $f^i = 0, i = 1, ..., 8$ locally describe the $8 - 2n$ cycle $C_i$ wrapped by the spacetime-filling $N$ M2-branes in $\widehat{X}_n$. Note that in the dual F-theory picture, this spacetime-filling M2-branes are T-dual to the spacetime-filling $N_{D3}$ D3-branes. Let's consider the implication in both Calabi-Yau four- and five-manifolds.

Integrating the above Bianchi identity over the Calabi-Yau four-folds $X_4$, where the $8 - 2n$-cycles $C_i$ are points, we have

$$-\frac{1}{2} \int_{\widehat{X}_n} G_4 \wedge G_4 + \frac{1}{24} \chi(\widehat{X}_4) = N_{D3}. \tag{2.167}$$

Here the $N_{D3}$ refers to the number of the spacetime-filling D3-branes.

Integrating it over the Calabi-Yau five-manifolds $\widehat{X}_5$, it yields to

$$[C] = \frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4 \cdot_{\widehat{X}_5} G_4)\,. \tag{2.168}$$

Given a complex structure on $X_n$, the cohomology space $H^{2,2}(\widehat{X}_n)$ enjoys a decomposition

$$H^{2,2}_{\mathbb{Z}}(\widehat{X}_n) = H^{2,2}_{\text{hor}}(\widehat{X}_n) \oplus H^{2,2}_{\text{vert}}(\widehat{X}_n) \oplus H^{2,2}_{\text{rem}}(\widehat{X}_n), \tag{2.169}$$

where each subspaces is mutually orthogonal with respect to the homological intersecting pairing. In the Calabi-Yau four-manifolds $\widehat{X}_4$, the horizontal piece arisen from the variation of the hodge structure, namely given a unique harmonic top $(4,0)$-form $\Omega_4$, the subspace $H^{2,2}_{\text{hor}}(\widehat{X}_n)$ by two successive variations of $\Omega_4$.

The primary vertical subspace $H^{2,2}_{\text{vert}}(\widehat{X}_n)$ is generated by elements of the form $H^{1,1}(\widehat{X}_n) \wedge H^{1,1}(\widehat{X}_n)$, namely the intersection of two divisors. The horizontal and the vertical subspaces are also mirror dual to each other. While $H^{2,2}_{\text{rem}}(\widehat{X}_n)$ denotes the rest parts which are neither vertical nor horizontal subspace, which was firstly introduced in [144]. Note that the Poincare dual cycles of both vertical subspace $H^{2,2}_{\text{vert}}(\widehat{X}_n)$ and remainder one $H^{2,2}_{\text{rem}}(\widehat{X}_n)$ are algebraic [47] in the whole space of complex structure moduli, whereas those dual to $H^{2,2}_{\text{hor}}(\widehat{X}_n)$ only on a subset of the complex structure moduli.

The vertical fluxes($G_4 \in H^{2,2}_{\text{vert}}(\widehat{X}_n)$) will be the main theme of this section, as it is the only ones for $G_4$ fluxes that can generate non-trivial chirality. We will go to a specific dimensional Calabi-Yau to illustrate various aspects of these vertical fluxes in the next subsection. Before that, let's first lay out some brief comments on the other two types of $G_4$ fluxes.

The horizon fluxes $H^{2,2}_{\text{hor}}(\widehat{X}_n)$ turns out also play a important role, as it would generate a F-term, so-called Gukov-Vafa-Witten (GVW) superpotential

$$W_{GVW} = \int_{\widehat{X}_4} \Omega_4 \wedge G_4. \tag{2.170}$$

The supersymmetric condition on this F-term enforce

$$\{W = 0\} \cup \{\partial W_{GVW} = 0\} \to G_4^{1,3} = G^{3,1} = 0 = G^{4,0} = G^{0,4}, \tag{2.171}$$

which (should) recapitulate the first supersymmetric result 2.161 obtained from the 11D supergravity compactification. Note in the degenerate limits, such GVW superpotentails in principle should reduce to the ones induced by the bulk fluxes $G_3 := F_3 + \tau H_3$, and also the superpotentials induced by the brane moduli (see e.g. [145]). However, an explicit correspondence has not yet obtained.

### 2.11.1. Vertical fluxes in F/M-theory compactifications on Calabi-Yau five-folds

After giving a general introduction on the $G_4$ fluxes in F/M-theory compactifications, we will provider further information on the vertical $G_4$ fluxes ($G_4 \in H^{2,2}_{\text{vert}}(\widehat{X}_n)$) in compactification of F/M-theory on elliptically fibered Calabi-Yau five-folds, in the anticipation of the detailed discussions of the anomaly cancellation of F-theory Calabi-Yau five-folds compactifcation in 4.

As we have said above, the horizon fluxes induce a GVW superpotential and further generate the F-term of the effective theories. The vertical ones, on the other hand, could also induce a

[47] Algebraic cycles can alway be described as vanishing loci of certain polynomials; According to Chow theorem, these are one-to-one correspondence with complex submanifolds of $\widehat{X}_n$.

superpotential-like form in terms of Kähler moduli as

$$W_2 = \int_{\widehat{X}_n} J^{n-2} \wedge G_4, \tag{2.172}$$

but rather it generates a D-term potential for F-theory effective theories

$$V_D = \int_{\widehat{X}_n} \pi^* J \wedge w_\Lambda \wedge G_4. \tag{2.173}$$

The supersymmetric condition again requires $V_D = 0$ hence enforce $J \wedge G = 0$, which reproduces the primitivity condition (2.161). Indeed, in the Calabi-Yau four-fold compactifications, this is exactly the result in 1.121. As for Calabi-Yau five-fold manifold compactifications, we will also provide a short comment in A.4.2 regarding this matter.

Note that based on the orthogonality in the decomposition (2.169), the only part of $G_4$ fluxes that could possible generate a non-vanishing $V_D$ is at the the subspaces $H^{2,2}_{\text{vert}}(\widehat{X}_n)$. The non-vanishing of $V_D$ is always accomplished, due to the supersymmetry, by the Stückelberg mechanism induced from the gauging of the axionic part of the Kähler moduli. Note that this type of Stückelberg mechanism depends on the fluxes, hence dubbed the flux stückeberg mechanism, which differs from another version-geometric Stückelberg mechanism mentioned previously. In terms of anomaly cancellations, we will invoke the flux version of Stückelberg mechanism in chapter 4 and the geometric version of the Stückelberg mechanism in 5, respectively.

Apart from inducing D-terms, the vertical fluxes can also induce non-trivial chiral indexes, whereas the other two parts of $G_4$ fluxes cannot. To see this, recall that in the Type IIB orientifold compactificatons, the gauge fluxes $F_A$ on the D7-branes can induce the chiral index for the localized charged matter. And the gauge fluxes $F_A$, together with the bulk 3-form fluxes $G_3 := F_3 - \tau H_3$, should be encapsulated in $G_4$ fluxes when uplifting Type IIB orientifold to F-theory. We refer the reader to the section 4.4 in [69] for more details on the connection between the $G_4$ fluxes and the type IIB fluxes $F_A, G_3$. Here we list several relevant facts for our discussions in chapter 3.

As we have said, the vertical $G_4$ fluxes are generated by products of the divisors of the $\widehat{X}_n$ i.e. $G_4 \in H^{2,2}_{\text{vert}}(\widehat{X}_n) \cong H^{1,1}(\widehat{X}_n) \wedge H^{1,1}(\widehat{X}_n)$. And according to the Tate-Shioda-Wazir theorem, the divisor group $H^{1,1}(\widehat{X}_n)$ can be decomposed as

$$H^{1,1}(\widehat{X}_n) = \langle [S_0], [S_A], [E_{i_I}], \pi^*[D^b_\alpha] \rangle, \tag{2.174}$$

hence $H^{2,2}_{\text{vert}}(\widehat{X}_n)$ can be generated by all linear combinations of products of above divisors.

In particular, we are interesting in the following combinations

$$G_4 = \sum_A \pi^* F_A \wedge w^A, F^A \in H^{1,1}(B_{n-1}), \tag{2.175}$$

where $w^A$ denote the hormonic 2-forms in $\widehat{X}_n$ which should have one leg in the fiber and one leg in the base, in order to satisfy the transversality conditions. We have seen that only two types of these harmonic 2-forms (or their Poincaré dual divisors) existing in an elliptic fibered Calabi-Yau $\widehat{X}_n$, namely the exceptional divisor $E_{i_I}$ and the rational section $S_A$ (or under the Shioda map $U_A$). If the class $[w^A]$ belongs to the exceptional divisor $E_{i_I}$ class, then the $F^A$ represents the D7-brane Cartan gauge fluxes localized on $W_I$ associated with the corresponding non-abelian algebra $\mathfrak{g}_I$. In such case, the non-abelian algebra $\mathfrak{g}_I$ typically breaks into the subalgebra $\mathfrak{h}_I$

which commutates with the abelian algebra $\mathfrak{u}(1)_{i_I}$.

In the other situation where $[w^A]$ coincides with the section divisor $U_A$, than the $F^A$ is interpreted as the so-called $U(1)_A$ flux. Note that this $U(1)_A$ flux typically cannot attribute to a single D7-brane like the fist situation, this is due to the same reason why the height pairing $b_{AA}$ cannot attribute to a single D7-brane divisor, as we elaborated at the end of 2.9.2.

## 2.12. Open String Descriptions

In the previous sections, we have a good understanding of the gauge and matter degrees of freedom of F-theory compactifications in terms of their dual descriptions in M-theory compactifications on resolved Calabi-Yau spaces $\widehat{X}_n$, viewed as the closed string (supergravity) descriptions.

In this section, we will switch the gear to the open string descriptions based on the gauge theory of 7-branes of F-theory compactifications, with the focus on how to count the zero modes and chirality. Instead working on the general elliptically fibred Calabi-Yau manifolds, we focus on the Calabi-Yau five-manifolds $X_5$. The general discussion here can be also applied to other dimensional Calabi-Yau spaces.

The goal within this section is to analyze the zero modes and its chirality under the world-volume gauge group of the 7-branes, whose information should be encoded locally. We consider the spacetime-filling 7-branes who further wraps on 6-cycles $\mathcal{W}_I$s in the $\widehat{X}_5$. If the 7-branes extended in a flat spacetime, as we have discussed in last chapter, the world-volume theory is the a simple 8D $\mathcal{N}=1$ minimal super Yang-Mills (SYM) with 16 supercharges and gauge group $G_I$, whose vector multiplet in the adjoint representation of $G_I$ contains a 8D gauge field $A$ and scalar field $\Phi$ parametrizing the normal fluctuation of the 7-branes, as well as their superpatner gaugino $\Psi$. However, if one wrap the 7-branes on a non-trivial cycles $W_I$ in the base $B_{n-1}$, supersymmetry of the resulting compactified effective theory typically demands that the field theory should be topologically twisted so that the resulting theory preservers certain supersymmetries. In Calabi-Yau five manifold compactifications, such a topological twist lead to a 2D $\mathcal{N}=(0,2)$ supersymmetric gauge theory (see more details on such supersymmetric theory in the appendix A.4). As a result, it enforces the following BPS equation

$$\begin{aligned} &\text{F-term: } F^{2,0}=F^{0,2}=0, \bar{\partial}_A\Phi=0, \\ &\text{D-term: } J\wedge J\wedge F+[\Phi,\bar{\Phi}]=0, \end{aligned} \tag{2.176}$$

where $\Phi$ is a 3-form on $W_I$ here. These are generalized Hitchin equations for the Higgs bundle $(A,\Phi)$ over the complex three-cycle $W_I$. And zero modes in the 2D $\mathcal{N}=(0,2)$ supersymmetric theories can be obtained by analyzing the above F-terms and D-terms. Due to the time reasons, we will not present the detailed analysis but only give the results below. For more details we refer to the pioneer papers [64, 65], as well as [107] in Calabi-Yau four-folds compactifications.

### 2.12.1. Zero modes and Chirality of the Calabi-Yau five compactification

As we have mentioned that the discriminant $\Delta$ will split into several irreducible divisors $W_I$s in the base $B_{n-1}$, and each divisor $W_I$ wrapped by 7-branes carries a $G_I$ symmetry determined by a Kodaira singularity of the fiber over $W_I$. In Calabi-Yau five-folds $X_5$ compactifications, each of these spacetime-filling 7-branes $W_I$ gives rise to a 2D $\mathcal{N}=(0,2)$ $G_I$ gauge theory coupled to the 2D $\mathcal{N}=(0,2)$ supergravities, which carries certain massless charged matter.

For our purpose, we only consider the chiral charged matter and its chirality. It turns out that there are three types of chiral matter with further discussions later: so-called bulk states, localized matter over $\Sigma_{IJ}$ intersecting with another 7-brane $W_J$ and further the 3-7 states [48]. See also the discussions in the picture of the weak coupled type IIB orientifold compactifications 1.9.1.

**Bulk matter** The bulk matter refers to states that propagate along the whole divisors $W_I$s. The bulk matter fields transform, in the absence of gauge flux, in the adjoint representations of $G_I$s. In the dual M-theory quantum mechanics, this matter arises from M2-branes wrapping suitable combinations of resolution $\mathbb{P}^1_{i_I}$ in the fibers over $W_I$s. For non-vanishing gauge backgrounds, which can be described by a non-trivial principal gauge bundle $L$, the original gauge group $G_I$ can be broken into a product of certain subgroups. The spectra decompose into irreducible representations $\mathbf{R}$ of the unbroken gauge factors

$$G_I \quad \to \quad H_I \tag{2.177}$$

$$\mathbf{Adj}(G_I) \quad \to \quad \mathbf{Adj}(H_I) \oplus \bigoplus_{\mathbf{R}} \mathbf{R}. \tag{2.178}$$

Note that if $\mathbf{R} \neq \bar{\mathbf{R}}$, each representation is accompanied by its complex conjugate. The matter fields organize into 2D $(0,2)$ chiral multiplets, which contain one complex boson and a complex chiral Weyl fermion, and also Fermi multiplets, which contain only one complex anti-chiral Weyl fermion. Each of these matter fields is counted by a certain cohomology group on $W_I$ involving a corresponding gauge bundle $L_{\mathbf{R}}$. The chiral index of massless matter in a given complex representation, defined as the difference of chiral and anti-chiral fermions in complex representation $\mathbf{R}$, is then given by [64, 65]

$$\chi(\mathbf{R}) = -\int_{W_I} c_1(W_I)\left(\frac{1}{12}\mathrm{rk}(L_{\mathbf{R}})\, c_2(W_I) + \mathrm{ch}_2(L_{\mathbf{R}})\right). \tag{2.179}$$

For real representations, this expression is to be multiplied with a factor of $\frac{1}{2}$. In particular, the chiral index of the adjoint representation depends purely on the geometry and takes the form $\chi(\mathbf{Adj}(H_I)) = -\frac{1}{24}\int_{W_I} c_1(W_I)c_2(W_I)$.

Note in $4n+2, \forall n \geqslant 0 \in \mathbb{Z}$ dimensional theories, a CPT conjugate of a chiral fields in representation $\mathbf{R}$ transforms under its complex conjugate representation $\bar{\mathbf{R}}$, but does not change its chirality. Namely the antiparticle $\mathbf{R}$ is still chiral, comparing to the anti-chiral in $4n$ dimensional theories. Taking into these account, we have

**Localized matter** The localized zero-modes refer to massless matter which are been trapped at the intersection loci $C_{\mathbf{R}}$ between two irreducible divisors as $W_I \cap W_J$. In the Calabi-Yau five-folds cases, the matter loci $C_{\mathbf{R}}$ typically are complex Kähler surfaces, which we further assume them to be smooth. In the topological twisting procedure above, the localized matter is taken as a 6D topological defect extending $\mathcal{R}^{1,1} \times C_{\mathbf{R}}$.

Again all of these states should be accompanied by their CPT conjugates. In summary, the localized zero modes on $C_{\mathbf{R}}$ contribute the following charged chiral multiplets to the 2d $\mathcal{N} = (0,2)$ effective gauge theory:

[48] In low dimensional Calabi-Yau compactification of F-theory, there are only the first two types of chiral matter, as the third one– 3-7 states are not chiral anymore. The thumb of rule to determine whether or not matter from intersecting D-branes in absence of fluxes is chiral, is that whether it can freely move along the base $B_{n-1}, \forall 0 \leqslant n < 5$. The intuitive interpretation is that if the intersection is fixed, then it introduces a preferred orientation in the base $B_{n-1}$ which violates the parity in the internal space $B_{n-1}$ and hence the matters in

| Multiplets | number |
|---|---|
| Vector mutiplets | $H^0_{\bar{\partial}}(W_I, L_{W_I}) \oplus H^0_{\bar{\partial}}(W_I, L_{W_I})^*$ |
| Chiral mutiplets | $H^1_{\bar{\partial}}(W_I, L_{W_I}) \oplus H^1_{\bar{\partial}}(W_I, L_{W_I})^*$ |
| Fermi mutiplets | $H^2_{\bar{\partial}}(W_I, L_{W_I}) \oplus H^2_{\bar{\partial}}(W_I, L_{W_I})^*$ |
| Chiral mutiplets | $H^3_{\bar{\partial}}(W_I, L_{W_I}) \oplus H^3_{\bar{\partial}}(W_I, L_{W_I})^*$ |

**Table 2.4.:** *Bulk zero modes in 2D $\mathcal{N} = (0,2)$ effective theories from F-theory Calabi-Yau five-manifolds compactifications.*

| Multiplets | number |
|---|---|
| Chiral mutiplets | $H^0_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}}) \oplus H^0_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}})^*$ |
| Fermi mutiplets | $H^1_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}}) \oplus H^1_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}})^*$ |
| Chiral mutiplets | $H^2_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}}) \oplus H^2_{\bar{\partial}}(C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes \mathcal{L}_{C_{\mathbf{R}}})^*$ |

**Table 2.5.:** *Localized zero modes n 2D $\mathcal{N} = (0,2)$ effective theories from F-theory Calabi-Yau five-manifolds compactifications.*

Now equipped with the these details, we can then use the Hirzebruch-Riemann-Roch theorem as well to calculate the net chiral index, which was firstly calculated in [64, 65]

$$\begin{aligned}
\xi_{C_{\mathbf{R}}}(\mathbf{R}) =& \sum_{i=0}^{2}(-1)^i h^i((C_{\mathbf{R}}, \sqrt{K_{C_{\mathbf{R}}}} \otimes L_{C_{\mathbf{R}}})) \\
=& \int_{C_{\mathbf{R}}} (\frac{1}{12}(c_1^2(C_{\mathbf{R}}) + c_2(C_{\mathbf{R}})) + \frac{1}{2}c_1(C_{\mathbf{R}})c_1(L_{C_{\mathbf{R}}} \otimes \sqrt{K_{C_{\mathbf{R}}}}) + ch_2(L_{C_{\mathbf{R}}} \otimes \sqrt{K_{C_{\mathbf{R}}}})) \\
=& \int_{C_{\mathbf{R}}} \left( c_1^2(C_{\mathbf{R}}) \left( \frac{1}{12} - \frac{1}{8}\mathrm{rk}(L_{\mathbf{R}}) \right) + \frac{1}{12}c_2(C_{\mathbf{R}}) + \left( \frac{1}{2}c_1^2(L_{\mathbf{R}}) - c_2(L_{\mathbf{R}}) \right) \right).
\end{aligned} \tag{2.180}$$

Note that the above counting depends on the assumption that the divisors $W_I$ are smooth. For the singular divisors $W_I$, there are certain sublteties associated with the countings and we refer to [64] for more details.

**3-7 matter** . As we have said, the D3-branes which is charged under the singlet $C_4$ of $SL(2,\mathbb{Z})$ are invariant when moving around the base [49]. For our purpose, let us focus on spacetime-filling D3-branes which thus wrap the curve class $[C]$ in $\widehat{X}_5$. Such D3-branes can transversely intersect with 7-branes in $\widehat{X}_5$ and hence contribute chiral matter to the 2D effective theory, located at the intersecting points with the 7-branes, dubbed a 3-7 sector matter, which comes in 2D (0,2) Fermi multiplets [64, 65]. In purely perturbative setups, each intersection point of $[C]$ with one of the D7-branes carries a single Fermi multiplet in the fundamental representation of the D7-brane gauge group. And hence the multiplicities of 3-7 matter are given by the intersecting numbers $[C] \cdot [W]$, note that $[C]$ needs to satisfy the D3-brane tadpole condition 2.168. In the strong coupling regime where $\mathrm{Im}(\tau)$ is large, the above perturbative analysis of the multiplicities

the flat spacetime inherits the chirality from such violation.

[49]One can easy to find that the complexified gauge coupling of the gauge theory on D3-branes coincides with the axio-dilaton $\tau$ by expanding the DBI and CS actions. Such a nice feature renders the D3-branes in F-theory as natural probe objects and leads to lots of fruitful results for supersymmetric field theories.

is still expected to remain valid as long as the size of $C$ is large enough. However, monodromy effects along the 3-brane worldvolume should be taken into accounts and in certain cases, would lead to fraction $\frac{1}{\mathrm{ord}\mathfrak{g}_I}$ of the above multiplicities [64]. A precise analysis in non-perturbative setting is still missing. However, as we will show in chapter 4, anomaly cancellations would sheds new light on the structure of 3-7 modes.

## 2.13. String Universalities in 8D?

It has been shown in [146] that in 8D $\mathcal{N}=1$ theories, the only consistent gauge algebras who does not have global anomaly are the following

$$\{\mathfrak{su}(n), \mathfrak{so}(2n), \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{g}_2\}. \tag{2.181}$$

As we learned from 2.4.1 in our golden example, all of these algebras admit F-theory realizations except the $\mathfrak{g}(2)$. Such $\mathfrak{g}(2)$ might be excluded from further studies on the consistency conditions as we expect.

# Chapter 3.

# 6D $\mathcal{N} = (1,0)$ Anomalies Cancellation in F-theory Compactifications

## 3.1. Introduction

In the previous chapter, we have seen that anomalies provide a unique tool to identify the 10d supergravity and string theory universality.

In six dimensional chiral theories, anomaly cancellation provides powerful constraints on the set of possible theories. For example in 6D $\mathcal{N} = (2,0)$, demanding absence of net anomalies uniquely determines the massless spectrum, namely 1 gravity multiplets and 21 tensor multiplets [147], which is exactly the type IIB spectrum compactified on a $K3$.

In the section, we mainly focus on anomaly cancellations in 6D $\mathcal{N} = (1,0)$ supergravities and their F-theory realizations, which is the most popular [1] framework for studying such theories. We will also present some basic facts for the F-theory Calabi-Yau three-fold compactification, as an excises of the discussions for generic dimensional Calabi-Yau compactification in 2. And later we will identify anomaly terms in 6D $\mathcal{N} = (1,0)$ theories with geometric quantities in F-theory compactifications.

## 3.2. Basics for 6D $N = (1,0)$ Supergravities

For completeness, we firstly review very basic facts about the massless spectrum and dynamics of 6D $\mathcal{N} = (1,0)$ supergravity. Generally speaking, the massless spectra of 6D $\mathcal{N} = (1,0)$ supergravity theories contain one graviton multiplet, $T$ tensor multiplets, $V$ vector multiplets and $H$ hypermultiplets [148–150]. With respect to the little group $SO(4) \approx SU(2)_L \times SU(2)_R$, one can label the massless spectra by integers or half-integers $(j_L, j_R)$. The details are as follows:

- Gravity multiplet:$(1,1) \oplus 2(1/2,1) \oplus (1,0)$, i.e. graviton $G_{\mu\nu}$, one Weyl right-handed gravitino $\psi^+_\mu$ and one self-dual two form $B^+_{\mu\nu}$.
- Tensor multiplet: $(0,1) \oplus 2(0,1/2) \oplus (0,0)$, i.e. one anti-self-dual two form $B^-_{\mu\nu}$, one real scalar $\phi$ and one Weyl left-handed tensorino $\chi^-$.
- Vector multiplet: $(1/2,1/2) \oplus 2(1/2,0)$, i.e. one vector $A_\mu$ and one Weyl right-handed gaugino $\lambda^+$. Note that there are no scalars included in the vector multiplets.
- Hypermultiplet : $4(0,0) \oplus 2(0,1/2)$, i.e. four real scalars $\phi$ and one Weyl left-handed Fermion $\psi^-$.

Let us denote the anti-symmetric two-forms collectively as $B_2^\alpha, \alpha = 0,1,...,T$. Theories with $T$ tensor multiplets have a moduli space locally taking the coset form $SO(1,T)/SO(T)$. A unit vector $j$ of $SO(1,T)$ parametrizes the $T$ scalar fields $\phi$ in the tensor multiplets. The additional

[1] by "popular" we mean lots of consistent 6D $\mathcal{N} = (1,0)$ supergravities can be realized in F-theory.

degrees of freedom of $j^\alpha$ are fixed by the condition

$$\Omega_{\alpha\beta} j^\alpha j^\beta = 1, \tag{3.1}$$

where $\Omega_{\alpha\beta}$ is a constant metric on the $SO(1,T)$ space.

The moduli space is endowed with a non-constant, positive definite metric $g_{\alpha\beta}$, which is given by

$$g_{\alpha\beta} = 2 j_\alpha j_\beta - \Omega_{\alpha\beta}. \tag{3.2}$$

Here we used $\Omega$ to lower the indices of $j^\alpha$: $j_\alpha = \Omega_{\alpha\beta} j^\beta$. Hence the $T$ scalars from the tensor multiplets parametrize the quotient space

$$SO(1,T)/SO(T). \tag{3.3}$$

Without loss of generalities, we assume the total gauge group of a generic 6D $\mathcal{N} = (1,0)$ supergravity to be

$$G^{\text{tot}} = \prod_{I=1}^{n_G} G_I \times \prod_{A=1}^{n_{U(1)}} U(1)_A \tag{3.4}$$

and matter fields in representations

$$\mathbf{R} = (\mathbf{r}^1, \ldots, \mathbf{r}^{n_G})_{\underline{q}}, \tag{3.5}$$

where $\mathbf{r}^I$ denotes an irreducible representation of the simple gauge group factor $G_I$ and $\underline{q} = (q_1, \ldots, q_{n_{U(1)}})$ are the charges under the Abelian gauge group factors.

Having settled the backgrounds, we list the bosonic part of the 6D $\mathcal{N} = 1$ effective action (see e.g. [151] and reference therein) (up to the two derivatives) as follows:

$$\begin{aligned} S_{6d} = \int_{\mathbb{R}^{1,5}} & \frac{1}{2} R * 1 - \frac{1}{2} g_{\alpha\beta} dj^\alpha \wedge * dj^\beta - \sum_\kappa (2j \cdot b_I) \frac{1}{\lambda_I} \text{tr} F_I \wedge * F_I - \sum_{A,B} (2j \cdot b_{ij}) \text{tr} F^A \wedge * F^B + S_{\text{hyper}} \\ & - \frac{1}{4} g_{\alpha\beta} H_3^\alpha \wedge * H_3^\beta - \frac{1}{2} \Omega_{\alpha\beta} B_2^\alpha \wedge X_4^\beta, \end{aligned} \tag{3.6}$$

where the 4-form $X_4$ was found to be the form

$$X_4^\alpha = \frac{1}{4} \left(-\frac{1}{2} a^\alpha \text{tr} R^2 + \sum_I (2 \frac{b_I^\alpha}{\lambda_I}) \text{tr} F_I^2 + \sum_{AB} 2 b_{AB}^\alpha F_A F_B\right). \tag{3.7}$$

Here $a^\alpha$ and $b^\alpha$'s also transform as $SO(1,T)$ vectors, "tr" of $F$ denotes the fundamental representation. And the normalization factors $\lambda_I$ denotes the Dynkin index in the fundamental representation and is tabulated in 4.1. Further the $\cdot$ operator acts on the vectors of the $SO(1,T)$ with the respect to metrics $\Omega_{\alpha\beta}$, i.e.

$$b \cdot j = \sum_{\alpha,\beta=0}^{T} b^\alpha \Omega_{\alpha\beta} j^\beta. \tag{3.8}$$

| $\mathfrak{g}$ | $A_n$ | $D_n$ | $B_n$ | $C_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 2 | 2 | 1 | 6 | 12 | 60 | 6 | 2 |

**Table 3.1.:** *Dynkin index of the fundamental representation for the simple Lie algebras.*

We also introduced the 3-form field strength $H_3^\alpha, \alpha = 0, 1, ..., T$ as

$$H_3^\alpha = dB_2^\alpha + \frac{1}{2}a^\alpha \omega_{3L} + 2\sum_A \frac{b_A^\alpha}{\lambda_A}\omega_{3Y}^A + 2\sum_{i,j} b_{ij}^\alpha \omega_{3Y}^{ij}, \tag{3.9}$$

where $\omega_{3L}$ and $\omega_{3Y}$ are the Chern-Simons 3-forms of the spin connection $\widehat{\omega}$ and the gauge fields $A$ (we omitted the subscripts for non-abelian and abelian cases), respectively,

$$\begin{aligned}\omega_{3L} &= \mathrm{tr}(\widehat{\omega}\wedge d\widehat{\omega} + \frac{2}{3}\widehat{\omega}\wedge\widehat{\omega}\wedge\widehat{\omega})\\ \omega_{3Y} &= \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A),\end{aligned} \tag{3.10}$$

and one has

$$dH_3^\alpha = X_4^\alpha\,. \tag{3.11}$$

We should stress that the above effective action is the so-called the pseudo action, as for cases with $T \neq 1$[2], there is no Lorentz covariant Lagrangian formula due to the (anti) self-duality conditions for 3-form strength $H^\alpha$. The pseudo action [148] should be completed by imposing the (anti) self-duality conditions for $H^\alpha$ by hand at the level of E.O.M by requiring

$$\Omega_{\alpha\beta} * H_3^\beta = G_{\alpha\beta}H_3^\beta. \tag{3.12}$$

$\Omega_{\alpha\beta}$ is an invariant symmetric bilinear form in $SO(1,T)$ associated with the Dirac paring between string charges, which form an integral lattice $\Gamma$. The consistency of quantum gravity imposes the integral lattice $\Gamma$ must further be embedding into a unimodular lattice.

Compactifying on $S^1$, the theory gives rise to a 5D $\mathcal{N}=1$ [3] supergravity

## 3.3. Anomalies and Constraints on 6D $\mathcal{N}=(1,0)$ Supergravity

As a non-renormalizable effective theory, the 6D $N=(1,0)$ supergravity does not require the various contributions of local anomalies received from chiral fields to be vanished by themselves. Instead, there exists a possibility of canceling a non-vanishing anomaly by adding non-

[2]For the case with $T=1$, one can formulate a classical Lagrangian, as the self-dual $B_2$ from the gravity multiplet and the anti self-dual $B_2$ from the tensor multiplet can be combined into a two-form without such property.

[3]Note that in parts of the literature, the 5D Minimal supergravity is dubbed as $\mathcal{N}=2$. These are just different conventions. As recalled, the spinors in the 5D Minkowski spaces are pseudo-real C.2. Hence one can always split the supercharges in 5D Minimal supergravity into a pair of spinors $Q_{\alpha i}, \alpha = 1, ..., 4; i = 1, 2$ in order to make the supersymmetric generators real. That is the reason why in 5D $\mathcal{N}=1$ theories in Minkowski space, the anti-commutation of the supercharges can contain a central charge $\xi$ (as a Lorentz scale).

$$\{Q_{\alpha i}, Q_{\beta j}\} = i\Gamma^M_{\alpha\beta}\epsilon_{ij}P_M + C_{\alpha\beta}\epsilon_{ij}\xi. \tag{3.13}$$

renormalizable, gauge-variant terms/operators in the tree level of Lagrangian. For example, the generalized Green-Schwarz mechanism states that the non-vanishing 1-loop anomaly polynomials of an effective theory generated by chiral matters should be factorized, and then could be cancelled by adding tree-level Green-Schwarz counter terms. In the context of 6D $N = (1,0)$ supergravity theories, such aspects have been investigated in great detail, most notably (see e.g. [152–156] and references therein).

Our conventions for the anomaly polynomial mostly follow [157]. In a general D-dimensional effective theory, the gauge and gravitational anomalies can be formulated by a guage invariant anomaly polynomial of degree $\frac{1}{2}(D+2)$ in terms of gauge field strength $F$ and the curvature two-from $R$,

$$I^{1-loop}_{D+2} = \sum_{\mathbf{R},s} n_s(\mathbf{R}) I_s(\mathbf{R})|_{D+2}\,, \tag{3.14}$$

where the sum is over all matter fields with spin $s$ which have zero-modes in representation $\mathbf{R}$ with multiplicity $n_s(\mathbf{R})$. In particular, a chiral fermion, corresponding to $s = 1/2$, contributes with

$$I^{1-loop}_{1/2}(\mathbf{R}) = -\mathrm{tr}_{\mathbf{R}}\, e^{-F}\, \widehat{A}(\mathrm{T})\,, \tag{3.15}$$

where $\widehat{A}(\mathrm{T})$ is the A-roof genus and $F$ denotes the hermitian gauge field strength. An anti-chiral fermion contributes with the opposite sign. For 6D $N = (1,0)$ theories, the 1-loop anomaly polynomials is an 8-form $I_8^{1-loop}$ and given by [155] [4],

$$\begin{aligned}
I_8^{1-loop} &= \frac{1}{5670}(H - V + 29T - 273)[\mathrm{tr}R^4 + \frac{5}{4}(\mathrm{tr}(R^2)^2)] \\
&+ \frac{1}{128}(9 - T)(\mathrm{tr}(R^2)^2) \\
&- \frac{1}{96}\mathrm{tr}R^2[\sum_I \mathrm{Tr}F_I^2 - \sum_{I,R_I} n_{R_I}\mathrm{tr}_{R_I}F_I^2] \\
&- \frac{1}{24}[\sum_I \mathrm{Tr}F_I^4 - \sum_{I,R_I} n_{R_I}\mathrm{tr}_{R_I}F_I^4 - 6\sum_{I,R_I,J,R_J} n_{R_I,R_J}(\mathrm{tr}_{R_I}F_I^2)(\mathrm{tr}_{R_J}F_J^2)] \\
&+ \sum_A F_A \wedge [\frac{1}{96}\mathrm{tr}R^2 \sum_{B,q_A,q_B} n_{q_A,q_B} q_A q_B F_B \\
&- \frac{1}{6}\sum_{I,q_A,R_I} n_{R_I,q_A} q_A (\mathrm{tr}_{R_I}F_I^3) - \frac{1}{4}\sum_{I,R_I,B,q_A,q_B} n_{R_I,q_A,q_B} q_A q_B (\mathrm{tr}_{R_I}F_I^2) F_B \\
&+ \frac{1}{24}\sum_{B,C,D,q_A,q_B,q_C,q_D} n_{q_A,q_B,q_C,q_D} q_A q_B q_C q_D F_B F_C F_D],
\end{aligned} \tag{3.16}$$

where we have factored out a factor of $F_A$ in the last three lines for later purpose. Namely note that considering the polynomials only involving the abelian gauge fields, then one notice that the relevant parts $I_8^{1-loop,U(1)}$ can be factorized as

$$I_8^{1-loop,U(1)_A} = \sum_A F_A \wedge X_6^A, \tag{3.17}$$

[4] The sign of the last term in (3.16) has been corrected

where $X_6$ is given by

$$
\begin{aligned}
X_6^A = [&\frac{1}{96}\mathrm{tr}R^2 \sum_{B,q_A,q_B} n_{q_A,q_B} q_A q_B F_B \\
&- \frac{1}{6}\sum_{I,q_A,R_I} n_{R_I,q_A} q_A (\mathrm{tr}_{R_I} F_I^3) - \frac{1}{4}\sum_{I,R_I,B,q_A,q_B} n_{R_I,q_A,q_B} q_A q_B (\mathrm{tr}_{R_I} F_I^2) F_B \\
&+ \frac{1}{24}\sum_{B,C,D,q_A,q_B,q_C,q_D} n_{q_A,q_B,q_C,q_D} q_A q_B q_C q_D F_B F_C F_D].
\end{aligned}
\tag{3.18}
$$

As usual, we denote "tr" as the trace in the fundamental representation and "Tr" in the adjoint representation. And $n_{...}$ represents the number of hypermultiplets (♡) in the given representation $\cdots$ as follows:

• $n_{R_I}$: ♡ in representation $R_I$ of the non-abelian gauge group $G_I$

• $n_{R_I,R_J}$: ♡ in representation $(R_I, R_J)$ of the non-abelian gauge group $G_I \times G_J$

• $n_{R_I,q_A}$: ♡ in representation $(R_I)$ of the non-abelian gauge group $G_I$ with charge $q_A$ under the abelian $U(1)_A$.

• $n_{R_I,q_A,q_B}$: ♡ in representation $(R_I)$ of the non-abelian gauge group $G_I$ with the charge $(q_A, q_B)$ under the abelian $U(1)_A \times U(1)_B$.

• $n_{q_A,q_B,q_C,q_D}$: ♡ with the charge $(q_A, q_B, q_C, q_D)$ under the abelian $U(1)_A \times U(1)_B \times U(1)_C \times U(1)_D$.

The generalized Green-Schwarz mechanism in six dimensions [158–160] states that one could add a local counter-term Green-Schwarz term [5]

$$S^{GS1} = -\frac{2\pi}{2}\int_{R^{1,5}} \Omega_{\alpha\beta} B_2^\alpha \wedge X_4^\beta, \tag{3.19}$$

as shown by the last two terms in the 6D pseduo action 3.6, to cancel the 1-loop anomaly $I_8^{1-loop}$, which instead does not vanish but remains as a suitable form of factorization

$$I_8^{1-loop} = \frac{1}{2}\Omega_{\alpha\beta} X_4^\alpha \wedge X_4^\beta, \tag{3.20}$$

where the 4-form $X_4^\alpha$ vector was listed in (3.7). In this sense, we dubbed the $a^\alpha$s and $b^\alpha$s as the anomaly coefficients of the theory.

The second term in the action constitutes the Green-Schwarz coupling, which is responsible for the non-standard Bianchi identity

$$dH_3^\alpha = X_4^\alpha\,. \tag{3.21}$$

One can then assign the following gauge transformation for the various gauge fields including

[5] Here the reason we label the GS counter term as GS1 is that, as we will show in chapter 5, there is another type of GS mechanism, which later we will denote as GS2 in (5.1).

the spin connection $\widehat{\omega}$ and the two-form $B_2^\alpha$

$$\begin{aligned} A_I &\to A_I + d\lambda_I + [A_I, \lambda_I], \\ A_A &\to A_A + d\lambda_A, \\ \widehat{\omega} &\to \widehat{\omega} + d\widehat{l} + [\widehat{l}, \widehat{\omega}], \\ B^\alpha &\to B^\alpha - \frac{1}{2} a^\alpha \mathrm{tr}(\widehat{l} d\widehat{\omega}) - 2b_I^\alpha \mathrm{tr}(\lambda_A dA_I) - 2b_{AB}^\alpha \lambda_I dA_A, \end{aligned} \tag{3.22}$$

such that the the covariant three-form $H_3^\alpha$ is gauge invariant. The above is essentially gauging the 1-form shift symmetry associated with the $A_I, A_A, \widehat{\omega}$, generalizing the 0-form shift symmetry, i.e. the shift symmetry of the axions. As a result, the pseudo-action picks up a gauge variation of the form

$$\delta S_{GS1} = \frac{2\pi}{2} \int \Omega_{\alpha\beta} (\frac{1}{2} a^\alpha \mathrm{tr}(\widehat{l} d\widehat{\omega}) + 2b_I^\alpha \mathrm{tr}(\lambda_A dA_I) + 2b_{AB}^\alpha \lambda_I dA_A) \wedge X_4^\beta =: 2\pi \int_{\mathbb{R}^{1,5}} I_6^{(1),\mathrm{GS}}(\lambda) \tag{3.23}$$

with $I_6^{(1),\mathrm{GS1}}$ a gauge invariant 6-form. By the standard descent procedure, it defines an anomaly-polynomial $I_8^{\mathrm{GS1}}$ encoding the contribution to the total anomaly from the Green-Schwarz sector. Concretely, the descent equations

$$I_8^{\mathrm{GS1}} = dI_7^{\mathrm{GS}}, \qquad \delta_\lambda I_7^{\mathrm{GS}} = dI_6^{(1),\mathrm{GS}}(\lambda) \tag{3.24}$$

imply

$$I_8^{\mathrm{GS1}} = -\frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta. \tag{3.25}$$

Consistency of the theory then requires that

$$I_8^{1-loop} + I_8^{\mathrm{GS1}} = 0\,. \tag{3.26}$$

Hence the 1-loop anomaly $I_8^{1-loop}$ should be subject to

$$I_8^{1-loop} = \frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta. \tag{3.27}$$

By substituting the 4-form $X_4$ in (3.7) and then comparing the explicit form of $I_8^{1-loop}$ listed in (3.16), we are led to the following anomaly cancellation equations in 6D $N = (1,0)$ supergravity:

$$273 = H - V + 29T, \tag{3.28}$$

$$0 = B_{adj,I} - \sum_{R_I} n_{R_I} B_{R_I}, \tag{3.29}$$

$$a \cdot a = 9 - T, \tag{3.30}$$

which refer to the pure gravitational anomalies. The Anomaly equations involving the non-abelian gauge group are given by

$$a \cdot b_I = \frac{1}{6} \lambda_I \left( A_{Adj,I} - \sum_{R_I} n_{R_I} A_{R_I} \right), \tag{3.31}$$

$$b_I \cdot b_I = -\frac{1}{3}\lambda_I^2\left(C_{Adj,I} - \sum_{R_I} n_{R_I} C_{R_I}\right), \tag{3.32}$$

$$b_I \cdot b_J = \lambda_I \lambda_J \sum_{R_I,R_J} n_{R_I R_J} A_{R_I} A_{R_J}, \quad I \neq J. \tag{3.33}$$

The anomaly equation involving the abelian $U(1)$s are given by

$$a \cdot b_{AB} = -\frac{1}{6}\sum_{AB q_A q_B} n_{q_A,q_B} q_A q_B, \tag{3.34}$$

$$0 = \sum_{R_I,q_A} n_{R_I,q_A} q_A E_{R_I}, \tag{3.35}$$

$$\frac{b_I}{\lambda_I} \cdot B_{AB} = \sum_{R_I,q_A,q_B} n_{R_I,q_A,q_B} q_A q_B A_{R_I}, \tag{3.36}$$

$$b_{AB} \cdot b_{CD} + b_{AC} \cdot b_{BD} + b_{AD} \cdot b_{BC} = \sum_{q_A,q_B,q_C,q_D} n_{q_A,q_B,q_C,q_D} q_A q_B q_C q_D, \tag{3.37}$$

where the group coefficients $A_{R_I}, B_{R_I}, C_{R_I}$ are defined as

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2; \qquad \mathrm{tr}_R F^3 = E_R \mathrm{tr} F^3; \qquad \mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R \left(\mathrm{tr} F^2\right)^2. \tag{3.38}$$

By looking at the parts of the above equations involving the abelian gauge symmetry and gravity, one notices that the RHS are integers, hence one would expect the inner products of the vector $a, b$ shall be integers. Indeed, as proved in [161] [6], it turns out that all the inner products, including the non-abelian ones, defines an integral lattice as

$$\Lambda \in \mathbb{R}^{1,5}, \tag{3.39}$$

namely $a, b_I, b_{AB} \in \Lambda$, which is dubbed the anomaly lattice for the theories.

In addition, there are constraints from other consistency conditions. For example, by inspecting the action in (3.6), one can see that the kinetic terms for gauge fields $A_A, A_I$ are proportional to $b_I \cdot j$ and $b_{AB} \cdot j$ (fixed by supersymmetry requirements). In order to have the correct sign for the kinetic terms, one should impose [159]

$$b_I \cdot j > 0; \quad b_{AB} \cdot j > 0. \tag{3.40}$$

One can interpret this as the statement that there has to exist a vector $j$ satisfying the above constraints, where $j$, as mentioned, is a unit vector in in $\mathbb{R}^{1,T}$ encoding the scalars in the tensor multiplets.

The above anomaly equations, together with other conditions impose strong constraints on the massless spectrum and gauge groups of the 6D $\mathcal{N}=(1,0)$ supergravity theory. To give a small taste, following [25], by inspection of the (3.28), one notices that (3.28) can bound the number of hypermultiplets for a given $T$ and gauge contents. For $T < 9$ and without abelian gauge factors, it was shown in [161,162] that $H \sim O(N^2)$ and $V \sim O(N)$, where $N$ here denotes the number of distinct non-abelian factors. Hence as $N \to \infty$, the pure gravitational anomaly

[6] The nice proof given in [161] was a bit technique and involved. Essentially, they follow from both local and global anomaly cancellations.

(3.28) would be violated. Furthermore [161,162] showed that the number of non-abelian theories with $T < 9$ tensor multiplets is finite and it still stays finite same in the presence of abelian gauge symmetries in [155] when ignoring the differences of the $U(1)$ charges [7].

If the 6D $\mathcal{N} = (1,0)$ supergravity theories have a consistent UV completion, as we are going to discuss in the next section, then one should notice that the anomaly lattice $\Lambda$ above attains a natural interpretation as a sublattice of a charge lattice $\Gamma$ for the dyonic string. Such dyonic strings are the BPS states, which are charged under the $B_2$ in the tensor multiplets. In the presence of non-abelian gauge factors, then such dyonic strings naturally can be viewed as the gauge instantons (as codimension four defects). To see that, recall that the form of the GS counter term (5.1), as well as $X_4^\alpha$ in (3.7), implies that the transverse directions support a gauge instanton profile $\int \mathrm{tr} F_I \wedge F_J$, with the charge given by the $b_I$. Similar arguments might also apply to the other two $a, b_{AB}$, hence it defines an embedding from the anomaly lattice $\Lambda$ to the charge lattice $\Gamma$ (a priori not all the charge comes from the $a, b$.). By applying the quantization conditions to the dyonic strings, one can see that the lattice $\Lambda$ is integral. Furthermore, consistency of the theories upon reduction to lower dimensions 2 and 4 further requires that the charge lattice $\Lambda$ for dyonic strings has to be unimodular (self-dual) [164]. Such embedding could impose further constraints on possible ranges of the theories. We refer to [25] for more details. For recent developments on the various consistent conditions on 6D $\mathcal{N} = (1,0)$ theories, including taking into the quantization of anomaly coefficients, we refer to [156,165] for more details.

## 3.4. Embedding to F-theory Compactifications

In this section, we are going to embed 6D $\mathcal{N} = (1,0)$ supergravity theories into the F-theory Calabi-Yau three-folds compactification. As we have shown in the previous chapter 2, various aspects of effective theory from F-theory compactification including the gauge groups and matter contents are determined by the geometry of the compactification. Hence it is reasonable to expect the various anomaly terms to have certain geometric correspondences. Indeed, we will show in this section the mappings from the terms in the 6D $\mathcal{N} = (1,0)$ theories to geometric structures of the F-theory compactification.

### 3.4.1. Calabi-Yau three-fold compactifications of F-theory

We start with a lighting review of a few of the most salient aspects of F-theory compactification to the six dimension. Following the discussion for generic manifolds in 2, an elliptically fibered Calabi-Yau three-fold $X_3$ with a global section can be described by a Weierstrass equation

$$y^2 = x^3 + fx + g, \tag{3.41}$$

where $f, g$ are local holomorphic functions on a complex surface base $B_2$. The discriminant locus

$$\Delta = 4f^3 + 27g^2 \tag{3.42}$$

[7] However, when considering the differences of the abelian $U(1)$ charges, it has been shown in [163] that there are indeed infinite number of different families of theories in terms of different $U(1)$ charges satisfying the anomaly equations and other known quantum consistence conditions, which might pose challenges to the string universalities in six dimension, at least F-theory universalities in six dimension as it can only give a finite number of the theories in terms of the matter spectra and gauge structures.

specifies the location of seven-branes. And globally speaking, $f, g, \Delta$ are sections of the following line bundles

$$f \in \Gamma(B_2, -4K_B), \qquad g \in \Gamma(B_2, -6K_B) \qquad [\Delta] \in \Gamma(B_2, -12K_B), \tag{3.43}$$

where $K_B$ is the canonical class of $B_2$.

F-theory compactified on this Calabi-Yau $X_3$ gives rise a 6D $\mathcal{N} = (1,0)$ supergravity theory. Let us assume that the total gauge group for this effective theory as $G_{tot}$, then according to the discussion in chapter 2, especially the sections involving the abelian and discrete symmetry, we have

$$\pi_0(G_{tot}) = \text{Ш}(X_3). \tag{3.44}$$

For our purpose in this chapter, let us focus the Calabi-Yau $X_3$ with the above homotopy group being trivial and focus on the cases with $G_{tot}$ being (3.4). We will discuss the relevant parts for non-trivial cases with $\pi_0(G_{tot})$ for chapter 5.

Based on the Kodaira condition (2.44) and the general discussion in 2.2.2, we know the first Chern class of base $c_1(B_2)$ is encoded by the divisor class $[W]$ as

$$c_1(B_2) = \frac{1}{12}[W] = \sum_I p_I[W_I] + W_0, \tag{3.45}$$

with the $p_I$ denoting the multiplicity 2.95. Each of these divisors $W_I$ support a non-abelian gauge group $G_I$. We assume that a flat resolution $\widehat{X}_3$ of the Calabi-Yau $X_3$ exists, with a further requirement that such a resolution does not involve the blow-ups of the base $B_2$. As we have discussed in 2, such blow-ups in the base $B_2$ (both in codimension-one and codimension-two loci) typically indicate the existences of a 6D SCFTs, and render the massless spectra more involved [166].

According to the Shioda-Tate-Wazir theorem (see the discussions for generic dimensional Calabi-Yau manifolds in 2.9.2), the number $V$ of vector multiplets, including the abelian ones, is given by

$$V = \sum_I \text{rk}(G_I) + \sum_A r_A = h^{1,1}\widehat{X}_3 - h^{1,1}(B_2) - 1. \tag{3.46}$$

From the discussion in (2.102), we know that the number of the tensor multiplets in the theory is given by

$$T = h^{1,1}(B_2) - 1. \tag{3.47}$$

By employing the dual M-theory picture, one can know that there are $h^{2,1}(\widehat{X}_3) + 1$ hypermultiplets in the dual 5D $\mathcal{N} = 1$ theory [8], and such hypermultiplets would uplift to the neutral hypermultiplets in the 6D in the F-theory limit and hence we have

$$H_{neutral} = h^{2,1}(\widehat{X}_3) + 1. \tag{3.48}$$

As we know that the Euler chracteristic of a smooth Calabi-Yau threefold is fully determined by two Hodge numbers $h^{1,1}, h^{1,2}$ hence we have $\chi(\widehat{X}_3)$ as

$$\chi(\widehat{X}_3) = 2(h^{1,1} - h^{2,1}) = 2(V - T - H_{neutral} + 3). \tag{3.49}$$

[8]This is because the neutral hypermultiplets in M-theory compactifications correspond to the $h^{2,1}(\widehat{X}_3)$ complex structure moduli of $\widehat{X}_3$ together with extra scalars from the reduction of $C_3$ along the corresponding 3-cycles.

How about the charged hypermultiplets? As for the **bulk matter**, if all the $G_I$ are simply-laced ADE algebras and further the corresponding divisors $W_I$ are smooth, then for each $W_I$, we have $g_I$ bulk hypermultiplets in the adjoint representation of $G_I$, where $g_I$ is the (geometric) *genus* of $W_I$, as a curve in the base $B_2$. However, if some of $G_I$ are not simple-laced ADE groups, the counting turns out to be a bit subtle. One can still use the Katz-Vafa picture [105] to determine. To this end, recall that the non-simply laced gauge algebra $\widetilde{\mathfrak{g}}_I$ arises by acting with an outer automorphism on a degenerating fiber and that this gives rise to the decomposition (2.109), namely

$$\mathbf{adj}(\widetilde{\mathfrak{g}}_I) = \mathbf{adj}(\mathfrak{g}_I) \oplus \widetilde{\rho}_0, \tag{3.50}$$

where $\widetilde{\rho}_0$ denotes the other representations of $\widetilde{\mathfrak{g}}_I$. To count the number, one may consider a branched cover $W_I'$ of $W_I$ with degree $d$. Then the number for the extra hypermultiplets in representation $\widetilde{\rho}_0$ is given by

$$n_{\widetilde{\rho_0}} = g(W_I') - g(W_I) = (d-1)(g(W_I)-1) + \frac{1}{2}\deg(R), \tag{3.51}$$

where $R$ denotes the ramification divisor of $W_I'$. Otherwise, if some of the divisors $W_I$ are singular, then there could be some extra exotic representations from the singular points [108].

The **Localized matter** in $\widehat{X}_3$, dictated by codimensional two singularities, are then localized over several points in the base $B_2$, and the number of the localized charged hypermultiplets could be counted by intersecting numbers between the $W_I$, among other factors. For more details, we refer to the review [21] and the references therein.

### 3.4.2. Mapping the anomaly terms

Now let us identify the terms in the anomaly equations, especially the anomaly polynomials, with geometric correspondences in the $\widehat{X}_3$ compactification.

From the perspective of the F-theory compactification, the 6D dyonic strings are given by the D3-branes wrapping along 2-cycles in the base $B_2$. This is not hard to figure out, as the dyonic strings are charged under the tensor multiplet $T$, and we already knew that then tensor multiplets in the F-theory compactification arise from the decomposition of 10d $C_4$ fields, which coupled to the D3-branes. Thus the charge lattice of dyonic strings $\Gamma$ is given by

$$\Gamma = H_2(B_2, \mathbb{Z}), \tag{3.52}$$

and the inner product is given by the intersection form on $B_2$. Notice that the unimodular conditions immediately follow from Poincaré duality. Similarly, the symmetric matrix $\Omega_{\alpha\beta}$, which can be viewed as an integer-valued quadratic form on $\Gamma$, is given by the intersecting form on $H_2(B_2, \mathbb{Z})$ as [9]

$$\Omega_{\alpha\beta} = \int_{B_2} \omega_\alpha \wedge \omega_\beta, \tag{3.53}$$

where $\omega_\alpha \in H^{1,1}(B_2, \mathbb{Z})$ is a basis.

The vector $j$, parametering the scalars in the tensor multiplet, is then given by the Kähler form $J$

$$J = j^\alpha \omega_\alpha, \tag{3.54}$$

which we normalized the volume to be $V = \frac{1}{2}J^2 = \frac{1}{2}j \cdot j = \frac{1}{2}$.

[9]Note that in the next chapter 4, $\Omega$ carries a prefactor $2\pi$ in (4.134) comparing with the one here.

Now let us move to the anomaly lattice $\Lambda$. A more detailed analysis shall follow the similar proofs in the section 5.2. Here we will use a heuristic argument. Recall that the anomaly coefficient $b_I$ can be viewed as the charge of the gauge instanton strings, where such instanton profile shall be given by the D3-brane wrapped on the the seven-brane divisor $W_I$, hence we have

$$b_I^\alpha = W_I^\alpha, \qquad \alpha = h^2(B_2) = T+1, \tag{3.55}$$

where $W_I^\alpha$ is the component of the divisor $W_I$ along the basis $\omega_\alpha$.

As for the abelian one $b_{AB}$, although the above picture is a bit elusive, one can still argue it should correspond to the height pairing $b_{AB}$ [10] associated with the rational sections, as we discussed in 2.9.3. Indeed, one can use the type IIB orientifold picture 5.2 to substantially prove it.

Before moving to the anomaly coefficient $a$, it is beneficial to recall that for any bases $B_2$ which support a elliptic fibration structure, one has the relation $K_B^2 = 10 - h^{1,1}(B_2)$. Noticing $h^{1,1}(B_2) = T+1$, we have

$$K_B^2 = 9 - T. \tag{3.56}$$

This exactly reproduce the anomaly equation (3.30). Hence one can identify $a^\alpha = K_B^\alpha$ with $K_B^\alpha$ being components of $K_B$ along the basis [11].

The maximal value for $T$ ever found for a consistent F-theory vacuum or any other 6D $\mathcal{N}=1$ constructions is 193 [167, 168].

### 3.4.3. Unifying with the 4D anomaly equation

We have identified the various terms with the geometric objects in $X_3$ in the F-theory compactification, which is not so surprising as we learned that the crucial date for anomaly cancellations such as gauge structures and matter content are encoding in the background geometry $X_3$ of F-theory. Now one might wonder whether such relations hold for other dimensional compactifications, and whether there are universal geometric equations in the F-theory Calabi-Yau compactifications encode the consistency conditions, particularly the anomaly cancellation conditions. To answer such questions, one should notice that in compactifications on higher-dimensional Calabi-Yaus, the $G_4$ background fluxes should be taken into accounts, especially the vertical $G_4$ fluxes which affect the chirality of the effective theories, as we discussed in the last chapter. Nevertheless, we know that one can express vertical $G_4$ fluxes in terms of algebraic cycles, and this suggests similar geometric relations. Indeed, in [169], the authors unify the (local) anomaly equations in Calabi-Yau four-folds and three-folds, which are captured by two relations, each valid in $H^4(\widehat{X}_3)$ or $H^4(\widehat{X}_4)$, of the form

$$\sum_{\mathbf{R},a} \beta^a_\Gamma(\mathbf{R})\, \beta^a_\Lambda(\mathbf{R})\, \beta^a_\Sigma(\mathbf{R}) S^a_{\mathbf{R}} - 3\, \mathfrak{F}_{(\Gamma} \cdot \pi^* \pi_* (\mathfrak{F}_\Lambda \cdot \mathfrak{F}_{\Sigma)}) \quad = \quad 0 \tag{3.57}$$

$$\sum_{\mathbf{R},a} \beta^a_\Lambda(\mathbf{R})\, S^a_{\mathbf{R}} + 6\, \mathfrak{F}_\Lambda \cdot c_1 \quad = \quad 0\,. \tag{3.58}$$

These two homological relations have been shown in [169] to be equivalent to the intersection theoretic identities derived from the requirement of gauge and mixed gauge-gravitational anomaly cancellation in 6D [123] and 4D [170] F-theory vacua. In addition the cancellation of purely gravitational anomalies in 6D F-theory vacua poses an extra constraint on the geometry of

[10] The reader may forgive us for sticking to this same notation.
[11] the other choice $a = -K_B$ would leads to wrong sign, as $c_1 = -K_B$ is effective.

$\widehat{X}_3$, which has no direct counterpart in 4D[12]. Interestingly enough, however, apart from this latter point anomaly cancellation in 6D and 4D F-theory vacua is based on the same type of homological relations.

While a general proof of these relations from first principles, and without relying on anomaly cancellation, is not yet available in the literature, these relations can be verified in explicit examples.[13] The details of such a verification appear to be completely independent of the choice of base of the elliptic fibration, including its dimension [169]. In the next chapter 4, we will discuss the relevant generation of these two equations on the Calabi-Yau five-manifolds, on which F-theory compactified give rise to 2d $\mathcal{N} = (0, 2)$ effective theories.

## 3.5. String Universality in 6D?

We have learned from chapter 1 and chapter 2 that all the consistent (supersymmetric) theories in 11d/10d and 8d (with one small potential caveat for gauge algebra $\mathfrak{g}_2$) can be realized by string theory, dubbed as string universalities. One may ask whether in 6D such statement exists, given that the anomaly equations constraints, together with other consistency conditions, are strong in 6D. As we mentioned at the beginning, for extended supersymmetric theories in 6D such as 6D $\mathcal{N} = (2, 0)$, the answer is positive. We now mainly focus on 6D $\mathcal{N} = (1, 0)$ theories.

To our knowledge, this question is still remaining open. Simply, for example in [163], the authors showed that there are indeed infinite number of different families of 6D $\mathcal{N} = (1, 0)$ theories in terms of different $U(1)$ charges satisfying the anomaly equations and other known quantum consistence conditions. However, the choices from the F-theory compactifications are finite [161]. In order to get positive answer, one may turn to other string realizations or identify new quantum consistency conditions on the low energy effective theory. From F-theory perspective, more progress is required to address this question. As we know, although the classifications of the codimensional one singularities in elliptically fibrations are understood, the ones for codimension-two singularities (and higher for other CYs) remains open. And as a result, we have not fully understood ranges of matter spectra associated with gauge groups. For example, some models with exotic representations such as 4-index antisymmetric representation of $SU(N)$ satisfy the known consistency conditions, however, such F-theory realizations are still missing, although there is progress along this line [109, 173, 174]. We refer more details on such topic to the review [25].

In any cases, exploring new quantum consistency conditions is beneficial for many aspects. Before we close this section, it is worthwhile to mention that, there are several new developments recently on the so-called Swampland conjectures (see e.g. the reviews [175, 176]), and hopefully one can see more substantiated results in the near future!

[12] This relation is given, for example, as equation (3.8) in [123], and proven generally in [171].

[13] On the other hand, [172] proves anomaly cancellation in 4D F-theory vacua by comparison with the dual M-theory. Combined with the above statement this is a physics proof of (3.57) and (3.58) on elliptic Calabi-Yau 4-folds.

# Part III.

# Developments and Results

# Chapter 4.

# The Green-Schwarz Mechanism and Geometric Anomaly Relations in 2D (0,2) F-theory Vacua

## 4.1. Introduction

After this preparation we are now in a position to present the main results of this thesis - the derivation of the structure of anomalies in F-theory compactifications on elliptic Calabi-Yau five-folds and their cancellation via a consistent Green-Schwarz mechanism. The following chapter closely follows our publication [23] in presentation and contents, where we have first described these results.

## 4.2. Motivation

We have introduced the anomaly cancellations in 10D and 6D supergravities, and the crucial point for the anomaly cancellations in such $4k+2$ dimensions is that in the presence of tensor fields, the celebrated Green-Schwarz-Sagnotti-West mechanism [8,158,159] can cancel such 1-loop anomalies provided the anomaly polynomial of the latter factorises suitably. A lower-dimensional analogue of these supergravities, similar in many respects, are chiral theories in two dimensions with $N=(0,2)$ supersymmetry. Such theories have sparked significant interest from various field theoretic perspectives, most notably concerning their RG flow to an SCFT point [177–181], in the context of computing elliptic genera and localisation [182], or with respect to novel types of dualities [183, 184]. Exploring the structure of anomalies of a class of 2D $N=(0,2)$ supergravities is the goal of this chapter.

If a supergravity theory is engineered by compactifying string theory, the consistency conditions from anomaly cancellation imply a rich set of constraints on the geometry defining the compactification. A prime example of this fruitful interplay between anomalies and geometry is provided by F-theory [20, 124, 185]. In this framework, 6D $N=(1,0)$ supergravities arise via compactification on elliptically fibered Calabi-Yau 3-folds. Anomaly cancellation then translates into various highly non-trivial relations between topological invariants of the latter [91,123,155,160,171], which would be hard to guess otherwise, and some of which are even harder to prove in full generality. Compactification of F-theory to four dimensions on a Calabi-Yau 4-fold gives rise to an $N=1$ supersymmetric theory which is chiral - and hence potentially anomalous - only in the presence of non-trivial gauge backgrounds. This makes it perhaps even more intriguing that the same types of topological relations [169] are responsible for the cancellation of gauge and mixed gauge-gravitational anomalies in six and four-dimensional [170] F-theory compactifications. If one is able to establish the cancellation of anomalies directly from a physical perspective, as has been achieved recently in [172] for four-dimensional F-theory vacua, such reasoning amounts to a physics proof of a number of highly non-trivial topological relations on elliptic fibrations of complex dimension three and four. One of the motivations for this work is to extend this list of topological identities to elliptic fibrations of higher dimension.

The 2D $(0,2)$ supergravity theories considered in this chapter are obtained by compatifying F-theory on an elliptically fibered Calabi-Yau 5-fold [64, 65]. As we will review in section 4.4 the theories contain three different coupled sub-sectors: The structure of the gauge theory sector is similar to the 2D $(0,2)$ GLSMs familiar from the worldsheet formulation of the heterotic string [42, 186]. It includes 2D $(0,2)$ chiral and Fermi multiplets charged under the in general abelian and non-abelian gauge group factors originating from a topologically twisted theory on 7-branes [64, 65]. D3-branes wrapped around curves on the base of the fibration give rise to additional degrees of freedom. These include a particularly fascinating, but largely mysterious sector of Fermi multiplets from the string excitations at the intersection of the D3-branes and the 7-branes [187].[1] These two sectors are coupled to a 2D $N = (0,2)$ supergravity sector [191]. The construction of 2D $N = (0,2)$ theories has received considerable attention also in other formulations of string theory, most notably via D1 branes probing singularities on Calabi-Yau 4-folds [192–197] and via orientifolds [198, 199].

Various aspects of the non-abelian gauge and the gravitational anomalies in the chiral 2D $(0,2)$ theory obtained via F-theory have already been addressed in [64, 65, 187, 191, 200]. The non-abelian anomalies induced by the chiral fermions in the 7-brane brane gauge sector must be cancelled by the anomalies of the 3-7 modes, as indeed verified in globally consistent examples in [64]. The cancellation of all gravitational anomalies for 2D (0,2) supergravities with a trivial gauge theory sector has been proven in [191] with the help of various index theorems. Such theories are obtained by F-theory compactification on smooth, generic Weierstrass models. On the other hand, the structure of gauge anomalies in the presence of abelian gauge theory factors is considerably more involved, and the subject of this chapter.

As in higher dimensions, abelian anomalies induced at 1-loop level need not vanish by themselves provided they are consistently cancelled by a two-dimensional version of the Green-Schwarz mechanism. In general 2D $(0,2)$ gauge theories, the structure of the Green-Schwarz mechanism has been laid out in [201–203] (see [192, 204] for early work). In the present situation, the Green-Schwarz mechanism operates at the level of real chiral scalar fields which are obtained by Kaluza-Klein reduction of the self-dual 4-form of Type IIB string theory. They enjoy a pseudo-action which is largely analogous to the pseudo-action of the self-dual 2-tensors in 6D $N = (1,0)$ supergravities and which we parametrise in general terms in section 4.3. As one of our main results we carefully derive this pseudo-action in section 4.6, thereby identifying the structure (and correct normalisation) of the anomalous Green-Schwarz couplings. The latter depend on the non-trivial gauge background and imply a classical gauge variance of the right form to cancel the 1-loop abelian gauge anomalies.

A challenge we need to overcome to show anomaly cancellation is that in absence of a perturbative limit the abelian charges of the 3-7 sector modes are notoriously hard to determine in a microscopic approach. Instead of computing the 3-7 anomaly from first principles we extract the anomaly inflow terms onto the worldvolume of the D3-branes in section 4.6. To this end we start from the Chern-Simons terms of the 10d effective pseudo-action in the presence of brane sources. Uplifting this result to F-theory allows us to quantify the contribution of the 3-7 modes in particular to the gauge anomalies and in turn also to deduce the net charge of the 3-7 modes.

[1]The theory on a D3-brane wrapping a curve [187, 188] or surface [189, 190] in F-theory is interesting by itself as an example of a gauge theory with varying gauge coupling. An $AdS_3$ gravity dual of an $N = (0,4)$ version has recently been constructed in [181].

One of our main results is to establish a closed expression for the complete gauge and gravitational anomalies of a 2D $(0,2)$ theory obtained by F-theory compactified on a Calabi-Yau 5-fold. The resulting conditions for anomaly cancellation are summarized in (4.61) and (4.78) of section 4.5. The structure of anomalies reflected in these equations interpolates between their analogue in 6D and 4D F-theory vacua: In 6D F-theory vacua the anomalies are purely dependent on properties of the elliptic fibration, while in 4D they vanish in absence of background flux and depend linearly on the flux background. In 2D F-theory vacua, we find both a purely geometric and a flux dependent contribution to the anomalies. For anomalies to be cancelled, the flux dependent and the flux independent parts of the topological identities (4.61) and (4.78) must in fact hold separately, on any elliptically fibered Calabi-Yau 5-fold and for any gauge background satisfying the consistency relations reviewed in section 4.4. We verify these highly non-trivial anomaly relations in a concrete example fibration for all chirality inducing gauge backgrounds in section 4.9.

It has already been pointed out that, despite their rather different structure at first sight, the gauge anomalies in 6D and 4D boil down to one universal relation in the cohomology ring of an elliptic fibration over a general base, and similarly for the mixed gauge-gravitational anomalies [169].[2] This prompts the question if the 2D anomaly relations (4.61) and (4.78) are also equivalent to this universal relation governing the structure of anomalies in four and six dimensions. As we will see in section 4.10, assuming the 4D/6D relation of [169] implies the flux dependent part of (4.61) and (4.78) for a special class of gauge background. However, it remains for further investigation whether the precise relations extracted in [169] on Calabi-Yau 3-folds and 4-folds follow in turn by anomaly cancellation on Calabi-Yau 5-folds in full generality.

## 4.3. Anomalies in 2D $(0,2)$ Supergravities

Consider an $N = (0,2)$ supersymmetric theory in two dimensions with gauge group

$$G^{\rm tot} = \prod_{I=1}^{n_G} G_I \times \prod_{A=1}^{n_{U(1)}} U(1)_A \tag{4.1}$$

and matter fields in representations

$$\mathbf{R} = (\mathbf{r}^1, \dots, \mathbf{r}^{n_G})_{\underline{q}}\,. \tag{4.2}$$

Here $\mathbf{r}^I$ denotes an irreducible representation of the simple gauge group factor $G_I$ and $\underline{q} = (q_1, \dots, q_{n_{U(1)}})$ are the charges under the Abelian gauge group factors. We are interested in the structure of the gauge and gravitational anomalies in such a theory. These are induced by chiral matter at the 1-loop level. In a general $D$-dimensional quantum field theory, the gauge and gravitational anomalies can be described by a gauge invariant anomaly polynomial of degree $D/2+1$ in the gauge field strength $F$ and the curvature two-form $R$,

$$I_{D+2} = \sum_{\mathbf{R},s} n_s(\mathbf{R}) I_s(\mathbf{R})|_{D+2}\,, \tag{4.3}$$

where the sum is over all matter fields with spin $s$ which have zero-modes in representation $\mathbf{R}$ with multiplicity $n_s(\mathbf{R})$. In particular, a chiral fermion, corresponding to $s = 1/2$, contributes

[2]By contrast, the purely gravitational anomaly in 6D has no direct counterpart in 4D. See, however, [205].

with

$$I_{1/2}(\mathbf{R}) = -\mathrm{tr}_{\mathbf{R}}\, e^{-F}\, \widehat{A}(\mathrm{T})\,, \tag{4.4}$$

where $\widehat{A}(\mathrm{T})$ is the A-roof genus and $F$ denotes the hermitian gauge field strength. An anti-chiral fermion contributes with the opposite sign. For more details on our conventions we refer to appendix A. In $D = 2$ dimensions, the 1-loop anomaly polynomial from the charged matter sector is hence a 4-form. Correspondingly, the anomaly contribution from chiral and anti-chiral fermions in the theory sums up to

$$I_4 = \sum_{\mathbf{R}} (n_+(\mathbf{R}) - n_-(\mathbf{R})) \left( -\frac{1}{2}\mathrm{tr}_{\mathbf{R}}(F)^2 + \frac{1}{24} p_1(\mathrm{T}) \dim(\mathbf{R}) \right)\,, \tag{4.5}$$

where the first Pontryagin class of the tangent bundle is defined as $p_1(\mathrm{T}) = -\frac{1}{2}\mathrm{tr}R^2$. For future purposes we express the anomaly polynomial for the non-abelian, the abelian and the gravitational anomaly as

$$I_4|_{G_I} = -\mathcal{A}_I\, \mathrm{tr}_{\mathbf{fund}} F_I^2 = -\frac{1}{2}\sum_{\mathbf{r}^I} c^{(2)}_{\mathbf{r}^I}\, \chi(\mathbf{r}^I)\, \mathrm{tr}_{\mathbf{fund}} F_I^2 \tag{4.6}$$

$$I_4|_{AB} = -\mathcal{A}_{AB}\, F^A F^B = -\frac{1}{2}\sum_{\mathbf{R}} q_A(\mathbf{R})\, q_B(\mathbf{R}) \dim(\mathbf{R})\, \chi(\mathbf{R})\, F^A F^B \tag{4.7}$$

$$I_4|_{\mathrm{grav}} = \frac{1}{24}\mathcal{A}_{\mathrm{grav}}\, p_1(T) = \frac{1}{24}\sum_{\mathbf{R}} \chi(\mathbf{R}) \dim(\mathbf{R})\, p_1(\mathrm{T})\,, \tag{4.8}$$

with $\chi(\mathbf{R})$ denoting the chiral index of zero-modes in representation $\mathbf{R}$. In the first line we have related the trace in a representation $\mathbf{r}^I$ of the simple gauge group factor $G_I$ to the trace in the fundamental representation via

$$\mathrm{tr}_{\mathbf{r}^I} F^2 = c^{(2)}_{\mathbf{r}^I}\, \mathrm{tr}_{\mathbf{fund}} F^2\,. \tag{4.9}$$

In general, the 1-loop induced quantum anomaly need not be vanishing in a consistent theory provided the tree-level action contains gauge variant terms, the Green-Schwarz counter-terms, which cancel the anomaly encoded by $I_{D+2}$. For this cancellation to be possible, the 1-loop anomaly polynomial $I_{D+2}$ of the matter sector must factorize suitably. In two dimensions, the Green-Schwarz counterterms derive from gauge variant interactions of scalar fields. The structure of the possible Green-Schwarz terms in a general 2D $N = (0, 2)$ supersymmetric field theory has been analyzed in [201–203] (see [192, 204] for early work). In this chapter, however, we are interested in the specific 2D $N = (0, 2)$ effective theory obtained by compactification of F-theory on an elliptically fibered Calabi-Yau 5-fold [64, 65]. In these theories a gauge theory with gauge group (4.1) is coupled to a 2D $N = (0, 2)$ supergravity sector.[3] The latter contains a set of real axionic scalar fields $c^\alpha$ arising from the Kaluza-Klein (KK) reduction of the F-theory/Type IIB Ramond-Ramond forms $C_4$ [191].[4] As we will derive in detail in section 4.6, their pseudo-action

[3] The gauge theory in question arises from spacetime-filling 7-branes. In addition, the compactification contains spacetime-filling D3-branes, but the associated gauge fields are projected out due to $SL(2, \mathbb{Z})$ monodromies along the D3-brane worldvolume [64, 187].

[4] As discussed in [191], these scalars split into $n_+$ chiral and $n_-$ anti-chiral real scalars. Out of these $n_+$ pairs of real chiral and anti-chiral scalars form non-chiral real scalars, which constitute the imaginary part of the bosonic component of a corresponding number of 2D (0, 2) chiral multiplets. The remaining $\tau = n_- - n_+$

can be parametrized as

$$S_{\rm GS} = -\frac{1}{4}\int_{\mathbb{R}^{1,1}} g_{\alpha\beta}\, H^\alpha \wedge *H^\beta - \frac{1}{2}\int_{\mathbb{R}^{1,1}} \Omega_{\alpha\beta}\, c^\alpha \wedge X^\beta\,. \tag{4.10}$$

The structure of this action is completely analogous to the well-familiar generalized Green-Schwarz action [158, 159] of self-dual tensor fields in $D = 6$ (see e.g. [151]) and, in fact, $D = 10$ dimensions, with the role of the gauge invariant self-dual field strengths being played here by the 1-forms $H^\alpha = Dc^\alpha$. These are subject to the self-duality condition

$$g_{\alpha\beta} * H^\alpha = \Omega_{\alpha\beta} H^\beta\,. \tag{4.11}$$

The second term in the action constitutes the Green-Schwarz coupling, which is responsible for the non-standard Bianchi identity

$$dH^\alpha = X^\alpha\,, \tag{4.12}$$

where we used that $\Omega_{\alpha\beta}$ is a constant matrix. The Green-Schwarz couplings will be found to take the form

$$X^\beta = \Theta^\beta_A\, F^A \tag{4.13}$$

with $F^A$ the field strength associated with the gauge group factor $U(1)_A$ and with $\Theta^\beta_A$ depending on the background flux. This identifies $H^\alpha$ as

$$H^\alpha = Dc^\alpha = dc^\alpha + \Theta^\alpha_A A^A\,. \tag{4.14}$$

The axionic shift symmetry of the chiral scalars is gauged by the abelian vector $A^A$ according to the transformation rule

$$\begin{aligned} A^A \to &\quad A^A + d\lambda^A \\ c^\alpha \to &\quad c^\alpha - \Theta^\alpha_A \lambda^A \end{aligned} \tag{4.15}$$

such that the covariant derivative $Dc^\alpha$ is gauge invariant. As a result, the pseudo-action picks up a gauge variation of the form

$$\delta S_{GS} = \frac{1}{2}\int \Omega_{\alpha\beta}\, \Theta^\alpha_A \lambda^A\, X^\beta =: 2\pi \int_{\mathbb{R}^{1,1}} I_2^{(1),\rm GS}(\lambda)\,, \tag{4.16}$$

with $I_2^{(1),\rm GS}$ a gauge invariant 2-form. By the standard descent procedure, it defines an anomaly-polynomial $I_4^{\rm GS}$ encoding the contribution to the total anomaly from the Green-Schwarz sector. Concretely, the descent equations

$$I_4^{\rm GS} = dI_3^{\rm GS}, \qquad \delta_\lambda I_3^{\rm GS} = dI_2^{(1),\rm GS}(\lambda) \tag{4.17}$$

imply

$$2\pi I_4^{\rm GS} = \frac{1}{2}\Omega_{\alpha\beta} X^\alpha X^\beta = \frac{1}{2}\Omega_{\alpha\beta}\Theta^\alpha_A \Theta^\beta_B\, F^A F^B\,. \tag{4.18}$$

anti-chiral real scalars form 2D $(0, 2)$ tensor multiplets and contribute, together with the gravitino, to the gravitational anomaly at 1-loop level according to the general formulae reviewed in appendix A. This contribution to the 1-loop anomaly is in addition to the classical gauge variance of the Green-Schwarz action discussed in this section.

Consistency of the theory then requires that

$$I_4 + I_4^{\rm GS} = 0\,. \tag{4.19}$$

This is possible only if the non-abelian and gravitational anomalies vanish by themselves and the abelian anomalies factorise suitably. The resulting constraints on the spectrum take the following form:

$$\text{Non-abelian}: \qquad \frac{1}{2}\sum_{\mathbf{R}^I} \chi(\mathbf{r}^I)\, c^{(2)}_{\mathbf{r}^I} = 0 \tag{4.20a}$$

$$\text{Abelian}: \qquad \frac{1}{2}\sum_{\mathbf{R}} \dim(\mathbf{R})\,\chi(\mathbf{R})\,q_A(\mathbf{R})\,q_B(\mathbf{R}) = \frac{1}{4\pi}\Omega_{\alpha\beta}\Theta_A^\alpha\Theta_B^\beta \tag{4.20b}$$

$$\text{Gravitational}: \qquad \sum_{\mathbf{R}} \dim(\mathbf{R})\,\chi(\mathbf{R}) = 0\,. \tag{4.20c}$$

Note that, unlike in higher dimensions, the 2D GS mechanism operates entirely at the level of the abelian gauge group factors: In $(4k+2)$ dimensions the analogue of (4.14) is the gauge invariant field strength associated with the self-dual rank $(2k+1)$-tensor fields, and the correction term in the covariant action involves the Chern-Simons $(2k+2)$-forms associated with the gauge and diffeomorphism group. In 2D the Chern-Simons form is proportional to the trace over the gauge connection and must hence be abelian. Therefore the 2D non-abelian and gravitational anomalies from the chiral sector at 1-loop must vanish by themselves; likewise there can be no mixed gravitational-gauge anomalies induced at 1-loop.

Furthermore, let us point out that in the 2D $(0,2)$ theories of the type considered here the gauging (4.15) of the scalars is directly related to the anomalous Green-Schwarz coupling (4.13). This is a notable difference to the implementation of the Green-Schwarz mechanism in the more general 2D $(0,2)$ gauge theories of [201], where these two are in principle independent.

Before we proceed, we would like to comment on the scalar fields $c^\alpha$. In principle, all of the axionic scalar fields $c^\alpha$ obtained from the Type IIB RR fields $C_p$ can contribute to the Green-Schwarz mechansim. However, as in 6D and 4D F-theory compactifications, the gauging of the scalar fields from $C_2$ is encoded via a geometric Stückelberg mechanism in terms of non-harmonic forms, at least in the description via the dual M-theory [131]. In this work we will we will only focus on the Green-Schwarz mechanism associated with the scalar fields arising from the RR potential $C_4$, which will be seen to depend on the background flux.

## 4.4. F-theory on Elliptically Fibered Calabi-Yau Five-manifolds

In this section we provide some background material on $N = (0,2)$ supersymmetric compactifications of F-theory to two dimensions. The reader familiar with this type of constructions from [64, 65] can safely skip this summary.

### 4.4.1. Gauge symmetries and gauge backgrounds, and 3-branes

We consider a 2D $(0,2)$ supersymmetric theory describing a vacuum of F-theory compactified on an elliptically fibered Calabi-Yau 5-fold $X_5$ [64, 65] with projection

$$\pi : X_5 \to B_4\,. \tag{4.21}$$

The base $B_4$ is a smooth complex 4-dimensional Kähler manifold, which is to be identified with the physical compactification space of F-theory. Via F/M-theory duality, F-theory on $B_4$ is related to the supersymmetric quantum mechanics [141] obtained by compactification of M-theory on $X_5$.

For simplicity we assume that $X_5$ has a global section $[z=0]$ so that it can be described by a Weierstrass equation

$$y^2 = x^3 + f\,x\,z^4 + g\,z^6\,. \tag{4.22}$$

Here the projective coordinates $[x:y:z]$ parametrise the fiber ambient space $\mathbb{P}_{2,3,1}$ and $f, g$ are sections of the fourth and sixth power of the anti-canonical bundle $\bar{K}$ of the base. The discriminant locus

$$\Delta = 4f^3 + 27g^2 = 0 \tag{4.23}$$

specifies the location of the 7-branes. The non-abelian gauge group factors $G_I$ in (4.1) are associated with 7-branes wrapping divisors $W_I$, which are complex 3-dimensional components of the discriminant locus $\Delta = 0$ in the base. We assume that the Kodaira singularities in the fibre above $W_I$ admit a crepant resolution[5]

$$\widehat{\pi} : \widehat{X}_5 \to B_4\,. \tag{4.24}$$

The resolution replaces the singularity over $W_I$ by a chain of rational curves. After taking into account monodromy effects, which appear for non-simply laced groups, this allows one to identify a collection $\mathbb{P}^1_{i_I}$, $i_I = 1, \ldots, \mathrm{rk}(\mathfrak{g}_I)$ of independent rational curves in the resolved fiber which can be associated with the simple roots $\alpha_{i_I}$ of the Lie algebra $\mathfrak{g}_I$ underlying $G_I$ in the following sense: The fibration of $\mathbb{P}^1_{i_I}$ over $W_I$ - more precisely of the image of $\mathbb{P}^1_{i_I}$ under monodromies in the non-simply laced case - defines a resolution divisor $E_{i_I}$ with the property that

$$[E_{i_I}] \cdot [\mathbb{P}^1_{j_J}] = -\delta_{IJ}\, C_{i_I j_J}\,. \tag{4.25}$$

Here $[E_{i_I}]$ denotes the homology class of the divisor $E_{i_I}$ and unless noted otherwise, all intersection products are taken on $\widehat{X}_5$. The matrix $C_{i_I j_I}$ is the Cartan matrix of $\mathfrak{g}_I$ (in conventions where the entries on its diagonal are $+2$). Via duality with M-theory, M2-branes wrapping the fibral curves $\mathbb{P}^1_{j_J}$ give rise to states associated with the simple roots $-\alpha_{i_I}$, and the Cartan $U(1)_{i_I}$ gauge field arises by KK reduction of the M-theory 3-form as

$$C_3 = A_{i_I} \wedge [E_{i_I}] + .... \tag{4.26}$$

In this sense the resolution divisors $[E_{i_I}]$ can be identified with the generators $\mathcal{T}_{i_I}$ of the Cartan subgroup of $G_I$ in the so-called co-root basis, whose trace over the fundamental representation of $G_I$ is normalised such that

$$\mathrm{tr}_{\mathrm{fund}} \mathcal{T}_{i_I} \mathcal{T}_{j_J} = \delta_{IJ}\, \lambda_I\, \mathfrak{C}_{i_I j_I} \qquad \text{with} \qquad \mathfrak{C}_{i_I j_I} = \frac{2}{\lambda_I} \frac{1}{\langle \alpha_{j_I}, \alpha_{j_I} \rangle} C_{i_I j_I}\,. \tag{4.27}$$

The quantity $\lambda_I$ denotes the Dynkin index in the fundamental representation and is tabulated in Table 4.1. Note that for simply-laced groups $\mathfrak{C}_{i_I j_I} = C_{i_I j_I}$. The geometric manifestation of

[5] To avoid clutter we will mostly avoid the hat above $\pi$ in the sequel.

| $\mathfrak{g}$ | $A_n$ | $D_n$ | $B_n$ | $C_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 2 | 2 | 1 | 6 | 12 | 60 | 6 | 2 |

**Table 4.1.:** *Dynkin index of the fundamental representation for the simple Lie algebras.*

this identification is the important relation

$$\pi_*([E_{i_I}] \cdot [E_{j_J}]) = -\delta_{IJ}\, \mathfrak{C}_{i_I j_I}\, [W_I] = -\mathrm{Tr}\, \mathcal{T}_{i_I} \mathcal{T}_{j_J}\, [W_I]\,, \tag{4.28}$$

where Tr is related to the trace in the fundamental representation via

$$\mathrm{Tr} = \frac{1}{\lambda_I}\, \mathrm{tr}_{\mathbf{fund}}\,. \tag{4.29}$$

The push-forward $\pi_*([E_{i_I}] \cdot [E_{j_J}])$ to the base of the fibration is defined by requiring that

$$[E_{i_I}] \cdot_{\widehat{X}_5} [E_{j_J}] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] = \pi_*([E_{i_I}] \cdot_{\widehat{X}_5} [E_{j_J}]) \cdot_{B_4} [D^{\mathrm{b}}_\alpha] \cdot_{B_4} [D^{\mathrm{b}}_\beta] \cdot_{B_4} [D^{\mathrm{b}}_\gamma] \tag{4.30}$$

for any basis of vertical divisors $[D_\alpha] = \pi^*[D^{\mathrm{b}}_\alpha]$, where $D^{\mathrm{b}}_\alpha$ is a divisor on $B_4$.

Each non-Cartan Abelian gauge group factor $U(1)_A$ is associated with a global rational section $S_A$ of $\widehat{X}_5$ in addition to the zero-section $S_0$. To each $S_A$ one can assign an element $[U_A] \in \mathrm{CH}^1(\widehat{X}_5)$ through the Shioda map

$$U_A = S_A - S_0 - D_A + \sum_{i_I} k^A_{i_I} E_{i_I}\,. \tag{4.31}$$

The vertical divisor $D_A$ and the in general fractional coefficients $k^A_{i_I}$ are chosen such that $U_A$ satisfies the transversality conditions

$$\begin{aligned} &[U_A] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] \cdot_{\widehat{X}_5} [D_\delta] = 0 \qquad [U_A] \cdot_{\widehat{X}_5} [S_0] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] = 0 \\ &[U_A] \cdot_{\widehat{X}_5} [E_{i_I}] \cdot_{\widehat{X}_5} [D_\alpha] \cdot_{\widehat{X}_5} [D_\beta] \cdot_{\widehat{X}_5} [D_\gamma] = 0\,, \end{aligned} \tag{4.32}$$

which must hold for every vertical divisor $[D_\alpha] = \pi^* D^{\mathrm{b}}_\alpha$.

In analogy with the relation (4.28), one can define the so-called height pairing [123, 206]

$$\pi_*([U_A] \cdot_{\widehat{X}_5} [U_B]) = -\mathrm{Tr}\, \mathcal{T}_A \mathcal{T}_B\, [D_{AB}]\,. \tag{4.33}$$

The objects $\mathcal{T}_A$, $\mathcal{T}_B$ are the generators of $U(1)_A$ and $U(1)_B$ and $D_{AB}$ is a divisor on the base of the fibration. Unlike the divisor $W_I$, even for $A = B$ this divisor is not one of the irreducible components of the discriminant $\Delta$ (in the sense that $\Delta$ would factorise into the union of various irreducible such $D_{AA}$). Nonetheless, we will see that it plays a very analogous role for the structure of anomalies also for F-theory compactifications to 2D.

A crucial ingredient in F/M-theory compactifications on Calabi-Yau five-folds is the gauge background for the field strength $G_4 = dC_3$ of the M-theory 3-form potential field. As in compactifications to four dimensions, the full gauge background is an element of the Deligne cohomology group $H^4_D(\widehat{X}_5, \mathbb{Z}(2))$ and can be parametrized by equivalence classes of rational complex codimension-2-cycles [137, 138], which form the second Chow group $\mathrm{CH}^2(\widehat{X}_5)$. The field strength of $G_4$ as such takes values in $H^4(\widehat{X}_5)$. It is subject to the Freed-Witten quantization condition [142]

$$G_4 + \frac{1}{2}c_2(Y_5) \in H^4(\widehat{X}_5, \mathbb{Z}) \,. \tag{4.34}$$

In order to preserve two supercharges in the M/F-theory compactification on $\widehat{X}_5$, the $(3,1)$ and $(1,3)$ Hodge components of $H^4(\widehat{X}_5)$ must vanish [141] and hence

$$G_4 + \frac{1}{2}c_2(Y_5) \in H^4(\widehat{X}_5, \mathbb{Z}) \cap H^{2,2}(\widehat{X}_5) \,. \tag{4.35}$$

By F/M-duality, the $G_4$ fluxes are subject to the transversality constraints

$$\int_{\widehat{X}_5} G_4 \wedge S_0 \wedge \pi^*\omega_4 = 0 \quad \text{and} \qquad \int_{\widehat{X}_5} G_4 \wedge \pi^*\omega_6 = 0 \,, \qquad \forall\, \omega_4 \in H^4(B_4),\ \omega_6 \in H^6(B_4) \,. \tag{4.36}$$

If this flux satisfies in addition the constraint

$$\int_{\widehat{X}_5} G_4 \wedge E_{i_I} \wedge \pi^*\omega_4 = 0 \tag{4.37}$$

it leaves the gauge group factor $G_I$ unbroken.

Higher curvature corrections in the M-theory effective action induce a curvature dependent tadpole for the M-theory 3-form $C_3$. In the dual F-theory these curvature corrections subsume the curvature contributions to the Chern-Simons action of the 7-branes (including, in the perturbative limit, the orientifold planes). In a consistent M-theory vacuum this tadpole must be cancelled by the inclusion of background flux $G_4$ and/or by M2-branes wrapping a curve class on $\widehat{X}_5$ determined by the tadpole equation [141]. The projection of this curve class to the base $B_4$ describes[6] , in the dual F-theory, the class wrapped by background D3-branes filling in addition the extended directions along $\mathbb{R}^{1,1}$. The projected class is given by [64, 141]

$$[C] = \frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4 \cdot_{\widehat{X}_5} G_4) \,. \tag{4.38}$$

### 4.4.2. Matter spectrum from F-theory compactification on CY 5-folds

The charged chiral matter fields whose contributions to the 1-loop anomalies we will be studying arise from three sources [64, 65]: 7-brane bulk matter propagating along the non-abelian divisors $W_I$, 7-brane codimension-two matter localised along the intersections of various discriminant components or self-intersections of the discriminant, and finally Fermi multiplets at the pointlike intersection of D3-branes with the 7-branes. Due to the chiral nature of the 2D $(0,2)$ theory, all three types of matter are chiral even for vanishing gauge backgrounds.

The bulk matter fields transform, in the absence of gauge flux, in the adjoint representation of $G_I$. In the dual M-theory quantum mechanics, this matter arises from M2-branes wrapping suitable combinations of resolution $\mathbb{P}^1_{i_I}$ in the fiber over $W_I$. For non-vanishing gauge backgrounds, which can be described by a non-trivial principal gauge bundle $L$, the original gauge group $G_I$ can be broken into a product of some sub-groups. The spectrum decomposea into irreducible representations $\mathbf{R}$ of the unbroken gauge factors

$$G_I \quad \rightarrow \quad H_I \tag{4.39}$$

[6] The M2-brane states along the fibral component of this class are related to momentum modes along the circle $S^1$ arising in F/M-theory duality [191].

$$\mathbf{Adj}(G_I) \quad \rightarrow \quad \mathbf{Adj}(H_I) \oplus \bigoplus_{\mathbf{R}} \mathbf{R} \tag{4.40}$$

Note that if $\mathbf{R} \neq \bar{\mathbf{R}}$, each representation is accompanied by its complex conjugate. The matter fields organise into 2D $(0,2)$ chiral multiplets, which contain one complex boson and a complex chiral Weyl fermion, as well as Fermi multiplets, which contain one complex anti-chiral Weyl fermion. Each of these matter fields is counted by a certain cohomology group on $W_I$ involving the vector bundle $L_{\mathbf{R}}$. The chiral index of massless matter in a given complex representation, defined as the difference of chiral and anti-chiral fermions in complex representation $\mathbf{R}$, is then given by [64, 65]

$$\chi(\mathbf{R}) = -\int_{W_I} c_1(W_I)\left(\frac{1}{12}\mathrm{rk}(L_{\mathbf{R}})\, c_2(W_I) + \mathrm{ch}_2(L_{\mathbf{R}})\right) . \tag{4.41}$$

For real representations, this expression is to be multiplied with a factor of $\frac{1}{2}$. In particular, the chiral index of the adjoint representation depends purely on the geometry and takes the form $\chi(\mathbf{Adj}(H_I)) = -\frac{1}{24}\int_{W_I} c_1(W_I)c_2(W_I)$.

Extra matter states in representation $\mathbf{R}$ of $G^{\mathrm{tot}}$ localizes on complex 2-dimensional surfaces $C_{\mathbf{R}}$ on $B_4$. This occurs whenever some of the rational curves $\mathbb{P}^1_{i_I}$ in the fiber split over $C_{\mathbf{R}}$. Group theoretically, this signifies the splitting of the associated simple roots into weights of representation $\mathbf{R}$.

The associated charged matter fields arise from M2-branes wrapped on suitable linear combinations of fibral curves over $C_{\mathbf{R}}$, which in fact span the weight lattice of the gauge theory. Hence to each state in representation $\mathbf{R}$ we can associate a matter 3-cycle $S^a_{\mathbf{R}}$ which is given by a linear combination of fibral curves over $C_{\mathbf{R}}$ and carries a weight vector $\beta^a_{i_I}, a = 1, ..., \dim(\mathbf{R})$, such that

$$\pi_*([E_{i_I}] \cdot [S^a_{\mathbf{R}}]) = \beta^a_{i_I}[C_{\mathbf{R}}] \,. \tag{4.42}$$

These matter states also organize both into chiral and Fermi multiplets and are counted by cohomology groups of a vector bundle $L_{\mathbf{R}}$ which derives from the gauge background. If the surface $C_{\mathbf{R}}$ on $B_4$ is smooth, the chiral index of this type of matter follows from an index theorem as [64, 65]

$$\chi(\mathbf{R}) = \int_{C_{\mathbf{R}}} \left( c_1^2(C_{\mathbf{R}})\left(\frac{1}{12} - \frac{1}{8}\mathrm{rk}(L_{\mathbf{R}})\right) + \frac{1}{12}c_2(C_{\mathbf{R}}) + \left(\frac{1}{2}c_1^2(L_{\mathbf{R}}) - c_2(L_{\mathbf{R}})\right)\right) . \tag{4.43}$$

Otherwise one has to perform a suitable normalisation in order to be able to apply the index theorem, and this will lead to correction terms as exemplified in [64].

The third type of massless matter arises from 3-7 string states at the intersection of the 7-branes with the spacetime-filling D3-branes wrapping the curve class $[C]$ in (4.38). Matter in the 3-7 sector comes in 2D (0,2) Fermi multiplets [64, 65]. In purely perturbative setups, each intersection point of $[C]$ with one of the D7-branes carries a single Fermi multiplet in the fundamental representation of the D7-brane gauge group. However, monodromy effects along the 3-brane worldvolume considerably obscure such a simple interpretation of the 3-7 modes in non-perturbative setups [64, 187]. As one of our results, we will see how the structure of 2D anomalies sheds new light on the structure of 3-7 modes, including, in particular, their charges under the non-Cartan abelian gauge factors.

## 4.5. Anomaly equations in F-theory on Calabi-Yau 5-folds

In this section we present closed expressions for the anomaly cancellation conditions in 2D $(0,2)$ F-theory vacua. We begin in section 4.5.1 by deriving a formula for the chiral index of charged matter states in the presence of 4-form flux $G_4$ in the dual M-theory, which is uniformly valid for the bulk and the localised 7-7 modes. We also shed some more light on the counting of 3-7 modes. Together with the Green-Schwarz counterterms this leads to formula (4.61) for the cancellation of all gauge anomalies. In section 4.5.2 we extend the gravitational anomaly cancellation conditions of [191] to situations with non-trivial 7-branes and fluxes, leading us to condition (4.77).

### 4.5.1. Gauge anomalies, Green-Schwarz terms and the 3-7 sector

Recall from the previous section that in this chapter we assume the existence of a smooth crepant resolution $\widehat{X}_5$, which describes the dual M-theory on its Coulomb branch. This forces us, as usual in this context, to restrict ourselves to Abelian gauge backgrounds $G_4$. In particular, the vector bundles appearing in the expressions (4.41) and (4.43) are complex line bundles.

For simplicity of presentation we first assume that the gauge flux $G_4$ does not break any of the non-abelian gauge group factors. The chiral index (4.43) of the localised matter can be split into a purely geometric and a flux dependent contribution

$$\begin{aligned}\chi(\mathbf{R}) &= \chi_{\text{geom}}(\mathbf{R}) + \chi_{\text{flux}}(\mathbf{R}) \\ \chi_{\text{geom}}(\mathbf{R}) &= -\frac{1}{12}\int_{C_{\mathbf{R}}} \text{ch}_2(C_{\mathbf{R}}) = \frac{1}{12}\int_{C_{\mathbf{R}}} c_2(C_{\mathbf{R}}) - \frac{1}{2}c_1^2(C_{\mathbf{R}}) \\ \chi_{\text{flux}}(\mathbf{R}) &= \int_{C_{\mathbf{R}}} \frac{1}{2}c_1^2(L_{\mathbf{R}})\,.\end{aligned} \tag{4.44}$$

We stress that this expression is correct provided the matter surfaces $C_{\mathbf{R}}$ on $B_4$ are smooth. The line bundle $L_{\mathbf{R}}$ on $C_{\mathbf{R}}$ to which a state with weight vector $\beta^a(\mathbf{R})$ couples is obtained from $G_4$ by first integrating $G_4$ over the fiber of the matter 3-cycle $S^a_{\mathbf{R}}$ and then projecting onto the surface $C_{\mathbf{R}}$. This gives rise to a divisor class on $C_{\mathbf{R}}$ which is to be identified, similarly to the procedure in F-theory on Calabi-Yau 4-folds [137, 138], with

$$c_1(L_{\mathbf{R}}) = \pi_*(G_4 \cdot S^a_{\mathbf{R}})\,. \tag{4.45}$$

Note that for gauge invariant flux, the result is the same for each of the matter 3-cycles $S^a_{\mathbf{R}}$ and hence correctly defines the line bundle associated with representation $\mathbf{R}$. This allows us to rewrite $\chi_{\text{flux}}(\mathbf{R})$ explicitly in terms of $G_4$ as

$$\chi_{\text{flux}}(\mathbf{R}) = \frac{1}{2}\pi_*(G_4 \cdot S^a_{\mathbf{R}}) \cdot_{C_{\mathbf{R}}} \pi_*(G_4 \cdot S^a_{\mathbf{R}})\,, \tag{4.46}$$

where $\cdot_{C_{\mathbf{R}}}$ denotes the intersection product on $C_{\mathbf{R}}$.

Next, consider the bulk modes. For gauge invariant flux, this sector contributes only states in the adjoint representation of $G_I$ (which due to the quadratic nature of the anomalies nonetheless contribute to the anomaly), and according to (4.41) their chiral index is given by

$$\chi_{\text{bulk}}(\mathbf{R} = \mathbf{adj}_I) = -\frac{1}{24}\int_{W_I} c_1(W_I)c_2(W_I)\,. \tag{4.47}$$

It is useful to note that $\chi_{\rm bulk}(\mathbf{R})$ is formally identical to the flux-independent part of the chirality of a localised state whose matter locus is given by the canonical divisor on $W_I$, i.e. the complex 2-cycle on $W_I$ in the class

$$[C_{\rm can}] = -c_1(W_I) = +c_1(K_{W_I})\,. \tag{4.48}$$

Indeed, by adjunction, using the short exact sequence

$$0 \to T_{C_{\rm can}} \to T_{W_I} \to N_{C_{\rm can}/W_I} \to 0 \tag{4.49}$$

and the resulting relation

$$c(T_{C_{\rm can}}) = c(T_{W_I})/c(N_{C_{\rm can}/W_I}) = (1 + c_1(W_I) + c_2(W_I))/(1 - c_1(W_I)), \tag{4.50}$$

one computes

$$c_1(C_{\rm can}) \;=\; 2c_1(W_I) \tag{4.51}$$

$$c_2(C_{\rm can}) \;=\; c_2(W_I) + 2c_1^2(W_I)\,. \tag{4.52}$$

This implies that

$$\int_{C_{\rm can}} \frac{1}{12}(c_2(C_{\rm can}) - \frac{1}{2}c_1^2(C_{\rm can})) = -\frac{1}{12}\int_{W_I} c_1(W_I)\cdot c_2(W_I). \tag{4.53}$$

The additional factor of $\frac{1}{2}$ in (4.47) is due to the fact that the adjoint is a real representation. More generally, and in complete analogy to the description of bulk modes in compactifications on Calabi-Yau 4-folds [138], we can associate to a bulk matter state associated with the root $\rho_I$ the 3-cycle

$$S^{\rho_I} = \sum_{i_I} \widehat{a}_{i_I} E_{i_I}|_{K_{W_I}}\,. \tag{4.54}$$

The parameters $\widehat{a}_{i_I}$ are related to the coefficients in the expansion of the root $\rho_I$ in terms of the simple roots $\alpha_{i_I}$.[7] Geometrically, the fiber of $S^{\rho_I}$ is given by the corresponding linear combination of fibral rational curves $\mathbb{P}^1_{i_I}$. An M2-brane wrapped along this linear combination of fibral curves gives rise to a state whose Cartan charges are given precisely by the root $\rho_I$. For gauge invariant flux satisfying (4.37), the line bundle $\pi_*(S^{\rho_I}\cdot G_4)$ vanishes by construction. Hence the expression for the bulk and the localised chirality are completely analogous and both types of matter will from now on be treated on the same footing.

This conclusion persists if the gauge background breaks some or all of the simple gauge group factors $G_I$. In this case, the adjoint representation for the bulk matter or the representations associated with the localised matter decompose into irreducible representations of the unbroken subgroup. The operation (4.45) now leads to a well-defined line bundle for each of these individual representations, for bulk and localised matter alike.

Next, we consider the contribution from the 3-7 modes. As it turns out, to each representation $\mathbf{R}$ one can associate a divisor $D_{37}(\mathbf{R})$ on $B_4$ such that the chiral index of 3-7 states in

[7] For simply laced Lie algebras, $\rho_I = \sum_{i_I} \widehat{a}_{i_I}\alpha_{i_I}$. For non-simply laced Lie algebras, fractional corrections must be included to take into account monodromy effects, as explained e.g. in appendix A of [123].

representation $\mathbf{R}$ is given by

$$\chi_{3-7}(\mathbf{R}) = -\Big(\frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4\cdot G_4)\Big)\cdot_{B_4} D_{37}(\mathbf{R})\,. \tag{4.55}$$

The expression in brackets is the curve class $[C]$, defined in (4.38), wrapped by the spacetime-filling D3-branes. For instance, for a perturbative gauge group $G_I = SU(N)$, each intersection point of $[C]$ with the 7-brane divisor $W_I$ hosts a (negative chirality) Fermi multiplet in representation $\mathbf{R} = (\mathbf{N})$ [64, 65] and therefore $D_{37}(\mathbf{R} = (\mathbf{N})) = W_I$. For non-perturbative gauge groups and for Abelian non-Cartan groups $U(1)_A$ determining the representation and charge of the 3-7 strings from first principles is more obscure due to subtle $SL(2,\mathbb{Z})$ monodromy effects on the worldvolume of the D3-brane along $C$ [187]. However, in the next section we will derive that in the presence of extra $U(1)_A$ gauge group factors the net contribution to the $U(1)_A - U(1)_B$ anomaly (4.7) from the 3-7 sector takes the form

$$\begin{aligned}\mathcal{A}_{AB}|_{3-7} &= \frac{1}{2}\sum_{\mathbf{R},3-7} q_A(\mathbf{R})\, q_B(\mathbf{R})\dim(\mathbf{R})\,\chi_{3-7}(\mathbf{R}) &(4.56)\\ &= \frac{1}{2}\Big(\frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4\cdot G_4)\Big)\cdot_{B_4}\pi_*(U_A\cdot U_B)\,. &(4.57)\end{aligned}$$

Here we recall that $U_A$ and $U_B$ generate the respective $U(1)$ factors via the Shioda map (4.31) and that the height-pairing $\pi_*(U_A\cdot U_B)$ had been introduced in (4.33). More generally, our results imply that the right-hand side correctly captures the contribution to the anomaly also of the Cartan $U(1)$ group for non-perturbative gauge groups. Let us introduce the notation

$$\mathrm{span}_{\mathbb{C}}\{\mathfrak{F}_\Sigma\} = \mathrm{span}_{\mathbb{C}}\{E_{i_I}, U_A\} \tag{4.58}$$

to collectively denote set of divisors generating any of the Cartan $U(1)_{i_I}$ or non-Cartan $U(1)_A$ gauge symmetries. Then our claim is that the contribution to the gauge anomaly due to 3-7 modes can be summarized as

$$\mathcal{A}_{\Lambda\Sigma}|_{3-7} = \frac{1}{2}\sum_{\mathbf{R},a}\beta^a_\Lambda(\mathbf{R})\,\beta^a_\Sigma(\mathbf{R})\,\chi_{3-7}(\mathbf{R}) = \frac{1}{2}\Big(\frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4\cdot G_4)\Big)\cdot_{B_4}\pi_*(\mathfrak{F}_\Lambda\cdot\mathfrak{F}_\Sigma) \tag{4.59}$$

If the index $\Lambda = i_I$ refers to a Cartan $U(1)_{i_I}$, the object $\beta^a_{i_I}(\mathbf{R})$ denotes the weights associated with representation $\mathbf{R}$ with respect this $U(1)_{i_I}$, and for $\Lambda = A$ we define $\beta^a_A(\mathbf{R}) = q_A(\mathbf{R})$. We will come back to the interpretation of this formula at the end of this section.

As the final ingredient we will derive, in section 4.6, the Green-Schwarz counterterms appearing on the righthand side of (4.20b). These are found to be purely flux dependent and of the form

$$\frac{1}{4\pi}\Omega_{\alpha\beta}\Theta^\alpha_\Sigma\Theta^\beta_\Lambda = \frac{1}{2}\pi_*(G_4\cdot\mathfrak{F}_\Sigma)\cdot_{B_4}\pi_*(G_4\cdot\mathfrak{F}_\Lambda)\,. \tag{4.60}$$

For instance, if we let $\mathfrak{F}_\Lambda = U_A$, $\mathfrak{F}_\Sigma = U_B$ refer to non-Cartan Abelian groups, then this describes the Green-Schwarz counterterms for the $U(1)_A - U(1)_B$ anomalies. For $\mathfrak{F}_\Lambda = E_{i_I}$, $\mathfrak{F}_\Sigma = E_{j_I}$, the right-hand side is non-vanishing only if the gauge background $G_4$ breaks the simple gauge group factors $G_I$ and $G_J$, in which case it computes the counterterms for the $U(1)_{i_I} - U(1)_{j_J}$ anomaly. For gauge invariant flux, on the other hand, no such Green-Schwarz terms are induced, in agreement with expectations.

With this preparation we can now rewrite the gauge anomaly equations (4.20a), (4.20b) in a rather suggestive form. Since the anomaly equations must hold for arbitrary gauge background $G_4$ and since the flux independent terms only give a constant off-set, the flux dependent and the flux independent contributions to the anomalies must vanish separately. The requirement (4.20a), (4.20b) of cancellation of all gauge anomalies therefore results in two independent identities:

$$0 = \sum_{\mathbf{R},a} \beta^a_\Lambda(\mathbf{R})\, \beta^a_\Sigma(\mathbf{R}) \int_{C_\mathbf{R}} \mathrm{ch}_2(C_\mathbf{R}) - \frac{1}{2}\, \pi_*(c_4(\widehat{X}_5)) \cdot \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) \tag{4.61a}$$

$$\begin{aligned} 0 = \sum_{\mathbf{R},a} & \beta^a_\Lambda(\mathbf{R})\, \beta^a_\Sigma(\mathbf{R})\, \pi_*(G_4 \cdot S^a_\mathbf{R}) \cdot_{C_\mathbf{R}} \pi_*(G_4 \cdot S^a_\mathbf{R}) \\ & - \big(\pi_*(G_4 \cdot G_4) \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) \\ & + \pi_*(G_4 \cdot \mathfrak{F}_\Sigma) \cdot_{B_4} \pi_*(G_4 \cdot \mathfrak{F}_\Lambda) + \pi_*(G_4 \cdot \mathfrak{F}_\Lambda) \cdot_{B_4} \pi_*(\mathfrak{F}_\Sigma \cdot G_4)\big)\,. \end{aligned} \tag{4.61b}$$

The two terms in (4.61a) respectively represent the flux independent anomaly contribution from the 7-7 sector, (4.44), and from the 3-7 sector, (4.59). In (4.61b) we have collected the flux dependent 3-7 and the Green-Schwarz contribution to the anomaly in the brackets in the second and third line to illustrate the striking formal similarity between them. We will understand this similarity in the next section.

Let us now come back to the interpretation of (4.59). For $\mathfrak{F}_\Lambda = E_{i_I}$, $\mathfrak{F}_\Sigma = E_{j_I}$ this equation allows us to deduce the net contribution to the anomalies due to 3-7 strings charged under the non-abelian gauge group factors, which, as noted already, can be rather obscure due to monodromy effects. To interpret this expression, recall the crucial identity (4.28). If we assume that each geometric intersection point $[C] \cdot_{B_4} W_I$ hosts an (anti-chiral) Fermi multiplet in representation $\mathbf{R}$, then for consistency this representation must satisfy

$$\sum_a \beta^a_{i_I}(\mathbf{R}) \beta^a_{j_I}(\mathbf{R}) \stackrel{!}{=} \mathfrak{C}_{i_I j_I}\,. \tag{4.62}$$

This is to be contrasted with the fact that for any representation $\mathbf{R}$ of a simple group $G_I$

$$\sum_a \beta^a_{i_I}(\mathbf{R}) \beta^a_{j_I}(\mathbf{R}) = \mathrm{tr}_\mathbf{R} \mathcal{T}_{i_I} \mathcal{T}_{j_I} = \lambda_I\, c^{(2)}_\mathbf{R}\, \mathfrak{C}_{i_I j_I} \tag{4.63}$$

with $\mathcal{T}_{i_I}$ denoting the Cartan generators in the coroot basis. The Dynkin index $\lambda_I$ for the fundamental representation of $G_I$ is collected, for all simple groups, in Table 4.1, and $c^{(2)}_\mathbf{R}$ normalizes the trace with respect to the fundamental representation as in (4.9). By definition, the smallest value of $c^{(2)}_\mathbf{R}$ occurs for the fundamental representation $c^{(2)}_\mathbf{fund} = 1$. Hence unless $\lambda_I = 1$ or $\lambda_I = 2$, the interpretation in terms of 3-7 modes necessarily involves 'fractional' Fermi multiplets.[8] This is in agreement with the observation of [64] that e.g. for $G_I = E_6$, the net contribution to the anomaly from the 3-7 sectors corresponds to that of a $\frac{1}{6}$-fractional Fermi multiplet per intersection point.

[8] The case $\lambda_I = 2$ requires, for consistency, that the fundamental representation be real and hence contributes with a factor of $\frac{1}{2}$ to compensate for $\lambda_I$. Table 4.1 confirms that this is indeed the case for all simple algebras with $\lambda_I = 2$.

### 4.5.2. Gravitational Anomaly

The gravitational anomaly for F-theory compactified on a smooth Weierstrass model $X_5$ without any 7-brane gauge group and background flux has already been discussed in [191]. The anomaly polynomial receives contributions from the moduli sector, from the 2D $(0,2)$ supergravity multiplet as well as from the 3-7 sector,

$$\begin{aligned} I_{4,\text{grav}} &= \frac{1}{24} p_1(T)\left(\mathcal{A}_{\text{grav}}|_{\text{mod}} + \mathcal{A}_{\text{grav}}|_{\text{uni}} + \mathcal{A}_{\text{grav}}|_{3-7}\right) && (4.64)\\ \mathcal{A}_{\text{grav}}|_{\text{mod}} &= -\tau(B_4) + \chi_1(X_5) - 2\chi_1(B_4), && (4.65)\\ \mathcal{A}_{\text{grav}}|_{\text{uni}} &= 24 && (4.66)\\ \mathcal{A}_{\text{grav}}|_{3-7} &= -6c_1(B_4)\cdot[C]\,. && (4.67)\end{aligned}$$

Note that $\mathcal{A}_{\text{grav}}|_{\text{mod}}$ includes what would be called in Type IIB language the contributions from the closed string moduli sector, from the moduli associated with the 7-branes (which however by assumption carry no gauge group), and from $\tau(B_4)$ many 2D $(0,2)$ tensor multiplets. Here

$$\chi_q(M) = \sum_{p=1}^{\dim(M)} (-1)^p h^{p,q}(M) \tag{4.68}$$

and

$$\chi_1(X_5) = -\frac{1}{24}\int_{X_5} c_5(X_5) = \int_{B_4} (90c_1^4 + 3c_1^2 c_2 - \frac{1}{2}c_1 c_3) \tag{4.69}$$

with $c_i = c_1(B_4)$. Furthermore the signature $\tau(B_4)$ counts the difference of self-dual and anti-self-dual 4-forms on $B_4$ and is related to the Hodge numbers of $B_4$ as

$$\tau(B_4) = b_4^+(B_4) - b_4^-(B_4) = 48 + 2h^{1,1}(B_4) + 2h^{3,1}(B_4) - 2h^{2,1}(B_4)\,. \tag{4.70}$$

The D3-brane class appearing fixed by the tadpole on a smooth Weierstrass model without flux is $[C] = \frac{1}{24}\pi_*(c_4(X_5))$. As shown in [191] with the help of various index theorems, the total anomaly can be evaluated as

$$I_{4,\text{grav}} = \frac{1}{24}p_1(T)\left(-24\chi_0(B_4) + 24\right) \equiv 0\,, \tag{4.71}$$

where the last equality holds because $h^{0,i}(B_4)$ for $i \neq 0$ if $B_4$ is to admit a smooth Calabi-Yau Weierstrass fibration over it.

Suppose now that the fibration contains in addition a non-trivial 7-brane gauge group and charged 7-7 matter, and let us also switch a non-trivial flux background $G_4$. For simplicity assume first that the supersymmetry condition that $G_4$ be of pure $(2,2)$ Hodge type [141] does not constrain the moduli of the compactification. In analogy with $G_4$ flux on Calabi-Yau 4-folds, this is guaranteed whenever $G_4 \in H^{2,2}_{\text{vert}}(\widehat{X}_5)$, the primary vertical subspace of $H^{2,2}(\widehat{X}_5)$ generated by products of $(1,1)$ forms.[9] In this situation the gravitational anomaly generalizes as follows: First, we must now work on the resolution $\widehat{X}_5$ of the singular Weierstrass model describing the more general 7-brane configuration. In particular the D3-brane curve class changes

[9] The space of $(2,2)$ forms on Calabi-Yau 5-folds deserves further study beyond the scope of this chapter. In particular it remains to investigate in more detail whether a similar split into horizontal and vertical subspaces exists as on Calabi-Yau 4-folds. In any event if $G_4$ is a sum of $(2,2)$ forms obtained as the product of two $(1,1)$ forms, the Hodge type does not vary.

to $[C] = \frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4 \cdot G_4)$ with $c_4(\widehat{X}_5)$ evaluated on the resolved space $\widehat{X}_5$. Second, we must add the anomaly contribution from the non-trivial 7-7 sector. This sector includes the localised matter in some representation $\mathbf{R}$ of the total gauge group as well as the bulk matter in the adjoint representation (or its decomposition if the flux background breaks the non-abelian gauge symmetry). Each massless multiplet in the bulk sector contributes $\dim(\mathbf{adj})$ many states to the anomaly. Of these, $\mathrm{rk}(G)$ many states are associated with the Cartan subgroup of the gauge group and are in fact encoded already in the contribution from the 'moduli sector'. More precisely, if we replace in (4.65) the contribution $\chi_1(X_5)$ by $\chi_1(\widehat{X}_5)$, the resulting expression $\mathcal{A}_{\text{grav}}|_{\text{mod}}$ now includes the anomaly from the $\mathrm{rk}(G) = h^{1,1}(\widehat{X}_5) - (h^{1,1}(B_4) - 1)$ many vector multiplets associated with the Cartan subgroup as well as the 'open string moduli' in the Cartan, which enter the values of $h^{1,p}(\widehat{X}_5)$. As a result, the total gravitational anomaly polynomial is now

$$I_{4,\text{grav}} = \frac{1}{24} p_1(T) \left(\mathcal{A}_{\text{grav}}|_{7-7} + \mathcal{A}_{\text{grav}}|_{\text{mod}} + \mathcal{A}_{\text{grav}}|_{\text{uni}} + \mathcal{A}_{\text{grav}}|_{3-7}\right) \tag{4.72}$$

with the individual contributions

$$\mathcal{A}_{\text{grav}}|_{7-7} = \sum_{\mathbf{R}} \dim(\mathbf{R})\chi(\mathbf{R}) - \mathrm{rk}(G)\chi(\mathbf{adj}) \tag{4.73}$$

$$\mathcal{A}_{\text{grav}}|_{\text{mod}} = -\tau(B_4) + \chi_1(\widehat{X}_5) - 2\chi_1(B_4), \tag{4.74}$$

$$\mathcal{A}_{\text{grav}}|_{\text{uni}} = 24 \tag{4.75}$$

$$\mathcal{A}_{\text{grav}}|_{3-7} = -6c_1(B_4) \cdot \left(\frac{1}{24}\pi_*(c_4(\widehat{X}_5)) - \frac{1}{2}\pi_*(G_4 \cdot G_4)\right). \tag{4.76}$$

Note that the topological invariants $\chi_1(\widehat{X}_5)$ and $c_4(\widehat{X}_5)$ contain correction terms in addition to the base classes appearing for the case of a smooth Weierstrass model which depend on the resolution divisors and extra sections (if present).

The vanishing of the total gravitational anomaly implies that these individual contributions must cancel each other,

$$\mathcal{A}_{\text{grav}}|_{7-7} + \mathcal{A}_{\text{grav}}|_{\text{mod}} + \mathcal{A}_{\text{grav}}|_{\text{uni}} + \mathcal{A}_{\text{grav}}|_{3-7} = 0\,. \tag{4.77}$$

This leads to a set of topological identities which must hold for every resolution $\widehat{X}_5$ of an elliptically fibered Calabi-Yau 5-fold, and for every consistent configuration of background fluxes thereon, as specified above. Note that the flux background enters not only through the 3-brane class in $\mathcal{A}_{3-7}$, but also because the chiral indices in the 7-brane sector split as $\chi(\mathbf{R}) = \chi(\mathbf{R})|_{\text{geom}} + \chi(\mathbf{R})|_{\text{flux}}$ as in (4.44). In principle, if the Hodge type of $G_4$ were to vary over the moduli space, the supersymmetry condition $G_4 \in H^{2,2}(\widehat{X}_5)$ would induce a potential for some of the moduli [141] and hence modify the number of uncharged massless fields. According to our assumptions, this does not occur for the choice of flux considered here and the uncharged sector contributes to the anomaly as above.

Then the anomaly equations split into the independent sets of equations

$$\sum_{\mathbf{R}} \dim(\mathbf{R})\chi(\mathbf{R})|_{\text{geom}} - \text{rk}(G)\chi(\mathbf{adj})|_{\text{geom}} - \tau(B_4) + \chi_1(\widehat{X}_5) - 2\chi_1(B_4) + 24 - \frac{1}{4}c_1(B_4)\cdot\left(\pi_* c_4(\widehat{X}_5)\right) = 0 \tag{4.78a}$$

$$-6c_1\cdot\pi_*(G_4\cdot G_4) = \sum_{\mathbf{R},a} \pi_*(G_4\cdot S^a_{\mathbf{R}})\cdot_{C_{\mathbf{R}}} \pi_*(G_4\cdot S^a_{\mathbf{R}}) \tag{4.78b}$$

In the second equation, which accounts for the flux dependent anomaly contribution, we do not need to treat the 7-brane states in the Cartan separately as their chirality is not affected by the flux background.

The flux independent contribution can be analysed further if the fibration $\widehat{X}_5$ is smoothly connected to a smooth Weierstrass model $X_5$. In the terminology of [207], this means that the F-theory model does not contain any non-Higgsable clusters and hence after the blowup of the resolution divisors the gauge symmetry can be completely Higgsed. In that case we know already from (4.71) that the anomalies on the resulting smooth Weierstrass model $X_5$ cancel for $G_4 = 0$. Let us therefore define

$$\Delta[C] = \frac{1}{24}\left(\pi_* c_4(\widehat{X}_5) - \pi_* c_4(X_5)\right) = \frac{1}{24}c_4(\widehat{X}_5)|_{B_4} - (15c_1^3 + \frac{1}{2}c_1c_2) \tag{4.79}$$

$$\Delta\chi_1 = -\frac{1}{24}\left(\pi_* c_5(\widehat{X}_5) - \pi_* c_5(X_5)\right) = -\frac{1}{24}\pi_* c_5(\widehat{X}_5) - (90c_1^4 + 3c_1^2c_2 - \frac{1}{2}c_1c_3)\,. \tag{4.80}$$

The anomaly equations can then be rewritten as

$$-6c_1\cdot\Delta[C] + \Delta\chi_1 = -\frac{1}{12}\sum_{\mathbf{R}}\dim(\mathbf{R})\int_{C_{\mathbf{R}}}\text{ch}_2(C_{\mathbf{R}}) + \frac{1}{12}\text{rk}(G)\int_{C(\mathbf{adj})}\text{ch}_2(C(\mathbf{adj})) \tag{4.81a}$$

$$-6c_1\cdot\pi_*(G_4\cdot G_4) = \sum_{\mathbf{R},a}\pi_*(G_4\cdot S^a_{\mathbf{R}})\cdot_{C_{\mathbf{R}}}\pi_*(G_4\cdot S^a_{\mathbf{R}}) \tag{4.81b}$$

It is interesting to speculate about the effect of $G_4$ fluxes which are not automatically of $(2,2)$ Hodge type. The supersymmetry condition (4.35) is reflected in a dynamical potential which is expected to render some of the supergravity moduli massive [141]. The resulting change in the gravitational anomaly compared to the fluxless geometry must be compensated by a suitable modification of the remaining uncharged spectrum. Indeed, the flux contributes at the same time to the D3-brane tadpole and hence changes the D3-brane curve class $[C]$ compared to the fluxless compactification. This changes the number of massless Fermi multiplets in the 3-7 sector. The net number of moduli stabilized in the presence of flux must equal the change in the number of 3-7 modes. This interesting effect has no analogue in 6D or 4D F-theory vacua:

In 6D there is no background flux, and in 4D there is no purely gravitational anomaly.

## 4.6. Derivation of the Green-Schwarz Terms and 3-7 Anomaly

In this section we derive the two key results of this chapter, the form and correct overall normalization of the 2D Green-Schwarz terms and the contribution to the gauge anomalies from the 3-7 string sector. As we will see, both can be obtained in a very compact manner directly from the gauging of the Type IIB Ramond-Ramond 4-form in the presence of source terms. The gauging of the Ramond-Ramond forms in the presence of brane sources is standard [34, 40, 208], and a similar ten-dimensional approach to determining the gauging in a compactification has been taken in [209, 210]. We will first derive this gauging in an orientifold limit and describe its implications for the Green-Schwarz terms and its relation to the 3-7 anomalies. We then uplift the result to F-theory on an elliptically fibered Calabi-Yau, which is valid beyond the perturbative limit. We close this section by making contact with the 2D effective action laid out in section 4.3.

### 4.6.1. 10D Chern-Simons terms

Consider a Type IIB orientifold compactification on a Calabi-Yau 4-fold $X_4$, with spacetime-filling D7-branes and O7-planes associated with a holomorphic orientifold involution $\sigma : X_4 \to X_4$. To simplify the presentation we omit orientifold invariant D7-branes and only consider D7-branes as pairs $\mathrm{D7}_a$, $\mathrm{D7}_{a'}$ wrapping effective divisors $D_a$ and $D_{a'} = \sigma_*(D_a) \neq D_a$ on $X_4$. The cohomology class Poincaré dual to $D_a$ will be denoted by $[D_a]$. The field strength on the $\mathrm{D7}_a$-brane is denoted as $\mathbf{F}_a$ with $\mathbf{F}_{a'} = -\sigma_*(\mathbf{F}_a)$. In addition we allow for spacetime-filling D3-branes and their image wrapping curves $C_i$ and $C_{i'}$ on $X_4$. Our conventions for the effective action of the supergravity fields and the branes are summarized in appendix A.1. The 10d supergravity action in the presence of 7-branes and 3-branes and after taking the orientifold quotient takes the form[10]

$$S = \frac{1}{2}\left(S_{\mathrm{IIB}} + \sum_a (S_a^{\mathrm{D7}} + S_{a'}^{\mathrm{D7}}) + S^{\mathrm{O7}} + \sum_i (S_i^{\mathrm{D3}} + S_{i'}^{\mathrm{D3}})\right) . \tag{4.82}$$

We are interested in the gauging of the RR 4-form potential $C_4$. Prior to taking into account the source terms due to the branes, its associated field strength is[11] $F_5 = dC_4$. It is gauge invariant and satisfies the Bianchi identity $dF_5 = 0$. Including the source terms, the relevant part of the action after taking the orientifold quotient becomes

$$\begin{aligned} S|_{C_4} \;=\; & 2\pi \int -\frac{1}{8} F_5 \wedge *F_5 + \\ & 2\pi \int C_4 \wedge \left(\frac{1}{2}\sum_a (Q_a(\mathbf{F}_a) + Q_{a'}(\mathbf{F}_{a'})) + \frac{1}{2} Q(\mathbf{R}) + \frac{1}{2}\sum_i (Q(\mathrm{D3}_i) + Q(\mathrm{D3}_{i'}))\right) \end{aligned} \tag{4.83}$$

[10] The overall factor of $\frac{1}{2}$ results from the orientifold quotient.

[11] Strictly speaking, the $SL(2,\mathbb{Z})$ invariant field strength in 10d is $\widetilde{F}_5 = dC_4 + \frac{1}{2}B_2 \wedge F_3 - \frac{1}{2}C_2 \wedge H_3$ with $H_3 = dB_2$ and $F_3 = dC_2$, but since we are only interested in the gauging of $C_4$ these corrections play no role for us.

The source terms linear in $C_4$ follow by summing up the $C_4$ dependent contributions to the Chern-Simons action of the 7-branes, the O7-plane and the D3-branes listed in Appendix A.1 as

$$Q_a(\mathbf{F}_a) = -\frac{1}{4}\mathrm{Tr}\,\mathbf{F}_a \wedge \mathbf{F}_a \wedge [D_a] \tag{4.84}$$
$$Q(\mathbf{R}) = -\frac{1}{16}\mathrm{tr}\,\mathbf{R} \wedge \mathbf{R} \wedge [O7] \tag{4.85}$$
$$Q(\mathrm{D3}_i) = \frac{1}{2}[C_i]\,. \tag{4.86}$$

Note the appearance of the trace Tr, defined in (4.29), in the expression (4.84). In the strict perturbative limit, in particular for gauge groups of type $SU(n)$, there is no difference compared to the trace in the fundamental representation. But more generally in F-theory, it is the object Tr, rather than tr, which appears in the Chern-Simons action.

As a result, the Bianchi identity for the field strength $F_5$ associated with $C_4$ now takes the non-standard form

$$dF_5 = \frac{1}{2}\sum_a (\mathrm{Tr}\,\mathbf{F}_a \wedge \mathbf{F}_a \wedge [D_a] + \mathrm{Tr}\,\mathbf{F}_{a'} \wedge \mathbf{F}_{a'} \wedge [D_{a'}]) + \mathrm{tr}\,\mathbf{R} \wedge \mathbf{R} \wedge \frac{1}{8}[O7] - \sum_i([C_i] + [C_{i'}])\,. \tag{4.87}$$

To proceed further, we introduce the Chern-Simons forms $\mathbf{w}_{3a}$ for the gauge group on the 7-brane along $D_a$ as well as $\mathbf{w}_{3Y}$ for the spin connection $\omega$ with the property

$$\mathrm{Tr}\,\mathbf{F}_a \wedge \mathbf{F}_a = d\,\mathbf{w}_{3a}, \qquad \mathrm{tr}\,\mathbf{R} \wedge \mathbf{R} = d\,\mathbf{w}_{3Y}\,. \tag{4.88}$$

Similarly, one can define an Euler form $e_{5,i}$ associated with the 6-form $[C_i]$ Poincaré dual to the curve $C_i$ such that $de_{5,i} = [C_i]$.[12] This allows us to express (4.87) as

$$d\left(F_5 - \frac{1}{2}\sum_a \big(\mathbf{w}_{3a} \wedge [D_a] + \mathbf{w}_{3a'} \wedge [D_{a'}]\big) - \frac{1}{8}\mathbf{w}_{3Y} \wedge [O7] + \sum_i (e_{5,i} + e_{5,i'})\right) = 0\,, \tag{4.89}$$

which is solved by setting

$$F_5 = dC_4 + \frac{1}{2}\sum_a \big(\mathbf{w}_{3a} \wedge [D_a] + \mathbf{w}_{3a'} \wedge [D_{a'}]\big) + \frac{1}{8}\mathbf{w}_{3Y} \wedge [O7] - \sum_i (e_{5,i} + e_{5,i'})\,. \tag{4.90}$$

Taking into account the backreation of the source terms means that it is now this form of $F_5$ which appears in the kinetic term in (4.87). The full action (4.83) is equivalent to

$$\begin{aligned} S|_{C_4} = {} & 2\pi \int -\frac{1}{8}\, F_5 \wedge *F_5 + \qquad (4.91) \\ & 2\pi \int F_5 \wedge \left(\frac{1}{8}\sum_a (\mathbf{w}_{3a} \wedge [D_a] + \mathbf{w}_{3a'} \wedge [D_{a'}]) + \frac{1}{32}\,\mathbf{w}_{3Y} \wedge [O7] - \frac{1}{4}\sum_i (e_{5,i} + (e_{5,i'}))\right)\,, \end{aligned}$$

again with $F_5$ as in (4.90).[13]

The form (4.90) for the gauge invariant field strength $F_5$ implies that $C_4$ must transform

[12]A careful definition can be found in [211]. A proper regularization of this term is necessary for a correct treatment of the normal bundle anomalies [40], but this will play no role for us in this chapter.

[13]Note that the cross-terms quadratic in the Chern-Simons terms vanish due their odd form degree.

non-trivially under gauge transformations associated with the 7-brane gauge group and the spin connection. In absence of any background values for the fields, if under a gauge and Lorentz transformation the gauge connection $\mathbf{A}_a$ and the spin connection $\omega$ change as

$$\mathbf{A}_a \to d\lambda_a + [\lambda_a, \mathbf{A}_a], \qquad \omega \to d\chi + [\chi, \omega]\,, \tag{4.92}$$

then the Chern-Simons forms vary as

$$\delta \mathbf{w}_{3a} = d(\lambda_a \, d\mathbf{A}_a), \qquad \delta \mathbf{w}_{3Y} = d(\chi \, d\omega)\,. \tag{4.93}$$

Since the field strength $F_5$ defined in (4.90) is gauge invariant, this induces a corresponding gauge transformation of the potential $C_4$. We are interested in situations in which both the gauge and the spin connection acquire non-trivial background values. Correspondingly we can decompose the field strength $\mathbf{F}$ into its fluctuation piece $F$ and a background component $\bar{F}$, and similarly for $\mathbf{R}$,

$$\mathbf{F} = F + \bar{F}, \qquad \mathbf{R} = R + \bar{R}\,. \tag{4.94}$$

The gauge dependence of $C_4$ then becomes[14]

$$\delta_{\rm gauge}\, C_4 = -\sum_a \mathrm{Tr}\,\lambda_a \left( (\bar{F}_a \wedge [D_a] - \bar{F}_{a'} \wedge [D_{a'}]) + \frac{1}{2}(dA_a \wedge [D_a] - dA_{a'} \wedge [D_{a'}]) \right) \tag{4.95}$$

$$\delta_{\rm spin}\, C_4 = -\mathrm{tr}\,\chi d\bar{\omega} \wedge \frac{1}{4}[O7] - \mathrm{tr}\, d\omega \wedge \frac{1}{8}[O7]\,. \tag{4.96}$$

Here we have used $\lambda_a = -\lambda_{a'}$, relating the gauge group on each brane along $D_a$ and its orientifold image. The relative factor of 2 in the first terms involving the background field strength and curvature results from expanding $\mathbf{F}_a^2 = 2F_a\bar{F}_a + F_a^2 + \bar{F}_a^2$, and similarly for $\mathbf{R}$. As we will see next, the terms on the righthand side involving the internal background flux $\bar{F}_a$ induce the Green-Schwarz counterterms in the two-dimensional effective action, while the terms depending on the fluctuations $F_a$ and $R$ contribute to the anomaly inflow counterterms for the anomaly from the 3-7 string modes.

### 4.6.2. Derivation of the GS term in Type IIB

In order to derive the Green-Schwarz counterterms, we first consider the flux-dependent piece in the gauge variation of $C_4$, (4.95),

$$\delta_{\rm gauge}\, C_4|_{\rm flux} = -\sum_a \mathrm{Tr}\,\lambda_a \left(\bar{F}_a \wedge [D_a] - \bar{F}_{a'} \wedge [D_{a'}]\right)\,. \tag{4.97}$$

Due to the appearance of $C_4$ in the action (4.83), while $F_5$ by itself is gauge invariant, this induces a gauge dependence of the effective action, which is precisely the manifestation of a Green-Schwarz counterterm. As we will see, the only relevant terms contributing to the Green-Schwarz terms are the couplings to $Q_a(\mathbf{F}_a)$ and $Q_{a'}(\mathbf{F}_{a'})$. If we focus on these, substituting the

[14] Strictly speaking, we are not taking into account variations of the spin connection in the direction of the normal bundle, which are more subtle [40, 211] but play no role for us. Note also that, as we will argue momentarily, only abelian fluxes are of relevance for us so that we are writing $\bar{F}_a$ instead of $d\bar{A}_a$.

variation (4.97) of $C_4$ into (5.21) gives

$$\begin{aligned}\delta S_{\rm GS} =& \frac{1}{2}\left(\sum_b \delta_{\rm gauge}\, S_b^{\rm D7} + \sum_{b'} \delta_{\rm gauge}\, S_{b'}^{\rm D7}\right)\Bigg|_{\rm flux} \\ =& \frac{2\pi}{8}\int_{\mathbb{R}^{1,1}\times X_4} \sum_{a,b} {\rm Tr}\lambda_a \left(\bar{F}_a\wedge[D_a] - \bar{F}_{a'}\wedge[D_{a'}]\right)\wedge\left({\rm tr}(\mathbf{F}_b\wedge\mathbf{F}_b)\wedge[D_b] + {\rm tr}(\mathbf{F}_{b'}\wedge\mathbf{F}_{b'})\wedge[D_{b'}]\right),\end{aligned} \tag{4.98}$$

where we are indicating that after compactification the spacetime is of the form $\mathbb{R}^{1,1}\times X_4$. If we identify the fluctuations $F$ with the 2D field strength $F^{\rm 2d}$, we see that for reasons of dimensionality only the last term in the decomposition

$${\rm Tr}(\mathbf{F}_b\wedge\mathbf{F}_b) = {\rm Tr}(F_b^{\rm 2d}\wedge F_b^{\rm 2d}) + {\rm Tr}(\bar{F}_b\wedge\bar{F}_b) + 2\,{\rm Tr}(F_b^{\rm 2d}\wedge\bar{F}_b) \tag{4.99}$$

makes a contribution. We thus find

$$\delta S_{\rm GS} = \frac{2\pi}{4}\sum_{ab}{\rm Tr}_a{\rm Tr}_b\,\lambda_a F_b^{\rm 2d}\int_{X_4}\left((\bar{F}_a\wedge[D_a]+\sigma^*(\bar{F}_a\wedge[D_a]))\wedge(\bar{F}_b\wedge[D_b]+\sigma^*(\bar{F}_b\wedge[D_b]))\right). \tag{4.100}$$

Here we have used the definition $\sigma^*({\rm Tr}\bar{F}_a\wedge[D_a]) = {\rm Tr}\bar{F}_{a'}\wedge[D_{a'}]$. Furthermore we have denoted the trace over the gauge group on brane $D_a$ with ${\rm Tr}_a$, and similarly for $D_b$. Through the descent equations, this gauge variance yields the Green-Schwarz contribution to the anomaly polynomial

$$I_4^{\rm GS} = \frac{1}{4}\sum_{a,b}{\rm Tr}_a{\rm Tr}_b F_a^{\rm 2d}\wedge F_b^{\rm 2d}\int_{X_4}\left((\bar{F}_a\wedge[D_a]+\sigma^*(\bar{F}_a\wedge[D_a]))\wedge(\bar{F}_b\wedge[D_b]+\sigma^*(\bar{F}_b\wedge[D_b]))\right). \tag{4.101}$$

Note that the trace is taken simultaneously over the external and the internal components of the field strength, both for the gauge groups associated with $D_a$ and with $D_b$. This implies that $I_4^{\rm GS}$ can only be non-vanishing for the abelian gauge symmetry factors in the two-dimensional effective action: Indeed, a contribution to a non-abelian gauge group would require at the same time non-abelian flux internally, but this would break the gauge group. The only option is that the flux is embedded along the direction of an abelian generator, which then acquires a Green-Schwarz anomaly term of the above form. This is a notable difference from the Green-Schwarz mechanism in six dimensions, which is well-known to operate also at the level of non-abelian gauge groups.

For a similar reason, the other source terms in (4.83) do not contribute to the gauge variance of the classical action. Also, there can be no Green-Schwarz contribution to the pure gravitational anomaly or even a mixed gauge-gravitational anomaly in two dimensions. This can be seen explicitly if one proceeds along the same lines with the background terms in (4.96) and uses the direct product structure of the Lorentz group as $SO(1,1)\times G_{\rm int}$ upon compactification. In summary, the complete effect of the gauge dependence associated with the background term in (4.95) is the Green-Schwarz anomaly polynomial (4.101), while the background term in (4.96) does not lead to any gauge dependence of the effective action.

The Green-Schwarz counterterm (4.101) and in particular its overall normalization will be checked in a prototypical brane setup in Appendix 4.7, where we will verify that it correctly cancels the 1-loop anomalies induced by the 3-7 and the 7-7 sector.

### 4.6.3. 3-7 anomaly from gauging in Type IIB

Let us now analyze the effect of the dependent piece of the gauging (4.95),

$$\delta C_4|_{\text{fluct.}} = -\frac{1}{2}\text{Tr}\sum_a \lambda_a(dA_a^{2\text{d}} \wedge [D_a] - dA_{a'}^{2\text{d}} \wedge [D_{a'}]) - \text{tr}\,\chi\, d\omega^{2\text{d}} \wedge \frac{1}{8}[O7]\,. \tag{4.102}$$

If we plug this expression into (4.83) we receive a contribution only from the internal components of the source terms. Summing over all source terms associated with the D3-branes, the D7-branes and the O7-plane gives a vanishing total result because the total $C_4$ charge along the internal space $X_4$ vanishes as a result of D3-brane tadpole cancellation. Nonetheless, each individual term by itself contains valuable information, namely (part of) the counterterms for the 1-loop gauge anomaly on the worldvolume of the respective branes. By construction of the Chern-Simons brane actions, these counterterms *locally* cancel the 1-loop anomaly associated with chiral modes on the worldvolume of the branes via the anomaly inflow mechanism [34, 40, 208]. Tadpole cancellation then implies that the sum of all counterterms vanishes globally, which equivalent to the statement of anomaly cancellation.

To extract the full anomaly inflow counterterm cancelling the 7-brane gauge anomalies from the 3-7 sector as well as the tangent bundle anomalies along the D3-brane, we follow the standard procedure [34, 40, 208] and rewrite the non-kinetic terms in the action (4.83) as

$$S|_{C_4} \supset S_1 + S_2 \tag{4.103}$$

$$S_1 = \frac{2\pi}{4}\int C_4 \wedge \sum_i([C_i] + [C_{i'}]) \tag{4.104}$$

$$S_2 = 2\pi \int F_5 \wedge \left(\frac{1}{8}\sum_a(\mathbf{w}_{3a} \wedge [D_a] + \mathbf{w}_{3a'} \wedge [D_{a'}]) + \frac{1}{32}\,\mathbf{w}_{3Y} \wedge [O7]\right)\,. \tag{4.105}$$

The anomaly inflow counterterms now have two contributions. The first contribution comes from plugging the gauge variation (4.102) into $S_1$,

$$\delta\, S_1|_{\text{inflow}} = -\frac{2\pi}{8}\int_{\mathbb{R}^{1,1}} \sum_{a,i} \text{Tr}\lambda_a\, dA_a^{2\text{d}} \int_{X_4} \big([D_a] + [D_{a'}]\big)\big([C_i] + [C_{i'}]\big) \tag{4.106}$$

$$-\frac{2\pi}{32}\int_{\mathbb{R}^{1,1}} \text{tr}\,\chi\, d\omega^{2\text{d}} \int_{X_4} [O7] \wedge \sum_i([C_i] + [C_{i'}])\,, \tag{4.107}$$

where in the first line we have used that $A_{a'}^{2\text{d}} = -A_a^{2\text{d}}$. In addition, the Chern-Simons forms appearing in $S_2$ vary according to (4.93).[15] After integration by parts we find a non-zero contribution because of the Bianchi identity (4.87). The relevant terms describing the anomaly inflow are obtained by plugging in only the last terms in (4.87), i.e. using $dF_5 = -\sum_i([C_i] + [C_{i'}]) + \ldots$. This gives a contribution of exactly the same form as (4.106) and hence altogether

$$\delta\, S|_{\text{inflow}} = \delta S_1|_{\text{inflow}} + \delta S_2|_{\text{inflow}} = 2\,\delta S_1|_{\text{inflow}}\,. \tag{4.108}$$

The terms (4.106) cancel the contribution to the 7-brane gauge group anomaly from the sector

[15] We are here only taking into account the contribution to (4.93) from the fluctuations of the fields; the contributions involving the background fields enter the Green-Schwarz terms and have hence already been taken into account in the previous section.

of 3-7 strings. By descent, the associated 1-loop anomaly polynomial is therefore

$$I^{3-7}_{4,\text{gauge}} = \frac{1}{4}\sum_{a,i} \text{Tr}F^{2\text{d}}_a \wedge F^{2\text{d}}_a \int_{X_4} ([D_a] + [D_{a'}])([C_i] + [C_{i'}]) \ . \qquad (4.109)$$

Note that we have included a minus sign in $I^{3-7}_4$ because (4.108) represents the inflow counterterms to the actual 1-loop anomaly. As the trace structure clearly shows, this contribution is non-vanishing also for simple gauge groups, in contrast to the Green-Schwarz terms derived earlier.

From the perspective of the effective 2D $(0, 2)$ theory, the gauging (4.102) translates into a gauging of the non-dynamical 2-forms obtained by dimensional reduction of $C_4$ in terms of internal 2-forms on $X_4$. This offers an interesting perspective on the contribution (4.109) to the total anomaly polynomial: Rather than interpreting it as due to chiral localised defect modes we can view it as the effect of gauging these non-dynamical top-forms in the effective supergravity theory. This makes the formal similarity between the Green-Schwarz terms, associated with the gauging of the scalars from $C_4$, and the 3-7 anomaly on the righthand side of (4.61b) more natural.

The remaining terms (4.107) cancel the contribution to the gravitational anomaly from all modes on the D3-brane worldvolume. This includes the 3-7 modes as well as the 3-brane bulk modes analyzed in detail in [187]. The associated anomaly polynomial is

$$I^{\text{D3}}_{4,\text{grav}} = \text{tr}\, R^{2\text{d}} \wedge R^{2\text{d}} \int_{X_4} \frac{1}{16}[O7] \wedge \sum_i ([C_i] + [C_{i'}])\,. \qquad (4.110)$$

## 4.7. Anomalies and Green-Schwarz Term in Type IIB Orientifolds

In this section we verify our intermediate results (4.101) for the Green-Schwarz terms in Type IIB orientifolds. Together with our confirmation of the final F-theoretic expressions in the explicit example of section 4.9, this also supports our rules explained in (4.8) for the correct uplift to F-theory.

The setup we analyze is identical to the one in appendix C.2 of [64], which we now briefly summarize. Consider a Type IIB orientifold on a general Calabi-Yau 4-fold $X_4$ with gauge group $(SU(n) \times U(1)_a) \times U(1)_b$. The brane configuration consists of $n$ 7-branes wrapping a divisor $W$ and one extra D7-brane along the divisor $V$, each accompanied by their orientifold images wrapped along $W'$ and $V'$, respectively. We assume that all brane divisors are smooth. In order to cancel the D7-tadpole, it is required that

$$n([W] + [W']) + ([V] + [V']) = 8[O7]\,. \qquad (4.111)$$

The D3-tadpole cancellation condition fully determines the spacetime-filling D3-brane system wrapped along a total curve class $[C]$ plus orientifold image brane $[C']$ as

$$[C] = \frac{n}{24}[W] \cdot c_2(W) + \frac{1}{12}[O7] \cdot c_2(O7) + n\, \text{ch}_2(L_W) \cdot [W] + \text{ch}_2(L_V) \cdot [V] \qquad (4.112)$$

$$[C'] = \frac{n}{24}[W'] \cdot c_2(W') + \frac{1}{12}[O7] \cdot c_2(O7) + n\, \text{ch}_2(L'_W) \cdot [W'] + \text{ch}_2(L'_V) \cdot [V']\,. \qquad (4.113)$$

Here $L_W$ and $L_V$ denote line bundles on $W$ and $V$ whose structure groups are identified with $U(1)_a$ and $U(1)_b$, respectively.

For simplicity, we require

$$[V] = [V'], \qquad [W] = [W'] \tag{4.114}$$

to prevent the gauge potentials associated with $U(1)_a$ and $U(1)_b$ from acquiring a mass, in absence of flux, via the geometric Stückelberg mechanism.[16] We simplify the calculation of the $U(1)_a$ anomaly contribution further by assuming

$$[W] \cdot [W'] = [W] \cdot [O7]\,. \tag{4.115}$$

This implies that there exists no intersection locus of $W$ and $W'$ away from the $O7$-plane, which would carry matter in the symmetric representation of $SU(n)$. This would lead to extra complications in the computation of the chiral spectrum, which we avoid by requiring (4.115). For the same reason we make the simplifying assumption that

$$[V] \cdot [V'] = [V] \cdot [O7]\,. \tag{4.116}$$

None of these assumptions is essential, but dropping them would require some modifications of the anomaly computation.

We are now in a position to determine the contribution to the $U(1)_a - U(1)_a$ and the $U(1)_b - U(1)_b$ anomaly due to the chiral matter states. Since our primary interest here is to check the Green-Schwarz counterterm (4.101) and its normalization relative to the 1-loop anomalies, it suffices to focus on the flux-dependent contribution of these states. The chiral spectrum from the D7-D7 brane sector and the flux dependent part of its contribution to the anomalies are listed in table 4.2, and similarly for the 3-7 sector in table 4.3. Note that we have omitted matter in the adjoint representation, which is not charged under $U(1)_a$ and $U(1)_b$. We adapt the convention (A.14) for the anomaly polynomial so that there is overall factor of $-1$ in front of every term in Table 4.2, while in Table 4.3 we have taken into account the anti-chiral nature of the 3-7 matter, which hence contributes with a $+1$. Merely to save some writing, we have assumed, in the column containing the $U(1)_a^2$ anomalies, that $L_V = 0$, and similarly in the column containing the $U(1)_b^2$ anomalies that $L_W = 0$. Furthermore, with our assumption (4.115) all matter on $W \cap W'$ transforms in the anti-symmetric representation of $U(n)$, while due to (4.116) the states on $V \cap V'$ are all projected out (as there exists no anti-symmetric representation of $U(1)_b$). The total anomaly from the 7-7 sector is then obtained by summing over all states in table Table 4.2 and dividing the final result by two. The division by two is due to the orientifold quotient. Table 4.2 contains sectors in this upstairs picture which are pairwise identified under the involution. To offset for this overall factor of $\frac{1}{2}$ in the invariant sector $W \cap W'$ we are including a factor of 2 for these states in Table 4.2.

From (4.112) we read off the flux-dependent term part of the 3-brane class $[C]$,

$$[C]|_{\text{flux}} = \frac{1}{2} n\, c_1^2(L_W) \cdot [W] + \frac{1}{2} c_1^2(L_V) \cdot [V]\,. \tag{4.117}$$

The 7-7 and 3-7 sector contribution to the $U(1)_a - U(1)_a$ anomaly is hence, for $c_1(L_V) = 0$ for simplicity,

$$I_4^{1-\text{loop}}\Big|_{U(1)_a^2} = F_a^{2\text{d}} \wedge F_a^{2\text{d}}\, \mathcal{A}_a \tag{4.118}$$

[16] Otherwise, a D5-bane tadpole cancellation must be imposed on the gauge background.

| Locus | Representation of $SU(n)_{q_a,q_b}$ | $U(1)_a^2$ anomaly $(c_1(L_V)=0)$ $= -\frac{1}{2}\int \mathrm{ch}_2(L) q_a^2 \dim(R)$ | $U(1)_b^2$ anomaly $(c_1(L_W)=0)$ $= -\frac{1}{2}\int \mathrm{ch}_2(L) q_b^2 \dim(R)$ |
|---|---|---|---|
| $W\bigcap V$ | $\bar{n}_{(-1,1)}$ | $-\frac{1}{2}[W]\cdot[V]\cdot\frac{1}{2}c_1^2(L_W)\,n$ | $-\frac{1}{2}[W]\cdot[V]\cdot\frac{1}{2}c_1^2(L_V)\,n$ |
| $W\bigcap V'$ | $\bar{n}_{(-1,-1)}$ | $-\frac{1}{2}[W]\cdot[V']\cdot\frac{1}{2}c_1^2(L_W)\,n$ | $-\frac{1}{2}[W]\cdot[V']\cdot\frac{1}{2}c_1^2(L_V)\,n$ |
| $W'\bigcap V'$ | $n_{(1,-1)}$ | $-\frac{1}{2}[W']\cdot[V']\cdot\frac{1}{2}c_1^2(L_W)\,n$ | $-\frac{1}{2}[W']\cdot[V']\cdot\frac{1}{2}c_1^2(L_V)\,n$ |
| $W'\bigcap V$ | $n_{(-1,-1)}$ | $-\frac{1}{2}[W']\cdot[V]\cdot\frac{1}{2}c_1^2(L_W)\,n$ | $-\frac{1}{2}[W']\cdot[V]\cdot\frac{1}{2}c_1^2(L_W)\,n$ |
| $W'\bigcap W$ | $\frac{1}{2}n(n-1)_{(2,0)}$ | $-2\times\frac{1}{2}[W]\cdot[W']\cdot\frac{1}{2}c_1^2(L_w^2)\times 2^2\times\frac{1}{2}n(n-1)$ | 0 |

**Table 4.2.:** *Charged chiral matter from the 7-7 string sector and its anomaly contributions.*

| Locus | Representation of $SU(n)_{q_a,q_b}$ | $U(1)_a^2$ $(c_1(L_V)=0)$ $= +\frac{1}{2}W\cdot Cq^2\dim(R)$ | $U(1)_b$ $(c_1(L_W)=0)$ $= +\frac{1}{2}V\cdot Cq^2\dim(R)$ |
|---|---|---|---|
| $W\bigcap C$ | $\bar{n}_{(-1,1)}$ | $+\frac{1}{2}[W]\cdot[C]\times 1^2\times n$ | 0 |
| $W\bigcap C'$ | $\bar{n}_{(-1,-1)}$ | $+\frac{1}{2}[W]\cdot[C]\times 1^2\times n$ | 0 |
| $W'\bigcap C'$ | $n_{(1,0)}$ | $+\frac{1}{2}[W]\cdot[C]\times 1^2\times n$ | 0 |
| $W'\bigcap C$ | $n_{(1,0)}$ | $+\frac{1}{2}[W]\cdot[C]\times 1^2\times n$ | 0 |
| $V\bigcap C$ | $1_{(0,-1)}$ | 0 | $+\frac{1}{2}[V]\cdot[C]\times 1^2\times 1$ |
| $V\bigcap C'$ | $1_{(0,-1)}$ | 0 | $+\frac{1}{2}[V]\cdot[C']\times 1^2\times 1$ |
| $V'\bigcap C$ | $1_{(0,1)}$ | 0 | $+\frac{1}{2}[V']\cdot[C]\times 1^2\times 1$ |
| $V'\bigcap C'$ | $1_{(0,1)}$ | 0 | $+\frac{1}{2}[V']\cdot[C']\times 1^2\times 1$ |

**Table 4.3.:** *Charged chiral matter from the 3-7 string sector and its anomaly contributions.*

with

$$\begin{aligned}\mathcal{A}_a &= -\frac{1}{2}\Bigg(\big(4\times\frac{1}{4}\big)c_1^2(L_W)\,n\,[W]\cdot[V] + 4\times\frac{1}{4}\frac{2}{2}n(n-1)\,c_1^2(L_W^2)\,[W]\cdot 4[O7] \\ &\quad -\,4\times\frac{1}{2}n[W]\cdot\frac{1}{2}nc_1^2(L_W)\cdot[W]\Bigg) \\ &= -\frac{1}{2}[W]\cdot c_1^2(L_W)\cdot\big(n[V]+4n^2\,[O7]-4n\,[O7]-n^2\,[W]\big) \\ &= -n^2\,[W]^2\cdot c_1^2(L_W)\,.\end{aligned} \tag{4.119}$$

In the last line we have used that

$$n[W]+[V]=4[O7],\qquad [W]\cdot[W]=[W]\cdot[O7]\,. \tag{4.120}$$

This 1-loop anomaly is precisely cancelled by the Green-Schwarz term contribution (4.101) because the trace over the diagonal $U(1)_a\subset U(n)$ evaluates to $\mathrm{Tr}\bar{F}_a=\mathrm{tr}\mathbb{I}_n\,\bar{F}_a$ and hence

$$I_4^{\mathrm{GS}}\big|_{U(1)_a^2}=\frac{1}{4}\mathrm{Tr}_a\mathrm{Tr}_a F_a^{2\mathrm{d}}\wedge F_a^{2\mathrm{d}}\left(4\,\bar{F}_a\cdot[W]\right)=F_a^{2\mathrm{d}}\wedge F_a^{2\mathrm{d}}\left(n^2c_1(L_W)\cdot[W]\right)\,. \tag{4.121}$$

Similarly, the 1-loop $U(1)_b^2$ anomaly induced by the chiral matter, for $c_1(L_W) = 0$,

$$\begin{aligned}\mathcal{A}_b &= -\frac{1}{2}\left((4\times\frac{1}{4})\, c_1^2(L_V)\, n\,[W]\cdot[V] - 4\times\frac{1}{2}[V]\cdot\frac{1}{2}\,c_1^2(L_V)\cdot[V]\right)\\ &= -[V]^2\cdot c_1^2(L_V)\end{aligned} \tag{4.122}$$

is correctly cancelled by the GS term contribution (4.101).

## 4.8. F-theory Lift

It remains to uplift the perturbative results for the Green-Schwarz terms and the 3-7 anomaly to a description in fully-fledged F-theory, defined via duality to M-theory on an elliptic fibration $\widehat{X}_5$. If a weakly coupled limit exists, the perturbative Type IIB Calabi-Yau $X_4$ is the double cover of the F-theory base $B_4$, with projection

$$\pi_+ : X_4 \to B_4\,. \tag{4.123}$$

The cohomology classes even under the holomorphic involution $\sigma$ on $X_4$ uplift to cohomology classes of the same bidegree on $B_4$. In particular, consider a divisor class $[D] \in H^{1,1}(X_4)$ and its image $\sigma^*[D]$ under the involution and define

$$[D_+] := [D] + \sigma_*[D] =: \pi_+^*[D^{\mathrm{b}}] \tag{4.124}$$

with $[D^{\mathrm{b}}] \in H^{1,1}(B_4)$. Then taking into account that $X_4$ is a double cover of $B_4$ the intersection numbers on both spaces are related as [212]

$$[D_{a+}]\cdot_{X_4}[D_{b+}]\cdot_{X_4}[D_{c+}]\cdot_{X_4}[D_{d+}] = 2\, D_a^{\mathrm{b}}\cdot_{B_4} D_b^{\mathrm{b}}\cdot_{B_4} D_c^{\mathrm{b}}\cdot_{B_4} D_d^{\mathrm{b}}\,. \tag{4.125}$$

With this in mind consider first the perturbative expression (4.109) for the 3-7 anomaly, with the aim of uplifting the sum over all brane stacks and their image to F-theory. A divisor on $X_4$ wrapped by a non-abelian stack of 7-branes on $X_4$ uplifts, together with its image under the involution, to a corresponding divisor on $B_4$ according to the above rule, and this divisor on $B$ is a component of the discriminant locus carrying the corresponding non-abelian gauge group. More subtle are the non-Cartan abelian gauge groups. In Type IIB language, $U(1)$ gauge symmetries which are massless in the absence of background flux are supported on linear combinations of divisors which are in the same class as their orientifold image. Hence each abelian gauge group factor $U(1)_A$ is associated with a linear combination of (typically several) divisor classes $[D_a] + \sigma^*[D_a]$ on $X_4$.

Let us assume first that a brane configuration gives rise to no massless (in absence of fluxes) abelian gauge symmetries, i.e. the gauge group is only a product of non-abelian factors $G_I$, Then the uplift of $\sum_a \mathrm{Tr} F_a^{2\mathrm{d}} \wedge F_a^{2\mathrm{d}}([D_a] + [D_{a'}])$ to F-theory is

$$\sum_{i_I, j_J} F_{i_I}^{2\mathrm{d}} \wedge F_{j_J}^{2\mathrm{d}}\, \mathrm{Tr}\, \mathcal{T}_{i_I}\mathcal{T}_{j_J} D_I^{\mathrm{b}} = \sum_{i_I, j_J} F_{i_I}^{2\mathrm{d}} \wedge F_{j_J}^{2\mathrm{d}}\, \left(-\pi_*(E_{i_I}\cdot E_{j_J})\right)\,. \tag{4.126}$$

Here we used (4.28) to express the correctly normalised trace to $\pi_*(E_{i_I}\cdot E_{j_J})$.

In the presence of non-Cartan abelian symmetries, we must include these in the sum. In F-theory language, a non-Cartan gauge group factor $U(1)_A$ is generated by a 2-form $U_A$, defined via the Shioda-map as in (4.31), but typically there is no separate component of the divisor $\Delta$

which we would associate with $U(1)_A$. This is because, form a 7-brane perspective, massless (in absence of gauge flux) $U(1)$s involve combinations of several divisor classes. However, the height-pairing (4.33) defines a completely analogous object on the base $B_4$ including the information about the trace appearing in (4.109). Hence, the correct uplift of the expression for the 3-7 anomaly is

$$\begin{aligned}
\sum_a \mathrm{Tr}\, F_a^{\mathrm{2d}} \wedge F_a^{\mathrm{2d}} \left([D_a] + [D_{a'}]\right) \quad &\longrightarrow \quad \sum_{\Lambda,\Sigma} F_\Lambda^{\mathrm{2d}} \wedge F_\Sigma^{\mathrm{2d}} \left(-\pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma)\right) \\
\sum_i ([C_i] + [C_{i'}]) \quad &\longrightarrow \quad [C] \\
\frac{1}{4}\sum_{a,i} \mathrm{Tr} F_a^{\mathrm{2d}} \wedge F_a^{\mathrm{2d}} \int_{X_4} ([D_a] + [D_{a'}])([C_i] + [C_{i'}]) \quad &\longrightarrow \quad \frac{1}{2}\sum_{\Lambda,\Sigma} F_\Lambda^{\mathrm{2d}} \wedge F_\Sigma^{\mathrm{2d}} \left(-\pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma)\right) \cdot_{B_4} [C]
\end{aligned} \tag{4.127}$$

Here $[C]$ is the total class of the D3-brane on $B_4$ and we summing over all generators $\mathfrak{F}_\Sigma$, Cartan and non-Cartan. The last line in addition uses (4.125). Hence

$$I_{4,\mathrm{gauge}}^{3-7} = \sum_{\Lambda,\Sigma} F_\Lambda^{\mathrm{2d}} F_\Sigma^{\mathrm{2d}} \left(\frac{1}{2}\ \left(-\pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma)\right)\ \cdot_{B_4} [C]\right) \tag{4.128}$$

in precise agreement with our claim (4.59) for the 3-7 gauge anomaly.

Note that in (4.109), there appear no mixed anomaly contributions because in the perturbative limit the 3-7 strings can only be charged under the diagonal $U(1)_a$ gauge group of at most one D7-brane stack. On the other hand, if we sum over all massless (in absence of flux) $U(1)_A$ group factors (which are linear combinations of the $U(1)_a$ if a perturbative limit exists), mixed anomaly terms in general do result.

To uplift the 3-7 contribution to the gravitational anomaly polynomial, we recall from [212] the general rule that $\pi_+^*(c_1(B_4)) = [O7]$, and therefore

$$\int_{X_4} [O7] \wedge \sum_i ([C_i] + [C_{i'}]) \quad \longrightarrow \quad 2c_1(B_4) \cdot_{B_4} [C]\,. \tag{4.129}$$

The resulting expression

$$I_{4,\mathrm{grav}}^{\mathrm{D3}} = \frac{1}{2}\mathrm{tr}\, R^{\mathrm{2d}} \wedge R^{\mathrm{2d}} \left(\frac{1}{4} c_1(B_4) \cdot_{B_4} [C]\right)\,. \tag{4.130}$$

had already been derived in [187].

It remains to uplift the Green-Schwarz anomaly polynomial (4.101) to F-theory. Consider an internal flux background associated with a line bundle whose structure group is identified with either a Cartan or a non-Cartan $U(1)$ subgroup. Such fluxes uplift in F-theory to expressions of the form $G_4 = \bar{F} \wedge \mathfrak{F}_\Lambda$ for the corresponding divisor generator that $U(1)$ symmetry. Employing once more (4.28) and (4.33), an expression of the form $\mathrm{Tr}_a\, F_a^{\mathrm{2d}}(\bar{F}_a \wedge [D_a] + \sigma^*(\bar{F}_a \wedge [D_a]))$ uplifts to

$$F_\Sigma^{\mathrm{2d}}(-\pi_*(G_4 \cdot \mathfrak{F}_\Sigma))\,. \tag{4.131}$$

This remains correct even if the flux, on the Type IIB side, is associated with a $U(1)$ that is geometrically massive even before switching on the flux. Such fluxes lift to more general elements of $G_4$ [212, 213]. Taking again into account the factor of 2 from (4.125) we therefore arrive at

$$I_4^{\mathrm{GS}} = \sum_{\Lambda,\Sigma} F_\Lambda^{\mathrm{2d}} F_\Sigma^{\mathrm{2d}} \left( \frac{1}{2} (\pi_*(G_4 \cdot \mathfrak{F}_\Lambda)) \cdot_{B_4} (\pi_*(G_4 \cdot \mathfrak{F}_\Sigma)) \right) \tag{4.132}$$

in agreement with our previous claim (4.60).

### 4.8.1. Relation to 2D effective action

For completeness, we can express our findings in the language of the 2D effective action and make contact with the formalism introduced in section 4.3. Indeed, straightforward dimensional reduction of the action (4.83) allows us to read off the kinetic metric $g_{\alpha\beta}$, the coupling matrix $\Omega_{\alpha\beta}$ and the gauging parameters $\Theta_A^\alpha$. This can be achieved by first performing the dimensional reduction in the language of Type IIB orientifolds and then uplifting the results according to the general rules described in section 4.8. We directly give the result in the language of F-theory: If we fix a basis $\omega_\alpha$ of $H^4(B_4, \mathbb{R})$, the real scalar fields are obtained as

$$C_4 = c^\alpha \, \omega_\alpha \, . \tag{4.133}$$

Matching the 10D and 2D kinetic terms in (4.83) and (4.10), respectively, as well as the 10D self-duality condition $F_5 = *F_5$ with its 2D analogue (4.11) then fixes

$$g_{\alpha\beta} = 2\pi \int_{B_4} \omega_\alpha \wedge *\omega_\beta \tag{4.134}$$

$$\Omega_{\alpha\beta} = 2\pi \int_{B_4} \omega_\alpha \wedge \omega_\beta =: 2\pi \, \widetilde{\Omega}_{\alpha\beta} \, . \tag{4.135}$$

Dimensional reduction of the interaction terms in (4.83) finally identifies the gauging parameters

$$\Theta_\Gamma^\alpha = \widetilde{\Omega}^{\alpha\beta} \int_{B_4} \pi_*(G_4 \cdot \mathfrak{F}_\Gamma) \wedge \omega_\beta \tag{4.136}$$

in terms of the inverse matrix $\widetilde{\Omega}$ satisfying $\widetilde{\Omega}^{\alpha\beta} \widetilde{\Omega}_{\beta\gamma} = \delta^\alpha_\gamma$. As a check, plugging this expression into (5.8) correctly reproduces our result (4.132) for the Green-Schwarz anomaly polynomial.

## 4.9. Example: $SU(5) \times U(1)$ Gauge Symmetry in F-theory

In this section we exemplify our general expressions for the anomaly relations in an F-theory compactification on a Calabi-Yau 5-fold with gauge group $SU(5) \times U(1)$. The four-dimensional version of this model and its flux backgrounds has been studied in great detail in the literature [138, 169, 212, 214], and its extension to Calabi-Yau five-folds has been discussed in [64]. The geometry is sufficiently intricate to exemplify all interesting aspects of abelian, non-abelian and gravitational anomaly cancellation, while at the same time it avoids extra complications which arise when the codimension-two matter loci on the base $B_4$ are singular.

### 4.9.1. Geometric background and 3-7 states

We will briefly recall the properties of this model relevant for our discussion, referring for more details to [64] as well as to [169, 214], whose notation we adopt.

We are considering the resolution of a Weierstrass model in Tate form defined by

$$y^2 s e_3 e_4 + a_1 xyzs + a_{3,2} yz^3 e_0^2 e_1 e_4 = x^3 s^2 e_1 e_2^2 e_3 + a_{2,1} x^2 z^2 s e_0 e_1 e_2 + a_{4,3} xz^4 e_0^3 e_1^2 e_2 e_4 \,, \quad (4.137)$$

where $[x : y : z]$ denote homogenous coordinates of the fibre ambient space $\mathbb{P}_{231}$ prior to resolution and $E_i : e_i = 0$, $i = 1, \ldots, 4$ represent the resolution divisors for the singularities associated with gauge group $SU(5)$. In addition to the zero section $S_0 : z = 0$, the fibration admits another independent rational section $S_A : s = 0$. The resolved $SU(5)$ singularity sits in the fibre over the divisor $W : w = 0$ on $B_4$ [17], with $\pi^{-1}W : e_0 e_1 e_2 e_3 e_4 = 0$. With the help of Sage, we find the projection of $c_5(\widehat{X}_5)$ and $c_4(\widehat{X}_5)$ of the resolved fibration $\widehat{X}_5$ on the base $B_4$ and evaluate

$$\begin{aligned} \pi_*(c_5(\widehat{X}_5)) &= -576c_1^4 + 1464c_1^3 W - 48c_1^2 c_2 - 1410c_1^2 W^2 + 46c_1 c_2 W + 12c_1 c_3 + 608c_1 W^3 \\ &\quad -18c_2 W^2 - 102W^4 \qquad (4.138) \\ \pi_*(c_4(\widehat{X}_5)) &= 144c_1^3 - 264c_1^2 W + 12c_1 c_2 + 162c_1 W^2 - 30W^3 \,. \qquad (4.139) \end{aligned}$$

Here and in the sequel, the Chern classes $c_i$ without any specification denote $c_i(B_4)$ and all of the intersection numbers between the divisors are evaluated on $B_4$. Finally, $a_{i,j}$ define the following divisor classes on the base $B_4$ with $c_1(B_4) =: c_1$,

$$[a_1] = c_1 \,, \quad [a_{2,1}] = 2c_1 - W \,, \quad [a_{3,2}] = 3c_1 - 2W \,, \quad [a_{4,3}] = 4c_1 - 3W \,. \quad (4.140)$$

The discriminant of the blowdown of this model (setting $e_i = 1$ for $i = 1, \ldots, 4$) is

$$\Delta = w^5 \left( a_1^4 a_{3,2} \left( a_{2,1} a_{3,2} - a_1 a_{4,3} \right) + O(w) \right) \quad (4.141)$$

and indicates that there are four codimension-two matter loci on $B_4$ with classes

$$\begin{aligned} C_{\mathbf{10}_1} : &\quad W \cdot [a_1] = W \cdot c_1 \\ C_{\mathbf{5}_3} : &\quad W \cdot [a_{3,2}] = W \cdot (3c_1 - 2W) \\ C_{\mathbf{5}_{-2}} : &\quad W \cdot [a_1 a_{4,3} - a_{2,1} a_{3,2}] = W \cdot (5c_1 - 3W) \\ C_{\mathbf{1}_5} : &\quad [a_{3,2}] \cdot [a_{4,3}] = (3c_1 - 2W) \cdot (4c_1 - 3W) \,. \end{aligned} \quad (4.142)$$

The subscripts denote the charges under the non-Cartan $U(1)_A$ associated with the divisor [214][18]

$$U_A = -\left(5(S_A - S_0 - c_1) + 2E_1 + 4E_2 + 6E_3 + 3E_4\right) \,. \quad (4.143)$$

Note that in this example all of the codimension-two loci are smooth, while in principle they could exhibit singularities. In this case the chirality formula (4.43) would receive corrections [64]. The height pairing associated with $U_A$ is

$$D_A = -\pi_*(U_A \cdot U_A) = -30W + 50c_1 \,. \quad (4.144)$$

[17] The divisor $W$ can be found from the fact the discriminant $\Delta$ factorises as $\Delta = w^5 \Delta'$ in the blowdown model as in (4.141)

[18] We are using the conventions of [138, 169], where in particular the fibre structure and the resulting charge assignments are detailed.

The D3-brane tadpole requires the inclusion of D3-branes wrapping a curve of total class $C$ constrained as in (4.38). In the present example, each intersection point between $C$ and the $SU(5)$ divisor $W$ carries one Fermi multiplet in the fundamental representation $\mathbf{5}_{q_1}$ of $SU(5)$ [64] with $U(1)_A$ charge $q_1$. The intersections with the remainder of the discriminant carry additional Fermi multiplets, whose determination is very subtle due to $SL(2,\mathbb{Z})$ monodromies along $C$. In general some of these will have a non-zero $U(1)_A$ charge, while some may be completely uncharged under $SU(5) \times U(1)_A$. Our knowledge of the net contribution (4.57) of the 3-7 sector to the abelian anomaly together with its contribution to the gravitational anomaly [191] allow us to constrain this matter as follows: Let us adopt from the discussion around (4.55) the notation $D_{37}(\mathbf{R})$ for the divisor class on $B_4$ such that the effective chiral index of 3-7 states in representation $\mathbf{R}$ is given by $\chi(\mathbf{R}) = -[C] \cdot D_{37}(\mathbf{R})$. Then $D_{37}(\mathbf{5}_{q_1}) = W$, and the remaining divisor classes are constrained by the abelian and gravitational anomaly as

$$5\, q_1^2\, D_{37}(\mathbf{5}_{q_1}) + \sum_i q_i^2\, D_{37}(\mathbf{1}_{q_i}) = D_A = -30W + 50c_1 \tag{4.145}$$

$$5\, D_{37}(\mathbf{5}_{q_1}) + \sum_i D_{37}(\mathbf{1}_{q_i}) = 8c_1\,. \tag{4.146}$$

These equations are consistent with the assertion that, in addition to the states $\mathbf{5}_{q_1}$, there is only one further type of 3-7 Fermi multiplets in representation $\mathbf{1}_{q_2}$ with charge assignments

$$|q_1| = \frac{1}{2}, \qquad |q_2| = \frac{5}{2} \tag{4.147}$$

such that

$$D_{37}(\mathbf{1}_{q_2}) = -5W + 8c_1. \tag{4.148}$$

These values are in complete agreement with the perturbative limit of the compactification: To see this, recall from [212] that the Type IIB limit consists of a brane stack (plus image) with gauge group $U(5)_a$ and another brane-image brane pair carrying gauge group $U(1)_b$. The geometrically massless $U(1)$ symmetry is given by the linear combination $U(1)_A = \frac{1}{2}(U(1)_a - 5U(1)_b)$, where $U(1)_a$ is the diagonal $U(1)$ of $U(5)_a$ (cf. equ. 4.3 of [212]) and the normalization conforms with the definition (4.143) of the $U(1)_A$ generator. The 3-7 modes at the intersection of $C$ with the $U(5)_a$ stack hence carry charge $|q_1| = \frac{1}{2}|(1+0)|$ and transform as $\mathbf{5}$ of $SU(5)_a$ and those at the intersection of $C$ with $U(1)_b$ are $SU(5)_a$ singlets with charge $|q_2| = \frac{1}{2}|(0-5)|$. The class (4.148) furthermore coincides with the class of the $U(1)_b$ brane as dictated by the 7-brane tadpole cancellation condition.

We stress that more generally the pattern of singlets in the 3-7 sector can be more intricate. What is uniquely determined, however, is the net contribution of the 3-7 states both to the gauge and the gravitational anomalies.

Now we are in the position to check our proposal (4.20) within this example. As we have discussed before, we expect that the curvature and the flux induced anomalies should each cancel among themselves. Therefore, in the following we split our proof into three parts: We begin with the flux independent contribution to the anomalies and verify their precise cancellation as a result of rather sophisticated relations between the topological invariants of the resolved 5-fold. Next we consider the two different types of $G_4$ flux spanning the space of fluxes within $H^{2,2}_{\rm vert}(\widehat{X}_5)$ with the purpose of verifying in particular our proposal for the Green-Schwarz term

(4.61), and it will be shown that the anomalies induced by the two $G_4$ fluxes are cancelled very neatly.

### 4.9.2. Curvature dependent anomaly relations

In this section, we verify that the conditions (4.20) for anomaly cancellation are satisfied in the absence of background flux, i.e. $G_4 = 0$. This amounts to evaluating (4.61a) for the gauge and (4.81a) for the gravitational anomalies.

The various 7-brane codimension-two matter loci $C_{\mathbf{R}}$ have been listed in (4.142), and in the present example they are all smooth [64] such that the index theorem can be applied as in (4.44). Noticing the matter surfaces (4.142) $C_{\mathbf{R}}$ can always be written as intersections of two divisors $A, B$ of the base $B_4$, With the adjunction formula we can obtain

$$\chi(C_{\mathbf{R}}) = \frac{1}{24} A \cdot B \cdot (2c_2 - c_1^2 + A^2 + B^2) \tag{4.149}$$

Applying to (4.142), we find the following flux independent part of the chiral indices for the matter surfaces,

$$\begin{aligned}
\chi(\mathbf{10}_1)|_{\text{geom}} &= \frac{1}{24} c_1 W \left(2c_2 + W^2\right) \\
\chi(\mathbf{5}_3)|_{\text{geom}} &= \frac{1}{24} W \left(3c_1 - 2W\right) \left(-12c_1 W + 8c_1^2 + 2c_2 + 5W^2\right) \\
\chi(\mathbf{5}_{-2})|_{\text{geom}} &= \frac{1}{12} W \left(5c_1 - 3W\right) \left(-15c_1 W + 12c_1^2 + c_2 + 5W^2\right) \\
\chi(\mathbf{1}_5)|_{\text{geom}} &= \frac{1}{24} \left(4c_1 - 3W\right)\left(3c_1 - 2W\right) \left(24c_1^2 + 2c_2 - 36c_1 W + 13W^2\right) .
\end{aligned} \tag{4.150}$$

The first equation in (4.20), i.e. the purely non-abelian $SU(5)$ gauge anomaly, has been verified in [64]. For this analysis to be self-contained, let us briefly recap the computation as a warmup. With the appropriate anomaly coefficients (4.9), $c^{(2)}_{\mathbf{10}} = 3$, $c^{(2)}_{\mathbf{5}} = 1$, the matter from the 7-brane codimension-two loci contributes to the non-abelian anomaly (4.6)

$$\mathcal{A}_{SU(5)}|_{\text{surface,geom}} = \frac{3}{2}\chi(\mathbf{10}_1)|_{\text{geom}} + \frac{1}{2}\chi(\mathbf{5}_3)|_{\text{geom}} + \frac{1}{2}\chi(\mathbf{5}_{-2})|_{\text{geom}} . \tag{4.151}$$

The chiral matter from the 7-brane bulk transforms in the adjoint with $c^{(2)}_{\mathbf{24}} = 10$ and contributes

$$\mathcal{A}_{SU(5)}|_{\text{bulk,geom}} = 5\chi(\mathbf{24}_0)|_{\text{geom}} = -\frac{5}{24} W \left(c_1 - W\right)\left(W\left(W - c_1\right) + c_2\right) , \tag{4.152}$$

where we have used (4.47). In addition, there is another contribution from anti-chiral fermions generated in the 3-7 sector. These modes transform in representation $\mathbf{5}_{q_1}$ and their chiral index is given by minus the point-wise intersection number $-[W] \cdot [C]$ with $[C] = \frac{1}{24}\pi_*(c_4(\widehat{X}_5))$ in the absence of flux. With the help of (4.139), their $SU(5)$ anomaly contribution follows as

$$\begin{aligned}
\mathcal{A}_{SU(5)}|_{3-7,\text{geom}} &= \frac{1}{2}\chi_{3-7}(\mathbf{5}_{q_1})|_{\text{geom}} = -\frac{1}{2} W \cdot \frac{1}{24}\pi_*(c_4(\widehat{X}_5)) \\
&= -\frac{1}{48} W \cdot (144c_1^3 - 264c_1^2 W + 12c_1 c_2 + 162c_1 W^2 - 30W^3) .
\end{aligned} \tag{4.153}$$

Then the pure non-abelian $SU(5)$ anomalies, in the absence of $G_4$ fluxes, indeed cancel,

$$\mathcal{A}_{SU(5)}|_{3-7,\text{geom}} + \mathcal{A}_{SU(5)}|_{\text{bulk,geom}} + \mathcal{A}_{SU(5)}|_{\text{surface,geom}} = 0\,. \tag{4.154}$$

Now we switch gear to check the cancellation of the $U(1)_A$ gauge anomaly. As we have discussed above, there are two types of charged matter states in the 3-7 sector with different $U(1)_A$ charges. With the help of (4.145), their combined contribution to the abelian anomalies is

$$\mathcal{A}_{U(1)}|_{3-7,\text{geom}} = \frac{5}{2}q_1^2\,\chi_{3-7}(\mathbf{5}_{q_1})|_{\text{geom}} + \frac{1}{2}q_2^2\chi_{3-7}(\mathbf{1}_{q_2})|_{\text{geom}} = -\frac{1}{48}\pi_*(c_4(\widehat{X}_5))\cdot(-30W+50c_1)\,. \tag{4.155}$$

This perfectly cancels the anomalies from the 7-7 sector,

$$\begin{aligned}\mathcal{A}_{U(1)}|_{\text{geom}} &= \frac{1}{2}\sum_{\mathbf{R}} q_A^2(\mathbf{R})\dim(\mathbf{R})\chi(\mathbf{R})|_{\text{geom}}\\ &= \frac{1}{2}\left[10\chi(\mathbf{10}_1)+20\chi(\mathbf{5}_{-2})+45\chi(\mathbf{5}_3)+25\chi(\mathbf{1}_5)+(5q_1^2\,\chi_{3-7}(\mathbf{5}_{q_1})+q_2^2\chi_{3-7}(\mathbf{1}_{q_2}))\right]|_{\text{geom}}\\ &= 0\end{aligned} \tag{4.156}$$

as it must since the Green-Schwarz counterterms vanish in absence of flux.

Finally, let us compute the gravitational anomalies. In absence of flux, gravitational anomaly cancellation is equivalent to (4.81a) over a generic base $B_4$. This equation involves the Chern class $c_5(\widehat{X}_5)$ and $c_4(\widehat{X}_5)$ of the resolved Calabi-Yau five-fold $\widehat{X}_5$. With the help of (4.138), we find

$$\begin{aligned}\Delta\chi_1(\widehat{X}_5) &= -66c_1^4 - 61c_1^3W - {c_1}^2c_2 + \frac{235c_1^2W^2}{4} - \frac{23c_1c_2W}{12} - \frac{76c_1W^3}{3} + \frac{3c_2W^2}{4} + \frac{17W^4}{4}\\ -6c_1\Delta[C] &= 54c_1^4 + 66c_1^3W - \frac{81c_1^2W^2}{2} + \frac{15c_1W^3}{2}\,.\end{aligned} \tag{4.157}$$

Summing both terms up perfectly matches the RHS of (4.81a),

$$\begin{aligned}&-(10\chi(\mathbf{10}_1)+5\chi(\mathbf{5}_{-2})+5\chi(\mathbf{5}_3)+\chi(\mathbf{1}_5)+24\chi(\mathbf{24}_0)-4\chi(\mathbf{24}_0))|_{\text{geom}}\\ =\;&-(12c_1^4 - 5c_1^3W + c_1^2c_2 - \frac{73c_1^2W^2}{4} + \frac{23c_1c_2W}{12} + \frac{107c_1W^3}{6} - \frac{3c_2W^2}{4} - \frac{17W^4}{4})\end{aligned} \tag{4.158}$$

In summary, we have checked that in this example with the absence of $G_4$ fluxes, all types of anomalies are cancelled by themselves and in agreement with (4.20).

### 4.9.3. Flux dependent anomaly relations

In the $SU(5)\times U(1)_A$ model defined by (4.137), there only exist two types of gauge invariant 4-form fluxes $G_4 \in H^{2,2}_{\text{vert}}(\widehat{X}_5)$ compatible with the $SU(5)\times U(1)_A$ gauge group [212]. We choose a basis of fluxes as

$$G_4^A = \pi^*(F)\cdot[U_A] \tag{4.159}$$

$$G_4^\lambda = -\lambda\left(E_2\cdot E_4 + \frac{1}{5}(2E_1 - E_2 + E_3 - 2E_4)\cdot c_1\right)\,. \tag{4.160}$$

Here $[U_A]$ is the 2-form class dual to the non-Cartan $U(1)_A$ divisor $U_A$ defined in (4.143), $\lambda$ is a constant and $F \in H^{1,1}(B_4)$ is an arbitrary class parametrizing the flux. Both $\lambda$ and $F$ are to be chosen such that $G_4 + \frac{1}{2}c_2(\widehat{X}_5) \in H^2(\widehat{X}_5, \mathbb{Z})$. We now analyze the anomaly relations, including the Green-Schwarz terms, for both of these flux backgrounds in turn.

### $G_4^A$ flux

We begin with the flux background (4.159). The cancellation of non-abelian $SU(5)$ gauge anomalies in the presence of $G_4^A$ has already been verified in [64] so that we can focus on evaluating (4.61b), or equivalently (4.20b), for the $U(1)_A$ anomaly. To compute the flux dependent chiral index of the 7-brane various matter states, we need to extract the line bundle $L_{\mathbf{R}}$ defined in (4.45) on the 7-brane codimension-two matter loci. Since $G_4^A$ is simply the gauge flux associated with the non-Cartan factor $U(1)_A$, we know that $\pi_*(G_4^A \cdot S^a_{\mathbf{R}}) = q_A(\mathbf{R}) F|_{C_{\mathbf{R}}}$. It follows that

$$c_1(L_{\mathbf{10}_1}) = F|_{C_{\mathbf{10}_1}}\,, \qquad c_1(L_{\mathbf{5}_3}) = 3\,F|_{C_{\mathbf{5}_3}}\,, \qquad c_1(L_{\mathbf{5}_{-2}}) = -2F|_{C_{\mathbf{5}_{-2}}}\,, \qquad c_1(L_{\mathbf{1}_5}) = 5\,F|_{C_{\mathbf{1}_5}} \tag{4.161}$$

and therefore

$$\chi(\mathbf{10}_1)|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{10}_1}} F^2, \qquad \chi(\mathbf{5}_{-2})|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{5}_{-2}}} (-2F)^2 \tag{4.162}$$

$$\chi(\mathbf{5}_3)|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{5}_3}} (3F)^2, \qquad \chi(\mathbf{1}_5)|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{1}_5}} (5F)^2. \tag{4.163}$$

The anomaly contribution (4.59) from the 3-7-brane sector is

$$\mathcal{A}_{U(1)}|_{3-7,\text{flux}} = -\frac{1}{4}F^2 \cdot_{B_4} \pi_*([U_A]\cdot[U_A]) \cdot_{B_4} \pi_*([U_A]\cdot[U_A]) \tag{4.164}$$

with $\pi_*([U_A]\cdot[U_A]) = -D_A$ as in (4.144). Altogether this gives for the LHS of (4.20b)

$$\begin{aligned} \mathcal{A}_{U(1)}|_{\text{flux}} &= \mathcal{A}_{U(1)}|_{7-7,\text{flux}} + \mathcal{A}_{U(1)}|_{3-7,\text{flux}} \\ &= \frac{1}{2}\left(10\chi(\mathbf{10}_1)|_{\text{flux}} + 20\chi(\mathbf{5}_{-2})|_{\text{flux}} + 45\chi(\mathbf{5}_3)|_{\text{flux}} + 25\chi(\mathbf{1}_5)|_{\text{flux}}\right) - \frac{1}{4}F^2 D_A^2 && (4.165) \\ &= \frac{1}{2}F^2(50c_1 - 30W)^2\,. && (4.166) \end{aligned}$$

This is to be compared to the RHS of (4.20b) given by the Green-Schwarz counterterms (4.60)

$$\begin{aligned} \frac{1}{4\pi}\Omega_{\alpha\beta}\Theta^\alpha_A\Theta^\beta_B &= \frac{1}{2}\pi_*(G_4\cdot G_4)\cdot_{B_4}\pi_*([U_A]\cdot[U_A]) = \frac{1}{2}F\cdot_{B_4}F\cdot_{B_4}\pi_*([U_A]\cdot[U_A])^2 \\ &= \frac{1}{2}F^2(50c_1 - 30W)^2\,. && (4.167) \end{aligned}$$

Hence (4.20b) and therefore (4.61b) hold.

Finally, let us switch to cancellation of the purely gravitational anomaly. Given the above expressions, the LHS of (4.81b) yields

$$-6c_1 \cdot \pi_*(G_4^A \cdot G_4^A) = -6c_1 \cdot F \cdot F \cdot (-D_A) = -6c_1 F^2(-50c_1 + 30W) \tag{4.168}$$

which perfectly matches the RHS of (4.81b) given by

$$2\left(10\chi(\mathbf{10}_1)|_{\text{flux}} + 5\chi(\mathbf{5}_3)|_{\text{flux}} + 5\chi(\mathbf{5}_{-2})|_{\text{flux}} + \chi(\mathbf{1}_5)|_{\text{flux}}\right) = 6c_1F^2(50c_1 - 30W)\,. \tag{4.169}$$

### $G_4^\lambda$ flux

Verifying the anomalies in the presence of flux of the form $G_4^\lambda$ is slightly more involved. In the sequel we heavily build on the analysis of [138], where this gauge background is described, in a compactification to four dimensions, as a 'matter surface flux'. Since the fiber structure is the same, we can extend these results to F-theory compactification on an elliptic 5-fold. Since we are now working over a base of complex dimension four, extra technical complications arise in the computation of the chiral index for the 7-brane ammeter, which we will solve in appendix A.3.

Key to computing the 7-brane matter chiralities is again the induced line bundle $L_{\mathbf{R}} = \pi_*(G_4^\lambda \cdot S^a_{\mathbf{R}})$, given this time by

$$c_1(L_{\mathbf{10}_1}) = \frac{-3\lambda}{5}[Y_1] + \frac{4\lambda}{5}[Y_2], \qquad c_1(L_{\mathbf{5}_3}) = \frac{-2\lambda}{5}[Y_2], \tag{4.170}$$

$$c_1(L_{\mathbf{5}_{-2}}) = \frac{3\lambda}{5}[Y_1] - \frac{2\lambda}{5}[Y_2], \qquad c_1(L_{\mathbf{1}_5}) = 0\,. \tag{4.171}$$

A derivation can be found in section 5 of [138]. By Poincaré duality, the objects $[Y_i]$ describe curve classes on the respective matter codimension-two loci on the base, defined as the intersection loci

$$\begin{aligned} C_{\mathbf{5}_3} \cap C_{\mathbf{10}_1} &= Y_2\,, \\ C_{\mathbf{5}_{-2}} \cap C_{\mathbf{10}_1} &= Y_1 + Y_2\,, \\ C_{\mathbf{5}_{-2}} \cap C_{\mathbf{5}_3} &= Y_2 + Y_3\,. \end{aligned} \tag{4.172}$$

The first Chern classes of the line bundles $L_{\mathbf{10}_1}$ and $L_{\mathbf{5}_3}$ can be expressed as the pullback of divisor classes from $W$ to the respective matter loci,

$$c_1(L_{\mathbf{10}_1}) = \frac{\lambda}{5}\left(-3([Y_2]+[Y_1]) + 7[Y_2]\right)|_{C_{\mathbf{10}_1}} = \frac{\lambda}{5}\left(6c_1 - 5W\right)|_{C_{\mathbf{10}_1}} \tag{4.173}$$

$$c_1(L_{\mathbf{5}_3}) = \frac{\lambda}{5}\left(-2[Y_2]\right)|_{C_{\mathbf{5}_3}} = \frac{\lambda}{5}\left(-2c_1\right)|_{C_{\mathbf{5}_3}}\,. \tag{4.174}$$

Hence we can straightforwardly compute the associated chiralities as integrals on $B_4$

$$\chi(C_{\mathbf{10}_1})|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{10}_1}} c_1^2(L_{\mathbf{10}_1}) = \frac{\lambda^2}{50}\, W \cdot c_1 \cdot (6c_1 - 5W)^2\,, \tag{4.175}$$

$$\chi(C_{\mathbf{5}_3})|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{5}_3}} c_1^2(L_{\mathbf{5}_3}) = \frac{\lambda^2}{50}\, W \cdot (3c_1 - 2W) \cdot 4c_1^2\,. \tag{4.176}$$

By contrast, $c_1(L_{\mathbf{5}_{-2}})$ cannot be interpreted as the class of a complete intersection of a base divisor with $C_{\mathbf{5}_{-2}}$ [138]. Each of the classes $Y_i$ defines a divisor class on $C_{\mathbf{5}_{-2}}$, dual to a curve. The technical difficulty is that $Y_1$ and $Y_2$ separately cannot be written as the pullback of a divisor class from the 7-brane divisor $W$ to $C_{\mathbf{5}_{-2}}$. Rather, on $W$, the curves $Y_i$ are given by intersections

$$Y_1 = a_1 \cap a_{2,1}|_W\,, \qquad Y_2 = a_1 \cap a_{3,2}|_W\,, \qquad Y_3 = a_{4,3} \cap a_{3,2}|_W\,, \tag{4.177}$$

where the class of these Tate coefficients have been listed in (4.140). In appendix A.3 we will discuss how to evaluate the chirality of $\mathbf{5}_{-2}$ despite this complication, our final result being

$$\chi(C_{\mathbf{5}_{-2}})|_{\text{flux}} = \frac{1}{2}\int_{C_{\mathbf{5}_{-2}}} c_1^2(L_{\mathbf{5}_{-2}}) = -\frac{\lambda^2}{25} c_1 \cdot W \cdot \left(60c_1^2 - 79c_1 W + 25W^2\right) . \tag{4.178}$$

In light of the discussion of section (4.9.1), the chiral indices for the 3-7 matter states as induced by $G_4^\lambda$ take the form

$$\chi_{3-7}(\mathbf{5}_{q_1}) = -[C]|_{\text{flux}} \cdot W \tag{4.179}$$
$$\chi_{3-7}(\mathbf{1}_{q_2}) = -[C]|_{\text{flux}} \cdot (-5W + 8c_1)\,, \tag{4.180}$$

where the flux dependent piece of the 3-brane class reads

$$[C]|_{\text{flux}} = -\frac{1}{2}\pi_*(G_4^\lambda \cdot G_4^\lambda) = -\frac{\lambda^2}{10} W \cdot c_1 \cdot (6c_1 - 5W)\,. \tag{4.181}$$

To derive this latter result, recall from section 4.3 of [138] that up to irrelevant correction terms $G_4^\lambda$ for $\lambda = 1$ is the class associated with one of the matter fibrations $S^a_{\mathbf{10}_1}$. The result for $\pi_*(G_4^\lambda \cdot G_4^\lambda) = \lambda\, \pi_*(G_4^\lambda \cdot S^a_{\mathbf{10}_1}$ can then be read off from (4.173).

We are finally in a position to check the cancellation of anomalies in the presence of $G_4^\lambda$, beginning with the pure non-abelian gauge anomaly. Note the $G_4^\lambda$ background does not induce any chirality for the 7-brane bulk matter. Together with the above explicit expressions for chiral indices in the 7-brane and the 3-7 sector, one can easily confirm that

$$\mathcal{A}_{SU(5)}|_{\text{flux}} = \frac{3}{2}\chi(C_{\mathbf{10}_1})|_{\text{flux}} + \frac{1}{2}\chi(C_{\mathbf{5}_3})|_{\text{flux}} + \frac{1}{2}\chi(C_{\mathbf{5}_{-2}})|_{\text{flux}} + \frac{1}{2}\chi_{3-7}(\mathbf{5}_{q_1})|_{\text{flux}} = 0\,. \tag{4.182}$$

Next we turn to the $G_4^\lambda$ dependent part of the abelian gauge anomalies. The combined 1-loop anomaly from the 7-7 and the 3-7 matter evaluates to

$$\begin{aligned}
\mathcal{A}_{U(1)_A}|_{\text{flux}} &= \frac{1}{2}\sum_{\mathbf{R}} \dim(\mathbf{R})\, q_A^2(\mathbf{R})\, \chi(\mathbf{R})|_{\text{flux}} \\
&= \frac{1}{2}\left(10\chi(\mathbf{10}_1) + 20\chi(\mathbf{5}_{-2}) + 45\chi(\mathbf{5}_3) + 25\chi(\mathbf{1}_5) + 5q_1^2\, \chi_{3-7}(\mathbf{5}_{q_1}) + q_2^2\chi_{3-7}(\mathbf{1}_{q_2})\right)|_{\text{flux}} \\
&= \frac{1}{2}\lambda^2\, c_1^2\, W^2\,.
\end{aligned} \tag{4.183}$$

For the 3-7 contribution we can either use (4.179) with the charge assignments (4.147), or directly evaluate the $G_4^\lambda$ dependent component of (4.57). The combined 1-loop anomaly forms the LHS of (4.20b) and must be cancelled by the Green-Schwarz terms (4.60) appearing on the RHS. To compute the latter, we make again use of the interpretation of $G_4^\lambda$ as one of the matter fibrations $S^a(\mathbf{10}_1)$. Intersection this with the $U(1)_A$ generator $U_A$ in the fiber reproduces the $U(1)_A$ charge of $\mathbf{10}_1$ and therefore

$$\pi_*(G_4 \cdot [U_A]) = \lambda\, C_{\mathbf{10}_1} \cdot W = \lambda\, c_1 \cdot W\,. \tag{4.184}$$

With this the Green-Schwarz terms are

$$\frac{1}{4\pi}\Omega_{\alpha\beta}\Theta_A^\alpha\Theta_B^\beta = \frac{1}{2}\pi_*(G_4^\lambda \cdot [U_A]) \cdot \pi_*(G_4^\lambda \cdot [U_A]) = \frac{1}{2}\lambda^2\, c_1^2 \cdot W^2\,. \tag{4.185}$$

This perfectly cancels the 1-loop anomalies (4.183) and hence verifies the $G_4^\lambda$ dependent part of

(4.20b) or equivalently (4.61b).

As for the cancellation of the gravitational anomalies, with the help of (4.181), the LHS of (4.81b) becomes

$$-6c_1 \cdot \pi_*(G_4^\lambda \cdot G_4^\lambda) = -\frac{6}{5}\lambda^2\, c_1^2 \cdot W \cdot (6c_1 - 5W), \tag{4.186}$$

which is again exactly equal to the RHS of (4.81b)

$$2\,(10\chi(\mathbf{10}_1) + 5\chi(\mathbf{5}_3) + 5\chi(\mathbf{5}_{-2}) + \chi(\mathbf{1}_5))|_{\text{flux}} = -\frac{6}{5}\lambda^2\, c_1^2 \cdot W \cdot (6c_1 - 5W)\,. \tag{4.187}$$

## 4.10. Comparison to 6D and 4D Anomaly Relations

In this final section we compare the 2D anomaly relations (4.61) and (4.78) to their analogue in a 6D or 4D F-theory compactification on an elliptic fibration $\widehat{X}_3$ or $\widehat{X}_4$ 3.4.3, respectively.

This raises the question if the same type of relations also holds on elliptically fibred Calabi-Yau 5-folds and if they play any role in anomaly cancellation in the associated 2D (0,2) theories.

The situation in compactifications to two dimensions looks rather more involved at first sight: As we have shown in section 4.5, there are two types of independent anomaly relations, (4.61), associated with the cancellation of the gauge anomaly, and another two, (4.78), for the pure gravitational anomaly. We will now see that the flux dependent part of these anomaly relations, (4.61b) and (4.78b), is in fact closely related in form to (3.57) and (3.58).

Consider first relation (4.61b) for the cancellation of the flux dependent part of the 2D gauge anomalies,

$$\begin{aligned}&\sum_{\mathbf{R},a} \beta^a_\Lambda(\mathbf{R})\beta^a_\Sigma(\mathbf{R})\pi_*(G_4 \cdot S^a_{\mathbf{R}}) \cdot_{C_{\mathbf{R}}} \pi_*(G_4 \cdot S^a_{\mathbf{R}}) \\ &= \pi_*(G_4 \cdot G_4) \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) + \pi_*(G_4 \cdot \mathfrak{F}_\Sigma) \cdot_{B_4} \pi_*(G_4 \cdot \mathfrak{F}_\Lambda) + \pi_*(G_4 \cdot \mathfrak{F}_\Lambda) \cdot_{B_4} \pi_*(\mathfrak{F}_\Sigma \cdot G_4)\,.\end{aligned} \tag{4.188}$$

A priori (4.188) holds for every transversal flux $G_4$, i.e. for every element $G_4 \in H^{2,2}(\widehat{X}_5)$ satisfying (4.36), including potentially non gauge invariant fluxes. Our first observation is that this relation can be generalized to

$$\begin{aligned}&\sum_{\mathbf{R},a} \beta^a_\Lambda(\mathbf{R})\beta^a_\Sigma(\mathbf{R})\pi_*(G_4^{(1)} \cdot S^a_{\mathbf{R}}) \cdot_{C_{\mathbf{R}}} \pi_*(G_4^{(2)} \cdot S^a_{\mathbf{R}}) \\ &= \pi_*(G_4^{(1)} \cdot G_4^{(2)}) \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) + \pi_*(G_4^{(1)} \cdot \mathfrak{F}_\Sigma) \cdot_{B_4} \pi_*(G_4^{(2)} \cdot \mathfrak{F}_\Lambda) + \pi_*(G_4^{(1)} \cdot \mathfrak{F}_\Lambda) \cdot_{B_4} \pi_*(\mathfrak{F}_\Sigma \cdot G_4^{(2)})\end{aligned} \tag{4.189}$$

valid for all transversal fluxes $G_4^{(1)}$ and $G_4^{(2)}$: To see this, insert the ansatz $G_4 = G_4^{(1)} + G_4^{(2)}$ into (4.188). This gives three types of contributions, one depending quadratically on $G_4^{(1)}$ and on $G_4^{(2)}$, respectively, and a cross-term involving $G_4^{(1)}$ and $G_4^{(2)}$. Since the quadratic terms vanish by themselves thanks to (4.188), this is enough to establish the more general relation (4.189).

Let us now specialise one of the fluxes appearing in (4.189) to

$$G_4^{(1)} = \pi^* D \cdot \mathfrak{F}_\Gamma \qquad \text{with} \quad D \in H^{1,1}(B_4) \tag{4.190}$$

and analyze the resulting identity further by repeatedly using the projection formulae

$$\pi_*(\pi^* A \cdot_{\widehat{X}_5} B) \;=\; A \cdot_{B_4} \pi_*(B) \tag{4.191}$$
$$\pi_*(E) \cdot_{B_4} F \;=\; E \cdot_{\widehat{X}_5} \pi^*(F) \tag{4.192}$$

for suitable cohomology classes on $B_4$ and $\widehat{X}_5$. In the sequel, unless specified explicitly, the symbol $\cdot$ denotes the intersection product on $\widehat{X}_5$. Then with (4.190) the first term on the RHS takes the form

$$\begin{aligned} \pi_*(G_4^{(1)} \cdot G_4^{(2)}) \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) &= \Big(D \cdot_{B_4} \pi_*(\mathfrak{F}_\Gamma \cdot G_4^{(2)})\Big) \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma) &(4.193)\\ &= G_4^{(2)} \cdot \mathfrak{F}_\Gamma \cdot \pi^*(D \cdot_{B_4} \pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma)) &(4.194)\\ &= \pi^* D \cdot G_4^{(2)} \cdot \mathfrak{F}_\Gamma \cdot \pi^*\pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_\Sigma)\,. &(4.195)\end{aligned}$$

Similar manipulations for the remaining two other terms on the RHS of (4.188) yield

$$\text{RHS of (4.188)} = 3\,\pi^* D \cdot G_4^{(2)} \cdot \mathfrak{F}_{(\Gamma} \cdot \pi^*\pi_*(\mathfrak{F}_\Lambda \cdot F_{\Sigma)})\,. \tag{4.196}$$

As for the LHS, observe that

$$\pi_*(G_4^{(1)} \cdot S^a_{\mathbf{R}}) = \pi_*(\pi^* D \cdot \mathfrak{F}_\Gamma \cdot S^a_{\mathbf{R}}) = \beta^a_\Gamma(\mathbf{R})\,(D \cdot_{B_4} C_{\mathbf{R}})\,. \tag{4.197}$$

Here we are using that in expressions of this form, the intersection of the divisor $\mathfrak{F}_\Gamma$ with the matter 3-cycle $S^a_{\mathbf{R}}$ in the fibre reproduces the charge $\beta^a_\Gamma$ of the associated state with respect to $U(1)_\Gamma$. As explained around (4.45), the expression on the right of (4.197) is the first Chern class of the line bundle induced by the specific flux $G_4^{(1)}$ to which the matter states on $C_{\mathbf{R}}$ couple. For the special choice (4.190) this line bundle is the pullback of a line bundle from $B_4$. With this understanding, the intersection product appearing on the LHS can be further simplified as

$$\pi_*(G_4^{(1)} \cdot S^a_{\mathbf{R}}) \cdot_{C_{\mathbf{R}}} \pi_*(G_4^{(2)} \cdot S^a_{\mathbf{R}}) = \beta^a_\Gamma(\mathbf{R})\,\pi^* D \cdot G_4^{(2)} \cdot S^a_{\mathbf{R}}\,. \tag{4.198}$$

Altogether we have thus evaluated (4.189), for the special choice (4.190), to

$$\pi^* D \cdot G_4^{(2)} \cdot \left( \sum_{\mathbf{R},a} \beta^a_\Gamma(\mathbf{R})\,\beta^a_\Lambda(\mathbf{R})\,\beta^a_\Sigma(\mathbf{R})\,S^a_{\mathbf{R}} - 3\,\mathfrak{F}_{(\Gamma} \cdot \pi^*\pi_*(\mathfrak{F}_\Lambda \cdot \mathfrak{F}_{\Sigma)}) \right) = 0\,. \tag{4.199}$$

Repeating the same steps for the flux dependent gravitational anomaly relation (4.78b) leads to

$$\pi^* D \cdot G_4^{(2)} \cdot \left( \sum_{\mathbf{R},a} \beta^a_\Lambda(\mathbf{R})\,S^a_{\mathbf{R}} + 6\,\mathfrak{F}_\Lambda \cdot c_1 \right) = 0\,. \tag{4.200}$$

The terms in brackets are identical in form with the linear combinations of 4-form classes which are guaranteed to vanish on an elliptically fibered Calabi-Yau 3-fold and 4-fold by anomaly cancellation according to (3.57) and (3.58). We conclude that *if* the relations (3.57) and (3.58) hold also within $H^4(\widehat{X}_5)$, as suggested by the results of [169], this implies cancellation of the flux dependent part of the anomalies in 2D F-theory vacua for the special choice of flux (4.190). For more general fluxes, however, the constraints imposed on anomaly cancellation on a Calabi-Yau 5-fold seem to be stronger. In particular, a direct comparison with (3.57) and (3.58) is made difficult by the fact that (4.61b) and (4.78b) are quadratic in fluxes and a priori involve the intersection product on the matter loci $C_{\mathbf{R}}$, not on $B_4$. For general $G_4$ backgrounds, this makes a difference, as we have seen in section 4.9.3. Furthermore, anomaly cancellation in 2D predicts the flux independent relations (4.61a) and (4.78a). Condition (4.78a) can be viewed as analogous, though very different in form, to the geometric condition on cancellation of the purely gravitational anomalies in 6D referred to in footnote 12. It would be very interesting to investigate if a deconstruction of the topological invariants appearing in (4.61a) and (4.78a), similar to the procedure applied for the Euler characteristic on Calabi-Yau 3-folds in [91, 171], can lead to a geometric proof of these identities.

## 4.11. Outlook

It is instructive to compare these 2D anomaly cancellation conditions to their analogue in 6D and 4D F-theory vacua in the form put forward in [123] and [170], respectively. The structure of anomalies as such becomes more and more constraining in higher-dimensional field theories. At the same time the engineering of the quantum field theory in terms of the internal geometry becomes more intricate as the dimension of the compactification space increases, and hence the number of large spacetime dimensions decreases. Correspondingly, the topological identities governing anomaly cancellation on elliptic 5-folds contain considerably more structure compared to their analogues in 4D and 6D F-theory compactifications. For once, the anomaly relations in 6D $N = (1, 0)$ F-theory vacua are only sensitive to the topology of the elliptic fibration, while in 4D $N = 1$ theories they are linearly dependent on a gauge flux. In 2D $N = (0, 2)$ F-theories, both a purely topological and a flux dependent contribution arises. The latter is, in fact, quadratic in the gauge background.

Despite differences in structure, the 6D and 4D gauge anomaly relations of [123] and [170] can be reduced to one single identity [169], valid in the cohomology ring $H^{2,2}(\widehat{X}_n)$ of an elliptically fibered Calabi-Yau $n$-fold, with $n = 3$ and 4, respectively. The same is true for their mixed gauge-gravitational counterparts. One motivation for the present work was to investigate these universal identities, (3.57) and (3.58), with respect to anomaly cancellation in 2D F-theories. The flux-dependent parts of (4.61) and (4.78) exhibit striking similarities to (3.57) and (3.58). We have shown that if the 6D and 4D universal relations hold also in the cohomology ring of an elliptic 5-fold, as suggested by the examples studied in [169], they imply the flux dependent anomaly relations at least for the subset of gauge backgrounds associated with massless $U(1)$ gauge groups. It would be very interesting to study further if also the converse is true, i.e. if the 2D relations allow us to establish a relation in the cohomlogy ring of elliptic 5-folds governing the 4D and 6D anomalies as well. The flux-independent anomaly relations, on the other hand, seem not to be related in a straightforward manner to the structure of anomalies in higher dimensions. In fact, already in 6D $N = (1, 0)$ F-theory vacua, cancellation of the purely gravitational anomalies implies another topological identity with no counterpart in 4D. This relation has been proven for generic Weierstrass model in [171] using a deconstruction of the Euler characteristic of elliptic 3-folds. It would be worthwhile exploring if a similar proof is possible on Calabi-Yau 5-folds.

The structure of anomalies in 6D and 4D F-theory vacua is closely related to the Chern-Simons terms in the dual M-theory in five [151, 215, 216] and three dimensions [170, 217, 218]. In [172] this reasoning has lead to a proof of anomaly cancellation in 4D $N = 1$ vacua obtained as F-theory on an elliptic Calabi-Yau 4-fold. It would be very interesting to extend such reasoning also to the 2D case. The Chern-Simons terms in the dual 1D $N = 2$ Super-Quantum-Mechanics have been analyzed in [64] and expressed geometrically in terms of data of the Calabi-Yau 5-fold. As expected, the similarities between the resulting identities such as (10.8) in [64] and the 2D anomaly conditions are striking.

At a more technical level, the expressions for the anomalies presented in this work are valid under the assumption that the loci on the base hosting massless matter are smooth. Quite frequently, this assumption is violated, and an application of the usual index theorems requires a normalization of the singular loci [64]. We leave it for future investigations to establish the anomaly relations in such more general situations. Likewise, in the presence of $\mathbb{Q}$-factorial terminal singularities in the fiber the precise counting of uncharged massless states in terms of topological invariants will change. In 6D F-theory vacua, this leads to a modification of the condition for cancellation of the gravitational anomaly [115, 116], and similar effects are expected to play a role in 2D models.

Our focus in this work has been on the implications of anomaly cancellation rather than on the structure of the effective 2D $N = (0, 2)$ supergravity per se. The axionic gaugings induced by the flux background, as derived in this context, give rise to a Kähler moduli dependent D-term, as noted already in [64]. What remains to be clarified is a careful definition of the chiral variables in the supergravity sector and a comparison of the Green-Schwarz action to the superspace formulation put forward in 2D (0,2) gauge theories in [201–203]. This will also determine the correct normalization of the D-term. At the level of the supersymmetry conditions induced by the flux, we have made, in passing, an interesting observation: Extrapolating from the situation on Calabi-Yau 4-folds we expect the existence of $G_4$ backgrounds which are not automatically of $(2, 2)$ Hodge type and would hence break supersymmetry [141]. More precisely, whenever $H^{2,2}(\widehat{X}_5)$ contains $(2, 2)$ forms which are not products of $(1, 1)$ forms, it is expected that the Hodge type of a 4-form varies over the complex structure moduli space. This would constrain some of the complex structure of the 5-fold [141]. This makes it tempting to speculate that the contribution of the supergravity sector to the purely gravitational anomaly should change compared to a background without flux. At the same time, the flux dependent contribution to the D3-brane tadpole modifies the class of the D3-branes in the background and therefore also the anomaly contribution from the sector of 3-7 string modes. For consistency, both effects have to cancel each other, which is in principle possible due to the opposite chirality of the fields involved. In this sense the net effect of complex moduli stabilization would be topological, in stark contrast to the situation in 4D $N = 1$ compactifications. More work on elliptically fibered 5-folds is needed to flesh out the details behind this phenomenon.

# Chapter 5.

# Discrete Anomaly Cancellation in F-theory Compactifications

In the previous chapters, we have discussed anomalies associated with continuous gauge symmetries and their cancellation in various dimensions. In addition to the anomaly cancellations of 10d supergravity, we have analyzed anomaly cancellation in F-theory compactifications on Calabi-Yau manifolds of various dimensions. The constraints from anomaly cancellations, as quantum consistency conditions, have manifested themselves in terms of geometric relations on elliptically fibered Calabi-Yau manifolds. Furthermore, they serve a very powerful guiding principle to explore the microscopic theory of gravity, in particular for higher dimensional theories. How about discrete symmetries?

To answer this question, one first needs to determine whether a discrete symmetry is a global symmetry or a gauge symmetry - the difference being that an anomalous global symmetry is perfectly consistent[1]. At first, the question whether a discrete symmetry is a global symmetry or a gauge symmetry might seem a bit elusive. Recall that when gauging a global continuous symmetries one needs to introduce a massless dynamical gauge field $A_p$ coupled to the Noether current associated with the global symmetry, and such a dynamical gauge field leads to observable consequences. However, gauging a discrete symmetry does not introduce any massless gauge fields. From this perspective it might seem that the difference between global or gauged discrete symmetries is artificial[2]. So then how to distinguish them? A discrete gauge symmetry is typically viewed as part of a continuous gauge symmetry in the UV (for instance a $U(1)$ gauge symmetry,) which is broken at a certain scale with a remaining discrete part in the effective theory. With this identification, it then is meaningful to study anomaly cancellation for discrete symmetries. This was the exact idea behind the pioneering work [219], which viewed a discrete symmetry in a four-dimensional effective theory as embedded into an (abelian) continuous gauge symmetry at high energies. The anomalies of the latter hence determine the anomalies of the remaining discrete symmetry at low energies, depending on of the specifics of how the symmetry is broken.

This way of viewing gauged discrete symmetries puts discrete and continuous symmetries on the same footing in the context of the quantum gravity and leads to the "folk theorem" in quantum gravity stating that any global symmetry (including the discrete ones) are forbidden in any consistent quantum gravity theories [133] - unless they are gauged at high energies. String theory, as the most promising candidate so far for the formulation of quantum gravity theory, provides a concrete and computable framework to study discrete symmetries. For example, discrete symmetries of type $\mathbb{Z}_N$ have been discussed in intersecting D-brane models in Type IIA in terms of torsional cohomology [134, 220, 221]. And in an even more geometric

[1]However, this does not mean that anomalies of global symmetries have no significance. Instead, anomalies themselves can reveal various properties of the theory, the typical example being the t' Hooft anomaly.

[2]However, the Hilbert space associated with a global versus a gauged discrete symmetry should be different.

picture, F-theory provides the stage to link a discrete $\mathbb{Z}_N$ symmetry to an $N$-section in F-theory genus-one compactifications [222]. This has been studied with respect to its phenomenological and formal aspects, for example in [132, 135, 136, 223–225].

With the above understanding of discrete gauge symmetries, it is natural to explore whether the anomaly constraints imposed on the underlying continuous gauge symmetries in the UV lead to constraints on the massless spectra charged under a discrete symmetry in the effective theory at lower energies. If yes, it is natural to treat these residual constraints as consistency conditions for the discrete gauge symmetry. Indeed, this is the exact the idea in the pioneering work [219] for studying the anomaly cancellation for discrete symmetries. Since then, many exciting works have followed, including [226, 227] and references therein. And other investigations such as [220, 225] apply the above ideas to model building in string theory.

In this chapter, we are going to study anomaly cancellation in the context of discrete gauge symmetries in F-theory compactifications. In particular we focus on a 6D $\mathcal{N} = (1, 0)$ supergravity theory. As we discussed in 2.10, the type of fibrations giving rise to such discrete symmetries shall be the genus-one fibrations whose fiber is a genus-one curve without any marked rational points such that the fibrations do not exhibit any global sections. Similarly to the ideas in chapter 3, we will identify the correspondence between the geometry of the genus-one fibrations and the anomaly coefficients in the anomaly equations of the discrete symmetries.

## 5.1. Anomaly Cancellation and GS Mechanisms in 6D

In chapter 3, we have described anomaly cancellation in 6D $\mathcal{N} = (1, 0)$ effective theories describing F-theory compactifications on Calabi-Yau three-folds $X_3$. As we have seen, anomaly cancellation for continuous gauge symmetries imposes very strong constraints on the spectrum of the massless fields, as well as the structure of the theory. In the context of F-theory compactifications, the anomaly coefficients $a, b$ in the 6D $\mathcal{N} = (1, 0)$ supergravity have a clear interpretation in terms of the geometry of the Calabi-Yau three-fold $X_3$. Anomaly cancellation heavily involves the generalized Green-Schwarz mechanism. The essence of the Green-Schwarz mechanism studied so far in the context of continuous gauge symmetries consists in gauging the shift symmetry of certain two-form fields $B_2$ with respect to a gauge transformation of the 1-form gauge fields under consideration.

However, there is another type of generalized Green-Schwarz mechanism for the cancellation of anomalies of abelian gauge symmetries in six dimensions. The distinguishing feature of this type of Green-Schwarz mechanism is, as pointed out in [155], that after anomaly cancellation the anomalous $U(1)$s receive a mass by the Stückelberg mechanism and are hence lifted from the massless spectrum. It might therefore seem that such an anomaly cancellation mechanism does not contain much information on the massless spectrum. However, as we will show, the GS (or rather the involved Stückelberg) mechanism may leave us with a discrete gauge symmetry under which parts of the massless spectrum may carry charge. Those charged states will then be subject to certain consistency conditions descending from the underlying abelian continuous gauge anomaly cancellation. In the UV, one should be able to match the anomalies of the continuous and its remnant discrete gauge symmetry. In this chapter, we are going to explore the anomaly equations associated with this type of anomaly cancellation and find the corresponding geometric quantities upon embedding the 6D $\mathcal{N} = (1, 0)$ theory into F-theory setting. To our knowledge, the detailed anomaly cancellation equations from this type of Green-Schwarz mechanism have not been studied in F-theory language in the literature before, though several related aspects have been covered in [131, 155, 225].

For simplicity of presentation, we first assume only one abelian $U(1)$ gauge field $A$ with field strengt $F$ participating in this type of Green-Schwarz mechanism. As alluded to above, in the Green-Schwarz mechanism one can invoke another tree-level gauge variant Green-Schwarz term, different from the type we discussed in 3.3, to cancel the anomaly associated with the corresponding $U(1)$ symmetry. It is given by

$$S^{GS2} = -\frac{2\pi}{2}\int_{R^{1,5}} \mathcal{K}_{ab}\phi^a \wedge \widetilde{X}_6^b\,, \tag{5.1}$$

where $\phi^a$ is a Stückelberg 0-form that belongs to a linear multiplet.[3] It has the following gauge transformation

$$\phi^a \to \phi^a + q^a\lambda, \qquad A \to A - d\lambda\,. \tag{5.3}$$

This amounts to gauging the shift symmetry associated with the axions $\phi^a$, with $q^a$ being the charge of $\phi^a$ under the gauge field $A$. The quantity $\widetilde{X}_6$ in our setting is given by

$$\widetilde{X}_6^a = -\sum(\frac{1}{6}F^3 + \frac{1}{96}F\mathrm{tr}R^2)c^a\,, \tag{5.4}$$

where $c^a$ is the anomaly coefficient. The precise form of (5.24) will be derived in next section 5.2.2. Furthermore, in order for the GS mechanism to cancel the anomalies, as usual, the 8D anomaly polynomial $I_8^{1-loop,U(1)}$ involving the abelian $U(1)$ gauge fields must factorize as

$$I_8^{1-loop,U(1)} = \frac{1}{2}F \wedge X_6\,. \tag{5.5}$$

Here $X_6$ denotes a 6-form anomaly polynomial, which we have listed in (3.18), and $F$ denotes the abelian gauge field strength participating in this type of Green-Schwarz mechanism.

Now, by following the standard procedure as in section 3.3, under the above gauge transformation (5.3) the GS term contributes a gauge variance to the effective action as

$$\delta S_{GS2} = -\frac{2\pi}{2}\int_{R^{1,5}} \mathcal{K}_{ab}q^a\lambda\widetilde{X}_6^b =: 2\pi\int_{\mathbb{R}^{1,5}} I_6^{(1),\mathrm{GS2}}(\lambda)\,, \tag{5.6}$$

where $I_6^{(1),\mathrm{GS2}}$ is a gauge invariant 6-form. By the standard descent procedure, it defines an anomaly-polynomial $I_8^{\mathrm{GS2}}$ encoding the contribution to the total anomaly from the Green-Schwarz sector. Concretely, the descent equations

$$I_8^{\mathrm{GS2}} = dI_7^{\mathrm{GS2}}, \qquad \delta_\lambda I_7^{\mathrm{GS2}} = dI_6^{(1),\mathrm{GS2}}(\lambda) \tag{5.7}$$

[3] 6D $\mathcal{N} = (1,0)$ theories (as well as four- and five dimensional theories with the same amount of supersymmetries) have two different kinds of scalar multiplets associated with different representations under the $SU(2)_R$ R-symmetry. The scalar components can transform either as **2** or $\mathbf{3}+\mathbf{1}$ under the $SU(2)_R$ R-symmetry. We refer to the first one as hypermultiplets and to the second as linear multiplets. The spinor components of the linear multiplet, opposite to the singlet in the hypermultiplet, transform as **2** under the R-symmetry. With the $\mathbf{3}+\mathbf{1}$ representation of $SU(2)_R$, the linear multiplets in general can not couple to the vector multiplets in the standard way due to no invariant $SU(2)$ term, except the cases with U(1) vector multiplets. Denoting $B, C$ as two chiral fields of a linear multiplet $L$, then the coupling to a $U(1)$ vector multiplete $V$ is given by

$$\int d^4xd^4\theta(\frac{1}{2}(iC - iC^\dagger - V)^2 + B^\dagger B) - \frac{1}{\sqrt{2}}\int d^4xd^2\theta B\Phi + (h.c.) \tag{5.2}$$

which is exactly a Stückelberg coupling form.

imply

$$I_8^{\mathrm{GS2}} = -\frac{1}{2} F \wedge \mathcal{K}_{ab} q^a \widetilde{X}_6^a \,. \tag{5.8}$$

As we will derive later in 5.2.2, it turns out that

$$I_8^{\mathrm{GS2}} = \frac{1}{2}[\frac{1}{6}(F^3F) + \frac{1}{96}\mathrm{tr}R^2(FF)](c\cdot c)\,. \tag{5.9}$$

Combining with the relevant $U(1)$ part in the first type of the GS mechanism $I_8^{\mathrm{GS1,\ U(1)}}$ listed in 3.3, consistency of the theory then requires that

$$I_8^{1-loop,U(1)} + I_8^{\mathrm{GS1,\ U(1)}} + I_8^{\mathrm{GS2}} = 0\,. \tag{5.10}$$

By comparing the coefficients, we obtain the following anomaly equations:

$$\boxed{\begin{aligned} &-2c\cdot c + 3(b\cdot b) = \sum_{q^a} n_{q^a}(q^a)^4, \\ &-\frac{c\cdot c}{2} - 6(a\cdot b) = \sum_{q^a} n_{q^a}(q^a)^2, \\ &(a\cdot a) = 9 - T, \\ &H_m - V + 29T = 273. \end{aligned}} \tag{5.11}$$

In these expressions, $q^a$s denote the charges under the abelian symmetry $U(1)$ and $n_{q^a}$ stands for the number of hypermultiplets with the charge $q^a$ under the $U(1)$.

At the level of the effective action, adding the counter-term (5.1) is equivalent to a Stückelberg-type modification of the kinetic term of $\phi$ as

$$S_{\mathrm{St}} = \frac{1}{2}\mathcal{K}_{ab}(d\phi^a - q^a A)\wedge *(d\phi^b - q^b A)\,. \tag{5.12}$$

As a consequence, the linear multiplet associated with the scalar $\phi^a$ combines with the abelian vector multiplet $V$ associated with the gauge field $A$ into forms a long multiplet, where the gauge field $A$ becomes massive. This is different from the first type of Green-Schwarz mechanism 3.3, where the Abelian gauge symmetries stay massless after the anomaly cancellation whereas the 2-form gauge symmetry associated with 2-form $B_{\mu\nu}$ becomes massive.

Note that this distinction between both types of generalized Green-Schwarz terms does not occur in four-dimensional theories: Here a 2-form is dual to a scalar field, and consequently the Green-Schwarz mechanisms gauging both types of fields are equivalent. In particular, while in six-dimensional theories only the Green-Schwarz term of type (5.1) gives a Stückelberg mass to a gauge field $A$, in four dimensions the Green-Schwarz mechanism for U(1)s always leads to massive gauge fields.

The above anomaly equations generalize to cases with generic gauge groups. For example in the case of one massless $U(1)_A$ (i.e. participating only in the GS mechanism of type (3.3)), a non-abelian gauge group $G_I$ and multiple massive $U(1)_i$ (i.e. participating both in (5.1) and possibly also in (3.3)) the anomaly equations are modified to

$$\boxed{\begin{aligned}
&-2c_i\cdot c_i+3(b_{ii}\cdot b_{ii})=\sum_{q_i} n_{q_i}q_i^4,\\
&-\frac{c_i\cdot c_j}{2}-6(a\cdot b_{ij})=\sum_{q_i,q_j} n_{q_i,q_j}q_iq_j,\\
&b_{ii}\cdot b_{jj}+2b_{ij}\cdot b_{ij}=\sum_{q_i,q_j} n_{q_i,q_j}q_i^2q_j^2,\\
&b_{ij}\cdot b_{kl}+b_{il}\cdot b_{kj}+b_{ik}\cdot b_{jl}=\sum_{q_i,q_j,q_k,q_l} n_{q_i,q_j,q_k,q_l}q_iq_jq_kq_l,\\
&3b_{ii}\cdot b_{AA}=\sum_{q_i,q_A} n_{q_i,q_A}q_i^2q_A^2,\\
&-c_i\cdot c_i+b_{ii}\cdot\frac{b_I}{\lambda_I}=\sum_{q_i,R_I} n_{q_i,R_I}A_{R_I}q_i^2.
\end{aligned}} \tag{5.13}$$

Here $q_i$ collectively denotes the charges $q_i^a$ under the massive $U(1)_i$, and $q_A$ under the massless $U(1)_A$; the remaining notation follows the conventions of chapter 3.

In the next section, we will mainly study this GS mechanism in the context of Type IIB orientifold/F-theory compactifications. In the context of type IIB orientifold/F-theory compactifications, the abelian gauge $U(1)$ stays as a massive $U(1)$ gauge symmetry at the perturbative level by participating in the Green-Schwarz mechanism (5.1) and will be broken to a discrete symmetry $\mathbb{Z}_N$ by instanton corrections if certain conditions are satisfied. Recall that in the F-theory compactification, the appearance of a discrete symmetry $\mathbb{Z}_N$ in the effective theory typically requires a genus-one fibration compactification, as we have introduced in 2.10,; one would therefore expect that the anomaly polynomials coefficients $a$,$b$ and $c$'s shall have some geometric interpretation in the background geometry of the genus-one fibration compactifications, similarly to what we have discussed in chapter 3 for the anomaly coefficients $a, b$ in elliptic fibrations. We will also show that even a $\mathbb{Z}_1$ symmetry, where the massive $U(1)$ is broken completely, leaves some imprints on the geometry, in the form of terminal singularities as discussion in chapter 2.8.

## 5.2. Embedding to the Type IIB Orientifold/F-theory

In this section, we will embed that 6D $\mathcal{N}=(1,0)$ supergravity into a Type IIB orientifold compactification and, we will show that the Green-Schwarz mechanism we are discussing essentially corresponds to a "geometric Stückelberg" mechanism.

### 5.2.1. Setup and notation

We consider a type IIB string theory orientifold compactification (see more general discussion in chapter 1) on a compact CY 2-fold $X_2$ (namely a K3), not necessarily being elliptic fibered. The orientifold involution $\sigma$ acts on the Kähler form $J$ and the holomorphic 2-form $\Omega_2$ of $X_2$ as

$$\sigma^*J=+J,\qquad \sigma^*\Omega_2=-\Omega_2\,. \tag{5.14}$$

The holomorphic involution $\sigma$ introduced above gives induces a splitting of the cohomology groups $H^{p,q}(X_2, Z)$ into even and odd eigenspaces of $\sigma^*$,

$$H^{p,q}(X_2, Z) = H_+^{p,q} \bigoplus H_-^{p,q} \,. \tag{5.15}$$

We thus introduce a basis $w_+^\alpha$, $\alpha = 1, ..., h_+^{1,1}(X_2)$ and $w_-^a$, $a = 1, ..., h_-^{1,1}(X_2)$ of 2-forms for the eigenspaces of the cohomology groups of $X_2$ with intersection $\int_{X_2} w_+^\alpha \wedge w_-^a = 0$. Note that on the Ramond-Ramond $p$-form potentials, $\Omega(-1)^{F_L} C_2 = -C_2$, $\Omega(-1)^{F_L} C_4 = C_4$, hence, we decompose the RR fields as

$$C_2 = c_2 + \sum_a \phi^a w_-^a, \qquad C_4 = c_4 + \sum_\alpha (B_2)^\alpha w_+^\alpha + \cdots \,, \qquad C_6 = c_6 + \sum c_4^a \wedge w_-^a + \cdots \tag{5.16}$$

where the subscript number $p$ in every field indicates a $p$-form field in 6D flat space $\mathbb{R}^{1,5}$, for example, $c_2$ denotes a 2-form field in $\mathbb{R}^{1,5}$. The non-trivial intersection numbers are given by

$$\mathcal{K}_{ab} = \int_X w_-^a \wedge w_-^b, \quad \mathcal{K}_{\alpha\beta} = \int_X w_+^\alpha \wedge w_+^\beta \,, \tag{5.17}$$

while all others vanish.

### 5.2.2. Abelian gauge symmmetries in 6D Type IIB orientifolds

For more details on the Type IIB orientifold compactification, we refer to the introductory parts in 1.9. Here for our purpose, we consider $n$ pairs of D7 branes $D_i$ and their image D7-branes $D_i'$ with the configuration following the case (1) in 1.138, which reads

$$[D_i] \neq [D_i'] \equiv [\sigma^* D_i] \,. \tag{5.18}$$

Here the class $[D_i] \in H^2(X_2)$ is Poincaré dual of the divisor class $D_i$, $i = 1, ..., n$. We introduce the following notation

$$D_i^\pm = D_i \cup (\pm D_i') \tag{5.19}$$

and their Poincaré dual objects $[D_i^\pm] \in H^2(X_2)^\pm$. Note that for orientifold invariant cycles one should include an extra factor of $\frac{1}{2}$ to ensure $D_i^+ = D_i$. For convenience, we introduce the wrapping numbers

$$b_i^\alpha = \int_{D_i^+} w_+^\alpha, \qquad c_i^a = \int_{D_i^-} w_-^a \,. \tag{5.20}$$

In the following we are going to deduce the explicit form of the anomaly coefficients $a, b, c$ in such type IIB orientifold/F-theory compactifications. We will work in the upstairs picture of the orientifold in the derivation.The Lagrangian in question is given by

$$S = \frac{1}{2}\left( S_{\mathrm{IIB}} + \sum_i (S_i^{\mathrm{D7}} + S_{i'}^{\mathrm{D7}}) + S^{\mathrm{O7}} \right) \,. \tag{5.21}$$

Let us first consider the simple case without non-abelian gauge fields, namely only one D7-brane wrapping each divisor $D_i$. The relevant terms of the Chern-Simons action of D-branes are

$$\frac{1}{2}\sum_i (S_i^{\mathrm{D7}} + S_{i'}^{\mathrm{D7}}) \supset -\frac{2\pi}{2}\sum_i \int_{R^{1,5}\times D_i} C_2 \wedge (\mathrm{ch}_3\mathcal{F} - \mathrm{ch}_1\mathcal{F} \wedge \frac{1}{48}(p_1(R))\,. \tag{5.22}$$

In this whole chapter, we set, for simplicity, the NS-NS two-form $B_2$ to zero, and thereby $\mathcal{F} = iF$ in the above CS actions. We can then reduce the CS action explicitly as

$$\begin{aligned} \frac{1}{2}\sum_i (S_i^{\mathrm{D7}} + S_{i'}^{\mathrm{D7}}) &\supset -\frac{2\pi}{2}\sum_i \int_{R^{1,5}} \sum_a \phi_a \wedge [\frac{-1}{6}(F_i)^3 + F_i \wedge \frac{1}{48}(p_1(R)] \int_{D_i^-} w_-^a \\ &= -\frac{2\pi}{2}\sum_i \int_{R^{1,5}} [\sum_a \phi_a \wedge -(\frac{1}{6}F_i^3 + \frac{1}{96}\mathrm{tr}R^2 F_i) \int_{D_i^-} w_-^a]\,. \end{aligned} \tag{5.23}$$

Here we have employed the fact that the gauge field strength $F'$ on the image D7-brane $D_i'$ has opposite sign with the $F$, as shown in (1.140). The above term exactly produces the Green-Schwarz counter term (5.1) and thereby the Stückelberg 0-form comes from RR fields $C_2$ zero modes. Essentially, the GS counterterm arises from gauging the RR fields in the presence of the sources.[4] The fact that the GS counter-terms originate in the Chern-Simons action of the D7-branes is of course no more surprise in view of the preceding results of this thesis.

To simplify the presentation further, let us consider only one stack of D7-brane and its image D7-brane ($i = 1$). Comparing with (5.1), we obtain the quantity $\widetilde{X}_6^a$ as

$$\widetilde{X}_6^a = -\sum_i (\frac{1}{6}F^3 + \frac{1}{96}F\mathrm{tr}R^2)c^a\,, \tag{5.24}$$

where we have substituted the wrapping number (5.20) $c^a = \int_{[D^-]} w_-^a$.

Now we are in the position to determine the charge $q^a$ of $\phi^a$ under the massive gauge field $A$ in order to uplift to the eight-form (5.8). To this end, we need to consider the reduction along $C_6$ of the Chern-Simon action of the D7-brane, which is given by

$$S \supset -\frac{2\pi}{2}\sum_a c^a \int_{R^{1,5}} c_4^a \wedge (-F) = -\frac{2\pi}{2}\sum_a c^a \int_{R^{1,5}} dc_4^a \wedge A = \frac{2\pi}{2}\sum_a c^a \int_{R^{1,5}} (*d\phi^a \wedge A), \tag{5.25}$$

where $F = dA$ and in the last equality, we have employed the 6D duality relation

$$dc_4^a = - * d\phi^a\,, \tag{5.26}$$

whose origin is the 10d duality on RR fields $dC_6 = - * dC_2$. We interpret the above action in terms of the kinetic term of a 6D massive U(1) effective theory,

$$-\frac{2\pi}{4}\mathcal{K}_{ab}\sum_a (d\phi^a - q^a A) \wedge *(d\phi^b - q^b A_i) = \frac{2\pi}{2}\mathcal{K}_{ab} q^a * d\phi^b \wedge A + \dots\,. \tag{5.27}$$

This allows us to read off

$$\mathcal{K}_{ab} q^b = c^a\,. \tag{5.28}$$

[4]One can use an alternative way to derive the GS terms in terms of gauging RR fields $C_2$ and $C_4$, as we showed in section 4.6.

As a result, the explicit form of the 6-form $X_6$ multiplying $-\frac{1}{2}F$ in (5.8) reads

$$\begin{aligned}\mathcal{K}_{ab}q^a\widetilde{X}_6^b &= -[\tfrac{1}{6}F^3+\tfrac{1}{96}F\text{tr}R^2]\textstyle\sum_{ab}(\mathcal{K}_{ab}q^ac^b) &(5.29)\\ &:= -[\tfrac{1}{6}F^3+\tfrac{1}{96}F\text{tr}R^2](c\cdot c)\,. &(5.30)\end{aligned}$$

One should notice that even in such a simple setting, the first type of GS mechanism (3.3) shall play a role in the anomaly cancellation as well. This can be seen from the fact that the divisor class $[D]$ wrapped by single D7-brane in this case can be split as $[D]=m_\alpha w_+^\alpha+n_a w_-^a$. Hence, in order to describe the GS mechanism of type (3.3), one reduces the 4-form anomaly $X_4$ first from the CS action of the 7-branes involving the Ramond-Ramond field $C_4$:

$$\frac{1}{2}S_{D7}+S_{D7'}\supset=-\frac{1}{2}\frac{2\pi}{2}\int_{R^{1,5}}B_2^\alpha\wedge(\frac{1}{2}(F)^2+(\frac{1}{96}\text{tr}R^2))\int_{D+D'}w_+^\alpha\,. \tag{5.31}$$

Note that the O7-plane can also have $C_4$ coupling, by decomposing its Chern-Simons action, we have

$$\begin{aligned}\frac{1}{2}S_{O7}\supset&\frac{8\pi}{2}\int_{R^{1,5}\times[O7]}C_4\wedge(\frac{1}{96}(p_1(R))\\ =&-\frac{2\pi}{2}\int_{R^{1,5}}\sum_\alpha B_2^\alpha\wedge(\frac{1}{192}\text{tr}R^2)\int_{O7}4w_+^\alpha\,.\end{aligned} \tag{5.32}$$

By noting that we have D7-brane tadpole cancellation $[D]+[D']+\Delta'=8[O7]$ in this simple setting, where $\Delta'$ denotes the residual discriminant without gauge fluxes, and by summing all of the D7-brane contributions, we find the explicit form of 4-form polynomial $X_4$ from each D7-brane

$$\begin{aligned}X_4^\alpha=&\frac{1}{2}[\sum_i\frac{1}{2}(F_i^2)\int_{D+D'}w_+^\alpha+\frac{1}{96}\text{tr}(R^2)\int_{\sum_i D+D'+\Delta'}w_+^\alpha+4\frac{1}{192}\text{tr}(R^2)\int_{[O7]}w_+^\alpha]\\ =&\frac{1}{4}[2\sum_i(F^2)b^\alpha-\frac{1}{2}\text{tr}(R^2)a^\alpha],\end{aligned} \tag{5.33}$$

where the prefactor $\frac{1}{2}$ comes from the orientifolding (5.21). Combing with (5.24) and (5.33), we have

$$a^\alpha=-\frac{1}{2}*\frac{1}{12}\int_{D+D'+\Delta'+2O7}w_+^\alpha,\ b^\alpha=\frac{1}{2}\int_{D^+}w_+^\alpha,\ \text{ and } c^a=\int_{[D^-]}w_-^a. \tag{5.34}$$

One can also consider generic configurations, namely with multiple D7-branes and its image branes. This can lead to the results (5.13).

As a last remark, we stress that the derivation of the Green-Schwarz terms in this section is a priori valid in settings where the background of the Type IIB orientifold is a smooth manifold - in this case a Calabi-Yau 2-fold - prior to orientifolding. We will come back to the importance of this caveat in section (5.3.1) in relation with the interpretation of the anomaly equations in an F-theory context.

### 5.2.3. Uplifting to 6D F-theory compactifications

In this section, we are going to determine the geometric quantities corresponding to the anomaly coefficients (5.34) $a, b, c$ in F-theory background geometry. Note that in our simplest setting there is only one D7-brane and image D7-brane with the wrapping divisor in different class. According to the discussion in chapter 2, this typically[5] leads to the conifold $I_1$ model in F-theory. Consequently, the massive $U(1)$ is completely broken due to the charge $q^a = 1$, namely the Calabi-Yau $X_3$ only has $I_1$ singularities along its codimension-one loci. Moreover, the elliptically fibred Calabi-Yau $X_3$ typically has terminal singularities. In the next section, we are mainly talking about the the conifold $I_1$ model in F-theory, here let us first focus on the anomaly coefficients $a, b, c$.

To evaluate (5.34), we need to recall the general relation between the upstairs geometry $X_2$ [6] and the downstairs geometry $B_2$. In the orientifold limit (if it exists), the IIB two-fold $X_2$ is the double cover of $B_2$ and relating by a 2-to-1 map

$$\pi : X_2 \to B_2 \,. \tag{5.35}$$

The uplift of the cohomology groups proceeds according to the familiar rules for comparing the invariant and anti-invariant cohomology groups on $X_2$ to the cohomology groups on the (resolved) elliptic fibration $\widehat{X}_3$. The even and odd cohomology group of $X_2$ lifts to cohomology of $\widehat{X}_3$ by the pushforward as

$$\pi_* : H^{p,q}_+(X_2) \;\; \to \;\; H^{p,q}(B_2) \tag{5.36}$$

$$\pi_* : H^{p,q}_-(X_2) \;\; \to \;\; H^{p+1,q}(\widehat{X}_3) - H^{p+1,q}(B_2), \tag{5.37}$$

respectively. The latter can be thought of as the lift of an anti-invariant $(p, q)$ form to a $(p+1, q)$ form on $\widehat{X}_3$ with one extra leg in the elliptic fiber. The idea is that as one encircles the O7-plane, the monodromy effect in the fiber compensates for the fact that the base component of the form is anti-invariant.

Following the above, it is easy to see that the divisor classes $[w^+_\alpha], \alpha \in h^{1,1}_+(X_2)$ with even eigenvalues under the involution $\sigma$ survive under this map, which we denote $[\omega^\alpha]$ in $H^2(B_2)$, namely we have the pullback

$$\pi^*(\omega^\alpha) = w^+_\alpha \,. \tag{5.38}$$

Note that the intersection numbers in $B_2$ are related to ones in $X_2$ as

$$\int_{X_2} [\pi^*(\omega^\alpha)] \wedge [\pi^*(\omega^\beta)] = \int_{\pi(X_2)} [\omega^\alpha] \wedge [\omega^\beta] = 2 \int_{B_2} \omega^\alpha \wedge \omega^\beta \,, \tag{5.39}$$

where the last equality is due to $\pi(X_2) = 2B_2$ as a consequence of the double covering.

Based on the Kodaira condition (2.44) and the general discussion in section 2.2.2, we know that the first Chern class $c_1(B_2)$ of the base $B_2$ is encoded by the discriminant $\Delta$ as

$$c_1(B_2) = \frac{1}{12}[\mathcal{D} + \Delta'], \tag{5.40}$$

where we have expanded the discriminant $\Delta$ explicitly in our simple setting in terms of the type $I_1$ component $\mathcal{D}$ wrapped by brane-and-image brane configuration and the residual component $\Delta'$ which generically does not support any gauge theory.

Now given these cohomology relations between the upstairs geometry and the downstairs

[5] But not always, such simple setting could also lead to discrete symmetry $\mathbb{Z}_N$ when the charges $q^a > 1$.

[6] By abuse of notations, we denote all the Calabi-Yau as $X_n$ no matter whether it carries an elliptical fibration or not. But as we stressed, the upstair geometry $X_2$ for Type IIB is not necessarily elliptically fibered.

geometry, we can expressed $c_1(B_2)$ in terms of the divisors in the upstairs geometry $X_2$ as

$$c_1(B_2) = 2[O7] + D + D' + \Delta\,. \tag{5.41}$$

Here the $O7$ comes from the $\Delta'$ as we learned from the general discussion in the sen limit 2.3.1 and the prefactor 2 is due to (5.39). Consequently,

$$c_1(B_2) = -a^\alpha w_\alpha, \qquad w_\alpha \in H^2(B_2)\,. \tag{5.42}$$

By the same logic, we argue that $b^\alpha$ is related to the coefficient of the 7-brane divisor $[\mathcal{D}]$ as

$$\mathcal{D} = b^\alpha w_\alpha, \qquad w_\alpha \in H^2(B_2). \tag{5.43}$$

As for as coefficient $c$, based on (5.37), it is typically hard to identify the corresponding physical meaning (if they exist) in the F-theory geometry as it depends on choice of base. Hence we will treat $c$ as a intermediate quantity and express the anomaly equations (5.11) only in terms of $a, b$.

Let us now recall the geometric details of the setup, leading to conifold $I_1$ model, and subsequently verify our anomaly equations in a specific example.

## 5.3. Conifold $I_1$ Model

The $U(1)$ symmetry discussed in previous section will uplift to a conifold $I_1$ model in F-theory, namely the Calabi-Yau $X_3$ only has $I_1$ singularities in codimensional one loci. The geometry is described by aTate model

$$y^2 + xyza_1 + yz^3a_3 = x^3 + x^2z^2a_2 + xz^4a_4 + z^6a_6, a_i \in \Gamma(B_2, -nK_{B_2}). \tag{5.44}$$

The functions $f, g$ of the Weierstrass form are related to $a_i$ by

$$\begin{aligned} f &= -\frac{1}{48}(b_2^2 - 24b_4), \qquad g = -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6) \\ \Delta &= -8b_4^3 + 9b_2b_6b_4 - 27b_6^2 + \frac{1}{4}b_2^2\left(b_2b_6 - b_4^2\right) \end{aligned} \tag{5.45}$$

in terms of

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = a_1a_3 + 2a_4, \qquad b_6 = a_3^2 + 4a_6 \tag{5.46}$$

with $b_n \in \Gamma(B_2, -nK_{B_2})$.

The conifold $I_1$ model is specified by taking $a_i = \widetilde{a}_i w^{k_i}$ with

$$\mathrm{ord}(a_1, a_2, a_3, a_4, a_6)|_{w=0} =: (k_1, k_2, k_3, k_4, k_6) = (0, 0, 1, 1, 1) \tag{5.47}$$

along a divisor $\Sigma_1 : w = 0$ on the base $B_2$. One can check that $\mathrm{ord}(f, g, \Delta)|_{w=0} = (0, 0, 1)$, thereby this is an $A_1$ singularity. The discriminant $\Delta$ can be factorized as

$$\Delta \propto w(\widetilde{a}_6(\widetilde{a}_1^2 + 4\widetilde{a}_2)^3 + ...) \tag{5.48}$$

with $\Sigma_0 := (\widetilde{a}_6(\widetilde{a}_1^2 + 4\widetilde{a}_2)^3 + ...)$. It is easy to follow that $[\Sigma_0] = -12K_{B_2} - [\Sigma_1]$.

As for the further enhancement in codimensional two loci in $B_2$, there are two types from

$\Sigma_1 \cap \Sigma_0$

$$P_1 : \{w = 0\} \cap \{\widetilde{a}_1^2 + 4\widetilde{a}_2\}, \qquad I_1 \to II \tag{5.49}$$

$$P_2 : \{w = 0\} \cap \{\widetilde{a}_6\} \qquad I_1 \to I_2 \tag{5.50}$$

the first locus (point) $P_1$ has a type $II$ cuspidal singularity in the fibre with $\mu_{P_1}(f,g) = 2, \chi_{top}(X_{P_1}) = 2$, consequently it does not trap any localized massless matter. The second locus (point) has a $I_2$ singularity with $\mu_{P_2}(f,g) = 0, \chi_{top}(X_{P_2}) = 1$, which supports localized matters.

The conifold $I_1$ model has been well studied (for examples see [115, 228]), and by "conifold" we mean that the $I_2$-singularity in codimensional two loci can be written in a conifold form

$$uv + xy = 0 \tag{5.51}$$

up to some recombinations of coordinates. Such a geometry does not admit a small, crepant resolution but only possible non-projective (non-Kähler ) small resolutions [115, 228]. This agrees with the argument made in [131] in the context of Calabi-Yau four-folds, that the geometric massive $U(1)$ in the dual M-theory perspective can be interpreted as decomposition of the $C_3$ field along a non-harmonic 2-form $\omega_{0i}$( hence $\mathrm{d}\omega_{0i} \neq 0$), which obstructs a Kähler resolution of the singular CY $X_4$:

$$C_3 = \sum_i A_0^i \omega_{0i}, \qquad J = v_0^i \omega_{0i}. \tag{5.52}$$

Such decomposition in M-theory have been discussed in 2.10.2 of chapter 2. We will come back to this point shortly.

### 5.3.1. The condition for presence of Sen limit

It is now extremely interesting to test our anomaly equations ((5.11)) for the 'massive U(1)' gauge symmetry. Recall that these have been derived in the framework of a Type IIB orientifold compactification on a smooth Calabi-Yau 2-fold $X_2$ (prior to orientifolding), which is the double cover of the base $B_2$ of the F-theory model. While the quantities $a$ and $b$ have a very clear interpretation in F-theory as cohomology elements on the base $B_2$ of the elliptic fibration, the object $c$ is harder to understand because it involves the uplift of orientifold-odd quantities. However, what we can readily check is the specific linear combination of the first two equations in ((5.11)) given by

$$3(b \cdot b) + 24(a \cdot b) \stackrel{?}{=} \sum_q n_{q^a}((q^a)^4 - 4(q^a)^2)\,. \tag{5.53}$$

Explicitly, in the Conifold $I_1$ model, all matter localizes at points of type $P_2$, with multiplicity and charge

$$n_{q^a} = (-6a - b) \cdot b\,, \qquad q^a = 1\,. \tag{5.54}$$

Here the charge refers to the putative charge with respect to a massive U(1) gauge symmetry, which we would like to interpret as the remnant of a Stückelberg breaking of an underlying $U(1)$ gauge symmetry. Plugging these values back into (5.53) we see that the anomaly equation is satisfied if and only if

$$a \cdot b = 0. \tag{5.55}$$

The multiplicity $a \cdot b$ is proportional to the multiplicity of intersections of the $I_1$ brane divisor in class $b$ with $a = \bar{K}$. By inspection, ((5.55)) hence implies that in order for ((5.53)) to hold, there must be no points of type $P_1$ in the model.

Given that (5.53) has been explicitly derived in the perturbative Type IIB framework on a smooth upstairs geometry $X_2$, we interpret this result as a necessary condition in order that the $I_1$ model has a smooth Type IIB or Sen orientifold limit.

To better understand this point and its relation to anomaly cancellation, recall that for an $I_n$ Tate model on an F-theory Calabi-Yau *four-fold*, a similar constraint is known to exist: Taking the Sen limit of such a Tate model, one encounters a codimension-three conifold singularity on the double cover of the base $B_3$, which is given by the Calabi-Yau three-fold of the Type IIB orientifold.[7] The conifold singularity appears at the intersection

$$a_1 = w = a_{2,1} = 0\,. \tag{5.56}$$

If we insist on working on a smooth Type IIB orientifold geometry, we must impose the constraint $a \cdot b \cdot (2a - b) = 0$, which is a necessary condition for the presence of a type IIB limit for $I_n$ models. For details, we refer to [212, 229]. If this constraint is violated, the resulting conifold singularity on the double cover of $B_3$ has an interesting interpretation, as explained in [?]: It corresponds to a singularity where an orientifold-odd 2-cycle has collapsed to a point. The orientifold-odd 2-cycle intersects the orientifold odd combination of the divisor and image wrapped by a stack of 7-branes, which are the Type IIB analogue of the $I_n$ divisor $w = 0$ in F-theory. An orientifold-odd cycle can be wrapped by a D1-instanton, and since the vanishing 2-cycle intersects the 7-brane and image system, the instanton is charged under the diagonal $U(1)$ gauge symmetry associated with the gauge group $U(n)$ on the stack of branes (and its orientifold image). Now, in absence of such an instanton, the $U(1)$ gauge symmetry, which becomes massive by the Stückelberg/Green-Schwarz mechanism, would remain in the effective theory as a perturbative global symmetry. However, the the D1-instanton at the singularity breaks the massive $U(1)$ gauge symmetry completely to a (vacuous) $\mathbb{Z}_1$ symmetry. A similar effect arises in the presence of suitable D3-brane instantons wrapping divisors, but in this case the instanton is exponentially suppressed; we can therefore view the $U(1)$ as a "perturbative' global $U(1)$, which is broken only by subleading effects. In particular, the D3-brane instanton effects are not geometrised in F-theory, and an anomaly equation for the perturbative global $U(1)$ must therefore be reflected in the F-theory geometry. With the D1-brane instantons the situation is different: Since the cycle wrapped by the instanton has zero size, the latter is unsuppressed and there exists therefore no sense in which the $U(1)$ symmetry can be viewed as a perturbative global symmetry in the theory. This is in fact the Type IIB realisation of the Yukawa couplings located at the points (5.56) in the four-dimensional F-theory model, which are known to completely break the Type IIB $U(1)$ global symmetry that would be expected in absence of instanton corrections.

These considerations suggest an analogous interpretation of the constraint (5.55) in compactifications to six dimensions: If (5.55) is violated, the Type IIB Calabi-Yau 2-fold $X_2$ prior to orientifolding has a pointlike singularity at the intersection of the 7-brane and the orientifold O7-plane. The most direct analogue would be that a D1-instanton wraps the underlying vanishing cycle and again completely breaks the massive $U(1)$ gauge symmetry to a trivial $\mathbb{Z}_1$ symmetry. In order for this to be the correct picture, the shrinking cycle has to be orientifold odd, as in compactifications to four dimensions. By contrast, if it were orientifold even, it could be wrapped by a D3 brane, giving rise to a tensionless string. Both objects can carry $U(1)$ charge (from intersection of the shrinking cycle with the orientifold odd, or, respectively, orientifold even combinations of 7-brane cycles and its image) and hence induce a breaking of the massive

[7] This conifold singularity on the Type IIB double cover of the base $B_3$ is not to be confused with the conifold in the fiber over the points of type $P_2$ in F-theory, which will be discussed in more detail in the next section.

$U(1)$. We leave it for future research to determine which of the two mechanisms is at work. In any event, since a $\mathbb{Z}_1$ symmetry is trivial there is no significance to states carrying charge under it, and hence we cannot give a meaning to an anomaly equation. Correspondingly, the fact that (5.53) is violated in F-theory models where (5.55) does not hold simply reflects the fact that there is no global symmetry in the effective action operating on the spectrum. By contrast, if (5.55) does hold, then we can think of assigning the matter states at the points $P_2$ a charge $q^a = 1$ under a massive $U(1)$ gauge symmetry, and its anomalies are correctly cancelled.

### 5.3.2. Topological invariants with Terminal singularities

As we have said, such a conifold $I_1$ model has codimension-two terminal singularities in the fiber over points of type $P_2$. In this subsection, we are going to list the main topological invariant for such Calabi-Yau $X_3$ for our later purpose.

According to the general discussion in 2.8, assuming $X_3$ being a Calabi-Yau three-fold with $\mathbb{Q}$-Factorial terminal singularities, we have

$$\begin{gathered}\mathrm{KaDef}(X_3) = b_2(X_3) = n_T + 2 + rk(\mathfrak{g}), \quad \mathrm{CxDef}(X_3) = \frac{1}{2}(b_3(X_3) + \sum_P m_P) - 1, \\ \chi_{top}(X_3) = 2 + 2b_2(X_3) - b_3(X_3).\end{gathered} \tag{5.57}$$

Combining with them, we obtain

$$\chi_{top}(X_3) = 2[\mathrm{KaDef}(X_3) - \mathrm{CxDef}(X_3) + \frac{1}{2}\sum_P m_P]. \tag{5.58}$$

Now let us calculate the Euler characteristics of $X_3$. According to [115], the Euler characteristics of $X_3$ $\chi_{top}(X_3)$ reads

$$\begin{aligned}\chi_{top}(X_3) =& (\sum_i B_i \cdot \chi_{top}(X_{P_i})) + m(2 - 2g - \sum_i B_i) - 132K_{B_2}^2 + mK_{B_2} \cdot \Sigma_1 \\ &+ m^2\Sigma_1^2 + 3C_c + \sum_i \epsilon_i B_i + 2m\Sigma_0 \cdot \Sigma_1,\end{aligned} \tag{5.59}$$

where $C_c$ is the number of cusps, $B_i$ is the number of matter points $P_i$ and $g$ is the genus of all $\Sigma_1$. Explicitly, $C_c$ is given by

$$C_c = 24K_{B_2}^2 + (4\mu_f + 6\mu_g)K_{B_2} \cdot \Sigma_1 + \mu_f\mu_g\Sigma_1^2 - \sum_i \mu_{P_i}(f, g)B_i. \tag{5.60}$$

We will discuss these expression in detail in the following example.

## 5.4. A working Example of Conifold $I_1$ Model

### 5.4.1. Type IIB geometry

In this section, we will verify the anomaly equation (5.11) in a non-trivial example. The starting point is a K3 surface-quartic $X_2$ : $\mathbb{CP}^3[4]$ , which is a degree 4 hypersurface constraint in $\mathbb{CP}^3$ with homogeneous coordinates $x_i, i = 1, ..., 4$. At generic points in the complex structure moduli space, this defines a smooth $K3$ surface. However, we choose a polynomial of the hypersurface such that all monomials containing $x_3^k, k > 2$ or $x_4^k, k > 2$ vanish. The corresponding quartic has two non-generic $CP^1$ singularities at the points $(x_1, x_2, x_3, x_4) = (0, 0, 0, 1)$ and $(x_1, x_2, x_3, x_4) = (0, 0, 1, 0)$. We then blow up these two singularities into $CP^1$ and then denotes them as two divisors $[s] : s = 0$ and $[t] : t = 0$. The toric data associated to the blow-up version of the quartic is

| Coord- -inates | GLSM Charge $Q^1$ | GLSM Charge $Q^2$ | GLSM Charge $Q^3$ | divisors |
|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | H |
| $x_2$ | 1 | 0 | 0 | H |
| $x_3$ | 1 | 0 | 1 | H+Y |
| $x_4$ | 1 | 1 | 0 | H+X |
| $s$ | 0 | 1 | 0 | X |
| $t$ | 0 | 0 | 1 | Y |

There are two corresponding triangulations. Here we focus on the first one, while the second one has an opposite intersection numbers among the divisors. The Stanley Reisner (SR) ideal in our triangulation is $\langle x_1x_2, x_3t, x_4s\rangle$. The triple intersection numbers in the ambient space with basis being $H, X, Y$ are

$$\begin{aligned} &HX^2 = HY^2 = -1, \quad XY^2 = X^2Y = 0, \\ &H^2X = H^2Y = 1, \quad X^3 = Y^3 = -H^3 = 1, \quad HXY = 0. \end{aligned} \tag{5.61}$$

Therefore the intersection numbers in the quartic $(4H + 2X + 2Y)$ can be obtained by the pullbacks and read as

$$X^2 = Y^2 = -2, \quad H^2 = 0, \quad XY = 0, \quad HX = HY = 2. \tag{5.62}$$

Now we consider an involution $\sigma : x_3 \leftrightarrow x_4, s \leftrightarrow t$. Consequently, a O7-plane is located at the fixed point locus of the involution $\sigma$, which is $\{x_3s - x_4t = 0\}$ and the divisor class is then $(H + X + Y)$. Now we wrap an D7-brane on $[s]$ and its image on $[t]$. The divisor associated with the Whitney brane can be obtained as $S = [8H + 7X + 7Y]$ by the D7-brane tadpole condition. The matter spectrum is given as follows

| matter points | class | multiplicities | $U(1)$ charge |
|---|---|---|---|
| $D_1 \bigcap S$ | [X][8H+7X+7Y] | 2 | -1 |

In order to identify the base $B_2$ as the quotient of $X_2/\sigma$, we replace $v \to st, w \to x_3x_4, h = x_3s + x_4t$. Then the base $B_2$ can be described by the toric data:

| Coords | GLSM charges $Q^1$ | $Q^2$ | divisors |
|---|---|---|---|
| $u_1$ | 1 | 0 | P |
| $u_2$ | 1 | 0 | P |
| $v$ | 2 | 1 | 2P+X |
| $h$ | 1 | 1 | P+X |
| $w$ | 0 | 1 | X |

Note that the intersection numbers in the base $B_2$ and K3 $X_2$ are related by

$$\int_{K3} \pi^*(\mathcal{D}_a) \wedge \pi^*(\mathcal{D}_b) = 2 \int_B \mathcal{D}_a \wedge \mathcal{D}_b, \tag{5.63}$$

where the prefactor 2 arises because of $\pi(X_2) = 2B$. Hence the intersecting numbers of toric divisors in $B_2$ read as

$$P^2 = 0,\ X^2 = -2,\ PX = 2. \tag{5.64}$$

### 5.4.2. Uplift to F-theory

The type IIB orientifold compactification in our model will lift to a "split $I_1$ singularity" over a divisor $\Sigma_1 : \{w = 0\}$ with $(\mu_f, \mu_g, m) := \mathrm{ord}(f, g, \Delta)|_{\Sigma_1} = (0, 0, 1)$ induced by specifying

$$a_2 = a_{21}, \qquad a_3 = a_{31}w, \qquad a_4 = a_{41}w, \qquad a_6 = a_{61}w. \tag{5.65}$$

Furthermore, we have

$$a_1 = \alpha h + wp_1(u), \qquad a_{21} = -\frac{1}{2}\alpha h p_1(u) - \alpha^2 v w - \frac{1}{4}p_1^2, \tag{5.66}$$

where $\alpha$ is a constant and $p_1(u)$ is a generic polynomials of degree 1 in $(u_1, u_2)$. In contrast, the other coefficients $a_{n1}, n = 3, 4, 6$ are taken to be the maximally generic polynomials of degree $(nP + (n-1)X)$. With the above equipments, the discriminant $\Delta$ splits into two components $\Sigma_1$ and $\Sigma_0$ as

$$\Delta = w\Delta'. \tag{5.67}$$

At the codimension-two loci, there are two types of enhancement located at the intersection $\Sigma_1 \cap \Sigma_0$, which consists of two points

$$P_1 = \{w = 0\} \cap \{h = 0\}, \qquad P_2 = \{w = 0\} \cap \{a_{61} = 0\}. \tag{5.68}$$

The first point $P_1$ has a type $II$ cuspidal singularity in the fibre with $\mu_{P_1}(f, g) = 2, \chi_{top}(X_{P_1}) = 2$ which does not host any localized massless matter. Moreove, the second point $P_2$ has a $I_2$ singularity with $\mu_{P_2}(f, g) = 0, \chi_{top}(X_{P_2}) = 1$, which supports localized matters.

### 5.4.3. Anomaly equation

Now we are going to to check our anomaly equations (5.11) in this example. In this setup of the 7-branes we have $c = Y - X$ and hence $c \cdot c = -4$. In the downstairs geometry $B_2$, we can read off $a, b$ as

$$a = -(P + X), \qquad b = X, \tag{5.69}$$

which leads to

$$-a \cdot b = 0, \qquad b \cdot b = -2. \tag{5.70}$$

Now, after matching the relevant data, we have

$$-2c \cdot c + 3(b \cdot b) = 8 + 3 * (-2) = 2 = \sum_q n_q q^4, \tag{5.71}$$

$$\frac{-c \cdot c}{2} - 6(a \cdot b) = 2 + 0 = 2 = \sum_q n_q q^2, \tag{5.72}$$

$$a \cdot a = 2 \to T = 7 \to h^{1,1}(B) = 8, \tag{5.73}$$

which **does** verify our anomaly equations (5.11).[8]

However, there is one last piece to be checked. As we discussed in 2.8, the presence of codimensional two terminal singularities indicates certain localized neutral hypermultiplets. Such neutral hypermultiplets definitely contribute to the pure gravitational anomaly in 6D. Hence, we need to verify that our gravitational anomaly equations with the presence of such terminal singularities.

From above, we know that $T = 7$ and the gauge group is trivial, $\mathrm{rk}(G) = 0$, and hence $V = 0$, we then have $KaDef(X_3) = T + 2 + \mathrm{rk}(G) = 9$. And with terminal singularities, we should modify the suitable Euler characteristics as [115]

$$\chi_{top}(X_3) = 2(\mathrm{KaDef}(X_3) - \mathrm{CxDef}(X_3) + \frac{1}{2}\sum_i B_i m_{P_i}). \tag{5.74}$$

With the fact that the Milnor number of the $A_1$ type singularity being $M_{P_2} = 1$ and $M_{P_1} = 0$, we obtain

$$\mathrm{CxDef}(X_3) = 9 - \frac{\chi_{top}(X_3)}{2} + \frac{2}{2}. \tag{5.75}$$

Since there are no charged hypermultiplets at all in the example, from the pure gravitational anomaly cancellation

$$H - V + 29T = 273, \tag{5.76}$$

we have to choose $H = 70$ due to $V = 0$. Furthermore, since in our example there are no charged hypermultiplets, we conclude

$$H = H^0 = 1 + \mathrm{CxDef}(X_3) = 70 \to \mathrm{CxDef}(X_3) = 69.$$

With the help of (5.75), $\chi_{top}(X_3)$ needs to be $-118$ in order to cancel the gravitational anomaly entirely.

Now let us check whether this is true in our example. We have

$$K_{B_2} = -(P + X), \qquad \Sigma_1 = X, \qquad \Sigma_0 = -12K_B - m\Sigma_1 \tag{5.77}$$

$$B_1 = 2, B_2 = 0, \qquad (\mu_f, \mu_g, m)|_{\Sigma_1} = (0, 0, 1), \qquad \epsilon_1 = \epsilon_2 = -1\,. \tag{5.78}$$

Substituting these values into (5.60) and (5.59), we obtain $C_c = 48$. Furthermore the genus $g$ of $\Sigma_1$ can be computed via the adjunction formula

$$\Sigma_1 \cdot \Sigma_1 + K_{B_2} \cdot \Sigma_1 = 2g - 2 \tag{5.79}$$

[8] The astute reader might wonder how $h^{1,1}(B) = 8$ is consistent with the fact that $B$ is a hypersurface in a toric ambient 3-fold $Y$ with $h^{1,1}(Y) = 2$. The answer is that six of the eight elements in $H^2(B, \mathbb{Z})$ are non-toric divisors. This is in perfect agreement with the Lefshetz hyperplane theorem, which asserts that the pullback map $H^2(Y, \mathbb{Z}) \to H^2(B, \mathbb{Z})$ is an injection for $Y$ a projective 3-fold and $B$ a hypersurface therein.

to be

$$g = 0\,. \tag{5.80}$$

As result, (5.59) explicitly yields

$$\chi_{top}(X_3) = -118\,, \tag{5.81}$$

which does satisfy the pure gravitational anomaly!

In summary, we have checked all of our anomaly equations (5.11) in this non-trivial example. This include the pure gravitational anomaly after taking into account the effects from the terminal singularities in this conifold $I_1$ model.

## 5.5. Anomaly Equation on Discrete $\mathbb{Z}_N$ Symmetry

In the previous sections, we have discussed with a particular F-theory model, namely the conifold $I_1$ model, where after the anomaly cancellation, the abelian symmetry $U(1)$ is totally broken by eating a axion $\phi$ through the Stückelberg coupling. This essentially corresponds to a Higgsing mechanism by turning on the vev for a scalar $\varphi$ with charge 1 whose phase is the axion $\phi$, see more details in 2.10.1.

However, in principle, the anomalous $U(1)$ can also be broken to a discrete $\mathbb{Z}_N$ symmetry, which similarly arises from the Higgs mechanism by turning on vev for a scalar $\varphi$ but with charge $N$. In terms of F-theory compactifications, these configurations in Type IIB orientifold compactification shall uplift to genus-one fibration so that the effective theory enjoys a discrete symmetry $\mathbb{Z}_N$. We exemplify this in 5.6.1. Furthermore, as we will prove in A.5, $c \cdot c$ becomes a torsional linking number once it uplifts to a genus-one fibration, which takes the value

$$c \cdot c \in \mathbb{Z} = 0 \text{ Mod } N, \tag{5.82}$$

Also notice that in a discrete symmetry $\mathbb{Z}_N$, a charge $q_a$ by definition satisfies $q^a \sim q^a + N$. Thus, by doing simple algebraic operations on 5.11, we propose that the anomaly cancellation conditions for the discrete symmetry $\mathbb{Z}_N$ in an F-theory genus-one fibration read [9]

$$3(b\cdot b)-\sum_q n_q(q)^4=0 \text{ Mod } N, \tag{5.85}$$

$$-6(a\cdot b)-\sum_q n_q(q)^2=0 \text{ Mod } \tfrac{N}{2}. \tag{5.86}$$

By abuse of notation, we also use $q$ to collectively denote each charge $q^a$ under the discrete symmetry $\mathbb{Z}_n$. The above anomaly equations could also generalize to cases with the presence of other gauge groups, and similar arguments can be readily yielded.

We would like to stress here that the constraints (5.85) are necessary but not sufficient conditions for the discrete symmetry $\mathbb{Z}_N$. This is because that they have been deduced from the conditions (5.11) of the embedding of the abelian $U(1)$ symmetry in the UV. This is thus only one way of UV completion of the discrete symmetry $\mathbb{Z}_N$. From a t'Hooft anomaly perspective, in order to obtain the complete constraints on the discrete symmetry, one needs to consider all the possible ways of UV completions. A good interesting question would then be how to connect it with the Dan-Freed anomaly cancellation [230,231] and study the geometric implications from F-theory perspective. we leave this for the future exploration.

## 5.6. Genus-one fibrations of F-theory compactification

For simplicity, let us focus only with the discrete symmetry $\mathbb{Z}_N$ in the F-theory compactification [10] and in this section we are going to identify the geometric correspondences for $a, b$ in the

[9] Note that there is a crucial step which we would like to comment on here. In [219], there is some charged chiral matter which gains the mass during the process of Higgsing a $U(1)$ through the Yukawa couplings, and they proved that in 4d cases the resulting massive matter contributes an $N-$multiple to the original $U(1)$ anomaly equation by virtue of the properties of $q^3$ and $q$. If there is similar chiral matter uplifting situations happening in 6D, where instead the anomaly contributions from 1-loop are proportional to $q^4$ and $q^2$, then we could not factor out a $N-$multiple and hence our proposal for anomaly equation is not entirely correct. However, it turns out this is not a problem since such types of Yukawa couplings in 4d do not exist in 6D F-theory contexts and the other channels for the matter gaining a mass in our non linear Higgsing process does not affect the chiral fermions concerning the anomaly equations.

To be more precise, let us recap the basic idea of [219] in 4d field theory. Specifically, let $[q_i, Q_i]$ be the $U(1)$ charges of a collections of left-handed Weyl fermions. To guarantee that the theory is $U(1)$ anomaly free, these charges must obey the following relations

$$\sum_i q_i^3+Q_i^3=0, \qquad \sum_i q_i+Q_i=0. \tag{5.83}$$

Let us turn on a vev of $\phi$ with charge $N$ under the $U(1)$ and subsequently, Higgs the $U(1)$ to a discrete $\mathbb{Z}_N$ symmetry. As a consequence, a Yukawa coupling between $\phi$ and charged $Q_i$ fields $\psi_i$ would render $\psi_i$ massive and hence are not treated as the massless spectra of the discrete symmetry. Based on the different species of $\psi_i$ in the Yukawa coupling, the charge of $Q_i$ can be determined up to a $N$ multiplet. For example, considering a field $\psi_1$ gaining the Dirac mass with another Weyl spinor $\psi_2$, then we have $Q_1^2+Q_2^2=nP_i, P_i\in\mathbb{Z}$. Or a Weyl spinor $\psi_i$ obtained a Majorana mass which leads to $2Q_i=nP_i', P_i'\in\mathbb{Z}$. Combining with the fact that a Weyl spinor have charge $q_i$ with identification $q_i=q_i+Np_i''$ under the discrete symmetry, we have

$$\sum_i q_i^3=pn+r\frac{n^3}{8}, \qquad \sum_i q_i=p'n+r'\frac{n}{2}, \qquad p,r,p',r'\in Z; p\in 3\mathbb{Z}, \quad \text{if } n\in 3\mathbb{Z}. \quad r,r'=0 \quad \text{if n is odd.} \tag{5.84}$$

Such uplifting of chiral matter does not to happen to a 6D theory from F-theory compactification. As we know, in such compactification, there is no codimensional three loci and hence no Yukawa couplings between the localized hypermultiplets (Though it does exist Yukawa couplings for different multiplets, but such couplings do not uplift the chiral matter which we are considering).

[10] As we discussed in 2, in such cases, the terminal singularities would also exists as in the codimensional two loci there are localized multiplets only charged under the discrete symmetries.

genus-one fibration.

As we introduced in 2.10, a 6D $\mathcal{N} = (1, 0)$ SUGRA with discrete $\mathbb{Z}_N$ symmetry usually arises from a genus-one fibration $\mathcal{X}_3$ in F-theory compactification. Such fibrations can be constructed by fibering a genus-one $\mathcal{C}$ on a Kähler base $B_2$ and there exists a divisor $S^{(N)}$ of $\mathcal{X}_3$, which is an $N$-covering of the base $B_2$, intersecting a generic fiber $N$ times locally but globally these $N$ intersection points are connected by the monodromy.

By the same token in chapter 3 and 5.2.3 for special cases with $N = 1$, we can identify that the anomaly coefficients $a, b$ in the genus-one fibration $\mathcal{X}_n$ as

$$a \to K_{B_2}, \tag{5.87}$$
$$b \to S^{(N)}. \tag{5.88}$$

Note that in the presence of other non-abelian gauge algebras $\mathfrak{g}_I$, the multi-section divisor $S^{(N)}$ should be replaced by a Shioda-like divisor $\phi(S^{(N)})$ as

$$\phi(S^{(N)}) = S^{(N)} + \sum_{ij} S^{(N)} \cdot \mathbb{P}^1_{i_I} (C^{-1})_{ij} E_{i_I}. \tag{5.89}$$

### 5.6.1. An Example with $\mathbb{Z}_3$ discrete symmetry

We are now going to verify our proposed anomaly equations 5.85 in an example with $\mathbb{Z}_3$ symmetry. This geometry has already been constructed in [223] (and also see [213]). It has been checked in [223] that such $\mathbb{Z}_3$ model can be Higgsing from a $U(1)$ model by turning on a vev for a charged 3 scalar.

The genus-one curve $\mathcal{C}$ in this example can be described as a hypersurface in the 2D toric variety $\mathbb{P}^2$, which is a Calabi-Yau one-fold. The only divisor class in $\mathbb{P}^2$ is the hyperplane class $H$, in order to be a Calabi-Yau hypersurface in $\mathbb{P}^2$, the degree of $\mathcal{C}$ must be 3 in $3H$ hence it can be realized as a generic cubic polynomial in $\mathbb{P}^2$ with coordinates $[u : v : w]$

$$\mathbb{P}_{F_1} = s_1u^3 + s_2u^2v + s_3uv^2 + s_4v^3 + s_5u^2w + s_6uvw + s_7v^2w + s_8uw^2 + s_9vw^2 + s_{10}w^3, \tag{5.90}$$

where $s_i$s are generic coefficients. In order to promote the genus-one curve to a hypersurface fibration over a base $B_2$, the coefficient $s_i$, as well as $u, v, w$ should be promoted to be sections of a various bundles. To determines the corresponding bundles, one can first construct a fibration of the 2D ambient space $\mathbb{P}^2$ over the a generic base $B_2$

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}(D, \widetilde{D}) \\ & & \downarrow \\ & & B_2, \end{array} \tag{5.91}$$

as suggested in [223]. Then one can impose the Calabi-Yau condition on $\mathbb{P}_{F_1}$ so that the $s_i$ and $[u : v : w]$ are sections of line bundles in terms of $D, \widetilde{D}, H, K_B$. It turns out in this case [223], $D, \widetilde{D} = S_7, S_9$ are the classes of the sections $s_7, s_9$. The anti-canonical bundle of $\mathbb{P}(D, \widetilde{D})$ can thus be read off as $K^{-1}_{\mathbb{P}(D,\widetilde{D})} = 3H + 2S_9 - S_7$. One can now choose the coordinates $u : v : w$ to be

$$u \in H^0(B, S_9 + K_B + H), \qquad v \in H^0(B, S_9 - S_7 + H), \qquad w \in H^0(B, H). \tag{5.92}$$

Finally, the Calabi-Yau condition of $\mathbb{P}_{F_1}$ imposes that the coefficients $s_i$ are sections of the following bundles

| section | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| bundle | $-3K_B - S_9 - S_7$ | $-2K_B - S_9$ | $-K_B + S_7 - S_9$ | $2S_7 - S_9$ | $-2K_B - S_7$ |
| section | $s_6$ | $s_7$ | $s_8$ | $s_9$ | $s_{10}$ |
| bundle | $-K_B$ | $S_7$ | $-K_B - S_7 + S_9$ | $S_9$ | $-S_7 + 2S_9$ |

(5.93)

The corresponding fibration $\mathbb{P}_{F_1}$ does not have a section, but a 3-section. Namely, the map $\widehat{S} : B_2 \to X$ maps $p \in X$ to 3 points in the fiber. Globally, the 3 points in the fiber can be connected by the monodromy action upon moving around the branch loci in $B_2$.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbb{P}_{F_1} \\ & & \downarrow \\ & & B_2. \end{array} \tag{5.94}$$

The fibration $X_{F_1}$ has a three-section that is given by

$$\widehat{s}_3 = X_{F_1} \bigcap \{u = 0\} : \qquad s_4 v^3 + s_7 v^2 w + s_9 v w^2 + s_{10} w^3, \tag{5.95}$$

whose divisor class $S^{(3)}$ according to the above table reads as $S^{(3)} := 3K_{B_2} + 2S_9 - S_7$

The matter spectrum under the $\mathbb{Z}_3$ symmetry was calculated in [223] and it turns out to be a singlet $\mathbf{1}_1$ with charge 1, whose multiplicity is given by

$$1_1 : 3(-6K_{B_2} - S_7^2 + S_7 S_9 - S_9^2 - K_{B_2}(S_7 + S_9)). \tag{5.96}$$

Substituting this multiplicity into (5.85) and noting (5.87), as well as $q = 1$, we can see

$$-6(K_B \cdot b) - 3(-6K_{B_2} - S_7^2 + S_7 S_9 - S_9^2 - K_{B_2}(S_7 + S_9) = 0 \text{ mod } \frac{3}{2}, \tag{5.97}$$

$$3(b \cdot b) - 3(-6K_{B_2} - S_7^2 + S_7 S_9 - S_9^2 - K_{B_2}(S_7 + S_9) = 0 \text{ mod } 3, \tag{5.98}$$

which **is** consistent with our proposal (5.85).

How about the type IIB orientifold picture of this model? Fortunately, the sen limit has been worked out in [213]. It can be achieved by the following procedure

$$s_i \to \epsilon^1 s_i, i \in \{5, 8, 10\}, s_j \to \epsilon^2 s_j, j \in \text{others} \tag{5.99}$$

and the locus of the D7-branes can be obtained as

$$\begin{aligned} \Delta^E = & -\frac{1}{4}(-s_3 s_6 s_7 + s_2 s_7^2 + s_3^2 s_9 + s_4(s_6^2 - 4 s_2 s_9)) \\ & \times [-s_{10}^2 s_2^3 + s_{10}(s_1 s_6^3 - s_2 s_6(s_5 s_6 + 3 s_1 s_9) + s_2^2(s_6 s_8 + 2 s_5 s_9)) \\ & + s_9(s_2^2 s_8^2 + s_2(-s_5 s_6 s_8 + s_5^2 s_9 - 2 s_1 s_8 s_9) + s_1(s_6^2 s_8 - s_5 s_6 s_9 + s_1 s_9^2))], \end{aligned} \tag{5.100}$$

which factorized into Whitney D7-brane $\Delta'$ and a pair of D7-brane and its image brane $D$ and $D'$, respectively. In particularly, the divisors $D$ and $D'$ are not in the same homology class, which exactly fits with our type IIB orientifold picture in (5.2.2).

# Appendix A.

# Conventions for Chapter 4 and Chapter 5

In this appendix we collect our conventions for chapter 4 and chapter 5.

## A.1. Type IIB 10D supergravity and Brane Chern-Simons Actions

The bosonic part of the 10d Type IIB supergravity pseudo-action in its democratic form is given by

$$\begin{aligned} S_{\mathrm{IIB}} = \;& 2\pi \Bigg( \int d^{10}x\; e^{-2\phi}(\sqrt{-g}R + 4\partial_M \phi\, \partial^M \phi) - \frac{1}{2}\int e^{-2\phi} H_3 \wedge *H_3 \\ & -\frac{1}{4}\sum_{p=0}^{4}\int F_{2p+1}\wedge *F_{2p+1} - \frac{1}{2}\int C_4 \wedge H_3 \wedge F_3 \Bigg)\,. \end{aligned} \tag{A.1}$$

Here we are working in conventions where the string length $\ell_s = 2\pi\sqrt{\alpha'} \equiv 1$ and the field strengths are defined as

$$\begin{aligned} &H_3 = dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2 - C_0\, dB_2, \\ &F_5 = dC_4 - \frac{1}{2}C_2 \wedge dB_2 + \frac{1}{2}B_2 \wedge dC_2, \end{aligned} \tag{A.2}$$

together with the duality relations $F_9 = *F_1$, $F_7 = -*F_3$, $F_5 = *F_5$, which hold at the level of equations of motion.

The Chern-Simons action for the D7-branes and the O7-plane takes the form

$$\begin{aligned} S^{D7} &= -\frac{2\pi}{2}\int_{D_7} \mathrm{Tr}\, e^{i\mathcal{F}} \sum_{2p} C_{2p}\sqrt{\frac{\widehat{A}(\mathrm{T}D_7)}{\widehat{A}(\mathrm{N}D_7)}} \\ S^{O7} &= \frac{16\pi}{2}\int_{O_7}\sum_{2p} C_{2p}\sqrt{\frac{L(\frac{1}{4}\mathrm{T}O_7)}{L(\frac{1}{4}\mathrm{N}O_7)}}\,. \end{aligned} \tag{A.3}$$

Since we are working in the democratic formulation, where each RR gauge potential is accompanied by its magnetic dual, the Chern-Simons action has to include a factor of $\frac{1}{2}$ [40], which we are making manifest in (A.3). This factor is crucial in order to obtain the correctly normalized anomaly inflow terms, and, as we find in the main text, also to reproduce the correctly normalised Green-Schwarz counterterms. As stressed already, the minus sign in front of the Chern-Simons action of the D7-branes ensures that in the above conventions for the supergravity fields the D7-brane couples magnetically to the axio-dilaton $\tau = C_0 + ie^{-\phi}$. Note furthermore that we are writing the brane action in terms of $\mathrm{Tr} = \frac{1}{\lambda}\mathrm{tr}_{\mathbf{fund}}$, where the Dynkin index $\lambda$ is given in Table 4.1. Finally, T$D_7$ and N$D_7$ denote the tangent and normal space to the 7-brane along $D_7$, and similarly for the O7-plane. The Chern-Simons action for a D3-brane carries a relative sign compared to the 7-brane action,

$$S^{D3} = \frac{2\pi}{2} \int_{D3} \mathrm{Tr}\, e^{i\mathcal{F}} \sum_{2p} C_{2p} \sqrt{\frac{\widehat{A}(\mathrm{T}D3)}{\widehat{A}(\mathrm{N}D3)}} \,. \tag{A.4}$$

The gauge invariant field strength $\mathcal{F}$ above is defined as

$$\mathcal{F} = i(\mathbf{F} + 2\pi\phi^* B_2 \mathbb{I}) \,. \tag{A.5}$$

Compared to expressions oftentimes used in the literature we have absorbed a factor of $\frac{-1}{2\pi}$ in the definition of $\mathcal{F}$, where the minus " $-$ " is consistent with the conventions of anomalyA.2. The NS-NS two-form field $B_2$ is pulled back to the brane via $\phi^*$. We will always set $B_2 = 0$ in this article, but one should bear in mind that it appears in various consistency conditions as detailed e.g. in [57]. We will sometimes decompose

$$\mathbf{F} = F + \bar{F} \tag{A.6}$$

so that $F$ denotes the gauge invariant field strength of the gauge field in non-compact flat space while $\bar{F}$ stands for the internal flux background. Note that it is the hermitian field strength $F$ which appears in the anomaly polynomial (A.14). Finally, the curvature terms in the above Chern-Simons actions enjoy the expansion

$$\sqrt{\frac{\widehat{A}(\mathrm{T}D_7)}{\widehat{A}(\mathrm{N}D_7)}} = 1 + \frac{1}{24} c_2(D_7) + \ldots \,, \qquad \sqrt{\frac{L(\frac{1}{4}\mathrm{T}O_7)}{L(\frac{1}{4}\mathrm{N}O_7)}} = 1 - \frac{1}{48} c_2(O_7) + \ldots \,. \tag{A.7}$$

Here we have used the definitions (A.16) together with the fact that $c_1(\mathrm{T}D) = -c_1(\mathrm{N}D)$ by adjunction on the Calabi-Yau space on which we compactify the Type IIB theory.

### A.1.1. Type IIB orientifold compactification with 7-branes

In a Type IIB orientifold compactification on a Calabi-Yau 4-fold $X_4$, the orientifold projection $\Omega(-1)^{F_L}\sigma$ acts as in the more familiar case of compactification on a 3-fold, as summarized e.g. in [60]. In particular, the $p$-form fields transform under the combined action of worldsheet parity $\Omega$ and left-moving femrion number $(-1)^{F_L}$ as

$$\Omega(-1)^{F_L} : \qquad (C_0, B_2, C_2, C_4, C_6) \;\rightarrow\; (C_0, -B_2, -C_2, C_4, -C_6) \,. \tag{A.8}$$

The holomorphic involution $\sigma$ acts only on the internal space $X_4$ such that the Kähler form $J$ and the holomorphic top-form $\Omega_{4,0}$ transform as

$$\sigma : \qquad J \rightarrow J \,, \qquad \Omega_{4,0} \rightarrow -\Omega_{4,0} \,. \tag{A.9}$$

The cohomology groups $H^{(p,q)}(X_4)$ split into two eigenspaces $H^{(p,q)}(X_4) = H^{(p,q)}_+(X_4) \bigoplus H^{(p,q)}_-(X_4)$ under the action of $\sigma$. In performing the dimensional reduction, the orientifold even and odd form fields are expanded along a basis of the invariant and anti-invariant cohomology groups.

Under the orientifold action the field strength on each brane is mapped to its cousin on the orientifold image brane

$$\mathcal{F}_i \to \mathcal{F}_i' = -\sigma^* \mathcal{F}_i \,, \tag{A.10}$$

where the minus sign is due to the worldsheet parity action.

## A.2. Conventions on Anomaly

Our conventions for the anomaly polynomial mostly follow [157]. Here we set the conventions for the anomaly polynomial associated with gauge anomaly. Consider a quantum field theory in $D = 2r$-dimensional Minkowski space $M_{2r}$ with quantum effective action $S[A]$, where $A_\alpha$ is the connection associated with a local symmetry of $S$ with gauge parameter $\epsilon^\alpha$. The anomaly $\mathfrak{A}_\alpha$ is defined as the gauge variation

$$\delta_\epsilon S[A] = \int_{M_{2r}} \epsilon^\alpha \mathfrak{A}_\alpha \,. \tag{A.11}$$

It is expressible as

$$\int_{M_{2r}} \epsilon^\alpha \mathfrak{A}_\alpha = 2\pi \int_{M_{2r}} I_{2r}^{(1)}(\epsilon) \,, \tag{A.12}$$

where the $2r$-form $I_{2r}^{(1)}(\epsilon)$ is related to $(2r+2)$-form $I_{2r+2}$ via the Stora-Zumino descent relations

$$I_{2r+2} = dI_{2r+1}, \qquad \delta_\epsilon I_{2r+1} = dI_{2r}^{(1)}(\epsilon) \,. \tag{A.13}$$

In our sign conventions, the anomaly polynomial $I_{2r+2}$ of a complex chiral Weyl fermion in representation $\mathbf{R}$ takes the form

$$I_{s=1/2}(\mathbf{R})|_{2r+2} = -\mathrm{tr}_{\mathbf{R}} e^{-F} \widehat{A}(\mathrm{T})|_{2r+2} \,. \tag{A.14}$$

Here $F$ is the hermitian field strength associated with the gauge potential $A$ and T denotes the tangent bundle to spacetime. Its curvature 2-form $R$ is the curvature associated with the spin connection. Furthermore, in $2r = 4k+2$ dimensions, a self-dual $r$-tensor contributes to the gravitational anomalies with

$$I_{\text{s.d.}}|_{2r+2} = -\frac{1}{8} L(\mathrm{T})|_{2r+2} \,. \tag{A.15}$$

The A-roof genus and the Hirzebruch L-genus (see more in B.1.1) above can be expressed as

$$\begin{aligned} \widehat{A}(\mathrm{T}) &= 1 - \frac{1}{24} p_1(\mathrm{T}) + \ldots = 1 - \frac{1}{24}(c_1^2(\mathrm{T}) - 2c_2(\mathrm{T})) + \ldots \\ L(\mathrm{T}) &= 1 + \frac{1}{3} p_1(\mathrm{T}) + \ldots = 1 + \frac{1}{3}(c_1^2(\mathrm{T}) - 2c_2(\mathrm{T})) + \ldots \,. \end{aligned} \tag{A.16}$$

We will oftentimes write the first Pontrjagin class of the tangent bundle as

$$p_1(\mathrm{T}) = -\frac{1}{2} \mathrm{tr} R \wedge R \,. \tag{A.17}$$

Note that we have included an overall minus sign in (A.14) and (A.15) compared to the conventions used in [157].

A left-handed Weyl gravitino in $2n$ dimension contribute $I_{3/2}(R)$ to the gravitational anomaly as

$$I_{3/2}(R) = (\sum_{j}^{n} 2\text{cosh} x_j - 1) \sum_{k=1}^{n} \frac{x_k/2}{sinh(x_k/2)} \tag{A.18}$$

where $x_i$ is the skew-eigenvalues of the curvature two-form $R_{ab}$.

The reason is that in the quantum field theory we are analyzing the chiral fermion fields arise as the zero-modes of strings on the worldvolume of 7- and 3-branes. The anomalies induced by these modes on the brane worldvolume must be cancelled via an anomaly inflow mechanism by the anomalous Chern-Simons action of the branes. This relates the sign of the 1-loop anomalies to the sign conventions used for the Chern-Simons brane actions. As we will discuss below, the sign of the 7-brane Chern-Simons action is fixed as in (A.3) by the convention that the 7-brane couples magnetically to the axio-dilaton, which is usually defined in F-theory as $\tau = C_0 + ie^{-\phi}$ (rather than $-C_0 + ie^{\phi}$). The sign chosen in (A.3) conforms with this convention. In order for the anomalies of chiral fermions in the worldvolume of a D7-brane to be cancelled by anomaly inflow, we must then adopt the convention (A.14).

## A.3. Chirality Computation for Matter Surface Flux

In this appendix we compute flux dependent part of the chiral index (4.178) induced for states in representation $\mathbf{5}_{-2}$ by the gauge background $G_4^{\lambda}$ in the $SU(5) \times U(1)_A$ model of section 4.9.3. The matter surface $C_{\mathbf{5}_{-2}} \subset W \subset B_4$ is cut out by the locus $P \cap W$ on $B_4$ with

$$P := \{a_1 a_{4,3} - a_{2,1} a_{3,2} = 0\} \,. \tag{A.19}$$

The classes in which the Tate polynomials $a_{i,j}$ take their value are listed in (4.140). As discussed in section 4.9.3, our task amounts to computing

$$\frac{1}{2} \int_{C_{\mathbf{5}_{-2}}} c_1^2(L_{\mathbf{5}_{-2}}) = \frac{\lambda^2}{50} \int_{C_{\mathbf{5}_{-2}}} (-2[Y_2] + 3[Y_1])^2 \,, \tag{A.20}$$

where $[Y_1]$ and $[Y_2]$ denote the classes of eponymous curves on the surface $C_{\mathbf{5}_{-2}} \subset W \subset B_4$. These curves cannot be expressed as the complete intersection of the surface $C_{\mathbf{5}_{-2}}$ with a divisor from $B_4$, but are defined by the complete intersection of 7-brane divisor $W$ with two divisors on $B_4$. Concretely, from (4.177) we read off

$$Y_i = A_i \cap B_i \tag{A.21}$$

with

$$[A_1] = [A_2] = [a_1]|_W \,, \qquad B_1 = [a_{2,1}]|_W \,, \qquad [B_2] = [a_{3,2}]|_W \,. \tag{A.22}$$

We hence need to evaluate the intersections $\int_{C_{\mathbf{5}_{-2}}} [Y_i] \cdot [Y_j]$ for $i = 1, 2$. The self-intersections of $Y_i$ on $C_{\mathbf{5}_{-2}}$ are computed via

$$\int_{C_{\mathbf{5}_{-2}}} [Y_i] \cdot [Y_i] = \int_{Y_i} [Y_i] = \int_{Y_i} c_1(N_{Y_i \subset C_{\mathbf{5}_{-2}}}) \,, \tag{A.23}$$

where the first Chern class of the normal bundle $N_{Y_i \subset C_{\mathbf{5}_{-2}}}$ is computed via the normal bundle short exact sequence

$$0 \to N_{Y_i \subset C_{\mathbf{5}_{-2}}} \to N_{Y_i \subset W} \to N_{C_{\mathbf{5}_{-2}} \subset W} \to 0\,. \tag{A.24}$$

The normal bundles are given as

$$N_{Y_i \subset W} = \mathcal{O}(A_i) \oplus \mathcal{O}(B_i) \tag{A.25}$$

$$N_{C_{\mathbf{5}_{-2}} \subset W} = \mathcal{O}(P)\,, \tag{A.26}$$

where $\mathcal{O}(A_i)$ defines a line bundle of first Chern class $[A_i]|_{C_{\mathbf{5}_{-2}}}$ on $C_{\mathbf{5}_{-2}}$ and $\mathcal{O}(P)$ is a line bundle on $W$ of first Chern class $[P]|_W$. This gives

$$c(N_{Y_i \subset C_{\mathbf{5}_{-2}}}) = \left.\frac{c(N_{Y_i \subset W})}{c(N_{C_{\mathbf{5}_{-2}} \subset W})}\right|_{C_{\mathbf{5}_{-2}}} = \left.\frac{1 + c_1(N_{Y_i \subset W})}{1 + c_1(N_{C_{\mathbf{5}_{-2}} \subset W})}\right|_{C_{\mathbf{5}_{-2}}} \tag{A.27}$$

$$= (1 + [A_i] + [B_i])(1 - [P] + [P]^2 + \ldots)\big|_{C_{\mathbf{5}_{-2}}}\,. \tag{A.28}$$

Collecting the terms of first order yields

$$c_1(N_{Y_i \subset C_{\mathbf{5}_{-2}}}) = (-[P] + [A_i] + [B_i])|_{C_{\mathbf{5}_{-2}}}\,. \tag{A.29}$$

The integral (A.23) can now be expressed as an integral directly on $W$,

$$\int_{Y_i} c_1(N_{Y_i \subset C_{\mathbf{5}_{-2}}}) = ([A_i] \cdot_W [B_i]) \cdot_W (-[P]|_W + [A_i] + [B_i])\,. \tag{A.30}$$

Since all involved classes are defined on or can be extended to $B_4$, this evaluates to

$$\int_{Y_1} [Y_1] = 2c_1 \cdot W \cdot (2c_1 - W) \cdot (W - c_1), \qquad \int_{Y_2} [Y_2] = c_1 \cdot W \cdot (3c_1 - 2W) \cdot (W - c_1)\,, \tag{A.31}$$

in terms of the intersection product on $B_4$, where we are using (A.22) and (4.140).

The remaining task is to compute the cross-term $\int_{C_{\mathbf{5}_{-2}}} [Y_1] \cdot [Y_2]$. We note that even though the curves $Y_i$ cannot individually be written as the complete intersection of a divisor with the divisor $P$ defining $C_{\mathbf{5}_{-2}}$, the combination $Y_1 + Y_2$ is of this simpler form: Indeed on $W$ we have that

$$Y_1 + Y_2 = C_{\mathbf{5}_{-2}} \cap C_{\mathbf{10}_1}\,. \tag{A.32}$$

Since $C_{\mathbf{10}_1} = \{a_1 = 0\}$ we can then write on $C_{\mathbf{5}_{-2}}$ for $Y_1 + Y_2$

$$Y_1 + Y_2 = \{a_1|_{C_{\mathbf{5}_{-2}}} = 0\} \subset C_{\mathbf{5}_{-2}}\,. \tag{A.33}$$

In particular, with $[a_1] = c_1$,

$$\int_{C_{\mathbf{5}_{-2}}} ([Y_1] + [Y_2])^2 = \int_{C_{\mathbf{5}_{-2}}} [a_1] \cdot [a_1] = c_1^2 \cdot W \cdot (5c_1 - 3W)\,, \tag{A.34}$$

where the last intersection is taken on $B_4$. The idea is then to express the cross-term as

$$\int_{C_{\mathbf{5}_{-2}}} [Y_1]\cdot[Y_2] = \int_{C_{\mathbf{5}_{-2}}} \frac{1}{2}\left(([Y_1]+[Y_2])^2 - [Y_1]^2 - [Y_2]^2\right) = c_1 \cdot W \cdot (6c_1^2 - 7c_1 W + 2W^2) \quad \text{(A.35)}$$

Plugging everything into (A.20) leads to the final result (4.178).

## A.4. 2D $\mathcal{N} = (0,2)$ Theories from F-theory Compactifications

### A.4.1. 2d $\mathcal{N} = (0,2)$ theories

In this section, we begin to review some basic aspects of 2D $\mathcal{N} = (0,2)$ theories, and out notations follow [42]. Through our discussion, we consider $\mathbb{R}^{1,1}$ with coordinates $(y^0, y^1)$ and the whole (0,2) superspaces $\mathbb{R}^{2|2}$ with coordinates $(y^+, y^-, \theta^+, \theta^-)$, where $y^+ = \frac{1}{2}(y^0 \pm y^1)$, as well as the correspondent derivatives $\partial_\pm = \partial_0 \pm \partial_1$. We also define the measure for Grassmann integration as $d^2\theta = d\bar{\theta}^+ d\theta^+$, so that $\int d^2\theta(\theta^+\bar{\theta}^+) = 1$.

An chiral $N = (0,2)$ supersymmetric theory in two dimensions is generated by two supercharges $Q_+$ and $\bar{Q}_+ = Q_+^\dagger$, as well as bosonic generators $H, P$ and $M$ of translations and rotations, and the generator $F_+$ of a U(1) R-symmetry. The algebras it admits are

$$\begin{aligned} &\bar{Q}_+^2 = Q_+^2 = 0, \qquad &&\{Q_+, \bar{Q}_+\} = 2(H-P), \\ &[M, Q_+] = -Q_+, \qquad &&[M, \bar{Q}_+] = -\bar{Q}_+, \\ &[F_+, Q_+] = -Q_+, \qquad &&[F_+, \bar{Q}_+] = +\bar{Q}_+. \end{aligned} \quad \text{(A.36)}$$

The superderivatives are

$$\begin{aligned} &D_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\partial_+, \qquad &&\bar{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\partial_+, \\ &\{D_+, D_+\} = \{\bar{D}_+, \bar{D}_+\} = 0, \qquad &&\{\bar{D}_+, D_+\} = 2i\partial_+. \end{aligned} \quad \text{(A.37)}$$

To constructe gauge theories, the superderivatives above should extend to gauge covariant superderivatives $\mathcal{D}_+, \bar{\mathcal{D}}_+$. The basic superfields in an $N = (0,2)$ theory are:

• The Chiral multiplets $\Phi$ with $\bar{D}_+\Phi = 0$, The component expansion contains $(\varphi, \chi_+)$ as

$$\Phi = \phi + \sqrt{2}\theta^+\chi_+ - i\theta^+\bar{\theta}^+\partial_+\phi, \quad \text{(A.38)}$$

and its conjugate chiral multiplet $\bar{\Phi}$ satisfying $D_+\bar{\Phi} = 0$: The component expansion includes $(\bar{\varphi}, \bar{\chi_+})$ as (note that $(\theta^+\bar{\theta}^+)^\dagger = \theta^+\bar{\theta}^+$):

$$\bar{\Phi} = \bar{\phi} - \sqrt{2}\bar{\theta}^+\bar{\chi}_+ + i\theta^+\bar{\theta}^+\partial_+\bar{\phi}. \quad \text{(A.39)}$$

• The Fermi multiplets $P$ with $\bar{D}_+P = E(\Phi)$, where $E(\Phi)$ measures the Fermi multiplets away from the chiral multiplets and is constructed as a holomorphic function of basic chiral superfields. The expansion goes as

$$P = \rho_- - \sqrt{2}\theta^+G - i\theta^+\bar{\theta}^+\partial_+\rho_- - \sqrt{2}\bar{\theta}^+E, \quad \text{(A.40)}$$

as well as it conjugated being $\bar{P} = (\bar{\rho}_-, \bar{G}, \bar{E})$ .

• If the 2D theory carries a gauge symmetry $U(1)^a$ [1], then the Vector Multiplet $V^a = (v_- = v_0 - v_1, \lambda_-, D)$ should expands as

$$V = v_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2i\theta^+\bar{\theta}^+ D, \quad \text{(A.41)}$$

where $\mathcal{D}$ is an auxiliary field, which we refers to a D-term. We also will denote $V_+ = \theta^+\bar{\theta}^+ v_+$. Note that then all derivates involved should be modified to the covariant derivatives $\mathcal{D}_\pm = \partial_\pm + iQv_\pm$. Under the $U(1)$ supergauge transformation, the above vector superfields transform as follows:

$$\delta_\Lambda V_+ = \frac{i}{2}(\bar{\Lambda} - \Lambda), \qquad \delta_\Lambda V = \frac{-1}{2}\partial_-(\bar{\Lambda} + \Lambda). \quad \text{(A.42)}$$

The gauge covariant field strength is defined as

$$\Upsilon = -2(\lambda_- - i\theta^+(D - iF_{01}) - i\theta^+\bar{\theta}^+\partial_+\lambda_-). \quad \text{(A.43)}$$

The kinetic term for the gauge fields then follows as [2]

$$\mathcal{L} = -\frac{1}{8e^2}\int d^2\theta^+\bar{\Upsilon}\Upsilon = \frac{1}{e^2}(\frac{1}{2}F_{01}^2 + i\bar{\lambda}_-\partial_+\lambda_- + \frac{1}{2}D^2). \quad \text{(A.44)}$$

The standard kinetic term for the charged chiral multiplets reads

$$\begin{aligned} &\frac{i}{2}\int d^2\theta\bar{\Phi}^i(\partial_- + iQV)\Phi^i \\ &= (-|D_\mu\phi^i|^2 + i\bar{\chi}_+^i\mathcal{D}_-\chi_+^i - iQ_i\sqrt{2}\bar{\phi}^i\lambda_-\chi_+^i + iQ_i\sqrt{2}\phi^i\lambda_-\bar{\chi}_+^i + Q_i\phi^i\bar{\phi}^i D). \end{aligned} \quad \text{(A.45)}$$

One can also introduce an FI term with complex coefficients $t_A = ir_A + \frac{\theta_A}{2\pi}$ with abelian gauge group $U(1)_A$

$$\frac{t_A}{4}\int d\theta^+\Upsilon_A + h.c. = -r_A + \frac{\theta_A}{2\pi}F_{01}. \quad \text{(A.46)}$$

In principle one can also make the FI terms $t$ field-dependent, we now consider modify $t$ as a chiral fields $X_i := x_i + \sqrt{2}\theta^+\chi_+ - i\theta^+\bar{\theta}^+\partial_+ x_i$ and couples to gauge fields as

$$\mathcal{L} \supset -\frac{i}{4}N_i^a\int d\theta^+ X_i\Upsilon_a + h.c. + \frac{1}{2}\int d^2\theta(\bar{X}_i + X_i + 2M_i^a V_+)(i\partial_-(X_i - \bar{X}_i - M_i^a V), \quad \text{(A.47)}$$

where the first term we dubbed as the Green-Schwarz term and the second term is dubbed as Stückerlberg term. They are the generic form for the $Y_i$ minimal coupled to the gauge fields. Note that under the $U(1)$ gauge transformation

$$X_i \to X_i - iM_i^a\Lambda_a, \quad \text{(A.48)}$$

the Green-Schwarz term goes as

$$\delta_\Lambda = \frac{1}{4}\int d\theta^+ M_i^b N_i^a \Lambda_b \Upsilon_a + \text{h.c.}. \quad \text{(A.49)}$$

[1] Here we assume the gauge group is direct product of abelian gauge group for simplicity, the generic situation with the examples from F-theory compactification carries non-abelian gauge group

[2] Note that this term is consistent with the convention that $d^2\theta = d\bar{\theta}^+\theta^+$

Denoting $x_i = \rho_i + i\theta_i$, we readily find that

$$X_i + \bar{X}_i = 2\rho_i + \sqrt{2}(\theta^+\chi_+ - \bar{\theta}^+\bar{\chi}_+) + 2\theta^+\bar{\theta}^+\partial_+\theta_i$$
$$i\partial_-(X_i - \bar{X}_i) = -2\partial_-\theta_i + \sqrt{2}i\partial_-(\theta^+\chi_+ + \bar{\theta}^+\bar{\chi}_+) + 2\theta^+\bar{\theta}^+\partial_-\partial_+\rho_i. \quad (A.50)$$

Together with

$$V_+ = \theta^+\bar{\theta^+}v_+$$
$$V = v_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2i\theta^+\bar{\theta^+}D. \quad (A.51)$$

We find in component the above Lagrangian reads as

$$\begin{aligned}
&+\frac{1}{2}\int d^2\theta(\bar{X}_i + X_i + 2M_i^aV_+)(i\partial_-(X_i - \bar{X}_i) - M_i^aV) \\
&= +\frac{1}{2}[(2\rho_i)(2\partial_-\partial_+r\rho_i - 2M_i^aD_a) + (2\partial_+\theta_i + 2M_i^av_{+,a})(-2\partial_-\theta_i - 2M_i^av_-) \\
&-(\sqrt{2}\chi_{+,i})(\sqrt{2}i\partial_-\bar{\chi}_+ - M_i^a(-2i\lambda_{-,a})) + (-\sqrt{2}\bar{\chi}_{+,i})(\sqrt{2}i\partial_-\chi_{+,i} - M_i^a(-2i\bar{\lambda}_-))] \\
&= -2k_i^2(\partial_-\rho_i)(\partial_+\rho_i) - 2k_i^2(\partial\theta_i + M_i^av)^2 - 2ik_i^2\bar{\chi}_{+,i}\partial_-\chi_{+,i} - 2k_i^2M_i^a\rho_iD_a \\
&+(-2k_i^2M_i^a\frac{i}{\sqrt{2}}\chi_{+,i}\lambda_{-,a} + \text{h.c.}).
\end{aligned} \quad (A.52)$$

Similarly, we obtain the components for

$$\begin{aligned}
&-\frac{i}{4}N_i^a\int d\theta^+X_i\Upsilon_a + h.c. \\
&= -\frac{i}{4}[4iN_i^a\rho_iD_a + 4iN_i^a\theta_iF_{01} + (-2\sqrt{2}N_i^a\chi_{+,i}\lambda_{-,a} + h.c)] \\
&= N_i^a\rho_iD + (N_i^a\frac{i}{\sqrt{2}}\chi_{+,i}\lambda_{-,a} + \text{h.c.}) + N_i^a\theta_iF_{01}.
\end{aligned} \quad (A.53)$$

Combining with them, we find that

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 - 2(\partial_-\rho_i)(\partial_+\rho_i) - 2(\partial\theta_i + M_i^av)^2 - 2i\bar{\chi}_{+,i}\partial_-\chi_{+,i} + N_i^a\theta_iF_{01} \\
&+(-2M_i^a + N_i^a)(\rho_iD_a + \frac{i}{\sqrt{2}}\chi_{+,i}\lambda_{-,a}) + ...,
\end{aligned} \quad (A.54)$$

where $\mathcal{L}_0$ includes the original Lagrangian in term of Kinetic term of various superfileds and poentials V(after integrating out auxiliary fields D):

$$V_0 = \frac{1}{2e^2}D^2 + \sum_a(|J_a|^2 + |E_a|^2), \qquad D_A = e^2(\sum_i Q_i\phi_i\bar{\phi}_i - r_A), \quad (A.55)$$

where $\phi_i$ stands for scalars in other chiral multiplets that not anticipate GS mechanism.

Now with the Green-Schwarz-Stückerlberg coupling, we have the following potential

$$V = \frac{1}{2e^2}\widetilde{D}^2 + \sum_a(|J_a|^2 + |E_a|^2) \qquad \widetilde{D}_A = e^2(\sum_i Q_i\phi_i\bar{\phi}_i - r_A + (-2M_i^a + N_i^a)\rho_i). \quad (A.56)$$

### A.4.2. The Corresponding Story in F-theory

First we also define various useful intersecting number as

$$K_{ijkp} = \int_{B_4} \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_p, \qquad K_{ij\alpha} = \int_{B_4} \omega_i \wedge \omega_j \wedge w_\alpha, \tag{A.57}$$

where $\omega_i \in H^{1,1}(B_4), w_\alpha \in H^4(B_4)$ and we also assume that $D_i$ is a Poincare dual of $\omega_i$.

The Kähler moduli associated with the chiral multiplet $\Phi_i$ could be written down as [64]

$$\phi = \rho_i + i\theta_i, \qquad \rho_i = \int_{D_i} J_B \wedge J_B \wedge J_B, \qquad \theta_i = \int_{D_i} J_B \wedge \widetilde{C_4}. \tag{A.58}$$

Here $\widetilde{C}_4 = C_4$ after orientifolding and $J$ denotes the kähler form of the Base $B_4$. Then with $J_B = v^i\omega_i$, we have

$$\rho_i = K_{ijkp} v^j v^k v^p, \qquad \theta_i = K_{ij\alpha} v^j c^\alpha. \tag{A.59}$$

Note that the kähler form $J \in H^2(\widehat{X}_5)$ could be expressed as

$$J = J_B + \sum_A t_A U_A + \sum t_i E_i + t_0 S_0. \tag{A.60}$$

Now the D-term $D_A$ associated with the $U(1)_A$ can be argued by identifying

$$r_A = \int_{B_4} J_B \wedge J_B \wedge U_A \wedge G_4. \tag{A.61}$$

Indeed, the dual M-theory compactification on the same Calabi-Yau five manifold $X_5$ admits a GVW type superpoential $W$, which is the real function of the kähler moduli,

$$W = \int_{\widehat{X}_5} J \wedge J \wedge J \wedge G_4. \tag{A.62}$$

The supersymmetric constraints is dubbed as F-term in [141], which reads

$$F_i = \frac{\partial W}{\partial v_i} = \int_{\widehat{X}_5} J \wedge J \wedge w_i \wedge G_4 = 0. \tag{A.63}$$

This is consistent with the transversality constraints on the $G_4$ fluxes (4.36).

This condition uplifting to F-theory should equivalent to the statement that identifies the FI parameter $r_A$ associated with the abelian gauge $U(1)_A$

$$r_A = \int_{\widehat{X}_5} J_B \wedge J_B \wedge U_A \wedge G_4 = 0. \tag{A.64}$$

Indeed, with the gauging transformation

$$\delta_\lambda c^\alpha = \Theta^\alpha_A \lambda^A \to \delta_\lambda \theta_i = K_{ij\alpha} v^j \Theta^\alpha_A \lambda^A, \tag{A.65}$$

we find that for the chiral multiplet defined as (A.58), the gauging $2M_i^a$ and the GS term $N_i^a$ should be proportional to our gauging term $\Theta^\alpha_A$ up to a factor, and more importantly, we have

$$2M_i^A = N_i^A = K_{ij\alpha} v^j \Theta^\alpha_A. \tag{A.66}$$

With this argument, we can get the D-term as

$$\widetilde{D}_A = e^2((-2M_i^A + N_i^A)\rho_i - r_A + Q_m\phi^m\bar{\phi}^m). \tag{A.67}$$

Here the standard kinetic term for $H^{1,1}(B_4) - 1$ chiral multiplet $\Phi$ associated with the kahler moduli $\rho_i$ has been replaced by the Stückerlberg term and $Q_m\phi^m\bar{\phi}^m$ arises from the standard kinetic term of other charged chiral multiplets (mainly from the open sector). Combined with (A.66) and (A.64) and assuming there is no VEV for the $\phi^m$, we get the supersymmetric condition that $\widetilde{D}_A = 0$. If we deform the FI term such that they are not vanishing, then the gauge symmetry $U(1)_A$ is broken by given the correspondent VEV for the charged scalar $\phi^m$ from the open sector.

Remarkably, the above argument suggest that the 1-loop gauge anomaly is accompanied by a corresponding supersymmetry anomaly [201, 202]. When the gauge anomaly cancels, the supersymmetry anomaly also cancels, as the D-term automatically vanish.

## A.5. Proof of $c_i \cdot c_i \in \mathbb{Z} = 0$ Mod $N$

For simplicity, let us focus on our simplest case in 5.2.2. Recall that $c \cdot c = \int_C w_2$ with $C := D - D'$ and its poincare dual class $w_2 := [D - D']$ and by definition it reads $c = N$ in the setting of the Type IIB orientifold compactification. The question is what is the fate of $C$ as an involution odd 2-cycles when uplifting to F-theory geometry $\mathcal{X}_3$. Similar question has already been explored in [210], when they argue that $NC = -\partial\Gamma$ where $\Gamma$ is a 3-chain in $\mathcal{X}_3$. With this in mind, together with the fact that $dw_2 = N\alpha_3$ where $\alpha_3\alpha_3 \in TorH^3(\mathcal{X}_3, \mathbb{Z})$ (see more details on the Torsional cohomology and the relevant parts in 2.10.2), we have

$$e^{2\pi i \int_C w_2} = e^{2\pi i N < \alpha_3, C>}, \tag{A.68}$$

where $N < \alpha_3, C > \in \mathbb{Z}$ (mod $N$). Hence we have $c \cdot c = \int_C w_2 \in \mathbb{Z}$ (mod $N$), combining with the fact that $c \cdot c = N^2$ before uplifting to F-theory, we conclude that

$$c \cdot c = \int_C w_2 = 0 \text{ (mod } N). \tag{A.69}$$

Generalized to cases with several D7-branes, It should satisfy

$$c_i \cdot c_j = \int_{C_i} w_2^j = 0 \text{ (mod } N), \tag{A.70}$$

with $N = gcd(q_i) = gcd(c_i)$.

# Appendix B.

# Mathematic Glossary

In order to make the context self-contained, we here gives some standard definitions and facts concerning the mathematical tools we used in the the thesis.

## B.1. Differential Geometry

We assume that the readers have background on differential geometry and algebraic topology such as fiber bundles, cohomology and homology groups, complex manifolds and Kähler manifolds, hence here we will not introduce the details but list two notations for our purposes. For more details we refer to the book [232].

### B.1.1. Characteristic classes and invariant polynomials

Here we list same salient aspects of Characteristic classes, for details we refer to mathematical references (e.g. [232]). Assume $X$ is a diagonalised matrix with $n$ distinct non-vanishing eigenvalues $x_i, i = 1, ..., n, ....$, then one has

$$\det(t\mathbb{I} + X) = \prod_{i}^{n}(t + x_i) = \sum_{i=0}^{n} c_{n-i}(x)t^i. \tag{B.1}$$

It turns out that the coefficients $c_i$s are symmetric polynomials:

$$c_0 = 1, c_1 = \sum_{i}^{n} x_i, c_2 = \sum_{i_1<i_2}^{n} x_{i_1}x_{i_2}, ..., c_n = x_1x_2....x_n. \tag{B.2}$$

Given this, one can introduce the total **Chern class** of a vector bundle $V$ with a curvature $F$ [1] as

$$c(V) = \det(t\mathbb{I} + \frac{1}{2\pi}F) = c_0 + c_1 + c_2 + \cdots , \tag{B.3}$$

where

$$\begin{aligned} &c_0(F) = 1, c_1(F) = \frac{i}{2\pi}\mathrm{Tr}F, \\ &c_2(F) = \frac{1}{2}(\frac{i}{2\pi})^2[\mathrm{Tr}F \wedge \mathrm{Tr}F - \mathrm{Tr}(F \wedge F)], ..., c_n(F) = (\frac{i}{2\pi})^n \det F. \end{aligned} \tag{B.4}$$

For the field strength in the $SU(n)$ group, since the generator are traceless, hence one has

$$c_2(F) = \frac{1}{8\pi}\mathrm{Tr}(F \wedge F). \tag{B.5}$$

[1] Note that in the main context, we have scale the field strength $F$ by $-\frac{1}{2\pi}$ on the $n$ dimensional manifold $M$. However, let us keep using the standard convention in this appendix.

Further, the Whitney product formula follows

$$c(V_1 \oplus V_2) = c(V_1)c(V_2). \tag{B.6}$$

More important, the Euler characteristic of the manifold $M$ is given by integrating the top Chern class of the tangent bundle $TM$ as

$$\chi(M) = \int_M c_{n/2}(TM). \tag{B.7}$$

In general, given a holomorphic vector bundle $V_h$ (necessarily requiring $M$ being complex manifold), the Euler characteristics of such a bundle is similarly given by

$$\chi(V_h) = \int_M c_{n/2}(V_h). \tag{B.8}$$

The Chern character by the definition is given by

$$ch(x) = \sum_i e^{r_i} = \sum_i (1 + r_i + \frac{r_i^2}{2} + ...) = n + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + .... \tag{B.9}$$

Here $r_i$ refers to the eigenvalues of $\frac{1}{2\pi}F$, known as **Chern roots**, and $n$ is the dimension of the manifold $M$ that the vector bundle $V$ lives.

It enjoys the following properties

$$ch(E \oplus F) = ch(E) + ch(F), \qquad ch(E \otimes F) = ch(E) \wedge ch(F). \tag{B.10}$$

The **Hirzebruch L-genus** is given by

$$L(F) = \prod_i \frac{r_i}{\tanh(r_i)} = 1 + \frac{1}{3}(c_1^2 - 2c_2) + \cdots . \tag{B.11}$$

Notice that on a manifold $M$ with even dimension, integrating the Hirzebruch L-genus gives the signature of the metric, which can be defined by the intersecting products of differential forms of middle dimensionality.

Correspondingly, the **A-roof genus** then is given as

$$\widehat{A}(F) = \prod_i \frac{r_i/2}{\sinh(r_i)} = 1 + \frac{1}{24}(c_1^2 - 2c_2) + \cdots . \tag{B.12}$$

And the **Todd class** is

$$Td(F) = \prod_i \frac{r_i}{1 - e^{-r_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + ..., \tag{B.13}$$

which gives rise to the **arithmetic genus** $\chi_0(M)$ when integrating it out

$$\chi_0(M) = \sum_{i=0}^{n} h^{i,0}(M) = 1 - h^{1,0} + h^{2,0} - \cdots . \tag{B.14}$$

### B.1.2. Calabi-Yau manifolds

A Calabi-Yau manifold of real dimensional $2n$ is a compact Käher manifold $(X_n, J, g)$ which can be characterized by several equivalent properties, which are

1. There exists a Ricci-flat metric $g$,
2. The first Chern class vanishes;
3. The holonomy group $hol(g) \subseteq SU(n)$
4. The canonical bundle is trivial;
5. It admits a globally defined and nowhere vanishing holomorphic $n-$form $\Omega_n$;

6.It admits a covariantly constant globally spinor $\eta$.

Any such Calabi-Yau manifolds (as Kähler manifolds) have two integrable structures: the complex structure and the symplectic structure and hence are equipped with the following nowhere-vanishing differential forms

$$\begin{aligned} \text{Kähler form}: \quad & J \in H^{1,1}(X_n), \\ \text{complex n-form}: \quad & \Omega_n \in H^{n,0}(X_n), \end{aligned} \tag{B.15}$$

which can be defined by the covariantly constant globally spinor $\eta$. For a fixed complex structure, they also subject to

$$J \wedge J \wedge J \propto \Omega \wedge \bar{\Omega}, \qquad J \wedge \Omega = 0. \tag{B.16}$$

The Dolbeault's generalization of the Hodge theorem provides the isomorphism

$$H^r(X_n, \mathbb{C}) : H^r(X_n) \otimes \mathbb{C} = \oplus_{p+1=r} H^{p,q}_{\bar{\partial}}(X). \tag{B.17}$$

Correspondingly, we denote the dimension as $h^{p,q}$, known as hodge number.

The hodge numbers in Calabi-Yau manifolds $X_n$ obeys

$$h^{p,0} = h^{n-p,0}, \qquad h^{p,q} = h^{n-p,n-q}, \tag{B.18}$$

where the fist one we used the contraction with the top form $\Omega_n$ and the second one is due to Kähler properties. For Calabi-Yau manifolds, from the top form we know that $h^{n,0} = h^{0,n} = 1$. The holonomy group $SU(n)$ enforce that $h^{p,0} = 0, 0 < p < n$. Taking the Calabi-Yau three-folds as examples, one can write the hodge numbers as the hodge diamond, which is given by

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{3,1} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{B.19}$$

## B.2. Algebraic Geometry

In this section, we are going to list the some definitions in the algebraic geometry. The materials in this section closely follows the textbook [233].

## B.2.1. Notations and conventions

**Analytic variety** An analytic subvariety $N$ of a complex manifold $M$ of dimension $m$ can be defined as a subset given locally as the zeros (vanishing locus) $V(f_1, f_2, ..., f_n)$ of a collection of meromorphic functions $f_1, f_2, ..., f_n$, where denote the zeros as

$$V(f_1, f_2, ..., f_n) := \{f_1 = 0\} \cap \{f_2 = 0\} \cap \cdots \cap \{f_n = 0\}, \tag{B.20}$$

when thus $m - n$ is the dimension of $N$. In particularly, If $n = 1$, then $N$ is dubbed an analytic hypersurace, i.e. for any point $p \in N \subset M$, $N$ can be given in the neighborhood of $p$ as the zeros of one single meromorphic function $f$ as $V(f)$. $f$ then is called a local defining function for $N$ near $p$, and is unique up to multiplication by a function nonzero at $p$.

A point $p \in N$ is called a smooth point of $N$ if $N$ is a submanifold of $M$ near $p$, namely in the neighborhood of $p$, $N$ can be given by $V(f_1, ..f_n)$ with the rank $J(f_i) = n$ where $J$ denotes the Jacobian of $f_i$. Then the locus of smooth points of $N$ is denoted as $N^*$. In other words, any points $p' \in N - N^*$ are singular. We denote the singular locus containing all these singular points in $N - N^*$ as $N_s$. In particular, $N_s$ is contained in an analytic subvariety of $M$ not equal to $N$.

**Proposition** An analytic subvariety $N$ is irreducible if and only if $N^*$ is connected.

Given a analytic hypersurface $N$ in $M$, it then can be expressed uniquely as the union of irreducible analytic hypersurfaces

$$N = N_1 \cup N_2 \cup \cdots \cup N_r, \tag{B.21}$$

where the irreducible analytic hypersurfaces $N_i$s are the closures of the connected components of $N^*$.

We list certain sheaves notations for the following discussions: Let $V \subset M$ being an analytic subvariety of $M$, we denote the following sheaves as

$\mathbf{Z}_U$: the constant sheaf on an open set $U \subset V$, whose stalk at any points in $U$ is the additive group of integers.

$\mathcal{O}_U$: Holomorphic functions on $U$

$\mathcal{O}_U^*$: Non-vanishing Holomorphic functions on $U$

$\mathcal{M}_U^*$: non-vanishing meromorphic functions on $U$

**Algebraic variety** An algebraic/projective variety can be viewed as a special analytic variety whose embedding space $M$ is a projective space $\mathbb{P}^m$. Namely, it is defined to be the set of complex zeros of **homogenous** polynomials $f_1, f_2, ..., f_n$ in $\mathbb{P}^m$. We will also call such algebraic variety as **algebraic cycles**, which is the main objects in the intersection theory [234] (see also the appendix in [235]).

Similarly, one can uniquely decompose an algebraic variety $V \subset \mathbb{P}^m$ into irreducible components $V_i$ with associated multiplicity $m_i$ as $V = \sum_i m_i V_i$. One can easily see it carries an additive structure by adding the multiplicities of two algebraic varieties $V_1$ and $V_2$ as $V_1 + V_2 := \sum_i (m_i^1 + m_i^2) V_i$. Now considering the homology class of algebraic varieties $V$ on $M$, which we denote as $A(M)$. Then $A(M)$ carries a multiplication induced by the intersection of these algebraic subvariety in $M$ and such multiplication induces a grading as $A(M) := \oplus_{i=0}^m A^i(M)$ with $m$ being the dimension of $M$. To see that, suppose we have two algebraic varieties $C_1$ and $C_2$ of codimensional $i_1$ and $i_2$ in $M$, if all components of intersection $C_1 \cap C_2$ have codimension $i_1 + i_2$, then we see they intersect transversely. Then given two classes $[C_1], [C_2]$

in $A(M)$ of codimension $i_1$ and $i_2$, there are always representatives $C_1$ and $C_2$, according to the so-called Moving Lemma, and hence define a intersection product for the class in $A(M)$ as $[C_1] \cdot [C_2] := C_1 \cup C_2$. Hence we can defines $[C_1] \times [C_2] \to [C_{1+2}]$, where $[C_{1+2}]$ denotes the cycle classes of codimension $i_1 + i_2$. Hence $A(M)$ carries a natural grading $A(M) := \oplus_{k=0}^{m} A^k(M)$.

The most important fact for this ring is that there is an isomorphism between the cohomology groups of type $(i, i)$:

$$\oplus_{i=0}^{m} H^{i,i}(M, \mathbb{Q}) \cong \oplus_{i=0}^{m} A^i(M)_{\mathbb{Q}}, \tag{B.22}$$

where $A^i(M)_{\mathbb{Q}}$ represents the coefficients $m_i$ are rational.

**Conventions for intersections** Given as algebraic cycles $C$ on $M$, in this thesis, we will denote the homology class and its cohomology class both as $[C]$, and denote the intersection among $C_1, C_2, ..C_n$ as

$$[C_1] \cdot [C_2] \cdot \cdots \cdot [C_n] = \int_M [C_1] \wedge [C_2] \cdots \wedge [C_n], \tag{B.23}$$

where the left hand side denotes the homology class and the right hand side is the cohomology class.

### B.2.2. Divisors

In this subsection, we will introduce the precise definition of divisors. We will specify them in a projective variety $M$ and the same definitions can also carry over to an analytic variety as did in the book [233]. We will follow the notations listed in the appendix of [21].

**Divisor** A (Weil) divisor $D$ on $M$ is a locally finite formal linear combination $D = \sum a_i N_i$ of irreducible algebraic hypersurfaces $N_i$ of $M$. A divisor is called **effective**, if all the coefficients $a_i \geqslant 0$. The set of divisors in $M$ is apparently an additive group and we denote is as $Div(M)$

**Principle Divisor** A principle divisor can be written as the zeroes and poles of a globally defined meremorphic function on $M$.

**Divisor class group** One can also define the divisor class group $Cl(M) = Div(M)/\sim$ where the $\sim$ equivalence is defined by linearly equivalent between two Weil divisors $D_1, D_2$ such that $D_1 \sim D_2$ means that they only differ by a principle divisor.

**Cartier divisor** A Cartier divisor is a Weil divisor which can be locally expressed as the zeroes or poles of a single meromorphic function on $M$. The Picard group $Pic(M)$ is the group of Cartier divisors modulo linear equivalence.

Note if $M$ is smooth algebraic variety, every Weil divisor is also Cartier and in this case we have $Cl(M) = Pic(X)$. Even more, this equivalent holds also for a complex algebraic variety $M$ with only factorial singularities.

For our purposes in the chapter 2, we need to define Néron-Severi group in a Calabi-Yau $\widehat{X}_n$, as a smooth algebraic variety. Namely,

**Neron-Severi group** The Néron-Severi group $NS(M)$ is defined as the group of Weil divisors modulo algebraic equivalence.

### B.2.3. Line bundle

Let's focus on holomorphic line bundle and its connection with divisor. We will follow the presentations in the appendix in [137, 236].

A holomorphic line bundle $\pi : L \to M$ is specified by a collections of transition functions $\{g_{\alpha\beta}\} \in \mathbb{C}^*$, where each $g_{\alpha\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cup U_\beta$. Here $U_\alpha$s is a set of open cover of $M$ with the trivializations

$$\phi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C} \tag{B.24}$$

of $L_{U_\alpha} = \pi^{-1}(U_\alpha)$ and $g_{\alpha\beta} := (\phi_\alpha \phi_\beta^{-1})|_{L_z}, z \in U_\alpha \cup U_\beta$. Further $g_\alpha$ must satisfy the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

Indeed, using the sheaf-theoretic languages, the transition functions associated with a line bundle $L$ can be viewed as a Cech 1-cochain on $M$ with coefficient in $\mathcal{O}_M^*$. Consequently, the set of line bundles on $M$ is an element in $H^1(M, \mathcal{O}^*)$, which is called the Picard group of $M$. To see the group structure, we can assume two line bundles $L_1, L_2$ specified by the sets of transition functions as $g^1_{\alpha\beta}$ and $g_[\alpha\beta]^2$, then we can immediately the group structure can be realized by

$$L_1 \otimes L_2 \sim \{g^1_{\alpha\beta} g^2_{\alpha\beta}\}, L^*_{1,2} \sim \{g^{1,2}_{\alpha\beta}\}, \tag{B.25}$$

where $L^*$ denotes the dual line bundle. Hence

Now we come to the important point on the equivalence between a line bundle $L$ and a divisor $D$ on $M$. Given any (Weil) divisor $D$, with local defining functions $f_\alpha \in \mathcal{M}^*(U_\alpha)$ over some open covers $\{U_\alpha\}$ of $M$. Then we can define functions

$$g_{\alpha\beta} = f_\alpha / f_\beta, \tag{B.26}$$

which is holomorphic and non-zero in the intersection $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$, and then also satisfy the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. Hence we define a line bundle from the divisor $D$, and written as $[D]$. Further such correspondence $[]$ defines a map:

$$[] : \mathrm{div}(M) \to \mathrm{Pic}(M) \tag{B.27}$$

by specifying the following

$$[D + D'] = [D] \otimes [D'], \tag{B.28}$$

where $D, D'$ are two divisors given by local datas $\{f_\alpha\}$ and $\{f'_\alpha\}$ and $D + D'$ is followed by $\{f_\alpha f'_\alpha\}$.

Given a line bundle on $M$, the first Chern class $c_1(L)$ of the $L$ is an integral $(1,1)$-form $c_1(L) \in H^{1,1}_{\mathbb{Z}}(M)$ where $H^{1,1}_{\mathbb{Z}}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$. According to properties of Chern class, we thenhave

$$c_1(L \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{B.29}$$

Hence it imply that there is a group homomorphism from the Picard group $\mathrm{Pic}(M)$ to $H^{1,1}_{\mathbb{Z}}(M)$ through the map $L \to c_1(L)$. it turns out such a map is a surjective and thus has a kernel, denoted as $\mathrm{Pic}^0(M)$. In terms of a exact sequence, we have

$$0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \mathrm{Pic}(M) \xrightarrow{c_1} H^{1,1}_{\mathbb{Z}}(M) \longrightarrow 0, \tag{B.30}$$

namely we have $\mathrm{Pic}^0(M) := \mathrm{Ker}(c_1 : \mathrm{Pic}(M) \to H^{1,1}_{\mathbb{Z}}(M))$.

Further, note that the following exponential exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \overset{exp}{\twoheadrightarrow} \mathcal{O}^* \longrightarrow 0, \tag{B.31}$$

where $\mathbf{Z}$ denotes the constant sheaf whose stalk at any point in $M$ is the additive group of integers $\mathbb{Z}$, $exp$ denotes the exponential map $exp : f \to e^{2\pi i f}$, whose kernel is exactly the $\mathbf{Z}$, it induces a long exact sequence in cohomology as

$$\cdots \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \overset{c_1}{\twoheadrightarrow} H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathcal{O}) \longrightarrow 0. \tag{B.32}$$

One can then see the kernal $\mathrm{Pic}^0(M)$ can be identified as $\mathrm{Pic}^0(M) := H^1(M, \mathcal{O}_M)/H^1(M, \mathbb{Z})$. The first sheaf cohomology $H^1(M, \mathcal{O}_M)$ can be identified with the Dolbeault cohomolgy $H^{0,1}(M)$ by hodge theory, hence we have $\mathrm{Pic}^0(M) = H^{0,1}(M)/H^1(M, \mathbb{Z})$, which topologically is a torus, known as the Jacobian $J^1(M)$ of $M$. Note that for any simply connected algebraic variety $M$, i.e. $\pi_1(M) = 0$, we have $J^1(M) = 0$.

In the case when $M$ is a smooth algebraic variety, the Néron-Severi group is defined as the quotient of the Picard group by the subgroup $Pic^0$

$$NS(M) = Pic(M)/Pic^0(M) = Pic(M)/ker(c_1) = H^2(M, \mathbb{Z}) \cap H^{1,1}(M). \tag{B.33}$$

Further when $M$ is simply connected, i.e. $Pic^0(M) = 0$, we have $NS(M) = \mathrm{Pic}(M) = \mathrm{Cl}(M)$.

**Lefschetz Theorem on $(1,1)$-classes** For $M \subset \mathcal{P}^N$ a algebraic subvariety, every cohomology class

$$\gamma \in H^{1,1}(M) \cap H^2(M, \mathbb{Z}) \tag{B.34}$$

is holomorphic(analytic).

### B.2.4. Adjunction formulas

In the thesis we have employed the adjunction formulas evaluated in a Calabi-Yau space $X_n$ many times. Here we give a necessary introduction for generic complex manifolds $M$.

**Adjunction formula $I$** Note that the normal bundle $N_V$ of a smooth analytic hypersurface $V$ in a compact complex manifold $M$ is a line bundle as the hypersurface is codimension-one object. Then the Adjunction formula $I$ states that

$$N_V = [V]|V, \tag{B.35}$$

where $[V]$ denotes the line bundle associated with the divisor $V$.

**Adjunction formula $II$** Given any a smooth analytic variety $V$ in a compact manifold $M$, the tangent bundle of $M$ can be decomposed as

$$TM|_V = TV \oplus NV. \tag{B.36}$$

If $V$ is further a analytic hypersurface, then by the first adjunction formula we have

$$TM|_V = TV \oplus [V]|_V. \tag{B.37}$$

## B.3. Cohomologies

Let $M$ be a smooth projective complex variety of dimension $n$. The hodge decomposition is a direct sum decomposition of complex cohomology groups

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad ,0 \leqslant k \leqslant 2n. \tag{B.38}$$

Here $H^{p,q}(M)$ is the Dolbeault cohomology for $X$, which consists of classes $[\alpha]$ of differential forms that represented by the closed form $\alpha$ of type $(p,q)$ meaning that locally

$$\alpha = \sum_{I,J\in\{1,\dots,n\},|I|=p,|q|=J} f_{I,J} dz^I \wedge d\bar{z}^J. \tag{B.39}$$

For some choices of local complex coordinates $z_1, z_n$. It also satisfy the Hodge symmetry

$$H^{q,p}(M) = \bar{H}^{p,q}(M). \tag{B.40}$$

## B.4. Resolutions of Conifold Singularities

In this section, we use the example of a conifold to briefly illustrate the basic aspects of two typical types of resolutions of a singularity: Deformation and (small) Blow-up. The conifold singularity refers to a singular point in a complex three-fold $Y_3$ that locally looks like (e.g, see [47])

$$AB = CD, \tag{B.41}$$

with $A, B, C, D$ are polynomials in the coordinate ring $R$ on $\mathbb{C}^4$. The singular point is at the origin $A = B = C = D = 0$ as one can easily see the origin is the solution of $P := AB - CD = 0 = dP$. For our purpose, we rewrite the coordinates such that $X = (A+B)/2, Y = i(A-B)/2, U = i(C+D)/2, V = (C-D)/2$ and the conifold has the new forms as

$$X^2 + Y^2 + U^2 + V^2 = 0, \tag{B.42}$$

which is also known for mathematicians as ordinary double points or node. The conifold has the topology $S^3 \times S^2$. To see this, one can set two vectors $\vec{a} = (\mathrm{Re}X, \mathrm{Re}Y, \mathrm{Re}U, \mathrm{Re}V)$ and $\vec{b} = (\mathrm{Im}X, \mathrm{Im}Y, \mathrm{Im}U, \mathrm{Im}V)$ and denote

$$\vec{a}^2 + \vec{b}^2 = 2r^2, \qquad \forall r^2 > 0. \tag{B.43}$$

Then the (B.42) says

$$\vec{a}^2 - \vec{b}^2 = 0, \qquad \vec{a}\cdot\vec{b}^2 = 0, \tag{B.44}$$

which indicates that $\vec{a}^2 = r^2 = \vec{b}^2$, so $\vec{a}$ parametrizes an $S^3$ and further $\vec{b}$ perpendicular to $\vec{a}$ which gives a $S^2$ with the fixed $r$ and $\vec{a}$. Hence we have a topologically $S^2 \times S^3$. At the origin, i.e. the singular point, the radius of the $S^3$ is 0. From the neighborhood of the conifold $Y_3$, it looks like a cone over $S^2 \times S^3$, when the apex denotes the singular point.

**Deformation** It is well-known that we can deform the complex structure moduli to resolve the singularity. In the cases of the conifold $Y_3$ above, this means that one can deform it as

$$X^2 + Y^2 + U^2 + V^2 = t^2, \tag{B.45}$$

which then becomes smooth at the origin. One can still apply the above procedure and arrive at the (B.44), and it stills look like $S^2 \times S^3$. However, the radius of the $S^3$ at the origin turns out to be $t$. This indicates that the complex structure has been modified as it characterize the "size" of the certain cycles. In fact the deformed conifold topologically is the cotangent bundle of $S^3$ $T^*S^3$, as a local Calabi-Yau threefold. To see this, one can normalize the vector $\vec{a}$ to $\vec{\tilde{a}} := \vec{a}/\sqrt{t^2 + \vec{b}^2}$ and then the (B.44) changes to

$$\vec{\tilde{a}} = 1, \vec{\tilde{a}} \cdot \vec{b} = 0, \tag{B.46}$$

then replacing the $b_i, i = 1, ..., 4$ (the component of the vector $\vec{b}$ ) by $d\tilde{a}_i$, the above two equation exactly defines the the cotangent bundle $\pi : T^*S^3 \to S^3$.

**Small resolution** One can also take another way to desingularization by small resolution [2]. To see this, one can write the conifold form as the determinant of a matrix

$$\det \begin{pmatrix} A & C \\ D & B \end{pmatrix}. \tag{B.47}$$

Now we can resolve the singular point by introducing a new space $Z \subset \mathbb{C}^4 \times \mathbb{P}^1$ defined by

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0. \tag{B.48}$$

Writing in the components, we have

$$A\lambda_1 + C\lambda_2 = 0, \qquad D\lambda_1 + B\lambda_2 = 0. \tag{B.49}$$

Denoting the map $\pi : Z \to Y_3$ As one can see the singular point at the origin is then replaced by $\pi^{-1}(0) = \mathbb{P}^1$. Similar to the deformed conifold, the small resolution of the conifold also defines a local Calabi-Yau three-fold $\widetilde{Y}_3 : \mathcal{O} \oplus \mathcal{O} \to \mathbb{P}^1$. To see this, one would go to two different local patch in $Z$ as $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ and each of them gives rise to a line (tautopological) bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Note that one can also blow up the ideal $(B, D)$ rather than the above ideal $(A, C)$ and do the small transition and also obtain a local Calabi-Yau $\widetilde{Y}_3$. In fact the two can be connected by a birational transformation known as a "flop".

[2] The small resolution, roughly speaking, is similar to the blow-up. The difference is that for the blow-up, it typically introduce (an) expectational divisors $E_i$, whereas the small resolution, will also introduce (a) new cycles $C$, but with higher codimension than one.

# Appendix C.

# Appendix

## C.1. Roots in Group Theory

Consider a simple Lie algebra $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ is defined as the maximal commuting subalgebra and its generators satisfy

$$H_i = H_i^\dagger, \qquad [H_i, H_j] = 0, i = 1, ..., r, \tag{C.1}$$

and can be normalized as

$$\mathrm{Tr}(H_i H_j) = \delta_{ij}. \tag{C.2}$$

The integer number $r$ is the rank of the lie algebra $\mathfrak{g}$. The Cartan generators can be diagonalized simultanously, then one can act them on the states of a representation $D$ of $\mathfrak{g}$ as

$$H_i|\mu, x, D >= \mu_i|\mu, x, D >, \tag{C.3}$$

where the eigenvalues $\mu_i$ are dubbed as weights of the representation $D$ of $\mathfrak{g}$, which can be resembled into the weight vector, $x$ are any other labels that is necessary for specifying the state.

The roots are the weights of the adjoint representation of $\mathfrak{g}$. More specifically, one can find a basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ such that

$$\begin{aligned} &[H_i, H_j] = 0, \\ &[H_i, E_\alpha] = \alpha_i E_\alpha. \end{aligned} \tag{C.4}$$

Here the $\alpha_i$ are the roots. One can further rearrange the basis $\{H_i, E_\alpha\}$ into the $\{H_i, E_{\alpha^+}, E_{\alpha^-}\}$ such that every positive root $\alpha_i^+$ is a non-negative linear combination of the simple roots

$$(\alpha_I)_i, I = 1, ..., r. \tag{C.5}$$

## C.2. Spinor and Clifford Algebra

A spinor in $D$ dimensional manifold with Minkowski signature is an irreducible representation of the Lorentz algebra $\mathfrak{so}(1, D-1)$ and has dimension $2^{[\frac{D+1}{2}]-1}$, where the bracket [] means taking the value down to the nearest integer. Depending on the dimension $D$, a spinor could also be real $\mathbb{R}$, complex $\mathbb{C}$, or quaternionic (pseduo-real) $\mathbb{H}$. More precisely, the rule is

$$\begin{aligned} &\mathbb{R}, \qquad \text{if D} = 1, 2, 3 \qquad (\text{mod } 8), \\ &\mathbb{C}, \qquad \text{if D} = 0 \qquad (\text{mod } 4), \\ &\mathbb{H}, \qquad \text{if D} = 5, 6, 7 \qquad (\text{mod } 8). \end{aligned} \tag{C.6}$$

One should note that a complex and quaternionic representation have twice as many as degrees of freedom as a real representation.

## C.3. Lefschetz Decomposition

Given a compact d-dimensional Kähler manifold $M$ with Kähler form $J$, one can define an $SU(2)$ action on the harmonic n-forms $w$ on $M$ by

$$\begin{aligned} J_3 :& w \to \frac{d-n}{2} w, \\ J_+ :& w \to J \wedge w, \\ J_- :& w \to (J \wedge w)^*, \end{aligned} \tag{C.7}$$

where $*$ action is the Hodge $*$-operator [1]. One can easily to see these operators $J_+, J_-, J_3$ satisfy an $\mathfrak{su}(2)$ algebra if one view the the number of the form as eigenvalue. **Primitive** forms are then defined as highest weight (spin) states under this $\mathfrak{su}(2)$ algebra, namely one defines

$$J_+ w_{primitive} = 0. \tag{C.8}$$

A $n$-form has at most spin $j = \frac{d-n}{2}$, which is exactly spin of the primitive $n$-form.. For middle cohomology this means that it has spin $j = 0$. Further one can define $J^2 = \sum_i^3 J_i^2$ and it commutate with the hodge $*$-operator. In other words, one can simultaneously diagonalize the Lefshetz spin $j$ and the Hodge $*$.

## C.4. Effective Action for 4D $\mathcal{N} = 1$ Theories

The action of 4D $\mathcal{N} = 1$ supersymetric field theories coupled to gravity is totally controlled by three quantities: a Kähler potential $K$, a superpotential $W$ and a coupling for gauge kinetic coupling $f_{\kappa\lambda}$. It typically reads

$$S^{(4)} = \int -\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \mathrm{Im} f_{\kappa\lambda} F^\kappa \wedge F^\lambda + \frac{1}{2} \mathrm{Re} f_{\kappa\lambda} F^\kappa \wedge * F^\lambda + K_{I\bar{J}} DM^I \wedge * D\bar{M}^{\bar{J}} + V * \mathbf{1}, \tag{C.9}$$

where the 4D $\mathcal{N} = 1$ supergravity potential are given by F-term and D-term together as

$$\begin{aligned} V =& V_F + V_D \\ :=& e^K (K^{I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2) + \frac{1}{2} (\mathrm{Re} f)^{-1\kappa\lambda} D_{\kappa\lambda}. \end{aligned} \tag{C.10}$$

Here $K_{I\bar{J}}$ is a Kähler metric satisfying $K_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(M, \bar{M})$ and $M^I$s denote all complex scalars. The covariant derivative with respect to the Kähler potential is defines as $D_I = \partial_I + \partial_I K$.

## C.5. Fronzen Singularities

We know that M-theory compactification of $C^4/\Gamma_G$, with $\Gamma_G$ being the discrete group of $SU(2)$, will give rise to a 7d Super Yang-Mills with the corresponding ADE group $G$. Now turning on a

[1] Note that a harmonic form wedging with the Kähler form does give rise to again a harmonic form, though the wedging of two harmonic form in general does not.

non-trivial torsion of $C_3$ along the boundary of $C^4/\Gamma_G$, i.e. topologically $S^3/\Gamma_G$, it will give rise to a rational number $r \in [0, 1)$ as

$$r = \int_{S^3/\Gamma_G} C_3. \tag{C.11}$$

And physically, the gauge algebra of 7d SYM would be modified to non-simply laced algebra. In the dual F-theory picture, it turns out that, for most of part, this additional data would be reflected into geometric phases of F-theory, as the rational number $r$ will be recaptured by the monodromy of the 2-cycles of elliptic fibrations. However, according to [72, 73], there is only one Kodaria types of elliptic fibration which could remain frozen are those of type $I_n^*$. Namely, the monodromy cannot distinguish between the case of $k + 8$ D7-branes and an $O7^-$ plane with $k$ D7-branes and an $O7^+$ planes, while the resulted gauge algebras are different. Moreover, the latter cases enjoy less deformations, for example, an $O7^+$ plane is described a $I_4^*$ singularities which for some reason cannot be deformed while the $O7^-$ plane with 8 D7-brane on its top can be deformed by pulling the D7-branes off. This is where the name "Frozen" comes from.

# Appendix D.

# Own Publications

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This material presented in this thesis concerns primarily the publication [3], which is closely applied to the chapter 4.

# Appendix E.

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