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Inverse-closed Banach subalgebras of higher-dimensional non-commutative tori

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Abstract. We give a systematic construction of inverse-closed (Banach) subalgebras in general higher-dimensional non-commutative tori.

1. Introduction. Let $\mathcal{A} \subseteq \mathcal{B}$ be two algebras with common identity. Then $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}$ if $a \in \mathcal{A}$ and $a^{-1} \in \mathcal{B}$ implies that $a^{-1} \in \mathcal{A}$. This property is a generalization of Wiener's Lemma for absolutely convergent Fourier series and occurs abundantly in many branches of mathematical analysis. The range of applications covers numerical analysis, pseudodifferential operators, frame theory, and, last but not least, non-commutative tori. See [8] for a survey of many versions of Wiener's Lemma and applications of inverse-closedness.

In this paper we study subalgebras of general non-commutative tori. Non-commutative tori are the founding examples of non-commutative geometry [4, 20], and are defined as the universal C^*-algebras generated by a finite number of unitary elements U_j with commutation relations of the form $U_jU_k = \theta_{jk}U_kU_j$ for $j, k = 1, \dots, n$, and $\theta_{jk} \in \mathbb{C}$. In non-commutative geometry inverse-closed subalgebras of non-commutative tori play an important role: on the one hand, as "smooth non-commutative manifolds", and on the other hand, in the K-theory of C^*-algebras. Perhaps the main result concerning inverse-closed subalgebras of non-commutative tori is the density theorem. It states that the K-groups of a non-commutative torus and of all its dense, inverse-closed subalgebras are isomorphic. Similarly the stable rank of a dense, inverse-closed subalgebra coincides with the stable rank of the ambient algebra [2].

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Usually the existence of an inverse-closed subalgebra is taken for granted and is the starting point for the theory. Also, mostly Fréchet subalgebras are considered rather than Banach subalgebras (because Fréchet algebras model "smooth" non-commutative tori). Our objective is the *systematic* construction of Banach subalgebras of non-commutative tori in higher dimensions. Indeed, we will characterize all those inverse-closed Banach subalgebras of the form $\ell^1_v(\mathbb{Z}^n)$, where v is a weight function on $\mathbb{Z}^n$. By choosing weights of subexponential growth, we even construct a Banach subalgebra that is contained in the ordinary smooth non-commutative torus. For certain non-commutative tori with an even number of generators these results were already obtained in [9]. An alternative proof for the case of two generators was subsequently given in [21]. The extension of our results was motivated by a question of N. C. Phillips who asked us whether the results in [9] also hold for arbitrary non-commutative tori in higher dimensions.

In the last part we investigate briefly the simplicity of the Banach subalgebras of non-commutative tori in terms of the parameters that define the commutation relations. As a consequence we obtain a new proof of the well-known characterization of the simplicity of the non-commutative tori.

Our methods are drawn from abstract harmonic analysis, in particular the investigation of projective representations and twisted convolution algebras in the school of Leptin and Ludwig.

Let us mention that in some areas an inverse-closed subalgebra is also called a spectral subalgebra, a local subalgebra, or a full algebra. If $\mathcal{A}$ is inverse-closed in $\mathcal{B}$, then $\mathcal{A}$ is called spectrally invariant in $\mathcal{B}$ or (under standard conditions) invariant under holomorphic calculus; $(\mathcal{A},\mathcal{B})$ is called a Wiener pair.

2. Higher-dimensional non-commutative tori. We first give a description of non-commutative tori in higher dimensions and explain the link to harmonic analysis.

Let $\mathbb{T}$ denote the unit circle. Let $U_1,\dots,U_n$ be unitary symbols satisfying the commutation relations

$$U_jU_k = \theta_{jk}U_kU_j,$$

where $\theta_{jk}\in\mathbb{T}$. Since $U_jU_k=\theta_{jk}U_kU_j=\theta_{jk}\theta_{kj}U_jU_k$, we have $\theta_{kj}=\overline{\theta_{jk}}$ and thus the matrix $\theta=(\theta_{jk})_{j,k=1,\dots,n}$ is hermitean.

The non-commutative torus $C^*(\theta)$ is the universal C^*-algebra generated by the unitaries U_j, $j=1,\dots,n$. To obtain a concrete and workable representation, we interpret $C^*(\theta)$ as a twisted group C^*-algebra of $\mathbb{Z}^n$.

Using multi-index notation with $U^l=U_1^{l_1}\cdots U_n^{l_n}$ for $l\in\mathbb{Z}^n$, we get

$$U^lU^m=\sigma(l,m)U^{l+m}\qquad\text{for } l,m\in\mathbb{Z}^n, \tag{2.1}$$

where $\sigma(l,m) \in \mathbb{T}$. In fact, repeated application of the commutation rules yields the expression

(2.2)
$$\sigma(l,m) = \Big(\prod_{j=1}^{n-1} \theta_{n,j}^{m_j}\Big)^{l_n} \Big(\prod_{j=1}^{n-2} \theta_{n-1,j}^{m_j}\Big)^{l_{n-1}} \cdots \Big(\prod_{j=1}^{1} \theta_{2,j}^{m_j}\Big)^{l_2} = \prod_{1\le j<k\le n} \theta_{k,j}^{l_k m_j}.$$

Since $U^0 = U_1^0 \cdots U_n^0 = \mathrm{I}$, (2.1) implies $\sigma(0,m) = \sigma(m,0) = 1$, which is consistent with (2.2). We also have $\sigma(-l,m) = \sigma(l,-m) = \overline{\sigma(l,m)}$.

Since we require the multiplication to be associative, we have

$$\sigma(l,m)\sigma(l+m,p) = \sigma(l,m+p)\sigma(m,p) \qquad \text{for } l,m,p \in \mathbb{Z}^n. \tag{2.3}$$

For $f,\, g \in \ell^1(\mathbb{Z}^n)$ we define the twisted convolution $f \natural_\theta g$ or simply $f \natural g$ by

$$f \natural_\theta g(x) = \sum_{y\in\mathbb{Z}^n} f(y)g(x-y)\,\sigma(y,x-y), \qquad x \in \mathbb{Z}^n.$$

The involution $f \mapsto f^*$ is defined by $f^*(x) = \overline{\sigma(x,-x)}\,\overline{f(-x)}$ for $x \in \mathbb{Z}^n$. For the special case of "Dirac" functions $\delta_y = \chi_{\{y\}}$ we have

$$\delta_y \natural \delta_z = \sigma(y,z)\delta_{y+z} \quad \text{and} \quad \delta_y^* = \overline{\sigma(-y,y)}\delta_{-y}, \qquad y,z \in \mathbb{Z}^n. \tag{2.4}$$

We also note that

$$\delta_y \natural \delta_y^* = \sigma(y,-y)\overline{\sigma(-y,y)}\delta_0 = \delta_0$$

and $\delta_y^* \natural \delta_y = \overline{\sigma(-y,y)}\sigma(-y,y)\delta_0 = \delta_0$, so δ_y is unitary for every $y \in \mathbb{Z}^n$.

Then $(\ell^1(\mathbb{Z}^n), \natural_\theta, {}^*)$ is a Banach $*$-algebra, which we denote by $\ell^1(\mathbb{Z}^n,\theta)$. This fact can be checked directly, but it also follows from the reasoning below.

Following [22] and [16], we define a central extension G of $\mathbb{T}$ by $\mathbb{Z}^n$ as follows. Let $G = \{(x,\xi) : x \in \mathbb{Z}^n,\ \xi \in \mathbb{T}\}$ with multiplication $(x,\xi)(y,\eta) = (x+y, \sigma(x,y)\xi\eta)$. Then G is a nilpotent group with neutral element $e = (0,1)$ and inverse $(x,\xi)^{-1} = (-x, \overline{\sigma(x,-x)\xi})$. The Haar measure on G is $\int_G f(a)\,da = \sum_{x\in\mathbb{Z}^n}\int_{\mathbb{T}} f(x,\xi)\,d\xi$, and the group convolution $\star$ on G is defined with respect to this measure.

For $f \in \ell^1(\mathbb{Z}^n)$ we define $f^\circ \in L^1(G)$ by $f^\circ(x,\xi) = f(x)\overline{\xi}$. This extension satisfies the following properties.

LEMMA 2.1. *The mapping* $\circ : \ell^1(\mathbb{Z}^n) \to L^1(G)$ *is an isometric $*$-homomorphism from* $\ell^1(\mathbb{Z}^n,\theta)$ *into* $L^1(G)$.

Proof. We have

$$\|f^\circ\|_1 = \int_G |f^\circ(a)|\,da = \sum_{x\in\mathbb{Z}^n}\int_{\mathbb{T}} |f(x)\overline{\xi}|\,d\xi = \sum_{x\in\mathbb{Z}^n} |f(x)| = \|f\|_1,$$

so $f \mapsto f^\circ$ is an isometry. This map is compatible with the involution, since

$$\begin{aligned}(f^*)^\circ(x,\xi) &= f^*(x)\bar{\xi} = \overline{\sigma(x,-x)f(-x)}\xi \\ &= \overline{f^\circ(-x,\overline{\sigma(x,-x)\xi})} = \overline{f^\circ((x,\xi)^{-1})} = (f^\circ)^*(x,\xi).\end{aligned}$$

For the homomorphism property we first write

$$(f \natural g)^\circ(x,\xi) = (f \natural g)(x)\bar{\xi} = \sum_{y\in\mathbb{Z}^n} f(y)g(x-y)\sigma(y,x-y)\bar{\xi}.$$

On the other hand, from

$$(y,\eta)^{-1}(x,\xi) = (-y,\overline{\sigma(y,-y)\eta})(x,\xi) = (x-y,\xi\overline{\eta\sigma(y,-y)}\sigma(-y,x))$$

we obtain

$$\begin{aligned}(f^\circ \star g^\circ)(x,\xi) &= \sum_y \int_{\mathbb{T}} f^\circ(y,\eta)g^\circ((y,\eta)^{-1}(x,\xi))\,d\eta \\ &= \sum_y \int_{\mathbb{T}} f(y)\bar{\eta}g(x-y)\bar{\xi}\eta\sigma(y,-y)\overline{\sigma(-y,x)}\,d\eta.\end{aligned}$$

Using (2.3) with $(l,m,p) = (y,-y,x)$ and $\sigma(0,x) = 1$, we have

$$\sigma(y,-y)\sigma(0,x) = \sigma(y,-y+x)\sigma(-y,x),$$

or $\sigma(y,-y)\overline{\sigma(-y,x)} = \sigma(y,x-y)$. Comparing the formulas, we see that

$$(f \natural g)^\circ = f^\circ \star g^\circ. \blacksquare$$

We may therefore think of $\ell^1(\mathbb{Z}^n,\theta)$ as a closed $*$-subalgebra of $L^1(G)$. In particular, it is a Banach $*$-algebra. Its enveloping C^*-algebra is the non-commutative torus $C^*(\theta)$.

To obtain a concrete realization of $C^*(\theta)$, we consider the regular representation λ of $\ell^1(\mathbb{Z}^n,\theta)$ on $\ell^2(\mathbb{Z})$ defined by

$$\lambda(f)g = f \natural_\theta g \quad \text{for } f \in \ell^1(\mathbb{Z}^n),\ g \in \ell^2(\mathbb{Z}^n).$$

This representation is faithful, and the closure of $\lambda(\ell^1)$ with respect to the operator norm is a C^*-algebra $\mathcal{C}$. By a special case of [12, Satz 6], $C^*(\theta)$ is isometrically isomorphic to $\mathcal{C}$. From now on, we will therefore not distinguish between the abstract algebra $C^*(\theta)$ and its concrete realization $\mathcal{C}$.

3. Inverse-closed subalgebras of $C^*(\theta)$. Next we construct a family of inverse-closed Banach subalgebras of the non-commutative torus $C^*(\theta)$. This construction relies on two important results in Banach algebra theory and abstract harmonic analysis.

First recall that a Banach $*$-algebra $\mathcal{A}$ is *symmetric* if the spectrum of every positive element is positive, i.e., $\sigma(a^*a) \subseteq [0,\infty)$ for all $a \in \mathcal{A}$. The connection between symmetry and inverse-closedness is folklore and implicit in many proofs of symmetry [10, 11, 13, 14]. The following proposition is contained in Palmer's book [17, Thm. 11.4.1]. (Since the regular representation of $\ell^1(\mathbb{Z}^n,\theta)$ is faithful and $\ell^1(\mathbb{Z}^n,\theta)$ is semisimple, we may quote a formulation that is already adapted to semisimple Banach algebras.)

PROPOSITION 3.1. *A unital semisimple Banach $*$-algebra $\mathcal{A}$ is symmetric if and only if it is inverse-closed in its enveloping C^*-algebra.*

Our second ingredient is a fundamental result of Ludwig [14].

PROPOSITION 3.2. *If G is a nilpotent group, then $L^1(G)$ is symmetric.*

By combining the explicit construction of non-commutative tori with these results, we obtain a fundamental inverse-closed subalgebra of $C^*(\theta)$.

THEOREM 3.3. *The Banach $*$-algebra $\ell^1(\mathbb{Z}^n,\theta)$ is inverse-closed in $C^*(\theta)$.*

Proof. By construction, the central extension G of $\mathbb{Z}^n$ is nilpotent, and consequently $L^1(G)$ is symmetric by Ludwig's result. Lemma 2.1 identifies $\ell^1(\mathbb{Z}^n,\theta)$ with a closed $*$-subalgebra of $L^1(G)$, and thus $\ell^1(\mathbb{Z}^n,\theta)$ is also a symmetric Banach $*$-algebra. By Proposition 3.1 this means that $\ell^1(\mathbb{Z}^n,\theta)$ is inverse-closed in $C^*(\theta)$, as claimed. ∎

REMARK 3.4. For even dimension and a special representation of the generators of $C^*(\theta)$ by phase-space shifts, Theorem 3.3 was proved in [9] when solving a problem in time-frequency analysis. An earlier result is contained in [1]. See also [15] and [7, Ch. 13] for connections with time-frequency analysis.

For the special case of two generators and irrational θ an elegant alternative proof of Theorem 3.3 was obtained by Rosenberg [21]. Whereas our approach yields the symmetry by identifying $\ell^1(\mathbb{Z},\theta)$ with a closed $*$-subalgebra of a symmetric algebra (given by L^1 of a nilpotent group), [21] uses the fact that $\ell^1(\mathbb{Z},\theta)$ can be interpreted as a $*$-quotient of a symmetric algebra. Undoubtedly, Rosenberg's proof can also be generalized to arbitrary non-commutative tori, but we found the approach in [9] more accessible.

To generate more examples of inverse-closed subalgebras of $C^*(\theta)$, we introduce weighted ℓ^1-algebras.

Let v be a submultiplicative and symmetric weight function on $\mathbb{Z}^n$, i.e., v satisfies the conditions

$$v(x+y) \le v(x)v(y) \quad \text{and} \quad v(-x) = v(x) \quad \text{for all } x, y \in \mathbb{Z}^n,$$

and let $\ell^1_v(\mathbb{Z}^n)$ be the corresponding weighted ℓ^1-space with norm $\|f\|_{\ell^1_v} = \|fv\|_1$. The pointwise inequality $|(f \natural_\theta g)(x)| \le (|f| * |g|)(x)$ for all $x \in \mathbb{Z}^n$ shows that $(\ell^1_v(\mathbb{Z}^n), \natural_\theta)$ is a Banach algebra, which we denote $\ell^1_v(\mathbb{Z}^n,\theta)$. Since v is symmetric, $\ell^1_v(\mathbb{Z}^n,\theta)$ is a $*$-subalgebra of $\ell^1(\mathbb{Z}^n,\theta)$.

The next proposition characterizes those submultiplicative symmetric weights for which $\ell^1_v(\mathbb{Z}^n, \theta)$ is inverse-closed in $C^*(\theta)$.

PROPOSITION 3.5. *The Banach algebra $\ell^1_v(\mathbb{Z}^n, \theta)$ is inverse-closed in $C^*(\theta)$ if and only if v satisfies the Gelfand–Raikov–Shilov condition (GRS-condition)*

$$\lim_{m\to\infty} v(mx)^{1/m} = 1 \quad \textit{for all } x \in \mathbb{Z}^n.$$

Proof. Assume first that v satisfies the GRS-condition. Then we may extend v to a weight on G by setting $\omega(x,\xi) = v(x)$ for all $x \in \mathbb{Z}^n$, $\xi \in \mathbb{T}$. The extended weight ω satisfies the GRS-condition on G, so the weighted version of Ludwig's Theorem, as proved in [6, Theorems 1.3 and 3.4], implies that $L^1_\omega(G)$ is symmetric. Since obviously $\|f^\circ\|_{L^1_\omega(G)} = \|f\|_{\ell^1_v}$, Lemma 2.1 shows that $\ell^1_v(\mathbb{Z}^n, \theta)$ can be identified with a closed subalgebra of $L^1_\omega(G)$ and thus is also symmetric. Consequently, by Proposition 3.1, $\ell^1_v(\mathbb{Z}^n, \theta)$ is inverse-closed in its enveloping C^*-algebra. To see that this C^*-algebra is $C^*(\theta)$, it suffices to note that $\ell^1_v(\mathbb{Z}^n, \theta)$ is dense in $\ell^1(\mathbb{Z}^n, \theta)$ and every $*$-representation π of $\ell^1_v(\mathbb{Z}^n, \theta)$ on a Hilbert space can be extended to $\ell^1(\mathbb{Z}^n, \theta)$. The latter follows from the fact that π is completely determined by the $\pi(\delta_x)$, $x \in \mathbb{Z}^n$, and those operators are unitary, so $\overline{\pi}(f) = \sum_{x\in\mathbb{Z}^n} f(x)\pi(\delta_x)$, $f \in \ell^1(\mathbb{Z}^n)$, is the desired extension of π to a $*$-representation of $\ell^1(\mathbb{Z}^n, \theta)$.

Conversely, assume that v violates the GRS-condition. This means that there exists an $x \in \mathbb{Z}^n$ such that $\lim_{m\to\infty} v(mx)^{1/m} > 1$. Since by (2.4) the mth power of δ_x is of the form $c_m\delta_{mx}$ with $|c_m| = 1$, the spectral radius of δ_x in $\ell^1_v(\mathbb{Z}^n, \theta)$ is

$$r_{\ell^1_v(\mathbb{Z}^n,\theta)}(\delta_x) = \lim_{m\to\infty} \|c_m\delta_{mx}\|^{1/m}_{\ell^1_v(\mathbb{Z}^n,\theta)} = \lim_{m\to\infty} v(mx)^{1/m} > 1.$$

On the other hand, since δ_x is unitary in $\ell^1_v(\mathbb{Z}^n, \theta)$, it is also unitary in $C^*(\theta)$. Consequently, the spectral radius of δ_x in $C^*(\theta)$ is 1. Therefore the spectrum of δ_x in $\ell^1_v(\mathbb{Z}^n, \theta)$ cannot be equal to the spectrum of δ_x in $C^*(\theta)$, and so $\ell^1_v(\mathbb{Z}^n, \theta)$ is not inverse-closed in $C^*(\theta)$. ∎

REMARK 3.6. A non-spectral subalgebra of the irrational rotation algebra (the non-commutative torus with two generators) and its simplicity were first discussed by Schweitzer [23].

Proposition 3.5 provides an abundance of examples of inverse-closed Banach subalgebras of a non-commutative torus in higher dimensions. By taking intersections of weighted ℓ^1-algebras, one may now construct inverse-closed Fréchet subalgebras of $C^*(\theta)$. In particular, fix $v(x) = 1 + |x|$ for some norm $|\cdot|$ on $\mathbb{Z}^n$ and set

$$\mathcal{S}(\mathbb{Z}^n, \theta) = \bigcap_{s\geq 0} \ell^1_{v^s}(\mathbb{Z}^n, \theta) = \{f \in \ell^1(\mathbb{Z}^n) : |f(x)| = \mathcal{O}(|x|^{-s})\ \forall s \geq 0\}. \tag{3.1}$$

Then $\mathcal{S}(\mathbb{Z}^n,\theta)$ consists of all rapidly decreasing sequences and coincides with the usual smooth non-commutative torus. Since an arbitrary intersection of inverse-closed subalgebras is again inverse-closed, $\mathcal{S}(\mathbb{Z}^n,\theta)$ is an inverse-closed Fréchet subalgebra of the non-commutative torus $C^*(\theta)$. This result goes back to Connes [3].

Proposition 3.5 also yields inverse-closed subalgebras of $C^*(\theta)$ that are even smaller than $\mathcal{S}(\mathbb{Z}^n,\theta)$. For this, fix a subexponential weight $v(x) = e^{a|x|^b}$ with $a > 0$ and $0 < b < 1$. Then v satisfies the GRS-condition, and thus $\ell^1_v(\mathbb{Z}^n,\theta)$ is inverse-closed in $C^*(\theta)$. On the other hand, $\ell^1_v(\mathbb{Z}^n,\theta)$ is a Banach subalgebra of the smooth non-commutative torus $\mathcal{S}(\mathbb{Z}^n,\theta)$. In the language of non-commutative geometry, one might say that $\ell^1_v(\mathbb{Z}^n,\theta)$ consists of "ultra-smooth" elements of $C^*(\theta)$.

4. Simplicity. The construction of inverse-closed subalgebras of non-commutative tori is completely independent of the fine structure of these tori. In particular, the simplicity of $\ell^1(\mathbb{Z}^n,\theta)$ is not related to its spectral properties.

In this section we treat the question of when the twisted ℓ^1-algebra $\ell^1(\mathbb{Z}^n,\theta)$ is simple. Making use of the symmetry of $\ell^1(\mathbb{Z}^n,\theta)$, one can derive Theorem 4.3 below from the characterization of the simplicity of higher-dimensional non-commutative tori $C^*(\theta)$ in [18], but one has to go back to [24] and [5] for its proof. We offer a simplified proof that works directly for $\ell^1(\mathbb{Z}^n,\theta)$, from which the known result about $C^*(\theta)$ follows. Our proof for the twisted ℓ^1-algebras is fairly elementary, but its idea is probably old.

Let δ_m, $m \in \mathbb{Z}^n$, denote the "Dirac" functions on $\mathbb{Z}^n$, and e_j, $j = 1,\dots,n$, the standard basis of $\mathbb{Z}^n$. Then δ_m is central in $\ell^1(\mathbb{Z}^n,\theta)$, if and only if $\delta_m \natural_\theta \delta_{e_j} = \delta_{e_j} \natural \delta_m$ for $j = 1,\dots,n$. Since

$$\delta_m \natural \delta_{e_j} = \sigma(m,e_j)\delta_{m+e_j} = \theta_{n,j}^{m_n}\cdots\theta_{j+1,j}^{m_{j+1}}\,\delta_{m+e_j}$$

and

$$\delta_{e_j} \natural \delta_m = \sigma(e_j,m)\delta_{m+e_j} = \theta_{j,1}^{m_1}\cdots\theta_{j,j-1}^{m_{j-1}}\,\delta_{m+e_j},$$

the following conditions are equivalent:

(i) δ_m is central in $\ell^1(\mathbb{Z}^n,\theta)$.

(ii) $\sigma(m,e_j) = \sigma(e_j,m)$ for $j = 1,\dots,n$.

(iii) $\prod_{j=1}^n \theta_{jk}^{m_j} = 1$ for $k = 1,\dots,n$.

If $\vartheta = (\vartheta_{jk})$ is a (non-unique) skew-symmetric real matrix with $e^{2\pi i\vartheta_{jk}} = \theta_{jk}$, then (iii) means that $\sum_{j=1}^n m_j\vartheta_{jk} \in \mathbb{Z}$ for $k = 1,\dots,n$. So, if we denote the skew-symmetric bilinear form $(m,l) \mapsto \vartheta(l,m) = m^T\vartheta l$ by ϑ again, a fourth equivalent property is

(iv) $\vartheta(l,m) \in \mathbb{Z}$ for all $l \in \mathbb{Z}^n$.

Definition 4.1. A cocycle σ is called *degenerate* if there exists a non-zero $m \in \mathbb{Z}^n$ satisfying one of the equivalent conditions (i)–(iv). Otherwise σ is called *non-degenerate.*

We note that σ can be degenerate even if ϑ is non-degenerate in the sense of linear algebra.

Remark 4.2. It is well known that a unital Banach algebra $\mathcal{A}$ with non-trivial center is not simple. For if the center $\mathcal{Z}$ is non-trivial, i.e., its dimension is at least two, then it contains an element a that is not invertible in $\mathcal{Z}$ by the Gelfand–Mazur Theorem. Since $\mathcal{Z}$ is inverse-closed in $\mathcal{A}$, we see that a is not invertible in $\mathcal{A}$. Consequently, the generated ideal $a\mathcal{A} = \mathcal{A}a$ is a proper two-sided ideal, and so is its closure $\overline{a\mathcal{A}}$. Thus $\mathcal{A}$ is not simple.

The following theorem characterizes the simplicity of twisted ℓ^1-algebras.

Theorem 4.3. *Let v be an arbitrary, submultiplicative weight function on $\mathbb{Z}^n$ (v need not satisfy the GRS-condition). Then the algebra $\ell^1_v(\mathbb{Z}^n,\theta)$ is simple if and only if the cocycle σ is non-degenerate.*

Proof. If σ is degenerate, then $\ell^1_v(\mathbb{Z}^n,\theta)$ has a non-trivial center and is not simple by Remark 4.2.

Now suppose that σ is non-degenerate. For each $j \in \{1,\dots,n\}$ the element δ_{e_j} is unitary, and its adjoint is $\overline{\sigma(-e_j,e_j)}\delta_{-e_j}$. For arbitrary $x \in \mathbb{Z}^n$ and $k \in \mathbb{N}$ we have

$$(\delta^*_{e_j})^k \natural \delta_x \natural \delta^k_{e_j} = \beta^k_x \delta_x$$

for some $\beta_x \in \mathbb{T}$. More precisely, $\beta_x = 1$ if and only if δ_x commutes with δ_{e_j}. We denote the "centralizer" of δ_{e_j} by

$$C_j = \{y \in \mathbb{Z}^n : \delta_y \natural \delta_{e_j} = \delta_{e_j} \natural \delta_y\}.$$

Now let I be a (closed) two-sided ideal of $\ell^1_v(\mathbb{Z}^n,\theta)$ and $f = \sum_{x\in\mathbb{Z}^n} \alpha_x\delta_x \in I \subseteq \ell^1_v(\mathbb{Z}^n,\theta)$. We consider the behavior of the averages

$$J_m(f) = \frac{1}{m}\sum_{k=1}^{m}(\delta^*_{e_j})^k \natural f \natural \delta^k_{e_j} = \sum_{x\in\mathbb{Z}^n} \alpha_x\Big(\frac{1}{m}\sum_{k=1}^{m}\beta^k_x\Big)\delta_x.$$

If $x \in C_j$, then $m^{-1}\sum_{k=1}^m \beta^k_x = 1$; if $x \notin C_j$, then $m^{-1}\sum_{k=1}^m \beta^k_x$ converges to zero for $m \to \infty$. Using dominated convergence, we conclude that

$$\lim_{m\to\infty} J_m(f) = \sum_{x\in C_j} \alpha_x\delta_x = f\chi_{C_j}$$

with convergence in the ℓ^1_v-norm. For $f \in I$, this means that also $f\chi_{C_j} \in I$. Since this is true for all $j = 1,\dots,n$, we $f \cdot \prod_{j=1}^n \chi_{C_j} = f\chi_{\bigcap_{j=1}^n C_j} \in I$.

Since σ is non-degenerate, we must have $\bigcap_{j=1}^n C_j = \{0\}$ and thus $f(0)\delta_0 \in I$. Either $I = \ell^1_v(\mathbb{Z}^n,\theta)$ or I is a proper ideal and $f(0) = 0$. By applying the argument to $\delta_x \natural f \in I$ for every $x \in \mathbb{Z}^n$, we find that $(\delta_x \natural f)(0) =$

$\sigma(x,-x)\,f(-x) = 0$, so $f(x) = 0$ for all $x \in \mathbb{Z}^n$. Consequently, either $I = \ell_v^1(\mathbb{Z}^n,\theta)$ or $I = \{0\}$, and thus $\ell_v^1(\mathbb{Z}^n,\theta)$ is simple. ∎

REMARK 4.4. We may also obtain an alternative proof of the well-known C^*-analogue of Theorem 4.3. The above proof works for $C^*(\theta)$ as well, because the finitely supported functions are dense in $C^*(\theta)$ and the inner automorphisms are also isometric in the $C^*(\theta)$-norm.

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