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A COMMUTATIVE BANACH ALGEBRA WHICH FACTORIZES BUT HAS NO APPROXIMATE UNITS

MICHAEL LEINERT

ABSTRACT. It is well known that any Banach algebra having bounded approximate units factorizes. For some time it was not clear if, conversely, factorization implied the existence of bounded approximate units. This was disproved by Paschke [3], but the problem remained open for commutative Banach algebras. We give an example of a commutative semisimple Banach algebra which factorizes but has not even unbounded approximate units.

If A is a commutative Banach algebra, we say that A factorizes if any element $a \in A$ can be written $a = bc$ with $b, c \in A$. We say that A has approximate units if for any $a \in A$ and $\varepsilon > 0$ there is some $b \in A$ such that $\|a - ba\| < \varepsilon$.

Let S be the semigroup (with pointwise addition) of all real sequences $b = \{b_n\}_{n \in \mathbf{N}}$ with $b_n > 0$ for almost all $n \in \mathbf{N}$, and let $A = l^1(S)$ be the corresponding semigroup algebra. Let $a \in A$ be given, $a = \sum_{n \in \mathbf{N}} \lambda_n \varepsilon_{f_n}$, where ε_{f_n} denotes the Dirac measure on S concentrated at $f_n = \{f_{nk}\}_{k \in \mathbf{N}}$. For each $m \in \mathbf{N}$ let $H_m = \{f_{nm} | 1 \leqslant n \leqslant m, f_{nm} > 0\}$ and define

$$h_m = \begin{cases} \frac{1}{2} \min H_m, & H_m \neq \varnothing, \\ 1, & H_m = \varnothing. \end{cases}$$

The sequence $h = \{h_m\}_{m \in \mathbf{N}}$ is in S, and by construction we have $g_n = f_n - h \in S$ for all $n \in \mathbf{N}$. This implies

$$a = \varepsilon_h \cdot \left(\sum_{n \in \mathbf{N}} \lambda_n \varepsilon_{g_n} \right),$$

so A factorizes.

For $g \in S$ and $a \in A$ we have $\|\varepsilon_g \cdot a - \varepsilon_g\| \geqslant 1$, since g is not contained in the support of $\varepsilon_g \cdot a$. So A has no approximate units. A is semisimple, being a subalgebra of a commutative group algebra.

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 48 BIELEFELD, POSTFACH 8640, WEST GERMANY