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SYMMETRY OF WEIGHTED L^1-ALGEBRAS AND THE GRS-CONDITION

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Abstract

Let G be a compactly generated, locally compact group of polynomial growth. Removing a restrictive technical condition from a previous work, we show that the weighted group algebra $L^1_\omega(G)$ is a symmetric Banach $*$-algebra if and only if the weight function ω satisfies the GRS-condition. This condition expresses in a precise technical sense that ω grows subexponentially.

1. *Introduction and results*

We investigate the symmetry of the weighted group algebra $L^1_\omega(G)$ for a locally compact group of polynomial growth. On the one hand, we are motivated by the old question in abstract harmonic analysis of how Wiener's lemma for absolutely convergent Fourier series can be generalized to non-commutative groups [**12, 13**]. On the other hand, we are motivated by concrete applications of harmonic analysis in signal analysis and numerical mathematics [**7, 8, 19**]. Here the concepts of symmetry and of inverse-closedness provide a useful form of symbolic calculus. In the applied context it is crucial to investigate *weighted* L^1-algebras, because the weight allows us to model decay conditions in a quantitative manner.

It turns out that the class of locally compact groups of polynomial growth is most suitable for our investigation. A locally compact group G is said to be compactly generated if there exists a compact neighbourhood U of the identity such that $G = \bigcup_{n=1}^{\infty} U^n$. A relatively compact symmetric neighbourhood $U \subseteq G$ with $G = \bigcup_{n=1}^{\infty} U^n$ will be called a *generating neighbourhood*. A compactly generated group G is said to have (at most) *polynomial growth*, if for some generating neighbourhood there exist constants $C > 0$ and $D \in \mathbb{N}$ such that $|U^k| \leqslant Ck^D$ for $k \in \mathbb{N}$. (Here and in the sequel $|U|$ denotes the Haar measure of the Borel set U.) We write [PG] for the class of compactly generated, locally compact groups of polynomial growth.

The detailed structure of groups of polynomial growth is now well understood and culminates in the structure results of Losert [**14, 15**].

In this paper we are interested in the symmetry of the group algebra $L^1(G)$ and certain subalgebras. Let $\mathcal{A}$ be an involutive Banach algebra. Writing $\sigma_{\mathcal{A}}(a)$ for the spectrum of $a \in \mathcal{A}$ and $r_{\mathcal{A}}(a)$ for its spectral radius, $\mathcal{A}$ is called *symmetric* if $\sigma_{\mathcal{A}}(a^*a) \subseteq [0, \infty)$ for all $a \in \mathcal{A}$. (Equivalently, $\mathcal{A}$ is symmetric if and only if $a = a^* \in \mathcal{A}$ implies that $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$.) If $L^1(G)$ is symmetric and G is amenable, then the spectrum of the convolution operator $L_f g = f * g$ acting on $L^p(G)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(G)$; see [**2, 7**].

It was a long-standing conjecture that groups in [PG] have a symmetric group algebra [**13**]. This conjecture was completely solved by Losert [**15**] by combining his structure theorem with a method of Ludwig [**16**].

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THEOREM 1.1 [**15**]. *If $G \in$ [PG], then $L^1(G)$ is symmetric.*

In our study of *weighted* group algebras we use locally bounded measurable weight functions $\omega : G \to \mathbb{R}^+$ with the following properties: $\omega(e) \geqslant 1$, where e is the identity element of G, $\omega(x) = \omega(x^{-1})$, and $\omega(xy) \leqslant \omega(x)\omega(y)$ for all $x, y \in G$, that is, ω is symmetric and submultiplicative. In particular, ω is bounded on compact sets of G and $\omega(x) \geqslant 1$ for all $x \in G$.

We may assume without loss of generality that ω is continuous (just replace ω by an equivalent weight of the form $\alpha\omega * \phi$, where $\phi \geqslant 0$ is a symmetric, continuous function with compact support and $\alpha > 0$ is suitably chosen; see [**4**]).

Let dx denote the Haar measure on G, then $L^1_\omega(G)$ is the weighted group algebra consisting of all measurable functions for which the norm

$$\|f\|_{L^1_\omega(G)} = \int_G |f(x)|\omega(x)\,dx \tag{1}$$

is finite. As a consequence of the properties of ω, $L^1_\omega(G)$ is a Banach $*$-algebra under convolution $f * g(x) = \int_G f(y)g(y^{-1}x)\,dy$ and the usual involution $f^*(x) = \overline{f(x^{-1})}$. Since $\omega(x) \geqslant 1$, $L^1_\omega(G)$ is a subalgebra of $L^1(G)$.

To study the symmetry of $L^1_\omega(G)$, we need additional conditions on the weight ω.

DEFINITION 1.2. (a) A weight ω on G is said to satisfy the *GRS-condition* if

$$\lim_{n\to\infty} \omega(x^n)^{1/n} = 1 \qquad \forall\, x \in G. \tag{2}$$

(In [**5**] this condition was called the GNR condition.)

(b) A weight ω on G is said to satisfy *condition* (S) if for some generating neighbourhood U of G

$$\lim_{n\to\infty} \sup_{y\in U^n} \omega(y)^{1/n} = \lim_{n\to\infty} \sup_{x_1,\dots,x_n\in U} \omega(x_1x_2\dots x_n)^{1/n} = 1. \tag{3}$$

Both conditions describe in a precise formal way the 'sub-exponential growth' of ω. The GRS-condition was introduced by Gelfand, Raikov and Shilov in [**6**] and nowadays is named after these authors. Condition (S) is a uniform version of the GRS-condition and clearly implies the GRS-condition. Furthermore, it is easy to see that if the condition (S) holds true for one generating neighbourhood U, then it holds for all generating neighbourhoods of G. For explicit examples of weights satisfying these conditions see [**5**].

Our main result characterizes the symmetry of $L^1_\omega(G)$ in terms of the weight ω.

THEOREM 1.3. *Let G be a locally compact, compactly generated group of polynomial growth and ω a weight on G. Then the following conditions are equivalent.*

(i) *ω satisfies the GRS-condition.*

(ii) *ω satisfies condition (S).*

(iii) *$L^1_\omega(G)$ is symmetric.*

(iv) *$\sigma_{L^1_\omega(G)}(f) = \sigma_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$.*

The equivalence of conditions (i)–(iv) was proved under a restrictive technical condition in [**5**] where ω was assumed to be 'tempered'. A careful investigation of our previous proof which is based on Ludwig's original method [**16**] reveals that the temperedness of ω is not necessary and that (ii), (iii) and (iv) are equivalent without any further restrictions on ω. In addition, these conditions are also equivalent to the GRS-condition which is probably the cleanest and easiest condition to check.

In particular, the two conditions on the weight ω, namely the GRS-condition and condition (S), are equivalent. Although the equivalence (i) $\iff$ (ii) is a statement about the weights, we can show it only via the symmetry of $L^1_\omega(G)$. On compactly generated locally compact abelian groups, we have a direct proof of this equivalence, see Lemma 2.2. In addition, we know a direct proof of this equivalence for connected nilpotent Lie groups and are confident that the results of Guivarc'h and Losert can be used for a direct proof of (i) $\Longrightarrow$ (ii) in the general case, but it is easier to pass through the symmetry of $L^1_\omega(G)$.

The statement of Theorem 1.3 is satisfying and aesthetically pleasing. The proof of implication (i) $\Longrightarrow$ (iii) is somewhat unpleasant, because it rests on a subtle refinement of our previous proof in [**5**], but does not yield a new conceptual insight. However, using Theorem 1.1, we will derive a direct and conceptually easier proof that condition (S) implies the symmetry of $L^1_\omega(G)$ (Section 3). The method is of interest in itself because it can be used in other situations to deduce the symmetry of a 'weighted Banach algebra' from the symmetry of the unweighted Banach algebra, see [**7**, **8**] for more results of this type.

2. *The GRS-condition and symmetry*

In this section we prove Theorem 1.3. We first show the equivalence of the GRS-condition and the condition (S) on compactly generated, locally compact abelian groups.

LEMMA 2.1. *Let $\omega : \mathbb{R} \to [1,\infty)$ be a weight function on $\mathbb{R}$. If $\lim_{n\to\infty} \omega(n)^{1/n} = 1$, then ω satisfies both the GRS-condition and condition (S).*

Proof. Since the function $\Phi(x) = \log \omega(x)$, $x \in \mathbb{R}$ is sub-additive, a standard lemma [**3**, Lemma VIII.1.4] implies that $\inf_{t>0} t^{-1}\Phi(t) = \lim_{t\to\infty} t^{-1}\Phi(t)$, so the condition $\lim_{n\to\infty} \omega(n)^{1/n} = 1$ implies the GRS-condition. Introducing the weight v on $\mathbb{N}$ by $v(k) = \sup\{\omega(x) : |x| \leqslant k\}$, we know by the same reasoning that $1 \leqslant \lim_{k\to\infty} v(k)^{1/k} =: c$ exists. If $c > 1$, then there exist $x_k \in \mathbb{R}$, with $x_k \leqslant k$ such that $\lim_{k\to\infty} \omega(x_k)^{1/k} > 1$. Since ω is bounded on compact sets, the sequence x_k must converge to ∞. However, since $x_k \leqslant k$, we have $1 \leqslant \omega(x_k)^{1/k} \leqslant \omega(x_k)^{1/x_k} \to 1$, a contradiction. □

LEMMA 2.2. *On a compactly generated, locally compact abelian group the GRS-condition and condition (S) are equivalent.*

Proof. By the structure theorem for compactly generated, locally compact abelian groups [**11**, Theorem 9.8], G splits as a direct product $G = \mathbb{R}^d \times \mathbb{Z}^e \times K$, where K is a compact group. Accordingly, we have $\omega(x,y,z) \leqslant \omega_{|\mathbb{R}^d}(x)\,\omega_{|\mathbb{Z}^e}(y)\,\omega_{|K}(z)$, $(x,y,z) \in \mathbb{R}^d \times \mathbb{Z}^e \times K$ and it suffices to consider each factor separately.

Clearly, $\omega_{|K}$ is bounded and so we need only consider the two other factors. Again we may dominate each of these factors by the product of the restrictions of the weight to the single coordinates, and consequently we are left with weights defined on $\mathbb{R}$ or on $\mathbb{Z}$.

For a weight on $\mathbb{R}$ we have just seen the validity of this implication in Lemma 2.1, and for a weight on $\mathbb{Z}$ the argument is even simpler: just omit the first part of the proof of Lemma 2.1. □

In a crucial part of the argument below, we need a result of Hebisch and Saloff-Coste [**10**] that was proved only for groups of *strict* polynomial growth. The following lemma shows that this is no restriction. The equivalence of polynomial growth and strict polynomial growth is a consequence of the structure results in [**9**, **15**]. We include the short proof, since the result is not stated in the literature.

LEMMA 2.3. *Every locally compact, compactly generated group G of polynomial growth has strict polynomial growth, that is, there exist a compact symmetric generating neighbourhood of the identity $U \subset G$ and constants $C_1, C_2 > 0$ and $D > 0$ such that*

$$C_1 k^D \leqslant |U^k| \leqslant C_2 k^D \qquad \text{for } k \in \mathbb{N}.$$

The exponent D is called the order of growth of G.

Proof. If G has polynomial growth, then it contains a maximal compact normal subgroup C such that G/C is a Lie group [**15**, Proposition 1]. Since G and G/C have the same order of growth [**9**, Theorem I.4], we may assume that G is a Lie group. By [**15**, Propositions 3 and 4] G possesses a maximal solvable normal subgroup R, such that G/R is compact. Once again G and R have the same order of growth. Furthermore, R is compactly generated by [**15**, Proposition 2]. Now [**9**, Corollary III.3] implies that R has strict polynomial growth. □

We now turn to the proof of Theorem 1.3. The equivalence of (iii) and (iv) is in [**5**, Theorem 3.6 and Lemma 3.8], where similar equivalent conditions are stated. The implication (ii) $\Longrightarrow$ (i) is trivial.

We first prove the implications (iv) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (ii).

PROPOSITION 2.4. *If $\sigma_{L^1_\omega(G)}(f) = \sigma_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$, then ω satisfies both the GRS-condition and condition (S).*

Proof. Given a relatively compact symmetric generating neighbourhood U of the identity, we choose a continuous symmetric function $p = p^* \geqslant 0$ with compact support satisfying the conditions $\inf\{p(x) : x \in U\} > 0$ and $\int p(x)\, dx = 1$.

Condition (iv) implies that the spectral radius of p in the algebra $L^1_\omega(G)$ equals its spectral radius in $L^1(G)$, that is, $r_{L^1_\omega(G)}(p) = r_{L^1(G)}(p) = \int p(x)\, dx = 1$. We now use pointwise Gaussian estimates for convolution powers of probability densities on groups of strict polynomial growth, which is possible by Lemma 2.3. By [**10**, Theorem 5.1] there exist positive constants C, C', C'' such that for $k < n/C''$ and any Borel set $E_k \subset U^k$

$$\begin{aligned}\int_G p^n(x)\omega(x)\, dx &\geqslant \int_{E_k} (Cn)^{-D/2} e^{-C'k^2/n} \omega(x)\, dx \\ &\geqslant (Cn)^{-D/2} e^{-C'k^2/n} \inf\{\omega(x) : x \in E_k\} |E_k|.\end{aligned}$$

Hence,

$$1 \leqslant \inf\{\omega(x) : x \in E_k\}^{1/k} \leqslant (Cn)^{D/(2k)} e^{C'k^2/(nk)} |E_k|^{-1/k} \left(\int_G p^n(x)\omega(x)\,dx \right)^{1/k}.$$

If we now choose $L > C''$ and $n = Lk$, then

$$\begin{aligned} 1 &\leqslant \limsup_{k\to\infty} \left(\inf\{\omega(x) : x \in E_k\}\right)^{1/k} \\ &\leqslant e^{C'/L} \limsup_{k\to\infty} |E_k|^{-1/k} r_{L^1_\omega(G)}(p)^L \\ &\leqslant e^{C'/L} \limsup_{k\to\infty} |E_k|^{-1/k}. \end{aligned}$$

Since $L \geqslant C''$ is arbitrary, we find that

$$1 \leqslant \limsup_{k\to\infty}(\inf\{\omega(x) : x \in E_k\})^{1/k} \leqslant \limsup_{k\to\infty}|E_k|^{-1/k}. \tag{4}$$

To show that (4) implies both the GRS-condition and condition (S), we will choose the sequence E_k appropriately.

For the GRS-condition let $x \in G$ arbitrary, $x \in U^l$, say. We set $j(k) = [(k-1)/l]$ (where $[r]$ denotes the integer part of r) and $E_k = x^{j(k)}U$. Then $E_k \subset U^k$ and $|E_k| = |U|$, and we obtain

$$\limsup_{k\to\infty}\left(\inf\left\{\omega(y) : y \in x^{j(k)}U\right\}\right)^{1/k} = 1.$$

If $y \in x^jU$, then $x^j = yu$ for some $u \in U$. Set $c = \sup_{u\in U}\omega(u) < \infty$, then

$$1 \leqslant \omega(x^j) \leqslant \sup_{u\in U}\omega(u) \inf_{y\in x^jU} \omega(y) = c\inf\{\omega(y) : y \in x^jU\}. \tag{5}$$

Since $j(k)l \sim k$ it follows that

$$1 \leqslant \lim_{j\to\infty} \omega(x^j)^{1/j} = \lim_{k\to\infty} \omega\left(x^{j(k)}\right)^{1/j(k)} \leqslant \lim_{k\to\infty} \left(\inf\left\{\omega(y) : y \in x^{j(k)}U\right\}\right)^{1/k} = 1.$$

Thus ω satisfies the GRS-condition.

For condition (S) we choose E_k as follows. For each $k \geqslant 1$ let $y_{k-1} \in U^{k-1}$ be such that $\omega(y_{k-1}) \geqslant \frac{1}{2}\sup\{\omega(y) : y \in U^{k-1}\}$ and set $E_k = y_{k-1}U$. Then $E_k \subseteq U^k$ and $|E_k| = |U|$. As above, we obtain

$$\begin{aligned} 1 &\leqslant \sup\{\omega(y) : y \in U^{k-1}\} \leqslant 2\omega(y_{k-1}) \\ &\leqslant 2\sup_{u\in U}\omega(u) \inf_{y\in y_{k-1}U} \omega(y) = 2c\inf\{\omega(y) : y \in y_{k-1}U\}. \end{aligned}$$

Taking roots and a limit, we obtain from (4)

$$\begin{aligned} 1 &\leqslant \lim_{k\to\infty} (\sup\{\omega(y) : y \in U^{k-1}\})^{1/k} \\ &\leqslant \lim_{k\to\infty} (2c)^{1/k} \lim_{k\to\infty} (\inf\{\omega(y) : y \in y_{k-1}U\})^{1/k} = 1. \end{aligned}$$

Thus ω satisfies condition (S). □

Finally, we prove the remaining implication (i) $\Longrightarrow$ (iii). For this we revisit and refine the proof of [**5**, Theorem 3.13] which was based on Losert's structure theorem [**15**, Theorem 2] and a method of Ludwig [**16**].

Definition 2.5. Let G be a locally compact group acting on the locally compact group H by automorphisms (for instance, if H is a normal subgroup of G or a quotient group of G). We say that H is an $[FC]_G^-$ group, if the G-orbits in H are relatively compact in H.

The group G acts on $L^1_\omega(G)$ by left translations, so that ${}_xf(y) = f(x^{-1}y)$ and $\|{}_xf\|_{L^1_\omega(G)} \leqslant \omega(x)\,\|f\|_{L^1_\omega(G)}$. We denote by S the bounded positive hermitian sesquilinear forms on $L^1_\omega(G)$. Then the group anti-acts on S by ${}_xB(f,g) = B({}_xf, {}_xg)$, that is, ${}_{(xy)}B = {}_y({}_xB)$. We use the following notation of [**16**]. For any subspace $F \subset L^1_\omega(G)$ and any subgroup $H \subset G$ we define the subspace $S_F^H \subset S$ by

$$S_F^H = \{B \in S|_hB = B\,\forall\, h \in H \text{ and } B(F,f) = 0\,\forall f \in L^1_\omega(G)\}.$$

Then the algebra $L^1_\omega(G)$ is symmetric if and only if $S_I^G \neq \{0\}$ for every proper modular left ideal $I \subset L^1_\omega(G)$ (same proof as in [**16**]). Since the closure of I is again a proper modular left ideal, the Hahn–Banach theorem guarantees the existence of a continuous linear functional $q \neq 0$ on $L^1_\omega(G)$ vanishing on I. Then the form $(f,g) \mapsto q(f)\overline{q(g)}$ is in $S_I = S_I^{\{e\}}$ and so

$$S_I = \{B \in S|B(I,f) = 0\;\forall\, f \in L^1_\omega(G)\} \neq \{0\}.$$

For a normal subgroup $N \subset G$ we write $\dot{g}$ for the canonical projection of $g \in G$ to G/N.

LEMMA 2.6. *Let $G \in$ [PG] and H and N be closed normal subgroups of G such that $N \subset H$ and such that H/N is $[FC]_G^-$. Given a weight ω on G, assume that the weight $(\dot{\omega_{|H}})$ defined by $(\dot{\omega_{|H}})(\dot{h}) = \inf_{n\in N} \omega_{|H}(hn)$ for $\dot{h} \in H/N$ satisfies condition (S) on H/N. Let I be a proper closed modular left ideal in $L^1_\omega(G)$ with modular right unit α. Then*

$$S_I^N \neq \{0\} \Longrightarrow S_I^H \neq \{0\}.$$

Proof. This lemma has been stated and proved in [**5**] under the assumption that ω satisfies the condition (S) on the whole group G. The only point where this is important is to ensure that a certain integral over an open subgroup V of H/N is finite. An inspection of the precise details of the proof in [**5**] shows that the lemma remains valid under the less restrictive assumption that $(\dot{\omega_{|H}})$ satisfies condition (S). □

PROPOSITION 2.7. *If ω satisfies the* GRS*-condition, then $L^1_\omega(G)$ is a symmetric Banach algebra.*

Proof. By Losert's structure theorem [**15**] the group $G \in$ [PG] contains a series of normal subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_{n-1} \supset G_n = \{e\}$$

such that G_0/G_1 and G_{n-1} are compact and G_{i-1}/G_i for $i = 1, \ldots, n-1$ are compactly generated torsion free abelian $[FC]_G^-$ groups. Given the weight ω on G satisfying the GRS-condition it is clear that the derived weights $(\dot{\omega_{|G_{i-1}}})$ (the dot is with respect to G_i) satisfy the GRS-condition on G_{i-1}/G_i too. As these quotients are either compact or abelian, it is either trivial or follows from Lemma 2.2 that these weights satisfy the condition (S).

Finally, if $I \subset L^1_\omega(G)$ is a modular ideal, we may apply Lemma 2.6 repeatedly to this normal series. Since $S_I \neq \{0\}$ we obtain $S_I^G \neq \{0\}$. Thus $L^1_\omega(G)$ is symmetric. □

3. *Condition* (S) *implies symmetry*

In this section we offer a direct proof that condition (S) implies the symmetry of $L^1_\omega(G)$.

In the following we consider a nested pair of involutive Banach $*$-algebras $\mathcal{A} \subseteq \mathcal{B}$ with common involution. We assume that $\mathcal{A}$ is continuously embedded in $\mathcal{B}$.

A nested pair $\mathcal{A} \subseteq \mathcal{B}$ with common identity is called a *Wiener pair* if $a \in \mathcal{A}$ and $a^{-1} \in \mathcal{B}$ implies that $a^{-1} \in \mathcal{A}$ [**6**]. In the recent literature, one sometimes says that $\mathcal{A}$ is *inverse closed* in $\mathcal{B}$ (see [**1**]) or that $\mathcal{A}$ is a *spectral subalgebra* of $\mathcal{B}$ (see [**18**]).

We denote the group of invertible elements in $\mathcal{A}$ by $G(\mathcal{A})$. Then $\mathcal{A}$ is inverse closed in $\mathcal{B}$ if and only if

$$G(\mathcal{A}) = G(\mathcal{B}) \cap \mathcal{A}.$$

Since invertibility in $\mathcal{A}$ and in $\mathcal{B}$ coincide, we obtain immediately the so-called spectral invariance property

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a) \qquad \forall\, a \in \mathcal{A}.$$

The following lemma is a useful tool for dealing with symmetry.

LEMMA 3.1. *Let $\mathcal{A} \subset \mathcal{B}$ be a nested pair of Banach algebras that either have a common identity element or both have no identity.*

(a) *Then the following are equivalent:*

(i) $\partial\sigma_{\mathcal{A}}(a) \subset \partial\sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$;

(ii) $\partial\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$;

(iii) $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$;

(iv) *for every* $a \in \mathcal{A}$ *with* $\|a\|_{\mathcal{B}} < 1$ *there exists* $n \in \mathbb{N}$ *(depending on* a*) such that* $\|a^n\|_{\mathcal{A}} < 1$.

(b) *If, in addition, $\mathcal{A}$ and $\mathcal{B}$ have a common involution and $\mathcal{B}$ is symmetric, then* (i)–(iv) *are equivalent to:*

(v) $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$.

In particular, in this case, if any of the conditions (i)–(iv) *is satisfied, then $\mathcal{A}$ is symmetric.*

Proof. (a) The implications (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) follow directly from the inclusion $\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$.

To prove (iii) $\Longrightarrow$ (ii) in the case that $\mathcal{A}$ and $\mathcal{B}$ have a common identity, assume that $\lambda \notin \sigma_{\mathcal{B}}(a)$, $\lambda_n \notin \sigma_{\mathcal{A}}(a)$, and $\lambda_n \to \lambda$. Since the inversion is continuous on $G(\mathcal{B})$ and $\lambda - a \in G(\mathcal{B})$, we have $\sup_{n\in\mathbb{N}} \|(\lambda_n - a)^{-1}\|_{\mathcal{B}} := C$ is finite. We write $\lambda - a = \lambda_n - a - (\lambda_n - \lambda) = (\lambda_n - a)(1 - (\lambda_n - \lambda)(\lambda_n - a)^{-1})$.

If $|\lambda_n - \lambda| < 1/2C$, then

$$r_{\mathcal{A}}((\lambda_n - \lambda)(\lambda_n - a)^{-1}) = r_{\mathcal{B}}((\lambda_n - \lambda)(\lambda_n - a)^{-1}) \leqslant \frac{1}{2C} \cdot C < 1,$$

so $\lambda - a$ is invertible in $\mathcal{A}$, and the inverse is $(\lambda - a)^{-1} = \sum_{k=0}^{\infty} (\lambda_n - \lambda)^k (\lambda_n - a)^{-k} (\lambda_n - a)^{-1}$, with absolute convergence in $\mathcal{A}$. Thus $\lambda \notin \partial\sigma_{\mathcal{A}}(a)$.

If $\mathcal{A}$ and $\mathcal{B}$ have no identity, it is not immediately clear that (iii) extends automatically to all elements $\lambda + a$ where $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$ after adjoining an identity. We therefore give a modified proof using the quasi-product $a \circ b = a+b-ab$ and the quasi-inverse a^q of a. Let λ and $\lambda_n \in \mathbb{C}$ as above, and note that $\lambda \neq 0$ since $0 \in \sigma_{\mathcal{B}}(a)$. As above, we check the identity

$$\frac{a}{\lambda} = \frac{a}{\lambda_n} \circ \left(\frac{\lambda - \lambda_n}{\lambda} \left(\frac{a}{\lambda_n} \right)^q \right).$$

Since quasi-inversion is continuous, we have $\sup_{n\in\mathbb{N}} \|(a/\lambda_n)^q\|_{\mathcal{B}} := D$ is finite. If $|\lambda_n - \lambda| < |\lambda|/2D$, we obtain

$$r_{\mathcal{A}}\left(\frac{\lambda - \lambda_n}{\lambda} \left(\frac{a}{\lambda_n} \right)^q \right) = r_{\mathcal{B}}\left(\frac{\lambda - \lambda_n}{\lambda} \left(\frac{a}{\lambda_n} \right)^q \right) \leqslant \frac{1}{2D} \cdot D < 1,$$

so $((\lambda - \lambda_n)/\lambda)(a/\lambda_n)^q$ is quasi-invertible in $\mathcal{A}$, hence a/λ is quasi-invertible in $\mathcal{A}$, because it is the quasi-product of two quasi-invertible elements. Thus $\lambda \notin \partial\sigma_{\mathcal{A}}(a)$.

(iii) $\Longrightarrow$ (iv) If $\|a\|_{\mathcal{B}} < 1$, then

$$\inf_{n\in\mathbb{N}} \|a^n\|_{\mathcal{A}}^{1/n} = \rho_{\mathcal{A}}(a) = \rho_{\mathcal{B}}(a) \leqslant \|a\|_{\mathcal{B}} < 1\,,$$

and (iv) follows.

(iv) $\Longrightarrow$ (iii) For $a \in \mathcal{A}$ and $\epsilon > 0$ there is a $k \in \mathbb{N}$ such that $\|a^k\|_{\mathcal{B}} < (\rho_{\mathcal{B}}(a)+\epsilon)^k$. Set $a_\epsilon = (a/(\rho_{\mathcal{B}}(a)+\epsilon))^k$, then $\|a_\epsilon\|_{\mathcal{B}} < 1$, and by assumption we have $\|a_\epsilon^n\|_{\mathcal{A}} < 1$ for some $n \in \mathbb{N}$. This implies that $\|a^{kn}\|_{\mathcal{A}} < (\rho_{\mathcal{B}}(a)+\epsilon)^{kn}$, and by taking roots we obtain $\rho_{\mathcal{A}}(a) \leqslant \rho_{\mathcal{B}}(a) + \epsilon$. The claim follows.

(b) We prove the implication (ii) $\Longrightarrow$ (v). First assume that $a = a^* \in \mathcal{A}$. Since $\mathcal{B}$ is symmetric, we have $\sigma_{\mathcal{B}}(a) \subset \mathbb{R}$. Consequently (ii) implies that $\sigma_{\mathcal{A}}(a) = \partial\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a) \subset \mathbb{R}$ and therefore $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.

For arbitrary elements $a \in \mathcal{A}$ we observe that $\lambda - a$ is invertible if and only if the self-adjoint elements $(\lambda - a)^*(\lambda - a)$ and $(\lambda - a)(\lambda - a)^*$ are both invertible. Thus $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ for all $a \in \mathcal{A}$.

The implications (v) $\Longrightarrow$ (i), (ii), (iii) are trivial. □

Remark 3.2. (1) The equivalence (ii) $\Longleftrightarrow$ (iii) follows also from [**17**, Corollary 2.5.10].

(2) For the case that $\mathcal{B}$ is the C*-algebra of bounded operators on a Hilbert space, the implication (iii) $\Longrightarrow$ (v) is usually called Hulanicki's lemma [**12**] and is crucial in many cases for the verification of the symmetry of a Banach $*$-algebra.

(3) Since conditions (i) and (ii) carry over to the algebras with an identity adjoined $\mathcal{A}_1$ and $\mathcal{B}_1$, so do (iii) and (iv) by part (a) of the lemma.

(4) If $\mathcal{A} \subseteq \mathcal{B}$ are involutive Banach algebras with common involution and $\mathcal{B}$ is symmetric, such that one of them has an identity which is not an identity for the other, then each of (i)–(iv) implies that $\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{B}}(a) \cup \{0\}$, for all $a \in \mathcal{A}$. (Apply the lemma to $\mathcal{A}_1$ and $\mathcal{B}_1$.)

As a preliminary result we need the symmetry of weighted $\ell^1_v(\mathbb{Z})$ which was characterized by Gelfand, Raikov and Shilov [**6**].

LEMMA 3.3. *Assume that v is a weight on $\mathbb{Z}$ satisfying the GRS-condition. If $b = (b_j)_{j\in\mathbb{Z}} \in \ell^1_v(\mathbb{Z})$ and $b_j \geqslant 0$, then*

$$r_{\ell^1_v(\mathbb{Z})}(b) = r_{\ell^1(\mathbb{Z})}(b) = \|b\|_1; \tag{6}$$

that is,

$$\lim_{n\to\infty}\left(\sum_{j_1,j_2,\ldots,j_n=-\infty}^{\infty} b_{j_1}b_{j_2}\ldots b_{j_n}\, v(j_1+j_2+\ldots+j_n)\right)^{1/n} = \lim_{n\to\infty}\left(\sum_{j_1,j_2,\ldots,j_n=-\infty}^{\infty} b_{j_1}b_{j_2}\ldots b_{j_n}\right)^{1/n} = \|b\|_1. \tag{7}$$

Proof. To provide some intuition about the meaning of the GRS-condition, we provide a sketch of the proof.

Since the Banach algebra $\ell^1_v(\mathbb{Z})$ is generated by the element δ_1 (where $\delta_1(1) = 1$ and $\delta_1(j) = 0$ for $j \neq 1$) and its inverse, any character (continuous multiplicative functional) of $\ell^1_v(\mathbb{Z})$ is completely determined by its value $z = \chi(\delta_1) \in \mathbb{C}$. Thus we may identify the Gelfand spectrum of $\ell^1_v(\mathbb{Z})$ with a compact subset K_v of $\mathbb{C}$ by writing χ_z for the character satisfying $\chi_z(\delta_1) = z$.

By Gelfand's theorem

$$r_{\ell^1_v(\mathbb{Z})}(b) = \sup\{|\chi_z(b)| : z \in K_v\}, \qquad \forall\, b \in \ell^1_v(\mathbb{Z}). \tag{8}$$

Given $z \in K_v$ we have for all $n \in \mathbb{Z}$

$$v(n) = \left\|\delta_1^n\right\|_{\ell^1_v(\mathbb{Z})} \geqslant |\chi_z(\delta_1^n)| = |z|^n.$$

Since $\lim_{n\to\infty} v(n)^{1/n} = 1$ by assumption, we must have $|z| \leqslant 1$. Likewise, $\lim_{n\to\infty} v(-n)^{1/n} = 1$ implies that $|z| \geqslant 1$. Consequently $K_v \subseteq \mathbb{T}$. Conversely $\mathbb{T} \subseteq K_v$ because every character of $\ell^1(\mathbb{Z})$ restricts to a character of $\ell^1_v(\mathbb{Z})$. Now (8) implies that $r_{\ell^1_v(\mathbb{Z})}(b) = r_{\ell^1(\mathbb{Z})}(b)$ for all $b \in \ell^1_v(\mathbb{Z})$.

If $b_j \geqslant 0$, then clearly $r_{\ell^1(\mathbb{Z})}(b) = \|b\|_1$ and so (6) holds. □

We can now prove the following theorem.

THEOREM 3.4. *Assume that $G \in [\mathrm{PG}]$. If ω satisfies condition (S), then $L^1_\omega(G)$ is symmetric, and*

$$\sigma_{L^1_\omega(G)}(f) = \sigma(L_f) \qquad \forall\, f \in L^1_\omega(G),$$

where L_f is the regular representation of $L^1(G)$ on $L^2(G)$.

Proof. In view of Lemma 3.1, we first show the identity of spectral radii $r_{L^1_\omega(G)}(f) = r_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$. We first calculate the $L^1_\omega(G)$-norm of $f^n = f * f * \cdots * f$ by induction and find that

$$\|f^n\|_{L^1_\omega(G)} \leqslant \int_G \cdots \int_G |f(x_1)|\,|f(x_2)| \ldots |f(x_n)|\,\omega(x_1\ldots x_n)\,dx_1\,dx_2\ldots dx_n. \tag{9}$$

Fix a generating neighbourhood of the identity U and assign a weight v on $\mathbb{Z}$ to the weight ω on G by setting

$$v(n) = \sup_{y\in U^{|n|}} \omega(y).$$

By definition, if ω satisfies condition (S) on G, then v satisfies the GRS-condition on $\mathbb{Z}$.

Since $G = \bigcup_{n=1}^{\infty} (U^n \setminus U^{n-1})$ as a disjoint union (where $U^0 = \emptyset$), we may split each integral in (9) as $\int_G = \sum_{n=1}^{\infty} \int_{U^n \setminus U^{n-1}}$. This yields

$$\|f^n\|_{L^1_\omega(G)} \leqslant \sum_{k_1,k_2,\ldots,k_n=1}^{\infty} \int_{U^{k_1}\setminus U^{k_1-1}} \cdots \int_{U^{k_n}\setminus U^{k_n-1}} |f(x_1)|\ldots|f(x_n)|\omega(x_1\ldots x_n)\,dx_1\ldots dx_n. \tag{10}$$

If $x_j \in U^{k_j} \setminus U^{k_j-1}$, then $x_1 \ldots x_n \in U^{k_1+\cdots+k_n}$ and so

$$\omega(x_1\ldots x_n) \leqslant \sup_{y\in U^{k_1+\cdots+k_n}} \omega(y) = v(k_1+\ldots+k_n).$$

Set $b_k := \int_{U^k\setminus U^{k-1}} |f(x)|\,dx$ and $b = (b_k)_{k\in\mathbb{N}}$. Then clearly we have $\|f\|_{L^1} = \|b\|_1$, and we can recast (10) as

$$\|f^n\|_{L^1_\omega(G)} \leqslant \sum_{k_1,k_2,\ldots,k_n=1}^{\infty} b_{k_1} b_{k_2} \ldots b_{k_n} v(k_1+k_2+\ldots+k_n) = \|b^n\|_{\ell^1_v}.$$

We now apply Lemma 3.3 and see that

$$\begin{aligned} r_{L^1_\omega(G)}(f) &= \lim_{n\to\infty} \|f^n\|_{L^1_\omega(G)}^{1/n} \\ &\leqslant \lim_{n\to\infty} \|b^n\|_{\ell^1_v}^{1/n} = r_{\ell^1_v}(b) = r_{\ell^1}(b) = \|b\|_1 \\ &= \|f\|_{L^1}. \end{aligned}$$

Thus for all $k \in \mathbb{N}$ we have

$$r_{L^1_\omega(G)}(f) = r_{L^1_\omega(G)}(f^k)^{1/k} \leqslant \|f^k\|_{L^1}^{1/k},$$

and by letting $k \to \infty$ we obtain

$$r_{L^1_\omega(G)}(f) \leqslant r_{L^1(G)}(f). \tag{11}$$

The converse inequality is elementary, and so we have proved that $r_{L^1_\omega(G)}(f) = r_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$.

By Lemma 3.1 we have $\sigma_{L^1_\omega(G)}(f) = \sigma_{L^1(G)}(f)$, and since $L^1(G)$ is symmetric by Losert's Theorem 1.1, we conclude that $L^1_\omega(G)$ is also symmetric. Since G is amenable and $L^1(G)$ is symmetric, we also have $\sigma_{L^1_\omega(G)}(f) = \sigma(L_f)$ for all $f \in L^1_\omega(G)$ (see [**2**, **7**]). □

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