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Independence of alternatives in ranking models

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Abstract

When Luce (1959) introduced his Choice Axiom, this raised immediate criticism by Debreu (1960), pointing out inconsistencies when items are ranked from inferior to superior (instead of ranking them from superior to inferior). As recently shown by Breitmoser (2019), Luce's *Independence of Irrelevant Alternatives* (IIA) is equivalent to Luce's Choice Axiom when positivity holds. This fact seems to have escaped attention so far and might suggest that Debreu's critique also applies to the notion of IIA, which is widely used in the literature. Furthermore, this notion could potentially be intuitively misleading, as the consequences of this axiom seem to be different than the name suggests. This might spill over to the intuitive interpretation of theoretical results that build on this axiom.
This paper motivates the introduction of the notion of *Independece of Alternatives* (IoA) in the context of ranking models. IoA postulates a property of independence which seems intuitively reasonable (as it exactly captures what Luce himself describes when speaking about IIA), but does not exclusively hold in models where Luce's Choice Axiom applies. Assuming IoA, expected ranks in the ranking of multiple alternatives can be determined from pairwise comparisons. The result holds in many models which do not satisfy IIA (e.g. certain Thurstone V models, Thurstone (1927)), can significantly simplify the calculation of expected ranks in practice and potentially facilitate analytic methods that build on more general approaches to model the ranking of multiple alternatives.

Keywords: Ranking models, IIA, IoA, Luce's Choice Axiom, Thurstone V

JEL-Codes: D03, C13

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1 Introduction

Luce's Choice Axiom (LCA, Luce (1959)) is one of the strong axioms in the field of ranking multiple alternatives. Breitmoser (2019) showed that, under the assumption that Positivity holds, it is equivalent to the property of Independence of Irrelevant Alternatives (IIA) introduced by Luce (1959) and restated by McFadden (1973).[1] Given the implications of LCA, which (among others) have been studied by Yellott Jr (1977) and Allison and Christakis (1994), one might wonder whether, in fact, the property of IIA could potentially be different from what its intuitive meaning suggests. On the one hand, one might wonder about the meaning of "irrelevant" in models where positivity holds. Furthermore, "independence" could potentially be interpreted differently, as Debreu's critique (Debreu, 1960) extends from LCA to IIA.

Consider the following situation for illustration. A government tries to aggregate voters' opinions in order to choose from different policies, but not all policies are available at all times. Now, one could ask whether the relative order of two policies depends on the availability of other policies. When Luce motivates IIA (Luce (1959), p. 9), he intuitively describes it as follows:

"The idea states that if one is comparing two alternatives according to some algebraic criterion, say preference, this comparison should be unaffected by the addition of alternatives or the subtraction of old ones (different from the two under consideration)."[2]

If the government assumes that this idea is reasonable, it might infer that IIA and (given positivity) LCA hold in this situation. In the opinion of the author, this is questionable, as IIA and LCA do not capture the intution described above.

On the one hand, imagine the government would set up a simple urn model

[1]Note that Luce in his original work showed that LCA and Positivity imply IIA, but did not show equivalence.

[2]Luce then argues that the "probabilistic analogue of this idea is not perfectly clear", and proposes to translate it as constant ratios of the probabilities to be ranked first.

to produce the ranking of policies. Let us assume that policies are represented by colors, with blue and red balls in the urn, and the number of blue and red balls determines the chance of one color being drawn first. The order in which the colors are drawn determine their ranking. Now, we can also add additional black balls to the urn, keeping the blue and red balls constant. In this situation LCA holds, as the ratio of blue and red balls in the urn does not change. Now imagine that we change the rules slightly, introducing that the ball which is drawn first is actually ranked last, the next distinct color drawn is ranked second-to-last and so on. This obviously changes the probabilities to be ranked first (which corresponds to being drawn third in this example) and, for any case of more than two colors, LCA (as well as IIA) is violated.[3] However, it is not straightforward to see why this model should violate "independence", as we do not change the composition of the urn with respect to blue and red balls while adding black balls.

On the other hand, note that IIA only imposes conditions on probabilities to rank first. Specifically, assume that policies X and Y are among the alternatives to be ranked, and X will be ranked either first or second, if Y is in the set to be ranked. At the same time X will rank either first or last, if Y is not in the set. In the light of Luce's description, it might seem unintuitive to say that the situation described can satisfy IIA, as the relative ranking of X and a third option, say Z, seems to depend on whether Y is to be ranked or not. Yet, a situation like this is not excluded by IIA, see the following example:

Example 1. *Let rankings be denoted by vectors, i.e.* (A, B, C, D) *implies item* A *is ranked first,* B *second, and so on. Let* $C = \{W, X, Y, Z\}$*. Now, define the following probabilities on rankings:*

$$Pr_C\left((W, X, Y, Z)\right) = Pr_C\left((X, Y, W, Z)\right) =$$
$$Pr_C\left((Y, X, W, Z)\right) = Pr_C\left((Z, X, Y, W)\right) = 0.25$$

[3]Instead, a modified version holds, where the probability ratios of being ranked last are constant across different sets. This modified version was discussed by Luce himself, see Theorem 8 of Luce (1959).

Note that X is ranked either first or second, while all elements have an equal chance to end up first. In sets of three alternatives, define

$$Pr_{C\setminus\{Z\}}((W,X,Y)) = Pr_{C\setminus\{Z\}}((X,Y,W)) = Pr_{C\setminus\{Z\}}((Y,X,W)) = \frac{1}{3}$$
$$Pr_{C\setminus\{Y\}}((W,Z,X)) = Pr_{C\setminus\{Y\}}((X,Z,W)) = Pr_{C\setminus\{Y\}}((Z,W,X)) = \frac{1}{3}$$
$$Pr_{C\setminus\{X\}}((W,Z,Y)) = Pr_{C\setminus\{X\}}((Z,Y,W)) = Pr_{C\setminus\{X\}}((Y,Z,W)) = \frac{1}{3}$$
$$Pr_{C\setminus\{W\}}((Z,X,Y)) = Pr_{C\setminus\{W\}}((X,Y,Z)) = Pr_{C\setminus\{W\}}((Y,X,Z)) = \frac{1}{3}$$

Again, all elements have an equal chance to end up first. However, X is exclusively ranked first or second, if Y is in the set. In the situation where Y is missing, X ends up either first or last. Finally, assume that in all pairwise comparisons, all items have equal probabilities of 0.5.

It is straightforward to see that probabilities to rank first are equal for all alternatives in any subset of C (i.e., constant probability ratios of exactly one). Therefore, IIA as introduced by Luce formally holds, while the situation might violate what Luce expresses to be his intuition of IIA.

In this paper, I propose the property of "Indepence of Alternatives" (IoA). It attempts to capture Luce's intuition, while providing structure that can potentially be beneficial in determining expected ranks of alternatives. Imagine the government wants to aggregate the rankings of available policies applying the "Rule of Borda". Assuming IoA, it can take a shortcut in ranking policies. Instead of eliciting full rankings of every voter and applying the rule, it is sufficient to know the proportion of voters that favors one policy over another in pairwise comparisons.

Note that other authors have promoted different notions of independence in the context of stochastic choice. Yet, IoA differs from *"i-independence"* introduced by Manzini and Mariotti (2014) and *"independence"* introduced by Gul, Natenzon, and Pesendorfer (2014). In this comparison, one might say that IoA is an axiom in the context of stochastic *ranking* rather than stochastic *choice*, as it does not (directly) impose restrictions on probabilities to be chosen, but to be ranked better. While stochastic choice and IIA helps to answer the question "Which alternative is the best?", stochastic ranking rather addresses the question "Which alternative is better?".

Given that IoA holds, I present a result to determine the expected rank of an item in the ranking of n alternatives from only pairwise comparisons. It relates to results by Luce (1959) and McFadden (1973), who established a similar link for LCA and IIA models, namely that the probability to be ranked first in any set can be inferred from pairwise comparisons. The differences of these results might become apparent when considering them in the context of chess ratings, see Chapter 3 for details.

The next chapter introduces mathematical properties associated with the ranking of n alternatives and describes their mutual relations. Chapter 3 derives the result for expected ranks and discusses its implications, and Chapter 4 concludes.

2 Mathematical properties of rankings

2.1 Notation

This paper studies rankings of multiple alternatives. Note that ties are excluded in these rankings. To fix notation, let

- C be a finite set of n objects,
- Σ be the power set of C,
- $(\Omega, \mathcal{A}, \mathbb{P})$ be a general abstract probability space.

Now, consider a family of rankings $\{r_S\}_{S\in\Sigma}$ on subsets of C (as $S \in \Sigma$). Rankings are random variables on Ω:

$$r_S : S \times \Omega \to \mathbb{N}$$

for all $S \in \Sigma$.

Example 2. *Define $\Omega = \{\omega_1, \omega_2\}$ and $C = \{x, y, z\}$. Now, define the rankings according to Table 1.*

$r_C(c,\omega)$

	x	y	z
ω_1	1	2	3
ω_2	3	1	2

$r_{\{x,y\}}(c,\omega)$

	x	y
ω_1	1	2
ω_2	2	1

$r_{\{x,z\}}(c,\omega)$

	x	z
ω_1	2	1
ω_2	2	1

$r_{\{y,z\}}(c,\omega)$

	y	z
ω_1	1	2
ω_2	2	1

Table 1: Ranking example

In this example, $r_C(z,\omega) > 1 \;\; \forall \;\; \omega \in \Omega$, i.e. alternative z is never ranked first when ranking the set C (but it will always be ranked first when ranking $\{x,z\}$).

Note that this notation allows to formulate the probability for item $c \in C$ to be ranked at position i when ranking the set S, i.e.

$$\mathbb{P}\left(\{\omega \in \Omega \mid r_S(c,\omega) = i\}\right).$$

2.2 Properties

The mathematical foundation of ranking models has a long history in different fields of science. Thurstone (1927) established a framework, which was later refined and extended by other researchers. The so-called Thurstone V models[4] represent a class of models in which items create a stochastic stimulus and are ranked accordingly. When correlation is set to zero, the stimulus is drawn independently from identical distributions, which only differ about their mean, see Figure 1 for an example.

Bradley and Terry (1952) established a Thurstone V model with underlying extreme value distributions, which is the foundation of logit analysis known today. While the authors focus on pairwise comparisons, this approach was extended by

[4]The Roman number V is a numeration of increasingly structured models.

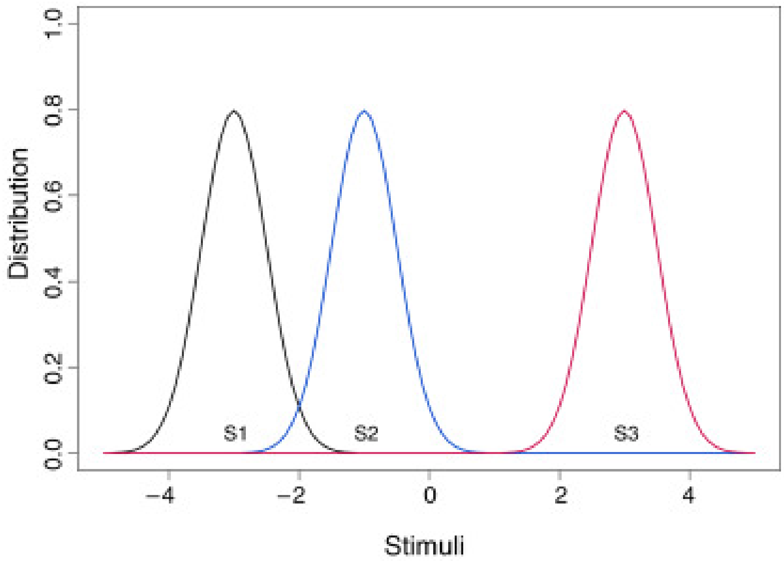

Figure 1: Thurstone V model, where stimuli of items S1, S2 and S3 are drawn from normal distributions that differ about their mean. Subsequently, items are ranked according to their stimuli.

Luce (1959) for rankings of n alternatives. In this work, he postulated what is today known as *Luce's Choice Axiom*,

$$p_S(x) = \frac{p_C(x)}{\sum_{y \in S} p_C(y)}. \tag{1}$$

Here, $p_S(x)$ denotes the probability of item x being ranked first when the set S is to be ranked,

$$p_S(x) = \mathbb{P}\left(\{\omega \in \Omega \mid r_S(x, \omega) = 1\}\right)$$

Intuitively speaking, LCA demands that the probability of being ranked first in a set equals the share of winning probabilities in the entire world. Note that it is crucial to define the relation on the probabilities to be ranked first. Reversing Luce's Choice Axiom in a way that "the probabilities of being ranked last in a set are caclulated using the probabilities of being last in the complete set of items" is a different assumption, and Luce himself proved that there is no distribution satisfying both his Choice Axiom and the "Reversed Choice Axiom" (see Luce (1959)).

Furthermore, he introduced the property of *Positivity* which is defined as

$$p_S(x) > 0 \qquad \forall S \subseteq C,\;\; \forall x \in S. \tag{2}$$

and states that each item has a positive probability to be chosen in any given choice set. Actually, one might be inclined to say that this property rules out the existence of any "irrelevant" option in any set. Specifically, as Luce (1959) pointed out himself, in any situation where LCA holds, $p_S(x) = 0$ for arbitrary S means that item x is never chosen in *any* set and could thus be eliminated.

Finally, Luce (1959) introduced the property of *Independence of Irrelevant Alternatives* (IIA) which, under the assumption of positivity, is formulated as

$$\frac{p_S(x)}{p_S(y)} = \frac{p_{S'}(x)}{p_{S'}(y)} \qquad \forall S, S' \subseteq C,\;\; \forall x, y \in S \cap S'. \tag{3}$$

Intuitively speaking, it states that the ratio of probabilities to be ranked first is constant for any two items in any two sets which contain both elements. It turns out that

Lemma 1. *The property of* IIA *is equivalent to* Luce's Choice Axiom *under the assumption of positivity.*

For a proof, see Breitmoser (2019). An alternative proof is provided in Appendix 5.1.

In the following, a novel independence property is defined. The property of *Independence of Alternatives* (IoA) assumes a certain structure on rankings, specifically on the probability of item x receiving a better rank than item y when ranking alternatives of the set S with $x, y \in S$. IoA states that this probability is independent of S.

Definition 2.1. *A model satisfies* Independence of Alternatives *if, for all* $S, S' \in \Sigma$ *and all* $x, y \in S \cap S'$,

$$\mathbb{P}\left(\{\omega \in \Omega \mid r_S(x,\omega) < r_S(y,\omega)\}\right) = \mathbb{P}\left(\{\omega \in \Omega \mid r_{S'}(x,\omega) < r_{S'}(y,\omega)\}\right).$$

Note that IoA and IIA are technically very different properties, as none implies the other. While IIA imposes a certain structure exclusively on probabilities to be ranked first, IoA relaxes this structure on rank one probabilities[5], but at the same time imposes additional structure on rank two to n. A simple example which satisfies IIA, but not IoA, as well as some further remarks can be found in section 2.3.

2.3 Relation between IIA and IoA

In order to see that IIA does not imply IoA, consider the following

Example 3. *Assume that the three alternatives $S = \{A, B, C\}$ are to be ranked. In pairwise comparisons, A is twice as likely to be first when compared to B or C, i.e.*

$$p_{\{A,B\}}(A) = \mathbb{P}\left(\{\omega \in \Omega \mid r_{\{A,B\}}(A,\omega) < r_{\{A,B\}}(B,\omega)\}\right) = \frac{2}{3},$$
$$p_{\{A,C\}}(A) = \mathbb{P}\left(\{\omega \in \Omega \mid r_{\{A,C\}}(A,\omega) < r_{\{A,C\}}(C,\omega)\}\right) = \frac{2}{3}.$$

In the comparison of B and C, both alternatives are equally likely to be first. Now, in the rating of all three alternatives, assume that

$$p_S(A) = \mathbb{P}\left(\{\omega \in \Omega \mid r_S(A,\omega) = 1, r_S(B,\omega) = 2, r_S(C,\omega) = 3\}\right) = 0.5,$$
$$p_S(B) = \mathbb{P}\left(\{\omega \in \Omega \mid r_S(B,\omega) = 1, r_S(A,\omega) = 2, r_S(C,\omega) = 3\}\right) = 0.25,$$
$$p_S(C) = \mathbb{P}\left(\{\omega \in \Omega \mid r_S(C,\omega) = 1, r_S(A,\omega) = 2, r_S(B,\omega) = 3\}\right) = 0.25.$$

One can easily verify that IIA holds, as A is still twice as likely to be first. On the other hand,

$$\mathbb{P}\left(\{\omega \in \Omega \mid r_S(A,\omega) < r_S(B,\omega)\}\right) = p_S(A) + p_S(C) = 0.75,$$

which is inconsistent with the pairwise comparison of A and B.

[5] Consider, for example, the urn models in Chapter 1, which obviously satisfy IoA, while one of them does not satisfy IIA (Luce, 1959).

This example discloses what IIA lacks in order to imply IoA: an extension of its (strong) structure of probabilities for ranking first to the probabilities of other ranks. If additional structure is assumed, for example, probabilities to rank second being connected to the probabilities to rank first, IIA can imply IoA. One way of achieving this is by introducing *Order Consistency* (OC). To my knowledge, this axiom is new in the context of stochastic rankings.

Definition 2.2. *A model satisfies* Order Consistency *if* $\forall S \subseteq C$, $\forall i, j \in S$*:*

$$\mathbb{P}\left(\{\omega \in \Omega \mid r_S(x,\omega)=1,\ r_S(y,\omega)=2\}\right) = \mathbb{P}\left(\{\omega \in \Omega \mid r_{S\setminus\{x\}}(y,\omega)=1\}\right).$$

Intuitively speaking, OC means that there is no information to be gained about the ranking of other alternatives by observing the best option, i.e. correlation is set to zero. Now, it can be shown that

Lemma 2. *Together, the properties of* IIA *and* OC *imply* IoA*.*

For a proof, see Appendix 5.2. Note that OC is not a necessary condition for IoA to hold, i.e. one can find examples of four or more alternatives where IIA and IoA hold, but OC does not.

One might be inclined to generalize OC in a way that observing *any* rank does not change ranking probabilities of the other alternatives, i.e. $\forall S \subseteq C,\ \ \forall i, j \in S:$

$$\mathbb{P}\left(\{\omega \in \Omega \mid r_S(x,\omega)=i,\ r_S(y,\omega)=j\}\right) = \begin{cases} \mathbb{P}\left(\{\omega \in \Omega \mid r_{S\setminus\{x\}}(y,\omega)=j-1\}\right) & \text{if } i<j \\ \mathbb{P}\left(\{\omega \in \Omega \mid r_{S\setminus\{x\}}(y,\omega)=j\}\right) & \text{if } j<i \end{cases} \tag{4}$$

However, while this property might be intuitively intriguing, note that there is no model that can satisfy it (other than trivial models, where all alternatives have an equal chance in every set). The argument is as follows: property (4) implies both OC and IoA[6], hence LCA. At the same time, it implies "reversed OC", which together with IoA implies "reversed LCA". Thus, no non-trivial model can satisfy

[6]See Appendix 5.3 for a formal proof.

(4). One might say that this property is equivalent to generalizing Luce's axiom, extending it to probabilities for any rank, which is known not to be feasible.

The next section derives a result which holds solely under the assumption of IoA.

3 Expected ranks from pairwise comparisons

The property of IoA in ranking models yields a relation between expected ranks in pairwise comparisons and expected ranks in sets of multiple alternatives. Let S be a set of k items to be ranked, i.e. $|S| = k$. Furthermore let $(\Omega, \mathcal{A}, \mathbb{P})$ be a general abstract probability space. Now, continue by defining the following random variables. Let $\hat{X}^S_{x,y}(\omega) : \Omega \to \{0, 1\}$ denote the random variable which (in state ω) yields a one if, in the overall ranking, x is ranked better than y, and zero otherwise.

$$\hat{X}^S_{x,y}(\omega) = \begin{cases} 1 & \text{for } r_S(x,\omega) < r_S(y,\omega) \\ 0 & \text{otherwise} \end{cases}$$

In addition, let $Y^S_x(\omega) : \Omega \to \{0, 1, ..., k-1\}$ be the random variable given by

$$Y^S_x(\omega) = \sum_{\substack{y \in S \\ y \neq x}} \hat{X}^S_{x,y}(\omega),$$

i.e. the random variable that states how many items have a higher rank than item x. Finally, recall that $r_S(x, \omega)$ denotes the rank of x.

Remark 1. *The rank $r_S(x, \omega)$ of item x when ranking set S equals (by definition) the difference between the total number of items and the number of items which are ranked higher than x, namely $Y^S_x(\omega)$:*

$$r_S(x, \omega) = k - Y^S_x(\omega) \tag{5}$$

Now, under the assumption of IoA, one can consider $\hat{X}^{\{x,y\}}_{x,y}(\omega)$, i.e. the random variable which yields a one if x is ranked better than y in the set $S = \{x, y\}$ (i.e., in the pairwise comparison of x and y) and claim that

$$\mathbb{P}\Big(\hat{X}^S_{x,y}(\omega) = 1\Big) \overset{(IoA)}{=} \mathbb{P}\Big(\hat{X}^{\{x,y\}}_{x,y}(\omega) = 1\Big) \tag{6}$$

From here, a link in expectation can be established, connecting the number of items that have a higher rank than x to the probabilities of pairwise comparison.

Lemma 3.

$$\mathbb{E}\left[Y_x^S(\omega)\right] = \sum_{\substack{y=1\\y\neq x}}^{k} \mathbb{P}\Big(\hat{X}_{x,y}^{\{x,y\}}(\omega)=1\Big)\,. \tag{7}$$

Proof. By definition, it holds that

$$\mathbb{E}\left[Y_x^S(\omega)\right] = \mathbb{E}\left[\sum_{\substack{y=1\\y\neq x}}^{k} \hat{X}_{x,y}^{S}(\omega)\right].$$

Now, by (6) and noting that probabilities imply expected values, it holds that

$$\mathbb{E}\left[\sum_{\substack{y=1\\y\neq x}}^{k} \hat{X}_{x,y}^{S}(\omega)\right] = \mathbb{E}\left[\sum_{\substack{y=1\\y\neq x}}^{k} \hat{X}_{x,y}^{\{x,y\}}(\omega)\right]$$

and subsequently,

$$\mathbb{E}\left[\sum_{\substack{y=1\\y\neq x}}^{k} \hat{X}_{x,y}^{\{x,y\}}(\omega)\right] = \sum_{\substack{y=1\\y\neq x}}^{k} \mathbb{E}\left[\hat{X}_{x,y}^{\{x,y\}}(\omega)\right] = \sum_{\substack{y=1\\y\neq x}}^{k} \mathbb{P}\Big(\hat{X}_{x,y}^{\{x,y\}}(\omega)=1\Big)\,.$$

□

Combining (5) and (7) yields the following

Theorem 1. *Under IoA, the expected rank of item i in the ranking of k alternatives is determined by it's pairwise comparisons with all other alternatives, i.e.*

$$\mathbb{E}\left[r_S(x,\omega)\right] = k - \sum_{\substack{j=1\\j\neq i}}^{k} \mathbb{P}\Big(\hat{X}_{x,y}^{\{x,y\}}(\omega)=1\Big)\,.$$

The value of this relation lies in its connection between pairwise comparison and ranking of $k > 2$ items. The left-hand-side corresponds to

$$\mathbb{E}[r_S(x,\omega)] = \sum_{j=1}^{k} j \cdot \mathbb{P}\left(\{\omega\in\Omega \mid r_S(x,\omega)=j\}\right)$$

where $\mathbb{P}(\{\omega \in \Omega \mid r_S(x, \omega) = j\})$ might be remarkably more complex to calculate.

The differences between the concepts and implications of IIA and IoA might best become apparent when considering chess ratings. Both the United States Chess Federation (USCF) and the Fédération Internationale des Échecs (FIDE) use rating systems to measure the strength of players and to predict outcomes. Notably, their formulas to calculate predictions differ. The USCF formulas correspond to a Thurstone V model with underlying extreme value distributions. On the other hand, FIDE assumes a Thurstone V model with underlying normal distributions. Yellott Jr (1977) shows that LCA (and therefore IIA) hold in the former, but not in the latter. Now, consider a chess tournament with multiple players. According to the results of Luce (1959) and McFadden (1973), one can infer the winning probabilities of the whole tournament from pairwise comparisons in the USCF system, but not in the FIDE system. On the other hand, as IoA holds, expected ranks can be derived from pairwise comparisons in both systems.

4 Conclusion

This paper motivates to consider the notion of IoA, which in some sense might seem competing with the notion of IIA. However, while their semantic appearance is similar, their formal definitions as well as their implications differ significantly. Debreu's critique about LCA and the recently discovered link between IIA and LCA demands for a closer re-evaluation. Intuitively speaking, one might wonder why random utility models such as Thurstone V as well as certain urn models would violate IIA. Potentially it could be fair to say that models which keep the stimulus of items constant (independent from other alternatives in the choice set), or models in which the reversed LCA holds, satisfy some independence property. The notion of IoA is in line with Luce's intuitive description of independence and could fix this issue, while at the same time allowing a cleaner interpretation of models that do or do not satisfy an independence property. In applications (such as chess tournaments), one might wonder whether some independence property holds, i.e. whether the level of performance is unaffected by other players who might or might not participate in the tournament. While there potentially can be arguements both for and against this property, it seems unintuitive to say that independence of irrelevant alternatives could hold in the USCF system, but (by definition) not in the FIDE system. As IoA holds in both, this issue does not exist when discussing independence of alternatives in chess tournaments.

In addition, this paper shows that, under the assumption of IoA, expected ranks can be determined solely from pairwise comparisons. This result holds specifically, but not exclusively, for Luce's model with underlying extreme value distributions. It constitutes a shortcut to calculate expected ranks in practice when the standard way of calculation could be significantly more cumbersome. The benefits of calculating expected ranks from pairwise comparisons might also simplify situations in which the comparison of multiple alternatives is significantly more difficult to comprehend.

5 Appendix

5.1 Proof of Lemma 1

Proof. First, note that positivity is necessary to guarantee that probability ratios are well-defined, i.e. none of the denominators is zero. Assuming that Luce's Choice Axiom (1) holds,

$$\frac{p_S(x)}{p_S(y)} \overset{(1)}{=} \frac{\frac{p_C(x)}{\sum_{z\in S}\ p_C(z)}}{\frac{p_C(y)}{\sum_{z\in S}\ p_C(z)}} = \frac{p_C(x)}{p_C(y)} = \frac{\frac{p_C(x)}{\sum_{z\in S'}\ p_C(z)}}{\frac{p_C(y)}{\sum_{z\in S'}\ p_C(z)}} \overset{(1)}{=} \frac{p_{S'}(x)}{p_{S'}(y)}.$$

On the other hand, assuming that IIA (3) holds and taking $S' = C$ yields

$$p_S(x) = p_C(x) \cdot \frac{p_S(y)}{p_C(y)}. \tag{8}$$

Summing (8) over all $x \in S$ yields

$$\sum_{x\in S} p_S(x) = 1 = \frac{p_S(y)}{p_C(y)} \sum_{x\in S} p_C(x)$$

and therefore,

$$p_S(y) = \frac{p_C(y)}{\sum_{x\in S} p_C(x)}.$$

□

5.2 Proof of Lemma 2

Proof. Assume IIA and OC hold and show that IoA holds by induction over $|S|$. Obviously, if $|S| = 2$, IoA holds. Now, assume it holds for $|S| = n$. Consider a set with $n+1$ alternatives, and rank one option to be in first place. This leads to three cases:

(i) x is ranked first

(ii) y is ranked first

(iii) some z different from x, y is ranked first

Consider cases (i) and (ii). Conditional on x or y being ranked first, it holds that

$$\begin{aligned}\mathbb{P}\left(\{\omega\in\Omega \mid r_S(x,\omega)<r_S(y,\omega)\}\right) &= \frac{p_S(x)}{p_S(x)+p_S(y)} = \frac{1}{1+\frac{p_S(y)}{p_S(x)}} \overset{\text{(IIA)}}{=} \frac{1}{1+\frac{p_{\{x,y\}}(y)}{p_{\{x,y\}}(x)}} \\ &= \frac{p_{\{x,y\}}(x)}{p_{\{x,y\}}(x)+p_{\{x,y\}}(y)} = p_{\{x,y\}}(x) \\ &= \mathbb{P}\left(\{\omega\in\Omega \mid r_{\{x,y\}}(x,\omega)<r_{\{x,y\}}(y,\omega)\}\right).\end{aligned}$$

Furthermore, in case (iii), we are left with n alternatives to rank from two to n. Given that OC holds, the probabilities to be ranked second in this set correspond to those of being ranked first in the set without z. This means, that both OC and IIA hold in this set of n alternatives, and therefore by induction assumption, $\mathbb{P}\left(\{\omega\in\Omega \mid r_S(x,\omega)<r_S(y,\omega)\}\right) = \mathbb{P}\left(\{\omega\in\Omega \mid r_{\{x,y\}}(x,\omega)<r_{\{x,y\}}(y,\omega)\}\right)$. Combining these arguments for the different cases finalizes the proof. □

5.3 Generalized order consistency proof

Proof. First, note that OC is a special case of property (4). Hence, it is left to show that the property implies IoA. Consider two alternatives, x and y and their order in pairwise comparison. Now note that, due to property (4), the addition of new elements to the set will not change the relative order of x and y, which implies that

$$\mathbb{P}\left(\{\omega\in\Omega \mid r_S(x,\omega)<r_S(y,\omega)\}\right) = \mathbb{P}\left(\{\omega\in\Omega \mid r_{\{x,y\}}(x,\omega)<r_{\{x,y\}}(y,\omega)\}\right)$$

for arbitrary S. Therefore, IoA holds. □

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