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Analysis and Numerical Approximation of Dielectrophoretic Force Driven Flow Problems

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Abstract

Dielectric fluids under the influence of an external electric field and temperature variations experience a body force that is a combination of buoyancy and dielectrophoresis. The resulting motion can be described by the so-called Thermal-Electro-Hydrodynamic (TEHD) Boussinesq equations. In this thesis, a variational formulation for these equations is proposed, with an emphasis on the mathematical modeling of the acting body force. Within this framework, existence and stability of steady and unsteady solutions is shown. Moreover, uniqueness of steady solutions under certain conditions is proven.

As second part of this thesis, a numerical method is proposed for approximately solving the TEHD Boussinesq equations in the stationary and instationary case. The spatial discretization is based on the conforming Finite Element Method (FEM) and temporal discretization is realized by a variant of the Backward Differentiation Formula (BDF). In both cases, a priori error estimates are derived and validated by numerical experiments in a 2D test problem.

Finally, the 3D flow inside a vertical annulus with applied temperature and electric potential gradient is simulated. The formation of vortex structures is analyzed and the obtained results are compared with experimental data. By means of the corresponding adjoint problem, the sensitivity of solutions w.r.t. perturbations is investigated and it is numerically shown, that steady solutions are not unique in this scenario.

Zusammenfassung

Werden dielektrische Fluide einem elektrischen Feld und Temperaturunterschieden ausgesetzt, so wirken Auftriebskraft und dielektrophoretische Kraft. Die resultierende Strömung kann mithilfe der sog. Thermisch-Elektrisch-Hydrodynamischen (TEHD) Boussinesq Gleichungen beschrieben werden. In dieser Arbeit wird eine variationelle Formulierung für diese Gleichungen vorgeschlagen, wobei das Hauptaugenmerk auf der mathematischen Modellierung der vorherrschenden Volumenkraft liegt. Für diese Formulierung wird Existenz und Stailität von Lösungen nachgewiesen. Weiterhin wird die Eindeutigkeit von stationären Lösungen unter gewissen Bedingungen gezeigt.

Der zweite Teil dieser Arbeit besteht in der numerischen Approximation von Lösungen der stationären und instationären TEHD Boussinesq Gleichungen. Die räumliche Diskretisierung basiert hierbei auf der konformen Finiten Elemente Methode (FEM), während eine Variante der sog. Backward Differentiation Formula (BDF) zur zeitlichen Diskretisierung verwendet wird. Im stationären, wie instationären Fall werden a priori Fehlerabschätzungen für die vorgeschlagene Diskretisierung hergeleitet und mittels numerischer Experimente validiert.

Abschließend wird die 3D Strömung in einem zylindrischen Spalt simuliert, wobei eine Potential- und Temperaturdifferenz zwischen innerem und äußerem Zylinder angenommen wird. Die Entstehung von Wirbelstrukturen wird untersucht und die Simulationsergebnisse werden mit experimentellen Daten verglichen. Mithilfe des zug. adjungierten Problems wird die Sensibilität der numerischen Lösung bzgl. Störungen analysiert. Außerdem werden numerische Lösungen konstruiert welche zeigen, dass in diesem Szenario die Eindeutigkeit von stationären Lösungen nicht gegeben ist.

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Contents

1. Introduction

In many technical devices, systems and facilities, fluids serve as medium for transferring thermal energy. Corresponding examples range from micro channel heat exchangers in combustion turbines, over domestic heating systems to cooling circuits in nucular power plants. The underlying principles are always the same: thermal energy is transferred by molecular interaction (conduction), by bulk motion of the fluid (convection), and by radiation, which occurs without interaction with the fluid.

This work addresses conduction and convection, with a strong emphasis on the later one. A simple scenario where these two mechanisms do occur is given by a vertical annulus, where a fluid is contained between two concentric cylinders of different temperature. By means of conduction, this temperature difference propagates through the fluid which becomes non-isothermal. As mass density is temperature dependent, the impact of gravity varies and the fluid experiences a spatially varying force. As result, a motion is induced in the initially resting fluid and heat transfer by convection arises. The acting force is called *buoyancy* and the overall process is denoted by *natural convection*, since gravity is the only external force being present. Natural convection can be observed in nature - as driving process behind atmospheric motions, for instance - and it is utilized to build heat exchanging systems without the need of an external power supply. However, the strength of buoyancy cannot be controlled and in many situations, the amount of transferred heat is not sufficiently high. This is apparent for space devices in Earth's orbit which do not experience any gravity at all. From a practical point of view, it is thus desirable to have another mechanism on hand which affects the fluid motion in a comparable way as buoyancy. Such an additional force can be obtained, if the fluid's molecules are polar or polarizable and if an external electric field is applied. In this situation, *dielectrophoresis* occurs and the fluid experiences an additional body force.

There has been growing interest of physicists and engineers in the investigation of these *dielectrophoretic force driven flows*. Physical experiments have been conducted for capacitors filled with dielectric liquids and in zero-gravity environments provided by parabolic flights and the *International Space Station (ISS)*. Moreover, stability analysis and simulations have been performed to gain more insights by theoretical means.

The underlying mathematical model is based on the *Navier-Stokes equations* for describing the motion of a non-isothermal fluid. These equations are enhanced by the dielectrophoretic (DEP) force and by Gauss' law for describing the internal electric field and they are simplified by using the *Boussinesq approximation*. The resulting equations are denoted by *Thermal-Electro-Hydrodynamic (TEHD) Boussinesq equations* and are given by [58]:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \alpha_e \left| \nabla \Phi \right|^2 \nabla \theta - \alpha_g \theta \mathbf{g} \\ \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \theta + (\mathbf{u} \cdot \nabla)\theta - \kappa \Delta \theta &= 0 \\ -\nabla \cdot (\epsilon(\theta) \nabla \Phi) &= 0. \end{aligned} \tag{1.1}$$

To the best of the author's knowledge, very few research has been done yet to investigate existence, stability and uniqueness for solutions of (1.1), and to analyze any discretization method for approximating solutions of (1.1) numerically. Therefore, it is the goal of this thesis to derive statements on well-posedness of the TEHD Boussinesq equations, to propose a discretization scheme and to derive a priori error estimates for this discretization.

1.1. Literature Overview

Dielectrophoretic force driven flows have been investigated by means of physical experiments in various works. [17] is one of the first of them. The authors conduct experiments with a silicone oil which is contained in the gap between two concentric cylinders of different temperature and with applied alternating voltage difference. It is shown that heat transfer between the inner, hot cylinder and the outer cooling circuit is indeed enhanced by application of an electric field. In particular, the Nusselt number as measure for heat transfer enhancement, grows with increasing voltage difference. The proposed vertical annulus as experimental geometry is investigated by many other works. In order to observe DEP driven flow under absence of Earth's gravity, experiments have been conducted in parabolic flight campaigns which allow micro-gravity conditions for approximately 22 s, see [18], [51], [55]. In [18], tracer particles and laser illumination is used to visualize fluid motion on an axial cut plane. Azimuthally aligned vortices are observed whose formation highly depends on the applied voltage and the curvature of the annulus. High curvature destabilizes the flow, i.e. the critical voltage, above which convection cells do arise, is decreased. Shadowgraph imaging and particle tracking as combined measurement technique is used in [51], [54], [55]. It is observed that the fluid can undergo a helical motion under micro-gravity conditions. In general, pure conductive, axisymmetric and non-axisymmetric states are observed; depending on the applied voltage. A detailed overview on micro-gravity experiments for DEP force driven flow is given by [58]. Here, experimental results for planar, cylindrical and spherical geometry are reported for parabolic flights and long-term micro-gravity conditions on the ISS. Experiments for the cylindrical geometry under the combined effect of gravity and DEP force are presented in [27] and [68]. There, particle tracking and shadowgraph imaging is used to characterize the flow field on axial cut planes and to visualize an axially averaged temperature distribution. The formation of axially oriented columnar flow structures and helicoidal flow structures is reported. Moreover, it is shown that heat transfer under natural convection is enhanced by applying an additional electric field.

Besides experiments, linear stability analysis has been an important method for investigating DEP force driven flows, see [48], [52], [74], [73], [76] and [81]. Most of these works consider the vertical annulus scenario of infinite length. In this case, an analytic solution of the stationary TEHD Boussinesq equations is known. Without gravity, this base solution exhibits no motion, i.e. $\mathbf{u} = 0$, see [81]. When gravity is axially aligned, velocity has a non-zero axial component, see [52]. By using cylindrical coordinates and exploiting symmetries, the governing equations (1.1) can be simplified such that all occurring quantities do only depend on the radial coordinate. Starting from this system, perturbations of the base flow are constructed in terms of axial and azimuthal wavenumbers. Then, stability of these perturbations is analyzed by computing the eigenvalues of the resulting systems. It is stated in [48] and [81], that the critical voltage difference highly depends on the annulus' curvature and the critical perturbation modes are stationary helices.

In addition, there are a few works on direct numerical simulation of system (1.1). Most of the proposed methods are applied on the vertical annulus geometry in order to compare the simulation results with available experimental data. In [78] and [79], the vertical annulus with periodic boundary conditions on top and bottom is considered. The so-called *gauge transform* method is used, where velocity is split according to $\mathbf{u} = \mathbf{a} - \nabla\psi$. This allows to derive decoupled equations for $\mathbf{a}$, ψ and p. Spatial discretization is done by a multiplicative ansatz w.r.t. the individual cylinder coordinates: Chebyshev polynomials are used for discretization in the radial direction and Fourier modes are employed for the periodic axial and azimuthal direction. In this way, the *Fast Fourier Transform* can be used for solving the arising algebraic equations. [53] and [70] employ a (not further specified) finite element method based on the software *Comsol Multiphysics*. In [42], [43], [44] and [82] the use of *Finite Volumes* is proposed for spatial discretization. The corresponding temporal discretizations are based on a fractional step method and Crank-Nicolson, respectively. In addition, a turbulence model is used for approximating subgrid scales in [82].

In contrast to the TEHD Boussinesq equations, well-posedness of the standard Boussinesq equations has been shown quite some time ago. In [56] and [57] results on existence and uniqueness of solutions are proven for the stationary and instationary Boussinesq equations with homogenous Dirichlet boundary conditions for velocity. Both works extend the corresponding results on the incompressible Navier-Stokes equations, see e.g. [77]. In these works, existence is shown by considering the problem in a finite-dimensional setting, which allows the use of a fixed-point theorem (stationary case) and existence results for systems of ordinary differential equations (instationary case). Solutions in the original function spaces are then obtained by a limit process. A crucial assumption in [56] and [57] concerns the Dirichlet boundary conditions for temperature. It is required that corresponding boundary liftings of arbitrarily small L^3-norm can be constructed in order to show that solutions of the Boussinesq equations are stable. Uniqueness is shown by similar arguments as for the incompressible Navier-Stokes problem. Thus, a small data condition is required in the stationary case and a temporal regularity has to supposed in the instationary case, which is higher than provided by the existence result. Further works on well-posedness consider the cases of temperature-dependent viscosity and thermal diffusion coefficients and non-homogenous or mixed velocity boundary conditions, see e.g. [4], [19], [33], [47] and [61].

Many works on numerical analysis of finite element discretization of the standard Boussinesq equations are extensions of the corresponding results on the incompressible Navier-Stokes equations. An overview on error estimates for incompressible flow problems is given in [32], [46], [41], for instance. Early works on finite element analysis for the standard Boussinesq equations are given by [10] and [11]. Later on, these results have been extended to the case of temperature dependent viscosity and thermal diffusion, e.g. [3], [59], [60], [62] and [75]. The underlying principles in the stationary case are typically very similar: first a best-approximation result is derived by using Galerkin orthogonality, comparable to Cea's lemma. Actual convergence rates are then obtained by using the approximation properties of interpolation and projection operators w.r.t. the given finite element spaces. A different principle is used in [5], where an approach based on the implicit function theorem is proposed. In [59], the use of $\mathbf{H}(div)$-conforming Brezzi-Douglas-Marini elements is proposed, which allows to obtain numerical velocities that are exactly divergence free. An approach based on the Discontinuous Galerkin method is proposed in [67].

The works [3] and [75] on the unsteady equations make use of the Rothe method for combining spatial and temporal discretization. Here, the error is measured at the temporal discretization points. Estimates are derived by using Galerkin orthogonality and approximation properties of the underlying finite element spaces for the spatial error contributions, and by Taylor expansion for the temporal error contributions. The application of Gronwall's inequality is crucial in the presented proofs.

1.2. Contributions of this Thesis

In this thesis, variational formulations of the stationary and instationary TEHD Boussinesq equations are proposed, which allow to derive statements on well-posedness. These formulations require a special treatment of the DEP force term $\mathbf{F}_{DEP} = |\nabla\Phi|^2\nabla\theta$, because it is a product of three gradients. For the typical regularity of solutions θ and Φ of heat equation and Gauss' law, respectively, one cannot assume that $\mathbf{F}_{DEP}$ is sufficiently regular to be an element of $\mathbf{H}^{-1}(\Omega)$. Thus, several modelizations of $\mathbf{F}_{DEP}$ are proposed that lead to well-defined problems. These models are either based on linearization, on the use of regularization operators or on cut-off functions.

Known results on the existence and stability of solutions of the stationary Boussinesq equations and instationary Navier-Stokes equations are then extended to the proposed problem formulations. In the stationary case, a uniqueness result under a small data condition is derived.

Besides analysis, discretization schemes are proposed for solving the TEHD Boussinesq equations numerically. The spatial discretization is based on the *Finite Element Method (FEM)* and for temporal discretization, a variant of the *Backward Differentiation Formula (BDF)* is employed. In both, stationary and instationary case, a priori error estimates are derived. In the former case, these estimates hold under a certain small data condition. The derived convergence rates are validated by numerical experiments.

Finally, the often considered scenario of a dielectric fluid confined in the gap between to concentric cylinders is numerically investigated. Here, emphasis is put on the formation of vortex structures and sensitivity of the solutions w.r.t. perturbations. To this end, the adjoint problem of (1.1) is derived and approximately solved. By numerical means it is shown, that solutions of the stationary problem are not unique. Moreover, different time stepping schemes and DEP formulations are compared in the framework of this 3D flow problem.

1.3. Outline

In Section 2, a brief description of the TEHD Boussinesq equations (1.1) is presented and the underlying assumptions are stated. This is followed by the derivation of the associated adjoint problem. A variational formulation of the stationary version of (1.1) is given in Section 3 and well-posedness of this problem is shown. In Section 4, a variational formulation of the instationary TEHD Boussinesq equations is given. Afterward, existence and stability of solutions is proven. Section 5 is concerned with the numerical solution of the previously introduced variational problems. First, a conforming finite element discretization for the stationary problem is proposed and a best-approximation result is derived. This result allows to state a priori convergence rates for the spatial error. In the second part of this section, a full discretization in space, based on FEM, and in time, based on a BDF scheme, is proposed. Then, a priori convergence rates are derived. Each of the Sections 3, 4 and 5 is concluded by considering the modelization of DEP force in the respective context. Section 6 on numerical experiments is split into two main parts. First, a 2D benchmark problem is considered to validate the previously derived convergence rates. Then, results for DEP driven flow in a vertical annulus are presented with emphasis on formation of vortices and adjoint sensitivity. This section is concluded with a comparison of experimental data and simulation results. The thesis is finalized by a short summary and outlook in Section 7.

2. Modelization of Dielectrophoretic Force Driven Flow

In this section, we give a brief overview on the underlying physical model which is basically an extension of the well-known Boussinesq equations for modeling natural convection flow. The physical scenario we have in mind is given by a dielectric fluid inside a closed, vertical annulus. Its inner and outer wall are kept on fixed temperatures and an electric voltage in the range of several kilovolts is applied between both walls, see Figure 1. This setup could be considered as an idealized heat exchanging system. It turns out, that the superposition of gravity and DEP force induces vertically aligned vortex structures which lead to an enhanced heat transfer between inner and outer wall, see Section 6.3.

In the second part of this section, the *adjoint* TEHD Boussinesq equations are derived, in order to investigate local sensitivity of solutions of system (2.28).

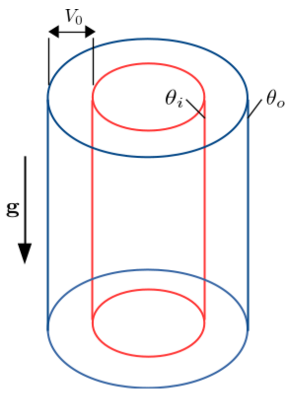


Figure 1: Schematic view on an idealized heat exchanging system. A vertical annulus is filled with a dielectric fluid and fixed temperatures θ_i and θ_o are applied on the inner and outer wall. Between both walls, an external electric voltage V_0 is applied. The boundaries on top and bottom of the annulus are assumed to be thermally and electrically insulating and $\mathbf{g}$ denotes gravity.

2.1. TEHD Boussinesq Equations

The state of a single-phase, non-isothermal fluid inside a closed container can be described by its velocity $\mathbf{u}(t, \mathbf{x})$, pressure $p(t, \mathbf{x})$, density $\rho(t, \mathbf{x})$ and temperature $\theta(t, \mathbf{x})$ at time $t \in [0, T]$ and at point $\mathbf{x} \in \Omega$. Here, $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ describes the fixed geometry of the container. The resulting governing equations are derived from the basic physical principles of mass, momentum and energy conservation and are given by the *compressible Navier-Stokes equations* for a *Newtonian* fluid in *non-conservative form*, see e.g. [63]:

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho(\nabla \cdot \mathbf{u}) \tag{2.1}$$

$$\rho\left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \mathbf{f} + \lambda \Delta \mathbf{u} + (\lambda + \mu) \nabla \left(\nabla \cdot \mathbf{u}\right) - \nabla p \tag{2.2}$$

$$\begin{aligned} \rho\left(\partial_t \theta + \mathbf{u} \cdot \nabla \theta\right) = {} & \lambda \frac{1}{4}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) : \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \\ & + \mu \left(\nabla \cdot \mathbf{u}\right)^2 - p\left(\nabla \cdot \mathbf{u}\right) + \alpha \Delta \theta + \rho Q, \end{aligned} \tag{2.3}$$

with dynamic viscosities λ, μ and thermal conduction coefficient α which themselves are temperature dependent.

Here, the respective left-hand sides denote the material derivatives of ρ, $\mathbf{u}$ and θ w.r.t. the convection field $\mathbf{u}$. The right-hand side of (2.2) collects the sum of all *internal* and *external* forces that act on the fluid. The external forces, also called body forces, are given by $\mathbf{f}$. If the container is placed in a gravitational field $\mathbf{g}(t, \mathbf{x})$, then the *buoyancy* force $\rho\mathbf{g}$ contributes to $\mathbf{f}$. The internal forces are given by the *viscous* forces which arise due to internal friction between the fluid's molecules and by spatial pressure differences within the fluid. The right-hand side of (2.3) consists of thermal conduction, $\alpha\Delta\theta$, external sources of heat, ρQ, and internal sources due to deformations of the fluid.

Since *continuity equation* (2.1), *momentum equation* (2.2) and *energy equation* (2.3) form an under-determined system for the unknowns $\mathbf{u}$, p, ρ, θ, they have to be accompanied with a state equation that defines the relationship between density, pressure and temperature. Often, the *ideal gas law* is assumed, which is given by

$$c_v\theta = \frac{p}{\rho(1-\beta)} \tag{2.4}$$

with *adiabatic exponent* β and *specific heat capacity* c_v at constant volume.

If the considered fluid is *dielectric*, then its molecules are polar or can be polarized by an applied outer electric field due to *induced polarization* or *orientation polarization* [82]. In case of induced polarization, the electric field causes a displacement between the negative electrons and the positive nuclei, such that the molecules become electric dipoles. Orientation polarization takes place if the molecules are permanent dipoles, which then get aligned along the external electric field. In any case, an internal electric field $\mathbf{E}(t, \mathbf{x})$ is induced which is determined by *Gauss's law*,

$$-\nabla \cdot (\epsilon\mathbf{E}) = \rho_E \tag{2.5}$$

where $\rho_E(t, \mathbf{x})$ denotes the free charge density. Moreover, $\epsilon = \epsilon_0\epsilon_r$ with *vacuum permittivity* ϵ_0 and *relative permittivity* ϵ_r. The later quantity becomes dependent on t and $\mathbf{x}$ through its temperature- and density-dependence, i.e. $\epsilon_r = \epsilon_r(\rho, \theta)$.

In absence of a magnetic field, the electric field can be expressed in terms of a scalar potential $\Phi(t, \mathbf{x})$,

$$\mathbf{E} = -\nabla\Phi. \tag{2.6}$$

In case of a non-zero electric field and free electric charges, additional electric forces arise that act on the fluid. Their densities are given by, see e.g. [50]:

$$\begin{aligned} \mathbf{f}_E &= \mathbf{f}_C + \mathbf{f}_{DEP} + \mathbf{f}_{ES} \\ &= \rho_E\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2\nabla\epsilon + \frac{1}{2}\nabla\left[\rho\left(\frac{\partial\epsilon}{\partial\rho}\right)_\theta |\mathbf{E}|^2\right], \end{aligned} \tag{2.7}$$

with *Coulombic* force density $\mathbf{f}_C$, *dielectrophoretic* force density $\mathbf{f}_{DEP}$, *electrostrictive* force density $\mathbf{f}_{ES}$. The Coulombic force acts on free charges within the fluid. The DEP force occurs in the presence of non-uniform electric fields. Then, induced or permanent dipoles align along this field and experience a resulting force that is non-zero since the electric field has different strength at the individual poles' position. Eletrostriction causes dielectrics to change their shape but has no effect on the velocity field when the fluid is inside a closed container. For details see [58] and the references therein.

The density of the resulting body force acting on the fluid is then given by

$$\mathbf{f} = \rho\mathbf{g} + \mathbf{f}_C + \mathbf{f}_{DEP} + \mathbf{f}_{ES}. \tag{2.8}$$

Moreover, orientation polarization induces *dielectric heating* if the applied outer electric field is alternating with some given frequency f. In this case, the term

$$Q_{DH} = \frac{2\pi f\epsilon_0}{\rho c_p} c_\epsilon |\mathbf{E}|^2 \tag{2.9}$$

enters the source term Q in the energy equation (2.3). Here, c_p denotes the specific heat capacity at constant pressure and c_ϵ denotes a permittivity-dependent factor, see e.g. [82] for details.

The unknowns $\mathbf{u}$, θ and Φ are further subjected to boundary conditions on $\Gamma = \partial\Omega$. As we consider a closed container with solid walls, we suppose that the fluid satisfies the *no-slip* condition on the entire boundary,

$$\mathbf{u} = 0 \text{ on } \Gamma. \tag{2.10}$$

Since some parts of the container's wall are kept on some predefined temperature or electric potential, Dirichlet boundary conditions of the form

$$\theta = \theta_D \text{ and } \Phi = \Phi_D \text{ on } \Gamma_D \subset \Gamma \tag{2.11}$$

are imposed. On the remaining part of the boundary, a certain amount of temperature or potential may penetrate through the wall. For simplicity, we assume that that container walls are thermally and electrically insulated, leading to the Neumann boundary conditions

$$\nabla\theta \cdot \mathbf{n} = 0 \text{ and } \nabla\Phi \cdot \mathbf{n} = 0 \text{ on } \Gamma_N = \Gamma \setminus \Gamma_D. \tag{2.12}$$

In practice, one often uses an alternating electric potential $V(t) = \sqrt{2}V_0 \sin(2\pi f t)$ of frequency f which is applied between two different parts of the boundary which are separated by an electrically insulating part. In this case, the Dirichlet boundary conditions for Φ are typically of the form

$$\Phi = V_0 \text{ on } \Gamma_{D,1} \text{ and } \Phi = 0 \text{ on } \Gamma_{D,2} \tag{2.13}$$

with $\Gamma_{D,1} + \Gamma_{D,2} = \Gamma_D$. Here, V_0 denotes the effective value of the electric potential, i.e. $V_0 = \sqrt{\langle V^2(t)\rangle}$ [81].

The full system of governing equations is now given by (2.1) - (2.9) with the boundary conditions (2.10) - (2.13).

At this point, several simplifications can be applied. First, we assume that the fluid is incompressible, meaning that

$$\nabla \cdot \mathbf{u} = 0 \tag{2.14}$$

holds everywhere on Ω. In this way, the third term in (2.2), as well as the second and third term in (2.3) cancel out. The next simplification involves the Coulombic force. According to [81], the free charge density ρ_E is negligible at sufficiently high frequency of the alternating outer electric field, i.e. if $f \gg \tau_E^{-1}$. Here, τ_E denotes the *charge relaxation time* which describes the time scale on which charge accumulation may occur in an initially electrically neutral fluid [81]. According to [68], $f \gg \tau_E^{-1}$ is satisfied for typical dielectric fluids if $f \geq 60$ Hz. In this case, $\mathbf{f}_C \approx 0$. We neglect dielectric heating, i.e. we consider fluids without permanent dipole moment, and heating due to friction, i.e. the first term in (2.3).

Finally, the well-known *Boussinesq Approximation* is applied, see e.g. [71]. To this end, the density is written as

$$\rho = \rho_r + \delta\rho \tag{2.15}$$

with fixed reference density $\rho_r \in (0, \infty)$ and density variation $\delta\rho(t, \mathbf{x})$. According to (2.4), there holds $\rho \propto \theta^{-1}$ for some fixed reference pressure $p = p_r$. If the temperature θ takes values in a small interval $[\theta_-, \theta_+]$, $\theta_+ - \theta_- \lessapprox 10$, then the density variation is typically small compared to the reference density, i.e. $\delta\rho \ll \rho_r$. Therefore, ρ on the left-hand side of (2.2) and (2.3) is approximated by ρ_r. Additionally, the continuity equation (2.1) is replaced by (2.14) since the left-hand side term $\partial_t\delta\rho + \mathbf{u} \cdot \nabla\delta\rho$ is assumed to be small compared to the right-hand side $\rho(\nabla \cdot \mathbf{u}) \approx \rho_r(\nabla \cdot \mathbf{u})$.

Plugging formulation (2.15) into the buoyancy force and assuming that gravitation is constant in space and time, yields for the pressure and bouyancy force in (2.2) and (2.8):

$$-\nabla p+\rho\mathbf{g}=-\nabla p+\rho_r\mathbf{g}+\delta\rho\mathbf{g}=-\nabla\tilde{p}+\delta\rho\mathbf{g}. \tag{2.16}$$

In (2.16), a generalized pressure $\tilde{p}:=p-\rho_r\tilde{\mathbf{g}}$ is introduced with $\tilde{\mathbf{g}}(\mathbf{x})_i:=\mathbf{g}_i\mathbf{x}_i$. A first order Taylor expansion applied to (2.4) yields

$$\rho(\theta,p)\approx\rho(\theta,p_r)\approx\rho(\theta_r,p_r)(1-\alpha_g(\theta-\theta_r))=:\rho_r(1-\alpha_g(\theta-\theta_r)) \tag{2.17}$$

for some reference temperature $\theta_r\in[\theta_-,\theta_+]$, *thermal expansion coefficient* $\alpha_g>0$ and reference density ρ_r. We thus replace $\delta\rho$ by $-\rho_r\alpha_g(\theta-\theta_r)$, resulting in an approximate buoyancy force density

$$\mathbf{f}_{buo}=-\rho_r\alpha_g(\theta-\theta_r)\mathbf{g}. \tag{2.18}$$

In an analogous way, relative permittivity is linearized around the reference temperature and density w.r.t. temperature, i.e.

$$\epsilon_r(\rho,\theta)\approx\epsilon_r(\rho_r,\theta)\approx\epsilon_r(\rho_r,\theta_r)\left(1-\gamma(\theta-\theta_r)\right)=:\epsilon_r(1-\gamma(\theta-\theta_r)), \tag{2.19}$$

with $\gamma=-\epsilon_r^{-1}\partial_\theta\epsilon(\theta,p_r)>0$. According to [18] and the references therein, there holds

$$\gamma=\alpha_g\frac{(\epsilon_r-1)(\epsilon_r+2)}{3\epsilon_r}. \tag{2.20}$$

In this way, the dielectrophoretic force density can be written as

$$\mathbf{f}_{DEP}=-\frac{1}{2}|\mathbf{E}|^2\nabla\epsilon\approx\frac{\epsilon_0\epsilon_r\gamma}{2}|\nabla\Phi|^2\nabla\theta. \tag{2.21}$$

Since the electrostrictive force is a gradient field, it can be hidden inside the new generalized pressure

$$\hat{p}:=\tilde{p}-\frac{1}{2}\rho\left(\frac{\partial\epsilon}{\partial\rho}\right)_\theta|\mathbf{E}|^2=p-\rho_r\tilde{\mathbf{g}}-\frac{1}{2}\rho\left(\frac{\partial\epsilon}{\partial\rho}\right)_\theta|\mathbf{E}|^2. \tag{2.22}$$

In this way, $\mathbf{f}_{DEP}$ is the only part of $\mathbf{f}_E$, that actually has an impact on the velocity field. The resulting force density is then given by

$$\mathbf{f}=\frac{\epsilon_0\epsilon_r\gamma}{2}\left|\nabla\Phi\right|^2\nabla\theta-\rho_r\alpha_g(\theta-\theta_r)\mathbf{g}. \tag{2.23}$$

In the literature, an alternative formulation of (2.21) is commonly used to stress the analogy between buoyancy and DEP force. To this end, note that $\mathbf{f}_{DEP}$ can be replaced by

$$\frac{\epsilon_0\epsilon_r\gamma}{2}\left|\nabla\Phi\right|^2\nabla\theta=\nabla\left(\frac{\epsilon_0\epsilon_r\gamma}{2}\left|\nabla\Phi\right|^2(\theta-\theta_r)\right)-\epsilon_0\epsilon_r\gamma\left(\nabla^2\Phi\nabla\Phi\right)(\theta-\theta_r), \tag{2.24}$$

where the first term in (2.24) can be hidden in another generalized pressure

$$P:=\hat{p}-\frac{\epsilon_0\epsilon_r\gamma}{2}|\nabla\Phi|^2(\theta-\theta_r). \tag{2.25}$$

Eventually, an alternative force density is obtained according to

$$\mathbf{f}_a=-\rho_r\alpha_g(\theta-\theta_r)\left(\mathbf{g}+\mathbf{g}_E\right), \tag{2.26}$$

with *electric gravitiy*

$$\mathbf{g}_E=\frac{\epsilon_0\epsilon_r\gamma}{\rho_r\alpha_g}\nabla^2\Phi\nabla\Phi. \tag{2.27}$$

By (2.26) it becomes clear that the DEP force can be considered as buoyancy force for the artificial gravity $\mathbf{g}_E$.

We further neglect the temperature dependence of viscosity and thermal conduction coefficient. Then, dividing momentum and energy equation by ρ_r yields the final, simplified system, also known as TEHD Boussinesq equations [58]:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \rho_r^{-1} \nabla \hat{p} &= \alpha_e \left| \nabla \Phi \right|^2 \nabla \theta - \alpha_g (\theta - \theta_r) \mathbf{g} \\ \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \theta + (\mathbf{u} \cdot \nabla)\theta - \kappa \Delta \theta &= 0 \\ -\nabla \cdot (\epsilon_0 \epsilon_r (1 - \gamma(\theta - \theta_r)) \nabla \Phi) &= 0 \end{aligned} \tag{2.28}$$

with *kinematic viscosity*, *thermal diffusion coefficient* and *DEP coeeficient* given by

$$\nu := \frac{\mu(\theta_r)}{\rho_r}, \ \kappa := \frac{\alpha(\theta_r)}{\rho_r}, \ \alpha_e := \frac{\epsilon_0 \epsilon_r \gamma}{2\rho_r}. \tag{2.29}$$

For the remaining of this work, we simply write p instead of $\rho_r^{-1}\hat{p}$.

In summary, the underlying physical model is an extension of the well-known Boussinesq equations for natural convection, with additional force term $\mathbf{f}_{DEP}$ and additional variable Φ.

2.2. Adjoint TEHD Boussinesq Equations

In order to obtain a better understanding of the physical behavior of DEP driven flows, we consider the local sensitivity of a given flow state w.r.t. small perturbations. Therefore, the goal of this section is the derivation of the *adjoint* problem associated to system (2.28). The corresponding *adjoint* or *dual* solution can be interpreted as derivative of a given quantity of interest evaluated at the *primal* solution, i.e. the solution of the *primal* problem (2.28).

First, the dual problem is derived on an abstract level, following the presentation in [37] and [8]. Then, the concrete set of equations is plugged into the abstract framework.

Abstract Adjoint Problem

Let normed spaces W and L be given and let $J \colon W \to \mathbb{R}$ denote some quantity of interest. Let the primal problem be given in form of a variational formulation, i.e. there is a primal operator $\rho \colon W \to L^*$ and the primal solution $u \in W$ is given as solution of

$$\rho(u)(\psi) = 0 \quad \text{ for all } \psi \in L. \tag{2.30}$$

We assume that ρ and J are continuously Frechet-differentiable with respective derivatives at point $u \in W$ given by $J'(u) \colon W \to \mathbb{R}$ and $\rho'(u) \colon W \to L^*$. Then, the dual operator w.r.t. J and ρ at u is defined as

$$\begin{aligned} \rho^*(u) \colon L &\to W^* \\ z &\mapsto J'(u; \cdot) + \rho'(u; \cdot)(z) \end{aligned} \tag{2.31}$$

and the dual solution $z \in L$ is given as solution of

$$\rho^*(u; z)(\phi) = 0 \quad \text{ for all } \phi \in W. \tag{2.32}$$

The connection between dual solution and sensitivity is now given by the following considerations, see [8]. Let $P \colon L \to L^*$ denote a continuously Frechet-differentiable perturbation operator. For some perturbation $p \in L$ denote with $u(p) \in W$ the solution of the perturbed problem

$$\rho(u)(\psi) = -P(p)(\psi) \text{ for all } \psi \in L.$$

Now, the derivative of $J(u(p))$ w.r.t. p can be stated in terms of the dual solution, for details see [37] and [8]:

$$\left\langle \frac{d}{dp} J(u(p)), q \right\rangle_{L^*} = \left\langle P'(p; q), z(p) \right\rangle_{L^*}, \tag{2.33}$$

with $z(p)$ denoting the solution of (2.32) for $u = u(p)$. In the special case of L being a Hilbert space with inner product $(\cdot,\cdot)_L$ and P being of the form $P(p) = (p, \cdot)_L$, there follows from (2.33):

$$\left(\frac{d}{dp} J(u(p)), q \right)_L = (z(p), q)_L \tag{2.34}$$

and, with u denoting the solution of the unperturbed system and z its corresponding dual, a first order Taylor expansion yields

$$J(u(p)) \approx J(u) + (z, p)_L . \tag{2.35}$$

According to (2.35), a dual solution of small L-norm means that any kind of perturbation of the primal system only leads to small changes in the quantity of interest. For typical applications, $(\cdot,\cdot)_L$ is an integral over the space-time domain $[0, T] \times \Omega$. Thus, large values of z at a specific period in time $\tilde{I} \subset [0, T]$ and in specific spatial areas $\tilde{\Omega} \subset \Omega$ imply that perturbations in $\tilde{I} \times \tilde{\Omega}$ may potentially lead to high variations of J. In this way, regions of low and high sensitivity of the primal solution w.r.t. a given quantity of interest can be characterized by means of the dual solution.

Adjoint Problem for TEHD Boussinesq Equations

Based on the abstract problem definition above, the adjoint problem of (2.28) is derived. To this end, first the pointwise formulation (2.28) is transformed into a variational one by multiplication with test functions, integration over Ω and $[0, T]$ and integrating by parts:

Let $\theta_b \in H^1_D(\Omega)$, $\Phi_b \in W^{1,6}(\Omega)$ denote liftings of the Dirichlet boundary conditions θ_D, Φ_D and $\mathbf{u}_0 \in \mathbf{H}^1_0(\Omega)$, $\theta_0 \in H^1_D(\Omega)$ denote initial conditions with $\nabla \cdot \mathbf{u}_0 = 0$. Then, the primal solution $u = (\mathbf{u}, p, \theta, \Phi) \in W$ is defined as solution of (2.30) with ansatz and test spaces

$$\begin{aligned} W &:= \mathcal{W}_0(0, T; 2, 2, \mathbf{H}^1_0, \mathbf{H}^{-1}) \times L^2(0, T; L^2_0) \times \mathcal{W}_0(0, T; 2, 2, H^1_D, H^{-1}_D) \times L^\infty(0, T; W^{1,6}_D) \\ L &:= L^2(0, T; \mathbf{H}^1_0) \times L^2(0, T; L^2_0) \times L^2(0, T; H^1_D) \times L^2(0, T; H^1_D) \end{aligned}$$

and primal operator

$$\begin{aligned} \rho(u)(\psi) := & \int_0^T \langle \partial_t \mathbf{u}, \mathbf{v} \rangle_{\mathbf{H}^{-1}} + ((\mathbf{u} + \mathbf{u}_0) \cdot \nabla(\mathbf{u} + \mathbf{u}_0), \mathbf{v}) + \nu\, (\nabla(\mathbf{u} + \mathbf{u}_0), \nabla \mathbf{v}) \; dt \\ & + \int_0^T - (p, \nabla \cdot \mathbf{v}) - \big(\alpha_e |\nabla(\Phi + \Phi_b)|^2 \nabla(\theta + \theta_0 + \theta_b) - \alpha_g(\theta + \theta_0 + \theta_b - \theta_r)\mathbf{g}, \mathbf{v}\big) \; dt \\ & + \int_0^T (\nabla \cdot (\mathbf{u} + \mathbf{u}_0), q) \; dt \\ & + \int_0^T \langle \partial_t \theta, \tau \rangle_{H^{-1}_D} + ((\mathbf{u} + \mathbf{u}_0) \cdot \nabla(\theta + \theta_0 + \theta_b), \tau) + \kappa\, (\nabla(\theta + \theta_0 + \theta_b), \nabla \tau) \; dt \\ & + \int_0^T \epsilon_0 \epsilon_r\, ((1 - \gamma(\theta + \theta_0 + \theta_b - \theta_r)) \nabla(\Phi + \Phi_b), \beta) \; dt \end{aligned} \tag{2.36}$$

for test functions $\psi = (\mathbf{v}, q, \tau, \beta) \in L$. For a definition of the involved function spaces and inner products see Section A.3 and A.4 in the Appendix.

We assume that the quantity of interest is of the form

$$
\begin{aligned}
J(u) &:= J^i(\mathbf{u},p,\theta,\Phi) + J^f(\mathbf{u},\theta) \\
&:= \int_0^T \int_\Omega j^i(\mathbf{u},p,\theta,\Phi)\,dx\,dt + \int_\Omega j^f(\mathbf{u}(T),\theta(T))\,dx
\end{aligned} \tag{2.37}
$$

with continuously differentiable functions $j^i\colon \mathbb{R}^{d+3} \to \mathbb{R}$ and $j^f\colon \mathbb{R}^{d+1} \to \mathbb{R}$. In this way, J covers both temporal averages and evaluations at the final time.

Actually, j^i and j^f could also depend on derivatives of the primal solution components, which is not considered here to shorten the presentation. According to (2.31), the dual operator $\rho^*(u;z)(\phi)$ for dual solution $z = (\hat{\mathbf{u}},\hat{p},\hat{\theta},\hat{\Phi}) \in L$ and dual test function $\phi = (\hat{\mathbf{v}},\hat{q},\hat{\tau},\hat{\beta}) \in W$ is then given by

$$
\rho^*(u;z) = \rho^*_{\mathbf{v}}(u;z) + \rho^*_q(u;z) + \rho^*_\tau(u;z) + \rho^*_\beta(u;z), \tag{2.38}
$$

with

$$
\begin{aligned}
\rho^*_{\mathbf{v}}(u;z)(\phi) = &\int_0^T \langle \partial_t\hat{\mathbf{v}},\hat{\mathbf{u}}\rangle_{\mathbf{H}^{-1}} + (\hat{\mathbf{v}}\cdot\nabla(\mathbf{u}+\mathbf{u}_0),\hat{\mathbf{u}}) + ((\mathbf{u}+\mathbf{u}_0)\cdot\nabla\hat{\mathbf{v}},\hat{\mathbf{u}}) \\
&+ \int_0^T \nu\,(\nabla\hat{\mathbf{v}},\nabla\hat{\mathbf{u}}) - (\hat{q},\nabla\cdot\hat{\mathbf{u}}) + \alpha_g\,(\hat{\tau}\mathbf{g},\hat{\mathbf{u}})\;dt \\
&- \int_0^T \alpha_e\left(|\nabla(\Phi+\Phi_b)|^2\nabla\hat{\tau},\hat{\mathbf{u}}\right) + 2\alpha_e((\nabla(\Phi+\Phi_b)\cdot\nabla\hat{\beta})\nabla(\theta+\theta_0+\theta_b),\hat{\mathbf{u}})\,dt \\
&+ \int_0^T \left(\partial_{\mathbf{u}} j^i(\mathbf{u}+\mathbf{u}_0,p,\theta+\theta_0+\theta_b,\Phi+\Phi_b),\hat{\mathbf{v}}\right)\;dt \\
&+ (\partial_{\mathbf{u}} j^f(\mathbf{u}(T)+\mathbf{u}_0,\theta(T)+\theta_0+\theta_b),\hat{\mathbf{v}}(T))
\end{aligned} \tag{2.39}
$$

$$
\rho^*_q(u;z)(\phi) = \int_0^T (\nabla\cdot\hat{\mathbf{v}},\hat{p})\;dt + \int_0^T \left(\partial_p j^i(\mathbf{u}+\mathbf{u}_0,p,\theta+\theta_0+\theta_b,\Phi+\Phi_b),\hat{q}\right)\;dt \tag{2.40}
$$

$$
\begin{aligned}
\rho^*_\tau(u;z)(\phi) = &\int_0^T \langle\partial_t\hat{\tau},\hat{\theta}\rangle_{H_D^{-1}} + (\hat{\mathbf{v}}\cdot\nabla(\theta+\theta_0+\theta_b),\hat{\theta}) + ((\mathbf{u}+\mathbf{u}_0)\cdot\nabla\hat{\tau},\hat{\theta})\,dt \\
&+ \int_0^T \kappa(\nabla\hat{\tau},\nabla\hat{\theta}) + (\partial_\theta j^i(\mathbf{u}+\mathbf{u}_0,p,\theta+\theta_0+\theta_b,\Phi+\Phi_b),\hat{\tau})\,dt \\
&+ (\partial_\theta j^f(\mathbf{u}(T)+\mathbf{u}_0,\theta(T)+\theta_0+\theta_b),\hat{\tau}(T))
\end{aligned} \tag{2.41}
$$

$$
\begin{aligned}
\rho^*_\beta(u;z)(\phi) = &\int_0^T \epsilon_0\epsilon_r(-\gamma\hat{\tau}\nabla(\Phi+\Phi_b),\nabla\hat{\Phi}) + \epsilon_0\epsilon_r((1-\gamma(\theta+\theta_0+\theta_b-\theta_r))\nabla\hat{\beta},\nabla\hat{\Phi})\,dt \\
&+ \int_0^T (\partial_\Phi j^i(\mathbf{u}+\mathbf{u}_0,p,\theta+\theta_0+\theta_b,\Phi+\Phi_b),\hat{\beta})\,dt.
\end{aligned} \tag{2.42}
$$

If primal and dual solution are sufficiently regular, then integrating by parts, such that all dual test functions occur without preceding differential operators, yields the following dual system in pointwise form:

$$
\begin{aligned}
-\partial_t\hat{\mathbf{u}} + \hat{\mathbf{u}}\cdot(\nabla\mathbf{u})^T - \mathbf{u}\cdot\nabla\hat{\mathbf{u}} - (\nabla\cdot\mathbf{u})\hat{\mathbf{u}} - \nu\Delta\hat{\mathbf{u}} - \nabla\hat{p} + \hat{\theta}\nabla\theta &= \hat{\mathbf{f}}_{\mathbf{u}} \\
-\nabla\cdot\hat{\mathbf{u}} &= \hat{f}_p \\
-\partial_t\hat{\theta} - \mathbf{u}\cdot\nabla\hat{\theta} - (\nabla\cdot\mathbf{u})\hat{\theta} - \kappa\Delta\hat{\theta} + \alpha_g\mathbf{g}\cdot\hat{\mathbf{u}} + \alpha_e\nabla\cdot(|\nabla\Phi|^2\hat{\mathbf{u}}) - \epsilon_0\epsilon_r\gamma(\nabla\Phi\cdot\nabla\hat{\Phi}) &= \hat{f}_\theta \\
-\nabla\cdot(\epsilon_0\epsilon_r(1-\gamma(\theta-\theta_r))\nabla\hat{\Phi}) + 2\alpha_e\nabla\cdot((\nabla\theta\cdot\hat{\mathbf{u}})\,\nabla\Phi) &= \hat{f}_\Phi,
\end{aligned} \tag{2.43}
$$

with $\hat{\mathbf{f}}_{\mathbf{u}} = -\partial_{\mathbf{u}} j^i(\mathbf{u}, p, \theta, \Phi)$, $\hat{f}_l = -\partial_l j^i(\mathbf{u}, p, \theta, \Phi)$ for $l \in \{p, \theta, \Phi\}$. The dual solution is subjected to initial conditions

$$\begin{aligned} \hat{\mathbf{u}}(T) &= -\partial_{\mathbf{u}} j^f(\mathbf{u}, p, \theta, \Phi) \\ \hat{\theta}(T) &= -\partial_{\theta} j^f(\mathbf{u}, p, \theta, \Phi) \end{aligned} \tag{2.44}$$

and boundary conditions

$$\begin{aligned} &\hat{\mathbf{u}} = 0 \text{ on } \Gamma \\ &\hat{\theta} = 0 \text{ on } \Gamma_D,\ \nabla\hat{\theta} \cdot \mathbf{n} \ = 0 \text{ on } \Gamma_N \\ &\hat{\Phi} = 0 \text{ on } \Gamma_D,\ \nabla\hat{\Phi} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N. \end{aligned} \tag{2.45}$$

One basic feature of the dual system (2.43)-(2.45) is the fact that all first order differential operators, in particular ∂_t, have a reverse sign. In combination with the prescribed value for dual velocity and dual temperature at the end of the time interval, this means that the dual problem is posed backwards in time. In contrast, the symmetric operators in the primal system, e.g. Δ, are not modified in the dual formulation. An interpretation of this fact is given by the physical meaning of Δ, which corresponds to a homogeneous diffusion process. Thus, there is no distinguished direction of information flow.

3. Analysis of the Stationary Problem

In this section, we consider the stationary TEHD Boussinesq equations on a bounded domain Ω:

$$\begin{aligned} \delta\mathbf{u} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{F}(\theta,\bar{\Phi}) + \mathbf{f_v} \\ \nabla\cdot\mathbf{u} &= 0 \\ \delta\theta + (\tilde{\mathbf{u}}\cdot\nabla)\theta - \kappa\Delta\theta &= f_\tau \\ -\nabla\cdot(\epsilon(\bar{\theta})\nabla\Phi) &= f_\beta. \end{aligned} \tag{3.1}$$

System (3.1) can be considered as that system, which is obtained after discretizing (2.28) in time. Here, δ, $\mathbf{f_v}$, f_τ, f_β denote contributions by some outer time-stepping scheme. This scheme may also determine the functions $(\bar{\mathbf{u}},\tilde{\mathbf{u}},\bar{\theta},\bar{\Phi})$. Depending on the degree of its implicitness, each of these variables could be either fixed or unknown. In particular, we allow the case $(\bar{\mathbf{u}},\tilde{\mathbf{u}},\bar{\theta},\bar{\Phi}) = (\mathbf{u},\mathbf{u},\theta,\Phi)$, $\delta = 0$ and $\mathbf{f_v} = f_\tau = f_\beta = 0$, which corresponds to the stationary version of the transient TEHD equations (2.28). Compared to (2.28), a general force term $\mathbf{F}(\theta,\bar{\Phi})$ and a general permittivity $\epsilon\colon \mathbb{R}\to(0,\infty)$ are introduced and θ is shifted by the constant reference temperature θ_r. The system of equations (3.1) is subjected to the boundary conditions

$$\begin{aligned} \mathbf{u} = 0 \quad &\text{on } \partial\Omega = \Gamma_D + \Gamma_N \\ \theta = \theta_D \ \text{ on } \Gamma_D,\ \ &\nabla\theta\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \\ \Phi = \Phi_D \ \text{ on } \Gamma_D,\ \ &\nabla\Phi\cdot\mathbf{n} = 0 \text{ on } \Gamma_N. \end{aligned} \tag{3.2}$$

Concerning well-posedness, we note that for given $\bar{\theta}$, the potential Φ is determined by Gauss's law. This fact motivates proving the existence of weak solutions of (3.1) by means of a fixed-point iteration. For this iteration, (3.1) is split into an hydrodynamical part, given by the stationary Boussinesq equations with buoyancy being replaced by $\mathbf{F}(\theta,\bar{\Phi})$ and Gauss' law with temperature-dependent permittivity.

The existence of solutions $(\mathbf{u},\theta)$ of the Boussinesq equations is shown by employing a Galerkin principle combined with a fixed-point argument to a series of finite-dimensional problems. To be more precise, we adapt the concept presented in [46] for the stationary incompressible Navier Stokes equations. This result can be seen as generalization of [57], where existence and uniqueness of solutions of the stationary Boussinesq equations is shown for the standard buoyancy force.

The corresponding proof requires $\mathbf{F}$ to satisfy some type of weak continuity property. Moreover, the proposed procedure demands stability of solutions $(\mathbf{u},\theta)$ w.r.t. the input data. This is established by combining ideas from the corresponding result for the stationary incompressible Navier-Stokes equations (see e.g. [46]) with the procedure proposed in [56] and [57] to cope with non-homogeneous Dirichlet boundary conditions for the temperature. To this end, one has to impose the assumption that there exists a family of boundary liftings for θ_D of arbitrarily small L^3-norm. Moreover, one has to require the force term $\mathbf{F}$ to fulfill a boundedness principle of the form $\|\mathbf{F}(\theta,\Phi)\| \le a_\mathbf{F}(\|\Phi\|)\|\nabla\theta\|$ for some non-decreasing function $a_\mathbf{F}$.

Eventually, a uniqueness result for the stationary TEHD equations (3.1) under rather strict assumptions on the involved data is derived by modifying existing techniques and one obtains the requirement that $\mathbf{F}$ is locally Lipschitz continuous in some sense.

Concerning the choice of $\mathbf{F}$, we suggest models that are based on linearization around a smooth reference potential or which make use of a regularization operator such as mollification. We give a heuristic justification for the proposed procedure in case of fluids with small permittivity variation $\frac{d\epsilon}{d\theta}$ and for small temperature variations.

The outline of this chapter is as follows: In Section 3.1, the Boussinesq problem with general force term $\mathbf{F}$ is formulated and the requirements posed on $\mathbf{F}$ are summarized. Afterward, stability and existence of solutions are proven. In Section 3.2, the variational formulation for the stationary TEHD Boussinesq equations is set up and well-posedness of these equations is considered by investigating existence, stability and uniqueness of solutions. In the final Section 3.3, several modelizations of the DEP force are proposed which fit into the framework of the previously considered general body force $\mathbf{F}$.

Throughout this work, $\|\cdot\|_{k,p}$ denotes the norm associated to the Sobolev space $W^{k,p}(\Omega)$ with $W^{0,p}(\Omega) := L^p(\Omega)$ for $k \in \mathbb{N}_0,\ p \in [1,\infty]$. If not stated otherwise, $\|\cdot\| := \|\cdot\|_{0,2}$ and the inner product on $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The constant in Friedrich's inequality for $\mathbf{H}_0^1(\Omega)$ is denoted by c_0, i.e. $\|\mathbf{v}\| \le c_0\|\nabla\mathbf{v}\|$ for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. A detailed description of all involved functions spaces, such as $\mathbf{H}_0^1(\Omega)$, $\mathbf{V}(\Omega)$, $L_0^2(\Omega)$ and $H_D^1(\Omega)$, is given in the Appendix, Section A.4.

As mentioned in the beginning of this section, we want to cover the cases of (3.1) being either a fully nonlinear system or a (partially) linearized system. Therefore, the introduced variables $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$, $\bar{\theta}$, $\bar{\Phi}$ can either denote known functions, in the following called *fixed*, or they are *unknown*. The later case means that $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = \mathbf{u}$, $\bar{\theta} = \theta$ and $\bar{\Phi} = \Phi$. The linearized case will be used in Section 5.2 for analyzing the proposed time discretization.

The following assumptions on the domain, boundary conditions and physical parameters are imposed for the remaining of this work.

Assumption 3.1. *(Domain and Boundary Conditions)*
Let $\Omega \subset \mathbb{R}^d,\ d \in \{2,3\}$ be given. Its boundary $\partial\Omega =: \Gamma =: \Gamma_D + \Gamma_N$ is split into a Dirichlet part Γ_D and Neumann part Γ_N. The corresponding Dirichlet boundary conditions for temperature and potential are denoted by $\theta_D\colon \Gamma_D \to \mathbb{R}$ and $\Phi_D\colon \Gamma_D \to \mathbb{R}$, respectively. The following conditions should hold:

(i) Ω is a bounded Lipschitz domain.

(ii) A Friedrich-type inequality is satisfied for functions vanishing on Γ_D, i.e. there is $c_D > 0$ such that

$$\|\tau\| \le c_D\|\nabla\tau\| \text{ for all } \tau \in H_D^1(\Omega).$$

(iii) There exists an extension $\Phi_b \in H^1(\Omega)$ of Φ_D.

(iv) There exists a family of extensions $\{\theta_b[\xi]\colon \xi \in (0,1)\} \subset H^1(\Omega)$ of θ_D with $\|\theta_b[\xi]\|_{0,3} \le \xi$ for all $\xi \in (0,1)$.

Assumption 3.2. *(Physical Parameters)*
Let $\delta \ge 0$ and $\nu,\kappa > 0$. The permittivity $\epsilon\colon \mathbb{R} \to [\epsilon_-,\epsilon_+]$ as a function of temperature is assumed to be Lipschitz continuous with constant L_ϵ and $0 < \epsilon_- \le \epsilon_+ < \infty$.

Remark 3.3. According to Remark 1.6 in [12], Assumption 3.1 *(ii)* is satisfied if the boundary Γ is piecewise smooth and the Dirichlet boundary part Γ_D has a positive $d-1$ dimensional measure. A more general sufficient condition implying *(ii)* is given by Proposition 7.5 in [21] and Theorem A.109. According to Lemma 2 in [56], *(iv)* is satisfied if Γ is of class C^1, Γ_D has positive $d-1$ dimensional measure, the intersection $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ is a $d-2$ dimensional C^1 manifold and $\theta_D \in C^1(\overline{\Gamma}_D)$.

The variational formulation of (3.1) is based on the function spaces for velocity, $\mathbf{U} := \mathbf{H}_0^1(\Omega)$

and $\mathbf{V} := \mathbf{V}(\Omega) = \overline{\mathcal{V}(\Omega)}^{\mathbf{H}_0^1}$ with $\mathcal{V}(\Omega) = \{\mathbf{u} \in \mathcal{D}(\Omega)^d\colon\ \nabla\cdot\mathbf{u} = 0\}$, for pressure, $M := L_0^2(\Omega)$, for temperature, $\Theta := H_D^1(\Omega)$ and for potential, $\Upsilon := H_D^1(\Omega)$. Moreover, the following bi- and trilinear forms are used:

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) &:= \nu(\nabla\mathbf{u},\nabla\mathbf{v}), & c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w}) &:= (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) \\ a_\tau(\theta,\tau) &:= \kappa(\nabla\theta,\nabla\tau), & c_\tau(\mathbf{u},\theta,\tau) &:= (\mathbf{u}\cdot\nabla\theta,\tau) \\ a_\beta(\theta,\Phi,\beta) &:= (\epsilon(\theta)\nabla\Phi,\nabla\beta), & b(\mathbf{u},q) &:= (\nabla\cdot\mathbf{u},q). \end{aligned} \tag{3.3}$$

For a precise definition of the involved Sobolev spaces see Definition A.89 and A.95.

Additionally, a set of domain dependent constants is frequently needed. Employing Friedrich's inequality A.106 and Assumption 3.1 (*ii*), the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, Theorem A.92, and Hölder's inequality, the following expressions take finite values

$$\mathbf{K}_p := \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{\|\mathbf{v}\|_{0,p}}{\|\nabla\mathbf{v}\|},\ K_p := \sup_{\tau\in H_D^1(\Omega)} \frac{\|\tau\|_{0,p}}{\|\nabla\tau\|},\ K_{p,q} := \sup_{\tau\in L^q(\Omega)} \frac{\|\tau\|_{0,p}}{\|\tau\|_{0,q}}, \tag{3.4}$$

for $p \in [1,p^*], q \geq p$ and $p^* \in [1,\infty]$ such that $-\frac{d}{p^*} \leq 1 - \frac{d}{2}$.

Note that $K_2 \leq c_D$ according to Assumption 3.1 (*ii*) and $\mathbf{K}_2 \leq c_0$ by Friedrich's inequality for $\mathbf{H}_0^1(\Omega)$.

Lemma 3.4. *(Properties of Trilinear Forms)*
There holds

$$\begin{aligned} K_{\mathbf{v}} &:= \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{U}\setminus\{0\}} \frac{c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|} < \infty \\ K_\tau &:= \sup_{\mathbf{u}\in\mathbf{U}\setminus\{0\},\theta,\tau\in\Theta\setminus\{0\}} \frac{c_\tau(\mathbf{u},\theta,\tau)}{\|\nabla\mathbf{u}\|\|\nabla\theta\|\|\nabla\tau\|} < \infty. \end{aligned}$$

Moreover, if $\mathbf{u} \in \mathbf{V}$*, then for* $\mathbf{v},\mathbf{w} \in \mathbf{U},\ \tau,\theta \in H^1(\Omega)$*:*

$$\begin{aligned} c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w}) &= -c_{\mathbf{v}}(\mathbf{u},\mathbf{w},\mathbf{v}) \\ c_\tau(\mathbf{u},\theta,\tau) &= -c_\tau(\mathbf{u},\tau,\theta). \end{aligned}$$

Proof. See Lemma II.1.1 and Lemma II.1.3 in [77]. □

3.1. The Boussinesq Problem with General Force Term

In this section, we investigate the standard Boussinesq equations for natural convection with buoyancy being replaced by a general force term $\mathbf{F}$. In contrast to previuous works with general force term, e.g. [5], $\mathbf{F}$ may depend on the temperature θ as an element of H^1 and not as scalar value. In this way, the formulation of $\mathbf{F}$ may also contain the gradient $\nabla\theta$. Moreover, it may take the potential Φ as additional argument.

The way for setting up the variational formulation and considering well-posedness is a generalization of the approach presented in [46] for investigating the incompressible Navier-Stokes equations. The main difficulty in doing so arises when proving stability of solutions, see Lemma 3.13. Here, we follow the idea of [57], where the standard Boussinesq equations are considered, and assume that θ_b can be chosen with arbitrarily small L^3-norm.

We start with setting up the variational formulation, given by Problem 3.5. Here, an additional coefficient $\lambda \in [0,1]$ is introduced to obtain a family of stationary problems. In this way, Problem 3.5 fits into the framework of a suitable fixed-point theorem. In the subsequent presentation, the problem is considered in solenoidal form.

Problem 3.5. *(Stationary Boussinesq Equations)*
Let $\theta_b \in H^1(\Omega)$ be a lifting of given Dirichlet boundary conditions θ_D and $\mathbf{F}\colon \Theta \to \mathbf{U}^*$, $\mathbf{f_v} \in \mathbf{U}^*$, $f_\tau \in \Theta^*$ given source terms. Let either $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ denote fixed elements of $\mathbf{V}$ or the unknown velocity $\mathbf{u}$. For $\lambda \in [0,1]$ find $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ such that for all $(\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$:

$$\begin{aligned} \delta(\mathbf{u},\mathbf{v}) + a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) + \lambda\left(c_{\mathbf{v}}(\bar{\mathbf{u}},\mathbf{u},\mathbf{v}) - \langle \mathbf{F}(\theta+\theta_b) + \mathbf{f_v}, \mathbf{v}\rangle_{\mathbf{U}^*}\right) &= 0 \\ \delta(\theta,\tau) + a_\tau(\theta,\tau) + \lambda\left(\delta(\theta_b,\tau) + a_\tau(\theta_b,\tau) + c_\tau(\tilde{\mathbf{u}},\theta+\theta_b,\tau) - \langle f_\tau, \tau\rangle_{\Theta^*}\right) &= 0. \end{aligned}$$

This problem can be written compactly in form of a fixed-point equation. To this end, a solution operator for the linear, elliptic part of Problem (3.5) is used.

Definition 3.6. *(Stokes Solution Operator)*
Let $\mathbf{W} \subset \mathbf{U}$ and $T \subset \Theta$ denote Hilbert spaces with inner products $(\cdot,\cdot)_{\mathbf{W}} := (\nabla\cdot,\nabla\cdot)$ and $(\cdot,\cdot)_T := (\nabla\cdot,\nabla\cdot)$, respectively. The solution operator for the Stokes equations in solenoidal form, combined with an additional heat equation, is defined as

$$\begin{aligned} \mathcal{K}[\mathbf{W},T]\colon \mathbf{W}^* \times T^* &\to \mathbf{W} \times T \\ (\mathbf{f}, g) &\mapsto (\mathbf{u}, \theta) \\ \text{such that for all } (\mathbf{v},\tau) \in \mathbf{W} \times T : \\ \delta(\mathbf{u},\mathbf{v}) + a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) &= -\langle \mathbf{f}, \mathbf{v}\rangle_{\mathbf{W}^*} \\ \delta(\theta,\tau) + a_\tau(\theta,\tau) &= -\langle g, \tau\rangle_{T^*}. \end{aligned}$$

The remaining terms, i.e. nonlinearities, source terms and boundary contributions, are collected in the operator $\mathcal{N}$.

Definition 3.7. *(Non-Stokes Terms)*
Let the notation and assumptions of Problem 3.5 and Definition 3.6 hold. Define

$$\begin{aligned} \mathcal{N}[\mathbf{W},T]\colon \mathbf{W} \times T &\to \mathbf{W}^* \times T^* \\ (\mathbf{u},\theta) &\mapsto \begin{pmatrix} c_{\mathbf{v}}(\bar{\mathbf{u}},\mathbf{u},\cdot) - \mathbf{F}(\theta+\theta_b) - \mathbf{f_v} \\ \delta(\theta_b,\cdot) + a_\tau(\theta_b,\cdot) + c_\tau(\tilde{\mathbf{u}},\theta+\theta_b,\cdot) - f_\tau \end{pmatrix}. \end{aligned}$$

The fixed-point version of Problem 3.5 is now given as follows.

Problem 3.8. *(Fixed-Point Version)*
Let the notation and assumptions of Problem 3.5, Definition 3.6 and Definition 3.7 hold. For $\mathcal{K} := \mathcal{K}[\mathbf{W},T]$, $\mathcal{N} := \mathcal{N}[\mathbf{W},T]$ and $\lambda \in [0,1]$ find $(\mathbf{u}_\lambda, \theta_\lambda) \in \mathbf{W} \times T$ such that

$$(\mathbf{u}_\lambda, \theta_\lambda) = \lambda \mathcal{K}(\mathcal{N}(\mathbf{u}_\lambda, \theta_\lambda)) = \lambda \mathcal{F}(\mathbf{u}_\lambda, \theta_\lambda).$$

with fixed-point operator

$$\mathcal{F} := \mathcal{F}[\mathbf{W},T] := \mathcal{K} \circ \mathcal{N} \colon \mathbf{W} \times T \to \mathbf{W} \times T.$$

In order to apply an appropriate existence theorem to Problem 3.8, one needs to show compactness of $\mathcal{F}$. For finite dimensional spaces, this can be accomplished by the next lemmas below.

Lemma 3.9. *(Properties of $\mathcal{K}$)*
Let the assumptions of Definition 3.6 hold. Then, the following statements hold.

(i) $\mathcal{K}[\mathbf{W}, T]$ *is well defined and linear*

(ii) There exists $C_{\mathcal{K}} \geq 0$ *such that* $(\mathbf{u}, \theta) = \mathcal{K}[\mathbf{W}, T](\mathbf{f}, g)$ *satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}} + \|\theta\|_T \leq C_{\mathcal{K}}(\|\mathbf{f}\|_{\mathbf{W}^*} + \|g\|_{T^*}) \text{ for all } (\mathbf{f}, g) \in \mathbf{W}^* \times T^*.$$

Proof. Follows by application of Lax-Milgram A.40, Friedrich's inequalities A.106 and Assumption 3.1 (ii). □

Lemma 3.10. *(Properties of $\mathcal{N}$)*
Let the assumptions of Problem 3.5 and Definition 3.7 hold. Assume that $\mathbf{F}(\cdot + \theta_b) \colon T \to \mathbf{W}^*$ *is continuous. Then,* $\mathcal{N}[\mathbf{W}, T]$ *is continuous w.r.t. the norms* $\|\cdot\|_{\mathbf{W}\times T} := \|\nabla \cdot\| + \|\nabla \cdot\|$ *and* $\|\cdot\|_{\mathbf{W}^*\times T^*} := \|\cdot\|_{\mathbf{W}^*} + \|\cdot\|_{T^*}$.

Proof. The proof is only presented for $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = \mathbf{u}$. Continuity of $\mathbf{W} \ni \mathbf{u} \mapsto c_{\mathbf{v}}(\mathbf{u}, \mathbf{u}, \cdot) \in \mathbf{W}^*$ is shown by using standard arguments, see e.g. Lemma 16 in [46]. Analogously, $\mathbf{W} \times T \ni (\mathbf{u}, \theta) \mapsto c_\tau(\mathbf{u}, \theta, \cdot) \in T^*$ is continuous. By continuity of $\mathbf{F}$, $\mathcal{N}[\mathbf{W}, T]$ is a composition of continuous functions and therefore continuous as mapping on $(\mathbf{W}\times T, \|\cdot\|_{\mathbf{W}\times T}) \to (\mathbf{W}^*\times T^*, \|\cdot\|_{\mathbf{W}^*\times T^*})$. □

Lemma 3.11. *(Properties of $\mathcal{F}$)*
Let the notation and assumptions of Problem 3.8 and of Lemma 3.9 and 3.10 hold. Then, Problem 3.5 and 3.8 are equivalent if $\mathbf{W} \times T = \mathbf{V} \times \Theta$. *Moreover, the following properties of the fixed-point operator* $\mathcal{F}[\mathbf{W}, T]$ *hold:*

(i) $\mathcal{F}[\mathbf{W}, T]$ *is continuous.*

(ii) $\mathcal{F}[\mathbf{W}, T]$ *is compact if* $\mathbf{W}$ *and* T *are finite dimensional, i.e. it maps bounded sets to sets with compact closure.*

Proof. (i) follows since $\mathcal{F}[\mathbf{W}, T]$ is a composition of continuous functions due to Lemma 3.9 and 3.10. If $\mathbf{W}$ and T are finite dimensional, (ii) follows since $\mathcal{F}[\mathbf{W}, T]$ is a continuous map on a finite dimensional Hilbert space, see Lemma A.13. To see the equivalence of Problem 3.5 and 3.8, let $(\mathbf{u}_\lambda, \theta_\lambda)$ denote a solution of Problem 3.8. By definition of $\mathcal{K}[\mathbf{V}, \Theta]$, this solution satisfies

$$\begin{pmatrix} \delta(\mathbf{u}, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) \\ \delta(\theta, \tau) + a_\tau(\theta, \tau) \end{pmatrix} = -\lambda \mathcal{N}[\mathbf{V}, \Theta](\mathbf{u}_\lambda, \theta_\lambda)(\mathbf{v}, \tau) \text{ for all } (\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$$

Inserting the definition of $\mathcal{N}[\mathbf{V}, \Theta]$ yields the assertion. □

The problem setup is finished by stating the following conditions on the body force $\mathbf{F}$. They are chosen to ensure certain well-posedness results for the generalized Boussinesq equations, Problem 3.5.

Assumption 3.12. *(General Body Force)*
Let $\mathbf{F}\colon H^1(\Omega)\times H^1(\Omega)\to\mathbf{U}^*$ *satisfy the following conditions for all* $\mathbf{v}\in\mathbf{U}$*:*

(i) $\mathbf{F}$ *is locally Lipschitz continuous: for all* $R>0$ *there is a non-decreasing function* $L_{\mathbf{F}}^{(\theta)}\colon[0,\infty)\to[0,\infty)$ *such that*

$$|\langle\mathbf{F}(\theta_1,\Phi)-\mathbf{F}(\theta_2,\Phi),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\theta)}(\|\Phi\|_{1,2})\|\theta_1-\theta_2\|_{1,2}\|\nabla\mathbf{v}\|.$$

for all $\theta_1,\theta_2\in B_R(0,H^1(\Omega))$*,* $\Phi\in H^1(\Omega)$ *and* $\mathbf{v}\in\mathbf{U}$.
Moreover, for all $R>0$ *there is* $L_{\mathbf{F}}^{(\Phi)}\ge 0$ *such that for all* $\theta\in H^1(\Omega)$*,* $\Phi_1,\Phi_2\in B_R(0,H^1(\Omega))$ *and* $\mathbf{v}\in\mathbf{U}$*,*

$$|\langle\mathbf{F}(\theta,\Phi_1)-\mathbf{F}(\theta,\Phi_2),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\Phi)}\|\theta\|_{1,2}\|\Phi_1-\Phi_2\|_{1,2}\|\nabla\mathbf{v}\|.$$

(ii) $\mathbf{F}$ *is bounded: There are non-decreasing functions*

$$a_{\mathbf{F}}\colon[0,\infty)\to[0,\infty)\quad\text{and}\quad b_{\mathbf{F}}\colon[0,\infty)\to[0,\infty),$$

such that

$$|\langle\mathbf{F}(\theta,\Phi),\mathbf{v}\rangle_{\mathbf{U}^*}|\le a_{\mathbf{F}}(\|\Phi\|_{1,2})\|\theta\|_{1,2}\|\nabla\mathbf{v}\|+b_{\mathbf{F}}(\|\Phi\|_{1,2})\|\nabla\mathbf{v}\|$$

for all $\theta\in H^1(\Omega),\Phi\in H^1(\Omega),\mathbf{v}\in\mathbf{U}$.

(iii) Let sequences $(\theta_n)_n$*,* $(\Phi_n)_n\subset H^1(\Omega)$ *and elements* θ_**,* $\Phi_*\in H^1(\Omega)$ *be given such that* $\theta_n\rightharpoonup\theta_*$ *in* $H^1(\Omega)$ *and* $\Phi_n\rightharpoonup\Phi_*$ *in* $H^1(\Omega)$*. Then, there holds*

$$|\langle\mathbf{F}(\theta_*,\Phi_*)-\mathbf{F}(\theta_n,\Phi_n),\mathbf{v}\rangle_{\mathbf{U}^*}|\to 0$$

The local Lipschitz condition (i) is used to show uniqueness of solutions under an appropriate small data condition and condition (ii) allows to prove stability of solutions of Problem 3.5. Note that the growth rate of $\mathbf{F}$ w.r.t. $\|\Phi\|_{1,2}$ is not restricted, while it may grow at most linearly w.r.t. $\|\theta\|_{1,2}$. This is due to the fact that $\|\Phi\|_{1,2}$ can be bounded in terms of the input data only. Assumption (iii) is needed in various cases to show that weak convergence of some kind of approximate solutions translates into pointwise convergence in $\mathbf{U}^*$ of the variational formulation.

Stability

The following lemma states that the norm of solutions of Problem (3.5) can be bounded in terms of input parameters, boundary liftings and source terms. For this reason, it is required that the boundary lifting θ_b can be chosen in such a way that $\|\theta_b\|_{0,3}$ is sufficiently small.

The subsequent result will play an important role in showing both existence (by virtue of a fixed-point argument) and uniqueness of solutions. For the former case, it is crucial that the stability bound is uniform w.r.t. $\lambda\in[0,1]$.

Lemma 3.13. *(Stability of Stationary Solutions)*
Let $\lambda\in[0,1]$*,* $\Phi\in H^1(\Omega)$ *and* $\mathbf{F}$ *satisfying Assumption 3.12 be given. Set* $\mathbf{F}(\theta):=\mathbf{F}(\theta,\Phi)$ *and* $a_{\mathbf{F}}:=a_{\mathbf{F}}(\|\Phi\|_{1,2})$*,* $b_{\mathbf{F}}:=b_{\mathbf{F}}(\|\Phi\|_{1,2})$*. Define*

$$d(s)=\begin{cases}\frac{1}{Ks}, & \text{if } \tilde{\mathbf{u}}=\mathbf{u}\\ \infty, & \text{if } \tilde{\mathbf{u}} \text{ is fixed}\end{cases},$$

with $K = \frac{\mathbf{K}_6}{\kappa\nu}\sqrt{8(K_2^2+1)}$. *Then, there are continuous functions*

$$g_i \colon \mathbb{R}^6 \cap \{x_1 < d(x_2)\} \to [0,\infty),\ i \in \{\mathbf{u},\theta\}$$

such that

$$\begin{aligned}\|\nabla\mathbf{u}\| &\leq g_{\mathbf{u}}(\|\theta_b\|_{0,3}, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) =: G_{\mathbf{u}}\\ \|\nabla\theta\| &\leq g_\theta(\|\theta_b\|_{0,3}, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) =: G_\theta\end{aligned}$$

for all solutions $(\mathbf{u},\theta)$ *of Problem 3.5 with boundary lifting* θ_b *satisfying* $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}})$. *In particular,* $g_{\mathbf{u}}$ *and* g_θ *do not depend on the parameter* $\lambda \in [0,1]$ *and are non-decreasing in their arguments* x_2 *and* x_3. *Moreover, if* $\tilde{\mathbf{u}} = \mathbf{u}$, *then* $g_i \to 0$ *for* $(x_4,x_5,x_6) \to 0$, *i.e.*

$$G_i \to 0 \text{ for } (\|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) \to 0.$$

Proof. Let $(\mathbf{u},\theta) \in \mathbf{V}\times\Theta$ denote a solution of the stationary problem. Inserting $\mathbf{v} = \mathbf{u}, \tau = \theta$ in 3.5 and noting that, by Lemma 3.4,

$$c_{\mathbf{v}}(\bar{\mathbf{u}},\mathbf{u},\mathbf{u}) = 0 \text{ and } c_\tau(\tilde{\mathbf{u}},\theta+\theta_b,\theta) = c_\tau(\tilde{\mathbf{u}},\theta_b,\theta) = -c_\tau(\tilde{\mathbf{u}},\theta,\theta_b)$$

since $\bar{\mathbf{u}}, \tilde{\mathbf{u}} \in \mathbf{V}$, one obtains

$$\delta\|\mathbf{u}\|^2 + \nu\|\nabla\mathbf{u}\|^2 = \lambda\langle \mathbf{F}(\theta+\theta_b) + \mathbf{f_v}, \mathbf{u}\rangle_{\mathbf{U}^*} \tag{3.5}$$

$$\delta\|\theta\|^2 + \kappa\|\nabla\theta\|^2 = -\lambda\left(\delta(\theta_b,\theta) + \kappa(\nabla\theta_b,\nabla\theta) - (\tilde{\mathbf{u}}\cdot\nabla\theta,\theta_b) - \langle f_\tau,\theta\rangle_{\Theta^*}\right). \tag{3.6}$$

Using the assumptions on $\mathbf{F}$ and Young's inequality, (3.5) yields

$$\begin{aligned}\nu\|\nabla\mathbf{u}\|^2 &\leq \underbrace{a_{\mathbf{F}}\sqrt{K_2^2+1}}_{=:\tilde{a}_{\mathbf{F}}}\|\nabla\theta\|\|\nabla\mathbf{u}\| + \underbrace{(b_{\mathbf{F}} + a_{\mathbf{F}}\|\theta_b\|_{1,2} + \|\mathbf{f_v}\|_{\mathbf{U}^*})}_{=:\tilde{b}_{\mathbf{F}}}\|\nabla\mathbf{u}\|\\ &\leq \left(\frac{1}{2\delta_1}\tilde{a}_{\mathbf{F}}^2\|\nabla\theta\|^2 + \frac{\delta_1}{2}\|\nabla\mathbf{u}\|^2\right) + \left(\frac{1}{2\delta_2}\tilde{b}_{\mathbf{F}}^2 + \frac{\delta_2}{2}\|\nabla\mathbf{u}\|^2\right)\end{aligned} \tag{3.7}$$

for $\delta_1,\delta_2 > 0$. (3.6) gives

$$\begin{aligned}\kappa\|\nabla\theta\|^2 &\leq \kappa\|\nabla\theta_b\|\|\nabla\theta\| + \delta K_2K_{23}\|\theta_b\|_{0,3}\|\nabla\theta\| + \mathbf{K}_6\|\nabla\tilde{\mathbf{u}}\|\|\nabla\theta\|\|\theta_b\|_{0,3} + \|f_\tau\|_{\Theta^*}\|\nabla\theta\|\\ &\leq \kappa\left(\frac{1}{2\delta_3}\|\nabla\theta_b\|^2 + \frac{\delta_3}{2}\|\nabla\theta\|^2\right) + \left(\frac{1}{2\delta_4}\mathbf{K}_6^2\|\theta_b\|_{0,3}^2\|\nabla\tilde{\mathbf{u}}\|^2 + \frac{\delta_4}{2}\|\nabla\theta\|^2\right)\\ &\quad + \left(\frac{1}{2\delta_5}\|f_\tau\|_{\Theta^*}^2 + \frac{\delta_5}{2}\|\nabla\theta\|^2\right) + \left(\frac{1}{2\delta_6}\delta^2K_2^2K_{23}^2\|\theta_b\|_{0,3}^2 + \frac{\delta_6}{2}\|\nabla\theta\|^2\right)\end{aligned} \tag{3.8}$$

for $\delta_3,\delta_4,\delta_5,\delta_6 > 0$. Rearranging terms in (3.7), (3.8) yields

$$(\nu - \frac{\delta_1}{2} - \frac{\delta_2}{2})\|\nabla\mathbf{u}\|^2 \leq \tilde{a}_{\mathbf{F}}^2\frac{1}{2\delta_1}\|\nabla\theta\|^2 + \frac{1}{2\delta_2}\tilde{b}_{\mathbf{F}}^2 \tag{3.9}$$

$$\begin{aligned}\bar{\kappa}\|\nabla\theta\|^2 &\leq \kappa\frac{1}{2\delta_3}\|\nabla\theta_b\|^2 + \frac{1}{2\delta_4}\mathbf{K}_6^2\|\theta_b\|_{0,3}^2\|\nabla\tilde{\mathbf{u}}\|^2\\ &\quad + \frac{1}{2\delta_5}\|f_\tau\|_{\Theta^*}^2 + \frac{1}{2\delta_6}\delta^2K_2^2K_{23}^2\|\theta_b\|_{0,3}^2.\end{aligned} \tag{3.10}$$

with $\bar{\kappa} = \kappa - \kappa\frac{\delta_3}{2} - \frac{\delta_4}{2} - \frac{\delta_5}{2} - \frac{\delta_6}{2}$. Setting $\delta_1 = \delta_2 = \frac{\nu}{2}$, $\delta_3 = \frac{1}{4}$, $\delta_4 = \delta_5 = \delta_6 = \frac{\kappa}{4}$, and dividing (3.9), (3.10) by $\frac{\nu}{2}$ and $\frac{\kappa}{2}$, respectively, yields

$$\|\nabla\mathbf{u}\|^2 \leq C_1\|\nabla\theta\|^2 + C_2 \tag{3.11}$$

$$\|\nabla\theta\|^2 \leq C_3\|\nabla\theta_b\|^2 + C_4\|\theta_b\|_{0,3}^2\|\nabla\tilde{\mathbf{u}}\|^2 + C_5 + C_6\|\theta_b\|_{0,3}^2 \tag{3.12}$$

with constants

$$C_1 = \frac{2}{\nu^2} a_{\mathbf{F}}^2 (K_2^2 + 1),\; C_2 = \frac{2}{\nu^2} \left(b_{\mathbf{F}} + a_{\mathbf{F}} \|\theta_b\|_{1,2} + \|\mathbf{f_v}\|_{\mathbf{U}^*}\right)^2,\; C_3 = 4$$
$$C_4 = \frac{4\mathbf{K}_6^2}{\kappa^2}, \qquad C_5 = \frac{4}{\kappa^2}\|f_\tau\|_{\Theta^*}^2, \qquad C_6 = 4\left(\frac{\delta K_2 K_{23}}{\kappa}\right)^2.$$

If $\tilde{\mathbf{u}} = \mathbf{u}$, set $d(a_{\mathbf{F}}) := (C_1 C_4)^{-\frac{1}{2}}$ and obtain for $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}})$ by combination of (3.11) and (3.12),

$$\|\nabla \mathbf{u}\|^2 \leq \frac{1}{1 - C_1 C_4 \|\theta_b\|_{0,3}^2} \left(C_2 + C_1 \left(C_3 \|\nabla \theta_b\|^2 + C_5 + C_6 \|\theta_b\|_{0,3}^2\right)\right) =: G_{\mathbf{u}}^2.$$

Now, θ can be bounded by

$$\|\nabla \theta\|^2 \leq C_3 \|\nabla \theta_b\|^2 + C_4 \|\theta_b\|_{0,3}^2 G_{\mathbf{u}}^2 + C_5 + C_6 \|\theta_b\|_{0,3}^2 =: G_\theta^2.$$

If $\tilde{\mathbf{u}}$ is fixed, one gets:

$$\begin{aligned} \|\nabla \theta\|^2 &\leq C_3 \|\nabla \theta_b\|^2 + C_4 \|\theta_b\|_{0,3}^2 \|\nabla \tilde{\mathbf{u}}\|^2 + C_5 + C_6 \|\theta_b\|_{0,3}^2 =: G_\theta^2 \\ \|\nabla \mathbf{u}\|^2 &\leq C_1 G_\theta^2 + C_2 =: G_{\mathbf{u}}^2. \end{aligned}$$

□

Remark 3.14. The previous proof shows that Lemma 3.13 is valid for arbitrary subspaces $\mathbf{W} \times T \subset \mathbf{V} \times \Theta$ with functions $d, g_{\mathbf{u}}, g_\theta$ that are independent of the specific choice of $\mathbf{W} \times T$.

The next lemma bounds the variation of solutions of the Boussinesq equations w.r.t. variations of the respective input data, i.e. the force term $\mathbf{F}$ and the terms $\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}$. These bounds will be used later on for showing uniqueness of solutions.

Lemma 3.15. *(Stability of Stationary Solutions w.r.t Varying Data)*
Let $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ denote solutions of Problem 3.5 for $\lambda = 1$, respective convection fields $(\bar{\mathbf{u}}_1, \tilde{\mathbf{u}}_1)$, $(\bar{\mathbf{u}}_2, \tilde{\mathbf{u}}_2)$ and external forces $\mathbf{F}_1 = \mathbf{F}_1(\cdot, \Phi_1)$, $\mathbf{F}_2 = \mathbf{F}_2(\cdot, \Phi_2)$ which do both satisfy Assumption 3.12. Assume

$$D_{\mathbf{F}} := \sup_{\mathbf{w} \in \mathbf{V}, \theta \in \Theta} \frac{|\langle \mathbf{F}_1(\theta + \theta_b) - \mathbf{F}_2(\theta + \theta_b), \mathbf{w}\rangle_{\mathbf{U}^*}|}{\|\nabla \mathbf{w}\| \|\theta + \theta_b\|_{1,2}} < \infty.$$

Then, the solutions satisfy

$$\begin{aligned} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\| &\leq D_1 \|\nabla(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2)\| + D_2 \|\nabla(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\| + D_3 D_{\mathbf{F}} \\ \|\nabla(\theta_1 - \theta_2)\| &\leq D_4 \|\nabla(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\| \end{aligned}$$

with constants $\{D_i\}_{i=1}^4$ given by (3.19).

Proof. Inserting $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ into Problem 3.5 and subtraction yields for all $(\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$

$$\begin{aligned} &\delta(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) + (\bar{\mathbf{u}}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}) - (\bar{\mathbf{u}}_2 \cdot \nabla \mathbf{u}_2, \mathbf{v}) \\ &= \langle \mathbf{F}_1(\theta_1 + \theta_b) - \mathbf{F}_2(\theta_2 + \theta_b), \mathbf{v}\rangle_{\mathbf{U}^*} \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\delta(\theta_1 - \theta_2, \tau) + a_\tau(\theta_1 - \theta_2, \tau) + (\tilde{\mathbf{u}}_1 \cdot \nabla \theta_1, \tau) - (\tilde{\mathbf{u}}_2 \cdot \nabla \theta_2, \tau) \\ &= -((\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2) \cdot \nabla \theta_b, \tau). \end{aligned} \tag{3.14}$$

Defining $\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2,\ \bar{\mathbf{u}} := \bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2,\ \tilde{\mathbf{u}} := \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2,\ \phi := \theta_1 - \theta_2$ and setting $\mathbf{v} = \mathbf{w},\ \tau = \phi$ yields

$$\delta\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 + (\bar{\mathbf{u}} \cdot \nabla\mathbf{u}_1, \mathbf{w}) = \langle \mathbf{F}_1(\theta_1 + \theta_b) - \mathbf{F}_2(\theta_2 + \theta_b), \mathbf{w}\rangle_{\mathbf{U}^*} \tag{3.15}$$

$$\delta\|\phi\|^2 + \kappa\|\nabla\phi\|^2 + (\tilde{\mathbf{u}} \cdot \nabla\theta_1, \phi) = -(\tilde{\mathbf{u}} \cdot \nabla\theta_b, \phi) = (\tilde{\mathbf{u}} \cdot \nabla\phi, \theta_b). \tag{3.16}$$

Let $G_{\mathbf{u}}^{(i)}$, $G_\theta^{(i)}$ denote the respective stability bounds provided by Lemma 3.13 for $i \in \{1, 2\}$. Define $G_\theta := \max\{G_\theta^{(1)}, G_\theta^{(2)}\}$ and $R := \|\theta_b\|_{1,2} + \sqrt{K_2^2 + 1}G_\theta$. Then, local Lipschitz continuity of $\mathbf{F}_1$ with constant $L_{\mathbf{F}_1}^{(\theta)} = L_{\mathbf{F}_1}^{(\theta)}(R)$ yields

$$\begin{aligned}
\langle \mathbf{F}_1(\theta_1 + \theta_b) - \mathbf{F}_2(\theta_2 + \theta_b), \mathbf{w}\rangle_{\mathbf{U}^*} &= \langle \mathbf{F}_1(\theta_1 + \theta_b) - \mathbf{F}_1(\theta_2 + \theta_b), \mathbf{w}\rangle_{\mathbf{U}^*} \\
&\quad + \langle \mathbf{F}_1(\theta_2 + \theta_b) - \mathbf{F}_2(\theta_2 + \theta_b), \mathbf{w}\rangle_{\mathbf{U}^*} \\
&\le L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2 + 1}\|\nabla(\theta_1 - \theta_2)\|\|\nabla\mathbf{w}\| + D_{\mathbf{F}}\|\theta_2 + \theta_b\|_{1,2}\|\nabla\mathbf{w}\|.
\end{aligned}$$

Thus, one obtains from (3.15)

$$\begin{aligned}
\nu\|\nabla\mathbf{w}\|^2 \le K_{\mathbf{v}}G_{\mathbf{u}}^{(1)}\|\nabla\bar{\mathbf{u}}\|\|\nabla\mathbf{w}\| + L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2 + 1}\|\nabla\phi\|\|\nabla\mathbf{w}\| \\
+ D_{\mathbf{F}}\left(G_\theta^{(2)}\sqrt{K_2^2 + 1} + \|\theta_b\|_{1,2}\right)\|\nabla\mathbf{w}\|,
\end{aligned} \tag{3.17}$$

and from (3.16),

$$\kappa\|\nabla\phi\|^2 \le K_\tau G_\theta^{(1)}\|\nabla\tilde{\mathbf{u}}\|\|\nabla\phi\| + \mathbf{K}_6\|\nabla\tilde{\mathbf{u}}\|\|\nabla\phi\|\|\theta_b\|_{0,3}. \tag{3.18}$$

Dividing by (3.17) and (3.18) by $\|\nabla\mathbf{w}\|$ and $\|\nabla\phi\|$, respectively, yields the desired result with constants

$$\begin{aligned}
D_1 &= \frac{1}{\nu}K_{\mathbf{v}}G_{\mathbf{u}}^{(1)} \\
D_2 &= \frac{1}{\nu\kappa}L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2 + 1}\left(K_\tau G_\theta^{(1)} + \mathbf{K}_6\|\theta_b\|_{0,3}\right) \\
D_3 &= \frac{1}{\nu}\left(G_\theta^{(2)}\sqrt{K_2^2 + 1} + \|\theta_b\|_{1,2}\right) \\
D_4 &= \frac{1}{\kappa}\left(K_\tau G_\theta^{(1)} + \mathbf{K}_6\|\theta_b\|_{0,3}\right).
\end{aligned} \tag{3.19}$$

□

Existence

Proving existence of solutions of Problem 3.5 can be achieved by application of the well-known Galerkin principle, following the work in [57] and the procedure proposed by [46]. Hereby, the application of the general fixed-point Theorem A.39 to a finite dimensional version of Problem 3.8 is combined with an approximation of the infinite dimensional problem by a series of finite dimensional systems.

Lemma 3.16. *(Existence of Solutions in Finite Dimensional Spaces)*
Let $\mathbf{W} \subset \mathbf{V}$, $T \subset \Theta$ denote finite dimensional Hilbert spaces and $\Phi \in H^1(\Omega)$ be given. Let Assumption 3.12 hold and assume that the boundary lifting θ_b is chosen such that $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}})$ for d, $a_{\mathbf{F}}$ defined in Lemma 3.13. Then, there exists a solution $(\mathbf{u}, \theta)$ of Problem 3.8 with $\lambda = 1$, i.e.

$$(\mathbf{u}, \theta) = \mathcal{F}[\mathbf{W}, T](\mathbf{u}, \theta).$$

Proof. Follows by application of Lemma 3.13 with Remark 3.14 (uniform stability), Lemma 3.11 (compactness of $\mathcal{F}[\mathbf{W}, T]\colon Y \to Y$ with $Y := \mathbf{W} \times T$) and Theorem A.39. □

The following theorem is based on the existence result in [57], with a slight modification to take into account the more general body force term $\mathbf{F}$.

Theorem 3.17. *(Existence of Solutions for Stationary Boussinesq Equations)*
Let $\Phi \in H^1(\Omega)$ be given, Assumption 3.12 be satisfied and assume that the boundary lifting θ_b is chosen such that $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}})$ for d, $a_{\mathbf{F}}$ defined in Lemma 3.13. Then, there exists a solution $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ of Problem 3.5 with $\lambda = 1$.

Proof. The proof for the case $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = \mathbf{u}$ works exactly like the one presented in [57]: Since $\mathbf{V}$ and Θ are closed subspaces of the separable normed spaces $W^{1,2}(\Omega)^d$ and $W^{1,2}(\Omega)$, respectively, they are separable as well according to Lemma A.4. Therefore, there a sequences $(\mathbf{W}_m)_m$, $(T_m)_m$ of finite dimensional subspaces satisfying

$$\mathbf{W}_m \subset \mathbf{V},\ \mathbf{W}_m \subset \mathbf{W}_{m+1} \text{ and } T_m \subset \Theta,\ T_m \subset T_{m+1}$$

with

$$\mathbf{V} = \overline{\bigcup_{n\in\mathbb{N}} \mathbf{W}_n} \text{ and } \Theta = \overline{\bigcup_{n\in\mathbb{N}} T_n}.$$

For each $n \in \mathbb{N}$ let $(\mathbf{u}_n, \theta_n) \in \mathbf{W}_n \times T_n$ denote the solution of Problem 3.5 with $\mathbf{V} \times \Theta$ replaced by $\mathbf{W}_n \times T_n$. These solutions exist due to Lemma 3.16. Moreover, they are uniformly bounded according to Lemma 3.13 and Remark 3.14,

$$\|\nabla \mathbf{u}_n\| + \|\nabla \theta_n\| \le G_{\mathbf{u}} + G_\theta \text{ for all } n \in \mathbb{N}.$$

Since $\mathbf{V}$ and Θ are Hilbert spaces, they are reflexive. Thus, there are $(\mathbf{u}_*, \theta_*) \in \mathbf{V} \times \Theta$ and a subsequence $(\mathbf{u}_k, \theta_k)_k \subset (\mathbf{u}_n, \theta_n)_n$ with $\mathbf{u}_k \rightharpoonup \mathbf{u}_*$ in $\mathbf{V}$ and $\theta_k \rightharpoonup \theta_*$ in Θ according to Theorem A.32. Using Assumption 3.12 (*iii*), one can show that $(\mathbf{u}_*, \theta_*)$ is indeed a solution of Problem 3.5 with $\lambda = 1$. For details see [57]. The assertion for $\bar{\mathbf{u}} \neq \mathbf{u}$ or $\tilde{\mathbf{u}} \neq \mathbf{u}$ follows analogously. □

3.2. The TEHD Boussinesq Problem

In this section, we consider the problem of finding a solution $(\mathbf{u}, \theta, \Phi)$ of the stationary TEHD Boussinesq equations. Existence of solutions is shown by applying a fixed-point iteration that is alternating between solutions $(\mathbf{u}, \theta)$ of the Boussinesq Problem 3.5 and solutions Φ of Gauss' law. Afterward, we show that solutions are unique under certain restrictions onto the data.

The variational formulation for the stationary TEHD equations is given as follows.

Problem 3.18. *(Stationary TEHD Equations)*
Let $\theta_b \in H^1(\Omega)$ and $\Phi_b \in H^1(\Omega)$ denote liftings of given boundary conditions and $\mathbf{F}\colon \Theta \times \Upsilon \to \mathbf{U}^*$, $\mathbf{f}_{\mathbf{v}} \in \mathbf{U}^*$, $f_\tau \in \Theta^*$, $f_\beta \in \Upsilon^*$ be given source terms. Each of the functions $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$, $\bar{\theta}$, $\bar{\Phi}$ can either denote a fixed element of $\mathbf{V}$, Θ and Υ, respectively, or the unknown variable $\mathbf{u}$, θ and Φ, respectively. Find $\mathbf{u} \in \mathbf{V}$, $\theta \in \Theta$, $\Phi \in \Upsilon$ such that for all $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$:

$$\begin{aligned}
\delta(\mathbf{u}, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) &= \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + \mathbf{f}_{\mathbf{v}}, \mathbf{v} \rangle_{\mathbf{U}^*} \\
\delta(\theta + \theta_b, \tau) + a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau \rangle_{\Theta^*} \\
a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta, \beta \rangle_{\Upsilon^*}.
\end{aligned}$$

Existence

The following theorem states existence of solutions of Problem 3.18. Moreover, it is shown that the H^1-norm of all solutions can be bounded by the problem data.

Theorem 3.19. *(Existence and Stability of Stationary TEHD Solutions)*
Let the assumptions of Problem 3.18 and Assumption 3.12 hold. Assume that θ_b is chosen such that $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}}(G_{\Phi,\Phi_b}))$ holds with

$$\begin{aligned} G_{\Phi,\Phi_b} &= \sqrt{K_2^2+1}\, G_\Phi + \|\Phi_b\|_{1,2} \\ G_\Phi &= \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}\|f_\beta\|_{\Upsilon^*}. \end{aligned}$$

Then, there exists a solution $(\mathbf{u}, \theta, \Phi)$ of Problem 3.18. Further, all solutions of 3.18 satisfy

$$\begin{aligned} \|\nabla\mathbf{u}\| &\le g_{\mathbf{u}}(\|\theta_b\|_{0,3}, a_{\mathbf{F}}(G_{\Phi,\Phi_b}), b_{\mathbf{F}}(G_{\Phi,\Phi_b}), \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) = G_{\mathbf{u}} \\ \|\nabla\theta\| &\le g_\theta(\|\theta_b\|_{0,3}, a_{\mathbf{F}}(G_{\Phi,\Phi_b}), b_{\mathbf{F}}(G_{\Phi,\Phi_b}), \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) = G_\theta \\ \|\nabla\Phi\| &\le G_\Phi, \end{aligned}$$

with functions $g_{\mathbf{u}}, g_\theta$ defined in Lemma 3.13.

Proof. As before, only the proof for the implicit case $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = \mathbf{u}, \bar{\Phi} = \Phi$ is shown.
Problem 3.18 is split into two parts: for given $\Phi_\# \in \Upsilon$ find $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ satisfying

$$(P_1): \begin{cases} \delta(\mathbf{u},\mathbf{v}) + a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) + c_{\mathbf{v}}(\mathbf{u},\mathbf{u},\mathbf{v}) - \langle \mathbf{F}(\theta+\theta_b, \Phi_\#+\Phi_b) + \mathbf{f_v}, \mathbf{v}\rangle_{\mathbf{U}^*} &= 0 \quad \forall \mathbf{v} \in \mathbf{V} \\ \delta(\theta+\theta_b,\tau) + a_\tau(\theta+\theta_b,\tau) + c_\tau(\mathbf{u},\theta+\theta_b,\tau) - \langle f_\tau, \tau\rangle_{\Theta^*} &= 0 \quad \forall \tau \in \Theta \end{cases}$$

and for given $\bar{\theta}_\# \in \Theta$ find $\Phi \in \Upsilon$ such that

$$(P_2): a_\beta(\bar{\theta}_\# + \theta_b, \Phi + \Phi_b, \beta) = \langle f_\beta, \beta\rangle_{\Upsilon^*} \quad \forall \beta \in \Upsilon.$$

Define the following fixed-point iteration: let $\Phi_0 \in \Upsilon$ be arbitrary and set for $n \in \mathbb{N}$

- $(\mathbf{u}_n, \theta_n)$ denotes the solution of (P_1) for $\Phi_\# = \Phi_{n-1}$
- if $\bar{\theta} = \theta$, then $\bar{\theta}_n := \theta_n$. Otherwise, $\bar{\theta}_n := \bar{\theta}$
- Φ_n denotes the solution of (P_2) for $\bar{\theta}_\# = \bar{\theta}_n$

Here, Theorem 3.17 garuantees the existence of $(\mathbf{u}_n, \theta_n)$. Moreover, under Assumption 3.2, the bilinear form $\Upsilon \times \Upsilon \ni (\Phi, \beta) \mapsto a_\beta(\bar{\theta}_n + \theta_b, \Phi, \beta) \in \mathbb{R}$ is bounded and coercive. Thus, there exists a unique solution $\Phi_n \in \Upsilon$ of (P_2) by Lax-Milgram, Theorem A.40, and it is bounded according to

$$\|\nabla\Phi_n\| \le \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}\|f_\beta\|_{\Upsilon^*} = G_\Phi.$$

On the other hand, for all $n \in \mathbb{N}$

$$\|\nabla\mathbf{u}_n\| \le G_{\mathbf{u}} \text{ and } \|\nabla\theta_n\| \le G_\theta,$$

according to Lemma 3.13 by using $\|\Phi_n + \Phi_b\|_{1,2} \le G_{\Phi,\Phi_b}$ and the monotonicity of $a_{\mathbf{F}}, b_{\mathbf{F}}, g_{\mathbf{u}}, g_\theta$. As in the proof of Theorem 3.17, there are $(\mathbf{u}_*, \theta_*, \Phi_*) \in \mathbf{V} \times \Theta \times \Upsilon$ and a subsequence $(\mathbf{u}_k, \theta_k, \Phi_k)_k \subset (\mathbf{u}_n, \theta_n, \Phi_n)_n$ with $\mathbf{u}_k \rightharpoonup \mathbf{u}_*$ in $\mathbf{V}$, $\theta_k \rightharpoonup \theta_*$ in Θ and $\Phi_k \rightharpoonup \Phi_*$ in Υ. The compact embedding

$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ for $d \in \{2,3\}$, Theorem A.92, additionally implies $\mathbf{u}_k \to \mathbf{u}_*$ in $L^4(\Omega)^d$, $\theta_k \to \theta_*$ in $L^4(\Omega)$ and $\Phi_k \to \theta_*$ in $L^4(\Omega)$. By Assumption 3.12 *(iii)*,

$$|\langle \mathbf{F}(\theta_* + \theta_b, \Phi_* + \Phi_b) - \mathbf{F}(\theta_n + \theta_b, \Phi_n + \Phi_b), \mathbf{v}\rangle_{\mathbf{U}^*}| \to 0 \text{ for all } \mathbf{v} \in \mathbf{U}.$$

Thus, as in the proof of Theorem 3.17, $(\mathbf{u}_*, \theta_*)$ solves (P_1) for $\Phi_\# = \Phi_*$.

It remains to show that Φ_* solves (P_2) for

$$\bar{\theta}_\# = \begin{cases} \theta_*, & \text{if } \bar{\theta} = \theta \\ \bar{\theta}, & \text{else} \end{cases}$$

In the former case, let $\beta \in C_D^\infty(\Omega)$ be arbitrary but fixed. Then, using Hölder's inequality and the Lipschitz continuity of ϵ,

$$\begin{aligned}
&|a_\beta(\theta_n + \theta_b, \Phi_n + \Phi_b, \beta) - a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta)| \\
\le\ & |((\epsilon(\theta_n + \theta_b) - \epsilon(\theta_* + \theta_b))\nabla(\Phi_n + \Phi_b), \nabla\beta)| + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_k - \Phi_*), \nabla\beta)| \\
\le\ & \|\epsilon(\theta_n + \theta_b) - \epsilon(\theta_* + \theta_b)\|_{0,3}\|\nabla(\Phi_n + \Phi_b)\|\|\nabla\beta\|_{0,6} + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_n - \Phi_*), \nabla\beta)| \\
\le\ & L_\epsilon G_{\Phi,\Phi_b}\|\theta_n - \theta_*\|_{0,3}\|\nabla\beta\|_{0,6} + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_n - \Phi_*), \nabla\beta)| \\
=:\ & A_n.
\end{aligned}$$

Here, $\lim_{n\to\infty} A_n = 0$ since $\theta_n \to \theta_*$ in $L^4(\Omega)$ and by $\Phi_n \rightharpoonup \Phi_*$ in Υ, respectively. Thus,

$$a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta) = \langle f_\beta, \beta\rangle_{\Upsilon^*} \text{ for all } \beta \in C_D^\infty(\Omega).$$

Since $C_D^\infty(\Omega)$ is dense in $H_D^1(\Omega)$ and the linear form $H_D^1(\Omega) \ni \beta \mapsto a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta)$ is continuous, Φ_* solves (P_2) for $\bar{\theta}_\# = \theta_*$. The case $\bar{\theta}_\# = \bar{\theta}$ follows analogously.

In order to show the stated energy norm estimate, let $(\mathbf{u}, \theta, \Phi) \in \mathbf{V}\times\Theta\times\Upsilon$ denote an arbitrary solution. As before, by setting $\beta = \Phi$,

$$\|\nabla\Phi\| \le \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}\|f_\beta\|_{\Upsilon^*} = G_\Phi,$$

which implies $\|\Phi + \Phi_b\|_{1,2} \le G_{\Phi,\Phi_b}$.

Moreover, by means of Lemma 3.13,

$$\begin{aligned}
\|\nabla\mathbf{u}\| &\le g_\mathbf{u}(\|\theta_b\|_{0,3}, a_\mathbf{F}, b_\mathbf{F}, \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) \\
\|\nabla\theta\| &\le g_\theta(\|\theta_b\|_{0,3}, a_\mathbf{F}, b_\mathbf{F}, \|\theta_b\|_{1,2}, \|\mathbf{f_v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*})
\end{aligned}$$

with constants $a_\mathbf{F} = a_\mathbf{F}(\|\Phi+\Phi_b\|_{1,2}) \le a_\mathbf{F}(G_{\Phi,\Phi_b})$ and $b_\mathbf{F} = b_\mathbf{F}(\|\Phi+\Phi_b\|_{1,2}) \le b_\mathbf{F}(G_{\Phi,\Phi_b})$. Since $g_\mathbf{u}, g_\theta$ are non-decreasing in their arguments x_2 and x_3, the stated result is obtained. □

So far, the TEHD equations have been considered in solenoidal form only. By means of a standard procedure, see e.g. Lemma IX.1.2 in [28], existence of solutions for the problem in mixed form can be shown.

Corollary 3.20. *(Recovering the Pressure)*
Let $(\mathbf{u}, \theta, \Phi) \in \mathbf{V}\times\Theta\times\Upsilon$ denote a solution of Problem 3.18. Then, there exists a pressure $p \in M$ such that $(\mathbf{u}, p, \theta, \Phi)$ satisfies for all $(\mathbf{v}, q, \tau, \beta) \in \mathbf{U}\times M\times\Theta\times\Upsilon$:

$$\begin{aligned}
\delta(\mathbf{u}, \mathbf{v}) + a_\mathbf{v}(\mathbf{u}, \mathbf{v}) + c_\mathbf{v}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) &= \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + \mathbf{f_v}, \mathbf{v}\rangle_{\mathbf{U}^*} \\
b(\mathbf{u}, q) &= 0 \\
\delta(\theta + \theta_b, \tau) + a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau\rangle_{\Theta^*} \\
a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta, \beta\rangle_{\Upsilon^*}.
\end{aligned}$$

Uniqueness

We now investigate uniqueness of solutions of the stationary TEHD Problem 3.18. To do so, we adapt the standard procedure for showing uniqueness of stationary solutions of the incompressible Navier Stokes equation, see e.g. [46], or of the stationary Boussinesq equations, see e.g. [57].

To this end, let the assumptions of the existence Theorem 3.19 hold and let $(\mathbf{u}_1, \theta_1, \Phi_1)$ and $(\mathbf{u}_2, \theta_2, \Phi_2)$ denote two solutions. Using the properties of $\mathbf{F}$ given by Assumption 3.12 (i), there is $L_{\mathbf{F}}^{(\Phi)} > 0$ such that

$$D_{\mathbf{F}} := \sup_{\mathbf{w}\in\mathbf{V},\theta\in\Theta} \frac{|\langle \mathbf{F}(\theta+\theta_b, \Phi_1+\Phi_b) - \mathbf{F}(\theta+\theta_b, \Phi_2+\Phi_b), \mathbf{w}\rangle_{\mathbf{U}^*}|}{\|\nabla\mathbf{w}\|\|\theta+\theta_b\|_{1,2}} \le L_{\mathbf{F}}^{(\Phi)}\|\Phi_1-\Phi_2\|_{1,2}. \tag{3.20}$$

Applying Lemma 3.15 with $\mathbf{F}_i = \mathbf{F}(\cdot, \Phi_i + \Phi_b)$ and introducing $d_x := x_1 - x_2$ with $x \in \{\mathbf{u}, \theta, \Phi, \bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ yields

$$\begin{aligned} \|\nabla d_{\mathbf{u}}\| &\le D_1\|\nabla d_{\bar{\mathbf{u}}}\| + D_2\|\nabla d_{\tilde{\mathbf{u}}}\| + D_5\|\nabla d_{\bar{\Phi}}\| && (3.21)\\ \|\nabla d_\theta\| &\le D_4\|\nabla d_{\tilde{\mathbf{u}}}\|. && (3.22)\end{aligned}$$

with $\{D_i\}_{i=1}^4$ given by (3.19) and

$$D_5 := \frac{1}{\nu}L_{\mathbf{F}}^{(\Phi)}\left(G_\theta(K_2^2+1) + \sqrt{K_2^2+1}\|\theta_b\|_{1,2}\right). \tag{3.23}$$

Here, $d_y = 0$ for $y \in \{\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ if the corresponding variable y is fixed. Otherwise, there holds $d_{\bar{\mathbf{u}}} = d_{\mathbf{u}},\ d_{\tilde{\mathbf{u}}} = d_\theta,\ d_{\bar{\theta}} = d_\theta$ and $d_{\bar{\Phi}} = d_\Phi$, respectively. According to the assumptions and results from Theorem 3.19, Gauss' law implies

$$\begin{aligned} \epsilon_-\|\nabla d_\Phi\|^2 &\le (\epsilon(\theta_2+\theta_b)\nabla d_\Phi, \nabla d_\Phi)\\ &= -((\epsilon(\theta_1+\theta_b) - \epsilon(\theta_2+\theta_b))\nabla(\Phi_1+\Phi_b), \nabla d_\Phi). \end{aligned} \tag{3.24}$$

If an additional regularity $\Phi_1 \in W^{1,3}(\Omega)$ is assumed, then the previous inequality together with $H^1(\Omega) \hookrightarrow L^6(\Omega)$ implies

$$\epsilon_-\|\nabla d_\Phi\|^2 \le L_\epsilon K_6\|\nabla d_{\bar{\theta}}\|\|\nabla(\Phi_1+\Phi_b)\|_{0,3}\|\nabla d_\Phi\|. \tag{3.25}$$

The previous estimates (3.21), (3.22) and (3.25) can be summarized as

$$\begin{aligned} \|\nabla d_{\mathbf{u}}\| &\le \alpha_1\|\nabla d_{\bar{\mathbf{u}}}\| + \alpha_2\|\nabla d_{\tilde{\mathbf{u}}}\| + \alpha_3\|\nabla d_{\bar{\Phi}}\|\\ \|\nabla d_\theta\| &\le \alpha_4\|\nabla d_{\tilde{\mathbf{u}}}\|\\ \|\nabla d_\Phi\| &\le \alpha_5(\|\nabla\Phi_1\|_{0,3})\|\nabla d_{\bar{\theta}}\|, \end{aligned} \tag{3.26}$$

with constants given by

$$\begin{aligned} \alpha_1 = D_1,\ \alpha_2 = D_2,\ \alpha_3 &= D_5,\ \alpha_4 = D_4\\ \alpha_5(s) &= \frac{L_\epsilon}{\epsilon_-}K_6(s + \|\nabla\Phi_b\|_{0,3}). \end{aligned} \tag{3.27}$$

Based on the set of inequalities (3.26), the following theorem yields conditions under which uniqueness of solutions for the stationary TEHD Problem 3.18 can be guaranteed. In doing so, one needs to differentiate w.r.t. the degree of implicitness in Problem 3.18.

Theorem 3.21. *(Uniqueness for Small Data)*
Let the assumptions of Theorem 3.19 hold and constants $\{\alpha_i\}_{i=1}^5$ *be given by (3.26). Assume that all solutions of Problem 3.18 satisfy* $\Phi \in W^{1,3}(\Omega)$ *and that the following condition holds:*

- (i) *if* $\tilde{\mathbf{u}} \in \mathbf{V}$ *fixed:*
  - $(i.i)$ *if* $\bar{\mathbf{u}} \in \mathbf{V}$ *fixed: nothing*
  - $(i.ii)$ *if* $\bar{\mathbf{u}} = \mathbf{u}$*:* $\alpha_1 < 1$
- (ii) *if* $\tilde{\mathbf{u}} = \mathbf{u}$*:*
  - $(ii.i)$ *if* $\bar{\theta} \in \Theta$ *fixed:* $\alpha_1 + \alpha_2 < 1$
  - $(ii.ii)$ *if* $\bar{\theta} = \theta$*:*
    - $(ii.ii.i)$ *if* $\bar{\Phi} \in \Upsilon$ *fixed:* $\alpha_1 + \alpha_2 < 1$
    - $(ii.ii.ii)$ *if* $\bar{\Phi} = \Phi$ *: there is* $R > 0$ *such that* $\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5(R) < 1$ *and* $\Phi \in B_R(0, W^{1,3}(\Omega))$

Then, there is at most one solution of Problem 3.18.

Proof. The assertions are proven by using (3.26) with $d_x = 0$ if $x \in \{\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ is fixed, or $d_{\bar{x}} = d_x$ for $x \in \{\mathbf{u}, \theta, \Phi\}$ otherwise. In case (i), i.e. $d_{\tilde{\mathbf{u}}} = 0$, (3.26) directly leads to $d_\theta = d_{\bar{\theta}} = d_\Phi = d_{\bar{\Phi}} = 0$. For $(i.i)$, $d_{\bar{\mathbf{u}}} = 0$ as well, implying $d_{\mathbf{u}} = d_\theta = d_\Phi = 0$ and therefore $(\mathbf{u}_1, \theta_1, \Phi_1) = (\mathbf{u}_2, \theta_2, \Phi_2)$. In case $(i.ii)$, one obtains

$$\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\|,$$

i.e. $d_{\mathbf{u}} = 0$ if $\alpha_1 < 1$. In $(ii.i)$, (3.26) leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq 0.
\end{aligned}$$

Thus, uniqueness of solutions is given if $\alpha_1 + \alpha_2 < 1$. In case $(ii.ii.i)$, i.e. $\bar{\theta} = \theta$ and $d_{\bar{\Phi}} = 0$, (3.26) leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq \alpha_5(\|\nabla \Phi_1\|_{0,3}) \|\nabla d_\theta\|.
\end{aligned}$$

Now, $\alpha_1 + \alpha_2 < 1$ implies $d_{\mathbf{u}} = 0$ and consequently, $d_\theta = 0$ and $d_\Phi = 0$. Finally, consider the case $(ii.ii.ii)$, i.e. the fully implicit problem. (3.26) leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| + \alpha_3\alpha_4\alpha_5(\|\nabla \Phi_1\|_{0,3}) \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq \alpha_5(\|\nabla \Phi_1\|_{0,3}) \|\nabla d_\theta\|.
\end{aligned}$$

In this case, $d_{\mathbf{u}} = 0$ if $\alpha_1 + \alpha_2 < 1$ and $\|\nabla \Phi_1\|_{0,3} < R$ such that

$$\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5(R) < 1.$$

□

The previous theorem shows that solutions are unique without any restriction on the problem data only, if the convection terms in both momentum and heat equation are explicitly given. Otherwise, uniqueness only holds under certain restrictions on the data. A closer look on the involved constants (3.27) reveals that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ tends to zero as $(\|\theta_b\|_{1,2}, \|\mathbf{f}_\mathbf{v}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*})$ tends to zero (see definition of D_i in (3.19)). Thus, if the energy that is put into the system is sufficiently small, uniqueness of solutions can be shown. This energy is given in terms of an applied external force, temperature source terms and temperature differences between the boundaries.

However, $\{\alpha_i\}_{i=1}^4$ are proportional to the inverse of viscosity ν and thermal diffusion coefficient κ, see (3.27) and (3.19). For most practical applications, ν and κ tend to take values of very small order of magnitude, e.g. for water there holds $\nu = \mathcal{O}\left(10^{-6}\right)$ and $\kappa = \mathcal{O}\left(10^{-7}\right)$. In such cases, uniqueness of solutions can only be guaranteed for very small values of $\|\theta_b\|_{1,2}$, i.e. for small temperature differences across the boundary and small volumes of the considered fluid container. Further, the constant α_3 is proportional to G_Φ, which in turn is proportional to $\|\Phi_b\|_{1,2}$ according to Theorem 3.19. Thus, with increasing voltage differences across the boundary of the container, the uniqueness conditions become even more restrictive.

In summary, for most applications the uniqueness statement is of little practical use as long as the system is considered with implicit convection fields $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$. However, if the instationary TEHD system is discretized in time such that $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$ are defined as numerical solution at a previous time step, then Theorem 3.21 can be applied without any restriction. This fact is used in Section 5.2 to show uniqueness of solutions of the spatially and temporally discretized system.

3.3. Modeling of DEP Force

In this final section on the stationary TEHD system, we propose several modelizations of the body force

$$\mathbf{F}_{DEP} + \mathbf{F}_{buo} = \alpha_e |\nabla\Phi|^2 \nabla\theta - \alpha_g \theta \mathbf{g} \tag{3.28}$$

that satisfy the main Assumption 3.12. By means of Theorem 3.19, one may state existence of solutions $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, θ, $\Phi \in H^1(\Omega)$ of the stationary TEHD Problem 3.18. Thus, Φ solves the Poisson equation

$$-\nabla \cdot (\epsilon \nabla\Phi) = f \tag{3.29}$$

with diffusion coefficient $\epsilon \in H^1(\Omega)$, subjected to mixed Dirichlet-Neumann boundary conditions and posed on a domain whose boundary is only Lipschitz. To the knowledge of the author it is unclear, whether more that L^2 regularity of $\nabla\Phi$ can be expected in this situation. For example, Theorem 1 in Chapter 6.3 of [26] provides interior H^2-regularity of Φ supposed that $\epsilon \in C^1(\Omega)$. On the other hand, H^2 regularity on the complete domain Ω can be shown if additionally $\partial\Omega \in C^2$ holds and pure Dirichlet boundary conditions are assumed, see Theorem 4. Whereas the case of pure Dirichlet conditions and smooth boundary could in principle be assured by the underlying application, $\epsilon \in C^1(\Omega)$ cannot be simply taken as a priori assumption, since the permittivity is temperature dependent.

In summary, it is not even guaranteed that $|\nabla\Phi|^2\nabla\theta$ is L^1-integrable. Thus, this term is not necessarily contained in $\mathbf{H}^{-1}(\Omega)$, meaning that $\langle \mathbf{F}_{DEP}, \mathbf{v} \rangle_{\mathbf{U}^*}$ is not well-defined for general $\mathbf{v} \in \mathbf{U}$. For this reason, it is necessary to replace $\mathbf{F}_{DEP}$ by an expression $\mathbf{F}(\theta, \Phi)$ that requires less regularity of θ and Φ to be an element of $\mathbf{H}^{-1}(\Omega)$. Moreover, Assumption 3.12 requires that the norm of $\mathbf{F}(\theta, \Phi)$ grows at most linearly with the norm of θ and weak H^1-convergence in the arguments (θ, Φ) must imply pointwise convergence of $\mathbf{F}(\theta, \Phi)$ in $\mathbf{U}^*$. In order to find an appropriate mathematical model of $\mathbf{F}_{DEP}$, we make use of the alternative DEP formulation

$$\mathbf{F}_{DEP,a} = -2\alpha_e (\nabla^2\Phi\nabla\Phi)\theta, \tag{3.30}$$

which has already been introduced in Section 2.

The models that are proposed in the following, rely on the idea of either replacing the potential Φ in $\mathbf{F}_{DEP,a}$ by some smooth, fixed function Φ_0, or on applying a smoothing operator to Φ.

Remark 3.22. When proving a priori error estimates for the fully discretized instationary TEHD equations, Section 5.2, it will be possible to work with another model force $\mathbf{F}_{s,K}$, that makes use of an cut-off operator. In this case, one can avoid the use of a predefined base potential Φ_0.

Definition 3.23. *(Approximations of $\mathbf{F}_{DEP} + \mathbf{F}_{buo}$ based on Linearization)*
Let $\Phi_0 \in W^{2,\infty}(\Omega)$ denote a fixed base potential. The standard body force with fixed base potential is defined as

$$\begin{aligned}\mathbf{F}_{s,0} \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto \alpha_e(|\nabla\Phi_0|^2 \nabla\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

The alternative body force with fixed potential is defined as

$$\begin{aligned}\mathbf{F}_{a,0} \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto -2\alpha_e((\nabla^2\Phi_0 \nabla\Phi_0)\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

The alternative body force with linearized potential is defined as

$$\begin{aligned}\mathbf{F}_{a,1} \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto -2\alpha_e((\nabla^2\Phi_0 \nabla\Phi)\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

The following Lemma shows that the previously defined approximations of $\mathbf{F}_{DEP} + \mathbf{F}_{buo}$ are suitable for proving existence and uniqueness of stationary solutions.

Lemma 3.24. *(Properties of the Body Forces)*
The body forces $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$ given by Definition 3.23 satisfy Assumption 3.12.

Proof. Assumption (i) and (ii) directly follow by the linearity of $\mathbf{F}_{s,0}, \mathbf{F}_{a,1}, \mathbf{F}_{a,1}$ w.r.t. θ and their growth rate w.r.t. $\|\Phi\|_{1,2}$ being linearly at most. Validness of (iii) is shown for $\mathbf{F}_{a,1}$ only: Let sequences $(\theta_n)_n \subset H^1(\Omega)$ and $(\Phi_n)_n \subset H^1(\Omega)$ be given that converge to $\theta_* \in H^1(\Omega)$ and $\Phi_* \in H^1(\Omega)$, respectively, in the following sense: $\theta_n \rightharpoonup \theta_*$ in $H^1(\Omega)$ and $\Phi_n \rightharpoonup \Phi_*$ in $H^1(\Omega)$. The compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$, Theorem A.92, implies $\theta_n \to \theta_*$ and $\Phi_n \to \Phi_*$ in $L^4(\Omega)$. Additionally, $\|\theta_n\|_{1,2} + \|\Phi_n\|_{1,2} \leq K$ for some $K > 0$ and all $n \in \mathbb{N}$ by Lemma A.33. Then, for arbitrary $\mathbf{v} \in \mathbf{U}$,

$$\begin{aligned}|\langle \mathbf{F}(\theta_*, \Phi_*) - \mathbf{F}(\theta_n, \Phi_n), \mathbf{v}\rangle| &\leq |\langle \mathbf{F}(\theta_*, \Phi_*) - \mathbf{F}(\theta_*, \Phi_n), \mathbf{v}\rangle| + |\langle \mathbf{F}(\theta_*, \Phi_n) - \mathbf{F}(\theta_n, \Phi_n), \mathbf{v}\rangle| \\ &\leq 2\alpha_e|((\nabla^2\Phi_0\nabla(\Phi_* - \Phi_n))\theta_*, \mathbf{v})| + 2\alpha_e|((\nabla^2\Phi_0\nabla\Phi_n)(\theta_* - \theta_n), \mathbf{v})| \\ &\quad + \alpha_g|((\theta_* - \theta_n)\mathbf{g}, \mathbf{v})| \\ &\leq C|((\nabla^2\Phi_0\nabla(\Phi_* - \Phi_n))\theta_*, \mathbf{v})| + CK\|\theta_* - \theta_n\|_{0,4}\|\mathbf{v}\|_{0,6} \\ &\quad + C\|\theta_* - \theta_n\|_{0,4}\|\mathbf{v}\|_{0,\frac{4}{3}}.\end{aligned}$$

Noting that $\theta_* \mathbf{v} \cdot \nabla^2\Phi_0 \in L^2(\Omega)$ follows by the assumption on Φ_0 and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, Theorem A.92, the first term converges to 0 due to $\Phi_n \rightharpoonup \Phi_*$ in H^1. Moreover, both other terms converge to 0 by $\theta_n \to \theta_*$ in L^4. $\square$

A heuristic justification for the proposed approximations can be given by employing Lemma 3.15. Therefore, let $\mathbf{F}^*(\theta,\Phi) := \langle (\nabla\Phi)^2 \nabla\theta, \cdot\rangle_{\mathbf{U}^*}$ denote the straightforward model of $\mathbf{F}_{DEP}$, and assume that a solution $(\mathbf{u}^*, \theta^*, \Phi^*)$ of Problem 3.18 with $\mathbf{F} = \mathbf{F}^*$ exists with the additional regularity $\theta^* \in L^\infty(\Omega), \Phi^* \in W^{1,6}(\Omega)$. In this case, $\mathbf{F}^*(\theta^*, \Phi^*) \in \mathbf{H}^{-1}(\Omega)$. Now, let $(\mathbf{u}, \theta, \Phi)$ denote another solution for $\mathbf{F} = \mathbf{F}_{s,0}$. If either $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ are fixed or if the input data (parameters $\{D_1, D_2, D_4\}$ in (3.19)) is sufficiently small, then Lemma 3.15 yields

$$\|\nabla(\mathbf{u}^* - \mathbf{u})\| + \|\nabla(\theta^* - \theta)\| \le C\, D_{\mathbf{F}}, \tag{3.31}$$

with

$$\begin{aligned} D_{\mathbf{F}} = \sup_{w\in\mathbf{V},\theta\in\Theta} & \frac{|\langle \mathbf{F}^*(\theta+\theta_b, \Phi^*+\Phi_b) - \mathbf{F}_{s,0}(\theta+\theta_b, \Phi+\Phi_b), w\rangle_{\mathbf{U}^*}|}{\|\nabla w\| \|\theta+\theta_b\|_{1,2}} \\ & \le C\|\nabla(\Phi^* - \Phi_0)\|_{0,6} \left(\|\nabla(\Phi^*+\Phi_b)\|_{0,6} + \|\nabla(\Phi_0+\Phi_b)\|_{0,6}\right). \end{aligned} \tag{3.32}$$

Thus, the difference between both solutions is proportional to the difference between the exact potential Φ^* and its approximation Φ_0. Such an a priori approximation Φ_0 could be defined as regularization of the solution Φ_{00} of Gauss's law for some given reference temperature $\theta_0 \in L^\infty(\Omega)$, i.e. Φ_{00} satisfies

$$(\epsilon(\theta_0)\nabla(\Phi_{00} + \Phi_b), \nabla\beta) = 0 \text{ for all } \beta \in \Upsilon. \tag{3.33}$$

Further note that

$$\|\nabla(\Phi_1 - \Phi_2)\| \le \frac{\|\epsilon^{(1)} - \epsilon^{(2)}\|_{0,\infty}}{\epsilon_-^{(2)}} \|\nabla(\Phi_1 + \Phi_b)\| \tag{3.34}$$

holds for potentials Φ_1, Φ_2 solving Gauss's law with respective permittivities $\epsilon^{(i)} \in L^\infty(\Omega)$, $\epsilon^{(i)} \ge \epsilon_-^{(i)} > 0$ a.e. If the permittivity is chosen as in [58], i.e. $\epsilon(\theta) = \epsilon_0\epsilon_r(1-\gamma\theta)$, we obtain for the relative H^1-deviation between Φ_{00} and the potential Φ^* determined by the correct temperature $\theta^* + \theta_b$:

$$\frac{\|\nabla(\Phi_{00} - \Phi^*)\|}{\|\nabla(\Phi^* + \Phi_b)\|} \le \frac{\gamma\|\theta_0 - (\theta^* + \theta_b)\|_{0,\infty}}{1 - \gamma\|\theta^* + \theta_b\|_{0,\infty}}. \tag{3.35}$$

When considering dielectric fluids with permittivity of low temperature sensitivity - typical values are $\gamma \approx 10^{-3} - 10^{-1}\ \mathrm{K}^{-1}$, [58] - and temperature regimes in which the Boussinesq approximation is fairly accurate, i.e. $\|\theta_0 - (\theta^* + \theta_b)\|_{0,\infty} \lessapprox 10$ K, then the right hand side term in (3.35) is of small order as well. If, in addition, the effect of regularization is moderate, i.e. $\|\nabla(\Phi_{00} - \Phi_0)\|$ is small, one may conclude that the difference between $(\mathbf{u}^*, \theta^*)$ and $(\mathbf{u}, \theta)$ is of moderate size. In Section 6.3 this reasoning is substantiated by numerical experiments.

Following the previously stated idea of regularization, we propose another approximation to $\mathbf{F}_{DEP,a}$ that is based on mollification of the electric potential. By replacing Φ by $S_{\psi,r}\Phi$ (see Definition A.110 for the mollification operator $S_{\psi,t}$) in the formulation of $\mathbf{F}_{DEP,a}$, we are able to construct a body force $\mathbf{F} = \mathbf{F}_r$ that satisfies Assumption 3.12. This implies existence of a family of solutions of Problem 3.18, $\{(\mathbf{u}_r, \theta_r, \Phi_r)\}_{r>0}$. Since $S_{\psi,r}$ converges pointwise to the identity operator on $L^p(\Omega)$ as $r \to 0$, see Lemma A.112, it is natural to ask whether a sequence of solutions $(\mathbf{u}_{r_n}, \theta_{r_n}, \Phi_{r_n})_n$ with $r_n \to 0$ converges in some sense. To answer this question, we need to introduce another modification of $\mathbf{F}_{DEP,a}$ to ensure that the associated growth parameters, $a_{\mathbf{F}} = a_{\mathbf{F}}(r)$ and $b_{\mathbf{F}} = b_{\mathbf{F}}(r)$, stay bounded for $r \to 0$.

Definition 3.25. *(Cut-Off Operator)*
For $K > 0$ let $\mathbf{m}_K \in \mathbf{L}^\infty(\mathbb{R}^d)$ denote a Lipschitz continuous function with $\mathbf{m}_K(x) = x$ if $|x| \leq K$. Define the cut-off operator

$$\begin{aligned} P_{\mathbf{m}_K} \colon \mathbf{L}^1(\Omega) &\to \mathbf{L}^\infty(\Omega) \\ \mathbf{g} &\mapsto \mathbf{m}_K \circ \mathbf{g}. \end{aligned}$$

By combining the previously defined operators, we may define a regularized electric gravity $\mathbf{g}_{E,r}$.

Definition 3.26. *(Regularized Electric Gravity and DEP Force)*
Let a mollifier ψ and a cut-off function $\mathbf{m}_K$ according to Definitions A.110 and 3.25 be given. For $r > 0$ define the regularized electric gravity

$$\begin{aligned} \mathbf{g}_{E,r} \colon H^1(\Omega) &\to \mathbf{L}^3(\Omega) \\ \Phi &\mapsto \mathbf{g}_{E,r}[\Phi] := P_{\mathbf{m}_K}\left[\nabla^2 S_{\psi,r}\Phi \cdot \nabla S_{\psi,r}\Phi\right]. \end{aligned}$$

The corresponding body force is defined by

$$\begin{aligned} \mathbf{F}_r \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto -2\alpha_e(\theta\, \mathbf{g}_{E,r}[\Phi], \cdot) - \alpha_g(\theta\, \mathbf{g}, \cdot). \end{aligned}$$

Lemma 3.27. *(Properties of Mollified Body Force)*
$\mathbf{F}_r$ satisfies Assumption 3.12 with growth rates $a_{\mathbf{F}}, b_{\mathbf{F}}$ being independent of r.

Proof. For $\Phi \in H^1(\Omega)$, note that the following estimates hold by means of Lemma A.112 and Definition 3.25

$$\begin{aligned} \|\nabla S_{\psi,r}\Phi\|_{0,6} &\leq C\|\nabla S_{\psi,r}\Phi\|_{0,\infty} \leq CC_\nabla(\psi, r)\|\Phi\| \\ \|\nabla^2 S_{\psi,r}\Phi\|_{0,6} &\leq C\|\nabla^2 S_{\psi,r}\Phi\|_{0,\infty} \leq CC_{\nabla^2}(\psi, r)\|\Phi\| \\ \|\mathbf{g}_{E,r}[\Phi]\|_{0,3} &\leq C\|\mathbf{m}_K\|_{0,\infty} \\ \|P_{\mathbf{m}_K}[f_1] - P_{\mathbf{m}_K}[f_2]\|_{0,p} &\leq L_{\mathbf{m}_K}\|f_1 - f_2\|_{0,p} \end{aligned} \tag{3.36}$$

with $L_{\mathbf{m}_K}$ denoting the Lipschitz constant of $\mathbf{m}_K$ and C denoting a generic constant that only depends on Ω and p. The proof is abbreviated by setting the involved physical parameters to 1 and only considering the DEP part of $\mathbf{F}_r$, since the stated assertions easily follow for $(\theta\mathbf{g}, \cdot)$. Assertion (i) of Assumption 3.12 follows from the estimates

$$\begin{aligned} |\langle \mathbf{F}_r(\theta_1, \Phi) - \mathbf{F}_r(\theta_2, \Phi), \mathbf{v}\rangle| &\leq \|\theta_1 - \theta_2\|\|\mathbf{g}_{E,r}[\Phi]\|_{0,3}\|\mathbf{v}\|_{0,6} \\ &\leq C\mathbf{K}_6\|\mathbf{m}_K\|_{0,\infty}\|\theta_1 - \theta_2\|_{1,2}\|\nabla\mathbf{v}\| \end{aligned}$$

and

$$\begin{aligned} |\langle \mathbf{F}_r(\theta, \Phi_1) - \mathbf{F}_r(\theta, \Phi_2), \mathbf{v}\rangle| &\leq \|\theta\|\|\mathbf{v}\|_{0,6}\|\mathbf{g}_{E,r}[\Phi_1] - \mathbf{g}_{E,r}[\Phi_2]\|_{0,3} \\ &\leq \mathbf{K}_6 L_{\mathbf{m}_K}\|\theta\|_{1,2}\|\nabla\mathbf{v}\| \\ &\quad \cdot \Big(\|\nabla^2 S_{\psi,r}(\Phi_1 - \Phi_2)\|_{0,6}\|\nabla S_{\psi,r}\Phi_1\|_{0,6} \\ &\quad + \|\nabla^2 S_{\psi,r}(\Phi_2)\|_{0,6}\|\nabla S_{\psi,r}(\Phi_1 - \Phi_2)\|_{0,6}\Big) \end{aligned}$$

for arbitrary $R > 0$, $\Phi_i, \theta_i \in B_R(0, H^1(\Omega))$, $\Phi, \theta \in H^1(\Omega), \mathbf{v} \in \mathbf{U}$ and by using (3.36). Assertion (ii) follows from

$$|\langle \mathbf{F}_r(\theta, \Phi), \mathbf{v}\rangle| = |(\theta \mathbf{g}_{E,r}[\Phi], \mathbf{v})| \leq \|\mathbf{g}_{E,r}[\Phi]\|_{0,3}\|\theta\|\,\|\mathbf{v}\|_{0,6} \leq C\|\mathbf{m}_K\|_{0,\infty}\|\theta\|_{1,2}\|\mathbf{v}\|_{0,6},$$

i.e. $b_\mathbf{F} = 0$ and $a_\mathbf{F} = C\|\mathbf{m}_K\|_{0,\infty}$. Finally, let weakly convergent sequences $(\theta_n)_n, (\Phi_n)_n$ be given with respective limits θ_*, Φ_* according to Assumption 3.12 (iii). Then,

$$\begin{aligned}
|\langle \mathbf{F}_r(\theta_*, \Phi_*) - \mathbf{F}_r(\theta_n, \Phi_n), \mathbf{v}\rangle| &\leq |((\theta_* - \theta_n)\mathbf{g}_{E,r}[\Phi_*], \mathbf{v})| + |(\theta_n\left(\mathbf{g}_{E,r}[\Phi_*] - \mathbf{g}_{E,r}[\Phi_n]\right), \mathbf{v})| \\
&\leq C\|\mathbf{m}_K\|_{0,\infty}\|\theta_* - \theta_n\|_{0,4}\|\mathbf{v}\|_{0,6} + CL_{\mathbf{m}_K}\|\theta_n\|\|\mathbf{v}\|_{0,6} \\
&\quad \cdot \Big(\|\nabla^2 S_{\psi,r}(\Phi_* - \Phi_n)\|_{0,6}\|\nabla S_{\psi,r}\Phi_*\|_{0,6} \\
&\qquad + \|\nabla^2 S_{\psi,r}(\Phi_n)\|_{0,6}\|\nabla S_{\psi,r}(\Phi_* - \Phi_n)\|_{0,6}\Big) \\
&=: A_n
\end{aligned}$$

Here, $\lim_{n\to\infty} A_n = 0$ follows from $\|\Phi_n - \Phi_*\|_{0,4} \to 0$, $\|\theta_n - \theta_*\|_{0,4} \to 0$, the uniform boundedness of $\|\Phi_n\|_{1,2}$, $\|\theta_n\|_{1,2}$ and the estimates (3.36). □

Due to Lemma 3.27, $\mathbf{F}_r$ satisfies the requirements of the existence Theorem 3.19 for all $r > 0$. Under the remaining conditions of 3.19, one may now state the existence of a family of solutions $\{(\mathbf{u}_r, \theta_r, \Phi_r)\}_{r>0}$ of Problem 3.18. Moreover, Theorem 3.19 provides energy bounds

$$\|\nabla \mathbf{u}_r\| \leq G_\mathbf{u}, \;\; \|\nabla\theta_r\| \leq G_\theta, \;\; \|\nabla\Phi_r\| \leq G_\Phi \tag{3.37}$$

with constants G_i that depend on $\mathbf{F}_r$ only via $a_\mathbf{F}(r) = \text{const}$, $b_\mathbf{F} = 0$. Thus, they are uniform w.r.t. r. Choosing an arbitrary sequence $r_n \to 0$ and using again the reflexivity of $\mathbf{U}, \Theta, \Upsilon$ (see Theorem A.90 and A.96), one obtains functions $\mathbf{u}_* \in \mathbf{U}$, $\theta_* \in \Theta$, $\Phi_* \in \Upsilon$ such that

$$\mathbf{u}_{r_n} \rightharpoonup \mathbf{u}_*, \;\; \theta_{r_n} \rightharpoonup \theta_*, \;\; \Phi_{r_n} \rightharpoonup \Phi_*. \tag{3.38}$$

By construction, $\|\mathbf{g}_{E,r_n}[\Phi_{r_n} + \Phi_b]\|_{0,3} \leq C\|\mathbf{m}_K\|_{0,\infty}$ and since $L^3(\Omega)$ is reflexive (Theorem A.84), there additionally exists some $\tilde{\mathbf{g}}_E \in \mathbf{L}^3(\Omega)$ such that

$$\mathbf{g}_{E,n} := \mathbf{g}_{E,r_n}[\Phi_{r_n}] \rightharpoonup \tilde{\mathbf{g}}_E. \tag{3.39}$$

For arbitrary test functions $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$ of Problem 3.18, one now obtains convergence of all bi- and trilinear forms as in the proof of Theorem 3.19, e.g. $a_\mathbf{v}(\mathbf{u}_{r_n}, \mathbf{v}) \to a_\mathbf{v}(\mathbf{u}_*, \mathbf{v})$, etc. Moreover, the term $\langle \mathbf{F}_{r_n}(\theta_{r_n} + \theta_b, \Phi_{r_n} + \Phi_b), \mathbf{v}\rangle_{\mathbf{U}^*}$ converges because of

$$\begin{aligned}
|(\theta_n \mathbf{g}_{E,n}, \mathbf{v}) - (\theta_* \mathbf{g}_{E,*}, \mathbf{v})| &\leq |(\theta_*(\mathbf{g}_{E,*} - \mathbf{g}_{E,n}), \mathbf{v})| + |((\theta_{r_n} - \theta_*)\mathbf{g}_{E,n}, \mathbf{v})| \\
&\leq |(\theta_*(\mathbf{g}_{E,*} - \mathbf{g}_{E,n}), \mathbf{v})| + C\|\theta_{r_n} - \theta_*\|_{0,4}\|\mathbf{g}_{E,n}\|_{0,3}\|\mathbf{v}\|_{0,6} \\
&=: A_n.
\end{aligned} \tag{3.40}$$

Here, $\lim_{n\to\infty} A_n = 0$ due to $\mathbf{g}_{E,n} \rightharpoonup \tilde{\mathbf{g}}_E$, and the second term by uniform $\mathbf{L}^3$-boundedness of $\mathbf{g}_{E,n}$ and the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$.

These considerations are summarized by the following theorem.

Theorem 3.28. *(Existence of Solutions with Weak Approximation of Electric Gravity)* *Let the requirements of Theorem 3.19 and Lemma 3.27 hold. Then, there are* $(\mathbf{u}, \theta, \Phi, \tilde{\mathbf{g}}_E) \in \mathbf{V} \times \Theta \times \Upsilon \times \mathbf{L}^3(\Omega)$ *such that for all* $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$*:*

$$\begin{aligned}
\delta(\mathbf{u}, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) &= -((\theta + \theta_b)(2\alpha_e \tilde{\mathbf{g}}_E + \alpha_g \mathbf{g}), \mathbf{v}) + \langle \mathbf{f}_{\mathbf{v}}, \mathbf{v} \rangle_{\mathbf{U}^*} \\
\delta(\theta + \theta_b, \tau) + a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau \rangle_{\Theta^*} \\
a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta, \beta \rangle_{\Upsilon^*}.
\end{aligned}$$

The connection between electric gravity $\tilde{\mathbf{g}}_E$ *and electric potential* Φ *is given in the sense that there is a sequence* $(r_n, \Phi_n)_n \subset (0, \infty) \times \Upsilon$ *with*

$$\begin{aligned}
r_n &\to 0 \\
\Phi_n + \Phi_b &\rightharpoonup \Phi + \Phi_b \ \text{ in } H^1(\Omega) \\
\mathbf{g}_{E,r_n}[\Phi_n + \Phi_b] &\rightharpoonup \tilde{\mathbf{g}}_E \qquad \text{in } \mathbf{L}^3(\Omega),
\end{aligned} \tag{3.41}$$

with approximate electric gravity $\mathbf{g}_{E,r_n}[\Phi_n + \Phi_b]$ *given by Definition 3.26.*

According to Theorem 3.28, we obtain a notion of a solution for the stationary TEHD equations 3.18, where the strong connection between electric gravity and potential (2.27),

$$\mathbf{g}_E(\Phi + \Phi_b) = \nabla^2(\Phi + \Phi_b) \cdot \nabla(\Phi + \Phi_b), \tag{3.42}$$

is replaced by the weaker form (3.41).

4. Analysis of the Instationary Problem

In this section, we consider the instationary TEHD Boussinesq equations, given by

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{F}(\theta, \Phi) + \mathbf{f_v} \\ \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \theta + (\mathbf{u} \cdot \nabla)\theta - \kappa \Delta \theta &= f_\tau \\ -\nabla \cdot (\epsilon(\theta) \nabla \Phi) &= f_\beta. \end{aligned} \tag{4.1}$$

System (4.1) directly follows from the TEHD Boussinesq system (2.28) through replacing $\mathbf{F}_{DEP} + \mathbf{F}_{buo}$ by the general body force term $\mathbf{F}(\theta, \Phi)$, as it is done in the stationary case. In contrast to the stationary system (3.1), (4.1) is only investigated in a completely nonlinear version. In particular, the convection fields in momentum and heat equation are not given by predefined functions $\bar{\mathbf{u}}, \tilde{\mathbf{u}}$.

The instationary equations are subjected to the same boundary conditions as their stationary counterparts, i.e.

$$\begin{aligned} \mathbf{u} &= 0 \quad \text{ on } \partial\Omega = \Gamma_D + \Gamma_N \\ \theta &= \theta_D \ \text{ on } \Gamma_D, \ \ \nabla\theta \cdot \mathbf{n} = 0 \text{ on } \Gamma_N \\ \Phi &= \Phi_D \text{ on } \Gamma_D, \ \nabla\Phi \cdot \mathbf{n} = 0 \text{ on } \Gamma_N. \end{aligned} \tag{4.2}$$

The aim of this section consists in setting up an approriate variational formulation and defining the notion of a weak solution for (4.1). Further, existence and stability of such solutions is shown by the main result of this chapter, Thereom 4.3. The corresponding proof is an extension of the proof of Theorem III.4.1 in [77] for showing existence of weak solutions for the incompressible Navier-Stokes equations. This result is obtained by a semi-discretization in time which leads to a series of stationary problems. Existence of such semi-discrete solutions can be guaranteed by the previously derived results for the stationary case. Using linear interpolation between the associated time steps, an approximate solution that is defined for every point in the entire time interval $[0, T]$ can be constructed. When the time step size k tends towards 0, one can show that the corresponding sequence of approximate solutions converges in weak topology towards a limit function that itself satisfies the variational formulation of the instationary system.

The outline of this section is as follows: first the variational formulation and the main result are stated. The proof of this theorem is split into several subsections. After some preliminaries in Section 4.1, the aforementioned semi-discrete problems are discussed in Section 4.2. In Section 4.3, the limit process for $k \to 0$ is investigated and the final proof of Theorem 4.3 is given. In the concluding Section 4.4, the modeling of the DEP force is once again considered, since one has to impose additional requirements compared to the stationary case.

For the remaining of this work, and if not stated otherwise, let $\|\cdot\|_{p;X} := \|\cdot\|_{L^p(0,T;X)}$ for some Banach space X and $p \in [1, \infty]$. Moreover, domain and physical parameters are subjected to the previous conditions, Assumption 3.1 and 3.2, and the same function spaces $\mathbf{U}$, $\mathbf{V}$, M, Θ, Υ as defined in Section 3 are used. In addition, let $\mathbf{V}_2 := \mathbf{V}_2(\Omega)$, $\mathbf{H} := \mathbf{H}(\Omega)$ denote the closure of $\{\mathbf{v} \in \mathcal{D}(\Omega)^d \colon \ \nabla \cdot \mathbf{v} = 0\}$ in $\mathbf{H}^2_0(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, and $\Theta_2 := H^2_D(\Omega)$, see Definition A.89 and A.95. The corresponding duals norms of these spaces naturally appear in the bounds of the temporal variation of the semi-discrete solutions. The underlying bi- and trilinear forms will be the same as in Section 3, i.e.

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) &:= \nu(\nabla\mathbf{u},\nabla\mathbf{v}), & c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w}) &:= (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) \\ a_\tau(\theta,\tau) &:= \kappa(\nabla\theta,\nabla\tau), & c_\tau(\mathbf{u},\theta,\tau) &:= (\mathbf{u}\cdot\nabla\theta,\tau) \\ a_\beta(\theta,\Phi,\beta) &:= (\epsilon(\theta)\nabla\Phi,\nabla\beta), & b(\mathbf{u},q) &:= (\nabla\cdot\mathbf{u},q). \end{aligned} \tag{4.3}$$

The following notion of a weak solution is comparable to Definition 7.8 in [41]. There, the test functions are split into purely space and time dependent contributions. The spatial variation is done analogously to the stationary case, whereas the temporal variational formulation is posed in $C^\infty([0,T])^*$. Moreover, integration by parts w.r.t. ∂_t is applied when deriving the weak formulation from the classical formulation (4.1). Thus, $\partial_t\mathbf{u}$ and $\partial_t\theta$ do not explicitly occur.

Problem 4.1. *(Instationary TEHD Equations in Solenoidal Form)*
Let initial conditions $(\mathbf{u}_0,\theta_0)\in\mathbf{V}\times\Theta$, boundary liftings $(\theta_b,\Phi_b)\in H^1(\Omega)\times H^1(\Omega)$ source terms $f_{\mathbf{v}}\in L^2(0,T;\mathbf{U}^*)$, $f_\tau\in L^2(0,T;\Theta^*)$, $f_\beta\in L^\infty(0,T;\Upsilon^*)$ and $T>0$ be given. Find

$$\begin{aligned} &\mathbf{u}\in L^2(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H}) \\ &\theta\in L^2(0,T;\Theta)\cap L^\infty(0,T;L^2) \\ &\Phi\in L^\infty(0,T;\Upsilon) \end{aligned}$$

such that for all $(\mathbf{v},\tau,\beta)\in\mathbf{V}\times\Theta\times\Upsilon$ and $\psi\in C^\infty([0,T])$ with $\psi(T)=0$:

$$\begin{aligned} &\int_0^T -(\mathbf{u},\mathbf{v})\psi' + \{a_{\mathbf{v}}(\mathbf{u},\mathbf{v})+c_{\mathbf{v}}(\mathbf{u},\mathbf{u},\mathbf{v})\}\,\psi\,dt \\ &\quad - \int_0^T \langle\mathbf{F}(\theta+\theta_b,\Phi+\Phi_b)+\mathbf{f}_{\mathbf{v}},\mathbf{v}\rangle_{\mathbf{U}^*}\psi\,dt &&= (\mathbf{u}_0,\mathbf{v})\psi(0) && (4.4)\\ &\int_0^T -(\theta,\tau)\psi' + \{a_\tau(\theta+\theta_b,\tau)+c_\tau(\mathbf{u},\theta+\theta_b,\tau)\}\,\psi\,dt \\ &\quad - \int_0^T \langle f_\tau,\tau\rangle_{\Theta^*}\psi\,dt &&= (\theta_0,\tau)\psi(0) && (4.5)\\ &\int_0^T \{a_\beta(\theta+\theta_b,\Phi+\Phi_b,\beta)-\langle f_\beta,\beta\rangle_{\Upsilon^*}\}\,\psi\,dt &&= 0. && (4.6) \end{aligned}$$

Similar to the stationary case, the general body force $\mathbf{F}$ has to satisfy certain requirements. First, since the instationary problem will be reduced to a series of stationary ones and the results of Section 3 are reused, $\mathbf{F}$ has to satisfy all conditions posed in Assumption 3.12. Further, an additional continuity result, similar to Assumption 3.12 (*iii*), is needed for passing to the limit $k\to 0$. In Section 4.4, it is shown that the DEP models $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$, $\mathbf{F}_{a,1}$, given by Definition 3.23, satisfy this additional assumption as well.

Assumption 4.2. *(Body Force for Instationary Problem)*
Let $\mathbf{F}\colon H^1(\Omega)\times H^1(\Omega)\to\mathbf{U}^*$ satisfy Assumption 3.12 and for bounded sequences $(\theta_n)_n\subset L^2(0,T;H^1)$, $(\Phi_n)_n\subset L^\infty(0,T;H^1)$ with

$$\begin{aligned} &\theta_n\rightharpoonup\theta \text{ in } L^2(0,T;H^1),\ \ \theta_n\to\theta \text{ in } L^2(0,T;L^2) \\ &\Phi_n\rightharpoonup\Phi \text{ in } L^2(0,T;H^1) \end{aligned}$$

assume that

$$\int_0^T \langle\mathbf{F}(\theta_n(t),\Phi_n(t)),\mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt \to \int_0^T \langle\mathbf{F}(\theta(t),\Phi(t)),\mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt$$

for all $\mathbf{v} \in \mathcal{V}(\Omega)$ *and* $\psi \in C^\infty([0,T])$.

The following theorem states the existence of solutions of Problem 4.1. For solutions that are constructed in the course of the proof of Theorem 4.3, further properties can be deduced. For instance, these solutions turn out to be weakly differentiable w.r.t. t according to Definition A.66 and the corresponding derivatives are L^1-integrable. Moreover, the norm of these solutions can be bounded by the input data.

Theorem 4.3. *(Existence of Instationary Solutions)*
Let Assumption 4.2 hold and assume that the boundary lifting θ_b *is chosen such that* $\|\theta_b\|_{0,3}$ *is sufficiently small in order to meet the requirements of Lemma 4.10 and 4.15. Then, there exists a solution* $(\mathbf{u}, \theta, \Phi)$ *of Problem 4.1. In addition,* $\mathbf{u}$ *and* θ *are weakly differentiable w.r.t.* t *and there holds* $\mathbf{u}' \in L^1(0,T;\mathbf{V}^*)$, $\theta' \in L^1(0,T;\Theta^*)$. *Furthermore, the following estimates hold*

$$\begin{aligned}
\|\mathbf{u}\|_{\infty;\mathbf{H}} &\leq G_{\mathbf{u},\infty,\mathbf{H}} \quad \text{and } \|\mathbf{u}\|_{2;\mathbf{V}} \leq G_{\mathbf{u},2,\mathbf{V}} \\
\|\mathbf{u}'\|_{2;\mathbf{V}_2^*} &\leq G_{\mathbf{w},2,\mathbf{V}_2^*} \\
\|\theta\|_{\infty;L^2} &\leq G_{\theta,\infty,L^2} \quad \text{and } \|\theta\|_{2;\Theta} \leq G_{\theta,2,\Theta} \\
\|\theta'\|_{2;\Theta_2^*} &\leq G_{\eta,2,\Theta_2^*} \\
\|\Phi\|_{2;\Upsilon} &\leq \sqrt{T} G_\Phi,
\end{aligned}$$

with constants G. *that only depend on the problem data and which are defined in Corollary 4.17.*

The proof of Theorem 4.3 is an extension to the proof of Theorem III.4.1 in [77] and it is split into several parts given in Section 4.1, 4.2 and 4.3. The summary of the proof is given in the end of Section 4.3.

4.1. Preliminaries

In this section, certain preliminary results are collected which are frequently used for proving Theorem 4.3.

Lemma 4.4. *(Lemma III.4.1 in [77])*
The trilinear forms $c_\mathbf{V}$ *and* $c_\mathcal{T}$ *satisfy*

$$\begin{aligned}
K_{\mathbf{v},2} &:= \sup_{\mathbf{u},\mathbf{v} \in \mathbf{V}\setminus\{0\}, \mathbf{w} \in \mathbf{V}_2\setminus\{0\}} \frac{c_\mathbf{V}(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\mathbf{u}\|\|\nabla\mathbf{v}\|\|\mathbf{w}\|_{2,2}} < \infty \\
K_{\tau,2} &:= \sup_{\mathbf{u} \in \mathbf{V}\setminus\{0\}, \theta \in \Theta\setminus\{0\}, \tau \in \Theta_2\setminus\{0\}} \frac{c_\mathcal{T}(\mathbf{u},\theta,\tau)}{\|\mathbf{u}\|\|\nabla\tau\|\|\tau\|_{2,2}} < \infty
\end{aligned}$$

Proof. The assertion for $c_\mathbf{V}$ is shown in Lemma III.4.1 in [77], the assertion for $c_\mathcal{T}$ follows analogously. □

Lemma 4.5.
Let $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, $\theta \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ *and* $\Phi \in L^\infty(0,T;H^1)$.

Moreover, let Assumption 4.2 hold. Then,

$$
\begin{aligned}
&[0,T] \ni t \mapsto a_{\mathbf{v}}(\mathbf{u}(t),\cdot) && \text{is an element of } L^2(0,T;\mathbf{U}^*)\\
&[0,T] \ni t \mapsto a_{\tau}(\theta(t),\cdot) && \text{is an element of } L^2(0,T;\Theta^*)\\
&[0,T] \ni t \mapsto a_{\beta}(\theta(t),\Phi(t),\cdot) && \text{is an element of } L^\infty(0,T;\Upsilon^*)\\
&[0,T] \ni t \mapsto c_{\mathbf{v}}(\mathbf{u}(t),\mathbf{u}(t),\cdot) && \text{is an element of } L^1(0,T;\mathbf{U}^*)\\
&[0,T] \ni t \mapsto c_{\tau}(\mathbf{u}(t),\theta(t),\cdot) && \text{is an element of } L^1(0,T;\Theta^*)\\
&[0,T] \ni t \mapsto \mathbf{F}(\theta(t),\Phi(t),\cdot) && \text{is an element of } L^2(0,T;\mathbf{U}^*).
\end{aligned}
$$

Proof. The result for a_β follows from $\|a_\beta(\theta,\Phi,\cdot)\|_{\Upsilon^*} \le \epsilon_+\|\nabla\Phi\|$ according to Assumption 3.2. The assertion for $c_{\mathbf{v}}$ is shown in Lemma III.3.1 and III.4.2 in [77], the assertion for c_τ follows analogously. According to Assumption 4.2, $\|\mathbf{F}(\theta,\Phi,\cdot)\|_{\mathbf{U}^*} \le a_{\mathbf{F}}(\|\Phi\|_{1,2})\|\theta\|_{1,2} + b_{\mathbf{F}}(\|\Phi\|_{1,2})$ with non-decreasing functions $a_{\mathbf{F}}$ and $b_{\mathbf{F}}$. □

The next lemma shows that for converging sequences of velocity and temperature, the corresponding convection terms converge pointwise, i.e. for a fixed choice of test functions.

Lemma 4.6. *(Convergence of Convection Terms, Lemma III.3.2 in [77])*
Let $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$ and $\theta \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$. Let sequences $(\mathbf{u}_n)_n \subset L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$ and $(\theta_n)_n \subset L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ be given with

$$
\begin{aligned}
\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;\mathbf{V}),\ \ \mathbf{u}_n \to \mathbf{u} \text{ in } L^2(0,T;\mathbf{H})\\
\theta_n \rightharpoonup \theta \text{ in } L^2(0,T;H^1),\ \theta_n \to \theta \text{ in } L^2(0,T;L^2).
\end{aligned}
$$

Then, there holds for all $\mathbf{v} \in C^1([0,T]\times\overline{\Omega})^d$ and $\tau \in C^1([0,T]\times\overline{\Omega})$

$$
\begin{aligned}
\int_0^T c_{\mathbf{v}}(\mathbf{u}_n(t),\mathbf{u}_n(t),\mathbf{v}(t))\,dt &\to \int_0^T c_{\mathbf{v}}(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}(t))\,dt\\
\int_0^T c_{\tau}(\mathbf{u}_n(t),\theta_n(t),\tau(t))\,dt &\to \int_0^T c_{\tau}(\mathbf{u}(t),\theta(t),\tau(t))\,dt
\end{aligned}
$$

Proof. The assertion for $c_{\mathbf{v}}$ is shown in Lemma III.3.2 in [77], the assertion for c_τ follows analogously. □

When deriving a semi-discrete, stationary formulation of Problem 4.1, the time-continuous source terms $\mathbf{f}_{\mathbf{v}},\ f_\tau,\ f_\beta$ need to be transformed to piecewise-constant sequences which converge towards the original source terms as the time step size converges towards 0. This is done by taking the average over small time intervals, as shown by the following lemmas.

Lemma 4.7. *(Lemma III.4.5 in [77])*
Let X denote a Banach space and $f \in L^2(0,T;X)$. Let $N \in \mathbb{N}$ and $k = \frac{T}{N}$ be given and define

$$
f^m := \frac{1}{k}\int_{(m-1)k}^{mk} f(t)\,dt \in X \text{ for } m = 1,\cdots,N.
$$

Then,

$$
k\sum_{m=1}^{N} \|f^m\|_X^2 \le \int_0^T \|f(t)\|_X^2\,dt.
$$

Lemma 4.8. *(Lemma III.4.9 in [77])*
Let X denote a Banach space and $f \in L^2(0,T;X)$. For $N \in \mathbb{N}$ and $k := \frac{T}{N}$ let elements $f^m \in X$ for $m = 1, \cdots, N$ be given according to Lemma 4.7. Define

$$f_N : [0,T] \to X,\ t \mapsto \sum_{m=1}^{N} f^m \chi_{[(m-1)k,mk)}(t).$$

Then, $f_N \to f$ in $L^2(0,T;X)$ as $N \to \infty$.

4.2. Semi-Discrete Problem

In this section, a semi-discrete formulation of Problem 4.1 is considered. Following the approach proposed in chapter III.4 in [77], this formulation is derived by applying the implicit Euler time stepping scheme to the continuous equations. In this way, the stability and existence results for the stationary case, Section 3, can be reused to analyze the sequence of semi-discrete solutions.

Problem 4.9. *(Semi-Discrete Problem)*
Let $T > 0$, initial conditions $(\mathbf{u}_0, \theta_0)$, boundary liftings (θ_b, Φ_b) and source terms $\mathbf{f_v}$, f_τ, f_β be given as in Problem 4.1. For $N \in \mathbb{N}$, $k = \frac{T}{N}$ a sequence of stationary problems is given for $1 \le m \le N$:
Find $(\mathbf{u}^m, \theta^m, \Phi^m) \in \mathbf{V} \times \Theta \times \Upsilon$ such that for all $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$:

$$\begin{aligned}
\frac{1}{k}(\mathbf{u}^m - \mathbf{u}^{m-1}, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}^m, \mathbf{v}) + c_{\mathbf{v}}(\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}) &= \langle \mathbf{F}(\theta^m + \theta_b, \Phi^m + \Phi_b) + \mathbf{f}^m_{\mathbf{v}}, \mathbf{v}\rangle_{\mathbf{U}^*} \\
\frac{1}{k}(\theta^m - \theta^{m-1}, \tau) + a_\tau(\theta^m + \theta_b, \tau) + c_\tau(\mathbf{u}^m, \theta^m + \theta_b, \tau) &= \langle f^m_\tau, \tau\rangle_{\Theta^*} \\
a_\beta(\theta^m + \theta_b, \Phi^m + \Phi_b, \beta) &= \langle f^m_\beta, \beta\rangle_{\Upsilon^*},
\end{aligned}$$

where $\mathbf{f}^m_{\mathbf{v}}, f^m_\tau, f^m_\beta$ are defined according to Lemma 4.7 and $(\mathbf{u}^0, \theta^0) := (\mathbf{u}_0, \theta_0)$.

The following lemma shows that a sequence of semi-discrete solutions exists for each $N \ge 1$. This result is a direct consequence of the stationary existence Theorem 3.19.

Lemma 4.10. *(Existence of Semi-Discrete Solutions)*
Let Assumption 4.2 hold and assume that θ_b is chosen such that $\|\theta_b\|_{0,3} \le d(a_{\mathbf{F}}(G_{\Phi,\Phi_b}))$ with d given by Lemma 3.13 and $a_{\mathbf{F}}$ given by Assumption 3.12, i.e.

$$d(s) = \frac{1}{Ks} \text{ with } K = \frac{\mathbf{K}_6}{\kappa\nu}\sqrt{8(K_2^2+1)}$$
$$a_{\mathbf{F}} : [0,\infty) \to [0,\infty) \text{ non-decreasing}$$

and

$$G_{\Phi,\Phi_b} = \sqrt{K_2^2+1}\, G_\Phi + \|\Phi_b\|_{1,2}, \quad G_\Phi = \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}\|f_\beta\|_{\infty;\Upsilon^*}.$$

Then, there exists a sequence of solutions $\{(\mathbf{u}^m, \theta^m, \Phi^m)\}_{m=1}^N$ of Problem 4.9.

Proof. For fixed $m \in \{1, \dots, N\}$, the corresponding stationary equations defined in Problem 4.9 fit into the framework of Problem 3.18 with $\delta = \frac{1}{k}$ and source terms (in the notation of Problem 3.18)

$$\begin{aligned} \mathbf{f}_{\mathbf{v}} &\mathrel{\widehat{=}} \mathbf{f}_{\mathbf{v}}^m + \frac{1}{k}\mathbf{u}^{m-1} \\ f_\tau &\mathrel{\widehat{=}} f_\tau^m + \frac{1}{k}(\theta^{m-1} + \theta_b) \\ f_\beta &\mathrel{\widehat{=}} f_\beta^m. \end{aligned}$$

Moreover, let

$$\begin{aligned} G_{\Phi,\Phi_b}^m &:= \sqrt{K_2^2 + 1}\, G_\Phi^m + \|\Phi_b\|_{1,2} \\ G_\Phi^m &:= \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}\|f_\beta^m\|_{\Upsilon^*}. \end{aligned}$$

Since $\|f_\beta^m\|_{\Upsilon^*} \leq \|f_\beta\|_{\infty;\Upsilon^*}$, there holds

$$d(a_{\mathbf{F}}(G_{\Phi,\Phi_b}^m)) \geq d(a_{\mathbf{F}}(G_{\Phi,\Phi_b})) \geq \|\theta_b\|_{0,3}.$$

Thus, the requirements of Theorem 3.19 are satisfied and a solution $(\mathbf{u}_m, \theta_m, \Phi_m)$ exists. □

Based on the solution of Problem 4.9 for some $N \geq 1$, one may now construct associated functions that are defined for all $t \in [0, T]$ by using characteristic functions and linear interpolation, see III.4.2 in [77].

Definition 4.11. *(Sequence of Approximate, Instationary Solutions)*
Let the assertions of Lemma 4.10 hold. For given $N \in \mathbb{N}$ let $\{(\mathbf{u}_N^m, \theta_N^m, \Phi_N^m)\}_{m=0}^N \subset \mathbf{V} \times \Theta \times \Upsilon$ denote the sequence of semi-discrete solutions of Problem 4.9. These solutions exist according to Lemma 4.10. Define interpolating functions

$$\begin{aligned} \mathbf{u}_N &\colon [0,T) \to \mathbf{V},\ \mathbf{u}_N(t) = \mathbf{u}_N^m \ \ \textit{for } t \in [(m-1)k, mk),\ 1 \leq m \leq N \\ \theta_N &\colon [0,T) \to \Theta,\ \theta_N(t) = \theta_N^m \ \ \textit{for } t \in [(m-1)k, mk),\ 1 \leq m \leq N \\ \Phi_N &\colon [0,T) \to \Upsilon,\ \Phi_N(t) = \Phi_N^m \ \ \textit{for } t \in [(m-1)k, mk),\ 1 \leq m \leq N \\ \mathbf{w}_N &\colon [0,T] \to \mathbf{V},\ \textit{continuous and piecewise linear with } \mathbf{w}_N(mk) = \mathbf{u}_N^m,\ 0 \leq m \leq N \\ \eta_N &\colon [0,T] \to \Theta,\ \textit{continuous and piecewise linear with } \eta_N(mk) = \theta_N^m,\ 0 \leq m \leq N. \end{aligned}$$

In addition, let functions

$$\begin{aligned} \mathbf{f}_{\mathbf{v},N} &\colon [0,T] \to \mathbf{U}^* \\ f_{\theta,N} &\colon [0,T] \to \Theta^* \\ f_{\Phi,N} &\colon [0,T] \to \Upsilon^* \end{aligned}$$

be given according to Lemma 4.8.

Using Lemma 4.10, we can suppose that the sequences given by Definition 4.11 do exist for all $N \geq 1$. Moreover, the notation $\{(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N, \mathbf{f}_{\mathbf{v},N}, f_{\tau,N}, f_{\beta,N})\}_N$ will always refer to Definition 4.11 and we assume that $N \geq T$, i.e. $k \leq 1$. The next lemma shows in which sense the constructed sequence of Definition 4.11 solves the instationary problem 4.1.

Lemma 4.12. *(Approximate Instationary Solution)*
For all $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$ *and* $\psi \in C^\infty([0,T])$ *with* $\psi(T) = 0$ *there holds*

$$\int_0^T -(\mathbf{w}_N, \mathbf{v})\psi' + \{a_\mathbf{v}(\mathbf{u}_N, \mathbf{v}) + c_\mathbf{v}(\mathbf{u}_N, \mathbf{u}_N, \mathbf{v})\}\, \psi \, dt \\ - \int_0^T \langle \mathbf{F}(\theta_N + \theta_b, \Phi_N + \Phi_b) + \mathbf{f}_{\mathbf{v},N}, \mathbf{v}\rangle_{\mathbf{U}^*} \psi \, dt \quad = (\mathbf{u}_0, \mathbf{v})\psi(0) \tag{4.7}$$

$$\int_0^T -(\eta_N, \tau)\psi' + \{a_\tau(\theta_N + \theta_b, \tau) + c_\tau(\mathbf{u}_N, \theta_N + \theta_b, \tau)\}\, \psi \, dt \\ - \int_0^T \langle f_{\tau,N}, \tau\rangle_{\Theta^*} \psi \, dt \quad = (\theta_0, \tau)\psi(0) \tag{4.8}$$

$$\int_0^T \{a_\beta(\theta_N + \theta_b, \Phi_N + \Phi_b, \beta)\psi - \langle f_{\beta,N}, \beta\rangle_{\Upsilon^*}\}\, \psi \, dt \quad = 0. \tag{4.9}$$

Proof. Let $t_m := mk$. There holds

$$\begin{aligned} &\int_0^T -(\mathbf{w}_N(t), \mathbf{v})\psi'(t)\, dt \\ &= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} -(\mathbf{w}_N(t), \mathbf{v})\psi'(t)\, dt \\ &= \sum_{m=1}^N \left\{ \int_{t_{m-1}}^{t_m} (\mathbf{w}_N'(t), \mathbf{v})\psi(t)\, dt - (\mathbf{w}_N(t_m), \mathbf{v})\psi(t_m) + (\mathbf{w}_N(t_{m-1}), \mathbf{v})\psi(t_{m-1}) \right\} \\ &= \sum_{m=1}^N \left\{ \int_{t_{m-1}}^{t_m} \frac{1}{k}(\mathbf{u}_N^m - \mathbf{u}_N^{m-1}, \mathbf{v})\psi(t)\, dt \right\} + (\mathbf{u}_0, \mathbf{v})\psi(0). \end{aligned}$$

Analogously,

$$\int_0^T -(\eta_N(t), \tau)\psi'(t)\, dt = \sum_{m=1}^N \left\{ \int_{t_{m-1}}^{t_m} \frac{1}{k}(\theta_N^m - \theta_N^{m-1}, \tau)\psi(t)\, dt \right\} + (\theta_0, \tau)\psi(0).$$

Moreover,

$$\int_0^T \langle \mathbf{f}_{\mathbf{v},N}(t), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt = \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \langle \mathbf{f}_{\mathbf{v},N}(t), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt = \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \langle \mathbf{f}_\mathbf{v}^m, \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt,$$

and, analogously,

$$\begin{aligned} \int_0^T \langle f_{\tau,N}(t), \tau\rangle_{\Theta^*}\psi(t)\, dt &= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \langle f_\tau^m, \tau\rangle_{\Theta^*}\psi(t)\, dt \\ \int_0^T \langle f_{\beta,N}(t), \beta\rangle_{\Upsilon^*}\psi(t)\, dt &= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \langle f_\beta^m, \beta\rangle_{\Upsilon^*}\psi(t)\, dt. \end{aligned}$$

Taking into account that $\mathbf{u}_N$, θ_N, Φ_N are piecewise constant, the assertion follows by multiplying the equations in Problem 4.9 by ψ, integrating over $[t_{m-1}, t_m]$ and summation over $1 \le m \le N$. □

The goal for the remainder of this subsection is to derive upper bounds on the norms of the approximate solutions $\{(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N, \mathbf{f}_{\mathbf{v},N}, f_{\tau,N}, f_{\beta,N})\}_N$ that are independent of k and N. This result will allow us to state the existence of a weakly convergent subsequence in Section 4.3. Afterward, one can pass to the limit in the variational formulation (4.7), (4.8), (4.9).

First, the following two lemmas show that these norms can be bounded by the respective norms of the semi-discrete solutions.

Lemma 4.13. *(L^∞-Norm Estimation)*
There holds

$$\begin{aligned}
\|\mathbf{u}_N\|_{\infty;\mathbf{H}} &= \max_{m\in\{0,\dots,N-1\}} \|\mathbf{u}_N^m\| \quad \textit{and} \quad \|\mathbf{w}_N\|_{\infty;\mathbf{H}} = \max_{m\in\{0,\dots,N\}} \|\mathbf{u}_N^m\|,\\
\|\theta_N\|_{\infty;L^2} &= \max_{m\in\{0,\dots,N-1\}} \|\theta_N^m\| \quad \textit{and} \quad \|\eta_N\|_{\infty;L^2} = \max_{m\in\{0,\dots,N\}} \|\theta_N^m\|,\\
\|\Phi_N\|_{\infty;\Upsilon} &= \max_{m\in\{0,\dots,N-1\}} \|\Phi_N^m\|.
\end{aligned}$$

Proof. The stated assertions hold due to the piecewise linear and piecewise constant in-time definition of $\mathbf{u}_N$, $\mathbf{w}_N$, θ_N, η_N and Φ_N. □

Lemma 4.14. *(L^2-Norm Estimation)*
There holds

$$\begin{aligned}
\|\mathbf{u}_N\|_{2;\mathbf{V}}^2 &= k\sum_{m=1}^N \|\nabla \mathbf{u}_N^m\|^2 && \textit{and} \quad \|\mathbf{w}_N\|_{2;\mathbf{V}}^2 \le k\sum_{m=0}^N \|\nabla \mathbf{u}_N^m\|^2,\\
\|\theta_N\|_{2;\Theta}^2 &= k\sum_{m=1}^N \|\nabla \theta_N^m\|^2 && \textit{and} \quad \|\eta_N\|_{2;\Theta}^2 \le k\sum_{m=0}^N \|\nabla \theta_N^m\|^2,\\
\|\mathbf{w}_N'\|_{2;\mathbf{V}_2^*}^2 &= \frac{1}{k}\sum_{m=1}^N \|\mathbf{u}_N^m - \mathbf{u}_N^{m-1}\|_{\mathbf{V}_2^*}^2 && \textit{and} \quad \|\eta_N'\|_{2;\Theta_2^*}^2 = \frac{1}{k}\sum_{m=1}^N \|\theta_N^m - \theta_N^{m-1}\|_{\Theta_2^*}^2.
\end{aligned}$$

Proof. Let $\psi\colon [0,T] \to \mathbb{R}$ denote a piecewise linear function with $\psi_i := \psi(ik)$ for $i = 0, \dots N$. Then, by the Simpson quadrature rule,

$$\begin{aligned}
\int_0^T |\psi(t)|^2\,dt = \sum_{i=1}^N \int_{(i-1)k}^{ik} |\psi(t)|^2\,dt &= \sum_{i=1}^N \frac{k}{6}\left(|\psi_{i-1}|^2 + |\psi_i|^2 + 4|\frac{1}{2}(\psi_{i-1}+\psi_i)|^2\right)\\
&\le \sum_{i=1}^N \frac{k}{2}\left(|\psi_{i-1}|^2 + |\psi_i|^2\right)\\
&\le k\sum_{i=0}^N |\psi_i|^2.
\end{aligned}$$

In this way, the assertions for $\mathbf{w}_N$ and η_N follow. The assertions for $\mathbf{u}_N$, θ_N, $\mathbf{w}_N'$ and η_N' directly follow by the fact, that these functions are piecewise constant in time. □

Now, the norm of the semi-discrete solution for some fixed $N \ge 1$ is bounded independently of

N. The first estimate, Lemma 4.15, is a modification of Lemma 3.13 in the stationary case. The latter one cannot be simply reused, since one has to ensure that the derived upper bounds are uniform w.r.t. N. A comparable result is given by Lemma III.4.4 in [77].

Lemma 4.15. *(Stability of Semi-Discrete Solutions I)*
Let $\{(\mathbf{u}^m, \theta^m, \Phi^m)\}_{m=0}^N$ denote the sequence of semi-discrete solutions of Problem 4.9. Let Assumption 4.2 hold and assume that θ_b is chosen such that $\|\theta_b\|_{0,3}$ is sufficiently small, i.e.

$$\|\theta_b\|_{0,3} \leq \frac{\nu\kappa}{\sqrt{8(K_2^2+1)}\mathbf{K}_6} \frac{1}{a_{\mathbf{F}}(G_{\Phi,\Phi_b})},$$

with G_{Φ,Φ_b} given by Lemma 4.10.

Then there exist nonnegative constants G_Φ (given by Lemma 4.10) and $D, \{\alpha_i\}_{i=1}^4$ (given by (4.25)) that only dependent on the input data but not on N, k such that

$$\begin{aligned} \|\nabla\Phi^m\| &\leq G_\Phi \text{ for } m = 1,\dots,N \\ \alpha_1\|\mathbf{u}^m\|^2 + \alpha_2\|\theta^m\|^2 &\leq D \quad \text{for } m = 1,\dots,N \\ \sum_{m=1}^N \left(\alpha_1\|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2 + \alpha_2\|\theta^m - \theta^{m-1}\|^2\right) &\leq D \\ k\sum_{m=1}^N \left(\alpha_3\|\nabla\mathbf{u}^m\|^2 + \alpha_4\|\nabla\theta^m\|^2\right) &\leq D. \end{aligned}$$

Proof. Setting $\beta = \Phi^m$ in Problem 4.9 and using $\epsilon \in [\epsilon_-, \epsilon_+]$ according to Assumption 3.2 yields

$$\|\nabla\Phi^m\| \leq \frac{1}{\epsilon_-}\left(\epsilon_+\|\nabla\Phi_b\| + \|f_\beta^m\|_{\Upsilon^*}\right) \leq \frac{1}{\epsilon_-}\left(\epsilon_+\|\nabla\Phi_b\| + \|f_\beta\|_{\infty;\Upsilon^*}\right) = G_\Phi. \tag{4.10}$$

Consequently,

$$\|\Phi^m + \Phi_b\|_{1,2} \leq G_{\Phi,\Phi_b} \text{ for } m = 1,\dots N, \tag{4.11}$$

with G_{Φ,Φ_b} defined in Lemma 4.10. Setting $\mathbf{v} = \mathbf{u}^m$, $\tau = \theta^m$ in Problem 4.9 yields

$$\begin{aligned} L_1 &:= (\mathbf{u}^m - \mathbf{u}^{m-1}, \mathbf{u}^m) + k\nu\|\nabla\mathbf{u}^m\|^2 \\ &= k\langle\mathbf{F}(\theta^m + \theta_b, \Phi^m + \Phi_b) + \mathbf{f}_{\mathbf{v}}^m, \mathbf{u}^m\rangle_{\mathbf{U}^*} =: R_1 \qquad (4.12) \\ L_2 &:= (\theta^m - \theta^{m-1}, \theta^m) + k\kappa\|\nabla\theta^m\|^2 \\ &= -k\kappa(\nabla\theta_b, \nabla\theta^m) + kc_\tau(\mathbf{u}^m, \theta^m, \theta_b) + k\langle f_\tau^m, \theta^m\rangle_{\Theta^*} =: R_2. \qquad (4.13) \end{aligned}$$

The respective left-hand sides of (4.12) and (4.13) can be reformulated to

$$L_1 = \frac{1}{2}\left(\|\mathbf{u}^m\|^2 - \|\mathbf{u}^{m-1}\|^2 + \|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2\right) + k\nu\|\nabla\mathbf{u}^m\|^2 \tag{4.14}$$

$$L_2 = \frac{1}{2}\left(\|\theta^m\|^2 - \|\theta^{m-1}\|^2 + \|\theta^m - \theta^{m-1}\|^2\right) + k\kappa\|\nabla\theta^m\|^2. \tag{4.15}$$

Using Assumption 4.2, (4.11) and Friedrich's inequality, the right-hand side of (4.12) can be estimated from above by

$$R_1 \leq kC_1^m\|\nabla\mathbf{u}^m\| + kC_2\|\nabla\mathbf{u}^m\|\|\nabla\theta^m\| \tag{4.16}$$

with

$$\begin{aligned} C_1^m &:= \|\mathbf{f}_\mathbf{v}^m\|_{\mathbf{U}^*} + a_\mathbf{F}\|\theta_b\|_{1,2} + b_\mathbf{F} \\ C_2 &:= a_\mathbf{F}\sqrt{K_2^2+1} \\ a_\mathbf{F} &:= a_\mathbf{F}(G_{\Phi,\Phi_b}) \\ b_\mathbf{F} &:= b_\mathbf{F}(G_{\Phi,\Phi_b}). \end{aligned}$$

On the other hand, R_2 can be estimated from above via

$$R_2 \le kC_3^m\|\nabla\theta^m\| + kC_4\|\theta_b\|_{0,3}\|\nabla\mathbf{u}^m\|\|\nabla\theta^m\| \tag{4.17}$$

with

$$\begin{aligned} C_3^m &:= \kappa\|\nabla\theta_b\| + \|f_\tau^m\|_{\Theta^*} \\ C_4 &:= \mathbf{K}_6. \end{aligned}$$

Combining (4.14) and (4.16) and using Young's inequality yields

$$\begin{aligned} \underbrace{\|\mathbf{u}^m\|^2 - \|\mathbf{u}^{m-1}\|^2 + \|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2}_{=:U^m} + k\nu\|\nabla\mathbf{u}^m\|^2 &\le \frac{2(C_1^m)^2}{\nu}k + \frac{2C_2^2}{\nu}k\|\nabla\theta^m\|^2 \\ &=: C_5^m k + C_6 k\|\nabla\theta^m\|^2. \end{aligned} \tag{4.18}$$

Analogously, combination of (4.15) and (4.17) yields

$$\begin{aligned} \underbrace{\|\theta^m\|^2 - \|\theta^{m-1}\|^2 + \|\theta^m - \theta^{m-1}\|^2}_{=:T^m} + k\kappa\|\nabla\theta^m\|^2 &\le \frac{2(C_3^m)^2}{\kappa}k + \frac{2C_4^2}{\kappa}k\|\theta_b\|_{0,3}^2\|\nabla\mathbf{u}^m\|^2 \\ &=: C_7^m k + C_8 k\|\theta_b\|_{0,3}^2\|\nabla\mathbf{u}^m\|^2. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) gives

$$U^m + k\nu\|\nabla\mathbf{u}^m\|^2 \le C_5^m k + \frac{C_6}{\kappa}\left(C_7^m k + C_8 k\|\theta_b\|_{0,3}^2\|\nabla\mathbf{u}^m\|^2 - T^m\right). \tag{4.20}$$

Choosing θ_b such that $\frac{C_6C_8}{\kappa}\|\theta_b\|_{0,3}^2 \le \frac{\nu}{2}$, (4.20) can reformulated to

$$U^m + \frac{C_6}{\kappa}T^m + k\frac{\nu}{2}\|\nabla\mathbf{u}^m\|^2 \le k\left(C_5^m + \frac{C_6C_7^m}{\kappa}\right) =: C_9^m k. \tag{4.21}$$

Combination of (4.21) and (4.19) yields

$$T^m + k\kappa\|\nabla\theta^m\|^2 \le C_7^m k + \frac{2C_8}{\nu}\|\theta_b\|_{0,3}^2\left(C_9^m k - U^m - \frac{C_6}{\kappa}T^m\right),$$

which can be rewritten as

$$F_1T^m + F_2U^m + k\kappa\|\nabla\theta^m\|^2 \le F_3^m k, \tag{4.22}$$

with

$$\begin{aligned} F_1 &:= 1 + \frac{2C_6C_8}{\nu\kappa}\|\theta_b\|_{0,3}^2 \\ F_2 &:= \frac{2C_8}{\nu}\|\theta_b\|_{0,3}^2 \\ F_3^m &:= C_7^m + \frac{2C_8}{\nu}C_9^m\|\theta_b\|_{0,3}^2. \end{aligned}$$

Adding (4.21) and (4.22) gives

$$(1+F_2)U^m + \left(\frac{C_6}{\kappa} + F_1\right) T^m + \frac{\nu}{2} k \|\nabla \mathbf{u}^m\|^2 + \kappa k \|\nabla \theta^m\|^2 \le (C_9^m + F_3^m) k.$$

With $\alpha_1 := 1 + F_2,\ \alpha_2 := \left(\frac{C_6}{\kappa} + F_1\right),\ \alpha_3 := \frac{\nu}{2},\ \alpha_4 := \kappa$ and $\alpha_5^m := C_9^m + F_3^m$, we finally obtain

$$\alpha_1 U^m + \alpha_2 T^m + k\alpha_3 \|\nabla \mathbf{u}^m\|^2 + k\alpha_4 \|\nabla \theta^m\|^2 \le \alpha_5^m k. \tag{4.23}$$

By carefully tracking the m-dependent contributions to α_5^m (C_1^m, C_3^m, C_5^m, C_7^m, C_9^m) one can state the existence of $d_1, d_2, d_3 \ge 0$, only depending on the input data, but not on m, N and k, such that

$$\alpha_5^m \le d_1 \|\mathbf{f}_\mathbf{v}^m\|_{\mathbf{U}^*}^2 + d_2 \|f_\tau^m\|_{\Theta^*}^2 + d_3.$$

Thus, by virtue of Lemma 4.7 one obtains

$$k \sum_{m=1}^{N} \alpha_5^m \le d_1 \|\mathbf{f}_\mathbf{v}\|_{2;\mathbf{U}^*}^2 + d_2 \|f_\tau\|_{2;\Theta^*}^2 + d_3\, T.$$

Taking the sum in (4.23) for $m = 1, \cdots, r$ with $r \in \{1, \cdots, N\}$, gives

$$\begin{aligned}
&\alpha_1 \|\mathbf{u}^r\|^2 + \alpha_1 \sum_{m=1}^{r} \|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2 + \alpha_3 k \sum_{m=1}^{r} \|\nabla \mathbf{u}^m\|^2 \\
&\quad + \alpha_2 \|\theta^r\|^2 + \alpha_2 \sum_{m=1}^{r} \|\theta^m - \theta^{m-1}\|^2 + \alpha_4 k \sum_{m=1}^{r} \|\nabla \theta^m\|^2 \\
&\le d_1 \|\mathbf{f}_\mathbf{v}\|_{2;\mathbf{U}^*}^2 + d_2 \|f_\tau\|_{2;\Theta^*}^2 + d_3\, T + \alpha_1 \|\mathbf{u}_0\|^2 + \alpha_2 \|\theta_0\|^2.
\end{aligned} \tag{4.24}$$

The assertion now follows from (4.10) and (4.24) with constants given by

$$\begin{aligned}
D &:= d_1 \|\mathbf{f}_\mathbf{v}\|_{2;\mathbf{U}^*}^2 + d_2 \|f_\tau\|_{2;\Theta^*}^2 + d_3\, T + \alpha_1 \|\mathbf{u}_0\|^2 + \alpha_2 \|\theta_0\|^2 \\
\alpha_1 &= 1 + 4\mathbf{K}_6^2 \|\theta_b\|_{0,3}^2 (\nu\kappa)^{-1} \\
\alpha_2 &= 1 + 2a_\mathbf{F}^2 (K_2^2 + 1)(\nu\kappa)^{-1} \left(1 + 4\mathbf{K}_6^2 \|\theta_b\|_{0,3}^2 (\nu\kappa)^{-1}\right) \\
\alpha_3 &= 0.5\nu \\
\alpha_4 &= \kappa \\
d_1 &= 8\nu^{-1} \left(1 + 4\mathbf{K}_6^2 \|\theta_b\|_{0,3}^2 (\nu\kappa)^{-1}\right) \\
d_2 &= 4\kappa^{-1} \left(1 + 2a_\mathbf{F}^2 (K_2^2 + 1)(\nu\kappa)^{-1} \left(1 + 4\mathbf{K}_6^2 \|\theta_b\|_{0,3}^2 (\nu\kappa)^{-1}\right)\right) \\
d_3 &= 4\kappa \|\nabla \theta_b\|^2 + \left(1 + 4\mathbf{K}_6^2 \|\theta_b\|_{0,3}^2 (\nu\kappa)^{-1}\right) \\
&\quad \cdot \left(8\nu^{-1}(a_\mathbf{F}^2 \|\theta_b\|_{1,2}^2 + b_\mathbf{F}^2) + 8a_\mathbf{F}^2 (K_2^2 + 1) \|\nabla \theta\|^2 \nu^{-1}\right).
\end{aligned} \tag{4.25}$$

□

Whereas the previous lemma provides bounds on the $L^2(0,T;H^1)$ and $L^\infty(0,T;L^2)$ norms of the semi-discrete solution, the following lemma shows that the $L^2(0,T;\mathbf{V}_2^*)$ and $L^2(0,T;\Theta_2^*)$ norm of $\mathbf{w}'$ and η', respectively, can be bounded independently of N. At this point, an H^2-regularity of test functions is needed to bound the convection terms $c_\mathbf{v}(\mathbf{u},\mathbf{u},\cdot),\ c_\tau(\mathbf{u},\theta,\cdot)$ in terms of a product of L^2- and H^1-norm.

Lemma 4.16. *(Stability of Semi-Discrete Solutions II, see Lemma III.4.6 in [77])*
Let the assumptions of Lemma 4.15 hold. Then, there exists a constant D_2, independent of N, k, such that

$$k\sum_{m=1}^{N}\left\{\|\frac{1}{k}(\mathbf{u}^m-\mathbf{u}^{m-1})\|^2_{\mathbf{V}_2^*}+\|\frac{1}{k}(\theta^m-\theta^{m-1})\|^2_{\Theta_2^*}\right\}\leq D_2.$$

Proof. In the following, C denotes a generic (varying) constant that is independent of k, m and N. Using Lemma 4.4 and Assumption 4.2, there holds for arbitrary $\mathbf{v}\in\mathbf{V}_2$,

$$\begin{aligned}\frac{1}{k}|(\mathbf{u}^m-\mathbf{u}^{m-1},\mathbf{v})|\leq{}& C\left(\|\nabla\mathbf{u}^m\|\|\nabla\mathbf{v}\|+\|\mathbf{u}^m\|\|\nabla\mathbf{u}^m\|\|\mathbf{v}\|_{2,2}+\|\mathbf{f}_{\mathbf{V}}^m\|_{\mathbf{U}^*}\|\nabla\mathbf{v}\|\right)\\ &+C\left(a_{\mathbf{F}}(\|\Phi^m+\Phi_m\|_{1,2})\|\theta^m+\theta_b\|_{1,2}\|\nabla\mathbf{v}\|+b_{\mathbf{F}}(\|\Phi^m+\Phi_m\|_{1,2})\|\nabla\mathbf{v}\|\right).\end{aligned}$$

With $\|\Phi^m+\Phi_b\|_{1,2}\leq G_{\Phi,\Phi_b}$ for all m according to Lemma 4.15, one obtains by taking the supremum over all $\mathbf{v}\in\mathbf{V}_2$ and Young's inequality,

$$\|\frac{1}{k}(\mathbf{u}^m-\mathbf{u}^{m-1})\|^2_{\mathbf{V}_2^*}\leq C\left(\|\nabla\mathbf{u}^m\|^2+\|\mathbf{u}^m\|^2\|\nabla\mathbf{u}^m\|^2+\|\mathbf{f}_{\mathbf{V}}^m\|^2_{\mathbf{U}^*}+\|\nabla\theta^m\|^2+1\right).$$

Using Lemma 4.7 and 4.15, multiplication by k and summation for $m=1,\ldots,N$, yields

$$\begin{aligned}k\sum_{m=1}^{N}\|\frac{1}{k}(\mathbf{u}^m-\mathbf{u}^{m-1})\|^2_{\mathbf{V}_2^*}\leq{}& C\left(k\sum_{m=1}^{N}\|\nabla\mathbf{u}^m\|^2+k\sum_{m=1}^{N}\left\{D\|\nabla\mathbf{u}^m\|^2\right\}+k\sum_{m=1}^{N}\|\mathbf{f}_{\mathbf{V}}^m\|^2_{\mathbf{U}^*}\right)\\ &+C\left(k\sum_{m=1}^{N}\|\nabla\theta^m\|^2+T\right)\\ \leq{}& C\left(D+D^2+\|\mathbf{f}_{\mathbf{V}}\|^2_{2;\mathbf{U}^*}+D+T\right),\end{aligned}$$

with constant D given by Lemma 4.15. Now, let $\tau\in\Theta_2$ be arbitrary. Then, by Lemma 4.4,

$$\begin{aligned}\frac{1}{k}|(\theta^m-\theta^{m-1},\tau)|\leq C\Big(&(\|\nabla\theta^m\|+\|\nabla\theta_b\|)\|\nabla\tau\|+\|\mathbf{u}^m\|(\|\nabla\theta^m\|+\|\nabla\theta_b\|)\|\tau\|_{2,2}\\ &+\|f_\tau^m\|_{\Theta^*}\|\nabla\tau\|\Big).\end{aligned}$$

Thus,

$$\|\frac{1}{k}(\theta^m-\theta^{m-1})\|^2_{\Theta_2^*}\leq C\left(\|\nabla\theta^m\|^2+\|\nabla\theta_b\|^2+\|\mathbf{u}^m\|^2(\|\nabla\theta^m\|^2+\|\nabla\theta_b\|^2)+\|f_\tau^m\|^2_{\Theta^*}\right)$$

and, similarly as before,

$$k\sum_{m=1}^{N}\|\frac{1}{k}(\theta^m-\theta^{m-1})\|^2_{\Theta_2^*}\leq C\left(D+T\|\nabla\theta_b\|^2+D^2+DT\|\nabla\theta_b\|^2+\|f_\tau\|^2_{2;\Theta^*}\right).$$

□

Combination of the previous lemmas now yields bounds on the sequence of approximate instationary solutions that are independent of N.

Corollary 4.17. *(Boundedness of Approximate Instationary Sequences)*
Let the assumptions of Lemma 4.15 hold. Then, there holds for all $N \in \mathbb{N}$

$$\begin{aligned}
&\|\mathbf{u}_N\|_{\infty;\mathbf{H}} \leq G_{\mathbf{u},\infty,\mathbf{H}}, \; \|\mathbf{u}_N\|_{2;\mathbf{V}} \leq G_{\mathbf{u},2,\mathbf{V}} \\
&\|\mathbf{w}_N\|_{\infty;\mathbf{H}} \leq G_{\mathbf{u},\infty,\mathbf{H}}, \; \|\mathbf{w}_N\|_{2;\mathbf{V}} \leq G_{\mathbf{u},2,\mathbf{V}} \\
&\|\mathbf{w}'_N\|_{2;\mathbf{V}_2^*} \leq G_{\mathbf{w},2,\mathbf{V}_2^*} \\
&\|\theta_N\|_{\infty;L^2} \leq G_{\theta,\infty,L^2}, \; \|\theta_N\|_{2;\Theta} \leq G_{\theta,2,\Theta} \\
&\|\eta_N\|_{\infty;L^2} \leq G_{\theta,\infty,L^2}, \; \|\eta_N\|_{2;\Theta} \leq G_{\theta,2,\Theta} \\
&\|\eta'_N\|_{2;\Theta_2^*} \leq G_{\eta,2,\Theta_2^*} \\
&\|\Phi_N\|_{\infty;\Upsilon} \leq G_\Phi,
\end{aligned}$$

with constants

$$\begin{aligned}
G_{\mathbf{u},\infty,\mathbf{H}} &:= \max\left\{\|\mathbf{u}_0\|, \sqrt{D\alpha_1^{-1}}\right\}, & G_{\theta,\infty,L^2} &:= \max\left\{\|\theta_0\|, \sqrt{D\alpha_2^{-1}}\right\}, \\
G_{\mathbf{u},2,\mathbf{V}} &:= \sqrt{\|\nabla \mathbf{u}_0\|^2 + D\alpha_3^{-1}}, & G_{\theta,2,\Theta} &:= \sqrt{\|\nabla \theta_0\|^2 + D\alpha_4^{-1}}, \\
G_{\mathbf{w},2,\mathbf{V}_2^*} &:= \sqrt{D_2}, & G_{\eta,2,\Theta_2^*} &:= \sqrt{D_2},
\end{aligned}$$

that are independent of k and N. Here, the constants D, D_2, $\{\alpha_i\}_{i=1}^4$ and G_Φ are given by the previous lemmas.

Proof. Follows by combination of Lemma 4.13, 4.14, 4.15 and 4.16. □

4.3. Passage to the Limit

In this section, we first show that the previously constructed sequences converge as $N \to \infty$ in various meanings. This is the result of Lemma 4.18, 4.19 and 4.20. Afterward, we investigate how this convergence translates into convergence of the individual terms in the variational formulation (4.7) - (4.9).

Lemma 4.18. *(Strong Limits I, see Lemma III.4.8 in [77])*
Let the assumptions of Lemma 4.15 hold. Then, there holds

$$\begin{aligned}
\mathbf{u}_N - \mathbf{w}_N &\to 0 \text{ in } L^2(0,T;\mathbf{H}) \\
\theta_N - \eta_N &\to 0 \text{ in } L^2(0,T;L^2).
\end{aligned}$$

Proof. According to Lemma III.4.8 in [77] there holds

$$\|\mathbf{u}_N - \mathbf{w}_N\|_{2;\mathbf{H}} = \sqrt{\frac{k}{3}} \left(\sum_{m=1}^{N} \|\mathbf{u}_N^m - \mathbf{u}_N^{m-1}\|^2\right)^{\frac{1}{2}}.$$

Thus, $\mathbf{u}_N - \mathbf{w}_N \to 0$ follows from $k \to 0$ as $N \to \infty$ and Lemma 4.15. $\theta_N - \eta_N \to 0$ follows analogously. □

Lemma 4.19. *(Weak Limits)*
Let the assumptions of Lemma 4.15 hold. Then, there are elements $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$ *with* $\mathbf{u}' \in L^2(0,T;\mathbf{V}_2^*)$, $\theta \in L^2(0,T;\Theta) \cap L^\infty(0,T;L^2)$ *with* $\theta' \in L^2(0,T;\Theta_2^*)$ *and* $\Phi \in L^\infty(0,T;\Upsilon)$ *and a (not relabeled) subsequence of* $\{(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N)\}_N$ *such that*

$$\begin{aligned}
\mathbf{u}_N &\rightharpoonup \mathbf{u} \;\; \textit{in } L^2(0,T;\mathbf{V}) \;\; \textit{and} \;\; \mathbf{u}_N \overset{*}{\rightharpoonup} \mathbf{u} \;\textit{in } L^\infty(0,T;\mathbf{H})\\
\mathbf{w}_N &\rightharpoonup \mathbf{u} \;\; \textit{in } L^2(0,T;\mathbf{V}) \;\; \textit{and} \;\; \mathbf{w}_N \overset{*}{\rightharpoonup} \mathbf{u} \;\textit{in } L^\infty(0,T;\mathbf{H})\\
\mathbf{w}_N' &\rightharpoonup \mathbf{u}' \;\textit{in } L^2(0,T;\mathbf{V}_2^*)\\
\theta_N &\rightharpoonup \theta \;\; \textit{in } L^2(0,T;\Theta) \;\; \textit{and} \;\; \theta_N \overset{*}{\rightharpoonup} \theta \;\textit{in } L^\infty(0,T;L^2)\\
\eta_N &\rightharpoonup \theta \;\; \textit{in } L^2(0,T;\Theta) \;\; \textit{and} \;\; \eta_N \overset{*}{\rightharpoonup} \theta \;\textit{in } L^\infty(0,T;L^2)\\
\eta_N' &\rightharpoonup \theta' \;\textit{in } L^2(0,T;\Theta_2^*)\\
\Phi_N &\rightharpoonup \Phi \;\; \textit{in } L^2(0,T;\Upsilon) \;\; \textit{and} \;\; \Phi_N \overset{*}{\rightharpoonup} \Phi \;\textit{in } L^\infty(0,T;\Upsilon).
\end{aligned}$$

Proof. By Theorem A.96, the spaces $\mathbf{V}, \mathbf{V}_2, \mathbf{H}, \Theta, \Theta_2, \Upsilon$ are reflexive and separable. According to Lemma A.24 and Lemma A.26 the same holds for $\mathbf{V}_2^*$ and Θ_2^*. Thus, all considered spaces $L^2(0,T;\cdot)$ are reflexive according to Theorem A.55. By the boundedness stated in Corollary 4.17 and the reflexivity of the spaces $L^2(0,T;\cdot)$, there exist elements $\mathbf{u}, \mathbf{w} \in L^2(0,T;\mathbf{V})$, $g_\mathbf{w} \in L^2(0,T;\mathbf{V}_2^*)$, $\theta, \eta \in L^2(0,T;\Theta)$, $g_\eta \in L^2(0,T;\Theta_2^*)$ and $\Phi \in L^2(0,T;\Upsilon)$ and corresponding subsequences (not relabeled) with

$$\begin{aligned}
\mathbf{u}_N &\rightharpoonup \mathbf{u} \;\; \text{in } L^2(0,T;\mathbf{V})\\
\mathbf{w}_N &\rightharpoonup \mathbf{w} \;\; \text{in } L^2(0,T;\mathbf{V})\\
\mathbf{w}_N' &\rightharpoonup g_\mathbf{w} \; \text{in } L^2(0,T;\mathbf{V}_2^*)\\
\theta_N &\rightharpoonup \theta \;\; \text{in } L^2(0,T;\Theta)\\
\eta_N &\rightharpoonup \eta \;\; \text{in } L^2(0,T;\Theta)\\
\eta_N' &\rightharpoonup g_\eta \;\; \text{in } L^2(0,T;\Theta_2^*)\\
\Phi_N &\rightharpoonup \Phi \;\; \text{in } L^2(0,T;\Upsilon).
\end{aligned}$$

Moreover, by Lemma A.61 and the $L^\infty(0,T;\cdot)$-boundedness, there are $\mathbf{u}_*, \mathbf{w}_* \in L^\infty(0,T;\mathbf{H})$, $\theta_*, \eta_* \in L^\infty(0,T;L^2)$ and $\Phi_* \in L^\infty(0,T;\Upsilon)$ (and another not relabeled subsequences) such that

$$\begin{aligned}
\mathbf{u}_N &\overset{*}{\rightharpoonup} \mathbf{u}_* \;\; \text{in } L^\infty(0,T;\mathbf{H})\\
\mathbf{w}_N &\overset{*}{\rightharpoonup} \mathbf{w}_* \; \text{in } L^\infty(0,T;\mathbf{H})\\
\theta_N &\overset{*}{\rightharpoonup} \theta_* \;\; \text{in } L^\infty(0,T;L^2)\\
\eta_N &\overset{*}{\rightharpoonup} \eta_* \;\; \text{in } L^\infty(0,T;L^2)\\
\Phi_N &\overset{*}{\rightharpoonup} \Phi_* \; \text{in } L^\infty(0,T;\Upsilon).
\end{aligned}$$

The weak convergence $\mathbf{u}_N \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathbf{V})$ implies

$$\int_0^T (\mathbf{u}_N(t), f(t))\,dt \to \int_0^T (\mathbf{u}(t), f(t))\,dt$$

for arbitrary $f \in L^2(0,T;\mathbf{H})$. Due to $\mathbf{u}_N \overset{*}{\rightharpoonup} \mathbf{u}_*$ in $L^\infty(0,T;\mathbf{H})$, there holds

$$\int_0^T (\mathbf{u}_N(t), g(t))\,dt \to \int_0^T (\mathbf{u}_*(t), g(t))\,dt \text{ for all } g \in L^1(0,T;\mathbf{H}).$$

Since $L^2(0,T;\mathbf{H}) \subset L^1(0,T;\mathbf{H})$, there holds $\mathbf{u} = \mathbf{u}_*$ in $L^2(0,T;\mathbf{H})$, i.e. $\mathbf{u} = \mathbf{u}_*$ for almost all $t \in (0,T)$. Analogously, $\mathbf{w} = \mathbf{w}_*$, $\theta = \theta_*$, $\eta = \eta_*$. A similar procedure (with both, $\mathbf{V}$ and $\mathbf{H}$, being replaced by Υ) shows that $\Phi_* = \Phi$.

Moreover, $\mathbf{u}_N - \mathbf{w}_N \overset{*}{\rightharpoonup} \mathbf{u} - \mathbf{w}$ in $L^\infty(0,T;\mathbf{H})$ implies $\overset{*}{\rightharpoonup}$ convergence in $L^2(0,T;\mathbf{H})$. On the other hand, $\mathbf{u}_N - \mathbf{w}_N \to 0$ in $L^2(0,T;\mathbf{H})$ according to Lemma 4.18, and therefore, $\mathbf{u}_N - \mathbf{w}_N \overset{*}{\rightharpoonup} 0$. Thus, $\mathbf{u} - \mathbf{w} = 0$. Analogously, $\theta = \eta$.

Finally, as $\mathbf{w}_N \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathbf{V})$ and $\mathbf{w}_N' \rightharpoonup g_{\mathbf{w}}$ in $L^2(0,T;\mathbf{V}_2^*)$, there holds $\mathbf{u}' = g_{\mathbf{w}}$ according to Lemma A.75 with $X = \mathbf{V}$ and $Y = \mathbf{V}_2^*$. Analogously, $g_\eta = \theta'$. □

Lemma 4.20. *(Strong Limits II)*
Let the assumptions of Lemma 4.15 hold. Then, there holds

$$\mathbf{w}_N \to \mathbf{u} \text{ and } \mathbf{u}_N \to \mathbf{u} \text{ in } L^2(0,T;\mathbf{H})$$
$$\eta_N \to \theta \text{ and } \theta_N \to \theta \quad \text{in } L^2(0,T;L^2)$$

for $(\mathbf{u},\theta)$ and not relabeled subsequences of the subsequences given by Lemma 4.19.

Proof. Lemma 4.17 implies that $\{\mathbf{w}_N\}_N$ is a bounded sequence in $\mathcal{W}(0,T;2,2,\mathbf{V},\mathbf{V}_2^*) =: \mathcal{W}$. According to Lemma A.74, $\mathcal{W}$ is reflexive and $\mathcal{W} \hookrightarrow\hookrightarrow L^2(0,T;\mathbf{H})$ according to Theorem A.71. Thus, there exists a subsequence (not relabeled) $\{\mathbf{w}_N\}_N$ that converges weakly to $\mathbf{w}_*$ in $\mathcal{W}$. By the compact embedding and Lemma A.35, $\mathbf{w}_N \to \mathbf{w}_*$ in $L^2(0,T;\mathbf{H})$. Moreover, weak convergence in $\mathcal{W}$ implies weak convergence in $L^2(0,T;\mathbf{V})$ since $\mathcal{W} \subset L^2(0,T;\mathbf{V})$. According to Lemma 4.19, $\mathbf{w}_N \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathbf{V})$. Thus, $\mathbf{w}_* = \mathbf{u}$. $\mathbf{u}_N \to \mathbf{u}$ in $L^2(0,T;\mathbf{H})$ follows from Lemma 4.18. The assertion for θ follows analogously. □

For the remainder of this section, $\{(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N)\}_N$ always denotes the final subsequence chosen in Lemma 4.20 and $(\mathbf{u}, \theta, \Phi)$ denotes the weak limits introduced in Lemma 4.19. Hereby, it is implicitly assumed that the necessary requirements, see Lemma 4.15, are satisfied.

The series of upcoming lemmas now show that all terms of the system provided by Lemma 4.12 do converge.

Lemma 4.21. *(Convergence of Time Derivative)*
Let $(\mathbf{v},\tau) \in \mathbf{V} \times \Theta$ and $\psi \in C^\infty([0,T])$ be arbitrary. Then,

$$\int_0^T -(\mathbf{w}_N(t),\mathbf{v})\psi'(t)\,dt \to \int_0^T -(\mathbf{u}(t),\mathbf{v})\psi'(t)\,dt$$
$$\int_0^T -(\eta_N(t),\tau)\psi'(t)\,dt \to \int_0^T -(\theta(t),\tau)\psi'(t)\,dt.$$

Proof. Follows from $\mathbf{w}_N \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathbf{V})$ and $\eta_N \rightharpoonup \theta$ in $L^2(0,T;\Theta)$. □

Lemma 4.22. *(Convergence of Bilinear Forms)*
Let $(\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$ and $\psi \in C^\infty([0,T])$ be arbitrary. Then,

$$\int_0^T a_{\mathbf{v}}(\mathbf{w}_N(t), \mathbf{v})\psi(t)\,dt \to \int_0^T a_{\mathbf{v}}(\mathbf{u}(t), \mathbf{v})\psi(t)\,dt$$
$$\int_0^T a_\tau(\eta_N(t), \tau)\psi(t)\,dt \to \int_0^T a_\tau(\theta(t), \tau)\psi(t)\,dt.$$

Proof. Follows analogously to the proof of Lemma 4.21. □

Lemma 4.23. *(Convergence of Convection Terms)*
Let $\mathbf{v} \in \mathcal{V}(\Omega)$, $\tau \in C_D^\infty(\Omega)$ and $\psi \in C^\infty([0,T])$ be arbitrary. Then,

$$\int_0^T c_{\mathbf{v}}(\mathbf{u}_N(t), \mathbf{u}_N(t), \mathbf{v})\psi(t)\,dt \to \int_0^T c_{\mathbf{v}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v})\psi(t)\,dt$$
$$\int_0^T c_\tau(\mathbf{u}_N(t), \theta_N(t) + \theta_b, \tau)\psi(t)\,dt \to \int_0^T c_\tau(\mathbf{u}(t), \theta(t) + \theta_b, \tau)\psi(t)\,dt.$$

Proof. The assertion follows directly by Lemma 4.6 with $\mathbf{v}(t) := \psi(t)\,\mathbf{v} \in C^1([0,T] \times \Omega)^d$. The assertion for c_τ follows analogously. □

Lemma 4.24. *(Convergence of Force Terms)*
Let Assumption 4.2 hold. Let $\mathbf{v} \in \mathcal{V}(\Omega)$, $\tau, \beta \in C_D^\infty(\Omega)$ and $\psi \in C^\infty([0,T])$ be arbitrary. Then,

$$\int_0^T \langle \mathbf{F}(\theta_N(t) + \theta_b, \Phi_N(t) + \Phi_b), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt \to \int_0^T \langle \mathbf{F}(\theta(t) + \theta_b, \Phi(t) + \Phi_b), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt$$
$$\int_0^T \langle \mathbf{f}_{\mathbf{v},N}(t), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt \to \int_0^T \langle \mathbf{f}_{\mathbf{v}}(t), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\,dt$$
$$\int_0^T \langle f_{\tau,N}(t), \tau\rangle_{\Theta^*}\psi(t)\,dt \to \int_0^T \langle f_\tau(t), \tau\rangle_{\Theta^*}\psi(t)\,dt$$
$$\int_0^T \langle f_{\beta,N}(t), \beta\rangle_{\Upsilon^*}\psi(t)\,dt \to \int_0^T \langle f_\beta(t), \beta\rangle_{\Upsilon^*}\psi(t)\,dt.$$

Proof. The assertion for $\mathbf{f}_{\mathbf{v}}$, f_τ, f_β is a direct consequence of Lemma 4.8. The assertion for $\mathbf{F}$ holds by Assumption 4.2. □

Lemma 4.25. *(Convergence of Gauss Bilinear Form)*
Let $\beta \in C_D^\infty(\Omega)$ and $\psi \in C^\infty([0,T])$ be arbitrary. Then,

$$\int_0^T a_\beta(\theta_N(t) + \theta_b, \Phi_N(t) + \Phi_b, \beta)\psi(t)\,dt \to \int_0^T a_\beta(\theta(t) + \theta_b, \Phi(t) + \Phi_b, \beta)\psi(t)\,dt.$$

Proof. There holds

$$
\begin{aligned}
&\left|\int_0^T a_\beta(\theta_N(t)+\theta_b, \Phi_N(t)+\Phi_b, \beta)\psi(t)\,dt - \int_0^T a_\beta(\theta(t)+\theta_b, \Phi(t)+\Phi_b, \beta)\psi(t)\,dt\right| \\
\leq &\int_0^T |((\epsilon(\theta_N(t)+\theta_b) - \epsilon(\theta(t)+\theta_b))\nabla(\Phi_N(t)+\Phi_b), \nabla\beta)||\psi(t)|\,dt \\
&+ \left|\int_0^T ((\epsilon(\theta(t)+\theta_b)\nabla(\Phi_N(t)-\Phi(t)), \nabla\beta)\psi(t)\,dt\right| \\
=: &\, I_1 + I_2
\end{aligned}
$$

Considering I_1,

$$
\begin{aligned}
|I_1| &\leq \int_0^T L_\epsilon \|\theta_N(t)-\theta(t)\| \|\nabla(\Phi_N(t)+\Phi_b)\| \|\nabla\beta\|_\infty |\psi(t)|\,dt \\
&\leq L_\epsilon \|\theta_N - \theta\|_{2;L^2} \left(\|\Phi_N\|_{\infty;\Upsilon} + \|\Phi_b\|_{\infty;\Upsilon}\right) \|\nabla\beta\|_\infty \|\psi\|_{2;\mathbb{R}} \\
&\leq L_\epsilon (G_\Phi + \|\nabla\Phi_b\|) \|\nabla\beta\|_\infty \|\psi\|_{2;\mathbb{R}}\, \|\theta_N - \theta\|_{2;L^2} \\
&=: A_N,
\end{aligned}
$$

where $\lim_{N\to\infty} A_N = 0$ since $\theta_N \to \theta$ according to Lemma 4.20. Moreover, let

$$
\begin{aligned}
L\colon L^2(0,T;\Upsilon) &\to \mathbb{R} \\
\Phi &\mapsto \int_0^T ((\epsilon(\theta(t)+\theta_b)\nabla\Phi(t), \nabla\beta)\psi(t)\,dt.
\end{aligned}
$$

Since $|L\Phi| \leq \epsilon_+ \|\psi\|_{2;\mathbb{R}} \|\nabla\beta\| \|\Phi\|_{2;\Upsilon}$, there holds $L \in L^2(0,T;\Upsilon)^*$. Thus, $I_2 \to 0$ since $\Phi_N \rightharpoonup \Phi$ in $L^2(0,T;\Upsilon)$. □

After considering the individual terms, the validness of the complete momentum and heat equation, as well as Gauss' law, are shown.

Lemma 4.26. *(Momentum and Heat Equation)*
Let Assumption 4.2 hold. Then, $(\mathbf{u}, \theta, \Phi)$ *satisfies (4.4) and (4.5) for all* $\mathbf{v} \in \mathbf{V}$, $\tau \in \Theta$ *and* $\psi \in C^\infty([0,T])$.

Proof. Let C denote a generic constant that only depends on problem data. For $\psi \in C^\infty([0,T])$ arbitrary but fixed, define the linear maps

$$
\begin{aligned}
L_{\mathbf{v}}\colon \mathbf{V} &\to \mathbb{R} \\
\mathbf{v} &\mapsto \int_0^T -(\mathbf{u}(t), \mathbf{v})\psi'(t) + \{a_{\mathbf{v}}(\mathbf{u}(t), \mathbf{v}) + c_{\mathbf{v}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v})\}\,\psi(t)\,dt \\
&\quad - \int_0^T \langle \mathbf{F}(\theta(t)+\theta_b, \Phi(t)+\Phi_b) + \mathbf{f}_{\mathbf{v}}(t), \mathbf{v}\rangle_{\mathbf{U}^*} \psi(t)\,dt \\
&\quad - (\mathbf{u}_0, \mathbf{v})\psi(0)
\end{aligned}
$$

and

$$
\begin{aligned}
L_\tau\colon \Theta &\to \mathbb{R} \\
\tau &\mapsto \int_0^T -(\theta(t),\tau)\psi'(t) + \{a_\tau(\theta(t)+\theta_b,\tau) + c_\tau(\theta(t),\theta(t)+\theta_b,\tau)\}\,\psi(t)\,dt \\
&\quad - \int_0^T \langle f_\tau(t),\tau\rangle_{\Theta^*}\psi(t)\,dt \\
&\quad - (\theta_0,\tau)\psi(0).
\end{aligned}
$$

Combining Lemma 4.12, 4.21, 4.22, 4.23 and 4.24, (4.4) and (4.5) hold for all $\mathbf{v} \in \mathcal{V}(\Omega)$, $\tau \in C_D^\infty(\Omega)$ and $\psi \in C^\infty([0,T])$, which is equivalent to

$$
\begin{aligned}
L_{\mathbf{v}}\mathbf{v} &= 0 \text{ for all } \mathbf{v} \in \mathcal{V}(\Omega) \\
L_\tau\tau &= 0 \text{ for all } \tau \in C_D^\infty(\Omega).
\end{aligned}
$$

Moreover, there holds for all $\mathbf{v} \in \mathbf{V} \setminus \{0\}$

$$
\begin{aligned}
\frac{|L_{\mathbf{v}}\mathbf{v}|}{\|\nabla\mathbf{v}\|} &\le C\left(\|\mathbf{u}\|_{2;\mathbf{V}}\|\psi'\|_{2;\mathbb{R}} + \|\mathbf{u}\|_{2;\mathbf{V}}\|\psi\|_{2;\mathbb{R}} + \|\mathbf{u}\|_{2;\mathbf{V}}^2\|\psi\|_{\infty;\mathbb{R}}\right) \\
&\quad + C\left(\|a_{\mathbf{F}}(\|\Phi+\Phi_b\|_{1,2})\|_{\infty;\mathbb{R}}\|\theta+\theta_b\|_{2;H^1}\|\psi\|_{2;\mathbb{R}}\right) \\
&\quad + C\left(\|b_{\mathbf{F}}(\|\Phi+\Phi_b\|_{1,2})\|_{\infty;\mathbb{R}}\|\psi\|_{2;\mathbb{R}} + \|\mathbf{f}_{\mathbf{v}}\|_{2;\mathbf{U}^*}\|\psi\|_{2;\mathbb{R}} + \|\mathbf{u}_0\||\psi(0)|\right) \\
&< \infty
\end{aligned}
$$

where we used that $\|\Phi\|_{\infty;\Upsilon} < \infty$ and the properties of $a_{\mathbf{F}}$, $b_{\mathbf{F}}$ according to Assumption 4.2. Moreover, for $\tau \in \Theta \setminus \{0\}$:

$$
\begin{aligned}
\frac{|L_\tau\tau|}{\|\nabla\tau\|} &\le C\left(\|\theta\|_{2;\Theta}\|\psi'\|_{2;\mathbb{R}} + \|\theta+\theta_b\|_{2;H^1}\|\psi\|_{2;\mathbb{R}}\right) \\
&\quad + C\left(\|\mathbf{u}\|_{2;\mathbf{V}}\|\theta+\theta_b\|_{2;H^1}\|\psi\|_{\infty;\mathbb{R}} + \|f_\tau\|_{2;\Theta^*}\|\psi\|_{2;\mathbb{R}} + \|\theta_0\||\psi(0)|\right) \\
&< \infty.
\end{aligned}
$$

Thus, both $L_{\mathbf{v}}$ and L_τ are bounded. Since $\mathcal{V}(\Omega)$ and $C_D^\infty(\Omega)$ are dense in $\mathbf{V}$ and Θ, respectively, there follows

$$
\begin{aligned}
L_{\mathbf{v}}\mathbf{v} &= 0 \text{ for all } \mathbf{v} \in \mathbf{V} \\
L_\tau\tau &= 0 \text{ for all } \tau \in \Theta.
\end{aligned}
$$

Since ψ was chosen arbitrarily, the assertion follows. □

Lemma 4.27. *(Gauss Law)*
$(\mathbf{u},\theta,\Phi)$ satisfies (4.6) for all $\beta \in \Upsilon$ and $\psi \in C^\infty([0,T])$.

Proof. Combining Lemma 4.12, 4.24 and 4.25, (4.6) holds for all $\beta \in C_D^\infty(\Omega)$ and $\psi \in C^\infty([0,T])$. Consider the linear map for an arbitrary, fixed $\psi \in C^\infty([0,T])$.

$$
\begin{aligned}
L\colon \Upsilon &\to \mathbb{R} \\
\beta &\mapsto \int_0^T ((\epsilon(\theta(t)+\theta_b)\nabla(\Phi(t)+\Phi_b),\nabla\beta)\psi(t)\,dt - \int_0^T \langle f_\beta(t),\beta\rangle_{\Upsilon^*}\psi(t)\,dt.
\end{aligned}
$$

Then,

$$|L\beta| \le \epsilon_+ \|\Phi + \Phi_b\|_{2;\Upsilon} \|\psi\|_{2;\mathbb{R}} \|\nabla\beta\| + \|f_\beta\|_{\infty;\Upsilon^*} \|\psi\|_{1;\mathbb{R}} \|\nabla\beta\|,$$

i.e. $L \in \Upsilon^*$. Since $L\beta = 0$ for all $\beta \in C_D^\infty(\Omega)$ which is dense in Υ by definition, (4.6) holds for all $\beta \in \Upsilon$. □

The proof of Theorem 4.3 is now given by combination of Lemma 4.26, 4.27 and can be completed by the following statement.

Proof of Theorem 4.3

Let $\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, $\theta \in L^2(0,T;\Theta) \cap L^\infty(0,T;L^2)$, $\Phi \in L^\infty(0,T;\Upsilon)$ denote the weak limits given by Lemma 4.19. The corresponding sequences do exist and the required stability properties hold due to Lemma 4.10, 4.15 and 4.16. Then, $(\mathbf{u}, \theta, \Phi)$ is a solution of Problem 4.1 by means of Lemma 4.26 and 4.27. In order to show the stated weak differentiability, define

$$\begin{aligned} h_{\mathbf{u}} \colon [0,T] &\to \mathbf{V}^* \\ t &\mapsto -a_{\mathbf{v}}(\mathbf{u}(t), \cdot) - c_{\mathbf{v}}(\mathbf{u}(t), \mathbf{u}(t), \cdot) + \mathbf{F}(\theta(t) + \theta_b, \Phi(t) + \Phi_b) + f_{\mathbf{v}}(t) \end{aligned}$$

$$\begin{aligned} h_\theta \colon [0,T] &\to \Theta^* \\ t &\mapsto -a_\tau(\theta(t) + \theta_b, \cdot) - c_\tau(\mathbf{u}(t), \theta(t) + \theta_b, \cdot) + f_\tau(t). \end{aligned}$$

According to Lemma 4.5, $h_{\mathbf{u}} \in L^1(0,T;\mathbf{V}^*)$ and $h_\theta \in L^1(0,T;\Theta^*)$. Since $(\mathbf{u}, \theta, \Phi)$ solves Problem 4.1, there holds for all $(\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$ and $\psi \in \mathcal{D}(0,T)$

$$\begin{aligned} \int_0^T -\langle \mathbf{u}(t), \mathbf{v} \rangle_{\mathbf{V}^*} \psi'(t)\, dt &= \int_0^T \langle h_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{V}^*} \psi(t)\, dt \\ \int_0^T -\langle \theta(t), \tau \rangle_{\Theta^*} \psi'(t)\, dt &= \int_0^T \langle h_\theta(t), \tau \rangle_{\Theta^*} \psi(t)\, dt. \end{aligned} \tag{4.26}$$

Using the reflexivity of $\mathbf{V}$ and Θ, Theorem A.96, (4.26) is equivalent to

$$\begin{aligned} \int_0^T -\langle \mathbf{v}^{**}, \mathbf{u}(t) \rangle_{\mathbf{V}^{**}} \psi'(t)\, dt &= \int_0^T \langle \mathbf{v}^{**}, h_{\mathbf{u}}(t) \rangle_{\mathbf{V}^{**}} \psi(t)\, dt \\ \int_0^T -\langle \tau^{**}, \theta(t) \rangle_{\Theta^{**}} \psi'(t)\, dt &= \int_0^T \langle \tau^{**}, h_\theta(t) \rangle_{\Theta^{**}} \psi(t)\, dt \end{aligned} \tag{4.27}$$

for all $\mathbf{v}^{**} \in \mathbf{V}^{**}$, $\tau^{**} \in \Theta^{**}$ and $\psi \in \mathcal{D}(0,T)$. Thus, by Theorem A.67, $\mathbf{u}' = h_{\mathbf{u}}$ and $\theta' = h_\theta$. Moreover, $\mathbf{u}$ and θ are a.e. equal to a continuous function from $[0,T]$ to $\mathbf{V}^*$ and Θ^*, respectively.

Finally, the norm bounds for $\{(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N)\}_N$ given by Corollary 4.17, the weak convergence of these sequences towards $(\mathbf{u}, \mathbf{u}', \theta, \theta', \Phi)$ stated by Lemma 4.19, together with Lemma A.34 and A.60 on the norm of weak limits yields the stated norm estimates.

□

4.4. Modeling of DEP Force

As in the stationary case, the formulation of the DEP force has to be given in such a way, that the requirements of the existence Theorem 4.3 are met. In Definition 3.23, we already introduced certain formulations for $\mathbf{F}$ that are based on linearization around a base potential and showed that these forces, $\mathbf{F}_{s,0},\ \mathbf{F}_{a,0},\ \mathbf{F}_{a,1},$ satisfy the crucial Assumption 3.12 for showing well-posedness of the stationary TEHD Boussinesq equations. In this section, we show that $\mathbf{F}_{s,0},\ \mathbf{F}_{a,0},\ \mathbf{F}_{a,1},$ satisfy the additional Assumption 4.2 as well.

Lemma 4.28. *(Properties of DEP Force Terms, Continued)*
The DEP force terms $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$ given by Definition 3.23, satisfy Assumption 4.2.

Proof. Due to Lemma 3.24 in only remains to show the additional requirement defined in Assumption 4.2. We only show the proof for $\mathbf{F}_{a,1}$. The assertion for $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ follows analogously.

To this end, let sequences $(\theta_n)_n$, $(\Phi_n)_n$ and elements θ, Φ be given as stated in Assumption 4.2. Let $\mathbf{v} \in \mathcal{V}(\Omega)$ and $\psi \in C^\infty([0,T])$ be arbitrary. For simplicity, we set all involved physical constants to 1. Then,

$$\begin{aligned}
&\left| \int_0^T \langle \mathbf{F}_{a,1}(\theta_n(t), \Phi_n(t)), \mathbf{v} \rangle_{\mathbf{U}^*} \psi(t)\, dt - \int_0^T \langle \mathbf{F}_{a,1}(\theta(t), \Phi(t)), \mathbf{v} \rangle_{\mathbf{U}^*} \psi(t)\, dt \right| \\
&\leq \left| \int_0^T \big((\nabla^2 \Phi_0 \nabla (\Phi_n(t) - \Phi(t)) \theta(t)), \mathbf{v} \big)\, \psi(t)\, dt \right| + \int_0^T \left| \big((\nabla^2 \Phi_0 \nabla \Phi_n(t) (\theta_n(t) - \theta(t)), \mathbf{v} \big)\, \psi(t) \right| dt \\
&\quad + \int_0^T \left| (\mathbf{g}(\theta_n(t) - \theta(t)), \mathbf{v})\, \psi(t) \right| dt \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Concerning I_1, let

$$\begin{aligned}
L \colon L^2(0,T;H^1) &\to \mathbb{R} \\
\Phi &\mapsto \int_0^T \big((\nabla^2 \Phi_0 \nabla \Phi(t) \theta(t)), \mathbf{v} \big)\, \psi(t)\, dt.
\end{aligned}$$

Since

$$|L\Phi| \leq \|\nabla^2 \Phi_0\|_{0,6} \|\mathbf{v}\|_{0,6} \|\psi\|_{\infty;\mathbb{R}} \|\theta\|_{2;L^6} \|\Phi\|_{2;H^1},$$

$L \in L^2(0,T;H^1)^*$ and $I_1 \to 0$ by the assumption $\Phi_n \rightharpoonup \Phi$ in $L^2(0,T;H^1)$. Moreover, the second term can be estimated from above via

$$I_2 \leq \|\nabla^2 \Phi_0\|_{0,\infty} \|\mathbf{v}\|_{0,\infty} \|\psi\|_{\infty;\mathbb{R}} \|\Phi_n\|_{2;H^1} \|\theta_n - \theta\|_{2;L^2}.$$

Then, $I_2 \to 0$ by the boundedness of $(\Phi_n)_n$ in $L^\infty(0,T;H^1)$ and $\theta_n \to \theta$ in $L^2(0,T;L^2)$. Analogously, $I_3 \to 0$. $\square$

We conclude this section on the instationary TEHD Boussinesq equations by reconsidering the DEP formulation

$$\mathbf{F}_r(\theta, \Phi) = -2\alpha_e(\theta\, \mathbf{g}_{E,r}[\Phi], \cdot) - \alpha_g(\theta\, \mathbf{g}, \cdot),\ r > 0, \tag{4.28}$$

which is based on a regularized electric gravity $\mathbf{g}_{E,r}[\Phi]$ with regularization parameter $r > 0$, see Definition 3.26. We now show that a similar procedure as presented in the end of Section 3.3 is also possible in the instationary case, i.e. we prove existence of instationary solutions of Problem 4.1 with a weakened connection between electric gravity and potential.

First, we set $\mathbf{F} = \mathbf{F}_{r=k}$ in the definition of the semi-discrete Problem 4.9, i.e. the regularization parameter is set to the time step size k. Note that $\mathbf{F}_r$ satisfies the requirements of the existence theorem in the stationary case with r-independent growth rates $a_\mathbf{F}$ and $b_\mathbf{F}$ according to Lemma

3.27 for all $r > 0$. Thus, all existence, stability and convergence results on the semi-discrete solutions $(\mathbf{u}_N, \mathbf{w}_N, \theta_N, \eta_N, \Phi_N)$ derived in Section 4.2 and 4.3 remain valid: Lemma 4.10 to 4.25. Therefore, one only has to investigate convergence of the term

$$\int_0^T \langle \mathbf{F}_k(\theta_N(t) + \theta_b, \Phi_N(t) + \Phi_b), \mathbf{v}\rangle_{\mathbf{U}^*} \psi(t)\, dt \tag{4.29}$$

for $\mathbf{v} \in \mathcal{V}(\Omega)$, $\psi \in C^\infty([0,T])$ which replaces Assumption 4.2. As noted in Section 3.3, there holds

$$\|\mathbf{g}_{E,\frac{T}{N}}[\Phi_N(t) + \Phi_b]\|_{0,3} \leq M \tag{4.30}$$

with M being independent of k, N, t, due to the cut-off operator in the Definition of $\mathbf{g}_{E,r}$. Thus, the sequence $\left(\mathbf{g}_{E,\frac{T}{N}}[\Phi_N + \Phi_b]\right)_N$ is bounded in $L^2(0,T;\mathbf{L}^3)$, which in turn is a reflexive Banach space, Theorem A.55. Thus, there is $\tilde{\mathbf{g}}_E \in L^2(0,T;\mathbf{L}^3)$ and a (not relabeled) subsequence such that

$$\mathbf{g}_{E,\frac{T}{N}}[\Phi_N + \Phi_b] \rightharpoonup \tilde{\mathbf{g}}_E \text{ in } L^2(0,T;\mathbf{L}^3). \tag{4.31}$$

for this subsequence, one obtains

$$\begin{aligned}
&\left|\int_0^T \langle \theta_N(t)\mathbf{g}_{E,\frac{T}{N}}[\Phi_N(t) + \Phi_b], \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t) - \int_0^T \langle \theta(t)\tilde{\mathbf{g}}_E(t), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt\right| \\
&\leq \left|\int_0^T \langle \theta(t)(\tilde{\mathbf{g}}_E(t) - \mathbf{g}_{E,\frac{T}{N}}[\Phi_N(t) + \Phi_b]), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt\right| \\
&\quad + \left|\int_0^T \langle (\theta_N(t) - \theta(t))\mathbf{g}_{E,\frac{T}{N}}[\Phi_N(t) + \Phi_b]), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt\right| \\
&\leq \left|\int_0^T \langle \theta(t)(\tilde{\mathbf{g}}_E(t) - \mathbf{g}_{E,\frac{T}{N}}[\Phi_N(t) + \Phi_b]), \mathbf{v}\rangle_{\mathbf{U}^*}\psi(t)\, dt\right| \\
&\quad + \|\theta_N - \theta\|_{2;L^2}\|\mathbf{g}_{E,\frac{T}{N}}[\Phi_N + \Phi_b])\|_{2;\mathbf{L}^3}\|\mathbf{v}\|_{0,6}\|\psi\|_{\infty;\mathbb{R}} \\
&=: A_N.
\end{aligned} \tag{4.32}$$

Here, $\lim_{N\to\infty} A_N = 0$, due to (4.31) (first term) and (4.30) and Lemma 4.20 (second term). Proceeding with Lemma 4.26, 4.27 and the proof of Theorem 4.3, the following result is obtained.

Theorem 4.29. *(Existence of Solutions with Weak Approximation of Electric Gravity)*
Let the requirements of Theorem 4.3 and Lemma 3.27 hold. Then, there is $\tilde{\mathbf{g}}_E \in L^2(0,T;\mathbf{L}^3)$ and $(\mathbf{u}, \theta, \Phi)$ with regularity given by Problem 4.1 and Theorem 4.3 such that heat equation (4.5), Gauss' law (4.6) and the following momentum equation (4.33) are satisfied for all $\mathbf{v} \in \mathbf{V}$, $\psi \in C^\infty([0,T])$ with $\psi(T) = 0$:

$$\begin{aligned}
&\int_0^T -(\mathbf{u}(t), \mathbf{v})\psi'(t) + \{a_{\mathbf{v}}(\mathbf{u}(t), \mathbf{v}) + c_{\mathbf{v}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v})\}\, \psi(t)\, dt \\
&- \int_0^T \{((\theta(t) + \theta_b)(2\alpha_e\tilde{\mathbf{g}}_E(t) + \alpha_g\mathbf{g}, \mathbf{v}) + \langle \mathbf{f}_{\mathbf{v}}(t), \mathbf{v}\rangle_{\mathbf{U}^*}\}\psi(t)\, dt = (\mathbf{u}_0, \mathbf{v})\psi(0).
\end{aligned} \tag{4.33}$$

The connection between electric gravity $\tilde{\mathbf{g}}_E$ and electric potential Φ is given in the sense that there is a sequence $(\frac{T}{N_n}, \Phi_{N_n})_n \subset (0,\infty) \times L^2(0,T;\Upsilon)$ with

$$\begin{aligned}
\frac{T}{N_n} &\to 0 \\
\Phi_{N_n} + \Phi_b &\rightharpoonup \Phi + \Phi_b \quad \text{in } L^2(0,T;H^1) \\
\mathbf{g}_{E,\frac{T}{N_n}}[\Phi_{N_n} + \Phi_b] &\rightharpoonup \tilde{\mathbf{g}}_E \qquad \text{in } L^2(0,T;\mathbf{L}^3),
\end{aligned} \tag{4.34}$$

with approximate electric gravity $\mathbf{g}_{E,k}[\cdot]$ *given by Definition 3.26.*

Similar to the stationary case, the system described by Theorem 4.29 can be considered as a weaker form of the TEHD Boussinesq equation (2.28) with alternative DEP formulation

$$\mathbf{f}_a = -\theta(2\alpha_e \mathbf{g}_E[\Phi + \Phi_b] + \alpha_g \mathbf{g}), \tag{4.35}$$

see (2.26). Here, the strong connection between electric gravity and potential,

$$\mathbf{g}_E[\Phi + \Phi_b] = \nabla^2(\Phi + \Phi_b)\nabla(\Phi + \Phi_b), \tag{4.36}$$

is replaced by the weaker condition (4.34).

5. Numerical Approximation

In this section, the discretization of the stationary and instationary TEHD Boussinesq equations, system (3.1) and (4.1), is addressed. The spatial discretization is based on the conforming Finite Element Method (FEM), whereas a variant of the Backward Differentiation Formula (BDF) is used for temporal approximation. We derive a priori error estimates in both cases. The result in the stationary case can be seen as direct generalization of the procedure presented in [46] for the incompresible Navier-Stokes equations and in [11] for the standard Boussinesq problem. The underlying approach is similar to the proof of uniqueness, Theorem 3.21. Thus, the corresponding result only holds under a suitable smallness condition posed on the problem data. The proposed error estimation for the instationary case is inspired by [75] where a full discretization of the instationary Boussinesq problem with temperature dependent viscosity is analyzed. This work is extended to take into account the DEP force and Gauss' law and to allow more general time stepping schemes.

The outline of this section is as follows. First, some required tools for spatial and temporal discretization are introduced. Afterward, the stationary and instationary problem are investigated separately in Section 5.1 and 5.2. Finally, the modeling of the DEP force has to be addressed again to cover additional aspects that come with the topic of discretization in the instationary case. This is done in Section 5.3.

For the remainder of this section, c denotes a generic constant that only depends on Ω, differentiability and integrability orders via embedding constants, but not on problem-dependent parameters such as ν or κ and discretization parameters such as time step size k and mesh width h. Moreover, the notation $a \lesssim b$ means that $a \leq Cb$ for some constant C that does not depend on h and k. As in the previous sections, the Assumptions 3.1 and 3.2 for domain, boundary conditions and the involved physical parameters should hold.

We start with setting up the basic tools that are needed to define the finite element discretization. In order to avoid technical difficulties in dealing with curved boundaries, the conditions on Ω become slightly more restrictive compared to the previous sections. Moreover, it is assumed that all cells in the mesh can be obtained by affine transformations of a single reference cell.

Assumption 5.1. *(Discretized Domain)*
Assume that Ω is a bounded, polyhedral domain with Lipschitz continuous boundary. For some $h_0 \in (0,1)$ let a regular family of triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0]}$ of Ω be given according to Definition A.118. The following conditions should hold for all $h \in (0, h_0]$:

(i) If $d = 2$, then $\mathcal{T}_h$ either consists of triangles or quadrilaterals only.

(ii) If $d = 3$, then $\mathcal{T}_h$ either consists of tetrahedrons or hexahedrons only.

(iii) There exists a compact set $K \subset \mathbb{R}^d$ with non-empty interior and piecewise smooth boundary such that for all $T \in \mathcal{T}_h$ there is an affine map $F_T(x) = A_T x + b_T$ with nonsingular matrix $A_T \in \mathbb{R}^{d\times d}$ such that $F_T(T) = K$.

(iv) $\Gamma_D = \bigcup_{f\in\mathcal{E}_{h,D}} \overline{f}$, where $\mathcal{E}_{h,D}$ denotes the set of all facets f, that have a non-empty intersection with Γ_D.

In the following, the spatial discretization parameter h, also called *mesh width*, denotes the maximal cell diameter relative to the diameter of the entire domain,

$$h := \max_{T\in\mathcal{T}_h} \frac{\operatorname{diam}(T)}{\operatorname{diam}(\Omega)}. \tag{5.1}$$

The subsequent definition relates the used finite element spaces for velocity, pressure, temperature and potential to some general finite element spaces. This is done to take care of additional restrictions, such as Dirichlet boundary conditions and zero average pressure. Afterward, Assumption 5.3 states sufficient conditions to ensure H^1-conformity, inf-sup stability, approximation properties and inverse estimates of the resulting finite element spaces.

Definition 5.2. *(Discrete Ansatz Spaces)*
Let Ω and a corresponding family of triangulations $\{\mathcal{T}_h\}_h$ satisfy Assumption 5.1. Let W_h, N_h, Y_h denote associated finite element spaces according to Definition A.120. The velocity ansatz space is defined by $\mathbf{U}_h := \left(W_h \cap H_0^1(\Omega)\right)^d$, the pressure space by $M_h := N_h \cap L_0^2(\Omega)$, the temperature and potential space by $\Theta_h = \Upsilon_h := X_h$ with $X_h := Y_h \cap H_D^1(\Omega)$.

Assumption 5.3. *(Finite Element Spaces)*
Let W_h, N_h and Y_h of Definition 5.2 satisfy for all $h \in (0, h_0]$:

(i) $W_h \subset H^1(\Omega)$, $M_h \subset L^2(\Omega)$ *and* $Y_h \subset H^1(\Omega)$.

(ii) *There is* $\beta > 0$ *such that*

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{1}{\|\mathbf{v}_h\|_{1,2}\|q_h\|} \int_\Omega q_h(\nabla \cdot \mathbf{v}_h)\, dx \geq \beta.$$

(iii) *There are* $m \in \mathbb{N}$ *and operators* $\Pi_{\mathbf{U}_h} \in \mathcal{L}\left(\mathbf{H}^2(\Omega), W_h^d\right) \cap \mathcal{L}\left(\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \mathbf{U}_h\right)$, $\Pi_{M_h} \in \mathcal{L}\left(L^2(\Omega), M_h\right)$ *such that for all integers* $1 \leq s \leq m$,

$$\begin{aligned} \|\mathbf{u} - \Pi_{\mathbf{U}_h}\mathbf{u}\| + h\|\mathbf{u} - \Pi_{\mathbf{U}_h}\mathbf{u}\|_{1,2} &\leq ch^{s+1}\|\mathbf{u}\|_{s+1,2} \quad \text{for all } \mathbf{u} \in \mathbf{H}^{s+1}(\Omega), \\ \|p - \Pi_{M_h}p\| &\leq ch^s\|p\|_{s,2} \qquad \text{for all } p \in H^s(\Omega). \end{aligned}$$

(iv) *There is* $n \in \mathbb{N}$ *and an operator* $\Pi_{X_h} \in \mathcal{L}(H^2(\Omega) \cap H_D^1(\Omega), X_h)$ *such that for all integers* $1 \leq s \leq n$

$$\|\theta - \Pi_{X_h}\theta\| + h\|\theta - \Pi_{X_h}\theta\|_{1,2} \leq ch^{s+1}\|\theta\|_{s+1,2} \quad \text{for all } \theta \in H^{s+1}(\Omega) \cap H_D^1(\Omega).$$

Moreover, the following stability estimate holds

$$\|\Pi_{X_h}\theta\|_{2,2} \leq c\|\theta\|_{2,2} \quad \text{for all } \theta \in H^2(\Omega) \cap H_D^1(\Omega).$$

(v) *For all* $s, t \in \{0, 1\}$ *with* $0 \leq t \leq s \leq 1$ *and* $q, r \in [1, \infty]$ *with* $1 \leq r \leq q \leq \infty$ *there holds*

$$\begin{aligned} \|\mathbf{u}_h\|_{s,q} &\leq ch^{t-s+d\left(\frac{1}{q}-\frac{1}{r}\right)}\|\mathbf{u}_h\|_{t,r} \quad \text{for all } \mathbf{u}_h \in \mathbf{U}_h, \\ \|\theta_h\|_{s,q} &\leq ch^{t-s+d\left(\frac{1}{q}-\frac{1}{r}\right)}\|\theta_h\|_{t,r} \quad \text{for all } \theta_h \in X_h. \end{aligned}$$

Assumption 5.3 (i) is used to obtain a conforming spatial discretization, i.e. $\mathbf{U}_h \subset \mathbf{U}$, $M_h \subset M$, $\Theta_h \subset \Theta$ and $\Upsilon_h \subset \Upsilon$. Condition (ii) assures that the discrete velocity-pressure pair is stable, i.e. the discretized equations for the incompressible flow part are solvable. Condition (iii) and (iv) state the existence of operators, that allow to approximate the exact solution by an element of the discrete ansatz space with an error that converges towards zero, as the mesh parameter h converges to zero. Typically, these operators are projection or interpolation operators. They are used to bound the error of the discrete solution in terms of h. The inverse estimate (v) allows to bound the norm of a finite element function by another, weaker norm in terms of differentiability and integrability. This goes to the expense of negative powers of h and is used to derive bounds on the pressure error.

One example of a combination of finite element spaces that satisfies Assumption 5.3 is based on the well-known Taylor-Hood element, see e.g. [32], for velocity and pressure.

Lemma 5.4. *(Taylor-Hood Element)*
Let Assumption 5.1 hold and define

$$\begin{aligned} W_h &:= \{u \in C(\overline{\Omega}) \colon\ u_{|_T} \circ F_T^{-1} \in \mathbb{I}_2\ \forall T \in \mathcal{T}_h\} \\ N_h &:= \{p \in C(\overline{\Omega}) \colon\ p_{|_T} \circ F_T^{-1} \in \mathbb{I}_1\ \forall T \in \mathcal{T}_h\} \\ Y_h &:= \{\theta \in C(\overline{\Omega}) \colon\ \theta_{|_T} \circ F_T^{-1} \in \mathbb{I}_2\ \forall T \in \mathcal{T}_h\}, \end{aligned}$$

with $\mathbb{I}_i = \mathbb{P}_i$, if $\mathcal{T}$ consists of triangles or tetrahedrons and $\mathbb{I}_i = \mathbb{Q}_i$ if $\mathcal{T}$ consists of quadrilaterals or hexahedrons. Then, Assumption 5.3 is satisfied for $m = 2$ and $n = 2$.

For the definition of $\mathbb{P}_i$ and $\mathbb{Q}_i$, $i \in \{1, 2\}$ see Definition A.113.

Proof. The underlying Finite Element $(K, \mathcal{P}, \mathcal{N})$ of the spaces W_h, N_h, Y_h is the well-known Lagrange element which satisfies Assumption A.123 for $(k, l) = (3, 0)$, $(k, l) = (2, 0)$ and $(k, l) = (3, 0)$, respectively. Let Π_{X_h} be given by the global interpolant $\mathcal{I}_h$ as in Definition A.122 w.r.t. Y_h. Using Assumption 5.1 (iv), Theorem A.124 and Corollary A.125 with $(k, l, p) = (3, 0, 2)$, Π_{X_h} is a bounded linear operator and satisfies condition (iv) for $n = 2$. By using the analog interpolation operators for $(W_h)^d$ and N_h, condition (iii) holds for $m = 2$.

Condition (v) follows by Theorem A.126. Finally, $W_h, Y_h \subset H^1(\Omega)$ due to their piecewise smoothness and global continuity of its member functions. Condition (ii) follows from Theorem A.127. □

When deriving a priori error estimates of finite element discretizations, convergence rates w.r.t. the mesh width h are typically obtained by projecting the unknown, exact solution onto the underlying finite element space. Whereas Assumption 5.3 (iii) and (iv) already provide projection operators $\Pi_{\mathbf{U}_h}$, Π_{M_h} and Π_{X_h}, it is beneficial to use a projection operator for velocity and pressure, that is tailored to the underlying equation. Such an operator is given by the *Stokes Projection*, see e.g. [30].

Definition 5.5. *(Stokes Projection)*
For $(\mathbf{u}, p) \in \mathbf{U} \times M$ with $\nabla \cdot \mathbf{u} = 0$ let $(\mathbf{w}_h, r_h) \in \mathbf{U}_h \times M_h$ denote the unique solution of

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{w}_h, \mathbf{v}_h) - b(\mathbf{v}_h, r_h) &= a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p) && \text{for all } \mathbf{v}_h \in \mathbf{U}_h \\ b(\mathbf{w}_h, q_h) &= 0 && \text{for all } q_h \in M_h. \end{aligned}$$

The Stokes Projection *is defined as*

$$\Pi_S \colon \mathbf{U} \times M \to \mathbf{U}_h \times M_h,\ (\mathbf{u}, p) \mapsto (\mathbf{w}_h, r_h).$$

The main advantage of Π_S compared to $\Pi_{\mathbf{U}_h}$ is given by the fact, that the projected velocity $\mathbf{w}_h$ is discretely divergence free and no additional consistency terms will occur, when replacing $a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p)$ by $a_{\mathbf{v}}(\mathbf{w}_h, \mathbf{v}_h) - b(\mathbf{v}_h, r_h)$ for deriving the error equation in the subsequent convergence proofs. The following lemmas provide additional properties of the Stokes projection, such as its approximation properties w.r.t. H^1- and L^2-norm and stability.

Lemma 5.6. *(Properties of Stokes Projection, Lemma 4.42 and 4.43 in [41])*
Let Assumption 5.3 (i) and (ii) hold. Then, Π_S is well-defined, linear and there holds for $(\mathbf{w}_h, r_h) = \Pi_S(\mathbf{u}, p)$

$$\begin{aligned} \|\nabla(\mathbf{u}-\mathbf{w}_h)\| &\le c\left(\left(1+\beta^{-1}\right)\inf_{\mathbf{v}_h\in\mathbf{U}_h}\|\nabla(\mathbf{u}-\mathbf{v}_h)\| + \nu^{-1}\inf_{q_h\in M_h}\|p-q_h\|\right) \\ \|p-r_h\| &\le c\beta^{-1}\left(\nu\left(1+\beta^{-1}\right)\inf_{\mathbf{v}_h\in\mathbf{U}_h}\|\nabla(\mathbf{u}-\mathbf{v}_h)\| + \inf_{q_h\in M_h}\|p-q_h\|\right) \\ \|\nabla\mathbf{w}_h\| &\le \|\nabla\mathbf{u}\| + \nu^{-1}\|p\|. \end{aligned}$$

Moreover, if $(\mathbf{u}, p) \in \left(\mathbf{H}^{l+1}(\Omega)\cap\mathbf{U}\right)\times\left(H^l(\Omega)\cap L^2_0(\Omega)\right)$ *for some* $1 \le l \le m$*, then*

$$\begin{aligned} \|\mathbf{u}-\mathbf{w}_h\|_{1,2} &\le C_S h^l\left(\|\mathbf{u}\|_{l+1,2} + \frac{1}{\nu}\|p\|_{l,2}\right) \\ \|p-r_h\| &\le C_S h^l\left(\nu\|\mathbf{u}\|_{l+1,2} + \|p\|_{l,2}\right), \end{aligned}$$

for some constant C_S that is independent of $(\mathbf{u}, p)$ and h.

If Ω possesses some additional regularity, then L^2-error estimates of order h^{l+1} can be derived for the Stokes projection, see the following Definition 5.7 and Lemma 5.8.

Definition 5.7. *(Regular Stokes Problem, Definition II.1.1 in [32])*
The Stokes problem is called regular*, if the mapping*

$$(\mathbf{u}, p) \mapsto -\nu\Delta\mathbf{u} + \nabla p$$

is an isomorphism from $\left(\mathbf{H}^2(\Omega)\cap\mathbf{V}\right)\times\left(H^1(\Omega)\cap L^2_0(\Omega)\right)$ *onto* $\mathbf{L}^2(\Omega)$.

Lemma 5.8. *(L^2-Error of Stokes Projection, Theorem II.1.9 in [32])*
Let Assumption 5.3 and the requirements of Lemma 5.6 hold and assume that the Stokes problem is regular. If $(\mathbf{u}, p) \in \left(\mathbf{H}^{l+1}(\Omega)\cap\mathbf{U}\right)\times\left(H^l(\Omega)\cap L^2_0(\Omega)\right)$ *for some* $1 \le l \le m$*, then* $(\mathbf{w}_h, r_h) = \Pi_S(\mathbf{u}, p)$ *satisfies*

$$\|\mathbf{u}-\mathbf{w}_h\| \le C_{L^2} h^{l+1}\left(\|\mathbf{u}\|_{l+1,2} + \|p\|_{l,2}\right).$$

Remark 5.9. According to Theorem I.5.4 in [32], the Stokes problem is regular if $\Gamma \in C^2$. According to Remark I.5.6, the Stokes problem is regular if $d = 2$ and Ω is a convex polygon.

The final point in this series of preparation is concerned with the trilinear forms $c_{\mathbf{v}}$ and c_τ. In previous stability results, e.g. Lemma 3.13, we made use of the identities $c_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{w}) = 0$ and $c_\tau(\mathbf{u}, \theta, \theta) = 0$ if $\nabla\cdot\mathbf{u} = 0$ holds. In general, these identities do not hold for discrete velocities $\mathbf{u}_h \in \mathbf{U}_h$, since the condition $(\nabla\cdot\mathbf{u}_h, q_h) = 0$ for all $q_h \in M_h$ does not imply $\nabla\cdot\mathbf{u}_h = 0$ a.e., unless $\nabla\cdot\mathbf{U}_h \subset M_h$. In order to cope with this issue, the convection terms $c_{\mathbf{v}}$ and c_τ are replaced by the well-known skew symmetrized trilinear forms, see e.g. Remark 6.25 in [41],

$$\begin{aligned} \tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) &:= \frac{1}{2}\left(c_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) - c_{\mathbf{v}}(\mathbf{u}, \mathbf{v}, \mathbf{w})\right), \\ \tilde{c}_\tau(\mathbf{u}, \theta, \tau) &:= \frac{1}{2}\left(c_\tau(\mathbf{u}, \theta, \tau) - c_\tau(\mathbf{u}, \tau, \theta)\right), \end{aligned} \tag{5.2}$$

for formulating the discretized problem. The skew symmetrized trilinear forms satisfy the following identities and estimates.

Lemma 5.10. *(Properties of Skew-Symmetrized Trilinear Forms)*
There holds for all $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $\theta, \tau \in H^1(\Omega)$

$$\begin{aligned}
&(i) && \tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -\tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{v}, \mathbf{w})\\
&(ii) && \tilde{c}_{\tau}(\mathbf{u}, \theta, \tau) = -\tilde{c}_{\tau}(\mathbf{u}, \tau, \theta)\\
&(iii) && |\tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v})| \le c\|\nabla\mathbf{u}\|\|\nabla\mathbf{w}\|\|\nabla\mathbf{v}\|\\
&(iv) && |\tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v})| \le c\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|\\
&(v) && |\tilde{c}_{\tau}(\mathbf{u}, \theta, \tau)| \le c\|\nabla\mathbf{u}\|\|\theta\|_{1,2}\|\tau\|_{1,2}\\
&(vi) && |\tilde{c}_{\tau}(\mathbf{u}, \theta, \tau)| \le c\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\theta\|_{1,2}\|\tau\|_{1,2}\\
&(vii) && |\tilde{c}_{\tau}(\mathbf{u}, \theta, \tau)| \le c\|\nabla\mathbf{u}\|\|\theta\|_{0,3}\|\tau\|_{1,2},
\end{aligned}$$

and, if $\nabla \cdot \mathbf{u} = 0$,

$$\begin{aligned}
&(viii) && \tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = c_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v})\\
&(ix) && \tilde{c}_{\tau}(\mathbf{u}, \theta, \tau) = c_{\tau}(\mathbf{u}, \theta, \tau),
\end{aligned}$$

with $\tilde{c}_{\mathbf{v}}$, $\tilde{c}_{\tau}$ *defined by (5.2). Additionally, if* $\mathbf{w} \in \mathbf{W}^{1,3}(\Omega) \cap \mathbf{L}^{\infty}(\Omega)$ *and* $\eta \in W^{1,3}(\Omega) \cap L^{\infty}(\Omega)$, *then*

$$\begin{aligned}
&(x) && |\tilde{c}_{\mathbf{v}}(\mathbf{u}, \mathbf{w}, \mathbf{v})| \le c\|\mathbf{u}\|\left(\|\nabla\mathbf{w}\|_{0,3} + \|\mathbf{w}\|_{0,\infty}\right)\|\mathbf{v}\|_{1,2}\\
&(xi) && |\tilde{c}_{\tau}(\mathbf{u}, \eta, \tau)| \le c\|\mathbf{u}\|\left(\|\nabla\eta\|_{0,3} + \|\eta\|_{0,\infty}\right)\|\tau\|_{1,2}.
\end{aligned}$$

Proof. The assertions follow by Hölder's inequality, the interpolation estimate (Theorem A.93), the Sobolev embeddings (Theorem A.92), integration by parts,

$$\begin{aligned}
\tilde{c}_{\tau}(\mathbf{u}, \theta, \tau) &= \frac{1}{2}\left(\int_{\Omega}(\mathbf{u}\cdot\nabla\theta)\,\tau\,dx - \int_{\Omega}(\mathbf{u}\cdot\nabla\tau)\,\theta\,dx\right)\\
&= \frac{1}{2}\left(-\int_{\Omega}(\nabla\cdot\mathbf{u})\theta\,\tau\,dx - 2\int_{\Omega}(\mathbf{u}\cdot\nabla\tau)\theta\,dx\right)
\end{aligned}$$

and the analog identity w.r.t. $\tilde{c}_{\mathbf{v}}$. □

5.1. Stationary Problem

In this section, a priori error estimates are derived for a conforming finite element discretization of the stationary TEHD Boussinesq equations. As in Section 3, the stationary system can also be considered as time-discrete version of the instationary equations. This continuous system in mixed form is given by Problem 5.11.

Problem 5.11. *(Continuous Stationary Problem in Mixed Form)*
Let $(\theta_b, \Phi_b) \in W^{1,2}(\Omega) \times W^{1,3}(\Omega)$ *denote boundary liftings and* $\mathbf{F}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{U}^*$, $f_{\mathbf{v}} \in \mathbf{U}^*$, $f_{\tau} \in \Theta^*$, $f_{\beta} \in \Upsilon^*$ *be given source terms. Let either* $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}} \in \mathbf{U}$, $\bar{\theta} \in \Theta$, $\bar{\Phi} \in \Upsilon$ *denote fixed functions or the unknown variables* $\mathbf{u}$, θ, Φ. *Find* $(\mathbf{u}, p, \theta, \Phi) \in \mathbf{U} \times M \times \Theta \times \Upsilon$ *such that for all* $(\mathbf{v}, q, \tau, \beta) \in \mathbf{U} \times M \times \Theta \times \Upsilon$:

$$\begin{aligned}
\delta(\mathbf{u}, \mathbf{v}) + a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) &= \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + \mathbf{f}_{\mathbf{v}}, \mathbf{v}\rangle_{\mathbf{U}^*}\\
b(\mathbf{u}, q) &= 0\\
\delta(\theta + \theta_b, \tau) + a_{\tau}(\theta + \theta_b, \tau) + c_{\tau}(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_{\tau}, \tau\rangle_{\Theta^*}\\
a_{\beta}(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_{\beta}, \beta\rangle_{\Upsilon^*}.
\end{aligned}$$

Compared to the previously stated stationary system, Problem 3.18, the boundary lifting Φ_b is supposed to be an element of $W^{1,3}$ instead of H^1 in order to derive a specific estimate from the bilinear form a_β in the proof of Lemma 5.14.

Discrete Problem

The corresponding discrete system is obtained by replacing the continuous functions spaces by their finite dimensional counterparts as given by Definition 5.2. Moreover, the trilinear forms $c_{\mathbf{v}}$, c_τ are replaced by the skew-symmetrized versions $\tilde{c}_{\mathbf{v}}$, $\tilde{c}_\tau$ defined by (5.2).

Problem 5.12. *(Discrete Stationary TEHD Equations)*
In the setting of Problem 5.11, let either $\bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h \in \mathbf{U}$, $\bar{\theta}_h \in \Theta$, $\bar{\Phi}_h \in \Upsilon$ denote fixed vector fields or the unknown variables $\mathbf{u}_h$, θ_h, Φ_h. Find $(\mathbf{u}_h, p_h, \theta_h, \Phi_h) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$ such that for all $(\mathbf{v}_h, q_h, \tau_h, \beta_h) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$:

$$
\begin{aligned}
\delta\left(\mathbf{u}_h, \mathbf{v}_h\right) + a_{\mathbf{v}}(\mathbf{u}_h, \mathbf{v}_h) + \tilde{c}_{\mathbf{v}}(\bar{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{v}_h) - b(p_h, \mathbf{v}_h) &= \langle F(\theta_h + \theta_b, \bar{\Phi}_h + \Phi_b) + f_{\mathbf{v}}, \mathbf{v}_h \rangle_{\mathbf{U}^*} \\
b(q_h, \mathbf{u}_h) &= 0 \\
\delta\left(\theta_h + \theta_b, \tau_h\right) + a_\tau(\theta_h + \theta_b, \tau_h) + \tilde{c}_\tau(\tilde{\mathbf{u}}_h, \theta_h + \theta_b, \tau_h) &= \langle f_\tau, \tau_h \rangle_{\Theta^*} \\
a_\beta(\bar{\theta}_h + \theta_b, \Phi_h + \Phi_b, \beta_h) &= \langle f_\beta, \beta_h \rangle_{\Upsilon^*}.
\end{aligned}
$$

If $\bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h$, $\bar{\theta}_h$, $\bar{\Phi}_h$ are fixed, suppose that they are uniformly bounded w.r.t. h, i.e. $\|\nabla \bar{\mathbf{u}}_h\|, \|\nabla \tilde{\mathbf{u}}_h\| \leq \overline{G}_{\mathbf{u}}$, $\|\nabla \bar{\theta}_h\| \leq \overline{G}_\theta$, $\|\nabla \bar{\Phi}_h\| \leq \overline{G}_\Phi$ for all $h \in (0, h_0]$

Before turning towards the error analysis, existence and stability of solutions of the discrete system are shown in the following Lemma 5.13. Due to the conformity of the involved finite element spaces and the skew-symmetrized trilinear forms, the corresponding proof works analogously as in the continuous setting.

Lemma 5.13. *(Existence and Stability of Discrete Stationary TEHD Solutions)*
Let the assumptions of Problem 5.12 and Assumption 3.12 and 5.3 hold. Assume that θ_b satisfies $\|\theta_b\|_{0,3} < d(a_{\mathbf{F}}(\tilde{G}_{\Phi,\Phi_b}))$ with

$$
\begin{aligned}
\tilde{G}_{\Phi,\Phi_b} &:= \sqrt{1 + c_D^2}\,\tilde{G}_\Phi + \|\Phi_b\|_{1,2} \\
\tilde{G}_\Phi &:= \frac{\epsilon_+}{\epsilon_-}\|\Phi_b\|_{1,2} + \frac{1}{\epsilon_-}\|f_\beta\|_{\Upsilon^*}
\end{aligned}
$$

and d be given by Lemma 3.13. Then, there exists a solution $(\mathbf{u}_h, p_h, \theta_h, \Phi_h)$ of Problem 5.12. Further, there exist constants $\tilde{G}_{\mathbf{u}}$, $\tilde{G}_\theta$, independent of h, such that all discrete solutions satisfy

$$
\|\nabla \mathbf{u}_h\| \leq \tilde{G}_{\mathbf{u}}, \quad \|\nabla \theta_h\| \leq \tilde{G}_\theta, \quad \|\nabla \Phi_h\| \leq \tilde{G}_\Phi.
$$

Proof. Since $\mathbf{U}_h$, M_h, Θ_h and Υ_h are closed subspaces of their infinite dimensional counterparts $\mathbf{U}$, M, Θ and Υ, respectively and the discrete inf-sup condition holds according to Assumption 5.3, existence and norm estimates of solutions follow analogously to the proof of Theorem 3.19 and Corollary 3.20 with minor modifications: the skew-symmetrized trilinear forms ensure existence and norm estimates of solutions of the Boussinesq equations for fixed potential. For proving stability, one needs to take into account that $\nabla \cdot \mathbf{u}_h \neq 0$ when estimating the term $\tilde{c}_\tau(\mathbf{u}_h, \theta_b, \theta_h)$ in the proof of Lemma 3.13. This can be done by using $|\tilde{c}_\tau(\mathbf{u}_h, \theta_b, \theta_h)| \leq c\|\nabla \mathbf{u}_h\|\|\nabla \theta_h\|\|\theta_b\|_{0,3}$ according to Lemma 5.10. $\square$

Error Estimate

The main result of this section is given by Theorem 5.15 which states that the discrete solution obtained by the standard Galerkin procedure satisfies a best-approximation property w.r.t. the chosen ansatz space. The approach for proving this result is similar to the standard Boussinesq case, e.g. in [11]: By using conformity, the error equation is obtained by subtracting the discrete equations from the continuous one with test functions chosen as $(\mathbf{v}, q, \tau, \beta) = (\mathbf{v}_h, q_h, \tau_h, \beta_h)$. Afterward, the individual error terms $e_{\cdot,h}$ can be split into discretization error $d_{\cdot,h} \in X_h$ and approximation error $z_{\cdot,h} \in X$. Then, the claimed result is obtained by bounding $d_{\cdot,h}$ in terms of $z_{\cdot,h}$.

As this procedure requires a case-by-case analysis similar to the uniqueness Theorem 3.21, a major of the calculation is put into a preliminary estimation given by Lemma 5.14.

Lemma 5.14. *(Preliminary Error Estimation)*
Let the assumptions stated in Problem 5.11 and 5.12 and in Lemma 5.13 hold. Assume that a solution $(\mathbf{u}, p, \theta, \Phi)$ of Problem 5.11 is given with $\Phi \in W^{1,3}(\Omega)$ and let $(\mathbf{u}_h, p_h, \theta_h, \Phi_h)$ denote a solution of Problem 5.12. For arbitrary $(\mathbf{w}_h, r_h, \eta_h, \Psi_h) \in \mathbf{V}_h \times M_h \times \Theta_h \times \Upsilon_h$ define the errors

$$\begin{aligned}
e_{\mathbf{u},h} &:= \mathbf{u} - \mathbf{u}_h &&= (\mathbf{u} - \mathbf{w}_h) + (\mathbf{w}_h - \mathbf{u}_h) =: \mathbf{z}_{\mathbf{u},h} + \mathbf{d}_{\mathbf{u},h}\\
e_{p,h} &:= p - p_h &&= (p - r_h) \;\;+ (r_h - p_h) \;\;=: z_{p,h} + d_{p,h}\\
e_{\theta,h} &:= \theta - \theta_h &&= (\theta - \eta_h) \;\;+ (\eta_h - \theta_h) \;\;=: z_{\theta,h} + d_{\theta,h}\\
e_{\Phi,h} &:= \Phi - \Phi_h &&= (\Phi - \Psi_h) + (\Psi_h - \Phi_h) =: z_{\Phi,h} + d_{\Phi,h}\\
e_{\bar{\mathbf{u}},h} &:= \bar{\mathbf{u}} - \bar{\mathbf{u}}_h\\
e_{\tilde{\mathbf{u}},h} &:= \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\\
e_{\bar{\theta},h} &:= \bar{\theta} - \bar{\theta}_h\\
e_{\bar{\Phi},h} &:= \bar{\Phi} - \bar{\Phi}_h.
\end{aligned}$$

Then, there exist nonnegative constants $\epsilon^*_\cdot$ and $C_\cdot$, independent of h, such that the discretization errors $d_{\cdot,h}$ satisfy

$$\begin{aligned}
\|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 \leq{}& \left(\epsilon^*_1 + \epsilon^*_{4,2}\|\nabla \bar{\mathbf{u}}_h\|^2\right)\|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 + \epsilon^*_{4,1}\|\nabla \mathbf{u}\|^2\|\nabla e_{\bar{\mathbf{u}},h}\|^2 + \epsilon^*_2\|z_{p,h}\|^2 \\
&+ \epsilon^*_{5,1}\epsilon^*_{7,1}\|\theta + \theta_b\|^2_{1,2}\|\nabla e_{\tilde{\mathbf{u}},h}\|^2 + \epsilon^*_{5,1}\left(1 + \epsilon^*_6 + \epsilon^*_{7,2}\|\nabla \tilde{\mathbf{u}}_h\|^2\right)\|\nabla z_{\theta,h}\|^2 \\
&+ \epsilon^*_{5,2}\|\nabla e_{\bar{\Phi},h}\|^2
\end{aligned} \tag{5.3}$$

$$\|\nabla d_{\theta,h}\|^2 \leq \left(\epsilon^*_6 + \epsilon^*_{7,2}\|\nabla \tilde{\mathbf{u}}_h\|^2\right)\|\nabla z_{\theta,h}\|^2 + \epsilon^*_{7,1}\|\theta + \theta_b\|^2_{1,2}\|\nabla e_{\tilde{\mathbf{u}},h}\|^2 \tag{5.4}$$

$$\|\nabla d_{\Phi,h}\|^2 \leq C^2_{0,1}\|\nabla z_{\Phi,h}\|^2 + C^2_{0,2}\|\nabla(\Phi + \Phi_b)\|^2_{0,3}\|\nabla e_{\bar{\theta},h}\|^2. \tag{5.5}$$

Proof. Using the fact $\nabla \cdot \mathbf{u} = 0$ with Lemma 5.10 and Assumption 5.3 (i), the following error equations hold for all $(\mathbf{v}_h, q_h, \tau_h, \beta_h) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$,

$$\delta\left(\mathbf{d}_{\mathbf{u},h}, \mathbf{v}_h\right) + a_{\mathbf{v}}(\mathbf{d}_{\mathbf{u},h}, \mathbf{v}_h) = -\langle R_1 + R_2 + R_3 + R_4 - R_5, \mathbf{v}_h\rangle \tag{5.6}$$

$$\delta\left(d_{\theta,h}, \tau_h\right) + a_\tau(d_{\theta,h}, \tau_h) = -\langle R_6 + R_7, \mathbf{v}_h\rangle \tag{5.7}$$

$$a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta_h) = a_\beta(\bar{\theta}_h + \theta_b, \Phi_h + \Phi_b, \beta_h), \tag{5.8}$$

with residuals

$$
\begin{aligned}
\langle R_1, \mathbf{v}_h\rangle &= \delta\left(\mathbf{z}_{\mathbf{u},h}, \mathbf{v}_h\right) + a_{\mathbf{v}}(\mathbf{z}_{\mathbf{u},h}, \mathbf{v}_h)\\
\langle R_2, \mathbf{v}_h\rangle &= b(\mathbf{v}_h, z_{p,h})\\
\langle R_3, \mathbf{v}_h\rangle &= b(\mathbf{v}_h, d_{p,h})\\
\langle R_4, \mathbf{v}_h\rangle &= \tilde{c}_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}_h) - \tilde{c}_{\mathbf{v}}(\bar{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{v}_h)\\
\langle R_5, \mathbf{v}_h\rangle &= \langle F(\theta+\theta_b, \bar{\Phi}+\Phi_b) - F(\theta_h+\theta_b, \bar{\Phi}_h+\Phi_b), \mathbf{v}_h\rangle_{\mathbf{U}^*}\\
\langle R_6, \tau_h\rangle &= \delta\left(z_{\theta,h}, \tau_h\right) + a_\tau(z_{\theta,h}, \tau_h)\\
\langle R_7, \tau_h\rangle &= \tilde{c}_\tau(\tilde{\mathbf{u}}, \theta+\theta_b, \tau_h) - \tilde{c}_\tau(\tilde{\mathbf{u}}_h, \theta_h+\theta_b, \tau_h).
\end{aligned}
$$

Now, all individual terms are estimated in order to bound $\|\nabla d_{\cdot,h}\|$ in terms of $\|\nabla z_{\cdot,h}\|$.

Estimation of Potential Error

Setting $\beta_h = d_{\Phi,h}$ in Gauss's law, one obtains

$$
\begin{aligned}
0 &= a_\beta(\bar{\theta}+\theta_b, \Phi+\Phi_b, d_{\Phi,h}) - a_\beta(\bar{\theta}_h+\theta_b, \Phi_h+\Phi_b, d_{\Phi,h})\\
&= a_\beta(\bar{\theta}+\theta_b, \Phi+\Phi_b, d_{\Phi,h}) - a_\beta(\bar{\theta}_h+\theta_b, \Phi+\Phi_b, d_{\Phi,h})\\
&\quad + a_\beta(\bar{\theta}_h+\theta_b, \Phi+\Phi_b, d_{\Phi,h}) - a_\beta(\bar{\theta}_h+\theta_b, \Psi_h+\Phi_b, d_{\Phi,h})\\
&\quad + a_\beta(\bar{\theta}_h+\theta_b, \Psi_h+\Phi_b, d_{\Phi,h}) - a_\beta(\bar{\theta}_h+\theta_b, \Phi_h+\Phi_b, d_{\Phi,h})\\
&=: L_1 + L_2 + L_3.
\end{aligned}
$$

The terms L_i can be estimated via

$$
\begin{aligned}
|L_1| &= |(\left(\epsilon(\bar{\theta}+\theta_b) - \epsilon(\bar{\theta}_h+\theta_b)\right)\nabla(\Phi+\Phi_b), \nabla d_{\Phi,h})|\\
&\le L_\epsilon K_6 \|\nabla e_{\bar{\theta},h}\| \|\nabla(\Phi+\Phi_b)\|_{0,3} \|\nabla d_{\Phi,h}\|
\end{aligned}
$$

$$
\begin{aligned}
|L_2| &= |(\epsilon(\bar{\theta}_h+\theta_b)\nabla(\Phi-\Psi_h), \nabla d_{\Phi,h})|\\
&\le \epsilon_+ \|\nabla z_{\Phi,h}\| \|\nabla d_{\Phi,h}\|
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= (\epsilon(\bar{\theta}_h+\theta_b)\nabla(\Psi_h-\Phi_h), \nabla d_{\Phi,h})\\
&\ge \epsilon_- \|\nabla d_{\Phi,h}\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\nabla d_{\Phi,h}\| &\le \frac{\epsilon_+}{\epsilon_-}\|\nabla z_{\Phi,h}\| + \frac{L_\epsilon K_6}{\epsilon_-}\|\nabla(\Phi+\Phi_b)\|_{0,3}\|\nabla e_{\bar{\theta}}\|\\
&=: \frac{1}{2}C_{0,1}\|\nabla z_{\Phi,h}\| + \frac{1}{2}C_{0,2}\|\nabla(\Phi+\Phi_b)\|_{0,3}\|\nabla e_{\bar{\theta}}\|. \qquad (5.9)
\end{aligned}
$$

Diffusive and Pressure Residual

Setting $\mathbf{v}_h = \mathbf{d}_{\mathbf{u},h} \in \mathbf{V}_h$ and using $d_{p,h} \in M_h$, there holds

$$
|\langle R_2, \mathbf{d}_{\mathbf{u},h}\rangle| \le c\|\nabla \mathbf{d}_{\mathbf{u},h}\|\|z_{p,h}\| =: C_2\|\nabla \mathbf{d}_{\mathbf{u},h}\|\|z_{p,h}\| \qquad (5.10)
$$

$$
\langle R_3, \mathbf{d}_{\mathbf{u},h}\rangle = 0. \qquad (5.11)
$$

Moreover, with $\tau_h = d_{\theta,h}$,

$$|\langle R_1, \mathbf{d}_{\mathbf{u},h}\rangle| \le \left(\delta c_0^2 + \nu\right) \|\nabla \mathbf{z}_{\mathbf{u},h}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\| =: C_1\|\nabla \mathbf{z}_{\mathbf{u},h}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\| \tag{5.12}$$

$$|\langle R_6, d_{\theta,h}\rangle| \le \left(\delta c_D^2 + \kappa\right) \|\nabla z_{\theta,h}\|\|\nabla d_{\theta,h}\| =: C_6\|\nabla z_{\theta,h}\|\|\nabla d_{\theta,h}\|. \tag{5.13}$$

Convection Residual

Using Lemma 5.10 (i), (ii), (iii), (v), $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and Friedrich's inequality,

$$\begin{aligned} |\langle R_4, \mathbf{d}_{\mathbf{u},h}\rangle| &= |\tilde{c}_{\mathbf{v}}(e_{\bar{\mathbf{u}},h}, \mathbf{u}, \mathbf{d}_{\mathbf{u},h}) + \tilde{c}_{\mathbf{v}}(\bar{\mathbf{u}}_h, e_{\mathbf{u},h}, \mathbf{d}_{\mathbf{u},h})| \\ &\le C_4\left(\|\nabla e_{\bar{\mathbf{u}},h}\|\|\nabla \mathbf{u}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\| + \|\nabla \bar{\mathbf{u}}_h\|\|\nabla \mathbf{z}_{\mathbf{u},h}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\|\right) \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} |\langle R_7, \mathbf{d}_{\mathbf{u},h}\rangle| &= |\tilde{c}_{\tau}(e_{\tilde{\mathbf{u}},h}, \theta + \theta_b, d_{\theta,h}) + \tilde{c}_{\tau}(\tilde{\mathbf{u}}_h, e_{\theta,h}, d_{\theta,h})| \\ &\le C_7\left(\|\nabla e_{\tilde{\mathbf{u}},h}\|\|\theta + \theta_b\|_{1,2}\|\nabla d_{\theta,h}\| + \|\nabla \tilde{\mathbf{u}}_h\|\|\nabla z_{\theta,h}\|\|\nabla d_{\theta,h}\|\right). \end{aligned} \tag{5.15}$$

Force Residual

Continuing with the force term, we have

$$\begin{aligned} |\langle R_5, \mathbf{d}_{\mathbf{u},h}\rangle| &\le |\langle F(\theta + \theta_b, \bar{\Phi} + \Phi_b) - F(\theta_h + \theta_b, \bar{\Phi} + \Phi_b), \mathbf{d}_{\mathbf{u},h}\rangle_{\mathbf{U}^*}| \\ &\quad + |\langle F(\theta_h + \theta_b, \bar{\Phi} + \Phi_b) - F(\theta_h + \theta_b, \bar{\Phi}_h + \Phi_b), \mathbf{d}_{\mathbf{u},h}\rangle_{\mathbf{U}^*}| \\ &=: L_4 + L_5. \end{aligned}$$

Using the local Lipschitz continuity of F, Assumption 3.12 (i), and the fact that $\|\theta + \theta_b\|_{1,2}$, $\|\theta_h + \theta_b\|_{1,2}$ and $\|\Phi_h + \Phi_b\|_{1,2}$ are uniformly bounded w.r.t. h, according to Theorem 3.19 and Lemma 5.13, one can state the existence of $L_{\mathbf{F}}^{(\theta)} = L_{\mathbf{F}}^{(\theta)}(\|\bar{\Phi} + \Phi_b\|_{1,2}) > 0$, $L_{\mathbf{F}}^{(\Phi)} > 0$ such that

$$\begin{aligned} L_4 &\le L_{\mathbf{F}}^{(\theta)}\|e_{\theta,h}\|_{1,2}\|\nabla \mathbf{d}_{\mathbf{u},h}\| \\ &\le L_{\mathbf{F}}^{(\theta)}\sqrt{1 + c_D^2}\left(\|\nabla z_{\theta,h}\| + \|\nabla d_{\theta,h}\|\right)\|\nabla \mathbf{d}_{\mathbf{u},h}\| \end{aligned}$$

$$\begin{aligned} L_5 &\le L_{\mathbf{F}}^{(\Phi)}\|\theta_h + \theta_b\|_{1,2}\|\bar{\Phi} - \bar{\Phi}_h\|_{1,2}\|\nabla \mathbf{d}_{\mathbf{u},h}\| \\ &\le L_{\mathbf{F}}^{(\Phi)}\left(\tilde{G}_\theta + \|\theta_b\|_{1,2}\right)\sqrt{1 + c_D^2}\|\nabla e_{\bar{\Phi},h}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\|. \end{aligned}$$

Thus,

$$|\langle R_5, \mathbf{d}_{\mathbf{u},h}\rangle| \le C_{5,1}\left(\|\nabla z_{\theta,h}\| + \|\nabla d_{\theta,h}\|\right)\|\nabla \mathbf{d}_{\mathbf{u},h}\| + C_{5,2}\|\nabla e_{\bar{\Phi},h}\|\|\nabla \mathbf{d}_{\mathbf{u},h}\|. \tag{5.16}$$

Left-Hand Sides

The respective left-hand sides in (5.6)-(5.8) can be estimated from below as

$$\delta\left(\mathbf{d}_{\mathbf{u},h}, \mathbf{d}_{\mathbf{u},h}\right) + a_{\mathbf{v}}(\mathbf{d}_{\mathbf{u},h}, \mathbf{d}_{\mathbf{u},h}) \ge \nu\|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 \tag{5.17}$$

$$\delta\left(d_{\theta,h}, d_{\theta,h}\right) + a_{\tau}(d_{\theta,h}, d_{\theta,h}) \ge \kappa\|\nabla d_{\theta,h}\|^2. \tag{5.18}$$

Combined Estimation

Combining the previous estimates (5.10) - (5.18), using Young's inequality, $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$, with parameters $\epsilon_\cdot > 0$ and stability bounds $G_{\mathbf{u}}$, G_θ, G_Φ derived for the exact solution according to Theorem 3.17, gives

$$\begin{aligned}
&\left(\nu - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2} - \frac{\epsilon_{4,1}}{2} - \frac{\epsilon_{4,2}}{2} - \epsilon_{5,1} - \frac{\epsilon_{5,2}}{2}\right) \|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 \\
&\leq \frac{1}{2\epsilon_1} C_1^2 \|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 + \frac{1}{2\epsilon_2} C_2^2 \|z_{p,h}\|^2 \\
&\quad + \frac{1}{2\epsilon_{4,1}} C_4^2 \|\nabla \mathbf{u}\|^2 \|\nabla e_{\bar{\mathbf{u}},h}\|^2 + \frac{1}{2\epsilon_{4,2}} C_4^2 \|\nabla \bar{\mathbf{u}}_h\|^2 \|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 \\
&\quad + \frac{1}{2\epsilon_{5,1}} C_{5,1}^2 \left(\|\nabla z_{\theta,h}\|^2 + \|\nabla d_{\theta,h}\|^2\right) + \frac{1}{2\epsilon_{5,2}} C_{5,2}^2 \|\nabla e_{\bar{\Phi},h}\|^2.
\end{aligned}$$

An appropriate choice of $\{\epsilon_i\}$ such that the factor on the left-hand side is positive, i.e. $\epsilon_i \propto \nu$, yields

$$\begin{aligned}
\|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 \leq\ & \epsilon_1^* \|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 + \epsilon_2^* \|z_{p,h}\|^2 + \epsilon_{4,1}^* \|\nabla \mathbf{u}\|^2 \|\nabla e_{\bar{\mathbf{u}},h}\|^2 + \epsilon_{4,2}^* \|\nabla \bar{\mathbf{u}}_h\|^2 \|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 \\
& + \epsilon_{5,1}^* \left(\|\nabla z_{\theta,h}\|^2 + \|\nabla d_{\theta,h}\|^2\right) + \epsilon_{5,2}^* \|\nabla e_{\bar{\Phi},h}\|^2,
\end{aligned} \tag{5.19}$$

with positive constants $\{\epsilon_i^*\}$. Concerning the error equation for e_θ, one obtains

$$\begin{aligned}
&\left(\kappa - \frac{\epsilon_6}{2} - \frac{\epsilon_{7,1}}{2} - \frac{\epsilon_{7,2}}{2}\right) \|\nabla d_{\theta,h}\|^2 \\
&\leq \frac{1}{2\epsilon_6} C_6^2 \|\nabla z_{\theta,h}\|^2 + \frac{1}{2\epsilon_{7,1}} C_7^2 \|\theta + \theta_b\|_{1,2}^2 \|\nabla e_{\tilde{\mathbf{u}},h}\|^2 + \frac{1}{2\epsilon_{7,2}} C_7^2 \|\nabla \tilde{\mathbf{u}}_h\|^2 \|\nabla z_{\theta,h}\|^2.
\end{aligned}$$

Again, choosing $\epsilon_i \propto \kappa$ such that the factor on the left-hand side is positive yields

$$\|\nabla d_{\theta,h}\|^2 \leq \epsilon_6^* \|\nabla z_{\theta,h}\|^2 + \epsilon_{7,1}^* \|\theta + \theta_b\|_{1,2}^2 \|\nabla e_{\tilde{\mathbf{u}},h}\|^2 + \epsilon_{7,2}^* \|\nabla \tilde{\mathbf{u}}_h\|^2 \|\nabla z_{\theta,h}\|^2 \tag{5.20}$$

for $\epsilon_i^* > 0$. Combining inequalities (5.19), (5.20) and (5.9) now yields the stated assertion. □

By means of the inequalities (5.3)-(5.5) it is possible to bound the discretization error $\|\nabla d_{\mathbf{u},h}\|$, $\|\nabla d_{\theta,h}\|$, $\|\nabla d_{\Phi,h}\|$ in terms of the approximation errors $\|\nabla z_{\mathbf{u},h}\|$, $\|z_{p,h}\|$, $\|\nabla z_{\theta,h}\|$, $\|\nabla z_{\Phi,h}\|$ and the error terms $\|\nabla e_{\bar{\mathbf{u}},h}\|$, $\|\nabla e_{\tilde{\mathbf{u}},h}\|$, $\|\nabla e_{\bar{\theta},h}\|$, $\|\nabla e_{\bar{\Phi},h}\|$. Depending on the degree of implicitness of Problem 5.12, each of the latter terms can be either fixed, or equal to the error in the solution, e.g. $e_{\bar{\mathbf{u}},h} = e_{\mathbf{u},h} = z_{\mathbf{u},h} + d_{\mathbf{u},h}$. In case of $\bar{\mathbf{u}}, \tilde{\mathbf{u}}$ and their respective discrete counterparts being fixed, (5.3)-(5.5) directly leads to some type of best approximation result of the discrete solution, up to the modeling errors $\|\nabla e_{\bar{\mathbf{u}},h}\|$, $\|\nabla e_{\tilde{\mathbf{u}},h}\|$, $\|\nabla e_{\bar{\theta},h}\|$, $\|\nabla e_{\bar{\Phi},h}\|$. However, if this is not the case, one has to impose certain smallness conditions onto the data in order to obtain a comparable result. The following theorem collects the different possibilities.

Theorem 5.15. *(Best Approximation of Finite Element Discretization)*
Let the assumptions of Lemma 5.14 hold. Assume that one of the following conditions hold:

(i) $\|\nabla \mathbf{u}\|$, $\|\theta + \theta_b\|_{1,2}$, $\|\Phi + \Phi_b\|_{1,3}$ *are sufficiently small.*

(ii) $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$ *and* $\bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h$ *are fixed.*

(iii) $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$, $\bar{\theta}$, $\bar{\Phi}$ *and* $\bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h$, $\bar{\theta}_h$, $\bar{\Phi}_h$ *are fixed.*

Then,

$$\begin{aligned}
\|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \|\nabla(\theta - \theta_h)\| + \|\nabla(\Phi - \Phi_h)\| \lessapprox\ & \inf_{\mathbf{v}_h \in \mathbf{U}_h} \|\nabla(\mathbf{u} - \mathbf{v}_h)\| + \inf_{q_h \in M_h} \|p - q_h\| \\
& + \inf_{\tau_h \in \Theta_h} \|\nabla(\theta - \tau_h)\| + \inf_{\beta_h \in \Upsilon_h} \|\nabla(\Phi - \beta_h)\| \\
& + \delta_{\bar{\mathbf{u}}} \|\nabla(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h)\| + \delta_{\tilde{\mathbf{u}}} \|\nabla(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)\| \\
& + \delta_{\bar{\theta}} \|\nabla(\bar{\theta} - \bar{\theta}_h)\| + \delta_{\bar{\Phi}} \|\nabla(\bar{\Phi} - \bar{\Phi}_h)\|,
\end{aligned}$$

with $\delta_{\bar{\mathbf{u}}} = \delta_{\tilde{\mathbf{u}}} = \delta_{\bar{\theta}} = \delta_{\bar{\Phi}} = 0$ in case (i), $\delta_{\bar{\mathbf{u}}} = \delta_{\tilde{\mathbf{u}}} = 1$, $\delta_{\bar{\theta}} = \delta_{\bar{\Phi}} = 0$ in case (ii) and $\delta_{\bar{\mathbf{u}}} = \delta_{\tilde{\mathbf{u}}} = \delta_{\bar{\theta}} = \delta_{\bar{\Phi}} = 1$ in case (iii).

Proof. Case (i):

Replace $\|\nabla e_{X,h}\|^2$ by $2\|\nabla d_{Y,h}\|^2 + 2\|\nabla z_{Y,h}\|^2$ for $(X,Y) \in \{(\bar{\mathbf{u}},\mathbf{u}), (\tilde{\mathbf{u}},\mathbf{u}), (\bar{\theta},\theta), (\bar{\Phi},\Phi)\}$ in the set of inequalities (5.3)-(5.5), estimate $\|\nabla\bar{\mathbf{u}}_h\| = \|\nabla\tilde{\mathbf{u}}_h\| = \|\nabla\mathbf{u}_h\| \leq \tilde{G}_{\mathbf{u}}$ and obtain for the velocity discretization error,

$$\begin{aligned}\Xi_1\|\nabla\mathbf{d}_{\mathbf{u},h}\|^2 \leq{}& 2\epsilon^*_{5,2}\|\nabla d_{\Phi,h}\|^2\\ &+ \left(\epsilon^*_1 + \epsilon^*_{4,2}\|\nabla\mathbf{u}\|^2 + 2\epsilon^*_{4,1}\|\nabla\mathbf{u}\|^2 + 2\epsilon^*_{5,1}\epsilon^*_{7,1}\|\theta+\theta_b\|^2_{1,2}\right)\|\nabla\mathbf{z}_{\mathbf{u},h}\|^2\\ &+ \epsilon^*_2\|z_{p,h}\|^2\\ &+ \epsilon^*_{5,1}\left(1 + \epsilon^*_6 + \epsilon^*_{7,2}\tilde{G}^2_{\mathbf{u}}\right)\|\nabla z_{\theta,h}\|^2\\ &+ 2\epsilon^*_{5,2}\|\nabla z_{\Phi,h}\|^2,\end{aligned} \tag{5.21}$$

with

$$\Xi_1 := 1 - 2\epsilon^*_{4,1}\|\nabla\mathbf{u}\|^2 - 2\epsilon^*_{5,1}\epsilon^*_{7,1}\|\theta+\theta_b\|^2_{1,2}. \tag{5.22}$$

If $\Xi_1 > 0$, then (5.21) leads to

$$\|\nabla\mathbf{d}_{\mathbf{u},h}\|^2 \leq \alpha^*_1\|\nabla d_{\Phi,h}\|^2 + \alpha^*_2\|\nabla\mathbf{z}_{\mathbf{u},h}\|^2 + \alpha^*_3\|z_{p,h}\|^2 + \alpha^*_4\|\nabla z_{\theta,h}\|^2 + \alpha^*_5\|\nabla z_{\Phi,h}\|^2, \tag{5.23}$$

with $\alpha^*_i \geq 0$. For the temperature and potential discretization error, (5.3)-(5.5) leads to

$$\begin{aligned}\|\nabla d_{\theta,h}\|^2 \leq{}& 2\epsilon^*_{7,1}\|\theta+\theta_b\|^2_{1,2}\|\nabla\mathbf{d}_{\mathbf{u},h}\|^2\\ &+ 2\epsilon^*_{7,1}\|\theta+\theta_b\|^2_{1,2}\|\nabla\mathbf{z}_{\mathbf{u},h}\|^2\\ &+ \left(\epsilon^*_6 + \epsilon^*_{7,2}\tilde{G}^2_{\mathbf{u}}\right)\|\nabla z_{\theta,h}\|^2\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\|\nabla d_{\Phi,h}\|^2 \leq{}& 2C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\|\nabla d_{\theta,h}\|^2\\ &+ 2C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\|\nabla z_{\theta,h}\|^2\\ &+ C^2_{0,1}\|\nabla z_{\Phi,h}\|^2.\end{aligned} \tag{5.25}$$

Inserting the upper bound for $\|\nabla d_{\Phi,h}\|^2$ (5.25) and then for $\|\nabla d_{\theta,h}\|^2$ (5.24) into (5.23), gives

$$\begin{aligned}\Xi_2\|\nabla\mathbf{d}_{\mathbf{u},h}\|^2 \leq{}& \left(\alpha^*_2 + 4\alpha^*_1\epsilon^*_{7,1}C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\|\theta+\theta_b\|^2_{1,2}\right)\|\nabla\mathbf{z}_{\mathbf{u},h}\|^2\\ &+ \alpha^*_3\|z_{p,h}\|^2\\ &+ \Big(\alpha^*_4 + 2\alpha^*_1C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\\ &\quad + 2\alpha^*_1C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\left(\epsilon^*_6 + \epsilon^*_{7,2}G^2_{\mathbf{u}}\right)\Big)\|\nabla z_{\theta,h}\|^2\\ &+ \left(\alpha^*_5 + \alpha^*_1C^2_{0,1}\right)\|\nabla z_{\Phi,h}\|^2.\end{aligned} \tag{5.26}$$

with

$$\Xi_2 := 1 - 4\alpha^*_1\epsilon^*_{7,1}C^2_{0,2}\|\nabla(\Phi+\Phi_b)\|^2_{0,3}\|\theta+\theta_b\|^2_{1,2}. \tag{5.27}$$

Imposing additionally $\Xi_2 > 0$, (5.26) can be written as

$$\|\nabla\mathbf{d}_{\mathbf{u},h}\|^2 \leq \beta^*_1\|\nabla\mathbf{z}_{\mathbf{u},h}\|^2 + \beta^*_2\|z_{p,h}\|^2 + \beta^*_3\|\nabla z_{\theta,h}\|^2 + \beta^*_4\|\nabla z_{\Phi,h}\|^2 \tag{5.28}$$

for constants $\beta_i^* > 0$. Inserting this expression into (5.24) yields

$$\begin{aligned}\|\nabla d_{\theta,h}\|^2 \leq\ & 2\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\left(1+\beta_1^*\right)\|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 \\ &+ \left(2\beta_2^*\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\right)\|\nabla z_{p,h}\|^2 \\ &+ \left(\epsilon_6^* + \epsilon_{7,2}^*\tilde{G}_{\mathbf{u}}^2 + 2\beta_3^*\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\right)\|\nabla z_{\theta,h}\|^2 \\ &+ \left(2\beta_4^*\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\right)\|\nabla z_{\Phi,h}\|^2.\end{aligned} \tag{5.29}$$

Finally, inserting (5.28) and (5.29) into (5.25) yields

$$\|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 + \|\nabla d_{\theta,h}\|^2 + \|\nabla d_{\Phi,h}\|^2 \lessapprox \|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 + \|z_{p,h}\|^2 + \|\nabla z_{\theta,h}\|^2 + \|\nabla z_{\Phi,h}\|^2. \tag{5.30}$$

Case (ii):
One can replace $\|\nabla e_{X,h}\|^2$ by $2\|\nabla d_{Y,h}\|^2 + 2\|\nabla z_{Y,h}\|^2$ for $(X,Y) \in \{(\bar\theta,\theta),(\bar\Phi,\Phi)\}$ in the set of inequalities (5.3)-(5.5) and obtains

$$\begin{aligned}\|\nabla \mathbf{d}_{\mathbf{u},h}\|^2 \leq\ & \left(\epsilon_1^* + \epsilon_{4,2}^*\|\nabla \bar{\mathbf{u}}_h\|^2\right)\|\nabla \mathbf{z}_{\mathbf{u},h}\|^2 + \epsilon_{4,1}^*\|\nabla \mathbf{u}\|^2\|\nabla e_{\bar{\mathbf{u}},h}\|^2 + \epsilon_2^*\|z_{p,h}\|^2 \\ &+ \epsilon_{5,1}^*\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\|\nabla e_{\tilde{\mathbf{u}},h}\|^2 + \epsilon_{5,1}^*\left(1+\epsilon_6^* + \epsilon_{7,2}^*\|\nabla \tilde{\mathbf{u}}_h\|^2\right)\|\nabla z_{\theta,h}\|^2 \\ &+ 2\epsilon_{5,2}^*\|\nabla d_{\Phi,h}\|^2 + 2\epsilon_{5,2}^*\|\nabla z_{\Phi,h}\|^2\end{aligned} \tag{5.31}$$

$$\|\nabla d_{\theta,h}\|^2 \leq \left(\epsilon_6^* + \epsilon_{7,2}^*\|\nabla \tilde{\mathbf{u}}_h\|^2\right)\|\nabla z_{\theta,h}\|^2 + \epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2\|\nabla e_{\tilde{\mathbf{u}},h}\|^2 \tag{5.32}$$

$$\|\nabla d_{\Phi,h}\|^2 \leq C_{0,1}^2\|\nabla z_{\Phi,h}\|^2 + 2C_{0,2}^2\|\nabla(\Phi+\Phi_b)\|_{0,3}^2\left(\|\nabla d_{\theta,h}\|^2 + \|\nabla z_{\theta,h}\|^2\right). \tag{5.33}$$

Inserting the upper bound on $\|\nabla d_{\theta,h}\|^2$ into the upper bound on $\|\nabla d_{\Phi,h}\|^2$ and inserting this new upper bound again into (5.31) finally yields

$$\begin{aligned}&\|\nabla d_{\mathbf{u}}\|^2 + \|\nabla d_\theta\|^2 + \|\nabla d_\Phi\|^2 \\ &\lessapprox \|\nabla z_{\mathbf{u}}\|^2 + \|z_p\|^2 + \|\nabla z_\theta\|^2 + \|\nabla z_\Phi\|^2 + \delta_{\bar{\mathbf{u}}}\|\nabla e_{\bar{\mathbf{u}}}\|^2 + \delta_{\tilde{\mathbf{u}}}\|\nabla e_{\tilde{\mathbf{u}}}\|^2.\end{aligned} \tag{5.34}$$

The analog result for case *(iii)*, directly follows from (5.3)-(5.5).

By means of the triangle inequality $\|\nabla e_{X,h}\| \leq \|\nabla d_{X,h}\| + \|\nabla z_{X,h}\|$, equivalence of l^1- and l^2-norm and taking the infimum over all $(\mathbf{v}_h, q_h, \tau_h, \beta_h) \in \mathbf{V}_h \times M_h \times \Theta_h \times \Upsilon_h$ the stated error estimate follows from (5.30) or (5.34), respectively, however, with $\inf_{\mathbf{v}_h\in\mathbf{V}_h}$ instead of $\inf_{\mathbf{v}_h\in\mathbf{U}_h}$. The latter term can be established by a standard procedure (see e.g. Theorem II.1.1 in [32]) under the use of the discrete inf-sup condition Assumption 5.3 *(ii)*. □

In combination with the interpolation properties of a specific finite element ansatz space and under sufficient regularity of the respective exact solution, Theorem 5.15 immediately yields an a priori error estimate, including h-convergence rates in H^1-norm.

Corollary 5.16. *(A Priori Convergence Rates)*
Let the assumptions and notation of Theorem 5.15 hold and assume that the exact solution satisfies the additional regularity requirements

$$\mathbf{u} \in \mathbf{H}^{l+1}(\Omega), \;\; p \in H^l(\Omega), \;\; \theta \in H^{l+1}(\Omega), \;\; \Phi \in H^{l+1}(\Omega),$$

for some $l \geq 1$. Then,

$$\begin{aligned}\|\nabla(\mathbf{u}-\mathbf{u}_h)\| + \|\nabla(\theta-\theta_h)\| + \|\nabla(\Phi-\Phi_h)\| \lessapprox\ & h^l\left(\|\mathbf{u}\|_{l+1,2} + \|p\|_{l,2} + \|\theta\|_{l+1,2} + \|\Phi\|_{l+1,2}\right) \\ &+ \delta_{\bar{\mathbf{u}}}\|\nabla(\bar{\mathbf{u}}-\bar{\mathbf{u}}_h)\| + \delta_{\tilde{\mathbf{u}}}\|\nabla(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h)\| \\ &+ \delta_{\bar\theta}\|\nabla\left(\bar\theta-\bar\theta_h\right)\| + \delta_{\bar\Phi}\|\nabla\left(\bar\Phi-\bar\Phi_h\right)\|.\end{aligned}$$

Proof. Follows directly from Theorem 5.15 and Assumption 5.3 (*iii*) and (*iv*) by estimating the inf-terms from above by chosing $\mathbf{v}_h = \Pi_{\mathbf{U}_h}\mathbf{u}$, $q_h = \Pi_{M_h}p$, $\tau_h = \Pi_{X_h}\theta$ and $\beta_h = \Pi_{X_h}\Phi$. $\square$

In the fully implicit case, the proof of Theorem 5.15 imposes the following smallness conditions on the problem data and exact solution, (5.22), (5.27):

$$\Xi_1 = 1 - 2\epsilon_{4,1}^*\|\nabla\mathbf{u}\|^2 - 2\epsilon_{5,1}^*\epsilon_{7,1}^*\|\theta+\theta_b\|_{1,2}^2 \qquad < 1 \tag{5.35}$$

$$\Xi_2 = 1 - 4\alpha_1^*\epsilon_{7,1}^* C_{0,2}^2\|\nabla(\Phi+\Phi_b)\|_{0,3}^2\|\theta+\theta_b\|_{1,2}^2 < 1. \tag{5.36}$$

A closer look onto the introduced constants reveals that

$$\begin{aligned}
&\epsilon_{4,1}^* \propto \nu^{-2},\ \epsilon_{5,1}^* \ \propto \nu^{-2}\left(L_{\mathbf{F}}^{(\theta)}\right)^2,\ \epsilon_{5,2}^* \propto \nu^{-2}\left(L_{\mathbf{F}}^{(\Phi)}\right)^2\left(\tilde{G}_\theta^2 + \|\theta_b\|_{1,2}^2\right)\\
&\epsilon_{7,1}^* \propto \kappa^{-2},\ C_{0,2} \propto L_\epsilon \epsilon_-^{-1}, \qquad\quad \alpha_1^* \ \propto \epsilon_{5,2}^*\Xi_1^{-1},
\end{aligned} \tag{5.37}$$

with $\propto$ meaning equal up to a multiplicative constant coming from Friedrich's inequality and Sobolev emebeddings. According to Lemma 3.13 and 5.13, $\|\nabla\mathbf{u}\| \to 0$, $\|\theta+\theta_b\|_{1,2} \to 0$ and $\tilde{G}_\theta \to 0$ for $(\|\theta_b\|_{1,2}, \|f_{\mathbf{v}}\|_{\mathbf{U}^*}, \|f_\tau\|_{\Theta^*}) \to 0$. Thus, the stated conditions can hold if the data is sufficiently small. However, similar to the the uniqueness Theorem 3.21, these conditions are very restrictive due to the terms ν^{-2} and κ^{-2}. Thus, the error estimates for the fully nonlinear, stationary system can only be applied in case of slow fluid flow, small temperature and potential differences and small domains. However, conditions (5.35),(5.36) are not sharp and might be too pessimistic in practice. For instance, the numerical experiments presented in Section 6.2 exhibit the previously stated convergence rates, even though (5.35),(5.36) are not satisfied.

On the other hand, if Problem 5.11 and 5.12 are derived by discretizing the instationary system in time with explicit convection fields $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}$, $\bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h$, then the best-approximation result holds without any restriction. Thus, Theorem 5.15 and Corollary 5.16 could be used for proving a priori convergence rates of a full discretization of the instationary equations by means of the Rothe method.

5.2. Instationary Problem

In this section, we propose a full discretization of the instationary TEHD equations by combining FEM and BDF with the Rothe method. We aim at deriving a priori error estimates. The error analysis is based on the procedure presented in [75] for the instationary Boussinesq equations with temperature dependent viscosity. However, it is modified to take into account Gauss' law and a general force term. Moreover, we do not restrict ourselves to a first order scheme in time. Instead, a general approximation of the ∂_t-terms is considered. The main result of this section is given by Theorem 5.24 which provides error bounds in terms of the spatial discretization parameter h under certain conditions on the spatial regularity of the exact solution. At this point, convergence w.r.t. the temporal discretization parameter k and temporal regularity have not been taken into account. This is done in Corollary 5.29, 5.30, 5.31 and 5.32, which provide temporal convergence rates for two different time stepping schemes. According to [35], strong temporal regularity assumptions on the exact solution may imply non-local compatibility conditions for the initial data that usually cannot by checked in realistic scenarios. For this reason, the convergence rates for each scheme are derived under two different sets of regularity conditions: a set of weak conditions, that do not imply the aforementioned compatibility problems (at the expense of reduced convergence rates), and a set of strong conditions that lead to optimal convergence rates. We first restate the continuous instationary system with increased temporal regularity requirements compared to Problem 4.1. This is done to obtain a set of equations that is pointwise well-defined for all $t \in [0,T]$.

Problem 5.17. *(Continuous Instationary TEHD Equations)*
Let initial conditions $(\mathbf{u}_0, \theta_0) \in \mathbf{V} \times \Theta$, boundary liftings $(\theta_b, \Phi_b) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$, body force $\mathbf{F}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{U}^*$, source terms $f_\mathbf{v} \in C(0,T;\mathbf{U}^*)$, $f_\tau \in C(0,T;\Theta^*)$, $f_\beta \in C(0,T;\Upsilon^*)$ and $T > 0$ be given. Find

$$\begin{aligned}
&\mathbf{u} \in L^2(0,T;\mathbf{V}) \text{ with } \partial_t \mathbf{u} \in L^1(0,T;\mathbf{H})\\
&\theta \in L^2(0,T;\Theta) \text{ with } \partial_t \theta \in L^1(0,T;L^2)\\
&p \in L^2(0,T;M)\\
&\Phi \in L^2(0,T;\Upsilon),
\end{aligned}$$

each of these functions having pointwise well-defined values on $[0,T]$, such that for all $t \in [0,T]$ and $(\mathbf{v}, q, \tau, \beta) \in \mathbf{U} \times M \times \Theta \times \Upsilon$:

$$\begin{aligned}
(\partial_t \mathbf{u}(t), \mathbf{v}) + a_\mathbf{v}(\mathbf{u}(t), \mathbf{v}) + \tilde{c}_\mathbf{v}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) &= \langle \mathbf{F}(\theta(t) + \theta_b, \Phi(t) + \Phi_b) + \mathbf{f}_\mathbf{v}(t), \mathbf{v} \rangle_{\mathbf{U}^*}\\
b(\mathbf{u}(t), q) &= 0\\
(\partial_t \theta(t), \tau) + a_\tau(\theta(t) + \theta_b, \tau) + \tilde{c}_\tau(\mathbf{u}(t), \theta(t) + \theta_b, \tau) &= \langle f_\tau(t), \tau \rangle_{\Theta^*}\\
a_\beta(\theta(t) + \theta_b, \Phi(t) + \Phi_b, \beta) &= \langle f_\beta(t), \beta \rangle_{\Upsilon^*}\\
\mathbf{u}(0) &= \mathbf{u}_0\\
\theta(0) &= \theta_0.
\end{aligned}$$

Discrete Problem

To set up the temporal discretization, let $0 = t_0 < t_1 < \ldots < t_N = T$ denote an equidistant partition of the time interval $[0,T]$, i.e. $t_i = ik$ with $k := \frac{T}{N}$ and $N \in \mathbb{N}$. For $0 \le m < M \le N$, a space of Z-valued sequences is defined by $l^p(m,M;Z) := \{(z^n)_n : \ n = m, \ldots, M, \ z^n \in Z\}$ and equipped with the norm

$$\|(z^n)_n\|^p_{l^p(m,M;Z)} := k \sum_{n=m}^{M} \|z^n\|^p_Z \qquad \text{for } p \in [1,\infty), \tag{5.38}$$

$$\|(z^n)_n\|_{l^\infty(m,M;Z)} := \max\{\|z^n\|_Z \colon \ n = m, \ldots, M\} \text{ for } p = \infty. \tag{5.39}$$

For a continuous function $z\colon [0,T] \to Z$ let $z^n := z(t_n)$ and $\|z\|_{l^p(m,M;Z)} := \|\left(z(t_n)\right)_n\|_{l^p(m,M;Z)}$.

Definition 5.18. *(q-Step Difference Operator)*
Let $q \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{R}^{q+1}$. The q-step difference operator *$J_\mathbf{c}$, applied to a sequence $(z^n)_n \subset Z$, is defined as*

$$J_\mathbf{c} z^n := \sum_{i=0}^{q} c_i z^{n-i}.$$

$J_\mathbf{c}$ is called explicit, *if $c_0 = 0$ and $J_\mathbf{1}$ denotes the 1-step operator determined by $\mathbf{c} = (1, -1)$.*

We now define a numerical scheme for approximating the solution of Problem 5.17 and which can be considered as modification of the one introduced in [75]. In contrast to [75], we investigate a temporal scheme based on general q-step difference operators as given by Definition 5.18.

Problem 5.19. *(Discretized Instationary TEHD Equations)*
Let $q \geq 1$, initial conditions $\left\{(\mathbf{u}_h^i, \theta_h^i)\right\}_{i=0}^{q-1}$ and q-step difference operators $J_\mathbf{c}, J_\mathbf{d}, J_\mathbf{f}, J_\mathbf{g}$ be given. The full discretization of Problem 5.17, is given by the following sequence of finite dimensional, stationary problems for $m \in \{q, \dots, N\}$: Find $(\mathbf{u}_h^m, p_h^m, \theta_h^m, \Phi_h^m) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$ such that for all $(\mathbf{v}_h, q_h, \tau_h, \beta_h) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$:

$$
\begin{aligned}
(k^{-1} J_\mathbf{d} \mathbf{u}_h^m, \mathbf{v}_h) + a_\mathbf{v}(\mathbf{u}_h^m, \mathbf{v}_h) + \tilde{c}_\mathbf{v}(J_\mathbf{c} \mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{v}_h) - b(\mathbf{v}_h, p_h^m) \qquad\qquad& \\
- \langle \mathbf{F}(J_\mathbf{f} \theta_h^m + \theta_b, \Phi_h^m + \Phi_b), \mathbf{v}_h \rangle_{\mathbf{U}^*} &= \langle \mathbf{f}_\mathbf{v}(t_m), \mathbf{v}_h \rangle_{\mathbf{U}^*} && (5.40) \\
b(\mathbf{u}_h^m, q_h) &= 0 && (5.41) \\
(k^{-1} J_\mathbf{d} \theta_h^m, \tau_h) + a_\tau(\theta_h^m + \theta_b, \tau_h) + \tilde{c}_\tau(J_\mathbf{c} \mathbf{u}_h^m, \theta_h^m + \theta_b, \tau_h) &= \langle f_\tau(t_m), \tau_h \rangle_{\Theta^*} && (5.42) \\
a_\beta(J_\mathbf{g} \theta_h^m + \theta_b, \Phi_h^m + \Phi_b, \beta_h) &= \langle f_\beta(t_m), \beta_h \rangle_{\Upsilon^*}. && (5.43)
\end{aligned}
$$

So far, the q-step difference operators $J_\mathbf{d}, J_\mathbf{c}, J_\mathbf{f}, J_\mathbf{g}$ for ∂_t, convection, force and Gauss terms have not been specified. The only requirement for proving the error estimates of Theorem 5.24 is given by Assumption 5.20.

Assumption 5.20. *(Difference Operators)*
The q-step difference operators introduced in Problem 5.19 should satisfy

(i) $J_\mathbf{c}, J_\mathbf{f}, J_\mathbf{g}$ *are explicit.*

(ii) $\sum_{i=1}^q f_i = \sum_{i=1}^q g_i = 1$.

(iii) For each Hilbert space Z, there exists a constant $o \geq 0$ such that for each sequence $(z^m)_{m\geq 0} \subset Z$ there are sequences $(Z_m)_{m\geq q-1}, (\overline{Z}_m)_{m\geq q} \subset \mathbb{R}$ with

$$(J_\mathbf{d} z^m, z^m)_Z \geq o J_\mathbf{1}(\|z^m\|_Z^2 + Z_m^2) + \overline{Z}_m^2 \quad \textit{for all } m \geq q.$$

The proposed time stepping scheme is implicit in the diffusion terms in order to avoid a time step size restriction due to the stiff nature of ODEs resulting from a conforming finite element discretization of the Laplace terms. Due to Assumption 5.20 (i), the nonlinear terms are treated semi-implicitly. In this way, the resulting system of algebraic equations is decoupled and can be solved by solving a series of linear problems in each time step:

TimeLoop

```
given N > 0 and (u_h^i, θ_h^i) for i = 0, ..., q − 1.
for m = q ... N do
    solve Gauss' law (5.43) for Φ_h^m
    solve Oseen equations (5.40), (5.41) for (u_h^m, p_h^m)
    solve heat equation (5.42) for θ_h^m
end for
```

Moreover, by Assumption 5.20 (i), the corresponding proof of a priori error estimates does not require a time step size restriction when applying Gronwall's inequality. Assumption 5.20 (ii) implies $J_\mathbf{f} x_b = J_\mathbf{g} x_b = x_b$ for the (not time-dependent) boundary liftings $x_b \in \{\theta_b, \Phi_b\}$. (iii) is used to obtain the L^2-norm of the discretization error in the proof of Theorem 5.24.

In order to derive a priori error estimates, it will be necessary to impose some kind of Lipschitz continuity for the general body force $\mathbf{F}$. Compared to the previously stated Assumption 4.2 for proving existence of continuous solutions, this new Assumption 5.21 is more restrictive in the sense that $\|\nabla(\theta_1 - \theta_2)\|$ is replaced by the weaker term $\|\theta_1 - \theta_2\|_{0,3}$. This modification turns out to be sufficient for avoiding a time step size restriction. It will be shown in Lemma 5.33 that Assumption 5.21 is satisfied by the previously defined body forces $\mathbf{F}_{s,0},\ \mathbf{F}_{a,0},\ \mathbf{F}_{a,1}$.

Assumption 5.21. *(Body Force Lipschitz Continuity I)*
Let $\mathbf{F}\colon H^1(\Omega)\times H^1(\Omega)\to \mathbf{U}^*$ *satisfy*

(i) $\mathbf{F}$ *is Lipschitz continuous in* θ*: there is a non-decreasing function* $L_{\mathbf{F}}^{(\theta,*)}\colon [0,\infty)\to[0,\infty)$ *such that for all* $\theta_1,\ \theta_2,\ \Phi\in H^1(\Omega)$ *and* $\mathbf{v}\in\mathbf{U}$,

$$|\langle \mathbf{F}(\theta_1,\Phi)-\mathbf{F}(\theta_2,\Phi),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\theta,*)}(\|\Phi\|_{1,2})\|\theta_1-\theta_2\|_{0,3}\|\nabla\mathbf{v}\|.$$

(ii) $\mathbf{F}$ *is locally Lipschitz continuous in* Φ*: for all* $R>0$ *there is* $L_{\mathbf{F}}^{(\Phi,*)}\ge 0$ *such that for all* $\theta\in H^1(\Omega),\ \Phi_1,\Phi_2\in B_R(0,H^1(\Omega))$ *and* $\mathbf{v}\in\mathbf{U}$,

$$|\langle \mathbf{F}(\theta,\Phi_1)-\mathbf{F}(\theta,\Phi_2),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\Phi,*)}\|\theta\|_{1,2}\|\Phi_1-\Phi_2\|_{1,2}\|\nabla\mathbf{v}\|.$$

(iii) $\mathbf{F}(0,\Phi)=0$ *for all* $\Phi\in H^1(\Omega)$.

The introduced DEP models $\mathbf{F}_{s,0},\ \mathbf{F}_{a,0},\ \mathbf{F}_{a,1}$ are based on linearization around a given base potential Φ_0. As pointed out in Section 3.3, this procedure can be justified in case of a small temperature dependence of the permittivity ϵ, since the resulting electric potential is then subjected to rather small spatial and temporal variations. However, a suitable base potential might not be available in general. In such a case, one might prefer a DEP model that does not require a priori information but is still close to the original term $|\nabla\Phi|^2\nabla\theta$. This requirement can be met by introducing a cut-off operator, that bounds the supremum norm of $\nabla\Phi$. The resulting DEP model will be defined in detail in the upcoming Section 5.3. At this point, we only state an alternative body force Assumption 5.22 that is tailored to this new model. For the DEP model we have in mind, it will be necessary to bound the variation of $\mathbf{F}(\theta,\Phi)$ w.r.t. θ in terms of $\|\nabla(\theta_1-\theta_2)\|$. As we will see in the proof of Theorem 5.24, a small data condition that involves the Lipschitz constant of $\mathbf{F}$ w.r.t. θ has to imposed when deriving error estimates in this case. However, this condition turns out to be less restrictive then the one which is imposed for proving error estimates in the stationary case, see Theorem 5.15 and the subsequent remarks.

Assumption 5.22. *(Body Force Lipschitz Continuity II)*
Let $\mathbf{F}\colon H^1(\Omega)\times H^1(\Omega)\to \mathbf{U}^*$ *satisfy*

(i) $\mathbf{F}$ *is Lipschitz continuous in* θ*: there are* $L_{\mathbf{F}}^{(\theta,H^1)}$, $L_{\mathbf{F}}^{(\theta,L_2)}$ *such that for all* $\theta_1,\ \theta_2,\ \Phi\in H^1(\Omega)$ *and* $\mathbf{v}\in\mathbf{U}$,

$$|\langle \mathbf{F}(\theta_1,\Phi)-\mathbf{F}(\theta_2,\Phi),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\theta,H^1)}\|\nabla(\theta_1-\theta_2)\|\,\|\mathbf{v}\|+L_{\mathbf{F}}^{(\theta,L_2)}\|\theta_1-\theta_2\|\,\|\mathbf{v}\|.$$

(ii) $\mathbf{F}$ *is locally Lipschitz continuous in* Φ*: for all* $R>0$ *there is* $L_{\mathbf{F}}^{(\Phi,**)}\ge 0$ *such that for all* $\theta\in W^{1,3}(\Omega),\ \Phi_1,\Phi_2\in B_R(0,H^1(\Omega))$ *and* $\mathbf{v}\in\mathbf{U}$,

$$|\langle \mathbf{F}(\theta,\Phi_1)-\mathbf{F}(\theta,\Phi_2),\mathbf{v}\rangle_{\mathbf{U}^*}|\le L_{\mathbf{F}}^{(\Phi,**)}\|\theta\|_{1,3}\|\nabla(\Phi_1-\Phi_2)\|\,\|\mathbf{v}\|_{0,6}.$$

(iii) $\mathbf{F}(0,\Phi)=0$ *for all* $\Phi\in H^1(\Omega)$.

The following Lemma 5.23 states that a unique discrete solution does exist and yields bounds on the solution's norm. The proof is similar to that of Lemma 4.10 and 4.15 which state an analogous result for the semi-discrete system, Problem 4.9. Here, it is crucial to obtain norm bounds which are independent of the discretization parameters h and k.

Lemma 5.23. *(Existence, Uniqueness and Stability of Discrete Solutions)*
Let Assumptions 5.3 and 5.20 hold and assume that $\mathbf{F}$ *satisfies either Assumption 5.21 or 5.22. In the latter case, assume additionally that*

$$qc_0|\mathbf{f}|_\infty L_{\mathbf{F}}^{(\theta,H^1)} < 2\sqrt{\nu\kappa} \tag{5.44}$$

with c_0 *denoting the constant of Friedrich's inequality w.r.t.* $\mathbf{H}_0^1(\Omega)$. *Then, there exists a unique sequence of solutions* $\{(\mathbf{u}_h^m, p_h^m, \theta_h^m, \Phi_h^m)\}_{m=q}^N$ *of Problem 5.19. Moreover, this sequence of discrete solutions is stable in the following sense: there exist constants* $\bar{\nu} \in (0,\nu)$, $\bar{\kappa} \in (0,\kappa)$ *and* $\tilde{G}_{\mathbf{u},\theta}$, $\tilde{G}_\Phi$ *only depending on model parameters and the initial conditions* $\left\{(\mathbf{u}_h^i, \theta_h^i, \Phi_h^i)\right\}_{i=0}^{q-1}$, *such that*

$$\begin{aligned}
\|(\mathbf{u}_h^m)_m\|_{l^\infty(0,N;\mathbf{L}^2)} &\le \tilde{G}_{\mathbf{u},\theta} \\
\|(\theta_h^m)_m\|_{l^\infty(0,N;L^2)} &\le \tilde{G}_{\mathbf{u},\theta} \\
\|(\Phi_h^m)_m\|_{l^\infty(0,N;H_D^1)} &\le \tilde{G}_\Phi \\
\bar{\nu}\|(\mathbf{u}_h^m)_m\|^2_{l^2(0,N;\mathbf{H}_0^1)} + \bar{\kappa}\|(\theta_h^m)_m\|^2_{l^2(0,N;H_D^1)} &\le \tilde{G}_{\mathbf{u},\theta}.
\end{aligned}$$

Proof. For fixed $m \in \{q,\dots,N\}$, the corresponding stationary equations defined in Problem 5.19 fit into the framework of the mixed formulation of Problem 3.18 (given by Corollary 3.20) with $\delta = \frac{d_0}{k}$ and the following terms (in the notation of Problem 3.18)

$$\begin{aligned}
(\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}) &\mathrel{\widehat{=}} (J_{\mathbf{c}}\mathbf{u}_h^m, J_{\mathbf{c}}\mathbf{u}_h^m, J_{\mathbf{g}}\theta_h^m, \Phi_h^m) \\
\mathbf{F}(\cdot,\cdot) &\mathrel{\widehat{=}} \mathbf{F}(J_{\mathbf{f}}\theta_h^m + \theta_b, \cdot) \\
\mathbf{f}_{\mathbf{v}} &\mathrel{\widehat{=}} \mathbf{f}_{\mathbf{v}}(t_m) + k^{-1}\sum_{i=1}^q d_i \mathbf{u}_h^{m-i} \\
f_\tau &\mathrel{\widehat{=}} f_\tau(t_m) + k^{-1}\sum_{i=1}^q d_i \theta_h^{m-i} \\
f_\beta &\mathrel{\widehat{=}} f_\beta(t_m),
\end{aligned}$$

with $c_{\mathbf{v}}$, c_τ being replaced by $\tilde{c}_{\mathbf{v}}$ and $\tilde{c}_\tau$, respectively. According to Assumption 5.20 (i), $\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}$ are fixed in the sense Problem 3.5, which implies $d(s) = \infty$, where d denotes the imposed upper bound on $\|\theta_b\|_{0,3}$, see Lemma 3.13. Moreover, the continuous spaces $\mathbf{U}$, M, Θ, Υ are replaced by their finite dimensional counterparts $\mathbf{U}_h$, M_h, Θ_h, Υ_h. These spaces are closed subspaces, the discrete inf-sup condition holds and the trilinear forms $\tilde{c}_{\mathbf{v}}$, $\tilde{c}_\tau$ have the property of being skew-symmetric. Moreover, since $\Theta_h = \Upsilon_h = X_h$ are finite dimensional, weak convergence implies strong convergence w.r.t these spaces, see Lemma A.36. Additionally, we have $X_h \subset W^{1,3}(\Omega)$. Therefore, the body force $\tilde{\mathbf{F}}(\cdot,\cdot) := \mathbf{F}(J_{\mathbf{f}}\theta_h^m + \theta_b, \cdot)$ with $\mathbf{F}$ satisfying Assumption 5.21 or 5.22 satisfies Assumption 3.12 and all the arguments used to prove Theorem 3.19, Corollary 3.20 and Theorem 3.21 remain valid. Here, note that the following modification in the proof of Lemma 3.13 is required: the used estimate $|c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \theta)| \le c\|\nabla\tilde{\mathbf{u}}\|\|\nabla\theta\|\|\theta_b\|_{0,3}$ is replaced by

$$|\tilde{c}_\tau(J_{\mathbf{c}}\mathbf{u}_h^m, \theta_h^m + \theta_b, \theta_h^m)| \le c\|\nabla J_{\mathbf{c}}\mathbf{u}_h^m\|\|\nabla\theta_h^m\|\|\theta_b\|_{0,3},$$

according to Lemma 5.10. Thus, existence of a solution $(\mathbf{u}_h^m, p_h^m, \theta_h^m, \Phi_h^m)$ follows from Corollary 3.20 and uniqueness from Theorem 3.21, using $\Phi_h^m \in X_h \subset W^{1,3}(\Omega)$.

The stability estimates follow similarly to the proof of Lemma 4.15: Let $m \in \{q, \dots, N\}$ be arbitrary. Setting $\beta_h = \Phi_h^m$ in Problem 5.19 and using $\epsilon \in [\epsilon_-, \epsilon_+]$ according to Assumption 3.2 yields

$$\|\nabla \Phi_h^m\| \leq \frac{1}{\epsilon_-} \left(\epsilon_+ \|\nabla \Phi_b\| + \|f_\beta(t_m)\|_{\Upsilon^*}\right) \leq \frac{1}{\epsilon_-} \left(\epsilon_+ \|\nabla \Phi_b\| + \|f_\beta\|_{\infty;\Upsilon^*}\right) =: \tilde{G}_\Phi.$$

Consequently,

$$\|\Phi_h^m + \Phi_b\|_{1,2} \leq c\left(\tilde{G}_\Phi + \|\Phi_b\|_{1,2}\right) =: \tilde{G}_\Phi \text{ for all } m \in \{q, \dots, N\}. \tag{5.45}$$

Setting $\mathbf{v}_h = \mathbf{u}_h^m$, $\tau_h = \theta_h^m$ in Problem 5.19 yields

$$\begin{aligned} L_1 &:= (J_{\mathbf{d}} \mathbf{u}_h^m, \mathbf{u}_h^m) + k\nu \|\nabla \mathbf{u}_h^m\|^2 \\ &= k\langle \mathbf{F}(J_{\mathbf{f}} \theta_h^m + \theta_b, \Phi_h^m + \Phi_b) + \mathbf{f}_{\mathbf{v}}(t_m), \mathbf{u}_h^m \rangle_{\mathbf{U}^*} =: R_1 \end{aligned} \tag{5.46}$$

$$\begin{aligned} L_2 &:= (J_{\mathbf{d}} \theta_h^m, \theta_h^m) + k\kappa \|\nabla \theta_h^m\|^2 \\ &= -k\kappa(\nabla \theta_b, \nabla \theta_h^m) - k\tilde{c}_\tau(J_{\mathbf{c}} \mathbf{u}_h^m, \theta_b, \theta_h^m) + k\langle f_\tau(t_m), \theta_h^m \rangle_{\Theta^*} =: R_2. \end{aligned} \tag{5.47}$$

The respective left-hand sides of (5.46) and (5.47) can be estimated from below by means of Assumption 5.20 (*iii*),

$$L_1 \geq oJ_{\mathbf{1}}\left(\|\mathbf{u}_h^m\|^2 + U_m^2\right) + \overline{U}_m^2 + k\nu \|\nabla \mathbf{u}_h^m\|^2 \tag{5.48}$$

$$L_2 \geq oJ_{\mathbf{1}}\left(\|\theta_h^m\|^2 + T_m^2\right) + \overline{T}_m^2 + k\kappa \|\nabla \theta_h^m\|^2, \tag{5.49}$$

for some real sequences $(U_m)_m, (\overline{U}_m)_m, (T_m)_m, (\overline{T}_m)_m$.

Using Assumption 5.21, (5.45) and Theorem A.93, the right-hand side of (5.46) can be estimated from above by

$$\begin{aligned} R_1 &\leq ck\|\nabla \mathbf{u}_h^m\| \left(\|\mathbf{f}_{\mathbf{v}}(t_m)\|_{\mathbf{U}^*} + L_{\mathbf{F}}^{(\theta,*)} \|J_{\mathbf{f}} \theta_h^m\|_{0,3} + L_{\mathbf{F}}^{(\theta,*)} \|\theta_b\|_{0,3}\right) \\ &\leq ck\left(\|\mathbf{f}_{\mathbf{v}}(t_m)\|_{\mathbf{U}^*} + L_{\mathbf{F}}^{(\theta,*)} \|\theta_b\|_{0,3}\right) \|\nabla \mathbf{u}_h^m\| \\ &\quad + ckL_{\mathbf{F}}^{(\theta,*)} |\mathbf{f}|_\infty \|\nabla \mathbf{u}_h^m\| \sum_{i=1}^q \left\{\|\theta_h^{m-i}\|^{\frac{1}{2}} \|\nabla \theta_h^{m-i}\|^{\frac{1}{2}}\right\}, \end{aligned} \tag{5.50}$$

with $L_{\mathbf{F}}^{(\theta,*)} := L_{\mathbf{F}}^{(\theta,*)}(\tilde{G}_\Phi)$. On the other hand, if $\mathbf{F}$ satisfies Assumption 5.22 instead of 5.21, then by using similar arguments,

$$\begin{aligned} R_1 &\leq k\|\mathbf{f}_{\mathbf{v}}(t_m)\|_{\mathbf{U}^*} \|\nabla \mathbf{u}_h^m\| \\ &\quad + kL_{\mathbf{F}}^{(\theta,H^1)} \left(\sum_{i=1}^q \left\{|f_i| \|\nabla \theta_h^{m-i}\|\right\} + \|\nabla \theta_b\|\right) \|\mathbf{u}_h^m\| \\ &\quad + kL_{\mathbf{F}}^{(\theta,L^2)} \left(\sum_{i=1}^q \left\{|f_i| \|\theta_h^{m-i}\|\right\} + \|\theta_b\|\right) \|\mathbf{u}_h^m\| \\ &\leq k\left(\|\mathbf{f}_{\mathbf{v}}(t_m)\|_{\mathbf{U}^*} + c_0 L_{\mathbf{F}}^{(\theta,H_1)} \|\nabla \theta_b\| + c_0 L_{\mathbf{F}}^{(\theta,L^2)} \|\theta_b\|\right) \|\nabla \mathbf{u}_h^m\| \\ &\quad + kc_0 L_{\mathbf{F}}^{(\theta,H^1)} |\mathbf{f}|_\infty \|\nabla \mathbf{u}_h^m\| \sum_{i=1}^q \|\nabla \theta_h^{m-i}\| \\ &\quad + kc_0 c_D^{\frac{1}{2}} L_{\mathbf{F}}^{(\theta,L^2)} |\mathbf{f}|_\infty \|\nabla \mathbf{u}_h^m\| \sum_{i=1}^q \left\{\|\theta_h^{m-i}\|^{\frac{1}{2}} \|\nabla \theta_h^{m-i}\|^{\frac{1}{2}}\right\}. \end{aligned} \tag{5.51}$$

Summing up (5.50) and (5.51), one obtains

$$R_1 \le kG_1\|\nabla \mathbf{u}_h^m\| + kG_2\|\nabla \mathbf{u}_h^m\| \sum_{i=1}^{q} \left\{ \|\theta_h^{m-i}\|^{\frac{1}{2}} \|\nabla \theta_h^{m-i}\|^{\frac{1}{2}} \right\} \tag{5.52}$$
$$+ kG_*\|\nabla \mathbf{u}_h^m\| \sum_{i=1}^{q} \|\nabla \theta_h^{m-i}\|,$$

with constants that are given in by

$$G_1 := c\|\mathbf{f_v}\|_{\infty;\mathbf{U}^*} + L_{\mathbf{F}}^{(\theta,*)}\|\theta_b\|_{0,3}$$
$$G_2 := cL_{\mathbf{F}}^{(\theta,*)}|\mathbf{f}|_\infty$$
$$G_* := 0,$$

if Assumption 5.21 holds and

$$G_1 := \|\mathbf{f_v}\|_{\infty;\mathbf{U}^*} + c_0 L_{\mathbf{F}}^{(\theta,H_1)}\|\nabla\theta_b\| + c_0 L_{\mathbf{F}}^{(\theta,L^2)}\|\theta_b\|$$
$$G_2 := c_0 c_D^{\frac{1}{2}} L_{\mathbf{F}}^{(\theta,L^2)}|\mathbf{f}|_\infty$$
$$G_* := c_0 L_{\mathbf{F}}^{(\theta,H^1)}|\mathbf{f}|_\infty,$$

otherwise. Using Lemma 5.10 (vi), R_2 can be estimated from above via

$$R_2 \le ck\|\nabla\theta_h^m\| \left(\kappa\|\nabla\theta_b\| + \|f_\tau(t_m)\|_{\Theta^*} + \|\theta_b\|_{1,2} \sum_{i=1}^{q} \left\{ |c_i| \|\mathbf{u}_h^{m-i}\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h^{m-i}\|^{\frac{1}{2}} \right\} \right)$$
$$\le kG_3\|\nabla\theta_h^m\| + kG_4\|\nabla\theta_h^m\| \sum_{i=1}^{q} \left\{ \|\mathbf{u}_h^{m-i}\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h^{m-i}\|^{\frac{1}{2}} \right\} \tag{5.53}$$

with constants

$$G_3 := c\kappa\|\nabla\theta_b\| + c\|f_\tau\|_{\infty;\Theta^*}, \quad G_4 := c\|\theta_b\|_{1,2}|\mathbf{c}|_\infty.$$

Combining (5.48) and (5.52) and using Young's inequality yields for arbitrary $\gamma \in (0,2)$, $\delta > 0$,

$$o\left(\|\mathbf{u}_h^m\|^2 - \|\mathbf{u}_h^{m-1}\|^2 + U_m^2 - U_{m-1}^2\right) + \overline{U}_m^2 + k\frac{\nu}{4}(2-\gamma)\|\nabla\mathbf{u}_h^m\|^2$$
$$\le \frac{2G_1^2}{\nu(2-\gamma)}k + \frac{2G_2^2 q}{\nu(2-\gamma)}k\sum_{i=1}^{q}\left\{\|\theta_h^{m-i}\|\|\nabla\theta_h^{m-i}\|\right\} + \frac{G_*^2 q}{2\gamma\nu}k\sum_{i=1}^{q}\|\nabla\theta_h^{m-i}\|^2$$
$$\le \frac{2G_1^2}{\nu(2-\gamma)}k + k\sum_{i=1}^{q}\left\{\frac{\kappa}{\delta q}\|\nabla\theta_h^{m-i}\|^2 + \frac{\delta G_2^4 q^3}{\nu^2\kappa(2-\gamma)^2}\|\theta_h^{m-i}\|^2 + \frac{G_*^2 q}{2\gamma\nu}\|\nabla\theta_h^{m-i}\|^2\right\}$$
$$= G_5 k + k\sum_{i=1}^{q}\left\{\frac{\kappa}{\delta q}\|\nabla\theta_h^{m-i}\|^2 + \frac{1}{\gamma q}G_{**}\|\nabla\theta_h^{m-i}\|^2 + G_6\|\theta_h^{m-i}\|^2\right\}, \tag{5.54}$$

with constants

$$G_5 := \frac{2G_1^2}{\nu(2-\gamma)}, \quad G_6 := \frac{\delta G_2^4 q^3}{\nu^2\kappa(2-\gamma)^2}, \quad G_{**} := \frac{G_*^2 q^2}{2\nu}.$$

Analogously, combination of (5.49) and (5.53) yields for arbitrary $\epsilon \in (0,1)$, $\sigma > 0$,

$$
\begin{aligned}
&o\left(\|\theta_h^m\|^2 - \|\theta_h^{m-1}\|^2 + T_m^2 - T_{m-1}^2\right) + \overline{T}_m^2 + k\kappa(1-\epsilon)\|\nabla\theta_h^m\|^2 \\
&\leq \frac{G_3^2}{2\epsilon\kappa}k + \frac{G_4^2 q}{2\epsilon\kappa}k\sum_{i=1}^{q}\left\{\|\mathbf{u}_h^{m-i}\|\|\nabla\mathbf{u}_h^{m-i}\|\right\} \\
&\leq \frac{G_3^2}{2\epsilon\kappa}k + k\sum_{i=1}^{q}\left\{\frac{\nu}{\sigma q}\|\nabla\mathbf{u}_h^{m-i}\|^2 + \frac{\sigma}{\epsilon^2}\frac{G_4^4 q^3}{16\kappa^2\nu}\|\mathbf{u}_h^{m-i}\|^2\right\} \\
&= G_7 k + k\sum_{i=1}^{q}\left\{\frac{\nu}{\sigma q}\|\nabla\mathbf{u}_h^{m-i}\|^2 + G_8\|\mathbf{u}_h^{m-i}\|^2\right\},
\end{aligned}
\tag{5.55}
$$

with

$$
G_7 := \frac{G_3^2}{2\epsilon\kappa}, \quad G_8 := \frac{\sigma}{\epsilon^2}\frac{G_4^4 q^3}{16\kappa^2\nu}.
$$

Adding (5.54) and (5.55) and summation over $m = q, \dots, M$ for arbitrary $M \in \{q, \dots, N\}$ yields

$$
\begin{aligned}
&o\left(\|\mathbf{u}_h^M\|^2 + \|\theta_h^M\|^2 + U_M^2 + T_M^2\right) + k\sum_{m=q}^{M}\left\{\bar{\nu}\|\nabla\mathbf{u}_h^m\|^2 + \bar{\kappa}\|\nabla\theta_h^m\|^2 + \frac{1}{k}\overline{U}_m^2 + \frac{1}{k}\overline{T}_m^2\right\} \\
&\leq k\sum_{m=0}^{M-1}\left\{G_8 q\|\mathbf{u}_h^m\|^2 + G_6 q\|\theta_h^m\|^2\right\} + k\sum_{m=q}^{M}\left\{G_5 + G_7\right\} \\
&\quad + o\left(\|\mathbf{u}_h^{q-1}\|^2 + \|\theta_h^{q-1}\|^2 + U_{q-1}^2 + T_{q-1}^2\right) \\
&\quad + k\sum_{i=0}^{q-1}\left\{\frac{\nu}{\sigma}\|\nabla\mathbf{u}_h^i\|^2 + \left(\frac{\kappa}{\delta} + \frac{G_{**}}{\gamma}\right)\|\nabla\theta_h^i\|^2\right\}.
\end{aligned}
\tag{5.56}
$$

with

$$
\bar{\nu} := \nu\left(\frac{1}{4}(2-\gamma) - \frac{1}{\sigma}\right) \tag{5.57}
$$

$$
\bar{\kappa} := \kappa\left(1 - \epsilon - \frac{1}{\delta}\right) - \frac{1}{\gamma}G_{**}. \tag{5.58}
$$

If $G_{**} < 2\kappa$, which is equivalent to (5.44), then $\gamma, \delta, \epsilon, \sigma$ can be chosen such that $\bar{\nu} > 0$ and $\bar{\kappa} > 0$. Then, application of the discrete Gronwall inequality, Lemma A.42, to (5.56) gives

$$
\begin{aligned}
&o\left(\|\mathbf{u}_h^M\|^2 + \|\theta_h^M\|^2 + U_M^2 + T_M^2\right) + k\sum_{m=q}^{M}\left\{\bar{\nu}\|\nabla\mathbf{u}_h^m\|^2 + \bar{\kappa}\|\nabla\theta_h^m\|^2 + \frac{1}{k}\overline{U}_m^2 + \frac{1}{k}\overline{T}_m^2\right\} \\
&\leq \exp\left((T+k)qo^{-1}\max\{G_6, G_8, 1\}\right) \\
&\quad\cdot\Bigg[(G_5 + G_7)T + o\left(\|\mathbf{u}_h^{q-1}\|^2 + \|\theta_h^{q-1}\|^2 + U_{q-1}^2 + T_{q-1}^2\right) \\
&\qquad + k\sum_{i=0}^{q-1}\left\{\frac{\nu}{\sigma}\|\nabla\mathbf{u}_h^i\|^2 + \left(\frac{\kappa}{\delta} + \frac{G_{**}}{\gamma}\right)\|\nabla\theta_h^i\|^2\right\}\Bigg].
\end{aligned}
\tag{5.59}
$$

Then, the stated estimate follows from (5.45) and (5.59). $\square$

Preliminary Error Estimate

The following Theorem 5.24 provides a basic a priori error estimate for the numerical scheme defined by Problem 5.19. The corresponding result provides convergence rates w.r.t. the spatial discretization parameter h. Moreover, the derived upper bounds depend on the error of the initial conditions and the approximation properties of the temporal difference operators. At this point, no regularity on temporal derivatives of the exact solution is imposed. Thus, no convergence rates w.r.t. k are shown yet. This will be done in a separate step.

Before stating Theorem 5.24, some notation has to be introduced. The exact solution at time t_n is denoted by $(\mathbf{u}^n, p^n, \theta^n, \Phi^n)$ and corresponding approximations in the underlying finite element spaces are obtained by using the Stokes projection Π_S and the projection Π_{X_h},

$$(\mathbf{w}_h^n, r_h^n) := \Pi_S(\mathbf{u}^n, p^n), \quad \eta_h^n := \Pi_{X_h}\theta^n, \quad \Psi_h^n := \Pi_{X_h}\Phi^n. \tag{5.60}$$

The discretization error between these projections and the discrete solution of Problem 5.19 is defined as

$$\mathbf{d}_{\mathbf{u},h}^n = \mathbf{w}_h^n - \mathbf{u}_h^n, \quad d_{p,h}^n = r_h^n - p_h^n, \quad d_{\theta,h}^n = \eta_h^n - \theta_h^n, \quad d_{\Phi,h}^n = \Psi_h^n - \Phi_h^n. \tag{5.61}$$

When applying the estimate stated in Assumption 5.20 (*iii*) for $z^m = \mathbf{d}_{\mathbf{u},h}^n$ and $z^m = d_{\theta,h}^n$, respectively, one further obtains real sequences $(U_n)_{n\geq q-1}$, $(T_n)_{n\geq q-1}$, $(\overline{U}_n)_{n\geq q}$, $(\overline{T}_n)_{n\geq q}$ such that the following estimates hold for all $n \geq q$

$$(J_{\mathbf{d}}\mathbf{d}_{\mathbf{u},h}^n, \mathbf{d}_{\mathbf{u},h}^n) \geq oJ_{\mathbf{1}}(\|\mathbf{d}_{\mathbf{u},h}^n\|^2 + U_n^2) + \overline{U}_n^2 \tag{5.62}$$

$$(J_{\mathbf{d}}d_{\theta,h}^n, d_{\theta,h}^n) \geq oJ_{\mathbf{1}}(\|d_{\theta,h}^n\|^2 + T_n^2) + \overline{T}_n^2. \tag{5.63}$$

The error will be bounded in terms of the subsequent expressions. The first set of expressions measures the error contributions by the q initial conditions:

$$\begin{aligned} E_1(q)^2 &:= k\sum_{n=0}^{q-1}\left\{\|\mathbf{u}^n - \mathbf{u}_h^n\|_{1,2}^2 + \|\theta^n - \theta_h^n\|_{1,2}^2\right\} \\ &\quad + \|\mathbf{u}^{q-1} - \mathbf{u}_h^{q-1}\|^2 + \|\theta^{q-1} - \theta_h^{q-1}\|^2 + U_{q-1}^2 + T_{q-1}^2 \end{aligned} \tag{5.64}$$

$$E_2(q) := \max\{\|\mathbf{u}^n - \mathbf{u}_h^n\|_{1,2} \colon\ n = 0, \ldots, q-1\} \tag{5.65}$$

$$E_3(q) := \max\{\|\theta^n - \theta_h^n\| \colon\ n = 0, \ldots, q-1\}. \tag{5.66}$$

Error contributions that arise due to the temporal discretization and which do not depend on the spatial discretization are given by

$$\begin{aligned} E(k)^2 &:= k\sum_{n=q}^{N}\left\{\|\partial_t\mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 + \|\partial_t\theta^n - k^{-1}J_{\mathbf{d}}\theta^n\|_{-1,2}^2\right\} \\ &\quad + k\sum_{n=q}^{N}\left\{\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + \|\theta^n - J_{\mathbf{g}}\theta^n\|^2\right\} \end{aligned} \tag{5.67}$$

$E(k)$ measures how well the time derivative ∂_t is approximated by the finite difference $k^{-1}J_{\mathbf{d}}$ and how well a function at time t_n can be extrapolated by the function values at time $t_{n-q}, \ldots, t_{n-1}$ through the operators $J_{\mathbf{c}}$, $J_{\mathbf{f}}$, $J_{\mathbf{g}}$. On the other hand, contributions by both temporal and spatial discretization are denoted by

$$E(h,k)^2 := h^{2l+2l^*} k \sum_{n=q}^{N} \{ \|k^{-1} J_{\mathbf{d}} \mathbf{u}^n\|_{l+1,2}^2 + \|k^{-1} J_{\mathbf{d}} p^n\|_{l,2}^2 \} \tag{5.68}$$
$$+ h^{2l} k \sum_{n=q}^{N} \|k^{-1} J_{\mathbf{d}} \theta^n\|_{l+1,2}^2,$$

with constants l, l^* that describe the spatial regularity of the exact solution and the approximation quality of the finite element spaces. Finally, the term

$$E_{\mathbf{d}}(h,k)^2 := k^{-1} \sum_{n=q}^{N} \|J_{\mathbf{d}} \mathbf{d}_{\mathbf{u},h}^n\|^2, \tag{5.69}$$

which involves the unknown velocity discretization error will be bounded by other terms and is used to bound the pressure discretization error. The actual error bounds are now composed by the previously defined terms and are separately given for velocity and temperature, pressure, potential and $E_{\mathbf{d}}(k,h)$ according to

$$\mathcal{E}_{\mathbf{u},\theta}(h,k,q) := h^l + E(k) + E(h,k) + E_1(q) \tag{5.70}$$
$$\mathcal{E}_{\Phi}(h,k,q) := E_3(q) + \mathcal{E}_{\mathbf{u},\theta}(h,k,q) \tag{5.71}$$
$$\mathcal{E}_p(h,k,q) := E(k) + E(h,k) + E_{\mathbf{d}}(k,h) + C_p(h,k,q)\mathcal{E}_{\mathbf{u},\theta}(h,k,q) \tag{5.72}$$
$$\mathcal{E}_{\mathbf{d}}(h,k,q) := k^{-\frac{1}{2}} \left(E(k) + E(h,k)\right) + C_p(h,k,q) \left(h^l + E_2(q) + k^{-\frac{1}{2}} \mathcal{E}_{\mathbf{u},\theta}(h,k,q)\right) \tag{5.73}$$
$$C_p(h,k,q) := 1 + E_2(q) + \min\{h^{-1}, k^{-\frac{1}{2}}\} \mathcal{E}_{\mathbf{u},\theta}(h,k,q). \tag{5.74}$$

Theorem 5.24. *(Error of Numerical Scheme)*
In the framework of Problem 5.19, let a solution of Problem 5.17 be given that satisfies the following additional regularity requirements for some $l \geq 1$:

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}^{l+1})$$
$$p \in L^\infty(0,T;H^l)$$
$$\theta \in L^\infty(0,T;H^{l+1})$$
$$\Phi \in L^\infty(0,T;W^{1,\infty}) \cap L^\infty(0,T;H^{l+1}).$$

Suppose that Assumption 5.3 holds with $m = n = l$, where m, n denote the respective finite element approximation orders, see (iii) and (iv). Moreover, let $l^* = 1$ if the underlying Stokes problem is regular and $l^* = 0$, otherwise. Let further the assumptions of Lemma 5.23 hold; in particular, assume that $\mathbf{F}$ satisfies Assumption 5.21 or 5.22. In the latter case, assume additionally that the following small data condition holds:

$$c_0 \sqrt{q} |\mathbf{f}|_2 L_{\mathbf{F}}^{(\theta,H^1)} < \sqrt{2\tilde{\nu}\tilde{\kappa}}, \tag{5.75}$$

with $\tilde{\nu} = \nu \frac{1}{2(1+c_0^2)}$, $\tilde{\kappa} = \kappa \frac{1}{2(1+c_D^2)}$ and c_0, c_D denoting the respective constants from Friedrich's *inequality w.r.t.* H_0^1 *and* H_D^1. *Then,*

$$
\begin{aligned}
&\| (\mathbf{u}^n - \mathbf{u}_h^n)_n \|_{l^\infty(q,N;\mathbf{L}_2)} + \| (\theta^n - \theta_h^n)_n \|_{l^\infty(q,N;L_2)} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q)\\
&\| (\mathbf{u}^n - \mathbf{u}_h^n)_n \|_{l^2(q,N;\mathbf{H}^1)} + \| (\theta^n - \theta_h^n)_n \|_{l^2(q,N;H^1)} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q)\\
&\| (U_n)_n \|_{l^\infty(q,N;\mathbb{R})} + \| (T_n)_n \|_{l^\infty(q,N;\mathbb{R})} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q)\\
&\| \left(\overline{U}_n\right)_n \|_{l^2(q,N;\mathbb{R})} + \| \left(\overline{T}_n\right)_n \|_{l^2(q,N;\mathbb{R})} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q)\, k^{\frac{1}{2}}\\
&\| (\Phi^n - \Phi_h^n)_n \|_{l^2(q,N;H^1)} \lesssim \mathcal{E}_{\Phi}(h,k,q)\\
&\| (p^n - p_h^n)_n \|_{l^2(q,N;L_2)} \lesssim \mathcal{E}_p(h,k,q)\\
&E_{\mathbf{d}}(h,k) \lesssim \mathcal{E}_{\mathbf{d}}(h,k,q).
\end{aligned}
$$

Theorem 5.24 states that the errors of velocity and temperature due to spatial discretizations, measured in $l^2(H^1)$ and $l^\infty(L^2)$ norm, converge with rate h^l for fixed time step size k. The analogous result in $l^2(H^1)$ holds for the potential error. Moreover, the purely temporal contributions $E(k)$ can be estimated by the series of corollaries in the end of this section. The remaining terms in $\mathcal{E}_{\mathbf{u},\theta}$ and $\mathcal{E}_\Phi$ stem from errors in the initial conditions, $E_1(q)$ and $E_3(q)$, and from the mixed terms $E(h,k)$. Depending on the temporal regularity of the exact solution, the latter ones may contain negative powers of k as pointed out in Lemma A.81 (v) and (vi). In this case, the resulting negative effect when decreasing k can be compensated by a simultaneous decay of h. Therefore, the convergence rate w.r.t. h is, at least in theory, reduced. Concerning the pressure error, the factor $k^{-\frac{1}{2}}$ enters $\mathcal{E}_p$ through the term $E_{\mathbf{d}}(k,h)$, thus yielding a reduced convergence rate compared to velocity, temperature and potential. Here, the term C_p denotes a bound on the $l^\infty(\mathbf{H}^1)$ norm of discrete velocity (5.116) and one can show that it stays bounded as $h,k \to 0$ supposed that the exact solution is sufficiently regular and the initial conditions are chosen such that $E_1(q) = \mathcal{O}(h)$, see Corollary 5.30 and 5.32.

Depending on the underlying body force assumption, the small data condition (5.75) has to be imposed. Compared to the smallness condition in the stationary case, see Theorem 5.15, (5.75) is much weaker though, since only the square root of inverse diffusion constants enters the condition. In Section 6.3, the validness of this assumption is shown in a practical scenario.

The underlying principle of the associated proof is inspired by comparable results on the error of fully discretized flow problems, see e.g. [75] for the Boussinesq problem with temperature-dependent coefficients and [41] for the incompressible Navier-Stokes equations: First, the error is split into discretization and approximation part, denoted by $d_{\cdot,h}^n$ and $z_{\cdot,h}^n$, respectively. Then, a corresponding error equation is derived by subtracting the discrete equations from the continuous ones. Afterward, one aims to bound $d_{\cdot,h}^n$ in terms of $z_{\cdot,h}^n$. This is done by exploiting coercivity of $a_{\mathbf{v}}$, a_τ, a_β to obtain H^1-norms of $d_{\cdot,h}^n$ and Assumption 5.20 (iii) to obtain L^2-norms from the $J_{\mathbf{d}}$ terms. Using Young's inequality multiple times, the remaining $d_{\cdot,h}^n$ norms can be hidden inside the ones on the left-hand side. The final estimate for velocity, temperature and potential is then obtained by employing a discrete Gronwall inequality. Afterward, it remains to bound the pressure error by a similar argument as used in [75].

Proof. The error of the respective variables is decomposed into approximation and discretization contributions:

$$
\begin{aligned}
e_{\mathbf{u},h}^n &:= \mathbf{u}^n - \mathbf{u}_h^n = \mathbf{u}^n - \mathbf{w}_h^n + \mathbf{w}_h^n - \mathbf{u}_h^n =: \mathbf{z}_{\mathbf{u},h}^n + \mathbf{d}_{\mathbf{u},h}^n\\
e_{p,h}^n &:= p^n - p_h^n = p^n - r_h^n + r_h^n - p_h^n =: z_{p,h}^n + d_{p,h}^n\\
e_{\theta,h}^n &:= \theta^n - \theta_h^n = \theta^n - \eta_h^n + \eta_h^n - \theta_h^n =: z_{\theta,h}^n + d_{\theta,h}^n\\
e_{\Phi,h}^n &:= \Phi^n - \Phi_h^n = \Phi^n - \Psi_h^n + \Psi_h^n - \Phi_h^n =: z_{\Phi,h}^n + d_{\Phi,h}^n.
\end{aligned}
$$

Error Equation

Subtracting the discrete equations 5.19 from the continuous ones 5.17 at each time t_n, $n \in \{q, \dots, N\}$ yields the following error equation for all $(\mathbf{v}_h, q_h, \tau_h, \beta_h) \in \mathbf{U}_h \times M_h \times \Theta_h \times \Upsilon_h$:

$$k^{-1}(J_{\mathbf{d}}\mathbf{d}^n_{\mathbf{u},h}, \mathbf{v}_h) + a_{\mathbf{v}}(\mathbf{d}^n_{\mathbf{u},h}, \mathbf{v}_h) - b(\mathbf{v}_h, d^n_{p,h}) = \langle R^n_1 + R^n_2 + R^n_3 + R^n_4, \mathbf{v}_h\rangle \tag{5.76}$$

$$b(\mathbf{d}^n_{\mathbf{u},h}, q_h) = \langle R^n_5, q_h\rangle \tag{5.77}$$

$$k^{-1}(J_{\mathbf{d}}d^n_{\theta,h}, \tau_h) + a_\tau(d^n_{\theta,h}, \tau_h) = \langle R^n_6 + R^n_7 + R^n_8, \tau_h\rangle \tag{5.78}$$

$$a_\beta(\theta^n + \theta_b, \Phi^n + \Phi_b, \beta_h) = a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Phi^n_h + \Phi_b, \beta_h) \tag{5.79}$$

with residuals

$$\begin{aligned}
\langle R^n_1, \mathbf{v}_h\rangle &= -(\partial_t \mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{w}^n_h, \mathbf{v}_h)\\
\langle R^n_2, \mathbf{v}_h\rangle &= -(a_{\mathbf{v}}(\mathbf{z}^n_{\mathbf{u},h}, \mathbf{v}_h) - b(\mathbf{v}_h, z^n_{p,h}))\\
\langle R^n_3, \mathbf{v}_h\rangle &= -(\tilde{c}_{\mathbf{v}}(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) - \tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{u}^n_h, \mathbf{u}^n_h, \mathbf{v}_h))\\
\langle R^n_4, \mathbf{v}_h\rangle &= \langle \mathbf{F}(\theta^n + \theta_b, \Phi^n + \Phi_b) - \mathbf{F}(J_{\mathbf{f}}\theta^n_h + \theta_b, \Phi^n_h + \Phi_b), \mathbf{v}_h\rangle\\
\langle R^n_5, q_h\rangle &= -b(\mathbf{z}^n_{\mathbf{u},h}, q_h)\\
\langle R^n_6, \tau_h\rangle &= -(\partial_t \theta^n - k^{-1}J_{\mathbf{d}}\eta^n_h, \tau_h)\\
\langle R^n_7, \tau_h\rangle &= -a_\tau(z^n_{\theta,h}, \tau_h)\\
\langle R^n_8, \tau_h\rangle &= -(\tilde{c}_\tau(\mathbf{u}^n, \theta^n + \theta_b, \tau_h) - \tilde{c}_\tau(J_{\mathbf{c}}\mathbf{u}^n_h, \theta^n_h + \theta_b, \tau_h)).
\end{aligned} \tag{5.80}$$

Now, all individual residual terms (5.80) are considered. By definition of $(\mathbf{w}^n_h, r^n_h)$ and the Stokes projection, there holds $\langle R^n_2, \mathbf{v}_h\rangle = 0$ and $\langle R^n_5, \mathbf{v}_h\rangle = 0$ and for residual R_7 one obtains

$$|\langle R_7, d^n_{\theta,h}\rangle| \le \kappa \|z^n_{\theta,h}\|_{1,2}\|d^n_{\theta,h}\|_{1,2}. \tag{5.81}$$

In the following, the introduced constants $\{C_{i,j}\}$ do not depend on h and k.

Estimation of Potential Error

Setting $\beta_h = d^n_{\Phi,h}$ in Gauss' law (5.79) gives

$$\begin{aligned}
0 &= a_\beta(\theta^n + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h}) &&- a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Phi^n_h + \Phi_b, d^n_{\Phi,h})\\
&= a_\beta(\theta^n + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h}) &&- a_\beta(J_{\mathbf{g}}\theta^n + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h})\\
&\quad + a_\beta(J_{\mathbf{g}}\theta^n + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h}) &&- a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h})\\
&\quad + a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Phi^n + \Phi_b, d^n_{\Phi,h}) &&- a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Psi^n_h + \Phi_b, d^n_{\Phi,h})\\
&\quad + a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Psi^n_h + \Phi_b, d^n_{\Phi,h}) &&- a_\beta(J_{\mathbf{g}}\theta^n_h + \theta_b, \Phi^n_h + \Phi_b, d^n_{\Phi,h})\\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Then, using Assumption 3.2 on ϵ,

$$
\begin{aligned}
|J_1| &= \left|\left((\epsilon(\theta^n+\theta_b)-\epsilon(J_{\mathbf{g}}\theta^n+\theta_b))\nabla(\Phi^n+\Phi_b),\nabla d^n_{\Phi,h}\right)\right| \\
&\le L_\epsilon\|\theta^n-J_{\mathbf{g}}\theta^n\|\|\Phi^n+\Phi_b\|_{1,\infty}\|\nabla d^n_{\Phi,h}\|
\end{aligned}
$$

$$
\begin{aligned}
|J_2| &= \left|\left((\epsilon(J_{\mathbf{g}}\theta^n+\theta_b)-\epsilon(J_{\mathbf{g}}\theta^n_h+\theta_b))\nabla(\Phi^n+\Phi_b),\nabla d^n_{\Phi,h}\right)\right| \\
&\le L_\epsilon\|J_{\mathbf{g}}(\theta^n-\theta^n_h)\|\|\Phi^n+\Phi_b\|_{1,\infty}\|\nabla d^n_{\Phi,h}\| \\
&\le L_\epsilon\|\Phi+\Phi_b\|_{\infty;W^{1,\infty}}\left(\|J_{\mathbf{g}}z^n_{\theta,h}\|+\|J_{\mathbf{g}}d^n_{\theta,h}\|\right)\|\nabla d^n_{\Phi,h}\|
\end{aligned}
$$

$$
\begin{aligned}
|J_3| &= \left|\left(\epsilon(J_{\mathbf{g}}\theta^n_h+\theta_b)\nabla(\Phi^n-\Psi^n_h),\nabla d^n_{\Phi,h}\right)\right| \\
&\le \epsilon_+\|z^n_{\Phi,h}\|_{1,2}\|\nabla d^n_{\Phi,h}\|
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= \left(\epsilon(J_{\mathbf{g}}\theta^n_h+\theta_b)\nabla(\Psi^n_h-\Phi^n_h),\nabla d^n_{\Phi,h}\right) \\
&\ge \epsilon_-\|\nabla d^n_{\Phi,h}\|^2.
\end{aligned}
$$

Combining these estimates with $\sum_{i=1}^4 J_i = 0$ yields

$$
\begin{aligned}
\|\nabla d^n_{\Phi,h}\| \le{}& \frac{L_\epsilon}{\epsilon_-}\|\Phi+\Phi_b\|_{\infty;W^{1,\infty}}\left(\|\theta^n-J_{\mathbf{g}}\theta^n\|+\|J_{\mathbf{g}}z^n_{\theta,h}\|+\|J_{\mathbf{g}}d^n_{\theta,h}\|\right)+\frac{\epsilon_+}{\epsilon_-}\|z^n_{\Phi,h}\|_{1,2} \\
=:{}& C_{0,1}\left(\|\theta^n-J_{\mathbf{g}}\theta^n\|+\|J_{\mathbf{g}}z^n_{\theta,h}\|+\|J_{\mathbf{g}}d^n_{\theta,h}\|\right)+C_{0,2}\|z^n_{\Phi,h}\|_{1,2}. \qquad (5.82)
\end{aligned}
$$

According to Lemma 5.23,

$$\|\Phi^n_h\|_{1,2} \le c\tilde{G}_\Phi \qquad (5.83)$$

and, by Assumption 5.3 (*iv*),

$$\|\eta^n_h\|_{1,2} \le \|\eta^n_h\|_{2,2} \le c\|\theta^n\|_{2,2} \le c\|\theta\|_{\infty;H^2} \qquad (5.84)$$

$$\|\Psi^n_h\|_{1,2} \le \|\Psi^n_h\|_{2,2} \le c\|\Phi^n\|_{2,2} \le c\|\Phi\|_{\infty;H^2}. \qquad (5.85)$$

Estimation of Body Force Residual

Residual R_4, induced by the force term $\mathbf{F}$, can be decomposed into five parts:

$$
\begin{aligned}
\langle R^n_4,\mathbf{v}_h\rangle ={}& \langle\mathbf{F}(\theta^n+\theta_b,\Phi^n+\Phi_b) - \mathbf{F}(J_{\mathbf{f}}\theta^n+\theta_b,\Phi^n+\Phi_b),\mathbf{v}_h\rangle \\
&+\langle\mathbf{F}(J_{\mathbf{f}}\theta^n+\theta_b,\Phi^n+\Phi_b)-\mathbf{F}(J_{\mathbf{f}}\theta^n+\theta_b,\Psi^n_h+\Phi_b),\mathbf{v}_h\rangle \\
&+\langle\mathbf{F}(J_{\mathbf{f}}\theta^n+\theta_b,\Psi^n_h+\Phi_b)-\mathbf{F}(J_{\mathbf{f}}\eta^n_h+\theta_b,\Psi^n_h+\Phi_b),\mathbf{v}_h\rangle \\
&+\langle\mathbf{F}(J_{\mathbf{f}}\eta^n_h+\theta_b,\Psi^n_h+\Phi_b)-\mathbf{F}(J_{\mathbf{f}}\eta^n_h+\theta_b,\Phi^n_h+\Phi_b),\mathbf{v}_h\rangle \\
&+\langle\mathbf{F}(J_{\mathbf{f}}\eta^n_h+\theta_b,\Phi^n_h+\Phi_b)-\mathbf{F}(J_{\mathbf{f}}\theta^n_h+\theta_b,\Phi^n_h+\Phi_b),\mathbf{v}_h\rangle \\
=:{}& \sum_{i=1}^5\langle L_i,\mathbf{v}_h\rangle.
\end{aligned}
$$

Then, by using the Lipschitz continuity of $\mathbf{F}$ as given by Assumption 5.21, together with the uniform boundedness of $(\eta_h^n, \Phi_h^n, \Psi_h^n)_n$ given by (5.83) - (5.85),

$$\begin{aligned}|\langle L_1, \mathbf{v}_h\rangle| &\le cL_{\mathbf{F}}^{(\theta,*)} \left(\|\Phi^n + \Phi_b\|_{1,2}\right) \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}\|\mathbf{v}_h\|_{1,2} \\ &\le cL_{\mathbf{F}}^{(\theta,*)} \left(\|\Phi\|_{\infty,H^1} + \|\Phi_b\|_{1,2}\right) \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}\|\mathbf{v}_h\|_{1,2} \\ &\quad (\text{ by } H^1(\Omega) \hookrightarrow L^3(\Omega) \text{ })\end{aligned}$$

$$\begin{aligned}|\langle L_2, \mathbf{v}_h\rangle| &\le L_{\mathbf{F}}^{(\Phi,*)} \|J_{\mathbf{f}}\theta^n + \theta_b\|_{1,2}\|z_{\Phi,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2} \\ &\le L_{\mathbf{F}}^{(\Phi,*)} \left(|\mathbf{f}|_1\|\theta\|_{\infty,H^1} + \|\theta_b\|_{1,2}\right) \|z_{\Phi,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2}\end{aligned}$$

$$\begin{aligned}|\langle L_3, \mathbf{v}_h\rangle| &\le cL_{\mathbf{F}}^{(\theta,*)} \left(\|\Psi_h^n + \Phi_b\|_{1,2}\right) \|J_{\mathbf{f}} z_{\theta,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2} \\ &\le cL_{\mathbf{F}}^{(\theta,*)} \left(c\|\Phi\|_{\infty;H^2} + \|\Phi_b\|_{1,2}\right) \|J_{\mathbf{f}} z_{\theta,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2}\end{aligned}$$

$$\begin{aligned}|\langle L_4, \mathbf{v}_h\rangle| &\le L_{\mathbf{F}}^{(\Phi,*)} \|J_{\mathbf{f}}\eta_h^n + \theta_b\|_{1,2}\|d_{\Phi,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2} \\ &\le L_{\mathbf{F}}^{(\Phi,*)} \left(c|\mathbf{f}|_1\|\theta\|_{\infty;H^2} + \|\theta_b\|_{1,2}\right) \|d_{\Phi,h}^n\|_{1,2}\|\mathbf{v}_h\|_{1,2}\end{aligned}$$

$$\begin{aligned}|\langle L_5, \mathbf{v}_h\rangle| &\le L_{\mathbf{F}}^{(\theta,*)} \left(\|\Phi_h^n + \Phi_b\|_{1,2}\right) \|J_{\mathbf{f}} d_{\theta,h}^n\|_{0,3}\|\mathbf{v}_h\|_{1,2} \\ &\le c\frac{\epsilon_+}{\epsilon_-} L_{\mathbf{F}}^{(\theta,*)} \left(c\tilde{G}_\Phi + \|\Phi_b\|_{1,2}\right) \|J_{\mathbf{f}} d_{\theta,h}^n\|^{\frac{1}{2}} \, \|J_{\mathbf{f}} d_{\theta,h}^n\|_{1,2}^{\frac{1}{2}} \, \|\mathbf{v}_h\|_{1,2} \\ &\quad (\text{ by Theorem A.93 }) \text{ .}\end{aligned}$$

On the other hand, if $\mathbf{F}$ satisfies Assumption 5.22 instead,

$$|\langle L_1, \mathbf{v}_h\rangle| \le c_0 L_{\mathbf{F}}^{(\theta,H^1)} \|\nabla(\theta^n - J_{\mathbf{f}}\theta^n)\|\|\mathbf{v}_h\|_{1,2} + c_0 L_{\mathbf{F}}^{(\theta,L^2)} \|\theta^n - J_{\mathbf{f}}\theta^n\|\|\mathbf{v}_h\|_{1,2}$$

$$\begin{aligned}|\langle L_2, \mathbf{v}_h\rangle| &\le L_{\mathbf{F}}^{(\Phi,**)} \|J_{\mathbf{f}}\theta^n + \theta_b\|_{1,3}\|\nabla z_{\Phi,h}^n\|\|\mathbf{v}_h\|_{0,6} \\ &\le cL_{\mathbf{F}}^{(\Phi,**)} \left(c|\mathbf{f}|_1\|\theta\|_{\infty;H^2} + \|\theta_b\|_{1,3}\right) \|\nabla z_{\Phi,h}^n\|\|\mathbf{v}_h\|_{1,2} \\ &\quad (\text{ by } H^2(\Omega) \hookrightarrow W^{1,3}(\Omega) \text{ })\end{aligned}$$

$$|\langle L_3, \mathbf{v}_h\rangle| \le c_0 L_{\mathbf{F}}^{(\theta,H^1)} \|\nabla J_{\mathbf{f}} z_{\theta,h}^n\|\|\mathbf{v}_h\|_{1,2} + c_0 L_{\mathbf{F}}^{(\theta,L^2)} \|J_{\mathbf{f}} z_{\theta,h}^n\|\|\mathbf{v}_h\|_{1,2}$$

$$\begin{aligned}|\langle L_4, \mathbf{v}_h\rangle| &\le L_{\mathbf{F}}^{(\Phi,**)} \|J_{\mathbf{f}}\eta_h^n + \theta_b\|_{1,3}\|\nabla d_{\Phi,h}^n\|\|\mathbf{v}_h\|_{0,6} \\ &\le cL_{\mathbf{F}}^{(\Phi,**)} \left(c|\mathbf{f}|_1\|\theta\|_{\infty;H^2} + \|\theta_b\|_{1,3}\right) \|\nabla d_{\Phi,h}^n\|\|\mathbf{v}_h\|_{1,2} \\ &\quad (\text{ by } H^2(\Omega) \hookrightarrow W^{1,3}(\Omega) \text{ and } H^1(\Omega) \hookrightarrow L^6(\Omega) \text{ })\end{aligned}$$

$$|\langle L_5, \mathbf{v}_h\rangle| \le c_0 L_{\mathbf{F}}^{(\theta,H^1)} \|\nabla J_{\mathbf{f}} d_{\theta,h}^n\|\|\mathbf{v}_h\|_{1,2} + c_0 L_{\mathbf{F}}^{(\theta,L^2)} \|J_{\mathbf{f}} d_{\theta,h}^n\|\|\mathbf{v}_h\|_{1,2}.$$

Using (5.82) to substitute $\|\nabla d^n_{\Phi,h}\|$, the force residual norm can be estimated from above via

$$\begin{aligned}\frac{|\langle R_4^n, \mathbf{v}_h\rangle|}{\|\mathbf{v}_h\|_{1,2}} \leq{}& C_{4,1}\|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2} + C_{4,2}\left(\|J_{\mathbf{f}} z^n_{\theta,h}\|_{1,2} + \|z^n_{\Phi,h}\|_{1,2}\right) \\ &+ C_{4,3}\left(C_{0,1}\left(\|\theta^n - J_{\mathbf{g}}\theta^n\| + \|J_{\mathbf{g}} z^n_{\theta,h}\| + \|J_{\mathbf{g}} d^n_{\theta,h}\|\right) + C_{0,2}\|z^n_{\Phi,h}\|_{1,2}\right) \\ &+ C_{4,i}\|J_{\mathbf{f}} d^n_{\theta,h}\|^{\frac{1}{2}}\ \|J_{\mathbf{f}} d^n_{\theta,h}\|^{\frac{1}{2}}_{1,2} + C_{4,ii}\|J_{\mathbf{f}} d^n_{\theta,h}\| + C_{4,iii}\|J_{\mathbf{f}} d^n_{\theta,h}\|_{1,2}.\end{aligned} \tag{5.86}$$

Depending on the underlying Assumptions on $\mathbf{F}$, the introduced constants take the followig form

- if Assumption 5.21 holds:

$$C_{4,ii} = C_{4,iii} = 0.$$

- if Assumption 5.22 holds:

$$C_{4,i} = 0,\ C_{4,ii} = c_0 L_{\mathbf{F}}^{(\theta,L^2)},\ C_{4,iii} = c_0 L_{\mathbf{F}}^{(\theta,H^1)}.$$

Estimation of Convection Residual

The convection residuals R_3 and R_8 can estimated as follows by using Lemma 5.10 (i) and (ii),

$$\begin{aligned}|\langle R_3, \mathbf{d}^n_{\mathbf{u},h}\rangle| &= |\tilde{c}_{\mathbf{v}}(\mathbf{u}^n, \mathbf{u}^n, \mathbf{d}^n_{\mathbf{u},h}) - \tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{u}^n_h, \mathbf{u}^n_h, \mathbf{d}^n_{\mathbf{u},h})| \\ &\leq |\tilde{c}_{\mathbf{v}}(\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n, \mathbf{u}^n, \mathbf{d}^n_{\mathbf{u},h})| + |\tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{z}^n_{\mathbf{u},h}, \mathbf{u}^n, \mathbf{d}^n_{\mathbf{u},h})| \\ &\quad + |\tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}, \mathbf{u}^n, \mathbf{d}^n_{\mathbf{u},h})| + |\tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{u}^n_h, \mathbf{z}^n_{\mathbf{u},h}, \mathbf{d}^n_{\mathbf{u},h})| \\ &=: \sum_{i=1}^{4} K_i\end{aligned}$$

and

$$\begin{aligned}|\langle R_8, d^n_{\theta,h}\rangle| &= |\tilde{c}_{\tau}(\mathbf{u}^n, \theta^n + \theta_b, d^n_{\theta,h}) - \tilde{c}_{\tau}(J_{\mathbf{c}}\mathbf{u}^n_h, \theta^n_h + \theta_b, d^n_{\theta,h})| \\ &\leq |\tilde{c}_{\tau}(\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n, \theta^n + \theta_b, d^n_{\theta,h})| + |\tilde{c}_{\tau}(J_{\mathbf{c}}\mathbf{z}^n_{\mathbf{u},h}, \theta^n + \theta_b, d^n_{\theta,h})| \\ &\quad + |\tilde{c}_{\tau}(J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}, \theta^n + \theta_b, d^n_{\theta,h})| + |\tilde{c}_{\tau}(J_{\mathbf{c}}\mathbf{u}^n_h, z^n_{\theta,h}, d^n_{\theta,h})| \\ &=: \sum_{i=5}^{8} K_i.\end{aligned}$$

Further, by Lemma 5.10 and the Sobolev embedding Theorem A.92, $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $H^2(\Omega) \hookrightarrow W^{1,3}(\Omega)$,

$$\begin{aligned}K_1 &\leq c\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}\ \|\mathbf{u}\|_{\infty;\mathbf{H}^2}\ \|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\| \\ K_2 &\leq c\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}\ \|\mathbf{u}\|_{\infty;\mathbf{H}^1}\ \|J_{\mathbf{c}}\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} \\ K_3 &\leq c\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}\ \|\mathbf{u}\|_{\infty;\mathbf{H}^1}\ \|J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}\|J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}_{1,2} \\ K_4 &\leq c\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}\ \|J_{\mathbf{c}}\mathbf{u}^n_h\|_{1,2}\ \|\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} \\ K_5 &\leq c\|d^n_{\theta,h}\|_{1,2}\ \|\theta + \theta_b\|_{\infty;H^2}\ \|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\| \\ K_6 &\leq c\|d^n_{\theta,h}\|_{1,2}\ \|\theta + \theta_b\|_{\infty;H^1}\ \|J_{\mathbf{c}}\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} \\ K_7 &\leq c\|d^n_{\theta,h}\|_{1,2}\ \|\theta + \theta_b\|_{\infty;H^1}\ \|J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}\|J_{\mathbf{c}}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}_{1,2} \\ K_8 &\leq c\|d^n_{\theta,h}\|_{1,2}\ \|J_{\mathbf{c}}\mathbf{u}^n_h\|_{1,2}\ \|z^n_{\theta,h}\|_{1,2}.\end{aligned}$$

Thus,

$$\frac{|\langle R_3, \mathbf{d}^n_{\mathbf{u},h}\rangle|}{\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}} \le C_{3,1}\left(\|\mathbf{u}^n - J_\mathbf{c}\mathbf{u}^n\| + \|J_\mathbf{c}\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} + \|J_\mathbf{c}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}\|J_\mathbf{c}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}_{1,2}\right) \\ + C_{3,2}\|J_\mathbf{c}\mathbf{u}^n_h\|_{1,2}\|\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} \tag{5.87}$$

$$\frac{|\langle R_8, d^n_{\theta,h}\rangle|}{\|d^n_{\theta,h}\|_{1,2}} \le C_{8,1}\left(\|\mathbf{u}^n - J_\mathbf{c}\mathbf{u}^n\| + \|J_\mathbf{c}\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} + \|J_\mathbf{c}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}\|J_\mathbf{c}\mathbf{d}^n_{\mathbf{u},h}\|^{\frac{1}{2}}_{1,2}\right) \\ + C_{8,2}\|J_\mathbf{c}\mathbf{u}^n_h\|_{1,2}\|z^n_{\theta,h}\|_{1,2}. \tag{5.88}$$

Estimation of Difference Quotient Residual

The residual norms of the difference quotients, R_1 and R_6, can be estimated via

$$\frac{|\langle R_1, \mathbf{v}_h\rangle|}{\|\mathbf{v}_h\|_{1,2}} \le \|\partial_t\mathbf{u}^n - k^{-1}J_\mathbf{d}\mathbf{u}^n\|_{-1,2} + k^{-1}\|J_\mathbf{d}\mathbf{z}^n_{\mathbf{u},h}\|_{-1,2} \tag{5.89}$$

$$\frac{|\langle R_6, \tau_h\rangle|}{\|\tau_h\|_{1,2}} \le \|\partial_t\theta^n - k^{-1}J_\mathbf{d}\theta^n\|_{-1,2} + k^{-1}\|J_\mathbf{d}z^n_{\theta,h}\|_{-1,2}. \tag{5.90}$$

Left-Hand Sides

By using Assumption 5.20 (*iii*) and the definition of $\mathbf{w}^n_h$ by means of the Stokes projection, the left-hand sides of (5.76),(5.78) can be estimated from below,

$$k^{-1}(J_\mathbf{d}\mathbf{d}^n_{\mathbf{u},h}, \mathbf{d}^n_{\mathbf{u},h}) + a_\mathbf{v}(\mathbf{d}^n_{\mathbf{u},h}, \mathbf{d}^n_{\mathbf{u},h}) - b(\mathbf{d}^n_{\mathbf{u},h}, d^n_{p,h}) \\ \ge ok^{-1}J_\mathbf{1}\left(\|\mathbf{d}^n_{\mathbf{u},h}\|^2 + U_n^2\right) + k^{-1}\overline{U}^2_n + \frac{1}{1+c_0^2}\nu\|\mathbf{d}^n_{\mathbf{u},h}\|^2_{1,2} \tag{5.91}$$

$$k^{-1}(J_\mathbf{d}d^n_{\theta,h}, d^n_{\theta,h}) + a_\tau(d^n_{\theta,h}, d^n_{\theta,h}) \\ \ge ok^{-1}J_\mathbf{1}\left(\|d^n_{\theta,h}\|^2 + T_n^2\right) + k^{-1}\overline{T}^2_n + \frac{1}{1+c_D^2}\kappa\|d^n_{\theta,h}\|^2_{1,2}. \tag{5.92}$$

Here, $(U_n)_n$, $(\overline{U}_n)_n$, $(T_n)_n$, $(\overline{T}_n)_n$ denote real sequences according to Assumption 5.20.

Application of Discrete Gronwall Inequality

Combination of the previous estimates (5.81), (5.86) - (5.92) with $\mathbf{v}_h = \mathbf{d}^n_{\mathbf{u},h}$ and $\tau_h = d^n_{\theta,h}$, application of Young's inequality $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ for all $a, b \in \mathbb{R}$, $\epsilon > 0$ and the inequality $\left(\sum_{i=1}^n a_i\right)^2 \le n\sum_{i=1}^n a_i^2$ for $a_i \ge 0$ yields for $n \ge q$:

$$\omega_l \le c(\omega_r^\mathbf{v} + \omega_r^\tau) \tag{5.93}$$

with

$$\omega_r^{\mathbf{v}} := \frac{1}{\epsilon_1}\left(\|\partial_t \mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 + k^{-2}\|J_{\mathbf{d}}\mathbf{z}_{\mathbf{u},h}^n\|_{-1,2}^2\right)$$
$$+ \frac{1}{\epsilon_{3,1}}\left(C_{3,1}^2\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + C_{3,1}^2\|J_{\mathbf{c}}\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + C_{3,2}^2\|J_{\mathbf{c}}\mathbf{u}_h^n\|_{1,2}^2\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2\right)$$
$$+ \frac{1}{\epsilon_{3,1}^2\epsilon_{3,2}}C_{3,1}^4\|J_{\mathbf{c}}\mathbf{d}_{\mathbf{u},h}^n\|^2$$
$$+ \frac{1}{\epsilon_4}\left(C_{4,1}^2\|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + C_{4,2}^2\left(\|J_{\mathbf{f}}z_{\theta,h}^n\|_{1,2}^2 + \|z_{\Phi,h}^n\|_{1,2}^2\right)\right)$$
$$+ \frac{1}{\epsilon_4}C_{4,3}^2\left(C_{0,1}^2\left(\|\theta^n - J_{\mathbf{g}}\theta^n\|^2 + \|J_{\mathbf{g}}z_{\theta,h}^n\|^2 + \|J_{\mathbf{g}}d_{\theta,h}^n\|^2\right) + C_{0,2}^2\|z_{\Phi,h}^n\|^2\right)$$
$$+ \left(\frac{1}{\epsilon_4^2\epsilon_{4,i}}C_{4,i}^4 + \frac{1}{\epsilon_{4,ii}}C_{4,ii}^2\right)\|J_{\mathbf{f}}d_{\theta,h}^n\|^2$$

$$\omega_r^{\tau} := \frac{1}{\epsilon_6}\left(\|\partial_t\theta^n - k^{-1}J_{\mathbf{d}}\theta^n\|_{-1,2}^2 + k^{-2}\|J_{\mathbf{d}}z_{\theta,h}^n\|_{-1,2}^2\right)$$
$$+ \frac{\kappa}{\epsilon_7}\|z_{\theta,h}^n\|_{1,2}^2$$
$$+ \frac{1}{\epsilon_{8,1}}\left(C_{8,1}^2\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + C_{8,1}^2\|J_{\mathbf{c}}\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + C_{8,2}^2\|J_{\mathbf{c}}\mathbf{u}_h^n\|_{1,2}^2\|z_{\theta,h}^n\|_{1,2}^2\right)$$
$$+ \frac{1}{\epsilon_{8,1}^2\epsilon_{8,2}}C_{8,1}^4\|J_{\mathbf{c}}\mathbf{d}_{\mathbf{u},h}^n\|^2$$

and

$$\omega_l := ok^{-1}J_{\mathbf{1}}\left(\|\mathbf{d}_{\mathbf{u},h}^n\|^2 + \|d_{\theta,h}^n\|^2 + U_n^2 + T_n^2\right) + k^{-1}\left(\overline{U}_n^2 + \overline{T}_n^2\right)$$
$$+ \left(2\tilde{\nu} - \frac{\epsilon_1}{2} - \frac{\epsilon_{3,1}}{2} - \frac{\epsilon_4}{2} - \frac{\epsilon_{4,ii}}{2} - \frac{\epsilon_{4,iii}}{2}\right)\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 - \frac{\epsilon_{3,2}+\epsilon_{8,2}}{2}\|J_{\mathbf{c}}\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2$$
$$+ \left(2\tilde{\kappa} - \frac{\epsilon_6}{2} - \frac{\epsilon_7\kappa}{2} - \frac{\epsilon_{8,1}}{2}\right)\|d_{\theta,h}^n\|_{1,2}^2 - \left(\frac{\epsilon_{4,i}}{2} + \frac{C_{4,iii}^2}{2\epsilon_{4,iii}}\right)\|J_{\mathbf{f}}d_{\theta,h}^n\|_{1,2}^2.$$

If $\epsilon_{4,iii} > 0$ is chosen such that

$$\frac{\epsilon_{4,iii}}{2} < \tilde{\nu}, \tag{5.94}$$

then ϵ_1, $\epsilon_{3,1}$, ϵ_4, $\epsilon_{4,ii} \propto \nu$ can be chosen such that

$$\left(2\tilde{\nu} - \frac{\epsilon_1}{2} - \frac{\epsilon_{3,1}}{2} - \frac{\epsilon_4}{2} - \frac{\epsilon_{4,ii}}{2} - \frac{\epsilon_{4,iii}}{2}\right) > \tilde{\nu}.$$

On the other hand, if there is some $\sigma \in (0,1)$ such that the condition

$$C_{4,iii}^2 < \sigma\frac{\tilde{\kappa}}{q|\mathbf{f}|_2^2}\epsilon_{4,iii} \tag{5.95}$$

holds, then

$$-\left(\frac{\epsilon_{4,i}}{2} + \frac{C_{4,iii}^2}{2\epsilon_{4,iii}}\right)\|J_{\mathbf{f}}d_{\theta,h}^n\|_{1,2}^2 \geq -\left(\frac{\epsilon_{4,i}}{2} + \frac{C_{4,iii}^2}{2\epsilon_{4,iii}}\right)|\mathbf{f}|_2^2\sum_{i=1}^{q}\|d_{\theta,h}^{n-i}\|_{1,2}^2$$
$$\geq -\left(\frac{\epsilon_{4,i}}{2}|\mathbf{f}|_2^2 + \sigma\frac{\tilde{\kappa}}{2q}\right)\sum_{i=1}^{q}\|d_{\theta,h}^{n-i}\|_{1,2}^2.$$

Thus, $\epsilon_{4,i} > 0$ can be chosen sufficiently small such that

$$-\left(\frac{\epsilon_{4,i}}{2} + \frac{C_{4,iii}^2}{2\epsilon_{4,iii}}\right) \|J_{\mathbf{f}} d_{\theta,h}^n\|_{1,2}^2 \geq -\frac{\tilde{\kappa}}{2q} \sum_{i=1}^{q} \|d_{\theta,h}^{n-i}\|_{1,2}^2.$$

In summary, if

$$C_{4,iii}^2 < \frac{2\tilde{\nu}\tilde{\kappa}}{q|\mathbf{f}|_2^2}, \tag{5.96}$$

then one can always select $\sigma \in (0,1)$ and $\epsilon_{4,iii} > 0$ such that both (5.94) and (5.95) do hold, since

$$\sigma \frac{\tilde{\kappa}}{q|\mathbf{f}|_2^2} \epsilon_{4,iii} \to \left(\frac{2\tilde{\nu}\tilde{\kappa}}{q|\mathbf{f}|_2^2}\right)^- \quad \text{for} \quad \sigma \to 1^-,\ \epsilon_{4,iii} \to (2\tilde{\nu})^-.$$

Note that (5.96) is precisely the assumed condition (5.75).

Choosing the remaining $\epsilon_7 > 0$, ϵ_6, $\epsilon_{8,1} \propto \kappa$ such that

$$\left(2\tilde{\kappa} - \frac{\epsilon_6}{2} - \frac{\epsilon_7 \kappa}{2} - \frac{\epsilon_{8,1}}{2}\right) \geq \tilde{\kappa}$$

and $\epsilon_{3,2}$, $\epsilon_{8,2} \propto \nu$ such that

$$\frac{\epsilon_{3,2} + \epsilon_{8,2}}{2} \leq \frac{\tilde{\nu}}{2q|\mathbf{c}|_2^2},$$

one obtains from (5.93),

$$\begin{aligned} & ok^{-1} J_{\mathbf{1}} \left(\|\mathbf{d}_{\mathbf{u},h}^n\|^2 + \|d_{\theta,h}^n\|^2 + U_n^2 + T_n^2\right) + k^{-1}\left(\overline{U}_n^2 + \overline{T}_n^2\right) \\ & \quad + \tilde{\nu}\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 + \tilde{\kappa}\|d_{\theta,h}^n\|_{1,2}^2 - \frac{\tilde{\nu}}{2q}\sum_{i=1}^{q} \|\mathbf{d}_{\mathbf{u},h}^{n-i}\|_{1,2}^2 - \frac{\tilde{\kappa}}{2q}\sum_{i=1}^{q}\|d_{\theta,h}^{n-i}\|_{1,2}^2 \\ & \leq \sum_{i=1}^{q}\left\{\alpha_{\mathbf{u}}\|\mathbf{d}_{\mathbf{u},h}^{n-i}\|^2 + \alpha_\theta \|d_{\theta,h}^{n-i}\|^2\right\} + \alpha^n + \beta^n + \gamma^n + \delta^n, \end{aligned} \tag{5.97}$$

with factors for $n \geq q$

$$\begin{aligned} \alpha_{\mathbf{u}} &:= c\left(\frac{1}{\nu^3}C_{3,1}^4 + \frac{1}{\kappa^2\nu}C_{8,1}^4\right)|\mathbf{c}|_\infty^2 \\ \alpha_\theta &:= c\left(\frac{1}{\nu^2\kappa}C_{4,i}^4 + \frac{1}{\nu}C_{4,ii}^2\right)|\mathbf{f}|_\infty^2 + c\frac{1}{\nu}C_{4,3}^2 C_{0,1}^2 |\mathbf{g}|_\infty^2 \end{aligned}$$

and

$$\begin{aligned} \frac{\alpha^n}{c} &\leq \left(\frac{1}{\nu}\|\partial_t \mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 + \left(\frac{1}{\nu}C_{3,1}^2 + \frac{1}{\kappa}C_{8,1}^2\right)\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2\right) \\ &\quad + \left(\frac{1}{\kappa}\|\partial_t\theta^n - k^{-1}J_{\mathbf{d}}\theta^n\|_{-1,2}^2 + \frac{1}{\nu}C_{4,1}^2\|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + \frac{1}{\nu}C_{4,3}^2C_{0,1}^2\|\theta^n - J_{\mathbf{g}}\theta^n\|^2\right) \\ \frac{\beta^n}{c} &\leq \left(\frac{1}{\nu}C_{3,1}^2 + \frac{1}{\kappa}C_{8,1}^2\right)\|J_{\mathbf{c}}\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + \frac{1}{\nu}C_{4,2}^2\|J_{\mathbf{f}}z_{\theta,h}^n\|_{1,2}^2 + \kappa\|z_{\theta,h}^n\|_{1,2}^2 + \frac{1}{\nu}C_{4,2}^2\|z_{\Phi,h}^n\|_{1,2}^2 \\ &\quad + \frac{1}{\nu}C_{4,3}^2C_{0,1}^2\|J_{\mathbf{g}}z_{\theta,h}^n\|^2 + \frac{1}{\nu}C_{4,3}^2C_{0,2}^2\|z_{\Phi,h}^n\|^2 \\ \frac{\gamma^n}{c} &\leq \frac{1}{\nu}k^{-2}\|J_{\mathbf{d}}\mathbf{z}_{\mathbf{u},h}^n\|_{-1,2}^2 + \frac{1}{\kappa}k^{-2}\|J_{\mathbf{d}}z_{\theta,h}^n\|_{-1,2}^2 \\ \frac{\delta^n}{c} &\leq \|J_{\mathbf{c}}\mathbf{u}_h^n\|_{1,2}^2\left(\frac{1}{\nu}C_{3,2}^2\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + \frac{1}{\kappa}C_{8,2}^2\|z_{\theta,h}^n\|_{1,2}^2\right). \end{aligned}$$

Multiplication of (5.97) by ko^{-1} and summation from $n = q, \dots, M$ for arbitrary $M \in \{q, \dots, N\}$ gives

$$\begin{aligned}
&\|\mathbf{d}_{\mathbf{u},h}^M\|^2 + \|d_{\theta,h}^M\|^2 + U_M^2 + T_M^2 + k\sum_{n=q}^{M}\frac{1}{o}\left\{k^{-1}\left(\overline{U}_n^2 + \overline{T}_n^2\right) + \frac{\tilde{\nu}}{2}\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 + \frac{\tilde{\kappa}}{2}\|d_{\theta,h}^n\|_{1,2}^2\right\} \\
&\leq k\sum_{n=0}^{M-1}\frac{2q}{o}\left\{\alpha_{\mathbf{u}}\|\mathbf{d}_{\mathbf{u},h}^n\|^2 + \alpha_\theta\|d_{\theta,h}^n\|^2\right\} + k\sum_{n=q}^{M}\frac{1}{o}\left\{\alpha^n + \beta^n + \gamma^n + \delta^n\right\} \qquad (5.98)\\
&\quad + k\sum_{i=0}^{q-1}\frac{1}{o}\left\{\frac{\tilde{\nu}}{2}\|\mathbf{d}_{\mathbf{u},h}^i\|_{1,2}^2 + \frac{\tilde{\kappa}}{2}\|d_{\theta,h}^i\|_{1,2}^2\right\} + \|\mathbf{d}_{\mathbf{u},h}^{q-1}\|^2 + \|d_{\theta,h}^{q-1}\|^2 + U_{q-1}^2 + T_{q-1}^2.
\end{aligned}$$

Application of the discrete Gronwall inequality, Lemma A.42, finally yields

$$\begin{aligned}
\|\mathbf{d}_{\mathbf{u},h}^M\|^2 + \|d_{\theta,h}^M\|^2 + U_M^2 + T_M^2 + k\sum_{n=q}^{M}\frac{1}{o}&\left\{k^{-1}\left(\overline{U}_n^2 + \overline{T}_n^2\right) + \frac{\tilde{\nu}}{2}\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 + \frac{\tilde{\kappa}}{2}\|d_{\theta,h}^n\|_{1,2}^2\right\} \\
&\leq \exp\left(T2qo^{-1}\max\{\alpha_{\mathbf{u}}, \alpha_\theta\}\right) \qquad (5.99)\\
&\quad \cdot\left(k\sum_{n=q}^{N}\frac{1}{o}\left\{\alpha^n + \beta^n + \gamma^n + \delta^n\right\} + \overline{C}_{q-1}^2\right),
\end{aligned}$$

with $\overline{C}_{q-1}^2$ denoting those terms in (5.98) that correspond to time steps $i \leq q-1$ only, i.e. the last row in (5.98).

Finite Element Approximation Error

The finite element approximation errors can be bounded by means of Assumption 5.3, Lemma 5.6, and the imposed regularity of the exact solution:

$$\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2} \leq C_S h^l\left(\|\mathbf{u}\|_{\infty;\mathbf{H}^{l+1}} + \frac{1}{\nu}\|p\|_{\infty;H^l}\right) \qquad (5.100)$$

$$\|z_{p,h}^n\| \leq C_S h^l\left(\nu\|\mathbf{u}\|_{\infty;\mathbf{H}^{l+1}} + \|p\|_{\infty;H^l}\right) \qquad (5.101)$$

$$\|z_{\theta,h}^n\|_{1,2} \leq ch^l\|\theta\|_{\infty;H^{l+1}} \qquad (5.102)$$

$$\|z_{\Phi,h}^n\|_{1,2} \leq ch^l\|\Phi\|_{\infty;H^{l+1}}. \qquad (5.103)$$

By using the linearity of Π_S and Π_{X_h}, Assumption 5.3 and Lemma 5.6, 5.8, one obtains for an arbitrary q-step difference operator $J_{\mathbf{h}}$,

$$\begin{aligned}
\|J_{\mathbf{h}}\mathbf{z}_{\mathbf{u},h}^n\|_{p,2} &= \|\sum_{i=0}^{q}h_i\mathbf{u}^{n-i} - \sum_{i=0}^{q}h_i\Pi_S(\mathbf{u}^{n-i}, p^{n-i})\|_{p,2} \\
&= \|J_{\mathbf{h}}\mathbf{u}^n - \Pi_S(J_{\mathbf{h}}\mathbf{u}^n, J_{\mathbf{h}}p^n)\|_{p,2} \\
&\leq C_S h^{l+l^*}\left(\|J_{\mathbf{h}}\mathbf{u}^n\|_{l+1,2} + \frac{1}{\nu}\|J_{\mathbf{h}}p^n\|_{l,2}\right) \qquad (5.104)
\end{aligned}$$

$$\begin{aligned}
\|J_{\mathbf{h}}z_{x,h}^n\|_{p,2} &= \|\sum_{i=0}^{q}h_i x^{n-i} - \sum_{i=0}^{q}h_i\Pi_{X_h}x^{n-i}\|_{p,2} \\
&= \|J_{\mathbf{h}}x^n - \Pi_{X_h}J_{\mathbf{h}}x^n\|_{p,2} \\
&\leq ch^l\|J_{\mathbf{h}}x^n\|_{l+1,2}, \qquad (5.105)
\end{aligned}$$

for $x \in \{\theta, \Phi\}$ and $p \in \{0,1\}$. Here, $l^* = 1$ if the underlying Stokes problem is regular and $p = 0$. Otherwise, $l^* = 0$.

Final Error Estimation for Velocity, Temperature and Potential

The term $\frac{k}{o}\sum_{n=q}^{N}\{\alpha^n + \beta^n + \gamma^n + \delta^n\}$ in (5.99) can estimated as follows:

$$\begin{aligned}\frac{k}{o}\sum_{n=q}^{N}\alpha^n \lesssim\, & k\sum_{n=q}^{N}\|\partial_t\mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 + k\sum_{n=q}^{N}\|\partial_t\theta^n - k^{-1}J_{\mathbf{d}}\theta^n\|_{-1,2}^2 \\ & + k\sum_{n=q}^{N}\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + k\sum_{n=q}^{N}\|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + k\sum_{n=q}^{N}\|\theta^n - J_{\mathbf{g}}\theta^n\|^2 \\ & = E(k)^2, \end{aligned} \tag{5.106}$$

and, by means of (5.100)-(5.103), there holds

$$\frac{k}{o}\sum_{n=q}^{N}\beta^n \lesssim h^{2l}. \tag{5.107}$$

Using (5.104) and (5.105) with $\mathbf{h} = \mathbf{d}$,

$$\begin{aligned}\frac{k}{o}\sum_{n=q}^{N}\gamma^n &\lesssim k^{-1}\left(h^{2l+2l^*}\sum_{n=q}^{N}\|J_{\mathbf{d}}\mathbf{u}^n\|_{l+1,2}^2 + h^{2l+2l^*}\sum_{n=q}^{N}\|J_{\mathbf{d}}p^n\|_{l,2}^2 + h^{2l}\sum_{n=q}^{N}\|J_{\mathbf{d}}\theta^n\|_{l+1,2}^2\right) \\ &= E(h,k)^2. \end{aligned} \tag{5.108}$$

Combination of (5.100),(5.102) and the discrete stability result, Lemma 5.23, yields

$$\begin{aligned}\frac{k}{o}\sum_{n=q}^{N}\delta^n &\le \frac{c}{o}\max_{n\in\{q,\ldots,N\}}\left(\frac{1}{\nu}C_{3,2}^2\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + \frac{1}{\kappa}C_{8,2}^2\|z_{\theta,h}^n\|_{1,2}^2\right)\cdot q|\mathbf{c}|_\infty^2 k\sum_{n=0}^{N}\|\mathbf{u}_h^n\|_{1,2}^2 \\ &\lesssim h^{2l}. \end{aligned} \tag{5.109}$$

Squaring and multiplying (5.82) by k, summing over $n = q, \ldots, N$ and using (5.103) gives

$$\begin{aligned}k\sum_{n=q}^{N}\|\nabla d_{\Phi,h}^n\|^2 &\lesssim k\sum_{n=q}^{N}\|\theta^n - J_{\mathbf{g}}\theta^n\|^2 + h^{2l} + k\sum_{n=0}^{N}\|d_{\theta,h}^n\|^2 \\ &\lesssim E(k)^2 + h^{2l} + E_3(q)^2 + k\sum_{n=q}^{N}\|d_{\theta,h}^n\|^2. \end{aligned} \tag{5.110}$$

Using (5.106) - (5.109), the right-hand side in (5.99) is equal to the stated term $\mathcal{E}_{\mathbf{u},\theta}(h,k,q)$, up to a multiplicative constant that is independent of h and k. In this way, the estimates

$$\|\left(\mathbf{d}_{\mathbf{u},h}^n\right)_n\|_{l^2(q,N,\mathbf{H}^1)} + \|\left(d_{\theta,h}^n\right)_n\|_{l^2(q,N,H^1)} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q) \tag{5.111}$$

$$\|\left(\mathbf{d}_{\mathbf{u},h}^n\right)_n\|_{l^\infty(q,N,\mathbf{L}^2)} + \|\left(d_{\theta,h}^n\right)_n\|_{l^\infty(q,N,L^2)} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q) \tag{5.112}$$

$$\|\left(U_n\right)_n\|_{l^\infty(q,N;\mathbb{R})} + \|\left(T_n\right)_n\|_{l^\infty(q,N;\mathbb{R})} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q) \tag{5.113}$$

$$\|\left(\overline{U}_n\right)_n\|_{l^2(q,N;\mathbb{R})} + \|\left(\overline{T}_n\right)_n\|_{l^2(q,N;\mathbb{R})} \lesssim \mathcal{E}_{\mathbf{u},\theta}(h,k,q)k^{\frac{1}{2}} \tag{5.114}$$

follow. Similarly, the square root of the right-hand side in (5.110) can be estimated from above by $\mathcal{E}_\Phi(h,k,q)$ (under the use of (5.111)). Thus,

$$\| \left(d^n_{\Phi,h}\right)_n \|_{l^2(q,N,H^1)} \lessapprox \mathcal{E}_\Phi(h,k,q) \tag{5.115}$$

is obtained. The stated assertions on the complete error of velocity, temperature and potential now follow by the decomposition $e^n_{\cdot,h} = z^n_{\cdot,h} + d^n_{\cdot,h}$, (5.111), (5.112), (5.115), estimations (5.100) - (5.103) for $z^n_{\cdot,h}$ and the triangle inequality.

Pressure Error Estimation

As next step, the pressure error $d^n_{p,h}$ for $n \geq q$ is considered. To this end, note that there holds for $n \geq q$ (see [75])

$$\begin{aligned}
\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2} &\leq \min\left\{ch^{-1}\|\mathbf{d}^n_{\mathbf{u},h}\|, \|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2}\right\} \\
&\leq \min\left\{ch^{-1}\| \left(\mathbf{d}^n_{\mathbf{u},h}\right)_n \|_{l^\infty(q,N;\mathbf{L}_2)},\, k^{-\frac{1}{2}}\| \left(\mathbf{d}^n_{\mathbf{u},h}\right)_n \|_{l^2(q,N;\mathbf{H}^1)}\right\} \\
&\lessapprox \min\left\{h^{-1}, k^{-\frac{1}{2}}\right\} \mathcal{E}_{\mathbf{u},\theta}(h,k,q) \\
&=: \tilde{D}(h,k,q)
\end{aligned}$$

by the inverse estimate, Assumption 5.3 (v), and (5.111), (5.112). Thus,

$$\| \left(\mathbf{d}^n_{\mathbf{u},h}\right)_n \|_{l^\infty(0,N;\mathbf{H}^1)} \lessapprox \max\left\{\tilde{D}(h,k,q),\, \|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2} \colon\ n = 0,\dots,q-1\right\} =: D(h,k,q).$$

Moreover, by

$$\begin{aligned}
\|\mathbf{u}^n_h\|_{1,2} &\leq \|\mathbf{u}^n\|_{1,2} + \|\mathbf{z}^n_{\mathbf{u},h}\|_{1,2} + \|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2} \\
&\lessapprox \|\mathbf{u}\|_{\infty;\mathbf{H}^1} + C_S\left(\|\mathbf{u}\|_{\infty;\mathbf{H}^{l+1}} + \frac{1}{\nu}\|p\|_{\infty;H^l}\right) h^l + D(h,k,q) \\
&\lessapprox 1 + h^l + \min\left\{h^{-1}, k^{-\frac{1}{2}}\right\} \mathcal{E}_{\mathbf{u},\theta}(h,k,q) + \max\{\|\mathbf{u}^n - \mathbf{u}^n_h\|_{1,2} \colon\ n = 0,\dots,q-1\} \\
&=: C_p(h,k,q),
\end{aligned}$$

one obtains

$$\| \left(\mathbf{u}^n_h\right)_n \|_{l^\infty(0,N;\mathbf{H}^1)} \lessapprox C_p(h,k,q). \tag{5.116}$$

Now, by the discrete inf-sup condition, Assumption 5.3 (ii), and the error equation (5.76),

$$\begin{aligned}
\beta\|d^n_{p,h}\| &\leq \sup_{\mathbf{v}_h\in\mathbf{U}_h} \frac{b(\mathbf{v}_h, d^n_{p,h})}{\|\mathbf{v}_h\|_{1,2}} \\
&\leq \sup_{\mathbf{v}_h\in\mathbf{U}_h} \frac{1}{\|\mathbf{v}_h\|_{1,2}}\Big(k^{-1}|(J_\mathbf{d}\mathbf{d}^n_{\mathbf{u},h}, \mathbf{v}_h)| + |a_\mathbf{v}(\mathbf{d}^n_{\mathbf{u},h}, \mathbf{v}_h)| + |\langle R^n_1, \mathbf{v}_h\rangle| \\
&\qquad\qquad + |\langle R^n_3, \mathbf{v}_h\rangle| + |\langle R^n_4, \mathbf{v}_h\rangle|\Big) \\
&\leq k^{-1}\|J_\mathbf{d}\mathbf{d}^n_{\mathbf{u},h}\| + \nu\|\mathbf{d}^n_{\mathbf{u},h}\|_{1,2} + \|R^n_1\|_{\mathbf{U}^*_h} + \|R^n_3\|_{\mathbf{U}^*_h} + \|R^n_4\|_{\mathbf{U}^*_h}.
\end{aligned} \tag{5.117}$$

$\|R^n_1\|_{\mathbf{U}^*_h}$ and $\|R^n_4\|_{\mathbf{U}^*_h}$ can be estimated from above by (5.89) and (5.86) respectively,

$$\|R^n_1\|_{\mathbf{U}^*_h} \leq \|\partial_t\mathbf{u}^n - k^{-1}J_\mathbf{d}\mathbf{u}^n\|_{-1,2} + k^{-1}\|J_\mathbf{d}\mathbf{z}^n_{\mathbf{u},h}\|_{-1,2} \tag{5.118}$$

$$\begin{aligned}
\|R^n_4\|_{\mathbf{U}^*_h} &\lessapprox \|\theta^n - J_\mathbf{f}\theta^n\|_{1,2} + \|\theta^n - J_\mathbf{g}\theta^n\| \\
&\quad + \|J_\mathbf{g}z^n_{\theta,h}\| + \|J_\mathbf{f}z^n_{\theta,h}\|_{1,2} + \|J_\mathbf{g}d^n_{\theta,h}\| + \|z^n_{\Phi,h}\|_{1,2} \\
&\quad + \|J_\mathbf{f}d^n_{\theta,h}\| + \|J_\mathbf{f}d^n_{\theta,h}\|^{\frac{1}{2}}\|J_\mathbf{f}d^n_{\theta,h}\|^{\frac{1}{2}}_{1,2} + \|J_\mathbf{f}d^n_{\theta,h}\|_{1,2}.
\end{aligned} \tag{5.119}$$

To estimate $\|R_3^n\|_{\mathbf{U}_h^*}$, use Lemma 5.10 (*iii*), (*x*) with $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, $H^2(\Omega) \hookrightarrow W^{1,3}(\Omega)$ and the bound (5.116) to obtain

$$\begin{aligned}
\|R_3^n\|_{\mathbf{U}_h^*} &= \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|\langle R_3^n, \mathbf{v}_h\rangle|}{\|\mathbf{v}_h\|_{1,2}} \\
&\leq \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{1}{\|\mathbf{v}_h\|_{1,2}} \Big(|\tilde{c}_{\mathbf{v}}(\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h)| + |\tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}(\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{u}^n, \mathbf{v}_h)| \\
&\qquad + |\tilde{c}_{\mathbf{v}}(J_{\mathbf{c}}\mathbf{u}_h^n, \mathbf{u}^n - \mathbf{u}_h^n, \mathbf{v}_h)| \Big) \\
&\leq \|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\| \|\mathbf{u}^n\|_{2,2} + |\mathbf{c}|_\infty \sum_{i=1}^{q} \left\{ \|\mathbf{z}_{\mathbf{u},h}^{n-i}\|_{1,2} + \|\mathbf{d}_{\mathbf{u},h}^{n-i}\|_{1,2} \right\} \|\mathbf{u}^n\|_{1,2} \\
&\quad + |\mathbf{c}|_\infty \left(\sum_{i=1}^{q} \|\mathbf{u}_h^{n-i}\|_{1,2} \right) \left(\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2} + \|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2} \right) \\
&\lesssim \|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\| + \sum_{i=1}^{q} \left\{ \|\mathbf{z}_{\mathbf{u},h}^{n-i}\|_{1,2} + \|\mathbf{d}_{\mathbf{u},h}^{n-i}\|_{1,2} \right\} \\
&\quad + C_p(h,k,q) \left(\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2} + \|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2} \right).
\end{aligned} \tag{5.120}$$

Squaring and multiplication by k of (5.117), summing from $n = q, \ldots, N$, using the estimates on the residual norms (5.118), (5.119), (5.120), the estimate of the approximation errors (5.100) - (5.105) and the estimates of the discretization errors (5.111), (5.112) yields

$$\begin{aligned}
k \sum_{n=q}^{N} \|d_{p,h}^n\|^2 &\lesssim k \sum_{n=q}^{N} \|\partial_t \mathbf{u}^n - k^{-1} J_{\mathbf{d}} \mathbf{u}^n\|_{-1,2}^2 \\
&\quad + k \sum_{n=q}^{N} \left\{ \|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + \|\theta^n - J_{\mathbf{g}}\theta^n\|^2 + \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 \right\} \\
&\quad + \left(1 + C_p(h,k,q)^2\right) h^{2l} k \sum_{n=0}^{N} \left\{ \|\mathbf{u}^n\|_{l+1,2}^2 + \|p^n\|_{l,2}^2 + \|\theta^n\|_{l+1,2}^2 + \|\Phi^n\|_{l+1,2}^2 \right\} \\
&\quad + h^{2l+2l^*} k^{-1} \sum_{n=q}^{N} \left\{ \|J_{\mathbf{d}}\mathbf{u}^n\|_{l+1,2}^2 + \|J_{\mathbf{d}} p^n\|_{l+1,2}^2 \right\} \\
&\quad + \left(1 + C_p(h,k,q)^2\right) k \sum_{n=0}^{q-1} \left\{ \|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 + \|d_{\theta,h}^n\|_{1,2}^2 \right\} \\
&\quad + k^{-1} \sum_{n=q}^{N} \|J_{\mathbf{d}} \mathbf{d}_{\mathbf{u},h}^n\|^2 \\
&\quad + \left(1 + C_p(h,k,q)^2\right) \mathcal{E}_{\mathbf{u},\theta}(h,k,q)^2 \\
&\lesssim E(k)^2 \\
&\quad + C_p(h,k,q)^2 h^{2l} \\
&\quad + E(h,k)^2 \\
&\quad + C_p(h,k,q)^2 \left(h^{2l} + E_1(q)^2 \right) \\
&\quad + E_{\mathbf{d}}(h,k)^2 \\
&\quad + C_p(h,k,q)^2 \mathcal{E}_{\mathbf{u},\theta}(h,k,q)^2.
\end{aligned}$$

Thus,

$$\| \left(d_{p,h}^n\right)_n \|_{l^2(q,N,L^2)} \lessapprox \mathcal{E}_p(h,k,q). \tag{5.121}$$

The stated assertion on $e_{p,h}^n$ now follows from (5.121) and the triangle inequality.

Bound on Temporal Variation of Estimation Error

It remains to derive an upper bound for $E_{\mathbf{d}}(h,k)$. Setting $\mathbf{v}_h = J_{\mathbf{d}}\mathbf{d}_{\mathbf{u},h}^n$ in (5.76) und using Young's inequality yields

$$\begin{aligned}
k^{-1}\|J_{\mathbf{d}}\mathbf{d}_{\mathbf{u},h}^n\|^2 \leq{}& \frac{\nu}{2}\sum_{i=0}^{q}|d_i|\left\{\|\nabla\mathbf{d}_{\mathbf{u},h}^n\|^2 + \|\nabla\mathbf{d}_{\mathbf{u},h}^{n-i}\|^2\right\} \\
&+ c\left(\|R_1^n\|_{\mathbf{U}_h^*}^2 + \|R_3^n\|_{\mathbf{U}_h^*}^2 + \|R_4^n\|_{\mathbf{U}_h^*}^2\right) + c\sum_{i=0}^{q}|d_i|^2\|\mathbf{d}_{\mathbf{u},h}^{n-i}\|_{1,2}^2.
\end{aligned}$$

Summing this inequality from $n = q,\ldots,N$ and using the previously derived bounds on $\left\{\|R_i^n\|_{\mathbf{U}_h^*}\right\}_{i=1}^4$, (5.118), (5.119), (5.120), the approximation estimates (5.100) - (5.105) and the error estimates for velocity and temperature, (5.111), (5.112), yields

$$\begin{aligned}
k^{-1}\sum_{n=q}^{N}\|J_{\mathbf{d}}\mathbf{d}_{\mathbf{u},h}^n\|^2 \lessapprox{}& (1 + C_p(h,k,q)^2)\sum_{n=0}^{N}\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2 + \sum_{n=q}^{N}\|d_{\theta,h}^n\|_{1,2}^2 + \sum_{n=q}^{N}\|\partial_t\mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 \\
&+ \sum_{n=q}^{N}\left\{\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + \|\theta^n - J_{\mathbf{g}}\theta^n\|^2\right\} \\
&+ k^{-2}\sum_{n=q}^{N}\|J_{\mathbf{d}}\mathbf{z}_{\mathbf{u},h}^n\|_{-1,2}^2 \\
&+ \left(1 + C_p(h,k,q)^2\right)\sum_{n=0}^{N}\left\{\|\mathbf{z}_{\mathbf{u},h}^n\|_{1,2}^2 + \|z_{\theta,h}^n\|^2\right\} \\
&+ \left(1 + C_p(h,k,q)^2\right)\sum_{n=0}^{N}\left\{\|z_{\theta,h}^n\|_{1,2}^2 + \|z_{\Phi,h}^n\|^2 + \|z_{\Phi,h}^n\|_{1,2}^2\right\} \\
\lessapprox{}& \left(1 + C_p(h,k,q)^2\right)\left(k^{-1}\mathcal{E}_{\mathbf{u},\theta}(h,k,q)^2 + \sum_{n=0}^{q-1}\|\mathbf{d}_{\mathbf{u},h}^n\|_{1,2}^2\right) \\
&+ \sum_{n=q}^{N}\|\partial_t\mathbf{u}^n - k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{-1,2}^2 \\
&+ \sum_{n=q}^{N}\left\{\|\mathbf{u}^n - J_{\mathbf{c}}\mathbf{u}^n\|^2 + \|\theta^n - J_{\mathbf{f}}\theta^n\|_{1,2}^2 + \|\theta^n - J_{\mathbf{g}}\theta^n\|^2\right\} \\
&+ h^{2l+2l^*}k^{-2}\sum_{n=q}^{N}\left\{\|J_{\mathbf{d}}\mathbf{u}^n\|_{l+1,2}^2 + \|J_{\mathbf{d}}p^n\|_{l,2}^2\right\} \\
&+ \left(1 + C_p(h,k,q)^2\right)h^{2l}\sum_{n=0}^{N}\left\{\|\mathbf{u}^n\|_{l+1,2}^2 + \|p^n\|_{l,2}^2 + \|\theta^n\|_{l+1,2}^2 + \|\Phi^n\|_{l+1,2}^2\right\}
\end{aligned}$$

Thus,

$$k^{-1}\sum_{n=q}^{N}\|J_{\mathbf{d}}\mathbf{d}_{\mathbf{u},h}^{n}\|^2 \leqslant C_p(h,k,q)^2\left(k^{-1}\mathcal{E}_{\mathbf{u},\theta}(h,k,q)^2+h^{2l}+E_2(q)^2\right) \\ +k^{-1}E^2(k) \\ +k^{-1}E^2(h,k) \\ +C_p(h,k,q)^2h^{2l}. \tag{5.122}$$

Taking the square root in (5.122) yields the final estimate

$$E_{\mathbf{d}}(h,k)\leqslant \mathcal{E}_{\mathbf{d}}(h,k,q). \tag{5.123}$$

□

Remark 5.25. In the previous proof, we required that $C_{4,iii}^2 < 2\tilde{\nu}\tilde{\kappa}|\mathbf{f}|_2^{-2}q^{-1}$, see (5.96). However, this assumption could be relaxed to $C_{4,iii}^2 < 4\tilde{\nu}\tilde{\kappa}|\mathbf{f}|_2^{-2}q^{-1}$, to the expense of a slightly more technical presentation.

In the proof of Theorem 5.24, a time step size restriction when applying Gronwall's inequality could be avoided, since the right-hand side in (5.98) only contains terms $\|\mathbf{d}_{\mathbf{u},h}^n\|$, $\|d_{\theta,h}^n\|$ for $n<M$, whereas the corresponding left-hand side involves $\|\mathbf{d}_{\mathbf{u},h}^M\|$, $\|d_{\theta,h}^M\|$. Thereby, $\|\mathbf{d}_{\mathbf{u},h}^n\|$ stems from $\|J_{\mathbf{c}}\mathbf{d}_{\mathbf{u},h}^n\|$ which in turn arises from the convection residual estimation (5.87), (5.88). For this reason, it is assumed that $J_{\mathbf{c}}$ is an explicit difference operator. Otherwise, the right-hand side in (5.98) would also depend on $\|\mathbf{d}_{\mathbf{u},h}^M\|$ and a restriction of the form $k\leqslant \alpha_{\mathbf{u}}^{-1}$ would be necessary, see Lemma A.42. An analogous argument applies for $\|d_{\theta,h}^n\|$ which occurs in the right-hand side through the terms $\|J_{\mathbf{f}}d_{\theta,h}^n\|$ and $\|J_{\mathbf{g}}d_{\theta,h}^n\|$.

One drawback of the presented proof lies in the exponential factor that arises due to the application of Gronwall's inequality. This exponent is proportional to negative powers of viscosity and thermal diffusion and may thus take very large values for typical fluids. Strictly speaking, one obtains an exponent $C_{\exp}$ with

$$C_{\exp}\propto T\max\{\alpha_{\mathbf{u}},\alpha_\theta\}\propto T\left(\nu^{-3}+\kappa^{-2}\nu^{-1}+\nu^{-2}\kappa^{-1}+\nu^{-1}\right). \tag{5.124}$$

The term $\nu^{-3}+\kappa^{-2}\nu^{-1}$ comes from the estimation of the convection terms $\tilde{c}_{\mathbf{v}}$, $\tilde{c}_{\tau}$, see (5.2), (5.87) and (5.88). In [66], an approach is shown that avoids the explicit occurrence of ν in the exponential factor when deriving error estimates for the instationary incompressible Navier-Stokes equations. This is achieved by the use of $\mathbf{H}(div)$-conforming elements that yield exactly divergence free discrete velocities in combination with $\nabla\mathbf{u}\in L^1(0,T;L^\infty)$ and a $W^{1,\infty}$-stability assumption on the Stokes projection. Conditions for this stability are given in [30], for instance.

The term $\nu^{-2}\kappa^{-1}$ is actually multiplied by the constant $C_{4,i}^4$, which is introduced in the estimation of the body force residual (5.86). As pointed out after equation (5.86), $C_{4,i}=0$ if Assumption 5.22 holds. The remaining factor ν^{-1} also arises due to the body force and is multiplied by the constant

$$C_{4,ii}^2+C_{4,3}^2C_{0,1}^2\propto\left(c_0L_{\mathbf{F}}^{(\theta,L^2)}\right)^2+\left(L_{\mathbf{F}}^{(\Phi,\cdot)}\left(\|\theta\|_{\infty;H^2}+\|\theta_b\|_{1,3}\right)\frac{L_\epsilon}{\epsilon_-}\|\Phi+\Phi_b\|_{\infty;W^{1,\infty}}\right)^2. \tag{5.125}$$

Thus, large values of ν^{-1} could be compensated if the domain has a small diameter, i.e. small c_0, and if the spatial variations of temperature and potential are rather low.

Full Error Estimate

We now consider two concrete temporal discretization schemes by choosing the difference operators introduced in Problem 5.19: 1-step BDF1 (which could be considered as semi-implicit Euler) and a 2-step BDF2 method. It is shown that the resulting difference operators for both schemes satisfy Assumption 5.20. Moreover, the k-dependent error contributions in Theorem 5.24, $E(k)$, $E(h,k)$ and $E_{\mathbf{d}}(h,k)$ are bounded from above in order to derive convergence rates w.r.t. k. This is done by using Taylor expansion under certain assumptions on the temporal regularity of the exact solution. However, it was pointed out in [35], that too strong regularity requirements on the solution of the incompressible Navier-Stokes equations for $t \to 0$ imply non-local compatibility conditions on the intial data which may be uncheckable in practice. Among others, it is critical to impose the following assumptions according to Corollary 2.1 in [35]:

$$\begin{aligned}
&\|\mathbf{u}\|_{\infty;\mathbf{H}^3} &&< \infty \\
&\|\partial_t \mathbf{u}\|_{\infty;\mathbf{H}^1} &&< \infty,\ \|\partial_t \mathbf{u}\|_{2;\mathbf{H}^2} < \infty, \\
&\|\partial_{tt} \mathbf{u}\|_{2;\mathbf{L}^2} &&< \infty.
\end{aligned} \tag{5.126}$$

On the other hand, the following conditions "are optimal in the sense that higher, not time-weighted regularity of the solution is equivalent to compatibility conditions on the problem's data" [24], according to Theorem 2.1 in [24] and [23]:

$$\begin{aligned}
&\partial_t \mathbf{u} &&\in L^2(0,T;\mathbf{H}^1) \\
&\partial_{tt} \mathbf{u} &&\in L^2(0,T;\mathbf{H}^{-1}),\ t^{\frac{1}{2}}\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{L}^2),\ t\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{H}^1) \\
&t\partial_{ttt}\mathbf{u} &&\in L^2(0,T;\mathbf{H}^{-1}),\ t^{\frac{3}{2}}\left(\partial_{tt}\mathbf{f} - \partial_{ttt}\mathbf{u}\right) \in L^2(0,T;\mathbf{L}^2),
\end{aligned} \tag{5.127}$$

where $\mathbf{f}$ denotes the entire right-hand side in the momentum equation. In (5.127), the integrability of $\partial_{tt}\mathbf{u}$ and $\partial_{ttt}\mathbf{u}$ is increased by weakening the spatial regularity requirements, i.e. going from $\mathbf{H}^1$ over $\mathbf{L}^2$ to $\mathbf{H}^{-1}$, and by time-weighting, i.e. multiplication by t^β. In doing so, a possibly non-smooth behavior of the solution for $t \to 0$ can be compensated. The higher β, the more regular the time-weighted terms become.

The proposed time stepping schemes are based on difference operators determined by the vectors

$$\begin{aligned}
\mathbf{a}^{(1)} &= (0,1), & \mathbf{a}^{(2)} &= (0,2,-1) \\
\mathbf{b}^{(1)} &= (1,-1), & \mathbf{b}^{(2)} &= (\frac{3}{2},-2,\frac{1}{2}).
\end{aligned} \tag{5.128}$$

Here, $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ define the standard one- and two-step backward finite differences $J_{\mathbf{d}}$ for approximating ∂_t. The remaining difference operators $J_{\mathbf{c}}$, $J_{\mathbf{f}}$, $J_{\mathbf{g}}$ are determined by $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$, leading to the following time stepping schemes.

Definition 5.26. *(BDF1 Scheme)*
The 1-step BDF scheme is given by $\mathbf{d} := (1,-1),\ \mathbf{c} := (0,1),\ \mathbf{f} := (0,1),\ \mathbf{g} := (0,1)$.

Definition 5.27. *(BDF2 Scheme)*
The 2-step BDF scheme is given by $\mathbf{d} := (\frac{3}{2},-2,\frac{1}{2}),\ \mathbf{c} := (0,2,-1),\ \mathbf{f} := (0,2,-1),\ \mathbf{g} := (0,2,-1)$.

Both, BDF1 and BDF2 are widely used for discretizing flow problems, see e.g. [75] for the use of

BDF1 for the Boussinesq equations and [24] and [31] for the use of BDF2 for the incompressible Navier-Stokes equations. In BDF1, the velocity $\mathbf{u}^n$ in the convection terms $\tilde{c}_{\mathbf{v}}$, $\tilde{c}_\tau$ is replaced by the first order extrapolation $\mathbf{u}^{n-1}$. In BDF2, the second order extrapolation $2\mathbf{u}^{n-1} - \mathbf{u}^{n-2}$ is used. The same holds for the temperature in the body force term $\mathbf{F}$ and in the Gauss bilinear form a_β.

It is easy to see that difference operators given by Definition 5.26 and 5.27 satisfy Assumption 5.20 (i) and (ii). The stability condition (iii) is stated by Lemma 5.28, see e.g. [75] and [24].

Lemma 5.28. *(Stability of Difference Operators)*
Let Z denote a Hilbert space, $z \in L^2(0,T;Z)$ with $z' \in L^1(0,T;Z)$, $z^n = z(t_n)$ for $n = 0,\ldots,N$. Then,

(i) for $n = 1,\ldots,N$,

$$(J_{\mathbf{b}^{(1)}} z^n, z^n) = \frac{1}{2} J_{\mathbf{1}} \|z^n\|_Z^2 + \frac{1}{2}\|z^n - z^{n-1}\|_Z^2,$$

i.e. $o = \frac{1}{2}$, $Z_n = 0$ and $\overline{Z}_n = \frac{1}{\sqrt{2}}\|J_{\mathbf{b}^{(1)}} z^n\|_Z$ in the notation of Assumption 5.20.

(ii) for $n = 2,\ldots,N$,

$$(J_{\mathbf{b}^{(2)}} z^n, z^n) = \frac{1}{4} J_{\mathbf{1}} \left(\|z^n\|_Z^2 + \|J_{\mathbf{a}^{(2)}} z^{n+1}\|_Z^2\right) + \frac{1}{4}\|z^{n+1} - 2z^n + z^{n-1}\|_Z^2,$$

i.e. $o = \frac{1}{4}$, $Z_n = \|J_{\mathbf{a}^{(2)}} z^{n+1}\|_Z$ and $\overline{Z}_n = \frac{1}{2}\|z^{n+1} - 2z^n + z^{n-1}\|_Z$.

The following corollaries now provide upper bounds for the k-dependent error contributions of Theorem 5.24. In this way, the errors can be bounded in terms of h, k and $E_l(q)$, $l \in \{1,2,3,4\}$. The later terms measure the error of the q initial conditions $(\mathbf{u}_h^i, \theta_h^i)$, $i = 0,\ldots,q-1$.

For each time stepping scheme, BDF1 and BDF2, two sets of estimates are derived. One set that only requires regularity conditions that are not stronger than the ones stated in (5.127) and another set which may involve the critical estimates (5.126) and, in return, allows for convergence rates of higher order. We start with BDF1.

Corollary 5.29. *(BDF1 Scheme with Weak Regularity Requirements)*
Let the assumptions of Theorem 5.24 hold with $l = 1$ and let additionally

$$\begin{aligned} &\partial_t \mathbf{u} \in L^2(0,T;\mathbf{L}^2), \quad \partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{H}^{-1}) \\ &\partial_t \theta \in L^2(0,T;H^{l+1}), \partial_{tt}\theta \in L^2(0,T;H_D^{-1}). \end{aligned}$$

Let the difference operators in Problem 5.19 be given according to Definition 5.26 and $h, k < 1$. Then, the upper error bounds derived in Theorem 5.24 satisfy

$$\begin{aligned} \mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim h^l + k + h^{l+l^*}k^{-1} + E_1(q) \\ \mathcal{E}_\Phi(h,k,q) &\lesssim h^l + k + h^{l+l^*}k^{-1} + E_1(q) + E_3(q) \\ \mathcal{E}_p(h,k,q) &\lesssim k^{\frac{1}{2}} + h^l k^{-\frac{1}{2}} + h^{l+l^*}k^{-\frac{3}{2}} + k^{-\frac{1}{2}}E_1(q) \\ &\quad + C_p(h,k,q)\left(h^l + k + h^{l+l^*}k^{-1} + E_1(q)\right) \\ C_p(h,k,q) &\lesssim 1 + E_2(q) + \min\{h^{-1}, k^{-\frac{1}{2}}\}\left(h^{l+l^*}k^{-1} + E_1(q)\right). \end{aligned}$$

Proof. Using Lemma A.81 (i), (iii), (v) with $\beta = 0$, the error terms given by Theorem 5.24 can be estimated as follows:

$$\begin{aligned} E(k)^2 &\lesssim k^2 \left(\|\partial_{tt}\mathbf{u}\|^2_{2;\mathbf{H}^{-1}} + \|\partial_{tt}\theta\|^2_{2;H_D^{-1}} \right) \\ &\quad + k^2 \left(\|\partial_t \mathbf{u}\|^2_{2;\mathbf{L}^2} + \|\partial_t\theta\|^2_{2;H^1} + \|\partial_t\theta\|^2_{2;L^2} \right) \end{aligned}$$

and

$$\begin{aligned} E(h,k)^2 &\lesssim h^{2l+2l^*} k^{-2} \left(k \sum_{n=1}^{N} \left\{ \|\mathbf{u}^n\|^2_{l+1,2} + \|p^n\|^2_{l,2} \right\} \right) + h^{2l} \|\partial_t\theta\|^2_{2;H^{l+1}} \\ &\lesssim h^{2l+2l^*} k^{-2} T \left(\|\mathbf{u}\|^2_{\infty;\mathbf{H}^{l+1}} + \|p\|^2_{\infty;H^l} \right) + h^{2l} \|\partial_t\theta\|^2_{2;H^{l+1}} \\ &\lesssim h^{2l+2l^*} k^{-2} + h^{2l}. \end{aligned}$$

By Lemma 5.28 (i),

$$E_{\mathbf{d}}(h,k)^2 = k^{-1} \sum_{n=1}^{N} \|J_{\mathbf{b}^{(1)}} \mathbf{d}^n_{\mathbf{u},h}\|^2 = 2k^{-1} \sum_{n=1}^{N} \overline{U}_n^2 = k^{-2} \| \left(\overline{U}_n\right)_n \|^2_{l^2(1,N;\mathbb{R})} \lesssim k^{-1} \mathcal{E}_{\mathbf{u},\theta}(h,k,q)^2.$$

Thus,

$$\begin{aligned} \mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim h^l + k + h^{l+l^*} k^{-1} + E_1(q) \\ E_{\mathbf{d}}(h,k)^2 &\lesssim k + h^{2l} k^{-1} + h^{2l+2l^*} k^{-3} + k^{-1} E_2^2(q). \end{aligned}$$

Then, the assertion follows by plugging these estimates into the terms $\mathcal{E}_p$, $\mathcal{E}_\Phi$, C_p. $\square$

Corollary 5.30. *(BDF1 Scheme with Strong Regularity Requirements)*
Let the assumptions of Corollary 5.29 hold for some $l \geq 1$ and let additionally

$$\partial_t \mathbf{u} \in L^2(0,T;\mathbf{H}^{l+1}) \quad \text{and} \quad \partial_t p \in L^2(0,T;H^l).$$

Then,

$$\begin{aligned} \mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim k + h^l + E_1(q) \\ \mathcal{E}_\Phi(h,k,q) &\lesssim k + h^l + E_1(q) + E_3(q) \\ \mathcal{E}_p(h,k,q) &\lesssim k^{\frac{1}{2}} + h^l k^{-\frac{1}{2}} + k^{-\frac{1}{2}} E_1(q) + C_p(h,k,q) \left(k + h^l + E_1(q) \right) \\ C_p(h,k,q) &\lesssim 1 + E_2(q) + \min\{h^{-1}, k^{-\frac{1}{2}}\} E_1(q). \end{aligned}$$

Proof. The assertion follows analogously to Corollary 5.29, however, with

$$\begin{aligned} E(h,k)^2 &\lesssim h^{2l+2l^*} \left(\|\partial_t\mathbf{u}\|^2_{2;\mathbf{H}^{l+1}} + \|\partial_t p\|^2_{2;H^l} \right) + h^{2l} \|\partial_t\theta\|^2_{2;H^{l+1}} \\ &\lesssim h^{2l+2l^*} + h^{2l}. \end{aligned}$$

according to Lemma A.81 (v) with $\beta = 0$. $\square$

The difference between Corollary 5.29 and 5.30 lies in the estimation of the term $E(h,k)$. In the strong regularity case, the finite difference $\|k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{l+1,2}$ can be estimated from above by using $\|\partial_t\mathbf{u}^n\|_{l+1,2}$ and thereby obtaining an upper bound that only depends on h. This is not possible in the weak regularity case, where $\|k^{-1}J_{\mathbf{d}}\mathbf{u}^n\|_{l+1,2}$ is simply estimated by using the triangle inequality. Thus one does not get rid of the preceding factor k^{-1}. At least, this factor is multiplied by h^{l+l^*} in the final estimate, such that the negative effect of an increased temporal resolution can be compensated by an increased spatial resolution. Apart from this factor and the initial condition contributions, the error of velocity, temperature and potential is proportional to $h^l + k$ in both cases, as expected by a first order method in time. In contrast, a factor $k^{\frac{1}{2}}$ is lost for the pressure error, similar to the estimates derived in [75].

We proceed with the BDF2 scheme.

Corollary 5.31. *(BDF2 Scheme with Weak Regularity Requirements)*
Let the assumptions of Theorem 5.24 hold with $l = 1$ and let additionally

$$\begin{aligned}
&\partial_t\mathbf{u} \in L^2(0,T;\mathbf{L}^2), \quad t^{\frac{1}{2}}\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{L}^2), \quad t\partial_{ttt}\mathbf{u} \in L^2(0,T;\mathbf{H}^{-1})\\
&\partial_t\theta \in L^2(0,T;H^{l+1}),\ t\partial_{tt}\theta \in L^2(0,T;H^{l+1}),\ t^{\frac{1}{2}}\partial_{tt}\theta \in L^2(0,T;H^1),\ t\partial_{ttt}\theta \in L^2(0,T;H_D^{-1}).
\end{aligned}$$

Let the difference operators in Problem 5.19 be given according to Definition 5.27 and $h,k<1$. Then, the upper error bounds derived in Theorem 5.24 satisfy

$$\begin{aligned}
\mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim h^l + k + h^{l+l^*}k^{-1} + E_1(q)\\
\mathcal{E}_{\Phi}(h,k,q) &\lesssim h^l + k + h^{l+l^*}k^{-1} + E_1(q) + E_3(q)\\
\mathcal{E}_{p}(h,k,q) &\lesssim k^{\frac{1}{2}} + h^{l+l^*}k^{-\frac{3}{2}}\\
&\quad + C_p(h,k,q)\left(k^{\frac{1}{2}} + h^l k^{-\frac{1}{2}} + h^{l+l^*}k^{-\frac{3}{2}} + k^{-\frac{1}{2}}E_1(q) + E_2(q)\right)\\
C_p(h,k,q) &\lesssim 1 + E_2(q) + \min\{h^{-1},k^{-\frac{1}{2}}\}\left(h^l + h^{l+l^*}k^{-1} + E_1(q)\right).
\end{aligned}$$

Proof. Using Lemma A.81 (ii) with $\beta = \frac{1}{2}$, (iv) with $\beta = 1$, (vi) with $\beta = 0,\ \gamma = 1$, the error terms given by Theorem 5.24 can be estimated as follows:

$$\begin{aligned}
E(k)^2 &\lesssim k^2\left(\|t\partial_{ttt}\mathbf{u}\|^2_{2;\mathbf{H}^{-1}} + \|t\partial_{ttt}\theta\|^2_{2;H_D^{-1}}\right)\\
&\quad + k^3\left(\|t^{\frac{1}{2}}\partial_{tt}\mathbf{u}\|^2_{2;\mathbf{L}^2} + \|t^{\frac{1}{2}}\partial_{tt}\theta\|^2_{2;H^1} + \|t^{\frac{1}{2}}\partial_{tt}\theta\|^2_{2;L^2}\right)\\
&\lesssim k^2 + k^3\\
E(h,k)^2 &\lesssim h^{2l+2l^*}k^{-2}\left(\|\left(\mathbf{u}^n\right)_n\|^2_{l^2(1,N;\mathbf{H}^{l+1})} + \|\left(p^n\right)_n\|^2_{l^2(1,N;H^l)}\right)\\
&\quad + h^{2l}\left(\|\partial_t\theta\|^2_{2;H^{l+1}} + \|t\partial_{tt}\theta\|^2_{2;H^{l+1}}\right)\\
&\lesssim h^{2l+2l^*}k^{-2}T\left(\|\mathbf{u}\|^2_{\infty;\mathbf{H}^{l+1}} + \|p\|^2_{\infty;H^l}\right)\\
&\quad + h^{2l}\left(\|\partial_t\theta\|^2_{2;H^{l+1}} + \|t\partial_{tt}\theta\|^2_{2;H^{l+1}}\right)\\
&\lesssim h^{2l+2l^*}k^{-2} + h^{2l}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim h^l + k + h^{l+l^*}k^{-1} + E_1(q) \\
\mathcal{E}_{\mathbf{d}}(h,k)^2 &\lesssim k^{-1}\left(k^2 + k^3 + h^{2l+2l^*}k^{-2}\right) \\
&\quad + C_p(h,k,q)^2\left(h^{2l} + k^{-1}\left(h^{2l} + k^2 + h^{2l+2l^*}k^{-2} + E_1(q)^2\right) + E_2(q)^2\right) \\
&\lesssim k + h^{2l+2l^*}k^{-3} + C_p(h,k,q)^2\left(k^{-1}h^{2l} + k + h^{2l+2l^*}k^{-3} + k^{-1}E_1(q)^2 + E_2(q)^2\right).
\end{aligned}
$$

Then, the assertion follows by plugging these estimates into the terms $\mathcal{E}_p$, $\mathcal{E}_\Phi$, C_p. □

Corollary 5.32. *(BDF2 Scheme with Strong Regularity Requirements)*
Let the assumptions of Corollary 5.31 hold for $l \geq 1$ and let additionally

$$
\begin{aligned}
&\partial_t\mathbf{u} \in L^2(0,T;\mathbf{H}^{l+1}),\ t\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{H}^{l+1}),\ \partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{L}^2),\ \partial_{ttt}\mathbf{u} \in L^2(0,T;\mathbf{H}^{-1}) \\
&\partial_t p \ \in L^2(0,T;H^l), \quad t\partial_{tt}p \in L^2(0,T;H^l) \\
&\partial_{tt}\theta \in L^2(0,T;H^1), \quad \partial_{ttt}\theta \ \in L^2(0,T;H_D^{-1}).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim k^2 + h^l + E_1(q) \\
\mathcal{E}_\Phi(h,k,q) &\lesssim k^2 + h^l + E_1(q) + E_3(q) \\
\mathcal{E}_p(h,k,q) &\lesssim k^{\frac{3}{2}} + h^{l+l^*}k^{-\frac{1}{2}} + C_p(h,k,q)\left(k^{\frac{3}{2}} + h^l k^{-\frac{1}{2}} + k^{-\frac{1}{2}}E_1(q) + E_2(q)\right) \\
C_p(h,k,q) &\lesssim 1 + E_2(q) + \min\{h^{-1}, k^{-\frac{1}{2}}\}E_1(q).
\end{aligned}
$$

Proof. The assertion follows analogously to Corollary 5.31, however, with

$$
\begin{aligned}
E(k)^2 &\lesssim k^4\left(\|\partial_{ttt}\mathbf{u}\|^2_{2;\mathbf{H}^{-1}} + \|\partial_{ttt}\theta\|^2_{2;H_D^{-1}}\right) \\
&\quad + k^4\left(\|\partial_{tt}\mathbf{u}\|^2_{2;\mathbf{L}^2} + \|\partial_{tt}\theta\|^2_{2;H^1} + \|\partial_{tt}\theta\|^2_{2;L^2}\right) \\
&\lesssim k^4 \\
E(h,k)^2 &\lesssim h^{2l+2l^*}\left(\|\partial_t\mathbf{u}\|^2_{2;\mathbf{H}^{l+1}} + \|\partial_t p\|^2_{2;H^l} + \|t\partial_{tt}\mathbf{u}\|^2_{2;\mathbf{H}^{l+1}} + \|t\partial_{tt}p\|^2_{2;H^l}\right) \\
&\quad + h^{2l}\left(\|\partial_t\theta\|^2_{2;H^{l+1}} + \|t\partial_{tt}\theta\|^2_{2;H^{l+1}}\right) \\
&\lesssim h^{2l+2l^*} + h^{2l},
\end{aligned}
$$

by using Lemma A.81 (ii), (iv), (vi) with $\beta = 0$, $\gamma = 1$. Thus,

$$
\begin{aligned}
\mathcal{E}_{\mathbf{u},\theta}(h,k,q) &\lesssim h^l + h^{l+l^*} + k^2 + E_1(q) \\
\mathcal{E}_{\mathbf{d}}(h,k)^2 &\lesssim k^{-1}\left(k^4 + h^{2l+2l^*}\right) + C_p(h,k,q)^2\left(h^{2l} + k^{-1}\left(k^4 + h^{2l} + E_1(q)^2\right) + E_2(q)^2\right) \\
&\lesssim k^3 + k^{-1}h^{2l+2l^*} + C_p(h,k,q)^2\left(k^3 + k^{-1}h^{2l} + k^{-1}E_1(q)^2 + E_2(q)^2\right).
\end{aligned}
$$

□

Corollary 5.31 shows that the BDF2 scheme yields the same convergence order as BDF1 in case of weak regularity requirements. To obtain the expected second order convergence in time, see Corollary 5.32, it is necessary to impose critical conditions, e.g. $\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{L}^2)$, as stated in (5.126). One way of coping with this issue is proposed in [36] and [24] for the incompressible Navier-Stokes equations. There, a time-weighted error is considered and optimal order convergence rates are derived.

In Section 6.2, we present numerical experiments that exhibit first and second order convergence rates of both proposed schemes. Moreover, we will see that a reduced temporal regularity for $t \to 0$ may actually lead to an observable reduced convergence rate when the BDF2 scheme is used.

In practice, the initial condition errors $E_l(q)$ can be controlled in the following way: First, $(\mathbf{u}_h^0, \theta_h^0)$ is computed by projecting or interpolating the analytically known functions $(\mathbf{u}_0, \theta_0)$ onto the respective discrete spaces. The resulting error can be bounded in terms of h^l. Then, the remaining initial conditions $(\mathbf{u}_h^i, \theta_h^i)$ for $i = 1, \dots q-1$ can be computed by using BDF1 with time step size k^2. In this way, the resulting errors can be bounded in terms of $h^l + k^2$ according to the previously derived convergence rates of BDF1.

5.3. Modeling of DEP Force

We conclude this section on discretization of the TEHD Boussinesq equations by reconsidering the modeling of DEP force w.r.t. the requirements posed in Section 5.2. The next lemma shows that the force terms that are defined in Section 3.3 and which are based on linearization do actually satisfy Assumption 5.21

Lemma 5.33. *(Properties of DEP Force Terms, Continued)*
The DEP force terms $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$ given by Definition 3.23 satisfy Assumption 5.21.

Proof. Consider $\mathbf{F}_{a,1}$: Let $\theta_1, \theta_2, \Phi \in H^1(\Omega)$ and $\mathbf{v} \in \mathbf{U}$. Then,

$$\begin{aligned}
|\langle \mathbf{F}_{a,1}(\theta_1, \Phi) - \mathbf{F}_{a,1}(\theta_2, \Phi), \mathbf{v}\rangle_{\mathbf{U}^*}| &\leq c2\alpha_e \|\nabla^2\Phi_0\|_{0,\infty}\|\Phi\|_{1,2}\|\theta_1 - \theta_2\|_{0,3}\|\nabla\mathbf{v}\| \\
&\quad + c\alpha_g\|\mathbf{g}\|_{0,\infty}\|\theta_1 - \theta_2\|_{0,3}\|\nabla\mathbf{v}\| \\
&=: L_{\mathbf{F}}^{(\theta,*)}\left(\|\Phi\|_{1,2}\right)\|\theta_1 - \theta_2\|_{0,3}\|\nabla\mathbf{v}\|,
\end{aligned}$$

thus (i) follows. Moreover, for $\theta,\ \Phi_1,\ \Phi_2 \in H^1(\Omega)$ there holds

$$\begin{aligned}
|\langle \mathbf{F}_{a,1}(\theta, \Phi_1) - \mathbf{F}_{a,1}(\theta, \Phi_2), \mathbf{v}\rangle_{\mathbf{U}^*}| &\leq c2\alpha_e \|\nabla^2\Phi_0\|_{0,6}\|\Phi_1 - \Phi_2\|_{1,2}\|\theta\|_{1,2}\|\nabla\mathbf{v}\| \\
&=: L_{\mathbf{F}}^{(\Phi,*)}\|\Phi_1 - \Phi_2\|_{1,2}\|\theta\|_{1,2}\|\nabla\mathbf{v}\|,
\end{aligned}$$

which implies (ii). The validness of (iii) is clear. The proof for $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ follows analogously. In the former case, use the identity

$$(|\nabla\Phi_0|^2\,\nabla\theta, \mathbf{v}) = -(|\nabla\Phi_0|^2\,\theta, \nabla\cdot\mathbf{v}) - (\nabla\,|\nabla\Phi_0|^2\,\theta, \mathbf{v}).$$

□

Finally, we give a DEP model that satisfies the alternative Assumption 5.22 and which does not require a priori information, such as the base potential in the definition of $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$. Instead, the integrability of the electric field term $|\nabla\Phi|^2$ in the DEP force is increased by replacing $|\nabla\Phi|^2$ by $|\mathbf{m}_K \circ \nabla\Phi|^2$, where $\mathbf{m}_K$ is a cut-off function. In this way, one obtains L^∞ integrability instead of L^2 and it is possible to define a DEP model that satisfies Assumption 5.22. This is shown in the subsequent Definition 5.34 and Lemma 5.35.

Definition 5.34. *(Cut-Off DEP Force)*
For $K > 0$ let $\mathbf{m}_K \in \mathbf{L}^\infty(\mathbb{R}^d)$ denote a Lipschitz continuous function with Lipschitz constant L_K and $\|\mathbf{m}_K\|_{0,\infty} \leq K$. Define

$$\begin{aligned}\mathbf{F}_{s,K} \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto \alpha_e(|\mathbf{m}_K \circ \nabla\Phi|^2 \nabla\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

Lemma 5.35. *(Properties of Cut-Off DEP Force Term)*
The force term $\mathbf{F}_{s,K}$ given by Definition 5.34 satisfies Assumption 5.22 with $L_{\mathbf{F}}^{(\theta,H^1)} = \alpha_e K^2$ and $L_{\mathbf{F}}^{(\theta,L^2)} = \alpha_g\|\mathbf{g}\|_{0,\infty}$.

Proof. Let $\theta_1, \theta_2, \Phi \in H^1(\Omega)$ and $\mathbf{v} \in \mathbf{U}$. Then,$(i)$ follows from

$$\begin{aligned}|\langle \mathbf{F}_{s,K}(\theta_1, \Phi) - \mathbf{F}_{s,K}(\theta_2, \Phi), \mathbf{v}\rangle_{\mathbf{U}^*}| &\leq \alpha_e\||\mathbf{m}_K \circ \nabla\Phi|^2\|_{0,\infty}\|\nabla(\theta_1 - \theta_2)\|\|\mathbf{v}\| \\ &\quad + \alpha_g\|\mathbf{g}\|_{0,\infty}\|\theta_1 - \theta_2\|\|\mathbf{v}\| \\ &\leq \alpha_e K^2\|\nabla(\theta_1 - \theta_2)\|\|\mathbf{v}\| + \alpha_g\|\mathbf{g}\|_{0,\infty}\|\theta_1 - \theta_2\|\|\mathbf{v}\|\end{aligned}$$

Moreover, for $\theta \in W^{1,3}(\Omega)$, $\Phi_1, \Phi_2 \in H^1(\Omega)$ there holds

$$\begin{aligned}|\langle \mathbf{F}_{s,K}(\theta, \Phi_1) - \mathbf{F}_{s,K}(\theta, \Phi_2), \mathbf{v}\rangle_{\mathbf{U}^*}| &\leq \alpha_e\||\mathbf{m}_K \circ \nabla\Phi_1|^2 - |\mathbf{m}_K \circ \nabla\Phi_2|^2\|\|\nabla\theta\|_{0,3}\|\mathbf{v}\|_{0,6} \\ &\leq \alpha_e\|\mathbf{m}_K \circ \nabla\Phi_1 + \mathbf{m}_K \circ \nabla\Phi_2\|_{0,\infty}\|\mathbf{m}_K \circ \nabla\Phi_1 - \mathbf{m}_K \circ \nabla\Phi_2\| \\ &\quad \cdot \|\nabla\theta\|_{0,3}\|\mathbf{v}\|_{0,6} \\ &\leq \alpha_e 2KL_K\|\nabla(\Phi_1 - \Phi_2)\|\|\nabla\theta\|_{0,3}\|\mathbf{v}\|_{0,6}\end{aligned}$$

Thus, (ii) is satisfied and (iii) follows trivially. □

An example of a suitable cut-off function $\mathbf{m}_K$ is given by the metric projection of a vector $\mathbf{w} \in \mathbb{R}^d$ onto the ball $\{\mathbf{x} \in \mathbb{R}^d\colon\ |\mathbf{x}| \leq K\}$.

Definition 5.36. *(Metric Projection)*
For $K > 0$ the metric projection *is defined by*

$$\mathbf{m}_K \colon \mathbb{R}^d \to \mathbb{R}^d, \quad \mathbf{w} \mapsto \begin{cases}\mathbf{w}, & |\mathbf{w}| \leq K \\ K|\mathbf{w}|^{-1}\mathbf{w}, & |\mathbf{w}| > K.\end{cases}$$

The following Lemma 5.37 states that the previously defined cut-off function satisfies the required properties as stated in Definition 5.34.

Lemma 5.37. *(Properties of Cut-Off Function)*
Let $\mathbf{m}_K$ be given by Definition 5.36. Then, $\|\mathbf{m}_K\|_{0,\infty} = K$ and $\mathbf{m}_K$ is Lipschitz continuous with constant $L_K = 1$.

Proof. The stated assertion $\|\mathbf{m}_K\|_{0,\infty} = K$ is clear and the Lipschitz continuity follows from the fact, that $\mathbf{m}_K$ can be written as $\mathbf{m}_K(\mathbf{w}) = \mathbf{w}_0$ where $\mathbf{w}_0$ is the unique solution of the minimization $\min_{\mathbf{r}\in B} |\mathbf{w} - \mathbf{r}|$, with the compact and convex set $B = \{\mathbf{x} \in \mathbb{R}^d\colon\ |\mathbf{x}| \leq K\}$, see e.g. [7]. □

Regarding Assumption 5.22 only, the actual value K does not play a role in the construction of $\mathbf{F}_{s,K}$. However, according to Lemma 5.23 and Theorem 5.24, a small data condition of the form

$$L_{\mathbf{F}}^{(\theta,H^1)} \leq c_\Omega \sqrt{\nu\kappa} \tag{5.129}$$

has to be supposed. Following Lemma 5.35, there holds $L_{\mathbf{F}}^{(\theta,H^1)} = \alpha_e K^2$ which yields an upper bound on K,

$$K^2 \leq \frac{c_\Omega}{\alpha_e}\sqrt{\nu\kappa}. \tag{5.130}$$

If $\mathbf{F}_{s,K}$ is based on the metric projection $\mathbf{m}_K$ of Definition 5.36, then the electric field at some point $x \in \Omega$ is not modified, as long as its magnitude is below a given threshold. Thus, the proposed approach can be justified for practical applications, if one can give an upper bound on $\max_{x\in\overline{\Omega}} |\mathbf{E}(x)| = \max_{x\in\overline{\Omega}} |\nabla\Phi(x)|$, e.g. by physical reasons. Moreover, these definitions of $\mathbf{F}_{s,K}$ and $\mathbf{m}_K$ allow to draw some conclusions about the error when $\mathbf{F}$ is set to the straightforward DEP formulation

$$\mathbf{F}_s(\theta,\Phi) = \alpha_e(|\nabla\Phi|^2\nabla\theta,\cdot) - \alpha_g(\theta\mathbf{g},\cdot). \tag{5.131}$$

Note that $\mathbf{F}_s$ does not satisfy the required Assumptions 5.21 and 5.22. However, if the computed discrete potential is bounded according to

$$\|\nabla(\Phi_h^n + \Phi_b)\|_{0,\infty} \leq K_* \text{ for all } n = 0,\dots,N, \tag{5.132}$$

uniformly w.r.t. h and k, then the obtained discrete solution is de facto the same as if it was computed by using $\mathbf{F} = \mathbf{F}_{s,K_*}$. If, in addition, K_* is small enough to fulfill (5.130), then Theorem 5.24 can be applied and the corresponding error estimates do hold. In practice, condition (5.132) probably can not be proven in a rigorous way, since it can only be checked for certain values of h and k. However, this uniform boundedness might be safely assumed, if a series of discrete potentials for various discretization parameters h, k do not exhibit a significant grow in the $\|\cdot\|_{0,\infty}$-norm.

Unfortunately, the new DEP model $\mathbf{F}_{s,K}$ does not satisfy Assumption 3.12 and 4.2 which are needed to show existence of solutions of the stationary and instationary TEHD Boussinesq equations. The problem is that the weak convergence $\Phi_n \rightharpoonup \Phi$ in $H^1(\Omega)$ does not translate to pointwise convergence in $\mathbf{U}^*$ as required by Assumption 3.12 (*iii*), since the term $|\mathbf{m}_K \circ \nabla\Phi|^2$ is nonlinear w.r.t. Φ.

6. Numerical Experiments

In this section, we present numerical experiments that are obtained by solving the stationary and instationary TEHD Boussinesq equations with the discretization proposed in Section 5. Two different scenarios are considered. First, a 2D benchmark problem is solved for which an analytic solution is known. This configuration is used to measure the error of the discrete solution and compare the empirical convergence rates with the theoretical ones that are derived in Section 5.1 and 5.2. The second scenario is given by a cylindrical configuration which is commonly used by engineers to investigate DEP-driven flow with physical experiments in the laboratory, see e.g. [68]. Before we turn to the numerical results in Section 6.2 and 6.3, the underlying solution method, implementation and computing details are given in Section 6.1.

As another preparation step, we make a note of the temperature boundary lifting θ_b. According to Assumption 3.1 (*iv*), a family of boundary liftings $\{\theta_b[\xi]\colon\ \xi \in (0,1)\}$ is supposed with the property $\|\theta_b[\xi]\|_{0,3} \leq \xi$ for all $\xi \in (0,1)$. In case of simple geometries, such a family can be constructed by using the following function for $\epsilon \in (0, \frac{1}{2})$, $b_i, b_o \in \mathbb{R}$:

$$b_{\epsilon,b_i,b_o}(x) := \begin{cases} b_{i,\epsilon}(x), & 0 \leq x < \epsilon \\ 0, & \epsilon \leq x < 1-\epsilon \\ b_{o,\epsilon}(x), & 1-\epsilon \leq x \leq 1 \end{cases} \tag{6.1}$$

with polynomials $b_{i,\epsilon}$, $b_{o,\epsilon}$ that satisfy

$$\begin{aligned} &b_{i,\epsilon} \in \mathbb{P}_3([0,\epsilon]), \qquad b_{i,\epsilon}(0) = b_i,\ b_{i,\epsilon}(\epsilon) = b^{'}_{i,\epsilon}(\epsilon) = b^{''}_{i,\epsilon}(\epsilon) = 0 \\ &b_{o,\epsilon} \in \mathbb{P}_3([1-\epsilon,1]),\ b_{o,\epsilon}(1) = b_o,\ b_{o,\epsilon}(1-\epsilon) = b^{'}_{o,\epsilon}(1-\epsilon) = b^{''}_{o,\epsilon}(1-\epsilon) = 0. \end{aligned} \tag{6.2}$$

Then, there holds $b_{\epsilon,b_i,b_o} \in C^2(\overline{\Omega})$ with $\|b_{\epsilon,b_i,b_o}\|_{0,3} \to 0$ for $\epsilon \to 0$ and $\|\nabla b_{\epsilon,b_i,b_o}\| \to 0$ for $(b_i, b_o) \to 0$.

6.1. Implementation

The discrete equations to be solved are stated in Problem 5.12 and 5.19 for the stationary and instationary case, respectively. The spatial discretization is based on Taylor-Hood finite elements for velocity and pressure, and continuous quadratic Lagrange elements for temperature and potential, see Lemma 5.4. The underlying mesh consists of quadrilaterals for 2D problems and hexahedrons for 3D problems, and is obtained from an initial coarse mesh by uniform refinement. For the cylindrical configuration in the second scenario, we use a formulation of the TEHD Boussinesq equations in cylindrical coordinates. Thus, the underlying domain is given by a cuboid.

The temporal discretization makes use of the difference operators given by Definition 5.26 and 5.27, i.e. BDF1 and BDF2 schemes are applied. In addition, we consider a Crank-Nicolson discretization of the instationary equations for the sake of comparison, see e.g. [41].

The resulting nonlinear system of algebraic equations for each time step is solved by the inexact Newton-Raphson method with Eisenstat-Walker forcing strategy [22]. This nonlinear iteration is terminated if the initial algebraic residual is decreased by a factor of 10^{-10} for instationary problems and 10^{-5} for stationary problems. Moreover, within each iteration, the pressure is shifted by a constant scalar to ensure $\int_\Omega p\, dx = 0$. The arising linear systems within each Newton iteration are solved by a preconditioned GMRES method [64]. We use a block Jacobi method as preconditioner whereat the inverse of each diagonal block is approximated by an incomplete LU factorization.

As noted in Section 2.2, the dual problem (2.43) is posed backwards in time with prescribed values at $t = T$. Thus, the temporal discretization can be realized in a time stepping manner, where the solution at time t_{n+1} is used to calculate the solution at t_n. Here, the Crank-Nicolson method is used as time stepping scheme. In each single time step, a linear system of PDEs has to be solved. The associated spatial discretization is based on the same mesh and finite element spaces as in the primal case and the resulting linear system of equations is solved similarly to the primal case. Further note that the primal solution occurs in the dual formulation (2.43). Since both systems are posed in opposite temporal directions, the primal problem has to be solved first and the corresponding solution has to be stored at each single time step t_n, $n = 0, \dots, N$. Only if this calculation is done, the dual system can be solved. As the amount of data for storing the complete primal solution is typically very high, the solution is not kept in memory, but stored on the hard drive by using the parallel file format *HDF5*.

The implementation is based on the open-source, general purpose FEM package *HiFlow*^{3} [29], which is written C++. In *HiFlow*^{3}, parallelization is mainly employed for assembling and solving linear systems and is obtained by domain decomposition of the physical domain Ω. The partitioning of the original mesh is performed by the *metis* library [45] and communication between the individual processes is based on *MPI*.

All simulations were performed on the HPC system *bwForCluster MLS & WISO*, located at Heidelberg University and based on *Intel Xeon E5* CPUs with 2.4 GHz and 20MB cache and 16 cores per node. For parallelization, we typically used 256 - 1024 cores, resulting in approximately 10,000 to 20,000 degrees of freedom per core. The application was compiled with *openmpi* 3.1.4 and the *GNU Compiler Collection gcc* 8.3.

6.2. Convergence of FEM-BDF Discretization

This section is devoted to substantiate the theoretically derived error estimates of Section 5.1 and 5.2 by numerical experiments. To this end, we consider a 2D test case where the right-hand sides $\mathbf{f_v}$, f_τ, f_β in Problem 5.11 and 5.17 are constructed in such a way, that a predefined analytical function does actually solve the equations in the classical, i.e. pointwise, sense. Hereby, $T = 1$ and the domain is given as unit square, $\Omega = (0,1)^2$ with left, right, top and bottom boundary defined by

$$\begin{aligned} &\Gamma_l := \{0\} \times [0,1],\ \Gamma_r := \{1\} \times [0,1] \\ &\Gamma_t := [0,1] \times \{1\},\ \Gamma_b := [0,1] \times \{0\}. \end{aligned} \tag{6.3}$$

We impose Dirichlet boundary conditions for temperature and potential on the left and right boundary, i.e. $\Gamma_D := \Gamma_l + \Gamma_r$, and homogeneous Neumann conditions on top and bottom, i.e. $\Gamma_N := \Gamma_t + \Gamma_b$:

$$\begin{aligned} &\theta = 1 \text{ on } \Gamma_l,\ \theta = 0 \text{ on } \Gamma_r \\ &\Phi = 1 \text{ on } \Gamma_l,\ \Phi = 0 \text{ on } \Gamma_r \\ &\mathbf{u} = 0 \text{ on } \Gamma. \end{aligned} \tag{6.4}$$

Here, the reference temperature θ_r is set to 0.5. The boundary liftings θ_b and Φ_b are defined according to $\theta_b[\xi](x,y) := b_{\zeta(\xi),1,0}(x)$ and $\Phi_b(x,y) := 1 - x$, with $\zeta(\xi) > 0$ chosen such that $\|\theta_b[\xi]\|_{0,3} \leq \xi$, see (6.1), (6.2). In practice, the boundary conditions are imposed by explicitly setting the degrees of freedom that are located on a boundary facet to the respective value. We further assume that the permittivity depends linearly on the temperature, see Section 2, and set for some parameter $\gamma \in (0,1)$

$$\epsilon\colon \mathbb{R} \to [1-\gamma, 1], \quad \epsilon(s) = \epsilon_r \cdot \begin{cases} 1, & s \le 0 \\ 1-\gamma s, & s \in (0,2), \\ 1-\gamma, & s \ge 2 \end{cases} \tag{6.5}$$

to meet the requirements of Assumption 3.2.

Moreover the stationary Problem 5.11 is considered as fully implicit, i.e. $\bar{\mathbf{u}} = \mathbf{u}$, $\tilde{\mathbf{u}} = \mathbf{u}$, $\bar{\theta} = \theta$, $\bar{\Phi} = \Phi$ and without contributions from outer time stepping, i.e. $\delta = 0$. In all considered cases, the fluid properties correspond to the liquid *Wacker AK5* , which is commonly used by physical experiments, see e.g. [51]. However, for the convergence analysis we increased the viscosity and thermal diffusion coefficient to improve the convergence of the used linear and nonlinear algebraic solvers. In Section 6.3, the fluid is used with its original properties. All fluid and experimental parameters are listed in Figure 2.

In the following, the right-hand side terms $\mathbf{f_v}$, f_τ, f_β are chosen such that

$$\begin{aligned} \mathbf{u}_{*,x}(t,x,y) &= g(t)\sin(\pi x)^2\, \sin(2\pi y) \\ \mathbf{u}_{*,y}(t,x,y) &= -g(t)\sin(2\pi x)\, \sin(\pi y)^2 \\ p_*(t,x,y) &= g(t)\cos(2\pi y) \\ \theta_*(t,x,y) &= 1 - x + g_\theta\, g(t)\left(1 - 4(x-0.5)^2\right) \\ \Phi_*(t,x,y) &= 1 - x + g_\Phi\, g(t)\sin(\pi x), \end{aligned} \tag{6.6}$$

solves the TEHD Boussinesq equations in strong form (2.28) with those source terms added. The temporal factor g is either chosen as $g_{\exp}(t) := \exp(-10\,t)$ or as $g_\alpha(t) := t^\alpha$ for some $\alpha \ge 0$ to cover different temporal regularities of the exact solution. In particular, $g = g_0 = 1$, $g_\theta = g_\Phi = 0$ corresponds to the stationary case, $g_{\exp} \in C^\infty([0,T])$ and there holds for $\alpha \in (\frac{3}{2}, \frac{5}{2}] \setminus \{2\}$,

$$\begin{aligned} &g_\alpha \;\in L^\infty(0,T;\mathbb{R}),\, g'_\alpha \in L^\infty(0,T;\mathbb{R}),\; g''_\alpha \in L^2(0,T;\mathbb{R}) \\ &tg'''_\alpha \in L^2(0,T;\mathbb{R}),\;\; g'''_\alpha \notin L^2(0,T;\mathbb{R}), \end{aligned} \tag{6.7}$$

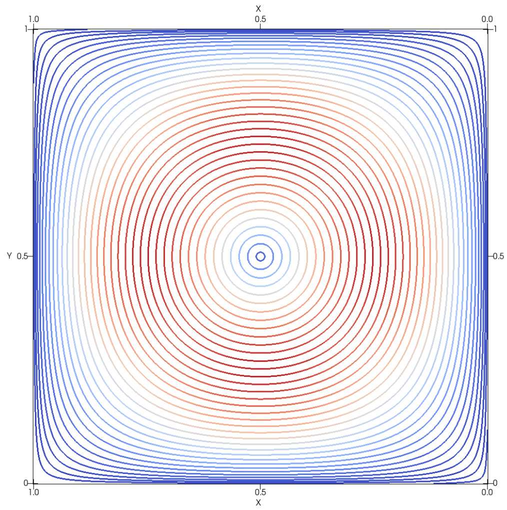
and for $\alpha \in (\frac{5}{2}, \infty) \cup \mathbb{N}$,

$$g_\alpha \in L^\infty(0,T;\mathbb{R}),\; g'_\alpha \in L^\infty(0,T;\mathbb{R}),\; g''_\alpha \in L^2(0,T;\mathbb{R}),\; g'''_\alpha \in L^2(0,T;\mathbb{R}). \tag{6.8}$$

Thus, for $g = g_{\exp}$ and $g = g_\alpha$ with $\alpha \in (\frac{5}{2}, \infty) \cup \mathbb{N}$, the exact solution satisfies the strong regularity conditions for both, BDF1 and BDF2, assumed by Corollary 5.30 and 5.32, respectively. On the other hand, if $\alpha \in (\frac{3}{2}, \frac{5}{2}] \setminus \{2\}$, then the strong regularity conditions for BDF1 are still satisfied, whereas the conditions for BDF2 do not hold. However, the weak conditions for BDF2 are still valid, see Corollary 5.31.

Stationary Case

We first consider error convergence of the discrete stationary solution, defined by Problem 5.12. In this case, the corresponding exact solution, given by (6.6) with $g(t) = 1$, satisfies the regularity conditions imposed by Corollary 5.16. This a priori estimation states that the H^1-error of velocity, temperature and potential is bounded from above by a constant times h^l. According to the underlying second order Taylor-Hood elements combined with quadratic elements for temperature and potential, we have $l = 2$, supposed that the small data conditions (5.35),



ν (st.)	5.0	$\cdot 10^{-1}$	$\left[\mathrm{m}^2\ \mathrm{s}^{-1}\right]$
ν (inst.)	5.0	$\cdot 10^{-3}$	$\left[\mathrm{m}^2\ \mathrm{s}^{-1}\right]$
κ (st.)	7.74	$\cdot 10^{-1}$	$\left[\mathrm{m}^2\ \mathrm{s}^{-1}\right]$
κ (inst.)	7.74	$\cdot 10^{-5}$	$\left[\mathrm{m}^2\ \mathrm{s}^{-1}\right]$
ϵ_r	2.391	$\cdot 10^{-11}$	$\left[\mathrm{kg}\ \mathrm{m}^{-3}\right]$
γ	1.065	$\cdot 10^{-3}$	$\left[\mathrm{K}^{-1}\right]$
α_g	1.08	$\cdot 10^{-3}$	$\left[\mathrm{K}^{-1}\right]$
α_e	1.379	$\cdot 10^{-17}$	$\left[\mathrm{A\ s\ m}^2(\mathrm{kg\,K\,V})^{-1}\right]$
$\mathbf{g}_x$	0		$\left[\mathrm{m}\ \mathrm{s}^{-2}\right]$
$\mathbf{g}_y$	-9.81		$\left[\mathrm{m}\ \mathrm{s}^{-2}\right]$
g_θ (st.)	0		
g_θ (inst.)	0.2		
g_Φ (st.)	0		
g_Φ (inst.)	0.1		

Figure 2: Left: Streamlines of exact flow field with colors indicating the velocity magnitude. Right: Physical parameters.

(5.36) holds. These conditions, however, is not satisfied because of $\frac{2}{\nu^2}\|\nabla \mathbf{u}\|^2 = 4\pi^2$, which implies $\Xi_1 > 1$. Nevertheless, the numerical experiments still exhibit the theoretically predicted convergence rates as shown in the following.

In Figure 3, the computed errors of the individual components are plotted over the mesh width h for the choice $\mathbf{F} = \mathbf{F}_{a,1}$ given by Definition 3.23 and the unmodified DEP formulation

$$\mathbf{F}_s(\theta, \Phi) = \alpha_e(|\nabla\Phi|^2\nabla\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot). \tag{6.9}$$

In contrast to $\mathbf{F}_{a,1}$, $\mathbf{F}_s$ does not satisfy the required Assmption 3.12. However, both choices of $\mathbf{F}$ lead to similar results.

One can observe that the velocity error converges with the theoretically predicted rate h^2, whereas temperature converges faster than expected. For large values of h, the potential error converges with similar rate as the temperature error, but is several orders of magnitude smaller. At a certain point, the potential error stagnates at a very low level. A possible reason is the finite accuracy of the underlying nonlinear solver. Perhaps, the algebraic residual of Gauss' law is dominated by the other equations and stagnation is a result of a non-accurate solution of the algebraic equations.

Instationary Case

We now investigate convergence in the instationary case. To this end, different configurations are considered: the temporal factor g is chosen from the set $\{g_{\exp}, g_{1.51}, g_2\}$, the temporal discretization is chosen as BDF1 or BDF2 and the DEP body force takes the forms $\mathbf{F}_{a,1}$ (Definition 3.23) or $\mathbf{F}_s$ given by (6.9).

Further, the initial condition is set to the exact solution at $t = 0$ and for the 2-step method BDF2, the approximate solution for the first time step is set to the exact solution at $t = k$. Thus, errors induced by the initial conditions, given by $E_1(q)$, $E_2(q)$, $E_3(q)$ in Theorem 5.24, can be neglected.

In case of $\mathbf{F}_{a,1}$, the base potential Φ_0 is chosen as the exact potential Φ_* defined in (6.6). Furthermore, $\mathbf{F}_s$ fits into the framework of $\mathbf{F}_{s,K}$ given by Definition 5.34, if the cut-off function is chosen as metric projection, see Definition 5.36, and if K is chosen sufficiently large to ensure $\mathbf{F}_s(\cdot, \Phi_h^n + \Phi_b) = \mathbf{F}_{s,K}(\cdot, \Phi_h^n + \Phi_b)$ for the computed discrete potentials, i.e. $\|\nabla(\Phi_h^n + \Phi_b)\|_{0,\infty} < K$.

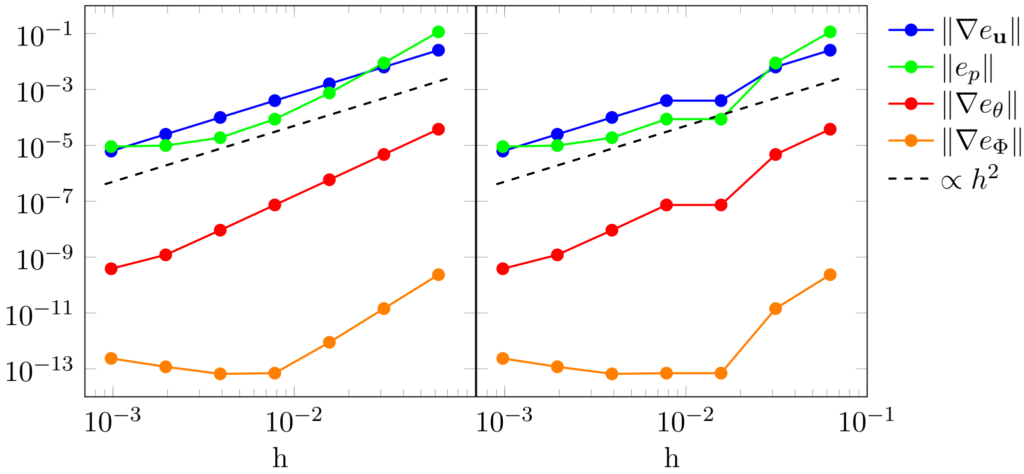


Figure 3: Error for $g = 1$ and $g_\theta = g_\Phi = 0$. Left: $\mathbf{F} = \mathbf{F}_{a,1}$, right: $\mathbf{F} = \mathbf{F}_s$.

On the other hand, K should be small enough such that the small data conditions supposed by Lemma 5.23 and Theorem 5.24 are satisfied. As pointed out in the end of Section 5.3, these conditions are of the form

$$\alpha_e K_*^2 \leq c_\Omega \sqrt{\nu\kappa}, \tag{6.10}$$

with domain dependent constant

$$c_\Omega = c_0^{-1} \min\left\{ \left(2q(1+c_0^2)(1+c_D^2)\right)^{-\frac{1}{2}} |\mathbf{f}|_2^{-1},\ 2q^{-1}|\mathbf{f}|_\infty^{-1} \right\}. \tag{6.11}$$

As c_0, c_D denote the respective constants in Friedrich's inequalities on a square domain, they can be bounded as $c_0 \leq 1$ and $c_D \leq 1$ according to Lemma A.107. Concerning the time stepping contributions, q and $\mathbf{f}$, we have

$$\begin{aligned} &\text{BDF1: } q = 1, |\mathbf{f}|_2 = 1, \quad |\mathbf{f}|_\infty = 1 \\ &\text{BDF2: } q = 2, |\mathbf{f}|_2 = \sqrt{5}, |\mathbf{f}|_\infty = 2. \end{aligned} \tag{6.12}$$

In summary, (6.11) and (6.12) lead to

$$\begin{aligned} &\text{BDF1: } c_\Omega \geq \frac{1}{\sqrt{8}} \\ &\text{BDF2: } c_\Omega \geq \frac{1}{4\sqrt{5}}. \end{aligned} \tag{6.13}$$

Using (6.10), (6.13), $\alpha_e = \frac{\epsilon_0 \epsilon_r \gamma}{2\rho}$ and the fluid parameters, an upper for K is given by

$$K^2 \leq c_\Omega \alpha_e^{-1} \sqrt{\nu\kappa} = c_\Omega \cdot 4.8 \cdot 10^{10} = \mathcal{O}(10^{10}), \tag{6.14}$$

which is several orders of magnitude larger than the computed values $\|\nabla(\Phi_h^n + \Phi_b)\|_{0,\infty} = \mathcal{O}(1)$ and the exact value $\|\nabla\Phi_*(t)\|_{0,\infty} = \mathcal{O}(1)$. Thus, K can be chosen small enough to ensure the validness of the small data condition (6.10) and large enough to guarantee $\mathbf{F}_s(\cdot, \Phi_*) = \mathbf{F}_{s,K}(\cdot, \Phi_*)$ and $\mathbf{F}_s(\cdot, \Phi_h^n + \Phi_b) = \mathbf{F}_{s,K}(\cdot, \Phi_h^n + \Phi_b)$ for all considered n, h and k.

Since two discretization parameters, spatial h and temporal k, can be selected, two different types of test series are performed: Either h is left constant, $h = 2^{-8}$, or h is chosen to be proportional to k, $h = 1.25k$. In either case, k is varied over a certain range. If $h = \text{const}$, then it is chosen sufficiently small to make the temporal contributions to the error dominant.

If $h \propto k$, the h^l terms in the error estimates given by Corollaries 5.29 - 5.32 are transformed to k^l. As pointed out in the stationary case, we obtain $l = 2$ due to the spatial regularity of the exact solution and due to the polynomial order of the underlying finite element space. Thus, the temporal contributions in the derived error estimates, which are of the form k and k^2, dominate the spatial contributions $h^l \propto k^l$ and the observed errors should resemble the temporal convergence order.

The results for $\mathbf{F} = \mathbf{F}_s$, $g = g_{\exp}$ and $h = \text{const}$ are shown in Figure 4. In this case, the exact solution has the full regularity as required by Corollary 5.32. Further, the errors of velocity and temperature approximation, given by $\| (\mathbf{u}_*^n - \mathbf{u}_h^n)_n \|_{l^2(0,N,\mathbf{H}^1)}$ and $\| (\theta_*^n - \theta_h^n)_n \|_{l^2(0,N,H^1)}$, respectively, exhibit first (BDF1) and second order (BDF2) convergence w.r.t. k, in accordance with Corollary 5.30 and Corollary 5.32. The same holds for $h \propto k$, see Figure 5. In each case, the pressure error $\| (p_*^n - p_h^n)_n \|_{l^2(0,N,L^2)}$ converges faster as predicted by the theory.

In fact, it converges with the same order as velocity and temperature and one does not loose a factor $k^{\frac{1}{2}}$. In the proof of Theorem 5.24, this factor $k^{-\frac{1}{2}}$ stems from the term $E_{\mathbf{d}}(h,k)$, (5.69), which basically measures the temporal variation of the discretization error $(\mathbf{d}_{\mathbf{u},h}^n)_n$. The similar quantity $E(h,k)$, which measures the temporal variation of the exact solution $(\mathbf{u}^n)_n$, does not cause the occurrence of $k^{-\frac{1}{2}}$, since the temporal regularity of $\mathbf{u}$ could be exploited. From a theoretical point of view, this temporal regularity could not be derived for $(\mathbf{d}_{\mathbf{u},h}^n)_n$. In the underlying test problem, however, this discretization error might have a similar temporal smoothness as the exact solution, resulting in comparable k-convergence rates for pressure and velocity.

The potential error $\| (\Phi_*^n - \Phi_h^n)_n \|_{l^2(0,N,H^1)}$ shows a behavior that deviates from the theory, as well. After decreasing with similar rate as velocity and temperature for large k, the error eventually stagnates for smaller k (BDF1) or even increases slightly (BDF2) for small k. This might be due to the fact, that the potential error is several orders of magnitude smaller than the errors in the other variables. Therefore, the contribution of Gauss' law to the nonlinear, algebraic residual is significantly smaller than the contributions of the other equations when employing Newton's method. Regarding the finite tolerance of Newton's method, the discrete potential may not be forced sufficiently strongly to further reduce the residual of Gauss' law, and thereby increase its accuracy.

When considering $g = g_2$, one can observe pretty much the same behavior as for $g = g_{\exp}$, see Figure 6 and 7. As before, the exact solution satisfies the regularity requirements supposed by Corollary 5.30 and 5.32 and the velocity and temperature error converge with the rate that is predicted by the theory. The only significant difference can be observed in Figure 6 (right), i.e. when BDF2 is applied with fixed h. In contrast to the previous case, compare Figure 4 (right), one can observe a stagnation of the velocity and temperature error around 10^{-4}. Apparently, at this point the spatial error contribution becomes dominant and h has to be decreased to further reduce the errors, see Figure 7 (right).

Two more scenarios are left to consider. In Figure 8, the error is shown for $g = g_2$, $h \propto k$ and the alternative DEP formulation $\mathbf{F} = \mathbf{F}_{a,1}$. Compared to the analog configuration with $\mathbf{F} = \mathbf{F}_s$, Figure 7, one can observe almost identical values for the respective errors.

Finally, Figure 9 illustrates the errors for the case $\mathbf{F} = \mathbf{F}_s$, BDF2, $h = \text{const}$ and $h \propto k$. In contrast to Figure 6 (right) and 7 (right), the temporal factor g is set to $g_{1.51}$ instead of $g = g_2$. As pointed out in the beginning of this section, the exact solution does not satisfy the strong regularity requirements of Corollary 5.32 any more, whereas the weaker assumptions supposed by Corollary 5.31 do still hold. Therefore, the theory predicts a linear convergence rate w.r.t. k. In practice, one can observe a temporal convergence order of 1.5 for velocity, temperature and pressure and an unchanged rate of 2 for potential. Thus, the regularity condition that is supposed for second order convergence appears not only sufficient, but also necessary. Since the

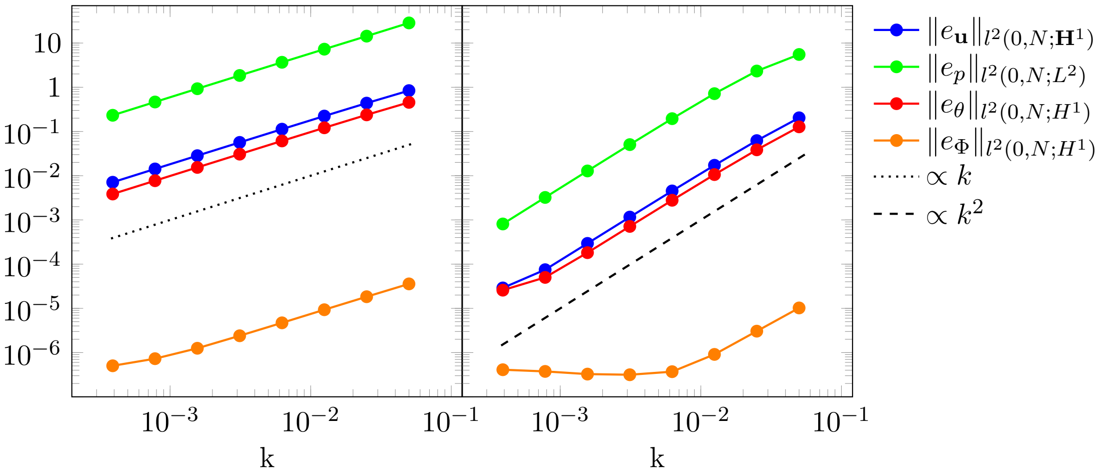


Figure 4: Error for $\mathbf{F} = \mathbf{F}_s$, $g = g_{\exp}$ and $h = \text{const}$. Left: BDF1, right: BDF2.

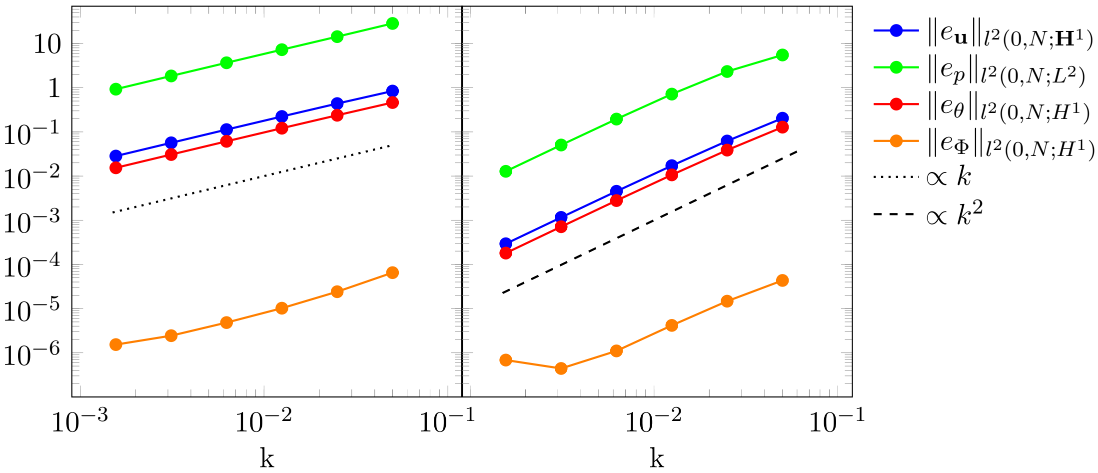


Figure 5: Error for $\mathbf{F} = \mathbf{F}_s$, $g = g_{\exp}$ and $h \propto k$. Left: BDF1, right: BDF2.

condition $\partial_{ttt}\mathbf{u}_* \in L^2(0,T;\mathbf{H}^{-1})$ in Corollary 5.32 is violated, but not $\partial_t\mathbf{u}_* \in L^2(0,T;\mathbf{H}^{l+1})$, we conclude that the overall convergence is reduced due to the purely temporal contributions $E(k)$, but not due to the mixed contributions $E(h,k)$.

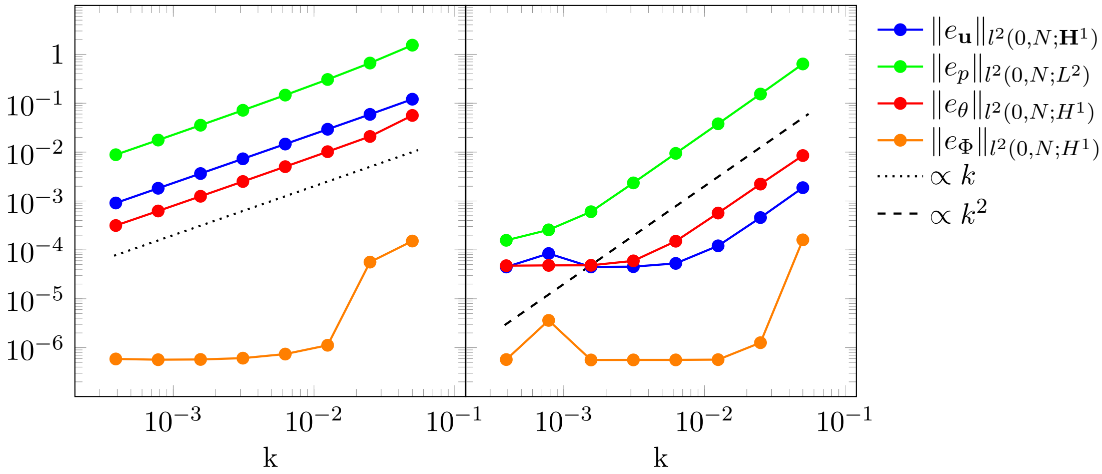


Figure 6: Error for $\mathbf{F} = \mathbf{F}_s$, $g = g_2$ and $h = \text{const}$. Left: BDF1, right: BDF2.

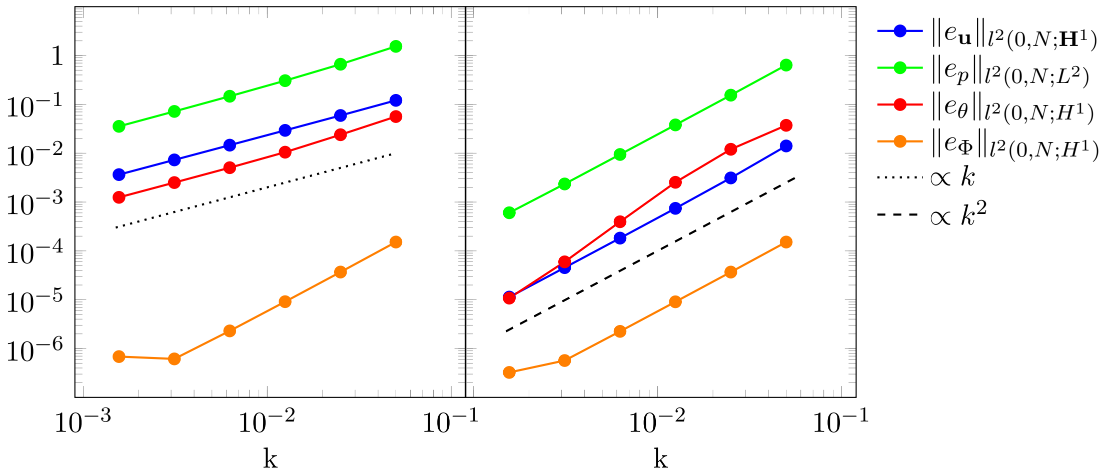


Figure 7: Error for $\mathbf{F} = \mathbf{F}_s$, $g = g_2$ and $h \propto k$. Left: BDF1, right: BDF2.

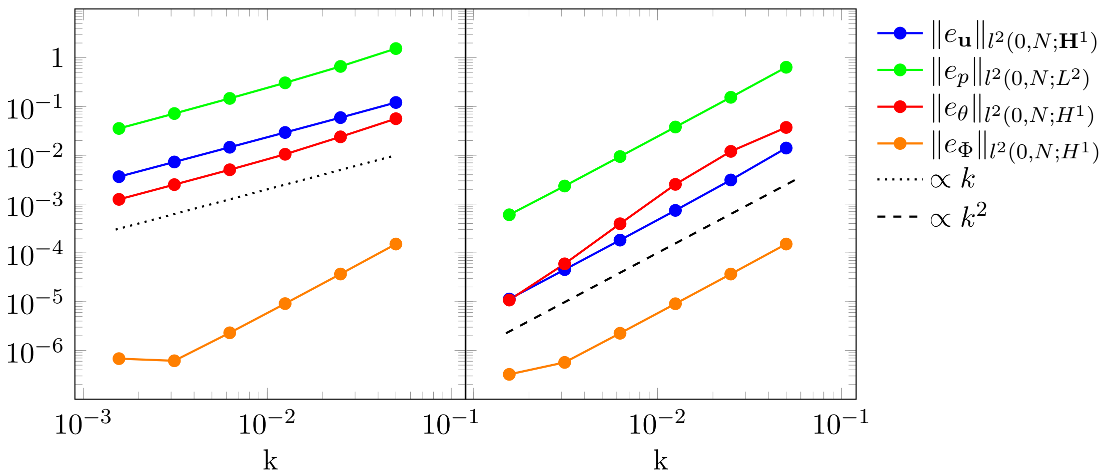


Figure 8: Error for $\mathbf{F} = \mathbf{F}_{a,1}$, $g = g_2$ and $h \propto k$. Left: BDF1, right: BDF2.

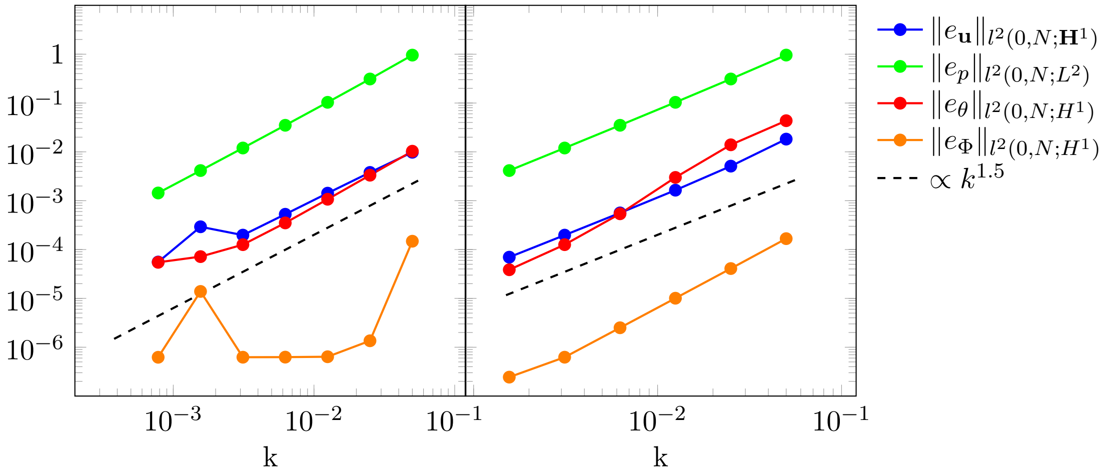


Figure 9: Error for $\mathbf{F} = \mathbf{F}_s$, $g = g_{1.51}$ and BDF2. Left: $h = \text{const}$, right: $h \propto k$.

6.3. Dielectrically Driven Flow in a Vertical Annulus

In this section, simulation results are presented for a scenario that often serves as test case for both numerical simulations and physical experiments of DEP-driven flow, see e.g. [42], [54], [58], [68], [78], [79], [81]. In this scenario, a dielectric fluid is contained in the gap between two concentric cylinders with inner radius r_i, outer radius r_o and height H:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3\colon\; r_i^2 < x^2 + y^2 < r_o^2,\, 0 < z < H\}. \tag{6.15}$$

In most cases considered in the aforementioned literature, the radii are in the range of $5-20$ mm and the height is typically in the range of $3-30$ cm.

We further assume that a temperature and potential difference is applied between the inner and outer wall of the annulus, whereas the top and bottom plate are supposed to be thermally and electrically insulated. In this way, the entire configuration can be considered as cylindrical capacitor where the fluid serves as dielectric.

To be precise, the boundary $\Gamma = \partial\Omega$ is decomposed into the following parts:

$$\begin{aligned} &\Gamma_i := \left\{\mathbf{x} \in \overline{\Omega}\colon\; x_1^2 + x_2^2 = r_i^2\right\}, \Gamma_o := \left\{\mathbf{x} \in \overline{\Omega}\colon\; x_1^2 + x_2^2 = r_o^2\right\} \\ &\Gamma_t := \left\{\mathbf{x} \in \overline{\Omega}\colon\; x_3 = H\right\}, \qquad \Gamma_b := \left\{\mathbf{x} \in \overline{\Omega}\colon\; x_3 = 0\right\}. \end{aligned} \tag{6.16}$$

Dirichlet boundary conditions for temperature and potential are imposed on the inner and outer cylinder, $\Gamma_D = \Gamma_i + \Gamma_o$, and homogeneous Neumann conditions on top and bottom, $\Gamma_N = \Gamma_t + \Gamma_b$. The Dirichlet boundary conditions are of the form

$$\begin{aligned} \mathbf{u} &= 0 \quad \text{on } \Gamma \\ \theta &= \theta_i \;\text{ on } \Gamma_i,\, \theta \;= \theta_o \text{ on } \Gamma_o \\ \Phi &= V_0 \text{ on } \Gamma_i,\, \Phi = 0 \;\text{ on } \Gamma_o. \end{aligned} \tag{6.17}$$

with inner temperature $\theta_i := \theta_r + \frac{1}{2}d\theta$ and outer temperature $\theta_o := \theta_r - \frac{1}{2}d\theta$. Here, θ_r denotes the reference temperature, for which the fluid's parameters, such as viscosity, density, thermal conduction coefficient, etc. are known, see Figure 11.

The presented configuration is relevant from a practical point of view for several reasons. On one hand, it can be regarded as an idealized heat exchanging system, where the fluid serves as medium for transporting heat from an inner, hot device to an outer cooling system. In this case,

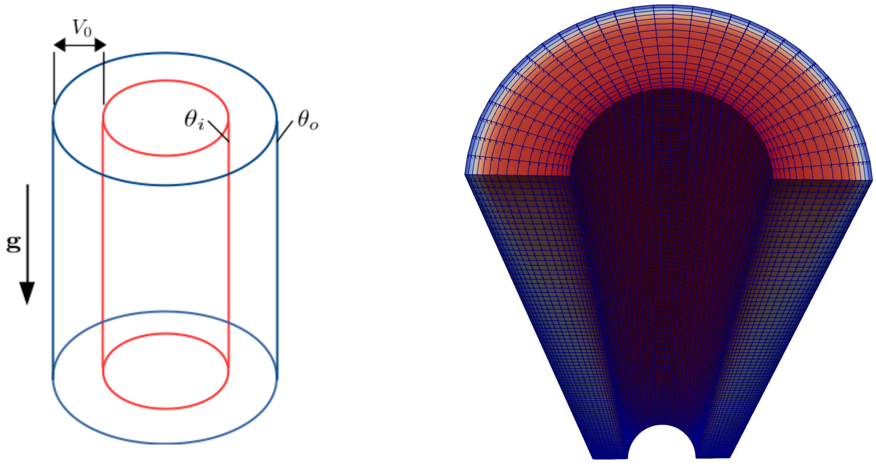


Figure 10: Left: Schematic view on domain and boundary conditions. Right: Illustration of finite element mesh and initial temperature distribution.

ν	5.0	$\cdot 10^{-6}$	$[\mathrm{m^2\ s^{-1}}]$
κ	7.74	$\cdot 10^{-8}$	$[\mathrm{m^2\ s^{-1}}]$
ρ	9.23	$\cdot 10^{2}$	$[\mathrm{kg\ m^{-3}}]$
ϵ_r	2.391	$\cdot 10^{-11}$	$[\mathrm{kg\ m^{-3}}]$
γ	1.065	$\cdot 10^{-3}$	$[\mathrm{K^{-1}}]$
α_g	1.08	$\cdot 10^{-3}$	$[\mathrm{K^{-1}}]$
α_e	1.379	$\cdot 10^{-17}$	$[\mathrm{A\ s\ m^2(\ kg\ K\ V)^{-1}}]$
θ_r	2.9815	$\cdot 10^{2}$	$[\mathrm{K}]$

$d\theta \in$	$[0, 7]$		$[\mathrm{K}]$
$V_0 \in$	$[0, 9]$	$\cdot 10^{3}$	$[\mathrm{V}]$
$\mathbf{g}_x$	0		$[\mathrm{m\ s^{-2}}]$
$\mathbf{g}_y$	0		$[\mathrm{m\ s^{-2}}]$
$\mathbf{g}_z$	-9.81		$[\mathrm{m\ s^{-2}}]$
r_i	5.0	$\cdot 10^{-3}$	$[\mathrm{m}]$
r_o	1.0	$\cdot 10^{-2}$	$[\mathrm{m}]$
H	1.0	$\cdot 10^{-1}$	$[\mathrm{m}]$

Figure 11: Fluid and geometry parameters

aspect ratio	A	$(r_0 - r_i)^{-1} H$
radius ratio	η	$r_i r_o^{-1}$
effective voltage	V_{eff}	$\left(\rho \kappa \nu \epsilon_r^{-1}\right)^{-0.5}$
base electric gravity	$\overline{\mathbf{g}}_E$	$\nabla^2 \overline{\Phi} \nabla \overline{\Phi}\vert_{r=0.5(r_i+r_o)}$
Prandtl number	Pr	$\nu \kappa^{-1}$
thermal Rayleigh number	Ra	$\alpha_g (r_o - r_i)^3 d\theta (\nu\kappa)^{-1} \vert\mathbf{g}\vert$
electric Rayleigh number	L	$\alpha_e (r_o - r_i)^3 d\theta (\nu\kappa)^{-1} \vert\overline{\mathbf{g}}_E\vert$

Figure 12: Characteristic numbers of non-dimensionalized equations

it is interesting to investigate to which extend the DEP force has an impact on the heat transfer between inner and outer cylinder. Ideally, the DEP force would lead to enhanced heat transfer, thus improving cooling efficiency.

On the other hand, this simple geometry offers an accessible way for analyzing DEP-driven flow by experimental means. In these experiments, the boundary conditions for temperature and potential can be controlled and the heat transfer can be determined by measuring the amount of electric power that is needed to keep the temperature at the inner boundary at a constant level, [27]. Moreover, it is possible to visualize the velocity field by means of particle tracing [18] and Schlieren technique [20]. In addition, one can draw some conclusions on the temperature field by using shadowgraph imaging, see e.g. [68]. In summary, experimental data is available for this configuration, making it possible to validate numerical simulations.

Concerning numerical simulations, the cylindrical geometry can be exploited when discretizing system (2.28) to obtain efficient solvers. Most reported numerical methods for solving the TEHD equations in a cylindrical domain impose periodic boundary conditions in the axial direction, see e.g. [42], [70], [78], [79]. By doing so, spectral methods can be applied and the Fast Fourier Transform can be used for solving the resulting discretized equations. An exception is presented in [53], where the same boundary conditions as in our case are supposed and a (not further specified) finite element solver is used.

In the literature, the TEHD system (2.28) is typically described by a set of non-dimensional numbers, see Table 12. Of particular interest are the thermal and electric Rayleigh numbers, Ra and L, since they measure the strength of the buoyancy and DEP force, respectively. The definition of L involves the electric gravity $\overline{\mathbf{g}}_E$ which is defined for some base potential $\overline{\Phi}$. As base potential, one typically uses the solution of the TEHD system (2.28) in a cylinder of infinite length and without natural gravity, $\mathbf{g} = 0$. A closed analytical expression in cylindrical coordinates is available in this case according to [81]:

$$
\begin{aligned}
\overline{\mathbf{u}}(\varphi, r, z) &= 0 \\
\overline{p}(\varphi, r, z) &= \overline{p}(r) \ = \int_{r_i}^{r} h(s)\, ds \\
\overline{\theta}(\varphi, r, z) &= \overline{\theta}(r) \ = \theta_o + d\theta \left(\ln(\eta)\right)^{-1} \left(\ln(r) - \ln(r_o)\right) \\
\overline{\Phi}(\varphi, r, z) &= \overline{\Phi}(r) = c + b \, \ln\left(1 - \gamma(\overline{\theta}(\varphi, r, z) - \theta_r)\right),
\end{aligned}
\tag{6.18}
$$

with

$$
\begin{aligned}
h(s) &= -(\overline{\theta}(s) - \theta_r)\partial^2_{rr}\overline{\Phi}(s)\partial_r\overline{\Phi}(s) \\
b &= V_0 \left(\ln \left(\frac{2 - \gamma d\theta}{2 + \gamma d\theta} \right) \right)^{-1} \\
c &= -b \ln \left(1 + \frac{1}{2}\gamma d\theta \right).
\end{aligned}
$$

Later on, we make use of $\overline{\Phi}$ to define the linearized DEP models $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$, $\mathbf{F}_{a,1}$.

The previously described configuration is usually investigated either under standard laboratory conditions, i.e. $\mathbf{g}$ corresponds to Earth's gravity [68], in the following denoted by $\mathbf{g}_n := -9.81 \cdot \mathbf{e}_z$, or under zero gravity conditions, i.e. $\mathbf{g} = 0$. The latter case can be experimentally implemented by means of a drop tower, a parabolic flight [18] or an orbital laboratory, e.g. on the ISS [58], in ascending order w.r.t. duration of the micro gravity state.

In this subsection, we give higher priority to the standard gravity case and most presented numerical results are obtained for a fixed temperature and potential difference, since a wide list of parametric studies can be found in the literature, see e.g. [68], [58] and the references therein. Instead, we want to put emphasis on the following aspects: temporal evolution of vortex formation, comparison of different time discretization schemes, comparison of different DEP models, sensitivity of the solution w.r.t. the initial condition and non-uniqueness of stationary solutions.

But before turning to the numerical results, we reconsider the small data condition that is imposed by the convergence result Theorem 5.24. As before, this condition is of the form

$$\alpha_e K_*^2 \le c_\Omega \sqrt{\nu\kappa}, \tag{6.19}$$

with domain-dependent constant

$$c_\Omega = c_0^{-1} \min\left\{ \left(2q(1+c_0^2)(1+c_D^2)\right)^{-\frac{1}{2}} |\mathbf{f}|_2^{-1},\ 2q^{-1}|\mathbf{f}|_\infty^{-1} \right\}. \tag{6.20}$$

As in the previous section, it possible to determine the domain-dependent constant due to the relatively simple geometry. According to Lemma A.107, there holds $c_0, c_D \le c_*$ with

$$c_* = \left(\frac{1}{2} \left(\ln(r_o) - \ln(r_i)\right) \left(r_o^2 - r_i^2\right) \right)^{\frac{1}{2}}. \tag{6.21}$$

Inserting the problem data, i.e. r_i, r_o and the time stepping parameters according to (6.12), into (6.20) and (6.21), leads to

$$\begin{aligned} &\text{BDF1: } c_\Omega \ge 138.69 \Rightarrow \frac{c_\Omega}{\alpha_e}\sqrt{\nu\kappa} \ge 6.255 \cdot 10^{12} \\ &\text{BDF2: } c_\Omega \ge 43.86 \quad \Rightarrow \frac{c_\Omega}{\alpha_e}\sqrt{\nu\kappa} \ge 1.978 \cdot 10^{12}. \end{aligned} \tag{6.22}$$

Thus, (6.19) is satisfied for

$$\begin{aligned} &\text{BDF1: } K_* \le 2.5 \cdot 10^6 \\ &\text{BDF2: } K_* \le 1.41 \cdot 10^6. \end{aligned} \tag{6.23}$$

Regarding the zero gravity solution, we obtain for the base potential $\nabla\overline{\Phi} = \partial_r \overline{\Phi}\, \mathbf{e}_r$ with

$$\begin{aligned} \partial_r \overline{\theta}(\varphi, r, z) &= \frac{d\theta}{\ln(\eta) r} \\ \partial_r \overline{\Phi}(\varphi, r, z) &= -b\left(1 - \gamma(\overline{\theta}(\varphi, r, z) - \theta_r)\right)^{-1} \gamma \partial_r \overline{\theta}(\varphi, r, z). \end{aligned} \tag{6.24}$$

Therefore,

$$|\partial_r \overline{\Phi}(\varphi, r, z)| = \frac{\gamma |b| d\theta}{|1 - \gamma(\overline{\theta} - \theta_r)||\ln(\eta)| r} \le 2V_0 d\theta \gamma \left| r_i \ln(\eta)(2 - \gamma d\theta) \ln\left(\frac{2 - \gamma d\theta}{2 + \gamma d\theta}\right) \right|^{-1}. \tag{6.25}$$

In the following, we only consider $d\theta \le 7$, which implies

$$|\partial_r \overline{\Phi}(\varphi, r, z)| \le 290 \cdot V_0.$$

In summary, $\|\nabla\overline{\Phi}\|_{0,\infty} \le K_*$ with K_* satisfying the small data condition (6.23) can be guaranteed if the potential difference between inner and outer cylinder wall V_0 satisfies

$$\begin{aligned} &\text{BDF1: } V_0 \le 8.62 \cdot 10^3 \text{ V} \\ &\text{BDF2: } V_0 \le 4.86 \cdot 10^3 \text{ V}. \end{aligned} \tag{6.26}$$

In one of the upcoming sections, it is numerically shown that the relative difference between $\Phi_h^n + \Phi_b$ and $\overline{\Phi}$ is only of order $\mathcal{O}(10^{-3})$. Further, we could observe that

$$\|\nabla(\Phi_h^n + \Phi_b)\|_{0,\infty} < 2 \cdot 10^6 \tag{6.27}$$

holds for $V_0 \leq 7 \cdot 10^3$ V. According to (6.23), (6.27) and the remarks at the end of Section 5.3, one can thus state that, at least for BDF1, the error estimates derived in Section 5.2 are applicable for the choice $\mathbf{F} = \mathbf{F}_s$ for $V_0 \leq 7 \cdot 10^3$ V. Moreover, a comparison between BDF1 and BDF2 shows that both schemes yield very similar results. For this reason, all conducted simulations are done by using BDF2 with time step size $k = 0.05$ and $\mathbf{F} = \mathbf{F}_s$, unless stated otherwise.

Concerning the spatial discretization, all simulations are performed on the same mesh consisting of 96,000 cells, see Figure 10; thus leading to approximately $4 \cdot 10^6$ spatial degrees of freedom. Again, if not stated otherwise, the initial condition is set to the stationary solution of the standard Boussinesq equations for natural convection, i.e. the stationary version of system (2.28) with $\alpha_e = 0$. Further, most simulations are conducted for $d\theta = 7$ K and $V_0 = 7 \cdot 10^3$ V which corresponds to a thermal Rayleigh number $Ra = 23946$, an electric Rayleigh number $L = 15220$ and a Prandtl number $Pr = 61$.

Temporal Evolution under Earth's Gravity

The temporal evolution of the fluid's velocity and temperature when both natural convection and DEP force are active is now considered. The occurrence of stationary, axially oriented vortex structures is reported by experiments [27], linear stability analysis [54] and simulations [42], if the electric Rayleigh number L is in a certain range that depends on η, A and Ra. Under this threshold, the fluid remains in a unicellular flow state that is close to the natural convection case. According to [27] and [70], the smaller the radius ratio η. i.e. the larger the curvature of the annulus, the lower the critical L becomes. For higher L, the linear stability analysis in [53] predicts the onset of stationary, helical modes with non-zero axial wavenumber. This parameter regime, however, is not covered by our simulations.

Our simulation results are illustrated by Figures 13, 14 and 15. In Figure 13, the temperature isosurface $\{(x, y, z) \in \Omega\colon\ \theta(x, y, z) = \theta_r\}$ w.r.t. the reference temperature is shown at times $t \in \{50, 60, 70, 80, 90, 100\}$ with colormap indicating the strength of axial vorticity. One can observe how the initially axisymmetric temperature distribution transforms to a state where a certain number of axially aligned plumes is present in the middle part of the annulus. Apparently, the temperature distribution does not deviate too much from the initial state in regions close to the top and bottom. This due to the fluid motion, which still keeps its initial characteristics of a unicellular convection field in those regions. This effect can be seen more clearly in Figure 14, where velocity is plotted on a vertical and horizontal cut plane. Here, color indicates the axial velocity component, i.e. red means upward movement and blue downward. Regarding the vertical plane, one can observe how the initial, symmetric convection cell is perturbed in the middle part of the cylinder, while it roughly keeps its shape close to top and bottom. Moreover, the velocity on the horizontal plane clearly illustrates an alternating behavior w.r.t the azimuthal direction. In summary, the fluid's velocity can be considered as superposition of a main, axially directed component, coming from natural convection, and rotational motion in the horizontal plane.

Eventually, the instationary solution appears to converge towards a stationary state. Figure 16 illustrates different aspects of the solution at the last computed step $t = 200$ s. Here, the streamlines (with color encoding axial velocity) on the left-hand side indicate a helicoidal motion. The visualization of velocity, body force, temperature and axial vorticity on the horizontal cut

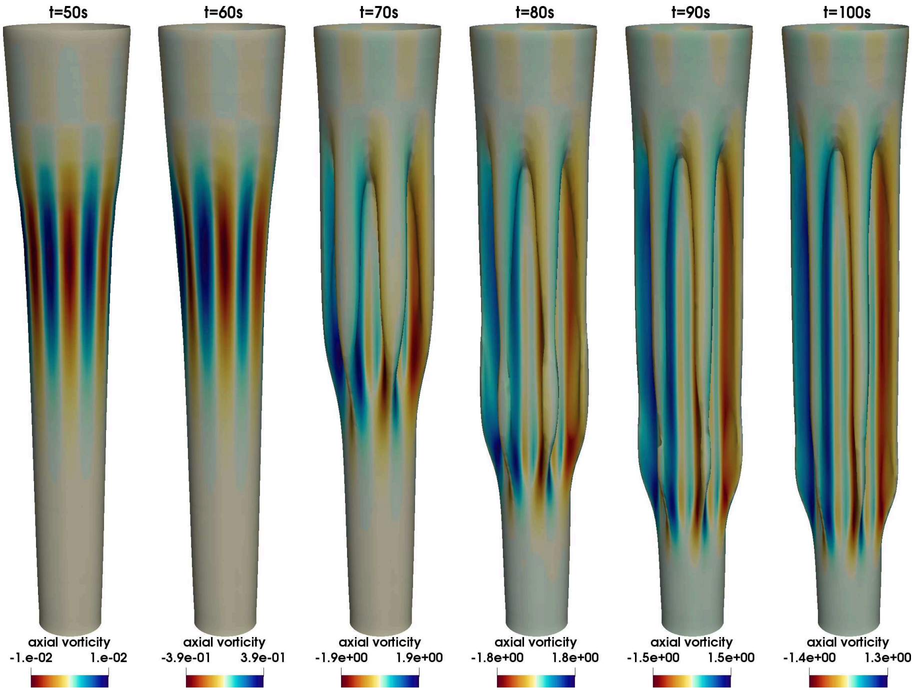


Figure 13: Temperature isosurface $\{\theta = \theta_r\}$ for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$ at multiple time instances.

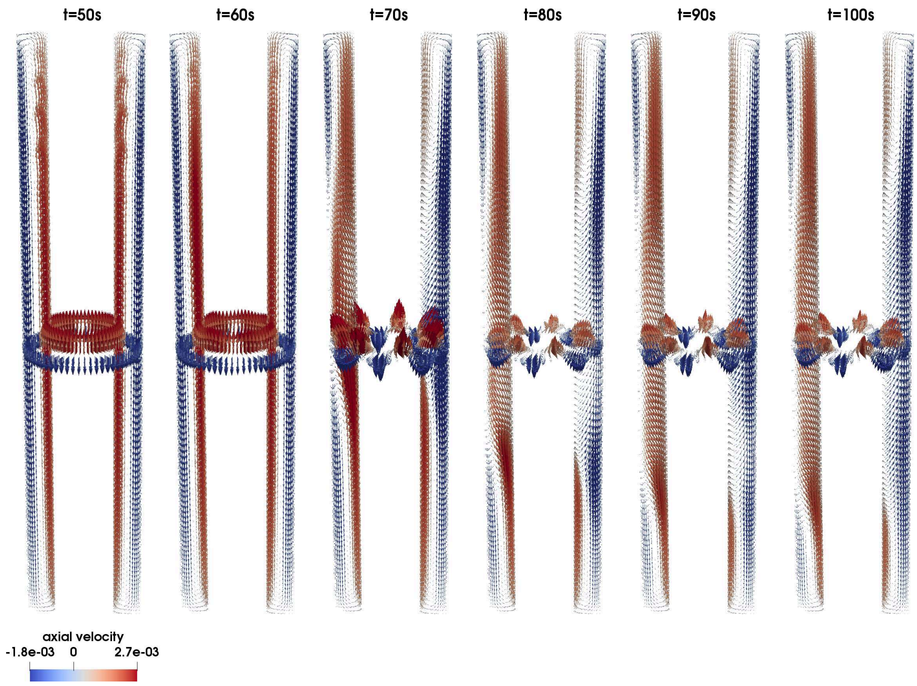


Figure 14: Velocity field for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$ at multiple time instances. The color map indicates axial velocity with red / blue denoting rising / falling fluid.

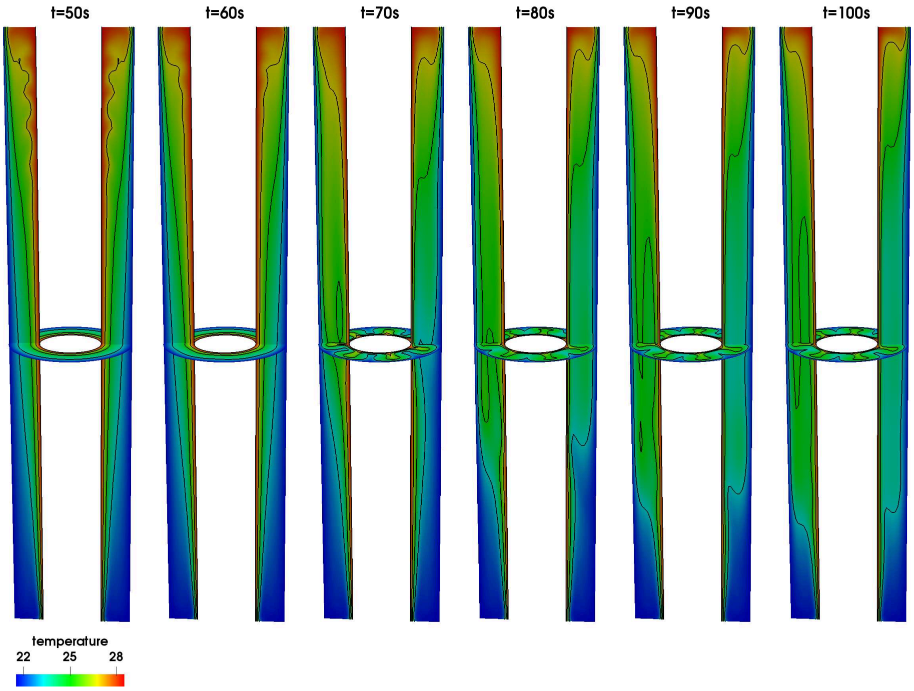


Figure 15: Temperature field for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$ at multiple time instances.

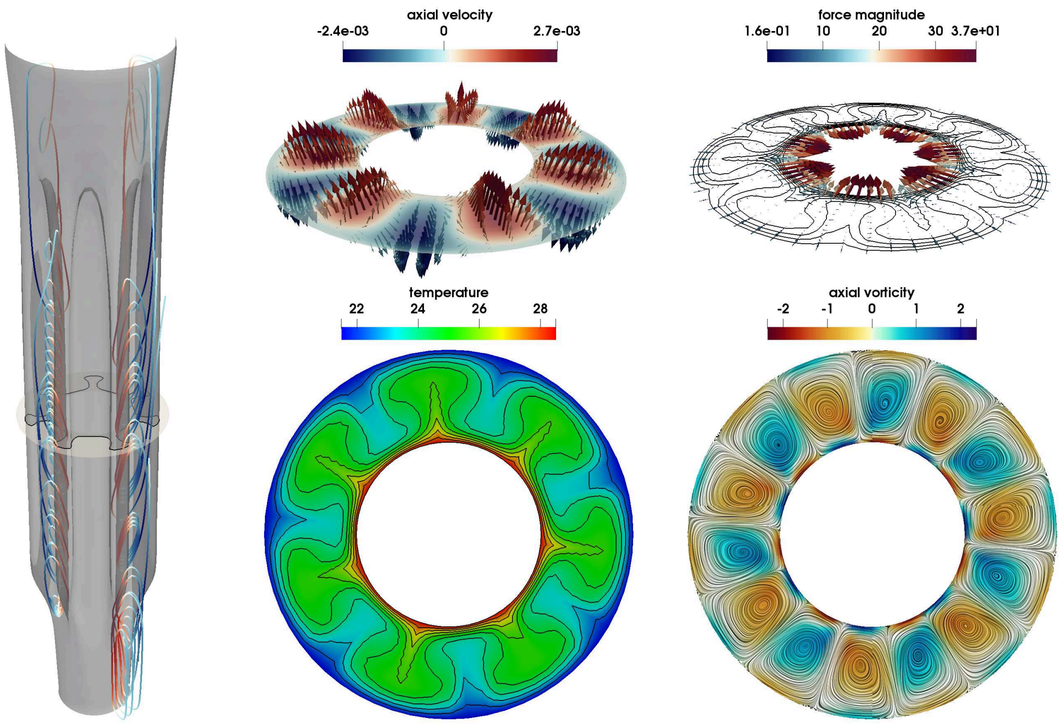


Figure 16: Simulation state for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$ at $t = 200$ s. Left: Temperature isosurface $\{\theta = \theta_r\}$ and streamlines with color indicating axial velocity. Middle upper: Axial velocity at $\{z = 0.5H\}$. Right upper: Body force $\mathbf{F}_{DEP} + \mathbf{F}_{buo}$. Middle lower: Temperature distribution. Right lower: Axial vorticity.

plane at $z = 0.5H$ clearly depicts a state that is periodic in the azimuthal direction with wave number $K = 7$. On one hand, the temperature plumes arise due to the counter-rotating, axially aligned convection rolls. On the other hand, this temperature distribution with strong gradient in radial and azimuthal direction further enhances the formation of axial vorticity, thus leading to a self-amplifying process with exponential growth of the axial vorticity component, see Figure 17. This can be made more clear by considering the vorticity equation: Applying the curl operator $\nabla\times$ to the momentum equation of (2.28) yields for the vorticity $\mathbf{w} = \nabla \times \mathbf{u}$:

$$\partial_t \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{w} - \mathbf{w} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{w} = \alpha_e \nabla \left(|\nabla \Phi|^2 \right) \times \nabla \theta - \alpha_g \nabla \theta \times \mathbf{g}. \tag{6.28}$$

The right-hand side terms in (6.28) denote the driving forces in the generation of vorticity. Supposing $\Phi \approx \overline{\Phi} = \overline{\Phi}(r)$, the DEP contribution to vorticity becomes

$$\alpha_e \nabla \left(|\nabla \Phi|^2 \right) \times \nabla \theta \approx \alpha_e \nabla \left(|\nabla \overline{\Phi}|^2 \right) \times \nabla \theta = \alpha_e \partial_r \left(\partial_r \overline{\Phi}(r) \right)^2 \begin{pmatrix} \partial_z \theta \\ 0 \\ -r^{-1} \partial_\varphi \theta \end{pmatrix} \tag{6.29}$$

and regarding buoyancy force, one obtains

$$-\alpha_g \nabla \theta \times \mathbf{g} = -\alpha_g \mathbf{g}_z \begin{pmatrix} \partial_r \theta \\ -r^{-1} \partial_\varphi \theta \\ 0 \end{pmatrix}. \tag{6.30}$$

Thus, azimuthal variations of temperature, $\partial_\varphi \theta$, enhance axial vorticity through DEP force and radial vorticity through buoyancy force, which in turn contribute to these azimuthal variations through the convection term $\mathbf{u} \cdot \nabla \theta$ in the heat equation.

Actually, one might expect a similar behavior for azimuthal vorticity, which is influenced by axial temperature variations according to (6.29). And indeed, one can observe the occurrence of slight convection rolls which are azimuthally aligned, see the upper third of Figure 14 and 15 at $t = 50$ s. These rolls origin at the lower part of the annulus and then travel upwards due to buoyancy. However, they quickly disappear with the onset of the columnar structures. This is in contrast to the zero-gravity scenario, where those rolls are much more present.

Multiplying (6.28) by $\mathbf{w}$ and integrating over Ω further yields an expression for $\frac{d}{dt}\|\nabla \times \mathbf{u}\|^2$ of the form

$$\frac{d}{dt}\|\mathbf{w}\|^2 = \mathbf{W}^{conv} + \mathbf{W}^{diss} + \mathbf{W}^{dep} + \mathbf{W}^{buo}, \tag{6.31}$$

with contributions by the convection terms, $\mathbf{W}^{conv}$, dissipation of vorticity, $\mathbf{W}^{diss}$, and source terms

$$\begin{aligned} \mathbf{W}^{dep} &= 2\alpha_e \left(\left(\nabla |\nabla \Phi|^2 \times \nabla \theta \right), \mathbf{w} \right) \\ \mathbf{W}^{buo} &= 2\alpha_g \left(\left(\nabla \theta \times \mathbf{g} \right), \mathbf{w} \right). \end{aligned} \tag{6.32}$$

The individual components of $\mathbf{W}^{dep}$ and $\mathbf{W}^{buo}$ are plotted in Figure 17. Apparently, the exponential growth of $\|(\nabla \times \mathbf{u})_z\|$ comes with an exponential growth of $\mathbf{W}_z^{dep}$ which is of similar order of magnitude. In contrast, the azimuthal and radial component of $\mathbf{W}^{dep}$ are several orders of magnitude smaller than the respective vorticity norm. It seems that for these vorticity contributions, buoyancy force is the main driving mechanism in accordance with (6.29) and (6.30).

To conclude the observation of vorticity, consider Figure 18 where all vorticity components are plotted over time, now on a linear vertical axis. The period when axial and radial vorticity become of comparable order of magnitude as the initially dominating azimuthal vorticity is

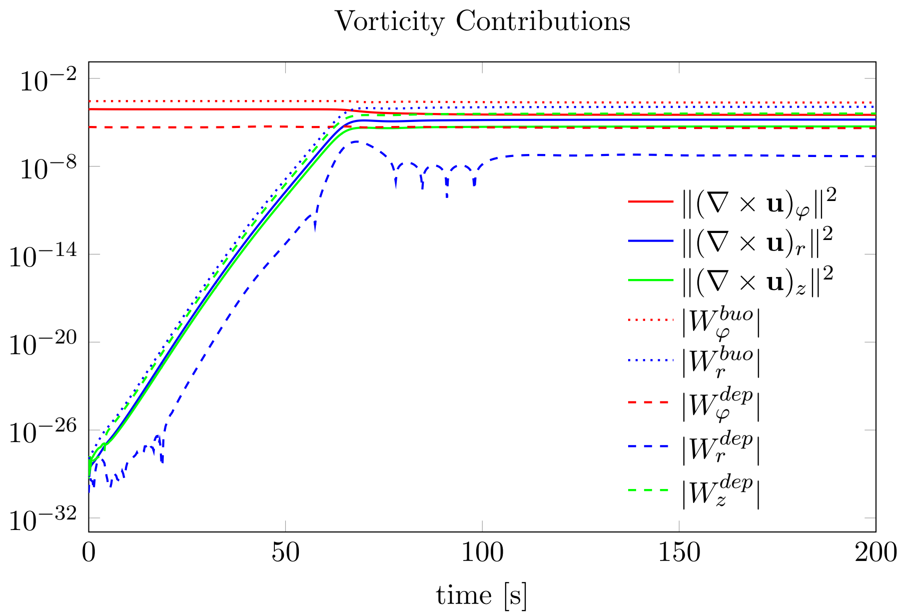


Figure 17: Temporal evolution of vorticity components and vorticity source terms for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$.

around 60 s − 70 s. During this period, also the first axially aligned, columnar structures become visible in Figure 13.

Eventually, we consider the heat transfer at the inner cylinder, which is given by

$$\mathrm{ht}(\theta, t) := \int_{\Gamma_i} \nabla\theta(t) \cdot \mathbf{n} \, d\sigma. \tag{6.33}$$

The corresponding Nusselt numbers are denoted by

$$\mathrm{Nu}^{cond}(t) := \mathrm{ht}(\theta, t)\, \mathrm{ht}(\overline{\theta}, t)^{-1} \tag{6.34}$$

$$\mathrm{Nu}^{conv}(t) := \mathrm{ht}(\theta, t)\, \mathrm{ht}(\theta(0), t)^{-1}. \tag{6.35}$$

They describe how strong the heat transfer is enhanced, compared to the initial state, Nu^{conv} or to the state of pure conduction, Nu^{cond}. Figure 18 clearly shows that the heat transfer is enhanced by the onset of the helicoidal fluid motion at time $t = 62$ s.

As final remark, we note that both kinetic energy and dissipation slightly decrease due the helicoidal motion, see Figure 19.

Temporal Evolution Without Gravity

In absence of Earth's gravity, the flow behavior significantly differs. The occurrence of axisymmetric, azimuthally aligned vortices is reported in [18] for numerical simulations and parabolic flight experiments. In that work, a time period of ≈ 20 s is considered, which corresponds to the duration of one micro-gravity phase during a parabolic flight. For an experimental cell of the same geometry as we consider and $d\theta = 10$ K, the transition from a single convective flow cell to multiple, small and azimuthally aligned convective cells is observed for $V_0 \geq 5300$ V. Moreover, it is noted that this critical voltage is significantly higher for cells of larger radii ratio η. In the linear stability analysis of [81], three different types of disturbances are considered, which develop into stationary convection rolls for sufficiently large L: azimuthally aligned (zero

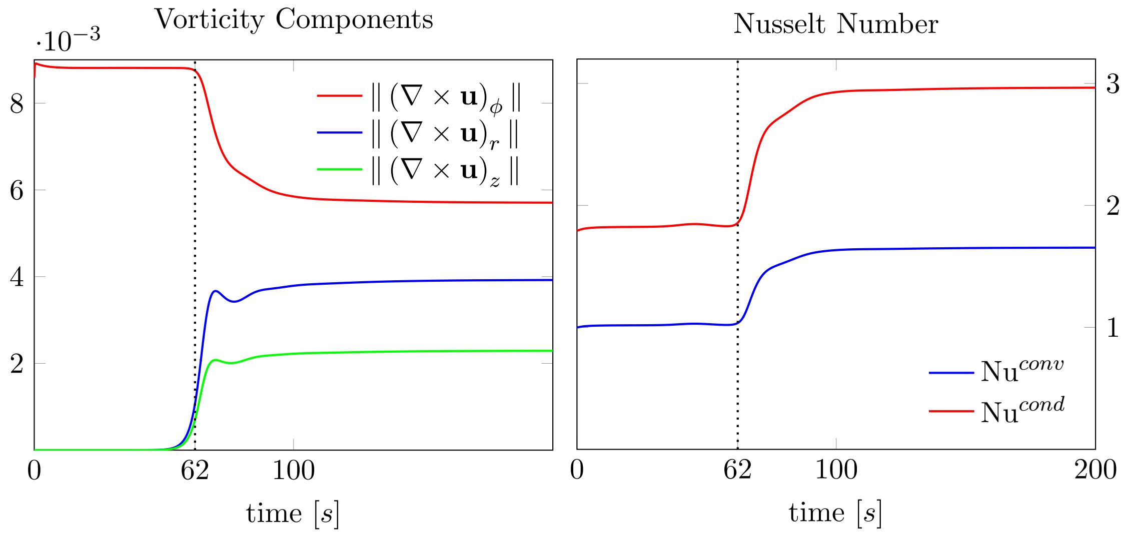


Figure 18: Temporal evolution of certain characteristics for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Left: L^2-norm of vorticity components. Right: Nusselt number.

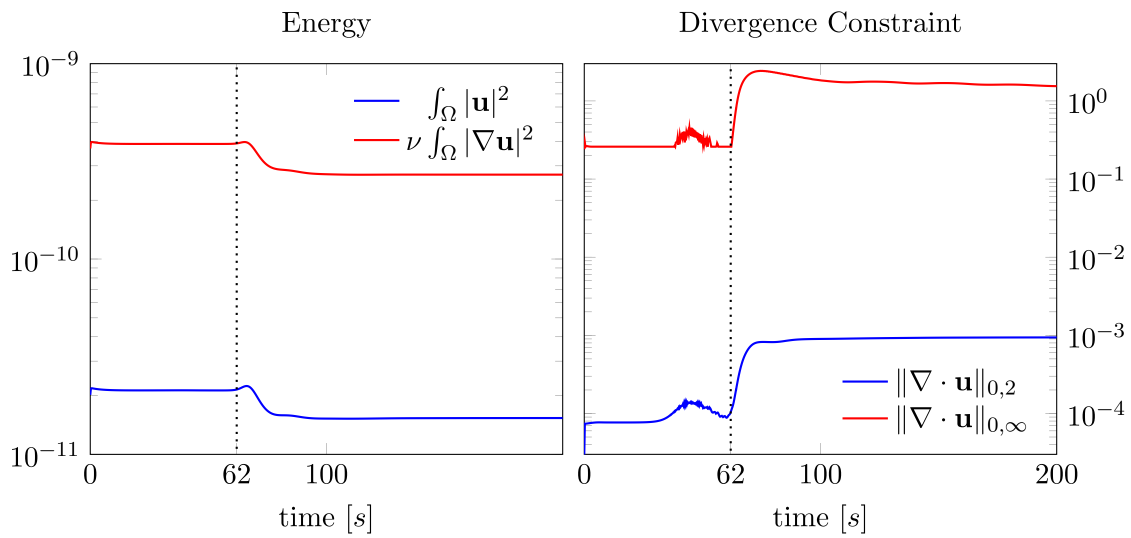


Figure 19: Temporal evolution of certain characteristics for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Left: Kinetic energy and dissipation. Right: Violation of incompressibility constraint.

azimuthal wavenumber), axially aligned (zero axial wavenumber) and helical ones (non-zero azimuthal and axial wavenumber). The later ones appear to yield the lowest critical L and "the predicted critical mode is made of stationary helices".

In our simulations, the initial condition is set to $\mathbf{u} = 0$, $p = 0$, $\theta = \overline{\theta}$ and $\Phi = 0$, i.e. the fluid is at rest and the temperature distribution is determined by conduction only. This corresponds to the standard, stationary Boussinesq equations with $\mathbf{g} = 0$.

Figure 22 and 23 depict the isosurface for $\theta = \theta_r$ with additional streamlines whose color indicate axial velocity, and the temperature distribution on an axial cut plane. After switching on the potential difference at $t = 0$ s, it takes around 25 s, until the first deviations from the initial state can be observed. Azimuthally aligned convection rolls are formating, in accordance with the only vorticity source term $\mathbf{W}^{dep}$, which has non-zero azimuthal and axial component. The first rolls have their origins at the outer regions of the annulus (without gravity, there is no "top" and "bottom" any more), then additional ones arise and extend towards the middle of the cylinder. This effect can be explained by (6.29): the azimuthal component of $\mathbf{W}^{dep}$ is proportional to $\partial_z \theta$. Thus, as soon as an initial temperature variation along the axial direction is present, azimuthal vorticity and thereby axial variation of velocity is enhanced, which in turn enhances axial variation of temperature due to the convection term $\mathbf{u} \cdot \nabla\theta$. As a result, exponential growth of $\|(\nabla \times \mathbf{u})_\varphi\|$ can be observed, see Figure 20. Moreover, this vorticity component is dominating the initial phase of the simulation. However, the φ-invariance breaks down around $t = 40$ s, starting in the middle of the cylinder, where azimuthal convection rolls from both ends come together, and at the very outer regions. Eventually, a rather irregular flow structure establishes and due to the non-zero axial component of $\mathbf{W}^{dep}$, axial vorticity takes comparable values and even higher values as the azimuthal one for $t \geq 360$ s.

Over the entire period, radial vorticity only plays a minor role, due to the absence of a corresponding source term; a fact that is also observed for the numerical experiments in [78]. Moreover, both kinetic energy and dissipation stay on a rather constant level after the first convection rolls have evolved, see Figure 21.

The last computed state is shown in Figure 24, which is not a stationary state yet. Thus, by comparing this simulation with the previous one, one can conclude that the presence of gravity has a stabilizing effect on the fluid's motion. Compared to the standard gravity case, the Nusselt number is of similar magnitude ≈ 3, see Figure 20. Thus, DEP-driven flow significantly enhances heat transfer, also in absence of gravity. Note that the heat transfer reaches its maximum at $t \approx 45$ s shortly after the onset of the azimuthally aligned vortices. This peak is followed by slow decay that correlates with the decay of azimuthal vorticity. As similar effect is observed for the simulations in [53], where a much shorter time period and different initial conditions are supposed in order to mimic the conditions of one micro-gravity parabola period.

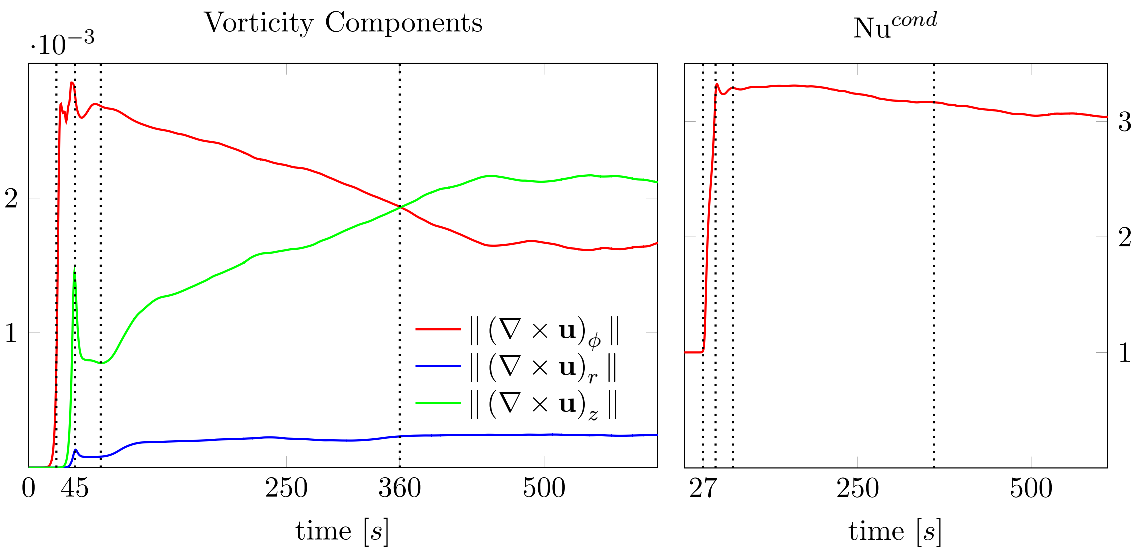


Figure 20: Temporal evolution of L^2-norm of vorticity components for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = 0$. The dotted vertical lines at $t \in \{27, 45, 70, 360\}$ mark the onset of azimuthally aligned vortices, peak of azimuthal vorticity, local minimum of axial vorticity and equilibrium of azimuthal and axial vorticity.

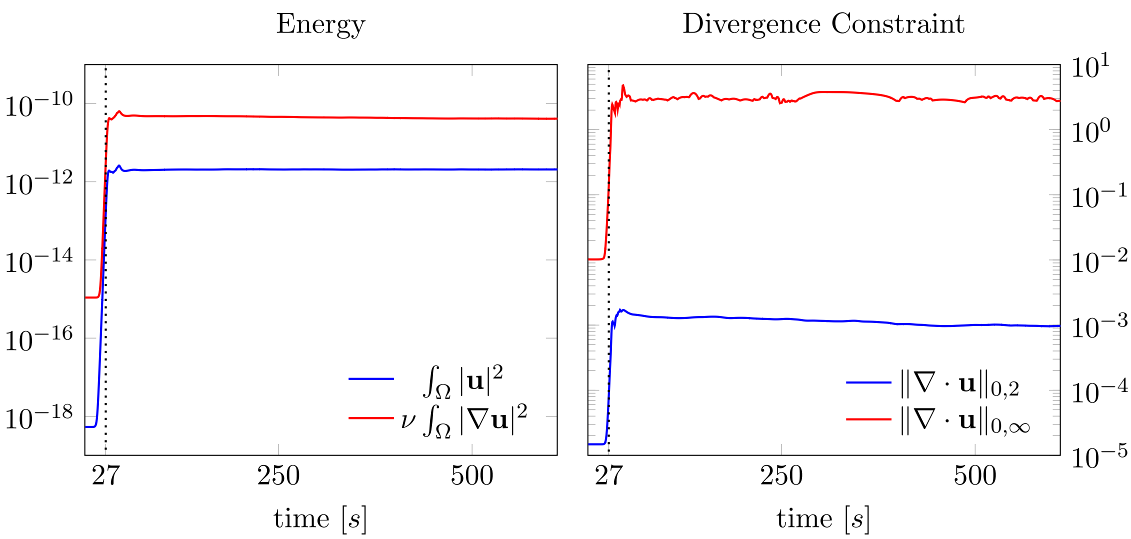


Figure 21: Temporal evolution of certain characteristics for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = 0$. Left: Kinetic energy and dissipation. Right: Violation of incompressibility constraint.

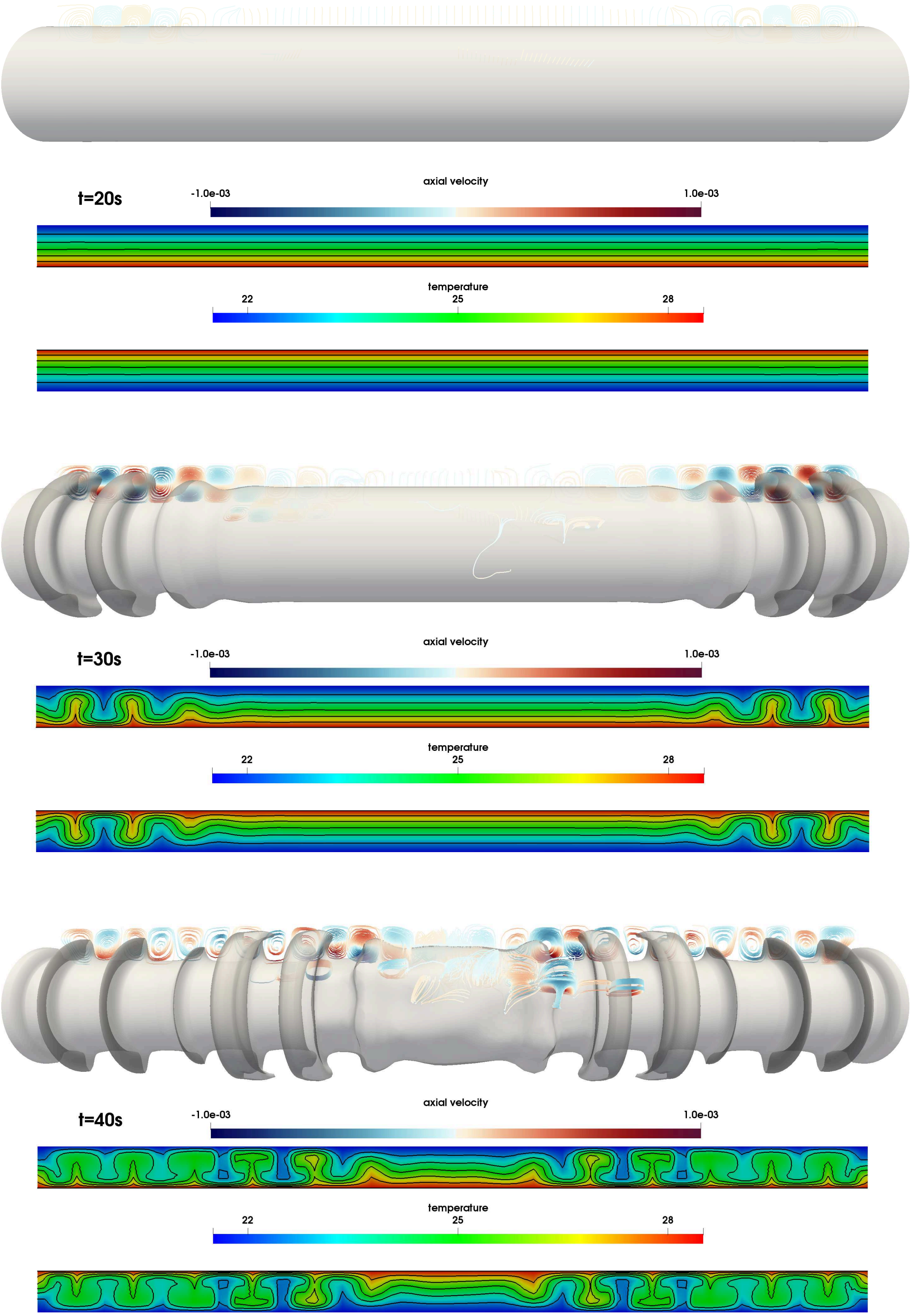


Figure 22: Simulation visualization for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = 0$ at certain time instances. Upper: Temperature isosurface $\{\theta = \theta_r\}$ and stream lines with color indicating axial velocity. Lower: Temperature field on axial cut plane.

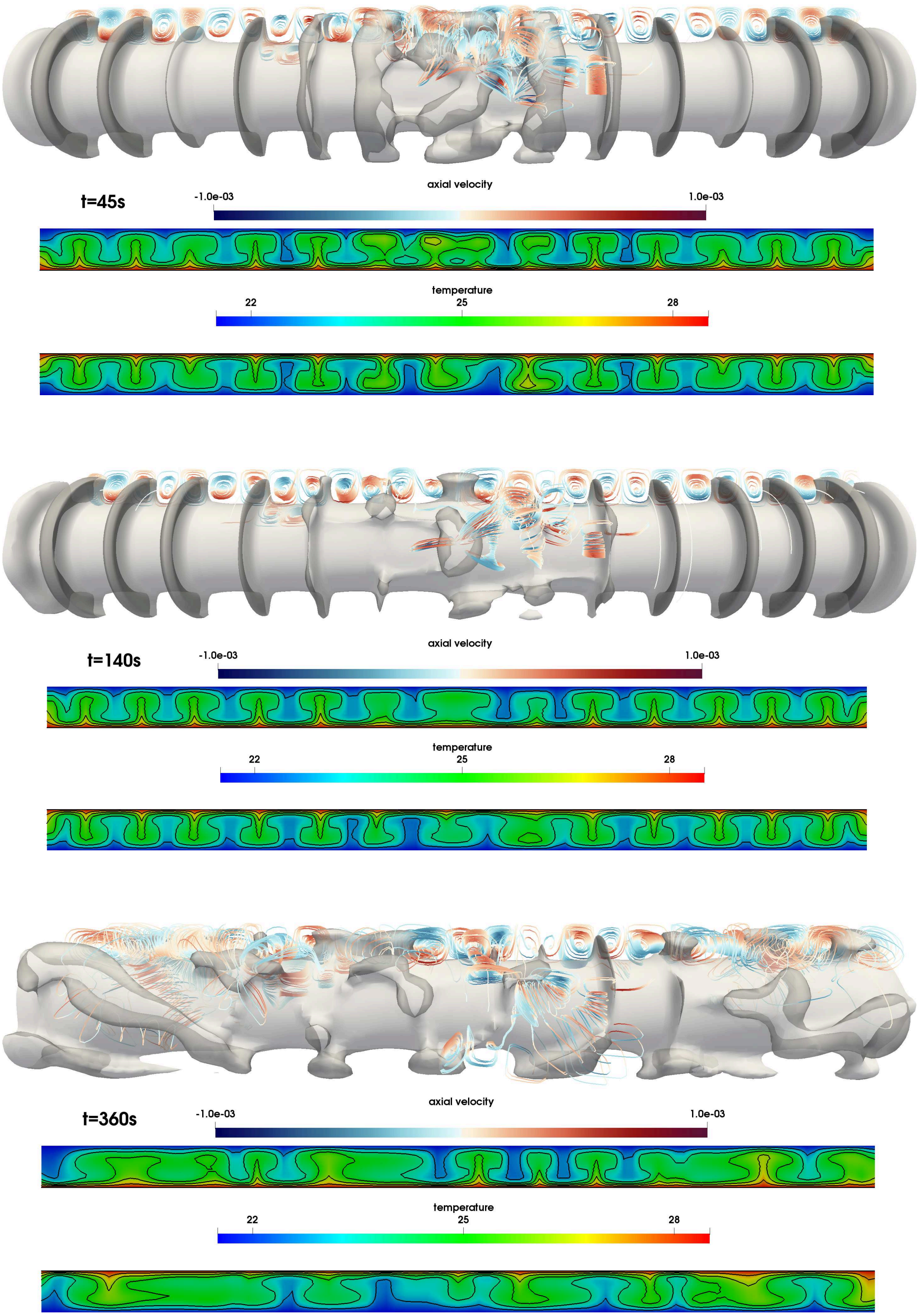


Figure 23: Simulation visualization for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = 0$ at certain time instances. Upper: Temperature isosurface $\{\theta = \theta_r\}$ and stream lines with color indicating axial velocity. Lower: Temperature field on axial cut plane.

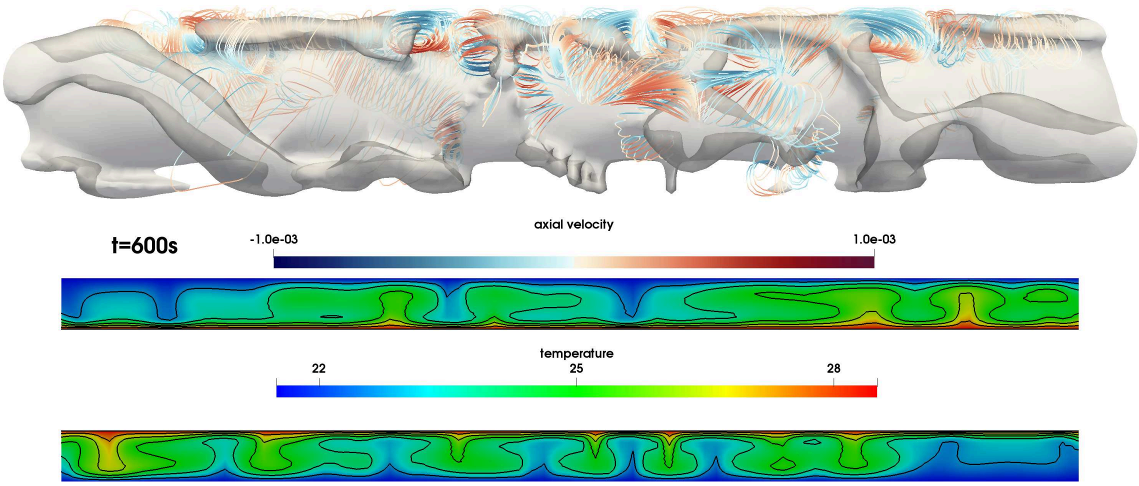


Figure 24: Simulation visualization for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = 0$ at final time $t = 600$ s. Upper: Temperature isosurface $\{\theta = \theta_r\}$ and stream lines with color indicating axial velocity. Lower: Temperature field on axial cut plane.

Comparison for Different Rayleigh Numbers

We now present results for a parametric study, where $d\theta = 7$ K is kept fixed and the voltage difference is varied over an interval $V_0 = 0 - 9000$ V. In addition, annuli of height $H \in \{3 \text{ cm}, 10 \text{ cm}, 30 \text{ cm}\}$ are considered, yielding aspect ratios $A \in \{6, 20, 60\}$.

Figure 25 shows the resulting Nusselt numbers and vorticity components at time $t = 200s$, plotted over electric Rayleigh number L. A first point to note is that in each test series, there exists a threshold L_0, such that for all electric Rayleigh numbers $L \le L_0$ there holds:

$$\begin{aligned} \mathrm{Nu}(\theta_{|L}, 200) &= \mathrm{Nu}(\theta_{|L=0}, 200) \\ \|(\nabla \times \mathbf{u}_{|L}(200))_l\| &= \|(\nabla \times \mathbf{u}_{|L=0}(200))_\varphi\|, \; l \in \{\varphi, r, z\} \end{aligned} \tag{6.36}$$

This means that the solution stays close to its initial state which is given as solution of the stationary Boussinesq equations without DEP force. This behavior could be explained by using the uniqueness result for the stationary TEHD Boussinesq equation. According to Theorem 3.21, there exists at most one solution, provided that the problem data is sufficiently small:

$$\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5(R) < 1, \tag{6.37}$$

with constants α_i defined in (3.27). Now, $L \to 0$, i.e. $V_0 \to 0$, implies $\nabla\Phi_b \to 0$ and $a_{\mathbf{F}}$ becoming smaller. This further implies that α_2 becomes smaller and $\alpha_5 \to 0$. Thus, (6.37) becomes more likely to hold for small L. In order to guarantee (6.37), one further has to impose that $d\theta$ is sufficiently small. However, the fact that it is possible to compute stationary solutions of the standard Boussinesq equations, might be a hint that α_1 and α_2 are indeed small enough to satisfy (6.37). In summary, there may exist unique solutions for the stationary TEHD equations at small L. On the other hand, if $(\mathbf{u}, p, \theta, \Phi)(\varphi, r, z)$ denotes a stationary solution of (2.28), then $(\mathbf{u}, p, \theta, \Phi)(\varphi + d\varphi, r, z)$ should be a solution as well for all $d\varphi \in [0, 2\pi]$ according to the φ-invariance of the geometry and boundary conditions. But this is a contradiction to uniqueness of solutions, unless $(\mathbf{u}, p, \theta, \Phi)$ is azimuthally invariant. Therefore, the occurrence of axially aligned, columnar structures is not possible for small L. In the end of the section, this reasoning is verified by comparison with experimental data.

Apart from that, Figure (25) shows the clear trend of increasing Nusselt number, radial and axial vorticity component and decreasing azimuthal vorticity component for increasing L and A.

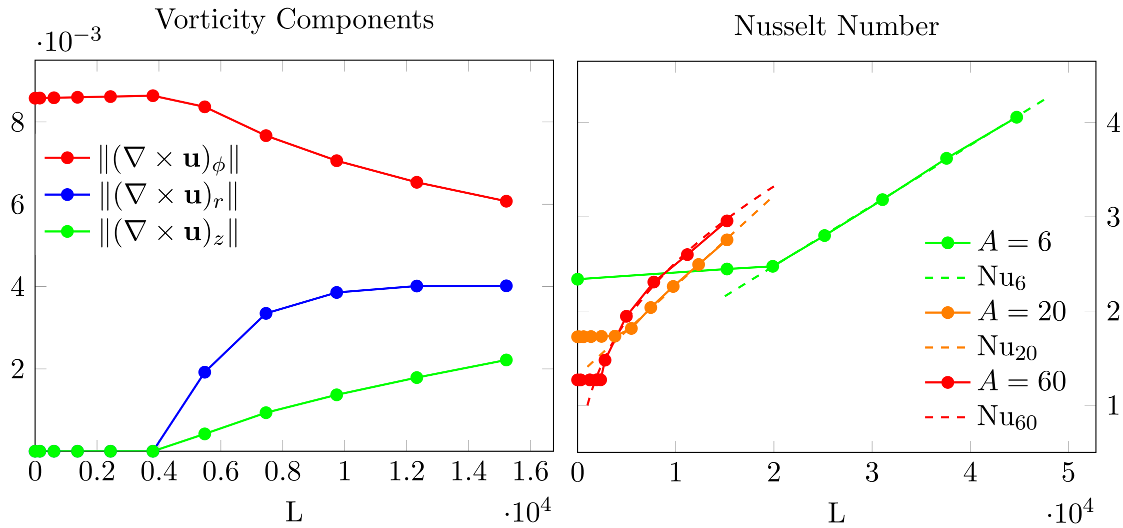


Figure 25: Left: Nusselt number for $Ra = 23946$ and $\mathbf{g} = \mathbf{g}_n$ over electric Rayleigh number L for different aspect ratios. Right: L^2-norm of vorticity components for $Ra = 23946$, $A = 20$ and $\mathbf{g} = \mathbf{g}_n$ over electric Rayleigh number L.

For $A = 6$ and $A = 20$ the Nusselt number grows linearly with L on the considered range:

$$\begin{aligned} \mathrm{Nu}_6(L) &= 1.20 + 6.42 \cdot 10^{-5} \cdot L, && L \in [19880, 44729] \\ \mathrm{Nu}_{20}(L) &= 1.31 + 9.57 \cdot 10^{-5} \cdot L, && L \in [5479, 15220] \\ \mathrm{Nu}_{60}(L) &= 0.116 \cdot L^{0.334}, && L \in [1242, 15220]. \end{aligned} \tag{6.38}$$

A comparable growth of Nusselt number w.r.t. V_0 is experimentally reported in [27].

The azimuthal wave number K appears to be independent of L, supposed that $L \geq L_0$, see Figure 26. A similar behavior is reported by linear stability analysis in [54]. However, the axial extent of the columnar structures increases with increasing L as depicted by Figure 27. In addition, these structures become less regular.

Figure 28 illustrates the exponential growth of axial vorticity for different values of V_0. One can observe a significant decay in the duration that is needed for the helicoidal motion to evolve. To be precise, there holds

$$\| \left(\nabla \times \mathbf{u}_{|L}(t)\right)_z \| \sim 10^{\alpha_L t} \text{ for } t \leq t_L \tag{6.39}$$

with t_L denoting the end time of exponential grow as depicted in Figure 28. Here, the exponential factor α_L shows a dependence on L that can be fairly well described by $\alpha_L \propto V_0^2 \propto L$, see Figure 28. Here, the simulation for $V_0 = 9000$ V was not taken into account since at some point, the Newton iteration did not converge any more. Also, as shown by 29, the violation of $\nabla \cdot \mathbf{u}_h = 0$ becomes more severe by increasing V_0. At this point, a higher spatial resolution or the use of stabilization techniques such as div-grad stabilization and SUPG should be considered for simulating scenarios of large V_0.

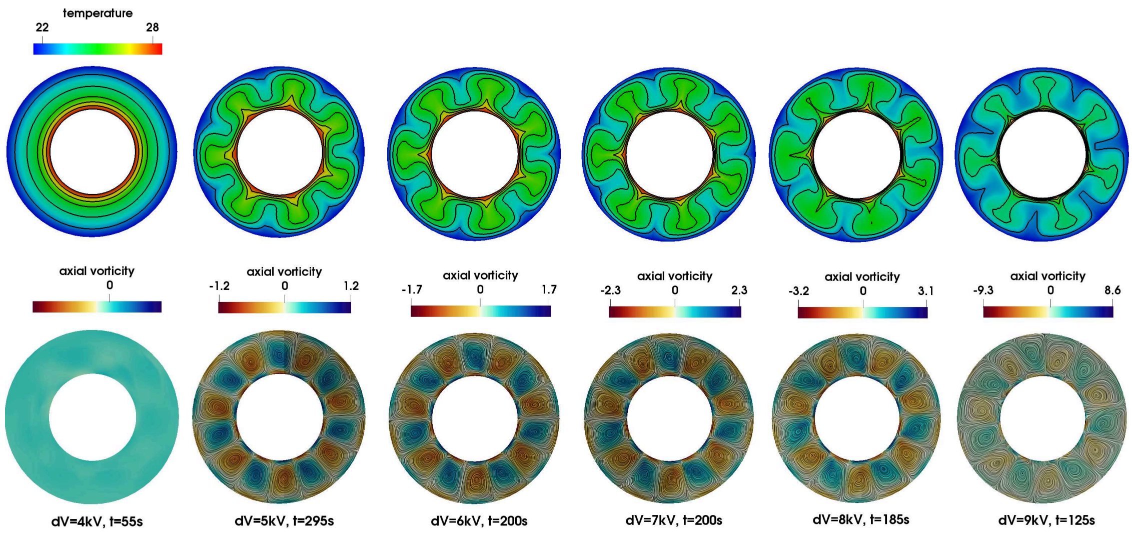


Figure 26: Upper: Temperature distribution on $\{z = 0.5H\}$ for $d\theta = 7$ K, $\mathbf{g} = \mathbf{g}_n$ for different potential differences V_0 and time instances. Lower: Axial vorticity.

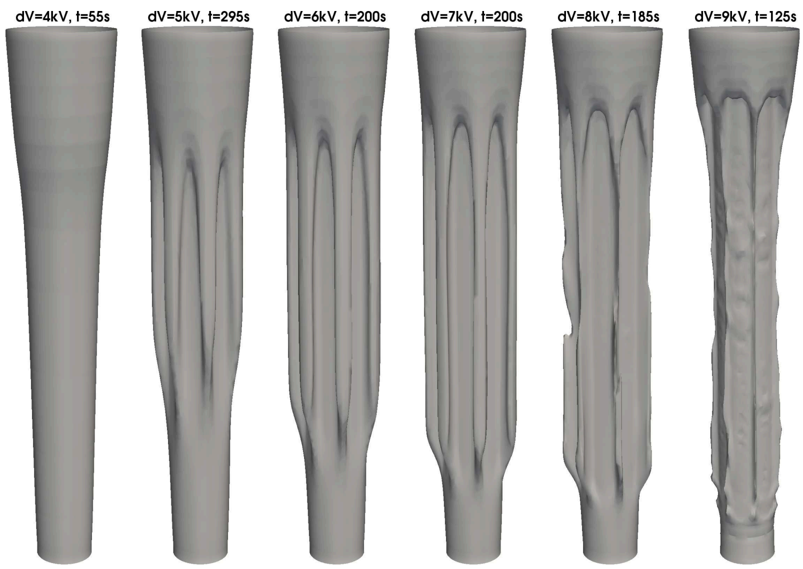


Figure 27: Temperature isosurface $\{\theta = \theta_r\}$ for $d\theta = 7$ K, $\mathbf{g} = \mathbf{g}_n$ for different potential differences V_0 and time instances.

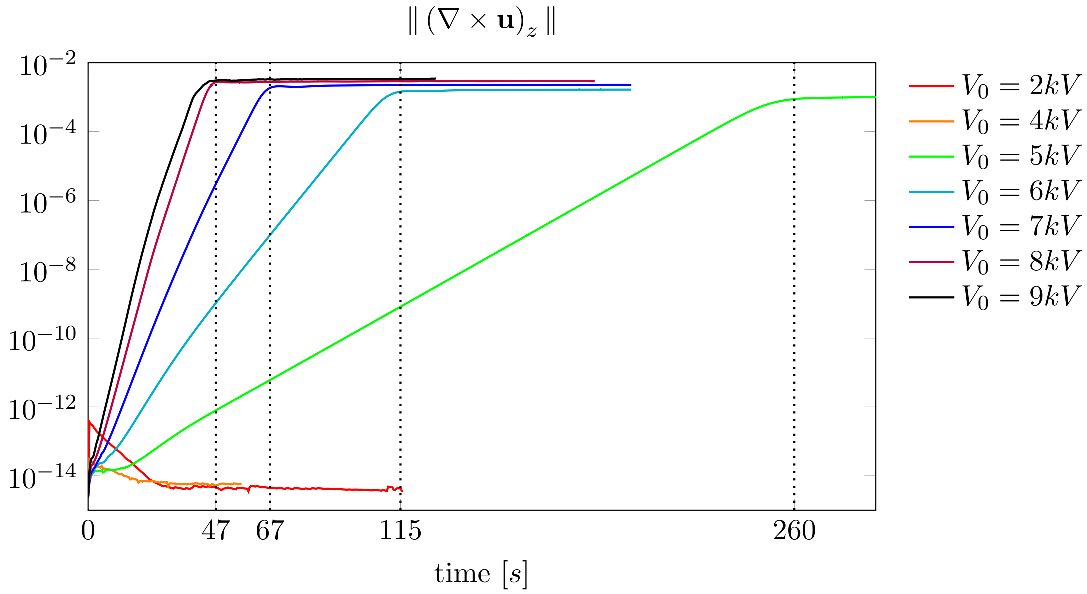


Figure 28: Temporal evolution of L^2-norm of axial vorticity for $d\theta = 7$ K, $\mathbf{g} = \mathbf{g}_n$ and different potential differences V_0.

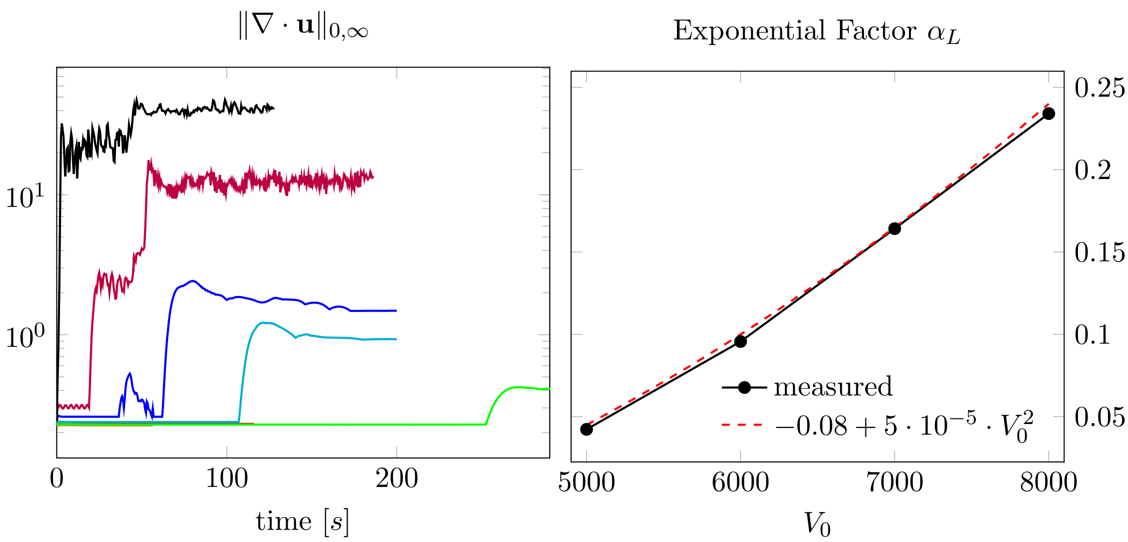


Figure 29: Left: Temporal evolution of divergence violation for $d\theta = 7$ K, $\mathbf{g} = \mathbf{g}_n$ and different potential differences V_0. Right: Factor α_L, defined in (6.39), for exponential growth of axial vorticity over voltage difference V_0.

Comparison of Time Stepping Schemes

We now compare the simulation results that are obtained by the time stepping schemes BDF1 and BDF2, as defined in Section 5.2, and the Crank-Nicolson method. The scenario under Earth's gravity is simulated by using each method and for time step sizes $k \in \{0.1, 0.05, 0.025\}$. Thereby, the norm of axial vorticity, Nusselt number and violation of the divergence constraint as function over time are considered as quantities of interest and plotted in Figure 31. In terms of these quantities, all schemes yield for all time step sizes very similar results, except of two outliers. For BDF2 with $k = 0.1$ and Crank-Nicolson with $k = 0.05$ a significant difference in all quantities can be observed. Apparently, BDF2 is not stable for too large time step sizes, as the very high values of $\|\nabla \cdot \mathbf{u}\|_{0,\infty}$ and Nusselt number indicate. From the literature, it is known that both BDF1 and BDF2 are A-stable, however, with BDF2 having a smaller stability region as BDF1.

The solution obtained by Crank-Nicolson for $k = 0.05$ could be a pathological case, since the respective results for $k = 0.1$ and $k = 0.025$ are very close to the majority of results.

Figure 33 illustrates the iso-surface for $\theta = \theta_r$ and axial vorticity at final time $t = 200$ s for all time stepping schemes and $k \in \{0.05, 0.025\}$. Except of the aforementioned Crank-Nicolson / $k = 0.05$ outliers, all solutions look very similar.

In Figure 30 the average CPU time for solving the arising linear systems is listed for each method and each time step size. As usual in the context of incompressible flow problems, the linear solver converges faster with decreasing time step size, since the well-conditioned mass matrix arising from the finite element discretization of $\partial_t \mathbf{u}$ becomes more dominant compared to the typically ill-conditioned stiffness matrix that comes from $\Delta \mathbf{u}$. For $k \leq 0.05$, BDF2 is apparently the fastest scheme in terms of linear systems. A clear advantage of BDF1 and BDF2 over Crank-Nicolson lies in the decoupling of the nonlinear system into three parts (momentum / continuity equation, heat equation, Gauss' law) which can be solved separately. However, the used nonlinear and linear solver are not optimized to exploit this fact. In doing so, a further reduction of CPU time could be possible.

k	BDF1	BDF2	CN
0.1	15.95	18.39	13.7
0.05	12.57	10.11	11.07
0.025	9.47	9.39	10.89

Figure 30: Average CPU time in seconds for solving the arising linear systems for different time stepping schemes and time step sizes k.

Comparison of DEP Formulations

So far, all presented simulation results are obtained for the choice $\mathbf{F} = \mathbf{F}_s$, given by (6.9) and which does not satisfy the requirements needed for the existence results. These conditions are met by the linearized DEP models $\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$, introduced in Definition 3.23.

We now compare the results that are obtained by the different formulations. As base potential in the definition of the linearized terms we set $\Phi_0 = \overline{\Phi}$, where $\overline{\Phi}$ denotes the analytical solution in the zero-gravity case with infinite cylinder, (6.18). In Section 3.3, the use of linearization was motivated by the fact that small values of γ in Gauss' law and small temperature differences $d\theta$ lead to a small corridor in which all possible potential solutions are contained. Figure 34 depicts the difference between $\overline{\Phi}$ and the potential Φ which is computed in the $\mathbf{F} = \mathbf{F}_s$ case with $d\theta = 7$ K and $V_0 = 7000$ V under Earth's gravity $\mathbf{g}_n$. Apparently, the relative difference between both potentials is of order $\mathcal{O}(10^{-3})$ over the entire time interval. We therefore expect

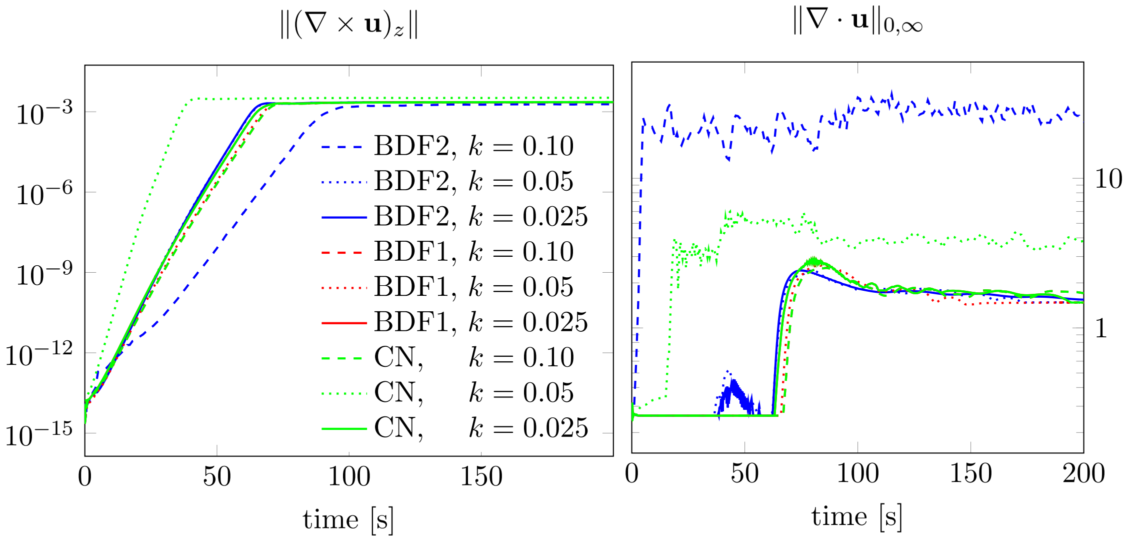


Figure 31: Temporal characteristics for $d\theta = 7$ K, $V_0 = 7000$ V and $\mathbf{g} = \mathbf{g}_n$ for different time stepping schemes and time step sizes k. Left: L^2-norm of axial vorticity. Right: Violation of divergence constraint

fairly similar results for all considered DEP formulations. This is indeed the case, as the following results show. According to Figure 34, the temporal evolution of axial vorticity exhibits pretty much the same characteristics, leading to very similar columnar vortex structures in the final state, see Figure 35. In particular, the number of those structures is the same in all cases. As single significant difference, one can observe that the resulting flow states are slightly rotated around the axial axis. This is illustrated in Figure 36, where the $\{\mathbf{u}_z = 0\}$ level set is used to highlight the location of the columnar structures.

In summary, one can conclude that the proposed linearized DEP models denote acceptable substitutions of the "exact" model $\mathbf{F}_s$ in case of small temperature dependence of permittivity and moderate temperature differences within the fluid. This is in agreement with the statements in [78]. According to them, "the thermo-electric coupling can be neglected in agreement with the linear stability theory" for "small values of $\gamma \approx 10^{-3} - 10^{-2}$ K^{-1} and wide gaps, i.e. $\eta \leq 0.6$".

Sensitivity and Non-Uniqueness of Stationary Solutions

As we will see in the end of this section, simulation results and experimental data are in good agreement in the sense that columnar structures are present in both cases. However, the actual number of these structure differs in some of the considered cases. In order to further investigate this issue, we now consider the sensitivity of the solution w.r.t. perturbations by using the adjoint framework presented in Section 2.2. For doing so, we first define the quantity of interest

$$J_{vort}(\mathbf{u}) := \int_{\tilde{\Omega}} \left(\partial_x \mathbf{u}_y(T) - \partial_y \mathbf{u}_x(T)\right)^2 d\mathbf{x} \tag{6.40}$$

with $\tilde{\Omega} = \{(x, y, z) \in \Omega\colon\ 0.6H \leq z \leq 0.7H\}$. J_{vort} measures the strength of axial vorticity in a certain area of the annulus at $T = 70$ s, see Figure 39. The idea behind this definition is that the formation of columnar structures is closely related to axial vorticity. Further, the previously presented simulations under earth gravity show that the spatial distribution of axial vorticity at $t = 70$ s correlates with the positions of thermal plumes, see Figure 13. Regarding the framework of Section 2.2, we have $J^i = 0$ and $J^f = J_{vort}$.

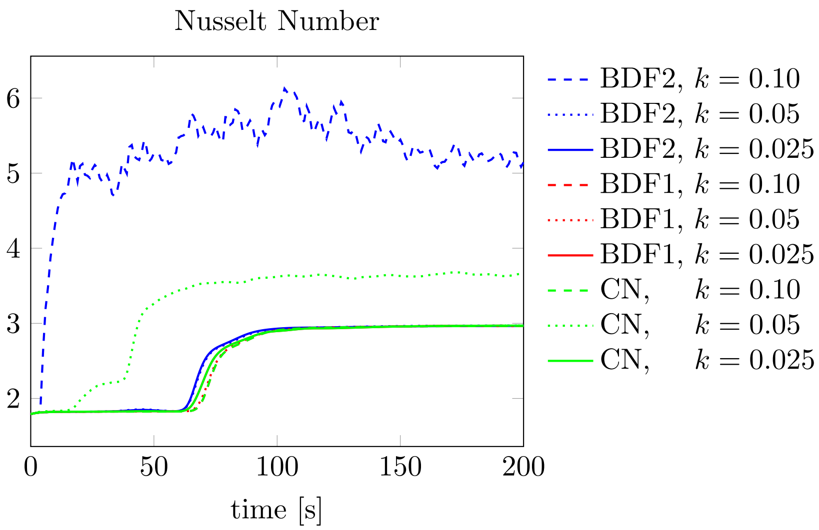


Figure 32: Nusselt number for $d\theta = 7$ K, $V_0 = 7000$ V and $\mathbf{g} = \mathbf{g}_n$ for different time stepping schemes and time step sizes k.

As pointed out in Section 2.2 the solution $z = (\hat{\mathbf{u}}, \hat{p}, \hat{\theta}, \hat{\Phi})$ of the dual system (2.43) with zero right-hand side terms $\partial.j^i$ and initial conditions $\hat{\mathbf{u}}(T) = \partial_{\mathbf{u}} J_{vort}(\mathbf{u})$, $\hat{\theta}(T) = 0$ can be interpreted as derivative of J_{vort} at the solution $u = (\mathbf{u}, p, \theta, \Phi)$ of the primal problem (2.28). Thus,

$$J_{vort}(\mathbf{u}(p)) \approx J(u) + (z, p)_L. \tag{6.41}$$

for small perturbations $p = (\delta\mathbf{u}, \delta p, \delta\theta, \delta\Phi) \in L := L^2(0, T; \mathbf{H}_0^1 \times L_0^2 \times H_D^1 \times H_D^1)$ on the right-hand side of (2.28). According to (6.41), perturbations at locations where the dual solution takes large values may potentially lead to large deviations of J_{vort} from the unperturbed state. On the other hand, small dual values indicate areas of low sensitivity.

Figure 38 illustrates the temporal evolution of the logarithm of the dual solution's magnitude. At final time $t = 70$ s, large dual values are restricted to the area given by $\tilde{\Omega}$. As time evolves, this area of influence is covering the entire annulus and the dual solution grows several orders of magnitude. At initial time $t = 0$ s, the largest values can be found in the lower part of the annulus. Apparently, perturbations which originate in the lower inner part and travel upwards due to natural convection have a larger impact than those who start in the upper outer part and travel downwards.

In Figure 37, the norms of the individual dual components are plotted over time for $V_0 = 7000$ V and $V_0 = 4000$ V. In the high voltage case, perturbations of Gauss' law have a comparable impact on axial vorticity as momentum perturbations have. Probably, the reason is the fact that Gauss' law perturbations lead to a perturbed potential field which in turn leads to a perturbed DEP force term. According to (6.29), DEP force is the only source of axial vorticity. In contrast, perturbations of heat equation and incompressibility constraint have a significantly smaller impact.

For $t \approx 55{-}70$ s, the norms of all dual components stay approximately at their level at $t = 70$ s. This is exactly that period, when the first columnar structures become visible, see Figure 13, and when axial vorticity reaches its final order of magnitude, see Figure 18. For $t < 55$ s, all dual components exhibit an exponential growth for decreasing t. Due to the resulting extremely large values, one can conclude that even tiny perturbations at the initial stage of the simulation may lead to large variations of the final axial vorticity. The sooner the flow state is perturbed, the larger the impact on the final state is. On the other hand, once the columnar structures are

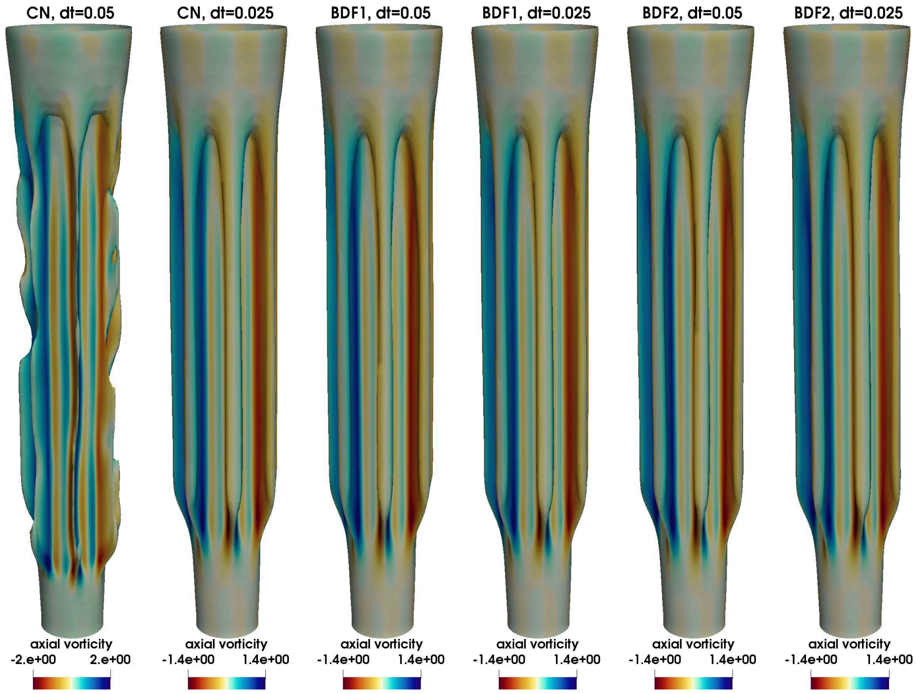


Figure 33: Temperature isosurface $\{\theta = \theta_r\}$ and axial vorticity for $d\theta = 7$ K, $V_0 = 7000$ V and $\mathbf{g} = \mathbf{g}_n$ at $t = 200$ s for different time stepping schemes and time step sizes k.

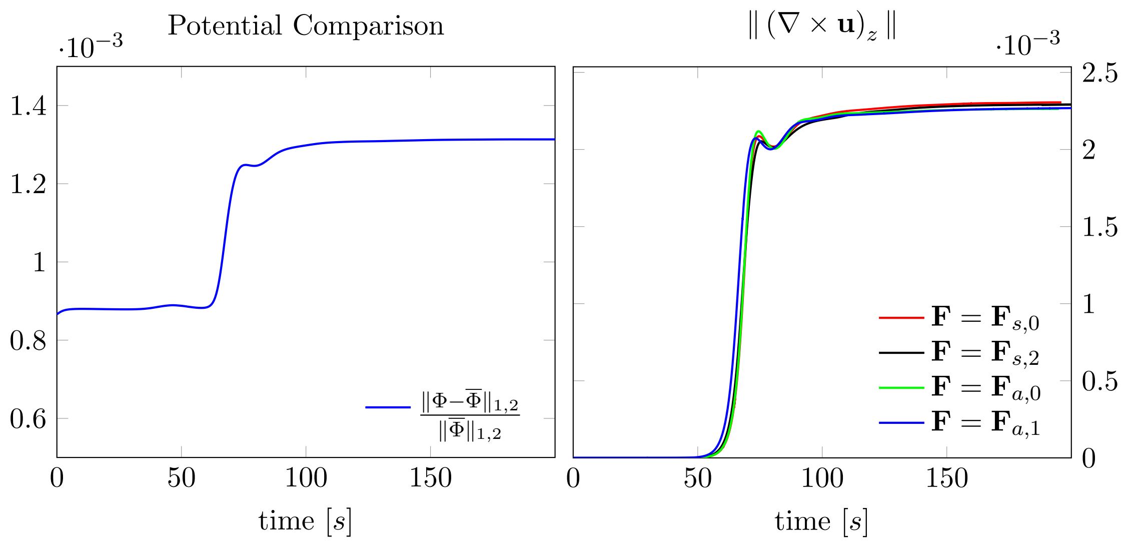


Figure 34: Temporal characteristics for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Left: Relative W^{12}-difference between analytic potential $\overline{\Phi}$ for infinite cylinder and computed potential Φ. Right: L^2-norm of axial vorticity for different choices of $\mathbf{F}$.

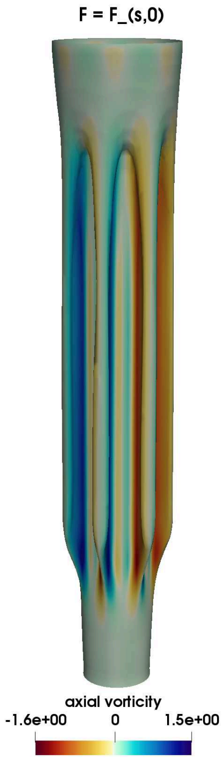


Figure 35: Temperature isosurface $\{\theta = \theta_r\}$ for $d\theta = 7$ K, $V_0 = 7000$ V and $\mathbf{g} = \mathbf{g}_n$ at $t = 200$ s for different choices of $\mathbf{F}$.

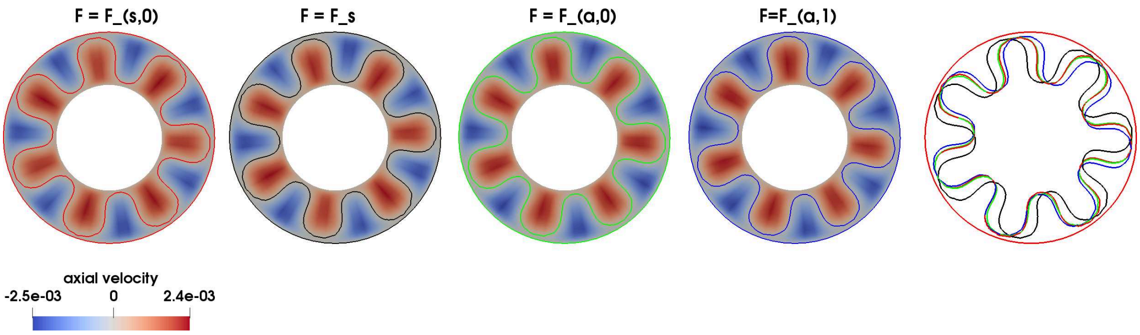


Figure 36: Left: Axial velocity on $\{z = 0.5H\}$ for $d\theta = 7$ K, $V_0 = 7000$ V and $\mathbf{g} = \mathbf{g}_n$ at $t = 200$ s for different choices of $\mathbf{F}$. Right: Comparison of $\{\mathbf{u}_z = 0\}$ level set on $\{z = 0.5H\}$.

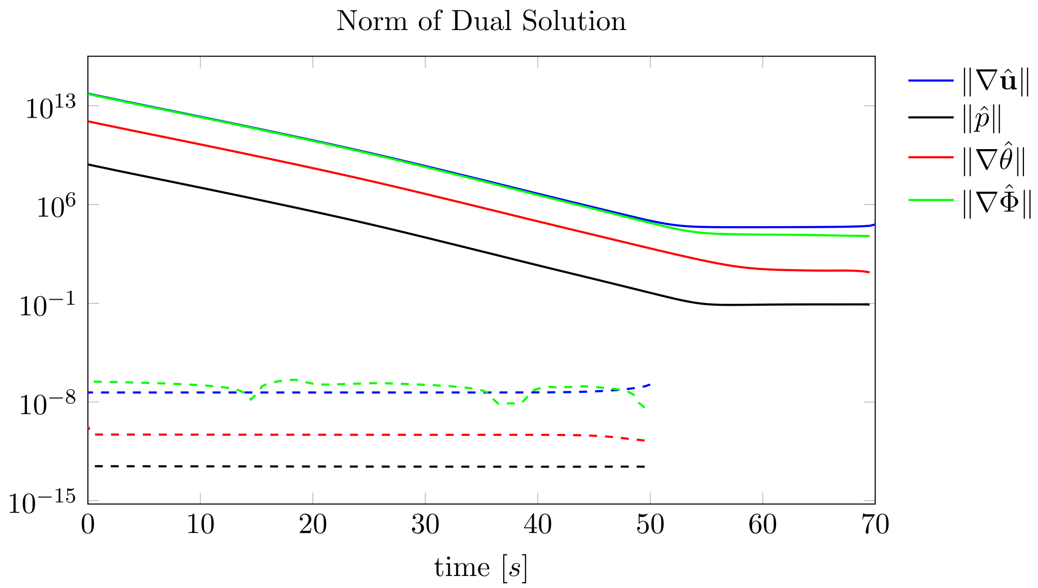


Figure 37: Temporal evolution of norms of dual solution for $d\theta = 7$ K, $\mathbf{F} = \mathbf{F}_s$, $\mathbf{g} = \mathbf{g}_n$ and for $V_0 = 7000$ V (solid), and $V_0 = 4000$ V (dashed).

almost fully developed, such severe amplification of perturbations is not given anymore.

As shown in Figure 28, the flow state for $V_0 = 4000$ V does not exhibit a transition from its unicellular initial state and its axial vorticity is zero. This matches the fact, that the corresponding dual solution stays at a very low level, see Figure 37. Apparently, axial vorticity is insensitive w.r.t. perturbations as long as the electric Rayleigh number is sufficiently small.

Motivated by these considerations, simulations with perturbed initial condition for temperature, $\tilde{\theta}_0 = \theta_0 + \delta\theta^{(j)}$, are conducted for two different types of perturbations:

$$\delta\theta^{(j)}(\varphi, r, z) := 0.1\cos(2\pi K\varphi)\sin\left(\pi\frac{r - r_i}{r_o - r_i}\right) \cdot \begin{cases} \sin\left(\pi\frac{z - z_{1,j}}{z_{2,j} - z_{1,j}}\right), & z \in [z_{1,j}, z_{2,j}] \\ 0, & \text{else} \end{cases}, \tag{6.42}$$

for $j \in \{l, h\}$. Thus, the perturbation is continuous and restricted to a cross-section of the cylinder, defined by lower and upper axial coordinate $z_{1,j}$, $z_{2,j}$. In a first scenario, $[z_{1,h}, z_{2,h}] = [0.1H, 0.16H]$, i.e. the perturbation is located in the area of highest dual magnitudes. The second scenario is defined by $[z_{1,l}, z_{2,l}] = [0.92H, 0.98H]$, where dual values are lower, but still of order $\mathcal{O}(10^5)$, see also Figure 39. In both cases, the azimuthal wavenumber K is set to 6. The results of these simulations are depicted in Figure 40 and 41. For both scenarios, one can observe that axial vorticity reaches its maximal value significantly sooner as in the unperturbed case. Generally, it appears that the solution converges faster towards its final, stationary state if the initial condition is perturbed in its high-sensitivity region, $j = h$, compared to the low-sensitivity case, $j = l$. The second scenario in turn converges faster than the unperturbed case. Furthermore, it is remarkable that the perturbed solutions exhibit an azimuthal wavenumber of 6, in contrast to 7 for the unperturbed solution.

We investigate this effect in more detail. In Figure 42, the temperature at $t = 100$ s is visualized on a horizontal cut-plane at $z = 0.5H$ for different perturbed initial conditions. All initial perturbations are of the form $d\theta^{(h)}$, however, with azimuthal wavenumbers $K \in \{0, 3, 4, 5, 6, 7, 8\}$. The resulting azimuthal wave numbers at final time are $K_T \in \{7, 6, 8, 5, 6, 7, 8\}$. In each case, the solution appears to have reached a stationary state. Thus, we have shown numerically, that the stationary TEHD equations exhibit several solutions, of similar shape but different amount of

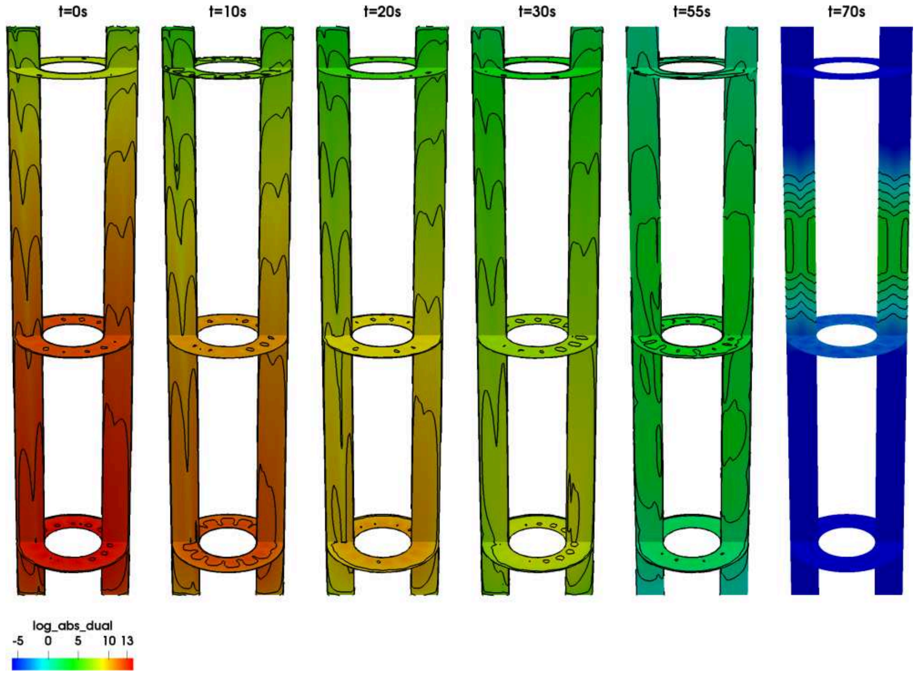


Figure 38: $\exp_z := \log_{10}\left(|\hat{\mathbf{u}}| + |\hat{p}| + |\hat{\theta}| + |\hat{\Phi}|\right)$ for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$ at different time instances.

columnar structures. According to the plots in Figure 43 and 44, several levels of heat transfer, axial vorticity, kinetic energy and dissipation can be observed, where solutions with coinciding wave number are on the same respective level. Interestingly, the solution for $K = 5$ is the only one, whose kinetic energy in the final state is larger as in the initial state. Generally, kinetic energy and dissipation decrease with increasing K_T, whereas heat transfer and vorticity increase.

In summary, one can state axial vorticity of the solution of the instationary TEHD equations is extremely sensitive to perturbations in the initial stage of the simulation. Moreover, by suitable, small perturbations of the initial condition, one can generate a variety of different stationary solutions and convergence towards these stationary states is enhanced.

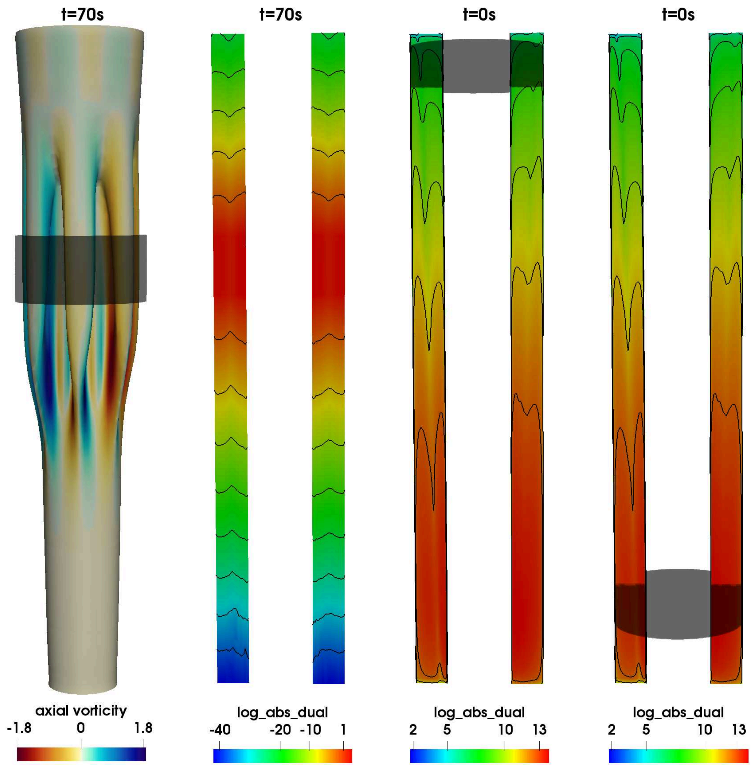


Figure 39: Initial / final state of primal / dual solution for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Left: Temperature isosurface $\{\theta = \theta_r\}$ of primal solution at $t = 70$ s with axial vorticity. The shaded part indicates the area of interest $\tilde{\Omega}$ in the definition of J_{vort}, (6.40). Second from the left: $\exp_z$ at $t = 70$ s. Second from the right: $\exp_z$ at $t = 0$ s with shaded area indicating the support of the initial condition perturbation $d\theta^{(l)}$. Right: $\exp_z$ at $t = 0$ s with shaded area indicating the support of $d\theta^{(h)}$.

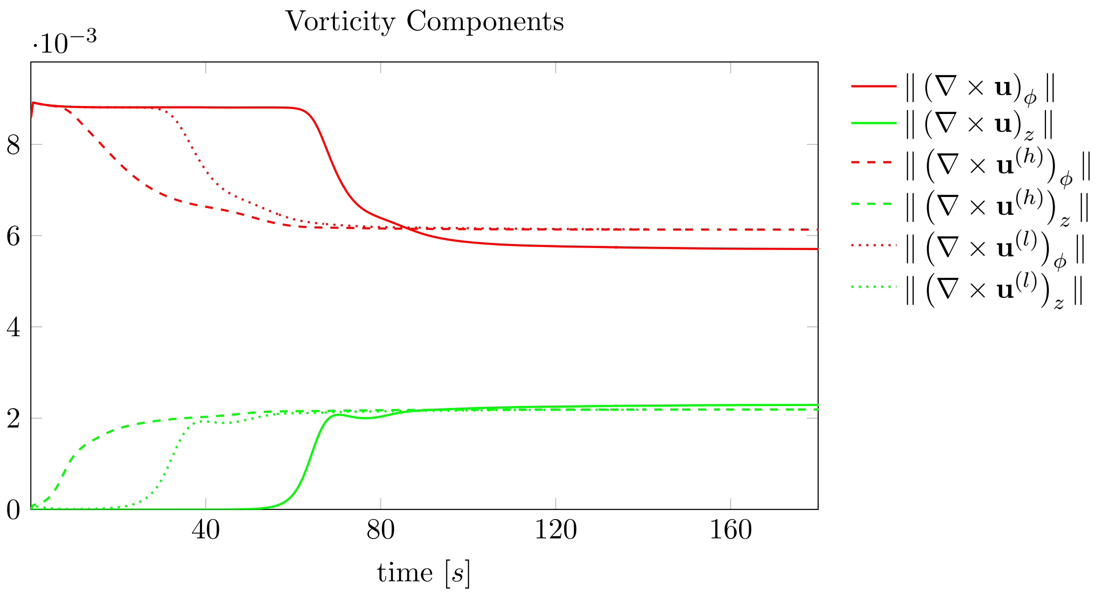


Figure 40: Temporal evolution of L^2-norm of azimuthal and axial vorticity for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. The unperturbed solution is denoted by $\mathbf{u}$; $\mathbf{u}^{(h)}$, $\mathbf{u}^{(l)}$ denote the solutions that are obtained by perturbing the initial temperature by $d\theta^{(h)}$ (high sensitivity region) and $d\theta^{(l)}$ (low sensitivity region), respectively.

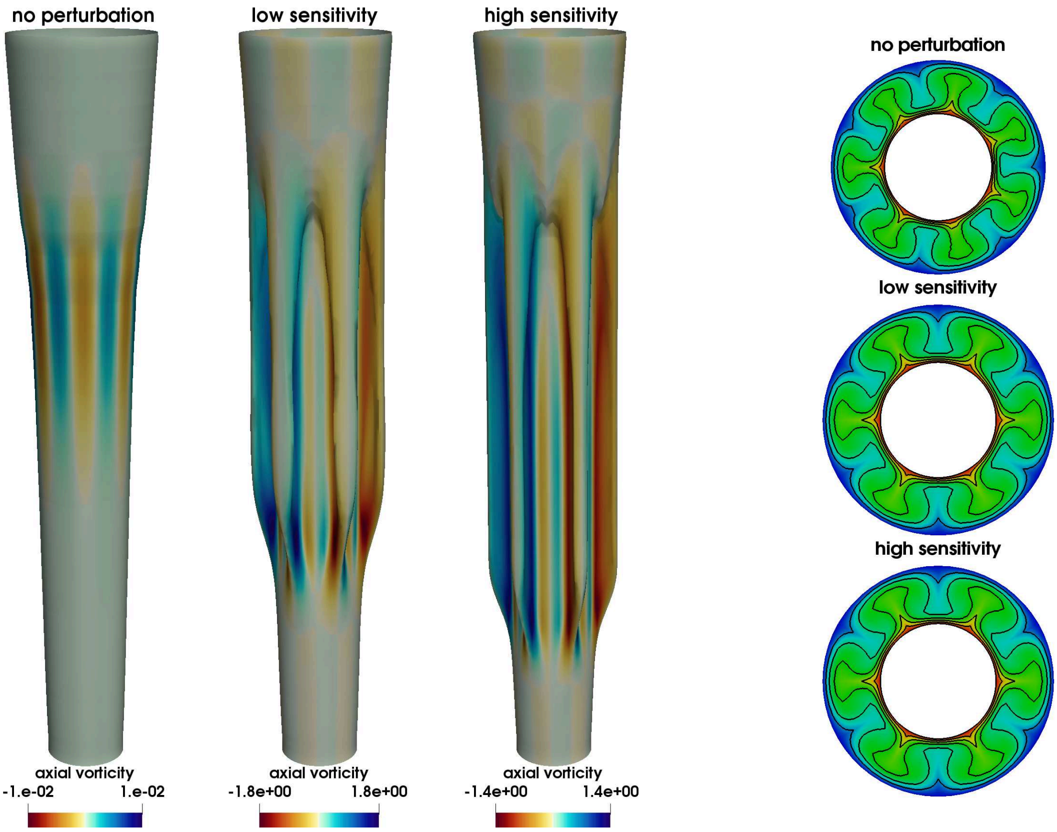


Figure 41: Unperturbed $\mathbf{u}$ and perturbed solutions $\mathbf{u}^{(l)}$, $\mathbf{u}^{(h)}$ for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Left: Temperature isosurface $\{\theta = \theta_r\}$ and axial vorticity of primal solutions at $t = 50$ s. Right: Temperature distribution on $\{z = 0.5H\}$ at $t = 140$ s.

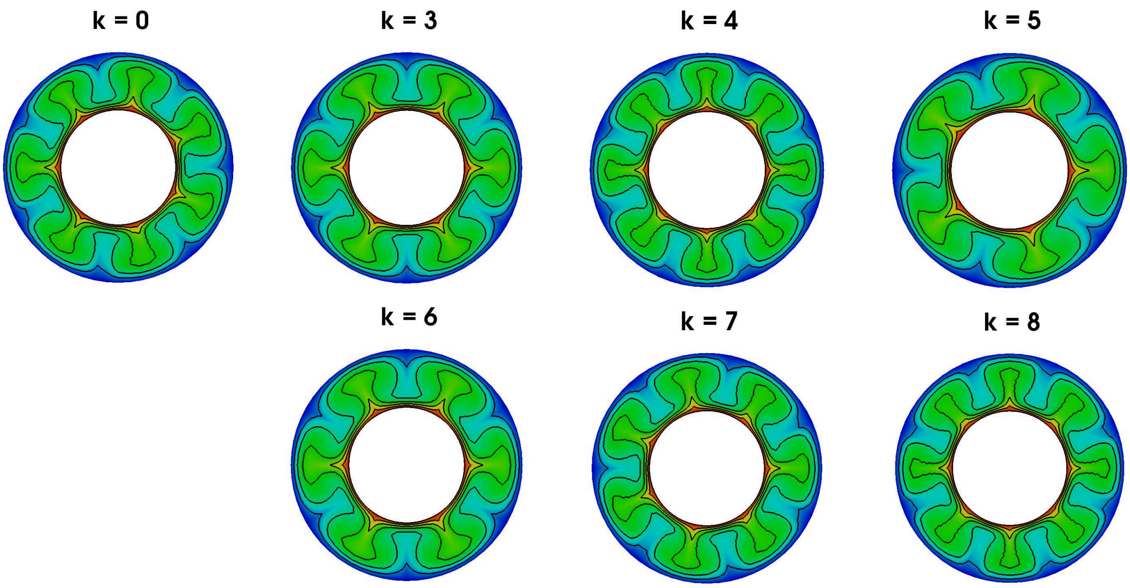


Figure 42: Temperature distribution on $\{z = 0.5H\}$ at $t = 100$ s for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. $K > 0$ denotes the azimuthal wavenumber of initial temperature perturbation $d\theta^{(h)}$ in (6.42). $K = 0$ denotes the unperturbed case.

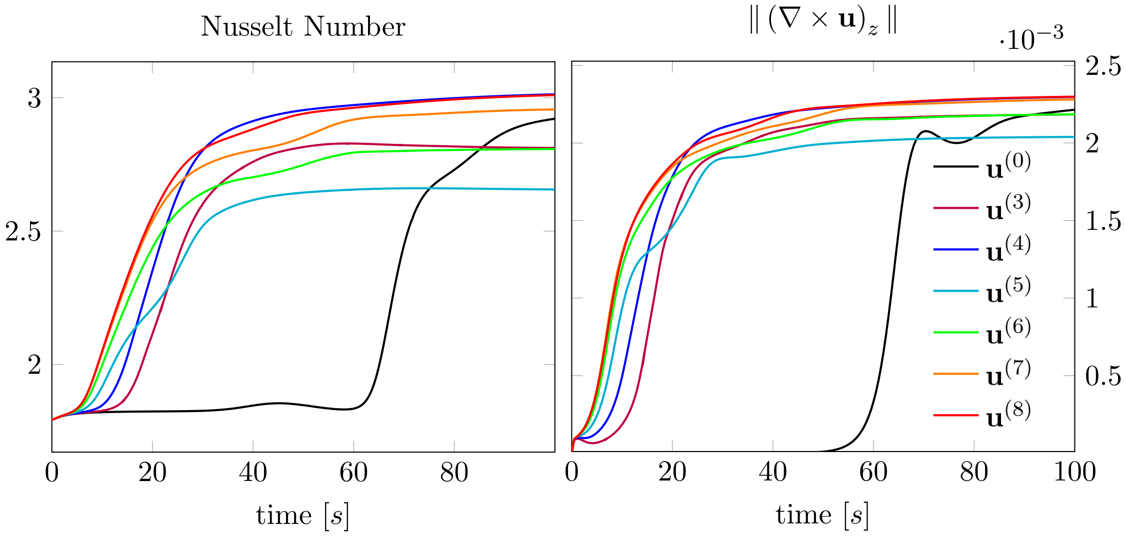


Figure 43: Temporal characteristics of perturbed solutions for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. $\mathbf{u}^{(K)}$ with $K > 0$ denotes the solution obtained for initial temperature perturbation $\delta\theta^{(h)}$ of azimuthal wavenumber K. $\mathbf{u}^{(0)}$ denotes the unperturbed solution. Left: Nusselt number. Right: L^2-norm of axial vorticity.

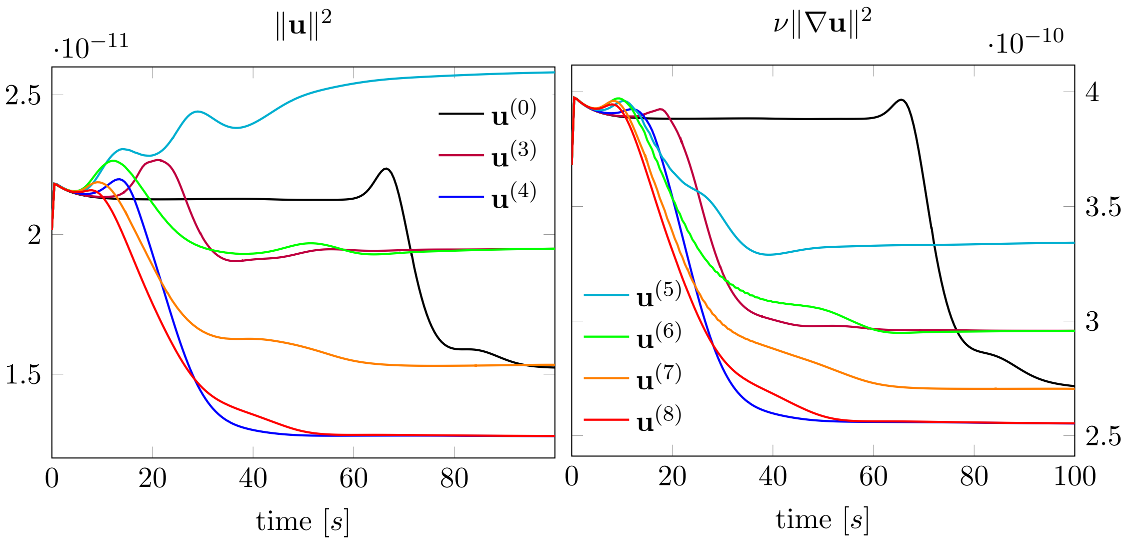


Figure 44: Temporal characteristics of perturbed solutions for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. $\mathbf{u}^{(K)}$ with $K > 0$ denotes the solution obtained for initial temperature perturbation $\delta\theta^{(h)}$ of azimuthal wavenumber K. $\mathbf{u}^{(0)}$ denotes the unperturbed solution. Left: Kinetic energy. Right: Dissipation.

Comparison with Experimental Data

We conclude this section on numerical results by comparing the simulation with experimental data in the standard gravity case with two different configurations of $d\theta$ and V_0. The physical experiments were conducted by Dr. Torsten Seelig at the *Lehrstuhl für Aerodynamik und Strömungslehre* at *BTU Cottbus*, which is headed by Prof. Christoph Egbers, see also [68].

Two different measurement techniques are employed: By means of the so-called *particle image velocimetry (PIV)* it is possible to measure the fluid's radial and axial velocity on a vertical cut plane. Here, particles are used that are transported by the fluid's motion. By means of short laser impulses, the positions of the particles can be recorded by a camera and correlation between positions at subsequent impulses allows to compute the particle velocities. As second technique, the so-called *shadowgraph imaging* is used. Here, a telecentric light is placed at the bottom of the annulus which has transparent top and ground plate. The light is directed along the axial axis and recorded by a camera. As the fluid's refraction index is density-, and therefore temperature-dependent, conclusions on the z-averaged fluid's temperature distribution can be drawn. For details concerning both techniques see [68]. Figure 45 depicts the experimental cell and a schematic view on the underlying measurement techniques.

The experiments are realized in the following way: first the temperature difference $d\theta$ is applied between the inner and outer cylinder. This state is kept unchanged for 60 minutes in order to obtain a fully developed, stationary, uni-cellular flow field, which corresponds to the solution of the stationary Boussinesq equations for natural convection. Then, the voltage difference V_0 is switched on and data is acquired for 15 minutes. In both considered cases, the experimental data shows convergence towards a stable state, [68]. The presented data always corresponds to the last recorded time instance.

The top row of Figure 46 shows the results of the shadowgraph measurement (left) and the $z-$averaged simulated temperature distribution (right) for $d\theta = 2$ K and $V_0 = 6000$ V. The shadowgraph image depicts the normalized light intensity, where blue / red regions refer to locations of denser / lighter fluid compared to the reference state with $V_0 = 0$ V, [68]. One can observe a good qualitative agreement in the sense that five, equally spaced temperature plumes are visible in both cases.

In the $d\theta = 7$ K, $V_0 = 7000$ V case, the experimental data indicates the occurrence of six thermal plumes, see the bottom row of Figure 46 (left). This is in contradiction to the simulation results, where seven plumes are visible, see Figure 16. However, it was shown in the previous section, that the solution of the stationary TEHD Boussinesq equations is not unique and different solutions can be generated by small perturbations of the initial condition. Figure 46 (right) illustrates the numerical solution for perturbed initial condition with azimuthal wave $K = 6$, (6.42). Then, experimental and simulation data show a similar qualitative agreement as in the previous case.

One should note that the physical experiments have been repeated multiple times for the same experimental configuration as the presented simulations and under fairly comparable laboratory conditions. The results of these experiments did not show any variation in the azimuthal wave number, although slight perturbations in the initial conditions can never be excluded in physical experiments and are likely to happen. In this sense, the mathematical model (or the numerical solution method) differs from the physical experiment, which appears to be more stable.

A more quantitative comparison is presented by Figure 48, where the flow field on a vertical cut-plane is shown. Here, the cut-plane for the simulation data is chosen to minimize the difference between experiment and simulation. The arrows depict the radial and axial velocity component with red / blue indicating positive / negative axial velocity. As for the shadowgraph measurement, one can observe a good agreement between experiment and simulation concerning

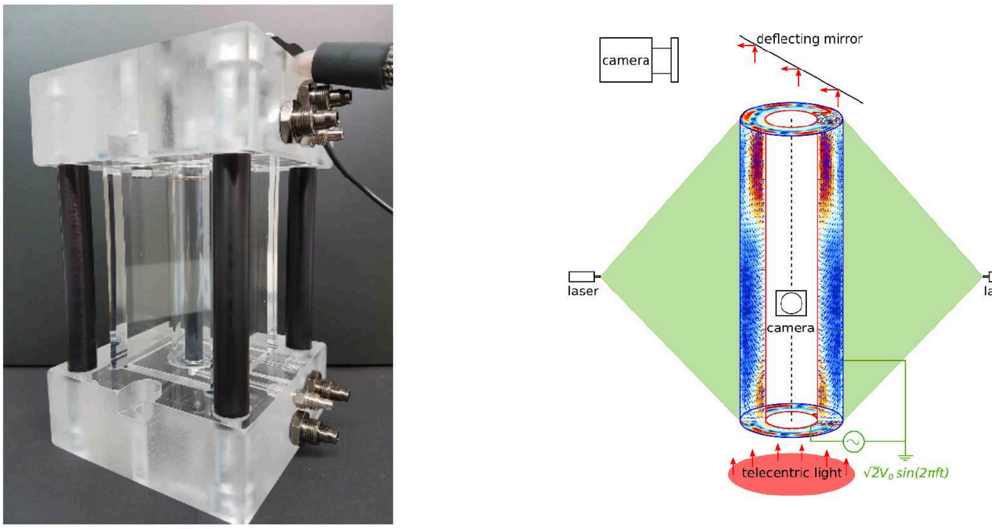


Figure 45: Left: Experimental cell of 100 mm height. The black cylinder in the middle of the cell corresponds to Γ_i. The larger, transparent cylinder enclosing the black one corresponds to Γ_o, [51]. Right: Schematic view on vertical annulus with required devices for PIV (laser on the left- and right-hand side and camera in the middle) and shadowgraph imaging (light source, mirror and camera on top), [68].

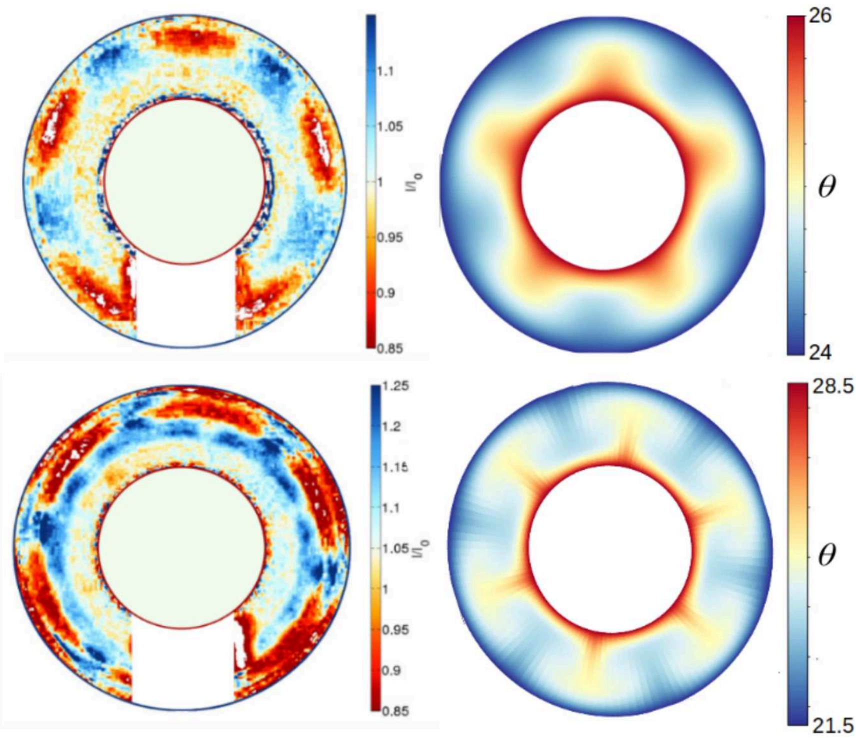


Figure 46: Comparison of simulation and experimental data for $\mathbf{F} = \mathbf{F}_s$ and $\mathbf{g} = \mathbf{g}_n$. Upper row: $d\theta = 2$ K, $V_0 = 6000$ V. Lower row: $d\theta = 7$ K, $V_0 = 7000$ V. Initial temperature perturbation $d\theta^{(h)}$ with $K = 6$ is used for simulation. Left: Shadowgraph image with color encoding normalized light intensity, where blue / red regions refer to locations of denser / lighter fluid compared to the reference state. Right: z-averaged temperature field obtained by simulation.

the qualitative flow structure. In the $d\theta = 2$ K, $V_0 = 6000$ V case, the axial velocity covers approximately the same range of values. For $d\theta = 7$ K, $V_0 = 7000$ V the simulation seems to slightly overestimate the magnitude of the velocity. Possible reasons for this discrepancy might be modeling errors (Boussinesq approximation), discretization inaccuracy and measurement errors.

As final comparison, we consider the critical value of V_0 at which the fluid departs from the stable, uni-cellular convection cell which is typical for natural convection at low Rayleigh numbers, to a state where columnar vortex structures do occur, see Figure 47. For three different configurations, critical values obtained by experiments [68] and intervals obtained by simulations are presented. Here, the respective lower interval bounds denote the largest considered effective voltage, for which no transition from the uni-cellular state was observed. The respective upper bounds denote the smallest considered effective voltage, for which this transition does occur in the numerical simulation.

In each case, the experimentally obtained critical values are either contained in the numerical interval, or very close to the respective upper bound.

configuration	experiment shadowgraph	PIV	simulation interval
$\Gamma = 20, d\theta = 2K$	≈ 1300	≈ 1300	$(1268, 1449)$
$\Gamma = 20, d\theta = 7K$	≈ 1000	≈ 1150	$(905, 1090)$
$\Gamma = 60, d\theta = 7K$		≈ 780	$(711, 776)$

Figure 47: Critical value $\frac{dV}{V_{eff}}$ above which non-zero azimuthal wavenumbers can be observed by shadowgraph imaging, PIV and simulation.

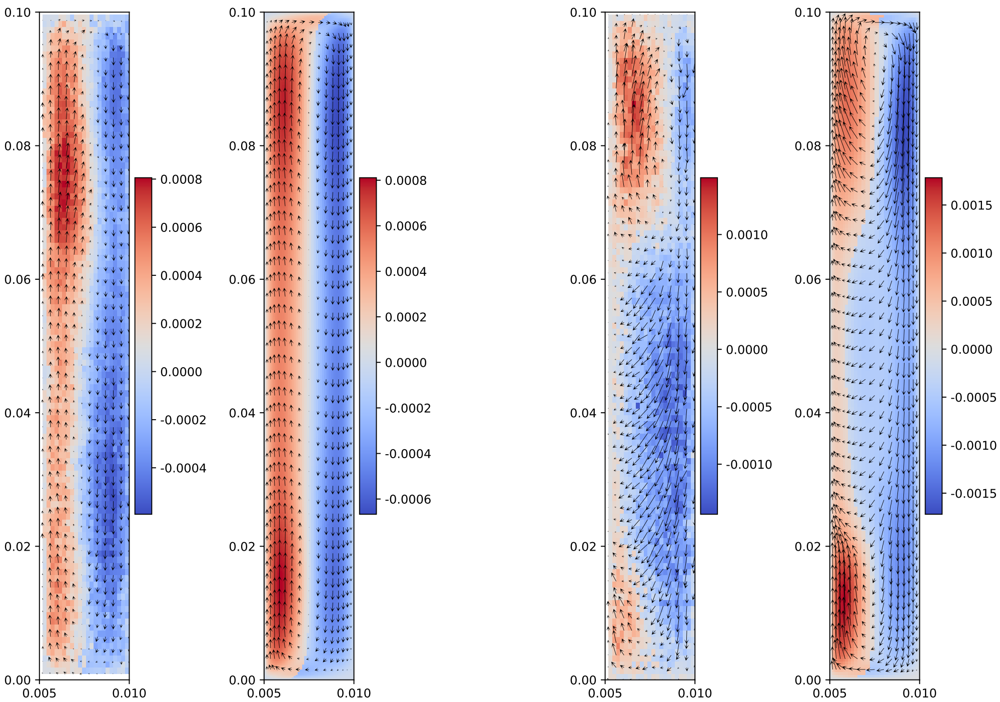


Figure 48: Comparison of experimental PIV data and simulation for radial-axial velocity field when the flow reached a stationary state, with $\mathbf{g} = \mathbf{g}_n$. Color encodes axial velocity. Left: PIV for $d\theta = 2$ K, $V_0 = 6000$ V. Second from the left: Simulation for $d\theta = 2$ K, $V_0 = 6000$ V and $\mathbf{F} = \mathbf{F}_s$. Second from the right: PIV for $d\theta = 7$ K, $V_0 = 7000$ V. Right: Simulation for $d\theta = 7$ K, $V_0 = 7000$ V, $\mathbf{F} = \mathbf{F}_s$ and initial temperature perturbation $d\theta^{(h)}$ with $K = 6$.

7. Conclusion

In this thesis, we analyzed the TEHD Boussinesq equations by analytical and numerical means. First, a variational formulation for the steady equations was given. Here, the crucial point is the mathematical modeling of the DEP force term $|\nabla\Phi|^2\nabla\theta$ due to the high regularity that is required on Φ and θ to make this term meaningful. Therefore, the DEP force is replaced by a general force term $\mathbf{F}$ which should satisfy certain conditions. For such $\mathbf{F}$, existence and stability of stationary solutions of the standard Boussinesq equations, i.e. with fixed potential Φ, was shown by extending existing results given by [57]. Afterward, existence and stability of solutions for TEHD Boussinesq equations was proven by employing a fixed-point iteration. Additionally, we showed uniqueness of solutions under a suitable small data condition.

Existence and stability of instationary solutions was shown by extending a proof approach given in [77] for the incompressible Navier-Stokes equations. Here, the unsteady problem is discretized in time, leading to a sequence of N steady problems with $k = \frac{T}{N}$ denoting the time step size. Existence and stability of solution sequences for those steady equations follows by the previously derived results. Unsteady solutions are then constructed by the taking the limit $k \to 0$.

So far, the body force $\mathbf{F}$ has been kept rather general. We then proposed several modelizations of the original DEP force $|\nabla\Phi|^2\nabla\theta$ which fit into the previously derived framework. These models are either based on linearization around a given base potential Φ_0, or on the use of a regularization operator. The later modelization is used to propose an alternative notion of steady and unsteady solutions, where the strong connection between potential and electric gravity $\mathbf{g}_E = \nabla^2\Phi\nabla\Phi$, is weakened.

The second main part of this thesis addressed the discretization of the stationary and instationary TEHD Boussinesq equations. A spatial discretization based on the conforming finite element method and a temporal discretization based on a variant of BDF was proposed. The main feature of the temporal discretization is the fact, that it is kept general, i.e. first and second order methods fit into the given time stepping scheme. Moreover, this temporal scheme allows to solve the resulting set of discretized equations in a decoupled way.

A priori error estimates were derived for the stationary and instationary problem. In the former case, another small data condition is supposed. For the unsteady error analysis, the proof approach given by [75] for the standard Boussinesq equations was extended to take into account the additional Gauss' law and DEP force. Additionally, we did not restrict ourselves to first order in time methods.

The derived error analysis works for the same DEP modelizations, that are proposed to show existence of solutions. Additionally, we presented another DEP formulation based on a simple cut-off function; thus being more suitable for practical realizations. In doing so, another small data condition has to supposed. However, this one is way less restrictive as the one that posed for the steady problem. Indeed, we showed that this condition is satisfied for a realistic scenario in the section on numerical experiments.

We concluded this thesis with numerical experiments. First, the theoretically derived convergence rates were validated for a 2D benchmark model. Then, the 3D flow of a dielectric fluid contained in a cylindrical gap was numerically investigated. We showed that heat transfer is enhanced by application of an electric field and visualized the resulting fluid states under Earth's gravity and zero-gravity conditions. It was pointed out that vorticity grows exponentially in time if the fluid experiences the DEP force. We further showed that a critical voltage difference exists, below which the fluid stays in its initial state. The results and computational effort of different time stepping schemes was compared. Moreover, the different DEP formulations were compared, which showed that linearized models are a good approximation to the exact DEP formulation, at least in the considered scenario. Afterward, we investigated the sensitivity of the solution w.r.t. perturbation by solving the associated adjoint problem. It was pointed out that the norm of the dual solution grows exponentially as $t \to 0$. We deduced that the resulting fluid motion is highly sensitive w.r.t. perturbations in the initial phase of the simulation. Using this information, we defined perturbed initial conditions and could thereby construct steady solutions of different azimuthal wave numbers. Thus, the corresponding steady problem exhibits more than one solution. Finally, we compared our numerical solution with experimental data, provided by our project partners at BTU Cottbus, and a good agreement was found.

There are several issues where future work could lead to new contributions. A more stable spatial discretization could lead to more accurate results and higher stability for high thermal and electrical Rayleigh numbers. Such a discretization could be given by $\mathbf{H}(\mathrm{div})$-conforming elements that provide exactly divergence-free solutions of the incompressible Navier-Stokes equations, e.g. *Raviart-Thomas* and *Brezzi-Douglas-Marini* elements, in combination with *Discontinuous Galerkin* techniques, such as upwinding. By means of these elements it could also be possible, to derive unsteady error estimates that are *Reynolds-semi-robust*, as it is done in [66] for the incompressible Navier-Stokes equations.

Concerning the underlying model, one could also take into account dielectric heating, as presented in [82]. For large temperature differences, the question arises whether the Boussinesq approximation is still accurate, or whether more advanced models, such as the Low-Mach approximation, should be used.

A. Appendix

A.1. General functional analytic results

Throughout, let X denote a non-empty vector space over $\mathbb{R}$.

Definition A.1. *(Normed Space and Banach Space)*
A map $\|\cdot\|_X \colon X \to [0,\infty)$ is called a norm *on X if the following conditions are satisfied*

(i) $\|\alpha x\|_X = |\alpha|\|x\|_X$ *for all* $\alpha \in \mathbb{R}, x \in X$.

(ii) $\|x+y\|_X \le \|x\|_X + \|y\|_X$ *for all* $x, y \in X$.

(iii) $\|x\|_X = 0 \Rightarrow x = 0$.

The space X equipped with norm $\|\cdot\|_X$, written as $(X, \|\cdot\|_X)$, is called normed space *. If any Cauchy sequence in $(X, \|\cdot\|_X)$ converges to an element of X, then $(X, \|\cdot\|_X)$ is called* Banach space*.*

For $R > 0$ and $x \in X$ let $B_R(x, X) := \{y \in X \colon\ \|x-y\|_X \le R\}$ denote the closed ball around x of radius R. In the following, we will often skip the index of the norm and write simply X instead of $(X, \|\cdot\|_X)$.

Definition A.2. *(Inner Product and Hilbert Space)*
A map $(\cdot,\cdot)_X : X \times X \to \mathbb{R}$ is called inner product on X, if

(i) $(\alpha x + \beta y, z)_X = \alpha\,(x,z)_X + \beta\,(y,z)_X$ *for all* $x, y, z \in X, \alpha, \beta \in \mathbb{R}$.

(ii) $(x,y)_X = (y,x)_X$ *for all* $x, y \in X$.

(iii) $(x,x)_X \ge 0$ *for all* $x \in X$.

(iv) $(x,x)_X = 0 \Rightarrow x = 0$.

The inner product space $(X, (\cdot,\cdot)_X)$ *is called* Hilbert space *, if $(X, \|\cdot\|_X)$ is a Banach space, where $\|\cdot\|_X = \sqrt{(\cdot,\cdot)_X}$.*

In the following, we will often skip the index of the inner product and denote the corresponding Hilbert space by X or $(X, \|\cdot\|)$.

Definition A.3. *(Separable Space)*
A normed space X is called separable*, if it contains a countable, dense subset.*

Lemma A.4. *(Subspaces of Separable Spaces, Aufgabe I.4.26 in [80])*
Let X denote a separable normed space. Let $U \subset X, U \neq \emptyset$ denote some subspace. Then U is separable.

Definition A.5. *(Bounded Linear Operator)*
Let X, Y denote normed spaces. A map $T \colon X \to Y$ is called bounded linear operator*, if*

(i) $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ *for all* $x, y \in X,\ \alpha, \beta \in \mathbb{R}$.

(ii) There is $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.

The space of all bounded linear operators from X to Y is denoted by

$$\mathcal{L}(X,Y) := \{T\colon X \to Y : \ T \text{ is bounded and linear } \}.$$

This space is equipped with the norm

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in X, \|x\|_X = 1} \|Tx\|_Y.$$

If Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space as well. If $X = Y$, we write $\mathcal{L}(X) := \mathcal{L}(X,X)$.

A linear operator $T \in \mathcal{L}(X,Y)$ is isometric, *if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. $T \in \mathcal{L}(X,Y)$ is an* isomorphism, *if T is bijective and $T^{-1} \in \mathcal{L}(Y,X)$.*

Theorem A.6. *(Closed Graf Theorem)*
Let X, Y denote Banach spaces. A linear operator $T\colon X \to Y$ is continuous if and only if its graph is closed in $X \times Y$. This implies that the inverse of a bounded, bijective linear operator $T \in \mathcal{L}(X,Y)$ is bounded as well.

Lemma A.7. *(Isometry and Closed Range)*
Let X, Y denote Banach spaces and $T \in \mathcal{L}(X,Y)$ an isometry. Then, the range $T(X)$ is closed in Y.

Proof. Let $(y_n)_n \subset T(X)$ with $y_n \to y$ in Y and $y_n = Tx_n$ for $x_n \in X$. Moreover,

$$\|y_n - y_m\|_Y = \|Tx_n - Tx_m\|_Y = \|x_n - x_m\|_X.$$

Thus, $(x_n)_n$ is a Cauchy sequence. Therefore, there exists $x \in X$ with $x_n \to x$ in X. Further,

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y,$$

i.e. $y \in T(X)$. □

Definition A.8. *(Embedding)*
Let X, Y be normed spaces and $T \in \mathcal{L}(X,Y)$ be injective. Then, T is called embedding. *If the identity operator $Id\colon X \to Y, x \mapsto x$ is an embedding, we write $X \hookrightarrow Y$. In this case, there is a constant $C > 0$ such that*

$$\|x\|_Y \leq C\|x\|_Y \text{ for all } x \in X.$$

If the embedding operator is additionally compact, we write $X \hookrightarrow\hookrightarrow Y$.

Definition A.9. *(Compact Set)*
Let X denote a normed space. A subset K of X is called compact, *if every open covering $\mathcal{C}$ of K contains a finite covering $\{O_1, \ldots, O_m\} \subset \mathcal{C}$ such that $K \subset O_1 \cup \ldots \cup O_n$. A subset C of X is called* sequentially compact, *if every sequence $(x_n) \subset C$ possesses a subsequence converging to some $x \in C$. According to 2.5 in [6], $K \subset X$ is compact if and only if K is sequentially compact.*

Theorem A.10. *(Compact Unit Ball, Satz I.2.7 in [80])*
Let X be a normed space. Then, $B := \overline{B}_X(0,1)$ with $B_X(0,1) := \{x \in X : \|x\|_X < 1\}$ is compact if and only if $\dim X < \infty$.

Definition A.11. *(Compact Linear Operator)*
Let X, Y denote Banach spaces. A linear operator $T \colon X \to Y$ is compact *if $\overline{T(B_X(0,1))}$ is compact in Y. Equivalently, for any bounded sequence $(x_n)_n \subset X$, the sequence $(Tx_n)_n \subset Y$ contains a converging subsequence. Moreover, a compact linear operator is bounded.*

Definition A.12. *(Compact Nonlinear Operator, Definition 25 in [46])*
Let X denote a Hilbert space. An operator $F \colon X \to X$ is compact *if it maps bounded sets to sets with compact closure in X.*

Lemma A.13. *(Compact Operator in Finite Dimensions)*
Let X denote a finite dimensional Hilbert space and $F \colon X \to X$ a continuous function. Then, F is compact.

Proof. Let $M \subset X$ denote a bounded set. Then, $\overline{M}$ is closed and bounded, thus compact since X is finite dimensional by using Theorem A.10. Let $K := F(\overline{M})$ and $\mathcal{C}$ denote an open covering of K. Since F is continuous, $F^{-1}(U) := \{x \in X \colon\ F(x) \in U\}$ is open for all open subset $U \subset X$. Therefore, $\mathcal{D} := \{F^{-1}(U) :\ U \in \mathcal{C}\}$ is an open cover of $\overline{M}$. Since $\overline{M}$ is compact, there exists $U_1, \ldots, U_m \in \mathcal{C}$ for some $m \geq 1$ such that $\overline{M} \subset F^{-1}(U_1) \cup \ldots \cup F^{-1}(U_m)$. Now

$$\begin{aligned}
F(\overline{M}) &\subset F\left(F^{-1}(U_1) \cup \ldots \cup F^{-1}(U_m)\right) \\
&\subset F(F^{-1}(U_1)) \cup \ldots \cup F(F^{-1}(U_m)) \\
&= U_1 \cup \ldots \cup U_m.
\end{aligned}$$

Therefore, $\mathcal{C}$ contains a finite covering of $F(\overline{M})$, i.e. $F(\overline{M})$ is compact. Thus, it is also bounded, implying that $F(M) \subset F(\overline{M})$ is bounded as well. Therefore, the closure of $F(M)$ is bounded and closed, thus compact in X. $\square$

Definition A.14. *(Linear Form and Dual Space)*
Let X denote a normed space. A bounded linear operator from X to $\mathbb{R}$ is called linear form. *The space of all linear forms on X is called* dual space *of X. One denotes $X^* = \mathcal{L}(X, \mathbb{R})$ with associated norm*

$$\|\phi\|_{X^*} = \sup_{x \in X, \|x\|_X = 1} |\phi(x)|.$$

We will use the following notation for the dual pairing of X and X^*

$$\langle \phi, x \rangle_{X^*,X} := \langle \phi, x \rangle_{X^*} := \langle \phi, x \rangle := \phi(x) \text{ for } \phi \in X^*, x \in X.$$

Definition A.15. *(Adjoint Operator)*
Let X, Y denote normed spaces and $T \in \mathcal{L}(X, Y)$. The adjoint *of T is defined as*

$$T^* \colon Y^* \to X^*, \ \langle T^* y^*, x \rangle_{X^*} := \langle y^*, Tx \rangle_{Y^*} \text{ for } y^* \in Y^*, x \in X.$$

There holds $T^* \in \mathcal{L}(Y^*, X^*)$ with $\|T^*\|_{\mathcal{L}(Y^*,X^*)} \leq \|T\|_{\mathcal{L}(X,Y)}$.

Theorem A.16. *(Schauder Theorem, Satz III.4.4 in [80])*
Let X, Y denote Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is compact if and only if $T^* \in \mathcal{L}(Y^*, X^*)$ is compact.

Lemma A.17. *(Adjoint Operator and Isomorphism)*
Let X, Y denote Banach spaces and $T \colon X \to Y$ an isomorphism. Then, $T^* \colon Y^* \to X^*$ is an isomorphism.

Proof. Let $T^* y^* = 0$ for some $y^* \in Y^*$. By definition, $\langle y^*, Tx \rangle = 0$ for all $x \in X$. By surjectivity of T, $\langle y^*, y \rangle = 0$ for all $y \in Y$, i.e. $y^* = 0$. Thus, T^* is injective. On the other hand, let $x^* \in X^*$ be arbitrary. Define $\langle y^*, y \rangle := \langle x^*, T^{-1} y \rangle$. Since $T^{-1} \in \mathcal{L}(Y, X)$, $y^* \in Y^*$. Moreover, for any $x \in X$,

$$\langle T^* y^*, x \rangle = \langle y^*, Tx \rangle = \langle x^*, T^{-1} Tx \rangle = \langle x^*, x \rangle,$$

i.e. $T^* y^* = x^*$. □

Definition A.18. *(Orthogonal Complement and Polar)*
Let X be a Hilbert space and $V \subset X$ a subspace. The orthogonal complement *and* polar *of V are defined as*

$$\begin{aligned} V^{\perp} &:= \{x \in X : \ (x, v)_X = 0 \ \forall v \in V\} \\ V^{\circ} &:= \{g^* \in X^* : \ \langle g, v \rangle_{X^*} = 0 \ \forall v \in V\}. \end{aligned}$$

Definition A.19. *(Reflexive Space)*
Let X be a normed space. The bidual *of X is defined as $X^{**} := (X^*)^*$. For each $x \in X$ define $J_X(x) \colon X^* \to \mathbb{R}$ by*

$$\langle J_X(x), x^* \rangle_{X^{**}} := \langle x^*, x \rangle_{X^*} \text{ for all } x^* \in X^*.$$

According to Satz III.3.1 in [80], $J_X(x) \in X^{**}$ and $J_X \colon X \to X^{**}$ is a linear isometry. The space X is called reflexive, *if J_X is surjective.*

Theorem A.20. *(Riesz Representation Theorem, Theorem V.3.6 in [80])*
Let H denote a Hilbert space. Then, for all $\phi \in H^*$ there exists a unique $g \in H$ such that $\langle \phi, f \rangle_{H^*} = (g, f)_H$ for all $f \in H$. Moreover, $\|g\|_H = \|\phi\|_{H^*}$.

Definition A.21. *(Riesz Identification)*
Let H denote a Hilbert space. Define

$$J_H\colon H \to H^*,\ g \mapsto (g,\cdot)_H.$$

According to Theorem A.20, J_H is bijective and isometric. Thus, H and H^* are isomorphic, written as $H \cong H^*$. Moreover their elements are identified in the following sense: For $x \in H$, $x \in H^*$ means that $J_H(x) \in H^*$.

Theorem A.22. *(Hilbert spaces are reflexive, Korollar V.3.7 in [80])*
Let X be a Hilbert space. Then X is reflexive.

Lemma A.23. *(Subspace of Reflexive Space, Satz III.3.4 in [80])*
Let X denote a reflexive Banach space and $Y \subset X$ a closed subspace. Then Y is reflexive.

Lemma A.24. *(Reflexive Dual Space, Satz III.3.4 in [80])*
Let X denote a reflexive Banach space. Then X^* is reflexive.

Lemma A.25. *(Theorem 2.4.6 in [16])*
Let X be a Banach space. If X^* is separable, then X is separable.

Lemma A.26. *(Reflexivity and Separability)*
Let X denote a reflexive and separable Banach space. Then, X^* is separable.

Proof. Let $M \subset X$ denote a dense and countable subset. Then, $M^{**} := \{J_X(x)\colon x \in M\}$ is countable. Let $x^{**} \in X^{**}$ be arbitrary. Let $x := J_X^{-1}(x^{**})$ and $(x_n)_n \subset M$ with $x_n \to x$. Then, there holds for $(x_n^{**} := J_X(x_n))_n \subset M^{**}$:

$$\|x^{**} - x_n^{**}\|_{X^{**}} = \|J_X(x - x_n)\|_{X^{**}} = \|x - x_n\|_X \to 0.$$

Thus, M^{**} is dense in X^{**}, i.e. X^{**} is separable. Using Lemma A.25, X^* is separable as well. $\square$

Lemma A.27. *(Reflexivity and Isomorphism)*
Let X, Y denote Banach spaces and $T \in \mathcal{L}(X,Y)$ be an isomorphism. If X is reflexive, then Y is reflexive.

Proof. Let $J_X\colon X \to X^{**}$ and $J_Y\colon Y \to Y^{**}$ be defined according to Definition A.19. Using Lemma A.17, $T^*\colon Y^* \to X^*$ is an isomorphism. Thus, again by Lemma A.17, $T^{**}\colon X^{**} \to Y^{**}$

is an isomorphism. Define $J := T^{**}J_X T^{-1} \in \mathcal{L}(Y, Y^{**})$. Let $y^* \in Y^*$, $y \in Y$ be arbitrary. Then,

$$\begin{aligned}\langle Jy, y^*\rangle_{Y^{**}} &= \langle T^{**}J_X T^{-1}y, y^*\rangle_{Y^{**}} \\ &= \langle J_X T^{-1}y, T^* y^*\rangle_{X^{**}} \\ &= \langle T^* y^*, T^{-1}y\rangle_{X^*} \\ &= \langle y^*, TT^{-1}y\rangle_{Y^*} \\ &= \langle y^*, y\rangle_{Y^*} \\ &= \langle J_Y y, y^*\rangle_{Y^{**}}.\end{aligned}$$

Thus, $J = J_Y$ and accordingly, J_Y is an isomorphism since T^{**}, J_X, T^{-1} are isomorphisms each. □

Theorem A.28. *(Consequence of Hahn-Banach, Korollar III.1.6 and III.1.7 in [80])*
Let X be a normed space, and $x \in X$. Then the following assertions hold

(i) If $x \neq 0$, there exists $x^* \in X^*$ with $\langle x^*, x\rangle_{X^*} = \|x\|$ and $\|x^*\| = 1$.

(ii) There holds

$$\|x\| = \max_{x^* \in X^*, \|x^*\| \le 1} |\langle x^*, x\rangle_{X^*}|.$$

Definition A.29. *(Weak Convergence)*
Let X be a normed space.

(i) A sequence $(x_n)_n \subset X$ converges weakly *to $x \in X$, written as $x_n \rightharpoonup x$, if*

$$\text{for all } x^* \in X^* : \langle x^*, x_n\rangle_{X^*} \to \langle x^*, x\rangle_{X^*} \text{ as } n \to \infty.$$

(ii) A sequence $(x_n^*)_n \subset X^*$ converges weakly* *to $x^* \in X^*$, written as $x_n^* \overset{*}{\rightharpoonup} x^*$, if*

$$\text{for all } x \in X : \langle x_n^*, x\rangle_{X^*} \to \langle x^*, x\rangle_{X^*} \text{ as } n \to \infty.$$

Lemma A.30. *(Weak Limits are Unique)*
Let $x_n \rightharpoonup x_1$ and $x_n \rightharpoonup x_2$. Then, $x_1 = x_2$.

Proof. Follows directly from Theorem A.28. □

Lemma A.31. *(Weak Limits and Embedding)*
Let X, Y denote Banach spaces with $X \hookrightarrow Y$ and $x_n \rightharpoonup x$ in X for some sequence $(x_n)_n \subset X$ and $x \in X$. Then, $x_n \rightharpoonup x$ in Y.

Proof. Let $\phi \in Y^*$ be arbitrary and define $\tilde{\phi} := \phi_{|X}$. Then, $\tilde{\phi} \in X^*$. Thus, $\phi(x_n) = \tilde{\phi}(x_n) \to \phi(x)$, i.e. $x_n \rightharpoonup x$ in Y. □

Theorem A.32. *(Weak Convergence in Reflexive Spaces, Theorem III.3.7 in [80])*
Let X be a reflexive Banach space. Then, every bounded sequence (x_n) in X has a weakly convergent subsequence.

Lemma A.33. *(Weak Convergence and Boundedness, Korollar IV.2.3 in [80])*
Let X denote a normed space and $x_n \rightharpoonup x$ in X. Then, $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$.

Lemma A.34. *(Weak Limits and Norm convergence)*
Let X be a normed space.

(*i*) *If $x_n \rightharpoonup x$ in X, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$*

(*ii*) *If $\phi_n \overset{*}{\rightharpoonup} \phi$ in X^*, then $\|\phi\|_{X^*} \leq \liminf_{n\to\infty} \|\phi_n\|_{X^*}$*

Proof. (*i*). Let $\phi \in X^*$ with $\langle \phi, x\rangle = \|x\|$ and $\|\phi\|_{X^*} \leq 1$, by Theorem A.28. Then, the assertion follows from

$$\|x\| = \langle \phi, x\rangle = \liminf_{n\to\infty} \langle \phi, x_n\rangle \leq \liminf_{n\to\infty} \|\phi\|_{X^*}\|x_n\| \leq \liminf_{n\to\infty} \|x_n\|.$$

(*ii*). Let $\epsilon > 0$ and $x \in X$ such that $\|x\| = 1$ and $|\langle \phi, x\rangle_{X^*}| \geq \|\phi\|_{X^*} - \epsilon$. Then,

$$\begin{aligned}
&\lim_{n\to\infty} |\langle \phi_n, x\rangle_{X^*}| = |\langle \phi, x\rangle_{X^*}| \geq \|\phi\|_{X^*} - \epsilon \\
&\lim_{n\to\infty} |\langle \phi_n, x\rangle_{X^*}| \leq \liminf_{n\to\infty} \|\phi_n\|_{X^*}\|x\| = \liminf_{n\to\infty} \|\phi_n\|_{X^*}
\end{aligned}$$

and therefore

$$\|\phi\|_{X^*} - \epsilon \leq \liminf_{n\to\infty} \|\phi_n\|_{X^*}.$$

Since $\epsilon > 0$ was chosen arbitrarily, the assertion follows. □

Lemma A.35. *(Weak Convergence and Compact Embedding Implies Strong Convergence)*
Let X, Y denote Banach spaces and $x_n \rightharpoonup x$ in X. If $T \in \mathcal{L}(X, Y)$ is compact, then $T(x_n) \to T(x)$ in Y.

Proof. According to Lemma A.33, the sequence $(x_n)_n$ is bounded. Thus, there is a subsequence $(x_k)_k \subset (x_n)_n$ and $y \in Y$ such that $Tx_k \to y$ by compactness of T. Now, let $\psi \in Y^*$ be arbitrary. Then, $\phi := \psi \circ T \in X^*$ and therefore $\psi(Tx_n) \to \psi(Tx)$ by $x_n \rightharpoonup x$. Since ψ was chosen arbitrarily, $Tx_n \rightharpoonup Tx$ in Y. Thus, $y = Tx$.

Now, assume that Tx_n does not converge to y in Y. Consequently, there is $\delta > 0$ and a subsequence $(x_l)_l \subset (x_n)_n$ such that $\|Tx_l - y\|_Y > \delta$ for all l. By compactness of T, there is another subsequence $(x_m)_m \subset (x_l)_l$ and $z \in Y$ with $Tx_m \to z$. Due to $\|Tx_m - y\|_Y > \delta$ for all m, $z \neq y$. Addtionally, $Tx_m \rightharpoonup z$. On the other hand, there still holds $Tx_m \rightharpoonup Tx = y$ since $Tx_l \rightharpoonup y$. This is a contradiction to $z \neq y$. Therefore, $Tx_n \to y = Tx$. □

Lemma A.36. *(Strong Convergence in Finite Dimensional Spaces)*
Let X denote a finite dimensional Banach space and $(x_n)_n \subset X$ with $\|x_n\|_X \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$. Then, there is $x^* \in X$ and a subsequence $(x_k)_k \subset (x_n)_n$ such that $x_k \to x^*$ in X.

Proof. As X is finite dimensional, it is isomorphic to the Hilbert space $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and thus reflexive by Lemma A.27. Thus, there is $x^* \in X$ and $(x_k)_k \subset (x_n)_n$ with $x_k \rightharpoonup x^*$ by Theorem A.32. Moreover, the identity map $Id\colon X \to X$ is compact since X is finite dimensional, use Theorem A.10. Using Lemma A.35, this implies $x_n \to x^*$. □

Theorem A.37. *(Banach-Alaoglu, Theorem 3.2.1 in [16])*
Let X denote a separable Banach space. Then, for each bounded sequence $(x_n^*)_n \subset X^*$ there exists $x^* \in X^*$ and a subsequence $(x_k^*)_k$ such that $x_k^* \overset{*}{\rightharpoonup} x^*$ in X^*.

Definition A.38. *(Gelfand Triple)*
let V, H denote separable Hilbert spaces with continuous and dense embedding. Then, the Gelfand triple *is defined as*

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*.$$

Theorem A.39. *(Leray-Schauder Fixed-Point Theorem, Theorem 6.16 in [46])*
Let Y be a Hilbert space and let $\mathcal{F}\colon Y \to Y$ be a compact map. Consider the fixed-point problem:
Find $y^* \in Y$ such that

$$y^* = \mathcal{F}(y^*) \tag{A.1}$$

Associate with (A.1) the family of fixed-point problems:
Find $y_\lambda \in Y$ such that

$$y_\lambda = \lambda \mathcal{F}(y_\lambda), 0 \le \lambda \le 1. \tag{A.2}$$

If there is a constant K such that all solutions of (A.2) are uniformly bounded, i.e. $\|y_\lambda\| \le K$ for all $0 \le \lambda \le 1$, then there exists a solution to (A.1).

Theorem A.40. *(Lax-Milgram, Satz 4.2 in [6])*
Let H denote a real Hilbert space with norm $\|\cdot\|_H$, $a\colon H \times H \to \mathbb{R}$ a bilinear form and $l\colon H \to \mathbb{R}$ a linear form. Assume that there exists $M, N, \alpha > 0$ such that for all $u, v \in H$:

$$a(u,v) \le M\|u\|_H\|v\|_H,\ a(v,v) \ge \alpha\|v\|_H^2,\ l(v) \le N\|v\|_H.$$

Then, there exists a unique solution u of

$$a(u,v) = l(v) \text{ for all } v \in H.$$

Moreover, this solution satisfies

$$\|u\|_H \le \frac{N}{\alpha}.$$

Theorem A.41. *(Well-Posedness Under Inf-Sup Condition, Lemma I.4.1 in [32])*
Let $(X, \|\cdot\|_X)$ and $(M, \|\cdot\|_M)$ denote Hilbert spaces and let a bilinear form $b\colon X \times M \to \mathbb{R}$ be given which is assumed to be bounded, i.e.

$$|b(v,p)| \le M\|v\|_X\|p\|_M \text{ for all } v \in X\ p \in M.$$

Define linear operators

$$B\colon X \to M^*,\ x \mapsto b(x,\cdot)$$
$$B^*\colon M \to X^*,\ p \mapsto b(\cdot,p)$$

and a subspace $V := \{v \in X : \ b(v,q) = 0\ \forall q \in M\}$. *Then, the following statements are equivalent:*

(*i*) *There is* $\beta > 0$ *such that*

$$\inf_{p\in M, p\neq 0} \sup_{v \in X, v\neq 0} \frac{b(v,p)}{\|v\|_X \|p\|_M} \geq \beta.$$

(*ii*) B^* *is an isomorphism from* M *to* V° *with*

$$\|B^* p\|_{X^*} \geq \beta \|p\|_M \text{ for all } p \in M.$$

(*iii*) B *is an isomorphism from* $V^\perp$ *to* M^* *with*

$$\|Bv\|_{M^*} \geq \beta \|v\|_X \text{ for all } v \in V^\perp.$$

Lemma A.42. *(Discrete Gronwall Inequality, Lemma 5.1 and Subsequent Remark in [36])*
Let k, $B \geq 0$ *and* $(a_l)_l$, $(b_l)_l$, $(c_l)_l$, $(\gamma_l)_l \subset [0,\infty)$ *such that*

$$a_n + k\sum_{l=0}^n b_l \leq k \sum_{l=0}^n \gamma_l a_l + k \sum_{l=0}^n c_l + B \text{ for } n \geq 0.$$

Suppose that $k\gamma_l < 1$ *for all* l *and set* $\sigma_l := (1-k\gamma_l)^{-1}$. *Then,*

$$a_n + k \sum_{l=0}^n b_l \leq \exp\left(k \sum_{l=0}^n \sigma_l \gamma_l\right)\left(k\sum_{l=0}^n c_l + B\right) \text{ for } n \geq 0.$$

If $\sum_{l=0}^n \gamma_l a_l$ *can be replaced by* $\sum_{l=0}^{n-1} \gamma_l a_l$, *then the restriction* $k\gamma_l < 1$ *is not necessary and* σ_l *can be set to 1.*

A.2. Bochner Spaces

The statements in this section are taken from [37], [38] and [77]. Let $a, b \in (-\infty,\infty)$ and X denote a Banach space with norm $\|\cdot\|$.

Definition A.43. *(Continuous Functions)*
A function $f\colon [a,b] \to X$ *is* continuous *at some point* $t_0 \in [a,b]$, *if* $\lim_{n\to\infty} \|f(t_0) - f(t)\|_X = 0$ *for all sequences* $(t_n)_n \subset [a,b]$ *with* $\lim_{n\to\infty} t_n = t_0$. *The space of continuous functions is defined as*

$$C(a,b;X) := \{f\colon [a,b] \to X,\ f \text{ is continuous for all } t_0 \in [a,b]\}$$

and equipped with the norm

$$\|f\|_{C(a,b;X)} := \sup_{t\in[a,b]} \|f(t)\|_X.$$

Definition A.44. *(Weakly Continuous Functions)*
A function $f\colon [a,b] \to X$ is weakly continuous*, if the function $[a,b] \ni t \mapsto \langle \phi, f(t)\rangle_{X^*} \in \mathbb{R}$ is continuous for all $\phi \in X^*$.*

Definition A.45. *(Simple Functions)*
A function $f\colon (a,b) \to X$ is called simple*, if it is of the form*

$$f(t) = \sum_{i=1}^{N} c_i \chi_{E_i}(t)$$

where $E_1, \ldots, E_N$ denote Lebesgue measurable subset of (a,b) and $c_1, \ldots, c_N \in X$.

Definition A.46. *(Measurability)*
A function $f\colon (a,b) \to X$ is called strongly measurable *if there is a sequence $\{f_n\}_n$ of simple functions such that $\|f_n(t) - f(t)\| \to 0$ for $n \to \infty$ for almost all t in (a,b).*

Lemma A.47. *(Remark in [38])*
If $f\colon (a,b) \to X$ is strongly measurable and $\phi\colon (a,b) \to \mathbb{R}$ is measurable, then $\phi f\colon (a,b) \to X$ is measurable.

Definition A.48. *(Integral of Simple Functions)*
Let $f = \sum_{i=1}^{N} c_i \chi_{E_i}(t)$ denote a simple function. The integral of f is defined by

$$\int_a^b f dt := \sum_{i=1}^{N} c_i |E_i| \in X.$$

Definition A.49. *(Bochner Integral)*
A strongly measurable function $f\colon (a,b) \to X$ is called Bochner integrable *if there is a sequence of simple functions $\{f_n\}_n$ such that $f_n(t) \to f(t)$ pointwise a.e. in (a,b) and*

$$\lim_{n\to\infty} \int_a^b \|f(t) - f_n(t)\| dt = 0.$$

In this case, the integral of f is defined by

$$\int_a^b f(t)dt := \lim_{n\to\infty} \int_a^b f_n(t)dt.$$

Theorem A.50. *(Bochner Integrability, Theorem 6.24 in [38])*
A function $f\colon (a,b) \to X$ is Bochner integrable if and only if it is strongly measurable and

$$\int_a^b \|f(t)\| dt < \infty.$$

Definition A.51. *(Bochner Space)*
For $p \in [1, \infty)$ define

$$L^p(a,b;X) := \{f\colon (a,b) \to X \text{ strongly measureable },\ \int_a^b \|f(t)\|^p < \infty\},$$

equipped with the norm

$$\|f\|_{L^p(a,b;X)} := \left(\int_a^b \|f(t)\|^p\right)^{\frac{1}{p}}.$$

The space of locally integrable functions is defined by

$$L^p_{loc}(a,b;X) := \{f\colon (a,b) \to X \text{ strongly measurbale }:\ f \in L^p(a',b';X) \textit{ for all } a < a' \le b' < b\}.$$

For $p = \infty$ define

$$L^\infty(a,b;X) := \{f\colon (a,b) \to X \text{ strongly measureable },\ \operatorname{esssup}_{t\in(a,b)}\|f(t)\| < \infty\},$$

equipped with the norm

$$\|f\|_{L^\infty(a,b;X)} := \operatorname{esssup}_{t\in(a,b)}\|f(t)\|.$$

If there is no danger of ambiguity, we write $L^p(X) := L^p(a,b;X)$ and $\|\cdot\|_{p;X} := \|\cdot\|_{L^p(a,b;X)}$. In particular, this is the case for $a = 0$ and $b = T$.

Theorem A.52. *(Banach Space Property, Theorem 7.5.6 in [16])*
If X is a Banach space, then $L^p(a,b;X)$ is a Banach space for $p \in [1,\infty]$.

Theorem A.53. *(Approximation of Bochner Spaces, Proposition 6.29 in [38])*
Let X be a Banach space and $p \in [1,\infty)$. Then the space

$$\{f(t) = \sum_{i=1}^n c_i\phi_i(t)\colon\ n \in \mathbb{N},\ c_i \in X,\ \phi_i \in C_c^\infty(a,b)\}$$

is dense in $L^p(a,b;X)$.

Theorem A.54. *(Separability Property, Korollar 2.11 in [65])*
Let X be a separable Banach space. Then, for $p \in [1,\infty)$ the space $L^p(X)$ is separable.

Theorem A.55. *(Reflexivity Property, Corollary 7.5.17 in [16])*
Let X be a reflexive Banach space. Then, for $p \in (1,\infty)$ the space $L^p(X)$ is reflexive.

Definition A.56. *(Radon-Nikodym Property, Definition 7.5.11 in [16])*
A Banach Space X has the Radon Nikodym property, *if every Lipschitz continuous function $f\colon [0,1] \to X$, is differentiable almost everywhere.*

Theorem A.57. *(Dunford-Pettis, Theorem 7.5.13 in [16])*
Let X denote a Banach space. If X^* is separable, then X^* has the Radon-Nikodym property.

Theorem A.58. *(Consequence of Dunford-Pettis, Theorem 7.5.13 in [16])*
Every reflexive Banach space has the Radon-Nikodym property.

Theorem A.59. *(Dual of Bochner Space, Theorem 1.3.10 in [39])*
Let $p \in [1, \infty)$ and X denote a Banach space with X^* having the Radon-Nikodym property. Then the dual of $L^p(a, b; X)$ is isometrically isomorphic to $L^{p^*}(a, b; X^*)$, i.e. the mapping

$$T\colon L^{p^*}(a, b; X^*) \to L^p(a, b; X)^*$$
$$g \mapsto \phi \text{ with } \langle \phi, f\rangle_{L^p(a,b;X)^*} = \int_a^b \langle g(t), f(t)\rangle_{X^*}\, dt \text{ for } f \in L^p(a, b; X)$$

is an isometric isomorphism.

Lemma A.60.
Let X be a reflexive Banach space. If $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(a, b; X^*)$, then

$$\|u\|_{L^\infty(a,b,X^*)} \leq \liminf_{n\to\infty} \|u_n\|_{L^\infty(a,b,X^*)}.$$

Proof. Due to the assumptions on X, the space $L^1(a, b, X)^*$ is isometrically isomorphic to $L^\infty(a, b; X^*)$, by Theorem A.58 and A.59. Let $T\colon L^\infty(a, b; X^*) \to L^1(a, b; X)^*$ denote the isometric isomorphism defined in Theorem A.59. Define $\phi_n := Tu_n$, $\phi = Tu \in L^1(a, b; X)^*$. Then, $u_n \overset{*}{\rightharpoonup} u$ is equivalent to $\phi_n \overset{*}{\rightharpoonup} \phi$ in $L^1(a, b; X)^*$. By Lemma A.34, have $\|\phi\|_{L^1(a,b;X)^*} \leq \liminf_{n\to\infty} \|\phi_n\|_{L^1(a,b;X)^*}$ and the assertion follows from the isometry property of T. □

Lemma A.61. *(Weak $*$ convergence in $L^\infty(a, b; X)$)*
Let X denote a separable, reflexive Banach space and a sequence $(u_n)_n \subset L^\infty(a, b; X^*)$ be given with $\|u_n\|_{L^\infty(a,b;X^*)} \leq M$ for all $n \in \mathbb{N}$. Then, there is $u \in L^\infty(a, b; X^*)$ and a subsequence $(u_k)_k \subset (u_n)_n$, such that $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(a, b; X^*)$. Moreover, there holds $\|u\|_{L^\infty(a,b;X^*)} \leq M$.

Proof. According to Theorem A.59, $L^\infty(X^*) \cong L^1(X)^*$. Define $\phi_n := Tu_n \in L^1(X)^*$ according to the isometric isomorphism T defined in Theorem A.59. Since $\|\phi_n\|_{L^1(X)^*} = \|u_n\|_{L^\infty(a,b;X^*)} \leq M$ and $L^1(X)$ is a separable Banach space due to Theorem A.52 and A.54, Theorem A.37 is applicable. Thus, there is $\phi \in L^1(X)^*$ and a subsequence $(\phi_k)_k \subset (\phi_n)_n$ with $\phi_k \overset{*}{\rightharpoonup} \phi$ in $L^1(X)^*$. Due to Theorem A.59, there is $u \in L^\infty(X^*)$ corresponding to ϕ, i.e. $u = T^{-1}\phi$. Thus,

$$\int_a^b \langle u_k(t), f(t)\rangle_{X^*}\, dt \to \int_a^b \langle u(t), f(t)\rangle_{X^*}\, dt \text{ for all } f \in L^1(X),$$

i.e. $u_n \overset{*}{\rightharpoonup} u$. Moreover, by Lemma A.60, $\|u\|_{L^\infty(a,b,X^*)} \leq \liminf \|u_n\|_{L^\infty(a,b,X^*)} \leq M$. □

Lemma A.62. *(Remark after Proposition 1.2.3 in [39])*
Let $f\colon (a,b) \to X$ be integrable and $A \in \mathcal{L}(X,Y)$ with Banach spaces X, Y. Then, $Af\colon (a,b) \to Y$ is integrable and

$$A\left(\int_a^b f(t)dt\right) = \int_a^b A(f(t))dt.$$

Lemma A.63. *(Corollary 6.33 in [38])*
Suppose that $f\colon (a,b) \to X$ is locally integrable and

$$\int_a^b \phi(t)f(t)dt = 0 \text{ for all } \phi \in C_c^\infty(a,b).$$

Then, $f(t) = 0$ a.e. in (a,b).

Lemma A.64.
Let X denote a Banach space and $A \in L^1(a,b;X^*)$ such that

$$\int_a^b \langle A(t), x\rangle_{X^*}\psi(t)\,dt = 0 \text{ for all } x \in X \text{ and } \psi \in C_c^\infty(a,b).$$

Then, $A(t) = 0$ a.e. in (a,b).

Proof. Let $\psi \in C_c^\infty(a,b)$ be fixed but arbitrary. Define $I := \int_a^b A(t)\psi(t)\,dt \in X^*$ and

$$J\colon X \to X^{**},\ x \mapsto \langle \cdot, x\rangle_{X^*}.$$

Using Lemma A.62, there holds for all $x \in X$:

$$\langle I, x\rangle_{X^*} = J(x)[I] = \int_a^b J(x)[A(t)\psi(t)]\,dt = \int_a^b \langle A(t), x\rangle_{X^*}\psi(t)\,dt = 0.$$

Thus,

$$0 = I = \int_a^b A(t)\psi(t)\,dt \text{ in } X^*.$$

Since ψ was chosen arbitrarily, Lemma A.63 now yields $A(t) = 0$ a.e. □

A.3. Vector-Valued Sobolev Spaces

The statements in this section are taken from [37], [38] and [77]. Throughout, let $a, b \in (-\infty, \infty)$ and X denote a Banach space with norm $\|\cdot\|$. Moreover, $\mathcal{D}(a,b) := C_c^\infty((a,b))$.

Definition A.65. *(Frechet Derivative)*
Let V, W denote normed spaces and $U \subset V$ be open. A function $f\colon U \to W$ is Frechet differentiable at $x \in U$, *if there is $A \in \mathcal{L}(V,W)$ such that*

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0.$$

In this case, A is called Frechet derivative *of f at x and denoted by $\partial_x f$. The function f is called* Frechet differentiable on U, *if f is Frechet differentiable at every point $x \in U$ and if the mapping*

$$Df\colon U \to \mathcal{V}, \mathcal{W},\ x \mapsto \partial_x f(x)$$

is continuous.

Definition A.66. *(Weak Time Derivative, Definition 6.31 in [38])*
Let $u, g \in L^1_{loc}(a, b; X)$. Then, g is called weak time derivative of u, if

$$\int_a^b u(t)\phi'(t)dt = -\int_a^b g(t)\phi(t)dt \text{ for all } \phi \in \mathcal{D}(a, b).$$

In this case, one writes

$$d_t u = g \text{ or } u' = g.$$

Theorem A.67. *(Characterization of Weak Time Derivative, Lemma III.1.1 in [77])*
Let $u, g \in L^1(a, b; X)$. Then, the following statements are equivalent

(i) $g = u'$.

(ii) *There exists $\xi \in X$ such that*

$$u(t) = \xi + \int_a^t g(s)ds, \ a.e.\, t \in [a, b].$$

(iii) *For each $\phi \in X^*$ there holds*

$$\frac{d}{dt}\langle \phi, u\rangle_{X^*} = \langle \phi, g\rangle_{X^*} \text{ in } \mathcal{D}(a, b)^*,$$

i.e.

$$-\int_a^b \langle \phi, u(t)\rangle_{X^*}\psi'(t)dt = \int_a^b \langle \phi, g(t)\rangle_{X^*}\psi(t)dt \text{ for all } \psi \in \mathcal{D}(a, b).$$

In this case, u is a.e. equal to a continuous function from $[a, b]$ to X.

Lemma A.68. *(Product Rule and Partial Integration)*
Let $u \in L^1(a, b; X)$ with derivative $u' \in L^1(a, b; X)$ and $\Phi \in C^\infty([a, b])$. Then, $(\Phi\, u)' = \Phi'\, u + \Phi\, u'$ and

$$\int_a^b \Phi(t)u'(t)\, dt = \Phi(b)u(b) - \Phi(a)u(a) - \int_a^b \Phi'(t)u(t)\, dt.$$

Proof. Let $\phi \in \mathcal{D}(a, b)$ be arbitrary. Then, $\Phi\,\phi \in \mathcal{D}(a, b)$ and by definition of u',

$$\int_a^b u'(t)(\Phi\phi)(t)\, dt = -\int_a^b u(t)\left(\Phi\phi\right)'(t)\, dt = -\int_a^b u(t)\Phi'(t)\phi(t) + u(t)\Phi(t)\phi'(t)\, dt.$$

Thus,

$$(u\Phi)' = u'\Phi + u\Phi'.$$

Moreover, using Theorem A.67 *(ii)*,

$$(u\Phi)(b) - (u\Phi)(a) = \int_a^b (u\Phi)'(t)\, dt = \int_a^b (u'\Phi)(t)\, dt + \int_a^b (u\Phi')(t)\, dt.$$

$\square$

Definition A.69. *(Vector-Valued Sobolev Space)*
Let $X_0,\ X_1$ denote Banach spaces with $X_0 \hookrightarrow X_1$ and $p_0,\ p_1 \in [1,\infty]$. Define

$$\mathcal{W} := \mathcal{W}(a,b;p_0,p_1;X_0,X_1) := \{v \in L^{p_0}(a,b;X_0),\ v' \in L^{p_1}(a,b;X_1)\},$$

equipped with the norm

$$\|u\|_{\mathcal{W}} := \|u\|_{L^{p_0}(a,b;X_0)} + \|u'\|_{L^{p_1}(a,b;X_1)}.$$

Theorem A.70. *(Banach Space Property and Continuous Injection, [69])*
Let $p_0, p_1 \in (1,\infty)$ and Banach spaces $X_0 \hookrightarrow X \hookrightarrow X_1$ be given. Then, $\mathcal{W}(a,b;p_0,p_1;X_0,X_1)$ is a Banach space and there holds

$$\mathcal{W}(a,b;p_0,p_1;X_0,X_1) \hookrightarrow L^{p_0}(a,b;X).$$

Theorem A.71. *(Compact injection I, Theorem III.2.1 in [77])*
Let $p_0, p_1 \in (1,\infty)$ and Banach spaces $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ be given with $X_0,\ X_1$ being reflexive. Then,

$$\mathcal{W}(a,b;p_0,p_1;X_0,X_1) \hookrightarrow\hookrightarrow L^{p_0}(a,b;X).$$

Theorem A.72. *(Compact injection II, Theorem 1.3 in [69])*
Let $p_0, p_1 \in [1,\infty)$ with $p_0 = 1$ if $p_1 = 1$ and Banach spaces $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ be given. Then,

$$\mathcal{W}(a,b;p_0,p_1;X_0,X_1) \hookrightarrow\hookrightarrow L^{p_0}(a,b;X).$$

Theorem A.73. *(Hilbert Space Setting, Theorem 1.32 in [37])*
Let $V \hookrightarrow H \hookrightarrow V^*$ denote a Gelfand triple. Then, $\mathcal{W}(a,b;2,2;V,V^*)$ is a Hilbert space and

$$\mathcal{W}(a,b;2,2;V,V^*) \hookrightarrow C(a,b;H).$$

In this setting, the space $\mathcal{W}_0 := \{v \in \mathcal{W}(a,b;2,2;V,V^*)\colon\ v(0) = 0\}$ is well-defined.
Moreover, for all $u, v \in \mathcal{W}(a,b;2,2;V,V^*)$, there holds

$$\int_a^b \langle u'(t), v(t)\rangle_{V^*} dt = (u(b), v(b))_H - (u(a), v(a))_H - \int_a^b \langle v'(t), u(t)\rangle_{V^*} dt.$$

Lemma A.74. *(Reflexive Property)*
Let $p_0, p_1 \in (1,\infty)$ and Banach spaces $X_0 \hookrightarrow X_1$ be given with $X_0,\ X_1$ being reflexive and separable. Then, $\mathcal{W}(a,b;p_0,p_1;X_0,X_1)$ is reflexive.

Proof. Let $\mathcal{W} := \mathcal{W}(a,b;p_0,p_1;X_0,X_1)$ and define

$$\begin{aligned} T\colon \mathcal{W} &\to L^{p_0}(a,b;X_0) \times L^{p_1}(a,b;X_1) \\ u &\mapsto (u,u'). \end{aligned}$$

Then T is an isometry, Moreover, $L^{p_0}(a,b;X_0)$ and $L^{p_1}(a,b;X_1)$ are reflexive according to Theorem A.55 and so is $L^{p_0}(a,b;X_0) \times L^{p_1}(a,b;X_1)$. Further, $T(\mathcal{W})$ is closed in $L^{p_0}(a,b;X_0) \times L^{p_1}(a,b;X_1)$ by Lemma A.7. Thus, $T(\mathcal{W})$ is reflexive by Lemma A.23. Since $T\colon \mathcal{W} \to T(\mathcal{W})$ is an isomorphism, $\mathcal{W}$ is reflexive as well according to Lemma A.27. □

Lemma A.75. *(Weak Limit of Weakly Differentiable Sequence)*
Let $X \hookrightarrow Y$ denote Banach spaces and a sequence $(w_k)_k \subset \mathcal{W}(a,b;1,1;X,Y)$ be given with $w_k \rightharpoonup u$ in $L^1(a,b;X)$ and $w_k' \rightharpoonup g$ in $L^1(a,b;Y)$. Then, $u \in \mathcal{W}(a,b;1,1;X,Y)$ with $u' = g$.

Proof. By Assumption, there holds for all $k \in \mathbb{N}$:

$$\int w_k(t)\phi'(t)\,dt = -\int w_k'(t)\phi(t)\,dt \text{ for all } \phi \in \mathcal{D}(a,b). \tag{A.3}$$

Let $\phi \in \mathcal{D}(a,b)$ be arbitrary but fixed. For arbitrary $f \in X^*$ and $t \in (a,b)$ define $\psi(t) := \phi'(t)\,f \in X^*$. Then, $\psi \in L^\infty(a,b;X^*) \cong \big(L^1(a,b;X)\big)^*$. Using Lemma A.62, Theorem A.59 and $w_k \rightharpoonup u$

$$\begin{aligned}\langle f, \int w_k(t)\phi'(t)dt\rangle_{X^*} = \int \langle f, w_k(t)\rangle_{X^*}\phi'(t)dt = \langle \psi, w_k\rangle_{L^1(X)^*}& \\ \to \langle \psi, u\rangle_{L^1(X)^*} = \langle f, \int u(t)\phi'(t)dt\rangle_{X^*}&.\end{aligned}$$

Thus,

$$\int w_k(t)\phi'(t)dt \rightharpoonup \int u(t)\phi'(t)dt \text{ in } X \text{ and thus in } Y \text{ since } X \hookrightarrow Y \text{ and Lemma A.31 }.$$

Analogously,

$$\int w_k'(t)\phi(t)dt \rightharpoonup \int g(t)\phi(t)dt \text{ in } Y.$$

Using (A.3),

$$\int u(t)\phi'(t)dt \leftharpoonup \int w_k(t)\phi'(t)dt = -\int w_k'(t)\phi(t)dt \rightharpoonup -\int g(t)\phi(t)dt \text{ in } Y.$$

By uniqueness of weak limits, Lemma A.30, and since ϕ was chosen arbitrarily, $u' = g \in L^1(a,b;Y)$. □

Lemma A.76. *(Taylor Expansion I)*
Let $a \geq 0$, $k > 0$ and $u \in L^2(a,a+k;X)$ with $u' \in L^1(a,a+k;X)$. For $\beta \in [0,\frac{1}{2})$ assume that $t^\beta u' \in L^2(a,a+k;X)$. Then,

if $a > 0$:

$$\|u(a+k) - u(a)\|_X \leq a^{-\beta} k^{\frac{1}{2}} \|t^\beta u'\|_{L^2(a,a+k;X)}.$$

if $a = 0$:

$$\|u(k) - u(0)\|_X \leq (1-2\beta)^{-\frac{1}{2}}\, k^{\frac{1}{2}-\beta} \|t^\beta u'\|_{L^2(0,k;X)}.$$

Proof. By Theorem A.67 there is $\xi \in X$ such that

$$u(t) = \xi + \int_a^t u'(s)\,ds \text{ for all } t \in [a, a+k].$$

Setting $t = a$ yields $\xi = u(a)$. Setting $t = a+k$, taking the norm and using Hölders inequality leads to

$$\begin{aligned}
\|u(a+k) - u(a)\|_X &\le \Big\| \int_a^{a+k} u'(s)\,ds \Big\|_X \\
&\le \int_a^{a+k} s^{-\beta} \|s^\beta u'(s)\|_X\,ds \\
&\le \left(\int_a^{a+k} s^{-2\beta}\,ds \right)^{\frac{1}{2}} \left(\int_a^{a+k} \|s^\beta u'(s)\|_X^2\,ds \right)^{\frac{1}{2}}.
\end{aligned}$$

The stated result follows from

$$\begin{aligned}
\int_a^{a+k} s^{-2\beta}\,ds &\le a^{-2\beta} k \qquad \text{for } \beta \ge 0 \text{ and } a > 0, \\
\int_0^k s^{-2\beta}\,ds &= \frac{1}{1-2\beta} k^{1-2\beta} \text{ for } \beta < \frac{1}{2}.
\end{aligned}$$

□

Lemma A.77. *(Taylor Expansion II)*
Let $a \ge 0$, $k > 0$ and $u \in L^2(a, a+k; X)$ with u', $u'' \in L^1(a, a+k; X)$ and $t^\beta u'' \in L^2(a, a+k; X)$ for some $\beta \in [0, \frac{3}{2})$. Then,

if $a > 0$:

$$\|u'(a+k) - k^{-1}\left(u(a+k) - u(a)\right)\|_X \le k^{\frac{1}{2}} a^{-\beta} \|t^\beta u''\|_{L^2(a,a+k;X)}.$$

if $a = 0$:

$$\|u'(k) - k^{-1}\left(u(k) - u(0)\right)\|_X \le (3-2\beta)^{-\frac{1}{2}}\, k^{\frac{1}{2}-\beta} \|t^\beta u''\|_{L^2(0,k;X)}.$$

Proof. Using integration by parts, Lemma A.68 and Theorem A.67 (ii),

$$\begin{aligned}
\int_a^{a+k} k^{-1}(t-a)u''(t)\,dt &= k^{-1}(ku'(a+k)) - \int_a^{a+k} k^{-1} u'(t)\,dt \\
&= u'(a+k) - k^{-1}\left(u(a+k) - u(a)\right).
\end{aligned}$$

Further, the left-hand side above can be estimated as

$$\begin{aligned}
\Big\| \int_a^{a+k} k^{-1}(t-a)u''(t)\,dt \Big\|_X &\le k^{-1} \int_a^{a+k} (t-a) t^{-\beta} \|t^\beta u''(t)\|_X\,dt \\
&\le k^{-1} \left(\int_a^{a+k} (t-a)^2 t^{-2\beta}\,dt \right)^{\frac{1}{2}} \|t^\beta u''\|_{L^2(a,a+k;X)}.
\end{aligned}$$

If $a > 0$, then

$$\int_a^{a+k} (t-a)^2 t^{-2\beta}\,dt \le k^3 a^{-2\beta},$$

and, if $a = 0$ and $\beta < \frac{3}{2}$,

$$\int_0^k t^2 t^{-2\beta}\, dt = \frac{1}{3-2\beta} k^{3-2\beta}.$$

$\square$

Lemma A.78. *(Taylor Expansion III)*
Let $a \geq 0$, $k > 0$ and $u \in L^2(a, a+2k; X)$ with u', $u'' \in L^1(a, a+2k; X)$ and $t^\beta u'' \in L^2(a, a+2k; X)$ for some $\beta \in [0, \frac{3}{2})$. Then,

if $a > 0$:

$$\|u(a+2k) - 2u(a+k) + u(a))\|_X \leq k^{\frac{3}{2}} \left((a+k)^{-\beta} + a^{-\beta}\right) \|t^\beta u''\|_{L^2(a,a+2k;X)}.$$

if $a = 0$:

$$\|u(2k) - 2u(k) + u(0)\|_X \leq k^{\frac{3}{2}-\beta} \left(1 + (3-2\beta)^{-\frac{1}{2}}\right) \|t^\beta u''\|_{L^2(0,2k;X)}.$$

Proof. Using integration by parts, Lemma A.68, one may verify that

$$u(a+2k) - 2u(a+k) + u(a) = \int_{a+k}^{a+2k} (a+2k-t)u''(t)\, dt + \int_a^{a+k} (t-a)u''(t)\, dt.$$

Moreover,

$$\begin{aligned}
\|\int_{a+k}^{a+2k} (a+2k-t)u''(t)\, dt\|_X &\leq \int_{a+k}^{a+2k} (a+2k-t)t^{-\beta}\|t^\beta u''(t)\|_X\, dt \\
&\leq \left(\int_{a+k}^{a+2k} (a+2k-t)^2 t^{-2\beta}\, dt\right)^{\frac{1}{2}} \|t^\beta u''\|_{L^2(a+k,a+2k;X)} \\
&\leq k^{\frac{3}{2}}(a+k)^{-\beta}\|t^\beta u''\|_{L^2(a+k,a+2k;X)}.
\end{aligned}$$

and

$$\begin{aligned}
\|\int_a^{a+k} (t-a)u''(t)\, dt\|_X &\leq \int_a^{a+k} (t-a)t^{-\beta}\|t^\beta u''(t)\|_X\, dt \\
&\leq \left(\int_a^{a+k} (t-a)^2 t^{-2\beta}\, dt\right)^{\frac{1}{2}} \|t^\beta u''\|_{L^2(a,a+k;X)}.
\end{aligned}$$

If $a = 0$, then

$$\int_0^k t^{2-2\beta}\, dt = \frac{1}{3-2\beta} k^{3-2\beta}.$$

Otherwise,

$$\int_a^{a+k} (t-a)^2 t^{-2\beta}\, dt \leq k^3 a^{-2\beta}.$$

$\square$

Lemma A.79. *(Taylor Expansion IV)*
Let $a \geq 0$, $k > 0$ and $u \in L^2(a, a+2k; X)$ with u', u'', $u''' \in L^1(a, a+2k; X)$ and $t^\beta u''' \in L^2(a, a+2k; X)$ for some $\beta \in [0, \frac{5}{2})$. Then,

if $a > 0$:

$$\|u'(a+2k) - k^{-1}\left(\frac{3}{2}u(a+2k) - 2u(a+k) + \frac{1}{2}u(a)\right)\|_X$$
$$\leq k^{\frac{3}{2}}\left((a+k)^{-\beta} + \frac{1}{16}a^{-\beta}\right)\|t^\beta u'''\|_{L^2(a,a+2k;X)}.$$

if $a = 0$:

$$\|u'(2k) - k^{-1}\left(\frac{3}{2}u(2k) - 2u(k) + \frac{1}{2}u(0)\right)\|_X$$
$$\leq k^{\frac{3}{2}-\beta}\left(1 + \frac{1}{16}(5-2\beta)^{-\frac{1}{2}}\right)\|t^\beta u'''\|_{L^2(0,2k;X)}.$$

Proof. Using integration by parts, Lemma A.68, one may verify that

$$u'(a+2k) - k^{-1}\left(\frac{3}{2}u(a+2k) - 2u(a+k) + \frac{1}{2}u(a)\right)$$
$$= \int_{a+k}^{a+2k} -(t-(a+2k))\left(\frac{3}{4k}(t-(a+2k))+1\right)u'''(t)\,dt + \int_a^{a+k}\frac{1}{4k}(t-a)^2u'''(t)\,dt.$$

Moreover, there holds

$$\int_{a+k}^{a+2k}(t-(a+2k))^2\left(\frac{3}{4k}(t-(a+2k))+1\right)^2 t^{-2\beta}\,dt \leq k^3(a+k)^{-2\beta}.$$

If $a > 0$, then

$$\int_a^{a+k}\frac{1}{16k^2}(t-a)^4t^{-2\beta}\,dt \leq \frac{1}{16}k^3a^{-2\beta}$$

and, otherwise,

$$\int_0^k \frac{1}{16k^2}t^{4-2\beta}\,dt = \frac{1}{16(5-2\beta)}k^{3-2\beta}.$$

□

Lemma A.80 and A.81 are the foundation of the final error estimates given in the end of Section 5.2. Their proofs make use of the previous estimates. Similar results can be found in [24].

Lemma A.80. *(Temporal Approximation)*
Let difference operators be given by

$$\begin{aligned}\mathbf{a}^{(1)} &= (0,1), \quad \mathbf{a}^{(2)} = (0,2,-1)\\ \mathbf{b}^{(1)} &= (1,-1), \ \mathbf{b}^{(2)} = (\frac{3}{2},-2,\frac{1}{2}).\end{aligned} \tag{A.4}$$

Let Z denote a Hilbert space, $z \in L^2(0,T;Z)$ with $z' \in L^1(0,T;Z)$, $z^n = z(t_n)$ for $n = 0,\dots,N$. Then

(i) If $t^\beta z' \in L^2(0,T;Z)$ for some $\beta \in [0,\frac{1}{2})$, then for $n = 1,\dots,N$:

$$\|J_{\mathbf{b}^{(1)}}z^n\|_Z = \|z^n - J_{\mathbf{a}^{(1)}}z^n\|_Z \lesssim k^{\frac{1}{2}-\beta}\|t^\beta z'\|_{L^2(t_{n-1},t_n;Z)}.$$

(ii) *If* $z'' \in L^1(0,T;Z)$, $t^\beta z'' \in L^2(0,T;Z)$ *for some* $\beta \in [0,\frac{3}{2})$, *then for* $n = 2,\dots,N$*:*

$$\|z^n - J_{\mathbf{a}^{(2)}} z^n\|_Z \leqslant k^{\frac{3}{2}-\beta} \|t^\beta z''\|_{L^2(t_{n-2},t_n;Z)}.$$

(iii) *If* $z'' \in L^1(0,T;Z)$, $t^\beta z'' \in L^2(0,T;Z)$ *for some* $\beta \in [0,\frac{3}{2})$, *then for* $n = 1,\dots,N$*:*

$$\|\partial_t z^n - k^{-1} J_{\mathbf{b}^{(1)}} z^n\|_Z \leqslant k^{\frac{1}{2}-\beta} \|t^\beta z''\|_{L^2(t_{n-1},t_n;Z)}.$$

(iv) *If* z'', $z''' \in L^1(0,T;Z)$, $t^\beta z''' \in L^2(0,T;Z)$ *for some* $\beta \in [0,\frac{5}{2})$, *then for* $n = 2,\dots,N$*:*

$$\|\partial_t z^n - k^{-1} J_{\mathbf{b}^{(2)}} z^n\|_Z \leqslant k^{\frac{3}{2}-\beta} \|t^\beta z'''\|_{L^2(t_{n-2},t_n;Z)}.$$

(v) *If* $z'' \in L^1(0,T;Z)$, $t^\beta z'$, $t^\gamma z'' \in L^2(0,T;Z)$ *for some* $\beta \in [0,\frac{1}{2}), \gamma \in [0,\frac{3}{2})$, *then for* $n = 2,\dots,N$*:*

$$\|J_{\mathbf{b}^{(2)}} z^n\|_Z \leqslant k^{\frac{1}{2}-\beta} \|t^\beta z'\|_{L^2(t_{n-2},t_n;Z)} + k^{\frac{3}{2}-\gamma} \|t^\gamma z''\|_{L^2(t_{n-2},t_n;Z)}.$$

Proof. (*i*), (*ii*), (*iii*), (*iv*) are direct consequences of Lemma A.76, A.78, A.77 and A.79, respectively. (*v*) follows by combination of Lemma A.76, (*ii*) and the identity $z^n - J_{\mathbf{a}^{(2)}} z^n = J_{\mathbf{b}^{(2)}} z^n - \frac{1}{2}\left(z^n - z^{n-2}\right)$. □

Lemma A.81. *(Estimation of Temporal Error Terms)*
Let the notation of Lemma A.80 hold. Then,

(i) *Under the assumption of Lemma A.80 (i),*

$$k \sum_{n=1}^{N} \|z^n - J_{\mathbf{a}^{(1)}} z^n\|_Z^2 \leqslant k^{2(1-\beta)} \|t^\beta z'\|_{2;Z}^2.$$

(ii) *Under the assumption of Lemma A.80 (ii),*

$$k \sum_{n=2}^{N} \|z^n - J_{\mathbf{a}^{(2)}} z^n\|_Z^2 \leqslant k^{2(2-\beta)} \|t^\beta z''\|_{2;Z}^2.$$

(iii) *Under the assumption of Lemma A.80 (iii),*

$$k \sum_{n=1}^{N} \|\partial_t z^n - k^{-1} J_{\mathbf{b}^{(1)}} z^n\|_Z^2 \leqslant k^{2(1-\beta)} \|t^\beta z''\|_{2;Z}^2.$$

(iv) *Under the assumption of Lemma A.80 (iv),*

$$k \sum_{n=2}^{N} \|\partial_t z^n - k^{-1} J_{\mathbf{b}^{(2)}} z^n\|_Z^2 \leqslant k^{2(2-\beta)} \|t^\beta z'''\|_{2;Z}^2.$$

(v) *Under the assumption of Lemma A.80 (i),*

$$k \sum_{n=1}^{N} \|k^{-1} J_{\mathbf{b}^{(1)}} z^n\|_Z^2 \leqslant k^{-2\beta} \|t^\beta z'\|_{2;Z}^2.$$

(vi) *Under the assumption of Lemma A.80 (v),*

$$k \sum_{n=2}^{N} \|k^{-1} J_{\mathbf{b}^{(2)}} z^n\|_Z^2 \leqslant k^{-2\beta} \|t^\beta z'\|_{2;Z}^2 + k^{2(1-\gamma)} \|t^\gamma z''\|_{2;Z}^2.$$

Proof. The stated assertions follow directly from Lemma A.80. □

A.4. Function spaces

In the following, let $\Omega \subset \mathbb{R}^d$ be bounded and open with Dirichlet boundary condition part $\Gamma_D \subset \partial\Omega$, where Γ_D is assumed to be closed.

Definition A.82. *(Smooth Functions)*
Let $M, O \subset \mathbb{R}^d$ with O being open, $k \in \mathbb{N}_0$, $\alpha \in (0,1]$ and $m \in \mathbb{N}$. The following sets of smooth functions are defined:

(i) Continuous:
$$C(M) := C^0(M) := \{u\colon M \to \mathbb{R}\colon\ u \text{ is continuous }\}.$$

(ii) Hölder continuous:
$$C^{0,\alpha}(M) := \left\{ u \in C^0(M)\colon\ \sup_{x,y\in M} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

(iii) Continuously differentiable:
$$C^k(M) := \{u \in C^0(M)\colon\ u \text{ is } k-\text{times cont. differentiable}\}.$$

(iv) Hölder continuously differentiable:
$$C^{k,\alpha}(M) := \{u \in C^k(M)\colon\ \text{all derivatives are in } C^{0,\alpha}(M)\}.$$

(v) Smooth functions:
$$C^\infty(M) := \bigcap_{l\geq 1} C^l(M).$$

(vi) Smooth functions with compact support:
$$C_c^\infty(M) := \{u \in C^\infty(M)\colon\ supp(u) \subset\subset M\}.$$

(vii) Smooth functions with with zero trace on Dirichlet boundary:
$$C_D^\infty(\Omega) := \{u_{|\Omega}\colon u \in C_c^\infty(\mathbb{R}^d),\ supp(u) \cap \Gamma_D = \emptyset\}.$$

(viii) Test functions:
$$\mathcal{D}(O) := C_c^\infty(O) \text{ and } \mathcal{D}(\overline{O}) := \{v_{|_O}\colon\ v \in \mathcal{D}(\mathbb{R}^d)\}.$$

Definition A.83. *(Lebesgue Spaces)*
For $p \in [1,\infty)$ and $u\colon \Omega \to \mathbb{R}$ define
$$\|x\|_{0,p,\Omega} := \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}},$$
$$\|x\|_{0,\infty,\Omega} := \operatorname{esssup}_{x\in\Omega} |u(x)|.$$

The corresponding Lebesgue spaces *are given as*
$$L^p(\Omega) := \{u\colon \Omega \to \mathbb{R} \text{ measurable },\ \|x\|_{0,p,\Omega} < \infty\}$$

and for vector-valued functions
$$\mathbf{L}^p(\Omega) := L^p(\Omega)^n := \{\mathbf{u}\colon \Omega \to \mathbb{R}^n \text{ measurable },\ u_i \in L^p(\Omega),\ i = 1,\ldots,n\}.$$

The scalar product of $L^2(\Omega)$ is denoted by

$$(u, v)_{2,\Omega} := \int_\Omega u(x)v(x)dx.$$

Moreover,

$$L^p_{loc}(\Omega) := \{v\colon \Omega \to \mathbb{R} \text{ measurable }, \exists K \subset\subset \Omega, v_{|_K} \in L^p(K)\}$$

and

$$L^p_0(\Omega) := \{v \in L^p(\Omega), \int_\Omega v = 0\}.$$

For bounded Ω, $L^p_0(\Omega)$ isomorphic to the quotient space $L^2(\Omega)/\mathbb{R}$ which is equipped with the norm

$$\|u\|_{0,2/\mathbb{R},\Omega} := \inf_{c\in\mathbb{R}} \|u + c\|_{0,2,\Omega}.$$

For $p \in [1, \infty]$ let $p^* \in [1, \infty]$ be defined such that $\frac{1}{p} + \frac{1}{p^*} = 1$.

If there is no potential unambiguity, we write $(\cdot, \cdot) := (\cdot, \cdot)_{2,\Omega}$, $\|\cdot\| := \|\cdot\|_{0,2,\Omega}$ and skip the subscript Ω in the norm notation.

Theorem A.84. *(Basic Properties of L^p Spaces)*

(i) $L^p(\Omega)$ is a Banach space for $p \in [1, \infty]$ (Theorem 2.16 in [1])

(ii) $L^2(\Omega)$ is a Hilbert space

(iii) $L^p(\Omega)$ is separable for $p \in [1, \infty)$ (Theorem 2.21 in [1])

(iv) $L^p(\Omega)$ is reflexive for $p \in (1, \infty)$ (Corollary 2.40 in [1])

(v) $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$ (Lemma I.1.1 in [32]).

Theorem A.85. *(Duals of L^p Spaces, Theorem 2..4 in [1])*
Let $p \in [1, \infty)$ and $\phi \in L^p(\Omega)^*$. Then there exists $v \in L^{p^*}(\Omega)$ such that for all $u \in L^p(\Omega)$

$$\langle \phi, u\rangle_{L^p(\Omega)^*} = (u, v) = \int_\Omega u(x)v(x)\, dx.$$

Moreover, $\|\phi\| = \|v\|_{p^*}$.

Definition A.86. *(Distributions, Chapter B.2 in [25])*
A linear mapping

$$u\colon \mathcal{D}(\Omega) \to \mathbb{R},\ \phi \mapsto \langle u, \phi\rangle_{\mathcal{D}'}$$

is called a Distribution on Ω, if and only if for all compact $K \subset \Omega$, there is an integer p and a constant c such that

$$\phi \in \mathcal{D}(\Omega), supp(\phi) \subset K \Rightarrow |\langle u, \phi\rangle_{\mathcal{D}'}| \le c \sup_{x\in K, |\alpha|\le p} |\partial^\alpha \phi(x)|.$$

$\mathcal{D}'(\Omega)$ denotes the set of all distributions on Ω.
A function $f \in L^1_{loc}(\Omega)$ can be identified with a distribution via

$$\langle f, \phi\rangle_{\mathcal{D}'} \cong \int_\Omega f(x)\phi(x)dx.$$

Definition A.87. *(Distributional Derivative, Chapter B.2 in [25])*
Let $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^d$ denote a multi-index. The distributional derivative $\partial^\alpha u \in \mathcal{D}'(\Omega)$ is defined as

$$\partial^\alpha u \colon \mathcal{D}(\Omega) \to \mathbb{R},\ \phi \mapsto (-1)^\alpha \langle u, \partial^\alpha \phi \rangle_{\mathcal{D}'}.$$

Lemma A.88. *(Fundamental Property, Chapter B.2 in [25])*
Let $f \in L^1_{loc}(\Omega)$ such that $\int_\Omega f\phi = 0$ for all $\phi \in \mathcal{D}(\Omega)$. Then, $f = 0$ a.e. in Ω.

Definition A.89. *(Sobolev Spaces, [25] and Chapter I.1.1 in [32])*
For $k \in \mathbb{N}$ and $p \in [1, \infty]$ denote

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) :\ \partial^\alpha u \in L^p(\Omega)\ \forall\ |\alpha| \le k\}$$

with the convention $W^{0,p}(\Omega) = L^p(\Omega)$. For $\sigma \in (0, 1)$ fractional Sobolev spaces are defined as

$$W^{\sigma,p}(\Omega) := \left\{u \in L^p(\Omega)\colon\ (u(x) - u(y))|x - y|^{-(\sigma+\frac{d}{p})} \in L^p(\Omega \times \Omega)\right\}.$$

and for $s > 1$, let $\sigma = s - \lfloor s \rfloor$ and define

$$W^{s,p}(\Omega) := \{u \in W^{\lfloor s \rfloor,p}(\Omega)\colon\ \partial^\alpha u \in W^{\sigma,p}(\Omega),\ \forall |\alpha| = \lfloor s \rfloor\}.$$

The corresponding norms are defined as

$$\begin{aligned}
\|u\|^p_{k,p,\Omega} &:= \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{0,p,\Omega} && \text{for } p \in [1, \infty)\\
\|u\|_{k,\infty,\Omega} &:= \max_{|\alpha| \le k} \|\partial^\alpha u\|_{0,\infty,\Omega} && \text{for } p = \infty\\
\|u\|^p_{s,p,\Omega} &:= \|u\|^p_{\lfloor s \rfloor,p,\Omega} + \sum_{|\alpha| = \lfloor s \rfloor} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+\sigma p}}\, dx\, dy && \text{for } s \in (0, \infty).
\end{aligned}$$

Furthermore, for $s \ge 0,\ p \in [1, \infty)$ set

$$W^{s,p}_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)} \text{ and } W^{s,p}_D(\Omega) := \overline{C^\infty_D(\Omega)}^{W^{s,p}(\Omega)}$$

and for $\frac{1}{p} + \frac{1}{p^*} = 1$ set

$$W^{-s,p^*}(\Omega) := W^{s,p}_0(\Omega)^*.$$

With $\|\cdot\|_{-s,p,\Omega}$ we denote the norm of $W^{s,p}_0(\Omega)^*$. Finally, a semi-norm is given by

$$|u|_{k,p,\Omega} := \left(\sum_{|\alpha|=k} \|\partial^\alpha u\|_{0,p,\Omega}\right)^{\frac{1}{p}}.$$

and for $p = 2$ an inner product can be defined as

$$(u, v)_{k,2,\Omega} := \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx.$$

For $p = 2$ we use the notation

$$
\begin{aligned}
H^s(\Omega) &:= W^{s,2}(\Omega),\ H^{-1}(\Omega) := W^{-1,2}(\Omega) \\
H^s_0(\Omega) &:= W^{s,2}_0(\Omega),\ \mathbf{H}^1_0(\Omega) \ := \left(H^1_0(\Omega)\right)^d,\ \mathbf{H}^{-1}(\Omega) := \left(\mathbf{H}^1_0(\Omega)\right)^* \\
H^s_D(\Omega) &:= W^{s,2}_D(\Omega),\ H^{-1}_D(\Omega) := \left(W^{1,2}_D(\Omega)\right)^*
\end{aligned}
$$

and skip the subscript Ω if there is no potential ambiguity.

Theorem A.90. *(Basic Properties of Sobolev Spaces)*

(i) $W^{k,p}(\Omega)$ *is a Banach space for* $k \in \mathbb{N}, p \in [1, \infty]$ *(Theorem 3.3 in [1]).*

(ii) $W^{k,2}(\Omega)$ *is a Hilbert space for* $k \in \mathbb{N}$.

(iii) $W^{k,p}(\Omega)$ *is separable for* $k \in \mathbb{N}, p \in [1, \infty)$ *(Theorem 3.6 in [1]).*

(iv) $W^{k,p}(\Omega)$ *is reflexive for* $k \in \mathbb{N}, p \in (1, \infty)$ *(Theorem 3.6 in [1]).*

Theorem A.91. *(Approximation by Smooth Functions, Theorem I.1.2 in [32])*
Let Ω *be open and Lipschitz.*

(i) $\mathcal{D}(\overline{\Omega})$ *is dense in* $W^{k,p}(\Omega)$ *for* $k \in \mathbb{N}$ *and* $p \in [1, \infty)$.

(ii) Let $u \in W^{k,p}(\Omega)$ *and* $\tilde{u}$ *its extension by zero outside* Ω. *If* $\tilde{u} \in W^{k,p}(\mathbb{R}^d)$, *then* $u \in W^{k,p}_0(\Omega)$.

(iii) If $\partial\Omega$ *is bounded and* $m \geq 1$, *there exists a continuous linear extension operator* $P\colon W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ *such that* $Pu_{|\Omega} = u$ *for all* $u \in W^{k,p}(\Omega)$.

Theorem A.92. *(Sobolev Embedding, Theorem I.1.3 in [32])*
Let $\Omega \subset \mathbb{R}^d$*be open and Lipschitz. Let* $p \in [1, \infty)$ *and* $m, n \in \mathbb{N}_0$ *with* $0 \leq n \leq m$. *Then, the following embeddings hold*

$$
\begin{aligned}
&W^{m,p}(\Omega) \hookrightarrow W^{n,q}(\Omega) && \text{if } \frac{1}{q} := \frac{1}{p} - \frac{m-n}{d} > 0 \\
&W^{m,p}(\Omega) \hookrightarrow W^{n,q}_{loc}(\Omega) \text{ for all } q \in [1, \infty) && \text{if } \frac{1}{p} = \frac{m-n}{d} \\
&W^{m,p}(\Omega) \hookrightarrow C^n(\Omega) && \text{if } \frac{1}{p} < \frac{m-n}{d}.
\end{aligned}
$$

Moreover, if Ω *is bounded, the last inclusion holds in* $C^n(\overline{\Omega})$ *and the emebedding of* $W^{m,p}(\Omega)$ *into* $W^{n,q'}(\Omega)$ *is compact for all real* q' *that satisfy*

$$
\text{or} \begin{cases} 1 \leq q' < \frac{dp}{d-(m-n)p} & \text{when } d > (m-n)p \\ 1 \leq q' < \infty & \text{when } d = (m-n)p \end{cases}.
$$

In addition, these compact emebddings are also valid for negative n *or* m.

Theorem A.93. *(Interpolation Theorem, Theorem 5.8 in [1])*
Let Ω be a domain satisfying the cone condition. If $mp > d$, let $p \le q \le \infty$; if $mp = d$, let $p \le q < \infty$; if $mp < d$, let $p \le q \le p^* = \frac{dp}{d-mp}$. Then there exists a constant K depending on m, n, p, q and Ω such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_{0,q} \le K\|u\|_{m,p}^{\theta}\|u\|_{0,p}^{1-\theta},$$

where $\theta = \frac{d}{mp} - \frac{d}{mq}$.

Theorem A.94. *(Continuity of Superposition Operator, Theorem 1 in [49])*
Let $f \in C^{0,1}(\mathbb{R})$. Then the following operator is continuous

$$\begin{aligned} T_f \colon W^{1,p}(G) &\to W^{1,p}(G) \\ v &\mapsto f \circ v. \end{aligned}$$

Definition A.95. *(Function Spaces for Flow Problems, Chapter I.1 in [77])*
Spaces of functions with weak divergence are defined as

$$\begin{aligned} \mathbf{E}(\Omega) &:= \{u \in L^2(\Omega)^d,\ \nabla \cdot u \in L^2(\Omega)\} \\ \mathbf{E}_0(\Omega) &:= \overline{\mathcal{D}(\Omega)^d}^{\mathbf{E}(\Omega)}, \end{aligned}$$

where $\mathbf{E}(\Omega)$ is equipped with the norm $\|\mathbf{u}\|_{\mathbf{E}(\Omega)} := \left(\|\mathbf{u}\|_{0,2}^2 + \|\nabla \cdot \mathbf{u}\|_{0,2}^2\right)^{\frac{1}{2}}$.
The space of divergence-free test functions is given by

$$\mathcal{V}(\Omega) := \{\mathbf{u} \in \mathcal{D}(\Omega)^d \colon\ \nabla \cdot \mathbf{u} = 0\}$$

and its closures by

$$\mathbf{V}(\Omega) := \overline{\mathcal{V}(\Omega)}^{\mathbf{H}_0^1(\Omega)},\ \ \mathbf{V}_2(\Omega) := \overline{\mathcal{V}(\Omega)}^{\mathbf{H}_0^2(\Omega)},\ \ \mathbf{H}(\Omega) := \overline{\mathcal{V}(\Omega)}^{\mathbf{L}^2(\Omega)}$$

equipped with respective inner products

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_{\mathbf{V}(\Omega)} &:= (\mathbf{u}, \mathbf{v})_{1,2} - (\mathbf{u}, \mathbf{v})_2 \\ (\mathbf{u}, \mathbf{v})_{\mathbf{V}_2(\Omega)} &:= (\mathbf{u}, \mathbf{v})_{2,2} - (\mathbf{u}, \mathbf{v})_2 \\ (\mathbf{u}, \mathbf{v})_{\mathbf{H}(\Omega)} &:= (\mathbf{u}, \mathbf{v})_2 \end{aligned}$$

Theorem A.96. *(Properties of Flow Spaces)*
The spaces $\mathbf{V}(\Omega), \mathbf{V}_2(\Omega), \mathbf{H}(\Omega), H_D^1(\Omega), H_D^2(\Omega)$ are complete, reflexive and separable.

Proof. Completeness, separability, reflexivity follow since all spaces are closed subspaces of the separable, reflexive Banach spaces $H_0^1(\Omega)^d,\ H_0^2(\Omega)^d,\ L^2(\Omega)^d,\ H^1(\Omega),\ H^2(\Omega)$, respectively, under the use of Lemma A.4 and A.23. □

Theorem A.97. *(Standard Trace Operator, Theorem I.1.5 in [32], Theorem 1.5.1.3 in [34])*
Let Ω be a bounded Lipschitz domain and $p \in (1,\infty)$. Define

$$\gamma_0 \colon \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\partial\Omega),\ v \mapsto v_{|\partial\Omega}.$$

Then, γ_0 can be extended to a linear, bounded operator

$$\gamma_0 \in \mathcal{L}(W^{1,p}(\Omega), W^{\frac{1}{p^*},p}(\partial\Omega)).$$

There holds $\ker(\gamma_0) = W_0^{1,p}(\Omega)$ *and γ_0 is surjective, i.e. for all $h \in W^{\frac{1}{p^*},p}(\partial\Omega)$ there is $v \in W^{1,p}(\Omega)$ such that*

$$\gamma_0 v = h \text{ and } \|v\|_{1,p,\Omega} \leq C\|h\|_{\frac{1}{p^*},p,\partial\Omega}$$

with C being independent of h and v.

Theorem A.98. *(Dirichlet Trace Operator)*
Let Ω be a bounded Lipschitz domain with $\Gamma_D \subset \partial\Omega$ having nonzero $d-1$ dimensional Haussdorff measure and $p \in (1,\infty)$. Let

$$\gamma_D \colon \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\Gamma_D),\ v \mapsto v_{|\Gamma_D}.$$

Then, γ_D can be extended to a linear, bounded operator

$$\gamma_D \in \mathcal{L}(W^{1,p}(\Omega), W^{\frac{1}{p^*},p}(\Gamma_D)).$$

There holds $W_D^{1,p}(\Omega) \subset \ker(\gamma_D)$.

Proof. Let $\tilde{\gamma}_D \colon \mathcal{D}(\partial\Omega) \to \mathcal{D}(\Gamma_D),\ v \mapsto v_{|\Gamma_D}$. Then, $\gamma_D = \tilde{\gamma}_D \circ \gamma_0$ and for $v \in \mathcal{D}(\overline{\Omega})$ there follows by use of Theorem A.97,

$$\begin{aligned} \|\gamma_D v\|_{\frac{1}{p^*},p,\Gamma_D} &= \|\tilde{\gamma}_D \gamma_0 v\|_{\frac{1}{p^*},p,\Gamma_D} \\ &\leq \|\gamma_0 v\|_{\frac{1}{p^*},p,\Gamma} \\ &\leq \|\gamma_0\|\|v\|_{1,p,\Omega}. \end{aligned}$$

Since $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, Theorem A.91, γ_D can be extended to a bounded operator in $\mathcal{L}(W^{1,p}(\Omega), W^{\frac{1}{p^*},p}(\Gamma_D))$.

Moreover, there holds $C_D^\infty(\Omega) \subset \mathcal{D}(\overline{\Omega})$ and $\gamma_D v = 0$ for all $v \in C_D^\infty(\Omega)$. Thus, $C_D^\infty(\Omega) \subset \ker(\gamma_D)$. Since $\ker(\gamma_D)$ is closed in $W^{1,p}(\Omega)$ as kernel of a bounded linear operator and $W_D^{1,p}(\Omega) = \overline{C_D^\infty(\Omega)}^{W^{1,p}}$, there follows $W_D^{1,p}(\Omega) \subset \ker(\gamma_D)$. □

Theorem A.99. *(Normal Trace Operator, Theorem I.1.1, I.1.2, I.1.3, Remark I.1.3 in [77])*
Let Ω be a bounded Lipschitz domain and $p \in (1,\infty)$. Define

$$\gamma_{\mathbf{n}} \colon C^0(\overline{\Omega})^d \to L^\infty(\partial\Omega),\ \mathbf{v} \mapsto \mathbf{v}_{|\partial\Omega} \cdot \mathbf{n}.$$

Then, $\gamma_{\mathbf{n}}$ can be extended to a linear, bounded operator

$$\gamma_{\mathbf{n}} \in \mathcal{L}(\mathbf{E}(\Omega), W^{-\frac{1}{2},2}(\partial\Omega)) \text{ with } \ker(\gamma_{\mathbf{n}}) = \mathbf{E}_0(\Omega).$$

Moreover, for all $\mathbf{u} \in \mathbf{E}(\Omega)$ *and* $w \in W^{1,2}(\Omega)$,

$$\int_\Omega \mathbf{u} \cdot \nabla w + \int_\Omega w \nabla \cdot \mathbf{u} = \langle \gamma_\mathbf{n} \mathbf{u}, \gamma_0 w \rangle_{-\frac{1}{2},2}.$$

According to Corollary B.57 in [25], the above result can be generalized to $p \in [1, \infty)$.

Proposition A.100. *(Characterization of* $\mathbf{H}(\Omega)$*, Theorem I.2.7 in [32])*
Let $\Omega \subset \mathbb{R}^d$ *be open, bounded, connected and Lipschitz. Then,*

$$\mathbf{H}(\Omega) = \{u \in L^2(\Omega)^d,\ \nabla \cdot u = 0, \gamma_\nu u = 0\}$$
$$\mathbf{H}(\Omega)^\perp = \{u \in L^2(\Omega)^d,\ u = \nabla p,\ p \in \mathbf{H}^1(\Omega)\}$$

Proposition A.101. *(Characterization of* $\mathbf{V}(\Omega)$*, Theorem I.1.6 in [77])*
Let $\Omega \subset \mathbb{R}^d$ *be open and Lipschitz. Then,*

$$\mathbf{V}(\Omega) = \{u \in \mathbf{H}^1_0(\Omega),\ \nabla \cdot u = 0\}.$$

Proposition A.102. *(De Rham, Theorem I.2.3 in [32])*
Let $\Omega \subset \mathbb{R}^d$ *be open, bounded and Lipschitz. If* $f \in H^{-1}(\Omega)^d$ *satisfies*

$$\langle f, v \rangle = 0 \text{ for all } v \in \mathcal{V}(\Omega),$$

then there exists $p \in L^2(\Omega)$ *such that*

$$f = \nabla p.$$

If Ω *is connected,* p *is unique up to an additive constant.*

Proposition A.103. *(Bijective Gradient and Divergence Operator, Corollary 2.4 in [32])*
Let $\Omega \subset \mathbb{R}^d$ *be open, bounded, connected and Lipschitz. Then,*

(i) $\nabla \colon L^2_0(\Omega) \to \mathbf{V}^\circ(\Omega)$ *is an isomorphism*

(ii) $\nabla \cdot \colon \mathbf{V}(\Omega)^\perp \to L^2_0(\Omega)$ *is an isomorphism*

By Proposition A.103 and Theorem A.41 one directly obtains the following proposition.

Proposition A.104. *(Inf-Sup Condition)*
Let $\Omega \subset \mathbb{R}^d$*,* $d \geq 2$ *be open, bounded and Lipschitz. Then, there is some* $\beta > 0$ *such that*

$$\inf_{q \in L^2_0(\Omega)} \sup_{v \in W^{1,2}_0(\Omega)^d} \frac{\int_\Omega q \nabla \cdot v}{\|\nabla v\|_2 \|q\|_2} \geq \beta.$$

Theorem A.105. *(Sobolev's Inequality, Lemma 4.3.4 in [13])*
Suppose Ω *is bounded and star-shaped w.r.t. a ball* B*. Let either* (i) $1 < p < \infty$ *and* $m \geq \frac{d}{p}$ *or* (ii) $p = 1$ *and* $m \geq d$*. Then, there exists a constant* $C = C(\Omega, d, m)$ *such that all* $u \in W^{m,p}(\Omega)$ *are continuous and*

$$\|u\|_{0,\infty} \leq C \|u\|_{m,p}.$$

Theorem A.106. *(Friedrich's Inequality for H_0^1, Theorem I.1.1 in [32])*
Let Ω be open, bounded and connected. Then, there is $C_{PF} > 0$ such that

$$\|u\|_{0,2} \le C_{PF} \|\nabla u\|_{0,2} \text{ for all } u \in H_0^1(\Omega).$$

Lemma A.107. *(Constant in Friedrich's Inequality for Simple Domains)*
Let

$$\begin{aligned} \Omega_{cub} &:= \bigotimes_{i=1}^{d} (a_i, b_i) \\ \Omega_{cyl} &:= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \colon\ r_i < \sqrt{x_1^2 + x_2^2} < r_o,\ 0 < x_3 < H \right\} \end{aligned}$$

with $a_i < b_i$, $i = 1, \dots, d$, $0 < r_i < r_o$ and $H > 0$. Moreover, let

$$\begin{aligned} \Gamma_{D,cub} &:= \{a_1\} \times \bigotimes_{i=2}^{d} (a_i, b_i) + \{b_1\} \times \bigotimes_{i=2}^{d} (a_i, b_i) \\ \Gamma_{D,cyl} &:= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \colon\ 0 \le x_3 \le H,\ \sqrt{x_1^2 + x_2^2} \in \{r_i, r_o\} \right\}. \end{aligned}$$

Let Ω denote either Ω_{cub} or Ω_{cyl}. In the former case, let $\Gamma_D := \Gamma_{D,cub}$ and in the latter case, $\Gamma_D := \Gamma_{D,cyl}$. Then, for all $\theta \in H_D^1(\Omega)$, there holds

$$\|\theta\|_{0,2} \le c_* \|\nabla \theta\|_{0,2}$$

with

$$\begin{aligned} c_* = c_{cub} &:= b_1 - a_1, && \text{if } \Omega = \Omega_{cub} \\ c_* = c_{cyl} &:= \left(\frac{1}{2} \left(\ln(r_o) - \ln(r_i) \right) \left(r_o^2 - r_i^2 \right) \right)^{\frac{1}{2}}, && \text{if } \Omega = \Omega_{cyl}. \end{aligned}$$

The same results hold for $\theta \in H_0^1(\Omega)$.

Proof. (*i*): Let $\Omega = \Omega_{cub}$ and $\theta \in C_D^\infty(\Omega)$. Define $(x_1, \dots, x_d) =: (x_1, x')$ and $\Omega_{cub} =: (a_1, b_1) \times \Omega'$. Then, $\theta(a_1, x') = \theta(b_1, x') = 0$ for $x' \in \Omega'$. Thus, for $x_1 \in (a_1, b_1)$ and $x' \in \bigotimes_{i=2}^{d} (a_i, b_i)$:

$$\begin{aligned} |\theta(x_1, x')|^2 &= \left| \int_{a_1}^{x_1} \partial_{x_1} \theta(s, x')\, ds \right|^2 \\ &\le \left(\int_{a_1}^{x_1} 1\, ds \right) \cdot \left(\int_{a_1}^{b_1} \left| \partial_{x_1} \theta(s, x') \right|^2 ds \right) \\ &\le (b_1 - a_1) \int_{a_1}^{b_1} |\nabla \theta(s, x')|^2\, ds. \end{aligned}$$

With Fubini's Theorem follows,

$$\begin{aligned}
\|\theta\|_{0,2}^2 = \int_\Omega |\theta(x)|^2\,dx &\le (b_1-a_1)\int_\Omega\int_{a_1}^{b_1} |\nabla\theta(s,x')|^2\,ds\,dx \\
&= (b_1-a_1)\int_{a_1}^{b_1} dx_1 \cdot \int_{\Omega'}\int_{a_1}^{b_1} |\nabla\theta(s,x')|^2\,ds\,dx' \\
&= (b_1-a_1)^2\|\nabla\theta\|_{0,2}^2 \\
&= c_{cub}^2\|\nabla\theta\|_{0,2}^2.
\end{aligned}$$

(ii): Let $\Omega = \Omega_{cyl}$ and $\theta \in C_D^\infty(\Omega)$. Consider θ in cylindrical coordinates, i.e. $\theta = \theta(\varphi, r, z)$, $\nabla\theta = r^{-1}\partial_\varphi\theta\mathbf{e}_\varphi + \partial_r\theta\mathbf{e}_r + \partial_z\theta\mathbf{e}_z$ and $\theta(\varphi, r_i, z) = \theta(\varphi, r_o, z) = 0$ for $(\varphi, z) \in [0, 2\pi] \times [0, H]$. Then, for arbitrary $\varphi \in (0, 2\pi)$, $r \in (r_i, r_o)$, $z \in (0, H)$, there holds

$$\begin{aligned}
|\theta(\varphi,r,z)|^2 &= \left|\int_{r_i}^{r} \partial_r\theta(\varphi,s,z)\,ds\right|^2 \\
&\le \left(\int_{r_i}^{r} \frac{1}{s}\,ds\right)\cdot\left(\int_{r_i}^{r} |\partial_r\theta(\varphi,s,z)|^2 s\,ds\right) \\
&\le (\ln(r_o)-\ln(r_i))\cdot\int_{r_i}^{r_o} |\nabla\theta(\varphi,s,z)|^2 s\,ds.
\end{aligned}$$

Thus, by Fubini,

$$\begin{aligned}
\|\theta\|_{0,2}^2 = \int_\Omega |\theta(\varphi,r,z)|^2 r\,d\varphi\,dr\,dz &\le (\ln(r_o)-\ln(r_i))\int_\Omega\int_{r_i}^{r_o} |\nabla\theta(\varphi,s,z)|^2 s\,ds\; r\,d\varphi\,dr\,dz \\
&= (\ln(r_o)-\ln(r_i))\int_{r_i}^{r_o} r\,dr \cdot \int_\Omega |\nabla\theta(\varphi,s,z)|^2 s\,ds\,d\varphi\,dz \\
&= (\ln(r_o)-\ln(r_i))\,\frac{1}{2}\left(r_o^2-r_i^2\right)\|\nabla\theta\|_{0,2}^2 \\
&= c_{cyl}^2\|\nabla\theta\|_{0,2}^2.
\end{aligned}$$

(iii): By combining (i) and (ii), there holds

$$\|\theta\|_{0,2} \le c_*\|\nabla\theta\|_{0,2} \text{ for all } \theta \in C_D^\infty(\Omega).$$

Now, let $\theta \in H_D^1(\Omega) = \overline{C_D^\infty(\Omega)}^{W^{1,2}}$ be arbitrary and $\{\theta_n\}_n \subset C_D^\infty(\Omega)$ with $\theta_n \to \theta$ in $W^{1,2}(\Omega)$. Thus, $\|\theta_n\|_{0,2} \to \|\theta\|_{0,2}$ and $\|\nabla\theta_n\|_{0,2} \to \|\nabla\theta\|_{0,2}$ and therefore,

$$\|\theta\|_{0,2} \le c_*\|\nabla\theta\|_{0,2}.$$

The assertion for $\theta \in H_0^1(\Omega)$ follows analogously. □

Theorem A.108. *(Generalized Friedrich's Inequality, Proposition 7.1 in [21])*
Let Ω be open, bounded and connected and $p \in (1,\infty)$. Assume that $(X, \|\cdot\|_X)$ is a closed subspace of $W^{1,p}(\Omega)$ that does not contain the function $f \equiv 1$ and for which the restriction of the canonical embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ to X is compact. Then, $\|\cdot\|_X$ is equivalent to $\|\nabla\cdot\|_p$.

Theorem A.109. *(Friedrich's Inequality for $H^1_D(\Omega)$)*
Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Assume that $\partial\Omega = \Gamma_N + \Gamma_D$ with Γ_D having positive $(d-1)-$Hausdorff measure. Then, there is $c_D > 0$ such that

$$\|u\|_2 \leq c_D \|\nabla u\|_2 \text{ for all } u \in H^1_D(\Omega).$$

Proof. Let $(X, \|\cdot\|_X) := (H^1_D(\Omega), \|\cdot\|_{1,2})$. By definition, $X = \overline{C^\infty_D(\Omega)}^{W^{1,2}}$, and $\gamma_D v = v_{|\Gamma_D}$ for all $v \in C^\infty_D(\Omega)$ by definition of γ_D with $\gamma_D \in \mathcal{L}(W^{1,2}(\Omega), W^{\frac{1}{2},2}(\Gamma_D))$ denoting the boundary trace operator w.r.t. Γ_D. Additionally, by Theorem A.98, $X \subset \ker(\gamma_D)$. Moreover, there holds

$$(H^1_D(\Omega), \|\cdot\|_{1,2}) \hookrightarrow W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$

where the first embedding is obvious and the second embedding follows from the Sobolev embedding Theorem 4.12 in [2]. Therefore,

$$(H^1_D(\Omega), \|\cdot\|_{1,2}) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Finally, the function $f\colon \Omega \to \mathbb{R},\ x \mapsto 1$ is not contained in $\ker(\gamma_D)$, since $\gamma_D(f) \equiv 1 \neq 0$ on Γ_D and Γ_D has non-zero measure. Therefore, $f \notin X$. Thus Theorem A.108 yields the existence of some $C > 0$ such that

$$\|\cdot\|_2 \leq \|\cdot\|_{1,2} := \|\cdot\|_X \leq C\|\nabla\cdot\|_2 \text{ on } X.$$

□

Definition A.110. *(Mollifier, Section 1.6 in [40])*
A nonnegative function $\psi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with $\operatorname{supp}(\psi) = \overline{B_1(0, \mathbb{R}^d)}$ and $\int_{\mathbb{R}^d} \psi(x)dx = 1$ is called mollifier. *For such kind of function and $t > 0$ define the* mollifying operator

$$\begin{aligned} S_{\psi,t}\colon L^1(\Omega) &\to C^\infty_c(\mathbb{R}^d) \\ f &\mapsto f_t := (\tilde{f} * \psi_t) = \int_{\mathbb{R}^d} \tilde{f}(\cdot - y)\psi_t(y)dy \end{aligned}$$

with $\psi_t(y) := t^{-d}\psi(\frac{y}{t})$ and $\tilde{f}$ denotes the extension of f by 0 outside of Ω.

Lemma A.111. *(Properties of Convolution)*
Let $\phi \in C^\infty_0(\mathbb{R}^d)$, $f \in L^1_{loc}(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then,

(i) $\|(g * \phi)\|_{0,\infty} \leq \|g\|_{0,p}\|\phi\|_{0,p^*}$.

(ii) $\frac{\partial}{\partial x_i}(f * \phi) = (f * \frac{\partial}{\partial x_i}\phi)$.

(iii) $\frac{\partial^2}{\partial x_i \partial x_j}(f * \phi) = (f * \frac{\partial^2}{\partial x_i \partial x_j}\phi)$.

Proof. (i) follows from

$$\begin{aligned} \|(g * \phi)\|_{0,\infty} = \sup_{x \in \mathbb{R}^d} \left|\int_{\mathbb{R}^d} g(x-y)\phi(y)dy\right| &= \sup_{x \in \mathbb{R}^d} \left|\int_{\mathbb{R}^d} g(y)\phi(x-y)dy\right| \\ &\leq \sup_{x \in \mathbb{R}^d} \|g\|_{0,p}\|\phi(x - \cdot)\|_{0,p^*} = \|g\|_{0,p}\|\phi\|_{0,p^*} \end{aligned}$$

(ii) is shown in Lemma 1.16 in [40]. (iii) follows by iteratively applying (ii). □

Lemma A.112. *(Properties of Mollifiers)*
Let $p \in [1,\infty)$ and p^* such that $\frac{1}{p}+\frac{1}{p^*}=1$ and $f \in L^p(\Omega)$. The following properties hold for the operator $S_{\psi,t}$ given by Definition A.110.

(i) $S_{\psi,t} \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ *with* $\|S_{\psi,t}\| \leq \|\psi_t\|_{0,p^*,\mathbb{R}^d}$.

(ii) $S_{\psi,t}f \to f$ *in* L^p *for* $t \to 0$.

(iii) $\|\nabla S_{\psi,t}f\|_{0,\infty} \leq C_\nabla(\psi,t)\|f\|_{0,p}$.

(iv) $\|\nabla^2 S_{\psi,t}f\|_{0,\infty} \leq C_{\nabla^2}(\psi,t)\|f\|_{0,p}$.

(v) *If* $f_n \to f$ *in* L^p*, then* $\nabla S_{\psi,t}f_n \to \nabla S_{\psi,t}f$ *in* $L^\infty(\Omega)$.

Proof. (i): Let $g \in L^p(\Omega)$ with $\|g\|_{0,p}=1$ and denote by $\tilde{g}$ its extension by 0 outside of Ω. Then, by Lemma A.111 (i),

$$\|S_{\psi,t}g\|_{0,\infty} = \|(\tilde{g} * \psi_t)\|_{0,\infty} \leq \|\tilde{g}\|_{0,p,\mathbb{R}^d}\|\psi_t\|_{0,p^*,\mathbb{R}^d} = \|\psi_t\|_{p^*,\mathbb{R}^d}.$$

Since $S_{\psi,t}$ is obviously linear, (i) follows.
For (ii), see e.g. Theorem 2.29 in [1]. (iii) is obtained by using Lemma A.111 in

$$\begin{aligned}
\|\nabla S_{\psi,t}f\|_{0,\infty} = \sup_{x\in\Omega}\left(\sum_{i=1}^d (|\partial_i S_{\psi,t}f(x)|)^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^d (\sup_{x\in\Omega}|\partial_i S_{\psi,t}f(x)|)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^d (\|\partial_i(\tilde{f} * \psi_t)\|_{0,\infty})^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^d (\|f\|_{0,p}\|\partial_i\psi_t\|_{0,p^*,\mathbb{R}^d})^2\right)^{\frac{1}{2}} \\
&=: C_\nabla(\psi,t)\|f\|_{0,p}.
\end{aligned}$$

(iv) follows analogously. (v) is a direct consequence of (iii) and the linearity of $S_{\psi,t}$. □

A.5. Finite Element Method

The statements in this section are taken from [13]. Let Ω denote a bounded domain in $\mathbb{R}^d$ with $d \in \{2,3\}$.

Definition A.113. *(Polynomial Spaces)*
For a multiindex $\alpha \in \mathbb{N}_0^d$ let

$$|\alpha|_1 := \sum_{i=1}^d \alpha_i \text{ and } |\alpha|_\infty := \max\{\alpha_1,\ldots,\alpha_d\}.$$

Polynomial spaces are defined as

$$\begin{aligned}
\mathbb{P}_k &:= \operatorname{span}\{x^\alpha \colon |\alpha|_1 \leq k\} \\
\mathbb{Q}_k &:= \operatorname{span}\{x^\alpha \colon |\alpha|_\infty \leq k\}.
\end{aligned}$$

Definition A.114. *(Star-Shaped Domain, Definition 4.2.2 in [13])*
Ω *is* star-shaped w.r.t. B *if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω.*

Definition A.115. *(Finite Element, Definition 3.1.1 and 3.1.2 in [13])*
Let

(i) $K \subset \mathbb{R}^d$ *be a bounded closed set with nonempty interior and piecewise smooth boundary (*element domain*).*

(ii) $\mathcal{P}$ *be a finite-dimensional space of functions on* K *(*shape functions*).*

(iii) $\mathcal{N} = \{N_1, \ldots, N_k\}$ *be a basis of* $\mathcal{P}^*$ *(*nodal variables*).*

Then, $(K, \mathcal{P}, \mathcal{N})$ *is called a* Finite Element*.*

The basis $\{\phi_1, \ldots, \phi_k\}$ *of* $\mathcal{P}$ *with* $N_i(\phi_j) = \delta_{ij}$ *is called the* nodal basis *of* $\mathcal{P}$*.*

Definition A.116. *(Subdivision of* Ω*, Definition 3.3.8 in [13])*
A subdivision *of* Ω *is a finite collection of element domains* $\{K_i\}$ *such that*

(i) $\mathrm{int} K_i \cap \mathrm{int} K_j = \emptyset$ *if* $i \neq j$*.*

(ii) $\bigcup K_i = \overline{\Omega}$*.*

Definition A.117. *(Triangulation, Generalization of Definition 3.3.11 in [13])*
A subdivision $\mathcal{T}$ *of a polygonal domain* Ω *is called* triangulation*, if the following is satisfied:*

(i) if $K_i \cap K_j = \{p\}$*, then* p *is a vertex of both* K_i *and* K_j

(ii) if $K_i \cap K_j$*,* $i \neq j$ *consists of more than one point, then* $K_i \cap K_j$ *is either an edge or a facet of* K_i *and* K_j*.*

Definition A.118. *(Quasi-Uniform and Regular Triangulation, Definition 4.4.13 in [13])*
Let $\{\mathcal{T}_h\}_{h \in (0,1]}$ *be a family of subdivisions of* Ω *such that*

$$\max\{\mathrm{diam}(K) \colon K \in \mathcal{T}_h\} \leq h \, \mathrm{diam}(\Omega).$$

The family is said to be quasi-uniform *if there is* $\rho > 0$ *such that*

$$\min\{\mathrm{diam}(B_K) \colon K \in \mathcal{T}_h\} \geq \rho h \, \mathrm{diam}(\Omega)$$

for all $h \in (0, 1]$*, where* B_K *is the largest ball contained in* K *such that* K *is star-shaped with respect to* B_K*. The family is called* regular*, if there is* $\rho > 0$ *such that for all* $K \in \mathcal{T}_h$ *and all* $h \in (0, 1]$*,*

$$\mathrm{diam}(B_K) \geq \rho \, \mathrm{diam}(K).$$

Definition A.119. *(Affine Equivalence, Definition 3.4.1 in [13])*
Let $(K, \mathcal{P}, \mathcal{N})$ *denote a finite element and* $F(x) := Ax + b$ *with nonsingular* $A \in \mathbb{R}^{d \times d}$*,* $b \in \mathbb{R}^d$*. For a function* $\hat{f} \colon \mathbb{R}^d \to \mathbb{R}$*, the* pull-back *is defined as* $F^*(\hat{f}) := \hat{f} \circ F$ *and the* push-forward *is defined by* $(F_* N)(\hat{f}) := N(F^*(\hat{f}))$*. The finite element* $(\widehat{K}, \widehat{\mathcal{P}}, \widehat{\mathcal{N}})$ *is* affine equivalent *to* $(K, \mathcal{P}, \mathcal{N})$ *if*

(i) $F(K) = \widehat{K}$*.*

(ii) $F^*\widehat{P} = \mathcal{P}$.

(iii) $F_*\mathcal{N} = \widehat{N}$.

Definition A.120. *(Finite Element Space)*
Let $\{\mathcal{T}_h\}_{h\in(0,1]}$ *be a family of triangulations of a polyhedral domain* Ω. *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a reference finite element and for* $T \in \mathcal{T}_h$ *let* $(T, \mathcal{P}_T, \mathcal{N}_T)$ *be affine-equivalent to the reference element. A corresponding family of finite element spaces is defined by*

$$V_h := \{v \colon \Omega \to \mathbb{R},\ v \text{ is measurable and } v_{|_T} \in \mathcal{P}_T\ \forall T \in \mathcal{T}_h\}.$$

Definition A.121. *(Local Interpolant, Definition 3.3.1 in [13])*
Given a finite element $(K, \mathcal{P}, \mathcal{N})$ *with nodal basis* $\{\phi_1, \ldots, \phi_k\}$. *If* $v \colon K \to \mathbb{R}$ *is a function for which all* $N_i \in \mathcal{N}$ *are defined, then the* local interpolant *is defined by*

$$\mathcal{I}_K v := \sum_{i=1}^{k} N_i(v)\phi_i.$$

Definition A.122. *(Global Interpolant, Definition 3.3.9 in [13])*
Let $\{\mathcal{T}_h\}_{h\in(0,1]}$ *denote a subdivision of* Ω. *Assume each* $K \in \mathcal{T}_h$ *is equipped with some type of shape functions* $\mathcal{P}$ *and nodal variables* $\mathcal{N}$ *such that* $(K, \mathcal{P}, \mathcal{N})$ *is a finite element. Let* m *be the order of the highest partial derivatives involved in the nodal variables. For* $f \in C^m(\overline{\Omega})$, *the* global interpolant *is defined by*

$$\mathcal{I}_h f_{|_{K_i}} := \mathcal{I}_{K_i} f \text{ for all } K_i \in \mathcal{T}_h.$$

Assumption A.123. *(Interpolation Estimate Assumption, see Theorem 4.4.4 in [13])*
Let $(K, \mathcal{P}, \mathcal{N})$ *denote a finite element satisfying*

(i) K *is star-shaped with respect to some ball.*

(ii) $\mathbb{P}_{k-1} \subset \mathcal{P} \subset W^{k,\infty}(K)$ *for some integer* $k \geq 1$.

(iii) $\mathcal{N} \subset \left(C^l(\overline{K})\right)^*$ *for some integer* $l \geq 0$.

Theorem A.124. *(Global Interpolation Estimate, Theorem 4.4.20 in [13])*
Let $\{\mathcal{T}_h\}_{h\in(0,1]}$ *be a regular family of subdivisions of* Ω. *Let* $(K, \mathcal{P}, \mathcal{N})$ *denote a reference finite element satisfying Assumption A.123 for some* k, l. *For all* $T \in \mathcal{T}_h$, *all* $h \in (0, 1]$ *let* $(T, \mathcal{P}_T, \mathcal{N}_T)$ *be affine-equivalent to the reference element. Suppose* $p \in [1, \infty]$ *and either* $k - l - \frac{d}{p} > 0$ *when* $p > 1$ *or* $k - l - d \geq 0$ *when* $p = 1$.

Then there exists a constant $C > 0$ *depending on the reference element,* d, k, p *and* ρ *of Definition A.118 such that for* $0 \leq s \leq k$,

$$\left(\sum_{T\in\mathcal{T}_h} \|v - \mathcal{I}_h v\|_{s,p,T}^p\right)^{\frac{1}{p}} \leq Ch^{k-s}|v|_{k,p,\Omega} \text{ if } p \in [1, \infty)$$
$$\max_{T\in\mathcal{T}_h} \|v - \mathcal{I}_h v\|_{s,\infty,T} \leq Ch^{k-s}|v|_{k,\infty,\Omega} \text{ if } p = \infty,$$

for all $v \in W^{k,p}(\Omega)$. Moreover, for all $0 \le s \le l$, there holds

$$\max_{T \in \mathcal{T}_h} \|v - \mathcal{I}_h v\|_{s,\infty,T} \le C h^{k-s-\frac{d}{p}} |v|_{k,p,\Omega},$$

for all $v \in W^{k,p}(\Omega)$.

Corollary A.125. *(Boundedness of Global Interpolant)*
Let the assumptions of Theorem A.124 hold. Then, $\mathcal{I}_h \in \mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\Omega))$.

Proof. Let $v \in W^{k,p}(\Omega)$. Then, by Theorem A.124,

$$\|\mathcal{I}_h v\|_{k,p} \le \|v - \mathcal{I}_h v\|_{k,p} + \|v\|_{k,p} \le (C+1)\|v\|_{k,p}.$$

□

Theorem A.126. *(Inverse Estimate, Theorem 4.5.11 in [13])*
Let $\{\mathcal{T}_h\}_{h\in(0,1]}$ be a quasi-uniform family of subdivisions of a polyhedral domain Ω. Let $(K, \mathcal{P}, \mathcal{N})$ be a reference finite element such that $\mathcal{P} \subset W^{l,p}(K) \cap W^{m,q}(K)$, where $1 \le p \le \infty$, $1 \le q \le \infty$ and $0 \le m \le l$. For $T \in \mathcal{T}_h$ let $(T, \mathcal{P}_T, \mathcal{N}_T)$ be affine-equivalent to the reference element and $V_h := \{v \colon \Omega \to \mathbb{R},\ v$ is measurable and $v_{|_T} \in \mathcal{P}_T\ \forall T \in \mathcal{T}_h\}$. Then there exists $C = C(l,p,q,\rho)$ such that for all $v \in V_h$:

$$\left[\sum_{T\in\mathcal{T}_h} \|v\|_{l,p,T}^p\right]^{\frac{1}{p}} \le C h^{m-l+\min\{0,\frac{d}{p}-\frac{d}{q}\}} \left[\sum_{T\in\mathcal{T}_h} \|v\|_{m,q,T}^q\right]^{\frac{1}{q}}.$$

When $p = \infty$ (resp., $q = \infty$), then $\left[\sum_{T\in\mathcal{T}_h} \|v\|_{l,p,T}^p\right]^{\frac{1}{p}}$ (resp., $\left[\sum_{T\in\mathcal{T}_h} \|v\|_{m,q,T}^q\right]^{\frac{1}{q}}$) is replaced by $\max_{T\in\mathcal{T}_h} \|v\|_{l,\infty,T}$ (resp., $\max_{T\in\mathcal{T}_h} \|v\|_{l,\infty,T}$).

Theorem A.127. *(Stability of Taylor-Hood Elements)*
Let the assertions and notation of Definition A.120 hold. For a given reference element K let an finite element space $\mathbf{V}_h = V_h^d$ be given that is based on shaped functions $\mathcal{P}_V$ and degrees of freedom such that $V_h \subset H^1(\Omega)$. Let another finite element space M_h based on shape functions $\mathcal{P}_M$ be given and degrees of freedom such that $M_h \subset H^1(\Omega)$ holds. Then, the discrete inf-sup condition

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{1}{\|\mathbf{v}_h\|_{1,2}\|q_h\|} \int_\Omega q_h(\nabla \cdot \mathbf{v}_h)\, dx \ge \beta,$$

for some constant β that is independent of h holds in the following cases:

(i) $d = 2$, $K = [0,1]^2$, $\mathcal{P}_V = \mathbb{Q}_2$, $\mathcal{P}_M = \mathbb{Q}_1$, *[14]*

(ii) $d = 3$, $K = [0,1]^3$, $\mathcal{P}_V = \mathbb{Q}_2$, $\mathcal{P}_M = \mathbb{Q}_1$, *[15]*

(iii) $d = 2$, $K = \{(x,y) \in [0,1]^2 \colon\ 0 \le x + y \le 1\}$, $\mathcal{P}_V = \mathbb{P}_2$, $\mathcal{P}_M = \mathbb{P}_1$, *[9]*

(iv) $d = 3$, $K = \{(x,y,z) \in [0,1]^3 \colon\ 0 \le x + y + z \le 1\}$, $\mathcal{P}_V = \mathbb{P}_2$, $\mathcal{P}_M = \mathbb{P}_1$, *[72]*

References

[1] R. Adams and J. Fournier. *Sobolev Spaces*. Pure and Applied Mathematics. Elsevier Science, 2003.

[2] R. A. Adams. *Sobolev spaces / Robert A. Adams*. Academic Press New York, 1975.

[3] R. Aldbaissy, F. Hecht, G. Mansour, and T. Sayah. A full discretisation of the time-dependent boussinesq (buoyancy) model with nonlinear viscosity. *Calcolo*, 55(4):44, 2018.

[4] G. V. Alekseev and A. B. Smishliaev. Solvability of the boundary-value problems for the boussinesq equations with inhomogeneous boundary conditions. *Journal of Mathematical Fluid Mechanics*, 3(1):18–39, 2001.

[5] K. Allali. A priori and a posteriori error estimates for boussinesq equations. *International Journal of Numerical Analysis and Modeling*, 2:179–196, 01 2005.

[6] H. Alt. *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung*. Masterclass. Springer Berlin Heidelberg, 2006.

[7] M. V. Balashov and M. O. Golubev. About the lipschitz property of the metric projection in the hilbert space. *Journal of Mathematical Analysis and Applications*, 394(2):545 – 551, 2012.

[8] M. Baumann, V. Heuveline, L. Scheck, and S. C. Jones. Goal-oriented adaptivity for idealised tropical cyclones: A binary interaction scenario. *Meteorologische Zeitschrift*, 24(3):269–292, 07 2015.

[9] M. Bercovier and O. Pironneau. Error estimates for finite element method solution of the stokes problem in the primitive variables. *Numerische Mathematik*, 33(2):211–224, 1979.

[10] J. Boland and W. Layton. An analysis of the finite element method for natural convection problems. *Numerical Methods for Partial Differential Equations*, 6(2):115–126, 1990.

[11] J. Boland and W. Layton. Error analysis for finite element methods for steady natural convection problems. *Numerical Functional Analysis and Optimization*, 11(5-6):449–483, 1990.

[12] D. Braess. *Finite elements. Theory, fast solvers and applications in elasticity theory. 4th revised and extended ed.* Berlin: Springer. xviii, 357 p. EUR 44.95; SFR 73.50 , 2007.

[13] S. Brenner and L. Scoot. *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics. Springer, New York, NY, 2008.

[14] F. Brezzi and R. S. Falk. Stability of higher-order hood–taylor methods. *SIAM Journal on Numerical Analysis*, 28(3):581–590, 1991.

[15] F. Brezzi and M. Fortin. Mixed and hybrid finite element method. *Springer Series In Computational Mathematics; Vol. 15*, page 350, 01 1991.

[16] T. Buehler and D. Salamon. *Functional analysis*. Graduate studies in mathematics. American Mathematical Society, 2018.

[17] B. Chandra and D. E. Smylie. A laboratory model of thermal convection under a central force field. *Geophysical Fluid Dynamics*, 3(1):211–224, 1972.

[18] N. Dahley, B. Futterer, C. Egbers, O. Crumeyrolle, and I. Mutabazi. Parabolic flight experiment "convection in a cylinder" – convection patterns in varying buoyancy forces. *Journal of Physics: Conference Series*, 318(8):082003, dec 2011.

[19] J. Diaz and G. Galiano. On the boussinesq system with non linear thermal diffusion. *Nonlinear Analysis: Theory, Methods & Applications*, 30(6):3255 – 3263, 1997. Proceedings of the Second World Congress of Nonlinear Analysts.

[20] A. Eder and M. Jordan. *The Schlieren Technique.* Optical Measurements. Heat and Mass Transfer. Springer New York, 2001.

[21] M. Egert, R. Haller-Dintelmann, and J. Rehberg. Hardy's inequality for functions vanishing on a part of the boundary. *Potential Analysis*, 43(1):49–78, 2015.

[22] S. C. Eisenstat and H. F. Walker. Globally convergent inexact newton methods. *SIAM Journal on Optimization*, 4(2):393–422, 1994.

[23] E. Emmrich. *Analysis von Zeitdiskretisierungen des inkompressiblen Navier-Stokes-Problems.* Cuvillier Verlag, Göttingen, 2001.

[24] E. Emmrich. Error of the two-step bdf for the incompressible navier-stokes problem. *ESAIM: M2AN*, 38(5):757–764, 2004.

[25] A. Ern and J. Guermond. *Theory and Practice of Finite Elements.* Applied Mathematical Sciences. Springer New York, 2004.

[26] L. Evans and A. M. Society. *Partial Differential Equations.* Graduate studies in mathematics. American Mathematical Society, 1998.

[27] B. Futterer, N. Dahley, and C. Egbers. Thermal electro-hydrodynamic heat transfer augmentation in vertical annuli by the use of dielectrophoretic forces through a.c. electric field. *International Journal of Heat and Mass Transfer*, 93:144–154, 2016.

[28] G. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Volume 2: Nonlinear Steady Problems.* Springer Monographs in Mathematics. Springer, 2011.

[29] S. Gawlok, P. Gerstner, S. Haupt, V. Heuveline, J. Kratzke, P. Lösel, K. Mang, M. Schmidtobreick, N. Schoch, N. Schween, J. Schwegler, C. Song, and M. Wlotzka. HiFlow3 – Technical Report on Release 2.0. *Preprint Series of the Engineering Mathematics and Computing Lab*, 0(6), 2017.

[30] V. Girault, R. H. Nochetto, and R. Scott. Stability of the finite element stokes projection in $w^{1,\infty}$. *Comptes Rendus Mathematique*, 338(12):957 – 962, 2004.

[31] V. Girault and P. Raviart. *Finite Element Approximation of the Navier-Stokes Equations.* Lecture Notes in Mathematics. Springer-Verlag, 1979.

[32] V. Girault and P. Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms.* Springer series in computational mathematics. Springer-Verlag, 1986.

[33] O. N. Goncharova. Unique solvability of a two-dimensional nonstationary problem for the convection equations with temperature-dependent viscosity. *Differential Equations*, 38(2):249–258, 2002.

[34] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, 2011.

[35] J. G. Heywood and R. Rannacher. Finite element approximation of the nonstationary navier–stokes problem. i. regularity of solutions and second-order error estimates for spatial discretization. *SIAM Journal on Numerical Analysis*, 19(2):275–311, 1982.

[36] J. G. Heywood and R. Rannacher. Finite-element approximation of the nonstationary navier–stokes problem. part iv: Error analysis for second-order time discretization. *SIAM Journal on Numerical Analysis*, 27(2):353–384, 1990.

[37] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints.* Mathematical Modelling: Theory and Applications. Springer Netherlands, 2010.

[38] J. Hunter. *Notes on Partial Differential Equations.* https://www.math.ucdavis.edu/ hunter/pdes/, 2014.

[39] T. Hytoenen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces*. Springer International Publishing, 2016.

[40] K. J. *Sobolev Spaces, Lecture Notes*. Aalto University, 2017.

[41] V. John. *Finite Element Methods for Incompressible Flow Problems*. Springer International Publishing, 2016.

[42] C. Kang, A. Meyer, H. N. Yoshikawa, and I. Mutabazi. Numerical simulation of circular couette flow under a radial thermo-electric body force. *Physics of Fluids*, 29(11):114105, 2017.

[43] C. Kang, A. Meyer, H. N. Yoshikawa, and I. Mutabazi. Thermoelectric convection in a dielectric liquid inside a cylindrical annulus with a solid-body rotation. *Phys. Rev. Fluids*, 4:093502, Sep 2019.

[44] C. Kang and I. Mutabazi. Dielectrophoretic buoyancy and heat transfer in a dielectric liquid contained in a cylindrical annular cavity. *Journal of Applied Physics*, 125(18):184902, 2019.

[45] G. Karypis and V. Kumar. A fast and high quality multilevel scheme for partitioning irregular graphs. *SIAM J. Sci. Comput.*, 20(1):359–392, Dec. 1998.

[46] W. Layton. *Introduction to the Numerical Analysis of Incompressible Viscous Flows*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.

[47] S. A. Lorca and J. L. Boldrini. Stationary solutions for generalized boussinesq models. *Journal of Differential Equations*, 124(2):389 – 406, 1996.

[48] S. V. Malik, H. N. Yoshikawa, O. Crumeyrolle, and I. Mutabazi. Thermo-electro-hydrodynamic instabilities in a dielectric liquid under microgravity. *Acta Astronautica*, 81(2):563 – 569, 2012.

[49] M. Marcus and V. J. Mizel. Every superposition operator mapping one sobolev space into another is continuous. *Journal of Functional Analysis*, 33(2):217 – 229, 1979.

[50] A. Mauro. Dielectrophoresis: The behavior of neutral matter in nonuniform electric fields. herbert a. pohl. *The Quarterly Review of Biology*, 55(1):68–69, 1980.

[51] M. Meier, M. Jongmanns, A. Meyer, T. Seelig, C. Egbers, and I. Mutabazi. Flow pattern and heat transfer in a cylindrical annulus under 1g and low-g conditions: Experiments. *Microgravity Science and Technology*, 2018.

[52] A. Meyer. *Active control of heat transfer by an electric field*. PhD thesis, Université du Havre, 2017.

[53] A. Meyer, O. Crumeyrolle, I. Mutabazi, M. Meier, M. Jongmanns, M.-C. Renoult, T. Seelig, and C. Egbers. Flow patterns and heat transfer in a cylindrical annulus under 1g and low-g conditions: Theory and simulation. *Microgravity Science and Technology*, 2018.

[54] A. Meyer, M. Jongmanns, M. Meier, C. Egbers, and I. Mutabazi. Thermal convection in a cylindrical annulus under a combined effect of the radial and vertical gravity. *Comptes Rendus Mécanique*, 345(1):11–20, 2017.

[55] A. Meyer, M. Meier, M. Jongmanns, T. Seelig, C. Egbers, and I. Mutabazi. Effect of the initial conditions on the growth of thermoelectric instabilities during parabolic flights. *Microgravity Science and Technology*, 31, 11 2019.

[56] H. Morimoto. On non-stationary boussinesq equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 67(5):159–161, 1991.

[57] H. Morimoto. On the existence and uniqueness of the stationary solution to the equations of natural convection. *Tokyo J. Math.*, 14(1):217–226, 06 1991.

[58] I. Mutabazi, H. N. Yoshikawa, M. T. Fogaing, V. Travnikov, O. Crumeyrolle, B. Futterer, and C. Egbers. Thermo-electro-hydrodynamic convection under microgravity: a review. *Fluid Dynamics Research*, 48(6):061413, 2016.

[59] R. Oyarzúa, T. Qin, and D. Schötzau. An exactly divergence-free finite element method for a generalized Boussinesq problem. *IMA Journal of Numerical Analysis*, 34(3):1104–1135, 10 2013.

[60] R. Oyarzúa and P. Zúñiga. Analysis of a conforming finite element method for the boussinesq problem with temperature-dependent parameters. *Journal of Computational and Applied Mathematics*, 323:71 – 94, 2017.

[61] C. E. Pérez, J.-M. Thomas, S. Blancher, and R. Creff. The steady navier–stokes/energy system with temperature-dependent viscosity—part 1: Analysis of the continuous problem. *International Journal for Numerical Methods in Fluids*, 56(1):63–89, 2008.

[62] C. E. Pérez, J.-M. Thomas, S. Blancher, and R. Creff. The steady navier–stokes/energy system with temperature-dependent viscosity—part 2: The discrete problem and numerical experiments. *International Journal for Numerical Methods in Fluids*, 56(1):91–114, 2008.

[63] V. R. *Finite Element Methoden für Navier-Stokes Gleichungen, Lecture Notes 2016 / 2017*. Ruhr-Universität Bochum, 2016.

[64] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Society for Industrial and Applied Mathematics, second edition, 2003.

[65] F. Schmitz, P. Nägele, and J. Daube. *Bochner-Räume, Lecture Notes WS 2015 / 2016*. Albert-Ludwigs-Universität Freiburg, 2016.

[66] P. W. Schröder, C. Lehrenfeld, A. Linke, and G. Lube. Towards computable flows and robust estimates for inf-sup stable fem applied to the time-dependent incompressible navier–stokes equations. *SeMA Journal*, 75(4):629–653, Dec 2018.

[67] P. W. Schröder and G. Lube. Stabilised dg-fem for incompressible natural convection flows with boundary and moving interior layers on non-adapted meshes. *Journal of Computational Physics*, 335:760 – 779, 2017.

[68] T. Seelig, A. Meyer, P. Gerstner, M. Meier, M. Jongmanns, M. Baumann, V. Heuveline, and C. Egbers. Dielectrophoretic force-driven convection in annular geometry under earth's gravity. *International Journal of Heat and Mass Transfer*, 139:386 – 398, 2019.

[69] R. Serrano. An alternative proof of the aubin–lions lemma. *Archiv der Mathematik*, 101(3):253–257, Sep 2013.

[70] M. Smiszek, O. Crumeyrolle, I. Mutabazi, and C. Egbers. Numerical simulation of thermoconvective instabilities of a dielectric liquid in a cylindrical annulus. *59^{th} International Austronautical Congress Glasgow, 29/09-3/10*, 2008.

[71] E. A. Spiegel and G. Veronis. On the boussinesq approximation for a compressible fluid. 1960.

[72] R. Stenberg. On some three-dimensional finite elements for incompressible media. *Computer Methods in Applied Mechanics and Engineering*, 63(3):261 – 269, 1987.

[73] P. Stiles and M. Kagan. Stability of cylindrical couette flow of a radially polarised dielectric liquid in a radial temperature gradient. *Physica A: Statistical Mechanics and its Applications*, 197(4):583–592, 1993.

[74] P. J. Stiles. Electro-thermal convection in dielectric liquids. *Chemical Physics Letters*, 179(3):311 – 315, 1991.

[75] M. Tabata and D. Tagami. Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients. *Numerische Mathematik*, 100(2):351–372, Apr 2005.

[76] M. Takashima. Electrohydrodynamic Instability in a Dielectric Fluid between two coaxial Cylinders. *The Quarterly Journal of Mechanics and Applied Mathematics*, 33(1):93–103, 02 1980.

[77] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* AMS/Chelsea publication. AMS Chelsea Pub., 2001.

[78] V. Travnikov, O. Crumeyrolle, and I. Mutabazi. Numerical investigation of the heat transfer in cylindrical annulus with a dielectric fluid under microgravity. *Physics of Fluids*, 27(5):054103, 2015.

[79] V. Travnikov, O. Crumeyrolle, and I. Mutabazi. Influence of the thermo-electric coupling on the heat transfer in cylindrical annulus with a dielectric fluid under microgravity. *Acta Astronautica*, 129:88–94, 2016.

[80] D. Werner. *Funktionalanalysis.* Springer-Lehrbuch. Springer Berlin Heidelberg, 2007.

[81] H. N. Yoshikawa, O. Crumeyrolle, and I. Mutabazi. Dielectrophoretic force-driven thermal convection in annular geometry. *Physics of Fluids*, 25(2):024106, 2013.

[82] F. Zaussinger, P. Haun, M. Neben, T. Seelig, V. Travnikov, C. Egbers, H. Yoshikawa, and I. Mutabazi. Dielectrically driven convection in spherical gap geometry. *Physical Review Fluids*, 3(9):093501, Sep 2018.