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Non-Hermitian $\mathcal{PT}$-Symmetric Theories

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Non-Hermitian $\mathcal{PT}$-Symmetric Theories

This dissertation concerns the concept of $\mathcal{PT}$ symmetry as a replacement of the conventional symmetry of the Dirac Hermiticity of the operators, in particular, for the Hamiltonian. After solving a controversial issue surrounding the claimed violation of the special relativity by $\mathcal{PT}$-symmetric quantum mechanics, we move further to discuss $\mathcal{PT}$ symmetry for the systems whose associated time reversal symmetry is odd, i.e., $\mathcal{T}^2 = -\mathbb{1}$. Our results, as published papers, are incorporated in the subsequent chapters. Then we conclude by suggestions for future work.

Nicht-Hermitesche $\mathcal{PT}$-symmetrische Theorien

Diese Dissertation befasst sich mit dem Konzept der $\mathcal{PT}$ Symmetrie als Ersatz für die konventionelle Symmetrie der Dirac Hermitität der Operatoren, insbesondere für die Hamiltonian. Nachdem wir ein kontroverses Problem gelöst haben, der behaupteten Verletzung der speziellen Relativitätstheorie durch die $\mathcal{PT}$-symmetrische Quantenmechanik betreffend, gehen wir weiter, um die $\mathcal{PT}$ Symmetrie für die Systeme zu diskutieren, deren zugehörige Zeitumkehrsymmetrie ungerade ist, d.h., $\mathcal{T}^2 = -\mathbb{1}$. Unsere Ergebnisse, als veröffentlichte Zeitschriftenartikel, fließen in die nachfolgenden Kapitel ein. Dann schließen wir mit Vorschlägen für die zukünftige Arbeit.

Contents

Prelude

This dissertation concerns itself with non-Hermitian quantum mechanics in the context of combined symmetry of parity and time reversal, namely, $\mathcal{PT}$ symmetry.

In the first chapter, we discuss the formulation of $\mathcal{PT}$ symmetry in two different directions: when time reversal is even and when it is odd. We also discuss the claimed violation of the no-signaling principle in $\mathcal{PT}$-symmetric quantum mechanics.

The subsequent three chapters are devoted to our published papers concerning $\mathcal{PT}$ symmetry for the case of odd time reversal as:

Chapter Two:
Two- and four-dimensional representations of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric fermionic algebras, Alireza Beygi, S. P. Klevansky, and Carl M. Bender, Phys. Rev. A **97**, 032128 (2018).

Chapter Three:
No-signaling principle and quantum brachistochrone problem in $\mathcal{PT}$-symmetric fermionic two- and four-dimensional models, Alireza Beygi and S. P. Klevansky, Phys. Rev. A **98**, 022105 (2018).

Chapter Four:
Relativistic $\mathcal{PT}$-symmetric fermionic theories in 1+1 and 3+1 dimensions, Alireza Beygi, S. P. Klevansky, and Carl M. Bender, arXiv:1904.00878.

In the following two chapters, we have studied coupled quantum systems. In the first paper, we have considered the coupling constant as pure imaginary and introduced the concept of *partial $\mathcal{PT}$* symmetry; And in the subsequent paper, we have considered a general coupling and investigated the implications of the analytic continuation of that:

Chapter Five:
Coupled oscillator systems having partial $\mathcal{PT}$ symmetry, Alireza Beygi, S. P. Klevansky, and Carl M. Bender, Phys. Rev. A **91**, 062101 (2015).

Chapter Six:
Analytic structure of eigenvalues of coupled quantum systems, Carl M. Bender, Alexander Felski, Nima Hassanpour, S. P. Klevansky, and Alireza Beygi, Phys. Scr. **92**, 015201 (2017).

Suggestions for future work are given in the final chapter.

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Chapter 1

Introduction

1.1 $\mathcal{PT}$ symmetry with even time reversal

Dirac Hermiticity [1], which states that the Hamiltonian H describing a physical system must be Hermitian, i.e., $H = H^\dagger$, is a purely mathematical requirement, but has physical consequences: the reality of the spectrum and the unitarity of time evolution are guaranteed.
To see that the spectrum is real, we note that if we consider $H|E_n\rangle = E_n|E_n\rangle$, and multiply its Dirac conjugate by $|E_n\rangle$, we obtain $\langle E_n|H^\dagger|E_n\rangle = \langle E_n|E_n^*|E_n\rangle$, where under the assumption that $H = H^\dagger$, we conclude the reality of the spectrum, i.e., $E_n = E_n^*$.
For the unitarity of the time evolution, first we mention that a state vector evolves in time as $|t\rangle = e^{-iHt}|0\rangle$, by exploiting $i\frac{d}{dt}|t\rangle = H|t\rangle$ (throughout this thesis we use natural units, $\hbar = c = 1$). So, we obtain $\langle t|t\rangle = \langle 0|e^{iH^\dagger t}e^{-iHt}|0\rangle$, where again if $H = H^\dagger$, it becomes $\langle t|t\rangle = \langle 0|0\rangle$. This implies that the norm of the state vector does not vary with time, or, in other words, the probability is conserved.

In 1998, C. M. Bender *et al.* [2] proposed the replacement of Dirac Hermiticity by a physical symmetry of combined parity and time reversal, namely, $\mathcal{PT}$ symmetry. We illustrate this proposed formalism with a number of examples.

First, we note that we can consider the effect of time reversal as complex conjugation. This can be seen in two ways: if we assume the canonical commutation relation, i.e., $[x, p] = i$, be invariant under time reversal, since $p \to -p$, i should also change sign; The second way to see this is to consider the time-dependent Schrödinger equation, $i\partial\psi/\partial t = \{-1/(2m)\nabla^2 + V\}\psi$. Here $\psi^*(-t, x)$ is a solution, not simply $\psi(-t, x)$. So, the time reversal operator is antilinear, and here, with the implication of $\mathcal{T}^2 = +1$.
The effect of parity is obtained similarly, where both $\psi(t, x)$ and $\psi(t, -x)$ satisfy the same equation. This implies that, as in the case of time reversal, $\mathcal{P}^2 = 1$.

As a first illustration of the concept of $\mathcal{PT}$ symmetry, we consider a two-dimensional matrix Hamiltonian as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{1.1}$$

where $H_1 = a + ib$ and $H_2 = a - ib$; Also, a and b are real numbers. If we assume that b is positive, the subsystem 1 described by H_1 is growing in time, however, the subsystem 2 is decaying.

Now, we assume the matrix representation of the parity operator, S, is given by $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where it interchanges the subsystems 1 and 2. Also, $S^2 = \mathbb{1}$, which means the system remains invariant if the parity operator is applied twice.

Although (1.1) is not Hermitian, it is $\mathcal{PT}$ symmetric: as the parity interchanges the subsystems, the time reversal (which changes $i \to -i$) switches the roles of gain and loss.

Now, we couple the two subsystems through the coupling strength s,

$$H = \begin{pmatrix} H_1 & s \\ s & H_2 \end{pmatrix}. \tag{1.2}$$

This system has eigenvalues: $E_\pm = a \pm \sqrt{s^2 - b^2}$. The spectrum is real if $b^2 < s^2$. Here, we note that if we define the difference of the eigenvalues as $\omega = E_+ - E_-$, we find: $\omega^2 = 4(s^2 - b^2)$, which is a positive quantity only if $b^2 < s^2$. The hyperbolic nature of ω^2 is one of the crucial characteristics of $\mathcal{PT}$-symmetric quantum mechanics. To make a comparison with the conventional Hermitian quantum mechanics, we recall that the most general two-dimensional Hermitian Hamiltonian has the form

$$H = \begin{pmatrix} s & H_1 \\ H_2 & u \end{pmatrix},$$

for which we find: $\omega^2 = (s-u)^2 + 4(a^2 + b^2)$. This illustrates the *elliptic* behavior of Hermitian quantum mechanics in contrast to the $\mathcal{PT}$-symmetric one which is *hyperbolic*.

In general, if H and $\mathcal{PT}$ commute, i.e., $[H, \mathcal{PT}] = 0$, the spectrum is real if the symmetry of the Hamiltonian is respected by its eigenvectors; This defines the *unbroken* phase of $\mathcal{PT}$ symmetry. This can be seen as: $H(\mathcal{PT}\psi) = \mathcal{PT}(H\psi) = E^*(\mathcal{PT}\psi)$, and if $\mathcal{PT}\psi = \psi$, we obtain $E = E^*$. For our Hamiltonian in (1.2), the eigenvectors read

$$|E_+\rangle = c \begin{pmatrix} \alpha + i\beta \\ \alpha - i\beta \end{pmatrix}, \qquad |E_-\rangle = ic \begin{pmatrix} \alpha - i\beta \\ -\alpha - i\beta \end{pmatrix}, \tag{1.3}$$

where $c = [s^2/\{4(s^2 - b^2)\}]^{1/4}$, $\alpha = [1/2 + (1/2)\{(s^2 - b^2)/s^2\}^{1/2}]^{1/2}$, and $\beta = [1/2 - (1/2)\{(s^2 - b^2)/s^2\}^{1/2}]^{1/2}$. The eigenvectors are $\mathcal{PT}$ symmetric if $b^2 < s^2$.
For (1.2), this means that the composite system of H_1 and H_2, although individually are not in equilibrium, but as a whole, the system is in a dynamical equilibrium and has real eigenvalues. So, $\mathcal{PT}$-symmetric systems, in the unbroken phase of symmetry, can be thought of as systems which are intermediate between closed and open systems. In the broken phase, i.e., $s^2 < b^2$, the spectrum is complex, and the system is no longer in equilibrium.

To examine the unitarity of time evolution, first we should determine the inner product in this new formulation of quantum mechanics. We replace the Dirac inner product by $(\phi, \psi)_{\mathcal{PT}} = (\mathcal{PT}\phi)^T\psi$, where T denotes the transposition. This choice, for (1.3), implies that

$$\langle E_+|E_+\rangle_{\mathcal{PT}} = 1, \qquad \langle E_-|E_-\rangle_{\mathcal{PT}} = -1, \qquad \langle E_\pm|E_\mp\rangle_{\mathcal{PT}} = 0. \tag{1.4}$$

The minus sign in the middle term suggests that our Hamiltonian possesses a hidden symmetry, called $\mathcal{C}$, which measures the sign of the norm, i.e., $\mathcal{C}|E_+\rangle = +|E_+\rangle$, however, $\mathcal{C}|E_-\rangle = -|E_-\rangle$. For the Hamiltonian in (1.2), the matrix representation of the $\mathcal{C}$ operator can be determined and has the form

$$K = \frac{1}{\sqrt{s^2 - b^2}} \begin{pmatrix} ib & s \\ s & -ib \end{pmatrix}. \tag{1.5}$$

It is easy to establish that: $[\mathcal{C}, \mathcal{PT}] = 0$, $[\mathcal{C}, H] = 0$, and $\mathcal{C}^2 = \mathbb{1}$. For a given problem, we obtain $\mathcal{C}$ by solving these equations simultaneously. By redefining the inner product as $(\phi, \psi)_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T\psi$, the norm in (1.4) becomes positive. This implies that in $\mathcal{PT}$-symmetric quantum mechanics each problem has its own inner product and the Hamiltonian determines its own adjoint. This is a completely new notion of the inner product as is understood in the conventional Hermitian quantum mechanics. Now, in terms of this dynamically determined inner product, the Hamiltonian in (1.2) is selfadjoint, i.e., $(H\phi, \psi)_{\mathcal{CPT}} = (\phi, H\psi)_{\mathcal{CPT}}$.

Now that we have the inner product at our disposal, we are able to study the evolution of the state vector, $|\psi_i\rangle = (x, y)^T$. First, we note that the time-evolution operator regarding (1.2) reads

$$e^{-iHt} = (2/\omega)e^{-iat}\begin{pmatrix} b\sin(\omega t/2) + (\omega/2)\cos(\omega t/2) & -si\sin(\omega t/2) \\ -si\sin(\omega t/2) & -b\sin(\omega t/2) + (\omega/2)\cos(\omega t/2) \end{pmatrix}. \quad (1.6)$$

For the evolved vector $|\psi_f\rangle = e^{-iHt}|\psi_i\rangle$, we have: $\langle\psi_f|\psi_f\rangle_{\mathcal{CPT}} = \langle\psi_i|\psi_i\rangle_{\mathcal{CPT}} = (2/\omega)\{s(|x|^2 + |y|^2) - 2b\,\mathrm{Im}(xy^*)\}$, which means that the time evolution is norm preserving or the probability is conserved.

$\mathcal{PT}$ symmetry enables us to consider complex Hamiltonians which in conventional Hermitian quantum mechanics would not be allowed. For example, the Hamiltonian $H = p^2 + ix^3$ is complex and not Hermitian. It can be thought of as a composite system where one of its subsystems which is gaining energy from the environment is extremely strong at $+\infty$, and the other one, is losing energy to the environment in the same fashion at $-\infty$. If $\mathcal{P}$ is applied, the subsystems are interchanged, and under $\mathcal{T}$, their roles are switched. So, the whole system is invariant under $\mathcal{PT}$. The question: is the system in dynamical equilibrium?, can be answered in multiple ways: The energy spectrum of H, based on numerical calculations, is real (positive): $E_0 = 1.16$ and $E_1 = 4.11$. This implies that the symmetry is unbroken, and the system is in equilibrium. As expected, the corresponding eigenfunctions are also symmetric, see Fig. 1.1.

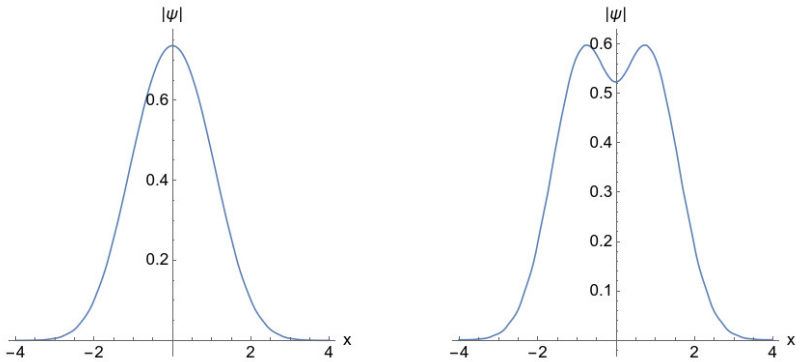

Figure 1.1: Absolute values of the eigenfunctions for the ground state (left panel) and the first excited state (right panel) of the potential ix^3.

If we consider the system at the classical level, the unbroken symmetry implies that the particle spends a finite time to traverse between the subsystems located at infinity: by exploiting, $p = \dot{x}$,

the travel time is

$$t = \int_{-\infty}^{+\infty} dx/p = \int_{-\infty}^{+\infty} dx/\sqrt{E - ix^3} = 2\sqrt{3/\pi}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)E^{-1/6}.$$

Indeed, as a general characteristic of the theory, the real spectrum at the quantum level is manifested by the confinement of the classical particle in the complex-x plane. For $H = p^2 + ix^3$, from Hamilton's equations we obtain: $\dot{x} = \sqrt{1 - ix^3}$. The classical paths are depicted in Fig. 1.2.

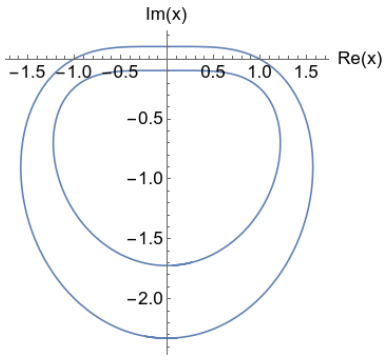

Figure 1.2: Complex trajectories of the classical particle under the influence of the potential ix^3.

These closed orbits suggest that we can identify the Hamiltonian as providing a description of a complex atom where the particle is traversing its orbits in the complex-x plane. The complexified Bohr-Sommerfeld formula, $\oint dxp = \oint dx\sqrt{E_n - ix^3} = (n + 1/2)\pi$, which should be integrated along the closed paths of the particle, gives:

$$E_n = \left[2\sqrt{\pi/3}\Gamma\left(\frac{11}{6}\right)(n + 1/2)/\Gamma\left(\frac{4}{3}\right)\right]^{6/5}.$$

From this, we find $E_0 = 1.09$ and $E_1 = 4.09$, which are good approximations to the exact eigenvalues.

As an example of a $\mathcal{PT}$-symmetric system which is not in equilibrium, we can consider $H = p^2 + ix$. For this system, the time needed for the classical particle to travel between the subsystems diverges; Indeed the trajectories of the particle as depicted in Fig. 1.3 are not localized in the complex-x plane which suggests that either the energy spectrum of the quantum system is complex or null, which in this case turns out to be null [4].

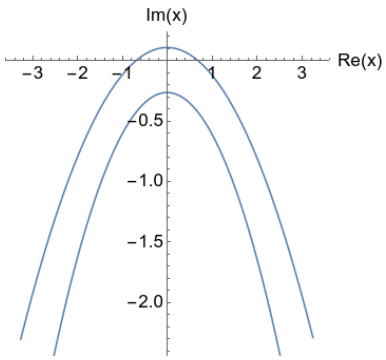

Figure 1.3: Classical trajectories of the particle in the potential ix.

1.1.1 No-signaling principle and $\mathcal{PT}$ symmetry

Y. Lee *et al.* [5], in 2014, based on their proposed thought experiment, claimed that (local) $\mathcal{PT}$ symmetry (in the case of even time reversal) is false as a fundamental theory. Here, we elucidate a different approach to the problem.

In their paper, they considered two spacelike separated parties, namely, Alice whose system is described by a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian as

$$H_{\text{Alice}} = s \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}, \tag{1.7}$$

where s and α are real, and on Bob's side, his system is governed by a Hamiltonian chosen to be an identity matrix. We note that this particular choice of Hamiltonian for Alice is a special case of (1.2), where $a = 0$ and $b = s\sin\alpha$. For the composite system of the two, we have $H_{\text{total}} = H_{\text{Alice}} \otimes I_{\text{Bob}}$.

Beforehand, Alice and Bob share a maximally entangled state, i.e., $|\psi_i\rangle = (1/\sqrt{2})(|+_x +_x\rangle + |-_x -_x\rangle)$, where $|\pm_k\rangle$ are the eigenstates of the Pauli matrices σ_k, $k = x, y, z$. Also, the two parties are not allowed to communicate classically.

If $\mathcal{PT}$ symmetry is a physically acceptable theory, Bob's probability distribution should not depend on Alice's choices of measurements; In other words, whatever Alice does on her side, should not affect the knowledge of Bob [6].

In order to examine this, the authors of [5] have considered two choices for the Alice's measurements: $A_+ = I$ and $A_- = \sigma_x$.
Then, by considering the joint final states as

$$|\psi_f^{\pm}\rangle = [e^{-iH_{\text{Alice}}t} A_{\pm} \otimes I_{\text{Bob}}]|\psi_i\rangle, \tag{1.8}$$

after $t = \pi/\omega$, where $\omega = E_+ - E_-$ and $E_\pm$ are the eigenvalues of (1.7), we obtain

$$|\psi_f^+\rangle = \begin{pmatrix} \sin\alpha \\ -i \\ -i \\ -\sin\alpha \end{pmatrix}, \qquad |\psi_f^-\rangle = \begin{pmatrix} -i \\ \sin\alpha \\ -\sin\alpha \\ -i \end{pmatrix}, \tag{1.9}$$

up to a normalization constant.

To consider Bob's knowledge, the authors calculated Bob's reduced density matrix for each case as: $\rho_B^+ = \text{Tr}_A(|\psi_f^+\rangle\langle\psi_f^+|)$ and $\rho_B^- = \text{Tr}_A(|\psi_f^-\rangle\langle\psi_f^-|)$. Direct calculations show that ρ_B^+ is different from ρ_B^-, that is, Bob can learn about Alice's choices of measurements; In other words, exploiting an entangled state, here, has resulted in a simultaneous communication between Alice and Bob. As a result, it violates the no-signaling principle.

In their calculations, as the title of their paper suggests, i.e., *Local $\mathcal{PT}$ Symmetry Violates the No-Signaling Principle*, the authors have considered Alice's system as local, and all calculations performed to obtain ρ_B^+ and ρ_B^- have been done by considering the Dirac inner product based on the assumption that Bob's system is Hermitian, and at the end, he is the one that is going to measure his system.
However, the time-evolved entangled state, i.e., $|\psi_f^{\pm}\rangle = [e^{-iH_{\text{Alice}}t} A_{\pm} \otimes I_{\text{Bob}}]|\psi_i\rangle$, has a share of Alice, and if we use the Dirac inner product, this implies that Alice's Hamiltonian is not selfadjoint with respect to this inner product; In other words, the time-evolution operator $e^{-iH_{\text{Alice}}t}$

is not unitary, and her system is nonlocal and open, in conflict to what has been claimed in the paper as Alice has a local $\mathcal{PT}$-symmetric system.

An appropriate treatment of the problem should consider the calculations with respect to the $\mathcal{CPT}$ inner product. Based on that, Alice's system becomes selfadjoint, and as a result, her Hamiltonian generates a unitary time evolution and describes a local system. To do so, first we note that the matrix representation of the $\mathcal{C}$ operator regarding (1.7) is

$$K = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \tag{1.10}$$

In order that (1.7) generates a unitary time evolution and represents a local system, we consider the time-evolution operator of Alice's system, which is norm preserving with respect to the Dirac inner product, as

$$U(t) = e^{-Q/2}e^{-iH_{\rm Alice}t}e^{Q/2}. \tag{1.11}$$

The operator Q is defined through $e^Q = KS$, where S is the matrix representation of parity, i.e., $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We obtain Q to be

$$Q = \tanh^{-1}(-\sin\alpha)\sigma_y. \tag{1.12}$$

Hence, for the time-evolution operator, we find

$$U(t) = \begin{pmatrix} \cos(\omega t/2) & m(\alpha)\sin(\omega t/2) \\ m(\alpha)\sin(\omega t/2) & \cos(\omega t/2) \end{pmatrix}, \tag{1.13}$$

where $m(\alpha) = (2/\omega)si(-1+\sin\alpha)[(1-\sin\alpha)/(1+\sin\alpha)]^{-1/2}$. For any given initial state $|\psi_i\rangle$, it is straightforward to show that: $\langle\psi_f|\psi_f\rangle = \langle\psi_i|\psi_i\rangle$, where $|\psi_f\rangle = U(t)|\psi_i\rangle$.

Now, by taking this into account, the joint final states are calculated through

$$|\psi_f^{\pm}\rangle = [U(t)A_{\pm}\otimes I_{\rm Bob}]|\psi_i\rangle. \tag{1.14}$$

This is to be compared with (1.8). After $t = \pi/\omega$, we find

$$|\psi_f^{+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ m(\alpha) \\ m(\alpha) \\ 0 \end{pmatrix}, \qquad |\psi_f^{-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} m(\alpha) \\ 0 \\ 0 \\ m(\alpha) \end{pmatrix}. \tag{1.15}$$

It is straightforward now to establish that

$$\rho_B^{+} = {\rm Tr}_A(|\psi_f^{+}\rangle\langle\psi_f^{+}|) = \rho_B^{-} = {\rm Tr}_A(|\psi_f^{-}\rangle\langle\psi_f^{-}|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.16}$$

This implies that Bob is unaware of Alice's measurements and still has a maximally entangled state; In other words, the no-signaling principle is respected.

1.2 $\mathcal{PT}$ symmetry with odd time reversal

In the previous section, we assumed that the effect of the time-reversal operator is to perform complex conjugation. In general, if we consider time reversal to be represented by a linear

operator followed by complex conjugation, i.e., $\mathcal{T}\psi = Z\psi^*$, we can show that the time-reversal operator comes in two different categories: even or odd. To see this, we note that if $\mathcal{T}$ is applied twice, we obtain $e^{i\theta} = ZZ^*$, where we have assumed that $\mathcal{T}^2$, in general, leaves the state invariant up to a factor $e^{i\theta}$. Now, if we take the complex conjugate of $e^{i\theta} = ZZ^*$, we end up with two equations for Z^* as: $Z^* = e^{i\theta}Z^{-1}$ and $Z^* = e^{-i\theta}Z^{-1}$. From these, we obtain $e^{i\theta} = \pm 1$. In other words, $\mathcal{T}^2 = +\mathbb{1}$ or $\mathcal{T}^2 = -\mathbb{1}$. For the Schrödinger equation, we observed that $\mathcal{T}^2 = +\mathbb{1}$. Now, we consider the Dirac electron coupled to electromagnetic fields as

$$i\partial\psi(t,\mathbf{x})/\partial t = [\alpha.\{-i\nabla - e\mathbf{A}(t,\mathbf{x})\} + \beta m + e\phi(t,\mathbf{x})]\psi(t,\mathbf{x}), \tag{1.17}$$

where $\alpha_i = \beta\gamma^i$, $\beta = \gamma^0$, with $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, and σ^i denote the Pauli matrices. Also, $\mathbf{A}$ is the vector potential and ϕ is the scalar one.

In (1.17), both $\psi(t,\mathbf{x})$ and $\gamma^1\gamma^3\psi^*(-t,\mathbf{x})$ are solutions. To obtain this, we have assumed that while $\mathbf{A}$ changes sign under time reversal, ϕ is invariant. So, we recognize the time-reversal operator to be $Z = \gamma^1\gamma^3$ followed by complex conjugation. This implies that $\mathcal{T}^2 = -\mathbb{1}$.

To obtain the parity operator, we note that both $\psi(t,\mathbf{x})$ and $\gamma^0\psi(t,-\mathbf{x})$ satisfy (1.17), where we have considered $\mathbf{A}(t,-\mathbf{x}) = -\mathbf{A}(t,\mathbf{x})$ and $\phi(t,-\mathbf{x}) = \phi(t,\mathbf{x})$. Like the expression for parity obtained in the previous section, the system is invariant if $S = \gamma^0$ is applied twice.

The most general four-dimensional traceless Hamiltonian which is invariant under $\mathcal{PT}$ takes the form

$$H = \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}, \tag{1.18}$$

where $B_\pm = b_1 \pm ib_2$ and $C_\pm = b_3 \pm ib_0$. All parameters a_0, b_0, b_1, b_2, and b_3 are real.
The eigenvalues of (1.18), $E_\pm = \pm(a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2)^{1/2}$, are real if $b_0^2 + b_1^2 + b_2^2 + b_3^2 < a_0^2$. The fact that the spectrum is twofold degenerate plays a crucial role here. When H and $\mathcal{PT}$ commute, in general there are two options to have a real spectrum. The first one, that we discussed in the previous section, was that the symmetry of the Hamiltonian be reflected in the eigenvectors, i.e., $\mathcal{PT}\psi = \psi$. The other option is that ψ and $\mathcal{PT}\psi$ be degenerate eigenvectors of H. As can be seen from $H(\mathcal{PT}\psi) = \mathcal{PT}(H\psi) = E^*(\mathcal{PT}\psi)$, the spectrum then becomes real. Here, with these particular choices for parity and time reversal, if we demand $\mathcal{PT}\psi = \psi$, this results in a vanishing ψ. So, in order to have real eigenvalues, the Hamiltonian has to be twofold degenerate with ψ and $\mathcal{PT}\psi$; In other words, the unbroken phase of symmetry, here, is defined primarily through the degeneracy of the Hamiltonian, not the symmetry of its eigenvectors. For (1.18), we have

$$|E_+^{(1)}\rangle = \frac{i}{\sqrt{2E_+}} \begin{pmatrix} \sqrt{a_0 + E_+}C_-/\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2} \\ \sqrt{a_0 + E_+}B_+/\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2} \\ \sqrt{a_0 - E_+} \\ 0 \end{pmatrix}, \qquad |E_+^{(2)}\rangle = \mathcal{PT}|E_+^{(1)}\rangle,$$

$$|E_-^{(1)}\rangle = \frac{i}{\sqrt{2E_+}} \begin{pmatrix} \sqrt{a_0 + E_-}C_-/\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2} \\ \sqrt{a_0 + E_-}B_+/\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2} \\ \sqrt{a_0 - E_-} \\ 0 \end{pmatrix}, \qquad |E_-^{(2)}\rangle = \mathcal{PT}|E_-^{(1)}\rangle.$$

In order to incorporate the characteristics of the time-reversal operator for fermions in the inner product, and make it consistent, we consider the inner product for $\mathcal{PT}$-symmetric fermions as

$(\phi,\psi)_{\mathcal{PT}} = (\mathcal{PT}\phi)^T Z\psi$ [7]. The application of this to the eigenvectors of (1.18) results in

$$\langle E_+^{(1)}|E_+^{(1)}\rangle_{\mathcal{PT}} = \langle E_+^{(2)}|E_+^{(2)}\rangle_{\mathcal{PT}} = 1, \qquad \langle E_-^{(1)}|E_-^{(1)}\rangle_{\mathcal{PT}} = \langle E_-^{(2)}|E_-^{(2)}\rangle_{\mathcal{PT}} = -1,$$

$$\langle E_\pm^{(1),(2)}|E_\mp^{(1),(2)}\rangle_{\mathcal{PT}} = 0.$$

The hidden symmetry of the Hamiltonian (1.18), i.e., $\mathcal{C}$, as the measure of the sign of the norm, is

$$K = \frac{1}{\sqrt{a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2}} \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}. \tag{1.19}$$

The operator $\mathcal{C}$ has been obtained by solving the equations: $[\mathcal{C},\mathcal{PT}] = 0$, $[\mathcal{C},H] = 0$, and $\mathcal{C}^2 = \mathbb{1}$. Here, it turns out that it is proportional to the Hamiltonian. Now, it is straightforward to verify that $\mathcal{C}|E_+^{(1),(2)}\rangle = +|E_+^{(1),(2)}\rangle$ and $\mathcal{C}|E_-^{(1),(2)}\rangle = -|E_-^{(1),(2)}\rangle$. The theory becomes consistent, e.g., to have a selfadjoint Hamiltonian, if we assume the inner product to be $(\phi,\psi)_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi$. This choice for the inner product also results in the unitarity of the time evolution: we can calculate the time-evolution operator, i.e., e^{-iHt}, for (1.18), then it is straightforward to show that for an initial state vector $|\psi_i\rangle = (x, x', y, y')^T$, we have

$$\begin{aligned} \langle\psi_f|\psi_f\rangle_{\mathcal{CPT}} &= \langle\psi_i|\psi_i\rangle_{\mathcal{CPT}} \\ &= \frac{1}{E_+}\big[a_0\{|x|^2 + |x'|^2 + |y|^2 + |y'|^2\} \\ &\quad - 2b_1\,\mathrm{Re}(xy'^*) - 2b_1\,\mathrm{Re}(x'y^*) + 2b_3\,\mathrm{Re}(x'y'^*) \\ &\quad + 2b_0\,\mathrm{Im}(x'y'^*) + 2b_2\,\mathrm{Im}(xy'^*) - 2b_2\,\mathrm{Im}(x'y^*)\big], \end{aligned}$$

where $|\psi_f\rangle = e^{-iHt}|\psi_i\rangle$. This confirms the unitarity of the time evolution.

In the previous section for even time reversal, we mentioned how the transition from Hermitian to $\mathcal{PT}$-symmetric quantum mechanics can be realized as the transition from an elliptic to a hyperbolic system. In $\mathcal{PT}$-symmetric *fermionic* quantum mechanics, this behavior manifests itself also in the anticommutation relation of the nilpotent $\mathcal{PT}$-symmetric fermionic operators, where $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$. In our first published paper dealing with $\mathcal{PT}$ symmetry for fermions, which is presented in the following chapter, we have demonstrated this property and constructed a model based on that. Later, we examine the no-signaling principle and quantum brachistochrone for the systems with odd time reversal. Therein, also, we propose a procedure to distinguish between two nonorthogonal states with a single measurement. In our latest paper on $\mathcal{PT}$ symmetry concerning fermionic systems, we study various extensions of the Dirac equation to incorporate non-Hermitian $\mathcal{PT}$-symmetric terms, which paves the path for future explorations. In the last two chapters, before Outlook, we come back to $\mathcal{PT}$ symmetry when time reversal is even. First, we study coupled quantum systems with a pure imaginary coupling constant and propose the notion of *partial* $\mathcal{PT}$ symmetry; In the subsequent paper, we consider a more general coupling and examine the analytic continuation of that. We sow the seeds of our future work in the final chapter.

Chapter 2

Two- and four-dimensional representations of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric fermionic algebras

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Motivation: Here, we invoke the proposed inner product for $\mathcal{PT}$-symmetric fermions to study the fermionic operator algebras and obtain the peculiar relation: $\eta\eta^{\mathcal{PT}}+\eta^{\mathcal{PT}}\eta=-\mathbb{1}$, regardless of the particular form of the Hamiltonian. We also construct an exactly solvable model based on this result.

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Two- and four-dimensional representations of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric fermionic algebras

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Fermionic systems differ from their bosonic counterparts, the main difference with regard to symmetry considerations being that $\mathcal{T}^2 = -1$ for fermionic systems. In $\mathcal{PT}$-symmetric quantum mechanics an operator has both $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints. Fermionic operators η, which are quadratically nilpotent ($\eta^2 = 0$), and algebras with $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints can be constructed. These algebras obey different anticommutation relations: $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$, where $\eta^{\mathcal{PT}}$ is the $\mathcal{PT}$ adjoint of η, and $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}$, where $\eta^{\mathcal{CPT}}$ is the $\mathcal{CPT}$ adjoint of η. This paper presents matrix representations for the operator η and its $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints in two and four dimensions. A $\mathcal{PT}$-symmetric second-quantized Hamiltonian modeled on quantum electrodynamics that describes a system of interacting fermions and bosons is constructed within this framework and is solved exactly.

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I. INTRODUCTION

A complex Hamiltonian that is $\mathcal{PT}$ symmetric (invariant under space-time reflection) may exhibit two phases separated by a phase-transition point: an unbroken-$\mathcal{PT}$-symmetric phase in which the energy spectrum is entirely real and a broken-$\mathcal{PT}$-symmetric phase in which the spectrum is partly real and partly complex [1]. Complex Hamiltonians have been studied extensively in quantum mechanics and in quantum field theory. Most of this work has been devoted to the study of *bosonic* theories, for which $\mathcal{T}^2 = 1$. However, $\mathcal{T}^2 = -1$ for fermionic theories, a crucial feature that leads to differences in the formalism. For example, if the Hamiltonian H has a real eigenvalue, then H has a corresponding degenerate pair of eigenvectors, ϕ and $\mathcal{PT}\phi$; this is a consequence of Kramer's theorem for ordinary quantum mechanics. Non-Hermitian fermionic systems have been studied within the wider framework of pseudo-Hermiticity [2].

A previous paper [3] investigated a matrix representation of a nilpotent fermionic operator η satisfying $\eta^2 = 0$ together with an adjoint nilpotent operator, denoted generically by $\bar{\eta}$. These operators satisfied a fermionic anticommutator relation $\eta\bar{\eta} + \bar{\eta}\eta = \epsilon\mathbb{1}$. The value $\epsilon = 0$ corresponds to a Grassmann algebra and the value $\epsilon = 1$ corresponds to a standard fermionic operator anticommutation relation. However, the value $\epsilon = -1$ was obtained for this anticommutation relation in a specific case of a four-dimensional matrix. Subsequently, Cherbal and Trifonov formalized this result [4], making use of the non-Hermitian formulation of quantum mechanics in Ref. [2] and the notation of Ref. [5].

The problem with determining the value of ϵ for the anticommutator lies in the definition of the adjoint nilpotent element $\bar{\eta}$. In Ref. [3] $\bar{\eta}$ was chosen to be the $\mathcal{PT}$ *reflection* of η; that is,

$$\bar{\eta} = \mathcal{PT}\eta\mathcal{T}^{-1}\mathcal{P}^{-1}.$$

This paper revises the definition of $\bar{\eta}$ in order to make it consistent with the concept of a fermionic inner product. With this revision, the fermionic algebra using $\eta^{\mathcal{PT}}$, the $\mathcal{PT}$ adjoint of η, always gives rise to an anticommutation relation with $\epsilon = -1$. However, if we use $\eta^{\mathcal{CPT}}$, the $\mathcal{CPT}$ adjoint, the fermionic algebra becomes the conventional Hermitian fermionic algebra $\epsilon = 1$.

Knowing the structural properties of the fermionic operators is a technical but important issue as it provides the basis for constructing theories of many-body systems in second quantization. It is particularly useful in the context of a given symmetry, such as $\mathcal{PT}$ symmetry, because the Hamilton or Lagrange functions constructed in this way automatically have the symmetry properties required. The second-quantized approach enables one to describe and analyze dynamic systems. We illustrate this formalism with an exactly solvable model of a $\mathcal{PT}$-symmetric Hamiltonian for fermions interacting with bosons. This model is based on the structure of quantum electrodynamics. We solve this Hamiltonian exactly for the eigenvalues and calculate the renormalized mass of the fermion.

This paper is organized as follows. In Sec. II we review the choice of the inner product in order to set our notation and we define the $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints using appropriate definitions of the $\mathcal{PT}$ and $\mathcal{CPT}$ inner products. In Secs. III and IV, we investigate two- and four-dimensional operator algebras and seek a general ansatz for the matrix representation of η and its respective $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints, which we denote as $\eta^{\mathcal{PT}}$ and $\eta^{\mathcal{CPT}}$. In Sec. V we present our calculation of a simple model of a $\mathcal{PT}$-symmetric Hamiltonian of fermions interacting with bosons. Concluding remarks are made in Sec. VI.

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II. $\mathcal{PT}$ AND $\mathcal{CPT}$ ADJOINTS OF FERMIONIC OPERATORS

In this section we follow the approach of Ref. [5] and describe the general (abstract) formulation. As the parity operator $\mathcal{P}$ is linear, its action on the wave function of a finite-dimensional system can be expressed by a matrix S, $\mathcal{P}\psi = S\psi$. Since parity applied twice yields the identity matrix, it follows that $S^2 = 1$ and that the matrix S must have the eigenvalues ± 1. In contrast, the time-reversal operator is antilinear, so its action on the system can be expressed by a matrix Z combined with the complex-conjugate operation on the function it operates on, $\mathcal{T}\psi = Z\psi^*$. It is assumed that $[\mathcal{P},\mathcal{T}] = 0$. In terms of these symbols, the $\mathcal{PT}$ inner product for fermions is defined as

$$(\phi,\psi)_{\mathcal{PT}} \equiv (\mathcal{PT}\phi)^T Z\psi. \tag{1}$$

Thus, the $\mathcal{PT}$ adjoint of any operator A is defined by

$$(A^{\mathcal{PT}}\phi,\psi)_{\mathcal{PT}} \equiv (\phi, A\psi)_{\mathcal{PT}}. \tag{2}$$

As was done in Ref. [5], we insert the definition (1) into the left and right sides of Eq. (2), set $A = \eta$, and extract the operator relation

$$\eta^{\mathcal{PT}} = S\eta^\dagger S. \tag{3}$$

This is the $\mathcal{PT}$ adjoint for fermionic systems.

Let us examine the anticommutator of η with $\eta^{\mathcal{PT}}$. According to [2], we obtain a fermionic algebra with a minus sign:

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}. \tag{4}$$

The minus sign is a signal that the $\mathcal{PT}$ inner product is not positive definite.

Following [1], one needs to introduce an additional operator $\mathcal{C}$ in order to change the $-$ sign in Eq. (4) to a $+$ sign. This operator thus reflects the sign of the norm.[1] The operator $\mathcal{C}$ is linear; it is thus represented by a matrix K. Then the $\mathcal{CPT}$ inner product is defined as [5]

$$(\phi,\psi)_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi = (KSZ\phi^*)^T Z\psi,$$

and, after some algebra, this takes the form

$$(\phi,\psi)_{\mathcal{CPT}} = \phi^\dagger SK\psi.$$

As a consequence, $A^{\mathcal{CPT}}$, the $\mathcal{CPT}$ adjoint of an operator A, is defined by

$$(A^{\mathcal{CPT}}\phi,\psi)_{\mathcal{CPT}} = (\phi, A\psi)_{\mathcal{CPT}},$$

and thus $A^{\mathcal{CPT}}$ is given by the operator relation

$$A^{\mathcal{CPT}} = KSA^\dagger SK.$$

The $\mathcal{CPT}$ adjoint is related to the $\mathcal{PT}$ adjoint by

$$A^{\mathcal{CPT}} = KA^{\mathcal{PT}}K.$$

[1]The mathematical properties of the $\mathcal{C}$ operator resemble those of the charge-conjugation operator of Dirac, but in this context $\mathcal{C}$ plays a completely different role, simply forcing the norm of the state vectors to be positive.

In accordance with [2], the anticommutator of a fermionic operator η with its $\mathcal{CPT}$ adjoint should satisfy a *conventional* fermionic algebra

$$\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}. \tag{5}$$

III. TWO-DIMENSIONAL η, $\eta^{\mathcal{PT}}$, $\eta^{\mathcal{CPT}}$

A. Real representations of η and $\eta^{\mathcal{PT}}$

We seek a two-dimensional matrix representation in which $\eta^{\mathcal{PT}}$ is the $\mathcal{PT}$ adjoint of η in accordance with Eq. (3). A general matrix

$$\eta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{6}$$

whose square vanishes, has a vanishing trace and determinant. Let us assume that a, b, and c are real numbers. The parameter a is fixed by the determinant condition

$$a^2 + bc = 0. \tag{7}$$

In two dimensions parity reflection $\mathcal{P}$ can be represented by σ_x, a real symmetric matrix whose square is unity:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then find that

$$\eta^{\mathcal{PT}} = \begin{pmatrix} -a & b \\ c & a \end{pmatrix}, \tag{8}$$

which satisfies the nilpotency condition $(\eta^{\mathcal{PT}})^2 = 0$. Now, evaluating the anticommutator of η with $\eta^{\mathcal{PT}}$, we find that

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \mathrm{diag}(-4a^2). \tag{9}$$

For nonvanishing values of a the anticommutator (9) is negative and with the choice $a^2 = 1/4$ it can be normalized to $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$.

Because the right side of Eq. (9) is nonpositive the standard fermionic algebra with $\epsilon = +1$ does not have a 2×2 representation. But the right side of Eq. (9) can vanish if we take $a = 0$. Thus, the Grassmann algebra has a nontrivial representation. For example, we may take

$$\eta = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

This result differs from the conventional Hermitian fermionic algebra, where the standard algebra has a representation but the Grassmann algebra does not [3].

B. $\mathcal{CPT}$ adjoint

We have not specified the Hamiltonian, which is required to calculate the $\mathcal{CPT}$ product. Nevertheless, we can still determine the $\mathcal{CPT}$ transformed operator $\eta^{\mathcal{CPT}}$ that yields the standard fermionic algebra (5). To do so, we first use the fact that $\mathcal{C}$ and $\mathcal{PT}$ commute to obtain a general form for the matrix K

$$K = \begin{pmatrix} g & B \\ A & -g \end{pmatrix}, \tag{10}$$

where g, A, and B are arbitrary real parameters. Since $K^2 = \mathbb{1}$, we obtain the constraint $g^2 + AB = 1$. Hence, the $\mathcal{CPT}$ adjoint of η is

$$\eta^{\mathcal{CPT}} = \begin{pmatrix} -ag^2 + bgA + cgB + aAB & -2agB + cB^2 - bg^2 \\ -2agA + bA^2 - cg^2 & ag^2 - bgA - cgB - aAB \end{pmatrix}.$$

The anticommutator of η and $\eta^{\mathcal{CPT}}$ is then

$$\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathrm{diag}(2a^2AB + c^2B^2 + b^2A^2).$$

By using the determinant relation (7) we eliminate a^2 and find that $(bA - cB)^2 = 1$, which links the parameters A, B to b,c. The choice $bA = cB$ seems to yield the Grassmann algebra $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = 0$. However, we shall see in Sec. III D that because $\mathcal{C}$ and the Hamiltonian commute, the choice $bA = cB$ is ruled out, and we arrive at the same result as in the conventional Hermitian fermionic algebra.

C. Ground state, excited state, and number operator

We can normalize the anticommutator $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta$ to $-\mathbb{1}$ by rescaling η and $\eta^{\mathcal{PT}}$ by $2a$. In this case Eqs. (6) and (8) become

$$\eta = \frac{1}{2a}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \eta^{\mathcal{PT}} = \frac{1}{2a}\begin{pmatrix} -a & b \\ c & a \end{pmatrix}.$$

We then define the ground state $|0\rangle$ as that state that is annihilated by η: $\eta|0\rangle = 0$. Using Eq. (6), we represent this state as

$$|0\rangle = \begin{pmatrix} -b \\ a \end{pmatrix}.$$

To create the $\mathcal{PT}$-symmetric state $|1\rangle$ we operate on $|0\rangle$ with $\eta^{\mathcal{PT}}$ and get

$$|1\rangle = \begin{pmatrix} b \\ a \end{pmatrix}.$$

We define the $\mathcal{PT}$ number operator as

$$N^{\mathcal{PT}} = \eta^{\mathcal{PT}}\eta$$

and establish by direct calculation that

$$N^{\mathcal{PT}}|0\rangle = 0, \qquad N^{\mathcal{PT}}|1\rangle = -|1\rangle.$$

Evidently, $N^{\mathcal{PT}}$ gives the *negative* of the state occupation number. We use this fact in Sec. V in constructing a second-quantized form of a $\mathcal{PT}$-symmetric fermionic Hamiltonian.

D. General two-dimensional $\mathcal{PT}$-symmetric Hamiltonian

A consistent fermionic $\mathcal{PT}$ quantum mechanics must satisfy three conditions: (i) The Hamiltonian must be self-adjoint with respect to the $\mathcal{PT}$ inner product for fermions; that is, the definition (1) must hold; (ii) H must commute with $\mathcal{PT}$; and (iii) the $\mathcal{PT}$ symmetry must be unbroken. The first two criteria give the following general form for a real Hamiltonian:

$$H = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \qquad (\alpha, \beta, \gamma \ \text{real}). \tag{11}$$

The matrix representations of the parity and time-reversal operators, that is, S and Z in Eq. (1), are given by σ_x.

In Sec. III B we obtained the matrix representation (10) associated with the $\mathcal{C}$ operator. A property of $\mathcal{C}$ not considered in Sec. III B is that $\mathcal{C}$ commutes with H. The commutation of K and H forces g to vanish, so the earlier constraint $g^2 + AB = 1$ reduces to $AB = 1$.

Sec. III B concludes that if $bA = cB$, one obtains a representation for a Grassmann algebra. However, the determinant condition $a^2 + bc = 0$ implies that bc is a nonpositive quantity. Therefore, to have $bA = cB$, AB must also be nonpositive, which contradicts the constraint $AB = 1$. Thus, as in the conventional Hermitian case, the Grassmann algebra does not have a nontrivial representation.

The eigenvalues of Eq. (11) are

$$\lambda_\pm = \alpha \pm \sqrt{\beta\gamma}, \tag{12}$$

and the corresponding eigenvectors are

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt[4]{\frac{\beta}{\gamma}} \\ \sqrt[4]{\frac{\gamma}{\beta}} \end{pmatrix}, \quad |\lambda_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt[4]{\frac{\beta}{\gamma}} \\ -\sqrt[4]{\frac{\gamma}{\beta}} \end{pmatrix}.$$

The formula (12) indicates that if β and γ are positive, the symmetry is unbroken; that is, the eigenvalues are real.

It is easy to establish that

$$\langle\lambda_+|\lambda_+\rangle_{\mathcal{PT}} = 1, \qquad \langle\lambda_-|\lambda_-\rangle_{\mathcal{PT}} = -1,$$
$$\langle\lambda_+|\lambda_-\rangle_{\mathcal{PT}} = \langle\lambda_-|\lambda_+\rangle_{\mathcal{PT}} = 0.$$

We introduce $\mathcal{C}$ as a measure of the sign of the norm:

$$\mathcal{C}|\lambda_+\rangle = |\lambda_+\rangle, \qquad \mathcal{C}|\lambda_-\rangle = -|\lambda_-\rangle.$$

The matrix representation of $\mathcal{C}$ is then

$$K = \begin{pmatrix} 0 & \sqrt{\beta/\gamma} \\ \sqrt{\gamma/\beta} & 0 \end{pmatrix}.$$

For the Hamiltonian (11), the annihilation operator now reads

$$\eta = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & -1 \end{pmatrix}.$$

As expected, η is nilpotent and

$$\eta|\lambda_-\rangle = 0, \qquad \eta|\lambda_+\rangle = |\lambda_-\rangle.$$

We now obtain the $\mathcal{PT}$ adjoint of η as

$$\eta^{\mathcal{PT}} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & 1 \end{pmatrix}.$$

Defining the $\mathcal{PT}$ number operator to be $N^{\mathcal{PT}} = \eta^{\mathcal{PT}}\eta$, we can show that

$$\{N,\eta\}_+ = -\eta, \qquad \{N,\eta^{\mathcal{PT}}\}_+ = -\eta^{\mathcal{PT}}.$$

The minus sign implies that the $\mathcal{PT}$ number operator $N^{\mathcal{PT}}$ gives the negative of the state occupation number, as discussed in Sec. III C.

In addition, we remark that the Hamiltonian of our $\mathcal{PT}$-symmetric interacting fermions can be recast as a free bosonic Hamiltonian:

$$H = \Delta\lambda(-N^{\mathcal{PT}}) + \lambda_- \mathbb{1},$$

where $\Delta\lambda = \lambda_+ - \lambda_-$.

The anticommutator $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$, but if instead we use the $\mathcal{CPT}$ adjoint of η,

$$\eta^{\mathcal{CPT}} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\beta/\gamma} \\ \sqrt{\gamma/\beta} & -1 \end{pmatrix},$$

we obtain the conventional anticommutator $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}$.

IV. FOUR-DIMENSIONAL η, $\eta^{\mathcal{PT}}$, $\eta^{\mathcal{CPT}}$

A general set of 12-parameter complex nilpotent matrices was proposed in Ref. [3] as

$$\eta = \begin{pmatrix} -ch - bg - af & f & g & h \\ -a(ch + bg + af) & af & ag & ah \\ -b(ch + bg + af) & bf & bg & bh \\ -c(ch + bg + af) & cf & cg & ch \end{pmatrix}, \tag{13}$$

where a, b, c, f, g, and h are arbitrary complex numbers. This form was constructed assuming that the trace of η as well as its determinant must vanish in order to guarantee nilpotency. We use the convention of Ref. [5] for the matrix representations of S and Z; that is,

$$S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad Z = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \tag{14}$$

where I is the 2×2 identity matrix, and e_2 is $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The $\mathcal{PT}$ adjoint of η reads

$$\eta^{\mathcal{PT}} = \begin{pmatrix} -F^* & -a^*F^* & b^*F^* & c^*F^* \\ f^* & a^*f^* & -b^*f^* & -c^*f^* \\ -g^* & -a^*g^* & b^*g^* & c^*g^* \\ -h^* & -a^*h^* & b^*h^* & c^*h^* \end{pmatrix},$$

where $F = ch + bg + af$. As required, $\eta^{\mathcal{PT}}$ is also nilpotent. One can evaluate the anticommutator of η and $\eta^{\mathcal{PT}}$. This is found to be

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \begin{pmatrix} J + |F|^2K & a^*J - F^*fK & -b^*J - F^*gK & -c^*J - F^*hK \\ aJ - f^*FK & |a|^2J + |f|^2K & -ab^*J + f^*gK & -ac^*J + f^*hK \\ bJ + g^*FK & ba^*J - g^*fK & -|b|^2J - |g|^2K & -bc^*J - g^*hK \\ cJ + h^*FK & ca^*J - h^*fK & -cb^*J - h^*gK & -|c|^2J - |h|^2K \end{pmatrix},$$

where $J = |F|^2 + |f|^2 - |g|^2 - |h|^2$ and $K = 1 + |a|^2 - |b|^2 - |c|^2$.

To obtain the fermionic algebras, it is necessary that the off-diagonal terms vanish. This gives the relations

$$a^*F = -f, \quad b^*F = g, \quad c^*F = h. \tag{15}$$

However, these relations force the diagonal terms to vanish. The particular choice of η in Eq. (13) proposed in Ref. [3] is only suitable for constructing a $\mathcal{PT}$-symmetric Grassmann algebra, where the anticommutator $\{\eta, \eta^{\mathcal{PT}}\}$ vanishes. An example of an η that satisfies the relations (15) and leads to a Grassmann algebra is

$$\eta = \begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ i & i & -1 & 1 \\ -i & -i & 1 & -1 \end{pmatrix}.$$

Let us examine another set of matrices that cannot be obtained from Eq. (13):

$$\eta = \begin{pmatrix} f & 0 & \alpha c & \alpha b \\ 0 & f & \alpha b^* & -\alpha c^* \\ \beta c^* & \beta b & -f & 0 \\ \beta b^* & -\beta c & 0 & -f \end{pmatrix}, \tag{16}$$

where b and c are complex and f, α, and β are real arbitrary parameters. This ansatz is a block-form construct with 2×2 matrices that ensures that the matrix is traceless in the simplest possible fashion. In addition, the off-diagonal elements have been chosen to be scaled Hermitian conjugates of one another, introducing a minimum number of parameters. Nilpotency of η must now be enforced and leads to the requirement that

$$f^2 + \alpha\beta(|b|^2 + |c|^2) = 0. \tag{17}$$

Using the matrix representations of S and Z in Eq. (14), we obtain the $\mathcal{PT}$ adjoint of η:

$$\eta^{\mathcal{PT}} = \begin{pmatrix} f & 0 & -\beta c & -\beta b \\ 0 & f & -\beta b^* & \beta c^* \\ -\alpha c^* & -\alpha b & -f & 0 \\ -\alpha b^* & \alpha c & 0 & -f \end{pmatrix}.$$

Equation (17) implies that $\eta^{\mathcal{PT}}$ is also nilpotent.

The anticommutator of η and $\eta^{\mathcal{PT}}$ is

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \mathrm{diag}\{2f^2 - (\alpha^2 + \beta^2)(|b|^2 + |c|^2)\},$$

and because of Eq. (17) this reduces to

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathrm{diag}\{(\alpha + \beta)^2(|b|^2 + |c|^2)\}.$$

Thus, the anticommutator is nonpositive. The choice $\alpha = -\beta$ gives rise to a nontrivial representation for the Grassmann algebra. However, when $\alpha \neq -\beta$, the above anticommutator with suitable normalization can be written as

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}.$$

To obtain the standard fermionic algebra we again consider the $\mathcal{CPT}$ adjoint of η instead of $\eta^{\mathcal{PT}}$. We construct the $\mathcal{C}$ operator as follows. We note that the commutation of $\mathcal{C}$ and $\mathcal{PT}$ gives

$$KSZ = SZK^*, \tag{18}$$

where K, S, and Z are the matrix representations of $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$. Another characteristic of the $\mathcal{C}$ operator is that it commutes with the Hamiltonian. The procedure to construct a general $\mathcal{PT}$-symmetric Hamiltonian for fermionic systems is described in Ref. [5]. A matrix K that satisfies the two criteria in Eq. (18) and $[\mathcal{C},H]=0$ is parametrized as

$$K=\begin{pmatrix} g & 0 & -\gamma c & -\gamma b \\ 0 & g & -\gamma b^* & \gamma c^* \\ \gamma c^* & \gamma b & -g & 0 \\ \gamma b^* & -\gamma c & 0 & -g \end{pmatrix},$$

where g and γ are real numbers.

The requirement $K^2=\mathbb{1}$ leads to the additional constraint

$$g^2-\gamma^2(|b|^2+|c|^2)=1. \tag{19}$$

Having found K, we can easily obtain the $\mathcal{CPT}$ adjoint of η:

$$\eta^{\mathcal{CPT}}=\begin{pmatrix} D & 0 & -cA & -bA \\ 0 & D & -b^*A & c^*A \\ c^*B & bB & -D & 0 \\ b^*B & -cB & 0 & -D \end{pmatrix},$$

where

$$\begin{aligned} D &= fg^2+(|b|^2+|c|^2)\gamma(\gamma f+\alpha g-\beta g), \\ A &= 2\gamma fg-\beta g^2+\alpha\gamma^2(|b|^2+|c|^2), \\ B &= 2\gamma fg+\alpha g^2-\beta\gamma^2(|b|^2+|c|^2). \end{aligned}$$

Finally, the anticommutation of η and $\eta^{\mathcal{CPT}}$ reads

$$\begin{aligned} &\eta\eta^{\mathcal{CPT}}+\eta^{\mathcal{CPT}}\eta \\ &\quad=\mathrm{diag}\{(|b|^2+|c|^2)[2\gamma f+(\alpha-\beta)g]^2\}, \end{aligned}$$

where Eqs. (17) and (19) have been used. Note that the anticommutator is *positive* and with a suitable normalization can be written as

$$\eta\eta^{\mathcal{CPT}}+\eta^{\mathcal{CPT}}\eta=\mathbb{1}.$$

For completeness, we remark that the ground state can be defined, as in Sec. III C, as being the state that is annihilated by η: $\eta|0\rangle=0$. Using Eq. (16), we represent this state as

$$|0\rangle=\begin{pmatrix} f \\ 0 \\ \beta c^* \\ \beta b^* \end{pmatrix}.$$

To create the $\mathcal{PT}$-symmetric state $|1\rangle$ we operate on $|0\rangle$ with $\eta^{\mathcal{PT}}$ and obtain

$$|1\rangle=\begin{pmatrix} \beta(|b|^2+|c|^2) \\ 0 \\ fc^* \\ fb^* \end{pmatrix}.$$

Following the procedure in Sec. III C, after normalizing $|0\rangle$ and $|1\rangle$ above, we ascertain by direct calculation that $N^{\mathcal{PT}}|0\rangle=0$ and $N^{\mathcal{PT}}|1\rangle=-|1\rangle$, where we have used $N^{\mathcal{PT}}=\eta^{\mathcal{PT}}\eta$ and Eq. (17), thus illustrating again that $N^{\mathcal{PT}}$ yields the *negative* of the state occupation number.

FIG. 1. The fermion (solid line) can emit or absorb bosons (wavy gray lines). These are the only possible interactions, so the fermion number is conserved.

V. SIMPLE MODEL HAMILTONIAN

In this section we construct a $\mathcal{PT}$-symmetric model of interacting fermions and bosons. The idea is based on the Lee model in which the lack of crossing symmetry makes the model exactly solvable [6]. We consider a single fermion that may emit and absorb bosons, as shown in Fig. 1, but the bosons may not produce a fermion-antifermion pair.

A Hamiltonian that describes this system is

$$H=ma^\dagger a-M\eta^{\mathcal{PT}}\eta-ga^\dagger\eta^{\mathcal{PT}}\eta-ga\eta^{\mathcal{PT}}\eta,$$

where the operator $a^\dagger$ creates (normal) bosons, but the fermionic operator $\eta^{\mathcal{PT}}$ creates a $\mathcal{PT}$-symmetric fermion. Here, m and M are the bare boson and fermion masses and g is the coupling amplitude. This Hamiltonian is not Hermitian but it is $\mathcal{PT}$ symmetric.

A state containing a single bare fermion and any number n of bare bosons can be written as $|E\rangle=\sum_{n=0}^{\infty}c_n|1,n\rangle$. We assume that this state is normalized; that is,

$$\langle E|E\rangle=\sum_{n=0}^{\infty}c_n^2<\infty. \tag{20}$$

The annihilation and creation operators for bosons obey $a|n\rangle=\sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$. In addition, $a^\dagger a$ is the boson number operator; that is, $a^\dagger a|n\rangle=n|n\rangle$. For the case of fermions we have the peculiar anticommutation relation $\eta\eta^{\mathcal{PT}}+\eta^{\mathcal{PT}}\eta=-\mathbb{1}$. However, as in Sec. III, we interpret η as a lowering operator and $\eta^{\mathcal{PT}}$ as a raising operator. Thus, the fermion number operator is $-\eta^{\mathcal{PT}}\eta$.

The time-independent Schrödinger equation $H|E\rangle=E|E\rangle$ takes the form

$$\begin{aligned} &\sum_{n=0}^{\infty}mnc_n|1,n\rangle+\sum_{n=0}^{\infty}Mc_n|1,n\rangle \\ &\quad+\sum_{n=0}^{\infty}g\sqrt{n+1}c_n|1,n+1\rangle \\ &\quad+\sum_{n=0}^{\infty}g\sqrt{n}c_n|1,n-1\rangle=\sum_{n=0}^{\infty}Ec_n|1,n\rangle. \end{aligned}$$

We shift indices and pick off the coefficients of $|1,n\rangle$ to obtain a recursion relation c_n:

$$(mn+M)c_n+g\sqrt{n}c_{n-1}+g\sqrt{n+1}c_{n+1}=Ec_n.$$

The substitution $c_n=d_n\sqrt{n!}$ gives the simpler recursion relation

$$(mn+M)d_n+gd_{n-1}+g(n+1)d_{n+1}=Ed_n. \tag{21}$$

For large n we can neglect the Md_n and Ed_n terms and obtain an approximate equation for d_n that is valid for large n:

$$mnd_n + gd_{n-1} + g(n+1)d_{n+1} \simeq 0.$$

There are two consistent asymptotic dominant balances for $n \gg 1$: If the first and second terms balance for large n, then

$$d_n \simeq (-g/m)^n/n!; \tag{22}$$

if the first and third terms balance, then

$$d_n \simeq (-m/g)^n. \tag{23}$$

(A dominant balance between the second and third terms is inconsistent.) The norm in Eq. (20) becomes $\sum_{n=0}^{\infty} d_n^2 n!$. Therefore, Eq. (22) is acceptable but Eq. (23) is not.

Next, we define a generating function $f(x) \equiv \sum_{n=0}^{\infty} d_n x^n$; if Eq. (22) holds, then $f(x)$ is an entire function of x, but if Eq. (23) holds, we see that $f(x)$ has a finite radius of convergence with a singularity in the complex-x plane at $x = -g/m$.

If we multiply Eq. (21) by x^n and sum from 0 to ∞, we obtain the first-order differential equation

$$(mx + g)f'(x) = (E - M - xg)f(x),$$

whose solution is

$$f(x) = Ke^{-gx/m}(mx + g)^{E/m - M/m + g^2/m^2}.$$

As predicted, there is a singularity at $x = -g/m$ unless the exponent in the second term on the right side is a non-negative integer $N = 0, 1, 2, \ldots$. This yields the *exact* spectrum of physical fermion states:

$$E_N = Nm + M - g^2/m \quad (N = 0, 1, 2, \ldots).$$

Note that as a consequence of the interaction, the mass $M - g^2/m$ of the physical fermion is *lower* than the mass M of the bare fermion.

VI. BRIEF CONCLUDING REMARKS

In this paper we have used the alternative formalism for the fermionic scalar product in Ref. [5] to reexamine the operator algebra for fermions in the context of $\mathcal{PT}$ symmetry. We have investigated general matrix representations of the $\mathcal{PT}$ and $\mathcal{CPT}$ fermionic creation and destruction operators without making direct reference to a Hamiltonian. Knowing the behavior of such operators, especially $\mathcal{PT}$ operators, can be important for many-body theory, which often uses the operator definitions to construct the Hamiltonian (in second-quantized form). It can also be important in understanding the nature of species oscillation in neutrinos [7].

We have examined the operator algebras in detail for 2×2 matrices and for the 4×4 case. Using the algebra that we have developed, we apply the peculiar anticommutation relations pertinent to the $\mathcal{PT}$ algebra to construct a second-quantized $\mathcal{PT}$-symmetric quantum field theory, namely, a solvable low-dimensional model of electrodynamics (a modified Lee model) for which the renormalized energy spectrum is calculated in a closed form and is found to be real.

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Chapter 3

No-signaling principle and quantum brachistochrone problem in $\mathcal{PT}$-symmetric fermionic two- and four-dimensional models

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Motivation: Here, we examine the validity of the no-signaling principle for general two- and four-dimensional Hamiltonians suitable for describing non-Hermitian $\mathcal{PT}$-symmetric fermionic systems. In contrast to the previous results obtained by others, we demonstrate that the no-signaling principle is upheld for $\mathcal{PT}$-symmetric systems with the caveat that the symmetry is broken (in the two-dimensional case). We also propose a feasible procedure to distinguish between two nonorthogonal states with a single measurement.

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No-signaling principle and quantum brachistochrone problem in $\mathcal{PT}$-symmetric fermionic two- and four-dimensional models

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Fermionic systems differ from bosonic ones in several ways, in particular the time-reversal operator $\mathcal{T}$ is odd, $\mathcal{T}^2 = -\mathbb{1}$. For $\mathcal{PT}$-symmetric bosonic systems, the no-signaling principle and the quantum brachistochrone problem have been studied to some degree, both of them controversially. In this paper, we apply the basic methods proposed for bosonic systems [Y. Lee *et al.*, Phys. Rev. Lett. **112**, 130404 (2014); C. M. Bender *et al.*, *ibid.* **98**, 040403 (2007)] to *fermionic* two- and four-dimensional $\mathcal{PT}$-symmetric Hamiltonians and obtain several surprising results: We find, in contrast to the bosonic case, that the no-signaling principle is upheld for two-dimensional fermionic Hamiltonians; however, the $\mathcal{PT}$ symmetry is broken. In addition, we find that the time required for the evolution from a given initial state, the spin up, to a given final state, the spin down, is a constant, independent of the parameters of the Hamiltonian, under the eigenvalue constraint. That is, it cannot, as in the bosonic case, be optimized. We do, however, also find a dimensional dependence: Four-dimensional $\mathcal{PT}$-symmetric fermionic Hamiltonians considered here again uphold the no-signaling principle, but it is not essential that the $\mathcal{PT}$ symmetry be broken. The symmetry is, however, broken if the measure of entanglement is conserved. In the four-dimensional systems, the evolution time between orthogonal states is dependent on the parameters of the Hamiltonian, with the conclusion that it again can be optimized and approach zero under certain circumstances. However, if we require the conservation of entanglement, the transformation time between these two states becomes the same constant as that found in the two-dimensional case, which coincides with the minimum time for such a transformation to take place in the Hermitian case.

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I. INTRODUCTION

Since the seminal work of Bender and Boettcher [1], the properties of systems with $\mathcal{PT}$ symmetry have been studied extensively and have led to important insights [2]. For the most part, these studies encompass bosonic systems, where time-reversal symmetry $\mathcal{T}$ is represented simply by complex conjugation. For fermionic systems, the situation is more complicated: The fact that $\mathcal{T}^2 = -\mathbb{1}$ leads to essential differences in the formulation and the possible outcomes. One notes, for example, that if a $\mathcal{PT}$-symmetric Hamiltonian H describing fermions has a real eigenvalue, then H has a corresponding degenerate pair of eigenvectors ψ and $\mathcal{PT}\psi$, which is a consequence of Kramer's theorem for conventional quantum mechanics. Non-Hermitian fermionic systems have been studied within the wider framework of pseudo-Hermiticity [3].

In a previous paper, we constructed two- and four-dimensional representations of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric fermionic algebras [4] and constructed a many-body second-quantized non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian modeled on quantum electrodynamics, which we were able to solve exactly. Based on the knowledge gained in that work, we now study two current problems of *fermionic* systems, which previously have been discussed only for bosons, notably quite controversially in the literature. These are (a) The no-signaling principle and (b) the quantum brachistochrone problem. Both have a common technical feature: The time-evolution operator for a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian must be considered. We discuss these problems in turn.

A. The no-signaling principle

Lee *et al.* [5] initiated the discussion of the no-signaling principle by studying a two-dimensional bosonic locally $\mathcal{PT}$-symmetric Hamiltonian $H_{2\times2}$ attributed to Alice combined with a two-dimensional Hermitian Hamiltonian attributed to Bob, the latter taken trivially to be $\mathbb{1}$. The combined system is then given as $H_{\rm tot} = H_{2\times2} \otimes \mathbb{1}$. Both parties start out with an initial maximally entangled state, given by $|\psi\rangle = (1/\sqrt{2})(|+_x +_x\rangle + |-_x -_x\rangle)$, where $|\pm_k\rangle$ are the eigenstates of the Pauli matrices σ_k, $k = x, y, z$. They then evaluate the no-signaling condition [6,7],

$$\sum_a P(a, b|A_+, B) = \sum_a P(a, b|A_-, B) = P(b|B), \quad (1)$$

where a and b are the measurement outcomes of our two space-like separated parties Alice and Bob and $A_\pm$ and B are different local measurements done by Alice and Bob on their respective sides. This condition means that the probability distribution of Bob over his measurement outcomes is unaffected by Alice's choice of measurements on her side.

The first assumption made is that a local $\mathcal{PT}$-symmetric (bosonic) Hamiltonian can coexist with a Hermitian Hamiltonian. The second, and perhaps more surprising assumption, is

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that the authors assume that the postmeasurement probabilities that must be computed in evaluating (1) should be performed within the framework of a conventional Hilbert space prescription, using a standard Dirac inner product. These authors find that if one requires the condition $\sum_a P(a,b|A_+,B) = \sum_a P(a,b|A_-,B)$ to be respected, Alice's Hamiltonian is forced to be Hermitian. They thus conclude that the no-signaling principle is violated for all 2×2 (nontrivial) $\mathcal{PT}$-symmetric Hamiltonians with even time reversal, $\mathcal{T}^2=+1$. Although they do not prove it explicitly, they claim that the use of a $\mathcal{CPT}$ inner product does not cure this problem.

Subsequent to this, in a detailed calculation, Japaridze *et al.* [8] revisited this problem and concluded that the calculations, redone using the $\mathcal{CPT}$ inner product for the evaluation of the probabilities, in fact do preserve the no-signaling principle. In the further literature, Brody [9] discussed the physical applicability of the claims of [5] and demonstrated the consistency of $\mathcal{PT}$-symmetric quantum mechanics with special relativity through the proposal that the metric operator on Hilbert space is not an observable. In other words, the author claimed that there is no statistical test that can be performed on the outcomes of measurements with the aim of distinguishing between Hermiticity and $\mathcal{PT}$ symmetry of a given Hamiltonian, at least for closed systems in finite dimensions.

In the work presented in this paper, we return to the ansatz of [5] and ask the question of how the outcomes will differ for fermionic systems. To this end, we perform calculations for both 2×2 and 4×4 $\mathcal{PT}$-symmetric fermionic matrix Hamiltonians. We arrive at the surprising results that the no-signaling principle, as discussed in the formalism of [5], is upheld, even with the unusual calculational constraints of using the conventional Dirac inner product. In addition, we also discuss this by calculating the marginal probabilities and also show that the measure of entanglement is conserved. However, we find that $\mathcal{PT}$ symmetry is broken in the two-dimensional case, while this symmetry breaking is not essential in four dimensions unless the requirement of conservation of the entanglement is imposed: In this case, the $\mathcal{PT}$ symmetry of the Hamiltonian is broken.

B. The $\mathcal{PT}$-symmetric fermionic quantum brachistochrone problem

The quantum brachistochrone problem is an attempt to find the minimal time required to transform a given initial state to a given final state in a system governed by a parametrized Hamiltonian H, while the difference between the largest and smallest eigenvalues is held fixed [10,11]. This has been studied by Bender *et al.* [10], who chose a (bosonic) $\mathcal{PT}$-symmetric matrix Hamiltonian and studied the optimal time required to evolve a spin-up state to a spin-down one. These authors found the intriguing result that the evolution time can approach zero, provided that the elements of the Hamiltonian are extremely large.

For our case, we find a surprising result, viz., that the time to transform a spin-up state to a spin-down one, under the same eigenvalue constraint, is a constant, independent of the parameters of the Hamiltonian. This constant is the same as the optimal time for such a transformation in the Hermitian case [10]. We make the crucial observation that in the two-dimensional case, the spin-up and spin-down states are, in fact, orthogonal to each other with respect to the $\mathcal{CPT}$ inner product. In four dimensions this is not the case, and then a dependence on the parameters of the Hamiltonian arises, so that the required time can be optimized and can be made arbitrarily small. However, if we take the conservation of the entanglement into account, the transformation time becomes the same constant as in the two-dimensional case.

This paper is structured as follows. In Sec. II, we discuss the no-signaling principle and the quantum brachistochrone problem for the 2×2 $\mathcal{PT}$-symmetric fermionic Hamiltonians. In Sec. III both are elucidated for the 4×4 case. We provide some further notes on $\mathcal{PT}$-symmetric quantum state discrimination in Sec. IV and make some concluding remarks in Sec. V.

II. TWO-DIMENSIONAL MODEL

A. No-signaling principle

A general $\mathcal{PT}$-symmetric fermionic two-dimensional Hamiltonian is described by [4]

$$H=\begin{pmatrix}\alpha & \beta\\ \gamma & \alpha\end{pmatrix}\qquad(\alpha,\ \beta,\ \gamma\ \text{real}),\tag{2}$$

which is self-adjoint with respect to the $\mathcal{PT}$ inner product for fermions, and it commutes with $\mathcal{PT}$.

We recall that the fermionic $\mathcal{PT}$ inner product is defined as [12]

$$\langle\phi|\psi\rangle_{\mathcal{PT}}=(\mathcal{PT}\phi)^T Z\psi,\tag{3}$$

where the parity $\mathcal{P}$, being a linear operator, can be represented by a matrix S as $\mathcal{P}\psi=S\psi$, and time-reversal $\mathcal{T}$, being an antilinear operator, can be represented by a matrix Z combined with the complex conjugation operation, i.e., $\mathcal{T}\psi=Z\psi^*$.

Alice and Bob are two spacelike separated parties who wish to communicate with each other without using any classical protocol. Without loss of generality, we can assume that Alice's system is governed by a special case of (2) as

$$H=\begin{pmatrix}1 & \sin\alpha\\ \cos\alpha & 1\end{pmatrix},\tag{4}$$

and Bob's is governed by the identity matrix. The two parties do not interact with each other.

The eigenvalues of (4) read

$$\lambda_\pm=1\pm\sqrt{\tfrac{1}{2}\sin 2\alpha},\tag{5}$$

with the corresponding eigenvectors

$$|\lambda_+\rangle=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\sqrt[4]{\tan\alpha}\\ \sqrt[4]{\cot\alpha}\end{pmatrix},$$

$$|\lambda_-\rangle=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\sqrt[4]{\tan\alpha}\\ -\sqrt[4]{\cot\alpha}\end{pmatrix}.$$

The eigenvalues of H in (5) are real, provided that $\sin 2\alpha>0$. This inequality defines the region of unbroken $\mathcal{PT}$ symmetry.

We note that the eigenvectors of H are not orthogonal to each other with respect to the conventional Dirac inner product, however, it is easy to establish that $\langle\lambda_-|\lambda_+\rangle_{\mathcal{PT}}=0$.

The time-evolution operator regarding Alice's Hamiltonian can be evaluated as

$$U = e^{-iHt} = -\frac{2i}{\sqrt{2\sin 2\alpha}} e^{-i\pi/\omega} \begin{pmatrix} 0 & \sin\alpha \\ \cos\alpha & 0 \end{pmatrix}, \tag{6}$$

where $\omega = \lambda_+ - \lambda_-$ and we have set $t = \pi/\omega$.

If Alice performs the measurement $\mathbb{1}$ with respect to the information that she wants to send to Bob, the state vector of the composite system of Alice and Bob after $t = \pi/\omega$ evolves to

$$|\psi_f^+\rangle = (U\mathbb{1} \otimes \mathbb{1})|\psi\rangle, \tag{7}$$

where $|\psi\rangle$ is the shared maximally entangled state described in terms of the eigenvectors of σ_x as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+_x\rangle \otimes |+_x\rangle + |-_x\rangle \otimes |-_x\rangle), \tag{8}$$

which is used by the two parties to discuss their communication protocol beforehand.

The measure of entanglement [13],

$$E = -\mathrm{tr}_A(\rho_A \log_2 \rho_A) = -\mathrm{tr}_B(\rho_B \log_2 \rho_B), \tag{9}$$

implies that $E(\psi) = 1$, where

$$\rho_A = \rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Thus, the final state reads

$$|\psi_f^+\rangle = -ie^{-i\pi/\omega} \begin{pmatrix} 0 \\ \sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}. \tag{11}$$

We note that (11) is normalized with regard to the conventional Dirac inner product for Hermitian quantum mechanics.

Now, if Alice performs the measurement σ_x, the final state of the composite system after $t = \pi/\omega$ becomes

$$|\psi_f^-\rangle = (U\sigma_x \otimes \mathbb{1})|\psi\rangle, \tag{12}$$

which is given explicitly as

$$|\psi_f^-\rangle = -ie^{-i\pi/\omega} \begin{pmatrix} \sin\alpha \\ 0 \\ 0 \\ \cos\alpha \end{pmatrix}, \tag{13}$$

where it is normalized as before.

Bob's density matrix when Alice performs the measurement $\mathbb{1}$ is

$$\rho_B^+ = \mathrm{Tr}_A(|\psi_f^+\rangle\langle\psi_f^+|), \tag{14}$$

which takes the form

$$\rho_B^+ = \begin{pmatrix} \cos^2\alpha & 0 \\ 0 & \sin^2\alpha \end{pmatrix}. \tag{15}$$

For Alice's second measurement, Bob's density matrix reads

$$\rho_B^- = \begin{pmatrix} \sin^2\alpha & 0 \\ 0 & \cos^2\alpha \end{pmatrix}. \tag{16}$$

In order for the no-signaling principle to be respected, Bob's density matrix should not be dependent on Alice's choice of measurements, that is,

$$\rho_B^+ = \rho_B^-, \tag{17}$$

which can be fulfilled if $\cos\alpha = -\sin\alpha$, implying that Alice's Hamiltonian, (4), is still non-Hermitian and $\mathcal{PT}$ symmetric. However, the symmetry is broken.

Now, if Alice and Bob measure their corresponding subsystems with the conventional quantum projectors $|\pm_y\rangle\langle\pm_y|$, we find

$$P(a, b|A_\pm, B) = \langle\psi_f^\pm|(|a\rangle\langle a| \otimes |b\rangle\langle b|)|\psi_f^\pm\rangle \tag{18}$$

for the joint probabilities, where $A_\pm$ correspond to the measurements $\mathbb{1}$ and σ_x, performed by Alice, and a and b are the possible outcomes $\pm_y$, i.e., the eigenvectors of σ_y.

The two marginal probabilities are found to be

$$\sum_{a=\pm y} P(a, +_y|A_+, B) = \sum_{a=\pm y} P(a, +_y|A_-, B) = \frac{1}{2}. \tag{19}$$

The above calculation shows that Bob's probability distribution over his local measurement outcomes is not altered by Alice's choice of measurements on her side; that is to say, the no-signaling principle is respected.

We conclude that whether the symmetry is broken or not, Alice's Hamiltonian remains $\mathcal{PT}$ symmetric without violating the no-signaling principle.

As a side remark, we note that the measure of entanglement is also conserved. Our starting point was a maximally entangled state, and in the end we still have a maximally entangled one. To see this, first, we obtain Bob's reduced density matrix ρ_B. To do so, we calculate the density matrix of the composite system after time $t = \pi/\omega$, that is,

$$\rho = \tfrac{1}{2}(|\psi_f^+\rangle\langle\psi_f^+| + |\psi_f^-\rangle\langle\psi_f^-|), \tag{20}$$

which reads

$$\rho = \frac{1}{2}\begin{pmatrix} \sin^2\alpha & 0 & 0 & \sin\alpha\cos\alpha \\ 0 & \sin^2\alpha & \sin\alpha\cos\alpha & 0 \\ 0 & \sin\alpha\cos\alpha & \cos^2\alpha & 0 \\ \sin\alpha\cos\alpha & 0 & 0 & \cos^2\alpha \end{pmatrix}. \tag{21}$$

By taking the partial trace over A, we obtain ρ_B as

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Equation (9) implies that the entanglement measure is still unity, although our time-evolution operator (6) is not unitary in the context of conventional Hermitian quantum mechanics.

B. Quantum brachistochrone problem

Given the initial and final states, we now investigate which $\mathcal{PT}$-symmetric fermionic two-dimensional matrix Hamiltonian H can achieve the transformation between these two states in the least time, provided that the difference between the largest and smallest eigenvalues of H is held fixed. To approach this problem, one can determine the optimal time for the Hamiltonian acting in the subspace spanned by the given initial and final states [14].

First, we note that the difference between the largest and the smallest eigenvalues of (2), that is, the eigenvalue constraint

$E_+ - E_- = \Omega$, reads

$$\Omega = 2\sqrt{\beta\gamma}. \tag{23}$$

Here Ω^2 is positive if the symmetry is unbroken.

The time-evolution operator with regard to (2) is

$$U = e^{-iHt} = e^{-i\alpha t}\begin{pmatrix} \cos\frac{1}{2}\Omega t & -i\sqrt{\frac{\beta}{\gamma}}\sin\frac{1}{2}\Omega t \\ -i\sqrt{\frac{\gamma}{\beta}}\sin\frac{1}{2}\Omega t & \cos\frac{1}{2}\Omega t \end{pmatrix}. \tag{24}$$

The initial state, chosen arbitrarily to be spin up, $|\psi_0\rangle = \binom{1}{0}$, evolves to the final state, $|\psi_f\rangle = \binom{a}{b}$, as

$$\begin{pmatrix} a \\ b \end{pmatrix} = U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\alpha t}\begin{pmatrix} \cos\frac{1}{2}\Omega t \\ -i\sqrt{\frac{\gamma}{\beta}}\sin\frac{1}{2}\Omega t \end{pmatrix}. \tag{25}$$

We note that the time-evolution operator preserves the $\mathcal{CPT}$ norm of the initial state, $\langle\psi_0|\psi_0\rangle_{\mathcal{CPT}} = \langle\psi_f|\psi_f\rangle_{\mathcal{CPT}} = \sqrt{\gamma/\beta}$, where the $\mathcal{CPT}$ inner product is defined as $\langle\phi|\psi\rangle_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi$ [12]. The $\mathcal{C}$ operator reflects the sign of the $\mathcal{PT}$ norm and forces the norm of the state vectors to be positive. Thus, the Hamiltonian plays a key role in determining the operator $\mathcal{C}$. For the problem at hand, its matrix representation K can be found to be

$$K = \begin{pmatrix} 0 & \sqrt{\beta/\gamma} \\ \sqrt{\gamma/\beta} & 0 \end{pmatrix}. \tag{26}$$

Now, let us assume that $a = 0$ and $b = 1$, that is, we flip the spin-up state to a spin-down one. To obtain the time required for this process, we solve for the first component of (25), finding

$$t = \frac{\pi}{\Omega}, \tag{27}$$

which is not dependent on the parameters of the Hamiltonian under the eigenvalue constraint. We also note that this constant is the minimum time for such a transformation in the Hermitian case, also called the *passage time* [10,15].

In addition, one can also show that $\binom{1}{0}$ and $\binom{0}{1}$ are indeed orthogonal to each other with respect to the $\mathcal{CPT}$ inner product, i.e., $\langle\psi_f|\psi_0\rangle_{\mathcal{CPT}} = 0$.

III. FOUR-DIMENSIONAL MODEL

A. Quantum brachistochrone problem

A four-dimensional five-parameter Hamiltonian which satisfies all the criteria of $\mathcal{PT}$-symmetric fermionic quantum mechanics, i.e., self-adjointness and invariance under $\mathcal{PT}$, can be written as [12,16]

$$H = \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}, \tag{28}$$

where $B_\pm = b_1 \pm ib_2$ and $C_\pm = b_3 \pm ib_0$. The parameters a_0, b_0, b_1, b_2, and b_3 are real.

Here the parity operator is taken to be the Dirac matrix γ_0, and the time-reversal operator is taken as the matrix Z followed by complex conjugation, where $Z = \mathrm{diag}[i\sigma_y]$. Note that with these choices $\mathcal{P}$ and $\mathcal{T}$ commute, $\mathcal{P}^2 = \mathbb{1}$, and $\mathcal{T}^2 = -\mathbb{1}$. These choices for the parity and time-reversal operators are similar to those of Bjorken and Drell [17] derived in the context of coupling the Dirac electron to electromagnetic fields.

The eigenvalues of (28) read

$$E_\pm = \pm\sqrt{a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2}, \tag{29}$$

which are twofold degenerate. The eigenvectors corresponding to the positive energy are

$$|\psi_1\rangle = \frac{i}{\sqrt{2E_+}}\begin{pmatrix} \frac{\sqrt{a_0+E_+}}{\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}}C_- \\ \frac{\sqrt{a_0+E_+}}{\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}}B_+ \\ \sqrt{a_0 - E_+} \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \mathcal{PT}|\psi_1\rangle, \tag{30}$$

while those corresponding to the negative energy are

$$|\psi_3\rangle = \frac{i}{\sqrt{2E_+}}\begin{pmatrix} \frac{\sqrt{a_0+E_-}}{\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}}C_- \\ \frac{\sqrt{a_0+E_-}}{\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}}B_+ \\ \sqrt{a_0 - E_-} \\ 0 \end{pmatrix}, \quad |\psi_4\rangle = \mathcal{PT}|\psi_3\rangle. \tag{31}$$

The above degeneracy is the $\mathcal{PT}$ analog of the phenomenon of Kramer's theorem in conventional Hermitian quantum mechanics, where the Hamiltonian is invariant under odd time reversal.

The eigenvalue constraint, $E_+ - E_- = \Omega$, given in terms of the parameters of the Hamiltonian, reads

$$\Omega^2 = 4\left(a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2\right),$$

which is a positive quantity when the symmetry is unbroken, that is, the eigenvalues are real, $a_0^2 > b_0^2 + b_1^2 + b_2^2 + b_3^2$.

Then the time-evolution operator for the Hamiltonian (28) is evaluated to be

$$U = e^{-iHt} = \begin{pmatrix} \cos\frac{1}{2}\Omega t - \frac{2ia_0}{\Omega}\sin\frac{1}{2}\Omega t & 0 & \frac{2iC_-}{\Omega}\sin\frac{1}{2}\Omega t & \frac{2iB_-}{\Omega}\sin\frac{1}{2}\Omega t \\ 0 & \cos\frac{1}{2}\Omega t - \frac{2ia_0}{\Omega}\sin\frac{1}{2}\Omega t & \frac{2iB_+}{\Omega}\sin\frac{1}{2}\Omega t & -\frac{2iC_+}{\Omega}\sin\frac{1}{2}\Omega t \\ -\frac{2iC_+}{\Omega}\sin\frac{1}{2}\Omega t & -\frac{2iB_-}{\Omega}\sin\frac{1}{2}\Omega t & \cos\frac{1}{2}\Omega t + \frac{2ia_0}{\Omega}\sin\frac{1}{2}\Omega t & 0 \\ -\frac{2iB_+}{\Omega}\sin\frac{1}{2}\Omega t & \frac{2iC_-}{\Omega}\sin\frac{1}{2}\Omega t & 0 & \cos\frac{1}{2}\Omega t + \frac{2ia_0}{\Omega}\sin\frac{1}{2}\Omega t \end{pmatrix}. \tag{32}$$

The initial state, which we arbitrarily choose to be

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

evolves to the final state

$$|\psi_f\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

through U as

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = U \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\Omega t - \frac{2ia_0}{\Omega}\sin\frac{1}{2}\Omega t \\ 0 \\ -\frac{2iC_+}{\Omega}\sin\frac{1}{2}\Omega t \\ -\frac{2iB_+}{\Omega}\sin\frac{1}{2}\Omega t \end{pmatrix}. \tag{33}$$

To investigate whether the norm of the initial state is conserved or not, we examine the $\mathcal{CPT}$ inner product for fermions as defined in Ref. [12], $\langle\phi|\psi\rangle_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi$. One can also obtain the matrix representation of $\mathcal{C}$ for the problem at hand as $2H/\Omega$. Then it is easy to establish that the probability is conserved, that is, $\langle\psi_0|\psi_0\rangle_{\mathcal{CPT}} = \langle\psi_f|\psi_f\rangle_{\mathcal{CPT}} = 2a_0/\Omega$.

Equation (33) indicates that the final state cannot be a spinor of a particle, so we consider it to correspond to that of an antiparticle and choose, say, $a = 0$, $b = 0$, $c = 0$, and $d = 1$. (This is reminiscent of the fact that the quantum states of a particle and an antiparticle can be interchanged by applying the charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, and time-reversal $\mathcal{T}$ operators.)

The first component implies that

$$t = \frac{2}{\Omega}\arctan\left(\frac{\Omega}{2a_0}\right). \tag{34}$$

To optimize this result over all positive a_0, t can approach zero as a_0 goes to infinity. This result requires that $|B_+|$ is also extremely large, as can be seen from the fourth component of (33). Thus, we can perform a spinor flip from a particle to that of an antiparticle in an arbitrarily short amount of time under the given eigenvalue constraint, provided that at least two parameters of the five-parameter $\mathcal{PT}$-symmetric Hamiltonian in (28) are extremely large.

We recall at this point that the time for evolution between two orthogonal states in conventional quantum mechanics is limited by the uncertainty principle [18]. We note, however, that the initial and final states, $|\psi_0\rangle$ and $|\psi_f\rangle$, are *not* orthogonal to each other with respect to the $\mathcal{CPT}$ inner product, that is, $\langle\psi_f|\psi_0\rangle_{\mathcal{CPT}} = -2B_+/\Omega$.

It is interesting to note that a tunable passage time could also be found numerically in the context of dissipative systems using the time-evolution operator associated with a non-Hermitian, non-$\mathcal{PT}$-symmetric Hamiltonian [19]. This study deals with bosonic systems.

B. No-signaling principle

For simplicity, we assume that Alice's system is governed by a special case of (28) as

$$H = \begin{pmatrix} a_0 & 0 & 0 & -B_- \\ 0 & a_0 & -B_+ & 0 \\ 0 & B_- & -a_0 & 0 \\ B_+ & 0 & 0 & -a_0 \end{pmatrix}. \tag{35}$$

After time $t = \pi/\Omega$, the time-evolution operator with regard to Alice's system reads

$$U = \begin{pmatrix} -\frac{2ia_0}{\Omega} & 0 & 0 & \frac{2iB_-}{\Omega} \\ 0 & -\frac{2ia_0}{\Omega} & \frac{2iB_+}{\Omega} & 0 \\ 0 & -\frac{2iB_-}{\Omega} & \frac{2ia_0}{\Omega} & 0 \\ -\frac{2iB_+}{\Omega} & 0 & 0 & \frac{2ia_0}{\Omega} \end{pmatrix}. \tag{36}$$

As in the two-dimensional case, we assume Alice and Bob share a maximally entangled state to discuss their communication protocol beforehand:

$$|\psi\rangle = \tfrac{1}{2}(|+_x\rangle_1 \otimes |+_x\rangle_1 + |+_x\rangle_2 \otimes |+_x\rangle_2 + |-_x\rangle_1 \otimes |-_x\rangle_1 + |-_x\rangle_2 \otimes |-_x\rangle_2), \tag{37}$$

where $|\pm_x\rangle_{1,2}$ are the eigenvectors of $\Sigma_x = \left(\begin{smallmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{smallmatrix}\right)$.

If Alice performs the measurement $\mathbb{1}$, after $t = \pi/\Omega$ the state vector of the composite system reads

$$|\psi_f^+\rangle = (U\mathbb{1} \otimes \mathbb{1})|\psi\rangle, \tag{38}$$

which becomes

$$|\psi_f^+\rangle = \sqrt{\frac{a_0^2 - |b|^2}{a_0^2 + |b|^2}} \begin{pmatrix} V^+ \\ W^+ \\ W^{+'} \\ V^{+'} \end{pmatrix}, \tag{39}$$

where $|b|^2 = b_1^2 + b_2^2$ and

$$V^+ = \begin{pmatrix} -\frac{ia_0}{\Omega} \\ 0 \\ 0 \\ \frac{iB_-}{\Omega} \end{pmatrix}, \qquad W^+ = \begin{pmatrix} 0 \\ -\frac{ia_0}{\Omega} \\ \frac{iB_+}{\Omega} \\ 0 \end{pmatrix}. \tag{40}$$

Also,

$$W^{+'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} W^{+*}, \tag{41}$$

and

$$V^{+'} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} V^{+*}. \tag{42}$$

Now, if Alice performs the measurement Σ_x, the final state of the composite system of Alice and Bob after time $t = \pi/\Omega$ becomes

$$|\psi_f^-\rangle = (U\Sigma_x \otimes \mathbb{1})|\psi\rangle, \tag{43}$$

which is given explicitly as

$$|\psi_f^-\rangle = \sqrt{\frac{a_0^2-|b|^2}{a_0^2+|b|^2}}\begin{pmatrix} W^- \\ V^- \\ V^{-'} \\ W^{-'} \end{pmatrix}, \tag{44}$$

where W^- is obtained by replacing B_+ in W^+ by B_- and V^- is obtained by replacing B_- by B_+ in V^+.

When Alice performs the first measurement, Bob's density matrix reads

$$\rho_B^+ = \mathrm{Tr}_A(|\psi_f^+\rangle\langle\psi_f^+|), \tag{45}$$

where

$$\rho_B^+ = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & \frac{2a_0B_+}{a_0^2+|b|^2} \\ 0 & 1 & \frac{2a_0B_-}{a_0^2+|b|^2} & 0 \\ 0 & \frac{2a_0B_+}{a_0^2+|b|^2} & 1 & 0 \\ \frac{2a_0B_-}{a_0^2+|b|^2} & 0 & 0 & 1 \end{pmatrix}. \tag{46}$$

For Alice's second measurement, Bob's density matrix becomes

$$\rho_B^- = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & \frac{2a_0B_-}{a_0^2+|b|^2} \\ 0 & 1 & \frac{2a_0B_+}{a_0^2+|b|^2} & 0 \\ 0 & \frac{2a_0B_-}{a_0^2+|b|^2} & 1 & 0 \\ \frac{2a_0B_+}{a_0^2+|b|^2} & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

The no-signaling principle is respected if $\rho_B^+ = \rho_B^-$. This requires that $B_+ = B_-$, which implies that b_2 must vanish. Under this constraint, the Hamiltonian that governs Alice's system, (35), is still non-Hermitian and $\mathcal{PT}$ symmetric, and its eigenvalues are also real, provided that $a_0^2 > b_1^2$.

To investigate the conservation of the entanglement measure, we first construct the density matrix of the composite system as before, according to (20). Then we calculate the reduced density matrix and by using (9) arrive at the measure of entanglement as being

$$E = 1 + \frac{2a_0b_1}{a_0^2+b_1^2}\log_4\frac{2(a_0^2+b_1^2)}{a_0b_1}. \tag{48}$$

The measure is no longer conserved; in fact, it has been increased. However, this measure can still be unity if a_0 approaches zero. This also implies that the eigenvalues are no longer real and thus that the $\mathcal{PT}$ symmetry is broken. Another implication of this is that now the time required to transform between the initial and final states mentioned in the brachistochrone problem, (34), approaches π/Ω as a_0 approaches zero. This value is again, as in the two-dimensional case (27), the optimal time for such a transformation in the Hermitian case.

IV. A NOTE ON $\mathcal{PT}$-SYMMETRIC QUANTUM STATE DISCRIMINATION

It is well known that if a system is in one of two nonorthogonal quantum states, $|\psi_1\rangle$ and $|\psi_2\rangle$, it is not possible to determine with absolute certainty which state the system is in with just one measurement [20]. This has been challenged by Bender *et al.* [21] by exploiting the features of a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian. A key point is that the inner product of such a problem is determined by the Hamiltonian at hand; that is, it is determined dynamically. Thus, it is possible to introduce a Hamiltonian in such a way that relative to its inner product the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ become orthogonal.

The general $\mathcal{PT}$-symmetric Hamiltonian which they considered is built on the assumption that the time-reversal operator is just complex conjugation, and as a result of this, they arrived at a *complex* Hamiltonian. And they concluded that this ability to distinguish between a pair of nonorthogonal states with a single measurement is due to the complex degrees of freedom made available by $\mathcal{PT}$ symmetry.

We show that their results are still valid for the fermionic case for which it turns out that the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian is real [see (2)].

First, we consider the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ on the Bloch sphere that are separated by the angular distance 2ϵ as

$$|\psi_1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}+\epsilon) \\ e^{i\phi}\sin(\frac{\theta}{2}+\epsilon) \end{pmatrix}. \tag{49}$$

For definiteness, we choose $\phi = \pi$ and $\theta = 2\pi/3 - \epsilon$.

These two states are not orthogonal in the conventional sense, i.e., $\langle\psi_1|\psi_2\rangle \neq 0$. Now, by considering (2), (26), and the inner product $\langle\psi_i|\psi_f\rangle_{\mathcal{CPT}} = (\mathcal{CPT}\psi_i)^T Z\psi_f$, we can construct the bra vector corresponding to $|\psi_1\rangle$ as

$$\langle\psi_1|_{\mathcal{CPT}} = \begin{pmatrix} \sqrt{\frac{\gamma}{\beta}}\cos(\frac{\pi}{3}-\frac{\epsilon}{2}) \\ -\sqrt{\frac{\beta}{\gamma}}\sin(\frac{\pi}{3}-\frac{\epsilon}{2}) \end{pmatrix}^T. \tag{50}$$

Then we require that $\langle\psi_1|\psi_2\rangle_{\mathcal{CPT}}$ vanishes, which results in the condition

$$\tan^2\frac{\epsilon}{2} = \frac{\gamma+3\beta}{3\gamma+\beta}. \tag{51}$$

Now, to distinguish between the two states, we need the projection operators

$$|\psi_1\rangle\langle\psi_1|_{\mathcal{CPT}}, \qquad |\psi_2\rangle\langle\psi_2|_{\mathcal{CPT}}. \tag{52}$$

Thus, by applying one of these projection measurements, we can distinguish between states $|\psi_1\rangle$ and $|\psi_2\rangle$ with absolute certainty.

V. CONCLUDING REMARKS

In this paper, we have applied the procedures suggested by [5,10] for studying the no-signaling principle and the quantum brachistochrone problem in $\mathcal{PT}$-symmetric fermionic two- and four-dimensional models. The results show several interesting properties. First, a dimensional dependence emerges. For the quantum brachistochrone problem, the time required to transform a spin-up state to a spin-down state in the two-dimensional case, unlike its bosonic counterpart, shows no dependence on the parameters of the Hamiltonian, and it is a constant under the eigenvalue constraint, where this constant coincides with the minimum time for such a transformation in the Hermitian case. A parameter dependence, however, reemerges as a feature of the analysis of the four-dimensional system, and it can approach zero provided that some parameters of the Hamiltonian are extremely large. In this case, however, one again recovers the same constant for the transformation

time as in the two-dimensional case by taking the conservation of entanglement into account. In general, the brachistochrone itself may be related to the orthogonality or alignment of the initial and final states within the chosen theory. Second, the no-signaling principle is upheld in the two-dimensional system, with the caveat that $\mathcal{PT}$ symmetry is broken. In four dimensions, however, it is again upheld, but $\mathcal{PT}$ symmetry is broken only if the conservation of entanglement is enforced.

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Chapter 4

Relativistic $\mathcal{PT}$-symmetric fermionic theories in 1+1 and 3+1 dimensions

This chapter presents work that is at the stage of being currently under review:

Title: Relativistic $\mathcal{PT}$-symmetric fermionic theories in 1+1 and 3+1 dimensions

Authors: Alireza Beygi, S. P. Klevansky, and Carl M. Bender

Reference: arXiv:1904.00878

Motivation: Here, we investigate the possible extensions of the Dirac equation to incorporate non-Hermitian $\mathcal{PT}$-symmetric terms that result in real eigenvalues. To what extent these systems could describe physical fermionic systems is still an open question.

Relativistic $\mathcal{PT}$-symmetric fermionic theories in 1+1 and 3+1 dimensions

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Relativistic $\mathcal{PT}$-symmetric fermionic interacting systems are studied in 1+1 and 3+1 dimensions. The noninteracting Dirac equation is separately $\mathcal{P}$ and $\mathcal{T}$ invariant. The objective here is to include non-Hermitian $\mathcal{PT}$-symmetric interaction terms that give rise to *real* spectra. Such interacting systems could be physically realistic and could describe new physics. The simplest such non-Hermitian Lagrangian density is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\gamma^5\psi$. The associated relativistic Dirac equation is $\mathcal{PT}$ invariant in 1+1 dimensions and the associated Hamiltonian commutes with $\mathcal{PT}$. However, the dispersion relation $p^2 = m^2 - g^2$ shows that the $\mathcal{PT}$ symmetry is broken (the eigenvalues become complex) in the chiral limit $m \to 0$. For field-theoretic interactions of the form $\mathcal{L}_{\rm int} = -g(\bar{\psi}\gamma^5\psi)^N$ with $N = 2, 3$, which we can only solve approximately, we also find that if the associated (approximate) Dirac equation is $\mathcal{PT}$ invariant, the dispersion relation always gives rise to complex energies in the chiral limit $m \to 0$. Other models are studied in which x-dependent $\mathcal{PT}$-symmetric potentials such as ix^3, $-x^4$, $i\kappa/x$, Hulthén, or periodic potentials are coupled to the fermionic field ψ using vector or scalar coupling schemes or combinations of both. For each of these models the classical trajectories in the complex-x plane are examined. Some combinations of these potentials can be solved numerically, and it is shown explicitly that a real spectrum can be obtained. In 3+1 dimensions, while the simplest system $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\gamma^5\psi$ resembles the 1+1-dimensional case, the Dirac equation is *not* $\mathcal{PT}$ invariant because $\mathcal{T}^2 = -\mathbb{1}$. This explains the appearance of complex eigenvalues as $m \to 0$. Other Lorentz-invariant two-point and four-point interactions are considered that give non-Hermitian $\mathcal{PT}$-symmetric terms in the Dirac equation. Only the axial vector and tensor Lagrangian interactions $\mathcal{L}_{\rm int} = -i\bar{\psi}\tilde{B}_\mu\gamma^5\gamma^\mu\psi$ and $\mathcal{L}_{\rm int} = -i\bar{\psi}T_{\mu\nu}\sigma^{\mu\nu}\psi$ fulfil both requirements of $\mathcal{PT}$ invariance of the associated Dirac equation and non-Hermiticity. The dispersion relations show that both interactions lead to complex spectra in the chiral limit $m \to 0$. The effect on the spectrum of the additional constraint of selfadjointness of the Hamiltonian with respect to the $\mathcal{PT}$ inner product is investigated.

Keywords: $\mathcal{PT}$ symmetry, relativistic fermionic theories, Dirac equation

I. INTRODUCTION

A non-Hermitian quantum-mechanical Hamiltonian H that is invariant under combined parity (space reflection) $\mathcal{P}$ and time reversal $\mathcal{T}$ can have real eigenvalues [1, 2]. If the spectrum is entirely real, we say that H has an *unbroken* $\mathcal{PT}$ symmetry. However, if H has complex eigenvalues, we say that H has a *broken* $\mathcal{PT}$ symmetry. Numerous theoretical studies of classical and quantum-mechanical $\mathcal{PT}$-symmetric systems have been done and many experiments on such systems have been performed. The remarkable features of $\mathcal{PT}$-symmetric include $\mathcal{PT}$ symmetry breaking in coupled wave guides, unidirectional invisibility, enhanced sensing at exceptional points, level bifurcation in superconducting wires, and robust wireless power transfer [3–10].

In quantum mechanics $x \to -x$ under parity $\mathcal{P}$ and $i \to -i$ under time reversal $\mathcal{T}$. Thus, the quantum-mechanical Hamiltonian $H = p^2 + x^2(ix^\varepsilon)$ (ε real) is $\mathcal{PT}$ invariant; H has a *real positive discrete* spectrum when $\varepsilon \geq 0$ [1]. This quantum theory generalizes to relativistic quantum field theory if the operator $x(t)$ is replaced by the pseudoscalar field $\phi(t, \mathbf{x})$ so that $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$ under $\mathcal{P}$ and $\phi(t, \mathbf{x}) \to \phi^*(-t, \mathbf{x})$ under $\mathcal{T}$. The analogous bosonic field-theoretic Hamiltonian density $(\partial\phi)^2 + \phi^2(i\phi)^\varepsilon$ also appears to have a real spectrum; this was shown to first order in ε for $0 \leq D < 2$ [11].

While $\mathcal{PT}$-symmetric bosonic systems have been studied heavily (there are over 4,000 papers on such systems), only a few papers have been written on $\mathcal{PT}$-symmetric fermionic systems. Early work on matrix models of fermionic systems can be found in Refs. [12–15]. The Lagrangian density for a free relativistic fermionic field with mass m was extended by including a non-Hermitian axial mass term $\mathcal{L}_{\rm int} = -g\bar{\psi}\gamma^5\psi$, where g is a real mass parameter [16]. Further developments were made in Ref. [17] in which quantum electrodynamics was extended to include such a term and the restoration of gauge symmetry was investigated. In Ref. [18] the relationship between conserved currents and invariances of the Lagrangian in the

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framework of non-Hermitian field theories was examined. An application of $\mathcal{PT}$-symmetric fermionic field theory to neutrino species oscillation was proposed in Ref. [19] in which an 8-dimensional Dirac equation was analyzed. Neutrino oscillations in the context of $\mathcal{PT}$ symmetry were studied further in Ref. [20].

$\mathcal{PT}$-symmetric fermionic field theories in 1+1 dimensions share the property with quantum-mechanical and bosonic field theories that $\mathcal{T}^2 = \mathbb{1}$ [16]. However, in Ref. [13] it was noted that $\mathcal{PT}$-symmetric fermionic systems in 3+1 dimensions have the propery that $\mathcal{T}^2 = -\mathbb{1}$. To explain this we first examine what happens in 1+1 dimensions, where the gamma matrices are [21]

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

Note that $\left(\gamma^0\right)^2 = \mathbb{1}$, $\left(\gamma^1\right)^2 = -\mathbb{1}$, and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$, where σ_3 is a Pauli matrix. Let us identify the discrete spatial symmetries of the free Dirac equation

$$\left[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m\right]\psi(t,x) = 0. \tag{2}$$

(Here $\partial_0 = \partial_t$ and $\partial_1 = \partial_x$.) To determine the effect of a space reflection we let $x \to -x$ and then multiply (2) on the left by γ^0 to get

$$\left[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m\right]\gamma^0\psi(t,-x) = 0.$$

Because this equation has the *same form* as (2) we identify that the action of parity reflection $\mathcal{P}$ on the spinor $\psi(t,x)$ is given by

$$\mathcal{P}:\ \psi(t,x) \to \mathcal{P}\psi(t,x)\mathcal{P}^{-1} = \gamma^0\psi(t,-x). \tag{3}$$

Next, to determine the effect of time reversal $\mathcal{T}$ we let $t \to -t$ in (2), take the complex conjugate of the resulting equation, and again multiply on the left by γ^0. We get

$$\left[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m\right]\gamma^0\psi^*(-t,x) = 0.$$

Again, from form invariance we conclude that time reversal for spinors in 1+1 dimensions is given by

$$\mathcal{T}:\ \psi(t,x) \to \mathcal{T}\psi(t,x)\mathcal{T}^{-1} = \gamma^0\psi^*(-t,x). \tag{4}$$

Since γ^0 is real we see that applying $\mathcal{P}$ or $\mathcal{T}$ twice leaves $\psi(t,x)$ invariant. Thus, $\mathcal{P}^2 = \mathbb{1}$ and $\mathcal{T}^2 = \mathbb{1}$. (Interestingly, this property of time reversal in 1+1 dimensions implies that the Dirac electron behaves like a boson [22].)

In 3+1 dimensions the Dirac representation of the gamma matrices is [23]

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{5}$$

where σ^i are the Pauli matrices. The actions of parity and time reversal obtained similarly, are now [23]

$$\begin{aligned} \mathcal{P}:\ \psi(t,\mathbf{x}) &\to \mathcal{P}\psi(t,\mathbf{x})\mathcal{P}^{-1} = \gamma^0\psi(t,-\mathbf{x}), \\ \mathcal{T}:\ \psi(t,\mathbf{x}) &\to \mathcal{T}\psi(t,\mathbf{x})\mathcal{T}^{-1} = i\gamma^1\gamma^3\psi^*(-t,\mathbf{x}). \end{aligned} \tag{6}$$

If we apply $\mathcal{T}$ twice, we observe a change of sign: $\mathcal{P}^2 = \mathbb{1}$, but now $\mathcal{T}^2 = -\mathbb{1}$. This underscores the different nature of fermions in 3+1 dimensions.

The purpose of this paper is to investigate the behavior of 1+1- and 3+1-dimensional relativistic $\mathcal{PT}$-invariant fermionic theories. An exploratory study in Ref. [24] examined in part the properties of a $\mathcal{PT}$-symmetric fermionic Lee model. This paper begins by reexamining the results in [16], where it was assumed that for real g the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m - g\gamma^5)\psi$ is $\mathcal{PT}$ symmetric. We find that including the axial term gives a dispersion relation $p^2 = m^2 - g^2$ that yields a real value for the physical mass only when $m^2 \geq g^2$. This implies that the spectrum is not real in the chiral limit $m \to 0$. This result holds for Lagrangians in both 1+1 and 3+1 dimensions. We ask, Why is this so and under what conditions is it not so? Obtaining spectral reality in the chiral limit is part of the motivation for this paper. One of our long-range goals to construct a $\mathcal{PT}$-symmetric version of the Thirring and Nambu-Jona-Lasinio models. The challenge is to identify additional non-Hermitian terms that are both $\mathcal{PT}$ symmetric and chiral and give rise to a real spectrum in the chiral limit [25].

Two ingredients are required for a precise analysis of fermionic systems: (i) Care must be taken in analyzing time reversal, which is nontrivial for fermionic systems; (ii) care is needed in deciding on the form of $\mathcal{PT}$-adjoint operators. In this paper we focus first on time reversal and then address the constraint of selfadjointness with respect to the $\mathcal{PT}$ inner product for fermions.

For various interactions in 1+1 and 3+1 dimensions we use the Euler-Lagrange equations to construct the Dirac equation that results from a Lagrangian density and investigate whether this (quantum-mechanical) Dirac equation is form invariant under the actions of $\mathcal{P}$ and $\mathcal{T}$. This enables us to identify the transformation properties of the interaction term and also to calculate the dispersion relation associated with plane-wave solutions of the Dirac equation. In addition, by rewriting the Dirac equation in the form $i\partial_t\psi = H\psi$, we identify the *effective Hamiltonian* H [23] associated with the interaction. We will see that the form invariance of the Dirac equation under $\mathcal{PT}$ is equivalent to the statement that H commutes with $\mathcal{PT}$.

In analyzing the case of 1+1 dimensions, we find the surprising result that for complex fermionic fields, the bilinear interaction form $-g\bar{\psi}\gamma^5\psi$ gives a Dirac equation that is *odd* under time reversal and also odd under parity. Thus, the Dirac equation with the interaction term is form invariant under $\mathcal{PT}$. The $\mathcal{PT}$ symmetry can also be verified by determining the Hamiltonian H associated with this Dirac equation $i\partial_t\psi = H\psi$. The 2×2 matrix representation clarifies this result. Comparing with the general result for a 2×2 $\mathcal{PT}$-symmetric fermionic Hamiltonian in 1+1 dimensions [24], it becomes evident that in 1+1 dimensions the $\mathcal{PT}$ symmetry is broken when m vanishes. In a second example, due to the similarities in the transformation properties of this interaction with

those of $\phi(t,\mathbf{x})$, we surmise that higher integer powers of the interaction Lagrangian density $-g(\bar{\psi}\gamma^5\psi)^N$ might lead to a spectral relation that has real energies; we investigate this for $N=2$ and 3. We find that the $\mathcal{PT}$ symmetry is always broken if we assume that the expectation value $\langle\bar{\psi}\gamma^5\psi\rangle$ is negative imaginary. There are no other matrix potentials in 1+1 dimensions.

We then turn to further examples for which x-dependent $\mathcal{PT}$-symmetric potentials ix^3, $-x^4$, and $i\kappa/x$ introduced via vector or scalar coupling various combinations, as well as the complex $\mathcal{PT}$-symmetric lattice potentials $i\kappa\cot(x)+i\gamma^0\sin(x)$ and the Hulthén potential are included in the Dirac equation of motion. In order to gain some understanding of these systems, we construct the analogous classical systems for which a classical phase structure can be obtained.

The situation in 3+1 dimensions is different because $\mathcal{T}^2=-1$. Studying the algebra in 3+1 dimensions, we confirm that the interaction term $-g\gamma^5\psi$ in the equation of motion is *even* under time reversal. Since the parity transformation is still odd in 3+1 dimensions, we conclude that the interaction term in the Dirac equation is not invariant under $\mathcal{PT}$. While the dispersion relation is superficially the same as for the 1+1-dimensional case, which implies that there is *no* region in which the spectrum is real in the chiral limit, the associated interaction Hamiltonian is anti-$\mathcal{PT}$ symmetric, which is consistent with the complex nature of the spectrum.

In 3+1 dimensions, we search for other bilinear combinations of fermionic fields with the aim of determining all possible combinations that give a Dirac equation that is form invariant under $\mathcal{PT}$ and that are not Hermitian. We find two types of terms having either an axial vector or a tensor structure. The spectra of both of the non-Hermitian $\mathcal{PT}$-symmetric interactions are analyzed. Here too we find that the $\mathcal{PT}$ symmetry is always broken in the chiral limit. We also look at the consequences of imposing an additional condition that the Hamiltonian be selfadjoint under the $\mathcal{PT}$ inner product for fermions [13, 14] and investigate the restrictions that this implies. We demonstrate that the $\mathcal{PT}$ symmetry is always broken in the chiral limit, a feature that prevails in the analysis of the Dirac equation in the dimensions studied.

This paper is organized as follows: In Sec. II we investigate possible $\mathcal{PT}$-symmetric interactions in 1+1 dimensions. We analyze $\mathcal{L}_{\rm int}=-g\bar{\psi}\gamma^5\psi$ in Subsec. II A and extensions to this as $-g(\bar{\psi}\gamma^5\psi)^N$ in Subsec. II B. We introduce the spatially dependent potentials ix^3, $-x^4$, and $i\kappa/x$, and the lattice and Hulthén potentials in Subsec. II C. In Sec. III we analyze 3+1-dimensional interactions, starting with $\mathcal{L}_{\rm int}=-g\bar{\psi}\gamma^5\psi$ in Subsec. III A and other two-body (four-point) interactions in Subsec. III B. Our conclusions and outlook are presented in Sec. IV.

II. NON-HERMITIAN $\mathcal{PT}$-SYMMETRIC FERMIONS IN 1+1 DIMENSIONS

A. Axial bilinear fermionic interaction

We start with the Lagrangian density for a conventional Hermitian free fermionic field theory,

$$\mathcal{L}_0=\bar{\psi}(i\partial\!\!\!/-m)\psi, \tag{7}$$

where $\bar{\psi}=\psi^\dagger\gamma^0$ and $\psi^\dagger$ is the Hermitian conjugate of ψ. In 1+1 dimensions the gamma matrices are given in (1). We have shown that the free Dirac equation (2) associated with (7) is form invariant under the operation of $\mathcal{P}$ in (3) and of $\mathcal{T}$ in (4). Note that (2) is also form invariant under the combined operations of $\mathcal{P}$ and $\mathcal{T}$ because the functions $\psi(t,x)$ and $\mathcal{PT}\psi(t,x)=\gamma^0\gamma^0\psi^*(-t,-x)=\psi^*(-t,-x)$ both satisfy (2). A plane-wave solution to (2) gives the dispersion relation $E^2=p^2+m^2$. Finally, we read off the *effective* or *quantum-mechanical Hamiltonian* H from the free Dirac equation $i\partial_t\psi=H\psi$ in (2): $H=-i\gamma^0\gamma^1\partial_1+m\gamma^0$. (This form is often written using the definitions $\alpha=\gamma^0\gamma^1$ and $\beta=\gamma^0$ [23]).

We observe that the form invariance of the Dirac equation under $\mathcal{PT}$ is equivalent to the statement that H commutes with $\mathcal{PT}$: $H(\mathcal{PT}\psi)=\mathcal{PT}(H\psi)$. This is so because the left hand side is

$$H(\mathcal{PT}\psi)=H(\gamma^0\gamma^0\psi^*)=H\psi^*,$$

and the right hand side is

$$\mathcal{PT}(H\psi)=\gamma^0\gamma^0\big(-i\gamma^0\gamma^1\partial_1+m\gamma^0\big)\psi^*=H\psi^*.$$

Next, we examine what happens if a pseudoscalar bilinear term in included in the Lagrangian density $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\rm int}$, where $\mathcal{L}_{\rm int}=-g\bar{\psi}\gamma^5\psi$ and g is a real parameter. Now the associated quantum-mechanical Dirac equation is altered to read

$$(i\partial\!\!\!/-m-g\gamma^5)\psi=0. \tag{8}$$

Parity transforms this equation into

$$(i\partial\!\!\!/-m+g\gamma^5)\gamma^0\psi(t,-x)=0, \tag{9}$$

and time reversal has the effect

$$(i\partial\!\!\!/-m+g\gamma^5)\gamma^0\psi^*(-t,x)=0. \tag{10}$$

This Dirac equation is not invariant under $\mathcal{P}$ or $\mathcal{T}$ separately but it is invariant under $\mathcal{PT}$ because the axial interaction term changes sign twice; it is odd under both $\mathcal{P}$ and $\mathcal{T}$. So this axial non-Hermitian term is $\mathcal{PT}$ symmetric.

We can formulate this differently: We identify the effective quantum-mechanical Dirac Hamiltonian associated with the Dirac equation as

$$H=H_0+H_{\rm int}=-i\gamma^0\gamma^1\partial_1+m\gamma^0+g\gamma^0\gamma^5,$$

where $H_0 = -i\gamma^0\gamma^1\partial_1 + m\gamma^0$ and $H_{\rm int} = g\gamma^0\gamma^5$. We have shown that H_0 commutes with $\mathcal{PT}$, and from the effect of $\mathcal{P}$ and $\mathcal{T}$ in 1+1 dimensions and the reality of $H_{\rm int}$, we see that $H_{\rm int}$ also commutes with $\mathcal{PT}$. Thus, the effective Hamiltonian H reflects the symmetry of the Dirac equation.

For this case the dispersion relation is obtained from a plane-wave solution $\psi(t,x)$, and multiplying (8) by $(\not{p} + m + g\gamma^5)$, where $\not{p} = \gamma^0 p_0 + \gamma^1 p_1$, yields the result [16] $p^2 = m^2 - g^2$, which is positive only when $m^2 \geq g^2$. Thus, in the chiral limit $m \to 0$ the spectrum is complex and the $\mathcal{PT}$ symmetry is broken in this limit.

The matrix representation makes this result clearer. Recall that a general two-dimensional $\mathcal{PT}$-symmetric fermionic Hamiltonian, which is selfadjoint with respect to the $\mathcal{PT}$ inner product for fermions and which commutes with $\mathcal{PT}$, can be written as [24]

$$H^{\mathcal{PT}} = \begin{pmatrix} a & b \\ f & a \end{pmatrix}, \tag{11}$$

where a, b, and f are real numbers. The eigenvalues are $E_\pm = a \pm \sqrt{bf}$. Thus, if b and f have the same sign, the spectrum is real and the $\mathcal{PT}$ symmetry is unbroken.

Now, if the interaction Lagrangian density is $-g\bar{\psi}\gamma^5\psi$, the quantum-mechanical interaction Hamiltonian is

$$H_{\rm int} = g\gamma^0\gamma^5 = g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Comparing this with (11), we confirm that $H_{\rm int}$ is $\mathcal{PT}$ symmetric and that this symmetry is always broken. Note that $H_{\rm int}$ is non-Hermitian.

If we add the conventional mass term to the interaction, the effective Hamiltonian in matrix form becomes

$$H = m\gamma^0 + g\gamma^0\gamma^5 = \begin{pmatrix} 0 & m+g \\ m-g & 0 \end{pmatrix}.$$

We see immediately that it is $\mathcal{PT}$ symmetric and that the $\mathcal{PT}$ symmetry is unbroken if $g^2 \leq m^2$.

Observe that the equation of motion resulting from the Dirac equation with an *imaginary* axial term,

$$(i\not{\partial} - m - ig\gamma^5)\psi = 0, \tag{12}$$

gives the dispersion relation $p^2 = m^2 + g^2$. So m is real for all g, including the chiral limit $m \to 0$. However, this axial term is not $\mathcal{PT}$ symmetric. In fact, it is *anti*-$\mathcal{PT}$ symmetric, so that (12) is not form invariant under $\mathcal{PT}$.

B. Approximate solution for higher-power field-theoretic interactions

This section explores the effect that higher-power interaction terms have in $1+1$-dimensional systems. Our starting point is the general Lagrangian density

$$\mathcal{L}(N) = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar{\psi}(i\not{\partial} - m)\psi - g(\bar{\psi}\gamma^5\psi)^N.$$

The Euler-Lagrange equations give the corresponding Dirac or single-particle equation of motion as

$$\left[i\not{\partial} - m - Ng\gamma^5(\bar{\psi}\gamma^5\psi)^{N-1}\right]\psi = 0,$$

which is nonlinear if $N > 1$. The case $N = 1$ reduces to that examined in Sec. II A. In the following we examine the cases $N = 2$ and $N = 3$.

1. $N = 2$

When $N = 2$, $\mathcal{L}(2) = \bar{\psi}(i\not{\partial} - m)\psi - g(\bar{\psi}\gamma^5\psi)^2$, and the interaction term is $\mathcal{PT}$ symmetric. The associated Euler-Lagrange equation is

$$\left[i\not{\partial} - m - 2g\gamma^5(\bar{\psi}\gamma^5\psi)\right]\psi = 0, \tag{13}$$

from which we deduce that the Hamiltonian H satisfying $i\partial_t\psi = H\psi$ is

$$H = -i\gamma^0\gamma^1\partial_1 + m\gamma^0 + 2g\gamma^0\gamma^5(\bar{\psi}\gamma^5\psi).$$

To solve (13) approximately we replace $\bar{\psi}\gamma^5\psi$ by its average value $\langle\bar{\psi}\gamma^5\psi\rangle = \langle\phi\rangle$. Furthermore, by operating on the approximate version of (13) by $(i\not{\partial} - m - 2g\gamma^5\langle\phi\rangle)$ we can solve for the spectrum. In the chiral limit $m \to 0$ this is

$$p^2 = -4g^2\langle\phi\rangle^2. \tag{14}$$

Now, noting that the expectation value of a bosonic pseudoscalar field should be negative imaginary [11],

$$\langle\phi\rangle = -iA,$$

where A is a constant, it follows from (14) that $p^2 = 4g^2A^2$ is real. However, with this choice of $\langle\phi\rangle$, $H_{\rm int} = 2g\gamma^0\gamma^5\langle\phi\rangle$ is anti $\mathcal{PT}$ symmetric, as is the interaction term in (13). Thus, the quantum-mechanical Dirac equation is no longer form invariant under $\mathcal{PT}$; also $\mathcal{PT}$ does not commute with H. Yet we obtain a real spectrum because now $H_{\rm int}$ is Hermitian. The opposite case, namely, when the Dirac equation is $\mathcal{PT}$ symmetric and H commutes with $\mathcal{PT}$, can be simulated by letting $g \to ig$. Then $p^2 < 0$ so, as in Sec. IIA, $\mathcal{PT}$ symmetry is again realized in the broken phase.

2. $N = 3$

When $N = 3$, $\mathcal{L}(3) = \bar{\psi}(i\not{\partial} - m)\psi - g(\bar{\psi}\gamma^5\psi)^3$. This resembles the case for $N = 1$. The Euler-Lagrange equation now reads

$$\left[i\not{\partial} - m - 3g\gamma^5(\bar{\psi}\gamma^5\psi)^2\right]\psi = 0. \tag{15}$$

It follows that the interaction part of the Hamiltonian is

$$H_{\rm int} = 3g\gamma^0\gamma^5(\bar{\psi}\gamma^5\psi)^2. \tag{16}$$

Again, to find an approximate solution we replace $(\bar{\psi}\gamma^5\psi)^2$ by its average value $\langle(\bar{\psi}\gamma^5\psi)^2\rangle$. Solving (15) we get

$$p^2 = -9g^2\langle(\bar{\psi}\gamma^5\psi)^2\rangle^2$$

in the chiral limit. We expect $\langle(\bar{\psi}\gamma^5\psi)^2\rangle$ to be real, so $p^2 < 0$ and the $\mathcal{PT}$ symmetry is always broken. We can confirm this explicitly by noting that (16) is simply proportional to γ^1 and thus only has off-diagonal values of opposite sign, see (1). Comparing this with (11), we note that (16) is manifestly $\mathcal{PT}$ symmetric.

We conclude that (i) If we construct a 1+1-dimensional Lagrangian density containing the axial $\mathcal{PT}$-symmetric interaction $(\bar{\psi}\gamma^5\psi)^N$ (N odd), our approximation scheme shows that we obtain an equation of motion that is form invariant under $\mathcal{PT}$, and correspondingly a $\mathcal{PT}$-symmetric Hamiltonian. The $\mathcal{PT}$ symmetry is broken in the chiral limit. (ii) For even N the equation of motion contains an anti-$\mathcal{PT}$-symmetric term and the associated interaction Hamiltonian is also anti-$\mathcal{PT}$ symmetric but we obtain a dispersion relation that has real masses as a result of Hermiticity. If we modify the interaction by replacing $g \to ig$, we obtain a $\mathcal{PT}$-symmetric system but once again the $\mathcal{PT}$ symmetry is broken.

C. Dirac particle in $\mathcal{PT}$-symmetric potentials

In 1+1 dimensions there are no other γ-matrix-based interactions. However, in addition to these, we can include $\mathcal{PT}$-symmetric potentials having a spatial dependence such as ix^3, $-x^4$, $i\kappa/x$, or even periodic potentials into the relativistic Dirac equation and study the effects of these. Unlike nonrelativistic potentials, which are scalars and can only be included as such in the Schrödinger equation, in the Dirac equation, such potentials can be incorporated either as the nonvanishing scalar part of the 4-vector potential (which we refer to as vector coupling), or as pure scalar interactions, or as combinations thereof. We consider some examples below.

1. *Vector coupling with ix^3*

The 1+1-dimensional Dirac equation that includes the non-Hermitian $\mathcal{PT}$-symmetric vector-coupled potential ix^3 reads

$$\big(i\partial\!\!\!/ - ix^3\gamma^0\big)\psi(t,x) = 0. \tag{17}$$

This is form invariant under $\mathcal{PT}$ and the associated relativistic Hamiltonian

$$H = -i\alpha\partial_x + ix^3 \quad (\alpha \equiv \gamma^0\gamma^1),$$

is also $\mathcal{PT}$ invariant. If we look for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$, we arrive at the corresponding eigenvalue problem

$$H\psi = \big(-i\alpha\partial_x + ix^3\big)\psi = E\psi.$$

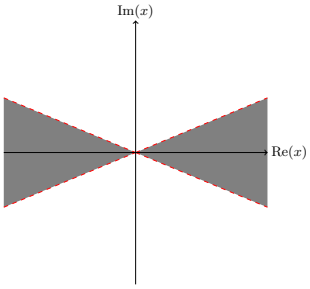

Figure 1: Stokes sectors in the complex-x plane for ψ_1 with an opening angle of $\pi/4$ for the massless Dirac particle in the vector coupled potential ix^3; ψ_1 vanishes exponentially as $|x| \to \infty$ inside these sectors.

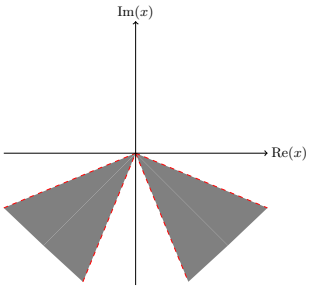

Figure 2: Stokes sectors in the complex-x plane for ψ_2 with an opening angle of $\pi/4$. In this case, the sectors rotate below the real-x axis; ψ_2 vanishes exponentially as $|x| \to \infty$ inside these sectors.

The eigenvectors $\psi_1(x)$ and $\psi_2(x)$ that solve this equation have the asymptotic behavior

$$\psi_1(x) \sim \begin{pmatrix} e^{-x^4/4} \\ 0 \end{pmatrix}, \qquad \psi_2(x) \sim \begin{pmatrix} 0 \\ e^{x^4/4} \end{pmatrix}.$$

The convergence domain for $\psi_1(x)$ and $\psi_2(x)$ in the complex-x plane are the $\mathcal{PT}$-symmetric Stokes sectors shown in Figs. 1 and 2 respectively. In these sectors $\psi_{1,2}(x)$ vanish exponentially as $|x| \to \infty$.

To obtain the selfenergy of the propagating particle we apply $i\partial\!\!\!/$ to (17) and obtain the differential equation

$$\big(E^2 + \partial_x^2\big)\psi = -\big(x^6 + 3x^2\gamma^1\gamma^0 + 2x^3\partial_x\gamma^1\gamma^0\big)\psi. \tag{18}$$

Since the matrix $\gamma^1\gamma^0 = \mathrm{diag}(1,-1)$ is diagonal, the two-component equations in (18) decouple. Although they are not Schrödinger-like, each is individually $\mathcal{PT}$ symmetric. We first examine the classical analog of these equations obtained by replacing $-i\partial_x$ by p,

$$\begin{pmatrix} E^2 - p^2 & 0 \\ 0 & E^2 - p^2 \end{pmatrix} = \begin{pmatrix} -x^6 - 2ipx^3 - 3x^2 & 0 \\ 0 & -x^6 + 2ipx^3 + 3x^2 \end{pmatrix}.$$

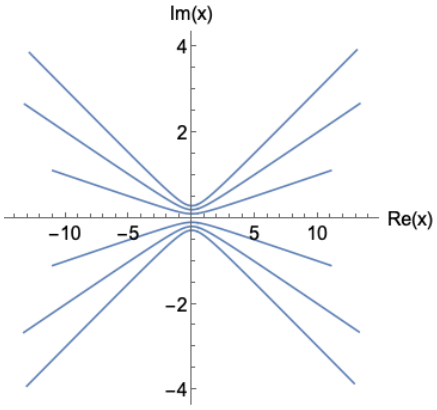

Figure 3: Classical trajectories in the complex-x plane described by $H_1 = \sqrt{p^2 - 2ix^3p - x^6 - 3x^2}$.

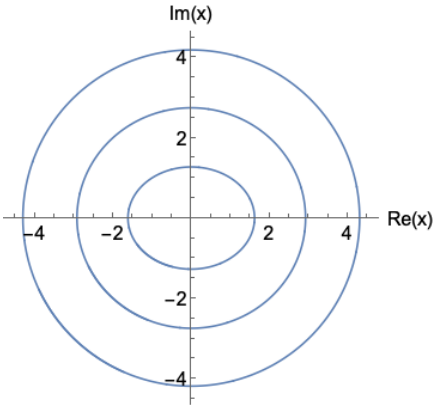

Figure 4: Classical trajectories in the complex-x plane described by $H_2 = \sqrt{p^2 + 2ix^3p - x^6 + 3x^2}$.

The classical Hamiltonian associated with ψ_1 is $H_1 = \sqrt{p^2 - 2ix^3p - x^6 - 3x^2}$. The equation of motion of a classical particle described by H_1 is obtained by combining Hamilton's equations $dx/dt = \partial H_1/\partial p$ and $dp/dt = -\partial H_1/\partial x$: $dx/dt = \pm\sqrt{1 + 3x^2/E^2}$. By rescaling both x and t this equation becomes

$$\frac{dx}{dt} = \pm\sqrt{1 + x^2}.$$

We find that $x(t)$ forms *open* trajectories in the complex-x plane, as shown in Fig. 3.

The open classical trajectories of the particle in the complex-x plane reflects the behavior seen in the quantum case: By setting $p = 0$, we observe that the selfenergy Σ_1 of the particle corresponding to ψ_1 is given by $\Sigma_1^2 = -x^6 - 3x^2$, which implies that Σ_1 cannot be real [28].

On the other hand, the trajectories of the classical particle in the complex-x plane that are associated with the classical Hamiltonian $H_2 = \sqrt{p^2 + 2ix^3p - x^6 + 3x^2}$, are closed, as can be seen in Fig. 4. In the quantum system, the selfenergy corresponding to ψ_2 is $\Sigma_2^2 = -x^6 + 3x^2$. By parametrizing x as $-i(\sqrt{1 + ir} - 1)$ where r is real, ψ_2 vanishes exponentially as $r \to \pm\infty$. We note that the ends of this path lie in the left and right Stokes sectors of Fig. 2 as $|x| \to \infty$. When $-\sqrt[4]{3} < x < \sqrt[4]{3}$, Σ_2^2 is positive. Thus, the selfenergy associated with the particle is real.

2. *Scalar coupling with ix^3 and vector coupling with $i\kappa/x$*

In the previous subsection we treated the $\mathcal{PT}$-symmetric potential ix^3 in a vector-coupling scheme; now we consider it as a *scalar* potential, where, in addition, the Dirac particle is also under the influence of a complex $\mathcal{PT}$-symmetric Coulomb potential. The non-Hermitian $\mathcal{PT}$-symmetric Dirac equation now reads

$$\big(i\partial\!\!\!/ - (i\kappa/x)\gamma^0 - ix^3\big)\psi(t,x) = 0, \tag{19}$$

where κ is a real parameter. The associated relativistic quantum-mechanical Hamiltonian is

$$H = -i\alpha\partial_x + i\kappa/x + \beta ix^3 \quad \big(\beta \equiv \gamma^0\big).$$

Again, looking for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$ leads to an eigenvalue problem

$$H\psi = \big(-i\alpha\partial_x + i\kappa/x + \beta ix^3\big)\psi = E\psi.$$

Writing the eigenfunction $\psi(x)$ in terms of its two spinor components, $\psi(x) = (\phi_1(x), \phi_2(x))$ [33] we find two coupled differential equations for the scalar functions $\phi_{1,2}(x)$,

$$i\phi_1' + i\kappa\phi_1/x + ix^3\phi_2 = E\phi_1, \tag{20}$$
$$-i\phi_2' + i\kappa\phi_2/x + ix^3\phi_1 = E\phi_2. \tag{21}$$

We can eliminate the second component ϕ_2 from (20) by exploiting (21), and after rescaling ϕ_1, and choosing κ to be $-3/2$ for convenience, we obtain the simple form

$$-\phi_1'' - x^6\phi_1 = E^2\phi_1, \tag{22}$$

which is a Schrödinger-like equation with a $-x^6$ potential. On the real-x axis this upside-down potential is unstable, but by imposing appropriate $\mathcal{PT}$-symmetric boundary conditions we can obtain a real spectrum. As in the previous subsection, we find that to have a convergent eigenfunction, we must treat the problem in the complex-x plane.

The WKB approximation for the solutions of (22) to leading order is [29]

$$\phi_{\rm WKB}(x) = C_\pm[Q(x)]^{-1/4}e^{\pm i\int^x ds\sqrt{Q(s)}}, \tag{23}$$

where $Q(x) = E^2 + x^6$. For large $|x|$ the exponential component of this asymptotic behavior is

$$\phi_1 \sim e^{\pm ix^4/4}. \tag{24}$$

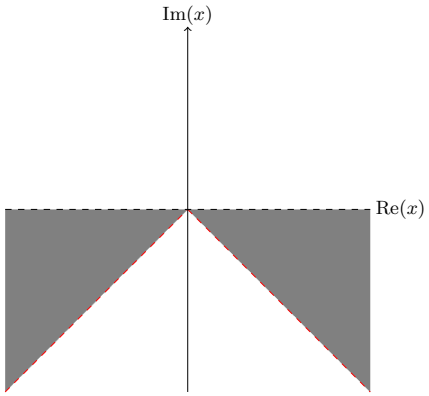

Figure 5: Stokes sectors in the complex-x plane for ϕ_1 in (22). ϕ_1 vanishes exponentially inside these sectors.

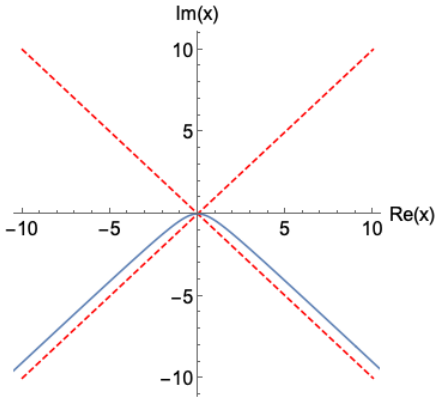

Figure 6: The contour (solid line) on which the eigenvalue problem in (22) is posed (blue online). The dashed lines (red online) denote the edges of the sectors.

There are eight Stokes sectors in the complex-x plane, each with an opening angle of $\pi/4$. To have a $\mathcal{PT}$-symmetric pair of Stokes sectors, we choose the minus sign in (24) for the right Stokes sector, which is located just below the positive-real-x axis. For the left Stokes sector we choose the positive sign in (24), which determines a sector located just below the negative-real-x axis. These two Stokes sectors are depicted in Fig. 5.

We can also approximate the eigenenergies of (22). To do so, we first find the two turning points which are determined by $E = -x^6$ and which lie in the Stokes sectors in Fig. 5. These two points are

$$x_1 = \sqrt[6]{E}e^{-5i\pi/6}, \qquad x_2 = \sqrt[6]{E}e^{-i\pi/6}.$$

The WKB quantization condition is

$$\int_{x_1}^{x_2} ds\sqrt{E_n^2 + s^6} = \left(n + \tfrac{1}{2}\right)\pi \quad (n \to \infty).$$

Thus,

$$E_n = \pm\left[4\sqrt{\pi/3}\Gamma(\tfrac{2}{3})(2n+1)/\Gamma(\tfrac{1}{6})\right]^{3/4} \quad (n \to \infty).$$

For $n = 0$ or 1, we obtain $E_0 = \pm 1.0$ and $E_1 = \pm 2.27$.

An exact calculation of the eigenvalues can be made on parametrizing x as $-i(\sqrt{1+ir}-1)$, where r is a real variable. As depicted in Fig. 6, the ends of this path lie inside the Stokes sectors as $|x| \to \infty$, so we pose the eigenvalue problem for the differential equation in (22) on this contour. We determine the ground-state and first-excited-state energies numerically as

$$E_0 = \pm 1.16, \qquad E_1 = \pm 2.29,$$

which illustrates the accuracy of the WKB approximation. Thus, the energy spectrum of the Dirac particle in the combined non-Hermitian $\mathcal{PT}$-symmetric potentials ix^3 and $i\kappa/x$ is real and discrete.

The trajectories of a classical particle in the complex-x plane described by the classical Hamiltonian $H = \sqrt{p^2 - x^6}$ obtained from (22) are shown in Fig. 7. These trajectories are closed, which reflects the reality and the discreteness of the spectrum at the quantum level.

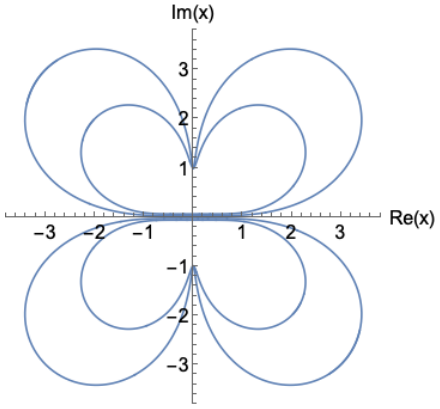

Figure 7: Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 - x^6}$.

3. *Vector coupling with $-x^4$*

Next, we consider a massless Dirac particle under the influence of the upside-down quartic potential $-x^4$. In the vector-coupling scheme, the relativistic Dirac equation is modified to read

$$(i\partial\!\!\!/ + x^4\gamma^0)\psi(t,x) = 0. \tag{25}$$

As in the previous examples, this equation is form invariant under $\mathcal{PT}$ and the associated Hamiltonian

$$H = -i\alpha\partial_x - x^4,$$

commutes with $\mathcal{PT}$. Looking for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$ leads to an eigenvalue equation

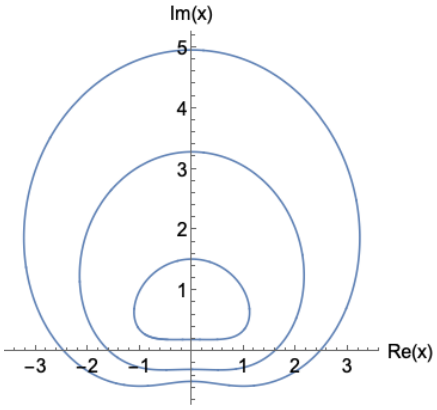

Figure 8: Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + 2x^4p + x^8 - 4ix^3}$.

$H\psi = E\psi$, whose eigenvectors behave asymptotically as

$$\psi_1 \sim \begin{pmatrix} e^{-ix^5/5} \\ 0 \end{pmatrix}, \qquad \psi_2 \sim \begin{pmatrix} 0 \\ e^{ix^5/5} \end{pmatrix}.$$

Note that ψ_1 vanishes exponentially in a Stokes sector with opening angle $\pi/5$. This sector contains the negative-imaginary-x axis, so it vanishes exponentially as $x \to -i\infty$. The function ψ_2 also vanishes exponentially in the same Stokes sector, but one that has rotated upward; that is, $\psi_2 \to 0$ as $x \to i\infty$.

Following the analysis given in Subsec. 1, we iteratively apply $i\partial\!\!\!/$ to the corresponding Dirac equation and find the decoupled system of equations

$$\begin{pmatrix} E^2 - p^2 & 0 \\ 0 & E^2 - p^2 \end{pmatrix} = \begin{pmatrix} x^8 + 2px^4 - 4ix^3 & 0 \\ 0 & x^8 - 2px^4 + 4ix^3 \end{pmatrix}. \quad (26)$$

The selfenergies Σ_1 and Σ_2 of the particle corresponding to ψ_1 and ψ_2 are given by $\Sigma_1^2 = x^8 - 4ix^3$ and $\Sigma_2^2 = x^8 + 4ix^3$. As ψ_1 and ψ_2 converge on $x = -ir$ and $x = ir$, the selfenergies become real.

The trajectories of the classical particle described by both of the classical Hamiltonians obtained from (26) are closed in the complex-x plane. In Fig. 8, this is shown for the classical Hamiltonian $H = \sqrt{p^2 + 2x^4p + x^8 - 4ix^3}$.

4. *Scalar coupling with $-x^4$ and vector coupling with $i\kappa/x$*

We now treat the upside-down potential $-x^4$ as a scalar potential and, in addition, we consider the effect of a complex $\mathcal{PT}$-symmetric Coulomb potential on the Dirac particle in a vector-coupling scheme, satisfying the modified Dirac equation

$$\big(i\partial\!\!\!/ - (i\kappa/x)\gamma^0 + x^4\big)\psi(t,x) = 0,$$

where κ is a real parameter. This equation is form invariant under $\mathcal{PT}$ and the associated Hamiltonian

$$H = -i\alpha\partial_x + i\kappa/x - \beta x^4,$$

commutes with $\mathcal{PT}$. The search for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$ requires solutions of the eigenvalue equation

$$H\psi = \big(-i\alpha\partial_x + i\kappa/x - \beta x^4\big)\psi = E\psi.$$

As in Subsec. 2, it is convenient to write $\psi(x)$ in terms of its (scalar) components, $\psi = (\phi_1, \phi_2)$ and derive the coupled equations that ϕ_1 and ϕ_2 satisfy. Following the procedure outlined in Subsec. 2, we eliminate ϕ_2 and arrive at a Schrödinger-like equation for ϕ_1,

$$-\phi_1'' + x^8\phi_1 = E^2\phi_1, \quad (27)$$

where, for convenience, we have set $\kappa = -2$. We have thus found an octic potential with positive sign. Hence, we pose the eigenvalue problem on the real-x axis. As before we use the WKB approximation to obtain the eigenvalues for large n,

$$E_n = \pm\Big[\sqrt{\pi}\,\Gamma\big(\tfrac{13}{8}\big)\big(n + \tfrac{1}{2}\big)/\Gamma\big(\tfrac{9}{8}\big)\Big]^{4/5} \quad (n \to \infty).$$

From this equation we find that $E_0 = \pm 0.87$ and $E_1 = \pm 2.10$. A direct numerical calculation gives $E_0 = \pm 1.11$ and $E_1 = \pm 2.18$. Thus, once again, we find that the energy spectrum of a Dirac particle in the presence of combined non-Hermitian $\mathcal{PT}$-symmetric vector and scalar potentials $i\kappa/x$ and $-x^4$ is real and discrete.

Here again, we see that the reality and discreteness of the spectrum is evident at the classical level with closed trajectories in the complex-x plane. We recognize the classical Hamiltonian of the system from (27) as being $H = \sqrt{p^2 + x^8}$. Figure 9 shows that the classical trajectories described by this Hamiltonian H are closed.

5. *Complex $\mathcal{PT}$-symmetric lattice potentials*

The methods in the previous subsections are general enough to be applied to a Dirac particle in complex $\mathcal{PT}$-symmetric lattices. The relativistic Dirac equation

$$(i\partial\!\!\!/ - i\kappa\cot(x)\gamma^0 - i\sin(x))\psi(t,x) = 0, \quad (28)$$

with κ real, has non-Hermitian interaction terms, but is form invariant with respect to $\mathcal{PT}$. The associated Hamiltonian,

$$H = -i\alpha\partial_x + i\kappa\cot(x) + i\beta\sin(x),$$

commutes with $\mathcal{PT}$.

As before, we can search for time-independent solutions of (28). Writing $\psi(t,x) = e^{-iEt}\psi(x)$, we obtain coupled equations for the components of the spinor eigenfunction ϕ_1 and ϕ_2, where $\psi = (\phi_1, \phi_2)$. Eliminating ϕ_2,

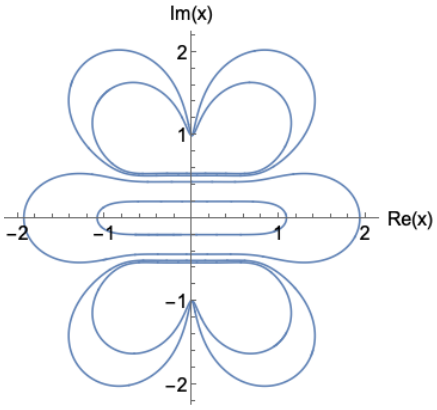

Figure 9: Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + x^8}$.

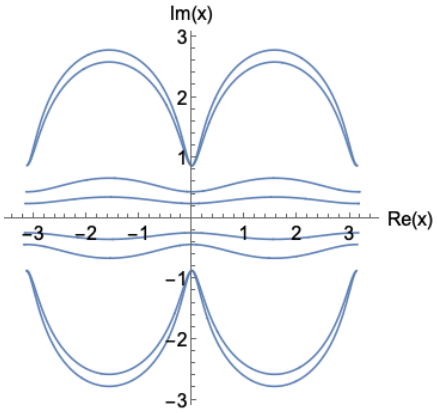

Figure 10: Classical paths for $H = \sqrt{p^2 - \sin^2(x)}$.

we find a Schrödinger-like equation for the ϕ_1, which after suitably rescaling, is

$$-\phi_1'' - \sin^2(x)\phi_1 = E^2\phi_1,$$

where we have set $\kappa = -1/2$.

The spectrum of the operator $-d^2/dx^2 - \sin^2(x)$ is real and consists of spectral bands separated by infinitely many spectral gaps [26]. The absence of discrete energies and the reality of the band-structure manifest itself via periodic, open trajectories of the classical particle described by $H = \sqrt{p^2 - \sin^2(x)}$ in the complex-x plane, as depicted in Fig. 10.

Before closing this subsection, we make a side remark: We note that the (quantum-mechanical, nonrelativistic) Hamiltonian $H = p^2 + i\sin(x)$ describes a particle subject to the periodic potential $V(x) = i\sin(x)$ in a $\mathcal{PT}$-symmetric crystal. As was shown in Ref. [27], by examining a discriminant, one can conclude that this Hamiltonian has real energy bands. However, to verify that the band structure is real, one can alternatively show that the *eigenfunctions* are $\mathcal{PT}$ symmetric; that is, that the $\mathcal{PT}$ symmetry of the Hamiltonian is unbroken. To this end we plot the absolute values of the eigenfunctions of the two states of $H = p^2 + i\sin(x)$ in Fig. 11 and observe that both are in fact symmetric. The energy bands are real, and are shown in Fig. 12. We use this technique in the next subsection.

6. *Scalar coupling with complex $\mathcal{PT}$-symmetric Hulthén potential*

The complex $\mathcal{PT}$-symmetric Hulthén potential is

$$V(x) = \frac{e^{-ix}}{1 - e^{-ix}}.$$

If we regard $V(x)$ as a potential in the *nonrelativistic* time-independent Schrödinger equation, $H\psi = E\psi$, with $H = p^2 + V(x)$, we find that the band structure for the energies is entirely complex, and, as is the case with $\mathcal{PT}$-symmetric potentials in the broken-symmetry phase, the eigenvalues occur in complex-conjugate pairs. We illustrate this by plotting the absolute values of the eigenfunctions of the two states of the Hamiltonian that correspond to the complex-conjugate pairs of the band-edge energies $E = 0.75 \pm 0.59i$, see Fig. 13. Note that the eigenfunctions display no symmetry, which implies the complex nature of the band structure.

We now consider the relativistic Dirac equation that includes the $\mathcal{PT}$-symmetric Hulthén potential in a scalar-coupling scheme, together with an additional $\mathcal{PT}$-symmetric vector potential:

$$\Big(i\partial\!\!\!/ - \kappa\frac{1}{1 - e^{-ix}}\gamma^0 - \frac{e^{-ix}}{1 - e^{-ix}}\Big)\psi(t, x) = 0, \qquad (29)$$

with κ being a real parameter. This equation has been constructed so as to be form invariant with respect to $\mathcal{PT}$ and the associated Hamiltonian

$$H = -i\alpha\partial_x + \kappa\frac{1}{1 - e^{-ix}} + \beta\frac{e^{-ix}}{1 - e^{-ix}},$$

once again commutes with $\mathcal{PT}$. Following the same procedure as in the last subsections, we search for time-independent solutions of the Dirac equation, and find the equations for the components of $\psi = (\phi_1, \phi_2)$. On eliminating ϕ_2, we obtain a Schrödinger-like equation for the first component of the two-component spinor eigenfunction as

$$-\phi_1'' + \frac{1}{(1 - e^{ix})^2}\phi_1 = E^2\phi_1, \qquad (30)$$

where we have set $\kappa = -1/2$ for convenience.

By using spectral methods, we determine numerically that the band structure in (30) is entirely real; that is, the symmetry is unbroken. We have shown the absolute values of the first two eigenfunctions in Fig. 14, which are

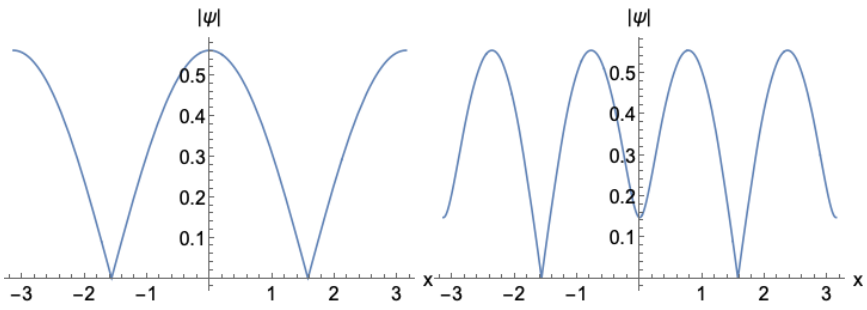

Figure 11: Absolute values of the eigenfunctions corresponding to the band-edge energies of 1.08 (left panel) and 3.97 (right panel) of $H = p^2 + i\sin(x)$.

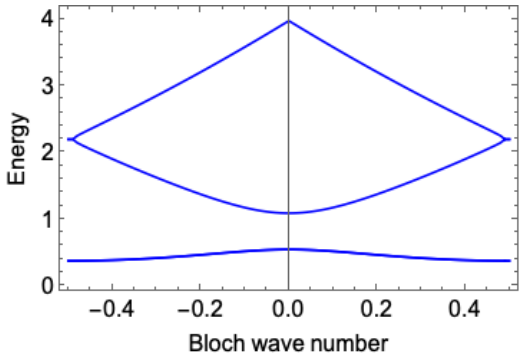

Figure 12: The energy bands associated with the potential $i\sin(x)$ in the first Brillouin zone.

clearly symmetric as expected. The (real) energy bands corresponding to this potential are shown in Fig. 15.

The classical Hamiltonian associated with this system is $H = \sqrt{p^2 + 1/(1 - e^{ix})^2}$. The trajectories of the classical particle, as shown in Fig. 16, are periodic and open. This appears to correspond to the fact that the quantum Hamiltonian has real energy bands but no discrete eigenvalues.

III. NON-HERMITIAN $\mathcal{PT}$-SYMMETRIC FERMIONS IN 3+1 DIMENSIONS

A. Axial bilinear fermionic interaction

In 3+1 dimensions, we again start with the free fermionic Lagrangian $\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ of (7) and the Dirac equation of motion,

$$(i\partial\!\!\!/ - m)\psi(t, \mathbf{x}) = 0, \tag{31}$$

and recall that the actions of $\mathcal{P}$ and $\mathcal{T}$ are given in (6), where the gamma matrices are given in (5). Equation (31) is form invariant under the combined operations $\mathcal{P}$ and $\mathcal{T}$ because the functions $\psi(t, \mathbf{x})$ and $\mathcal{PT}\psi(t, \mathbf{x}) = \gamma^0(i\gamma^1\gamma^3)\psi^*(-t, -\mathbf{x})$ satisfy the same equation. For the free Dirac equation, this is true for $\mathcal{P}\psi = \gamma^0\psi(t, -\mathbf{x})$ and $\mathcal{T}\psi = i\gamma^1\gamma^3\psi^*(-t, \mathbf{x})$ individually. By setting $\mathbf{x} \to -\mathbf{x}$ in (31), it becomes

$$(i\gamma^0\partial_0 - i\gamma^i\partial_i - m)\psi(t, -\mathbf{x}) = 0,$$

where $i = 1, 2, 3$ denote the spatial components. Multiplying this result from the left with γ^0 and using the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ results in

$$(i\partial\!\!\!/ - m)\gamma^0\psi(t, -\mathbf{x}) = 0.$$

On the other hand, taking the complex conjugate of (31) and replacing $t \to -t$ gives

$$\big[-i\big(-\gamma^0\partial_0 + \gamma^1\partial_1 - \gamma^2\partial_2 + \gamma^3\partial_3\big) - m\big]\psi^*(-t, \mathbf{x}) = 0$$

because $\big(\gamma^2\big)^* = -\gamma^2$. Multiplying this equation from the left by $i\gamma^1\gamma^3$ and using the anticommutation relations for the gamma matrices then gives

$$(i\partial\!\!\!/ - m)i\gamma^1\gamma^3\psi^*(-t, \mathbf{x}) = 0.$$

The form invariance of the equation satisfied by $\mathcal{PT}\psi(t, \mathbf{x})$ then follows.

Next we include an axial non-Hermitian bilinear term into the Lagrangian density, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}$, where $\mathcal{L}_{\rm int} = -g\bar{\psi}\gamma^5\psi$. The corresponding Euler-Lagrange equation

$$(i\partial\!\!\!/ - m - g\gamma^5)\psi(t, \mathbf{x}) = 0, \tag{32}$$

superficially resembles the 1+1-dimensional case. However, here, while parity transforms this equation into

$$(i\partial\!\!\!/ - m + g\gamma^5)\gamma^0\psi(t, -\mathbf{x}) = 0, \tag{33}$$

time reversal transforms it into

$$(i\partial\!\!\!/ - m - g\gamma^5)i\gamma^1\gamma^3\psi^*(t, -\mathbf{x}) = 0. \tag{34}$$

Note the minus sign before the last term in (34): While parity flips the sign of the axial term, time reversal in 3+1 dimensions does not. Parity is odd, but time reversal is

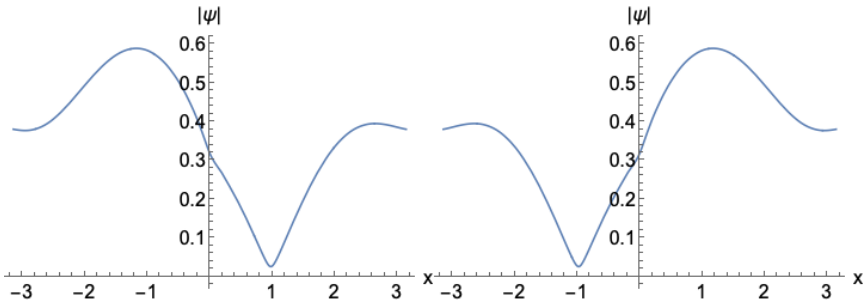

Figure 13: Absolute values of the eigenfunctions corresponding to the band-edge energies $0.75+i0.59$ (left panel) and $0.75-0.59i$ (right panel) for $H = p^2 + e^{-ix}/(1-e^{-ix})$.

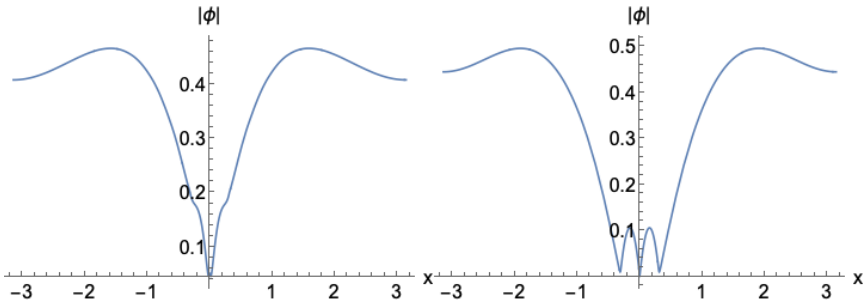

Figure 14: Absolute values of the eigenfunctions corresponding to the band-edge energies of $E = 0.65$ (left panel) and $E = 0.98$ (right panel), obtained from (30). The symmetry of the eigenfunctions implies the reality of the energy band.

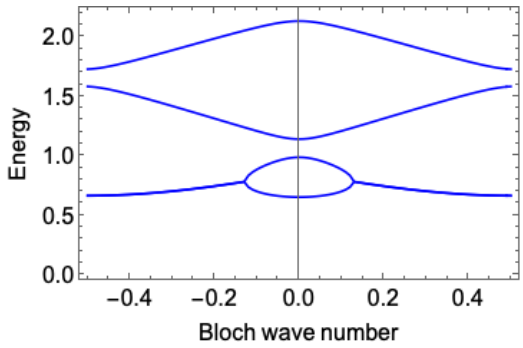

Figure 15: The energy bands for the potential $(1-e^{ix})^{-2}$ in the first Brillouin zone.

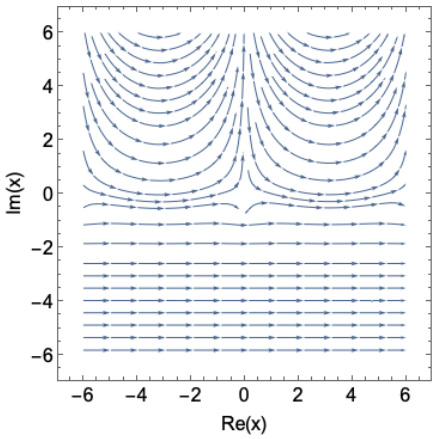

Figure 16: Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + 1/(1-e^{ix})^2}$.

even in 3+1 dimensions. So the combination of $\mathcal{PT}$ does not lead to a form-invariant Dirac equation. The axial term by itself is anti-$\mathcal{PT}$ symmetric. This differs from the 1+1-dimensional case [see (9) and (10)].

The dispersion relation that one obtains from (32) is formally the same as in the 1+1-dimensional case; assuming plane-wave solutions of the form $\psi = e^{-ip^\mu x_\mu}$ and multiplying (32) by $(\not{p} + m + g\gamma^5)$, we arrive at the same spectral relation as in 1+1 dimensions,

$$p^2 = m^2 - g^2,$$

which is positive only when $m^2 \geq g^2$ and is complex in the chiral limit $m \to 0$.

As before, the form invariance of the Dirac equation unter $\mathcal{PT}$ imples that $H(\mathcal{PT}\psi) = \mathcal{PT}(H\psi)$, where H is the Dirac Hamiltonian identified through $i\partial_t\psi = H\psi$. Thus, we can ascertain the properties of various interaction terms by testing them with this commutation rela-

tion. For (32) the associated Hamiltonian is

$$H = \alpha \cdot (-i\nabla) + \beta m + \beta g \gamma^5.$$

Let us check the symmetry of the axial interaction term $H_{\rm int} = g\gamma^0\gamma^5$ under $\mathcal{P}$ and $\mathcal{T}$. Using (6), we evaluate $\mathcal{PT}\psi(t,\mathbf{x}) = \gamma^0 i\gamma^1\gamma^3\psi^*(-t,-\mathbf{x})$ and apply $H_{\rm int}$:

$$\begin{aligned} H_{\rm int}\big(\mathcal{PT}\psi\big) &= g\gamma^0\gamma^5\gamma^0 i\gamma^1\gamma^3\psi^* = -\gamma^0 i\gamma^1\gamma^3 g\gamma^0\gamma^5\psi^* \\ &= -P\mathcal{T}H_{\rm int}\psi^* = -\mathcal{PT}H^*_{\rm int}\psi \\ &= -\mathcal{PT}\big(H_{\rm int}\psi\big). \end{aligned} \tag{35}$$

$H_{\rm int}$ *anti*-commutes with $\mathcal{PT}$, confirming that this term is not $\mathcal{PT}$ symmetric. It thus explains the complex nature of the dispersion relation in the chiral limit. By contrast, if $H_{\rm int}$ is imaginary, that is $H_{\rm int} = ig\gamma^0\gamma^5$, we have a $\mathcal{PT}$-symmetric Hamiltonian, which is also Hermitian, and does have a real spectrum for all g, $p^2 = g^2$ in the chiral limit.

Once again, to clarify this point, we turn to an explicit matrix representation. Then $H_{\rm int}$ becomes

$$H_{\rm int} = g\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{36}$$

which is not Hermitian. By comparison, a general four-dimensional $\mathcal{PT}$-symmetric fermionic Hamiltonian that is invariant under $\mathcal{PT}$ and also selfadjoint under the $\mathcal{PT}$ inner product has a matrix form [13, 14, 30],

$$H = \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}, \tag{37}$$

where $B_\pm = b_1 \pm ib_2$ and $C_\pm = b_3 \pm ib_0$. The parameters a_0, b_0, b_1, b_2, and b_3 are real. This matrix has twofold degenerate real eigenvalues

$$E_\pm = \pm\sqrt{a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2}, \tag{38}$$

for $a_0^2 \geq \sum_{i=0}^3 b_i^2$ [32]. Equation (36) is not a special case of (37), so it is does not represent a $\mathcal{PT}$-symmetric fermionic Hamiltonian.

Evidently, the symmetry properties of the axial term $-g\gamma^5\psi$ in the Dirac equation in 1+1 dimensions differ from those in 3+1 dimensions. The Dirac equation is form invariant in 1+1 dimensions under $\mathcal{PT}$, but not in 3+1 dimensions. This corresponds to a relativistic $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian in 1+1 dimensions, but not in in 3+1 dimensions. This difference is caused by the different effect of time reversal in 1+1 and 3+1 dimensions. The spectrum obtained in both cases is formally the same, so we conclude that the $\mathcal{PT}$ symmetry is always broken in 1+1 dimensions when $m \to 0$. However, in 3+1 dimensions the Hamiltonian is anti-$\mathcal{PT}$ symmetric in the chiral limit, which explains the complex nature of the spectrum when $m \to 0$.

Interestingly, if we include the conventional mass term $m\gamma^0$, (36) becomes

$$H_{\rm int} = \begin{pmatrix} m & 0 & g & 0 \\ 0 & m & 0 & g \\ -g & 0 & -m & 0 \\ 0 & -g & 0 & -m \end{pmatrix}, \tag{39}$$

which is neither Hermitian nor $\mathcal{PT}$-symmetric. However, $H_{\rm int}$ is *pseudo*-Hermitian in the sense of [31] because $H^\dagger_{\rm int} = \gamma^0 H_{\rm int}(\gamma^0)^{-1}$. Hence, this Hamiltonian can be used to describe pseudo-Hermitian fermions.

We can construct fermionic creation and annihilation operators which are quadratically nilpotent, and investigate their anticommutation relations. First, we note that the eigenvalues of (39) are

$$E_\pm = \pm\omega = \pm\sqrt{m^2 - g^2},$$

with corresponding eigenvectors

$$\big|E^{(1)}_-\big\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} 0 \\ -\sqrt{m+w}(m-w)/g \\ 0 \\ \sqrt{m+w} \end{pmatrix} \qquad \big|E^{(2)}_-\big\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} -\sqrt{m+w}(m-w)/g \\ 0 \\ \sqrt{m+w} \\ 0 \end{pmatrix},$$

$$\big|E^{(1)}_+\big\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} 0 \\ -\sqrt{m-w}(m+w)/g \\ 0 \\ \sqrt{m-w} \end{pmatrix} \qquad \big|E^{(2)}_+\big\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} -\sqrt{m-w}(m+w)/g \\ 0 \\ \sqrt{m-w} \\ 0 \end{pmatrix}.$$

The spectrum is twofold degenerate and is real if $g^2 \leq m^2$. This degeneracy is the analog of the phenomenon

of Kramer's theorem in conventional Hermitian quantum mechanics, where the Hamiltonian is invariant under odd time reversal, as is the case with (39).

We introduce the annihilation operator for the Hamiltonian (39) as

$$\eta = \frac{1}{2w}\begin{pmatrix} g & 0 & m-w & 0 \\ 0 & g & 0 & m-w \\ -m-w & 0 & -g & 0 \\ 0 & -m-w & 0 & -g \end{pmatrix},$$

which is nilpotent ($\eta^2 = 0$) as required. We verify that

$$\eta\big|E_-^{(1)}\big\rangle = \eta\big|E_-^{(2)}\big\rangle = 0,$$

$$\eta\big|E_+^{(1)}\big\rangle = \big|E_-^{(1)}\big\rangle, \qquad \eta\big|E_+^{(2)}\big\rangle = \big|E_-^{(2)}\big\rangle.$$

The creation operator reads

$$\eta' = \frac{1}{2w}\begin{pmatrix} g & 0 & m+w & 0 \\ 0 & g & 0 & m+w \\ -m+w & 0 & -g & 0 \\ 0 & -m+w & 0 & -g \end{pmatrix}.$$

One can now establish the anticommutation relations

$$\{N, \eta\} = -\eta \qquad \{N, \eta'\} = -\eta',$$

where N is the number operator, $N = \eta'\eta$, as well as the peculiar anticommutation relation $\eta\eta' + \eta'\eta = -\mathbb{1}$. The minus sign indicates that the number operator gives the negative of the state occupation number. For further illustrations of this in the context of $\mathcal{PT}$ symmetry see Refs. [15, 24].

Finally, we comment that in terms of the number operator N, we can write the four-dimensional pseudo-Hermitian fermionic Hamiltonian in (39) in the form of a free (bosonic) harmonic oscillator as

$$H = \Delta\omega(-N) + \omega_-\mathbb{1},$$

where $\Delta\omega = \omega_+ - \omega_-$ and $\mathbb{1}$ is the four-dimensional identity matrix.

B. Other matrix-type two-body (four-point) $\mathcal{PT}$- and anti-$\mathcal{PT}$-symmetric interactions and the resulting $\mathcal{PT}$-symmetric Hamiltonians

Having determined that an axial non-Hermitian interaction Lagrangian density of the form $-g\bar{\psi}\gamma^5\psi$ in 3+1 dimensions does not give rise to a Dirac equation that is form invariant with respect to $\mathcal{PT}$, we seek other types of interactions that are $\mathcal{PT}$ symmetric but non-Hermitian. Usually, the standard method of analyzing two-body (four-point) interactions involves constructing the 16 independent bilinears from the 16 4×4 independent matrices and considering the Lagrangian density associated with each of these. The standard Hermitian combinations are (1) $\bar{\psi}\psi$, (2) $\bar{\psi}\gamma^\mu\psi$, (3) $\bar{\psi}\sigma^{\mu\nu}\psi$, (4) $\bar{\psi}\gamma^5\gamma^\mu\psi$, and (5) $i\bar{\psi}\gamma^5\psi$. This Lagrangian-density approach is suitable for a discussion of symmetries that lead to conserved currents through Noether's theorem, but the analysis of $\mathcal{PT}$ symmetry is most simply done by examining the form-invariance of the appropriate Dirac-like equation that can be derived using the Euler-Lagrange equations. Since this in turn tranlates into a commutation relation of the Hamiltonian with $\mathcal{PT}$, in a form of *reverse engineering*, we only need to identify possible $\mathcal{PT}$-symmetric forms of the interaction Hamiltonians. Thus, we consider the five interaction Hamiltonians below and show that these combinations are all $\mathcal{PT}$ symmetric:

$$\begin{aligned} H_{\rm int,1} &= g\gamma^0, \\ H_{\rm int,2} &= B_\mu\gamma^0\gamma^\mu, \\ H_{\rm int,3} &= iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}, \\ H_{\rm int,4} &= i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu, \\ H_{\rm int,5} &= ig_A\gamma^0\gamma^5, \end{aligned}$$

where g, B_μ, $T_{\mu\nu}$, $\tilde{B}_\mu$, and g_A are taken to be real.

Using the procedure in (35) in which $H_{{\rm int},i}$ is applied to $\mathcal{PT}\psi$, we evaluate the commutator of $H_{{\rm int},i}$ and $\mathcal{PT}$ using (6), and where necessary, make use of the relation $\gamma^\mu i\gamma^1\gamma^3 = i\gamma^1\gamma^3\gamma^*_\mu$. Then

$$H_{\rm int,1}(\mathcal{PT}\psi) = g\gamma^0\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3 g\gamma^0\psi^* = PTH_{\rm int,1}\psi^* = \mathcal{PT}(H_{\rm int,1}\psi), \tag{40}$$

$$H_{\rm int,2}(\mathcal{PT}\psi) = B_\mu\gamma^0\gamma^\mu\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3 B_\mu\gamma^0\gamma^{\mu*}\psi^* = PTH^*_{\rm int,2}\psi^* = \mathcal{PT}(H_{\rm int,2}\psi), \tag{41}$$

$$H_{\rm int,3}(\mathcal{PT}\psi) = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}\gamma^0 i\gamma^1\gamma^3\psi^* = -\gamma^0 i\gamma^1\gamma^3 iT_{\mu\nu}\gamma^0\sigma^{\mu\nu*}\psi^* = -PTi\gamma^0\sigma^{\mu\nu*}T_{\mu\nu}\psi^* = \mathcal{PT}(H_{\rm int,3}\psi), \tag{42}$$

$$H_{\rm int,4}(\mathcal{PT}\psi) = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3(-i)\tilde{B}_\mu\gamma^0\gamma^5\gamma^{\mu*}\psi^* = PTH^*_{\rm int,4}\psi^* = \mathcal{PT}(H_{\rm int,4}\psi), \tag{43}$$

$$H_{\rm int,5}(\mathcal{PT}\psi) = ig_A\gamma^0\gamma^5\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3(-i)g_A\gamma^0\gamma^5\psi^* = PTH^*_{\rm int,5}\psi^* = \mathcal{PT}(H_{\rm int,5}\psi). \tag{44}$$

We conclude that

$$[\mathcal{PT}, H_{{\rm int},i}] = 0 \quad (i = 1, \cdots, 5).$$

Thus, the general form of a relativistic quantum-mechanical Dirac equation, which is form invariant under

$\mathcal{PT}$ transformations, reads

$$(i\partial\!\!\!/ - g - B_\mu\gamma^\mu - iT_{\mu\nu}\sigma^{\mu\nu} - i\tilde{B}_\mu\gamma^5\gamma^\mu - ig_A\gamma^5)\psi(t,\mathbf{x}) = 0.$$

A brief analysis shows that $H_{\rm int,3}$ and $H_{\rm int,4}$ are anti-Hermitian, while $H_{\rm int,1}$, $H_{\rm int,2}$, and $H_{\rm int,5}$ are Hermitian. So we have identified two types of terms that give rise to non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonians. We consider each of these in turn.

1. $H_{\rm int,3} = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}$

To understand the structure of $H_{\rm int,3}$ we write it in matrix form:

$$H_{\rm int,3} = \begin{pmatrix} iq_4 & -q_5+iq_6 & -q_3 & -q_1+iq_2 \\ q_5+iq_6 & -iq_4 & -q_1-iq_2 & q_3 \\ q_3 & q_1-iq_2 & -iq_4 & q_5-iq_6 \\ q_1+iq_2 & -q_3 & -q_5-iq_6 & iq_4 \end{pmatrix}, \tag{45}$$

where the coefficients q_i, $i = 1,\ldots,6$, are abbreviations for combinations of the $T_{\mu\nu}$,

$$q_1 = T_{01}-T_{10}, \quad q_2 = T_{02}-T_{20}, \quad q_3 = T_{03}-T_{30},$$
$$q_4 = T_{12}-T_{21}, \quad q_5 = T_{13}-T_{31}, \quad q_6 = T_{23}-T_{32}.$$

The eigenvalues of (45) are

$$\pm\big\{-Q^2 \pm 2\big[(q_1^2+q_2^2)q_4^2 + (q_1^2+q_3^2)q_5^2 + (q_2^2+q_3^2)q_6^2 + 2q_2q_3q_4q_5 + 2q_1q_2q_5q_6 - 2q_1q_3q_4q_6\big]^{1/2}\big\}^{1/2},$$

where $Q^2 = \sum_{i=1}^6 q_i^2$. Thus, the eigenvalues are complex and the $\mathcal{PT}$ symmetry is broken. Including a finite mass term $m\gamma^0$ in general does not change this result. The eigenvalues of $H_{\rm int,3} + m\gamma^0$ are modified to read

$$\pm\big\{m^2 - Q^2 \pm 2\big[(q_1^2+q_2^2-m^2)q_4^2 + (q_1^2+q_3^2-m^2)q_5^2 + (q_2^2+q_3^2-m^2)q_6^2 + 2q_2q_3q_4q_5 + 2q_1q_2q_5q_6 - 2q_1q_3q_4q_6\big]^{1/2}\big\}^{1/2}.$$

As we have already argued, only if the spectrum is twofold degenerate, can the eigenvalues be real [32].

If we compare (45) with (37), we see that both have a quaternionic structure. However, in addition to being $\mathcal{PT}$ symmetric, (37) fulfills the additional condition that this Hamiltonian is selfadjoint with regard to the $\mathcal{PT}$ inner product according to [13]. This means that, in addition, $H_{\rm int,3}$ should fulfill the condition $H_{\rm int,3}^{\mathcal{PT}} = PH_{\rm int,3}^\dagger P = H_{\rm int,3}$. If we construct $H_{\rm int,3}^{\mathcal{PT}}$, we find that

$$q_4 = q_5 = q_6 = 0,$$

for this condition to hold. The eigenvalues are twofold degenerate and if a mass term is included, they are

$$E_\pm = \pm\sqrt{m^2 - q_1^2 - q_2^2 - q_3^2},$$

which is real provided that $m^2 \geq q_1^2+q_2^2+q_3^2$. Thus, $\mathcal{PT}$ symmetry is broken in the chiral limit. The regions of unbroken $\mathcal{PT}$ symmetry for the Hamiltonian $H_{\rm int,3}+m\gamma^0$ for some specific parameters are shown in Fig. 17.

2. $H_{\rm int,4} = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu$

We now consider the equation of motion resulting from the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian $H_{\rm int,4}$ (as well as its corresponding Lagrangian $\mathcal{L}_{\rm int,4}$),

$$(i\partial\!\!\!/ - i\gamma^5\tilde{B}\!\!\!/)\psi = 0.$$

The spectrum associated with this equation can be obtained by calculating the poles of the associated Green function in momentum space, which satisfies

$$(p\!\!\!/ - i\gamma^5\tilde{B}\!\!\!/)S(p) = 1.$$

Rationalizing this expression for $S(p)$, we identify the dispersion relation as

$$(p^2 - \tilde{B}^2)^2 + 4(p\cdot\tilde{B})^2 = 0.$$

This has no real solutions for all p_0. Thus, again we find that the $\mathcal{PT}$ symmetry of the Hamiltonian is broken. We also notice that an anti-$\mathcal{PT}$-symmetric but Hermitian Hamiltonian would give a real spectrum with dispersion relation $(p^2-\tilde{B}^2)^2 - 4(p\cdot\tilde{B})^2 = 0$.

Note that the matrix form of the Hamiltonian $H_{\rm int,4}$, with components $\tilde{B}_\mu = (\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ is

$$H_{\rm int,4} = \begin{pmatrix} -i\tilde{B}_3 & -i\tilde{B}_1 - \tilde{B}_2 & -i\tilde{B}_0 & 0 \\ \tilde{B}_2 - i\tilde{B}_1 & i\tilde{B}_3 & 0 & -i\tilde{B}_0 \\ -i\tilde{B}_0 & 0 & -i\tilde{B}_3 & -i\tilde{B}_1 - \tilde{B}_2 \\ 0 & -i\tilde{B}_0 & \tilde{B}_2 - i\tilde{B}_1 & i\tilde{B}_3 \end{pmatrix},$$

which has complex eigenvalues for all $\tilde{B}_\mu$ real,

$$E_{1,2} = i\tilde{B}_0 \pm i\sqrt{\tilde{B}_1^2+\tilde{B}_2^2+\tilde{B}_3^2},$$
$$E_{3,4} = -i\tilde{B}_0 \pm i\sqrt{\tilde{B}_1^2+\tilde{B}_2^2+\tilde{B}_3^2}.$$

If, as in Subsec. IIIB1, we demand that the Hamiltonian $H_{\rm int,4}$ satisfies the selfadjointness condition according to [13, 14, 24], that is, $H_{\rm int,4}^{\mathcal{PT}} = PH_{\rm int,4}^\dagger P = H_{\rm int,4}$, we calculate that $\tilde{B}_0 \neq 0$ and $\tilde{B}_1 = \tilde{B}_2 = \tilde{B}_3 = 0$. The resulting twofold degenerate energies are

$$E_\pm = \pm\sqrt{m^2 - \tilde{B}_0^2}, \tag{46}$$

where we have included a mass term. This implies a real spectrum for $m^2 \geq \tilde{B}_0^2$. Once again, in the chiral limit the $\mathcal{PT}$ symmetry is broken.

IV. MAIN CONCLUSIONS AND OUTLOOK

Our focus in this paper has been on investigating non-Hermitian $\mathcal{PT}$-symmetric extensions to fermionic systems in 1+1 and 3+1 dimensions. The main findings are the following:

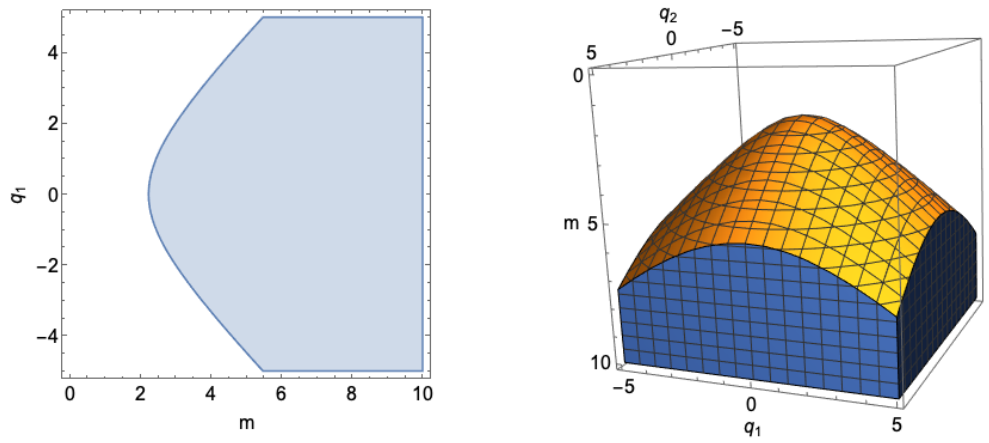

Figure 17: Parametric regions of unbroken $\mathcal{PT}$ symmetry (shaded regions) for the Hamiltonian $H_{\text{int},3} + m\gamma^0$, where $q_4 = q_5 = q_6 = 0$. Left panel: in the (m, q_1) plane, $q_2 = 1$ and $q_3 = 2$. Right panel: $q_3 = 2$.

a) Usually, we explore the symmetries of a field theory by examining the Lagrangian density. However, the properties associated with $\mathcal{PT}$ symmetry are more easily found by forming the Euler-Lagrange equations and demanding form-invariance of the relativistic equation of motion with respect to $\mathcal{PT}$. This is equivalent to constructing the quantum-mechanical relativistic Hamiltonian and investigating its commutation relation with $\mathcal{PT}$.

b) For a pure axial interaction the symmetry properties in 1+1 dimensions differ from those in 3+1 dimensions even though the formal structure of the energy relation is unchanged. This can be traced back to the different transformation properties of time reversal in 1+1 and 3+1 dimensions and is ultimately due to the fact that $\mathcal{T}^2 = -\mathbb{1}$ in 3+1 dimensions.

c) In 1+1 dimensions including a complex $\mathcal{PT}$-symmetric position-dependent potential in both scalar- and vector-coupling schemes and combinations thereof can result in real and discrete eigenvalues, when searching for plane wave solutions. For appropriately chosen combinations of scalar and vector couplings, a Schrödinger-like equation can be found and the spectrum can be determined numerically. The analogous classical systems give information about the nature of the spectrum. They display closed contours when the eigenvalues are real and discrete and they are periodic and open if there is a real band structure. If the eigenvalues are complex, the paths are open and nonperiodic.

d) In 3+1 dimensions only two possible Lorentz-invariant two-body combinations are $\mathcal{PT}$ symmetric and not Hermitian. These, however, give rise to a complex spectrum in the chiral limit. Including a mass term can result in a real spectrum. In addition, further constraints are placed on the parameters if the condition of selfadjointness with respect to the $\mathcal{PT}$ inner product is placed on the Hamiltonian. This does not change the conclusion.

It remains an open question as to whether including non-Hermitian $\mathcal{PT}$-symmetric terms can play a role in physical fermionic systems, for example, affecting chiral symmetry restoration within the Nambu-Jona-Lasinio or Thirring models, or in weak interactions.

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[25] In this paper most of the Hamiltonians that are considered are bilinear in the fermionic field. One result of this paper is that it is not possible to construct a $\mathcal{PT}$-symmetric bilinear fermionic Hamiltonian that has a real spectrum in the chiral limit. However, we firmly believe that a Hamiltonian having higher-power interactions will achieve this goal. A simple bosonic model that illustrates this point is the Hamiltonian $H = p^2 + \left(m^2 - \mu^2\right)x^2 - gx^4$. In the quadratic case $g = 0$ we see that the spectrum becomes complex in the $m \to 0$ limit. However, if $g \neq 0$ it is shown in C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie, Phys. Rev. D **74**, 025016 (2006) that H is equivalent to the Hermitian Hamiltonian $\tilde{H} = p^2 - 2\hbar x\sqrt{g} + 4g\left[x^2 - \left(m^2 - \mu^2\right)/(4g)\right]^2$, which has a spectrum that is real and bounded below for *all* values of m and μ.
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[31] G. W. Mackey, *Commutative Banach Algebras*, Instituto de Matematica pura e Aplicada do Conselho Nacional de Pesquisa, Rio De Janeiro (1959).
[32] When H and $\mathcal{PT}$ commute, in general, in order to have a real spectrum, we have two possibilities. One option is ψ to be $\mathcal{PT}$ symmetric, i.e., it has the symmetry of the Hamiltonian. The other option is that ψ and $\mathcal{PT}\psi$ be degenerate eigenvectors of H. For $\mathcal{T}^2 = -\mathbb{1}$ the first option is ruled out since, if we require $\mathcal{PT}\psi = \psi$, then ψ must vanish; So the degeneracy of the Hamiltonian is a necessary condition for the spectrum to be real.
[33] We reserve the notation ψ_1, ψ_2 for two different spinor solutions of the same equation.

Chapter 5

Coupled oscillator systems having partial $\mathcal{PT}$ symmetry

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Motivation: Here, we propose the notion of *partial* $\mathcal{PT}$ symmetry and examine that in exactly solvable coupled oscillator systems both in the quantum-mechanical and classical sense.

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Coupled oscillator systems having partial $\mathcal{PT}$ symmetry

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This paper examines chains of N coupled harmonic oscillators. In isolation, the jth oscillator ($1 \leqslant j \leqslant N$) has the natural frequency ω_j and is described by the Hamiltonian $\frac{1}{2}p_j^2 + \frac{1}{2}\omega_j^2 x_j^2$. The oscillators are coupled adjacently with coupling constants that are purely imaginary; the coupling of the jth oscillator to the $(j+1)$th oscillator has the bilinear form $i\gamma x_j x_{j+1}$ (γ real). The complex Hamiltonians for these systems exhibit *partial* $\mathcal{PT}$ symmetry; that is, they are invariant under $i \to -i$ (time reversal), $x_j \to -x_j$ (j odd), and $x_j \to x_j$ (j even). [They are also invariant under $i \to -i$, $x_j \to x_j$ (j odd), and $x_j \to -x_j$ (j even).] For all N the quantum energy levels of these systems are calculated exactly and it is shown that the ground-state energy is real. When $\omega_j = 1$ for all j, the full spectrum consists of a real energy spectrum embedded in a complex one; the eigenfunctions corresponding to real energy levels exhibit partial $\mathcal{PT}$ symmetry. However, if the ω_j are allowed to vary away from unity, one can induce a phase transition at which *all* energies become real. For the special case $N = 2$, when the spectrum is real, the associated classical system has localized, almost-periodic orbits in phase space and the classical particle is confined in the complex-coordinate plane. However, when the spectrum of the quantum system is partially real, the corresponding classical system displays only open trajectories for which the classical particle spirals off to infinity. Similar behavior is observed when $N > 2$.

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I. INTRODUCTION

There are many experimental and theoretical studies of $\mathcal{PT}$-symmetric coupled-oscillator Hamiltonians [1–6]. In most cases the starting point is either a coupled set of $\mathcal{PT}$-symmetric equations of motion, or a $\mathcal{PT}$-symmetric Hamiltonian that governs such a system. It has been established that such $\mathcal{PT}$-symmetric systems exhibit a rich phase structure with phase boundaries depending on the number of oscillators, how they are coupled, and the values of the coupling parameters [5].

In a recent paper on radiative coupling and weak lasing of exciton-polariton condensates, Aleiner *et al.* [7] considered a Hamiltonian function that governs condensation centers that are bilinearly coupled by a term of the form $igzz^*$, where each center is described by the complex coordinate z and g is a coupling strength. They investigated the classical dynamics of the system. While there is no obvious underlying symmetry, the authors found closed paths in their spin trajectories. This intriguing result motivates the current study of an unusual type of oscillator system, namely, a chain of N harmonic oscillators with *pure imaginary coupling*. The Hamiltonian for the jth oscillator ($1 \leqslant j \leqslant N$) has the form $\frac{1}{2}p_j^2 + \frac{1}{2}\omega_j^2 x_j^2$, where the natural frequency ω_j is real and positive. The jth oscillator is coupled to the $(j+1)$th oscillator by an imaginary coupling constant $i\gamma$, where γ is real and independent of j. The coupling term is *bilinear*; that is, it has the form $i\gamma x_j x_{j+1}$. The Hamiltonian that governs this system of N adjacently coupled oscillators has the form

$$H_N = \frac{1}{2}\sum_{j=1}^{N}\left(p_j^2 + \omega_j^2 x_j^2\right) + i\gamma\sum_{j=1}^{N-1} x_j x_{j+1} \quad (N \geqslant 2). \quad (1)$$

This complex Hamiltonian is not $\mathcal{PT}$ symmetric because i changes sign under time reversal $\mathcal{T}$ and it is assumed that every coordinate x_j changes sign under parity $\mathcal{P}$. However, H_N is *partially* $\mathcal{PT}$ symmetric; that is, it remains invariant if we change the sign of i and simultaneously reverse the sign of only the odd-numbered or only the even-numbered coordinates. To illustrate, we define $\mathcal{P}_j$ as the operator that reverses the sign of x_j but does not affect any other coordinate. Then, H_2 is partially $\mathcal{PT}$ symmetric with respect to $\mathcal{P}_1\mathcal{T}$ and also with respect to $\mathcal{P}_2\mathcal{T}$. Similarly, H_3 is partially $\mathcal{PT}$ symmetric with respect to $\mathcal{P}_1\mathcal{P}_3\mathcal{T}$ and also with respect to $\mathcal{P}_2\mathcal{T}$. Note that reversing the signs of an even number of coordinates is achievable by a rotation but reversing the signs of an odd number of coordinates is not achievable by a rotation. For example, for $N = 2$, $x_1 \to -x_1$, $x_2 \to -x_2$ is merely a rotation by an angle of π in the x_1,x_2 plane, but $x_1 \to -x_1$, $x_2 \to x_2$ cannot be achieved by a rotation. For $N = 3$, $\mathcal{P}_1\mathcal{P}_3$ is a rotation but $\mathcal{P}_2$ and also $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ are not.

Systems having partial $\mathcal{PT}$ symmetry have remarkable properties. In Sec. II we set $\omega_j = 1$ for all j and show that for small N and for all values of the coupling parameter γ the ground-state energy of the quantum system is real and positive. Then, in Sec. III we present the exact solution for the complete quantum spectrum for all N. We find that the ground-state energy is always real, but that the full spectrum is partly real and partly complex. For each energy, we calculate the corresponding eigenfunction and demonstrate that simultaneous eigenfunctions of the Hamiltonian and the partial $\mathcal{PT}$ operator have real energies, while those that are not partially $\mathcal{PT}$ symmetric are associated with complex energies.

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Thus, partial $\mathcal{PT}$ symmetry is associated with a partially real energy spectrum. In Sec. IV we relax the constraint that $\omega_j = 1$. We show that for $N = 2$ it is possible to choose the natural oscillator frequencies to make the energy spectrum completely real. Thus, there is a phase transition from a partially real to a completely real spectrum. This result is shown to hold in a modified form for $N = 3$ and $N = 4$. Next, in Sec. V, we investigate the classical solutions for the $N = 2$ and $N = 3$ systems and find no remnant of the partially $\mathcal{PT}$-symmetric phase; that is, all classical orbits are open unless the quantum spectrum is entirely real, in which case the orbits are all closed and periodic. Brief concluding remarks are given in Sec. VI.

II. GROUND-STATE ENERGIES OF N COUPLED OSCILLATORS WITH $\omega_j = 1$

In this section we show that the ground-state energy of a quantum system of N coupled oscillators with natural frequency $\omega_j = 1$ is real and positive.

A. Two coupled oscillators

Let us consider the quantum-mechanical Hamiltonian of two coupled oscillators [8,9] $H_2 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + i\gamma xy$, where x and y are the coordinates, p and q are the conjugate momenta, γ is a coupling strength, and $\omega_1 = \omega_2 = 1$. This Hamiltonian is partially $\mathcal{PT}$ symmetric because it is invariant under the transformations $\mathcal{P}_x\mathcal{T}$ or $\mathcal{P}_y\mathcal{T}$, where $\mathcal{P}_x : (x,y) \to (-x,y)$, $\mathcal{P}_y : (x,y) \to (x,-y)$, and $\mathcal{T} : i \to -i$. The Schrödinger equation associated with H_2 is

$$\left(-\tfrac{1}{2}\partial_x^2 - \tfrac{1}{2}\partial_y^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + i\gamma xy\right)\psi(x,y) = E\psi(x,y). \tag{2}$$

The ground-state eigenfunction has the (non-nodal) Gaussian form

$$\psi_0(x,y) = \exp\left(-\tfrac{1}{2}ax^2 - \tfrac{1}{2}ay^2 + bxy\right), \tag{3}$$

where a and b are constants. Note that $\psi_0(x,y)$ is $\mathcal{PT}$ symmetric in either x or y. Inserting (3) into (2) and matching powers of x and y gives the three equations $E_0 = a, a^2 + b^2 = 1$, and $2ab = -i\gamma$.

The physically acceptable solution to these equations requires that b be imaginary, $b = -i\frac{\gamma}{2a}$, and that $E_0 = a$ be the real and positive solution to $a^4 - a^2 - \gamma^2/4 = 0$,

$$E_0 = a = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}. \tag{4}$$

Note that because b is imaginary and a is real and positive, $\psi_0(x,y)$ vanishes as $x^2 + y^2 \to \infty$.

B. Three coupled oscillators

For three oscillators the Hamiltonian H_3 in (1) with $\omega_j = 1$ has the form

$$H_3 = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \tfrac{1}{2}r^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2 + i\gamma(xy + yz). \tag{5}$$

Again, H_3 is partially $\mathcal{PT}$ symmetric; it is invariant under $\mathcal{P}_y\mathcal{T}$ (and also $\mathcal{P}_x\mathcal{P}_z\mathcal{T}$). The lowest-energy eigenstate has the form $\psi_0(x,y,z) = \exp[-\frac{1}{2}a(x^2+z^2) - \frac{1}{2}by^2 + c(xy + yz) + dxz]$, where a, b, c, and d are constants. Solving the Schrödinger equation $H_3\psi_0(x,y,z) = E\psi_0(x,y,z)$ and comparing powers in x, y, and z gives the five equations $E_0 = a + \frac{1}{2}b$, $1 = a^2 + d^2 + c^2$, $1 = 2c^2 + b^2$, $i\gamma = c(d - a - b)$, and $2ad = c^2$. We solve these equations and verify that the eigenfunction is normalizable [$\psi_0(x,y,z)$ vanishes as $x^2 + y^2 + z^2 \to \infty$] and that, even though H_3 is complex, the ground-state energy is real and positive,

$$E_0 = \tfrac{1}{2} + \tfrac{1}{2}\left(2 + 2\sqrt{1+2\gamma^2}\right)^{1/2}. \tag{6}$$

The ground-state eigenfunction $\psi_0(x,y,z)$ has partial $\mathcal{PT}$ symmetry. Also, in the limit $\gamma \to 0$ the oscillators decouple and we recover the expected result that $E_0 = 3/2$.

C. Four coupled oscillators

For four coupled oscillators the coordinates are x,y,z,w, the canonical momenta are p,q,r,s, the Hamiltonian H_4 with $\omega_j = 1$ is partially $\mathcal{PT}$ symmetric in the variables x,z or y,w, and reads

$$H_4 = \tfrac{1}{2}(p^2 + q^2 + r^2 + s^2) + \tfrac{1}{2}(x^2 + y^2 + z^2 + w^2) + i\gamma(xy + yz + zw). \tag{7}$$

We solve the Schrödinger equation $H_4\psi_0 = E\psi_0$ with the ansatz for a partially $\mathcal{PT}$-symmetric ground-state wave function of Gaussian form

$$\psi_0(x,y,z,w) = \exp\left[-\frac{a}{2}x^2 - \frac{b}{2}y^2 - \frac{b}{2}z^2 - \frac{a}{2}w^2 + c(xy + zw) + d(xz + yw) + exw + fyz\right], \tag{8}$$

where a,b,c,d,e,f are six arbitrary constants. This leads to the conditions $E_0 = a + b$, $f^2 + d^2 + c^2 + b^2 = 1$, $e^2 + d^2 + c^2 + a^2 = 1$, $i\gamma = 2cd - 2bf$, $cf + ce = bd + ad$, $i\gamma = df + de - cb - ca$, and $ae = cd$. Clearly, the complexity of the coupled nonlinear system of equations increases rapidly as the number of coupled oscillators increases.

For the case of N coupled oscillators with $\omega_j = 1$, we show in Sec. III that the ground-state energy E_0 is

$$E_0 = \frac{1}{2}\sum_{j=1}^{N}\sqrt{1 + 2i\gamma\cos[j\pi/(N+1)]}. \tag{9}$$

By setting $N = 2$ or $N = 3$, we readily recover (4) and (6). For $N = 4$, (9) yields the value

$$E_0 = \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \gamma^2(3+\sqrt{5})/2}\right]^{1/2} + \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \gamma^2(3-\sqrt{5})/2}\right]^{1/2}. \tag{10}$$

Closer inspection of (9) reveals that the ground-state energy of such coupled oscillators is always real. Indeed, (9) can be rewritten as

$$E_0 = \frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\gamma^2\cos^2[j\pi/(N+1)]}\right)^{1/2}. \tag{11}$$

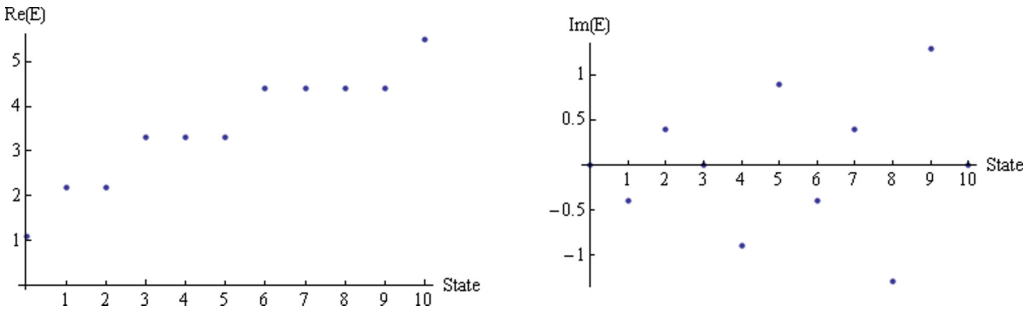

FIG. 1. (Color online) Real parts (left) and imaginary parts (right) of the first 11 energy levels of H_2 ($N = 2$, $\gamma = 1$).

III. EXACT EIGENFUNCTIONS AND SPECTRA OF N COUPLED OSCILLATORS

A. Two coupled oscillators

Let us return to the two-coupled-oscillator system governed by the Hamiltonian H_2 with $\omega_j = 1$. The transformation $x_1 = (x + y)/\sqrt{2}, x_2 = (x - y)/\sqrt{2}$ decouples the oscillators, leading to the Hamiltonian $H = \frac{1}{2}p_1^2 + \frac{1}{2}(1 + i\gamma)x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}(1 - i\gamma)x_2^2$, which has complex-conjugate frequencies $\nu_1^2 = 1 + i\gamma$ and $\nu_2^2 = 1 - i\gamma$. Apart from a normalization constant, the eigenfunctions are

$$\Psi_{n_1,n_2}(x_1,x_2) = \mathcal{H}_{n_1}(\sqrt{\nu_1}x_1)\mathcal{H}_{n_2}(\sqrt{\nu_2}x_2)e^{-\nu_1 x_1^2/2}e^{-\nu_2 x_2^2/2} \tag{12}$$

with corresponding energy eigenvalues $E_{n_1,n_2} = \nu_1(n_1 + \frac{1}{2}) + \nu_2(n_2 + \frac{1}{2})$. (Here, $\mathcal{H}_n$ are Hermite polynomials [10].) In terms of the coupling parameter γ the frequencies are

$$\nu_1 = \nu_2^* = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2} + i\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2},$$

whose real parts are positive. The general result for the energy spectrum is

$$E_{n_1,n_2} = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}(n_1 + n_2 + 1) + i\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}(n_1 - n_2).$$

Note that the spectrum is real if $n_1 = n_2$. If $n_1 = n_2 = 0$, we recover the ground-state energy in (4). In addition, we obtain the corresponding eigenfunction from (12),

$$\Psi_{0,0}(x,y) = \exp\left[-\tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}(x^2 + y^2) - i\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}xy\right],$$

which verifies the ansatz (3) and explicitly demonstrates that an eigenfunction having the partial $\mathcal{PT}$ symmetry of the Hamiltonian is associated with a real eigenvalue. Note also that the real spectrum is part of a larger spectrum containing complex-conjugate pairs. This can be illustrated by the choice $n_1 = 1$ and $n_2 = 0$ or $n_1 = 0$ and $n_2 = 1$:

$$E_{1,0} = \left(2 + 2\sqrt{1+\gamma^2}\right)^{1/2} + i\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+\gamma^2}\right)^{1/2}$$

and $\Psi_{1,0}(x,y) = \sqrt{2}(1 + i\gamma)^{1/4}(x + y)\Psi_{0,0}$, which is neither $\mathcal{P}_x\mathcal{T}$ nor $\mathcal{P}_y\mathcal{T}$ symmetric. In addition, $E_{0,1} = E_{1,0}^*$ and $\Psi_{0,1}(x,y) = \Psi_{1,0}(x,y)^*$. The real parts of the energies are $(n_1 + n_2 + 1)$-fold degenerate, as shown in Fig. 1.

The nature of the eigenfunctions associated with the first few energy levels is depicted in Fig. 2. The ground-state ($n_1 = n_2 = 0$) and the third-excited-state ($n_1 = n_2 = 1$) eigenfunctions are partially $\mathcal{PT}$ symmetric, as can be seen in the left-hand upper and lower diagrams, while the eigenfunctions corresponding to the first and third (complex) eigenvalues ($n_1 = 0, n_2 = 1$ and $n_1 = 0, n_2 = 2$), which are not partially $\mathcal{PT}$ symmetric, are shown on the right-hand diagrams.

B. Three coupled oscillators

To find the exact solution to the Schrödinger equation for H_3 in (5) with $\omega_j = 1$ we make the transformation $x_1 = (x - z)/\sqrt{2}$, $x_2 = y/\sqrt{2} + (x + z)/2$, $x_3 = -y/\sqrt{2} + (x + z)/2$, which decouples the oscillators, giving $H = \frac{1}{2}p_1^2 + \frac{1}{2}\nu_1^2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}\nu_2^2x_2^2 + \frac{1}{2}p_3^2 + \frac{1}{2}\nu_3^2x_3^2$ with $\nu_1^2 = 1$, $\nu_2^2 = 1 + i\gamma\sqrt{2}$, and $\nu_3^2 = 1 - i\gamma\sqrt{2}$. Thus, the unnormalized eigenfunctions are

$$\Psi(x_1,x_2,x_3) = \mathcal{H}_{n_1}(x_1)\mathcal{H}_{n_2}(\sqrt{\nu_2}x_2)\mathcal{H}_{n_3}(\sqrt{\nu_3}x_3) \times \exp\left[-\left(x_1^2 + \nu_2x_2^2 + \nu_3x_3^2\right)/2\right]$$

and the energies are $E = n_1 + \frac{1}{2} + \nu_2(n_2 + \frac{1}{2}) + \nu_3(n_3 + \frac{1}{2})$, where

$$\nu_{2,3} = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+2\gamma^2}\right)^{1/2} \pm i\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+2\gamma^2}\right)^{1/2}.$$

Thus, the energy spectrum can be expressed as

$$E = n_1 + \tfrac{1}{2} + \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+2\gamma^2}} + \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+2\gamma^2}} \times (n_2 + n_3) + i\sqrt{-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+2\gamma^2}}\,(n_2 - n_3).$$

Evidently, if the second and third oscillators are in the same state ($n_2 = n_3$), the energy is real and the corresponding eigenfunctions are partially $\mathcal{PT}$ symmetric. In particular, the ground-state energy (6) is recovered and the ground-state eigenfunction is

$$\Psi_{0,0,0}(x,y,z) = \exp[-(1 + a)(x^2 + z^2)/4 - ay^2/2 - (a - 1)xz/2 - ib(xy + yz)/\sqrt{2}],$$

where $a = \mathrm{Re}\,\nu_2$, $b = \mathrm{Im}\,\nu_2$. The first ten energies of H_3 for $\gamma = 1$ are shown in Fig. 3.

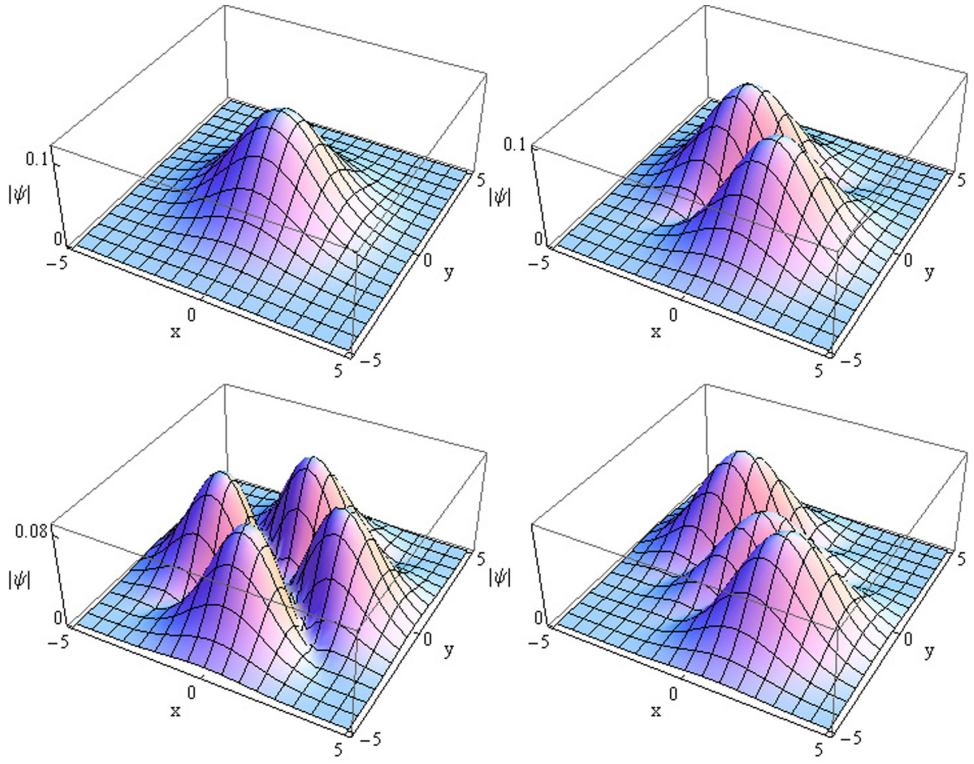

FIG. 2. (Color online) Absolute value of the eigenfunction for the ground state (upper left), $n_1 = n_2 = 1$ (lower left), $n_1 = 0$, $n_2 = 1$ (upper right), and $n_1 = 0$, $n_2 = 2$ (lower right) for $\gamma = 1$.

C. Four coupled oscillators

For the Hamiltonian (7), which governs four linearly coupled oscillators, the transformation

$$x_{1,2} = \frac{1}{2\sqrt{5}}\left[\sqrt{5-\sqrt{5}}\,(x \pm w) \pm \sqrt{5+\sqrt{5}}\,(y \pm z)\right],$$
$$x_{3,4} = \frac{1}{2\sqrt{5}}\left[\sqrt{5+\sqrt{5}}\,(x \mp w) \pm \sqrt{5-\sqrt{5}}\,(y \mp z)\right]$$

exactly decouples the oscillators. The new Hamiltonian takes the form

$$H = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}\nu_1^2 x_1^2 + \tfrac{1}{2}p_2^2 + \tfrac{1}{2}\nu_2^2 x_2^2 + \tfrac{1}{2}p_3^2 + \tfrac{1}{2}\nu_3^2 x_3^2 + \tfrac{1}{2}p_4^2 + \tfrac{1}{2}\nu_4^2 x_4^2,$$

where $\nu_1^2 = \nu_2^{2*} = 1 + \frac{1}{2}i\gamma(1+\sqrt{5})$ and $\nu_3^2 = \nu_4^{2*} = 1 + \frac{1}{2}i\gamma(-1+\sqrt{5})$ are complex frequencies. Let $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2 = A$, $\mathrm{Re}\,\nu_3 = \mathrm{Re}\,\nu_4 = C$, $\mathrm{Im}\,\nu_1 = -\mathrm{Im}\,\nu_2 = B$, $\mathrm{Im}\,\nu_3 = -\mathrm{Im}\,\nu_4 = D$, where

$$(A,B) = \sqrt{\pm\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{1}{2}\gamma^2(3+\sqrt{5})}},$$
$$(C,D) = \sqrt{\pm\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{1}{2}\gamma^2(3-\sqrt{5})}}.$$

In terms of these variables and the quantum numbers n_1, n_2, n_3, n_4, the total energy is

$$E_{n_1,n_2,n_3,n_4} = A(n_1+n_2+1) + C(n_3+n_4+1) + iB(n_1-n_2) + iD(n_3-n_4).$$

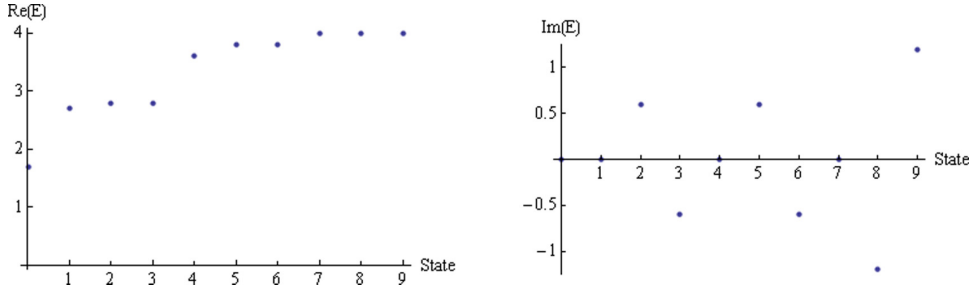

FIG. 3. (Color online) Real parts (left) and imaginary parts (right) of the first ten energies for H_3 ($\gamma = 1$, $N = 3$).

Thus, if $n_1 = n_2$ and $n_3 = n_4$, the energy is real. When the energy is real, the corresponding eigenfunction is always $\mathcal{P}_{xz}\mathcal{T}$ or $\mathcal{P}_{yw}\mathcal{T}$ symmetric. For example, the ground-state energy is $E_{0,0,0,0} = A + C$ and the corresponding eigenfunction is

$$\Psi_{0,0,0,0} = \exp\left[-\frac{1}{4}\left(A + C + \frac{C-A}{\sqrt{5}}\right)(x^2+w^2) - \frac{1}{4}\left(A + C + \frac{A-C}{\sqrt{5}}\right)(y^2+z^2) - \frac{1}{\sqrt{5}}(A-C)(xz+yw)\right.$$
$$\left. - \frac{i}{2}\left(B - \frac{B}{\sqrt{5}} - D - \frac{D}{\sqrt{5}}\right)xw - \frac{i}{2}\left(B + \frac{B}{\sqrt{5}} - D + \frac{D}{\sqrt{5}}\right)yz - \frac{i}{\sqrt{5}}(B+D)(xy+wz)\right]. \quad (13)$$

This eigenfunction displays the symmetries assumed in the ansatz (8).

As another illustration, we consider the case in which the first two oscillators are in the first excited state, and the other two are in the ground state ($n_1 = n_2 = 1$, $n_3 = n_4 = 0$). The energy is $E_{1,1,0,0} = 3A + C$ and the eigenfunction is

$$\Psi_{1,1,0,0} = \tfrac{1}{5}\sqrt{A^2+B^2}\big[(5-\sqrt{5})(x^2-\omega^2) - (5+\sqrt{5})(y^2-z^2) + 4\sqrt{5}xz - 4\sqrt{5}yw\big]\Psi_{0,0,0,0}.$$

Once again, the energy is real and the eigenfunction is $\mathcal{P}_{xz}\mathcal{T}$ or $\mathcal{P}_{yw}\mathcal{T}$ symmetric. A complex energy $E_{1,0,0,0} = 2A + C + iB$ arises for the choice $n_1 = 1$, $n_2 = n_3 = n_4 = 0$.

D. Five coupled oscillators

The Hamiltonian for five coupled oscillators is

$$H_5 = \tfrac{1}{2}(p^2+q^2+r^2+s^2+t^2) + \tfrac{1}{2}(x^2+y^2+z^2+w^2+u^2) + i\gamma(xy+yz+zw+wu).$$

Rather than decoupling the oscillators we treat this case by constructing the secular equation

$$\det(M_{jk} - \nu^2\delta_{jk}) = 0, \quad (14)$$

where M_{jk} is the tridiagonal matrix defined as

$$M_{jk} \equiv \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0 = \begin{pmatrix} 1 & i\gamma & 0 & 0 & 0 \\ i\gamma & 1 & i\gamma & 0 & 0 \\ 0 & i\gamma & 1 & i\gamma & 0 \\ 0 & 0 & i\gamma & 1 & i\gamma \\ 0 & 0 & 0 & i\gamma & 1 \end{pmatrix}, \quad (15)$$

U is the potential, and q_j and q_k are coordinates.

The solution to the secular equation (14) gives complex-conjugate pairs of frequencies and one real frequency: $\nu_{1,2}^2 = 1 \pm i\gamma\sqrt{3}$, $\nu_{3,4}^2 = 1 \pm i\gamma$, $\nu_5 = 1$. Thus, the decoupled Hamiltonian is

$$H = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}\nu_1^2x_1^2 + \tfrac{1}{2}p_2^2 + \tfrac{1}{2}\nu_2^2x_2^2 + \tfrac{1}{2}p_3^2 + \tfrac{1}{2}\nu_3^2x_3^2 + \tfrac{1}{2}p_4^2 + \tfrac{1}{2}\nu_4^2x_4^2 + \tfrac{1}{2}p_5^2 + \tfrac{1}{2}x_5^2$$

and the energy of the system reads

$$E = \nu_1\left(n_1 + \tfrac{1}{2}\right) + \nu_2\left(n_2 + \tfrac{1}{2}\right) + \nu_3\left(n_3 + \tfrac{1}{2}\right) + \nu_4\left(n_4 + \tfrac{1}{2}\right) + n_5 + \tfrac{1}{2},$$

where n_1, n_2, n_3, n_4, n_5 are non-negative integers.

E. General case: N coupled oscillators with $\omega_j = 1$

In this section we consider the Hamiltonian (1) for N linearly coupled oscillators with $\omega_j = 1$. To obtain the frequencies of the decoupled oscillators we use (14) to construct the $N \times N$ tridiagonal matrix secular equation, $\det(\mathbf{M} - \nu^2\mathbf{I}) = 0$, which has the form

$$D_N = \begin{vmatrix} 1-\nu^2 & i\gamma & & & & & \\ i\gamma & 1-\nu^2 & i\gamma & & & & \\ & i\gamma & 1-\nu^2 & i\gamma & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & i\gamma & 1-\nu^2 & i\gamma & \\ & & & & i\gamma & 1-\nu^2 & \end{vmatrix} = 0.$$

Because this matrix equation is tridiagonal, D_N satisfies the three-term recurrence relation

$$D_k + (\nu^2 - 1)D_{k-1} - \gamma^2 D_{k-2} = 0 \quad (k = 1,2,\ldots,N),$$

where $D_0 = 1$ and $D_{-1} = 0$. We solve this difference equation to obtain the frequencies

$$\nu^2 = 1 + 2i\gamma\cos[j\pi/(N+1)] \quad (j = 1,2,\ldots,N).$$

Thus, the exact expression for the total energy of the system of N oscillators is given by

$$E = \sum_{j=1}^{N}\sqrt{1 + 2i\gamma\cos[j\pi/(N+1)]}\left(n_j + \tfrac{1}{2}\right),$$

where $n_j \geqslant 0$ ($j = 1,\ldots,N$). Choosing $n_j = 0$ for all j, we find the exact ground-state energy in (11), which has been shown to be real.

IV. COUPLED OSCILLATORS WITH ARBITRARY FREQUENCIES

In Secs. II and III the oscillator frequencies multiplying x_j^2 were set to unity. Our conclusion in the foregoing analysis was that a real spectrum is embedded in a complex spectrum containing complex-conjugate pairs of energies. We now relax this constraint on the natural frequencies. For the two-, three-, and four-coupled-oscillator systems, we demonstrate that for an appropriate choice of ω_j the spectrum can be entirely real.

A. Two coupled oscillators with general natural frequencies ω_x and ω_y

The Hamiltonian H_2 in (1) reads $H_2 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{2}\omega_x^2x^2 + \frac{1}{2}\omega_y^2y^2 + i\gamma xy$. The frequencies of the decoupled

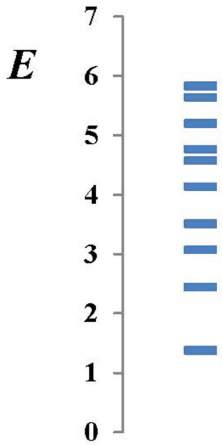

FIG. 4. (Color online) First ten energies of H_2 for the parameter choice $\omega_x^2 = 3$, $\omega_y^2 = 1$, and $\gamma = 1/2$. These states have the quantum numbers $(n_1,n_2) = (0,0),(0,1),(1,0),(0,2),(1,1),(0,3),(2,0),(1,2),(0,4),(2,1)$.

oscillators in this case are

$$\nu_{1,2}^2 = \tfrac{1}{2}\left(\omega_x^2 + \omega_y^2 \pm \sqrt{\left(\omega_x^2 - \omega_y^2\right)^2 - 4\gamma^2}\right) \qquad (16)$$

and the energies of the system are $E_{n_1,n_2} = \nu_1(n_1 + \frac{1}{2}) + \nu_2(n_2 + \frac{1}{2})$, where $n_1,n_2 \geqslant 0$.

In contrast to the results found in Sec. III, the entire energy spectrum can be real for specific values of ω_x, ω_y, and γ. For this to be so, the parameters must satisfy the condition

$$\left|\omega_x^2 - \omega_y^2\right| \geqslant 2|\gamma|. \qquad (17)$$

The case considered in Sec. III had $\omega_x = \omega_y = 1$ and $\gamma = 1$, which does not satisfy this condition and, as we saw, the energy spectrum was only partially real.

To have real eigenvalues the associated eigenfunctions must all be partially $\mathcal{PT}$ symmetric. This can be seen explicitly by decoupling the oscillators with the transformation

$$x_1 = \sqrt{A + B}(Dx + Ex + iCy)/(2CE),$$
$$x_2 = \sqrt{A - B}(-Dx + Ex - iCy)/(2CE),$$

where $A = 8\gamma^2 - 2(w_x^2 - w_y^2)^2$, $B = 2(w_x^2 - w_y^2)[(w_x^2 - w_y^2)^2 - 4\gamma^2]^{1/2}$, $C = 2\gamma$, $D = w_x^2 - w_y^2$, $E = [(w_x^2 - w_y^2)^2 - 4\gamma^2]^{1/2}$, leading to the Hamiltonian $H = \frac{1}{2}p_1^2 + \frac{1}{2}\nu_1^2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}\nu_2^2x_2^2$, where ν_1 and ν_2 are given in (16). Up to a normalization constant, the eigenfunctions are

$$\Psi_{n_1,n_2}(x_1,x_2) = \mathcal{H}_{n_1}(\sqrt{\nu_1}x_1)\mathcal{H}_{n_2}(\sqrt{\nu_2}x_2) \times \exp\left(-\tfrac{1}{2}\nu_1x_1^2 - \tfrac{1}{2}\nu_2x_2^2\right).$$

Note that $A + B$ and $A - B$ have opposite signs in the $\mathcal{PT}$-symmetric phase and that $A - B < 0$. Rewriting $\Psi_{n_1,n_2}(x_1,x_2)$ in terms of the original variables x and y, one can show that the eigenfunction $\Psi_{n_1,n_2}(x,y)$ has partial $\mathcal{PT}$ symmetry because

$$(\mathcal{P}_x\mathcal{T})\Psi_{n_1,n_2}(x,y) = (-1)^{n_1}\Psi_{n_1,n_2}(x,y),$$
$$(\mathcal{P}_y\mathcal{T})\Psi_{n_1,n_2}(x,y) = (-1)^{n_2}\Psi_{n_1,n_2}(x,y).$$

To illustrate, we consider the case $\omega_x^2 = 3$, $\omega_y^2 = 1$, and $\gamma = 1/2$. The relation (17) is satisfied and we find the purely real nondegenerate spectrum shown in Fig. 4. When $|\omega_x^2 - \omega_y^2| < 2|\gamma|$, the energy spectrum is only partially real. Thus, there is a phase transition from the unbroken partially $\mathcal{PT}$-symmetric phase to the broken one. For example, keeping $\omega_y = 1$ and $\gamma = 1/2$, but adjusting ω_x^2 so that it passes 2, the first-excited-state energy becomes complex.

B. Three coupled oscillators with general natural frequencies ω_x, ω_y, and ω_z

For the three-oscillator Hamiltonian $H_3 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{2}r^2 + \frac{1}{2}\omega_x^2x^2 + \frac{1}{2}\omega_y^2y^2 + \frac{1}{2}\omega_z^2z^2 + i\gamma(xy + yz)$ the

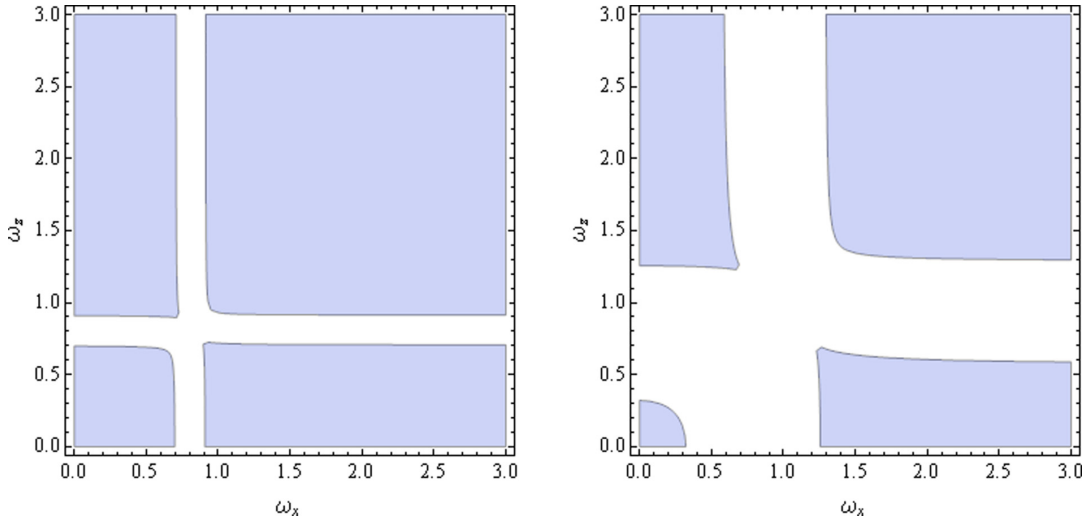

FIG. 5. (Color online) Unbroken partial $\mathcal{PT}$-symmetric phases of H_3 depicted as shaded areas for the parameters $\gamma = 1/12$ and $\omega_y^2 = 2/3$ (left) and $\gamma = 1/3$ and $\omega_y^2 = 1$ (right).

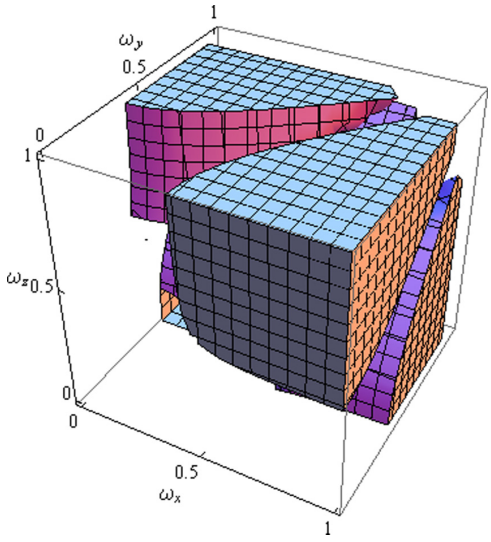

FIG. 6. (Color online) Unbroken partial $\mathcal{PT}$-symmetric phases of H_3 for $\gamma = 1/12$ depicted as colored volumes.

frequencies $\nu^2 = \lambda$ of the decoupled oscillators satisfy the cubic equation $f(\lambda) = 0$, where

$$f(\lambda) = \lambda^3 - \left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)\lambda^2 + \left(\omega_x^2\omega_y^2 + \omega_x^2\omega_z^2 + \omega_y^2\omega_z^2 + 2\gamma^2\right)\lambda - \omega_x^2\omega_y^2\omega_z^2 - \left(\omega_x^2 + \omega_z^2\right)\gamma^2.$$

If the discriminant associated with this equation is positive, three real distinct roots can emerge, giving a real spectrum. To guarantee that the roots are positive, it is necessary that

$$f(0) < 0, \quad \lambda_{\max} > 0, \quad \lambda_{\min} > 0,$$
$$f(\lambda_{\max}) > 0, \quad f(\lambda_{\min}) < 0$$

are all fulfilled, with $\lambda_{\min}$ and $\lambda_{\max}$ being the extrema of the polynomial. Figure 5 displays the regions in which the frequencies of the decoupled oscillators are all real (blue shaded areas) in the parametric space of ω_x and ω_z for fixed values of γ and ω_y^2. This figure shows that several regions of unbroken symmetry exist. This is in contrast to the case of the two coupled oscillators.

For example, Fig. 5 shows that $\omega_x^2 = 1/3$, $\omega_y^2 = 2/3$, $\omega_z^2 = 1$, and $\gamma = 1/12$ gives an unbroken symmetry phase. We obtain three different real positive (decoupled) frequencies

$$\nu_1 = \sqrt{2/3}, \quad \nu_2 = \sqrt{(8+\sqrt{14})/12}, \quad \nu_3 = \sqrt{(8-\sqrt{14})/12}.$$

Thus, the spectrum is entirely real with energies given by $E_{n_1,n_2,n_3} = \nu_1(n_1 + \frac{1}{2}) + \nu_2(n_2 + \frac{1}{2}) + \nu_3(n_3 + \frac{1}{2})$. By fixing only γ we can find regions in the three-dimensional parameter space of ω_x, ω_y, and ω_z for which unbroken symmetry (and therefore a real spectrum) exists. This is shown in the colored volumes depicted in Fig. 6 for the specific choice $\gamma = 1/12$.

C. Four coupled oscillators with general natural frequencies ω_x, ω_y, ω_z, and ω_w

The previous analysis can be applied to the four-coupled-oscillator Hamiltonian

$$H_4 = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \tfrac{1}{2}r^2 + \tfrac{1}{2}s^2 + \tfrac{1}{2}\omega_x^2x^2 + \tfrac{1}{2}\omega_y^2y^2 + \tfrac{1}{2}\omega_z^2z^2 + \tfrac{1}{2}\omega_w^2w^2 + i\gamma(xy + yz + zw),$$

where ω_x, ω_y, ω_z, and ω_w are real frequencies and γ is a real coupling parameter.

The eigenvalues of the matrix

$$M = \begin{pmatrix} \omega_x^2 & i\gamma & 0 & 0 \\ i\gamma & \omega_y^2 & i\gamma & 0 \\ 0 & i\gamma & \omega_z^2 & i\gamma \\ 0 & 0 & i\gamma & \omega_w^2 \end{pmatrix}$$

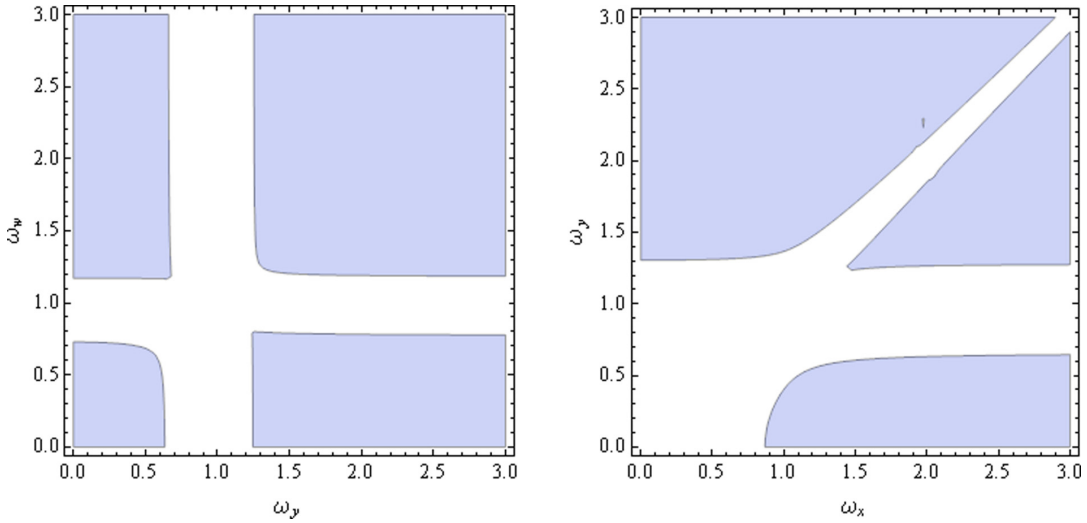

FIG. 7. (Color online) Unbroken partial $\mathcal{PT}$-symmetric phases of H_4 depicted as shaded area. Left: $\gamma = 1/5$, $\omega_x^2 = 1$, and $\omega_z^2 = 1$. Right: $\gamma = 3/10$, $\omega_z^2 = 1$, and $\omega_w^2 = 4$.

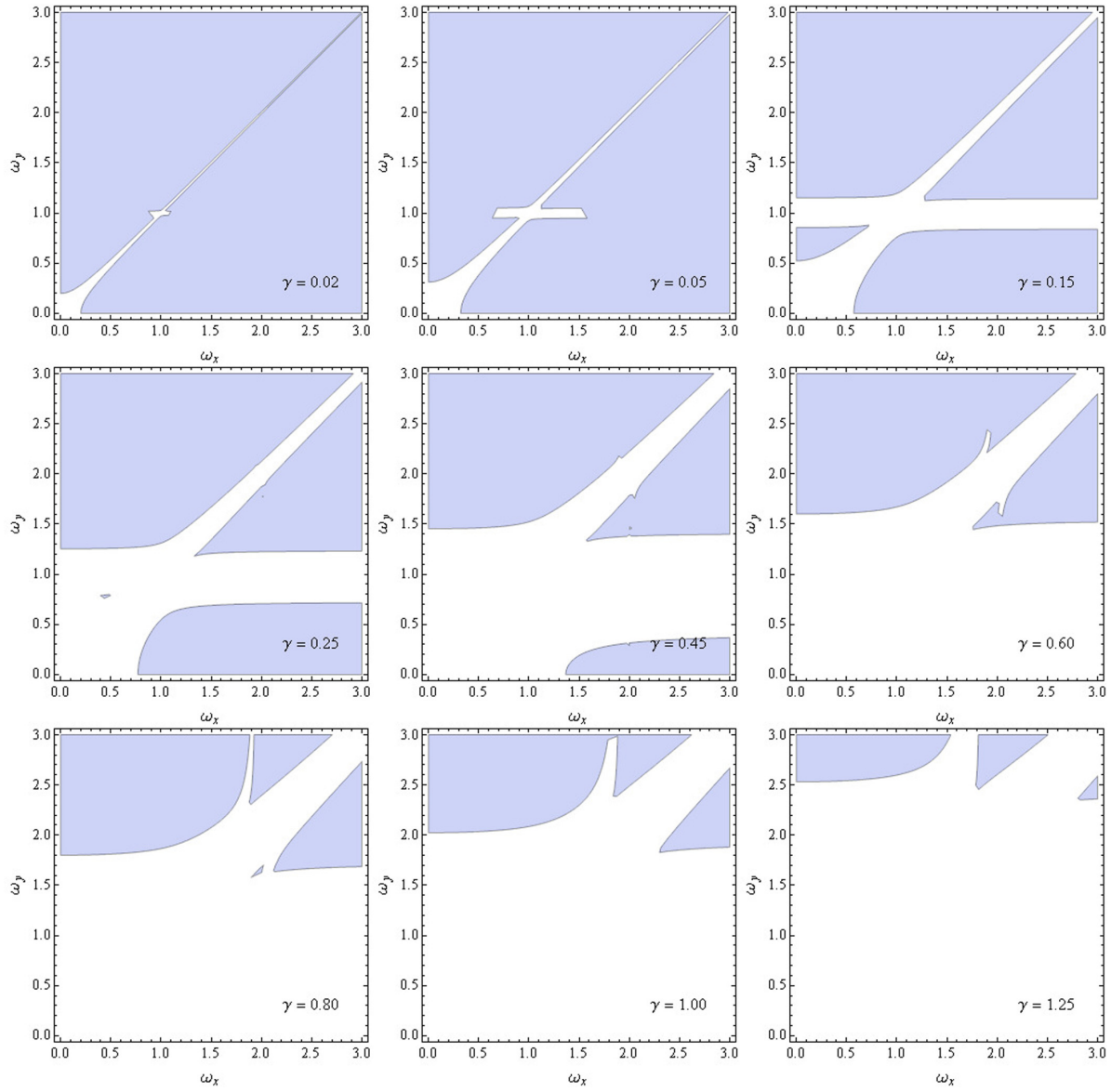

FIG. 8. (Color online) Unbroken partial $\mathcal{PT}$-symmetric phases of H_4 in the (ω_x,ω_y) plane depicted as shaded areas for the parameter choices $\omega_z^2 = 1$ and $\omega_w^2 = 4$ for nine values of γ.

are the squares of the corresponding decoupled-oscillator frequencies. The eigenvalues $\nu^2 = \lambda$ satisfy the fourth-order equation $f(\lambda) = \lambda^4 - a\lambda^3 + b\lambda^2 - c\lambda + d = 0$, where

$$
\begin{aligned}
a &= \omega_x^2 + \omega_y^2 + \omega_z^2 + \omega_w^2,\\
b &= \omega_x^2\omega_y^2 + \omega_x^2\omega_z^2 + \omega_x^2\omega_w^2 + \omega_y^2\omega_z^2 + \omega_y^2\omega_w^2 + \omega_z^2\omega_w^2 + 3\gamma^2,\\
c &= \omega_x^2\omega_y^2\omega_z^2 + \omega_x^2\omega_y^2\omega_w^2 + \omega_x^2\omega_z^2\omega_w^2 + \omega_y^2\omega_z^2\omega_w^2 + 2\gamma^2\omega_x^2\\
&\quad + 2\gamma^2\omega_w^2 + \gamma^2\omega_y^2 + \gamma^2\omega_z^2,\\
d &= \omega_x^2\omega_y^2\omega_z^2\omega_w^2 + \gamma^2\omega_x^2\omega_y^2 + \gamma^2\omega_x^2\omega_w^2 + \gamma^2\omega_z^2\omega_w^2 + \gamma^4.
\end{aligned}
$$

Regions in which all decoupled oscillator frequencies are real give a completely real energy spectrum, which means that partial $\mathcal{PT}$ symmetry is unbroken. This requires that $f(\lambda)$ have four positive roots, which is the case if $f(0) > 0$. In addition, if $f'(\lambda)$ has three positive roots, the extrema of $f(\lambda)$ lie on the positive abscissa. To have four real roots the minimum value of $f(\lambda)$ must be negative, and the maximum value must be positive. Figure 7 shows the regions in which these conditions are fulfilled (that is, the regions in which the partial $\mathcal{PT}$ symmetry is unbroken) for specific choices of the parameters. Fixing the values of $\omega_z^2 = 1$ and $\omega_w^2 = 4$ as in the right-hand panel of Fig. 7, we can investigate the

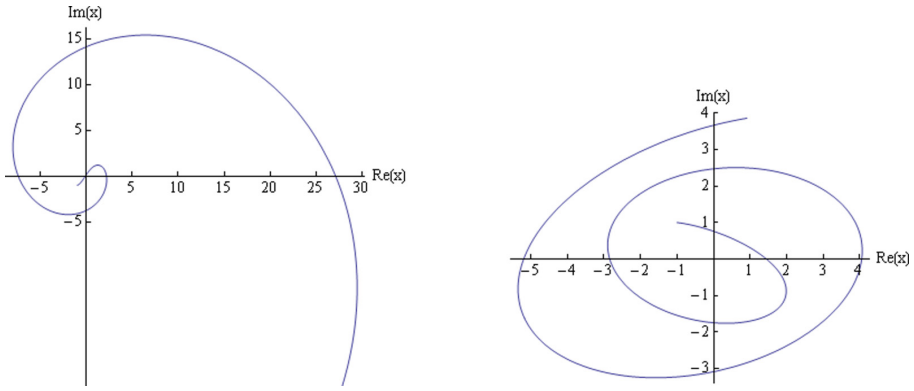

FIG. 9. (Color online) Left: Classical trajectory in the complex-x plane for H_2 with $\omega_x^2 = \omega_y^2 = 1$ with the initial conditions $x(0) = y(0) = -1 - i$ and $\dot{x}(0) = \dot{y}(0) = 1 + i/2$, and $\gamma = 1$. Right: Classical trajectory in the complex-x plane for H_2 with $\omega_x^2 = 2$, $\omega_y^2 = 1$, and $\gamma = 1/2$ for the initial conditions $x(0) = y(0) = -1 + i$ and $\dot{x}(0) = \dot{y}(0) = 2 - i/4$.

development of the phase boundaries as a function of the coupling strength γ. This is shown in Fig. 8.

V. CORRESPONDING CLASSICAL THEORY

In this section we investigate the features of partially $\mathcal{PT}$-symmetric classical theories. We begin with the two-coupled-oscillator Hamiltonian H_2 with frequencies $\omega_x = \omega_y = 1$. Hamilton's classical equations of motion lead to

$$x''(t) + x(t) + i\gamma y(t) = 0, \quad y''(t) + y(t) + i\gamma x(t) = 0$$

and combining these equations gives the fourth-order differential equation

$$x''''(t) + 2x''(t) + (1 + \gamma^2)x(t) = 0.$$

We seek solutions $x(t) = e^{i\nu t}$ and find that $\lambda = \nu^2$ satisfies the quadratic equation $\lambda^2 - 2\lambda + 1 + \gamma^2 = 0$, so $\nu = \pm\sqrt{1 \pm i\gamma}$. Thus, the characteristic frequencies are always complex. By decomposing $\sqrt{1 + i\gamma} = a + ib$ into its real and imaginary parts we can write the general solution as

$$x(t) = [(A + D)e^{-bt} + (B + C)e^{bt}]\cos(at) + i[(A - D)e^{-bt} + (B - C)e^{bt}]\sin(at),$$

where A, B, C, and D are arbitrary constants. Therefore, for any initial conditions, the real and imaginary parts of $x(t)$ are oscillatory and growing (or decreasing). As a consequence, the trajectories in the complex-x plane spiral outward (or inward). Hence, the classical paths are open (see Fig. 9, left-hand panel). Thus, although the quantum spectrum is partially real, this partial reality does not give rise to closed classical trajectories.

More generally, if the coupled oscillators described by H_2 have natural frequencies ω_x and ω_y, we obtain the equation

$$x''''(t) + \left(w_x^2 + w_y^2\right)x''(t) + \left(w_x^2 w_y^2 + \gamma^2\right)x(t) = 0.$$

Again seeking solutions $x(t) = e^{i\nu t}$, we find that $\nu^2 = \frac{1}{2}w_x^2 + \frac{1}{2}w_y^2 \pm \frac{1}{2}[(\omega_x^2 - \omega_y^2)^2 - 4\gamma^2]^{1/2}$. We deduce that four real values of ν exist when $|\omega_x^2 - \omega_y^2| \geqslant 2|\gamma|$, which is precisely the condition that guarantees a fully real spectrum in the quantum system [see (17)]. Thus, the transition from the broken partial-$\mathcal{PT}$-symmetric phase to the unbroken phase occurs at the same point as for the quantum case. For the parameter choice $\omega_x^2 = 2$, $\omega_y^2 = 1$, and $\gamma = 1/2$ the classical trajectory depicted in Fig. 9 (right-hand panel) spirals outward, which indicates that $\mathcal{PT}$ symmetry is broken even though the system is partially $\mathcal{PT}$ symmetric. The behavior of the trajectories in the unbroken phase is illustrated in Fig. 10. Here, one sees that while the classical trajectory is not closed it is confined to a compact region in the complex-x plane. We thus observe the phase transition at the classical level; that is, we observe the transition from spirals (broken phase) to localized trajectories in the complex-x plane (unbroken phase), which happens at $|\omega_x^2 - \omega_y^2| = 2|\gamma|$.

In addition to studying the trajectory in the complex-x plane, one can study the trajectories in phase space by plotting

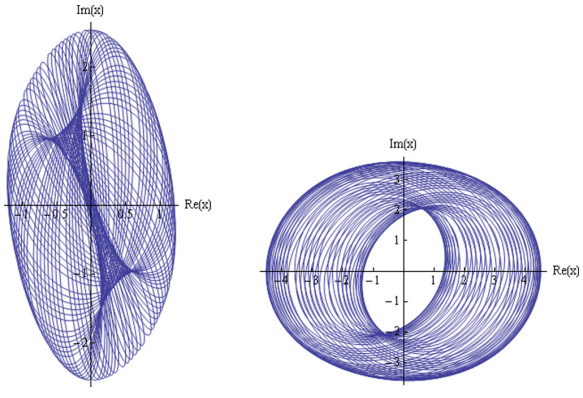

FIG. 10. (Color online) Classical trajectory in the complex-x plane for H_2 with the parameter choice $\omega_x^2 = 3$, $\omega_y^2 = 1$, $\gamma = 1/2$, with the initial conditions $x(0) = -1 + i$, $y(0) = 2 - 2i$, $\dot{x}(0) = 1 + i/2$, $\dot{y}(0) = 3/2 + i$ (left) and $x(0) = 1 + 2i$, $y(0) = 2 + i/4$, $\dot{x}(0) = -3 + i$, $\dot{y}(0) = 1/2 + 5i$ (right).

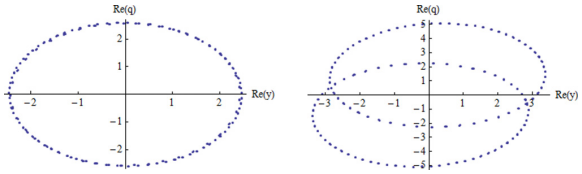

FIG. 11. (Color online) Poincaré section of the classical trajectory in phase space for H_2 with the parameter choice $\omega_x^2 = 3, \omega_y^2 = 1$, $\gamma = 1/2$ and initial conditions $x(0) = 1 + i/2$, $y(0) = -2 + 2i$, $\dot{x}(0) = 1 + i/2$, $\dot{y}(0) = -3/2 - 3i/2$ (left) and $x(0) = 2 + 5i/2$, $y(0) = 2 - 3i$, $\dot{x}(0) = 1 - i/2$, $\dot{y}(0) = 2 - 2i$ (right). The structure of this plot indicates that the orbits are almost periodic.

a Poincaré section. From the structure of the Poincaré plot we conclude that the confined trajectories are *almost periodic* [11]. This is illustrated in Fig. 11. The appearance of open almost-periodic trajectories occurs because the number of degrees of freedom exceeds one; if $N = 1$, the classical orbits associated with real quantum energies are closed [12].

A similar analysis can be done for the three-coupled-oscillator Hamiltonian H_3 with general natural frequencies. The classical equations of motion are

$$x''(t) + \omega_x^2 x(t) + i\gamma y(t) = 0,$$
$$y''(t) + \omega_y^2 y(t) + i\gamma(x(t) + z(t)) = 0,$$
$$z''(t) + \omega_z^2 z(t) + i\gamma y(t) = 0.$$

Seeking solutions of the form $x(t) = Ae^{ivt}$, $y(t) = Be^{ivt}$, and $z(t) = Ce^{ivt}$, we obtain a cubic equation for $\lambda = v^2$:

$$\lambda^3 - \left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)\lambda^2 + \left(\omega_x^2\omega_y^2 + \omega_x^2\omega_z^2 + \omega_y^2\omega_z^2 + 2\gamma^2\right)\lambda - \omega_x^2\omega_y^2\omega_z^2 - \left(\omega_x^2 + \omega_z^2\right)\gamma^2 = 0.$$

The characteristic frequencies are real if and only if the corresponding quantum system is in an unbroken $\mathcal{PT}$-symmetric phase (all eigenvalues are real); the criteria for real eigenvalues are given in Sec. IV. To illustrate, recall that in Sec. IV the parameter choice $\omega_x^2 = 1/3, \omega_y^2 = 2/3, \omega_z^2 = 1, \gamma = 1/12$ lies

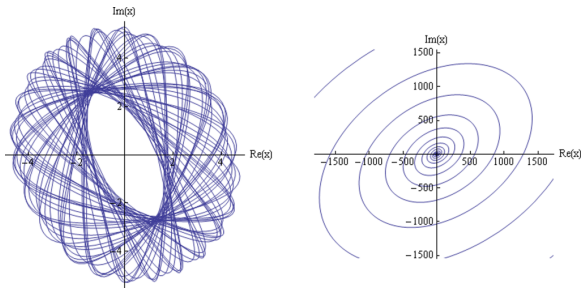

FIG. 12. (Color online) Classical trajectory in the complex-x plane for H_3 with the parameter choice $\omega_x^2 = 1/3$, $\omega_y^2 = 2/3, \gamma = 1/12$, $\omega_z^2 = 1$ (left) and $\omega_z^2 = 13/20$ (right) with the initial conditions $x(0) = -2 + i, y(0) = 3 - 3i, z(0) = 3 + 2i, \dot{x}(0) = -1 + 3i$, $\dot{y}(0) = 3 + 2i, \dot{z}(0) = -2 + i$.

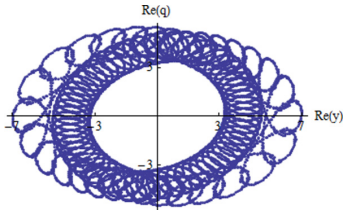

FIG. 13. (Color online) Poincaré section of the classical trajectory for H_3 with the parameter choice $\omega_x^2 = 1/3, \omega_y^2 = 2/3, \omega_z^2 = 1$, $\gamma = 1/12$ and the initial conditions $x(0) = -2 + i$, $y(0) = 3 - 3i$, $z(0) = 3 + 2i, \dot{x}(0) = -1 + 3i, \dot{y}(0) = 3 + 2i, \dot{z}(0) = -2 + i$. This figure indicates that the orbit is almost periodic.

in the unbroken phase and gives a real energy spectrum. Figure 12 shows that for this parameter choice the classical trajectory is confined to a compact region in the complex-x plane, whereas for the choice $\omega_x^2 = 1/3$, $\omega_y^2 = 2/3$, $\omega_z^2 = 13/20$, and $\gamma = 1/12$, for which the quantum symmetry is broken, the classical trajectory spirals outward to infinity. A Poincaré section is given in Fig. 13 for the parameter choice of Fig. 12 (left-hand panel).

VI. BRIEF CONCLUDING REMARKS

In this paper we have examined systems of N linearly coupled oscillators that are partially $\mathcal{PT}$ symmetric. In the quantum-mechanical analysis we have found that the ground state of each of these systems is always real. We have shown that the entire spectrum may in fact be completely real depending on the values of the natural frequencies ω_x, ω_y, ω_z, ... and their relation to the coupling strength γ. This happens even though the system is only partially $\mathcal{PT}$ symmetric. We have studied this in detail for systems of two and three coupled oscillators. A phase transition point exists beyond which the energy spectrum is only partially real.

For the two and three classical oscillator systems, we find a phase transition at exactly the same point as the quantum-mechanical oscillator systems. When the eigenvalues of the quantum system are all real, the classical trajectories are confined and almost periodic, but when the quantum eigenvalues are partly real and partly complex, the corresponding classical system always has open trajectories that spiral out to infinity.

Finally, we comment that while it is not obvious what kinds of experiments can be performed to verify the results obtained in this paper, it is very likely that in the area of experimental optics it will be possible to mimic partially $\mathcal{PT}$-symmetric Hamiltonians, to study their behavior directly, and to access their physical properties (see, for example, Ref. [9]).

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Chapter 6

Analytic structure of eigenvalues of coupled quantum systems

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Motivation: The coupling constant of the coupled theory in the previous chapter was assumed to be pure imaginary. Here, we consider a more general coupling and demonstrate that we can obtain different phases of the uncoupled theory by analytic continuation of the coupling.

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Analytic structure of eigenvalues of coupled quantum systems

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Abstract

By analytically continuing the coupling constant g of a coupled quantum theory, one can, at least in principle, arrive at a state whose energy is lower than the ground state of the theory. The idea is to begin with the uncoupled $g = 0$ theory in its ground state, to analytically continue around an exceptional point (square-root singularity) in the complex-coupling-constant plane, and finally to return to the point $g = 0$. In the course of this analytic continuation, the uncoupled theory ends up in an unconventional state whose energy is lower than the original ground-state energy. However, it is unclear whether one can use this analytic continuation to extract energy from the conventional vacuum state; this process appears to be exothermic but one must do work to vary the coupling constant g.

Keywords: analytic continuation, coupled quantum systems, unconventional quantum states

(Some figures may appear in colour only in the online journal)

1. Introduction

The analytic structure of eigenvalues of self-coupled systems, such as the quantum anharmonic oscillator, has been studied in great depth. Singularities in the coupling-constant plane have been identified as the cause of the divergence of perturbation theory [1, 2]. These singularities are typically square-root branch points and are associated with the phenomenon of level crossing. These singularities are sometimes referred to as exceptional points [3]. Studies of coupling-constant analyticity have revealed a remarkable and generic phenomenon, namely, that the eigenvalues belonging to the spectrum of the Hamiltonian are analytic continuations of one another as functions of the complex coupling constant. Thus, the energy levels of a quantum system, which are discrete when the coupling constant is real and positive, are actually smooth continuations of one another in the complex-coupling-constant plane [4], and a simple geometric picture of quantization emerges: The discrete eigenvalues of a quantum system are in one-to-one correspondence with the sheets of the Riemann surface. The different energy levels of the Hamiltonian are merely different branches of a multivalued energy function.

While this picture of quantization has emerged from studies of coupling-constant singularities of self-coupled systems, this paper argues that an even more elaborate picture arises from studies of coupled quantum systems. Consider, for example, the simple case of two coupled quantum harmonic oscillators, one having natural frequency $\nu > 0$ and the other having natural frequency $\omega > 0$. For definiteness, we assume that $\nu > \omega$. The Hamiltonian for such a system has the form

$$H = p^2 + \nu^2 x^2 + q^2 + \omega^2 y^2 + gxy, \qquad (1)$$

where g is the coupling parameter. For sufficiently large $|g|$ the eigenvalues of H become singular. To demonstrate this we

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rewrite the potential $V(x, y) = \nu^2x^2 + \omega^2y^2 + gxy$ as

$$V(x, y) = \nu^2\left(x + \frac{gy}{2\nu^2}\right)^2 + y^2\left(\omega^2 - \frac{g^2}{4\nu^2}\right). \quad (2)$$

We see immediately that on the line $x + gy/2\nu^2 = 0$ in the (x, y) plane $V(x, y)$ becomes unbounded below if $g^2 > 4\nu^2\omega^2$. Thus, while the potential has a positive discrete spectrum when the coupling constant g lies in the range

$$-2\nu\omega < g < 2\nu\omega, \quad (3)$$

we expect there to be singular points at $g = \pm 2\nu\omega$ in the coupling-constant plane. This result raises the question, What is the nature of the singular points at $\pm 2\nu\omega$?

Coupled-oscillator models have been studied in great detail in many papers [5–11] and in particular for oscillator models of the type in (1). The presence of singularities at $g = \pm 2\nu\omega$ was noted in [6]; however, the nature of singularities and the Riemann sheet structure was not identified in any of these papers.

In this paper we show that the Riemann surface for the coupled-oscillator Hamiltonian (1) consists of four sheets. The singularities at $g = \pm 2\nu\omega$ are square-root singularities, like the exceptional-point singularities of self-coupled oscillators. However, if we cross either of the square-root branch cuts, we enter a second sheet of the Riemann surface on which two *new* square-root branch points appear. These new branch points are located at $g = \pm i(\omega^2 - \nu^2)$. If we cross either of the branch cuts emanating from these new branch points, we enter a third sheet of the Riemann surface where there are yet another pair of square-root singularities at $g = \pm 2\nu\omega$, unconnected with the singularities on sheets one and two. Crossing either of the branch cuts emanating from these singularities at $g = \pm 2\nu\omega$ takes us to a fourth sheet of the Riemann surface. Not all energy levels of the coupled harmonic oscillator mix among themselves as g varies on this four-sheeted Riemann surface. Rather, each energy level belongs to a quartet of energies that are analytic continuations of one another. We find that the four sheets of the Riemann surface correspond to four distinct *spectral phases* of the coupled oscillator system (1).

We give a detailed description of these spectral phases in section 2. We explain below how such spectral phases arise. Let us consider a single harmonic oscillator, whose dynamics are defined by the Hamiltonian

$$H = p^2 + \nu^2x^2. \quad (4)$$

This simple quantum system actually has two spectral phases characterized by two distinct spectra. To understand why, we assume that ν is a positive parameter and we note that the nth eigenvalue E_n, which is defined by the eigenvalue problem

$$-\frac{d^2}{dx^2}\psi(x) + \nu^2x^2\psi(x) = E_n\psi(x) \quad (\psi \to 0 \text{ as } x \to \pm\infty), \quad (5)$$

is given by

$$E_n = (2n + 1)\nu \quad (n = 0, 1, 2, 3, \ldots).$$

In [12] it was observed that if we analytically continue ν in a semicircle in the complex-ν plane, that is, if we let $\nu = re^{i\phi}$ (r real) and allow ϕ to run from 0 to π, the eigenvalues change sign even though the Hamiltonian remains unchanged. By this analytic continuation we reach a new phase of the harmonic oscillator whose spectrum is *negative* and *unbounded below*. Thus, the Hamiltonian (4) of the harmonic oscillator has two distinct and independent real spectra that are related by analytic continuation in the natural frequency ν of the oscillator.

How can one Hamiltonian (4) have two different spectra? The answer to this question is that the positive spectrum is obtained by imposing the boundary conditions in (5) in a pair of Stokes wedges [4, 13–15] centered about the positive-real-x and negative-real-x axes. We refer to the positive spectrum as the *conventional* one. These wedges have angular opening $\pi/2$. The negative spectrum is defined by imposing the boundary conditions in a pair of Stokes wedges containing and centered about the upper and lower imaginary-x axes. We refer to the negative spectrum as the *unconventional* spectrum of the harmonic oscillator. These Stokes wedges also have angular opening of $\pi/2$. To understand the configuration of the wedges we examine the WKB geometrical-optics approximation [4]

$$\psi \sim e^{\pm\nu x^2/2} \quad (6)$$

to the solutions of the harmonic-oscillator eigenvalue equation (5). On the basis of (6) we can see that the 90° wedges in which the eigenfunctions vanish rotate clockwise through an angle of $\pi/2$ as ν rotates anticlockwise through an angle of π. Thus, these two phases are analytic continuations of one another and are analytically connected by rotations in the complex-frequency plane.

A principal result of this paper is that, if we analytically continue the physical system consisting of two coupled harmonic oscillators described by the Hamiltonian in (1) in the coupling constant parameter g, we obtain all four possibilities for the phases of the two oscillators in which each oscillator is in a conventional or an unconventional phase. Thus, all four phases are analytically connected on the Riemann surface of the complex coupling constant g, even though the frequencies ν and ω are held fixed and positive.

In section 2 we construct analytically the four-sheeted Riemann surface for the coupled harmonic oscillator model (1). In section 3 we examine the Riemann surface defined by the partition function for some zero-dimensional quantum field theories. In general, the number of sheets in the complex Riemann surface for these theories is smaller than the number of sheets for the corresponding quantum-mechanical problem. For example, for the zero-dimensional quantum field theory that is analogous to the quantum-mechanical oscillator model of (1), the Riemann surface only has two sheets and not four sheets. For a coupled pair of sextic models, the Riemann surface has six sheets, indicating that this theory has six different phases. Section 4 gives some brief concluding remarks.

2. Energy levels of the coupled harmonic oscillator

In this section we examine the analytic structure of the eigenvalues of the coupled harmonic oscillator Hamiltonian (1). We begin by examining the ground state, whose eigenfunction has the general form

$$\psi(x, y) = \mathrm{e}^{-ax^2/2 - by^2/2 + cxy}, \tag{7}$$

where a, b, and c are constants to be determined. We substitute (7) into the eigenvalue equation $H\psi = E\psi$, which has the explicit form

$$-\psi_{xx} + \nu^2 x^2\psi - \psi_{yy} + \omega^2 y^2\psi + gxy\psi = E\psi. \tag{8}$$

We then equate the coefficients of x^2, y^2, xy, and x^0y^0 and obtain the four equations

$$\begin{aligned} \nu^2 &= a^2 + c^2, \\ \omega^2 &= b^2 + c^2, \\ 0 &= 2ac + 2bc + g, \end{aligned} \tag{9}$$

$$E = a + b. \tag{10}$$

Subtracting the first equation from the second and combining the result with the third and fourth equations allows us to calculate a, b, and c, which we then eliminate in favor of a single quartic polynomial equation for the eigenvalue E:

$$E^4 - 2(\nu^2 + \omega^2)E^2 + (\nu^2 - \omega^2)^2 + g^2 = 0. \tag{11}$$

The solution to this equation involves nested square roots,

$$E(g) = [\nu^2 + \omega^2 + (4\nu^2\omega^2 - g^2)^{1/2}]^{1/2} \tag{12}$$

and from this equation we see that $E(g)$ is a four-valued function of the coupling constant g.

Let us make a grand tour of the Riemann surface on which $E(g)$ is defined. We begin on Sheet 1, where both square-root functions are real and positive when their arguments are real and positive. There are two obvious square-root branch points (zeros of the inner square root) and these are located at $g = \pm 2\nu\omega$. Square-root branch cuts emerge from each of these branch points and, as shown on figure 1, we have chosen to draw these branch cuts as vertical lines going downward. On Sheet 1

$$E(0) = \nu + \omega \tag{13}$$

and because we assume that ν and ω are real and positive we see that both oscillators are in their conventional ground states.

There are no other singularities on Sheet 1 that allow us to change the sign of the outer square root. This is because at such a singular point the argument of the outer square root function would have to vanish:

$$\nu^2 + \omega^2 + \sqrt{4\nu^2\omega^2 - g^2} = 0. \tag{14}$$

The solution of this equation is obtained by squaring $\nu^2 + \omega^2 = -\sqrt{4\nu^2\omega^2 - g^2}$:

$$-g^2 = (\nu^2 - \omega^2)^2, \tag{15}$$

so $-g^2$ is positive. The solution in (15) is *spurious* because both terms in (14) are positive.

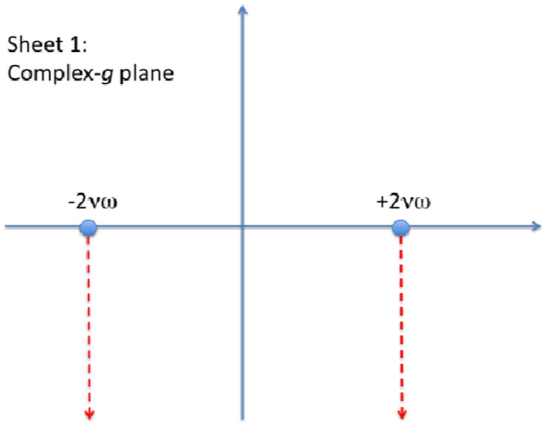

Figure 1. Sheet 1 of the complex Riemann surface of $E(g)$ in (12). On this sheet both the inner and outer square roots are positive when their arguments are positive. Branch points are indicated by blue dots and branch cuts by red dashed lines. On this sheet $E(0) = \nu + \omega$.

If we analytically continue $E(g)$ through either of the branch cuts on Sheet 1, we arrive on Sheet 2, where the inner square root changes sign. Therefore, on this sheet

$$E(0) = \nu - \omega, \tag{16}$$

assuming that $\nu > \omega$. Thus, the x oscillator is in its conventional ground state but the y oscillator is in its unconventional ground state. Because the inner square root returns negative values when its argument is positive, the solution for $-g^2$ in (15) is *not* spurious. Therefore, there are new branch cuts associated with the sign change of the outer square root; these branch cuts emanate from branch points located at

$$g = \pm\mathrm{i}(\nu^2 - \omega^2). \tag{17}$$

All four branch cuts on Sheet 2 are shown on figure 2. If we now pass through a branch cut emanating from $\pm 2\nu\omega$, we return to Sheet 1 but if we pass through a branch cut emanating from either branch point in (17), we enter Sheet 3.

On Sheet 3 there are two pairs of square-root branch cuts. The branch points on the imaginary axis coincide with those on Sheet 2. However, there is a new pair of branch points on the real axis at $g = \pm 2\nu\omega$. Although these branch points appear at the same locations as on Sheets 1 and 2, they are unrelated to those branch points. We show this explicitly in figure 3 by drawing the associated branch cuts differently. On this sheet both the inner and outer square-root functions in (12) are negative and

$$E(0) = -\nu + \omega \tag{18}$$

when $\nu - \omega$ is positive. Now the x oscillator is in an unconventional ground state and the y oscillator is in a conventional ground state.

If we now pass through a branch cut emanating from (17), we return from Sheets 3 to 2. However, if pass through a branch cut emanating from $\pm 2\nu\omega$, we enter Sheet 4. On this sheet there are only two branch points, which are located at

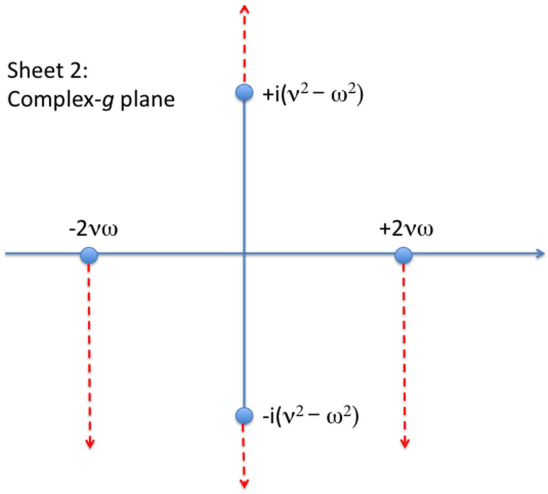

Figure 2. Sheet 2 of the complex Riemann surface of $E(g)$ in (12). On this sheet the inner square root in (12) is negative and the outer square root is positive when their arguments are positive. On this sheet $E(0)$ is $\nu - \omega$ (assuming that $\nu - \omega$ is positive).

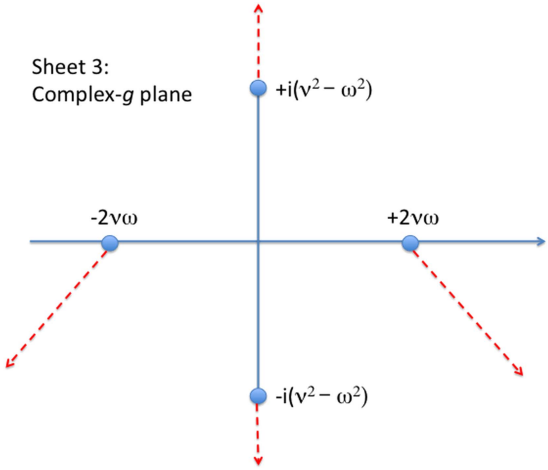

Figure 3. Sheet 3 of the complex Riemann surface of $E(g)$ in (12). On this sheet both the inner and outer square roots are negative when their arguments are positive and thus $E(0) = -\nu + \omega$.

$\pm 2\nu\omega$ (see figure 4). On Sheet 4

$$E(0) = -\nu - \omega. \tag{19}$$

Both oscillators are now in unconventional ground states.

To summarize, figures 1–4 describe each of the four branches of the function $E(g)$ in (12). On these four branches $E(0)$ takes the values given in (13), (16), (18), and (19). From these four values of $E(0)$ we infer that by analytically continuing the two-coupled-oscillator system in (1) through the entire Riemann surface we access both phases, conventional and unconventional, of both oscillators, even though the two frequency parameters ν and ω are held fixed.

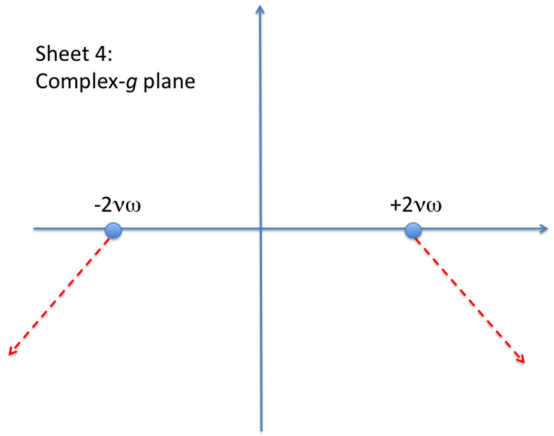

Figure 4. Sheet 4 of the complex Riemann surface of $E(g)$ in (12). On this sheet the inner square root is positive while the outer square root is negative when their arguments are positive. On this sheet $E(0) = -\nu - \omega$.

The four-fold structure of the ground-state energy is repeated for all of the energy levels. To verify this, we construct the eigenfunctions associated with the other energy levels of the theory. These eigenfunctions consist of the exponential in (7) multiplied by a polynomial $P(x, y)$. If $P(x, y)$ has the form

$$P(x, y) = Ax + By + Cxy + D, \tag{20}$$

the eigenvalue equation (8) leads to the three coupled equations (9) together with four alternatives for E:

$$ED = (a + b)D, \tag{21}$$

$$EA = A(3a + b) - 2Bc, \tag{22}$$

$$EB = B(a + 3b) - 2Ac, \tag{23}$$

$$EC = D(g + 2bc + 2ac) + 3C(a + b). \tag{24}$$

For the quartet of ground-state energy levels described above, $D = 1$, $A = B = C = 0$, so that $P(x, y) = 1$. We assign the label (0, 0) to this quartet because it reduces to the (conventional and unconventional) ground states of the x and y oscillators when $g = 0$ and $c = 0$. We use the designation (0, 1) for the quartet $P(x, y) = y$, (1, 0) for the quartet $P(x, y) = x$, and (1, 1) for the quartet $P(x, y) = xy$ that give rise to spectra in the decoupling limit $g = 0$, $c = 0$. In this limit, it follows again that $a^2 = \nu^2$ and $b^2 = \omega^2$, leading to four quartets with the additional three spectra arising from (22) for (1,0) when $B = C = D = 0$, (23) for (0,1) when $A = C = D = 0$ and (24) for (1,1) when $A = B = D = 0$. These four quartets are illustrated in figure 5 for the case $\nu = 2$ and $\omega = 1$. We emphasize that the energy levels of different quartets are *not* analytic continuations of one another but the elements of each quartet are analytic continuations of one another and branches of a four-valued function defined on exactly the same the Riemann surface pictured in figures 1–4.

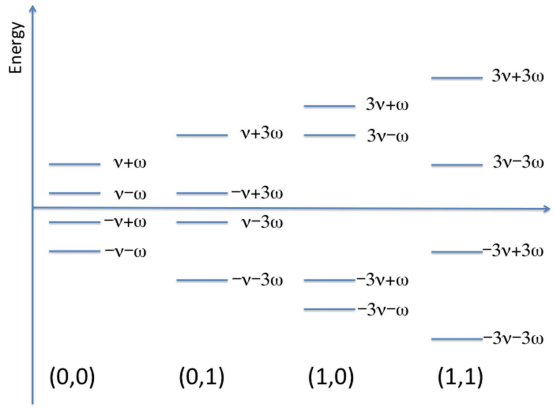

Figure 5. First four quartets of energy levels associated with the Hamiltonian (1). The quartets are labeled (m, n), and the quartets shown are for $m = 0, 1$ and $n = 0, 1$. We have chosen the values $\nu = 2$ and $\omega = 1$ and have plotted the values of $E(0)$ to scale. Note that each energy eigenvalue corresponds to the lowest such state on a different Riemann sheet.

3. Partition functions for zero-dimensional field theories

3.1. Interacting quadratic field theory

Let us examine the zero-dimensional field-theoretic equivalent of the Hamiltonian (1). The partition function for this field theory is given by the integral

$$Z(g) = \int\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{e}^{-\nu^2x^2-\omega^2y^2-gxy}, \tag{25}$$

where both integration paths run from $-\infty$ to ∞. We can evaluate the integral exactly by rearranging the terms in the exponential as we did in (2):

$$Z(g) = \int\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{e}^{-\nu^2[x+gy/(2\nu^2)]^2-y^2[\omega^2-g^2/(4\nu^2)]}. \tag{26}$$

Simple transformations then reduce this to a product of two gaussian integrals,

$$Z(g) = \int\int \frac{2\ \mathrm{d}x\ \mathrm{d}y}{\sqrt{4\nu^2\omega^2 - g^2}}\mathrm{e}^{-x^2-y^2}, \tag{27}$$

which evaluate to

$$Z(g) = \frac{2\pi}{\sqrt{4\nu^2\omega^2 - g^2}}. \tag{28}$$

This partition function is a *double*-valued function of g and is defined on a two-sheeted Riemann surface. Like the coupled harmonic oscillator discussed in section 2 the square-root singularities are located at $g = \pm 2\nu\omega$. However, unlike the case of the coupled harmonic oscillator, the Riemann surface has two sheets and not four; these sheets correspond to the two possible signs of $Z(g)$ and these two sheets correspond to the analogs of the conventional–conventional theory and the unconventional–unconventional theory. (To obtain the unconventional–unconventional theory from the conventional–conventional theory we replace x by ix and y by iy and this changes the sign of the partition function.) There is no analytic continuation to the partition function for a mixed unconventional–conventional theory. This is because the path of integration is included with the integral that defines the partition function. Given an eigenvalue differential equation we are free to choose the boundary conditions (we can require that the eigenfunctions vanish as $x \to \pm\infty$ or as $x \to \pm\mathrm{i}\infty$) but there is no such freedom in the case of an integral. To obtain other phases we would have to change the path of integration in the definition of the partition function.

We can generalize this calculation by including in the partition function external fields J and K coupled to the x and y fields:

$$Z(J, K; g) = \int\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{e}^{-\nu^2x^2-\omega^2y^2-gxy+Jx+Ky}.$$

Evaluating this integral by following the same procedure as above, we now find a more elaborate singularity structure,

$$Z(g) = \frac{2\pi}{\sqrt{4\nu^2\omega^2 - g^2}}\exp\left(\frac{J^2\omega^2 + K^2\nu^2 - gKJ}{4\omega^2\nu^2 - g^2}\right),$$

which is again defined on a two-sheeted Riemann surface but in addition has essential singularities at the square-root branch points. Consequently, all of the Green's functions, which are obtained by taking derivatives with respect to the external sources, have increasingly stronger singularities at $g = \pm 2\nu\omega$.

3.2. Interacting sextic field theory

A higher-power selfinteracting field theory that possesses a conventional real spectrum and in addition possesses a real $\mathcal{PT}$-symmetric spectrum has a sextic interaction of the form ϕ^6. We thus examine a field theory that describes the coupling of two sextic oscillators and we choose a symmetric form for the coupling. The partition function for the zero-dimensional version of this coupled quantum field theory is

$$Z(g) = \int\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{e}^{-x^6-y^6-gx^3y^3}. \tag{29}$$

This sextic theory is more difficult to examine analytically. We begin by expanding the coupling term as a series in powers of g:

$$Z(g) = \sum_{n=0}^{\infty}\frac{(-g)^n}{n!}\int\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{e}^{-x^6-y^6}x^{3n}y^{3n}. \tag{30}$$

Since the x and y integrals run from $-\infty$ to ∞, only even values of n contribute to the partition function. When n is even, we have

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-x^6}x^{3n} = \frac{1}{3}\Gamma\left(\frac{n}{2} + \frac{1}{6}\right),$$

but if n is odd, the integral vanishes. Thus, we make the replacement $n = 2m$ and re-express the partition function as a sum over m:

$$Z(g) = \frac{1}{9}\sum_{m=0}^{\infty}\frac{g^{2m}}{(2m)!}\Gamma^2\left(m + \frac{1}{6}\right). \tag{31}$$

This sum is a hypergeometric series:

$$Z(g) = \frac{1}{9}\Gamma^2(1/6)\,{}_2F_1\left(\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; \frac{g^2}{4}\right). \tag{32}$$

In general, the hypergeometric series has a radius of convergence of 1. (This is easy to verify by using the Stirling approximation for the Gamma function.) This implies that $Z(g)$ has a singularity on the circle $|g| = 2$.

It is important to identify the precise location and nature of this singularity. To do so we use the linear transformation formula [16]

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &\times {}_2F_1(a, b; a+b-c+1; 1-z) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ &\times {}_2F_1(c-a, c-b; c-a-b+1; 1-z). \end{aligned}$$

This transformation makes the singularity explicit because the hypergeometric function is analytic in the unit circle. Applying this transformation gives

$$\begin{aligned} Z(g) = \frac{\sqrt{\pi}\,\Gamma^3(1/6)}{9\Gamma^2(1/3)}\,{}_2F_1\left(\frac{1}{6}, \frac{1}{6}; \frac{5}{6}; 1-\frac{g^2}{4}\right) \\ + \left(1-\frac{g^2}{4}\right)^{1/6}\frac{\sqrt{\pi}\,\Gamma(-1/6)}{9} \\ \times {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{7}{6}; 1-\frac{g^2}{4}\right), \end{aligned} \tag{33}$$

from which we conclude that $Z(g)$ is defined on a six-sheeted Riemann surface and that the branch points on all six sheets of the Riemann surface are located at $g = \pm 2$, which corresponds with the singularities of the coupled harmonic oscillator model at $\pm 2\nu\omega$ with $\nu = \omega = 1$.

More generally, we can examine the Green's functions $G_{\alpha,\beta}$ of the theory, which are defined as integrals of the form

$$G_{\alpha,\beta} \equiv \int\int \mathrm{d}x\ \mathrm{d}y\ x^\alpha y^\beta \mathrm{e}^{-x^6-y^6-gx^3y^3}, \tag{34}$$

where α and β are integers. It is necessary that $\alpha + \beta$ is even for the Green's function to be nonvanishing. Following the same analysis as above, we find that all Green's functions are defined on a six-sheeted Riemann surface and that the singularity in the complex-g plane has the form

$$\left(1-\frac{g^2}{4}\right)^{(1-\alpha-\beta)/6}.$$

Thus, like the Green's functions for the coupled harmonic oscillator, we see that the singularity becomes stronger with increasing α and β, but the Green's functions are always six-valued functions of g.

4. Conclusions

We have shown that a coupled quantum theory has a rich analytic structure as a function of the coupling constant. By analytically continuing in the coupling constant we can obtain different spectral phases of the *uncoupled* theory. Indeed, if we think of the coupling constant as an external classical source, then by varying this external source in a closed loop in the complex-coupling-constant plane we can even imagine extracting energy from the conventional ground state of such a theory, at least in principle. For example, we can begin with the uncoupled harmonic-oscillator system (1) in its conventional ground state (13). We then turn on the source g, smoothly and continuously vary g, and finally turn off g again when the system is in the unconventional ground state (19). Such a process appears to be exothermic because it extracts an amount of energy equal to $2\nu + 2\omega$. However, varying the coupling constant may require that we do work on the system. Until now, it is not clear what it means to vary a coupling constant through complex values. However, remarkable progress on this is currently being made from an experimental point of view. It is experimentally possible to vary the parameters of a system and by doing so to analytically continue from one energy state to another. Such a process has actually been achieved in the laboratory by smoothly varying the parameters of a microwave cavity [17] and, in doing so, going continuously from one frequency mode to another. More recently, experiments have been performed in which an exceptional point is *dynamically* encircled [18, 19]. That is, a combination of physical parameters is varied in real time, and the system response is measured, allowing one to access different Riemann surfaces. While [19] emphasizes robust switching, [18] concerns itself with energy transfer between different states of a system, such as has been considered here in our illustrative prototypical system. An experimental approach, whether optomechanical or using light, acoustic, matter waves, or microwaves may in the future yield experimental verification of the analytic continuation discussed in this work.

Finally, these studies have been performed for *linear* couplings between the oscillators, which led to the four-fold structure shown here. It is to be expected that other types of couplings lead to different, possibly more complicated Riemann surfaces.

Acknowledgments

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Chapter 7

Outlook

In the previous chapters, many non-Hermitian $\mathcal{PT}$-symmetric systems which we studied were complex. We can consider these systems as extension of real theories to the complex plane; This can be viewed as *one* of the applications of $\mathcal{PT}$ symmetry, that is, to consider complex deformation of the known theories which are described on the real axis: $\mathcal{PT}$ symmetry provides this freedom to alter the formalism by, for instance, including terms which are complex and non-Hermitian but $\mathcal{PT}$ symmetric; So, we could build more general physically acceptable complex theories, where the conventional Hermitian case is a limit of that. In this dissertation, we mainly examined the characteristics of the $\mathcal{PT}$-symmetric fermionic (and bosonic) systems but still it poses an open question how this formalism could be exploited to affect the known physical fermionic (and bosonic) theories. One such example is given in the following.

Complex Nambu-Jona-Lasinio model

One of the fermionic models studied in the literature is the Nambu-Jona-Lasinio model [8], where its Lagrangian density in $SU(2)$ reads

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma^5 \tau \psi)^2], \tag{7.1}$$

where G is the coupling strength and τ's are the Pauli matrices. This model enables one to study how the mechanism of chiral symmetry breaking functions within a theory of interacting fermions.

In the covariant regularization scheme the gap equation, which rises to the dynamically generated mass of the particle, takes the form [8]

$$2\pi^2/(N_c N_f G\Lambda^2) = 1 - (m^*/\Lambda)^2 \ln[1 + 1/(m^*/\Lambda)^2], \tag{7.2}$$

where Λ and m^* are the regulatory cut-off and the constant value of the self-energy, respectively.

In the conventional (real) Nambu-Jona-Lasinio gap equation, m^* is real. This demands that the left hand side of (7.2) should not be greater than one; In other words, $2\pi^2/(N_c N_f) = \pi^2/3 \leq G\Lambda^2$. Thus, the real solutions lie in the strong coupling regime. The usual choices for the regulatory cut-off as $\Lambda = 1015$ MeV and the coupling as $G\Lambda^2 = 3.93$ [8] satisfy this inequality. With these parameters we obtain for m^* the values ± 238.486 MeV, which give a good estimate for a constituent quark mass.

The objective here is to generalize the real gap equation to the complex plane through treating the coupling strength G as a variable and look for the solutions of the gap equation as a function

of that. Although, it is not yet feasible to vary the coupling strength of the interaction for fermions at the level of quarks, however, it still is a curious question to study the implications of such a procedure. This is the first step to understanding how to handle a complex gap equation.

If we assume G to be a parameter which can vary *arbitrarily*, we are forced to consider m^* in general to be a complex number and denote $(m^*/\Lambda)^2$ as w, where $w = u + iv$.
Thus, (7.2) can be written as

$$C = 1 - |w|e^{i\phi}[\ln \frac{|1+w|}{|w|} + i(\theta - \phi) + 2i\pi s], \tag{7.3}$$

where the left hand side of (7.2) is designated as C, $\tan\phi = v/u$, and $\tan\theta = v/(1+u)$. The value of s determines the Riemann sheets, i.e., $s = 0$: the first sheet, $s = -1$: the second sheet, and so on. We note that the branch cut has been chosen on the u-axis which connects -1 and 0. Also, ϕ and θ vary between 0 and 2π.
As it is shown in Table 7.1 and illustrated in Figure 7.1, decreasing the coupling strength, G, results in a decrease in the dynamically generated mass on the first sheet, then, after reaching a critical point, the mass becomes complex on the second sheet.

$G\Lambda^2$	C	u_r	v_r	$m^* = m - i\gamma$
$\pi^2/3$	1.00	0.0000	0.0000	$0.0000 - i0.0000$
2.84	1.16	−0.0195	−0.0224	$72.4807 - i159.1940$
2.19	1.50	−0.0471	−0.0916	$169.6900 - i278.0620$
1.64	2.01	−0.0654	−0.1970	$270.6190 - i374.9820$
1.32	2.49	−0.0719	−0.2946	$345.2820 - i439.6510$
1.10	2.99	−0.0736	−0.3904	$408.3260 - i492.4980$
0.94	3.50	−0.0728	−0.4846	$463.5990 - i538.4470$
0.82	4.01	−0.0709	−0.5766	$512.5710 - i579.4590$
0.73	4.51	−0.0684	−0.6636	$555.3430 - i615.5270$
0.66	4.98	−0.0658	−0.7462	$593.2740 - i647.8910$

Table 7.1: Quark masses, m^*'s, in the weak coupling region, i.e., $1 \leq C$, for different C's in (7.3) on the second sheet of the Riemann surface: $s = -1$. Λ has been set to 1015 MeV.

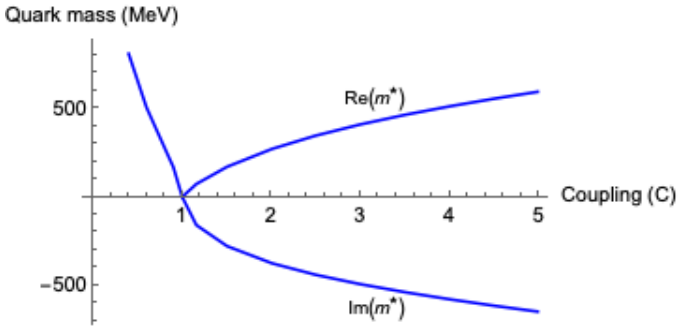

Figure 7.1: Quark mass versus $C = \pi^2/(3G\Lambda^2) \geq 1$ on the second sheet of the Riemann surface. For comparison the normal solution for $C < 1$ on the first sheet is also given. The mass scale has been set using $\Lambda = 1015$ MeV.

In order to investigate the stability of solutions against the choice of the regularization scheme, we study the gap equation this time in the Pauli-Villars regularization scheme.
In this regularization scheme the gap equation takes the form

$$2\pi^2/(N_c N_f G\Lambda^2) = [2 + (m^*/\Lambda)^2]\ln[1 + 2/(m^*/\Lambda)^2] - 2[1 + (m^*/\Lambda)^2]\ln[1 + 1/(m^*/\Lambda)^2]. \tag{7.4}$$

The right hand side of (7.4) has a global maximum at $2\ln 2$, so in order to have a real mass, the left hand side should not be greater than this maximum, that is, $\pi^2/(6\ln 2) \leq G\Lambda^2$, in other words, if we assume that we are in the strong coupling regime. The usual parameters' choices for the regulatory cut-off and the coupling strength as $\Lambda = 859$ MeV and $G\Lambda^2 = 2.84$ satisfy this inequality, so we find a real mass which it reads $m^* = 240.334$ MeV.
As done before, we consider the quark mass in general to be a complex number and we denote $(m^*/\Lambda)^2$ as $w = u + iv$. Thus, we write (7.4) as

$$\begin{aligned} C = &|2+w|e^{i\xi}[\ln|2+w| + i(\xi + 2\pi q)] \\ &- 2|1+w|e^{i\theta}[\ln|1+w| + i(\theta + 2\pi m)] \\ &+ |w|e^{i\phi}[\ln|w| + i(\phi + 2\pi n)], \end{aligned} \tag{7.5}$$

where C, θ, and ϕ are as before. Also $0 < \xi < 2\pi$ which $\tan\xi = v/(2+u)$. The values of q, m, and n determine the Riemann sheets, i.e., $q = m = n = 0$: the first sheet, $q = m = -1$ and $n = 0$: the second sheet, and so on. The branch cut connects -2 and 0 on the u-axis.
It can be seen in Table 7.2 and Figure 7.2 that the masses have the same behavior both in the Pauli-Villars and covariant regularization schemes.

$G\Lambda^2$	C	u_r	v_r	$m^* = m - i\gamma$
2.37	1.39	0.0000	0.0000	$0.0000 - i0.0000$
2.05	1.60	−0.0269	−0.0311	$72.4305 - i158.4150$
1.58	2.08	−0.0648	−0.1273	$169.6860 - i276.7820$
1.19	2.76	−0.0890	−0.2698	$268.2920 - i371.0150$
0.95	3.46	−0.0981	−0.4094	$345.1480 - i437.6220$
0.79	4.16	−0.1004	−0.5434	$408.4530 - i490.8330$
0.68	4.84	−0.0995	−0.6676	$460.7770 - i534.5420$
0.59	5.58	−0.0969	−0.7999	$511.4290 - i577.1130$
0.53	6.21	−0.0941	−0.9107	$550.5140 - i610.3280$
0.47	7.00	−0.0901	−1.0475	$595.5260 - i648.9480$

Table 7.2: Quark masses, m^*'s, in the weak coupling region, i.e., $2\ln 2 \leq C$, for different C's in (7.5) on the second sheet of the Riemann surface: $q = m = -1$ and $n = 0$. The cut-off Λ has been set to 859 MeV.

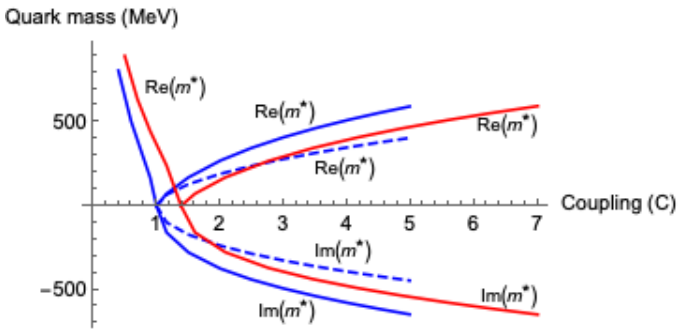

Figure 7.2: Quark mass versus C for both the covariant and Pauli-Villars regularization schemes. The blue curves correspond to the covariant regularization scheme and the red one to the Pauli-Villars. The solution on the third sheet in the case of the covariant regularization scheme, as a dashed curve, is also shown.

The mass of the pseudoscalar isovector meson associated with the Nambu-Jona-Lasinio Lagrangian is determined in [8]. We replace m^* by $m - i\gamma$ in Equations (4.17) and (4.18) of [8],

and find

$$(1/i)\Pi_{\rm ps}(k^2) = 4N_cN_f \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2-(m-i\gamma)^2} - 2N_cN_fk^2I(k^2), \tag{7.6}$$

where

$$I(k^2) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{[(p+k/2)^2-(m-i\gamma)^2][(p-k/2)^2-(m-i\gamma)^2]}. \tag{7.7}$$

By exploiting the gap equation, we find

$$1 - 2G\Pi_{\rm ps}(k^2) = 4iGN_cN_fk^2I(k^2). \tag{7.8}$$

When k^2 vanishes, the above has a real root corresponding to a real pseudoscalar mass. Thus, the Goldstone mode is not affected by the possible complex nature of the constituent quarks' masses.

The mass of the scalar isoscalar meson is given in [8]. As before, by the usage of the gap equation we arrive at

$$1 - 2G\Pi_s(k^2) = 4iGN_cN_f[k^2 - 4(m-i\gamma)^2]I(k^2). \tag{7.9}$$

Thus, the mass of the scalar meson is given as

$$m_\sigma = \pm 2(m - i\gamma). \tag{7.10}$$

Hence, if the constituent quarks have complex masses, the scalar meson mass gains a width.

Now that we have a sketch of complex Nambu-Jona-Lasinio model at our disposal, it is an interesting and open question how we could extend this model to incorporate complex non-Hermitian $\mathcal{PT}$-symmetric interaction terms and study the effects of this extension on the spectrum; In Chapter 4, we presented non-Hermitian $\mathcal{PT}$-symmetric fermionic interaction terms like $H_{\rm int} = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}$ and $H_{\rm int} = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu$. The effects of these terms on the gap equation and dynamically generated mass could lead to strange and interesting results.

Final comments

$\mathcal{PT}$ symmetry as a complex extension of conventional Hermitian quantum mechanics is still in its infancy. It is difficult to imagine what exciting new complex theories have yet to be discovered.

List of own publications

1. A. Beygi, S. P. Klevansky, and C. M. Bender, Phys. Rev. A **97**, 032128 (2018).

2. A. Beygi and S. P. Klevansky, Phys. Rev. A **98**, 022105 (2018).

3. A. Beygi, S. P. Klevansky, and C. M. Bender, arXiv:1904.00878.

4. A. Beygi, S. P. Klevansky, and C. M. Bender, Phys. Rev. A **91**, 062101 (2015).

5. C. M. Bender, A. Felski, N. Hassanpour, S. P. Klevansky, and A. Beygi, Phys. Scr. **92**, 015201 (2017).

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