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ON SOME TOPOLOGIES WHICH COINCIDE ON THE UNIT SPHERE OF THE FOURIER-STIELTJES ALGEBRA B(G) AND OF THE MEASURE ALGEBRA M(G)

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Introduction. Let G be an arbitrary locally compact group [$A(G)$], $B(G)$ the [Fourier] Fourier-Stieltjes algebra of G and $M(G)$ the Banach algebra of bounded Radon measures on G (see definitions in what follows).

We prove in §1 of this paper that we w^*-topology τ_{w^*} and the multiplier topology τ_M coincide on the unit sphere $S = \{u \in B(G);\ \|u\| = 1\}$ of $B(G)$, where $u_\alpha \to u$ in τ_M if and only if $\|(u_\alpha - u)v\| \to 0$ for each $v \in A(G)$. This result proves a conjecture of McKennon [**10**, p. 49]. It improves a result of Derighetti [**1**] and McKennon [**10**] (that $\tau_{w^*} = \tau_{uc}$ on S, where τ_{uc} is the topology of uniform convergence on compacta) which in turn improves a theorem of Raikov [**13**] and Yoshizawa [**17**] (that $\tau_{w^*} = \tau_{uc}$ on the positive definite face of S). Applying this result we show in theorem B_1 that for any compact $K \subset G$ the Banach space $A_K(G) = \{u \in A(G);$ supp $u \subset K\}$ has the Radon-Nikodym property and consequently a strong Krein-Milman theorem, for closed bounded convex subsets of $A_K(G)$, follows. Theorem B_2 of this section consists of a long list of topologies which coincide on S.

§3 consists of a measure theoretical selfcontained proof of a result of McKennon [**10**] which states that the w^* and the L^p-multiplier topology on $S = \{\mu \in M(G);\ \|\mu\| = 1\}$ coincide ($\mu_\alpha \to \mu$ in the latter if and only if $\|(\mu_\alpha - \mu)*f\|_p \to 0$ for each $f \in L^p$). The reader familiar with [**10**, pp. 21–25 and 32–33] will find, we think, that our proof is simpler, more natural and self-contained. Finally we investigate in §2, subsets of $B_p^M(G)$ (the space of multipliers of $A_p(G)$) on which the topologies τ_M and $\sigma(B_p^M, L^1)$ coincide. As a consequence a necessary and sufficient condition for a subset of $A_p(G)$ to be norm compact is given (in case G is amenable). In view of [**8**] the results seem to be of interest even for the nonamenable case.

Definitions and notations. Let G be a locally compact group with unit e. $C(G)$ ($C_{00}(G)$) [$C_0(G)$] will denote the space of complex bounded continuous functions (with compact support) [which vanish at infinity]. λ or dx will denote a left Haar measure on G. $\|f\|_p = (\int |f|^p dx)^{1/p}$ will denote the

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$L^p(G)$ norm of f. $\Delta(x)$ will denote the modular function on G and if h is a complex function on G, then we define as usual ([**4**]) $\bar{h}(x) = \overline{h(x)}$, $h^\vee(x) = h(x^{-1})$, $h^\sim(x) = \overline{h(x^{-1})}$, and $h^*(x) = (1/\Delta(x))\, h^\sim(x)$.

We follow Eymard [**4**] in the definitions and notations for the spaces $A(G)$, $B(G)$, $C^*(G)$, etc., and for the norms $\| \|_\rho$, $\| \|_\Sigma$, etc. Different norms will sometimes occur and we write $\|u\|_{B(G)} = \|u\|_B$, $\|u\|_{A(G)} = \|u\|_A$, etc., to emphasize which norm we consider.

If X, Y are normed spaces in duality then $\sigma(X, Y)$ will denote as usual the weakest topology on X which makes all linear functionals in Y continuous. If X^* is the conjugate Banach space of X then $\sigma(X, X^*)$ is denoted by w, the weak topology of X and $\sigma(X^*, X)$ is the w^* (weak star) topology of X^*.

If X, Y are normed spaces in duality and if for each $x \in X$ $\|x\| = \sup\{|\langle x, y\rangle|, y \in Y, \|y\| \leqq 1\}$, then $x_\alpha \to x$ in $\sigma(x, y)$ implies $\liminf \|x_\alpha\| \geqq \|x\|$. If in addition $\sup\|x_\alpha\| < \infty$, then $|\langle x_\alpha - x, y\rangle| \to 0$ uniformly on norm compact subsets of Y. These properties are known and easily proved.

If τ is a topology on X and $K \subset X$, then $\tau\mathrm{cl}K$ will denote the τ closure of K in X. 1_K will denote the function which is one on K and zero outside K.

The rest of the definitions are given in the following sections.

1. **Various topologies on the unit sphere of** $B(G)$**.** The basic notations in this section are as in Eymard [**4**]. If h is a continuous linear functional on a C^*-algebra, we denote by $|h|$ the positive linear functional determined by the conditions $\| |h| \| = \|h\|$ and $|h(a)|^2 \leqq \|h\|\, |h|(aa^*)$. The extension of $|h|$ to the algebra with adjoined unit is again denoted by $|h|$. $B(G)$ is the dual of the C^*-algebra $C^*(G)$ as defined in [**4**]. We define the topologies τ_{uc}, τ_{w^*}, τ_{bw^*}, τ_{nw^*} τ_M on $B(G)$ by the statement that a net u_α converges to u in τ_{uc} if $u_\alpha \to u$ uniformly on compacta; τ_{w^*} if $u_\alpha \to u$ in w^*, i.e., $\sigma(B(G)), C^*(G))$; τ_{bw^*} if u_α is norm bounded and $u_\alpha \to u$ weakly*; τ_{nw^*} if $\|u_\alpha\| \to \|u\|$ and $u_\alpha \to u$ in w^*; and τ_M if $\|(u_\alpha - u)v\| \to 0$ for all $v \in A(G)$. (M stands for multiplier).

Note that, on norm bounded sets of $B(G)$, τ_{w^*} coincides with $\sigma(B(G), L^1(G))$ since $L^1(G)$ is dense in $C^*(G)$.

The main result of this section is theorem A which shows that τ_{w^*} coincides with τ_M on $S = \{u \in B(G), \|u\| = 1\}$. It proves a conjecture of McKennon [**10**, p. 49] (it improves theorem 5.5, [**10**, p. 47]) which in turn improves a theorem of Derighetti [**1**] (that $\tau_{w^*} = \tau_{uc}$ on S) which in turn improves a theorem of Raikov [**13**] and Yoshizawa [**17**] (that $\tau_{w^*} = \tau_{uc}$ on the positive definite face of S).

The next result is theorem B_1 which states that, for any compact $K \subset G$, the Banach space $A_K(G) = \{u \in A(G); \operatorname{supp} u \subset K\}$ is a dual Banach space with the Radon-Nikodym property (definitions are given later) and consequently a strong Krein Milman theorem for closed bounded convex subsets of $A_K(G)$ follows. If G is compact abelian, $A_G(G) = A(G) \cong \ell_1(\hat{G})$ and this result is known (Phelps [**12**, p. 87]). If G is abelian and noncompact then $A(G)$ does not have the Radon-Nikodym property. We use theorem A in the proof of theorem B_1.

The last result in this section is theorem B_2 which consists of a long list of topologies which coincide on S.

The reader familiar with [**10**] will note the simplicity of the proofs that follow.

The following is a particular case of lemma (3.2) of McKennon [**10**, p. 23] with a much simpler proof.

LEMMA 1. *Let u_β be a net in $B(G)$ such that $u_\beta \to u_0 \in B(G)$ in τ_{nw^*}. Let $e_\alpha \in L^1(G)$ be a positive (in the $C^*(G)$ sense) approximate identity for $L^1(G)$ consisting of real valued functions such that $\|e_\alpha\|_1 \leqq 1$. Then, for any $\varepsilon > 0$ there exist α_0 and β_0 such that $\|e_{\alpha_0} * u_\beta - u_\beta\|_B < \varepsilon$ for all $\beta \geqq \beta_0$ and $\|e_{\alpha_0} * u_0 - u_0\|_B < \varepsilon$.*

REMARK. Our assumption implies that $e_\alpha = f_\alpha * f_\alpha^*$ for $f_\alpha \in L^1$. Since $\|e_\alpha\|_{C^*(G)} \leqq \|e_\alpha\|_1 \leqq 1$, it follows that $0 \leqq e_\alpha \leqq 1$ (in the C^*-algebra $C^*(G)$ sense) and therefore $0 \leqq (1 - e_\alpha)*(1 - e_\alpha) \leqq 1 - e_\alpha$ in $C^*(G)$ with adjoined identity.

PROOF. We can assume that $u_0 \neq 0$. If G is nondiscrete, we may adjoin a unit 1 to $L^1(G)$. If $f \in L^1(G)$, then since $\bar{e}_\alpha = e_\alpha$, we have for any $u \in B(G)$

$$\begin{aligned} |\langle e_\alpha * u - u, f\rangle|^2 &= |\langle u, (\bar{e}_\alpha - 1)*f\rangle|^2 \\ &\leqq \|u\| \, |\langle |u|, (e_\alpha - 1)*f*f^{*}*(e_\alpha - 1)\rangle| \\ &\leqq \|u\| \, \|f*f^*\|_{C^*(G)} |\langle |u|, (1 - e_\alpha)*(1 - e_\alpha)\rangle| \\ &\leqq \|u\| \, \|f\|^2_{C^*(G)}(\langle |u|, 1 - e_\alpha\rangle). \end{aligned}$$

We have used the fact that if p is a positive linear functional on a C^*-algebra A then $p(a^*ba) \leqq \|b\| \, p(a^*a)$ which follows readily from representing $p(c) = (\pi(c)\xi, \xi)$ where π is a representation of A on some Hilbert space H and $\xi \in H$. The last inequality is true by the remark above and since $|u|$ is a positive functional on $C^*(G)$.

Now $|u|(e_\alpha) \to |u|(1) = \|u\|$. Let α_0 be such that $\|u_0\|(\langle |u_0|, 1 - e_{\alpha_0}\rangle) < \varepsilon$. Then

$$\|e_{\alpha_0} * u_\beta - u_\beta\|_B \leqq \|u_\beta\|(\langle |u_\beta|, 1 - e_{\alpha_0}\rangle) \to \|u_0\|(\langle |u_0|, 1 - e_{\alpha_0}\rangle) < \varepsilon$$

since by Effors [**3**, lemma 3.5], $|u_\beta| \to |u_0|$. Choose β_0 such that $\|u_\beta\| (\langle |u_\beta|, 1 - e_{\alpha_0}\rangle) < \varepsilon$ if $\beta \geqq \beta_0$.

The following is lemma 13.5.1 in [**2**, p. 260].

LEMMA 2. *Let $A \subset L^\infty(G)$ be norm bounded and $f \in L^1(G)$. If ϕ_α is a net in A such that $w^*\lim \phi_\alpha = \phi$, then $f * \phi_\alpha \to f * \phi$ uniformly on compacta.*

For proof just note as in [2] that $f*\phi_\alpha(s) = \langle \phi_{\alpha'} {}_s f\rangle \to \langle \phi, {}_s f\rangle$ uniformly for s in a compact K since $\{{}_s f; s \in K\}$ is a compact subset of $L^1(G)$.

We note here that τ_{w^*} coincides with $\sigma(B(G), L^1(G))$ on bounded subsets of $B(G)$.

LEMMA 3. *Let $F = f*g$ for $g, f \in C_{00}(G)$. Then $u \to F*u$ is continuous from $(B(G), \tau_{bw^*})$ to $(B(G), \tau_M)$.*

PROOF: Let $u_\alpha \to u$ in the w^* topology of $B(G)$ be such that $\sup_\alpha \|u_\alpha\|_B = \gamma < \infty$ and $u \in B(G)$. Let $v \in A \cap C_{00}(G)$ and let $K \subset G$ be compact such that $S_f^{-1}S_v \subset K$ where S_w is the support of w. Then for any $w \in L^\infty$ and $h \in L^1$ one has

$$\langle (f*w)v, h\rangle = \langle w, \bar{f}^* * (vh)\rangle = \langle w1_K, \bar{f}^* * vh\rangle = \langle [f*(w1_K)]v, h\rangle.$$

Also, f and $(w1_K)^\sim$ are in $L^2(G)$; hence $f*(w1_K) \in L^2(G)*L^2(G)^\sim = A(G)$, [4, p. 218] and

$$\|(f*w)v\|_{A(G)} = \|(f*w1_K)v\|_{A(G)} \leqq \|f\|_2 \|(w1_K)^\sim\|_2 \|v\|_{A(G)}.$$

Let $w = g*(u_\alpha - u)$. Then

$$\|(f*g*(u_\alpha - u))v\| \leqq \|f\|_2 \|[(g*(u_\alpha - u))1_K]^\sim\|_2 \|v\|_{A(G)} \to 0,$$

since by lemma 2, $g*(u_\alpha - u) \to 0$ uniformly on K. We have shown that $\|[F*(u_\alpha - u)]v\|_{A(G)} \to 0$ for any $v \in A \cap C_{00}(G)$. To finish the proof it is enough to show that $F*u_\alpha \in B(G)$ and $\sup_\alpha \|F*u_\alpha\| < \infty$, both of which follow from Eymard [4, p. 198 (2.18)] by which

$$\|F*u_\alpha\|_{B(G)} \leqq \|F\|_\Sigma \|u_\alpha\|_{B(G)} \leqq \|F\|_1 \gamma.$$

REMARK. We have only used in the proof that $g \in L^1(G)$ and that $f \in L^\infty$ is 0 a.e. except on some compact set. It is not hard to show, using corollary 1 to theorem A (which follows), that F can be chosen to be any element of $L^1(G)$ and still lemma 3 remains true.

THEOREM A. *$\tau_{nw^*} \supset \tau_M$. In particular τ_{w^*} and τ_M coincide on $S = \{u \in BG); \|u\| = 1\}$.*

PROOF. Let $u_\alpha, u \in B(G)$ satisfy $u_\beta \to u$ in w^* and $\|u_\beta\| \to \|u\|$ and $\varepsilon > 0$. Let U_α be a relatively compact neighborhood base at e, $e \in V_\alpha = V_\alpha^{-1}$ be open and such that $V_\alpha^2 \subset U_\alpha$ and $e_\alpha = f_\alpha * f_\alpha^*$ where $f_\alpha = \lambda(V_\alpha)^{-1}1_{V_\alpha}$. Then e_α satisfies the conditions of lemma 1. Hence there exists α_0 and β_0 such that $\|e_{\alpha_0}*u_\beta - u_\beta\|_{B(G)} < \varepsilon/3$ if $\beta \geqq \beta_0$ and $\|e_{\alpha_0}*u - u\|_{B(G)} < \varepsilon/3$. Thus, if $v \in A(G)$ and $\beta \geqq \beta_0$ we have

$$\|(u_\beta - u)v\|_{A(G)} \leqq \frac{\varepsilon}{3} + \|[e_{\alpha_0}*(u_\beta - u)]v\|_{A(G)} + \frac{\varepsilon}{3}.$$

Take now $e_{\alpha_0} = F$ in lemma 3. Then, there is some $\beta_1 \geqq \beta_0$ such that $\|(e_{\alpha_0}*(u_\beta - u))v\|_{A(G)} < \varepsilon/3$ if $\beta \geqq \beta_1$.

The rest of the proof is immediate since $\tau_M \supset \tau_{uc} \supset \tau_{w^*}$ on bounded sets. In fact if $u_\alpha \to u$ in τ_M and we let $v \in A(G)$ be 1 on the compact K, then $\|(u_\alpha - u)1_K\|_\infty \leqq \|(u_\alpha - u)v\|_A \to 0$.

Remark. McKennon has proved in [**10**, theorem 5.5] that if u, $u_\beta \in B(G)$ and u is positive definite, then $u_\beta \to u$ in τ_{nw^*} implies that $u_\beta \to u$ in τ_M. He conjectured that the assumption tnat u is positive definite might not be needed [**10**, p. 49]. Theorem A proves this conjecture.

Corollary 1. (Raikov [**13**], McKennon [**10**]. Derighetti [**1**]). $\tau_{nw^*} \supset \tau_{uc}$. *In particular* τ_{w^*} *coincides with* τ_{uc} *on the unit sphere of* $B(G)$.

One only has to note that $\tau_M \supset \tau_{uc}$ since for any compact K there is some $v \in A(G)$ such that $v = 1$ on K.

Definition. Let $K \subset G$ be closed. Then $A_K(G) = \{f \in A(G), \operatorname{supp} f \subset K\}$ where $\operatorname{supp} f = \operatorname{cl}\{x \in G; f(x) \neq 0\}$. It is readily seen that $A_K(G) = \{f \in A(G); f = 0 \text{ on } \operatorname{cl}(G \sim K)\}$.

Corollary 2. *Consider* $A_K(G)$ *as a subset of* $B(G)$. *If* K *is compact, then* τ_{w^*} *coincides with the norm topology on the unit sphere of* $A_K(G)$.

Proof. Let v_α, $v \in A_K$ be such that $\|v_\alpha\| = 1 = \|v\|$ and $v_\alpha \to v$ in τ_{w^*}. Let $w \in A(G)$ be such that $w = 1$ on K (See [**4**, p. 208]). Then $(v_\alpha - v)w = v_\alpha - v$ and by theorem A, $\|(v_\alpha - v)w\| \to 0$.

Remark. If G is metric nondiscrete, it is easy to find a positive definite $u \in A \cap C_{00}$ such that $0 \leqq u \leqq 1$ and $\{x; u(x) = 1\} = \{e\}$. Then $v_n = u^n$ will satisfy $v_n(x) \to 0$ a.e., thus $v_n \to 0$ in the w^* topology of $B(G)$. Yet $\|v_n\|_A = u^n(e) = 1 \nrightarrow 0$.

Remark. Let G be compact abelian. Then $A(G) \approx \ell_1(\Gamma)$ (isometric isomorphism) where $\Gamma = \hat{G}$ is the discrete dual of G. In this case $A(G) = B(G)$. Corollary 2 reduces to the known fact that the norm and w^* topologies on the unit sphere of $\ell_1(\Gamma)$ coincide. If G is compact nonabelian, then $A(G)$ is just the dual of the noncommutative C^*-algebra $C^*(G)$. Then Corollary 2 applied to $A(G)$ yields another family of Banach spaces with this same property (which is just property (**) of I. Namioka [**11**, p. 530]).

Definition. A Banach space X has the Radon-Nikodym property (RNP) if every bounded subset C of X is dentable, i.e., for each $\varepsilon > 0$ there is some $x \in C$ such that (*) $x \notin$ norm cl Co$[C \sim (x + \varepsilon U)]$ where $U = \{x \in X; \|x\| \leqq 1\}$. A point $x \in C$ for which (*) holds for each $\varepsilon > 0$ is said to be a denting point of C.

It has been proved by M.A. Rieffel that vector valued measures with range in a Banach space with the RNP satisfy a Radon Nikodym theorem implenented by Bochner integrable functions [**41**]. (see also [**12**] [**16**]).

THEOREM B_1. *Let $K \subset G$ be compact. Then $A_K(G)$ is a dual Banach space with the Radon Nykodym property. Consequently every bounded closed convex subset C of $A_K(G)$ has strongly exposed points and moreover C is the norm closed convex-hull of its strongly exposed points.*

REMARK. Let G be abelian and noncompact. Then $A(G)$ does not have the RNP. In fact $A(G) \approx L^1(H)$ where $H = \hat{G}$ is not discrete. If $f \in L^1(H)$, $\int |f| dx = 1$, let $\mu(A) = \int_A |f| dx$. Then, as is well known, there is some Borel set A_0 such that $\mu(A_0) = 1/2 = \mu(G \sim A_0)$. It readily follows that if $g = f 1_{A_0} - f l_{G \sim A_0}$, then $\int |f \pm g| dx \leqq 1$, which shows that the closed unit ball of $L^1(H)$ (hence of $A(G)$) does not have extreme points and afortiori [**12**, p. 80] does not have the RNP. This seems to indicate, at least for abelian noncompact G, that if $K \subset G$ is closed with interior which is not relatively compact, then $A_K(G)$ does not have the RNP. It is possible that the proof of this fact is quite easy.

PROOF. R. Phelps has proved in [**12**, p. 85] that any Banach space with the RNP satisfies the above consequence. Hence it is enough to prove that any bounded subset $C \subset A_K(G)$ has a denting point (see [**12**, p. 79] in the definition and the remark thereafter). If K has empty interior, then $A_K(G) = \{0\}$. Hence we assume that int $K \neq \emptyset$.

We claim at first that $A_K(G) = A_K$ is w^* closed in $B(G)$.

In fact, if $u_\alpha \to u$ in w^*, $u_\alpha \in A_K$ and $v \in B(G)$ is such that $v = 0$ on K, then $0 = u_\alpha v \to uv$ in $\sigma(B(G), L^1(G))$. Hence $uv = 0$. Now for any $x \notin K$ there is some $v \in A(G)$ such that $v(K) = 0$ and $v(x) \neq 0$. Thus $u(x) = 0$ if $x \notin K$; hence $\{y \in G; u(y) \neq 0\} \subset K$. This readily shows that $u \in A_K$.

We show now that A_K is a dual Banach space. In fact, if $M = (A_K)_\perp = \{\phi \in C^*(G); \langle \phi, v \rangle = 0 \text{ for all } v \in A_K\}$ then, by [**15**, p. 92 thm 4.9(b)], the Banach space $(C^*(G)/M)^*$ is isometric to $M^\perp = \{u \in B(G); \langle u, \phi \rangle = 0$ for all $\phi \in M\}$. But $((A_K)_\perp)^\perp = M^\perp = A_K$ since A_K is w^* closed in $B(G)$, which is the dual of $C^*(G)$. This show that A_K is the dual of a Banach space and has property (**) of I. Namioka [**11**, p. 530] by our Corollary 2. Prop 4.11 of [**11**, p. 530] implies that each bounded norm closed convex subset of A_K has a denting point and hence, by Phelps [**12**, p. 79], A_K has the RNP.

THEOREM B_2. *Let $S = \{u \in B(G); \|u\|_B = 1\}$. Let $u_\beta \in S$ and $u \in S$. The following properties are equivalent.*

(a) $u_\beta \to u$ *in* τ_{w^*}(*i.e.*, $\sigma(B(G), C^*(G))$.

(b) $u_\beta \to u$ *in* τ_{uc}.

(c) $u_\beta T \to uT$ *in* $\| \ \|_{C_\rho^*(G)}$ norm for all $T \in C_\rho^*(G)$.

(c′) $u_\beta T \to uT$ *in* $\sigma(VN(G), A(G))$ *for all* $T \in VN(G)$.

(d) $T(u_\beta v) \to T(uv)$ *in* $\| \ \|_{A(G)}$ *norm for all* $T \in C_\rho^*(G)$, $v \in A(G)$.

(d′) $T(u_\beta v) \to T(uv)$ *weakly (i.e., $\sigma(A(G), VN(G))$ for all $T \in C_\rho^*(G)$, $v \in A(G)$).*

(e) $u_\beta v \to uv$ *in $A(G)$ norm for all $v \in A(G)$.*

(e′) $u_\beta v \to uv$ *weakly (i.e., $\sigma(A(G), VN(G))$ for all $v \in A(G)$).*

PROOF. $(a) \Leftrightarrow (b) \Leftrightarrow (e)$ follows from theorem A. $(e) \Rightarrow (c')$ is clear.

To show $(c') \Rightarrow (a)$, let $f \in C_{00}$, $v \in A$ with $v = 1$ on supp f. Thus $vf = f$ and $\langle u_\beta, f\rangle = \langle u_\beta f, v\rangle \to \langle uf, v\rangle = \langle u, f\rangle$ where $uf \in VN(G)$. Hence $u_\beta \to u$ in $\sigma(B, C_{00})$ which by density implies (a).

$(e) \Rightarrow (c)$ since any $T \in C_\rho^*(G)$ with compact support is expressible as vT where $v \in A(G)$ with $v = 1$ on supp T.

$(c) \Rightarrow (a)$, $(d') \Rightarrow (a)$ and $(e') \Rightarrow (a)$ are all shown in the same way as $(c') \Rightarrow (a)$. The implications $(e) \Rightarrow (d)$, $(d) \Rightarrow (d')$ and $(e) \Rightarrow (e')$ are all evident.

Another application of the above methods which will be proved in greater generality in the next section is the following theorem.

THEOREM B_3. *Let G be a amenable group. A set $E \subset A(G)$ is relatively norm compact if and only if the following hold:*

(*a*) *E is norm bounded;*

(*b*) *For each $v \in A(G)$ and $\varepsilon > 0$ there is a neighborhood V of e such that $\|(\ell_x u - u)v\| < \varepsilon$ for each $u \in E$ and $x \in V$ ($(\ell_x u)(y) = u(xy)$); and*

(*c*) *For each $\varepsilon > 0$ there is some $v \in A(G)$ such that $\|u - uv\| < \varepsilon$ for each $u \in E$.*

2. **Subsets of $B_p(G)$ on which τ_M and $\sigma(B_p, L^1)$ coincide.** Let $A_p(G)$ be (as in Herz [**5**, p. 96]) the Banach algebra of all functions f on G which can be represented as $f = \sum_1^\infty v_n * \check{u}_n$, an absolutely and uniformly convergent sum, such that $\sum_n \|v_n\|_{p'} \|u_n\|_p < \infty$, $1/p + 1/p' = 1$. We define the norm $\|f\|_{A_p} = \inf \sum \|v_n\|_{p'} \|u_n\|_p$ over all such representations. The space $B_p(G)$ defined in Herz [**6**, p. 146] is denoted by us by $B_p^H(G)$ or B_p^H ($\|u\|_H$ will denote the norm in B_p^H).

We define by $B_p^M(G)$, or B_p^M, the space of all functions u such that $uv \in A_p$ for each $v \in A_p$. It then follows by the closed graph theorem that $\|u\|_M = \sup\{\|uv\|_{A_p}; \|v\|_{A_p} \leqq 1\}$ is finite. We equip B_p^M with this multiplier norm. $B_p^M(G) \subset C(G)$ becomes in this way a translation invariant Banach algebra.

It has been proved by Herz in [**6**, p. 147] that for any G, $A_p \subset B_p^H \subset B_p^M$, and $\|u\|_M \leqq \|u\|_H$ if $u \in B_p^H$, and $\|u\|_H \leqq \|u\|_{A_p}$ if $u \in A_p$. If G is amenable, then $B_p^H = B_p^M$ and the norms coincide. In this case $B_2^M(G) = B(G)$ where $B(G)$ is defined in [**4**]. If G is the free group on two generators, then $B_2^H(G) \neq B(G)$ as shown by Leinert in [**8**]. For any G one has $B(G) \subset B_2^H(G)$. The τ_M topology on B_p^M is defined so that a net $u_\alpha \in B_p^M$ converges τ_M to $u \in B_p^M$ if $\|(u_\alpha - u)v\|_{A_p} \to 0$ for each $v \in A_p$. Define $\ell_x u(y) = u(xy)$ for all $x, y \in G$ and $u \in B_p^M$.

Our main result in this section is the following theorem.

THEOREM C. *Let $E \subset B_p^M(G)$ be norm bounded. If for each $\varepsilon < 0$ and $v \in A_p(G)$ there exists a neighborhood V of e such that $\|(\ell_x u - u)v\|_{A_p} < \varepsilon$ for each $u \in E$ amd $x \in B$, then τ_M and $\sigma(B_b^M, L^1)$ convergence coincide on E.*

B_p^M is not known to be a dual Banach space (even though B_p^H is one, as proved by Herz in [**6**]). Translation, i.e., the map $x \to \ell_x u$ is not known to be norm continuous in B_p^M. In fact I. Khalil stated as an open question in his thesis whether translation is norm continuous in $B_p(R)$ for $p \neq 2$ (where R is the real line). M. Cowling informs us that for amenable G translation is norm continuous in B_p^M (this uses a deep theorem on tensor products due to John E. Gilbert). In spite of these difficulties, translation is continuous in $(A_p(G)$, norm) and also in (B_p^M, τ_M). In fact one has the following trivial lemma.

LEMMA 4. *For any $u \in B_p^M$, $a \in G$, $\|u\|_\infty \leqq \|u\|_M = \|\ell_a u\|_M$ and $x \to \ell_x u$ is continuous from G to (B_p^M, τ_M). Hence for any compact $K \subset G$, the set $\{\ell_x u; x \in K\}$ is compact in (B_p^M, τ_M).*

PROOF. $(B_p^M, \| \ \|_M)$ is a commutative Banach algebra of continuous bounded functions on G and G is included in the maximal ideal space of B_p^M, hence $\|u\|_\infty \leqq \|u\|_M$. Furthermore

$$\|\ell_a u\|_M = \sup\{\|(\ell_a u)v\|_{A_p}; \|v\|_{A_p} \leqq 1\}.$$

Now $\|\ell_x v\|_{A_p} = \|v\|$ as easily checked. Thus

$$\|\ell_a u\|_M = \sup\{\|u\ell_{a^{-1}}v\|_{A_p}; \|\ell_{a^{-1}}v\|_{A_p} \leqq 1\} = \|u\|_M.$$

As to the continuity of $x \to \ell_x u$ in τ_M, one has for $v \in A_p$ that

$$\begin{aligned}\|(\ell_x u - u)v\|_{A_p} &\leqq \|\ell_x(uv) - uv\|_{A_p} + \|(\ell_x u)v - \ell_x(uv)\|_{A_p} \\ &\leqq \|\ell_x(uv) - uv\|_{A_p} + \|\ell_x u\|_M \|\ell_x v - v\|_{A_p} \\ &= \|\ell_x(uv) - uv\|_{A_p} + \|u\|_M \|\ell_x v - v\|_{A_p} \to 0\end{aligned}$$

since translation is continuous in A_p.

Denote by δ_a the point mass at a, by Co L the convex hull of the set L and by $\tau_M$cl A the closure of $A \subset B_p^M$ in the τ_M topology.

LEMMA 5. *Let $K \subset G$ be compact and μ a probability measure on the Borel subsets of K. Let μ_α be a net in* Co$\{\delta_x; x \in K\}$ *such that $\mu_\alpha \to \mu$ in $\sigma(M(G), C(G))$. If $u \in B_p^M$, then $\mu * u \in B_p^M$ and $\mu_\alpha * u \to \mu * u$ in τ_M. Consequently $\|\mu * u\|_M \leqq \|u\|_M$.*

PROOF. For each $x \in G$ we have

$$\mu_\alpha * u(x) = \int u(y^{-1}x)d\mu_\alpha(y) \to \int u(y^{-1}x)d\mu(y) = \mu * u(x)$$

since for fixed x the function $y \to u(y^{-1}x)$ is continuous and bounded. If $v \in A_p$ and $\nu = \sum_1^n \beta_j \delta_{a_j}$, then $(\nu * u)v(x) = \sum \beta_j u(a_j^{-1}x)v(x) \in A_p$ and

$$\|(\nu * u)v\|_{A_p} \leqq (\sum_1^n |\beta_j|)\|u\|_M \|v\|_{A_p}.$$

Hence $\|(\mu_\alpha * u)v\|_{A_p} \leqq \|u\|_M \|v\|_{A_p}$ and $\mu_\alpha * u \in \text{Co } L$ where $L = \{\ell_x u; x \in K^{-1}\}$. We show now that (B_p^M, τ_M) is a complete locally convex space. In fact if u_α is a τ_M Cauchy net, then for each $v \in A_p$, $u_\alpha v \to w_v \in A_p$ in A_p norm and hence pointwise. Now for each x there is some $v \in A_p$ with $v(x) \neq 0$. Hence there is some function u on G such that $u_\alpha v \to uv = w_v \in A_p$ for each $v \in A_p$. Thus $uA_p \subset A_p$; hence $u \in B_p^M$. We apply now [7, p. 133 (13.4)] and get that $\tau_M \text{cl Co} L$ is τ_M compact.

In conclusion, there exists a subnet and some $w \in \tau_M \text{cl Co} L$ such that for each $v \in A_p$, $\|(\mu_{\alpha_\beta} * u - w)v\|_{A_p} \to 0$. Since $\mu_\alpha * u \to \mu * u$ pointwise, it follows that $\mu * u = w \in B_p^M$. But every subnet μ_{α_β} has a further subnet ν_γ such that $\nu_\gamma * u \to \mu * u$ in τ_M. This immediately implies that $\mu_\alpha * u \to \mu * u$ in τ_M. Clearly $\|\mu_\alpha * \mu\|_M \leqq \|u\|_M$ and since $\|w\|_M = \sup\{\|wv\|; \|v\|_{A_p} \leqq 1\}$ we get $\|\mu * u\|_M \leqq \|u\|_M$.

The proof of the next lemma is a slight modification of the proof of lemma 3.

LEMMA 6. *Let u_α, $u \in B_p^M(G)$ be such that $\sup_\alpha \|u_\alpha\|_M = \gamma < \infty$ and $u_\alpha \to u$ in $\sigma(B_p^M, L_1)$. If $F = f * g$ where $g \in L^1$, $f \in L^\infty$ and f has compact support. Then $F * (u_\alpha - u) \to 0$ in τ_M.*

PROOF. $F * (u_\alpha - u)$ belongs to B_p^M by lemma 5. Let $v \in C_{00} \cap A_p$ and $K \subset G$ be compact such that $s_f^{-1} s_v \subset K$ where S_w is the support of w. Then as in the proof of lemma 3 one has for each $w \in L^\infty$, $h \in L^1$ that $(f * w)v = [f * (w 1_K)]v$. Now $f \in L^{p'}$ and $(w 1_K) \in L^p$ thus $(f * (w 1_K))v \in A_p$ and

$$\|[f * (w 1_K)]v\|_{A_p} \leqq \|f\|_{p'} \|(w 1_K)^{\vee}\|_p \|v\|$$

(see [5, p. 97]).

Choose now $w = g * (u_\alpha - u)$. Then

$$\|[f * g * (u_\alpha - u)]v\|_{A_p} \leqq C\|((g * (u_\alpha - u))1_K]^{\vee}\|_p \to 0$$

since by lemma 2, $g * (u_\alpha - u) \to 0$ uniformly on K. To finish the proof it is enough to show that $\sup_\alpha \|F * u_\alpha\|_M < \infty$. However by lemma 5,

$$\|F * u_\alpha\|_M \leqq \|F\|_1 \|u_\alpha\|_M \leqq \gamma \|F\|_1.$$

DEFINITION. $E \subset B_p^M$ is said to be τ_M left equicontinuous if for each $\varepsilon > 0$ and $v \in A_p$, there is some neighborhood V of e such that $\|(\ell_x u - u)v\|_{A_p} < \varepsilon$ for all $x \in V$ and $u \in E$.

THEOREM C. *Let $E \subset B_p^M$ be norm bounded and τ_M left equicontinuous. Then $\sigma(B_p^M, L^1)$ and τ_M coincide E.*

PROOF. Let u_α, $u \in E$ be such that $\int u_\alpha h dx \to \int uh dx$ for each $h \in L^1$ and $v \in A_p$. Let $\sup\{\|w\|_M; w \in E\} = \gamma < \infty$. Let $V^{-1} = V$ be a compact neighborhood of e such that $\|(w - \ell_x w)v\|_{A_p} < \varepsilon/3$ if $x \in V$ and $w \in E$. Let $U = U^{-1}$ be a compact neighborhood of e such that $U^2 \subset V$. Let $g = f = \lambda(U)^{-1}1_U$ and $F = f*g$. Then $\|F\|_1 = 1$ and $F \geqq 0$. Let $\mu_\alpha \in \mathrm{Co}\{\delta_x; x \in V\}$ be such that $\mu_\alpha \to F dx$ in $\sigma(M(G), C(G))$. Then by lemma 5 $\mu_\alpha * w \to F*w \in B_p^M$, τ_M convergence, for each $w \in B_p^M$. Now $\|(w - \ell_x w)v\|_{A_p} < \varepsilon/3$ implies $\|(w - \mu_\alpha * w)v\|_{A_p} < \varepsilon/3$ for each α and each $w \in E$. By lemma 5 one has $\|(w - F*w)v\|_{A_p} \leqq \varepsilon/3$ for each $w \in E$. Thus

$$\|(u_\alpha - u)v\|_{A_p} \leqq \|(u_\alpha - F*u_\alpha)v\|_{A_p} + \|(u - F*u)v\|_{A_p} + \|(F*(u_\alpha - u))v\|_{A_p}$$
$$\leqq \frac{2}{3}\varepsilon + \|(F*(u_\alpha - u))v\|_{A_p} < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon$$

if $\alpha \geqq \alpha_0$ where α_0 is chosen using lemma 6. The converse consists in noting that τ_M implies uniform convergence on compacta hence $\sigma(B_p^M, L^1)$ convergence.

COROLLARY. *Let $E \subset A_p(G)$. If*

(a) *E is τ_M left equicontinuous, A_p norm bounded and*

(b) *for each $\varepsilon > 0$ there is some $v \in A_p$ such that $\|uv - u\|_{A_p} < \varepsilon$ for each $u \in E$,*

then E is relatively norm compact (i.e., its norm closure is norm compact.)

If G is amenable and $E \subset A_p(G)$ is relatively norm compact, then (a) and (b) hold.

PROOF. Assume (a) and (b). Then $\|u\|_H \leqq \|u\|_{A_p}$ for all $u \in A_p \subset B_p^H$ (as defined in [**6**]). Thus E is a norm bounded subset of B_p^H which is the dual Banach space of the normed space $L^1(G)$, normed with the (complicated) norm QF_p, see Herz [**6**, p. 153]. We show now that the w^* closure of E in B_p^H (which is certainly w^* compact) is τ_M left equicontinuous. Let $\varepsilon > 0$, $v \in A_p$ and $w \in A_p \cap C_{00}$ be such that $\|v - w\|_{A_p} < \varepsilon(3\gamma)^{-1}$ where $\gamma = \sup\{\|u\|_{A_p}; u \in E\} < \infty$. Then

$$\|(\ell_x u - u)v\|_{A_p} \leqq \|(\ell_x u - u)(v - w)\|_{A_p} + \|(\ell_x u - u)w\|_{A_p}$$
$$< \varepsilon(3\gamma)^{-1}2\gamma + \|(\ell_x u - u)w\|_{A_p}.$$

Hence it is enough to prove that for each $\varepsilon > 0$ and $w \in A_p \cap C_{00}$ there is a neighborhood V of e such that $\|(\ell_x u - u)w\|_{A_p} < \varepsilon$ for each $u \in w^*\mathrm{cl}\, E$ and $x \in V$. Let $\varepsilon < 0$ and $w \in A_p \cap C_{00}$ be fixed. Then $w = v_1 v_2$ where v_1, $v_2 \in A_p$ (in fact take $v_1 = w$ and $v_2 \in A_p \cap C_{00}$ which is 1 on the support of w). There is a neighborhood V of e such that

$$\|(\ell_x u - u)v_1\|_{A_p} < \varepsilon \|v_2\|_{A_p}^{-1}$$

for each $x \in V$, $u \in E$. Let now $u \in w^*\mathrm{cl}\, E$, and u_α a net in E such that $u_\alpha \to u$ in w^*. Then $(\ell_x u_\alpha - u_\alpha)v_1 \to (\ell_x u - u)v_1$ in w^* (in B_p^H). Thus

$$\liminf \|(\ell_x u_\alpha - u_\alpha)v_1\|_H \geqq \|(\ell_x u - u)v_1\|_H.$$

Hence

$$\|(\ell_x u - u)w\|_{A_p} \leqq \|(\ell_x u - u)v_1\|_H \|v_2\|_{A_p} \leqq (\varepsilon \|v_2\|_{A_p}^{-1})\|v_2\|_{A_p} = \varepsilon$$

for each $x \in V$ and $u \in w^*\mathrm{cl}\, E \subset B_p^H$. We have shown that $w^*\mathrm{cl}\, E \subset B_p^H$ is $\| \; \|_H$ norm bounded (and afortiori $\| \; \|_M$ norm bounded) and τ_M left equicontinuous. Let now u_α be a net in E. A subnet $w_\beta = u_{\alpha_\beta}$ will converge w^* (and by theorem C even in τ_M), to some $u \in B_p^H$. Let $\varepsilon > 0$ be given and choose some $v \in A_p$ such that $\|w_\beta - w_\beta v\|_{A_p} < \varepsilon/3$ for each β (by (b)). Then

$$\|w_\beta - w_\gamma\|_{A_p} \leqq \|(w_\beta - w_\gamma)v\|_{A_p} + \|(w_\beta - w_\gamma)(1 - v)\|_{A_p}$$
$$< \|(w_\beta - w_\gamma)v\|_{A_p} + \frac{2}{3}\varepsilon$$

for any β, γ. However $\|(w_\beta - u)v\|_{A_p} \to 0$. Hence

$$\|(w_\beta - w_\gamma)v\|_{A_p} \leqq \|(w_\beta - u)v\|_{A_p} + \|(w_\gamma - u)v\|_{A_p} < \varepsilon/3$$

if $\beta, \gamma \geqq \beta_0$. Thus w_β is a norm Cauchy net in the Banach algebra $A_p(G)$. Hence for some $u_1 \in A_p$, $\|w_\beta - u_1\|_{A_p} \to 0$. Thus $u = u_1 \in A_p$ and $w^*\mathrm{cl}\, E$ is in fact a subset of A_p, which is norm compact. The care involved in the above proof is warranted by [**8**].

We prove now the second part. Let G be amenable and $v_\alpha \in A_p$ be a bounded approximate identity for $A_p(G)$. If E is a norm compact subset of $A_p(G)$, then $\|v_\alpha u - u\| \to 0$ uniformly in $u \in E$ since $\sup \|v_\alpha\|_{A_p} < \infty$. Also $\|\ell_x u - u\|_{A_p} \to 0$ as $x \to e$ uniformly in $u \in E$. Thus stronger conditions than (a) and (b) hold. (Only $\sup \|v_\alpha\|_M < \infty$ has been used. Haagerup has shown that if G is the free group on 2 generators, then $A(G)$ has an approximate identity v_n such that $\sup \|v_n\|_M < \infty$).

REMARK. This Corollary applied to $A(G)$ yields theorem B_3 of the previous section.

3. **Various topologies on the unit sphere of** $M(G)$. The main result of this section is a measure theoretical selfcontained proof of a result of McKennon [**10**, p. 32 theorem (4.2)]. McKennon relies heavily in his proof on theorem (3.3) [**10**, p. 25] which relies heavily on an intricate result on approximate identities in C^* algebras [**10**, lemma (3.2)] which in turn uses results of E. Effros on C^* algebras. The reader who will peruse through pp. 21–25 and 32–33 of [**10**] will find, we think, that our proof is much simpler and more natural.

LEMMA 3.9. *Let μ_α, $\mu \in M(G)$ with $\mu_\alpha \to \mu$ weakly* and $\|\mu_\alpha\| \to \|\mu\|$. Then for every $\varepsilon > 0$ there is a compact set C and an index α_0 such that $\int_{G/C} d(|\mu_\alpha| + |\mu|) < \varepsilon$ for $\alpha > \alpha_0$.*

PROOF. Let $\varepsilon > 0$ and choose $h \in C_{00}(G)$ with $\|h\|_\infty \leqq 1$ and $|\langle h, \mu\rangle - \|\mu\|\,| < \varepsilon$. For $C = \operatorname{supp} h$ this implies $\int_{G/C} d|\mu| < \varepsilon$. Choose α_0 such that $|\langle h, \mu_\alpha\rangle - \langle h, \mu\rangle| < \varepsilon$ and $|\|\mu_\alpha\| - \|\mu\|| < \varepsilon$ for $\alpha > \alpha_0$. We have

$$\|\mu\| \leqq |\langle h, \mu\rangle| + \varepsilon$$
$$\leqq |\langle h, \mu - \mu_\alpha\rangle| + |\langle h, \mu_\alpha\rangle| + \varepsilon \leqq 2\varepsilon + \int_C d|\mu_\alpha|$$

for $\alpha > \alpha_0$. Hence

$$\int_{G\setminus C} d|\mu_\alpha| = \|\mu_\alpha\| - \int_C d|\mu_\alpha| \leqq \|\mu\| + \varepsilon - \int_C d|\mu_\alpha| \leqq 3\varepsilon$$

for $\alpha > \alpha_0$. We thus have $\int_{G/C} d(|\mu_\alpha| + |\mu|) < 4\varepsilon$ for $\alpha > \alpha_0$.

REMARK. Let μ_α, $\mu \in M(G)$ be such that $\mu_\alpha \to \mu$ in w^* and $|\mu_\alpha|(G) \to |\mu|(G)$. Then $|\mu_\alpha| \to |\mu|$ in w^*. Assume in fact that a subnet μ_{α_β} is such that $|\mu_{\alpha_\beta}| \to \nu \neq |\mu|$ in w^* (the unit ball of $M(G)$ is w^* compact). If $0 \leqq f \in C_0(G)$ and $|g| \leqq f$, then $|\mu_{\alpha_\beta}|(f) \geqq |\mu_{\alpha_\beta}(g)| \to |\mu(g)|$. Thus $\nu(f) \geqq \sup\{|\mu(g)|; |g| \leqq f\} = |\mu|(f)$, i.e., $(\nu - |\mu|) \in M(G)^+$. But $(\nu - |\mu|)(G) = 0$. Thus $\nu = |\mu|$ which cannot be.

THEOREM D. (a) *Let μ_α, $\mu \in M(G)$ be such that $\mu_\alpha \to \mu$ in $\sigma(M(G), C_0(G))$ and $\|\mu_\alpha\| \to \|\mu\|$. Then $\|(\mu_\alpha - \mu)*f\|_p \to 0$ for each $f \in L^p(G)$ where $1 \leqq p < \infty$. (If $f \in UCB_r(G)$, then $\|(\mu_\alpha - \mu)*f\|_\infty \to 0$).*

(b) *If μ_α is a norm bounded net in $M(G)$, $\mu \in M(G)$ and if $\|(\mu_\alpha - \mu)*f\|_p \to 0$ for each $f \in C_{00}(G)$ for fixed $1 \leqq p < \infty$, then $\mu_\alpha \to \mu$* in w^*.

REMARK. $f \in UCB_r(G)$ if and only if $f \in C(G)$ and $x \to \ell_x f$ from G to $(C(G), \|\ \|_\infty)$ is continuous.

PROOF. It is enough to prove that for any $f \in C_{00}(G)$ with $0 \leqq f \leqq 1$, $\|(\mu_\alpha - \mu)*f\|_p \to 0$. Since then, it will be true for any $f \in C_{00}(G)$. Thus, by the density of $C_{00}(G)$ in $L^p(G)$ and since $\|\mu_\alpha * f\|_p \leqq \|\mu_\alpha\|\,\|f\|_p$ where $\|\mu_\alpha\| = |\mu_\alpha|(G) \to |\mu|(G) = \|\mu\|$ is a bounded net (past some α), it will readily follow for all $f \in L^p(G)$. Hence fix $0 \leqq f \leqq 1$ in $C_{00}(G)$. For any compact set $K \subset G$ we have

(a) $$\|(\mu_\alpha - \mu)*f\|_p^p \leqq \int_K |(\mu_\alpha - \mu)*f|^p + \int_{G/K} (|\mu_\alpha|*f + |\mu|*f).^p$$

Now let $g_\alpha = |\mu_\alpha|*f$, $g = |\mu|*f$. One has g_α, $g \in L^1(G)$ and $\|g_\alpha\|_1 = \|\mu_\alpha\|\,\|f\|_1 \to \|\mu\|\,\|f\|_1 = \|g\|_1$. Also $g_\alpha \to g$ weakly* since (by the remark preceding theorem D) $|\mu_\alpha| \to |\mu|$ weakly*, hence by Lemma 3.9 for every

$\varepsilon > 0$ there is a compact set $K \subset G$ and an index α_0 such that $\int_{G/K}(|\mu_\alpha|*f + |\mu|*f) < \varepsilon$ for $\alpha > \alpha_0$. Since the integrand is dominated in sup-norm by $(\|\mu_\alpha\| + \|\mu\|)\|f\|_\infty \leqq M < \infty$ for $\alpha > \alpha_1$. we obtain

$$\int_{G/K}(|\mu_\alpha|*f + |\mu|*f)^p \leqq M^{p-1}\int_{G/K}(|\mu_\alpha|*f + |\mu|*f) < M^{p-1}\cdot\varepsilon$$

for $\alpha > \alpha_0, \alpha_1$. Since $\{_{x^{-1}}\check{f} \mid x \in K\}$ is compact in $C_0(G)$, we have $\mu_\alpha*f \to \mu*f$ uniformly on K, hence $\int_K|(\mu_\alpha - \mu)*f|^p < \varepsilon$ for $\alpha > \alpha_2$. By (a) we obtain for $\alpha > \alpha_0, \alpha_1, \alpha_2$ the inequality

$$\|(\mu_\alpha - \mu)*f\|_p^p < \varepsilon(1 + M^{p-1})$$

which proves that $\|(\mu_\alpha - \mu)*f\|_p \to 0$, for all $f \in L^p$ (for fixed $1 \leqq p < \infty$). If $f \in UCB_r(G)$, then $f = g*h$ with $g \in L^1$, $h \in L^\infty$ as is well known. Thus $\|(\mu_\alpha - \mu)*f\|_\infty \leqq \|(\mu_\alpha - \mu)*g\|_1 \|h\|_\infty \to 0$ since $g \in L^1$.

For the proof of (b) let $f, g \in C_{00}(G)$. Then

$$\langle \mu_\alpha - \mu, g*f^\sim\rangle = \langle(\mu_\alpha - \mu)*\bar{f}, g\rangle \to 0$$

since $f, g \in L^2$. Thus $\langle\mu_\alpha, h\rangle \to \langle\mu, h\rangle$ for all h in a norm dense subspace of $A(G)$ [**4**, p. 208]. Since $A(G)$ is sup norm dense in $C_0(G)$ (Eymard [**4**, p. 210]). $\mu_\alpha \to \mu$ in w^*.

Remark. Let G be nondiscrete and x_α a net in G such that $x_\alpha \to x$ in G with $x_\alpha \neq x$ for each α. Then the point masses $\delta_{x_\alpha} \to \delta_x$ in $\sigma(M(G), C_0(G))$ (e.i., in w^*) and $\|\delta_{x_\alpha}\| = \|\delta_x\| = 1$. Clearly $\|\delta_{x_\alpha} - \delta_x\| = 2$; hence the assumptions of theorem $D(a)$ do not imply norm convergence of μ_α to μ.

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