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THE GRS-CONDITION AND SYMMETRY OF WEIGHTED L^1-ALGEBRAS

MICHAEL LEINERT
Institut für Angewandte Mathematik
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 294
D-69120 Heidelberg, Germany
E-mail: leinert@math.uni-heidelberg.de

This note is about joint work with Gero Fendler and Karlheinz Gröchenig. Complete details will be given in [FGL]. Symmetry of Bannach $*$-algebras is of interest for several reasons, one of them being its relation to inverse-closedness in a bigger algebra. This for instance has concrete applications in signal analysis and numerical analysis (see [GL1], [GL2], [S]). In the applied context, weighted L^1-algebras are important, since the weight allows one to model a specific decay at infinity according to what one wants or needs.

Looking at the case of the trivial weight $\omega \equiv 1$ first, one sees that for groups with exponential growth the L^1-algebra "usually" is non-symmetric. For groups with polynomial growth, non-symmetry can happen, too (see [FRW]), but for compactly generated groups of polynomial growth, using his structure theorem and [Lu], Losert [Lo] has shown that the L^1-algebra is symmetric. So it seems reasonable to consider weighted L^1-algebras on such groups.

A locally compact group G is called *compactly generated* if there is a compact neighbourhood U of the identity e with $G = \bigcup_{k=1}^{\infty} U^k$. We then call U a *generating neighbourhood.* We say that a compactly generated group G has (at most) *polynomial growth* if there is a generating neighbourhood U and constants $C > 0$ and $D \in \mathbb{N}$ such that $|U^k| \leq Ck^D$ for all $k \in \mathbb{N}$. Here and in the following, $|U|$ denotes the Haar measure of U. The L^1-algebra of the group G with the usual norm $\|f\|_1 = \int |f(x)|\, dx$, where dx denotes the Haar measure on G, and the usual $*$-algebra structure is denoted by $L^1(G)$. By a *weight* on G we mean a locally bounded measurable function $\omega : G \to \mathbb{R}^+$ satisfying $\omega(e) \geq 1$, where e is the identity element of G, $\omega(x^{-1}) = \omega(x)$, and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. The

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corresponding weighted L^1-algebra $L^1_\omega(G) = \{f \in L^1(G) \mid \int |f(x)|\omega(x)\,dx < \infty\}$ with norm $\|f\|_{1,\omega} = \int |f(x)|\,\omega(x)\,dx$ is an (actually dense) Banach $*$-subalgebra of $L^1(G)$. The weight ω is said to satisfy the *Gelfand-Raikov-Shilov condition (GRS-condition)* if $\omega(x^n)^{\frac{1}{n}} \to 1$ for every $x \in G$. We say that ω satisfies *condition* (**S**) if $\sup_{y\in U^n} \omega(y)^{\frac{1}{n}} \to 1$ for some generating neighbourhood U of G. Here $U^n = \{x_1x_2\ldots x_n \,\big|\, x_1,\ldots,x_n \in U\}$. If condition (S) holds with respect to some generating neighbourhood U, it holds for every generating neighbourhood of G. Since every $x \in G$ is contained in some generating neighbourhood U, condition (S) implies the GRS-condition. For explicit examples of weights satisfying these conditions see [FGLLM].

If A is a Banach $*$-algebra, for $a \in A$ we denote by $\sigma_A(a)$ and $r_A(a)$ the spectrum and the spectral radius of a, respectively. A is called symmetric, if $\sigma_A(a^*a) \subset [0,\infty)$ for all $a \in A$. This is equivalent to saying that $\sigma_A(a) \subset \mathbb{R}$ for all $a = a^* \in A$ (see [SF]).

THEOREM. *Let G be a locally compact, compactly generated group of polynomial growth and ω a weight on G. The following conditions are equivalent:*

(i) *ω satisfies the GRS-condition.*
(ii) *ω satisfies condition (S).*
(iii) *$L^1_\omega(G)$ is symmetric.*
(iv) *$\sigma_{L^1_\omega(G)}(f) = \sigma_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$.*

In [FGLLM], this was proved under the somewhat awkward technical assumption that the weight ω be "tempered". But this condition can be dispensed with, as we shall see.

LEMMA 1. *Let ω be a weight on $\mathbb{R}$. If $\omega(n)^{\frac{1}{n}} \to 0$, then ω satisfies condition (S).*

Proof. Assuming $\omega(n)^{\frac{1}{n}} \to 0$, from the fact that the logarithm of ω is subadditive, one can derive $\lim_{t\to\infty} t^{-1}\log\omega(t) = 0$. To check condition (S) using the generating neighbourhood $[-1,1]$, choose $x_k \in [0,k]$ such that $|\sup_{x\in[-k,k]} \omega(x)^{1/k} - \omega(x_k)^{1/k}| < \frac{1}{k}$. Since $1 \le \limsup_{k\to\infty} \omega(x_k)^{\frac{1}{k}} \le \limsup_{k\to\infty} \omega(x_k)^{\frac{1}{x_k}} = 1$, we obtain that condition (S) is satisfied. ∎

LEMMA 2. *If ω is a weight on a compactly generated locally compact abelian group G, condition (S) and the GRS-condition are equivalent.*

Proof. Since G is of the form $\mathbb{R}^d \times \mathbb{Z}^\ell \times K$, where K is a compact group, and since ω is dominated by the product of its restrictions to each coordinate, it suffices to consider weights on $\mathbb{R}, \mathbb{Z}$, and K. Assuming the GRS-conditon, we obtain condition (S) on $\mathbb{R}$ by Lemma 1. A similar argument works for $\mathbb{Z}$. Since ω is bounded on K, condition (S) holds on K, too. ∎

If G is a compactly generated locally compact group, from results of Losert [Lo] and Guivarc'h [G] one can derive that G has *strict polynomial growth*, i.e. there are a symmetric generating neighbourhood U of G and constants $C_1, C_2, D > 0$ such that $C_1k^D \le |U^k| \le C_2k^D$ for all $k \in \mathbb{N}$. This allows one to use a result of Hebisch and Saloff-Coste [HS] on pointwise Gaussian estimates for convolution powers of probability densities. We need this for

PROPOSITION 3. *If $\sigma_{L^1_\omega(G)}(f) = \sigma_{L^1(G)}(f)$ for all $f \in L^1_\omega(G)$, then ω satisfies the GRS-condition and even condition (S).*

For the proof of this, one makes use of the above-mentioned estimate for a continuous symmetric probability density with compact support. This yields an inequality also involving an integral over U^k, where U is a generating neighbourhood. The point now is to estimate this integral from below by the integral over a carefully chosen subset E_k of U^k. The rest then follows by suitable estimates and limit operations.

Sketch of the proof of the theorem:
(iii) $\Leftrightarrow$ (iv): see [FGLLM], Theorem 3.6 and Lemma 3.8.
(iv) $\Rightarrow$ (ii) is Proposition 3.
(ii) $\Rightarrow$ (i) has been noted after the definition of condition (S).
(i) $\Rightarrow$ (iii): The general line of proof is as in the proof of [FGLLM], Theorem 13. In Lemma 3.12 of [FGLLM] replace the assumption that the weight ω on G satisfies condition (S) by the weaker one that the quotient weight $\dot{\omega}_{|_H}$ on H/N satisfies condition (S). Then the proof given there is still valid. In the proof of the theorem, this Lemma is applied repeatedly to pairs of subgroups $G_{i-1} \supset G_i$ (the G_j coming from Losert's structure theorem [Lo]). In this situation, the quotient weight $\dot{\omega}_{|_{G_{i-1}}}$ clearly satisfies the GRS-condition, so by Lemma 2, condition (S) is satisfied on G_{i-1}/G_i. So everything is fine, and (iii) follows.

REMARK 1. A few more equivalent conditions can be added to the theorem (see [FGLLM]). Let us mention just two of them:

(v) $r_{L^1_\omega(G)}(f) = r_{B(H)}(Lf)$ for all $f \in L^1_\omega(G)$
(vi) $\sigma_{L^1_\omega(G)}(f) = \sigma_{B(H)}(Lf)$ for all $f \in L^1_\omega(G)$.

Here, $L : f \mapsto Lf$ denotes the left regular representation of $L^1_\omega(G)$ on $H = L^2(G)$ and $B(H)$ is the algebra of all bounded linear operators on H.

It would be sufficient, to ask (v) and (vi) for selfadjoint elements $f \in L^1_\omega(G)$ (see [FGLLM]). Condition (v) then reads:

(vii) $r_{L^1_\omega(G)}(f) = \|Lf\|_{B(H)}$ for all $f = f^* \in L^1_\omega(G)$.

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