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Weighted group algebras on groups of polynomial growth

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Abstract Let G be a compactly generated group of polynomial growth and ω a weight function on G. For a large class of weights we characterize symmetry of the weighted group algebra $L^1(G, \omega)$. In particular, if the weight ω is sub-exponential, then the algebra $L^1(G, \omega)$ is symmetric. For these weights we develop a functional calculus on a total part of $L^1(G, \omega)$ and use it to prove the Wiener property.

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1 Introduction

Weighted group algebras play an important role in different areas of harmonic analysis. For abelian groups, the properties of these algebras are well known since the works of Beurling [Be39], [Be47], Domar [Do56] and Vretblad [Vr73]. For non-abelian groups, however, the results are sparse. The aim of the present paper is to develop a corresponding non-abelian theory. We first give the definition of a weight and introduce the notion of weighted group algebras.

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1.1 Definition. *Let G be a locally compact group. A weight ω on G is a Borel function*

$$\omega : G \longrightarrow [1, \infty[$$

such that

$$\omega(st) \leq \omega(s)\omega(t) \quad \forall s, t \in G.$$

In this paper all weights are assumed to be symmetric, i.e.,

$$\omega(s^{-1}) = \omega(s) \quad \forall s \in G.$$

By changing to an equivalent Banach algebra norm on $L^1(G, \omega)$, we may assume without loss of generality that the weight ω is upper semi-continuous, see [Rei67, Ch. 3 § 7]. Moreover, we shall always require the function ω to be bounded on compact sets. Hence, if G is compact, every weight ω is equivalent to a constant one. In the remainder of this paper we shall exclude the trivial case of a compact group G. Let us give some examples of weights:

1) Let (T, E) be a strongly continuous representation of the locally compact group G on the Banach space E. Then

$$\omega(s) = \max\{\|T(s)\|_{op}, \|T(s^{-1})\|_{op}\}$$

and

$$\omega_1(s) = \|T(s)\|_{op} + \|T(s^{-1})\|_{op}$$

are weights on G. These weights are in general not upper semi-continuous, but can be suitably modified as mentioned above.
2) The function $\omega(s) = 1$ for all $s \in G$ is the trivial weight on G.
3) The function $t \mapsto e^{|t|}$ is a weight on $\mathbb{R}$. Likewise, the function $t \mapsto e^{\gamma|t|^\alpha}$, with $0 < \alpha < 1$ and $\gamma > 0$ fixed, is a weight function.

Given a weight ω on G, we define the weighted group algebra $L^1(G, \omega)$ to be the set of all measurable functions f from G to $\mathbb{C}$ such that

$$\|f\|_\omega = \int_G |f(x)|\omega(x)dx < \infty.$$

Then $L^1(G, \omega)$ is a Banach $*$-algebra for the usual convolution and involution. One motivation to study weighted group algebras stems from Gabor analysis, where it is important to understand the invertibility of convolution operators and related operators on function spaces different from L^1 or L^2. See [GL01, Jan95] for problems of this type.

If G is a connected, simply connected, nilpotent Lie group and if ω is a polynomial weight on G (in the sense that ω is bounded by a polynomial function in the canonical coordinates of the first or second kind), then the harmonic analysis properties of $L^1(G, \omega)$ are well understood. In particular, $L^1(G, \omega)$ is symmetric, it has the Wiener property, and there exist minimal ideals of a given hull. The closed proper prime ideals of $L^1(G, \omega)$ (in particular the kernels of algebraically or topologically irreducible representations of $L^1(G, \omega)$) coincide with the kernels $\ker \pi \cap L^1(G, \omega)$, $\pi \in \hat{G}$, and the algebraically and topologically irreducible

$*$-representations of $L^1(G, \omega)$ may be characterized [MMB98]. On the other hand, if ω increases more rapidly, it is not known whether, for every algebraically or topologically irreducible representation (T, E) of $L^1(G, \omega)$, there exists $\pi \in \hat{G}$ such that $\ker T = \ker \pi \cap L^1(G, \omega)$.

First we study the symmetry of weighted group algebras for compactly generated groups of polynomial growth. For the group $G = \mathbb{Z}$ this question is equivalent to the validity of a weighted version of Wiener's lemma on absolutely convergent Fourier series and a complete answer was obtained by Naimark [Nai72]. For groups of polynomial growth and polynomial weights the symmetry of $L^1(G, \omega)$ was shown by Pytlik [Pyt73]. The most difficult case, namely the symmetry of the full L^1-algebra was proved only recently by Losert [Los01] as a consequence of a detailed structure theorem for groups of polynomial growth.

In the locally compact group G let e denote the neutral element and $|U|$ the left Haar measure of a Borel set $U \subset G$.

1.2 Definition. *A locally compact, compactly generated group G is said to have (at most)* polynomial growth, *if there exists a compact symmetric neighbourhood $U \subset G$ and constants $C > 0$ and $D \in \mathbb{N}$ (the smallest D is called the order of growth of G) such that $G = \bigcup_{n=1}^{\infty} U^n$ and $|U^k| \leq Ck^D$ for $k \in \mathbb{N}$. We write* [PG] *for the class of compactly generated, locally compact groups of polynomial growth.*

We say that G has strict polynomial growth, *if there exists a compact symmetric neighbourhood $U \subset G$ and constants $C_1, C_2 > 0$ and $D > 0$ (again called the order of growth of G) such that*

$$C_1 k^D \leq |U^k| \leq C_2 k^D \quad \textit{for } k \in \mathbb{N}.$$

Note that we always assume that $G \in$ [PG] is compactly generated. By replacing U by a suitable power of itself, one may assume in this definition that the interior $\mathrm{int}(U)$ of U is a symmetric neighbourhood of e and generates G, i.e., $G = \bigcup_{n=1}^{\infty} \mathrm{int}(U)^n$. For brevity we shall call an open, symmetric and relatively compact neighbourhood of the identity that generates G a *generating neighbourhood.*

It is known that connected Lie groups [Gui73] and finitely generated discrete groups [Gro81] of polynomial growth have strict polynomial growth.

For a given generating neighbourhood U we define a function $\tau_U : G \to [1, \infty[$ by

$$\tau_U(x) = \inf\{n \in \mathbb{N} \mid x \in U^n\} \quad \text{for } x \neq e, \quad \tau_U(e) = 1$$

Then τ_U serves as a "metric" on G and the function $\omega_U(x) = 1 + \tau_U(x)$ is a natural weight on G (see [Hul71,Lud87]). An arbitrary weight ω on G satisfies the inequality

$$\omega(x) \leq e^{C\tau_U(x)}$$

where $C = \ln \sup_{x \in U} \omega(x)$. Moreover, for every α such that $0 \leq \alpha < 1$ and every $C > 0$, the function

$$\omega(x) = e^{C\tau_U(x)^\alpha}, \quad \forall x \in G,$$

is a weight on G. Such weights appear naturally in the following situation: Let G be a connected, nilpotent Lie group. Let G_1 be the derived group of G, i.e., the closed subgroup generated by the elements of the form $[x, y] = x^{-1}y^{-1}xy, x, y \in G$. Let U be a generating neighbourhood in G and $V = U \cap G_1$ the corresponding neighbourhood of e in G_1. Then it is shown in [Ale00] that

$$(1+\tau_U|_{G_1}(x)) \leq K \cdot (1+\tau_V(x))^{\frac{1}{2}}, \quad \forall x \in G_1,$$

for some positive constant K. Consequently, if ω is any weight on G, then

$$\omega|_{G_1}(x) \leq e^{C\tau_V(x)^{\frac{1}{2}}}, \quad \forall x \in G_1,$$

for some constant C.

To any weight ω and any generating neighbourhood U we associate a weight $v_U^\omega : \mathbb{Z} \to [1, \infty[$ by

$$v_U(k) = v_U^\omega(k) = \sup\{\omega(y) \mid y \in U^{|k|}\}.$$

Now we list some possible properties of weights with increasing strength.

(i) We say that a weight ω satisfies the Gelfand-Naimark-Raikov condition *if*

$$\lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} = 1 \quad \forall x \in G.$$

(ii) The weight ω satisfies the condition (S) *if*

$$\lim_{k\to\infty} v_U^\omega(k)^{\frac{1}{k}} = 1.$$

(iii) We call the weight ω sub-exponential *of degree at most α, $0 < \alpha < 1$, if there exists $C > 0$ such that*

$$\omega(x) \leq e^{C\tau_U(x)^\alpha} \; \forall x \in G.$$

In this paper, we shall prove the following results:

Theorem (3.13). *Let $G \in$ [PG]. If the weight ω satisfies condition (S), then $L^1(G, \omega)$ is symmetric.*

For radial weights, i.e., weights which depend only on τ_U for some neighbourhood U, we obtain a converse on groups of strict polynomial growth. In fact, this converse is true for the larger class of tempered weights which are defined as follows: A weight $\omega : G \to [1, \infty[$ on $G \in$ [PG] is called *tempered* if there exist a sequence $\varepsilon_k > 0$, $k \in \mathbb{N}$ with $\lim_k \varepsilon_k^{\frac{1}{k}} = 1$, an $l \in \mathbb{N}$ and a generating neighbourhood U such that for all $k \in \mathbb{N}$:

$$\omega(x) \geq \varepsilon_k \sup\{\omega(y) \mid y \in U^k\}, \quad \forall x \in G \setminus U^{kl}.$$

Under this technical condition we prove the following converse.

Theorem (3.18). *Let G be as in the above theorem, but of strict polynomial growth and assume that ω is tempered. If $L^1(G,\omega)$ is symmetric, then ω satisfies condition (S).*

The main tools in proving these theorems are the structure theorem for groups of polynomial growth of Losert [Los01], the Gaussian estimates of Hebisch and Saloff-Coste [HSC93] and the methods of Ludwig [Lud79].

For sub-exponential weights we develop a functional calculus on a total part of $L^1(G,\omega)$ in Section 4. This functional calculus and the symmetry of $L^1(G,\omega)$ are essential tools for proving the Wiener property in Section 5.

Theorem (5.6). *Let $G \in$ [PG] and ω be a sub-exponential weight on G. Then $L^1(G,\omega)$ has the Wiener property, i.e., for every proper, closed, two-sided ideal I of $L^1(G,\omega)$ there exists a topologically irreducible $*$-representation π of $L^1(G,\omega)$ on a Hilbert space such that $I \subset \ker \pi$.*

2 Some results on weights

2.1 Polynomial weights

a) Let G be a compactly generated, locally compact group of polynomial growth with generating neighbourhood U. Let $\tau_U(x) = \inf\{k \mid x \in U^k\}$. A weight $\omega : G \to [1,\infty[$ is said to be polynomial if there exist $\alpha > 0$ and $C > 0$ such that

$$\omega(x) \leq C(1+\tau_U(x))^\alpha, \quad \forall x \in G.$$

b) In particular the weight $\omega_U(x) = 1+\tau_U(x)$ is polynomial. For every $\alpha > 0$ this weight satisfies the inequality

$$\omega^\alpha(xy) \leq c_\alpha(\omega^\alpha(x)+\omega^\alpha(y)), \quad \forall x, y \in G$$

for some $c_\alpha > 0$ [Lud87].

c) In [MMB98] a weight on a connected nilpotent Lie group is said to be polynomial if, for any Jordan-Hölder basis $\{X_0, X_1, \ldots, X_n\}$ of the Lie algebra, $\omega(x)$ is bounded by a polynomial in the coordinates $(x_1, x_2, \ldots, x_n)$ of $x = \exp(x_0X_0)\exp(x_1X_1)\cdots\exp(x_nX_n)$ [resp. in the coordinates $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ of $x = \exp(\tilde{x}_0X_0 + \tilde{x}_1X_1 + \ldots + \tilde{x}_nX_n)$]. This definition is equivalent to the one given in a), by arguments of [Lud87].

d) In [Pyt82] Pytlik defines a weight ω to be polynomial if

$$\sup_{x,y\in G} \frac{\omega(xy)}{\omega(x)+\omega(y)} < \infty,$$

and he shows that such a weight is polynomial in the sense of a). Thus a) seems to be the most general definition of a polynomial weight.

e) Polynomial weights are of course sub-exponential. Hence all the results of this paper are in particular true for polynomial weights.

2.2 For a weight ω and a generating neighbourhood U, we define $v_U^\omega : \mathbb{Z} \to [1, \infty[$ by

$$v_U(k) = v_U^\omega(k) = \sup\{\omega(y) \mid y \in U^{|k|}\}.$$

Then v_U^ω is a symmetric weight on the additive group $\mathbb{Z}$ and increasing on $\mathbb{Z}_+$. Since the "metric" τ_U satisfies $\tau_U(xy) \le \tau_U(x) + \tau_U(y)$, the function $\tilde{\omega}$ defined by

$$\tilde{\omega} = v_U^\omega \circ \tau_U$$

is again a weight. Then $\omega \le \tilde{\omega}$ and $\tilde{\omega}$ is an increasing function of $\tau_U(x)$. If ω is sub-exponential of degree at most α, then the same is true for $\tilde{\omega}$. More generally, if $\omega(x) \le \Psi(\tau_U(x))$, $\forall x \in G$ for some increasing function Ψ, $\tilde{\omega}(x) = v_U(\tau_U(x)) \le \Psi(\tau_U(x))$, $\forall x \in G$.

2.3 We investigate the symmetry of weighted group algebras. In case of the most basic group $\mathbb{Z}$ the algebra $\ell^1(\mathbb{Z}, \omega)$ is symmetric if and only if $\lim_{k\to\infty} \omega(k)^{\frac{1}{k}} = 1$ [Nai72]. This result motivates the following definition.

Definition. *A weight ω is said to satisfy the Gelfand-Naimark-Raikov (G-N-R) condition if*

$$\lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} = 1 \quad \forall x \in G.$$

Since $k \mapsto \omega(x^k)$ is sub-multiplicative, the above limit always exists and is actually an infimum.

2.4 We notice that

$$\lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} \le \left(\lim_{k\to\infty} v_U(k)^{\frac{1}{k}}\right)^{\tau_U(x)} \qquad \forall x \in G.$$

In fact, if $\tau_U(x) = l$, then $x^k \in U^{kl}$, and so $1 \le \omega(x^k) \le v_U(lk) \le v_U(k)^l$. Hence, $1 \le \lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} \le \left(\lim_{k\to\infty} v_U(k)^{\frac{1}{k}}\right)^l$.

2.5 A uniform analogue of the condition in (2.3) is to require the G-N-R condition for the weight v_U^ω, that is $\lim_{k\to\infty} v_U^\omega(k)^{\frac{1}{k}} = 1$. Since this limit is just the inverse of the radius of convergence of the power series $\sum_{n\ge 0} v_U^\omega(n) z^n$ and since $v_U^\omega(k) \ge 1$, $\forall k \in \mathbb{N}$, we may reformulate this condition as follows.

Definition. *A weight ω satisfies* condition (S) *if for all $\varepsilon > 0$*

$$\sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon)^k} v_U^\omega(k) < \infty.$$

Remarks.

a) Condition (S) is independent of the choice of the generating neighbourhood. In fact, if V is another one there exist n, m such that $U \subset V^n$ and $V \subset U^m$. Then

$$\begin{aligned} v_U(k) = \sup_{x \in U^k} \omega(x) &\leq \sup_{x \in V^{nk}} \omega(x) \\ &= v_V(kn) \leq v_V(k)^n. \end{aligned}$$

Similarly, $v_V(k) \leq v_U(k)^m$ and both inequalities together imply

$$\lim_{k\to\infty} v_U(k)^{\frac{1}{k}} = 1 \iff \lim_{k\to\infty} v_V(k)^{\frac{1}{k}} = 1.$$

b) Recall that $\tilde{\omega}(x) = \sup\{\omega(y) \mid \tau_U(y) \leq \tau_U(x)\} = v_U^\omega(\tau_U(x))$. Hence, $v_U^{\tilde{\omega}} = v_U^\omega$ and we see that $\tilde{\omega}$ satisfies condition (S) if and only if ω does.

c) If $G \in [\mathrm{PG}]$ then for some constants $C > 0$, $D > 0$ we have $|U^{k+1} \setminus U^k| \leq C(k+1)^D$. Hence, if ω satisfies condition (S), then

$$\begin{aligned} &\sum_{k=1}^{\infty} \left(\int_{U^{k+1}\setminus U^k} \omega(x)\, dx \right) \frac{1}{(1+\varepsilon)^k} \\ &\leq C \sum_{k=1}^{\infty} v_U(k+1) \frac{(k+1)^D}{(1+\varepsilon)^k} < \infty, \quad \forall \varepsilon > 0. \end{aligned}$$

d) Assume that there exists an increasing function $\Phi : \mathbb{Z}_+ \to \mathbb{R}_+$ with $\lim_{n\to\infty} \frac{\Phi(n)}{n} = 0$ such that the weight ω satisfies for some $C > 0$

$$\omega(x) \leq C e^{\Phi(\tau_U(x))}.$$

Then ω satisfies condition (S). In fact,

$$\sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon)^k} v_U^\omega(k) \leq C \sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon)^k} e^{\Phi(k)} < \infty,$$

because the radius of convergence of the series $\sum e^{\Phi(k)} z^k$ is $\lim_{n\to\infty} e^{-\frac{\Phi(n)}{n}} = 1$.

2.6 Examples

a) If $\omega(x) \leq e^{C\tau_U(x)^\alpha}$, $\quad 0 < \alpha < 1$, then $\Phi(s) = Cs^\alpha$. By (2.5.d), ω satisfies condition (S). Consequently, every sub-exponential weight, and in particular every polynomial weight, satisfies condition (S).

b) It is easy to check that the function

$$\omega(x) = e^{[\ln(\tau_U(x)+1)]^\alpha}, \quad 0 < \alpha < 1,\ \forall x \in G,$$

is a weight satisfying $\omega(x) \leq \tau_U(x) + 1$ for all $x \in G \setminus U$. Since ω is dominated by a polynomial weight, it satisfies condition (S).

c) If $\omega(x) = e^{C\tau_U(x)}$ for some $C > 0$, then $v_U^\omega(n) = e^{Cn}, n \in \mathbb{N}$, and $\lim_{n\to\infty} v_U^\omega(n)^{\frac{1}{n}} = e^C \neq 1$. For this weight the condition (S) is not satisfied.

d) If $\omega(x) = e^{\sum_n c_n \tau_U(x)^{\gamma_n}}$, with $0 < \gamma_n < 1$, $\gamma_n \uparrow 1$, $c_n > 0$, $\sum c_n < \infty$, we may take $\Phi(s) = \sum_n c_n s^{\gamma_n}$ in (2.5.d), and thus ω satisfies condition (S).

2.7 We need a lemma on the existence of special (discrete) one parameter subgroups in groups of polynomial growth.

Lemma. *If $G \in [\mathrm{PG}]$ and U is a generating neighbourhood of e, then there is some $x \in G$ and $\beta > 0$ such that $\tau_U(x^n) \geq \beta n$ for all $n \in \mathbb{N}$.*

Proof. To prove this statement, we use the following characterization by Losert [Los01, Thm. 2]: A compactly generated group G has polynomial growth if and only if there is a finite descending sequence of normal subgroups of G, $G = G_0 \supset G_1 \supset \ldots \supset G_{n-1} \supset G_n = \{e\}$, such that G_0/G_1 and G_{n-1} are compact and $G_j/G_{j+1} \cong \mathbb{R}^{k_j} \times \mathbb{Z}^{l_j}$ for $j = 1, \ldots, n-2$, and every G_j/G_{j+1} is an $[FC]^-_G$-group. (see (3.11) for the definition).

(a) Assume first that $G = \mathbb{R}^k \times \mathbb{Z}^l$, define the generating neighbourhood K by $K = \{(z, m) \in G : \|z\|_2 < 1, m = 0 \text{ or } \pm 1\}$ and let $\tau(x) = \min\{n | x \in nK\}$ for $x \in G$. If $x \neq e$ and $r = \min\{m | mx \notin K\}$, then $\tau(nx) \geq \frac{1}{r} n$ for all $n \in \mathbb{N}$.
(b) Next assume that G has a closed normal subgroup H such that $G/H \cong \mathbb{R}^k \times \mathbb{Z}^l$ and choose $x \in G \setminus H$ arbitrary. Let V be a generating neighbourhood of e in G with corresponding metric τ. We claim that $\underline{\lim}_{n\to\infty} \frac{\tau(x^n)}{n} > 0$. To see this, project everything into G/H by the canonical projection $y \mapsto \dot{y} = yH$. Then $\dot{V}$ is a generating neighbourhood of G/H and the corresponding metric $\dot{\tau}$ satisfies $\tau(y) \geq \dot{\tau}(\dot{y})$. Since $\dot{x} \neq \dot{e}$, step (a) implies that $\underline{\lim}_{n\to\infty} \frac{\tau(x^n)}{n} \geq \underline{\lim}_{n\to\infty} \frac{\dot{\tau}(\dot{x}^n)}{n} > 0$.
(c) Finally assume that G is non-compact and $G_0, \ldots, G_n$ are as in Losert's theorem above. Let W be a generating neighbourhood of e in G_1 and τ_1 the corresponding metric. Using (a) and (b) we see that $\underline{\lim}_{n\to\infty} \frac{\tau_1(x^n)}{n} > 0$ for $x \in G_1 \setminus G_2$.

In order to obtain the same conclusion for a metric τ_0 on G_0, we show that $\tau_0\big|_{G_1}$ is equivalent to τ_1 by using an argument of Guivar'ch [Gui73].

Choose a generating neighbourhood V of e in G_0 such that $V \supset W$ and $p(V) = G_0/G_1$ where p is the canonical projection from G_0 onto G_0/G_1. This can be done because G_0/G_1 is compact.

If $x \in G_1$ and $\tau_1(x) = n$, i.e., $x \in W^n \setminus W^{n-1}$, then $x \in V^n$, so $\tau_0(x) \leq n = \tau_1(x)$.

On the other hand, since $p(V) = G_0/G_1$, we have $G_0 = G_1 \cdot V = \bigcup_n W^n \cdot V$, so there is $r \in \mathbb{N}$ with $V^2 \subset W^r V$. By induction we have $V^{n+1} \subset W^{rn} V$. Now assume $\tau_0(x) = n$ for $x \in G_1$, i.e., $x \in V^n \setminus V^{n-1}$. Then $x \in (W^{r(n-1)} V) \cap G_1 = W^{r(n-1)}(V \cap G_1) \subset W^{r(n-1)} W^a = W^{r(n-1)+a}$ for suitable $a \in \mathbb{N}$. So $\tau_1(x) \leq rn + (a - r) \leq An = A\tau_0(x)$ where $A > 0$ is some constant. Hence for $x \in G_1 \setminus G_2$

$$\underline{\lim}_{n\to\infty} \frac{\tau_0(x^n)}{n} \geq \frac{1}{A} \underline{\lim}_{n\to\infty} \frac{\tau_1(x^n)}{n} > 0$$

If we use a different generating neighbourhood $\tilde{V}$ with corresponding metric $\tilde{\tau}$, then by equivalence of τ_0 and $\tilde{\tau}$ the results holds true for $\tilde{\tau}$ (with a possibly different constant). □

2.8 We call a weight ω radial with respect to some generating neighbourhood U, if $\omega(x)$ is a function of $\tau_U(x)$. An example of a radial weight function is given by $\omega(x) = e^{\Phi(\tau_U(x))}$ where $\Phi : \mathbb{Z}_+ \to [0, \infty[$ is sub-additive. We notice that Φ is sub-additive if Φ is the restriction of a concave function $\tilde{\Phi} : \mathbb{R}_+ \to \mathbb{R}_+$.

Conversely, if ω is radial with respect to U (and G is not compact), then the sequence $(U^n)_{n\in\mathbb{N}}$ never stabilizes, and thus τ_U is surjective onto $\mathbb{N}$, and $\omega(x) = e^{\Phi(\tau_U(x))}$, for some function $\Phi : \mathbb{N} \to [0, \infty[$. If $m \in \mathbb{N}$ and $m = n + k$, then there exist $x, y, z \in G$ such that $xy = z$, $\tau_U(x) = n$, $\tau_U(y) = k$, and $\tau_U(z) = m$. Consequently, $\Phi(m) = \ln \omega(z) \leq \ln(\omega(x)\omega(y)) = \ln \omega(x) + \ln \omega(y) = \Phi(n) + \Phi(k)$. That is, Φ is necessarily subadditive.

Corollary. *Assume that the weight ω is radial with respect to some generating neighbourhood U. Then $\omega = e^{\Phi\circ\tau_U}$ fulfills the G-N-R condition if and only if $\lim_{k\to\infty} \frac{1}{k}\Phi(k) = 0$.*

Proof. Assume that $\frac{\Phi(n)}{n} \to 0$. For $x \in G$ set $n_k = \tau_U(x^k)$, $k \in \mathbb{N}$. If n_k is a bounded sequence, then clearly $\frac{\Phi(n_k)}{k} \to 0$ and $1 \leq \lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} = \lim_{k\to\infty} e^{\frac{\Phi(n_k)}{k}} = 1$. If n_k is unbounded, then there exists a subsequence, which by abuse of notation we denote by n_k again, such that both $n_k \to \infty$ and $n_k \leq k\tau_U(x)$. Consequently $\frac{\Phi(n_k)}{k} \leq \frac{\Phi(n_k)}{n_k}\tau_U(x)$ and $1 \leq \lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} = \inf_{k\in\mathbb{N}} \omega(x^k)^{\frac{1}{k}} = \inf_{k\to\infty} e^{\frac{\Phi(n_k)}{k}} \leq e^{\inf_k \frac{\Phi(n_k)}{n_k}\tau_U(x)} = 1$. For the converse implication we note that Lemma (2.7) ensures the existence of $\beta > 0$ and $x \in G$ with $n_k = \tau_U(x^k) \geq \beta k$. If $1 = \lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}}$, then $\lim_{k\to\infty} \frac{\Phi(k)}{k} \leq \lim_{k\to\infty} \frac{\Phi(n_k)}{n_k} \leq \beta^{-1} \lim_{k\to\infty} \frac{\Phi(n_k)}{k} = 0$. $\square$

2.9 Let ω be a weight on $(\mathbb{Z}, +)$ and set $v(k) = \sup\{\omega(l) \mid |l| \leq k\}$.

a) By definition it is clear that both $\lim_{k\to\infty} \omega(k)^{\frac{1}{k}} = c_1 \geq 1$ and $\lim_{k\to\infty} v(k)^{\frac{1}{k}} = c_2 \geq 1$ exist and that $c_1 \leq c_2$. On the other hand, for some $n_k \leq k$ we have $v(k) = \omega(n_k)$. Now, if $(n_k)_{k\in\mathbb{N}}$ is a bounded sequence then $\lim_{k\to\infty} v(k)^{\frac{1}{k}} = 1$ and c_1 and c_2 coincide. If $(n_k)_{k\in\mathbb{N}}$ is unbounded then

$$c_2 \leq v(k)^{\frac{1}{k}} = \omega(n_k)^{\frac{1}{k}} \leq \omega(n_k)^{\frac{1}{n_k}} \to c_1, \quad \text{as } k \to \infty.$$

That is $c_1 = c_2$ in this case too.

b) If $c = \lim_{k\to\infty} \omega(k)^{\frac{1}{k}} > 1$, then ω is almost increasing in the following sense: There exists l such that for all $k \in \mathbb{N}$, $|n| \geq lk$ implies $\omega(n) \geq \omega(k)$.

For, otherwise we find sequences k_l, n_l with $n_l \geq lk_l$ such that $\omega(n_l) < \omega(k_l)$. Then

$$c \leq \omega(n_l)^{\frac{1}{n_l}} < \omega(k_l)^{\frac{1}{n_l}} \leq \omega(k_l)^{\frac{1}{lk_l}}.$$

If the sequence k_l remains bounded then the right hand side tends to one as $l \to \infty$, and $c = 1$ follows. If $k_l \to \infty$ then we estimate for $l > 2$ the right hand side by $\omega(k_l)^{\frac{1}{lk_l}} \leq \omega(k_l)^{\frac{1}{2k_l}} \to c^{\frac{1}{2}}$, as $l \to \infty$. Then $1 \leq c \leq c^{\frac{1}{2}}$, and this again yields the contradiction $c = 1$.

2.10 Definition. *We call a weight* $\omega : G \to [1, \infty[$ *on a locally compact group* G tempered *if there exist a sequence* $\varepsilon_k > 0$, $k \in \mathbb{N}$ *with* $\lim_k \varepsilon_k^{1/k} = 1$, *an* $l \in \mathbb{N}$ *and a generating neighbourhood* U *such that for all* $k \in \mathbb{N}$:

$$\omega(x) \geq \varepsilon_k \sup\{\omega(y) \mid y \in U^k\}, \quad \forall x \in G \setminus U^{kl}.$$

Remarks.

a) Every weight ω satisfying condition (S) is tempered. To see this, let $\varepsilon_k = v_U(k)^{-1}$, then we have $\lim_k \varepsilon_k^{\frac{1}{k}} = 1$ and $\varepsilon_k v_U(k) = 1 \leq \omega(x), \quad \forall x \in G$, as claimed.

b) On the other hand, if G has polynomial growth and ω is a tempered weight that fulfills the G-N-R condition, then it satisfies condition (S). To see this, we choose $x \in G$ and $L \in \mathbb{N}$ as in Lemma (2.7) so that $\tau_U(x^n) > \frac{1}{L}n \quad \forall n$. If ε_k, l, U are as in the above definition then $\tau_U(x^{klL}) > kl$, and we have

$$\varepsilon_k v_U(k) = \varepsilon_k \sup\{\omega(y) \mid y \in U^k\} \leq \omega(x^{klL}).$$

Hence,

$$1 \leq \lim_{k\to\infty} v_U(k)^{\frac{1}{k}} \leq \lim_{k\to\infty} \big(\frac{1}{\varepsilon_k}\,\omega(x^{klL})\big)^{\frac{1}{k}} \leq \big(\lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}}\big)^{lL} = 1.$$

We summarize these observations in a lemma.

Lemma. *Let* $G \in$ [PG]. *Then a weight* ω *satisfies condition (S), if and only if* ω *is tempered and satisfies the G-N-R condition.*

c) A radial weight on any compactly generated group is tempered. To prove this, we may assume that ω does not satisfy condition (S), i.e., $c = \lim_{k\to\infty} v_U^{\omega}(k)^{\frac{1}{k}} > 1$. Let ω' denote the weight on $\mathbb{Z}$ for which $\omega'(\tau_U(x)) = \omega(x)$ for a suitable generating neighbourhood U. Then $v(k) = \sup\{\omega'(l) \mid |l| \leq k\} = \sup\{\omega(x) \mid \tau_U(x) \leq k\} = v_U^{\omega}(k)$ and (2.9.a) implies that $\lim_{k\to\infty} \omega'(k)^{\frac{1}{k}} = \lim_{k\to\infty} v(k)^{\frac{1}{k}} = c > 1$. Let $\varepsilon_k = \frac{\omega'(k)}{v(k)}$, then $\lim_{k\to\infty} \varepsilon_k^{\frac{1}{k}} = 1$. We take l as in (2.9.b) and see that for $x \in G$ with $\tau_U(x) > lk$:

$$\omega(x) = \omega'(\tau_U(x)) \geq \omega'(k) = \varepsilon_k v(k) = \varepsilon_k v_U^{\omega}(k).$$

2.11 Let H be a compactly generated subgroup of the compactly generated, locally compact group G. Let K and U be generating neighbourhoods in H and G respectively with $K \subset U$. For $x \in H$ we have $\tau_U(x) \leq \tau_K(x)$. Hence, if $\omega(x) \leq Ce^{\Phi(\tau_U(x))}$, $\forall x \in G$, for some increasing function Φ then

$$\omega|_H(x) \leq Ce^{\Phi(\tau_K(x))}, \quad \forall x \in H.$$

In particular, if ω is sub-exponential of degree at most α, then the same is true for $\omega|_H$. Moreover,

$$v_{H,K}^{\omega|_H}(n) = \sup\{\omega|_H(x) \mid x \in K^n\} \leq \sup\{\omega(x) \mid x \in U^n\} = v_{G,U}^{\omega}(n), \quad n \in \mathbb{N},$$

and $\omega|_H$ satisfies the condition (S) whenever ω does.

2.12 Let N be a closed normal subgroup of G with canonical projection $p : G \to G/N$ and let U be a generating neighbourhood of G. Then $V = p(U) = UN \subset G/N$ is a generating neighbourhood in G/N such that $\tau_V(\dot{x}) \leq \tau_U(x)$ for all $x \in G$. To any weight ω on G/N we associate a weight w on G by $w(x) = \omega \circ p(x) = \omega(\dot{x})$. If $\omega(\dot{x}) \leq Ce^{\Phi(\tau_V(\dot{x}))}$ on G/N for some increasing function Φ, then also

$$w(x) \leq Ce^{\Phi(\tau_U(x))}, \quad \forall x \in G.$$

So, if ω is sub-exponential of degree at most α on G/N, then the same is true for w on G. Moreover,

$$v^w_{G,U}(n) = \sup\{w(x) \mid x \in U^n\} = \sup\{\omega(\dot{x}) \mid \dot{x} \in V^n\} = v^\omega_{G/N,V}(n)$$

and w satisfies condition (S) on G if and only if ω satisfies condition (S) on G/N.

2.13 Conversely, given a weight ω on G and a closed normal subgroup N we define a weight $\dot{\omega}$ on the quotient group G/N by

$$\dot{\omega}(\dot{x}) = \inf\{\omega(xn) \mid n \in N\}, \quad \forall \dot{x} \in G/N.$$

It is easy to check that for all $\dot{x}, \dot{y} \in G/N$: $\dot{\omega}(\dot{x}\dot{y}) \leq \dot{\omega}(\dot{x})\dot{\omega}(\dot{y})$. Since ω is upper semi-continuous, $\dot{\omega}$ is upper semi-continuous, too, hence measurable.

Let U be a generating neighbourhood in G, $V = p(U)$, and let τ_U and τ_V be the corresponding "metrics" on G and on G/N respectively. If $\inf_{n\in N} \tau_U(xn) \leq k$, then $xn \in U^k$ for some $n \in N$. Hence $\dot{x} \in V^k$ and $\tau_V(\dot{x}) \leq k$. Conversely, if $\tau_V(\dot{x}) \leq k$ then there are $x_1, \ldots, x_k \in U$ such that $\dot{x} = \dot{x}_1 \ldots \dot{x}_k$, and thus $xn = x_1 \ldots x_k$ for some $n \in N$. It follows that $\inf_{n\in N} \tau_U(xn) \leq k$ and thus $\tau_V(\dot{x}) = \inf_{n\in N} \tau_U(xn), \forall x \in G$.

An inequality of the form

$$\omega(x) \leq Ce^{\Phi(\tau_U(x))}, \quad \forall x \in G,$$

with increasing Φ therefore implies

$$\dot{\omega}(\dot{x}) \leq Ce^{\Phi(\tau_V(\dot{x}))}, \quad \forall \dot{x} \in G/N.$$

In particular, if ω is a sub-exponential weight on G of degree at most α, then the same is true for $\dot{\omega}$ on G/N. Clearly,

$$v^{\dot{\omega}}_{G/N,V}(n) = \sup\{\dot{\omega}(\dot{x}) \mid \dot{x} \in V^n\} \leq \sup\{\omega(x) \mid x \in U^n\} = v^\omega_{G,U}(n).$$

Thus $\dot{\omega}$ on G/N inherits condition (S), temperedness, and the G-N-R condition from ω on G.

3 Symmetry of $L^1(G, \omega)$

3.1 We first mention some properties of the Banach algebra $L^1(G, \omega)$ that hold for arbitrary locally compact groups and arbitrary weights ω. The algebra $L^1(G, \omega)$ is a $*$-algebra for the involution defined by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$. The left translations $a \longmapsto {}_af$, where ${}_af(x) = f(a^{-1}x)$, are strongly continuous from G to $L^1(G, \omega)$. The same is true for the right translations. The algebra $L^1(G, \omega)$ admits bounded approximate identities. This property ensures that the closed left, right, and two-sided ideals in $L^1(G, \omega)$ are just the closed left, right, and two-sided translation invariant subspaces. Let π be a strongly continuous representation of G on a Banach space such that for all $x \in G$, $\|\pi(x)\|_{op} \leq C \cdot \omega(x)$ for some positive constant C. Then π defines a representation of $L^1(G, \omega)$ by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

If the representation π of G is irreducible, the same is true for the corresponding representation of $L^1(G, \omega)$. Conversely, let π' be a continuous representation of $L^1(G, \omega)$ on a Banach space E. Suppose that π' is non-degenerate, i.e., that $\pi'(L^1(G, \omega))E$ is dense in E. Because of the existence of bounded approximate identities, the classical proof shows that there exists a representation π of G satisfying $\|\pi(x)\|_{op} \leq C \cdot \omega(x)$ and such that

$$\pi'(f) = \int_G f(x)\pi(x)dx, \qquad \text{for all } f \in L^1(G, \omega).$$

3.2 We are only interested in $*$-representations on Hilbert spaces. Using [Lep67, Satz 5] it is easy to see that we have in this case:

If π' is a $*$-representation of $L^1(G, \omega)$ on a Hilbert space $\mathcal{H}$, then it is the restriction of a $*$-representation of $L^1(G)$.

The previous remarks apply in particular to $\pi \in \hat{G}$ (the set of equivalence classes of topologically irreducible unitary representations of G). There is a bijection between $\hat{G}$ and the equivalence classes of topologically irreducible, continuous $*$-representations of $L^1(G, \omega)$.

3.3 A Banach-$*$-algebra $\mathcal{A}$ is called symmetric if for all $a \in \mathcal{A}$ the spectrum of a^*a is positive. An equivalent condition is that for all $a = a^* \in \mathcal{A}$ the spectrum is real. Leptin showed in [Lep73] that this is equivalent to the fact that every proper modular left ideal is contained in the kernel of a positive hermitian functional. If the algebra $\mathcal{A}$ contains bounded two–sided approximate identities, the positive functional may be taken to be continuous [BD73].

3.4 The symmetry of the group algebra $L^1(G)$ has been studied extensively. For instance, in [Pog77] Poguntke shows that connected, nilpotent Lie groups have symmetric group algebras. In [Lud79] Ludwig proves that the same is true for compact extensions of nilpotent groups and for connected groups of polynomial growth. Recently, Losert [Los01] showed the symmetry for every compactly generated, locally compact group of polynomial growth. For weighted group algebras only few results are known. In [Pyt82] Pytlik proves that, if G is a connected locally compact group with polynomial growth and if ω is a polynomial weight on G such that $\omega^{-1} \in L^p(G)$ for some p, $0 < p < \infty$, then $L^1(G, \omega)$ is symmetric.

On the other hand, $L^1(\mathbb{R}, e^{|t|})$ with the exponential weight $e^{|t|}$ is not symmetric, because it admits non-unitary characters [Nai72].

3.5 In this section we shall show that for a compactly generated, locally compact group of polynomial growth and for a sub-exponential weight, or more generally for a weight satisfying condition (S), the weighted group algebra $L^1(G, \omega)$ is symmetric. For this we use the notation and the proof of [Lud79] and the structure theorem of [Los01]. For groups of strict polynomial growth we shall show a partial converse.

3.6 First we give some equivalent spectral descriptions of the symmetry of weighted L^1-algebras. Denote by L the left regular representation of G (and of $L^1(G)$) on $L^2(G)$. For an element a of a Banach-$*$-algebra $\mathcal{A}$ let $\sigma(a)$ and $\nu(a)$, or more precisely $\sigma_{\mathcal{A}}(a)$ and $\nu_{\mathcal{A}}(a)$, denote its spectrum and its spectral radius, respectively.

Theorem. *Assume that* $G \in$ [PG] *and that* ω *is a weight on* G*. Then the following are equivalent:*

(*i*) $L^1(G, \omega)$ *is symmetric.*
(*ii*) $\nu_{L^1(G,\omega)}(f) = \| L(f) \|_{op}$ *for all* $f = f^* \in L^1(G, \omega)$.
(*iii*) $\nu_{L^1(G,\omega)}(f) = \nu_{L^1(G)}(f)$ *for all* $f = f^* \in L^1(G, \omega)$.
(*iv*) $\sigma_{L^1(G,\omega)}(f) = \sigma(L(f))$ *for all* $f = f^* \in L^1(G, \omega)$.
(*v*) $\sigma_{L^1(G,\omega)}(f) = \sigma_{L^1(G)}(f)$ *for all* $f = f^* \in L^1(G, \omega)$.

Proof. We prove the following scheme of implications:

$$\begin{array}{ccc} (i) \Rightarrow & (ii) & \Leftarrow (iii) \\ \nwarrow & \Downarrow & \Uparrow \\ & (iv) \Rightarrow & (v) \end{array}$$

(*i*) $\Rightarrow$ (*ii*) Let $f = f^* \in L^1(G, \omega)$. Since $L^1(G, \omega)$ is symmetric, there is a bounded $*$-representation π of $L^1(G, \omega)$ on a Hilbert space $\mathcal{H}$ with $\sigma(\pi(f)) = \sigma_{L^1(G,\omega)}(f)$ (see [Nai72, p. 312], Corollary; note that for selfadjoint elements the left spectrum is the full spectrum), consequently

$$\nu_{L^1(G,\omega)}(f) = \nu(\pi(f)) = \| \pi(f) \|_{op} . \tag{1}$$

We may assume π to be non-degenerate because restricting π to the essential subspace of $\mathcal{H}$ does not affect $\| \pi(f) \|_{op}$. Now by (3.2) π is the restriction to $L^1(G, \omega)$ of a $*$-representation $\widetilde{\pi}$ of $L^1(G)$. We therefore have $\| \pi(f) \|_{op} = \| \widetilde{\pi}(f) \|_{op} \leq \| L(f) \|_{op}$ because G is amenable. Hence by (1) we obtain $\nu_{L^1(G,\omega)}(f) \leq \| L(f) \|_{op}$. The reverse inequality always holds.

(*ii*) $\Rightarrow$ (*iv*) follows from [Hul72, Prop. 2.5] (see also the Appendix below).

(*iv*) $\Rightarrow$ (*i*) is obvious.

$(iv) \Rightarrow (v)$ We have $\sigma_{L^1(G,\omega)}(f) = \sigma(L(f)) \subset \sigma_{L^1(G)}(f)$. The reverse inclusion $\sigma_{L^1(G)}(f) \subset \sigma_{L^1(G,\omega)}(f)$ always holds.

$(v) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (ii)$ Since by Losert's fundamental result $L^1(G)$ is symmetric [Los01], we have $\nu_{L^1(G)}(f) = \| L(f) \|_{op}$ (this follows by the implication $(i) \Rightarrow (ii)$ for $\omega \equiv 1$, but can also be seen directly). Consequently, (iii) implies $\nu_{L^1(G,\omega)}(f) = \| L(f) \|_{op}$. □

The reader may have noticed that the first five implications hold for arbitrary locally compact groups and weights ω, except that in $(i) \Rightarrow (ii)$ we also have used the amenability of G.

3.7 The following conditions are also equivalent to those in the theorem:

$(iii)'$ $\nu_{L^1(G,\omega)}(f) = \nu_{L^1(G)}(f)$ for all $f \in L^1(G, \omega)$.

$(iv)'$ $\sigma_{L^1(G,\omega)}(f) = \sigma(L(f))$ for all $f \in L^1(G, \omega)$.

$(iv)''$ $\sigma_{L^1(G,\omega)}(f) = \sigma(L(f))$ for all $f = g^* * g$ where $g \in L^1(G, \omega)$.

$(v)'$ $\sigma_{L^1(G,\omega)}(f) = \sigma_{L^1(G)}(f)$ for all $f \in L^1(G, \omega)$.

$(v)''$ $\sigma_{L^1(G,\omega)}(f) = \sigma_{L^1(G)}(f)$ for all $f = g^* * g$ where $g \in L^1(G, \omega)$.

In fact one has the following scheme of implications:

$$\begin{array}{ccccc}
(iv)' & \Longrightarrow & (v)' & \Longrightarrow & (iii)' \\
\Uparrow & & \Downarrow & & \Downarrow \\
(iv) & \Longrightarrow & (v) & \Longrightarrow & (iii) \\
\Downarrow & & \Downarrow & & \\
(iv)'' & \Longrightarrow & (v)'' & \Longrightarrow & (i) .
\end{array}$$

With the exception of $(v)'' \Rightarrow (i)$ the horizontal implications are as in the above proof and, except $(iv) \Rightarrow (iv)'$, the vertical ones are trivial. Now, $(v)'' \Rightarrow (i)$ is a consequence of the fact that $L^1(G)$ is symmetric and $(iv) \Rightarrow (iv)'$ may be seen applying the following lemma.

Lemma. *Assume that $\mathcal{B}$ is a Banach-$*$-algebra and $\mathcal{A}$ a not necessarily norm closed $*$-subalgebra. If $\sigma_{\mathcal{A}}(g) = \sigma_{\mathcal{B}}(g)$ for all $g = g^* \in \mathcal{A}$, then $\sigma_{\mathcal{A}}(f) = \sigma_{\mathcal{B}}(f)$ holds for all $f \in \mathcal{A}$.*

Proof. For $f \in \mathcal{A}$, the inclusion $\sigma_{\mathcal{A}}(f) \supset \sigma_{\mathcal{B}}(f)$ clearly holds true and it suffices to show that $\sigma_{\mathcal{A}}(f) \subset \sigma_{\mathcal{B}}(f)$.

If $\mathcal{A}$ has an identity, p say, then $p^* = p = p^2$ is true in $\mathcal{B}$. Further $0 \notin \sigma_{\mathcal{A}}(p) = \sigma_{\mathcal{B}}(p)$ and p is invertible in $\mathcal{B}$. It follows that p is the identity for $\mathcal{B}$.

If $f \in \mathcal{A}$ is invertible in $\mathcal{B}$, then $f^* f$ is invertible in $\mathcal{B}$. Since $0 \notin \sigma_{\mathcal{B}}(f^* f) = \sigma_{\mathcal{A}}(f^* f)$, it follows that $f^* f$ is invertible in $\mathcal{A}$. Similarly, $(f f^*)^{-1}$ exists in $\mathcal{A}$. But then f is invertible in $\mathcal{A}$, because it has the left inverse $(f^* f)^{-1} f^*$ and the right inverse $f^*(f f^*)^{-1}$.

Now, for $\lambda \in \mathbb{C} \setminus \sigma_{\mathcal{B}}(f)$ we apply this argument to $h = f - \lambda p$ and obtain $\lambda \notin \sigma_{\mathcal{A}}(f)$.

If $\mathcal{A}$ has no identity, we may assume that $\mathcal{B}$ contains an identity e, possibly adjoining one to $\mathcal{B}$.

Next for $f \in \mathcal{A}$ we have $0 \in \sigma_{\mathcal{B}}(f)$, because otherwise, by the above argument, $f^* f$ would be invertible in $\mathcal{A}$.

If $\lambda \neq 0$ and $\lambda \notin \sigma_{\mathcal{B}}(f)$, then $(f-\lambda e)^*(f-\lambda e) = f^* f - \overline{\lambda} f - \lambda f^* + |\lambda|^2 e$ is invertible in $\mathcal{B}$. Hence $-|\lambda|^2 \notin \sigma_{\mathcal{B}}(f^* f - \overline{\lambda} f - \lambda f^*) = \sigma_{\mathcal{A}}(f^* f - \overline{\lambda} f - \lambda f^*)$ and $(f-\lambda)^*(f-\lambda)$ is invertible in $\mathcal{A}_e$, the algebra obtained by adjoining an identity to $\mathcal{A}$. So $f - \lambda$ has the left inverse $((f-\lambda)^*(f-\lambda))^{-1}(f-\lambda)^*$ in $\mathcal{A}_e$. Similarly we obtain a right inverse. Hence $\lambda \notin \sigma_{\mathcal{A}}(f)$.

Thus we obtained $\sigma_{\mathcal{A}}(f) \subset \sigma_{\mathcal{B}}(f)$ in this case too. □

3.8 Recall that the group G acts on $L^1(G,\omega)$ by left translations ${}_x f(y) = f(x^{-1}y)$ and that $\| {}_x f \|_\omega \le \omega(x) \| f \|_\omega$. Denote by S the bounded positive hermitian sesquilinear forms on $L^1(G,\omega)$. Then the group anti-acts on S by ${}_x B(f,g) = B({}_x f, {}_x g)$, i.e., ${}_{(xy)}B = {}_y({}_x B)$. For $f, g \in L^1(G,\omega)$ we have

$$|{}_x B(f,g)| \le \omega(x)^2 \| B \| \| f \|_\omega \| g \|_\omega \quad \text{for all } x \in G$$

and the resulting estimate

$$\begin{aligned} B(f*g, f*g) &= \int_G f(x) B({}_x g, f*g)\, dx \\ &\le \int_G |f(x)|^{\frac{1}{2}} \omega(x)^{\frac{1}{2}} |f(x)|^{\frac{1}{2}} \omega(x)^{-\frac{1}{2}} B({}_x g, {}_x g)^{\frac{1}{2}} B(f*g, f*g)^{\frac{1}{2}}\, dx \\ &\le \Big(\int_G |f(x)| \omega(x)\, dx \Big)^{\frac{1}{2}} \Big(\int_G |f(x)| \omega(x)^{-1} B({}_x g, {}_x g)\, dx \Big)^{\frac{1}{2}} \\ &\quad \times B(f*g, f*g)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$B(f*g, f*g) \le \| f \|_\omega \int_G |f(x)|\, {}_x B(g,g) \omega(x)^{-1}\, dx.$$

3.9 We use the following notation of [Lud79]. Let F be a subspace of $L^1(G,\omega)$ and $H \subset G$ a subgroup. Define $S_F^H \subset S$ by

$$S_F^H = \{ B \in S \mid {}_h B = B \; \forall h \in H \text{ and } B(F, f) = 0 \; \forall f \in L^1(G,\omega) \}.$$

As in [Lud79] one obtains that the algebra $L^1(G,\omega)$ is symmetric if and only if $S_I^G \neq \{0\}$ for every proper modular left ideal $I \subset L^1(G,\omega)$. Since the closure of I is again a proper modular left ideal, the Hahn-Banach theorem guarantees the existence of a continuous linear functional $q \neq 0$ on $L^1(G,\omega)$ vanishing on I. Then $(f,g) \mapsto q(f)\overline{q(g)}$ is in S_I. Hence

$$S_I = \{ B \in S \mid B(I, f) = 0 \; \forall f \in L^1(G,\omega) \} \neq \{0\}.$$

3.10 Let I be a closed proper modular left ideal of $L^1(G,\omega)$ with the modular right unit α. As in [Lud79], one sees that $B(\alpha,\alpha) > 0$ for all $B \in S_I^G \setminus \{0\}$. For any non-zero $B \in S_I$ we have

$$0 < \sup_{x\in G} \frac{{}_xB(\alpha,\alpha)}{\omega^2(x)} < +\infty.$$

This follows from the estimate

$$\begin{aligned} B(f,f) &= B(f*\alpha, f*\alpha) \\ &\le \|f\|_\omega \int_G |f(x)|\omega(x)^{-1}\, {}_xB(\alpha,\alpha)dx \\ &\le \|f\|_\omega^2 \sup_{x\in G} \frac{{}_xB(\alpha,\alpha)}{\omega^2(x)} \\ &\le C\|f\|_\omega^2\ \|\alpha\|_\omega^2. \end{aligned}$$

3.11 Before stating and proving the main lemma necessary for the symmetry of $L^1(G,\omega)$, we recall the following definition.

Definition. *Let G be a locally compact group acting on the locally compact group H by automorphisms (for instance, if H is a normal subgroup of G or a quotient group of G). We say that H is an $[FC]_G^-$ group, if the G-orbits in H are relatively compact in H.*

3.12 This concept is useful in the following lemma:

Lemma. *Let $G \in$ [PG] and ω a weight satisfying condition (S). Let H and N be closed normal subgroups of G such that $N \subset H$ and such that H/N is $[FC]_G^-$. Let I be a proper closed modular left ideal in $L^1(G,\omega)$ with modular right unit α. Then:*

$$S_I^N \neq \{0\} \Longrightarrow S_I^H \neq \{0\}.$$

Proof. We adapt the proof of [Lud79] to our situation. Assume that $S_I^N \neq \{0\}$ and choose a non-zero form $B \in S_I^N$. Let $K \subset H/N$ be compact and $\varepsilon > 0$ arbitrary. We shall show the existence of $\tilde{B} \in S_I^N$ such that $\tilde{B}(\alpha,\alpha) \geq \frac{1}{2}$,

$$\|\tilde{B}\| = \sup_{\|f\|_\omega=\|g\|_\omega=1} |\tilde{B}(f,g)| \le 1$$

and

$$|\,{}_k\tilde{B}(f,f) - \tilde{B}(f,f)| \le \varepsilon\|f\|_\omega^2 \tag{2}$$

for all $f \in L^1(G,\omega)$ and all $k \in K$.

As H/N is an $[FC]_G^-$ group and its inner automorphisms are contained in the image of G under the homomorphism $c : G \to \mathrm{Aut}(H/N)$ induced by conjugation, the structure theorem 3.20 of [GM71] implies that H/N has a compact G-invariant neighbourhood of the identity (see Lemma 2 of [Los01]). It follows that the G-conjugacy classes of K in H/N, which are relatively compact, are contained in a compact symmetric neighbourhood U of the identity $\dot{e}$ in H/N. Let $V = \bigcup_{k=1}^{+\infty} U^k$ be the group generated by U. Then V is open in H/N. We use the following function $\rho = \rho_\varepsilon$ introduced by Jenkins [Jen76] and defined on V by

$$\rho(s) = \rho_\varepsilon(s) = (1+\varepsilon)^{-k} \quad \text{if } s \in U^{k+1} \setminus U^k.$$

It is easy to check that

$$|\rho(st) - \rho(t)| \leq \varepsilon\rho(t) \quad \text{and} \quad |\rho(ts) - \rho(t)| \leq \varepsilon\rho(t)$$

for all $s \in U$ and all $t \in V$. We use ρ to define $B' \in S_I^N$ by

$$B'(f,g) = \int_V \rho(\dot{v})\ {}_{\dot{v}^{-1}}B(f,g)d\dot{v}$$

for all $f, g \in L^1(G,\omega)$, where $d\dot{v}$ is the Haar measure on H/N. To show that this integral is well-defined and convergent, we first notice that G/N acts on S_I^N. In fact, for $v \in G$ and $n \in N$

$${}_{vn}B = {}_{vnv^{-1}\cdot v}B = {}_{v}({}_{vnv^{-1}}B) = {}_{v}B.$$

By assumption, ω satisfies condition (S) on G, hence condition (S) holds also for $\dot{\omega}$ on G/N by (2.13), and for $\dot{\omega}|_V$ on the compactly generated subgroup V of G/N by (2.11). In other words, if

$$s_k = \sup_{\dot{v} \in U^k} \dot{\omega}(\dot{v}),$$

then $\lim_{k\to\infty} s_k^{\frac{1}{k}} = 1$. Moreover, (3.8) implies that

$$\begin{aligned} |\ {}_{\dot{v}^{-1}}B(f,g)| &\leq C \inf_{n\in N} \omega(v^{-1}n)^2 \| f \|_\omega \| g \|_\omega \\ &= C\dot{\omega}(\dot{v}^{-1})^2 \|f\|_\omega \|g\|_\omega = C\dot{\omega}(\dot{v})^2 \|f\|_\omega \|g\|_\omega \,. \end{aligned}$$

Combining this with the polynomial growth of H/N, i.e., $|U^k| \leq Ak^D$ for some $D \in \mathbb{N}$ and some $A > 0$, we estimate

$$\begin{aligned} |B'(f,g)| &\leq \sum_{k=0}^{\infty} \int_{U^{k+1}\setminus U^k} |\rho(\dot{v})|\,|\ {}_{\dot{v}^{-1}}B(f,g)|d\dot{v} \\ &\leq CA\|f\|_\omega \|g\|_\omega (1+\varepsilon) \sum_{k=0}^{\infty} (k+1)^D s_{k+1}^2 (1+\varepsilon)^{-(k+1)}. \end{aligned}$$

The last series converges by (2.5.c). It is easy to check that B' is non-zero and in S_I^N, and (3.10) shows that

$$0 < b = \sup_{x\in G} \frac{{}_{x}B'(\alpha,\alpha)}{\omega^2(x)} < \infty.$$

Now choose $y \in G$ such that

$$\frac{{}_y B'(\alpha, \alpha)}{\omega^2(y)} \geq \frac{1}{2} b$$

and define $\tilde{B} \in S_I^N$ by

$$\tilde{B} = \frac{1}{b} \cdot \frac{1}{\omega^2(y)} \cdot {}_y B'.$$

Then by definition $\tilde{B}(\alpha, \alpha) \geq \frac{1}{2}$ and by (3.10) we have

$$\begin{aligned}
|\tilde{B}(f, g)| &\leq \tilde{B}(f, f)^{\frac{1}{2}} \tilde{B}(g, g)^{\frac{1}{2}} \\
&\leq \|f\|_\omega \|g\|_\omega \sup_{x \in G} \frac{{}_x \tilde{B}(\alpha, \alpha)}{\omega^2(x)} \\
&\leq \|f\|_\omega \|g\|_\omega \cdot \frac{1}{b} \cdot \sup_{x \in G} \Big[\frac{1}{\omega^2(yx)} B'({}_{yx}\alpha, {}_{yx}\alpha)\Big] \\
&= \|f\|_\omega \|g\|_\omega.
\end{aligned}$$

This proves that $\|\tilde{B}\| \leq 1$. To show the continuity property (2) of $\tilde{B}$, we estimate for all $u \in K$ and all $f \in L^1(G, \omega)$ that

$$\begin{aligned}
&|{}_u \tilde{B}(f, f) - \tilde{B}(f, f)| \\
&= \frac{1}{b} \cdot \frac{1}{\omega^2(y)} |B'({}_{yu} f, {}_{yu} f) - B'({}_y f, {}_y f)| \\
&= \frac{1}{b} \cdot \frac{1}{\omega^2(y)} \cdot \Big| \int_V \rho(\dot{v}) \big[{}_{\dot{v}^{-1}} B({}_{yu} f, {}_{yu} f) - {}_{\dot{v}^{-1}} B({}_y f, {}_y f) \big] d\dot{v} \Big| \\
&\leq \frac{1}{b} \cdot \frac{1}{\omega^2(y)} \cdot \int_V |\rho((yuy^{-1}v)^{\cdot}) - \rho(\dot{v})| {}_{(v^{-1}y)^{\cdot}} B(f, f) d\dot{v}.
\end{aligned}$$

As $(yuy^{-1})^{\cdot} \in U$ and $\dot{v} \in V$,

$$\begin{aligned}
|{}_u \tilde{B}(f, f) - \tilde{B}(f, f)| &\leq \frac{\varepsilon}{b} \cdot \frac{1}{\omega^2(y)} \cdot \int_V \rho(\dot{v}) \, {}_{\dot{v}^{-1}} B({}_y f, {}_y f) d\dot{v} \\
&= \frac{\varepsilon}{b} \cdot \frac{1}{\omega^2(y)} B'({}_y f, {}_y f) \\
&= \varepsilon \tilde{B}(f, f) \\
&\leq \varepsilon \|f\|_\omega^2.
\end{aligned}$$

Now we may finish the proof as in [Lud79]. For $K \subset H/N$ compact and for $\varepsilon > 0$, let

$$\begin{aligned}
A_{K,\varepsilon} = \{B \in S_I^N \mid \|B\| \leq 1, \ B(\alpha, \alpha) \geq \frac{1}{2}, \\
|{}_k B(f, f) - B(f, f)| \leq \varepsilon \|f\|_\omega^2, \ \forall f \in L^1(G, \omega), \forall k \in K\}.
\end{aligned}$$

Then $\tilde{B} \in A_{K,\varepsilon}$ and $A_{K,\varepsilon} \neq \emptyset$. The intersection of finitely many sets $A_{K,\varepsilon}$ is non-empty and each $A_{K,\varepsilon}$ is weak $*$-closed in

$$\left(S_I^N\right)_1 = S_I^N \cap \{B \in S \mid \|B\| \leq 1\}.$$

Since $(S_I^N)_1$ is compact in the weak $*$-topology, there exists $B_1 \in \bigcap \{A_{K,\varepsilon} \mid K \text{ compact}, \varepsilon > 0\}$. Then $B_1(\alpha, \alpha) \geq \frac{1}{2}$ and so $B_1 \neq 0$. Moreover $\|B_1\| \leq 1$ and ${}_hB_1(f, f) = B_1(f, f)$ for all $f \in L^1(G, \omega)$ and all $h \in H$. By the polarization identity, ${}_hB_1(f, g) = B_1(f, g)$ for all $f, g \in L^1(G, \omega)$ and all $h \in H$. Hence $B_1 \in S_I^H$ and $S_I^H \neq \{0\}$. □

3.13 Theorem. *Let $G \in$ [PG]. If a weight ω on G satisfies condition (S), then the algebra $L^1(G, \omega)$ is symmetric. In particular, if ω is a sub-exponential weight on G, then $L^1(G, \omega)$ is symmetric.*

Proof. We argue as in [Lud79]. We apply Lemma 3.12 inductively to the normal series of Losert's structure theorem [Los01, Thm. 2] which we have stated in the proof of Lemma 2.7. Since $S_I \neq \emptyset$, we conclude that $S_I^G \neq \emptyset$. Thus $L^1(G, \omega)$ is symmetric. □

3.14 For discrete groups we draw the following more explicit and useful consequence of the theorems (3.6) and (3.13) in the spirit of Wiener's lemma for absolutely convergent Fourier series.

Corollary. *Assume that G is a discrete, finitely generated group of polynomial growth and that ω satisfies condition (S). If $f \in \ell^1(G, \omega)$ and the convolution operator $L(f)$ is invertible on $\ell^2(G)$, then f is invertible in $\ell^1(G, \omega)$ and as a consequence $L(f)^{-1}$ is bounded simultaneously on all $\ell^p(G, \omega)$, $1 \leq p \leq \infty$.*

In this form, the theorems have found applications in signal analysis [GL01].

3.15 Next let $\mathcal{S}_\omega = \bigcap_{C \geq 0} L^1(G, \omega^C)$. If G is discrete, $U \subset G$ a generating neighbourhood and $\omega \geq (1 + \tau_U)^\delta$ for some $\delta > 0$, then $\mathcal{S}_\omega$ consists of all functions f satisfying $f(x) = o(\omega(x)^{-C})$ for all $C \geq 0$. Therefore we may call $\mathcal{S}_\omega$ the Frechet algebra of ω-rapidly decreasing functions on G.

Corollary. *If the weight ω satisfies condition (S), then*

$$\sigma_{\mathcal{S}_\omega}(f) = \sigma(L(f)) \quad \textit{for all} \quad f \in \mathcal{S}_\omega.$$

In particular, if G is discrete and $f \in \mathcal{S}_\omega$ is such that $L(f)$ is invertible, then f is invertible in $\mathcal{S}_\omega$.

If $\omega(x) = 1 + \tau_U(x)$, then $\mathcal{S}_\omega$ coincides with the space $\mathcal{S}$ of rapidly decreasing functions in the sense of Hulanicki [Hul72]: Let $f \in \mathcal{S}_\omega$, then we have for every $C > 0$ that

$$\begin{aligned} n^C \cdot & \int_{G \setminus U^n} |f(x)|\, dx \\ & \leq \int_{G \setminus U^n} |f(x)|\, \omega^C(x)\, dx \to 0\,, \quad \text{since} \quad f \in L^1(G, \omega^C). \end{aligned} \tag{3}$$

Thus $\int_{G\setminus U^n}|f|(x)dx = o(n^{-C})$ for every $C > 0$, and so $f \in \mathcal{S}$ by Hulanicki's definition.

Conversely, assume that $\int_{G\setminus U^n}|f|(x)dx = o(n^{-C})$ holds for all $C \geq 0$, i.e., $f \in \mathcal{S}$. Then, for every $C' > 0$, the sum $\sum_{n=1}^{\infty}\int_{G\setminus U^n}|f(x)|dx \cdot n^{C'+2} \cdot n^{-2} \leq const \sum_{n=1}^{\infty} n^{-2}$ converges and so $\int_G |f|(x)\omega^{C'}(x)dx < \infty$, i.e., $f \in \mathcal{S}_\omega$. As a consequence, the above corollary applied to the special case $\omega = 1 + \tau_U$ provides a sharpened version of the theorem in [Hul72].

3.16 For every generating neighbourhood U of a non-compact group G the sequence $|U^k|$ is increasing and divergent. Hence

$$\limsup_{k\to\infty}(|U^{k+1}| - |U^k|)^{\frac{1}{k}} \geq 1.$$

This holds true because otherwise the estimate $|G| = \lim_{n\to\infty}|U^n| = (\sum_{k=1}^{\infty}|U^{k+1}| - |U^k|) + |U| < \infty$ would yield a contradiction.

3.17 Lemma. *Assume that ω is a tempered weight on a locally compact, compactly generated group G of strict polynomial growth. Let U be a generating neighbourhood and $p \in L^1(G,\omega)$ be compactly supported, non-negative, symmetric with $\int_G p(x)dx = 1$ and $\inf\{p(x)|x \in U\} > 0$. Then for every $\alpha > 1$ there exists $L > 0$ such that for all $x \in G$:*

$$\lim_k \omega(x^k)^{\frac{1}{k}} \leq \left(\lim_k v_U(k)^{\frac{1}{k}}\right)^{\tau_U(x)} \leq \alpha^{\tau_U(x)} \nu_{L^1(G,\omega)}(p)^{L\tau_U(x)}.$$

Proof. For ω let ε_k, l, U be as in Definition (2.10). We apply the lower Gaussian estimate of Hebisch and Saloff–Coste [HSC93, Thm. 5.1] to p. (This is where we need the strict polynomial growth of G.) This estimate then yields for all n, k with $(k+1)l < \frac{n}{C''}$:

$$\begin{aligned}\int_G p^{*n}(x)\omega(x)dx &\geq \int_{U^{kl+l}\setminus U^{kl}} p^{*n}(x)\omega(x)dx \\ &\geq \int_{U^{kl+l}\setminus U^{kl}} (Cn)^{-\frac{D}{2}} e^{-C'\frac{\tau_U(x)^2}{n}}\omega(x)\,dx \\ &\geq (Cn)^{-\frac{D}{2}} e^{-C'\frac{((k+1)l)^2}{n}} \varepsilon_k v_U(k)|U^{kl+l}\setminus U^{kl}|.\end{aligned}$$

Choosing any integer $L_0 > 2C''$ and $n = L_0 kl$ we obtain

$$\varepsilon_k v_U(k) \leq e^{C'\frac{((k+1)l)^2}{L_0 kl}}(CL_0 kl)^{\frac{D}{2}} \frac{1}{|U^{kl+l}| - |U^{kl}|}\int_G p^{*L_0 kl}(x)\omega(x)dx.$$

It follows that

$$\liminf_k v_U(k)^{\frac{1}{k}} \leq e^{\frac{C'l}{L_0}} \liminf_k \left(\frac{1}{|(U^l)^{k+1}| - |(U^l)^k|}\right)^{\frac{1}{k}} \nu_{L^1(G,\omega)}(p)^{L_0 l}.$$

Hence, by (3.16), applied to the generating neighbourhood U^l:

$$\lim_k v_U(k)^{\frac{1}{k}} = \inf_k v_U(k)^{\frac{1}{k}} \le e^{\frac{C'l}{L_0}} \nu_{L^1(G,\omega)}(p)^{L_0 l}.$$

Finally we choose L_0 large enough so that $e^{\frac{C'l}{L_0}} \le \alpha$ and we set $L = L_0 l$. Then for any $x \in G$

$$\lim_{k\to\infty} \omega(x^k)^{\frac{1}{k}} \le \lim_{k\to\infty} v_U(\tau_U(x^k))^{\frac{1}{k}} \le \lim_{k\to\infty} v_U(\tau_U(x)k)^{\frac{1}{k}} \le \left(\lim_k v_U(k)^{\frac{1}{k}}\right)^{\tau_U(x)},$$

and the assertion follows. □

3.18 Theorem. *Let G be a compactly generated, locally compact group of strict polynomial growth. Assume that ω is a tempered weight on G. If $L^1(G,\omega)$ is symmetric, then ω fulfills the condition (S). This applies in particular to radial weights.*

Proof. Since ω is symmetric and the group of polynomial growth G is unimodular, any real-valued symmetric L^1-function is selfadjoint. By theorem (3.6) the symmetry of $L^1(G,\omega)$ implies that for $f \in L^1(G,\omega)$ with $f = f^* \ge 0$ we have $\nu_{L^1(G,\omega)}(f) = \nu_{L^1(G)}(f) = \int_G f(x)dx$. Let U be a generating neighbourhood, $f = f^* \ge 0$ be compactly supported with $\inf\{f(x)|x \in U\} > 0$ and $\int_G f(x)dx = 1$. For $x \in G$ and any $\alpha > 1$ we have by Lemma 3.17.

$$1 \le \lim_k v_U(k)^{\frac{1}{k}} \le \alpha\, \nu_{L^1(G,\omega)}(f)^L = \alpha.$$

Since $\alpha > 1$ was arbitrary, ω satisfies condition (S). □

4 Functional calculus

4.1 In this section we shall develop a functional calculus on a total part of $L^1(G,\omega)$ for compactly generated groups of polynomial growth and sub-exponential weights. It is similar to the one given for $L^1(G)$ and $\mathcal{S}(G)$ in [Dix60], [Pyt73] and [Hul84].

4.2 For a generating neighbourhood U and corresponding metric τ_U, $0 < \alpha < 1$, and $C > 0$, we set $w_\alpha^C(x) = e^{C\tau_U(x)^\alpha}$ and define the (Fréchet) algebra $L_\alpha(G)$ to be

$$L_\alpha(G) = \bigcap_{C>0} L^1(G, w_\alpha^C).$$

The following inclusions are obvious: If $0 < \alpha < \beta < 1$ and $C, C' > 0$, then

$$L_\beta(G) \subset L^1(G, w_\beta^C) \subset L_\alpha(G) \subset L^1(G, w_\alpha^{C'}).$$

4.3 Let $f = f^* \in L^1(G,\omega) \subset L^1(G)$ be a hermitian element. Then f operates on $L^2(G)$ by

$$\begin{aligned} L(f) : L^2(G) &\longrightarrow L^2(G) \\ g &\longmapsto L(f)(g) = f * g. \end{aligned}$$

This defines a self-adjoint operator on $L^2(G)$. For any $\lambda \in \mathbb{C}$ we now consider

$$u(i\lambda f) = \sum_{k=1}^{+\infty} \frac{1}{k!}(i\lambda f)^{*k} = \Big(\sum_{k=0}^{+\infty} \frac{1}{(k+1)!}(i\lambda f)^{*k}\Big) * (i\lambda f),$$

where we denoted h^{*k} the k-th convolution power of $h \in L^1(G)$. As $\|u(i\lambda f)\|_\omega \leq e^{|\lambda| \|f\|_\omega}$, $u(i\lambda f) \in L^1(G,\omega)$. Motivated by an argument in [Pyt73] and imposing some additional condition on f, we will obtain a sub-exponential bound for $\|u(i\lambda f)\|_\omega$ in (4.4) below. If we denote by Ψ the function

$$\Psi(t) = \frac{e^{it} - 1}{it} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(it)^k,$$

then

$$u(i\lambda f) = \Psi(\lambda f) * (i\lambda f)$$

and for $\lambda \in \mathbb{R}$

$$\|L(\Psi(\lambda f))\|_{op} = \sup_{\mu \in \sigma(L(\lambda f))} \Big|\frac{e^{i\mu} - 1}{i\mu}\Big| \leq 1,$$

where $\sigma(L(\lambda f))$ denotes the spectrum of the operator $L(\lambda f)$ on $L^2(G)$.

4.4 Let U be an open relatively compact symmetric neighbourhood of e with associated "metric" $\tau = \tau_U$. Since G has polynomial growth, there exists $D \in \mathbb{N}$ and $A > 0$ such that $|U^n| \leq An^D$. Let ω be a sub-exponential weight of degree at most α, $0 < \alpha < 1$, that is, $\omega(x) \leq e^{C\tau(x)^\alpha}$ for some $C > 0$.

Choose $f = f^* \in L^2(G) \cap L_\beta(G)$ for $\beta \in]\alpha, 1[$. For $\lambda \in \mathbb{R}$ we compute $\|u(i\lambda f)\|_\omega$ through the following decomposition:

$$\|u(i\lambda f)\|_\omega = \int_{U^n} |u(i\lambda f)(x)|\omega(x)dx + \int_{G \setminus U^n} |u(i\lambda f)(x)|\omega(x)dx\,.$$

The first integral is estimated by

$$\begin{aligned} \int_{U^n} |u(i\lambda f)(x)|\omega(x)dx &= \int_{U^n} |\Psi(\lambda f) * (i\lambda f)(x)|\omega(x)dx \\ &\leq \|\Psi(\lambda f) * (i\lambda f)\|_{L^2(U^n)} \cdot \|\omega|_{U^n}\|_{L^2(U^n)} \\ &\leq \|L(\Psi(\lambda f))\|_{op} \cdot |\lambda| \cdot \|f\|_2 \cdot \sup_{x \in U^n} |\omega(x)| \cdot |U^n|^{\frac{1}{2}} \\ &\leq A^{\frac{1}{2}} \|f\|_2 \cdot |\lambda| \cdot e^{Cn^\alpha} n^{\frac{D}{2}}. \end{aligned}$$

Setting $w(x) = e^{C\tau(x)^\beta}$, the second integral is bounded as follows:

$$\int_{G\setminus U^n} |u(i\lambda f)(x)|\omega(x)dx = \int_{G\setminus U^n} \frac{1}{w(x)} |u(i\lambda f)(x)|w(x)\omega(x)dx$$
$$\leq \big(\sup_{x\notin U^n} w(x)^{-1}\big)\, e^{|\lambda|\|f\|_{w\omega}} < \infty.$$

Since $(w\omega)(x) \leq e^{C\tau(x)^\beta} \cdot e^{C\tau(x)^\alpha} \leq e^{2C\tau(x)^\beta}$ for $0<\alpha<\beta<1$ and $f \in L_\beta(G)$, we conclude that $f \in L^1(G, w\omega)$ and the previous estimate makes sense. Furthermore, since $\tau(x) \geq n$ for $x \notin U^n$, we obtain $w(x) \geq e^{Cn^\beta}$ and $\sup_{x\notin U^n}\big(\frac{1}{w(x)}\big) \leq e^{-Cn^\beta}$. This proves that

$$\|u(i\lambda f)\|_\omega \leq A^{\frac{1}{2}} \|f\|_2 \cdot |\lambda| \cdot e^{Cn^\alpha} \cdot n^{\frac{D}{2}} + e^{-Cn^\beta} \cdot e^{|\lambda|\|f\|_{w\omega}} \text{ for all } n \in \mathbb{N}.$$

To minimize this expression with respect to n, we choose n to be

$$n = \lfloor (\frac{1}{C}|\lambda|\|f\|_{w\omega})^{\frac{1}{\beta}} + 1) \rfloor,$$

where $\lfloor x \rfloor$ is the integer part of $x \geq 0$. For this choice of n, we have $-Cn^\beta + |\lambda|\|f\|_{w\omega} \leq 0$, and

$$e^{Cn^\alpha} \leq e^{C\left((\frac{1}{C}|\lambda|\,\|f\|_{w\omega})^{\frac{1}{\beta}}+1\right)^\alpha} \leq C_1 e^{C_2|\lambda|^{\frac{\alpha}{\beta}}}$$

for some positive constants C_1 and C_2. Similarly

$$n^{\frac{D}{2}} \leq \left((\frac{1}{C}|\lambda|\|f\|_{w\omega})^{\frac{1}{\beta}} + 1\right)^{\frac{D}{2}} \leq C_3(1 + |\lambda|^{\frac{1}{\beta}})^{\frac{D}{2}},$$

for some positive constant C_3. Hence

$$\|u(i\lambda f)\|_\omega \leq A^{\frac{1}{2}} \|f\|_2 \cdot |\lambda| \cdot C_1 \cdot e^{C_2|\lambda|^{\frac{\alpha}{\beta}}} \cdot C_3 \cdot (1 + |\lambda|^{\frac{1}{\beta}})^{\frac{D}{2}} + 1$$
$$\leq C'(1 + |\lambda|^{\frac{1}{\beta}})^{\frac{D+2}{2}} \cdot e^{C''|\lambda|^{\frac{\alpha}{\beta}}},$$

where $0 < \frac{\alpha}{\beta} < 1$ and the constants $C', C'' > 0$ depend on $\|f\|_2$, $\|f\|_{w\omega}$, C, and β.

4.5 We define A_γ to be the space of all periodic C^∞-functions with Fourier coefficients in $\ell^1(\mathbb{Z}, w_\gamma)$, i.e., $\varphi \in A_\gamma$ if $\varphi(x) = \sum_{n\in\mathbb{Z}} \hat\varphi(n) e^{inx}$ and $\sum_{n\in\mathbb{Z}} |\hat\varphi(n)|\, e^{|n|^\gamma} < \infty$. Then A_γ is an algebra under pointwise multiplication. Since w_γ is sub-exponential, a result of Beurling [Be39] and Domar [Do56, Thm. 2.11] implies that A_γ contains functions of arbitrary small support. As a consequence of [Do56, L. 1.24] for every $\varepsilon > 0$ and every interval $[p, q] \in (0, 2\pi)$ such that $p + \varepsilon < q - \varepsilon$, there exists a function $\varphi \in A_\gamma$ satisfying

$$0 \leq \varphi \leq 1,$$
$$\operatorname{supp} \varphi \subset [p, q],$$
$$\varphi(x) = 1 \text{ for } x \in [p + \varepsilon, q - \varepsilon].$$

4.6 The algebra A_γ is a sufficiently rich algebra to act on certain subspaces of $L^1(G,\omega)$. Let ω be a sub-exponential weight of degree at most α, $0<\alpha<\beta<1$, and $0<\frac{\alpha}{\beta}<\gamma<1$. If $f=f^*\in L^2(G)\cap L_\beta(G)$, then $\varphi\in A_\gamma$ with $\varphi(0)=0$ operates on f through

$$\varphi\{f\}=\sum_{n\in\mathbb{Z}}\hat{\varphi}(n)u(inf).$$

The resulting function $\varphi\{f\}$ is in $L^1(G,\omega)$. To see this, we use the estimate of (4.4) and obtain that

$$\begin{aligned}\|\varphi\{f\}\|_\omega &\le \sum_{n\in\mathbb{Z}}\|u(inf)\|_\omega|\hat{\varphi}(n)|\\ &\le \sum_{n\in\mathbb{Z}} C'(1+|n|^{\frac{1}{\beta}})^{\frac{D+2}{2}}e^{C''|n|^{\frac{\alpha}{\beta}}}\cdot|\hat{\varphi}(n)|\\ &\le C\sum_{n\in\mathbb{Z}}e^{|n|^\gamma}\,|\hat{\varphi}(n)|\;<\;\infty,\end{aligned}$$

since $0<\frac{\alpha}{\beta}<\gamma<1$.

4.7 If $\varphi,\psi\in A_\gamma$, then $\varphi\cdot\psi$ also operates on f, since A_γ is an algebra. Moreover,

$$(\varphi\cdot\psi)\{f\}=\varphi\{f\}*\psi\{f\}.$$

To see this, it suffices to check that

$$L\big((\varphi\cdot\psi)\{f\}\big)=L\big(\varphi\{f\}\big)\circ L\big(\psi\{f\}\big).$$

This identity follows from the fact that for any $*$-representation ρ of $L^1(G,\omega)$,

$$\rho(\varphi\{f\})=\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\infty}\frac{(in)^k}{k!}\rho(f)^k\hat{\varphi}(n)=\varphi(\rho(f)).$$

Here $\varphi(\rho(f))$ is obtained by the usual functional calculus applied to the hermitian operator $\rho(f)$. In fact, the spectral measure E of $\rho(f)$ is compactly supported, and using $\varphi(0)=0$ we obtain

$$\begin{aligned}\varphi(\rho(f)) &= \int_{\mathbb{R}}\varphi(x)dE(x)\\ &= \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}e^{inx}\,\hat{\varphi}(n)\,dE(x)\\ &= \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}(e^{inx}-1)\hat{\varphi}(n)\,dE(x)\\ &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}(e^{inx}-1)dE(x)\hat{\varphi}(n)\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k=1}^{\infty}\frac{(in)^k}{k!}\rho(f)^k\hat{\varphi}(n)$$

$$= \rho\left(\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\infty}\frac{(in)^k}{k!}f^{*k}\hat{\varphi}(n)\right) = \rho(\varphi\{f\}).$$

4.8 If φ is a C^∞-function on $\mathbb{R}$ with compact support, such that $\varphi(0) = 0$ and $\int_{\mathbb{R}} |\hat{\varphi}(\lambda)|e^{|\lambda|^\gamma}\, d\lambda < \infty$, then the functional calculus on $L^2(G) \cap L_\beta(G)$ may also be defined by

$$\varphi\{f\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(i\lambda f)\hat{\varphi}(\lambda)d\lambda$$

The properties are the same as before.

5 Wiener property

5.1 Let us recall the following definition:

Definition. *Let $\mathcal{A}$ be a Banach-$*$-algebra. We say that $\mathcal{A}$ has the Wiener property if for every proper closed two-sided ideal I of $\mathcal{A}$, there exists a topologically irreducible $*$-representation π of $\mathcal{A}$ such that $I \subset \ker\pi$. If $\mathcal{A}$ is of the form $L^1(G)$ for some locally compact group G, we say that the group G has the Wiener property.*

5.2 Examples

a) The algebra $L^1(\mathbb{R})$ has the Wiener property. In this case the Wiener property means that for every proper closed ideal I of $L^1(\mathbb{R})$ there exists $a \in \mathbb{R}$ such that

$$I \subset \{f \in L^1(\mathbb{R}) \mid \hat{f}(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-iax}dx = 0\}.$$

b) More generally, if G is a locally compact group with polynomial growth, hence compactly generated by our definition, then $L^1(G)$ has the Wiener property [Los01].
c) Compact extensions of nilpotent groups have the Wiener property [Lud79]. In particular, nilpotent Lie groups possess the Wiener property.
d) There are connected, simply connected exponential Lie groups that fail to have the Wiener property. One example is the group $G_{4,9}(0) = \exp \mathfrak{g}_{4,9}(0)$ whose Lie algebra is generated by X, Y, Z, T satisfying $[T, X] = -X, \quad [T, Y] = Y, \quad [X, Y] = T$, see [Lep73,Pog77].
e) The affine group has the Wiener property. More generally, this is true for every semi-direct product of abelian groups [Pyt82].

5.3 In this section we shall study the Wiener property for algebras of the form $L^1(G,\omega)$, where $G \in$ [PG] and ω is a sub-exponential weight of degree at most α, $0 < \alpha < 1$. Let $(f_j)_{j \in J}$ be a bounded approximate identity in $L^1(G,\omega)$ such that

$$f_j = f_j^* \quad \text{for all } j\,,$$
$$\sup_{j \in J} \|f_j\|_\omega = C_0 < \infty\,,$$
$$\bigcup_{j \in J} \operatorname{supp} f_j \subset K\,,$$

where K a fixed compact, symmetric neighbourhood of e.

We shall show that there exists a periodic function $\varphi \in A_\gamma$ for suitable $\gamma < 1$ with $\varphi(1) = 1$, $\varphi \equiv 0$ in a neighbourhood of 0, such that

$$\varphi\{f_j\} = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) u(inf_j)$$

converges for all j, and such that

$$\|\varphi\{f_j\} * g - g\|_\omega \to 0 \tag{4}$$

for all continuous functions g with compact support in G.

If $\varphi(1) = 1$ and $\varphi(0) = 0$, then

$$\|\varphi\{f_j\} * g - g\|_\omega = \|\sum_{n \in \mathbb{Z}} \hat{\varphi}(n)[e^{inf_j} * g - e^{in} g]\|_\omega\,.$$

In the sequel we shall use the same techniques as in the construction of the functional calculus in 4.4 to show that this second expression tends to 0.

5.4 First we show that for a fixed n and $j \to \infty$

$$e^{inf_j} * g = \sum_{k=0}^{\infty} \frac{(in)^k}{k!} f_j^{*k} * g \xrightarrow{\|\cdot\|_\omega} e^{in} g\,.$$

In fact, for any $\varepsilon > 0$ we may find $M \in \mathbb{N}$ (independent of j) such that

$$\|\sum_{k=M}^{\infty} [\frac{(in)^k}{k!} f_j^{*k} * g - \frac{(in)^k}{k!} g]\|_\omega \le \sum_{k=M}^{\infty} \frac{n^k}{k!} C_0^k \|g\|_\omega + \sum_{k=M}^{\infty} \frac{n^k}{k!} \|g\|_\omega < \varepsilon\,.$$

On the other hand, as $(f_j)_j$ is an approximate identity, we have

$$f_j^{*k} * g \xrightarrow{\|\cdot\|_\omega} g$$

as $j \to \infty$ for all k, and hence

$$\|\sum_{k=0}^{M-1} \Big(\frac{(in)^k}{k!} f_j^{*k} * g - \frac{(in)^k}{k!} g\Big)\|_\omega \longrightarrow 0.$$

Combining these estimates we have shown that

$$e^{inf_j} * g \to e^{in} g$$

converges in $L^1(G, \omega)$ for any fixed n. Hence, for any function φ with $\varphi(1) = 1$ and $\varphi(0) = 0$ and any $N \in \mathbb{N}$ fixed, we have

$$\sum_{|n| \le N} \hat{\varphi}(n)[e^{inf_j} * g - e^{in} g] \to 0.$$

5.5 Next we show that we may choose φ such that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} \| \sum_{|n|>N} \hat{\varphi}(n)[e^{inf_j} * g - e^{in} g]\|_\omega &\le \sum_{|n|>N} |\hat{\varphi}(n)| \|e^{inf_j} * g\|_\omega \\ &\quad + | \sum_{|n|>N} \hat{\varphi}(n) e^{in} | \|g\|_\omega < \varepsilon, \end{aligned}$$

independently of j. Suppose that we have already determined φ and N_1 such that

$$\sum_{|n|>N_1} |\hat{\varphi}(n)| \|e^{inf_j} * g\|_\omega < \frac{\varepsilon}{2}, \tag{5}$$

for all j, then (as $\sum_{n\in\mathbb{Z}} \hat{\varphi}(n) e^{in} = \varphi(1) = 1$ converges) we can choose $N \ge N_1$ such that

$$| \sum_{|n|>N} \hat{\varphi}(n) e^{in} | \, \|g\|_\omega < \frac{\varepsilon}{2}.$$

Thus it suffices to show (5). For this purpose, we decompose the $L^1(G, \omega)$-norm as

$$\|e^{inf_j} * g\|_\omega = \int_{U^M} |e^{inf_j} * g(x)| \, \omega(x) \, dx + \int_{G \setminus U^M} |e^{inf_j} * g(x)| \, \omega(x) \, dx$$

where $M \in \mathbb{N}$. Then

$$\begin{aligned} \int_{U^M} |e^{inf_j} * g(x)| \omega(x) dx &\le \|e^{inf_j} * g|_{U^M}\|_{L^2(U^M)} \cdot \|\omega|_{U^M}\|_{L^2(U^M)} \\ &\le A^{\frac{1}{2}} \|e^{inf_j} * g\|_2 \, M^{\frac{D}{2}} \, e^{C' M^\alpha} \\ &\le A^{\frac{1}{2}} \|g\|_2 \, M^{\frac{D}{2}} \, e^{C' M^\alpha}, \end{aligned}$$

because the norm of the convolution operator by e^{inf_j} in $L^2(G)$ equals 1 for all j, and because $|U^M| \le A M^D$ by polynomial growth. Choose $0 < \alpha < \beta < 1$ and let $w(x) = e^{C' \tau_U(x)^\beta}$. Then

$$\begin{aligned} \int_{G \setminus U^M} |e^{inf_j} * g(x)| \omega(x) dx &\le \sup_{x \notin U^M} \frac{1}{w(x)} \|e^{inf_j} * g\|_{w\omega} \\ &\le e^{-C' M^\beta} \|e^{inf_j}\|_{w\omega} \|g\|_{w\omega}. \end{aligned}$$

Since all f_j have their support in the fixed compact set K, we estimate

$$\|f_j\|_{w\omega} \leq \sup_{x\in K} w(x)\|f_j\|_\omega \leq C < \infty\,.$$

Hence

$$\|e^{inf_j}\|_{w\omega} \leq e^{|n|\|f_j\|_{w\omega}} \leq e^{|n|C}.$$

Combining all estimates we obtain

$$\|e^{inf_j} * g\|_\omega \leq A^{\frac{1}{2}}\|g\|_2 M^{\frac{D}{2}} e^{C'M^\alpha} + e^{-C'M^\beta} e^{|n|C} \|g\|_{w\omega}.$$

Similar to (4.4) we now choose M to be

$$M = \lfloor (\frac{C}{C'}|n|)^{\frac{1}{\beta}} + 1 \rfloor\,,$$

where $\lfloor x \rfloor$ is the integer part of x. Then $|n|C - C'M^\beta \leq 0$ and

$$\|e^{inf_j} * g\|_\omega \leq A^{\frac{1}{2}}\|g\|_2 \, e^{C'\left((\frac{C}{C'}|n|)^{\frac{1}{\beta}}+1\right)^\alpha} \cdot \left((\frac{C}{C'}|n|)^{\frac{1}{\beta}} + 1\right)^{\frac{D}{2}} + \|g\|_{w\omega}$$
$$\leq B|n|^b e^{B'|n|^{\frac{\alpha}{\beta}}}$$

for some new constants B, B', b independent of j.

Now choose γ, such that $0 < \frac{\alpha}{\beta} < \gamma < 1$. By (4.5) there exists a function $\varphi \in A_\gamma$ such that

$$\varphi \equiv 0 \text{ in a neighbourhood of } 0 \tag{6}$$
$$0 \leq \varphi \leq 1 \tag{7}$$
$$\varphi(1) = 1 \tag{8}$$
$$\operatorname{supp}\varphi \subset]0, 2\pi[\,. \tag{9}$$

For this $\varphi \in A_\gamma$, we have

$$\sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)|\, \|e^{inf_j} * g\|_\omega \leq C \sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)|\, e^{|n|^\gamma} < \infty\,.$$

Consequently, there exists N_1 such that

$$\sum_{|n|>N_1} |\hat{\varphi}(n)|\, \|e^{inf_j} * g\|_\omega < \frac{\varepsilon}{2}.$$

5.6 Theorem. *Let $G \in$ [PG] and let ω be a sub-exponential weight on G. Then $L^1(G, \omega)$ has the Wiener property.*

Proof. Assume that $I \subset L^1(G, \omega)$ is a closed two-sided ideal not contained in any kernel of a topologically irreducible $*$-representation of $L^1(G, \omega)$. Let $M = \{\varphi\{f_j\} \mid j \in J\}$, where φ, $(f_j)_{j \in J}$ are as in (5.3). By (4) in (5.3) any closed ideal containing M will also contain the dense subspace of continuous compactly supported functions and thus coincide with $L^1(G, \omega)$. In the language of ideal theory, the hull $h(M)$ of M is empty where $h(M)$ is defined as $h(M) = \{\ker \rho \mid M \subset \ker \rho\}$ and ρ ranges over all topologically irreducible $*$-representation of $L^1(G, \omega)$.

Now choose $\psi \in A_\gamma$ satisfying the conditions (6) and (7) (with ψ in place of φ) and $\psi \equiv 1$ on $\operatorname{supp} \varphi \cap [0, 2\pi]$. Clearly $h(M) = \emptyset \subset h(\psi\{f_j\})$ for each $j \in J$. By (4.7) we have for each $j \in J$:

$$\psi\{f_j\} * \varphi\{f_j\} = (\psi \cdot \varphi)\{f_j\} = \varphi\{f_j\}$$

Now we apply Lemma 2 of [Lud80] and we conclude that $M \subset I$. As shown above I coincides with $L^1(G, \omega)$. □

Let us point out that the result of [Lud80] used in the proof makes crucial use of the symmetry of the underlying algebra. In our case the symmetry of $L^1(G, \omega)$ for $G \in [\mathrm{PG}]$ and sub-exponential weight ω is an important and necessary, though somewhat hidden ingredient in the proof of Theorem 5.6.

6 Appendix

In the proof of Theorem 3.6 we have used the following statement of [Hul72].

6.1 Proposition. *[Hul72, Prop. 2.5] Let $\mathcal{A}$ be a Banach-$*$-algebra and $\mathcal{S}$ a (not necessarily closed) $*$-subalgebra of $\mathcal{A}$. Let T be a faithful $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ satisfying*

$$\| T_x \|_{op} = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} \quad \text{for all } x = x^* \in \mathcal{S}.$$

If $\mathcal{A}$ has a unit, e say, assume in addition that $T_e = id_{\mathcal{H}}$. Then for every $x = x^*$ in $\mathcal{S}$ we have

$$\sigma_{\mathcal{A}}(x) = \sigma(T_x).$$

It seems that the proof in [Hul72] yields only the equality $\sigma_{\mathcal{A}}(x) \setminus \{0\} = \sigma(T_x) \setminus \{0\}$. This is sufficient for all purposes of symmetry. However, since we need the full result as stated, we include a modification which covers zero too. We use the notation of [Hul72] except that we still denote the spectrum of a by $\sigma(a)$. For the proof we use the following lemma.

6.2 Lemma. *Let $\mathcal{B}$ be the $\| \quad \|_{\mathcal{A}}$-closure of some commutative $*$-subalgebra of $\mathcal{A}$. If $id_{\mathcal{H}}$ is in the operator norm closure of the image of $\mathcal{B}$ under T, there is some $e \in \mathcal{B}$ with $T_e = id_{\mathcal{H}}$. It follows that $\mathcal{A}$ has a unit, namely e.*

Proof of the Lemma. As in [Hul72] and by continuity, $\lambda : x \mapsto \| T_x \|_{op}$ and $\nu : x \mapsto \lim_n \| x^n \|^{1/n}$ are equivalent norms on $\mathcal{B}$. The completion $\mathcal{B}^\lambda$ is a commutative C^*-algebra and isomorphic to $\overline{T(\mathcal{B})}^\lambda$, and by assumption $\mathcal{B}^\lambda$ contains a unit.

As $\mathcal{B}$ is dense in $\mathcal{B}^\lambda$ and every $\varphi \in X(\mathcal{B})$ can be extended to $\widetilde{\varphi} \in X(\mathcal{B}^\lambda)$ because $\nu \sim \lambda$ on $\mathcal{B}$, the Gelfand spaces $X(\mathcal{B})$ and $X(\mathcal{B}^\lambda)$ are homeomorphic via the map $\tilde{\varphi} \mapsto \tilde{\varphi}|_{\mathcal{B}}$. Since the unit of $\mathcal{B}^\lambda$ has Gelfand transform 1, there is $f \in \mathcal{B}$ such that $\|\widehat{f}-1\|_\infty < \frac{1}{2}$. As $|\widehat{f}| \geq \frac{1}{2}$ on $X(\mathcal{B})$, there is a unit element e in $\mathcal{B}$ (see [BD73]), and $T_e = id_{\mathcal{H}}$. For $a \in \mathcal{A}$ we have $T_{a-ae} = T_a - T_a id_{\mathcal{H}} = 0$ and similarly $T_{a-ea} = 0$. Since T is faithful, $a = ae = ea$, so e is a unit for $\mathcal{A}$. □

Proof of the Proposition. For $x = x^* \in \mathcal{S}$ let $\mathcal{B}$ be a commutative $\| \|_{\mathcal{A}}$-closed $*$-subalgebra of $\mathcal{A}$ containing x. We distinguish several cases.

Case I. If the assumptions of the Lemma hold for $\mathcal{B}$, i.e. $id_{\mathcal{H}} \in \mathcal{B}^\lambda$, we have

$$\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x) = \{\varphi(x)|\varphi \in X(\mathcal{B}) = X(\mathcal{B}^\lambda)\} = \sigma_{\mathcal{B}^\lambda}(x) = \sigma_{B(\mathcal{H})}(T_x), \tag{10}$$

where the outer equality signs hold, because the spectrum in the "middle" does not separate the complex plane and $\mathcal{A}$ and $\mathcal{B}$ as well as $\mathcal{B}^\lambda$ and $B(\mathcal{H})$ have a common unit element.

Case II. If $id_{\mathcal{H}} \notin \mathcal{B}^\lambda$, there are two further cases:

(i) $\mathcal{A}$ has no unit. Then $0 \in \sigma_{\mathcal{A}}(x)$ by definition. Since $\mathcal{B}^\lambda + \mathbb{C} id_{\mathcal{H}} \cong \mathcal{B}^\lambda \oplus \mathbb{C}$, we have $0 \in \sigma_{\mathcal{B}^\lambda + \mathbb{C} id_{\mathcal{H}}}(x) = \sigma_{B(\mathcal{H})}(T_x)$, because $\mathcal{B}^\lambda + \mathbb{C} id_{\mathcal{H}}$ and $B(\mathcal{H})$ have the common unit element $id_{\mathcal{H}}$. For the nonzero spectral values, (10) still applies, so

$$\sigma_{\mathcal{A}}(x) = \sigma_{B(\mathcal{H})}(T_x).$$

(ii) $\mathcal{A}$ has a unit e, and by assumption $T_e = id_{\mathcal{H}}$. It follows $e \notin \mathcal{B}$ (as $id_{\mathcal{H}} \notin \mathcal{B}^\lambda$). Because of $\mathcal{B} + \mathbb{C}e \cong \mathcal{B} \oplus \mathbb{C}$ we obtain $0 \in \sigma_{\mathcal{B}+\mathbb{C}e}(x) = \sigma_{\mathcal{A}}(x)$, as $\mathcal{B} + \mathbb{C}e$ and $\mathcal{A}$ have the common unit element e. We also have $0 \in \sigma_{B(\mathcal{H})}(T_x)$ as in (i). For the nonzero spectral values, (10) applies, so

$$\sigma_{\mathcal{A}}(x) = \sigma_{B(\mathcal{H})}(T_x)\,.$$ □

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