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Uniform Shadow Limit Reduction for Reaction-Diffusion-ODE Systems

Gutachter

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To my Mum

Abstract

Reaction-diffusion equations coupled with ordinary differential equations (ODEs) are used to model various biological, chemical and ecological processes. In case some diffusion coefficients tend to infinity, the reaction-diffusion-ODE system can be approximated by a reduced system. This system is called *shadow limit* and is used to facilitate model analysis. A convergence result is well-known for time intervals which are finite compared to the large diffusion parameter.
This research investigates the relation between a reaction-diffusion-ODE system endowed with zero flux boundary conditions and its shadow limit on long-time scales. Such long-time intervals scale with the diffusion coefficient and tend to infinity as diffusion tends to infinity. Solutions of both systems are compared with respect to the L^∞ norm and errors are estimated in terms of the inverse of the large diffusion parameter. This work shows that an extension of uniform error estimates to large time intervals may fail without additional stability assumptions. Error estimates are derived by using a uniform stability condition for the evolution of the linearized subsystem of ODEs and of the linearized shadow system. The method is based on previous results for short-time intervals which use a cut-off technique applied to the system linearized at the shadow solution. The partial lack of diffusion implies low regularity in space of solutions to both systems. Hence, mild solutions are considered in this work. Moreover, two analytical ways of verifying the stability conditions are discussed in detail: dissipativity of evolution systems and linearized stability of stationary shadow solutions using a spectral analysis.
The general framework applied in this thesis allows to study the uniform shadow limit approximation for reaction-diffusion systems and reaction-diffusion-ODE systems, under low regularity of the solutions and of the domain. The explicit error estimates provide information on the long-term dynamics of such models from results obtained for their shadow limit. Additionally, this detailed study shows that the shadow limit reduction exhibits characteristic time scales. Validity of the approximation on these time ranges can be verified under certain stability assumptions on the shadow system.

Zusammenfassung

In vielen biologischen, chemischen und ökologischen Prozessen finden Reaktions-Diffusions-Gleichungen Anwendung, welche an gewöhnliche Differentialgleichungen (ODEs) gekoppelt sind. Falls einer der Diffusionskoeffizienten sehr groß ist, lässt sich das Reaktions-Diffusions-ODE-System durch ein reduziertes System – *shadow limit* genannt – approximieren. Dieses System wird zur Vereinfachung der Modell-Analyse benutzt. Ein entsprechendes Konvergenzresultat existiert für Zeitintervalle, welche endlich sind verglichen mit der sehr großen Diffusion.
Diese Forschungsarbeit untersucht die Beziehung eines Reaktions-Diffusions-ODE-Systems mit homogener Neumann-Randbedingung zu dessen shadow limit für große Zeitintervalle. Solche Langzeitintervalle skalieren mit der Diffusion und wachsen mit dieser gegen Unendlich. Lösungen beider Systeme werden bezüglich der L^∞-Norm verglichen und die Fehlerterme werden mithilfe des inversen Diffusionskoeffizienten abgeschätzt. Diese Arbeit zeigt, dass die Fehlerabschätzungen ihre Gültigkeit auf Langzeitintervallen ohne zusätzliche Stabilität möglicherweise verlieren. Die Abschätzungen werden mittels Stabilitätsbedingungen an die linearisierten Evolutionssysteme des shadow limit und des ODE-Teilsystems erzielt. Dabei wird eine bereits bekannte Abschneidemethode angewendet, welche nichtlineare Terme nach einer Linearisierung um die shadow Lösung lokalisiert. Aufgrund partieller Diffusion können Lösungen beider Systeme geringe räumliche Regularität aufweisen. Infolgedessen werden milde Lösungen betrachtet. Zur analytischen Überprüfung der Stabilität werden dissipative Evolutionssysteme sowie linearisierte Stabilität stationärer Lösungen des shadow limit mittels Spektralanalyse untersucht.
Der allgemeine Rahmen dieser Arbeit lässt sich gleichermaßen auf die shadow limit Approximation von Reaktions-Diffusions- und Reaktions-Diffusions-ODE-Systemen anwenden – in beiden Fällen unter geringen Regularitätsanforderungen an die Lösungen und das Gebiet. Mithilfe der expliziten Fehlerabschätzungen lassen sich Informationen über das Langzeitverhalten solcher Modelle aus dem des shadow limit gewinnen. Darüber hinaus weist diese Modell-Reduktion charakteristische Zeitskalen auf, deren Gültigkeit sich unter gewissen Stabilitätsbedingungen nachweisen lässt.

Related Publications

C. Kowall, A. Marciniak-Czochra, A. Mikelić, Long-time shadow limit for a reaction-diffusion-ODE system. Applied Mathematical Letters, Vol. 112 (2021), 106790.

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Notation

$\mathbb{N}; \mathbb{N}_0$	Set of all natural numbers; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}; \mathbb{R}_{>0}; \mathbb{R}_{\geq 0}$	Set of all real numbers; set of all positive real numbers; set of all non-negative real numbers
Ω	Open, bounded and connected subset of $\mathbb{R}^n$
Ω_T	Space-time domain $\Omega \times (0, T)$
$\partial\Omega$	Boundary of $\Omega \subset \mathbb{R}^n$ locally being the graph of a Lipschitz continuous function, $\partial\Omega \in C^{0,1}$, see [1, Paragraph 4.9]
$\mathbf{n}$	Outward unit normal vector on the boundary $\partial\Omega$
$\lvert\Omega\rvert$	Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$
$\langle z \rangle_\Omega$	Spatial mean value of a function $z \in L^1(\Omega)$
$\lVert\cdot\rVert_{p,q}$	Mixed norm of the Lebesgue space $L_{p,q}(\Omega_T)$ defined in (4.18)
D	Least positive entry of the diagonal diffusion matrix $\mathbf{D}^v \in \mathbb{R}^{k \times k}_{>0}$
$\log D$	Natural logarithm of a real number $D \in \mathbb{R}_{>0}$
χ_A	Characteristic function being identical 1 on a set A
$(\mathbf{u}, \mathbf{v})$	Solution to the shadow problem (1.4)–(1.6)
$(\mathbf{u}_D, \mathbf{v}_D)$	Solution to the diffusive problem (1.1)–(1.3)
ψ_D	Mean value correction defined by system (2.10)–(2.11)
$(\mathbf{U}_D, \mathbf{V}_D)$	Error functions $\mathbf{U}_D = \mathbf{u}_D - \mathbf{u}$, $\mathbf{V}_D = \mathbf{v}_D - \mathbf{v} - \psi_D$ defined in (3.1)
∇; Δ	Nabla operator; Laplace operator
$(S_\Delta(\tau))_{\tau \in \mathbb{R}_{\geq 0}}$	Heat semigroup for zero flux boundary conditions, see Lemma 2.1

λ_j, w_j	Eigenvalue λ_j, eigenfunction w_j from a spectral basis of $-\Delta$, see Proposition A.1
I	Identity operator on corresponding Banach space
$\mathcal{L}(B)$	Set of all linear, bounded operators on a Banach space B
$\mathcal{D}(\mathbf{L})$	Domain of a linear (unbounded) operator $\mathbf{L}$
$\sigma(\mathbf{L}); \rho(\mathbf{L})$	Spectrum of a linear operator $\mathbf{L}$; resolvent set $\rho(\mathbf{L}) = \mathbb{C} \setminus \sigma(\mathbf{L})$
$\sigma_p(\mathbf{L}); \sigma_{\text{ess}}(\mathbf{L})$	Point spectrum of a linear operator $\mathbf{L}$; Wolf essential spectrum of $\mathbf{L}$ defined in Proposition C.1
$\mathcal{U}, \tilde{\mathcal{U}}$	Evolution system induced by $\mathbf{D}^u\Delta + \mathbf{A}_*(\cdot, t)$ and $\mathbf{A}_{11}(\cdot, t)$, respectively, see (4.6) and Assumption L0
$\mathbf{L}_0(t)$	Linearized shadow operator defined in (4.11) and used for (4.39)
$\mathcal{W}$	Evolution system induced by $\mathbf{D}^S\Delta + \mathbf{L}_0(t)$, see (4.12) and (4.39)
A1	Assumption A1: regularity of nonlinearities for existence, uniqueness and first-order truncation, see p. 12
A2	Assumption A2: essentially bounded initial conditions, see p. 12
A3	Assumption A3: decay estimate for the mean value correction ψ_D, see p. 44
A4	Assumption A4: regularity of nonlinearities for second-order truncation, see p. 52
B	Assumption B: uniform boundedness of shadow solution, see p. 51
L; **L0**	Assumption L; L0: uniform boundedness of evolution subsystem $\mathcal{U}; \tilde{\mathcal{U}}$, see pp. 35; 49
L1p	Assumption L1p: uniform boundedness of evolution system $\mathcal{W}$, see p. 43
Dp	Assumption Dp: dissipativity of subsystem $\mathbf{A}_*$ and $\mathbf{A}_{11}$, see p. 68
D1p	Assumption D1p: dissipativity of linearized shadow system $\mathbf{L}_0(t)$, see p. 73

Contents

1 Introduction

Many biological, chemical and ecological processes can be mathematically described using a model consisting of reaction-diffusion-type equations endowed with zero flux boundary conditions which are typical for closed physical systems. Examples include the activator-inhibitor model of Gierer and Meinhardt [28], the Gray-Scott model [29], the Lengyel-Epstein model [60] and many others [83]. A special case of reaction-diffusion-type systems consists of a class of models coupling semilinear reaction-diffusion equations with ordinary differential equations (ODEs) acting on some function space. Such reaction-diffusion-ODE models arise, for example, in the context of cell biology: Intercellular dynamics regulated by a diffusive signaling factor can be described by so called receptor-based models [44, 52, 73, 74]. Those models are also applied to a range of ecological and chemical processes, see [68, 82, 94, 111] and [99], respectively.

Reaction-diffusion-type systems may exhibit quite complex structures. From a mathematical point of view, it is desirable to reduce their complexity as far as possible in order to understand the behavior of the full system while simultaneously maintaining its main properties. In many applications there is one diffusion coefficient (or even more) that is significantly larger compared to the others. Concerning this case, a model reduction has been successfully applied to classical reaction-diffusion systems for the last four decades, starting from the work of Keener [51], and Nishiura [86] who referred to the reduced system as a shadow system. Nowadays, the term shadow systems is used for a more general class of large diffusion limits of reaction-diffusion-type systems with the largest diffusion coefficient tending to infinity. The relation of the original partly diffusive model to its reduced system is the main objective of this thesis. A detailed description is given in the next section *Aims of the thesis*.

The shadow limit was originally considered in the case of two coupled reaction-diffusion equations, where one diffusion is fixed and the other tends to infinity. It has been used to investigate stationary problems of reaction-diffusion systems with a large ratio of diffusion rates. A stationary solution of the shadow problem allows finding a stationary solution of the original problem provided that the shadow operator linearized around the stationary shadow solution is invertible [103]. Stability properties of those stationary solutions are also inherited [78, 109]. Not only are steady states close to each other in a suitable norm, but in [32, 79] it is shown that compact attractors for both the reaction-diffusion system and its shadow limit are closely related. However, dynamics of non-stationary solutions of the shadow system and of the reaction-diffusion system do not need to be related in general. It may even happen that solutions of the shadow system blow-up in finite time although the original system has solutions which exist globally in time [62].

Aims of the thesis

The shadow limit approximation for classical reaction-diffusion systems is usually performed for regular domains [32, 79]. However, some applications may require a lower boundary regularity of the underlying domain such that existence of solutions within the class of regular functions is not guaranteed. Consequently, mild solutions given by an implicit integral equation of the reaction-diffusion-type system and its shadow limit are considered in this thesis. Reaction-diffusion-ODE systems may also exhibit singular patterns with jump-discontinuities [38, 54] or patterns of mass concentration in finite or infinite time [37, 69, 70, 72]. The main aim of this dissertation is to investigate the relation between a reaction-diffusion-type system with large diffusive components and its shadow limit under low regularity assumptions. Apart from mild solutions, we allow for non-smooth domains and initial conditions. Moreover, this study focuses on finding conditions under which the shadow limit is an adequate reduction of the partly diffusive system on long-time scales. The thesis extends the uniform convergence results from [7, 75], which already exist for short-time intervals, to large time intervals including global error estimates.

Let Ω be a given bounded domain (open and connected set) in $\mathbb{R}^n$, $n \in \mathbb{N}$, with a Lipschitz boundary $\partial\Omega \in C^{0,1}$. We focus on the reaction-diffusion-type problem

$$\frac{\partial \mathbf{u}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{u}_D = \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{u}_D(\cdot, 0) = \mathbf{u}^0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\frac{\partial \mathbf{v}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{v}_D = \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{v}_D(\cdot, 0) = \mathbf{v}^0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} = \mathbf{0}, \quad \frac{\partial \mathbf{v}_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.3}$$

where $\mathbf{u}_D : \overline{\Omega_T} \to \mathbb{R}^m$, $\mathbf{v}_D : \overline{\Omega_T} \to \mathbb{R}^k$ with $m, k \in \mathbb{N}$ are vector-valued functions on the domain $\Omega_T := \Omega \times (0, T)$ endowed with zero Neumann boundary conditions for each diffusive component. Although the function $\mathbf{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$, condition (1.3) is defined implicitly for mild solutions of the above system. The Laplace operator Δ is applied component by component and is multiplied by diagonal diffusion matrices $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$ and $\mathbf{D}^v \in \mathbb{R}^{k\times k}_{>0}$ with non-negative and positive entries, respectively. For reaction-diffusion-ODE systems, boundary condition (1.3) only applies to components of $\mathbf{u}_D$ for which the entry on the diagonal of $\mathbf{D}^u$ is positive. In this thesis, the splitting of the model components $\mathbf{u}_D$ and $\mathbf{v}_D$ depends on the size of diffusion coefficients. All components of $\mathbf{D}^v$ are much larger (or tend to infinity) compared to components of $\mathbf{D}^u$. Furthermore, we consider model nonlinearities

$$\mathbf{f} : \mathbb{R}^{m+k} \times \overline{\Omega} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m \qquad \text{and} \qquad \mathbf{g} : \mathbb{R}^{m+k} \times \overline{\Omega} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^k$$

which are given functions depending on the unknown solution, space and time. Detailed properties are provided in assumptions A1–A4 below.

An asymptotic analysis of problem (1.1)–(1.3) was considered in [100] in the case when all diffusion coefficients tend to infinity, i.e., $\mathbf{D}^u, \mathbf{D}^v \to \infty$. If some diffusion coefficients, denoted by $\mathbf{D}^u$, are fixed and bounded, the classical shadow limit for $\mathbf{D}^v \to \infty$ was studied, for instance, in [51]. A shadow limit reduction of the reaction-diffusion-ODE case was performed in [75]. In general, the shadow system of equations (1.1)–(1.3) for all entries of $\mathbf{D}^v$ tending to infinity reads

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}^u \Delta \mathbf{u} = \mathbf{f}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{u}(\cdot, 0) = \mathbf{u}^0 \quad \text{in} \quad \Omega, \tag{1.4}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \langle \mathbf{g}(\mathbf{u}(\cdot, t), \mathbf{v}(t), \cdot, t) \rangle_\Omega \quad \text{in} \quad (0, T), \qquad \mathbf{v}(0) = \langle \mathbf{v}^0 \rangle_\Omega, \tag{1.5}$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0, T). \tag{1.6}$$

Here the (componentwise) spatial mean value for $\mathbf{z} \in L^1(\Omega)^k$ is abbreviated by

$$\langle \mathbf{z} \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \mathbf{z}(x) \, \mathrm{d}x.$$

To simplify the notation in the remainder of this thesis, we use expression (1.6) but mention that the boundary condition only applies to diffusive components of $\mathbf{u}$.

As there exists a huge variety of results concerning approximation of stationary patterns or nearby stationary solutions by the classical shadow limit (see [53, 78, 86, 109] or [32, 79], respectively), the current work focuses on reaction-diffusion-ODE systems. Nevertheless, the framework which is presented in the following allows to consider both cases simultaneously with only minor adaptions.
Error estimates given in [75] justify the underlying equations (1.4)–(1.6) of the shadow system but are useful only for a model reduction of system (1.1)–(1.3) on finite time intervals. In the published proof, bounds depend exponentially on the length of the interval, hence the error estimate deteriorates significantly for larger time scales. The aim of this thesis is to find sufficient assumptions which guarantee a uniform approximation of the original dynamics by the solutions of the shadow system (1.4)–(1.6) – on long-time intervals and on the asymptotic time scale. Such long-time intervals $(0, T)$ have a length proportional to a power of the least positive entry D of the diffusion matrix $\mathbf{D}^v$, i.e., $T \sim D^\ell$ for some $\ell > 0$, and errors are estimated by a bound proportional to a power of the inverse D^{-1} as $D \to \infty$. These estimates are uniform in the sense that we use the $L^\infty(\Omega_T)$ norm to compare model solutions. Such estimates provide understanding of the long-term dynamics of reaction-diffusion-type models from results obtained for their associated shadow limit. To obtain a comprehensive picture of this limit process, the thesis includes a critical reflection of the made assumptions with various examples and applications from natural sciences.

Mathematical challenges and applied methods

In this dissertation, I present a detailed study of the limit process by comparing solutions of the reaction-diffusion-type system (1.1)–(1.3) and its shadow limit (1.4)–(1.6). The spatial mean values in the integro-differential system (1.4)–(1.6) implies that the shadow system is a singular limit of the partly diffusive system. Hence, in the error estimates, we involve a correction term ψ_D. This term includes the initial layer which originates from different initial values $\mathbf{v}^0$ and $\langle \mathbf{v}^0 \rangle_\Omega$. Moreover, there is a discrepancy between nonlinearities of both systems, which we also include in this correction term ψ_D to simplify linearization.
Due to the low regularity of solutions to reaction-diffusion-ODE systems [38, 54], another difficulty is related to finding a suitable norm for estimating solutions. The choice of $L^\infty(\Omega_T)$ is suitable for bounded, discontinuous solutions of a wide range of nonlinear problems which can be solved using the method of Rothe [94]. This

approach requires a deeper investigation of the heat semigroup which is not strongly continuous on $L^\infty(\Omega)$. Nevertheless, it is possible to show existence and uniqueness of mild solutions to both systems (1.1)–(1.3) and (1.4)–(1.6), and to estimate these with respect to $L^\infty(\Omega)$. At the same time, boundary regularity of Ω can be relaxed to a Lipschitz boundary which is necessary for validity of Sobolev embeddings, existence of the heat semigroup and a proper notion of a boundary condition.

When comparing both solutions, estimates for the error $\mathbf{u}_D - \mathbf{u}$ and $\mathbf{v}_D - \mathbf{v} - \psi_D$ are established with respect to the $L^\infty(\Omega_T)$ norm where the correction term ψ_D is negligible for larger times. The system of errors is linearized around a globally defined, uniformly bounded shadow limit. To control the growth of the nonlinear parts, we introduce a cut-off, similar to [75]. This cut-off does not affect the nonlinear part in a small neighborhood of $\mathbf{0}$, which is a ball of radius $D^{-\delta_0}$ for some $\delta_0 > 0$. The truncation allows to obtain uniform estimates of the truncated errors. For fixed time T and sufficiently big diffusion $D \geq D(T)$, the estimates show that the truncated solution does not leave the latter neighborhood on Ω_T and we gain an estimate for the original errors. In general, however, this approach provides uniform error estimates which are valid only on finite time intervals $(0, T)$.

In order to extend these results to long-time intervals, we have to assume a stability condition on the linearization around the time-dependent shadow solution. This approach is due to the work with Mikelić, who refined estimates from [75] and who suggested to consider shadow systems with an L^2 dissipative linearization in order to obtain long-time estimates. Since dissipative systems form a particular class of uniformly stable evolution systems, the dissipativity assumption could be relaxed in the work at hand. If the evolutionary system is uniformly stable with respect to $L^\infty(\Omega)^m \times \mathbb{R}^k$, we obtain error estimates with explicit dependence on the time interval length T. Stability implies estimates that are valid on Ω_T for $T \sim D^\ell$ and some $0 < \ell < 1$. Assuming uniform exponential stability of the linearized shadow system even yields global error estimates on $\Omega \times \mathbb{R}_{\geq 0}$.

The latter stability condition for the shadow system can also be considered in $L^p(\Omega)^m \times \mathbb{R}^k$ for sufficiently large $p < \infty$. In this case, parabolic L^p estimates in combination with the truncation method of Stampacchia are employed to turn to L^∞ bounds for the diffusive components, with explicit dependence on T. An additional uniform stability of the ODE subsystem in L^∞ as in [55] yields an estimate for the non-diffusive components.

In the stationary case, stability properties of the linerization can be deduced from the knowledge of the spectrum of the corresponding linear operator. This is a consequence of the well-known spectral mapping theorem for analytic semigroups. The spectrum of the linearized shadow operator as well as the diffusive operator is characterized for bounded steady states. In the reaction-diffusion-ODE case, however, both linear operators need not to have a pure point spectrum. To determine the essential spectrum, we apply a spectral decomposition for block operator matrices [5]. This is based on properties of bounded multiplication operators, which are induced by the ODE subsystem on $L^p(\Omega)^m$ for $1 \leq p \leq \infty$.

Scientific contribution

The general semigroup framework applied in this thesis allows to study the uniform shadow limit approximation for classical reaction-diffusion systems and reaction-diffusion-ODE systems. Qualitative and quantitative convergence results on finite time scales are well established, see [7, Section 3.1] for shadow solutions that are continuous in space, and [69, Appendix A], [75, Theorem 3] for bounded solutions. On long-time scales, however, there are only few qualitative results including [62] or [75, Theorem 4]. This dissertation approaches the problem of extending estimates to long-time intervals, also heading for a global approximation result. Model examples in this work show that a shadow approximation has to be considered on different characteristic time scales. Using linearization around a time-dependent shadow solution, sufficient conditions are derived to validate the quality of the approximation for solutions on the following time ranges:

- short-time intervals taking account for the initial time layer,
- long-time intervals for times $T \sim D^\ell$ scaling with diffusion for $0 < \ell < 1$, and
- asymptotic state for times up to $T = \infty$

While the initial time layer, due to the singular shadow limit, dominates for small times, the correction term ψ_D decays exponentially in time and a transient state is approached. This intermediate period $(0, T)$ for $T \sim D^\ell$ has an already large time range since the diffusivity D is large. Uniform stability of the evolution system induced by the linearization around the shadow limit yields a natural condition for solutions of the partly diffusive system (1.1)–(1.3) to stay nearby the solution of the shadow system (1.4)–(1.6) for all large diffusion. Unfortunately, as examples in Chapter 6 show, accuracy of the approximation for transient states does not imply a valuable asymptotic approximation as $T \to \infty$. It becomes apparent that uniform

exponential stability of the evolution system induced by the linearization at the shadow limit is a sufficient (but not necessary) concept for global estimates.
Within this study, low regularity assumptions are imposed to guarantee a uniform approximation of the original dynamics by the shadow system (1.4)–(1.6). This is motivated by solutions of reaction-diffusion-ODE systems which are bounded but may be discontinuous in space. Using a non-smooth domain, the results generalize the works [7, 75] on shadow limit approximation for finite time intervals to non-smooth solutions. Fundamental ideas of this work which apply to a linear case of the low-regularity setting have been published in [55].
As a possible way of verifying stability conditions for the linearized shadow problem in the stationary case, we characterize the spectrum of the corresponding linear shadow operator. The knowledge of the spectrum not only allows answering questions concerning the shadow approximation of system (1.1)–(1.3). Also stability properties of stationary patterns of the shadow system can be derived from a linearized stability analysis of the nonlinear shadow problem [14]. This characterization of the spectrum shows that the instability result in [69, Appendix B] which restricts to the point spectrum is valid under much more general conditions.
A similar spectral decomposition is valid for the partly diffusive operator linearized around a bounded, stationary solution of system (1.1)–(1.3). This is in accordance with stability considerations in [71, 106]. However, the characterization in this thesis generalizes results obtained in [71] for a reaction-diffusion-ODE system with one ODE component to arbitrary systems.

Outline of the thesis

A brief summary of basic results for the diffusive system (1.1)–(1.3) and its shadow problem (1.4)–(1.6), including existence, uniqueness and definition of mild solutions, is given in Chapter 2. Moreover, since comparing solutions of the diffusive system and its singular limit requires correction terms, a suitable initial layer correction is introduced. The analysis is based on fundamental properties of the heat semigroup which are presented in more detail in Appendix B.
The core of the thesis consists of three parts corresponding to different time ranges on which the approximation result is valid: short-time intervals (Chapter 3), long-time intervals (Chapter 4) and asymptotic behavior of solutions (Chapter 5). Chapter 3 is devoted to a brief but essential comparison of solutions to the diffusive system and its shadow counterpart.

The aim of Chapter 4 is to find criteria for an extension of the uniform error estimates to long-time intervals $(0, T)$ of the length proportional to a power of D, i.e., $T \sim D^\ell$ for $\ell > 0$. With increasing complexity, the focus transfers from linear problems, which already incorporate all main difficulties of the approximation problem on long-time scales, to nonlinear problems using linearization. The last section of Chapter 4 is dedicated to a particular class of problems, namely time-dependent systems with a dissipative linearization. Those provide a possibility to verify assumptions of the proven convergence results.
Chapter 5 briefly considers global uniform error estimates up to $T = \infty$ as a natural consequence of the detailed study in the foregoing chapter under stronger assumptions. To verify assumptions of the established theorems close to stationary solutions of the shadow problem, the spectrum of the corresponding linearized, stationary shadow operator is characterized. In virtue of the proof, a similar decomposition of the spectrum is shown for the reaction-diffusion-ODE system linearized around a stationary solution. Finally, asymptotic behavior around steady states is deduced from linearized stability considerations in both cases.
Particular applications are studied in Chapter 6 in more detail. A Lotka-Volterra-type system from ecology shows global convergence results. The well-known Lengyel-Epstein model exhibits various patterns and exemplifies the convergence results while a system modeling stem cell dynamics shows asymptotic discrepancies between the diffusive system and its shadow limit.

2 Preliminary results

Both the diffusive and the shadow problem are of great interest but their analysis requires slightly different methods. Before starting analysis of existence, uniqueness and regularity of solutions for each problem separately, we begin with a property that is fundamental for this work. The latter property is distinctive of zero flux boundary conditions which are considered in this thesis.

The solution of the one-dimensional heat equation

$$\partial_t z - D\Delta z = 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \qquad z(\cdot, 0) = z^0 \in L^\infty(\Omega) \tag{2.1}$$

with zero flux boundary conditions can be expressed by

$$z(x,t) = S_\Delta(Dt)z^0(x), \qquad (x,t) \in \Omega \times \mathbb{R}_{>0},$$

using the heat semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ defined in Proposition B.2. Due to the maximum principle, see [18, Theorem 1.3.9] or [89, Corollary 4.10], the semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ is contractive in $L^\infty(\Omega)$, i.e., it satisfies

$$\|S_\Delta(\tau)z^0\|_{L^\infty(\Omega)} \leq \|z^0\|_{L^\infty(\Omega)} \qquad \forall\, z^0 \in L^\infty(\Omega), \tau \in \mathbb{R}_{\geq 0}.$$

Moreover, we have the following smoothing property as $\tau \to \infty$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$ and $\lambda_1 > 0$ the first non-zero eigenvalue of $-\Delta$ endowed with zero Neumann boundary conditions. Then there exists a constant $C > 0$, depending on Ω only, such that for all $1 \leq q \leq p \leq \infty$*

$$\|S_\Delta(\tau)z^0\|_{L^p(\Omega)} \leq Cm(\tau)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\mathrm{e}^{-\lambda_1\tau}\|z^0\|_{L^q(\Omega)} \qquad \forall\, \tau \in \mathbb{R}_{>0}$$

holds for all $z^0 \in L^q(\Omega)$ satisfying $\langle z^0\rangle_\Omega = 0$. Here, we denote $m(\tau) = \min\{1, \tau\}$. Moreover, $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ is a contraction semigroup on $L^p(\Omega)$ for each $1 \leq p \leq \infty$, which is strongly continuous for $1 \leq p < \infty$ and analytic for $1 < p < \infty$.

Proof. See Lemma B.4. □

A direct consequence of this lemma is the estimate

$$\|S_\Delta(\tau)(z^0 - \langle z^0\rangle_\Omega)\|_{L^\infty(\Omega)} \leq \overline{C}\mathrm{e}^{-\lambda_1\tau}\|z^0 - \langle z^0\rangle_\Omega\|_{L^\infty(\Omega)} \qquad \forall\, \tau \in \mathbb{R}_{\geq 0} \tag{2.2}$$

for some $\overline{C} > 0$ which only depends on Ω. Remember that constant functions stay invariant under the action of the Neumann heat semigroup. Considering the scaled time $\tau = Dt$, this implies that the solution of the above heat equation is averaged in space for large diffusion $D \to \infty$ and a fixed time $t > 0$:

$$\|z(\cdot,t) - \langle z^0\rangle_\Omega\|_{L^\infty(\Omega)} \to 0 \qquad (D \to \infty)$$

In this work, the smoothing property (2.2) is crucial for proofs and will be used for the fast diffusing components of system (1.1)–(1.3). Remark that this decay estimate enables us to derive the shadow limit for some simple, decoupled linear systems in its particular form (1.4)–(1.6). Accuracy of the shadow limit can already be deduced from the works [7, 32, 75] which make use of a higher boundary regularity and more regular nonlinearities. Uniform estimates will be shown in Chapter 3 in its generality under low-regularity assumptions on the domain and the solutions.

2.1 The partly diffusive problem

In this section, we study the diffusive system (1.1)–(1.3) which reads

$$\begin{aligned}
\frac{\partial \mathbf{u}_D}{\partial t} - \mathbf{D}^u\Delta\mathbf{u}_D &= \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{u}_D(\cdot,0) = \mathbf{u}^0 \ \text{ in } \ \Omega,\\
\frac{\partial \mathbf{v}_D}{\partial t} - \mathbf{D}^v\Delta\mathbf{v}_D &= \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{v}_D(\cdot,0) = \mathbf{v}^0 \ \text{ in } \ \Omega,\\
\frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} = \mathbf{0}, \quad &\frac{\partial \mathbf{v}_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \ \ \partial\Omega \times (0,T).
\end{aligned}$$

There are several notions of a solution to this system and methods to solve it, amongst which I should mention the books [59, 64] for the theory of weak solutions or the more abstract semigroup theory by [8, 40, 76, 92] or [94]. As this work shall be dedicated mainly to nonlinear problems and will allow for non-smooth solutions, we consider L^∞ estimates similar to Rothe [94].

For the sake of completeness, we recollect definitions and the main idea of the proof of [94, Part II, Theorem 1]. Therefore, we rewrite system (1.1)–(1.3) as a system of

$m+k$ reaction-diffusion equations

$$\frac{\partial \Psi}{\partial t} - \mathbf{D}\Delta\Psi = \mathbf{h}(\Psi, x, t) \quad \text{in} \quad \Omega_T, \qquad \Psi(\cdot, 0) = (\mathbf{u^0}, \mathbf{v}^0) =: \Psi^0 \quad \text{in} \quad \Omega, \tag{2.3}$$

where $\Psi = (\mathbf{u}_D, \mathbf{v}_D), \mathbf{h} = (\mathbf{f}, \mathbf{g})$, and

$$\mathbf{D} := \operatorname{diag}(\mathbf{D}^u, \mathbf{D}^v) \in \mathbb{R}^{m+k}_{\geq 0}$$

is a diagonal matrix decomposed of $\mathbf{D}^u, \mathbf{D}^v$. For each component Ψ_i with a non-vanishing diffusion coefficient, we impose zero flux boundary conditions

$$\frac{\partial \Psi_i}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{2.4}$$

In this setting, only components of $\mathbf{u}_D$ are allowed not to diffuse, i.e., $\mathbf{D}^u \in \mathbb{R}^m_{\geq 0}$, whereas all components of $\mathbf{v}_D$ are diffusive since $\mathbf{D}^v \in \mathbb{R}^k_{>0}$.
If we consider this system for the simplest decoupled case $\mathbf{h} = \mathbf{0}$, the solution $(\mathbf{u}_D, \mathbf{v}_D)$ is given by the corresponding components. Using the heat semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ from Lemma 2.1, we define a semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$ on $L^p(\Omega)^{m+k}$ by $\mathbf{S}(t) = (\mathbf{S}^u(t), \mathbf{S}^v(t))$. The components are given by

$$S^u_i(t) = \begin{cases} S_\Delta(D^u_i t) & \text{if} \quad D^u_i > 0, \\ I & \text{if} \quad D^u_i = 0, \end{cases} \qquad \text{and} \qquad S^v_j(t) = S_\Delta(D^v_j t) \tag{2.5}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, k$ provided I is the identity operator on $L^p(\Omega)$,

$$\mathbf{D}^u := \operatorname{diag}(D^u_1, \ldots, D^u_m) \in \mathbb{R}^{m\times m}_{\geq 0}, \quad \text{and} \quad \mathbf{D}^v := \operatorname{diag}(D^v_1, \ldots, D^v_k) \in \mathbb{R}^{k\times k}_{>0}.$$

Analyticity of the constant semigroup I and the heat semigroup according to Lemma 2.1 induces an analytic semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$ on $L^p(\Omega)^{m+k}$ for each $p \in (1, \infty)$. This semigroup can be restricted to the invariant subspace $L^\infty(\Omega)^{m+k}$ independently of p, compare [18, Theorems 1.3.9, 1.4.1]. The resulting operators yield a formal semigroup on $L^\infty(\Omega)^{m+k}$ which is not strongly continuous [94, Part I, Lemma 2]. Nevertheless, it is a contraction semigroup on $L^\infty(\Omega)^{m+k}$ due to the maximum principle from [18, Theorem 1.3.9];

$$\|\mathbf{S}(t)\Psi^0\|_{L^\infty(\Omega)^{m+k}} \leq \|\Psi^0\|_{L^\infty(\Omega)^{m+k}} \qquad \forall\, t \in \mathbb{R}_{\geq 0}.$$

We proceed similar to [94, Part II, Definition 2] using this semigroup approach.

Definition 2.2. Let $0 < T \leq \infty$. An $E_{\infty,0,T}$-*mild solution* of problem (2.3)–(2.4) for initial data $\Psi^0 = (\mathbf{u}^0, \mathbf{v}^0) \in L^\infty(\Omega)^{m+k}$ on the interval $[0, T)$ is a measurable function

$$\Psi : \Omega \times [0, T) \to \mathbb{R}^{m+k}$$

satisfying for all $t \in (0, T)$

(i) $\Psi(\cdot, t) \in L^\infty(\Omega)^{m+k}$ and $\sup_{s\in(0,t)} \|\Psi(\cdot, s)\|_{L^\infty(\Omega)^{m+k}} < \infty$,

(ii) the integral representation

$$\Psi(\cdot, t) = \mathbf{S}(t)\Psi^0 + \int_0^t \mathbf{S}(t-\tau)\mathbf{h}(\Psi(\cdot, \tau), \cdot, \tau)\, \mathrm{d}\tau,$$

where the integral is an absolutely converging Bochner integral in $L^\infty(\Omega)^{m+k}$.

Following the proof of [94, Part II, Theorem 1] and its assumptions which yield an $E_{\infty,0,T}$-mild solution for system (2.3)–(2.4), we assume

A1 Let the function $\mathbf{h} = (\mathbf{f}, \mathbf{g})$ be measurable in $(x, t) \in \overline{\Omega} \times \mathbb{R}_{\geq 0}$ for every fixed $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{m+k}$. Moreover, for every bounded set $B \subset \mathbb{R}^{m+k} \times \overline{\Omega} \times \mathbb{R}_{\geq 0}$, let there exist a constant $L(B) > 0$ such that for all $(\mathbf{u}, \mathbf{v}, x, t), (\mathbf{y}, \mathbf{z}, x, t) \in B$

$$\begin{aligned} |\mathbf{h}(\mathbf{u}, \mathbf{v}, x, t)| &\leq L(B), \\ |\mathbf{h}(\mathbf{u}, \mathbf{v}, x, t) - \mathbf{h}(\mathbf{y}, \mathbf{z}, x, t)| &\leq L(B)\left(|\mathbf{u} - \mathbf{y}| + |\mathbf{v} - \mathbf{z}|\right). \end{aligned}$$

A2 Initial values $(\mathbf{u}^0, \mathbf{v}^0) \in L^\infty(\Omega)^{m+k}$.

These assumptions are due to [94, p. 109] while Assumption A1 reflects a local Lipschitz continuity of $\mathbf{h}$ with respect to the variable Ψ. Clearly, the domain of definition of the nonlinearities can be relaxed depending on the problem. It has to be mentioned that the assumption $\partial\Omega \in C^{2,\alpha}$ on the regularity of the boundary made in [94] is not fulfilled. However, as the proof works along the same lines using a Picard iteration with the semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$ induced by Proposition B.2 for $\partial\Omega \in C^{0,1}$, we omit details in the following proposition.

Proposition 2.3. *Let assumptions A1–A2 hold for a boundary $\partial\Omega \in C^{0,1}$. Then for each diffusion matrix $\mathbf{D}^v \in \mathbb{R}^{k\times k}_{>0}$ there exists a maximal time $T_1 = T_1(\mathbf{D}^v) > 0$, such* *that problem* (1.1)–(1.3) *has a unique* $E_{\infty,0,T_1}$*-mild solution* $\Psi = (\mathbf{u}_D, \mathbf{v}_D)$ *satisfying* $\Psi \in L^\infty(\Omega_T)^{m+k}$ *for each* $T < T_1$*. Non-diffusive components* $u_{D,i}$ *of* $\mathbf{u}_D$ *additionally satisfy* $u_{D,i} \in C([0,T]; L^\infty(\Omega))$*. Furthermore, the solutions for different diffusion* $\mathbf{D}^v$ *initially exist on some joint time interval.*

Proof. As explained in the setting (2.3)–(2.4), we define $\mathbf{h}$ by the vector with the two components $\mathbf{f}, \mathbf{g}$. The integral representation from Definition 2.2 is used for a fixed-point iteration on the Banach space $L^\infty(\Omega)^{m+k}$ according to the theorem of Picard-Lindelöf. This Picard iteration yields a unique mild solution. A non-diffusive component $u_{D,i}$ of $\mathbf{u}_D$ satisfies higher regularity in time due to the integral representation, i.e., $u_{D,i} \in C([0,T]; L^\infty(\Omega))$. Recall that the contraction property of the semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$ is independent of the diffusion matrix $\mathbf{D}^v$. If one looks carefully in the proof of [94, Part II, Theorem 1], this independence implies that all local-in-time solutions $\Psi = \Psi(\mathbf{D}^v)$ exist on the same small time interval independent of $\mathbf{D}^v \in \mathbb{R}^{k\times k}_{>0}$. □

To obtain classical solutions, one has to choose a more regular boundary $\partial\Omega \in C^{2,\alpha}$, Hölder continuous initial data and regular nonlinearities [94, Part II, Theorem 1]. In accordance to the blow-up theory for ordinary differential equations, the solution may blow up in finite time. If the maximal existence time satisfies $T_1 < \infty$, then by [94, Part II, Theorem 1]

$$\lim_{t\nearrow T_1} \|\Psi(\cdot,t)\|_{L^\infty(\Omega)^{m+k}} = \infty.$$

Beside these spatially homogeneous blow-up effects, there are cases of diffusion induced blow-up [25, 46, 65, 81]; while the corresponding system of ordinary differential equations possesses global, uniformly bounded solutions, the diffusive problem blows-up for some diffusion parameters and initial conditions. This kind of blow-up may occur if one or more diffusions are introduced as shown in a collection of examples in [25, §4]. For completeness, the following example adapted from [81] is presented.

Example 2.4. Let $D^v = \lambda_1^{-1}$ be the inverse of the first positive eigenvalue λ_1 from a spectral basis $(\lambda_j, w_j)_{j\in\mathbb{N}_0}$ of $-\Delta$ in Proposition A.1 and $a := w_1(x_0) \neq 0$ for some $x_0 \in \Omega$. Consider the system

$$\frac{\partial u_D}{\partial t} = u_D^2 - (a - v_D)^2 u_D^3 \quad \text{in} \quad \Omega_T, \qquad u_D(\cdot,0) = u^0 \;\text{ in }\; \Omega,$$

$$\frac{\partial v_D}{\partial t} - \lambda_1^{-1}\Delta v_D = v_D \qquad \text{in} \quad \Omega_T, \qquad v_D(\cdot,0) = v^0 \ \text{ in } \ \Omega,$$
$$\frac{\partial v_D}{\partial \mathbf{n}} = 0 \ \text{ on } \ \partial\Omega \times (0,T).$$

Proposition A.1 and $v^0 := w_1$ ensure that the heat equation has the stationary mild (actually weak) solution $v_D = w_1$. The component u_D satisfies an ordinary differential equation in each point $x \in \Omega$. Hence, for sufficiently smooth u^0,

$$u_D(x_0,t) = \frac{1}{u^0(x_0)^{-1} - t}$$

blows-up for $u^0(x_0) > 0$ in finite time.
Concerning space-independent solutions of the corresponding ordinary differential equations, the cubic term implies uniform bounds for the solution since $a \neq 0$. Such bounds can be found as in [25] using the method of invariant rectangles.

2.2 The shadow problem

In this section, we consider the shadow limit introduced in equations (1.4)–(1.6),

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}^u\Delta\mathbf{u} = \mathbf{f}(\mathbf{u},\mathbf{v},x,t) \qquad \text{in} \quad \Omega_T, \qquad \mathbf{u}(\cdot,0) = \mathbf{u}^0 \quad \text{in} \quad \Omega,$$
$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \langle \mathbf{g}(\mathbf{u}(\cdot,t),\mathbf{v}(t),\cdot,t)\rangle_\Omega \quad \text{in} \quad (0,T), \qquad \mathbf{v}(0) = \langle \mathbf{v}^0\rangle_\Omega,$$
$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0,T).$$

Since we are interested in low-regularity solutions, we search for solutions of the following integral representation

$$\mathbf{u}(\cdot,t) = \mathbf{S}^u(t)\mathbf{u}^0 + \int_0^t \mathbf{S}^u(t-\tau)\mathbf{f}(\mathbf{u}(\cdot,\tau),\mathbf{v}(\tau),\cdot,\tau)\,\mathrm{d}\tau \quad \text{in} \quad \Omega, \tag{2.6}$$
$$\mathbf{v}(t) = \langle \mathbf{v}^0\rangle_\Omega + \int_0^t \langle \mathbf{g}(\mathbf{u}(\cdot,\tau),\mathbf{v}(\tau),\cdot,\tau)\rangle_\Omega\,\mathrm{d}\tau, \tag{2.7}$$

with the same notation (2.5) for the semigroup $(\mathbf{S}^u(t))_{t\in\mathbb{R}_{\geq 0}}$. For a precise definition of a mild solution, we rewrite system (1.4)–(1.6) as a system of m reaction-diffusion equations coupled to k non-local differential equations. Let $\Phi = (\mathbf{u},\mathbf{v})$, then the shadow system (1.4)–(1.6) becomes

$$\frac{\partial \Phi}{\partial t} - \mathbf{D}^S\Delta\Phi = \mathbf{h}^S(\Phi,x,t) \quad \text{in} \quad \Omega_T, \qquad \Phi(\cdot,0) = (\mathbf{u^0},\langle \mathbf{v}^0\rangle_\Omega) =: \Phi^0 \quad \text{in} \quad \Omega \tag{2.8}$$

where $\mathbf{h}^S = (\mathbf{f}, \langle \mathbf{g} \rangle_\Omega)$, and $\mathbf{D}^S := \mathrm{diag}(\mathbf{D}^u, \mathbf{0}) \in \mathbb{R}^{m+k}_{\geq 0}$ is a diagonal matrix decomposed of $\mathbf{D}^u$ and k zeroes. For each component Φ_i with a non-vanishing diffusion coefficient, we again impose zero flux boundary conditions

$$\frac{\partial \Phi_i}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{2.9}$$

Definition 2.5. Let $0 < T \leq \infty$. An $E_{\infty,0,T}$*-mild solution* of the shadow problem (2.8)–(2.9) for initial datum $\Phi^0 \in L^\infty(\Omega)^m \times \mathbb{R}^k$ on the interval $[0, T)$ is a measurable function

$$\Phi : \Omega \times [0, T) \to \mathbb{R}^{m+k}$$

satisfying for all $t \in (0, T)$

(i) $\Phi(\cdot, t) \in L^\infty(\Omega)^m \times \mathbb{R}^k$ and $\sup_{s\in(0,t)} \|\Phi(\cdot, s)\|_{L^\infty(\Omega)^m \times \mathbb{R}^k} < \infty$,

(ii) the integral representation (2.6)–(2.7), i.e.,

$$\Phi(\cdot, t) = \mathbf{S}(t)\Phi^0 + \int_0^t \mathbf{S}(t - \tau)\mathbf{h}^S(\Phi(\cdot, \tau), \cdot, \tau)\, \mathrm{d}\tau$$

where the integral is a Bochner integral in $L^\infty(\Omega)^m \times \mathbb{R}^k$ and we set $\mathbf{D}^v = \mathbf{0}$ in definition (2.5) of $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$.

Applying a Picard iteration as in [94, Part II, Theorem 1] to the corresponding integral equation yields a local-in-time solution.

Proposition 2.6. *Let assumptions A1–A2 hold for a boundary* $\partial\Omega \in C^{0,1}$. *Then there is a maximal time* $T_0 > 0$ *such that the shadow problem* (1.4)–(1.6) *has a unique* $E_{\infty,0,T_0}$*-mild solution* Φ *satisfying* $\Phi = (\mathbf{u}, \mathbf{v}) \in L^\infty(\Omega_T)^m \times C([0, T])^k$ *for each* $T < T_0$. *Furthermore, non-diffusing components fulfill* $u_i \in C([0, T]; L^\infty(\Omega))$ *for each* $T < T_0$.

Proof. Using the method of proof of [94, Part II, Theorem 1], we establish a unique $E_{\infty,0,T_0}$-mild solution Φ, compare to Proposition 2.3. This mild solution possibly blows up with

$$\lim_{t\nearrow T_0} \|\Phi(\cdot, t)\|_{L^\infty(\Omega)^m\times\mathbb{R}^k} = \infty \qquad \text{if} \quad T_0 < \infty.$$

Furthermore, we have $\Phi = (\mathbf{u}, \mathbf{v}) \in L^\infty(\Omega_T)^m \times C([0, T])^k$ for each $T < T_0$, where the continuity for the non-diffusing components can be deduced from the integral representation (2.6)–(2.7). $\square$

Certainly, spatially homogeneous solutions to the shadow problem (1.4)–(1.6) for space-independent nonlinearities are solutions to the corresponding system of ordinary differential equations. Beside spatially homogeneous blow-up effects, there are cases of integro-driven blow-up which is illustrated by examples from [62, 63, 69]. Their proof of blow-up is much more complex compared to Example 2.4. The classical integro-driven blow-up is discussed in [69] in which the solution of the shadow system blows up in finite time but the corresponding ordinary differential equations exhibit global solutions. The example of Li shows even more [62, 63]; the shadow system for $\mathbf{D}^u \in \mathbb{R}_{>0}^{m\times m}$ undergoes integro-driven blow-up in finite time while the solution to the corresponding diffusive system (1.1)–(1.3) exists for all times. So far, it is not known if there is an analogon for the case of $\mathbf{D}^u = \mathbf{0}$, see for example [48], where both systems exhibit blow-up in contrast to [63].

2.3 The mean value correction

When comparing solutions of both problems the diffusive system (1.1)–(1.3) and the shadow system (1.4)–(1.6), we have to face two problems linked to the mean values in the shadow system. For one thing, the discrepancy between the initial condition for the equation of $\mathbf{v}_D$ and $\mathbf{v}$ implies an initial time layer corresponding to $\mathbf{v}^0 - \langle \mathbf{v}^0\rangle_\Omega$ which we have to incorporate. For another thing, we have to take care of a similar difference induced by $\mathbf{g} - \langle \mathbf{g}\rangle_\Omega$. In [75], the authors distinguish between the initial time layer and the mean value correction for $\mathbf{g}$ but, since in many applications uniformly bounded solutions are considered, I will combine them unless an explicit dependence is needed.

Suppose that there exists a unique shadow limit $(\mathbf{u}, \mathbf{v}) \in L^\infty(\Omega_T)^{m+k}$ on some time interval $[0, T]$. In the remainder of this work, we use the mean value correction ψ_D which solves the inhomogeneous system

$$\frac{\partial \psi_D}{\partial t} - \mathbf{D}^v \Delta \psi_D = \mathbf{g}(\mathbf{u}, \mathbf{v}, x, t) - \langle \mathbf{g}(\mathbf{u}, \mathbf{v}, \cdot, t)\rangle_\Omega \quad \text{in} \quad \Omega_T, \tag{2.10}$$

$$\frac{\partial \psi_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0, T), \qquad \psi_D(\cdot, 0) = \mathbf{v}^0 - \langle \mathbf{v}^0\rangle_\Omega \quad \text{in} \quad \Omega. \tag{2.11}$$

The right-hand side of (2.10) is bounded by Assumption A1. Using the same notation from (2.5) for the semigroup $(\mathbf{S}^v(t))_{t\in\mathbb{R}_{\geq 0}}$, the well-known Duhamel formula yields the unique mild solution

$$\psi_D(\cdot,t) = \mathbf{S}^v(t)(\mathbf{v}^0 - \langle \mathbf{v}^0 \rangle_\Omega) + \int_0^t \mathbf{S}^v(t-\tau)\Big\{\mathbf{g}(\mathbf{u},\mathbf{v},\cdot,\tau) - \langle \mathbf{g}(\mathbf{u},\mathbf{v},\cdot,\tau)\rangle_\Omega\Big\}\,\mathrm{d}\tau$$

as in Proposition 2.3. The function ψ_D fulfills $\langle \psi_D \rangle_\Omega = \mathbf{0}$ in view of Proposition B.3. Hence, the exponential decay estimate (2.2) applies and we estimate

$$\begin{aligned}\|\psi_D(\cdot,t)\|_{L^\infty(\Omega)^k} \leq{}& \overline{C}\mathrm{e}^{-\lambda_1 D t}\|v^0 - \langle v^0 \rangle_\Omega\|_{L^\infty(\Omega)^k} \\ &+ \overline{C}\int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)}\|\mathbf{g}(\mathbf{u},\mathbf{v},\cdot,\tau) - \langle \mathbf{g}(\mathbf{u},\mathbf{v},\cdot,\tau)\rangle_\Omega\|_{L^\infty(\Omega)^k}\,\mathrm{d}\tau\end{aligned}$$

utilizing the least entry $D = \min_{j=1,\dots,k} D_j^v > 0$ of $\mathbf{D}^v$. Since we assumed a bounded shadow limit, the mean value correction satisfies

$$\|\psi_D(\cdot,t)\|_{L^\infty(\Omega)^k} \leq C_{v^0}\mathrm{e}^{-\lambda_1 D t} + C_g D^{-1} \qquad \forall\, D > 0, t \in [0,T] \tag{2.12}$$

for some constants $C_{v^0}, C_g > 0$ that do not depend on D, but on bounds of $\mathbf{v}^0$ resp. $\mathbf{g}, \mathbf{u}, \mathbf{v}$ and time $T > 0$. Boundedness of Ω implies the same estimates for the Lebesgue spaces $L^p(\Omega)^k$ with $1 \leq p < \infty$. Further remark that, by estimate (2.12), the exponentially decaying initial time layer $\mathbf{S}^v(t)(\mathbf{v}^0 - \langle \mathbf{v}^0 \rangle_\Omega)$ tends to zero as t grows and it becomes negligible for larger time scales:

$$\|\psi_D(\cdot,t)\|_{L^\infty(\Omega)^k} \leq C D^{-1} \qquad \text{for} \qquad t \geq \log(D)/(\lambda_1 D)$$

As a consequence of low initial regularity, the correction term ψ_D need not to be an element of $C([0,T);L^\infty(\Omega)^k)$ because this could only be possible if $\mathbf{v}^0 \in C(\overline{\Omega})^k$ as shown in [94, Part I, Lemma 2]. Nevertheless, the mean value correction ψ_D is continuous for $t > 0$ due to the regularizing effect of the heat equation, see Proposition B.2. If one additionally assumes $\mathbf{v}^0 \in H^1(\Omega)^k = W^{1,2}(\Omega)^k$, then differentiating the Fourier expansion from Proposition B.3 yields $\psi_D \in L^\infty(0,T;H^1(\Omega)^k)$ with a similar estimate

$$\|\psi_D(\cdot,t)\|_{H^1(\Omega)^k} \leq C_{v^0}\mathrm{e}^{-\lambda_1 D t} + C_g D^{-1} \qquad \forall\, D > 0, t \in [0,T].$$

3 Short-time intervals

The shadow limit (1.4)–(1.6) can be used as a reduction of the entire diffusive system (1.1)–(1.3) for large diffusion $\mathbf{D}^v$. In case of $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{>0}$, the reduced system was considered by [32, 51, 86] and [103] who gave birth to shadow systems. Their results have been used and further developed since the 1980s. For the case of $\mathbf{D}^u = \mathbf{0}$ there are fewer results, yet the works [6, 69, 75] from the last decade are noteworthy. In all these papers, the regularity of solutions as well as of the boundary is abundant and can be reduced to obtain uniform estimates with respect to the L^∞ norm.

In this chapter, I examine the relation between the partly diffusive system and its shadow system for large diffusion $\mathbf{D}^v$, i.e., the least positive diagonal entry D of $\mathbf{D}^v$ is assumed to be large. In view of low regularity of solutions, we consider mild solutions of the partly diffusive system and its shadow limit. The difference of solutions to both systems is estimated including the mean value correction ψ_D, which takes account of the singular limit. The length of the time interval on which the error estimates are derived is not prescribed within this study. Nevertheless, due to a possible deterioration of constants depending on time, estimates in Theorem 3.3 below are meaningful in general only if the length T of the time interval is finite, i.e., $T = \mathcal{O}(1)$ as $D \to \infty$. Apart from explicit uniform error estimates in the next sections, based on the ideas of [75], I discuss several limitations of the result and the used method of proof, especially in the reaction-diffusion-ODE case $\mathbf{D}^u = \mathbf{0}$.

The approach used in [75, Section 3] is based on a linearization around the bounded shadow solution that exists locally in time. Due to low spatial regularity of the shadow solution, the linearization might contain zero order terms that are discontinuous in space as well. To obtain estimates with respect to the $L^\infty(\Omega_T)$ norm, the nonlinearities of the system of errors are cut off and uniform estimates for the localized solutions are derived in Section 3.1. For a suitable cut-off, the truncation can be removed as the least positive entry D of the diffusion matrix $\mathbf{D}^v$ grows. In this way, estimates for the original errors are obtained in Section 3.2. The following sections also show that the assumptions made in [75] can be weakened to achieve uniform convergence results for the shadow approximation.

3.1 First-order truncation

As Proposition 2.3 shows, the partly diffusive system (1.1)–(1.3) possesses, for each diffusion matrix $\mathbf{D}^v$, a unique local solution $(\mathbf{u}_D, \mathbf{v}_D)$ on some time interval $[0, T_1)$, where $T_1 = T_1(\mathbf{D}^v)$ depends on the diffusion matrix. The corresponding shadow system (1.4)–(1.6) admits the local-in-time solution $(\mathbf{u}, \mathbf{v})$ on $[0, T_0)$ by Proposition 2.6. Clearly, both problems differ drastically at $t = 0$ and the initial time layer of $\mathbf{v}_D - \mathbf{v}$, which is incorporated in the mean value correction ψ_D, is used to obtain estimates up to $t = 0$. For a comparison of solutions, we start with a given shadow limit $(\mathbf{u}, \mathbf{v})$ from Proposition 2.6 and introduce the error functions

$$\mathbf{U}_D = \mathbf{u}_D - \mathbf{u} \quad \text{and} \quad \mathbf{V}_D = \mathbf{v}_D - \mathbf{v} - \psi_D. \tag{3.1}$$

Since we consider nonlinear systems, we take $L^\infty(\Omega_T)$ as a basis to estimate the error functions. We shall show convergence

$$(\mathbf{U}_D, \mathbf{V}_D) \to \mathbf{0} \qquad \text{as} \quad D := \min_{j=1,\dots,k} D_j^v \to \infty$$

with respect to the norm induced by $L^\infty(\Omega_T)$. Such estimates allow to link the time interval of existence of solutions to the partly diffusive system (1.1)–(1.3) to some time interval of existence for the shadow problem (1.4)–(1.6). More precisely, using the maximal existence times T_0, T_1 above, for each $T < T_0$ we find some lower bound $D_T > 0$ such that $T_1(\mathbf{D}^v) > T$ for each sufficiently large $\mathbf{D}^v$ satisfying $D \geq D_T$.

Due to the mean value correction ψ_D defined in equations (2.10)–(2.11), the error functions fulfill the system

$$\frac{\partial \mathbf{U}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{U}_D = \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{f}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T, \tag{3.2}$$

$$\frac{\partial \mathbf{V}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{V}_D = \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{g}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T, \tag{3.3}$$

$$\mathbf{U}_D(\cdot, 0) = \mathbf{0}, \mathbf{V}_D(\cdot, 0) = \mathbf{0} \ \text{ in } \ \Omega, \quad \frac{\partial \mathbf{U}_D}{\partial \mathbf{n}} = \mathbf{0}, \frac{\partial \mathbf{V}_D}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0, T). \tag{3.4}$$

In view of Assumption A1, we are able to linearize the error system (3.2)–(3.4) around the shadow solution $(\mathbf{u}, \mathbf{v})$. Let us write this problem as one equation

$$\frac{\partial \Psi_D}{\partial t} - \mathbf{D} \Delta \Psi_D = \mathbf{h}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{h}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T,$$

where we used $\Psi_D = (\mathbf{U}_D, \mathbf{V}_D), \mathbf{D} = \operatorname{diag}(\mathbf{D}^u, \mathbf{D}^v)$ and $\mathbf{h} = (\mathbf{f}, \mathbf{g})$. A Taylor expansion yields the following system

$$\frac{\partial \Psi_D}{\partial t} - \mathbf{D}\Delta\Psi_D = \mathbf{H}(\mathbf{U}_D, \mathbf{V}_D + \psi_D, x, t) \quad \text{in} \quad \Omega_T, \tag{3.5}$$

where $\mathbf{H} = (\mathbf{F}, \mathbf{G})$ is given by the linearized term

$$\mathbf{H}(\mathbf{y}, \mathbf{z} + \psi_D, x, t) = \mathbf{h}^u(\mathbf{y}, \mathbf{z} + \psi_D, x, t)\mathbf{y} + \mathbf{h}^v(\mathbf{y}, \mathbf{z} + \psi_D, x, t)\,(\mathbf{z} + \psi_D)\,. \tag{3.6}$$

Herein, we use the matrix-valued functions $\mathbf{h}^u = (h^u_{\ell i})_{\substack{i=1,\dots,m \\ \ell=1,\dots,m+k}}, \mathbf{h}^v = (h^v_{\ell j})_{\substack{j=1,\dots,k \\ \ell=1,\dots,m+k}}$. The entries $h^u_{\ell i}, h^v_{\ell j}$ are given by difference quotients. Their structure can be adapted from the scalar-valued case $k = 1 = m$ where

$$h^u_{11}(y, z + \psi_D, x, t) = \begin{cases} \dfrac{f(u+y, v+z+\psi_D, x, t) - f(u, v+z+\psi_D, x, t)}{y}, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

$$h^v_{11}(y, z + \psi_D, x, t) = \begin{cases} \dfrac{f(u, v+z+\psi_D, x, t) - f(u, v, x, t)}{z + \psi_D}, & z + \psi_D \neq 0, \\ 0, & z + \psi_D = 0. \end{cases}$$

Notice that, a priori, the errors are defined only on Ω_T for times

$$T \leq \inf_{\mathbf{D}^v} \min\{T_1(\mathbf{D}^v), T_0\},$$

i.e., on a common time interval of existence for the partly diffusive solution and the shadow solution. Following ideas of center manifold theory (see for example [11, 34] or [113, Chapter 9, Section 5]), we construct a suitable cut-off for the possibly unbounded remainder $\mathbf{H}$. In general, this procedure localizes the problem and is also useful in proving local existence or positivity of solutions [113, Chapter 9, Section 2]. In order to show that the errors actually exist on a common time interval $[0, T]$ for all sufficiently large diffusion $\mathbf{D}^v$, we identify the bounded solutions of the truncated problems with the original error functions for large $D = \min_{j=1,\dots,k} D^v_j$ depending on the interval length T.

For the cut-off procedure, let $\Theta \in C^{0,1}(\mathbb{R})$ be defined by

$$\Theta(z) = \begin{cases} \operatorname{sgn}(z) \cdot D^{-\delta_0} & \text{for } |z| > D^{-\delta_0}, \\ z & \text{for } |z| \leq D^{-\delta_0} \end{cases} \tag{3.7}$$

for some $\delta_0 \in (0,1)$. Then Θ is uniformly bounded and satisfies the estimate

$$|\Theta(z)| \leq \min\{D^{-\delta_0}, |z|\} \qquad \forall\, z \in \mathbb{R}.$$

For simplicity, we write $\Theta(\mathbf{y}) := (\Theta(y_1), \ldots, \Theta(y_m))$ in the vector-valued case, too. To assure boundedness of the model variables in the subsequent analysis, we truncate the remainder $\mathbf{H}$. Using the cut-off function Θ, we define $\mathbf{H}_D = (\mathbf{F}_D, \mathbf{G}_D)$ by

$$\begin{aligned} \mathbf{H}_D(\mathbf{y}, \mathbf{z}, x, t) = {} & \mathbf{h}^u(\Theta(\mathbf{y}), \Theta(\mathbf{z}) + \psi_D, x, t)\mathbf{y} \\ & + \mathbf{h}^v(\Theta(\mathbf{y}), \Theta(\mathbf{z}) + \psi_D, x, t)\,(\mathbf{z} + \psi_D)\,. \end{aligned}$$

Since the functions $\mathbf{h}^u, \mathbf{h}^v$ are locally bounded due to Lipschitz continuity stated in Assumption A1, there exists a uniform bound for these coefficients depending only on bounds of the shadow limit $(\mathbf{u}, \mathbf{v})$, the mean value corrrection ψ_D and the cut-off Θ. This assures that the truncated function $\mathbf{H}_D$ grows at most linearly in the variables $\mathbf{y}$ and $\mathbf{z} + \psi_D$.

The idea for using the cut-off is as follows. Let (α_D, β_D) be the solution to the truncated problem

$$\frac{\partial \alpha_D}{\partial t} - \mathbf{D}^u \Delta \alpha_D = \mathbf{F}_D(\alpha_D, \beta_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \alpha_D(\cdot, 0) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{3.8}$$

$$\frac{\partial \beta_D}{\partial t} - \mathbf{D}^v \Delta \beta_D = \mathbf{G}_D(\alpha_D, \beta_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \beta_D(\cdot, 0) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{3.9}$$

$$\frac{\partial \alpha_D}{\partial \mathbf{n}} = \mathbf{0}, \quad \frac{\partial \beta_D}{\partial \mathbf{n}} = \mathbf{0} \qquad \text{on} \quad \partial\Omega \times (0, T). \tag{3.10}$$

Using suitable estimates, we will show that the truncated solution (α_D, β_D) is located within the small neighborhood of $\mathbf{0}$ with radius $D^{-\delta_0}$ for sufficiently large diffusions D. However, in this small neighborhood of the origin, the cut-off Θ does not affect the right-hand sides if one restricts to the trajectory of the solution (α_D, β_D). As solutions to problem (3.5) supplemented with boundary conditions (3.4) are unique, this implies $(\alpha_D, \beta_D) = (\mathbf{U}_D, \mathbf{V}_D)$. Before showing that this is the case for the particular choice of Θ, let us summarize some properties of the truncated right-hand side.

Lemma 3.1. *For each time $T < T_0$ there is a constant $C_T > 0$, independent of $D \geq 1$ but which depends on time T, such that*

$$|\mathbf{F}_D(\mathbf{y}, \mathbf{z}, x, t)|, |\mathbf{G}_D(\mathbf{y}, \mathbf{z}, x, t)| \leq C_T\Big(|\mathbf{y}| + |\mathbf{z} + \psi_D(x, t)|\Big) \tag{3.11}$$

holds for all $(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{m+k}, x \in \overline{\Omega}$ and $t \in [0, T]$. Especially, $\mathbf{F}_D, \mathbf{G}_D$ satisfy the local Lipschitz condition A1 in $(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{m+k}$ on bounded sets of $\mathbb{R}^{m+k} \times \overline{\Omega} \times [0, T_0)$.

Proof. We assume $D \geq 1$ for simplicity and consider the constants C_{v^0}, C_g from estimate (2.12) which we derived for ψ_D. Note that C_g depends on time T while C_{v^0} is independent of time. Using this estimate for ψ_D, we obtain

$$\begin{aligned} \|\mathbf{u} + \Theta(\mathbf{y})\|_{L^\infty(\Omega_T)^m} &\leq \|\mathbf{u}\|_{L^\infty(\Omega_T)^m} + 1, \\ \|\mathbf{v} + \Theta(\mathbf{z}) + \psi_D\|_{L^\infty(\Omega_T)^k} &\leq \|\mathbf{v}\|_{L^\infty(\Omega_T)^k} + 1 + C_{v^0} + C_g. \end{aligned}$$

By Assumption A1 and the Lipschitz continuity of Θ, we are able to estimate all difference quotients in the matrices $\mathbf{h}^u_\ell(\Theta(\mathbf{y}), \Theta(\mathbf{z}) + \psi_D, x, t), \mathbf{h}^v_\ell(\Theta(\mathbf{y}), \Theta(\mathbf{z}) + \psi_D, x, t)$ for bounded arguments. Hence, the function $\mathbf{H}_D$ is defined on $\mathbb{R}^{m+k} \times \overline{\Omega} \times [0, T]$ and there exists a constant $C_T > 0$ such that

$$|\mathbf{F}_D(\mathbf{y}, \mathbf{z}, x, t)|, |\mathbf{G}_D(\mathbf{y}, \mathbf{z}, x, t)| \leq C_T\Big(|\mathbf{y}| + |\mathbf{z} + \psi_D(x, t)|\Big)$$

holds. Furthermore, as a composition of local Lipschitz functions, the function $\mathbf{H}_D = (\mathbf{F}_D, \mathbf{G}_D)$ is locally Lipschitz in the variable $(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{m+k}$ in the sense of Assumption A1 for all finite times $T < T_0$. □

Having the above growth bound for the truncated right-hand sides of the system (3.8)–(3.10) in mind, we consider mild solutions to the cut-off problem.

Proposition 3.2. *Let assumptions A1–A2 hold. Then there exists a unique mild solution (α_D, β_D) of the cut-off problem* (3.8)–(3.10). *The solution is defined on the same maximal time interval of existence $[0, T_0)$ as the shadow solution from Proposition 2.6. For each time $T < T_0$ there is a constant $C_T > 0$, independent of $D \geq 1$ but which depends on T, such that the solution satisfies the estimate*

$$\|\alpha_D\|_{L^\infty(\Omega_T)^m} + \|\beta_D\|_{L^\infty(\Omega_T)^k} \leq C_T D^{-1}. \tag{3.12}$$

Proof. As in the partly diffusive case in Proposition 2.3, we solve the problem using the method of Rothe [94, Part II, Theorem 1]. Similarly to the setting (2.3), we rewrite system (3.8)–(3.10) as a system of $m+k$ reaction-diffusion equations

$$\frac{\partial \Psi_D}{\partial t} - \mathbf{D}\Delta\Psi_D = \mathbf{H}_D(\Psi_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \Psi_D(\cdot, 0) = \mathbf{0} \quad \text{in} \quad \Omega$$

where $\Psi_D = (\alpha_D, \beta_D), \mathbf{H}_D = (\mathbf{F}_D, \mathbf{G}_D)$ and possibly not all components of α_D diffusive. Observe that the right-hand side $\mathbf{H}_D$ of the above system fulfills a local Lipschitz condition in Ψ_D by Lemma 3.1. Moreover, the growth estimate

$$\|\mathbf{H}_D(\mathbf{w}, \cdot, t)\|_{L^\infty(\Omega)^{m+k}} \leq k_1(t) + k_2(t)|\mathbf{w}| \qquad \forall\, \mathbf{w} \in \mathbb{R}^{m+k}, t \in [0, T_0)$$

holds for some non-negative continuous functions k_1, k_2 due to the bound (2.12) for ψ_D and Lemma 3.1. We verify assumptions of [94, Part II, Theorem 1] and its proof to get an $E_{\infty,0,\tau}$-mild solution in the sense of Definition 2.2. Furthermore, we have the integral representation

$$\begin{aligned} \alpha_D(\cdot, t) &= \int_0^t \mathbf{S}^u(t-\tau)\mathbf{F}_D(\Psi_D(\cdot, \tau), \cdot, \tau)\, \mathrm{d}\tau, \\ \beta_D(\cdot, t) &= \int_0^t \mathbf{S}^v(t-\tau)\mathbf{G}_D(\Psi_D(\cdot, \tau), \cdot, \tau)\, \mathrm{d}\tau. \end{aligned}$$

The solution can be extended to some maximal time interval $[0, T_{\max})$ for which we will show $T_{\max} \geq T_0$, where T_0 is the maximal time interval of existence of the shadow solution from Proposition 2.6. In fact, if $T_{\max} < T_0 \leq \infty$, we use the growth condition for $\mathbf{H}_D$ to obtain

$$\|\Psi_D(\cdot, t)\|_{L^\infty(\Omega)^{m+k}} \leq \int_0^t k_1(\tau) + k_2(\tau)\|\Psi_D(\cdot, \tau)\|_{L^\infty(\Omega)^{m+k}}\, \mathrm{d}\tau \qquad \forall\, t \in [0, T_{\max})$$

due to the contraction property of the semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$. Gronwall's inequality implies boundedness of the solution uniformly in $t \in [0, T_{\max})$. This is a contradiction to the well-known blow-up effect of the solution when $t \nearrow T_{\max}$, i.e.,

$$\lim_{t \nearrow T_{\max}} \|\Psi_D(\cdot, t)\|_{L^\infty(\Omega)^{m+k}} = \infty \qquad \text{if} \quad T_{\max} < \infty.$$

Thus, we obtain a unique $E_{\infty,0,T_{\max}}$-mild solution for some $T_{\max} \geq T_0$ satisfying $\Psi_D \in L^\infty(\Omega_T)^{m+k}$ for all $T < T_{\max}$ by Definition 2.2. Recall that above truncated problem depends on the shadow solution and is at most well-defined on the interval $[0, T_0)$, i.e., $T_{\max} = T_0$. Using again the integral representation, the contraction property of the semigroup $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$ and the estimate (3.11) for $\mathbf{H}_D$ yields

$$\begin{aligned}\|\Psi_D(\cdot,t)\|_{L^\infty(\Omega)^{m+k}} &\leq C_1\int_0^t \|\Psi_D(\cdot,\tau)\|_{L^\infty(\Omega)^{m+k}}\,\mathrm{d}\tau + C_2\int_0^t \|\psi_D(\cdot,\tau)\|_{L^\infty(\Omega)^k}\,\mathrm{d}\tau\\ &\leq C_1\int_0^t \|\Psi_D(\cdot,\tau)\|_{L^\infty(\Omega)^{m+k}}\,\mathrm{d}\tau + C_3D^{-1}\end{aligned}$$

for all $t\in[0,T]$ and $D\geq 1$. Gronwall's inequality then implies the existence of a constant $C_T>0$ depending (in general exponentially) on time T such that estimate (3.12) holds. □

3.2 Convergence results

We are now able to draw a conclusion for the original errors $(\mathbf{U}_D,\mathbf{V}_D)$ using the estimates for the truncated problem in the last section. We infer that the cut-off introduced in system (3.8)–(3.10) is not required for sufficiently large diffusions D. Consequently, the same estimates from the last section for the truncated errors (α_D,β_D) hold true for the original error functions $(\mathbf{U}_D,\mathbf{V}_D)$.

Theorem 3.3. *Let assumptions A1–A2 hold and let T_0 be the maximal existence time of the shadow limit $(\mathbf{u},\mathbf{v})$ from Proposition 2.6. Then for each time $T<T_0$ there are constants $C_T>0$ and $D_T\geq 1$, independent of D but which depend on T, such that for $D\geq D_T$ we have*

$$\|\mathbf{u}_D-\mathbf{u}\|_{L^\infty(\Omega_T)^m}\leq C_TD^{-1}, \tag{3.13}$$
$$\|\mathbf{v}_D-\mathbf{v}-\psi_D\|_{L^\infty(\Omega_T)^k}\leq C_TD^{-1}, \tag{3.14}$$
$$\|\langle\mathbf{v}_D\rangle_\Omega-\mathbf{v}\|_{L^\infty(0,T)^k}\leq C_TD^{-1}. \tag{3.15}$$

Consequently, the diffusive solution $(\mathbf{u}_D,\mathbf{v}_D)$ from Proposition 2.3 exists on the same time interval $[0,T]$ for each $D\geq D_T$.

Proof. Since $C_TD^{-1}\leq D^{-\delta_0}$ for large $D\geq D_T$ and $\delta_0\in(0,1)$, we conclude that $\Theta(\alpha_D)=\alpha_D,\Theta(\beta_D)=\beta_D$ on the interval $[0,T]$. Hence, (α_D,β_D) satisfies the same equations as $(\mathbf{U}_D,\mathbf{V}_D)$ on $[0,T]$. Uniqueness implies the equality $(\alpha_D,\beta_D)=(\mathbf{U}_D,\mathbf{V}_D)$ and the corresponding first two estimates for the error functions are verified. The remaining estimate (3.15) is a consequence of $\langle\psi_D\rangle_\Omega=\mathbf{0}$ deduced from equations (2.10)–(2.11). One has to observe that the solution $(\mathbf{u}_D,\mathbf{v}_D)$ of the diffusive problem (1.1)–(1.3) is not involved explicitly in the linearized problem (3.5). Therefore, the uniform estimates imply that $(\mathbf{u}_D, \mathbf{v}_D)$ exist on a common time interval $[0, T]$ for each large diffusion $D \geq D_T$. □

Theorem 3.3 is the analogon of [75, Theorem 3] but the former requires less boundary regularity and employs optimized assumptions on the nonlinearities $\mathbf{f}, \mathbf{g}$. In view of low-regularity of mild solutions, Theorem 3.3 generalizes results from [6, Section 3.1] in the particular case of reaction-diffusion-type equations with zero flux boundary conditions. The latter method requires a strongly continuous heat semigroup generated on $C(\overline{\Omega})$ and hence, it is only working for initial conditions that are continuous in space [94, Part I, Lemma 2]. Allowing for low-regularity solutions, assumptions A1–A2 are sufficient for the rough uniform error estimate given by (3.13)–(3.15) on short-time intervals. Unfortunately, the employed truncation method does not provide information on the relation of the maximal time interval of existence of solutions to the partly diffusive problem and its shadow problem.

The above result is applicable to many models from natural sciences, see [51, 53, 69, 86], including the references and examples given in Chapter 6. The closest to our result is [53, Theorems 4]. The authors develop L^∞ error estimates and focus on time-dependence of the constants involved, especially on D_T from Theorem 3.3. The requirements A1–A2 are met in [53] and thus, Theorem 3.3 is applicable. To compare our results with [53, Theorem 4], it has to be mentioned that

$$\|\psi_D(\cdot, t)\|_{L^\infty(\Omega)} \leq C D^{-1} \qquad \forall\, t \in [T(D), \infty) \tag{3.16}$$

holds for the mean value correction where we used (without Sobolev's inequality)

$$T(D) := \max\left\{0, \frac{\log(\|w_0 - \langle w_0\rangle_\Omega\|_{L^\infty(\Omega)} D)}{\lambda_1 D}\right\}$$

which is different than in [53]. Moreover, using the decay estimate (2.12) for ψ_D, the constant $C > 0$ in above estimate (3.16) only depends on uniform bounds of the shadow solution $(u^\infty, v^\infty, \xi)$ from [53, Theorem 2]. Although Theorem 3.3 extends the time range in which the error estimate is valid without considering the initial time layer, the lower bound D_T does depend (possibly exponentially) on time T. In [53], the lower bound D_T of the diffusion is independent of T. On the downside, the authors obtain a lower convergence rate as $D \to \infty$ as well as a time interval $[T(D), T]$ with a much larger $T(D)$ on which the estimates hold.

Although we use a low-regularity setting, we can obtain estimates for first-order derivatives of the errors. A natural consequence of the weak formulation, also used for energy estimates in the L^2 setting of [53], are the following estimates for spatial and temporal derivatives.

Corollary 3.4. *Let assumptions A1–A2 hold and let $(\mathbf{U}_D, \mathbf{V}_D)$ be given by equation* (3.1). *Then, additionally to the conclusions of Theorem 3.3, we obtain that for each time $T < T_0$ there exist constants $C_T > 0$ and $D_T \geq 1$, independent of D but which depend on T, such that for $D \geq D_T$ we have for all $t \in [0, T]$*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{U}_D\rangle_\Omega(t)\right|, \left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{V}_D\rangle_\Omega(t)\right| \leq C_T(D^{-1} + e^{-\lambda_1 D t}),$$
$$\|\nabla \mathbf{V}_D\|_{L^2(\Omega_T)^{n\times k}} \leq C_T D^{-3/2}.$$

Proof. According to Proposition B.3, the mild solution $(\alpha_D, \beta_D) = (\mathbf{U}_D, \mathbf{V}_D)$ is a weak solution of the truncated problem (3.8)–(3.10). Testing the weak formulation with the constant function $|\Omega|^{-1}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{U}_D(\cdot, t)\rangle_\Omega = \langle \mathbf{F}_D(\mathbf{U}_D, \mathbf{V}_D, \cdot, t)\rangle_\Omega,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{V}_D(\cdot, t)\rangle_\Omega = \langle \mathbf{G}_D(\mathbf{U}_D, \mathbf{V}_D, \cdot, t)\rangle_\Omega.$$

Both expressions can be bounded using estimate (3.11) for $\mathbf{F}_D, \mathbf{G}_D$. For non-diffusing components of $\mathbf{U}_D$, the time derivative $\partial_t U_{D,i}$ can be estimated as well. For an estimate of the spatial gradient, we consider the purely diffusing equation for $\mathbf{V}_D$ which may be written as

$$\frac{\partial \mathbf{V}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{V}_D = \mathbf{G}_D(\mathbf{U}_D, \mathbf{V}_D, x, t) \quad \text{in} \quad \Omega_T,$$
$$\mathbf{V}_D(\cdot, 0) = \mathbf{0} \ \text{ in } \ \Omega, \qquad \frac{\partial \mathbf{V}_D}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0, T),$$

where $\mathbf{G}_D$ is a bounded function by Lemma 3.1 for large enough $D \geq D_T$. By Proposition B.3, $\mathbf{V}_D \in L^\infty(0, T; H^1(\Omega)^k)$ is a weak solution with $\partial_t \mathbf{V}_D \in L^2(\Omega_T)^k$. Taking $\mathbf{V}_D$ as a test function yields, using [98, Proposition III.1.2],

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\mathbf{V}_D|^2 \,\mathrm{d}x + \sum_{j=1}^k D_j^v \int_\Omega |\nabla V_{D,j}|^2 \,\mathrm{d}x &= \int_\Omega \mathbf{G}_D(\mathbf{U}_D, \mathbf{V}_D, x, t) \cdot \mathbf{V}_D \,\mathrm{d}x \\ &\leq C_T D^{-1}\left(D^{-1} + e^{-\lambda_1 D t}\right).\end{aligned}$$

For the latter energy estimate, we used the estimates (3.11), (2.12) and (3.14) for $\mathbf{G}_D, \psi_D$ and $\mathbf{V}_D$, respectively. Since an integration over $(0,T)$ yields

$$\sum_{j=1}^{k} D_j^v \|\nabla V_{D,j}\|^2_{L^2(\Omega_T)^n} \le C_T D^{-2},$$

the definition of $D = \min_{j=1,\ldots,k} D_j^v$ implies the assertion. Additionally, energy estimates for diffusing components of $\mathbf{U}_D$ imply $\|\nabla U_{D,i}\|_{L^2(\Omega)^n} \le C_T D^{-1}$. □

Proposition 3.2 might suggest that the diffusive solution $(\mathbf{u}_D, \mathbf{v}_D)$ exists at least as long as the shadow solution. However, Example 2.4 shows that this is not true for all diffusion matrices $\mathbf{D}^v$ in general since D_T may be quite large.

Example 3.5. Let $D^v = \lambda_1^{-1}$ be the inverse of λ_1 from a spectral basis $(\lambda_j, w_j)_{j\in\mathbb{N}_0}$ of $-\Delta$ in Proposition A.1 and $a := w_1(x_0) \neq 0$ for some $x_0 \in \Omega$. Consider the system

$$\begin{aligned}
\frac{\partial u_D}{\partial t} &= u_D^2 - (a - v_D)^2 u_D^3 \quad \text{in} \quad \Omega_T, \qquad u_D(\cdot,0) = u^0 \ \text{in} \ \Omega,\\
\frac{\partial v_D}{\partial t} - \lambda_1^{-1}\Delta v_D &= v_D \qquad\qquad\qquad\quad\ \text{in} \quad \Omega_T, \qquad v_D(\cdot,0) = v^0 \ \text{in} \ \Omega,\\
\frac{\partial v_D}{\partial \mathbf{n}} &= 0 \ \text{ on } \ \partial\Omega \times (0,T).
\end{aligned}$$

As already seen in Example 2.4, $v^0 := w_1$ and sufficiently smooth, positive u^0 imply a blow-up of u_D in x_0 in finite time. Concerning the shadow limit, it is clear that $\langle v^0\rangle_\Omega = 0$ and $v(t) = \langle v^0\rangle_\Omega \mathrm{e}^t = 0$. The corresponding solution u is given by

$$\frac{\partial u}{\partial t} = u^2 - a^2 u^3$$

and (u,v) is uniformly bounded by $\max\{\|u^0\|_{L^\infty(\Omega)}, |a|^{-2}\}$ as in the case of ordinary differential equations.

The latter example can be adapted to different diffusion parameters using a multiple of the stationary solution w_1, e.g., by $D^v = d\lambda_1^{-1}$. Note that Theorem 3.3 reveals a relation between the error functions $(\mathbf{U}_D, \mathbf{V}_D)$ and the truncated solutions (α_D, β_D) of Proposition 3.2 only for sufficiently large diffusion D. Not only the constant C_T but also the lower diffusion bound D_T may grow exponentially as $T \nearrow T_0$ and the estimates deteriorate as the next example shows. Hence, a consideration of the shadow limit as an approximation of the full system (1.1)–(1.3) is, in general, only useful for short-time intervals.

Example 3.6. Let us take an eigenfunction w_j of $-\Delta$ for some $j \in \mathbb{N}$ from Proposition A.1. We consider initial values $u^0 = 0, v^0 = w_j$ for the linear problem

$$\begin{aligned}
\frac{\partial u_D}{\partial t} - D^u \Delta u_D &= a u_D + b v_D \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad u_D(\cdot, 0) = u^0 \;\; \text{in} \;\; \Omega, \\
\frac{\partial v_D}{\partial t} - D \Delta v_D &= c u_D + d v_D \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad v_D(\cdot, 0) = v^0 \;\; \text{in} \;\; \Omega, \\
\frac{\partial u_D}{\partial \mathbf{n}} &= \frac{\partial v_D}{\partial \mathbf{n}} = 0 \;\; \text{on} \;\; \partial\Omega \times \mathbb{R}_{>0}
\end{aligned}$$

with constant coefficients $a, b, c, d \in \mathbb{R}$. Let $a > \lambda_j D^u \geq 0$ such that the corresponding matrix

$$\mathbf{M} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is stable, hence it satisfies $\operatorname{tr}(\mathbf{M}) = a + d < 0$ and $\det(\mathbf{M}) = ad - bc > 0$. It is clear from the initial conditions that $\langle v^0 \rangle_\Omega = 0$. The corresponding shadow limit is $(u, v) = (0, 0)$ as u is the unique solution of the corresponding heat equation resp. ODE for each fixed $D^u \geq 0$, see Proposition B.2 after rescaling with e^{-at}. Moreover, the mean value correction ψ_D reduces to

$$\psi_D(\cdot, t) = S_\Delta(Dt) v^0 = \mathrm{e}^{-D\lambda_j t} w_j.$$

If we consider the error $U_D = u_D - u = u_D$, it remains to solve the linear equation for (u_D, v_D) by Galerkin's ansatz as in Proposition B.3. The solution is given by the projection on the eigenspace spanned by w_j since the initial values are multiples of w_j, i.e.,

$$u_D(\cdot, t) = u_{j,D}(t) w_j \qquad \text{and} \qquad v_D(\cdot, t) = v_{j,D}(t) w_j.$$

The coefficients satisfy the following system of ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}u_{j,D}}{\mathrm{d}t} + D^u \lambda_j u_{j,D} &= a u_{j,D} + b v_{j,D} \quad \text{in} \quad \mathbb{R}_{>0}, \qquad u_{j,D}(0) = 0, \\
\frac{\mathrm{d}v_{j,D}}{\mathrm{d}t} + D \lambda_j v_{j,D} &= c u_{j,D} + d v_{j,D} \quad \text{in} \quad \mathbb{R}_{>0}, \qquad v_{j,D}(0) = 1.
\end{aligned}$$

Solving this differential equation yields the simple representation

$$\begin{pmatrix} u_D(\cdot, t) \\ v_D(\cdot, t) \end{pmatrix} = \mathrm{e}^{\mathbf{M}_{D,j} t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot w_j \qquad \text{with} \quad \mathbf{M}_{D,j} = \begin{pmatrix} a - \lambda_j D^u & b \\ c & d - \lambda_j D \end{pmatrix}.$$

The exponential of the shifted matrix $\mathbf{M}_{D,j} = \mathbf{M} - \lambda_j \mathbf{D}$ is given by

$$\mathrm{e}^{\mathbf{M}_{D,j} t} = \mathrm{e}^{\tau_D t/2} \left(\frac{1}{\gamma_D} \sinh(\gamma_D t) \mathbf{M}_{D,j} + \left(\cosh(\gamma_D t) - \frac{\tau_D}{2\gamma_D} \sinh(\gamma_D t) \right) I \right)$$

for $\tau_D = \operatorname{tr}(\mathbf{M}_{D,j}) < 0, \delta_D = \det(\mathbf{M}_{D,j}) < 0$ and $\gamma_D > 0$ with $4\gamma_D^2 = \tau_D^2 - 4\delta_D$. This is due to Putzer's formula [93, Theorem 2]. Then the error $U_D = u_D$ is given by

$$U_D(\cdot, t) = \frac{b}{\gamma_D} \sinh(\gamma_D t) \mathrm{e}^{\tau_D t/2} w_j = \frac{b}{2\gamma_D} \left(\mathrm{e}^{(\tau_D/2 + \gamma_D)t} - \mathrm{e}^{-(\tau_D/2 - \gamma_D)t} \right) w_j.$$

Recall that the positive eigenvalue converges, $\tau_D/2 + \gamma_D \to a - \lambda_j D^u$ as $D \to \infty$. Furthermore, we find some constants satisfying

$$C_1 D^{-1} \leq \gamma_D^{-1} \leq C_2 D^{-1}$$

and we infer $\|U_D(\cdot, t)\|_{L^2(\Omega)} \geq C_3 D^{-1} \mathrm{e}^{(a - \lambda_j D^u)t/2} - C_4 D^{-1}$ for some constants $C_i > 0$ independent of time t and diffusion D. The latter inequality implies exponential growth of the error $\mathbf{U}_D$ in $L^\infty(\Omega)$ by boundedness of Ω and $a > \lambda_j D^u$.

The last example also illustrates diffusion-driven instability, both for $D^u = 0$ and $D^u > 0$, see [106] and references therein. This type of instability arises if the corresponding system of ordinary differential equations possesses a spatially constant steady state which is stable to spatially homogeneous perturbations but unstable to spatially heterogeneous perturbations, i.e., space-dependent perturbations. In fact, the above example has asymptotically stable steady states in the ODE case but diffusion destabilizes the system; the determinant δ_D switches its sign for bigger diffusion D in view of the unstable subsystem corresponding to $a > \lambda_j D^u$. This causes that solutions grow exponentially in time, even if we choose small initial perturbations such as $v^0 = \delta w_j$ for small parameters $\delta > 0$. Classically considered for $D^u > 0$, this mechanism is frequently called Turing instability used in the context of pattern formation, see also [38, 69, 71] and references therein. A complete spectral analysis of the above linear operator induced by $\mathbf{M} + \mathbf{D}\Delta$ is deferred to Propositions 5.11 and 5.13, consult also Example 5.14 for further discussions on linearized stability.

4 Long-time intervals

A comparison of dynamics of the diffusive system and its shadow limit, as done for instance in [53, Theorem 4], [69, Theorem A.2] or Example 3.6, indicates that error estimates may deteriorate significantly for larger time scales. In this thesis, long-time intervals $(0, T)$ have a length proportional to a power of the least positive entry D of the diffusion matrix $\mathbf{D}^v$, i.e., $T \leq CD^\ell$ for some $0 < \ell < 1$. The error functions are estimated by a bound proportional to a power of D^{-1} as $D \to \infty$. In order to obtain a valuable approximation of the full system (1.1)–(1.3), the aim of the present chapter is to establish long-range estimates on $(0, T)$ for $T \leq CD^\ell$, subject to the existence of a global shadow limit. As Example 3.6 shows, we have to precise assumptions A1–A2 made in the last chapter to prevent from (integro- and diffusion-driven) instabilities. To understand the full nonlinear system and, furthermore, under which conditions convergence can be shown via linearization around the shadow solution, we consider first linear systems in Section 4.1 and subsequently proceed with the nonlinear case in Section 4.2.

The linear case already includes all key aspects of extending estimates in a valuable manner. Starting from a stability condition for the evolution of the ODE subsystem in the space-independent case as concerned in [55], we consider space-dependent linear problems which require a further stability condition on the entire shadow system for long-time estimates. The last stability condition is chosen with respect to some L^p space. The case $p = \infty$ implies error estimates at once in Proposition 4.4 below. A bootstrap argument for parabolic equations is used to achieve a similar result for finite, sufficiently big p; see further the brief description of the employed method preceding Theorem 4.5 and results therein.
The nonlinear case finally combines stability properties of the linearized shadow system with the truncation method from Chapter 3. Using a more complex cut-off procedure for the second derivatives similar to [75, Theorem 3] yields estimates for the localized solutions. The truncation may be removed as $D \to \infty$ and estimates for the original solutions are obtained on long-time intervals with an upper bound

proportional to D^ℓ for some $0 < \ell < 1$, see Theorem 4.10. The cut-off approach is due to the work with Mikelić who refined the estimates from [75, Theorem 3] and who suggested to consider shadow systems with an L^2 dissipative linearization to obtain long-time estimates. Section 4.3 is devoted to a brief discussion of dissipative systems which are a particular class of stable evolution systems used in this work.

4.1 The linear case

To begin with, we will focus on a linear version of the partly diffusive system (1.1)–(1.3), where the right-hand side of the full problem

$$\begin{aligned}
&\frac{\partial \mathbf{u}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{u}_D = \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{u}_D(\cdot, 0) = \mathbf{u}^0 \ \text{ in } \ \Omega,\\
&\frac{\partial \mathbf{v}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{v}_D = \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{v}_D(\cdot, 0) = \mathbf{v}^0 \ \text{ in } \ \Omega,\\
&\qquad\qquad \frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} = \mathbf{0}, \quad \frac{\partial \mathbf{v}_D}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0, T)
\end{aligned}$$

is given by linear terms

$$\mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) = \mathbf{A}_*(x,t)\mathbf{u}_D + \mathbf{B}_*(x,t)\mathbf{v}_D + \mathbf{r}(x,t), \tag{4.1}$$

$$\mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) = \mathbf{C}_*(x,t)\mathbf{u}_D + \mathbf{D}_*(x,t)\mathbf{v}_D + \mathbf{s}(x,t). \tag{4.2}$$

Such linear models including their shadow limits have been recently applied in the context of control theory in [42] for constant coefficients. However, the primary function of studying linear problems is to understand complex, nonlinear problems by linearization. According to assumptions A1–A2 and in view of long-time estimates, we assume bounded coefficients

$$\mathbf{A}_* \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^{m \times m}, \mathbf{B}_*, \mathbf{C}_*^T \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^{m \times k}, \mathbf{D}_* \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^{k \times k}$$

and $\mathbf{r} \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^m, \mathbf{s} \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^k$. The mild solution $\Psi = (\mathbf{u}_D, \mathbf{v}_D)$ from Proposition 2.3 can be extended to a global solution on $\mathbb{R}_{\geq 0}$, possibly unbounded, due to the linear growth of the right-hand side $\mathbf{h} = (\mathbf{f}, \mathbf{g})$. The shadow limit reduction of system (1.1)–(1.3) as $D = \min_{j=1,\dots,k} D_j^v \to \infty$ using linear terms (4.1)–(4.2) yields the following system of integro-differential equations

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}^u \Delta \mathbf{u} = \mathbf{A}_*(x,t)\mathbf{u} + \mathbf{B}_*(x,t)\mathbf{v} + \mathbf{r}(x,t) \qquad \text{in} \quad \Omega_T,$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \langle \mathbf{C}_*(\cdot,t)\mathbf{u}\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\mathbf{v}\rangle_\Omega + \langle \mathbf{s}(\cdot,t)\rangle_\Omega \quad \text{in} \quad (0,T),$$
$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \partial\Omega \times \mathbb{R}_{>0}, \quad \mathbf{u}(\cdot,0) = \mathbf{u}^0 \quad \text{in} \quad \Omega, \quad \mathbf{v}(0) = \langle \mathbf{v}^0\rangle_\Omega.$$

The right-hand side which grows at most linearly implies a global shadow limit defined on $\Omega \times \mathbb{R}_{\geq 0}$ as the mild solution of the integral representation (2.6)–(2.7), see Proposition 2.6. Remark that also the shadow solution may be (exponentially) unbounded as $t \to \infty$. The mean value correction ψ_D defined in equations (2.10)–(2.11) satisfies

$$\frac{\partial \psi_D}{\partial t} - \mathbf{D}^v \Delta \psi_D = \mathbf{C}_*(x,t)\mathbf{u} - \langle \mathbf{C}_*(\cdot,t)\mathbf{u}\rangle_\Omega + \mathbf{D}_*(x,t)\mathbf{v} - \langle \mathbf{D}_*(\cdot,t)\mathbf{v}\rangle_\Omega$$
$$+ \mathbf{s}(x,t) - \langle \mathbf{s}(\cdot,t)\rangle_\Omega \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0},$$
$$\frac{\partial \psi_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times \mathbb{R}_{>0}, \qquad \psi_D(\cdot,0) = \mathbf{v}^0 - \langle \mathbf{v}^0\rangle_\Omega \quad \text{in} \quad \Omega.$$

The equations for the error functions $\mathbf{U}_D = \mathbf{u}_D - \mathbf{u}$ and $\mathbf{V}_D = \mathbf{v}_D - \mathbf{v} - \psi_D$ are given by system (3.2)–(3.4), i.e., by the linear inhomogeneous system

$$\frac{\partial \mathbf{U}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{U}_D = \mathbf{A}_*(x,t)\mathbf{U}_D + \mathbf{B}_*(x,t)(\mathbf{V}_D + \psi_D) \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \tag{4.3}$$
$$\frac{\partial \mathbf{V}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{V}_D = \mathbf{C}_*(x,t)\mathbf{U}_D + \mathbf{D}_*(x,t)(\mathbf{V}_D + \psi_D) \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \tag{4.4}$$
$$\mathbf{U}_D(\cdot,0) = \mathbf{0}, \mathbf{V}_D(\cdot,0) = \mathbf{0} \text{ in } \Omega, \qquad \frac{\partial \mathbf{U}_D}{\partial \mathbf{n}} = \mathbf{0}, \frac{\partial \mathbf{V}_D}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \partial\Omega \times \mathbb{R}_{>0}. \tag{4.5}$$

In order to understand difficulties on long-time ranges indicated by Example 3.6, even in the case of coefficients which are independent of space and time, we first focus on the case of space-independent coefficients. This situation simplifies due to zero mean values $\langle \mathbf{U}_D\rangle_\Omega = \mathbf{0}$, $\langle \mathbf{V}_D\rangle_\Omega = \mathbf{0}$. Subsequently, we consider space- and time-dependent coefficients $\mathbf{A}_*$, $\mathbf{B}_*$, $\mathbf{C}_*, \mathbf{D}_*$ for which the estimation of the errors is more complex but already captures all key aspects of the nonlinear case.

4.1.1 Space-independent coefficients

The main part of this section is published in the scientific work [55] with focus on the reaction-diffusion-ODE case $\mathbf{D}^u = \mathbf{0}$ as well as on the classical scalar case $D^u \in \mathbb{R}_{>0}$.

We aim for an explicit formula of the error $\mathbf{U}_D$ satisfying equation (4.3) which only depends on $\mathbf{V}_D$ and ψ_D. This in turn will be used to obtain an implicit equation

for the error $\mathbf{V}_D$ which only depends on $\mathbf{V}_D$ and which allows to estimate this error function. Finally, the explicit dependence of the error $\mathbf{U}_D$ on estimated quantities such as $\mathbf{V}_D$ and ψ_D yields estimates for the solution of system (4.3)–(4.5).
Let us first establish an estimate for the mean value correction ψ_D similarly to inequality (2.12). One infers from the above setting that ψ_D only depends on the difference of initial values $\mathbf{v}^0 - \langle \mathbf{v}^0 \rangle_\Omega$, $\mathbf{C}_*(t)$ and $\xi := \mathbf{u} - \langle \mathbf{u} \rangle_\Omega$. The last term is given by the initial value problem

$$\frac{\partial \xi}{\partial t} - \mathbf{D}^u \Delta \xi = \mathbf{A}_*(t)\xi \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad \xi(\cdot, 0) = \mathbf{u}^0 - \langle \mathbf{u}^0 \rangle_\Omega \quad \text{in} \quad \Omega,$$

endowed with zero Neumann boundary conditions, if necessary. The solution can also be expressed by a corresponding evolution system. To recognize this, we recall that $\mathbf{D}^u \Delta$ generates a contraction semigroup $(\mathbf{S}^u(t))_{t \in \mathbb{R}_{\geq 0}}$ on $L^p(\Omega)^m$ for each $1 \leq p \leq \infty$ by Lemma 2.1. Moreover, the family $(\mathbf{A}_*(t))_{t \in \mathbb{R}_{\geq 0}}$ induces bounded multiplication operators on $\mathbb{R}^m$ resp. $L^p(\Omega)^m$ for each $1 \leq p \leq \infty$ since $\mathbf{A}_* \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^{m \times m}$ [104, Proposition 2.2.14]. Using a well-known perturbation result for evolution equations [92, Chapter 6, Theorem 1.2], there is, for each finite $p < \infty$, a unique mild solution $\xi \in C(\mathbb{R}_{\geq 0}; L^p(\Omega)^m)$ given implicitly by the Duhamel formula

$$\xi(\cdot, t) = \mathbf{S}^u(t-s)\xi(\cdot, s) + \int_s^t \mathbf{S}^u(t-\tau)\mathbf{A}_*(\tau)\xi(\cdot, \tau)\, \mathrm{d}\tau \qquad \forall\, 0 \leq s \leq t. \tag{4.6}$$

Continuity in $t = 0$ does not carry over to $L^\infty(\Omega)^m$ for $\mathbf{D}^u \in \mathbb{R}^{m \times m}_{>0}$ in general [94, Part I, Lemma 2]. However, the solution can be found by the same method using Proposition 2.3 instead. We obtain $\xi \in C(\mathbb{R}_{>0}; L^\infty(\Omega)^m)$ by Proposition B.2 and the implicit formula (4.6). We thus define an evolution system $\mathcal{U}$ consisting of a family of evolution operators $\mathbf{U}(t, s)$ for $s, t \in \mathbb{R}_{\geq 0}, s \leq t$, on $L^p(\Omega)^m$ induced by the unique solution ξ of the implicit equation (4.6);

$$\xi(\cdot, t) = \mathbf{U}(t, s)\xi(\cdot, s) \qquad \forall\, s, t \in \mathbb{R}_{\geq 0}, s \leq t.$$

Let us write $\mathbf{U}(t, 0) =: \mathbf{U}(t)$ for short. Each bounded, linear operator $\mathbf{U}(t, s)$ satisfies the usual conditions

$$\mathbf{U}(t, t) = I, \quad \mathbf{U}(t, r) = \mathbf{U}(t, s)\mathbf{U}(s, r) \qquad \forall\, r, s, t \in \mathbb{R}_{\geq 0}, r \leq s \leq t$$

of an evolution system and Gronwall's inequality implies the a priori estimate

$$\|\mathbf{U}(t,s)\xi(\cdot,s)\|_{L^p(\Omega)^m} \leq \exp\left(\int_s^t \|\mathbf{A}_*(\tau)\| \,\mathrm{d}\tau\right) \|\xi(\cdot,s)\|_{L^p(\Omega)^m}.$$

In the case of $\mathbf{D}^u = \mathbf{0}$, the evolution system $\mathcal{U}$ can be defined on $L^\infty(\Omega)^m$ using standard techniques of ordinary differential equations in Banach spaces [14, Chapter III, §1]. Notice that, in general, the convolution property

$$\mathbf{U}(t,s) = \mathbf{U}(t-s,0) \qquad \forall\, s,t \in \mathbb{R}_{\geq 0}, s \leq t$$

is not fulfilled. This can be guaranteed only in the case of time-independent operators $\mathbf{A}_*(t) \equiv \mathbf{A}_*$ where $\mathbf{U}(t,0) = \mathbf{U}(t)$ play the role of semigroup operators [92, Section 5.1].

It turns out that uniform boundedness of the evolution system $\mathcal{U}$ defined by equation (4.6) resp. uniform stability of the corresponding differential equation is essential for showing convergence results. This condition is optimal in the sense that there are examples where uniform convergence on long-time intervals may not be achievable in the absence of uniform boundedness in $L^\infty(\Omega)^m$. For instance, take $m = 1$ and $A_*(x,t) = \chi_{A(x)}(t)$ with indicator function $\chi_{A(x)}$ on some interval $A(x)$ with measure $|A| \in L^p(\Omega) \setminus L^\infty(\Omega)$ or compare to Example 3.6.
Since the same reasoning above holds for space-dependent coefficients $\mathbf{A}_*(\cdot,t)$ as well, we formulate the following stability assumption

L Let the evolution system $\mathcal{U}$ be uniformly bounded in $L^\infty(\Omega)^m$, i.e., there is a constant $C > 0$ independent of time $s,t \in \mathbb{R}_{\geq 0}$ such that

$$\|\mathbf{U}(t,s)\xi^0\|_{L^\infty(\Omega)^m} \leq C\|\xi^0\|_{L^\infty(\Omega)^m} \qquad \forall\, \xi^0 \in L^\infty(\Omega)^m, s,t \in \mathbb{R}_{\geq 0}, s \leq t.$$

Note that boundedness (stability) of the operator family $(\mathbf{U}(t))_{t\in\mathbb{R}_{\geq 0}}$ is not enough to estimate integrals uniformly if $\mathbf{A}_*$ additionally depends on time, compare the example of Perron in [14, p.123]. As a consequence of the uniform assumption L, we are able to estimate

$$\left\|\int_0^T \mathbf{U}(t,\tau)\mathbf{f}(\tau)\,\mathrm{d}\tau\right\|_{L^\infty(\Omega)^m} \leq C\int_0^T \|\mathbf{f}(\tau)\|_{L^\infty(\Omega)^m}\,\mathrm{d}\tau \qquad \forall\, \mathbf{f} \in L^1(0,T;L^\infty(\Omega)^m).$$

It has to be mentioned that a uniform bound for $\mathcal{U}$ independent of p for all large exponents $p < \infty$ yields bounds in $L^\infty(\Omega)^m$ in the limit $p \to \infty$ by continuity of the L^p norm with respect to p [1, Theorem 2.14].

Assuming uniform boundedness of $\mathcal{U}$, we receive an estimate for the mean value correction ψ_D similar to inequality (2.12) on long-time intervals. More precisely, let us write the solution ψ_D of the inhomogeneous equation

$$\frac{\partial \psi_D}{\partial t} - \mathbf{D}^v \Delta \psi_D = \mathbf{C}_*(t)\mathbf{U}(t)(\mathbf{u}^0 - \langle \mathbf{u}^0 \rangle_\Omega) \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}$$

as an explicit integral via formula (B.3). In view of Lemma 2.1, this can be estimated to obtain

$$\|\psi_D(\cdot,t)\|_{L^\infty(\Omega)^k} \leq C_{v^0} \mathrm{e}^{-\lambda_1 D t} + C_g D^{-1} \qquad \forall\, D > 0, t \in \mathbb{R}_{\geq 0}$$

with some constants $C_{v^0}, C_g > 0$ independent of time t and diffusion D.
These facts can be used to write the errors which solve equations (4.3)–(4.5) in the following implicit integral form

$$\mathbf{U}_D(\cdot,t) = \int_0^t \mathbf{U}(t,\tau)\mathbf{B}_*(\tau)\Big(\mathbf{V}_D(\cdot,\tau) + \psi_D(\cdot,\tau)\Big)\,\mathrm{d}\tau, \tag{4.7}$$

$$\mathbf{V}_D(\cdot,t) = \int_0^t \mathbf{S}^v(t-\tau)\Big(\mathbf{C}_*(\tau)\mathbf{U}_D(\cdot,\tau) + \mathbf{D}_*(\tau)\big(\mathbf{V}_D(\cdot,\tau) + \psi_D(\cdot,\tau)\big)\Big)\,\mathrm{d}\tau. \tag{4.8}$$

Estimations yield the following result.

Theorem 4.1. *Let assumptions A1–A2 and L hold for linearities* (4.1)–(4.2) *with space-independent, globally bounded coefficients. Then for any $\alpha \in (0,1]$ there exist constants $C, D_0 > 0$ independent of T, D such that for all $T \leq D^{1-\alpha}$ and all $D \geq D_0$*

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega_T)^m} \leq C D^{-\alpha},$$
$$\|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega_T)^k} \leq C D^{-1}.$$

Moreover, the spatial mean values are independent of D in the sense that

$$\langle \mathbf{u}_D \rangle_\Omega = \langle \mathbf{u} \rangle_\Omega, \qquad \langle \mathbf{v}_D \rangle_\Omega = \mathbf{v} \qquad \textit{in} \quad \mathbb{R}_{\geq 0}.$$

Proof. In what follows, $C > 0$ is a generic constant which does neither depend on time t nor on D, but on the particular system with presumed bounds. We test the weak formulation of the error system (4.3)–(4.5) with the constant function $|\Omega|^{-1}$ to infer $\langle \mathbf{U}_D \rangle_\Omega = \mathbf{0}$, $\langle \mathbf{V}_D \rangle_\Omega = \mathbf{0}$ on $\Omega \times \mathbb{R}_{\geq 0}$. Indeed, we obtain an ordinary differential equation for the spatial mean values and, since $\langle \psi_D \rangle_\Omega = \mathbf{0}$, the initial values imply the result. This leads to a specific situation in the space-independent case in which we are able to apply directly decay estimate (2.2) of $(\mathbf{S}^v(t))_{t \in \mathbb{R}_{\geq 0}}$ to the integral representation (4.8):

$$\begin{aligned}\|\mathbf{V}_D(\cdot,t)\|_{L^\infty(\Omega)^k} \leq C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \Big(\|\mathbf{U}_D(\cdot,\tau)\|_{L^\infty(\Omega)^m} + \|\mathbf{V}_D(\cdot,\tau)\|_{L^\infty(\Omega)^k} \Big) \, \mathrm{d}\tau \\ + C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \big(\mathrm{e}^{-\lambda_1 D\tau} + D^{-1} \big) \, \mathrm{d}\tau\end{aligned}$$

Since the non-negative function $t \mapsto t \exp(-\lambda_1 D t)$ is bounded on $\mathbb{R}_{\geq 0}$ by its maximum at $\lambda_1 D t = 1$, the last integral is uniformly bounded by CD^{-1} for some $C > 0$. According to Assumption L, the boundedness of the evolution system $\mathcal{U}$ leads to an estimate for $\mathbf{U}_D$ given by equation (4.7). The aforementioned estimate for ψ_D implies

$$\|\mathbf{U}_D(\cdot,t)\|_{L^\infty(\Omega)^m} \leq C \int_0^t \|\mathbf{V}_D(\cdot,\tau)\|_{L^\infty(\Omega)^k} \, \mathrm{d}\tau + CD^{-1}(1+t). \tag{4.9}$$

Combining both estimates for $\mathbf{U}_D$ and $\mathbf{V}_D$ and evaluating the double integral yields

$$\begin{aligned}\|\mathbf{V}_D(\cdot,t)\|_{L^\infty(\Omega)^k} &\leq C \int_0^t \int_s^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \|\mathbf{V}_D(\cdot,s)\|_{L^\infty(\Omega)^k} \, \mathrm{d}\tau \, \mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \|\mathbf{V}_D(\cdot,\tau)\|_{L^\infty(\Omega)^k} \, \mathrm{d}\tau + CD^{-1} \\ &\leq CD^{-1}(1+T) \|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} + CD^{-1}\end{aligned}$$

for all $t \leq T$. For each $\alpha \in (0,1]$, we consider $T \leq D^{1-\alpha}$ and obtain the estimate

$$\|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} \leq CD^{-1}$$

by absorption for sufficiently large $D \geq D_0(\alpha)$. Using inequality (4.9) implies a similar estimate multiplied by T due to integration. $\square$

The linear growth in time in estimates of Theorem 4.1 is due to the fact that the evolution system $\mathcal{U}$ is only bounded. Example 4.2 below shows optimality of this stability assumption. Let us consider the case in which the evolution system $\mathcal{U}$ satisfies an improved stability condition, so called uniform exponential stability (compare [9, 21] and references therein). This means that there are some time-independent constants $\eta, C > 0$ such that

$$\|\mathbf{U}(t,s)\xi^0\|_{L^\infty(\Omega)^m} \leq C\mathrm{e}^{-\eta t}\|\xi^0\|_{L^\infty(\Omega)^m} \qquad \forall\, \xi^0 \in L^\infty(\Omega)^m, s,t \in \mathbb{R}_{\geq 0}, s \leq t.$$

In this case, we obtain

$$\|\mathbf{U}_D(\cdot,t)\|_{L^\infty(\Omega_T)^m} \leq C\int_0^t \mathrm{e}^{-\eta(t-\tau)}\|\mathbf{V}_D(\cdot,\tau)\|_{L^\infty(\Omega_T)^k}\,\mathrm{d}\tau + CD^{-1}$$

instead of estimate (4.9). This leads to the best possible, global estimate

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})^m} \leq CD^{-1}, \qquad \|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})^k} \leq CD^{-1}. \quad (4.10)$$

See further Chapter 5 for results concerning exponential stability.

As already mentioned above, the lower convergence rate for $\mathbf{U}_D$ and the restriction of the valid time interval $[0,T]$ in Theorem 4.1 are not only for technical reasons. The latter are optimal in the sense that estimates may not be available for $\alpha = 1$ and $T = \infty$, respectively. I will complete the analysis of the space-independent case with the following example, even endowed with time-independent coefficients, which demonstrates optimality.

Example 4.2. Consider again Example 3.6 and take initial values $u^0 = v^0 = w_j$ for some $j \in \mathbb{N}$ from Proposition A.1 for the following linear problem with parameters $a = d = 0, bc \neq 0$, and $D^u = 0$:

$$\begin{aligned}
\frac{\partial u_D}{\partial t} &= bv_D \quad \text{in} \quad \Omega\times\mathbb{R}_{>0}, \qquad u_D(\cdot,0) = u^0 \ \text{ in } \ \Omega,\\
\frac{\partial v_D}{\partial t} - D\Delta v_D &= cu_D \quad \text{in} \quad \Omega\times\mathbb{R}_{>0}, \qquad v_D(\cdot,0) = v^0 \ \text{ in } \ \Omega,\\
\frac{\partial v_D}{\partial \mathbf{n}} &= 0 \qquad \text{on} \quad \partial\Omega\times\mathbb{R}_{>0}
\end{aligned}$$

It is clear from the initial condition that $\langle u\rangle_\Omega = v = 0$. Hence, the shadow component u is constant in time and the corresponding shadow limit is $(u,v) = (u^0, 0)$. The solution (u_D, v_D) is given by the projection on the eigenspace spanned by w_j because initial values are multiples of w_j and the same holds for the error functions

$$U_D(\cdot,t) = U_{j,D}(t)w_j \qquad \text{and} \qquad V_D(\cdot,t) = V_{j,D}(t)w_j.$$

The coefficients satisfy the following ODE where we use the shifted matrix $\mathbf{M}_{D,j}$ and the explicit form of $b\psi_D$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}U_{j,D}\\ V_{j,D}\end{pmatrix} = \mathbf{M}_{D,j}\begin{pmatrix}U_{j,D}\\ V_{j,D}\end{pmatrix} + \begin{pmatrix}b\mathrm{e}^{-\lambda_j D t} + bc\int_0^t \mathrm{e}^{-\lambda_j D(t-\tau)}\,\mathrm{d}\tau\\ 0\end{pmatrix} \qquad \forall\, t\in\mathbb{R}_{>0}$$

This system is endowed with initial conditions $U_{j,D}(0) = 0 = V_{j,D}(0)$ and the variation of constants formula yields the representation

$$\begin{pmatrix}U_{j,D}(t)\\ V_{j,D}(t)\end{pmatrix} = \int_0^t \mathrm{e}^{\mathbf{M}_{D,j}(t-s)}\begin{pmatrix}b\mathrm{e}^{-\lambda_j D s} + bc\int_0^s \mathrm{e}^{-\lambda_j D(s-\tau)}\,\mathrm{d}\tau\\ 0\end{pmatrix}\mathrm{d}s.$$

Using the matrix exponential given in Example 3.6 yields

$$\begin{aligned}U_{j,D}(t) = \int_0^t &\mathrm{e}^{\tau_D(t-s)/2}\left(\cosh(\gamma_D(t-s)) - \frac{\tau_D}{2\gamma_D}\sinh(\gamma_D(t-s))\right)\\ &\cdot\left(b\mathrm{e}^{-\lambda_j D s} + bc\int_0^s \mathrm{e}^{-\lambda_j D(s-r)}\,\mathrm{d}r\right)\mathrm{d}s\end{aligned}$$

for $\tau_D = -\lambda_j D < 0, \delta_D = -bc \neq 0$ and $\gamma_D > 0$ with $4\gamma_D^2 = \tau_D^2 - 4\delta_D$. Both eigenvalues $\mu_\pm = \tau_D/2 \pm \gamma_D$ of the matrix $\mathbf{M}_{D,j}$ are real for sufficiently big diffusion D. We obtain

$$U_{j,D}(t) = \int_0^t \left(C_+\mathrm{e}^{\mu_+(t-s)} + C_-\mathrm{e}^{\mu_-(t-s)}\right)\cdot\left(b(1+c(\lambda_j D)^{-1})\mathrm{e}^{-\lambda_j D s} + bc(\lambda_j D)^{-1}\right)\mathrm{d}s$$

where we used $C_\pm = 1 \pm \tau_D/(2\gamma_D)$. Clearly, there holds $C_\pm \to 2$ as $D\to\infty$.
The negative eigenvalue $\mu_- \leq \tau_D/2$ tends to $-\infty$ as $D\to\infty$ in such a way that there are constants with $C_1 D \leq |\mu_-| \leq C_2 D$. As a consequence, the integration over exponents including μ_- yields only terms of order D^{-1}.
For the critical eigenvalue μ_+, which is either positive or negative and tends to 0 as $D\to\infty$ since $\mu_+\mu_- = -bc$, one can find constants satisfying

$$C_1 D^{-1} \leq |\mu_+| \leq C_2 D^{-1}.$$

If $\mu_+ < 0$, i.e., $bc < 0$, we obtain linear growth in time by $|U_{j,D}(t)| \leq CD^{-1}(1+t)$. If $\mu_+ > 0$, we infer that integration over exponents including μ_+ yields always bad terms like $CD^{-1}\mathrm{e}^{\mu_+ t}$ or even worse if $t \geq D$. To get convergence results, we have to restrict our considerations to $t \leq CD^{1-\alpha}$ for some $\alpha > 0$ and the proof works as presented for Theorem 4.1.

4.1.2 Space-dependent coefficients

As figured out in the foregoing section, uniform stability of the evolution subsystem for fixed diffusion $\mathbf{D}^u$ is essential to show accuracy of the shadow approximation for long-time intervals. Another main feature used in the latter proof is the fact that spatial mean values of the errors are zero in the space-independent case. This, of course, may not be true in space-dependent situations where $\langle \mathbf{U}_D \rangle_\Omega, \langle \mathbf{V}_D \rangle_\Omega$ might even grow exponentially in the linear case as the following example shows.

Example 4.3. Take the eigenfunction $v^0 := w_1$ from Example A.4 corresponding to the first positive eigenvalue λ_1 of $-\Delta$ on $\Omega = (0,1)$, i.e., $w_1(x) = \sqrt{2}\cos(\pi x)$. Let us focus on an equation for v_D only,

$$\begin{aligned} \frac{\partial v_D}{\partial t} - D\Delta v_D = d(x) v_D \quad &\text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad v_D(\cdot,0) = v^0 \ \text{ in } \ \Omega, \\ \frac{\partial v_D}{\partial \mathbf{n}} = 0 \qquad\qquad &\text{on} \quad \partial\Omega \times \mathbb{R}_{>0}, \end{aligned}$$

for a space-dependent coefficient $d := w_1 + w_1^2 \in L^\infty(\Omega)$. The corresponding shadow limit is given by $v = 0$ since $\langle v^0 \rangle_\Omega = 0$ and $\langle d \rangle_\Omega = 1$. Hence, the mean value correction ψ_D reduces to

$$\psi_D(\cdot,t) = S_\Delta(Dt) v^0 = \mathrm{e}^{-D\lambda_1 t} w_1.$$

If we consider the spatial mean of the error $V_D = v_D - \psi_D$, it remains to show exponential growth of $\langle V_D \rangle_\Omega = \langle v_D \rangle_\Omega$. The function v_D is given by

$$v_D(x,t) = S_\Delta(Dt) v^0(x) + \int_0^t S_\Delta(D(t-\tau)) d(x) v_D(x,\tau) \, \mathrm{d}\tau$$

and the implicit integral equation can be solved by a Picard iteration as done in the proof of existence of mild solutions [94, Part II, Theorem 1]. According to this iteration, we define approximations $v_D^{(j)}(\cdot,t) \in L^\infty(\Omega)$ recursively given by

$$\begin{aligned} v_D^{(1)}(\cdot,t) &= S_\Delta(Dt) v^0, \\ v_D^{(j+1)}(\cdot,t) &= S_\Delta(Dt) v^0 + \int_0^t S_\Delta(D(t-\tau)) \left[d(\cdot) v_D^{(j)}(\cdot,\tau) \right] \, \mathrm{d}\tau. \end{aligned}$$

Using trigonometry, we rewrite $d = w_1 + w_1^2 = w_0 + w_1 + \sqrt{2}^{-1} w_2$. We iteratively multiply the coefficient d with $v_D^{(j)}$ and use that products $w_j w_i$ can be rewritten as a linear combination of w_{j+i} and $w_{|j-i|}$. This procedure yields

$$\begin{aligned} v_D^{(2)}(\cdot,t) &= \mathrm{e}^{-\lambda_1 Dt} w_1 + \int_0^t S_\Delta(D(t-\tau)) f_1^{(1)}(\tau) \left[w_1 + w_1^2 + \sqrt{2}^{-1} w_1 w_2 \right] \,\mathrm{d}\tau \\ &= \left(\int_0^t f_1^{(1)}(\tau)\,\mathrm{d}\tau \right) w_0 + \left(\mathrm{e}^{-\lambda_1 Dt} + \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} f_1^{(1)}(\tau)\,\mathrm{d}\tau \right) w_1 \\ &\quad + \int_0^t S_\Delta(D(t-\tau)) f_1^{(1)}(\tau) h^{(2)} \,\mathrm{d}\tau, \end{aligned}$$

where $v_D^{(1)}(\cdot,t) = \mathrm{e}^{-\lambda_1 Dt} w_1 =: f_1^{(1)}(t) w_1$, $h^{(2)} = (w_1 + \sqrt{2} w_2 + w_3)/2$ and $w_0 \equiv 1$. To understand the next step, let us rewrite the second approximation as

$$v_D^{(2)}(\cdot,t) = f_0^{(2)}(t) w_0 + f_1^{(2)}(t) w_1 + f_2^{(2)}(t) h^{(2)}$$

and note that the coefficients of the eigenfunctions in $h^{(2)}$ are all positive and $h^{(2)}$ includes w_1 as well. Considering spatial means, $\langle w_j \rangle_\Omega = 0$ for all $j \in \mathbb{N}$ implies $\langle v_D^{(1)} \rangle_\Omega = 0$ and

$$\langle v_D^{(2)}(\cdot,t) \rangle_\Omega = \int_0^t \mathrm{e}^{-\lambda_1 D\tau} \,\mathrm{d}\tau.$$

Using again $d = w_0 + w_1 + \sqrt{2}^{-1} w_2$, this leads to the third approximation

$$\begin{aligned} v_D^{(3)}(\cdot,t) &= \left(\int_0^t f_0^{(2)}(\tau) + f_1^{(2)}(\tau)\,\mathrm{d}\tau \right) w_0 \\ &\quad + \left(\mathrm{e}^{-\lambda_1 Dt} + \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} (f_0^{(2)}(\tau) + f_1^{(2)}(\tau))\,\mathrm{d}\tau \right) w_1 + f_3^{(3)}(t) h^{(3)}, \end{aligned}$$

where $h^{(3)}$ is a sum of positive multiples of w_j for $j = 0, \dots, 4$ and $f_3^{(3)} \geq 0$ is a continuous function in time. Estimating from below, we successively gain for all $j \in \mathbb{N}$ (by setting $f_0^{(1)} \equiv 0$)

$$f_0^{(j+2)}(t) \geq \int_0^t f_0^{(j+1)}(\tau) + f_1^{(j+1)}(\tau)\,\mathrm{d}\tau \geq \int_0^t \left(\mathrm{e}^{-\lambda_1 D\tau} + \int_0^\tau f_0^{(j)}(r) + f_1^{(j)}(r)\,\mathrm{d}r \right) \mathrm{d}\tau.$$

Starting from the innermost double integral and applying Fubini's rule inductively, this yields

$$f_0^{(j+2)}(t) \geq \int_0^t f_0^{(j+1)}(\tau) + f_1^{(j+1)}(\tau)\,\mathrm{d}\tau \geq \int_0^t \sum_{i=0}^{j} \frac{(t-\tau)^i}{i!} \mathrm{e}^{-\lambda_1 D\tau} \mathrm{d}\tau.$$

Since $v_D^{(j)}$ converges to v_D in $L^\infty(\Omega_T)$, we obtain a lower bound due to

$$\int_0^t \sum_{i=0}^{j} \frac{(t-\tau)^i}{i!} \mathrm{e}^{-\lambda_1 D\tau} \mathrm{d}\tau \leq f_0^{(j+2)}(t) \leq \langle v_D^{(j+2)}(\cdot,t)\rangle_\Omega \to \langle v_D(\cdot,t)\rangle_\Omega.$$

Finally, the theorem of monotone convergence and the evaluation of the integral leads to exponential growth of $\langle V_D \rangle_\Omega$ for all large diffusions D:

$$\langle v_D(\cdot,t)\rangle_\Omega \geq \int_0^t \mathrm{e}^{t-\tau}\mathrm{e}^{-\lambda_1 D\tau}\, \mathrm{d}\tau \geq CD^{-1}\left(\mathrm{e}^t - 1\right) \geq 0$$

This induces exponential growth of $t \mapsto \|v_D(\cdot,t)\|_{L^\infty(\Omega)}$ by Hölder's inequality.

The above example shows that space-dependence requires more assumptions than just uniform stability for the subsystem of $\mathbf{u}$. Indeed, we require stability for the evolution of the entire shadow system to which $\mathbf{U}_D, \langle \mathbf{V}_D\rangle_\Omega$ are solutions. Let us define this concept for the shadow system similarly to the evolution system $\mathcal{U}$ defined at the beginning of Subsection 4.1.1. Regard the homogeneous shadow problem

$$\begin{aligned} \frac{\partial \xi_1}{\partial t} - \mathbf{D}^u \Delta \xi_1 &= \mathbf{A}_*(x,t)\xi_1 + \mathbf{B}_*(x,t)\xi_2 && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \quad \xi_1(\cdot,0) = \xi_1^0 \quad \text{in} \quad \Omega, \\ \frac{\mathrm{d}\xi_2}{\mathrm{d}t} &= \langle \mathbf{C}_*(\cdot,t)\xi_1\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\xi_2\rangle_\Omega && \text{in} \quad \mathbb{R}_{>0}, \qquad \xi_2(0) = \langle \xi_2^0\rangle_\Omega \end{aligned}$$

endowed with zero Neumann boundary conditions for ξ_1 if necessary. Note that $\mathbf{D}^u\Delta$ as well as the identity I generates a (not necessarily strongly continuous) contraction semigroup on $L^p(\Omega)^m$ for each $1 \leq p \leq \infty$ by Lemma 2.1. Using notation (2.8), let us consider the shadow problem as the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = \mathbf{D}^S\Delta\xi + \mathbf{L}_0(t)\xi \qquad \text{for} \quad t \in \mathbb{R}_{>0}, \qquad \xi(0) = \begin{pmatrix} \xi_1^0 \\ \langle \xi_2^0\rangle_\Omega \end{pmatrix}.$$

The linear operators $\mathbf{L}_0(t)$ are defined on the Banach space $L^p(\Omega)^m \times \mathbb{R}^k$ by their action on $\xi = (\xi_1, \xi_2)^T$ induced by the linear right-hand side of the shadow problem:

$$\begin{aligned} &\mathbf{L}_0(t) : L^p(\Omega)^m \times \mathbb{R}^k \to L^p(\Omega)^m \times \mathbb{R}^k, \\ &\mathbf{L}_0(t)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} \mathbf{A}_*(x,t)\xi_1(x) + \mathbf{B}_*(x,t)\xi_2 \\ \langle \mathbf{C}_*(\cdot,t)\xi_1\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\xi_2\rangle_\Omega \end{pmatrix} \end{aligned} \tag{4.11}$$

Since the coefficients are uniformly bounded, $(\mathbf{L}_0(t))_{t\in\mathbb{R}_{\geq 0}}$ is a family of bounded operators on $L^p(\Omega)^m \times \mathbb{R}^k$ for each $1 \leq p \leq \infty$. As in the space-independent case in

Subsection 4.1.1, the sum of both operators generates a strongly continuous evolution system $\mathcal{W}$ consisting of bounded, linear operators $\mathbf{W}(t,s)$ for $s,t \in \mathbb{R}_{\geq 0}, s \leq t$, on $L^p(\Omega)^m \times \mathbb{R}^k$ for each finite $p < \infty$. The evolution operators $\mathbf{W}(t,s)$ are defined by the unique solution

$$\xi(\cdot,t) = \mathbf{W}(t,s)\xi(\cdot,s), \qquad \xi(\cdot,0) = \begin{pmatrix} \xi_1^0 \\ \langle \xi_2^0 \rangle_\Omega \end{pmatrix} \tag{4.12}$$

of the above shadow limit. We refer to Proposition 2.6 for an implicit integral representation similar to formula (4.6) and deduce $\xi = (\xi_1, \xi_2)^T \in C(\mathbb{R}_{\geq 0}; L^p(\Omega)^m \times \mathbb{R}^k)$ from the fact that $\mathbf{L}_0(t)\xi$ is uniformly bounded and hence integrable. Continuity does not carry over to $L^\infty(\Omega)^m$ for $\mathbf{D}^u \neq \mathbf{0}$ in general [94, Part I, Lemma 2]. However for the case of $p = \infty$, the evolution operator can be defined in the same way as we did for $\mathcal{U}$ using $\xi \in C(\mathbb{R}_{>0}; L^\infty(\Omega)^m \times \mathbb{R}^k)$, see Subsection 4.1.1. In the case of $\mathbf{D}^u = \mathbf{0}$, the evolution system $\mathcal{W}$ can directly be defined on $L^p(\Omega)^m \times \mathbb{R}^k$ using standard techniques of ordinary differential equations [14, Chapter III, §1].

Example 4.3 indicates that uniform boundedness of the evolution system $\mathcal{W}$ (resp. uniform stability of the corresponding evolution equation, see [14, p. 112], [16, Definition 3]) is essential for showing convergence results for space-dependent problems. This can be seen by using an extension of Example 4.3 to a full shadow system, e.g., by $A_*, B_*, C_* = 0$ and $D_* = d$. For this reason let us assume

L1p Let the evolution system $\mathcal{W}$ be uniformly bounded for some $1 \leq p \leq \infty$, i.e., there is a constant $C > 0$ independent of time such that for all $s,t \in \mathbb{R}_{\geq 0}, s \leq t$,

$$\|\mathbf{W}(t,s)\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \leq C\|\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \qquad \forall\, \xi^0 \in L^p(\Omega)^m \times \mathbb{R}^k.$$

An example using measurable coefficients for A_*, C_* of the form $\chi_{J(x)}(t)$ with indicator function $\chi_{J(x)}$ on some interval $J(x) \subset \mathbb{R}$ with measure $|J| \in L^p(\Omega) \setminus L^q(\Omega)$ shows that Assumption L1p has to be checked for each index p separately. Though, from Riesz-Thorin interpolation theorem in [8, Theorem 4.32], uniform boundedness in $L^p(\Omega)^m \times \mathbb{R}^k$ and $L^q(\Omega)^m \times \mathbb{R}^k$ implies uniform boundedness in $L^r(\Omega)^m \times \mathbb{R}^k$ for each $1 \leq p \leq r \leq q \leq \infty$. Another interesting case is the limit $p \to \infty$, which yields uniform bounds in $L^\infty(\Omega)^m \times \mathbb{R}^k$ if one has a uniform bound for $\mathcal{W}$ independent of p for all large exponents $p < \infty$ [1, Theorem 2.14]. See [9, 21] and references therein for further characterizations of uniform (exponential) stability of evolution systems.

Since we are looking for long-time behavior of models, we additionally assume that the mean value correction ψ_D satisfies the following continuation of inequality (2.12), already used in the space-independent case.

A3 The mean value correction ψ_D satisfies the estimate

$$\|\psi_D(\cdot,t)\|_{L^\infty(\Omega)^k} \le C_{v^0}\mathrm{e}^{-\lambda_1 Dt} + C_g D^{-1} \qquad \forall\, D>0, t\in\mathbb{R}_{\ge 0} \tag{4.13}$$

for some constants $C_{v^0}, C_g > 0$ that do not depend on time t or diffusion parameter D, but on bounds of $\mathbf{g}$ resp. $\mathbf{v}^0$.

Clearly, if $\mathbf{g} - \langle \mathbf{g}\rangle_\Omega$ is uniformly bounded in the time variable $t \in \mathbb{R}_{\ge 0}$ on bounded subsets of $\mathbb{R}^{m+k} \times \overline{\Omega}$, then Assumption A3 is satisfied. Especially nonlinearities $\mathbf{g}$ that do not depend explicitly on time fulfill A3 if A1–A2 are satisfied and the global shadow solution is uniformly bounded.

These assumptions lead to a proof of convergence for the general linear case (4.1)–(4.2) valid for long-time ranges. Starting from equations (4.3)–(4.5), we infer the following a priori L^p error estimate as a natural consequence of Assumption L1p.

Proposition 4.4. *Consider linearities* (4.1)–(4.2) *with globally bounded coefficients and let assumptions A1–A3, and L1p hold for some* $1 \le p \le \infty$. *Then for any* $\alpha \in (0,1]$ *there exist constants* $C, D_0 > 0$ *independent of* T, D *such that for all* $T \le D^{1-\alpha}$ *and all* $D \ge D_0$ *there holds*

$$\sup_{t\in[0,T]} \Big(\|\mathbf{U}_D(\cdot,t)\|_{L^p(\Omega)^m} + \|\mathbf{V}_D(\cdot,t)\|_{L^p(\Omega)^k} \Big) \le CD^{-\alpha}. \tag{4.14}$$

Proof. To make use of the decay estimate (2.2) for the heat semigroup, it is convenient to split up the error $\mathbf{V}_D$ and consider its spatial mean value and the remainder [32, Theorem 1]. Hence, we define the functions

$$\mathbf{W}_D := \mathbf{V}_D - \langle \mathbf{V}_D\rangle_\Omega \qquad \text{and} \qquad \mathbf{b}_D := \langle \mathbf{V}_D\rangle_\Omega$$

which satisfy the differential equations

$$\begin{aligned} \frac{\partial \mathbf{W}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{W}_D &= \mathbf{C}_*(x,t)\mathbf{U}_D - \langle \mathbf{C}_*(\cdot,t)\mathbf{U}_D\rangle_\Omega + \mathbf{D}_*(x,t)(\mathbf{V}_D+\psi_D) \\ &\quad - \langle \mathbf{D}_*(\cdot,t)(\mathbf{V}_D+\psi_D)\rangle_\Omega \qquad\qquad \text{in} \quad \Omega\times\mathbb{R}_{>0}, \end{aligned} \tag{4.15}$$

$$\frac{\mathrm{d}\mathbf{b}_D}{\mathrm{d}t} = \langle \mathbf{C}_*(\cdot,t)\mathbf{U}_D\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)(\mathbf{V}_D+\psi_D)\rangle_\Omega \quad \text{in} \quad \Omega\times\mathbb{R}_{>0}, \tag{4.16}$$

$$\mathbf{W}_D(\cdot,0) = \mathbf{0} \quad \text{in} \quad \Omega, \quad \mathbf{b}_D(0) = \mathbf{0}, \quad \frac{\partial \mathbf{W}_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times \mathbb{R}_{>0}. \tag{4.17}$$

We start by estimating the term $\mathbf{W}_D$ by means of Lemma 2.1. Denoting the right-hand side of the system (4.15) by $\mathbf{R}_D$, the solution $\mathbf{W}_D$ may be written as

$$\mathbf{W}_D(\cdot,t) = \int_0^t \mathbf{S}^v(t-\tau)\mathbf{R}_D(\cdot,\tau)\,\mathrm{d}\tau.$$

For each $p \geq 1$ we infer from decay estimate (4.13) for ψ_D and $\langle \mathbf{R}_D \rangle_\Omega = \mathbf{0}$ that

$$\begin{aligned}\|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \Big(&\|\mathbf{U}_D(\cdot,\tau)\|_{L^p(\Omega)^m} + \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k} \\ &+ |\Omega|^{1/p} \left(|\mathbf{b}_D(\tau)| + D^{-1} + \mathrm{e}^{-\lambda_1 D\tau} \right) \Big)\,\mathrm{d}\tau\end{aligned}$$

with a similar estimate for $p = \infty$. Notice that the term $\mathbf{W}_D$ always implies a factor D^{-1} in above estimates while the terms $\mathbf{U}_D$ and $\mathbf{b}_D$ have to be controlled by Assumption L1p. These components satisfy a shadow problem whose solution is given by

$$\begin{pmatrix} \mathbf{U}_D(\cdot,t) \\ \mathbf{b}_D(t) \end{pmatrix} = \int_0^t \mathbf{W}(t,\tau) \begin{pmatrix} \mathbf{B}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) \\ \langle \mathbf{D}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) \rangle_\Omega \end{pmatrix} \mathrm{d}\tau.$$

According to Assumption L1p, we have the estimate

$$\begin{aligned}\|\mathbf{U}_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)| &\leq C \int_0^t \|(\mathbf{W}_D + \psi_D)(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau \\ &\leq C \int_0^t \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau + C|\Omega|^{1/p} D^{-1}(1+t).\end{aligned}$$

Combining the latter two estimates leads to an estimate for $\mathbf{W}_D$. More precisely,

$$\begin{aligned}\|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq C &\int_0^t \left(\mathrm{e}^{-\lambda_1 D(t-\tau)} + D^{-1} \right) \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau \\ &+ C|\Omega|^{1/p} D^{-1} \left(1 + D^{-1}t \right),\end{aligned}$$

since the non-negative function $t \mapsto t\exp(-\lambda_1 Dt)$ is bounded on $\mathbb{R}_{\geq 0}$ by its maximum at $\lambda_1 Dt = 1$. Taking the supremum over t yields

$$\sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq CD^{-1}(1+T) \sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} + CD^{-1}(1+D^{-1}T).$$

An absorption of $\mathbf{W}_D$ terms on the left-hand side for all $T \leq D^{1-\alpha}$ implies

$$\sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq CD^{-1}$$

for all $D \geq D_0$. Finally, we infer estimates for the errors $\mathbf{U}_D$ resp. $\mathbf{V}_D$ from the above inequality for $\mathbf{U}_D, \mathbf{b}_D$ since $\mathbf{V}_D = \mathbf{W}_D + \mathbf{b}_D$. □

If Assumption L1p holds with $p = \infty$, we already obtain a corresponding estimate in $L^\infty(\Omega_T)$ from Proposition 4.4. Recall that we used $\mathbf{b}_D \equiv \mathbf{0}$ in the space-independent case and it suffices to use condition L instead of L1p for $p = \infty$. Proposition 5.7 indicates that Assumption L1p for $p = \infty$ might imply Assumption L (and L0 defined below) in many cases.

Let us further develop the case $p < \infty$. Results of Proposition 4.4 provide a priori bounds for norms of the error functions $\mathbf{U}_D, \mathbf{V}_D$ in the parabolic space $L_{p,\infty}(\Omega_T)$, where $L_{p,r}(\Omega_T)$ is given by all measurable functions ψ on Ω_T with finite norm

$$\|\psi\|_{p,r} := \left(\int_0^T \left(\int_\Omega |\psi(x,t)|^p \,\mathrm{d}x\right)^{r/p} \mathrm{d}t\right)^{1/r} \qquad \text{for} \quad 1 \leq p, r < \infty \tag{4.18}$$

and an obvious modification for $r = \infty$ [59, Chapters I, II, §1 in both cases]. From results of Proposition 4.4 and Hölder's inequality we derive a priori estimates for each $1 \leq r < \infty$ of the form

$$\|\mathbf{V}_D\|_{p,r} \leq T^{1/r}\|\mathbf{V}_D\|_{p,\infty} \leq CD^{-\alpha+\frac{1}{r}(1-\alpha)},$$

and similarly for $\mathbf{U}_D$. Those estimates imply an $L^\infty(\Omega_T)$ estimate of the error $\mathbf{V}_D$ and components of $\mathbf{U}_D$ that diffuse by using a bootstrap argument for parabolic equations similar to [59, Chapter III, §7]. For a diffusive component z_d of $\mathbf{U}_D$ or $\mathbf{V}_D$ we use the fact that the right-hand side $R_d \in L_{p,r}(\Omega_T)$ of the corresponding parabolic equation

$$\frac{\partial z_d}{\partial t} - d\Delta z_d = R_d(x,t)$$

can be estimated in powers of the inverse D^{-1} for long times $T \leq D^{1-\alpha}$ by

$$\|R_d\|_{p,r} \leq C\left(\|\mathbf{U}_D\|_{p,r} + \|\mathbf{V}_D\|_{p,r} + \|\psi_D\|_{p,r}\right).$$

An $L^\infty(\Omega_T)$ estimate of z_d is established in Proposition B.10 with explicit dependence on the length T of the time interval and the norm $\|R_d\|_{p,r}$. In doing so, the value p from Assumption L1p is restricted due to Sobolev's embedding to $p \geq 1 = n$ and $p > n/2$ for $n \geq 2$ and $1 < r \leq \infty$ is chosen according to (B.14).
After applying stability assumption L to the ODE subsystem of non-diffusing components, we obtain explicit estimates for all components of $\mathbf{U}_D, \mathbf{V}_D$.

Theorem 4.5. *Consider linearities* (4.1)–(4.2) *with globally bounded coefficients and let assumptions A1–A3, L, and L1p hold for some finite* $p \geq 1 = n$ *or* $p > n/2$ *for* $n \geq 2$ *and choose* r *as in the parameter setting* (B.14). *Then there exists an* $\alpha_0 = \alpha_0(r) > 0$ *and constants* $C, D_0 > 0$ *independent of* T, D *such that for all* $\alpha \in (\alpha_0, 1], T \leq D^{1-\alpha}$ *and* $D \geq D_0$ *there holds*

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega_T)^m} \leq CD^{-\gamma}, \tag{4.19}$$

$$\|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega_T)^k} \leq CD^{-\gamma+(\alpha-1)}, \tag{4.20}$$

$$\|\langle \mathbf{v}_D \rangle_\Omega - \mathbf{v}\|_{L^\infty((0,T))^k} \leq CD^{-\gamma+(\alpha-1)} \tag{4.21}$$

for some $\gamma = \gamma(\alpha, r) > 0$. *Moreover, for diffusing components* $u_{D,i}$ *of* $\mathbf{u}_D$, *we have the same convergence rate as for* $\mathbf{v}_D$, *i.e.,* $D^{-\gamma+(\alpha-1)}$.
If Assumption L1p holds with $p = \infty$, *estimate* (4.14) *is true, and we may choose* $D^{-\alpha}$ *as a convergence rate for each component and* $\alpha_0 \equiv 0$, *see Proposition 4.4.*

Proof. In view of Proposition 4.4, we confine ourselves to the case $p < \infty$. To make use of the L^∞ estimate (B.23) for the diffusive component z_d, it remains to further estimate the right-hand side R_d of the corresponding inhomogeneous heat equation. The L^p estimate (4.14) from Proposition 4.4 yields for each $r > 1$ and $1 \leq T \leq D^{1-\alpha}$

$$\|R_d\|_{p,r} \leq C\left(\|\mathbf{U}_D\|_{p,r} + \|\mathbf{V}_D\|_{p,r} + \|\psi_D\|_{p,r}\right) \leq C\left(T^{1/r}D^{-\alpha} + D^{-1/r}\right) \tag{4.22}$$

where we used the decay estimate (4.13) for ψ_D. Note that the constant $C > 0$ in this estimate depends on the parameters of the system but only depends on a lower bound of the diffusion d and is independent of T, D. As a consequence of Proposition B.10, for r defined by (B.14), diffusing components satisfy

$$\|z_d\|_{L^\infty(\Omega_T)} \leq CT^{1-1/r}\left(T^{1/r}D^{-\alpha} + D^{-1/r}\right) \leq CD^{(1-\alpha)-\min\{\alpha,(2-\alpha)/r\}}.$$

To obtain an analog estimate for non-diffusing components of $\mathbf{U}_D$, we use Assumption L and rewrite $\mathbf{U}_D$ as

$$\mathbf{U}_D(\cdot,t) = \int_0^t \mathbf{U}(t,\tau)\mathbf{B}_*(\cdot,\tau)\Big(\mathbf{V}_D(\cdot,\tau) + \psi_D(\cdot,\tau)\Big)\,\mathrm{d}\tau.$$

Non-diffusing components $U_{D,i}$ of $\mathbf{U}_D$ thus can be estimated by

$$\begin{aligned}\|U_{D,i}\|_{L^\infty(\Omega_T)} &\le C\left(T\|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} + \int_0^T \|\psi_D(\cdot,\tau)\|_{L^\infty(\Omega)^k}\,\mathrm{d}\tau\right)\\ &\le CT\left(D^{(1-\alpha)-\min\{\alpha,(2-\alpha)/r\}} + D^{-1}\right).\end{aligned}$$

If we take $T \le D^{1-\alpha}$ and $\alpha < 1$ into account, we have

$$\|U_{D,i}\|_{L^\infty(\Omega_T)} \le CD^{2(1-\alpha)-\min\{\alpha,(2-\alpha)/r\}}, \quad \|z_d\|_{L^\infty(\Omega_T)} \le CD^{(1-\alpha)-\min\{\alpha,(2-\alpha)/r\}}.$$

A valuable convergence rate is of order $D^{-\gamma}$ for some $\gamma > 0$. Hence, considering the worse estimate for non-diffusive components, we need

$$2(1-\alpha) - (2-\alpha)/r < 0 \quad \Leftrightarrow \quad \alpha > 2\frac{1-\xi}{2-\xi} =: \alpha_1(\xi) \qquad \text{with} \quad \xi = \frac{1}{r}$$

and $2(1-\alpha) - \alpha < 0$, i.e., $\alpha > 2/3$. From the parameter setting (B.14) it is clear that $\xi \in (0,1/2)$ for $n = 1$ and $\xi \in (0,1)$ for $n \ge 2$. Thus, the monotone decreasing function α_1 satisfying $\alpha_1(0) = 1, \alpha_1(1/2) = 2/3$ and $\alpha_1(1) = 0$ yields convergence rates of the form $D^{-\gamma}$ for some $\gamma > 0$. Especially for $n = 1$, only the curve α_1 is restrictive. For $n \ge 2$ we have to check

$$\alpha > \alpha_1(\xi) \quad \text{for} \quad \xi \in (0,1/2), \qquad \alpha > 2/3 \quad \text{for} \quad \xi \ge 1/2$$

and $\alpha_0 = \max\{\alpha_1(\xi), 2/3\}$ may be chosen. □

The above theorem provides a quite natural stabilization criterion on the shadow system and its ODE subsystem under which a shadow approximation of the full system (1.1)–(1.3) with linearities (4.1)–(4.2) is valuable on extended time intervals. Sections 4.3, 5.2 are devoted to the question on how to check these criteria while the elaborated means are applied exemplarily in Chapter 6 to several nonlinear models.

Let us briefly discuss one generalization having the above proof in mind. Assumption L can be omitted if all components of $\mathbf{u}$ diffuse. If there are some components which do not diffuse and some which do, Assumption L can be relaxed. In the proof of Theorem 4.5 we only used L^∞ estimates of L for non-diffusing components since all diffusing components can be treated in the same way with Ladyzenskaja's method. Let us delete all rows and columns of $\mathbf{A}_*(x,t)$ for which the corresponding component is diffusive to obtain $\mathbf{A}_{11}(x,t) \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$ for some $\tilde{m} \le m$. This submatrix generates an evolution system $\tilde{\mathcal{U}}$ in $L^\infty(\Omega)^{\tilde{m}}$ for which we assume (instead of Assumption L)

L0 Let the evolution system $\tilde{\mathcal{U}}$ be uniformly bounded, i.e., there is a constant $C > 0$ independent of time such that

$$\|\tilde{\mathbf{U}}(t,s)\xi^0\|_{L^\infty(\Omega)^{\tilde{m}}} \le C\|\xi^0\|_{L^\infty(\Omega)^{\tilde{m}}} \qquad \forall\, \xi^0 \in L^\infty(\Omega)^{\tilde{m}}, s,t \in \mathbb{R}_{\ge 0}, s \le t.$$

Remark that, without requiring Assumption L1p in the space-independent case, Assumption L might not be relaxed to L0 in Theorem 4.1. This is due to diffusion-driven instability effects as presented in Example 4.12.

Before turning to the nonlinear case, two modifications of Theorem 4.5 are in order: uniform polynomial stability and exponential stability may be considered instead of uniform stability of the evolution systems in assumptions L1p, L resp. L0. Polynomial growth in time in estimates of Theorem 4.5 is due to the fact that the evolution systems $\mathcal{U}$ resp. $\tilde{\mathcal{U}}$ and $\mathcal{W}$ are uniformly bounded. Similar estimates can be derived if the evolution systems are only uniformly bounded by some polynomial (see [21, Definition 1.15] for semigroups or [31, Definition 2.7] for general evolution systems). More precisely, the proof of Proposition 4.4 works if there are constants $C > 0, d \ge 0$ independent of time such that for all $s,t \in \mathbb{R}_{\ge 0}, s \le t$ there holds

$$\|\mathbf{W}(t,s)\xi^0\|_{L^p(\Omega)^m\times\mathbb{R}^k} \le C\left(1 + (t-s)^d\right)\|\xi^0\|_{L^p(\Omega)^m\times\mathbb{R}^k} \qquad \forall\, \xi^0 \in L^p(\Omega)^m \times \mathbb{R}^k.$$

The condition given here is slightly more general than uniform polynomial boundedness given in [31, Definition 2.7]. Statements of Proposition 4.4 and Theorem 4.5 remain essentially the same apart from different time restrictions $T \le D^{(1-\alpha)/(d+1)}$ and $T \le D^{(1-\alpha)/(\tilde{d}+1)}$, assuming polynomial growth with degree $\tilde{d} \ge 0$ instead of L0 for $\tilde{\mathcal{U}}$ too. By way of illustration, one may consider Example 4.2 with $bc \le 0$ to obtain estimates with polynomial growth in time.

Next, consider the case where both or either evolution system, $\mathcal{U}$ and $\mathcal{W}$, satisfies a so called uniform exponential stability condition, in $L^\infty(\Omega)^m$ and $L^p(\Omega)^m \times \mathbb{R}^k$, respectively. Let the evolution system $\mathcal{U}$ satisfy a uniform exponential stability condition (see [9, 21] and references therein), i.e., there are some time-independent constants $\eta, C > 0$ such that

$$\|\mathbf{U}(t,s)\xi^0\|_{L^\infty(\Omega)^m} \leq C\mathrm{e}^{-\eta t}\|\xi^0\|_{L^\infty(\Omega)^m} \qquad \forall\, \xi^0 \in L^\infty(\Omega)^m, s, t \in \mathbb{R}_{\geq 0}, s \leq t.$$

This yields

$$\|\mathbf{U}_D(\cdot,t)\|_{L^\infty(\Omega_T)^m} \leq C\int_0^t \mathrm{e}^{-\eta(t-\tau)}\|\mathbf{V}_D(\cdot,\tau)\|_{L^\infty(\Omega_T)^k}\,\mathrm{d}\tau + CD^{-1}$$

and the estimate for non-diffusing components is as good as the ones for diffusing components. The conditions on γ in the last theorem can be adapted accordingly since now $\alpha > 1/2$ and

$$\alpha > \alpha_1(\xi) = \frac{1-2\xi}{1-\xi} \qquad \text{with} \quad \alpha_1(\xi) \leq 0 \quad \forall\, \xi \in [1/2, 1)$$

have to be satisfied for convergence. The same modifications remain valid if we merely consider the ODE subsystem $\tilde{\mathcal{U}}$ on $L^\infty(\Omega)^{\tilde{m}}$.

Assuming uniform exponential stability of $\mathcal{W}$ in $L^p(\Omega)^m \times \mathbb{R}^k$ for some exponent $\sigma > 0$ in Assumption L1p, this yields

$$\|\mathbf{U}_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)| \leq C\int_0^t \mathrm{e}^{-\sigma(t-\tau)}\|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau + C|\Omega|^{1/p}D^{-1}$$

and the proof of Proposition 4.4 leads to global estimates

$$\sup_{t\in\mathbb{R}_{\geq 0}} \Big(\|\mathbf{U}_D(\cdot,t)\|_{L^p(\Omega)^m} + \|\mathbf{V}_D(\cdot,t)\|_{L^p(\Omega)^k}\Big) \leq CD^{-1}. \tag{4.23}$$

From this estimate we infer similar exponents as in the foregoing discussion, more precisely estimate (4.22) holds for $\alpha = 1$ and one ascertains a parameter $\gamma > 0$. Global estimates for nonlinear problems will be further discussed in Chapter 5 with regard to asymptotic behavior.

4.2 The nonlinear case

As figured out in the discussion of the linear case, convergence has been shown under several additional stability conditions referred to as Assumption L resp. L0 and L1p. Let us consider the semilinear diffusive case (1.1)–(1.3) following the same ideas as in the case of short-time intervals: truncation of the corresponding linearized problem. It turns out in Theorem 4.10 below that assuming conditions L0 and L1p for the linearized shadow system yields estimates which are valid for the nonlinear system on long-time ranges, similar to the results in the linear case in Theorem 4.5. As standing assumptions for long-time intervals, we use the local Lipschitz condition A1, bounded initial values from Assumption A2, the decay estimate (4.13) for the mean value correction ψ_D from Assumption A3 and the existence of a globally bounded shadow limit:

B The solution $(\mathbf{u}, \mathbf{v})$ of the shadow system (1.4)–(1.6) is globally defined and satisfies $\mathbf{u} \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^m$ and $\mathbf{v} \in L^\infty(\mathbb{R}_{\geq 0})^k$.

Further assumptions will be made in the following section.

4.2.1 Second-order truncation

Let us start from system (3.2)–(3.4) for the errors $\mathbf{U}_D, \mathbf{V}_D$, i.e.,

$$\begin{aligned}
\frac{\partial \mathbf{U}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{U}_D &= \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{f}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T, \\
\frac{\partial \mathbf{V}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{V}_D &= \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{g}(\mathbf{u}, \mathbf{v}, x, t) \quad \text{in} \quad \Omega_T, \\
\mathbf{U}_D(\cdot, 0) = \mathbf{0}, \mathbf{V}_D(\cdot, 0) = \mathbf{0} \ \text{ in } \ \Omega, \quad & \frac{\partial \mathbf{U}_D}{\partial \mathbf{n}} = \mathbf{0}, \frac{\partial \mathbf{V}_D}{\partial \mathbf{n}} = \mathbf{0} \ \text{ on } \ \partial\Omega \times (0, T).
\end{aligned}$$

Using Taylor's expansion, we write for $\mathbf{h} = (\mathbf{f}, \mathbf{g})$

$$\begin{aligned}
\mathbf{h}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{h}(\mathbf{u}, \mathbf{v}, x, t) = \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{u}, \mathbf{v}, x, t) \mathbf{U}_D + \nabla_{\mathbf{v}} \mathbf{h}(\mathbf{u}, \mathbf{v}, x, t)(\mathbf{V}_D + \psi_D) \\
+ \mathbf{H}(\mathbf{U}_D, \mathbf{V}_D + \psi_D, x, t), \qquad (4.24)
\end{aligned}$$

where the remainder $\mathbf{H} = (\mathbf{F}, \mathbf{G})$ is given componentwise by

$$H_\ell(\mathbf{y}, \mathbf{z} + \psi_D, x, t) = \begin{pmatrix} \mathbf{y} & \mathbf{z} + \psi_D \end{pmatrix}^T \mathbf{h}_\ell^{(u,v)}(\mathbf{y}, \mathbf{z} + \psi_D, x, t) \begin{pmatrix} \mathbf{y} \\ \mathbf{z} + \psi_D \end{pmatrix}$$

for $\ell = 1, \ldots, m + k$. Each matrix-valued function $\mathbf{h}_\ell^{(u,v)}$ is decomposed of difference quotients for the first derivatives of $\mathbf{h}$ similar to the linearization in (3.6). For this to be valid and for the following procedure let us assume, in addition to assumptions A1–A3, B a differentiability assumption.

A4 Let $\mathbf{f}, \mathbf{g}$ be continuously differentiable with respect to $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{m+k}$ and let their derivatives $\nabla_u f_i, \nabla_v f_i, i = 1, \ldots, m$, and $\nabla_u g_j, \nabla_v g_j, j = 1, \ldots, k$, satisfy the local Lipschitz condition A1. Let the linearized parts $\nabla_{(\mathbf{u},\mathbf{v})}\mathbf{f}$, $\nabla_{(\mathbf{u},\mathbf{v})}\mathbf{g}$, evaluated at the shadow limit, be uniformly bounded in $(x,t) \in \overline{\Omega} \times \mathbb{R}_{\geq 0}$.

For instance, the local Lipschitz continuity is satisfied for an autonomous right-hand side $(\mathbf{f}, \mathbf{g})$ which is of class C^2 with respect to the unknown variables $(\mathbf{u}, \mathbf{v})$. If, in addition, we have a uniformly bounded shadow limit, then Assumption A4 is fulfilled.

Following the idea of truncation developed in Chapter 3, we construct a suitable cut-off for the possibly unbounded right-hand side $\mathbf{H} = (\mathbf{F}, \mathbf{G})$. As before, we will modify their arguments using the cut-off function Θ defined in (3.7) and consider the function $\overline{\mathbf{H}}_D = (\overline{\mathbf{F}}_D, \overline{\mathbf{G}}_D)$ given by

$$\overline{H}_{D,\ell}(\mathbf{y}, \mathbf{z} + \psi_D, x, t) := \begin{pmatrix} \Theta(\mathbf{y}) & \mathbf{z} + \psi_D \end{pmatrix}^T \mathbf{h}^{(u,v)}_{\ell,\Theta} \begin{pmatrix} \Theta(\mathbf{y}) \\ \mathbf{z} + \psi_D \end{pmatrix} \tag{4.25}$$

for each $\ell = 1, \ldots, m+k$. Herein, we abbreviate

$$\mathbf{h}^{(u,v)}_{\ell,\Theta} = \mathbf{h}^{(u,v)}_{\ell}(\Theta(\mathbf{y}), \Theta(\mathbf{z}) + \psi_D, x, t).$$

Since the arguments of this matrix-valued function are uniformly bounded by assumptions A3, B and the definition of Θ, we may apply the Lipschitz assumption A4 and infer a constant $C > 0$ such that

$$|\overline{\mathbf{H}}_D(\mathbf{y}, \overline{\mathbf{z}}, x, t)| \leq C|(\Theta(\mathbf{y}), \overline{\mathbf{z}})|^2 \tag{4.26}$$

holds for $\overline{\mathbf{z}} := \mathbf{z} + \psi_D$. The constant C depends on the time-independent bounds on $\mathbf{u}, \mathbf{v}$ in Assumption B, on the one for ψ_D in Assumption A3, and on Lipschitz bounds in Assumption A4 for the derivatives, but neither on diffusion $D \geq 1$ nor on time t. In order to control the $\mathbf{z}$-component of the truncation which we estimated in the last inequality (4.26), we introduce another function. Define

$$\rho(z) = \begin{cases} 1 & \text{for } |z| \leq L, \\ 0 & \text{for } |z| \geq 2L \end{cases} \tag{4.27}$$

as a smooth and symmetric cut-off function $\rho \in C_c^\infty(\mathbb{R}; [0,1])$ for $L := C_{v^0} + 2$, where $C_{v^0} > 0$ is the same constant from the time decay estimate (4.13) of ψ_D. This is possible by mollifying the characteristic function $\chi_{(-r,r)}$ where $L < r < 2L$ [1, Theorem 2.29]. Using this cut-off, we control the $\mathbf{z}$-component in $\overline{\mathbf{H}}_D$ by setting

$$\begin{aligned}\mathbf{H}_D(\mathbf{y},\mathbf{z},x,t) := \rho\left(\frac{D^{\delta_0}|\overline{\mathbf{z}}|}{L}\right)\overline{\mathbf{H}}_D(\mathbf{y},\overline{\mathbf{z}},x,t)\left(1-\rho\left(\frac{2\lambda_1 Dt}{\log D}\right)\right)\\ +\rho\left(\frac{|\overline{\mathbf{z}}|}{L}\right)\overline{\mathbf{H}}_D(\mathbf{y},\mathbf{z},x,t)\rho\left(\frac{2\lambda_1 Dt}{\log D}\right).\end{aligned} \tag{4.28}$$

A similar modification is done in [75, Lemma 2]. Properties of the truncated function $\mathbf{H}_D = (\mathbf{F}_D, \mathbf{G}_D)$ is given in

Lemma 4.6. *For each $\delta_0 \le 1/2$ there is a constant $C > 0$, independent of $D \ge 1$ but which depends on L defined in* (4.27) *and assumptions A3–A4, B, such that for $\overline{\mathbf{z}} = \mathbf{z} + \psi_D$ we have*

$$|\mathbf{F}_D(\mathbf{y},\mathbf{z},x,t)|, |\mathbf{G}_D(\mathbf{y},\mathbf{z},x,t)| \le C\Big(D^{-2\delta_0} + \chi_{\{t\le \log D/(\lambda_1 D)\}}(t)\cdot \min\{1, |\overline{\mathbf{z}}|\}\Big) \tag{4.29}$$

for all $(\mathbf{y},\mathbf{z}) \in \mathbb{R}^{m+k}, t \in \mathbb{R}_{\ge 0}$ and for a.e. $x \in \Omega$. Here, $\chi_{\{t\le c\}}$ is the characteristic function on the time interval $[0,c]$.

Proof. The first term in definition (4.28) can be estimated as $\overline{\mathbf{H}}_D$, see inequality (4.26). However, only $D^{\delta_0}|\overline{\mathbf{z}}| \le 2L$ has to be considered due to the compact support of ρ defined in (4.27). For the second term in definition (4.28), we use the estimate

$$\left|\rho\left(\frac{|\overline{\mathbf{z}}|}{L}\right)\overline{\mathbf{H}}_D(\mathbf{y},\mathbf{z},x,t)\rho\left(\frac{2\lambda_1 Dt}{\log D}\right)\right| \le C(D^{-2\delta_0} + |\overline{\mathbf{z}}|^2)\rho\left(\frac{|\overline{\mathbf{z}}|}{L}\right)\chi_{\{t\le \log D/(\lambda_1 D)\}}$$

which results once again from inequality (4.26). The right-hand side is at most non-zero if $|\overline{\mathbf{z}}| \le 2L$ and the quadratic term is (linearly) bounded. □

In the following, we study the localized problem using the truncation $\mathbf{F}_D, \mathbf{G}_D$ associated with the error system (3.2)–(3.4). Starting for instance from a substitution of $\mathbf{H}$ in equation (4.24), we obtain

$$\frac{\partial \alpha_D}{\partial t} - \mathbf{D}^u\Delta\alpha_D = \nabla_{\mathbf{u}}\mathbf{f}\cdot\alpha_D + \nabla_{\mathbf{v}}\mathbf{f}\cdot(\beta_D + \psi_D) + \mathbf{F}_D(\alpha_D,\beta_D,x,t) \quad \text{in } \Omega_T, \tag{4.30}$$

$$\frac{\partial \beta_D}{\partial t} - \mathbf{D}^v\Delta\beta_D = \nabla_{\mathbf{u}}\mathbf{g}\cdot\alpha_D + \nabla_{\mathbf{v}}\mathbf{g}\cdot(\beta_D + \psi_D) + \mathbf{G}_D(\alpha_D,\beta_D,x,t) \quad \text{in } \Omega_T, \tag{4.31}$$

$$\alpha_D(\cdot,0) = \mathbf{0},\ \beta_D(\cdot,0) = \mathbf{0} \ \text{ in } \Omega, \quad \frac{\partial\alpha_D}{\partial\mathbf{n}} = \mathbf{0}, \frac{\partial\beta_D}{\partial\mathbf{n}} = \mathbf{0} \ \text{ on } \partial\Omega\times(0,T). \tag{4.32}$$

To get an overview, we abbreviate the Jacobians $\nabla_{\mathbf{u}}\mathbf{f}, \nabla_{\mathbf{v}}\mathbf{f}, \nabla_{\mathbf{u}}\mathbf{g}, \nabla_{\mathbf{v}}\mathbf{g}$ which are evaluated at the shadow solution $(\mathbf{u}, \mathbf{v})$ and depend in general on space and time.

The focal idea using cut-offs is to find estimates for the solution (α_D, β_D) of the truncated problem (4.30)–(4.32) to show that its solution actually is located for sufficiently large diffusions D within a small neighborhood of $\mathbf{0}$, with radius $D^{-\delta}$ for some $\delta > 0$, where the cut-offs have no effect if $\delta \geq \delta_0$. Similarly to Proposition 3.2, we find global solutions.

Proposition 4.7. *Let assumptions A1–A4, B hold and let $D \geq 1$. Then there exists a unique mild solution (α_D, β_D) of the truncated problem* (4.30)–(4.32) *which is global. Furthermore, for all finite times $T > 0$ we have $(\alpha_D, \beta_D) \in L^\infty(\Omega_T)^{m+k}$ and diffusing components are weak solutions, especially $\beta_D \in L^\infty(0, T; H^1(\Omega)^k)$ with weak derivative $\partial_t \beta_D \in L^2(\Omega_T)^k$.*

Proof. To apply Rothe's method as in Proposition 3.2, we write system (4.30)–(4.32) as a system of $m + k$ differential equations

$$\frac{\partial \Psi_D}{\partial t} - \mathbf{D}\Delta\Psi_D = \mathbf{h}_D(\Psi_D, x, t) \text{ in } \Omega_T, \quad \Psi_D(\cdot, 0) = \mathbf{0} \text{ in } \Omega.$$

The function $\Psi_D = (\alpha_D, \beta_D)$ is endowed with zero Neumann boundary conditions for diffusing components and $\mathbf{h}_D$ is given by

$$\mathbf{h}_D(\Psi_D, x, t) = \mathbf{J}(x, t) \cdot \begin{pmatrix} \alpha_D \\ \beta_D + \psi_D(x, t) \end{pmatrix} + \begin{pmatrix} \mathbf{F}_D(\alpha_D, \beta_D, x, t) \\ \mathbf{G}_D(\alpha_D, \beta_D, x, t) \end{pmatrix}.$$

Here and in the sequel, we use the notation $\mathbf{J}$ for the Jacobian

$$\mathbf{J}(x, t) = \begin{pmatrix} \nabla_{\mathbf{u}}\mathbf{f}(\mathbf{u}(x, t), \mathbf{v}(t), x, t) & \nabla_{\mathbf{v}}\mathbf{f}(\mathbf{u}(x, t), \mathbf{v}(t), x, t) \\ \nabla_{\mathbf{u}}\mathbf{g}(\mathbf{u}(x, t), \mathbf{v}(t), x, t) & \nabla_{\mathbf{v}}\mathbf{g}(\mathbf{u}(x, t), \mathbf{v}(t), x, t) \end{pmatrix} \tag{4.33}$$

evaluated at the shadow solution $(\mathbf{u}, \mathbf{v})$. By Assumption A4, $\mathbf{h}_D$ is bounded on bounded subset of $\mathbb{R}^{m+k} \times \overline{\Omega} \times \mathbb{R}_{\geq 0}$. Local Lipschitz continuity of $\mathbf{h}_D$ in the variable Ψ_D on bounded sets in $\overline{\Omega} \times \mathbb{R}_{\geq 0}$ carries over from $\mathbf{h}$ since $\mathbf{F}_D$ and $\mathbf{G}_D$ are locally Lipschitz in the sense of Assumption A1, see definition (4.28). Following the proof of Proposition 2.3, there exists an $E_{\infty,0,\tau}$-mild solution in the sense of Definition 2.2. We obtain the integral representation

$$\Psi_D(\cdot, t) = \int_0^t \mathbf{S}(t - \tau)\mathbf{h}_D(\Psi_D(\cdot, \tau), \cdot, \tau) \, \mathrm{d}\tau.$$

The function $\mathbf{h}_D$ is linearly bounded in the variable Ψ_D due to Lemma 4.6. By the same reasoning as in Proposition 3.2, this implies that no blow-up is possible and we obtain a unique $E_{\infty,0,\infty}$-mild solution with $\Psi_D \in L^\infty(\Omega_T)^{m+k}$ for all $T < \infty$.
In order to improve regularity for diffusing components which are denoted by z_d, we apply parabolic L^2 theory performed in Proposition B.3. Recall for this that z_d solves the equation

$$\frac{\partial z_d}{\partial t} - d\Delta z_d = R_d \in L^\infty(\Omega_T)$$

for some diffusion $d > 0$ and the initial datum for z_d is zero. □

To get estimates for very long time intervals of order D^ℓ for some $\ell > 0$, we need a more detailed estimation of the quantities which are involved and follow the steps of Proposition 4.4. We decompose β_D into its mean $\mathbf{b}_D = \langle \beta_D \rangle_\Omega$ and the residual $\mathbf{W}_D = \beta_D - \langle \beta_D \rangle_\Omega$ with $\langle \mathbf{W}_D \rangle_\Omega = \mathbf{0}$. Equations (4.31)–(4.32) may be replaced by

$$\begin{aligned} \frac{\partial \mathbf{W}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{W}_D &= \nabla_{\mathbf{u}} \mathbf{g} \cdot \alpha_D - \langle \nabla_{\mathbf{u}} \mathbf{g} \cdot \alpha_D \rangle_\Omega + \nabla_{\mathbf{v}} \mathbf{g} \cdot \mathbf{b}_D - \langle \nabla_{\mathbf{v}} \mathbf{g} \cdot \mathbf{b}_D \rangle_\Omega \\ &\quad + \nabla_{\mathbf{v}} \mathbf{g} \cdot (\mathbf{W}_D + \psi_D) - \langle \nabla_{\mathbf{v}} \mathbf{g} \cdot (\mathbf{W}_D + \psi_D) \rangle_\Omega \\ &\quad + \mathbf{G}_D - \langle \mathbf{G}_D \rangle_\Omega \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \end{aligned} \tag{4.34}$$

$$\begin{aligned} \frac{\mathrm{d}\mathbf{b}_D}{\mathrm{d}t} &= \langle \nabla_{\mathbf{u}} \mathbf{g} \cdot \alpha_D \rangle_\Omega + \langle \nabla_{\mathbf{v}} \mathbf{g} \cdot \mathbf{b}_D \rangle_\Omega + \langle \nabla_{\mathbf{v}} \mathbf{g} \cdot (\mathbf{W}_D + \psi_D) \rangle_\Omega \\ &\quad + \langle \mathbf{G}_D \rangle_\Omega \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \end{aligned} \tag{4.35}$$

$$\mathbf{W}_D(\cdot, 0) = \mathbf{0} \quad \text{in} \quad \Omega, \quad \mathbf{b}_D(0) = \mathbf{0}, \quad \frac{\partial \mathbf{W}_D}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times \mathbb{R}_{>0}. \tag{4.36}$$

We start estimating the term $\mathbf{W}_D$ using Lemma 2.1. Denoting the right-hand side of equation (4.34) by $\mathbf{R}_D$, the solution $\mathbf{W}_D$ may be written as

$$\mathbf{W}_D(\cdot, t) = \int_0^t \mathbf{S}^v(t - \tau) \mathbf{R}_D(\cdot, \tau) \, \mathrm{d}\tau.$$

By Assumption A4, the Jacobian $\mathbf{J} \in L^\infty(\Omega \times \mathbb{R}_{\geq 0})^{(m+k)\times(m+k)}$ is uniformly bounded. We infer from decay estimate (4.13) for ψ_D and $\langle \mathbf{R}_D \rangle_\Omega = \mathbf{0}$ that

$$\begin{aligned} \|\mathbf{W}_D(\cdot, t)\|_{L^p(\Omega)^k} \leq C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \Big(&\|\alpha_D(\cdot, \tau)\|_{L^p(\Omega)^m} + |\Omega|^{1/p} |\mathbf{b}_D(\tau)| \\ &+ \|(\mathbf{W}_D + \psi_D)(\cdot, \tau)\|_{L^p(\Omega)^k} + \|\mathbf{G}_D(\cdot, \tau)\|_{L^p(\Omega)^k} \Big) \, \mathrm{d}\tau \end{aligned} \tag{4.37}$$

holds for each finite $p \geq 1$ with a similar estimate for $p = \infty$. As in Proposition 4.4, we have to control the terms α_D and $\mathbf{b}_D$. This can be done by Assumption L1p as in the linear case using a condition for the Jacobian $\mathbf{J}$ instead. For convenience, let us use the same notation as in the linear case, compare to notation (4.11), and abbreviate the Jacobian defined by (4.33) in the form

$$\mathbf{J}(x,t) = \begin{pmatrix} \mathbf{A}_*(x,t) & \mathbf{B}_*(x,t) \\ \mathbf{C}_*(x,t) & \mathbf{D}_*(x,t) \end{pmatrix}.$$

Then components $\alpha_D, \mathbf{b}_D$ satisfy a shadow problem whose solution is given by

$$\begin{pmatrix} \alpha_D(\cdot,t) \\ \mathbf{b}_D(t) \end{pmatrix} = \int_0^t \mathbf{W}(t,\tau) \begin{pmatrix} \mathbf{B}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) + \mathbf{F}_D(\cdot,\tau) \\ \langle \mathbf{D}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) + \mathbf{G}_D(\cdot,\tau)\rangle_\Omega \end{pmatrix} \mathrm{d}\tau \tag{4.38}$$

where we used the evolution system $\mathcal{W}$ induced by the linearization. This system is given by evolution operators $\mathbf{W}(t,s)$ for $t, s \in \mathbb{R}_{\geq 0}, s \leq t$, defined by

$$\xi(\cdot,t) = \mathbf{W}(t,s)\xi(\cdot,s), \qquad \xi(\cdot,0) = \begin{pmatrix} \xi_1^0 \\ \langle \xi_2^0 \rangle_\Omega \end{pmatrix}, \tag{4.39}$$

where $\xi \in C(\mathbb{R}_{\geq 0}; L^p(\Omega)^m \times \mathbb{R}^k)$ is the unique solution of the homogeneous linear shadow problem

$$\begin{aligned} \frac{\partial \xi_1}{\partial t} - \mathbf{D}^u \Delta \xi_1 &= \mathbf{A}_*(x,t)\xi_1 + \mathbf{B}_*(x,t)\xi_2 && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \\ \frac{\mathrm{d}\xi_2}{\mathrm{d}t} &= \langle \mathbf{C}_*(\cdot,t)\xi_1\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\xi_2\rangle_\Omega && \text{in} \quad \mathbb{R}_{>0}, \\ \xi_1(\cdot,0) = \xi_1^0 \quad \text{in} \quad \Omega, & \qquad \xi_2(0) = \langle \xi_2^0 \rangle_\Omega \end{aligned}$$

endowed with zero Neumann boundary conditions for ξ_1 if necessary. As in the linear case considered in (4.11), let us assume that the shadow evolution system $\mathcal{W}$ is uniformly bounded in $L^p(\Omega)^m \times \mathbb{R}^k$ for some $1 \leq p \leq \infty$, i.e., there exists a constant $C > 0$ independent of time such that

$$\|\mathbf{W}(t,s)\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \leq C\|\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \qquad \forall\, \xi^0 \in L^p(\Omega)^m \times \mathbb{R}^k, s, t \in \mathbb{R}_{\geq 0}, s \leq t.$$

We will still name this Assumption L1p having in mind that it is induced by the linearization $\mathbf{J}$ of the nonlinear problem.

Proposition 4.8. *Let assumptions A1–A4, B, and L1p hold for some* $1 \leq p \leq \infty$*. Then for any* $\alpha \in (0,1], \delta_0 \in (0,1/2]$ *with* $\gamma := 2\delta_0 + (\alpha - 1) \in (0,1]$ *there exist constants* $C, D_0 > 0$ *independent of time* T *and diffusion* D *such that for all times*

$T \leq D^{1-\alpha}$ and all $D \geq D_0$ the solution (α_D, β_D) of system (4.30)–(4.32) satisfies

$$\sup_{t\in[0,T]} \left(\|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + \|\beta_D(\cdot,t)\|_{L^p(\Omega)^k}\right) \leq CD^{-\gamma}. \tag{4.40}$$

Proof. We already estimated $\mathbf{W}_D$ and received a relation to $\alpha_D, \mathbf{b}_D$ in inequality (4.37) above. In view of estimate (4.29) for $\mathbf{F}_D, \mathbf{G}_D$, where

$$\overline{\beta_D} = \beta_D + \psi_D = \mathbf{b}_D + \mathbf{W}_D + \psi_D,$$

and Assumption A3 on ψ_D, we observe for each $\delta_0 \geq 0$

$$\begin{aligned} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \left(\|\alpha_D(\cdot,\tau)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(\tau)|\right) \,\mathrm{d}\tau \\ + C \int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)} \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k} \,\mathrm{d}\tau + C|\Omega|^{1/p} D^{-1} \end{aligned} \tag{4.41}$$

with a similar estimate for $p = \infty$. To obtain a corresponding inequality for $\alpha_D, \mathbf{b}_D$, we consider the explicit formula

$$\begin{pmatrix} \alpha_D(\cdot,t) \\ \mathbf{b}_D(t) \end{pmatrix} = \int_0^t \mathbf{W}(t,\tau) \begin{pmatrix} \mathbf{B}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) + \mathbf{F}_D(\cdot,\tau) \\ \langle \mathbf{D}_*(\cdot,\tau)(\mathbf{W}_D + \psi_D)(\cdot,\tau) + \mathbf{G}_D(\cdot,\tau)\rangle_\Omega \end{pmatrix} \,\mathrm{d}\tau.$$

Applying the stability assumption L1p on $\mathcal{W}$, Assumption A4 and estimate (4.29) for truncations $\mathbf{F}_D, \mathbf{G}_D$ to the latter integral, this yields

$$\begin{aligned} \|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)| &\leq C \int_0^t \|(\mathbf{W}_D + \psi_D)(\cdot,\tau)\|_{L^p(\Omega)^k} \,\mathrm{d}\tau \\ &\quad + C \int_0^t \|\mathbf{F}_D(\cdot,\tau)\|_{L^p(\Omega)^m} + \|\mathbf{G}_D(\cdot,\tau)\|_{L^p(\Omega)^k} \,\mathrm{d}\tau \\ &\leq C \int_0^t \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k} \,\mathrm{d}\tau \\ &\quad + C \int_0^t \chi_{\{\tau \leq \log D/(\lambda_1 D)\}} |\Omega|^{1/p}|\mathbf{b}_D(\tau)| \,\mathrm{d}\tau \\ &\quad + C|\Omega|^{1/p} \left(D^{-1}(1+t) + D^{-2\delta_0} t\right). \end{aligned}$$

For obvious reasons, we restrict ourselves to $\delta_0 \leq 1/2$. Since $\log D/D \to 0$ as $D \to \infty$ by L'Hospital's rule, we absorb $\mathbf{b}_D$ on the left-hand side and obtain

$$\begin{aligned} \sup_{t\in[0,T]} \left(\|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)|\right) \leq C \int_0^T \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k} \,\mathrm{d}\tau \\ + C|\Omega|^{1/p} \left(D^{-1} + D^{-2\delta_0} T\right). \end{aligned} \tag{4.42}$$

Inserting the above estimate (4.42) into inequality (4.41) for $\mathbf{W}_D$ leads to an estimate for $\mathbf{W}_D$, more precisely, we obtain

$$\sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq C \sup_{t\in[0,T]} \int_0^t \left(\mathrm{e}^{-\lambda_1 D(t-\tau)} + D^{-1}\right) \|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau$$
$$+ C|\Omega|^{1/p} D^{-1}\left(1 + D^{-2\delta_0}T\right).$$

Taking the supremum over t yields

$$\sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq CD^{-1}(1+T) \sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} + CD^{-1}(1+D^{-2\delta_0}T)$$

and absorbing terms on the left-hand side for all $T \leq D^{1-\alpha}$ implies for $D \geq D_0$

$$\sup_{t\in[0,T]} \|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq CD^{-\min\{1,2\delta_0+\alpha\}}.$$

Using estimate (4.42) for α_D, for each $\gamma := 2\delta_0 + (\alpha - 1) \in (0,1]$ there holds

$$\sup_{t\in[0,T]} \left(\|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)|\right) \leq C\left(D^{-2\delta_0}T + D^{-1}T\right) \leq CD^{-\gamma}$$

for $D \geq D_0 = D_0(\gamma)$. As a consequence, $\gamma > 0$ implies $2\delta_0 + \alpha > 1$ and $\|\mathbf{W}_D\|_{L^p(\Omega)^k} \leq CD^{-1}$. The relation $\beta_D = \mathbf{W}_D + \mathbf{b}_D$ implies an estimate for β_D. $\square$

If Assumption L1p holds with $p = \infty$, we already obtain a corresponding estimate in $L^\infty(\Omega_T)$ for solutions to the truncated problem (4.30)–(4.32). If this is not the case, Hölder's inequality yields bounds for the norms of α_D, β_D in the parabolic space $L_{p,r}(\Omega_T)$ for each $1 \leq r < \infty$, for instance,

$$\|\alpha_D\|_{p,r} \leq T^{1/r}\|\alpha_D\|_{p,\infty} \leq CD^{-\gamma+\frac{1}{r}(1-\alpha)}$$

and similarly for β_D. Actually the latter inequalities imply, either for $p = 1 = n$ or for $p > n/2$, an $L^\infty(\Omega_T)$ estimate for diffusing components using a bootstrap argument for parabolic equations. This can be done in the same manner as for the linear case, compare to Proposition B.10 and especially inequality (B.23). For non-diffusing components we apply Assumption L0 to the corresponding ODE subsystem induced by a submatrix $\mathbf{A}_{11}(x,t)$ of $\mathbf{A}_*(x,t)$ of the above Jacobian $\mathbf{J}$ defined in (4.33). Combining both methods, we reach at

Proposition 4.9. *Let assumptions A1–A4, B as well as L0 and L1p hold for some finite $p \geq 1 = n$ or $p > n/2$ for $n \geq 2$ and choose r as in* (B.14). *Then there exist triples $(\alpha, \delta_0, r) \in (0,1] \times (0,1/2] \times (1,\infty)$ and constants $C, D_0 > 0$ independent of T, D such that for all $T \leq D^{1-\alpha}$ and $D \geq D_0$ there holds*

$$\|\alpha_D\|_{L^\infty(\Omega_T)^m} \leq CD^{2(1-\alpha)}\left(D^{-\gamma} + D^{-(2-\alpha)/r}\right), \tag{4.43}$$

$$\|\beta_D\|_{L^\infty(\Omega_T)^k} \leq CD^{(1-\alpha)}\left(D^{-\gamma} + D^{-(2-\alpha)/r}\right) \tag{4.44}$$

for $\gamma = 2\delta_0 + (\alpha - 1) \in (0,1]$ from Proposition 4.8. Moreover, for diffusing components $\alpha_{D,i}$ we have the same convergence rate as for β_D.
If Assumption L1p holds with $p = \infty$, estimate (4.40) *is true without requiring Assumption L0, and we may choose $D^{-\gamma}$ as a convergence rate for each component, see Proposition 4.8.*

Proof. If $p = \infty$, there is nothing to show, see Proposition 4.8 and estimate (4.40). Note that uniform boundedness of the evolution system $\mathcal{W}$ of the entire shadow system in $L^\infty(\Omega)^m \times \mathbb{R}^k$ is sufficient, thus no boundedness of the ODE subsystem stated in Assumption L0 is needed.
Let us assume $p < \infty$ and start with a diffusive component z_d of the truncated problem (4.30)–(4.32) written as in equation (B.8), i.e.,

$$\frac{\partial z_d}{\partial t} - d\Delta z_d = R_d(x,t).$$

Proposition B.10 yields a constant $C > 0$ such that

$$\|z_d\|_{L^\infty(\Omega_T)} \leq CT^{1-1/r}\|R_d\|_{p,r}$$

for some $r > 1$ defined by (B.14). This constant C is independent of time T and only depends on a lower bound for the diffusion d and parameters of the systems. Thus, it remains to find an estimate for $\|R_d\|_{p,r}$. We infer from the right-hand side of the truncated system (4.30)–(4.32) that

$$\|R_d\|_{p,r} \leq C\left(\|\alpha_D\|_{p,r} + \|\beta_D\|_{p,r} + \|\psi_D\|_{p,r} + \max\{\|\mathbf{F}_D\|_{p,r}, \|\mathbf{G}_D\|_{p,r}\}\right).$$

Assumption A3 on ψ_D and Lemma 4.6 with $\overline{\beta}_D = \beta_D + \psi_D$ yields

$$\|R_d\|_{p,r} \leq C\left(\|\alpha_D\|_{p,r} + \|\beta_D\|_{p,r} + D^{-1/r} + D^{-2\delta_0}T^{1/r}\right).$$

As a consequence of Proposition 4.8, above a priori estimates for α_D, β_D in $L_{p,r}(\Omega_T)$ and $T \leq D^{1-\alpha}$ for some $\alpha \in (0, 1]$ imply

$$\|z_d\|_{L^\infty(\Omega_T)} \leq CD^{(1-\frac{1}{r})(1-\alpha)} \left(D^{-\gamma+\frac{1}{r}(1-\alpha)} + D^{-1/r} + D^{-2\delta_0+\frac{1}{r}(1-\alpha)}\right).$$

Since $\gamma = 2\delta_0 + (\alpha - 1) \leq 2\delta_0$, we further estimate

$$\|z_d\|_{L^\infty(\Omega_T)} \leq CD^{1-\alpha} \left(D^{-\gamma} + D^{-(2-\alpha)/r}\right).$$

To obtain an analog estimate for α_D, we restrict our consideration to condition L0 for the subsystem of non-diffusing components. In the following, $\tilde{\alpha}_D$ represents the corresponding vector of non-diffusing components. Then, by using the notion of Assumption L0 and equation (4.30), this error component is given by

$$\tilde{\alpha}_D(\cdot, t) = \int_0^t \tilde{\mathbf{U}}(t, \tau)\tilde{\mathbf{R}}_D(\cdot, \tau) \, \mathrm{d}\tau$$

for some right-hand side $\tilde{\mathbf{R}}_D \in L^\infty(\Omega_T)^{\tilde{m}}$ that only depends linearly on diffusing components of α_D, $\beta_D + \psi_D$ and components of $\mathbf{F}_D$ which we already estimated. Using uniform boundedness of the evolutionary subsystem $\tilde{\mathcal{U}}$ stated in Assumption L0, we obtain

$$\|\tilde{\alpha}_D\|_{L^\infty(\Omega_T)^{\tilde{m}}} \leq C \int_0^T \|\tilde{\mathbf{R}}_D(\cdot, \tau)\|_{L^\infty(\Omega)^{\tilde{m}}} \, \mathrm{d}\tau \leq CD^{2(1-\alpha)} \left(D^{-\gamma} + D^{-(2-\alpha)/r}\right).$$

Specifically, we employed the corresponding estimate for diffusing components z_d and that $\mathbf{F}_D$ can be estimated by $\overline{\beta}_D = \beta_D + \psi_D$, too. $\square$

4.2.2 Convergence results

We are now in a position to draw a conclusion for the original errors $(\mathbf{U}_D, \mathbf{V}_D)$ using estimates for the truncated problem in the last section. In order to dispose of truncation, we infer from results of Proposition 4.9 that the truncated solution (α_D, β_D), for sufficiently large diffusion D, is located in a neighborhood of $\mathbf{0}$ where the cut-off is not required.

Theorem 4.10. *Let the assumptions A1–A4, B, L0, and L1p hold for some p with $p \geq 1 = n$ or $p > n/2$ if $n \geq 2$ and let $r \in (1, \infty)$ given by* (B.14). *Then there exist lower bounds $\alpha_0 = \alpha_0(r) \in (0, 1), D_0 > 0$ and a constant $C > 0$ such that for any*

$\alpha \in [\alpha_0, 1)$, $D \geq D_0$ and times $T \leq D^{1-\alpha}$ we have the uniform estimates

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega_T)^m} \leq CD^{-3(1-\alpha)}, \tag{4.45}$$

$$\|\langle \mathbf{v}_D \rangle_\Omega - \mathbf{v}\|_{L^\infty((0,T))^k} \leq CD^{-4(1-\alpha)}, \tag{4.46}$$

$$\|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega_T)^k} \leq CD^{-4(1-\alpha)}. \tag{4.47}$$

All diffusive components have a convergence rate of the same order, i.e., $D^{-4(1-\alpha)}$. For $p = \infty$, without requiring Assumption L0, similar estimates hold true with a convergence rate $D^{-(1-\alpha)}$ for any component.

Proof. To get rid of truncation, it suffices to show

$$\|\alpha_D\|_{L^\infty(\Omega_T)^m} + \|\beta_D\|_{L^\infty(\Omega_T)^k} \leq D^{-\delta_0} \qquad \forall\, D \geq D_0 \tag{4.48}$$

since in that case Θ is not needed anymore in equation (4.25) for $\overline{\mathbf{H}}_D$ and we obtain $\overline{\mathbf{H}}_D = \mathbf{H}$. By considering the two cases, above and below the critical time $t^* := \delta_0 \log D/(\lambda_1 D)$ for construction (4.28), we deduce $\mathbf{H}_D = \mathbf{H}$ in the following.

- If $t \leq t^*$, we obtain $\rho\left(\frac{2\lambda_1 Dt}{\log D}\right) = 1$ by $2\delta_0 \leq L$ and thus,

  $$\mathbf{H}_D(\alpha_D, \beta_D, x, t) = \rho\left(\frac{|\overline{\beta_D}|}{L}\right)\mathbf{H}(\alpha_D, \beta_D, x, t).$$

  Additionally, there holds $\rho\left(\frac{|\overline{\beta_D}|}{L}\right) = 1$ since by definition of $L = C_{v^0} + 2$

  $$|\overline{\beta_D}| \leq |\beta_D| + |\psi_D| \leq D^{-\delta_0} + C_g D^{-1} + (L-2)\mathrm{e}^{-\lambda_1 Dt} \leq L$$

  for all large enough $D \geq D_0$.

- If $t > t^*$, then $\mathrm{e}^{-\lambda_1 Dt} \leq D^{-\delta_0}$ and thus, for large D

  $$|\overline{\beta_D}| \leq |\beta_D| + |\psi_D| \leq D^{-\delta_0} + C_g D^{-1} + (L-2)D^{-\delta_0} \leq LD^{-\delta_0}.$$

  Clearly, $|\overline{\beta}_D| \leq |D^{\delta_0}\overline{\beta}_D| \leq L$ and we find once again by definition (4.28) that

  $$\mathbf{H}_D(\alpha_D, \beta_D, x, t) = \overline{\mathbf{H}}_D(\alpha_D, \beta_D, x, t) = \mathbf{H}(\alpha_D, \beta_D, x, t).$$

This means that the cut-off does not affect right-hand sides of the truncated problem (4.30)–(4.32) if one restricts to the trajectory of the solution (α_D, β_D). By uniqueness of solutions to problem (4.30)–(4.32), we conclude $(\alpha_D, \beta_D) = (\mathbf{U}_D, \mathbf{V}_D)$ and estimates (4.43)–(4.44) from Proposition 4.9 are also valid for the original error functions $(\mathbf{U}_D, \mathbf{V}_D)$ on the domain Ω_T.

In the case of $p = \infty$, estimate (4.40) holds true and it remains to satisfy $\gamma > \delta_0$ such that inequality (4.48) is valid. The latter is equivalent to $\alpha > 1 - \delta_0$ and choosing $\alpha_0 = 1 - \delta_0$ yields the estimate

$$\|\mathbf{U}_D\|_{L^\infty(\Omega_T)^m}, \|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} \leq CD^{-(1-\alpha)}.$$

For the case of $p < \infty$, in view of the results of Proposition 4.9, it remains to find triples (α, δ_0, r) such that

$$\delta_0 < 2(\alpha - 1) + \gamma \qquad \text{and} \qquad \delta_0 < 2(\alpha - 1) + (2 - \alpha)\frac{1}{r}$$

is satisfied where $\gamma = 2\delta_0 + (\alpha - 1)$. For existence of such triples, we define the following two restrictive curves (one depends on the parameter r as well)

$$\alpha > \ell(\delta_0) := 1 - \frac{1}{3}\delta_0 \qquad \text{and} \qquad \alpha > \ell_r(\delta_0) := \frac{r}{2r-1}\delta_0 + 1 - \frac{1}{2r-1}.$$

While ℓ is strictly monotone decreasing with $\ell(1/2) = 5/6 < 1$, the function ℓ_r is strictly increasing with $\ell_r(0) \in (0, 1)$ since $1 < r < \infty$. Thus, we always find such triples for small enough $\delta_0 > 0$.
More precisely, the intersection of ℓ_r and ℓ is given by one point for $\overline{\delta}_0 = \frac{3}{5r-1}$. We have at least $1 > \alpha > 5/6$ for both functions and we have a triangular restrictive area for $r \leq 7/5$ given by $1 > \alpha > \ell(\delta_0)$. For $7/5 < r < 2$ there is a quadrilateral area where both lines restrict the possible values of $\alpha < 1$. A triangular area induced by $1 > \alpha > \ell_r(\delta_0)$ restricts for $r \geq 2$.
Since these conditions are quite opaque, our goal is to further simplify inequalities (4.43)–(4.44) under consideration of the particular case $\delta_0 \leq 1/(2r)$. In this case we have $\delta_0 < \overline{\delta}_0$ and the only restriction is given by $\ell(\delta_0) < \alpha < 1$. Notice that $\alpha > \ell(\delta_0)$ is equivalent to $\delta_0 > 3(1 - \alpha)$ and the assertion follows from estimates in Proposition 4.9. $\square$

Theorem 4.10 is an extension of [75, Theorem 3] to intermediate time ranges that scale with the diffusion parameter D. A natural extension of Assumption L1p leading to global-in-time estimates is deferred to Chapter 5. It is clear from Corollary

3.4 that there are similar estimates for the temporal and spatial derivatives using above estimates resp. the weak formulation for $\mathbf{V}_D$, hence we omit details.

The polynomial growth in time in estimates of Theorem 4.10 is due to the fact that the evolution systems $\mathcal{U}$ and $\mathcal{W}$ are uniformly bounded. As already mentioned in the linear case following on the definition of condition L0, similar estimates can be derived if the evolution systems are only uniformly bounded by some polynomial, compare [21, Definition 1.15], [31, Definition 2.7]. The proof of Theorem 4.10 can be adapted if there are constants $C > 0, d \geq 0$ independent of time such that for $s, t \in \mathbb{R}_{\geq 0}, s \leq t$ there holds

$$\|\mathbf{W}(t,s)\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \leq C\left(1 + (t-s)^d\right)\|\xi^0\|_{L^p(\Omega)^m \times \mathbb{R}^k} \qquad \forall\, \xi^0 \in L^p(\Omega)^m \times \mathbb{R}^k,$$

and similarly some $C > 0, \tilde{d} \geq 0$

$$\|\mathbf{U}(t,s)\xi^0\|_{L^\infty(\Omega)^{\tilde{m}}} \leq C\left(1 + (t-s)^{\tilde{d}}\right)\|\xi^0\|_{L^\infty(\Omega)^{\tilde{m}}} \qquad \forall\, \xi^0 \in L^\infty(\Omega)^{\tilde{m}}$$

if necessary. The statement of Proposition 4.8 remains the same apart from a different restriction $T \leq D^{(1-\alpha)/(d+1)}$. Proposition 4.9 holds with

$$\begin{aligned}
\|\alpha_D\|_{L^\infty(\Omega_T)^m} &\leq CD^{(1-\alpha)\left[\frac{1}{d+1}+\frac{1}{\tilde{d}+1}\right]}\left(D^{-\gamma} + D^{-\frac{1}{r}\left[1+\frac{1-\alpha}{d+1}\right]}\right), \\
\|\beta_D\|_{L^\infty(\Omega_T)^k} &\leq CD^{(1-\alpha)/(d+1)}\left(D^{-\gamma} + D^{-\frac{1}{r}\left[1+\frac{1-\alpha}{d+1}\right]}\right)
\end{aligned}$$

using both restrictions $T \leq D^{(1-\alpha)/(d+1)}$ and $T \leq D^{(1-\alpha)/(\tilde{d}+1)}$. Theorem 4.10 is a direct consequence with modified estimates

$$\begin{aligned}
\|\mathbf{U}_D\|_{L^\infty(\Omega_T)^m} &\leq CD^{-(1-\alpha)\left[1+\frac{1}{d+1}+\frac{1}{\tilde{d}+1}\right]}, \\
\|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} &\leq CD^{-(1-\alpha)\left[1+\frac{1}{d+1}+\frac{2}{\tilde{d}+1}\right]}.
\end{aligned}$$

The above result of Theorem 4.10 enables us to check various model solutions for uniform convergence on long-time scales, see for instance the model examples and references in Chapter 6. Let us consider again [53] as a particular application, especially [53, Theorem 4]. Assumption L1p can be checked using numerical simulations and Thereom 4.10 accordingly applies on long-time scales, e.g., around stable stationary solutions of the shadow system.

Furthermore, we can improve the convergence rate in [53, Theorem 3] to be of order D^{-1} by the following basic arguments.

Corollary 4.11. *Let* $(\mathbf{u}_D, \mathbf{v_D})$ *and* $(\mathbf{u}, \mathbf{v})$ *be a uniformly bounded, globally defined diffusive solution of system* (1.1)–(1.3) *and shadow solution of system* (1.4)–(1.6), *respectively. If* $\mathbf{g} - \langle \mathbf{g} \rangle_\Omega$ *is uniformly bounded in* $\overline{\Omega} \times \mathbb{R}_{\geq 0}$*, the error* $\mathbf{V}_D = \mathbf{v}_D - \mathbf{v} - \psi_D$ *satisfies the uniform estimate*

$$\|\mathbf{V}_D - \langle \mathbf{V}_D \rangle_\Omega \|_{L^\infty(\Omega \times \mathbb{R}_{\geq 0})^k} \leq C D^{-1} \tag{4.49}$$

for some $C > 0$ *independent of diffusion* D.

Proof. Recall that $\mathbf{W}_D := \mathbf{V}_D - \langle \mathbf{V}_D \rangle_\Omega$ satisfies system (3.3)–(3.4), i.e.,

$$\begin{aligned} \frac{\partial \mathbf{W}_D}{\partial t} - \mathbf{D}^v \Delta \mathbf{W}_D = \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) - \mathbf{g}(\mathbf{u}, \mathbf{v}, x, t) \\ - \langle \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, \cdot, t) - \mathbf{g}(\mathbf{u}, \mathbf{v}, \cdot, t) \rangle_\Omega \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0} \end{aligned}$$

endowed with homogeneous zero flux boundary and zero initial conditions. Uniform boundedness of the right-hand side in the latter equation yields the uniform estimate

$$\|\mathbf{V}_D - \langle \mathbf{V}_D \rangle_\Omega \|_{L^\infty(\Omega \times \mathbb{R}_{\geq 0})^k} \leq C D^{-1}$$

for some $C > 0$ independent of diffusion by Proposition B.3 and Lemma 2.1. $\square$

In the case of autonomous problems, where $\mathbf{f}, \mathbf{g}$ do not depend explicitly on time, uniform boundedness of the diffusive solution $(\mathbf{u}_D, \mathbf{v}_D)$ and the shadow solution $(\mathbf{u}, \mathbf{v})$ as stated in [53, Theorems 1, 2] implies uniform boundedness of the function $\mathbf{g} - \langle \mathbf{g} \rangle_\Omega$. Hence, Corollary 4.11 applies to [53] with $\mathbf{V}_D - \langle \mathbf{V}_D \rangle_\Omega = \mathbf{v}_D - \langle \mathbf{v}_D \rangle_\Omega - \psi_D$. Recall that estimate (3.16) holds for ψ_D which implies that the component $\mathbf{v}_D$ becomes almost spatially homogeneous as time grows;

$$\|(\mathbf{v}_D - \langle \mathbf{v}_D \rangle_\Omega)(\cdot, t)\|_{L^\infty(\Omega)^k} \leq C D^{-1} \qquad \forall\, t \in [T(D), \infty)$$

where $T(D)$ is defined consequent on estimate (3.16).

Remark that Proposition 4.8 and Theorem 4.10 can be proven in a similar way for the Hilbertian case $p = 2$ using energy estimates. Such energy estimates are also employed in [53, 75]. The method which can be used to prove the convergence result in Theorem 4.10 uses error estimates for L^2 dissipative shadow systems – a smaller class of systems for which uniform boundedness of the corresponding evolution system is often easier to verify. General dissipativity conditions imposed on the linearized shadow operator in L^p, which imply Assumptions L0 and L1p, are discussed in the subsequent section.

4.3 Dissipative systems

We already established in Example 3.6 that introducing diffusion in a system of ordinary differential equations or in a reaction-diffusion-type system may cause instability of the corresponding stationary solution. As a natural consequence, the same holds concerning (uniform) boundedness of the evolution system $\mathcal{U}$ and $\mathcal{W}$ defined by equation (4.6) and (4.39), respectively. Although assumptions L0 and L1p are not necessary for long-time convergence results, the following example shows again that violating the assumptions might imply no reasonable long-time error estimates.

Example 4.12. Take an eigenfunction w_j of $-\Delta$ for some $j \in \mathbb{N}$ from Proposition A.1 and consider initial values $\mathbf{u}^0 = \mathbf{0} \in \mathbb{R}^2, v^0 = w_j$ for the linear problem

$$\begin{aligned} \frac{\partial \mathbf{u}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{u}_D &= \mathbf{A}_* \mathbf{u}_D + \mathbf{B}_* v_D \quad &&\text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad &&\mathbf{u}_D(\cdot, 0) = \mathbf{u}^0 \ \text{ in } \ \Omega, \\ \frac{\partial v_D}{\partial t} - D \Delta v_D &= 0 &&\text{in} \quad \Omega \times \mathbb{R}_{>0}, &&v_D(\cdot, 0) = v^0 \ \text{ in } \ \Omega, \\ \frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} = \mathbf{0}, \quad \frac{\partial v_D}{\partial \mathbf{n}} &= 0 &&\text{on} \quad \partial\Omega \times \mathbb{R}_{>0} \end{aligned}$$

for constant coefficients $\mathbf{A}_* \in \mathbb{R}^{2\times 2}, \mathbf{B}_* \in \mathbb{R}^{2\times 1}$. The corresponding shadow limit is given by $(\mathbf{u}, v) = (\mathbf{0}, 0)$ for each diagonal $\mathbf{D}^u \in \mathbb{R}^{2\times 2}_{\geq 0}$. Since the mean value correction ψ_D reduces to

$$\psi_D(\cdot, t) = S_\Delta(Dt) v^0 = \mathrm{e}^{-D\lambda_j t} w_j$$

and $V_D \equiv 0$, we focus on the error system

$$\begin{aligned} \frac{\partial \mathbf{U}_D}{\partial t} - \mathbf{D}^u \Delta \mathbf{U}_D &= \mathbf{A}_* \mathbf{U}_D + \mathbf{B}_* \psi_D \quad &&\text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad &&\mathbf{U}_D(\cdot, 0) = \mathbf{0} \ \text{ in } \ \Omega, \\ \frac{\partial \mathbf{U}_D}{\partial \mathbf{n}} &= \mathbf{0} &&\text{on} \quad \partial\Omega \times \mathbb{R}_{>0}. \end{aligned}$$

Projecting the solution on the eigenspace spanned by w_j, we infer

$$\mathbf{U}_D(\cdot,t) = \int_0^t \mathrm{e}^{\mathbf{A}_{D,j}(t-\tau)}\mathbf{B}_*\psi_D(\cdot,\tau)\,\mathrm{d}\tau$$

where we used the shifted matrix

$$\mathbf{A}_{D,j} = \mathbf{A}_* - \lambda_j\mathbf{D}^u = \begin{pmatrix} a-\lambda_j D_1^u & b \\ c & d-\lambda_j D_2^u \end{pmatrix}.$$

In order to get a bounded semigroup for $\mathbf{D}^u = \mathbf{0}$, we choose $a>0$, $\mathrm{tr}(\mathbf{A}_*)<0$ and $\det(\mathbf{A}_*)>0$, i.e., $\mathbf{A}_*$ is a stable matrix in the ODE case. To find unstable modes in the case of $\mathbf{D}^u \neq \mathbf{0}$, we consider the eigenvalues $\mu_\pm$ of $\mathbf{A}_{D,j}$ which are real for $\delta_D = \det(\mathbf{A}_{D,j})<0$ since $\tau_D = \mathrm{tr}(\mathbf{A}_{D,j})<0$. Clearly, $\mu_- \leq \mathrm{tr}(A) < 0$ but $\mu_+>0$ causes instability as the following shows. For fixed $\lambda_j>0$ choose $\delta_D<0$, which is possible for $D_1^u \geq 0$ small and D_2^u large since

$$\delta_D = \lambda_j^2 D_1^u D_2^u - \lambda_j(aD_2^u - |d|D_1^u) + \det(\mathbf{A}_*).$$

Fixing $\lambda_j, \mathbf{D}^u$ as above, we find a real diagonal matrix and a corresponding real invertible matrix $\mathbf{S}$ such that

$$\mathbf{A}_{D,j} = \mathbf{S}\begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix}\mathbf{S}^{-1}.$$

The matrix exponential thus is given by

$$\mathrm{e}^{\mathbf{A}_{D,j}t}\mathbf{B}_* = \mathbf{S}\begin{pmatrix} \mathrm{e}^{\mu_+ t} & 0 \\ 0 & \mathrm{e}^{\mu_- t} \end{pmatrix}\mathbf{S}^{-1}\mathbf{B}_*$$

and choosing the eigenvector $\mathbf{B}_* = \mathbf{S}\begin{pmatrix}1 & 0\end{pmatrix}^T$ yields exponential growth of the error

$$\mathbf{U}_D(\cdot,t) = \int_0^t \mathrm{e}^{\mathbf{A}_{D,j}(t-\tau)}\mathbf{B}_*\psi_D(\cdot,\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{\mu_+(t-\tau)-D\lambda_j\tau}\mathbf{B}_* w_j\,\mathrm{d}\tau.$$

In a similar way, we conclude that the evolution system $\mathcal{U}$ which is simply induced by a semigroup $(\mathbf{U}(t))_{t\in\mathbb{R}_{\geq 0}}$ cannot be bounded since for $\mathbf{u}^0 = \mathbf{B}_* w_j$ with $\mathbf{B}_*$ defined above we find

$$\mathbf{U}(t)\mathbf{u}^0 = \mathrm{e}^{\mu_+ t}\mathbf{B}_* w_j = \mathrm{e}^{\mu_+ t}\mathbf{u}^0.$$

The preceding example shows that boundedness of the evolution system $\mathcal{U}_0$ induced by the matrix multiplication $\mathbf{A}_*$ if $\mathbf{D}^u = \mathbf{0}$ does not carry over to the case of $\mathbf{D}^u \neq \mathbf{0}$, though the one dimensional case might be misleading. The aim of the present section is to discuss several notions of dissipativity which imply Assumption L resp. L0 and L1p. A further, much simpler characterization of dissipativity which makes use of a quadratic form is given in the case of the subsystem considered in condition L resp. L0.

Let us first consider the evolutionary subsystem $\mathcal{U}$ from Assumption L to describe the principle of dissipativity. Notice that Assumption L0 does not involve any diffusion but the same method equally applies to diffusion matrices $\mathbf{D}^u$ with non-negative entries. The corresponding evolution operators of $\mathcal{U}$ are induced by the solution ξ of

$$\frac{\partial \xi}{\partial t} - \mathbf{D}^u \Delta \xi = \mathbf{A}_*(\cdot, t)\xi \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad \xi(\cdot, 0) \in L^p(\Omega)^m, \tag{4.50}$$

compare to equation (4.6) and Assumption L. In terms of evolution systems resp. semigroups, the above problem is a problem of perturbation theory in which context we disturb the Laplacian (which generates a contraction semigroup) by a bounded evolution operator $\mathbf{A}_*(\cdot, t)$ (which generates a strongly continuous evolution system). Considering the case of constant matrices $\mathbf{A}_*$ as in Example 4.12, [106] provides several conditions to prove stability of solutions to the partly diffusive system (4.50) and thus uniform boundedness of the corresponding evolution operators.
In the more general case, where $\mathbf{A}_*$ is still time-independent but depends on the space variable x, boundedness (even contractivity) of the perturbed semigroup induced by the sum $\mathbf{A}_* + \mathbf{D}^u\Delta$ may be shown if $\mathbf{A}_*$ fulfills a dissipativity condition. Concerning contractivity, [23, Chapter III, Theorem 2.7] applies the condition

$$\|\mathbf{y}\|_{L^p(\Omega)^m} \leq \|(I - \lambda \mathbf{A}_*)\mathbf{y}\|_{L^p(\Omega)^m} \qquad \forall\, \lambda \in \mathbb{R}_{>0}, \mathbf{y} \in \mathbf{L}^p(\Omega)^m. \tag{4.51}$$

It should be mentioned that some authors assume accretivity or monotonicity of $-\mathbf{A}_*$ instead. Nevertheless, these conditions imply the same estimates, see for instance [8, Theorems 7.4, 7.8] and references therein.
The time-dependent case can be treated in the same way, compare [49, Theorem 1] for contractive evolution systems. Since we only need boundedness of the perturbed evolutionary system in Assumption L, we follow the more general approach used in [66]. Therefore, consider the bounded multiplication operators $\mathbf{A}_*(\cdot, t)$, of either the linear case or the linearized nonlinear case in the setting (4.33), and let us assume

Dp Let $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous function with $\kappa \in L^1(\mathbb{R}_{\geq 0})$ such that

$$(1 - \lambda\kappa(t))\|\mathbf{y}\|_{L^p(\Omega)^m} \leq \|(I - \lambda\mathbf{A}_*(\cdot, t))\mathbf{y}\|_{L^p(\Omega)^m} \tag{4.52}$$

is satisfied for all $\mathbf{y} \in L^p(\Omega)^m$, $\lambda \in \mathbb{R}_{>0}$, and $t \in \mathbb{R}_{\geq 0}$.

Note that $\kappa \equiv 0$ corresponds to [49, Theorem 1] already mentioned above. There are several equivalent formulations of dissipativity condition (4.52), see [90, Remark 1.2] or [23, Chapter II, Proposition 3.23] and references therein. Especially, condition (4.52) is equivalent to dissipativity of $\mathbf{A}_*(\cdot, t) - \kappa(t)I$ on $L^p(\Omega)^m$ in the sense of inequality (4.51) for each time $t \in \mathbb{R}_{\geq 0}$. This can be seen using a characterization from [90, Remark 1.2] via the duality map J on $L^p(\Omega)^m$, which we will describe after having a look at the implication of condition (4.52). Actually, dissipativity of $\mathbf{A}_*$ in the sense of inequality (4.52) yields uniform boundedness of the corresponding evolution system $\mathcal{U}$ for each non-negative, diagonal diffusion matrix $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$.

Proposition 4.13. *Let $\mathbf{A}_* : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times m}$ be a measurable, locally bounded matrix-valued function satisfying Assumption Dp for some $1 \leq p \leq \infty$. Then the corresponding evolution system $\mathcal{U}$ induced by $\mathbf{D}^u\Delta + \mathbf{A}_*(\cdot, t)$ is uniformly bounded on $L^p(\Omega)^m$ for the same exponent p.*

Proof. The same reasoning as in the proof of Proposition 2.3 applies and the mild solution ξ of problem (4.50), which defines the evolution operators $\mathbf{U}(t, s)$ of $\mathcal{U}$, is given by

$$\xi(\cdot, t) = \mathbf{U}(t, s)\xi(\cdot, s) = \mathbf{S}^u(t - s)\xi(\cdot, s) + \int_s^t \mathbf{S}^u(t - \tau)\mathbf{A}_*(\cdot, \tau)\xi(\cdot, \tau)\, \mathrm{d}\tau. \tag{4.53}$$

We will follow the ideas of [90, Remark 2.2] to obtain an estimate for the non-smooth integral solution ξ. Since the integral is an absolutely converging Bochner integral in $L^\infty(\Omega)^m$, the same holds in $L^q(\Omega)^m$ for each $1 \leq q < \infty$. Hence, for each $t > h > 0$

$$\begin{aligned}\xi(\cdot, t) &= \mathbf{S}^u(h)\xi(\cdot, t - h) + \int_{t-h}^t \mathbf{S}^u(t - \tau)\mathbf{A}_*(\cdot, \tau)\xi(\cdot, \tau)\, \mathrm{d}\tau \\ &= \mathbf{S}^u(h)\xi(\cdot, t - h) + h\mathbf{A}_*(\cdot, t)\xi(\cdot, t) + h\mathbf{r}(\cdot, t; h)\end{aligned}$$

where $\|\mathbf{r}(\cdot, t; h)\|_{L^p(\Omega)^m} \to 0$ holds by Lebesgue's differentiation theorem for the remainder $\mathbf{r}$ as $h \to 0$ [3, Proposition 1.2.2]. Using dissipativity condition (4.52) with $\lambda = h > 0$ yields

$$(1 - h\kappa(t))\|\xi(\cdot, t)\|_{L^p(\Omega)^m} \leq \|\mathbf{S}^u(h)\xi(\cdot, t - h) + h\mathbf{r}(\cdot, t; h)\|_{L^p(\Omega)^m}$$

$$\leq \|\xi(\cdot, t-h)\|_{L^p(\Omega)^m} + h\|\mathbf{r}(\cdot, t; h)\|_{L^p(\Omega)^m}$$

since $\mathbf{D}^u\Delta$ generates a contraction semigroup $(\mathbf{S}^u(t))_{t\in\mathbb{R}_{\geq 0}}$ on $L^q(\Omega)^m$ for each exponent $1 \leq q \leq \infty$ by Lemma 2.1. Rewriting above inequality and letting $h \to 0$, we gain an estimate for the upper left Dini derivative of the norm:

$$D^- \|\xi(\cdot, t)\|_{L^p(\Omega)^m} \leq \kappa(t)\|\xi(\cdot, t)\|_{L^p(\Omega)^m} \qquad \forall\, t > 0$$

In view of Proposition B.2, $\xi \in C(\mathbb{R}_{>0}; L^q(\Omega)^m)$ for each $1 \leq q \leq \infty$, and moreover $\xi \in C(\mathbb{R}_{\geq 0}; L^q(\Omega)^m)$ for $q < \infty$. A well-known result of monotonicity from [95, Appendix I, Theorem 2.1] implies

$$\|\xi(\cdot, t)\|_{L^p(\Omega)^m} \leq \exp\left(\int_s^t \kappa(\tau)\, \mathrm{d}\tau\right) \|\xi(\cdot, s)\|_{L^p(\Omega)^m} \qquad \forall\, s, t \in \mathbb{R}_{>0}, s \leq t.$$

Uniform boundedness of the evolution system $\mathcal{U}$ is a consequence of $\kappa \in L^1(\mathbb{R}_{\geq 0})$. For $p < \infty$, continuity of the solution yields estimates up to $s = 0$. For $p = \infty$, the latter estimate holds for all $s, t \in \mathbb{R}_{>0}, s \leq t$. Estimating formula (4.53) for $s = 0$ with the help of Gronwall's inequality, we obtain a similar estimate which relates time $t \geq 0$ to time $s = 0$. The obtained constant might differ from $\exp(\|\kappa\|_{L^1(\mathbb{R}_{\geq 0})})$ but we choose their maximum in the case $p = \infty$. Hence, we established uniform boundedness of the evolution system $\mathcal{U}$. □

The result of Proposition 4.13 states that dissipativity of the operators $\mathbf{A}_*$ implies uniform boundedness of the corresponding evolution system induced by $\mathbf{D}^u\Delta + \mathbf{A}_*$. However, the converse is not true in general, compare Example 3.6 and 5.3 below.
In practice, dissipativity condition (4.52) is not easy to be verified but as mentioned above, by [90, Remark 1.2], (4.52) is equivalent to dissipativity of $\mathbf{A}_*(\cdot, t) - \kappa(t)I$. The latter means that for all $\lambda \in \mathbb{R}_{>0}$, $\mathbf{y} \in L^p(\Omega)^m$, and $t \in \mathbb{R}_{\geq 0}$ there holds

$$\|\mathbf{y}\|_{L^p(\Omega)^m} \leq \|(I - \lambda(\mathbf{A}_*(\cdot, t) - \kappa(t)I))\mathbf{y}\|_{L^p(\Omega)^m}. \tag{4.54}$$

This equivalence can be verified via the duality map J, see [23, Chapter II, Proposition 3.23] or [91, Chapter I, Proposition 2.1]. The characterization (4.54) is useful in particular if the duality set is just a singleton:
For $1 < p < \infty$, inequality (4.52) resp. (4.54) is equivalent to

$$\int_\Omega (\mathbf{y}^*)^T \left(\mathbf{A}_*(\cdot, t) - \kappa(t)I\right) \mathbf{y}\, \mathrm{d}x \leq 0 \qquad \forall\, \mathbf{y} \in L^p(\Omega)^m, t \in \mathbb{R}_{\geq 0}. \tag{4.55}$$

Indeed, by [23, Chapter II, Example 3.26 (ii)], the duality map satisfies $J(\mathbf{y}) = \{\mathbf{y}^*\}$ for $\mathbf{y}^* \in L^q(\Omega)^m$ where $\mathbf{y}^* = \mathbf{0}$ for $\mathbf{y} = \mathbf{0}$ and

$$\mathbf{y}^* = \frac{\mathbf{y}|\mathbf{y}|^{p-2}}{\|\mathbf{y}\|_{L^p(\Omega)^m}^{p-2}} \qquad \text{for} \quad \mathbf{y} \neq \mathbf{0}.$$

Remember that the dual pairing satisfies $\langle \mathbf{y}^*, \mathbf{y} \rangle = \|\mathbf{y}^*\|_{L^q(\Omega)^m}^2 = \|\mathbf{y}\|_{L^p(\Omega)^m}^2$ with conjugate exponent $q = (p-1)/p$, and $L^p(\Omega)^m$ is again uniformly convex choosing the (squared) vector norm $|\mathbf{y}(x)|^2 = \sum_{i=1}^m |y_i(x)|^2$ [13, Theorem 1].

It turns out that there is even a simpler criterion for dissipativity of multiplication operators on $L^p(\Omega)^m$ which is independent of the exponent p. More precisely, above inequalities (4.52)–(4.55) can be checked via pointwise estimates of the corresponding quadratic form

$$q(x,t) : \mathbb{R}^m \to \mathbb{R}, \qquad \overline{\mathbf{y}} \mapsto \overline{\mathbf{y}}^T(\mathbf{A}_*(x,t) - \kappa(t)I)\overline{\mathbf{y}}. \tag{4.56}$$

Such a condition is already used in the time-independent case [88, Propositions 6, 7]. Moreover, pointwise estimates of the latter quadratic form are a well-known technique in the context of classical solutions to preserve contractivity of the corresponding evolution system [56, Theorem 2.3].

Lemma 4.14. *Let κ be given by Assumption Dp for some measurable, bounded function $\mathbf{A}_* : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times m}$ and q be defined as above in* (4.56)*. Then, dissipativity condition* (4.52) *on $L^p(\Omega)^m$ for some $1 \leq p \leq \infty$ is equivalent to $q(x,t) \leq 0$ on $\mathbb{R}^m$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_{\geq 0}$. Moreover, inequality* (4.52) *holds for all $1 \leq p \leq \infty$ if and only if it holds for one exponent p.*

Proof. Let $q \leq 0$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_{\geq 0}$. Since conditions (4.52)–(4.55) are all equivalent for $1 < p < \infty$, we integrate $q \leq 0$ for a symmetric choice of vectors $\overline{\mathbf{y}}(x)$ instead of $\mathbf{y}^*$ and $\mathbf{y}$ and obtain inequality (4.55). For $p \in \{1, \infty\}$, we use continuity of the L^p norm with respect to p since we already established estimate (4.54) for all $1 < p < \infty$. Uniform boundedness of coefficients of $\mathbf{A}_*$ implies boundedness of $\mathbf{A}_* - \kappa I$ and thus, $I - \lambda(\mathbf{A}_*(t) - \kappa(t)I)$ is invertible for small $\lambda > 0$ due to Neumann's series. By dissipativity, compare [23, Chapter II, Proposition 3.14], the latter operator is invertible for all $\lambda > 0$ and estimate (4.54) yields

$$\|(I - \lambda(\mathbf{A}_*(\cdot,t) - \kappa(t)I))^{-1}\mathbf{y}\|_{L^p(\Omega)^m} \leq \|\mathbf{y}\|_{L^p(\Omega)^m}$$

for all $\mathbf{y} \in L^p(\Omega)^m$ and $1 < p < \infty$. The result follows by letting $p \to 1$ or $p \to \infty$ where [1, Theorem 2.14] applies since $|\Omega| < \infty$. Thus, $q \leq 0$ implies dissipativity. Now, let dissipativity inequality (4.52) be fulfilled and assume q is not non-positive, i.e., there is a set $\Omega_1 \subset \Omega$ with $|\Omega_1| > 0$ and some time point $t \geq 0$ as well as $\overline{\mathbf{y}} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that

$$q(x,t)\overline{\mathbf{y}} = \overline{\mathbf{y}}^T(\mathbf{A}_*(x,t) - \kappa(t)I)\overline{\mathbf{y}} > 0$$

holds for a.e. $x \in \Omega_1$. Using measure theory, we find uniform bounds

$$0 < q_0 \leq q(\cdot,t)\overline{\mathbf{y}} \leq q_1 < \infty$$

almost everywhere on a possibly smaller set $\Omega_2 \subset \Omega_1$ with positive measure. Let us consider $\mathbf{y}_p \in L^p(\Omega)^m$ for $p < \infty$ given by

$$\mathbf{y}_p(x) := \frac{\chi_{\Omega_2}(x)}{(q(x,t)\overline{\mathbf{y}})^{1/p}}\overline{\mathbf{y}}.$$

This bounded vector-valued function satisfies

$$\begin{aligned}\int_\Omega (\mathbf{y}_p^*)^T \left(\mathbf{A}_*(\cdot,t) - \kappa(t)I\right)\mathbf{y}_p \,\mathrm{d}x &= \int_{\Omega_2} |\overline{\mathbf{y}}|^{p-2}\|\mathbf{y}_p\|^{2-p}_{L^p(\Omega_2)^m}\,\mathrm{d}x \\ &= \|(q(\cdot,t)\overline{\mathbf{y}})^{-1}\|^{2-p}_{L^p(\Omega_2)^m} > 0\end{aligned}$$

which is a contradiction to condition (4.55), and thus to (4.52) for $1 < p < \infty$. For $p = 1, \infty$, choosing again the squared vector norm $|\mathbf{y}(x)|^2 = \sum_{i=1}^m |y_i(x)|^2$, let us consider $\mathbf{y}_2 \in L^\infty(\Omega)^m$ in preceding definition. One infers that

$$|(I - \lambda(\mathbf{A}_*(\cdot,t) - \kappa(t)I))\mathbf{y}_2|^2 = |\mathbf{y}_2|^2 + \lambda^2 \left|(\mathbf{A}_*(\cdot,t) - \kappa(t)I)\mathbf{y}_2\right|^2 - 2\lambda$$

holds on the set Ω_2. For small enough $\lambda > 0$, the right-hand side of the latter equation is smaller than $|\mathbf{y}_2|^2$ since $\mathbf{A}_*, \kappa, \mathbf{y}_2$ are bounded functions on Ω_2. This leads to a contradiction to dissipativity condition (4.54) also for the cases $p \in \{1, \infty\}$ since

$$\left\|\left(|\mathbf{y}_2|^2\right)^{1/2}\right\|_{L^p(\Omega_2)} \leq \left\|\left(|\mathbf{y}_2|^2 - \lambda\right)^{1/2}\right\|_{L^p(\Omega_2)}$$

□

Lemma 4.14 shows that dissipativity assumption Dp is independent of the exponent p and it provides a simple way to check Assumption L resp. L0 for use in Theorems

4.1, 4.5 and Theorem 4.10, respectively. Notice that condition L0 only considers a subsystem $\mathbf{A}_{11}(x,t) \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$ of $\mathbf{A}_*$ for which diffusion is not relevant and for which we can check dissipativity in $L^p(\Omega)^{\tilde{m}}$ for some (and hence all) $1 \le p \le \infty$. Since $\mathbf{A}_{11}$ is non-symmetric in general, one can verify definiteness of the corresponding quadratic form q defined in (4.56) by looking equivalently on the real eigenvalues of the symmetric part

$$\frac{1}{2}\left(\mathbf{A}_{11}(x,t) + \mathbf{A}_{11}(x,t)^T\right) - \kappa(t)I \in \mathbb{R}^{\tilde{m}\times\tilde{m}}.$$

Non-positivity of its eigenvalues $\lambda(x,t)$ pointwise for almost every $(x,t) \in \Omega \times \mathbb{R}_{\ge 0}$ implies condition L0. As a precaution, let us recall a consequence of linear algebra also demonstrated in Example 5.3 below: a non-positive quadratic form q implies $\mathrm{Re}(\lambda) \le 0$ for all eigenvalues $\lambda(x,t)$ of $\mathbf{A}_{11}(x,t)$ but in general there is no equivalence for non-symmetric matrices.

Now let us turn to the evolution system $\mathcal{W}$ induced by the linear resp. linearized shadow system written as

$$\begin{aligned}
\frac{\partial \xi_1}{\partial t} - \mathbf{D}^u \Delta \xi_1 &= \mathbf{A}_*(x,t)\xi_1 + \mathbf{B}_*(x,t)\xi_2 && \text{in} \quad \Omega \times \mathbb{R}_{>0},\\
\frac{\mathrm{d}\xi_2}{\mathrm{d}t} &= \langle \mathbf{C}_*(\cdot,t)\xi_1\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\xi_2\rangle_\Omega && \text{in} \quad \mathbb{R}_{>0},\\
\xi_1(\cdot,0) = \xi_1^0 \quad \text{in} \quad \Omega, &\qquad \xi_2(0) = \langle \xi_2^0\rangle_\Omega
\end{aligned}$$

endowed with zero Neumann boundary conditions for ξ_1 if necessary, see definition (4.12) resp. (4.39). Using the operator notation (4.11), let us rewrite the shadow problem as an ordinary differential equation in the Banach space $L^p(\Omega)^m \times \mathbb{R}^k$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = \mathbf{D}^S \Delta \xi + \mathbf{L}_0(t)\xi \qquad \text{in} \quad \mathbb{R}_{>0}, \qquad \xi(0) = \begin{pmatrix} \xi_1^0 \\ \langle \xi_2^0\rangle_\Omega \end{pmatrix},$$

where $\mathbf{L}_0(t)$ is a linear operator given by its action on $\xi = (\xi_1, \xi_2)^T$ induced by the linear right-hand side of the shadow problem. By Assumption A1 resp. A4, $(\mathbf{L}_0(t))_{t\in\mathbb{R}_{\ge 0}}$ is a family of bounded operators on $L^p(\Omega)^m \times \mathbb{R}^k$, and the full operator $\mathbf{D}^S\Delta + \mathbf{L}_0(t)$ can be seen as a perturbation of the matrix operator $\mathbf{D}^S\Delta = \mathrm{diag}(\mathbf{D}^u\Delta, \mathbf{0})$ which generates a contraction semigroup on $L^p(\Omega)^m \times \mathbb{R}^k$ for all $1 \le p \le \infty$. Let us apply the principle of dissipativity for $\mathbf{L}_0(t)$ as we already did for the evolution system $\mathcal{U}$. The associated evolution system $\mathcal{W}$ on $L^p(\Omega)^m \times \mathbb{R}^k$ is determined by the unique mild solution ξ of the shadow problem as in Proposition 2.6. More precisely,

$$\xi(\cdot,t) = \mathbf{W}(t,s)\xi(\cdot,s) = \mathbf{S}(t-s)\xi(\cdot,s) + \int_s^t \mathbf{S}(t-\tau)\mathbf{L}_0(\tau)\xi(\cdot,\tau)\,\mathrm{d}\tau$$

where the integral is an absolutely converging Bochner integral in $L^\infty(\Omega)^m \times \mathbb{R}^k$ and we set $\mathbf{D}^v = \mathbf{0}$ in definition (2.5) of $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$. Let us assume in analogy to Assumption Dp

D1p Let $\varrho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous function with $\varrho \in L^1(\mathbb{R}_{\geq 0})$ such that

$$(1-\lambda\varrho(t))\|\mathbf{y}\|_{L^p(\Omega)^m\times\mathbb{R}^k} \leq \|(I-\lambda\mathbf{L}_0(t))\mathbf{y}\|_{L^p(\Omega)^m\times\mathbb{R}^k} \tag{4.57}$$

is satisfied for all $\mathbf{y} \in L^p(\Omega)^m \times \mathbb{R}^k$, $\lambda \in \mathbb{R}_{>0}$, and $t \in \mathbb{R}_{\geq 0}$.

Recall that using the duality map one obtains an equivalent integral inequality similar to estimate (4.55) for $1 < p < \infty$. The latter has a quite convenient form for L^2 energy estimates:

$$\int_\Omega \mathbf{y}^T\left(\mathbf{L}_0(\cdot,t) - \rho(t)I\right)\mathbf{y}\,\mathrm{d}x \leq 0 \qquad \forall\, \mathbf{y} \in L^2(\Omega)^m \times \mathbb{R}^k, t \in \mathbb{R}_{\geq 0}$$

Following the proof of Proposition 4.13, we reach at

Proposition 4.15. *Let the linear operators $\mathbf{L}_0(t) : L^p(\Omega)^m \times \mathbb{R}^k \to L^p(\Omega)^m \times \mathbb{R}^k$ defined above for each $t \in \mathbb{R}_{\geq 0}$ satisfy Assumption D1p for some $1 \leq p \leq \infty$ and uniformly bounded coefficients $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*$ and $\mathbf{D}_*$. Then the corresponding shadow evolution system $\mathcal{W}$ induced by $\mathbf{D}^S\Delta+\mathbf{L}_0(t)$ is uniformly bounded on $L^p(\Omega)^m\times\mathbb{R}^k$ for the same exponent p, thus $\mathcal{W}$ satisfies Assumption L1p. Moreover, the evolutionary subsystem $\mathcal{U}$ and $\tilde{\mathcal{U}}$ satisfies Assumption L and L0, respectively.*

Proof. By the same reasoning as in the proof of Proposition 4.13, we obtain the estimate

$$D^-\|\xi(\cdot,t)\|_{L^p(\Omega)^m\times\mathbb{R}^k} \leq \varrho(t)\|\xi(\cdot,t)\|_{L^p(\Omega)^m\times\mathbb{R}^k} \qquad \forall\, t > 0$$

for the upper left Dini derivative of the norm. In view of Proposition B.2, we have $\xi \in C(\mathbb{R}_{\geq 0}; L^p(\Omega)^m \times \mathbb{R}^k)$ for each $p < \infty$ since the integrand in the above implicit integral equation of ξ is bounded locally-in-time and thus integrable. Moreover, $\xi \in C(\mathbb{R}_{>0}; L^p(\Omega)^m \times \mathbb{R}^k)$ for each $1 \leq p \leq \infty$ and the same implication as in the proof of Proposition 4.13 follows for each $1 \leq p \leq \infty$, i.e.,

$$\|\xi(\cdot,t)\|_{L^p(\Omega)^m\times\mathbb{R}^k} \leq \exp\left(\int_s^t \varrho(\tau)\,\mathrm{d}\tau\right)\|\xi(\cdot,s)\|_{L^p(\Omega)^m\times\mathbb{R}^k}.$$

Since $\varrho \in L^1(\mathbb{R}_{\geq 0})$, this yields uniform boundedness of the evolution system $\mathcal{W}$. Considering the subsystem $\mathbf{A}_*(\cdot, t)$ of the linear operator $\mathbf{L}_0(t)$, dissipativity is inherited by the full system $\mathbf{L}_0(t)$. Clearly for $1 < p < \infty$, we infer from condition (4.55) that dissipativity of $\mathbf{L}_0(t) - \varrho(t)I$ in $L^p(\Omega)^m \times \mathbb{R}^k$ implies dissipativity of the corresponding subsystem $\mathbf{A}_*(\cdot, t) - \varrho(t)I$ in $L^p(\Omega)^m$ since $\mathbf{y}^* = (\mathbf{y}_1^*, \mathbf{y}_2)^T$ for

$$\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)^T \in L^p(\Omega)^m \times \mathbb{R}^k \qquad \text{with} \qquad \|\mathbf{y}\|^2_{L^p(\Omega)^m \times \mathbb{R}^k} := \|\mathbf{y}_1\|^2_{L^p(\Omega)^m} + |\mathbf{y}_2|^2.$$

For $p \in \{1, \infty\}$, the same can be shown as in the proof of Lemma 4.14 by contradiction, assuming $q \leq 0$ does not hold almost everywhere. Due to the results of Lemma 4.14 and Proposition 4.13, Assumption D1p for some $1 \leq p \leq \infty$ implies Assumption L resp. L0 for all $1 \leq p \leq \infty$. □

Unfortunately, the author is not aware of a simple characterization of dissipativity of the shadow operators $\mathbf{L}_0(t)$, similar to the quadratic form in Lemma 4.14. Nevertheless, I want to mention a simple consequence in the Hilbertian case $p = 2$. Dissipativity of the linearized part

$$\mathbf{J}(x,t) = \begin{pmatrix} \mathbf{A}_*(x,t) & \mathbf{B}_*(x,t) \\ \mathbf{C}_*(x,t) & \mathbf{D}_*(x,t) \end{pmatrix}$$

in the sense of condition (4.52) in $L^2(\Omega)^{m+k}$ or the equivalent formulation due to Lemma 4.14, i.e.,

$$\overline{\mathbf{y}}^T(\mathbf{J}(x,t) - \kappa(t)I)\overline{\mathbf{y}} \leq 0 \qquad \forall\, \overline{y} \in \mathbb{R}^{m+k}, \text{a.e. } (x,t) \in \Omega \times \mathbb{R}_{\geq 0},$$

implies dissipativity of $\mathbf{L}_0(t) - \kappa(t)I$ for the shadow system in $L^2(\Omega)^m \times \mathbb{R}^k$. However, the converse does not hold true. Choosing for instance an eigenfunction w_j of $-\Delta$ from Proposition A.1 for some $j \in \mathbb{N}$, it can be seen by means of Lemma 4.14 that

$$\mathbf{J}(\cdot,t) = \begin{pmatrix} 0 & 0 \\ 0 & w_j \end{pmatrix} \qquad \text{with} \qquad \mathbf{L}_0(t) = \mathbf{0}.$$

Nevertheless, using the well-known interpolation theorem of Riesz-Thorin [8, Theorem 4.32], we want to record the following helpful lemma concerning dissipativity of the linear shadow operator

$$\mathbf{L}_0(t) : L^p(\Omega)^m \times \mathbb{R}^k \to L^p(\Omega)^m \times \mathbb{R}^k,$$

$$\mathbf{L}_0(t)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}(x) = \begin{pmatrix}\mathbf{A}_*(x,t)\xi_1(x) + \mathbf{B}_*(x,t)\xi_2\\ \langle \mathbf{C}_*(\cdot,t)\xi_1\rangle_\Omega + \langle \mathbf{D}_*(\cdot,t)\xi_2\rangle_\Omega\end{pmatrix}.$$

which is bounded for each $1 \le p \le \infty$, for instance, by Assumption A4.

Lemma 4.16. *Let the bounded linear operator $\mathbf{L}_0(t)$ defined above satisfy Assumption D1p$_i$ for some $1 \le p_1, p_2 \le \infty$ with $p_1 < p_2$ and uniformly bounded coefficients $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*$ and $\mathbf{D}_*$. Then $\mathbf{L}_0(t)$ satisfies the dissipativity condition D1p for all $p_1 \le p \le p_2$ with the same function ϱ. Moreover, if $\mathbf{L}_0(t)$ satisfies the dissipativity condition D1p for all $1 \le p_1 < p < p_2 \le \infty$ with the same ϱ, then D1p is satisfied for all $p_1 \le p \le p_2$ with the same function ϱ.*

Proof. Uniform boundedness of coefficients of the Jacobian $\mathbf{J}$ implies boundedness of $\mathbf{L}_0 - \rho I$ and thus, $I - \lambda(\mathbf{L}_0(t) - \rho(t)I)$ is invertible for small $\lambda > 0$ due to Neumann's series. By dissipativity, compare [23, Chapter II, Proposition 3.14], the latter operator is invertible for all $\lambda > 0$ and we obtain the estimate (4.54) for p_i:

$$\|(I - \lambda(\mathbf{L}_0(t) - \rho(t)I))^{-1}\mathbf{y}\|_{L^{p_i}(\Omega)^m\times\mathbb{R}^k} \le \|\mathbf{y}\|_{L^{p_i}(\Omega)^m\times\mathbb{R}^k}, \qquad i = 1,2$$

An application of the Theorem of Riesz-Thorin implies condition D1p for all values p with $p_1 \le p \le p_2$.

If Assumption D1p is satisfied on an open interval $p_1 < p < p_2$, then continuity of the norm $\|\cdot\|_{L^p(\Omega)^m}$ implies for $p \to p_1$ resp. $p \to p_2$ that

$$\|(I - \lambda(\mathbf{L}_0(t) - \rho(t)I))^{-1}\mathbf{y}\|_{L^{p_i}(\Omega)^m\times\mathbb{R}^k} \le \|\mathbf{y}\|_{L^{p_i}(\Omega)^m\times\mathbb{R}^k}$$

holds for all $\mathbf{y} \in L^{p_i}(\Omega)^m \times \mathbb{R}^k$ and $i = 1,2$. Note that this is also valid for $p_2 = \infty$ since $|\Omega| < \infty$ [1, Theorem 2.14]. $\square$

It is clear from Proposition 4.13 and Proposition 4.15 that, once shown dissipativity condition D1p for the full shadow system, Theorem 4.10 can be applied for a sufficiently big exponent p.

5 Asymptotic behavior

Solutions to both problems, the diffusive problem (1.1)–(1.3) and its shadow problem (1.4)–(1.6), may exhibit various limiting behavior as time tends to infinity. Instabilities such as exponential growth in time were already illustrated in Example 3.6. Similar to ordinary differential equations, even blow-up phenomena occur, e.g., in Example 2.4 and in the discussion of integro-driven blow-up following on Proposition 2.6, respectively. Concerning global existence, we infer from Theorem 3.3 that a global shadow solution of system (1.4)–(1.6) implies that diffusive solutions of system (1.1)–(1.3) exist almost globally – in the sense that the solution exists on every large finite time interval at least for all diffusivities D which are large enough (see Example 3.5). This chapter is devoted to the question whether the shadow solution is a reasonable approximation of the diffusive solution concerning asymptotics.

To prove convergence results, we already applied a stabilizing effect to ensure that diffusive solutions of system (1.1)–(1.3) stay nearby the shadow limit. Such a result is achieved via linearization around the shadow solution in Theorem 4.10. It is well known that exponential stabilization of the linearized system often leads to global estimates for the corresponding nonlinear system. The next section shows that this concept is in fact expedient to obtain a valuable shadow approximation on the global time scale. Uniform global error estimates are derived in Section 5.1.

Moreover, in searching for a way to check stability assumptions L0 and L1p, the spectrum of a linear shadow operator is characterized in Section 5.2. This also allows to verify stability or instability results for stationary solutions of the (nonlinear) shadow system, in the reaction-diffusion-ODE case of shadow limits [69] and the classical shadow limit case [80], respectively. As a byproduct, linearization around steady states of the partly diffusive system (1.1)–(1.3) yields a similar spectral decomposition. Using this characterization, destabilizing effects already established in Examples 3.6 and 4.12 arising from an unstable subsystem become reasonable. The spectral analysis is completed with several nonlinear examples from [69, 71] to which the obtained results can be applied in order to study stability of stationary solutions.

5.1 Convergence results

This section is again based on the truncation method used in the foregoing chapters to prove convergence results. The reader is requested to consult Section 4.2 for further details. A natural assumption for global estimates is some exponential stability of the corresponding shadow evolution system induced by its linearization. We already used this concept in the linear case for Assumption L1p, compare to the global estimates (4.10) and (4.23).

Consider the evolution system $\mathcal{W}$ defined by equation (4.12) resp. (4.39) which is generated on $L^p(\Omega)^m \times \mathbb{R}^k$ by the linearization around the shadow solution. Let the evolution system $\mathcal{W}$ be uniformly exponentially stable. This means that there exists a constant $C > 0$ and an exponent $\sigma > 0$, both independent of time, such that for all $s, t \in \mathbb{R}_{\geq 0}$, $s \leq t$ there holds

$$\|\mathbf{W}(t,s)\xi^0\|_{L^p(\Omega)^m\times\mathbb{R}^k} \leq C\mathrm{e}^{-\sigma(t-s)}\|\xi^0\|_{L^p(\Omega)^m\times\mathbb{R}^k} \qquad \forall\, \xi^0 \in L^p(\Omega)^m \times \mathbb{R}^k.$$

For the case of $p = \infty$ this assumption leads to global estimates as the following analog of Proposition 4.8 already indicates.

Corollary 5.1. *Let assumptions A1–A4, B, and L1p hold with some $1 \leq p \leq \infty$ and uniform exponential stability exponent $\sigma > 0$. Then, for any $\delta_0 \in (0, 1/2]$, there exist constants $C, D_0 > 0$ independent of diffusion D such that for all $D \geq D_0$ the solution (α_D, β_D) of the truncated problem (4.30)–(4.32) satisfies*

$$\|\alpha_D\|_{L^p(\Omega\times\mathbb{R}_{\geq 0})^m} + \|\beta_D\|_{L^p(\Omega\times\mathbb{R}_{\geq 0})^k} \leq CD^{-2\delta_0}. \tag{5.1}$$

Proof. We already estimated the function $\mathbf{W}_D = \mathbf{V}_D - \langle \mathbf{V}_D\rangle_\Omega$ in inequality (4.41), where we reached at

$$\begin{aligned}\|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \leq C\int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)}\left(\|\alpha_D(\cdot,\tau)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(\tau)|\right)\,\mathrm{d}\tau \\ + C\int_0^t \mathrm{e}^{-\lambda_1 D(t-\tau)}\|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau + C|\Omega|^{1/p}D^{-1},\end{aligned}$$

with a similar estimate for $p = \infty$. To obtain a corresponding inequality for $\alpha_D, \mathbf{b}_D$, we consider the representation

$$\begin{pmatrix}\alpha_D(\cdot,t)\\ \mathbf{b}_D(t)\end{pmatrix} = \int_0^t \mathbf{W}(t,\tau)\begin{pmatrix}\mathbf{B}_*(\cdot,\tau)(\mathbf{W}_D+\psi_D)(\cdot,\tau) + \mathbf{F}_D(\cdot,\tau)\\ \langle\mathbf{D}_*(\cdot,\tau)(\mathbf{W}_D+\psi_D)(\cdot,\tau) + \mathbf{G}_D(\cdot,\tau)\rangle_\Omega\end{pmatrix}\,\mathrm{d}\tau.$$

Applying the uniform exponential stability condition of Assumption L1p and estimate (4.29) for the truncated nonlinearities $\mathbf{F}_D, \mathbf{G}_D$ yield

$$\begin{aligned}\|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)| \le C\int_0^t \mathrm{e}^{-\sigma(t-\tau)}\Big(&\|(\mathbf{W}_D+\psi_D)(\cdot,\tau)\|_{L^p(\Omega)^k} \\ &+ \|\mathbf{F}_D(\cdot,\tau)\|_{L^p(\Omega)^m} + \|\mathbf{G}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\Big)\,\mathrm{d}\tau \\ \le C\int_0^t &\mathrm{e}^{-\sigma(t-\tau)}\|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau + C|\Omega|^{1/p}D^{-2\delta_0} \\ &+ C\int_0^t \mathrm{e}^{-\sigma(t-\tau)}\chi_{\{\tau\le \log D/(\lambda_1 D)\}}|\Omega|^{1/p}|\mathbf{b}_D(\tau)|\,\mathrm{d}\tau.\end{aligned}$$

Absorption of the supremum of the function $\mathbf{b}_D$ on the left-hand side can be done in the same way as in Proposition 4.8. This results in

$$\begin{aligned}\sup_{t\in[0,T]}\Big(\|\alpha_D(\cdot,t)\|_{L^p(\Omega)^m} + |\Omega|^{1/p}|\mathbf{b}_D(t)|\Big) \le C\int_0^T \mathrm{e}^{-\sigma(t-\tau)}\|\mathbf{W}_D(\cdot,\tau)\|_{L^p(\Omega)^k}\,\mathrm{d}\tau \\ + C|\Omega|^{1/p}D^{-2\delta_0}\end{aligned}$$

where we restrict ourselves to exponents $\delta_0 \le 1/2$ and $D \ge D_0$. Combining above inequalities leads to an estimate for $\mathbf{W}_D$, more precisely,

$$\sup_{t\in[0,T]}\|\mathbf{W}_D(\cdot,t)\|_{L^p(\Omega)^k} \le CD^{-1}.$$

Since $T>0$ is not specified in the latter inequality, we obtain a global estimate for the truncated solution. Since $\beta_D = \mathbf{W}_D + \mathbf{b}_D$, we infer

$$\|\alpha_D\|_{L^p(\Omega\times\mathbb{R}_{\ge 0})^m} + \|\beta_D\|_{L^p(\Omega\times\mathbb{R}_{\ge 0})^k} \le CD^{-2\delta_0}$$

from the above inequality for $\alpha_D, \mathbf{b}_D$. □

The usual way of disposing of truncation in the nonlinear case and getting $L^\infty(\Omega_T)$ estimates yields an analog of Theorem 4.10.

Theorem 5.2. *Let the assumptions A1–A4, B, L0, and L1p hold for some p with $p \ge 1 = n$ or $p > n/2$ if $n \ge 2$ and some uniform exponential stability exponent $\sigma > 0$ of the evolution system $\mathcal{W}$. Let $r \in (1,\infty)$ be given by* (B.14)*. Then there exist lower bounds $\alpha_0 = \alpha_0(r) \in (0,1)$, $D_0 > 0$ and a constant $C>0$ such that for any $\alpha \in [\alpha_0, 1)$, $D \ge D_0$ and $T \le D^{1-\alpha}$ we have*

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega_T)^m} \le CD^{-3(1-\alpha)}, \tag{5.2}$$

$$\|\langle \mathbf{v}_D\rangle_\Omega - \mathbf{v}\|_{L^\infty((0,T))^k} \le CD^{-4(1-\alpha)}, \tag{5.3}$$
$$\|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega_T)^k} \le CD^{-4(1-\alpha)}. \tag{5.4}$$

Concerning $p=\infty$, we have global estimates

$$\|\mathbf{u}_D - \mathbf{u}\|_{L^\infty(\Omega\times\mathbb{R}_{\ge 0})^m} + \|\mathbf{v}_D - \mathbf{v} - \psi_D\|_{L^\infty(\Omega\times\mathbb{R}_{\ge 0})^k} \le CD^{-1} \tag{5.5}$$

without requiring Assumption L0.

Proof. To get rid of truncation, it is sufficient to show

$$\|\alpha_D\|_{L^\infty(\Omega_T)^m} + \|\beta_D\|_{L^\infty(\Omega_T)^k} \le D^{-\delta_0} \qquad \forall\, D \ge D_0$$

as in the proof of Theorem 4.10. If $p=\infty$, we can choose $2\delta_0 = 1$ in estimate (5.1) and remove the cut-off due to $2\delta_0 > \delta_0$ to obtain the global estimate

$$\|\alpha_D\|_{L^p(\Omega\times\mathbb{R}_{\ge 0})^m} + \|\beta_D\|_{L^p(\Omega\times\mathbb{R}_{\ge 0})^k} \le CD^{-1}.$$

For the case of $p<\infty$, we restrict our time interval in view of the bootstrap method of Ladyzenskaja employed in the proof of Proposition 4.9. Since

$$\|\alpha_D\|_{p,r}, \|\beta_D\|_{p,r} \le CT^{1/r}D^{-2\delta_0} \qquad \text{and} \qquad \|R_d\|_{p,r} \le C\left(T^{1/r}D^{-2\delta_0} + D^{-1/r}\right),$$

there holds along the same lines of the proof of Proposition 4.9

$$\|\beta_D\|_{L^\infty(\Omega_T)} \le D^{(1-\alpha)}\left(D^{-2\delta_0} + D^{-(2-\alpha)/r}\right), \quad \|\alpha_D\|_{L^\infty(\Omega_T)} \le D^{(1-\alpha)}\|\beta_D\|_{L^\infty(\Omega_T)}$$

for all times $T \le D^{1-\alpha}$. In order to dispose of truncation, it remains to find triples (α, δ_0, r) such that the inequalities

$$\delta_0 < 2(\alpha-1) + 2\delta_0 \quad \text{and} \quad \delta_0 < 2(\alpha-1) + (2-\alpha)\frac{1}{r}$$

are satisfied. For existence of such triples, we again define two restrictive curves

$$\alpha > \ell(\delta_0) := 1 - \frac{1}{2}\delta_0 \qquad \text{and} \qquad \alpha > \ell_r(\delta_0) := \frac{r}{2r-1}\delta_0 + 1 - \frac{1}{2r-1}.$$

As in the proof of Theorem 4.10, the function ℓ_r is strictly increasing and ℓ is decreasing. Thus, we always find such triples for small enough $\delta_0 > 0$.

To further simplify above inequalities, we consider the particular case $\delta_0 \le 1/(2r)$. In this case, since $\delta_0 < \overline{\delta}_0 = \frac{2}{4r-1}$ is smaller than the value $\overline{\delta}_0$ of intersection of ℓ and ℓ_r, the only restriction is given by $\ell(\delta_0) < \alpha < 1$. Notice that $\alpha > \ell(\delta_0)$ is equivalent to $\delta_0 > 2(1-\alpha)$ and the results above lead to

$$\|\mathbf{U}_D\|_{L^\infty(\Omega_T)^m} \le C D^{-3(1-\alpha)} \qquad \text{and} \qquad \|\mathbf{V}_D\|_{L^\infty(\Omega_T)^k} \le C D^{-4(1-\alpha)}.$$

Recall the definition of the error functions in (3.1). Since $|\Omega| < \infty$, the spatial mean value $\langle \mathbf{V}_D \rangle_\Omega = \langle \mathbf{v}_D \rangle_\Omega - \mathbf{v}$ can be estimated uniformly too. $\square$

We omitted the fact that diffusive components of the error $\mathbf{U}_D$ possess the same convergence rate as $\mathbf{V}_D$, see Proposition 4.9. If the evolution system $\mathcal{U}$ satisfies an exponential stability condition in Assumption L0 in addition, then all components of $\mathbf{U}_D$ have the same convergence rate as $\mathbf{V}_D$. Recall that the time-restriction in above uniform estimates is due to exponential stability for finite $p < \infty$ but the case $p = \infty$ can be regarded in many applications, see Chapter 6.

Theorem 5.2 enables us to check various models for a global convergence result of the shadow approximation. Particular applications are discussed in Chapter 6 including models from [53, 60, 82]. Since the subsystem inducing $\tilde{\mathcal{U}}$ is simply a system of ordinary differential equations on $L^\infty(\Omega)^{\tilde{m}}$, it is quite standard to check Assumption L0, also for the case of $p = \infty$. Recall that we do not require condition L0 if Assumption L1p holds with $p = \infty$. Nevertheless, verification of Assumption L1p is not an easy task and, possibly, numerical simulations have to be applied.
Let us briefly review dissipative systems already discussed in Section 4.3 which imply an exponential stability condition. Assume that for some $\eta \in \mathbb{R}_{>0}$ and non-negative $\kappa \in C(\mathbb{R}_{\ge 0}) \cap L^1(\mathbb{R}_{\ge 0})$ the stronger condition

$$(1 - \lambda(\kappa(t) - \eta))\|\mathbf{y}\|_{L^p(\Omega)^{\tilde{m}}} \le \|(I - \lambda \mathbf{A}_{11}(\cdot, t))\mathbf{y}\|_{L^p(\Omega)^{\tilde{m}}} \tag{5.6}$$

is satisfied for all $\mathbf{y} \in L^p(\Omega)^{\tilde{m}}$, $\lambda \in \mathbb{R}_{>0}$, and $t \in \mathbb{R}_{\ge 0}$. Following the proof of Proposition 4.13, one verifies that this condition implies a uniform exponential stability of the evolutionary subsystem $\tilde{\mathcal{U}}$ in $L^p(\Omega)^{\tilde{m}}$ with exponent $\eta > 0$ in Assumption L0. Note that, similar to estimate (5.6), the inequality

$$(1 - \lambda(\varrho(t) - \sigma))\|\mathbf{y}\|_{L^p(\Omega)^m \times \mathbb{R}^k} \le \|(I - \lambda \mathbf{L}_0(t))\mathbf{y}\|_{L^p(\Omega)^m \times \mathbb{R}^k} \tag{5.7}$$

leads to an application of Theorem 5.2 using an exponential stability condition for the shadow evolution system $\mathcal{W}$ in Assumption L1p with exponent $\sigma > 0$.
Let us recall that dissipativity of an operator in the sense of condition (4.52) or (4.57) in Section 4.3 is not enough to have exponential stability of the corresponding evolution system. Both notions are not related to each other for general systems, e.g., for $\mathcal{U}$ and $m \geq 2$, as the following example shows.

Example 5.3. Consider the constant matrix operators

$$\mathbf{A}_1 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(i) $\mathbf{A}_1$ is not dissipative in $L^p(\Omega)^2$ for a positive eigenvalue of its symmetric part, see Lemma 4.14. However, the semigroup with elements $\mathbf{U}_1(t) = \mathrm{e}^{\mathbf{A}_1 t}$ exhibits exponential stability since $\mathbf{A}_1$ possesses only the eigenvalue -1, thus with negative real part [23, Chapter I, Theorem 3.14].

(ii) Since the symmetric part of $\mathbf{A}_2$ equals $\mathbf{0}$, the operator is dissipative by Lemma 4.14. However, by [23, Chapter I, Example 2.7], $\mathbf{A}_2$ induces the semigroup $(\mathbf{U}_2(t))_{t\in\mathbb{R}_{\geq 0}}$ with

$$\mathbf{U}_2(t) = \mathrm{e}^{\mathbf{A}_2 t} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

which features no exponential stability and is just bounded.

Finally, I want to emphasize that stability condition L1p is sufficient but not necessary for long-time convergence results. The decoupled system of Example 3.5 shows that global estimates may exist without (exponential) stability.

Example 5.4. Consider the partly diffusive system

$$\begin{aligned} \frac{\partial u_D}{\partial t} &= u_D^2 - (a - v_D)^2 u_D^3 && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \quad u_D(\cdot, 0) = u^0 \ \text{ in } \ \Omega, \\ \frac{\partial v_D}{\partial t} - D\Delta v_D &= v_D && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \quad v_D(\cdot, 0) = v^0 \ \text{ in } \ \Omega, \\ \frac{\partial v_D}{\partial \mathbf{n}} &= 0 && \text{on} \quad \partial\Omega \times \mathbb{R}_{>0} \end{aligned}$$

for $a \neq 0$. The shadow limit (u, v) is uniformly bounded for $v^0 := w_1, u^0 > 0$ a.e. in Ω as already shown in Example 3.5, satisfying $v \equiv 0$. The component u is given by

$$\frac{\partial u}{\partial t} = u^2 - a^2 u^3$$

and $a^2 u(\cdot, t) \to 1$ as $t \to \infty$ by monotonicity of the solution pointwise in space. The unstable subsystem of v implies absence of stability of the evolution system induced by the linearized shadow operators

$$\mathbf{L}_0(t) = \begin{pmatrix} u(\cdot,t)(2 - 3a^2 u(\cdot,t)) & 2au(\cdot,t)^3 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} -|a|^{-2} & 2|a|^{-5} \\ 0 & 1 \end{pmatrix} \quad \text{as} \quad t \to \infty.$$

Even Theorem 4.10 would not apply to this situation. Nevertheless, using Bohl exponents as in [14, Corollary 4.2] for the subsystem of u, we obtain exponential stability of the corresponding evolution system $\mathcal{U}$ for some exponent $\eta > 0$. The cut-off procedure (4.30)–(4.32) yields

$$\beta_D = (\mathrm{e}^t - 1)\psi_D \qquad \text{with} \quad \langle \beta_D \rangle_\Omega = 0$$

where β_D is globally bounded by CD^{-1} due to $\psi_D(\cdot, t) = \mathrm{e}^{-\lambda_1 D t} w_1$. The corresponding equation (4.38) for α_D may be written as

$$\alpha_D(\cdot, t) = \int_0^t U(t, \tau) \Big[B_*(\cdot, \tau)(\beta_D + \psi_D)(\cdot, \tau) + F_D(\alpha_D, \beta_D, \cdot, \tau) \Big] \,\mathrm{d}\tau$$

for bounded $B_*(\cdot, t) = 2au(\cdot, t)^3$. An application of Lemma 4.6 yields the estimate

$$\|\alpha_D(\cdot, t)\|_{L^\infty(\Omega)} \le C \int_0^t \mathrm{e}^{-\eta(t-\tau)} \|(\beta_D + \psi_D)(\cdot, \tau)\|_{L^\infty(\Omega)} \,\mathrm{d}\tau + CD^{-2\delta_0}.$$

The special form of β_D implies $\|\alpha_D\|_{L^\infty(\Omega \times \mathbb{R}_{\ge 0})} \le CD^{-2\delta_0}$ and $2\delta_0 = 1$ is chosen to remove the truncation. We finally obtain global error estimates with a convergence rate of order D^{-1}.

A natural question concerning long-time behavior is whether stability carries over from the shadow system (1.4)–(1.6) to the full diffusive system (1.1)–(1.3). In the case of classical reaction-diffusion systems, this question was studied in [32, 79] for (global) attractors. However, due to the lack of diffusion terms in case of partly diffusive systems, the solution map is not compact anymore. Reaction-diffusion-ODE systems thus can be seen as a partially degenerated reaction-diffusion system for which the methods used in [32, 79] to ensure attractors are not applicable. Nevertheless, using linearized stability analysis for the corresponding diffusive system around a steady state constructed from a shadow solution, [78, 103, 109] or [38] succeeded and (asymptotic) stability carries over in these cases.

5.2 Linearized stability

In many applications of differential and integro-differential equations, asymptotic behavior is governed by steady states, i.e., time-independent solutions of the corresponding system. Using a linearization around some steady state, it is often possible to verify stability properties via the spectrum of the linearized operator. Hence, to check the stability condition L1p, knowing the location of the spectrum of the linearized shadow operator is of prior interest. In the following section, I will present a complete characterization of the latter spectrum for shadow systems of the form (1.4)–(1.6). As a byproduct, the same method of proof leads to a similar representation of the spectrum of the partly diffusive operator linearized around a stationary solution of system (1.1)–(1.3).
Using linearization principles, we are able to apply our results to detect stability of steady states as, for instance, [80, Appendix] for a classical shadow system or [38, Theorem 3.9] for a reaction-diffusion-ODE system. Also cases in which steady states of the partly diffusive system are constructed from a shadow solution are treatable [38, 78, 103, 109]. Instabilities resulting from an unstable subsystem as we already established in Examples 3.6 and 4.12 are merely a consequence of Proposition 5.7 or 5.13 stated below. It has to be mentioned that the following characterization of the spectra also can be applied to instability results presented in [69] for shadow systems and in [71] for partly diffusive systems. Concerning non-smooth bounded steady states, the latter characterization generalizes [69, Theorem B.1] and [71, Theorem 2.11], respectively.

5.2.1 Steady states of the shadow system

Let us consider a bounded steady state solution of the shadow system (1.4)–(1.6), i.e., a solution $(\overline{\mathbf{u}}, \overline{\mathbf{v}}) \in L^\infty(\Omega)^m \times \mathbb{R}^k$ of the integro-differential problem

$$\begin{aligned} -\mathbf{D}^u \Delta \mathbf{u} &= \mathbf{f}(\mathbf{u}, \mathbf{v}, x) \quad \text{in} \quad \Omega, \qquad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} \;\text{ on }\; \partial\Omega, \\ \mathbf{0} &= \langle \mathbf{g}(\mathbf{u}, \mathbf{v}, \cdot) \rangle_\Omega. \end{aligned}$$

Stability or instability properties are often verified using linearization around a steady state of the nonlinear problem, see [14, Chapter VII, Theorem 2.1], [40, Theorem 5.1.1], [108, Proposition 4.17] and their subsequent statements about instability. Further references concerning instability can be found in [69, Theorem B.1] and [71, Theorems 2.1, 2.11]. Certainly, their results crucially depend on the underlying Banach space on which the operator acts, e.g., $L^p(\Omega)^{m+k}, W^{1,p}(\Omega)^{m+k}$ or, in more abstract words, some domain of the fractional operator $(aI - \mathbf{D}\Delta)^\alpha$ for some $a, \alpha > 0$ [40, Definition 1.5.4].

Let us exemplarily consider the case of a linearization in $L^p(\Omega)^m \times \mathbb{R}^k$ which we need to verify Assumption L1p in the time-independent case $(\mathbf{u}, \mathbf{v}) = (\overline{\mathbf{u}}, \overline{\mathbf{v}})$. In doing so, we receive an impression of the methods used to analyze the spectrum of the linearized operator and the reader may adapt them to other function spaces if necessary. Let $\mathbf{f}, \mathbf{g}$ be once continuously differentiable with respect to the unknown variables $\mathbf{u}, \mathbf{v}$ according to Assumption A4. Hence, it can be verified that the linearization is induced by the linear operator

$$\mathbf{L}\xi(x) = \mathbf{D}^S \Delta \xi(x) + \begin{pmatrix} \mathbf{A}_*(x)\xi_1(x) + \mathbf{B}_*(x)\xi_2 \\ \langle \mathbf{C}_*(\cdot)\xi_1 \rangle_\Omega + \langle \mathbf{D}_*(\cdot)\xi_2 \rangle_\Omega \end{pmatrix}.$$

Here, we used the same notation as in the Jacobian (4.33) for the uniformly bounded entries $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, \mathbf{D}_*$ being the parts of the Jacobian of $(\mathbf{f}, \mathbf{g})$ evaluated at the shadow steady state $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$. Let us resort equations to obtain the following form of the diffusion matrix

$$\mathbf{D}^S = \operatorname{diag}(\mathbf{D}^u, \mathbf{0}) = \operatorname{diag}(\mathbf{0}, \mathbf{D}_+, \mathbf{0})$$

for some diagonal matrix $\mathbf{D}_+ \in \mathbb{R}^{\ell \times \ell}_{>0}$ and denote $\tilde{m} = m - \ell \geq 0$ the number of zeroes on the diagonal of $\mathbf{D}^u \in \mathbb{R}^{m \times m}_{\geq 0}$. Then the linearized operator $\mathbf{L}$ has the form

$$\mathbf{L}\begin{pmatrix} \xi_{11}(x) \\ \xi_{12}(x) \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{D}_+\Delta\xi_{12}(x) \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{11}(x)\xi_{11}(x) + \mathbf{A}_{12}(x)\xi_{12}(x) + \mathbf{B}_1(x)\xi_2 \\ \mathbf{A}_{21}(x)\xi_{11}(x) + \mathbf{A}_{22}(x)\xi_{12}(x) + \mathbf{B}_2(x)\xi_2 \\ \langle \mathbf{C}_1\xi_{11} \rangle_\Omega + \langle \mathbf{C}_2\xi_{12} \rangle_\Omega + \langle \mathbf{D}_* \rangle_\Omega \xi_2 \end{pmatrix}, \quad (5.8)$$

where $\xi_{11} \in L^p(\Omega)^{\tilde{m}}, \xi_2 \in \mathbb{R}^k$. Owing to Lemma 2.1, $(S_\Delta(\tau))_{\tau \in \mathbb{R}_{\geq 0}}$ is a strongly continuous contraction semigroup on $L^p(\Omega)$ for each finite $1 \leq p < \infty$. Defining $(\mathbf{S}_+(t))_{t \in \mathbb{R}_{\geq 0}}$ similarly to definition (2.5) yields a strongly continuous contraction semigroup on $L^p(\Omega)^\ell$ generated formally by $\mathbf{D}_+\Delta$. The generator of $(\mathbf{S}_+(t))_{t\in\mathbb{R}_{\geq 0}}$,

$$\mathbf{H}_p : \mathcal{D}(\mathbf{H}_p) \subset L^p(\Omega)^\ell \to L^p(\Omega)^\ell,$$

is a densely defined, closed, linear, unbounded operator [23, Chapter II, Theorem 1.4]. We additionally choose $\xi_{12} \in \mathcal{D}(\mathbf{H}_p)$ such that the operator $\mathbf{L}$ given by (5.8) is well-defined. Using the domain $\mathcal{D}(H_p)$ of the generator H_p of the heat semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ on $L^p(\Omega)$ from the scalar-valued case in Lemma 2.1, we identify $\mathcal{D}(\mathbf{H}_p) = \mathcal{D}(H_p)^\ell \subset W^{1,p}(\Omega)^\ell$. Since $\mathbf{f}, \mathbf{g}$ are continuously differentiable, boundedness of the steady state $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$ yields boundedness of the second operator in definition (5.8) induced by the Jacobian evaluated at $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$. Thus, the full operator $\mathbf{L}$ is unbounded for $\ell > 0$, closed, linear, and densely defined [23, Chapter III, Theorem 1.3]. It even generates an analytic semigroup for $1 < p < \infty$ in view of Lemma 2.1 and [23, Chapter III, Proposition 1.12].
Concerning the stability condition we are faced with in Assumption L1p, it is important to understand the location of the spectrum $\sigma(\mathbf{L})$ in the complex plane. Recall that analyticity of the semigroup generated by $\mathbf{L}$ implies validity of the spectral mapping theorem for $1 < p < \infty$ [23, Chapter IV, Corollary 3.12]. This implies, for instance, that uniform exponential stability can be deduced from a negative spectral bound $s(\mathbf{L}) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathbf{L})\}$ [23, Chapter V, Theorem 1.10]. As in the finite-dimensional case, one has to be careful if the spectral bound of $\mathbf{L}$ is zero. In this case uniform boundedness of the semigroup is not derivable in an easy way, see [21, Chapter III, Theorem 1.11]. Subsequently, we observe that the spectrum of the shadow operator $\mathbf{L}$ has a quite different decomposition depending on zero entries of the matrix $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$.

For $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{>0}$ and $m = 1$, [80, Appendix] already considered the spectrum and showed its discreteness. Let us prove this for completeness in case of systems.

Proposition 5.5. *Let $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{>0}$ and $\mathbf{L}$ be the shadow operator defined in* (5.8) *on $\mathcal{D}(H_p)^m \times \mathbb{R}^k \subset L^p(\Omega)^m \times \mathbb{R}^k$ for matrices $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, \mathbf{D}_*$ with bounded entries in $L^\infty(\Omega)$. Then the spectrum $\sigma(\mathbf{L})$ is a discrete set of eigenvalues of $\mathbf{L}$ for each finite $1 < p < \infty$ and for the particular case $p = 1 = n$.*

Proof. We will apply [50, Chapter III, Theorem 6.29] by showing compactness of the resolvent $(\lambda I - \mathbf{L})^{-1}$ for some real, large λ. Since $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, \mathbf{D}_*$ are uniformly bounded in $L^\infty(\Omega)$ and $\xi_2 \in \mathbb{R}^k$, we can solve the first equation of

$$(\lambda I - \mathbf{L})\xi = \psi \quad \Leftrightarrow \quad \begin{cases} (\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)\xi_1(x) - \mathbf{B}_*(x)\xi_2 = \psi_1(x), \\ -\langle \mathbf{C}_*\xi_1\rangle_\Omega + (\lambda I - \langle \mathbf{D}_*\rangle_\Omega)\xi_2 = \psi_2 \end{cases}$$

with respect to ξ_1 for sufficiently large real $\lambda > \|\mathbf{A}_*\|_{L^\infty(\Omega)^{m\times m}}$ as an elliptic problem in ξ_1 for each $(\psi_1, \psi_2) \in L^p(\Omega)^m \times \mathbb{R}^k$. Existence and uniqueness in $W^{1,p}(\Omega)^m$ for large λ follows from Lemma A.3. By Lemma B.5, a weak solution $\xi_1 \in W^{1,p}(\Omega)^m$ is an element of $\mathcal{D}(\mathbf{H}_p) \subset W^{1,p}(\Omega)^m$. The resulting equation for $\xi_2 \in \mathbb{R}^k$ is

$$\mathbf{H}(\lambda)\xi_2 = \psi_2 + \langle \mathbf{C}_*(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}\psi_1\rangle_\Omega$$

where we used the matrix-valued function

$$\mathbf{H} : \rho(\mathbf{D}^u\Delta + \mathbf{A}_*) \to \mathbb{C}^{k\times k}, \qquad \lambda \mapsto \lambda I - \langle \mathbf{D}_*\rangle_\Omega - \langle \mathbf{C}_*(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}\mathbf{B}_*\rangle_\Omega.$$

Recall that the resolvent $(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}$ is holomorphic in $\lambda \in \rho(\mathbf{D}^u\Delta + \mathbf{A}_*)$ [23, Chapter IV, Proposition 1.3]. Hence, we may apply [77, Theorem] (wherein local boundedness assumption A_3 is derived from the first resolvent identity) to infer holomorphy of the operator $\mathbf{H}$. The determinant $\det(\mathbf{H})$ is the sum of products of holomorphic functions in λ and, consequently, it is holomorphic itself. By the identity theorem for holomorphic functions, it has only a discrete set of zeroes or is identically zero. The latter case is impossible since the above equation is solvable with respect to ξ_2 at least for all large real λ. To see this, we recall the resolvent estimate

$$\|(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}\| \leq \frac{M}{\lambda - w}$$

for some $w, M \in \mathbb{R}_{\geq 0}$ [23, Chapter II, Theorem 3.8]. In combination with the triangle inequality for matrix norms, this yields that the matrix $\mathbf{H}(\lambda)$ is invertible for all sufficiently large real λ, hence $\rho(\mathbf{L}) \neq \emptyset$.

It remains to show compactness of the resolvent for one $\lambda \in \rho(\mathbf{L})$. The resolvent is given by the solution ξ,

$$\begin{aligned} \xi_1 &= (\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}\left(\psi_1 + \mathbf{B}_*\xi_2\right), \\ \xi_2 &= \mathbf{H}(\lambda)^{-1}\left(\psi_2 + \langle \mathbf{C}_*(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}\psi_1\rangle_\Omega\right), \end{aligned}$$

of the equation $(\lambda I - \mathbf{L})\xi = \psi$. As usual, the compact embedding $W^{1,p}(\Omega) \hookrightarrow^c L^p(\Omega)$ from [1, Theorem 6.3] implies compactness of the resolvent $(\lambda I - \mathbf{D}^u\Delta - \mathbf{A}_*)^{-1}$ with image $\mathcal{D}(H_p)^m \subset W^{1,p}(\Omega)^m$ [23, Chapter II, Proposition 4.25]. Since matrix operators such as $\mathbf{H}(\lambda)^{-1}$ are compact on finite-dimensional spaces such as $\mathbb{R}^k$, compactness of the resolvent $(\lambda I - \mathbf{L})^{-1}$ follows from [8, Proposition 6.3]. $\square$

Now let us assume that $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$ has some zero on its diagonal. For simplicity, let us assume the above form (5.8) of the shadow operator $\mathbf{L}$ defined on the domain $L^p(\Omega)^{\tilde{m}} \times \mathcal{D}(H_p)^\ell \times \mathbb{R}^k$. To study invertibility of the operator $\lambda I - \mathbf{L}$ for some $\lambda \in \mathbb{C}$, we focus on the following system of equations

$$(\lambda I - \mathbf{L})\xi = \psi \quad \Leftrightarrow \quad \begin{cases} (\lambda I - \mathbf{A}_{11})\xi_{11} - \mathbf{A}_{12}\xi_{12} - \mathbf{B}_1\xi_2 = \psi_{11}, \\ -\mathbf{A}_{21}\xi_{11} + (\lambda I - \mathbf{D}_+\Delta - \mathbf{A}_{22})\xi_{12} - \mathbf{B}_2\xi_2 = \psi_{12}, \\ -\langle \mathbf{C}_1\xi_{11}\rangle_\Omega - \langle \mathbf{C}_2\xi_{12}\rangle_\Omega + (\lambda I - \langle \mathbf{D}_*\rangle_\Omega)\xi_2 = \psi_2 \end{cases}$$

for $\psi \in L^p(\Omega)^m \times \mathbb{R}^k$. Let $\mathbf{A}_{11}$ also denote the bounded multiplication operator induced by the matrix $\mathbf{A}_{11}(x)$ on $L^p(\Omega)^{\tilde{m}}$ [104, Proposition 2.2.14]. If $\lambda \notin \sigma(\mathbf{A}_{11})$, the first equation can be solved resulting in

$$\xi_{11} = (\lambda I - \mathbf{A}_{11})^{-1}\left[\psi_{11} + \mathbf{A}_{12}\xi_{12} + \mathbf{B}_1\xi_2\right] =: \mathbf{f}_{11}(\lambda, \xi_{12}, \xi_2). \tag{5.9}$$

Let us first characterize the spectral values of $\mathbf{L}$ in $\rho(\mathbf{A}_{11})$. Equation (5.9) shows that the above system can be reduced to a spectral problem which can be treated similarly to the classical shadow limit case $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{>0}$ in Proposition 5.5.

Lemma 5.6. *Let* $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$, $\mathbf{D}_+ \in \mathbb{R}^{\ell\times\ell}_{>0}$ *for some* $0 \leq \ell < m$ *and let* $\mathbf{L}$ *be the shadow operator defined in* (5.8) *on* $L^p(\Omega)^{\tilde{m}} \times \mathcal{D}(H_p)^\ell \times \mathbb{R}^k \subset L^p(\Omega)^m \times \mathbb{R}^k$ *for bounded coefficient matrices* $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, \mathbf{D}_*$ *and a finite* $1 < p < \infty$. *Then*

$$\Sigma := \sigma(\mathbf{L}) \cap \rho(\mathbf{A}_{11}) \subset \sigma_p(\mathbf{L})$$

is a discrete (probably empty) set of eigenvalues of $\mathbf{L}$. *Moreover,*

$$\sigma(\mathbf{L}) \subset \sigma(\mathbf{A}_{11}) \,\dot{\cup}\, \Sigma.$$

If $\ell = 0$ *or space dimension* $n = 1$, *the above assertion for* $\mathbf{L}$ *also holds for the particular cases* $p \in \{1, \infty\}$ *or* $p = 1$, *respectively.*

Proof. Provided $\lambda \in \rho(\mathbf{A}_{11})$ we have already

$$(\lambda I - \mathbf{L})\xi = \psi \quad \Leftrightarrow \quad \begin{cases} \xi_{11} = \mathbf{f}_{11}(\lambda), \\ (\lambda I - \mathbf{D}_+\Delta - \mathbf{f}_{12}(\lambda))\xi_{12} - \mathbf{f}_2(\lambda)\xi_2 = \mathbf{h}_{12}(\lambda), \\ -\langle \mathbf{g}_{12}(\lambda)\xi_{12}\rangle_\Omega + (\lambda I - \langle \mathbf{D}_*\rangle_\Omega - \mathbf{g}_2(\lambda))\xi_2 = \mathbf{h}_2(\lambda) \end{cases}$$

for $\mathbf{f}_{11}$ defined in (5.9) and

$$\begin{aligned} \mathbf{f}_{12}(\lambda) &= \mathbf{A}_{22} + \mathbf{A}_{21}(\lambda I - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}, & \mathbf{f}_2(\lambda) &= \mathbf{B}_2 + \mathbf{A}_{21}(\lambda I - \mathbf{A}_{11})^{-1}\mathbf{B}_1, \\ \mathbf{g}_{12}(\lambda) &= \mathbf{C}_2 + \mathbf{C}_1(\lambda I - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}, & \mathbf{g}_2(\lambda) &= \langle \mathbf{C}_1(\lambda I - \mathbf{A}_{11})^{-1}\mathbf{B}_1\rangle_\Omega, \\ \mathbf{h}_{12}(\lambda) &= \psi_{12} + \mathbf{A}_{21}(\lambda I - \mathbf{A}_{11})^{-1}\psi_{11}, & \mathbf{h}_2(\lambda) &= \psi_2 + \langle \mathbf{C}_1(\lambda I - \mathbf{A}_{11})^{-1}\psi_{11}\rangle_\Omega. \end{aligned}$$

The subsystem for (ξ_{12}, ξ_2) can be treated in a similar way as in the proof of Proposition 5.5. The aim of the proof is to apply an analytic Fredholm theorem [33, Theorem 4.34] to show discreteness of the remaining spectrum Σ. Let us first show solvability for some $\lambda \in \rho(\mathbf{A}_{11})$. The resolvent mapping $\lambda \to (\lambda I - \mathbf{A}_{11})^{-1}$ of the multiplication operator $\mathbf{A}_{11}$ is a well-defined function with values in $L^\infty(\Omega)^{m\times m}$ [35, Proposition 2.2]. Since $\mathbf{A}_{11}$ generates a uniformly continuous semigroup, we obtain a resolvent estimate of $(\lambda I - \mathbf{A}_{11})^{-1}$ [23, Chapter II, Theorem 3.8]. The latter implies that $\mathbf{f}_{12}$ is uniformly bounded for all large λ. Hence, the elliptic problem

$$(\lambda I - \mathbf{D}_+\Delta - \mathbf{f}_{12}(\lambda))\xi_{12} - \mathbf{f}_2(\lambda)\xi_2 = \mathbf{h}_{12}(\lambda)$$

can be solved in the weak sense with respect to $\xi_{12} \in \mathcal{D}(H_p)^\ell$ depending on the large parameter λ and ξ_2, see Lemma A.3. Invertibility of $\lambda I - \mathbf{L}$ is finally equivalent to solving equation

$$(\lambda I - \langle \mathbf{D}_*\rangle_\Omega - \mathbf{M}(\lambda))\xi_2 = \mathbf{h}(\lambda)$$

with respect to $\xi_2 \in \mathbb{R}^k$ where

$$\begin{aligned} \mathbf{M}(\lambda) &= \mathbf{g}_2(\lambda) + \langle \mathbf{g}_{12}(\lambda)\,[\lambda I - \mathbf{D}_+\Delta - \mathbf{f}_{12}(\lambda)]^{-1}\,\mathbf{f}_2(\lambda)\rangle_\Omega, \\ \mathbf{h}(\lambda) &= \mathbf{h}_2(\lambda) + \langle \mathbf{g}_{12}(\lambda)\,[\lambda I - \mathbf{D}_+\Delta - \mathbf{f}_{12}(\lambda)]^{-1}\,\mathbf{h}_{12}(\lambda)\rangle_\Omega. \end{aligned}$$

For sufficiently large λ, resolvent estimates of $(\lambda I - \mathbf{A}_{11})^{-1}$ and $(\lambda I - \mathbf{D}_+\Delta - \mathbf{A}_{22})^{-1}$ for small perturbations via $\mathbf{f}_{12}(\lambda)$ such as in [23, Chapter III, Theorem 1.3] imply that $\mathbf{M}(\lambda)$ is uniformly bounded in λ and is small with respect to the matrix norm on $\mathbb{R}^k$. Estimating the matrix $\lambda I - \langle \mathbf{D}_*\rangle_\Omega - \mathbf{M}(\lambda)$ from below then yields invertibility

of the latter matrix, hence $\lambda \in \rho(\mathbf{L}) \neq \emptyset$.
Finally, solving the subsystem for (ξ_{12}, ξ_2) is equivalent to studying invertibility of the operator $I - \mathbf{k}(\lambda)$ for the operator-valued function

$$\begin{aligned}\mathbf{k} :& \rho(\mathbf{A}_{11}) \to \mathcal{L}(L^p(\Omega)^\ell \times \mathbb{R}^k),\\ &\lambda \mapsto \begin{pmatrix} (lI - \mathbf{D}_+\Delta)^{-1}\left[(\lambda - l)I + \mathbf{f}_{12}(\lambda)\right] & -(lI - \mathbf{D}_+\Delta)^{-1}\mathbf{f}_2(\lambda) \\ -(lI - \langle \mathbf{D}_* \rangle_\Omega)^{-1}\langle \mathbf{g}_{12}(\lambda)\cdot\rangle_\Omega & (lI - \langle \mathbf{D}_* \rangle_\Omega)^{-1}\left[(\lambda - l)I + \mathbf{f}_{12}(\lambda)\right] \end{pmatrix}\end{aligned}$$

for some $l > 0$. Similar to the proof of Proposition 5.5, holomorphy of resolvents implies that $\mathbf{k}$ is analytic in $\lambda \in \rho(\mathbf{A}_{11})$. Let us recall that, for sufficiently large $l > 0$, the resolvents $(lI - \mathbf{D}_+\Delta)^{-1}$ and $(lI - \langle \mathbf{D}_* \rangle_\Omega)^{-1}$ are compact in $L^p(\Omega)^\ell$ and $\mathbb{R}^k$, respectively, and $\mathbf{k}(\lambda)$ is compact. Application of an analytic Fredholm theorem [33, Theorem 4.34] yields that the subsystem is uniquely solvable with respect to $(\xi_{12}, \xi_2) \in \mathcal{D}(H_p)^\ell \times \mathbb{R}^k$ for all $\lambda \in \rho(\mathbf{A}_{11}) \setminus \Sigma$, where Σ is a discrete set in $\mathbb{C}$. For values $\lambda \in \Sigma$ we infer an eigenfunction $(\xi_{12}, \xi_2) \neq \mathbf{0}$ of the eigenvalue equation of the subsystem, using $\mathbf{h}_{12} = \mathbf{0}, \mathbf{h}_2 = \mathbf{0}$. Determining ξ_{11} via equation (5.9) yields that all $\lambda \in \Sigma$ are eigenvalues of $\mathbf{L}$, too.
In the degenerated case $\ell = 0$, without any diffusive component, Σ is determined by all $\lambda \in \rho(\mathbf{A}_*)$ such that

$$\mathbf{H}(\lambda) = \lambda I - \langle \mathbf{D}_* \rangle_\Omega - \langle \mathbf{C}_*(\lambda I - \mathbf{A}_*)^{-1}\mathbf{B}_* \rangle_\Omega$$

is not invertible in $\mathbb{R}^k$ (compare to Proposition 5.5 by formally setting $\mathbf{D}^u \equiv \mathbf{0}$). □

Let us remark that the discrete set Σ is not necessarily closed. However, all accumulation points are included in $\sigma(\mathbf{A}_{11})$ by the following argument. A sequence of eigenvalues $\mu_j \in \Sigma \subset \sigma_p(\mathbf{L})$ has corresponding eigenfunctions such that the singular sequence of normalized eigenfunctions implies $\lim_{j\to\infty} \mu_j \in \sigma(\mathbf{L})$ which is a subset of $\sigma(\mathbf{A}_{11}) \cup \Sigma$, hence $\lim_{j\to\infty} \mu_j \in \sigma(\mathbf{A}_{11})$.

The multiplication operator induced by the subsystem $\mathbf{A}_{11}$ on $L^p(\Omega)^{\tilde{m}}$ may cause problems while inverting the operator $\lambda I - \mathbf{L}$ [69, Theorem B.1]. In general, the third equation of the above eigenvalue problem is not uniquely solvable with respect to ξ_{11}. To recognize this, let us show $\sigma_p(\mathbf{A}_{11}) \subset \sigma_p(\mathbf{L})$ in case of $\mathbf{D}^u \equiv \mathbf{0}$ using the characterization given in [35, Corollary 2.6]. Accordingly, for each $\lambda \in \sigma_p(\mathbf{A}_{11})$ there is a subset $\Omega_1 \subset \Omega$ with positive measure $|\Omega_1| > 0$ such that for arbitrary functions

$\phi \in L^p(\Omega_1)$ we have eigenfunctions $\xi_{1,\phi} \in L^p(\Omega)^{\tilde{m}}$ of the form

$$\xi_{1,\phi}(x) = \begin{cases} \phi(x)\mathbf{y}(x) & \text{for} \quad x \in \Omega_1, \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega_1 \end{cases} \qquad \text{with} \quad (\lambda I - \mathbf{A}_{11})\xi_{1,\phi} = \mathbf{0},$$

for a fixed $\mathbf{y} \in L^\infty(\Omega_1)^{\tilde{m}}$ and $1 \leq p \leq \infty$. From the eigenvalue problem

$$(\lambda I - \mathbf{L})\xi = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} (\lambda I - \mathbf{A}_*)\xi_{1,\phi}(x) - \mathbf{B}_*(x)\xi_2 = \mathbf{0}, \\ -\langle \mathbf{C}_*\xi_{1,\phi}\rangle_\Omega + (\lambda I - \langle \mathbf{D}_*\rangle_\Omega)\xi_2 = \mathbf{0} \end{cases}$$

it is deduced that the operator $\lambda I - \mathbf{L}$ cannot be injective in case of $\mathbf{D}^u \equiv \mathbf{0}$. Indeed, concerning $\xi_2 = \mathbf{0}$, the linear integral operator

$$\tilde{\mathbf{C}}_* : L^p(\Omega_1) \to \mathbb{R}^k, \qquad \phi \mapsto \langle \mathbf{C}_*\mathbf{y}\phi\rangle_\Omega$$

cannot be injective due to the rank theorem, hence $\lambda \in \sigma_p(\mathbf{L})$.

A complete characterization of the spectrum of the multiplication operator $\mathbf{A}_{11}$ is given in Proposition C.1. Moreover, it is shown that the spectrum is essential. This fact enables us to verify that $\sigma(\mathbf{A}_{11})$ is a part of the essential spectrum of the shadow operator $\mathbf{L}$.

Proposition 5.7. *Let $\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, \mathbf{D}_*$ be matrix-valued functions with entries in $L^\infty(\Omega)$ according to shadow operator $\mathbf{L}$ defined by* (5.8) *on $L^p(\Omega)^m \times \mathbb{R}^k$ for some $1 < p < \infty$. Then there holds*

$$\sigma(\mathbf{L}) = \sigma(\mathbf{A}_{11}) \,\dot{\cup}\, \Sigma,$$

where $\Sigma \subset \sigma_p(\mathbf{L})$ is the discrete (possibly empty) set defined in Lemma 5.6. The same is true for the particular case $p = 1 = n$. If $\mathbf{D}^u \equiv \mathbf{0}$, i.e., $\ell = 0$ and $\mathbf{A}_{11} = \mathbf{A}_*$, we have $\sigma_p(\mathbf{A}_*) \subset \sigma_p(\mathbf{L})$, and the assertion holds for $p \in \{1, \infty\}$ too.

Proof. It remains to show $\sigma(\mathbf{A}_{11}) \subset \sigma(\mathbf{L})$ since from the considerations in Lemma 5.6 we already have

$$\rho(\mathbf{A}_{11}) \cap \Sigma = \Sigma \subset \sigma(\mathbf{L}) \subset \sigma(\mathbf{A}_{11}) \,\dot{\cup}\, \Sigma.$$

As $\mathbf{L}$ is a 3×3 operator matrix defined in (5.8), this situation corresponds to [47, Theorem 4.1 (i)]. However, in view of Proposition 5.5, we apply [5, Theorem 2.2] to

show $\sigma_{\text{ess}}(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{L})$, and follow the ideas of the proof of [35, Theorem 4.2] which uses similar methods. From Proposition C.1 we infer $\sigma(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{L})$, once shown above equality. Let us show $\sigma_{\text{ess}}(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{L})$:
In order to apply [5, Theorem 2.2] we permute the operator matrix $\mathbf{L}$ in (5.8). Let us consider a permutation matrix $\mathbf{P} \in \mathbb{R}^{(m+k)\times(m+k)}$ with $\mathbf{P}^2 = I$ which is an isomorphism from $L^p(\Omega)^{\tilde{m}} \times (L^p(\Omega)^{\ell} \times \mathbb{R}^k)$ to $(L^p(\Omega)^{\ell} \times \mathbb{R}^k) \times L^p(\Omega)^{\tilde{m}}$. Then $\lambda I - \mathbf{L}$ is a Fredholm operator if and only if $\lambda I - \tilde{\mathbf{L}}$ is Fredholm where $\tilde{\mathbf{L}} = \mathbf{P}^{-1}\mathbf{L}\mathbf{P}$, hence $\sigma_{\text{ess}}(\mathbf{L}) = \sigma_{\text{ess}}(\tilde{\mathbf{L}})$. This is a consequence of the fact that the invertible operator $\mathbf{P}$ is Fredholm and $\lambda I - \tilde{\mathbf{L}} = \mathbf{P}^{-1}(\lambda I - \mathbf{L})\mathbf{P}$ is a composition of Fredholm operators [8, Chapter 6]. We apply the results of [5] to the closed operator $\tilde{\mathbf{L}}$ given by

$$\tilde{\mathbf{L}} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We take the bounded multiplication operator $D := \mathbf{A}_{11}$ on $X_2 := L^p(\Omega)^{\tilde{m}}$ and A is given by the $\ell + k$ equations of (5.8) induced by reaction-diffusion and shadow system on $X_1 := L^p(\Omega)^{\ell} \times \mathbb{R}^k$ with domain $\mathcal{D}(H_p)^{\ell} \times \mathbb{R}^k$. In view of Proposition 5.5 and previous discussions, the operator A is densely defined and closed with non-empty resolvent set and compact resolvent. Consequently, the operators B, C in notation of [5] consist of bounded multiplication operators as well as integral (shadow) operators. Note that $S(\mu)$ for $\mu \in \rho(A)$ in assumption (e) of their paper is given by

$$S(\mu) = D - C(A - \mu I)^{-1}B,$$

where we can choose $S_0 = D$ and compactness of $M(\mu) = S(\mu) - S_0$ follows from a standard perturbation result for the compact resolvent $(\mu I - A)^{-1}$ [50, Chapter III, Theorem 4.8]. Then $\sigma_{\text{ess}}(\tilde{\mathbf{L}}) = \sigma_{\text{ess}}(S_0)$ is a consequence of [5, Theorem 2.2] since $\mathbf{L}$ is a closed operator. This shows $\sigma_{\text{ess}}(\mathbf{L}) = \sigma_{\text{ess}}(\mathbf{A}_{11})$. □

Let us remind ourselves of remarks after [35, Proposition 4.4]; values in $\sigma_p(\mathbf{A}_{11})$ need not to be elements of the point spectrum of $\mathbf{L}$ for $\mathbf{D}^u \neq \mathbf{0}$ in general. Moreover, since $\sigma_{\text{ess}}(\mathbf{L})$ possibly also contains eigenvalues, the discrete set Σ in the above splitting of the spectrum might not contain all eigenvalues of $\mathbf{L}$.

Remark that time-dependent linearizations which are asymptotically comparable to linearized systems evaluated at the steady state have similar stability properties, see [14, Corollary 4.2] or [16, Theorem 5]. I will give several examples in Chapter 6 for which Assumption L1p will be checked using this method. It is needless to say that the result of Proposition 5.7 is not limited to just verify Assumption L1p for Theorems 4.10 and 5.2. In many cases, stability and instability of stationary solutions to the shadow problem (1.4)–(1.6) can be examined using this characterization for its linearization. Let us return to the simple Example 3.6 where instability of steady states is inherited from the unstable subsystem with $a > 0$.

Example 5.8. Steady states of the linear shadow system of Example 3.6 satisfy

$$\begin{aligned} -D^u\Delta\overline{u} &= a\overline{u} + b\overline{v} \qquad \text{in} \quad \Omega, \qquad \frac{\partial \overline{u}}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \\ 0 &= c\langle\overline{u}\rangle_\Omega + d\overline{v}. \end{aligned}$$

Integration over Ω yields $\langle\overline{u}\rangle_\Omega = \overline{v} = 0$ since the matrix $\mathbf{M}$ induced by a, b, c, d is invertible. $D^u = 0$ would already imply $(\overline{u}, \overline{v}) = \mathbf{0}$ since $\overline{u}$ is spatially homogeneous for $\overline{v}$ is so. If $D^u > 0$, there might be additional non-trivial solutions $(\overline{u}, \overline{v}) = (w_j, 0)$ for several values $a, D^u > 0$ satisfying $a = D^u\lambda_j > 0$, $j \in \mathbb{N}$.
Concerning instability of steady states, let us consider the linear system of Example 3.6 around some $(\overline{u}, \overline{v})$. Then solutions are given by the analytic semigroup induced by $\mathbf{L}$. By the spectral mapping theorem, the spectral bound $s(\mathbf{L})$ equals the growth bound of the generated semigroup [23, Chapter IV, Corollary 3.11]. Hence, instability of the steady state due to exponentially growing solutions follows from a positive spectral bound $s(\mathbf{L}) > 0$. If $D^u = 0$, then $\sigma(A_*) = \{a\}$ immediately yields instability of $(\overline{u}, \overline{v}) = \mathbf{0}$ by Proposition 5.7 since $a > 0$. If $D^u > 0$, we know from Proposition 5.5 that $\sigma(\mathbf{L}) = \sigma_p(\mathbf{L})$ and we search possible eigenfunctions, i.e., non-trivial solutions $\xi = (\xi_1, \xi_2) \in \mathcal{D}(H_p) \times \mathbb{R}$ of

$$(\lambda I - \mathbf{L})\xi = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} (\lambda - \mathbf{D}^u\Delta - a)\xi_1(x) - b\xi_2 = 0, \\ \qquad -c\langle\xi_1\rangle_\Omega + (\lambda - d)\xi_2 = 0. \end{cases}$$

Integration over Ω yields $(\lambda I - \mathbf{M})\langle\xi\rangle_\Omega = \mathbf{0}$. If $\lambda \in \sigma(\mathbf{M})$, then we can choose constant functions $\xi = \langle\xi\rangle_\Omega \neq \mathbf{0}$ and deduce $\sigma(\mathbf{M}) \subset \sigma(\mathbf{L})$. If $\lambda \in \rho(\mathbf{M})$, then $\langle\xi_1\rangle_\Omega = 0 = \xi_2$ and the above problem reduces to find $\xi_1 \neq 0$ satisfying Neumann boundary conditions and $(\lambda - \mathbf{D}^u\Delta - a)\xi_1 = 0$. The last problem is only solvable with $\xi_1 \neq 0$ if $\lambda = a - \lambda_j D^u$ for $j \in \mathbb{N}$ by Proposition A.1. All in all,

$$\sigma(\mathbf{L}) = \sigma(\mathbf{M}) \cup \bigcup_{j\in\mathbb{N}} \{a - \lambda_j D^u\},$$

and instability arises from $a > \lambda_{j_0} D^u$ for some $j_0 \in \mathbb{N}$ since $\mathbf{M}$ is stable by assumptions in Example 3.6. Take into account that parameter values $0 < a < \lambda_1 D^u$ would not lead to instability of the classical shadow limit.

Let us conclude this section with two nonlinear examples. The following model originating from [69, Appendix C] was recently also targeted in control theory [41].

Example 5.9. Bounded steady states of the shadow Gray-Scott model

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -(B+k)u + u^2 v \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad && u(\cdot,0) = u^0 \quad \text{in} \quad \Omega,\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= B(1-v) - \langle u^2 \rangle_\Omega v \qquad \text{in} \quad \mathbb{R}_{>0}, && v(0) = \langle v^0 \rangle_\Omega
\end{aligned}
$$

are given by $(\overline{u}, \overline{v}) \in L^\infty(\Omega) \times \mathbb{R}$ with $\overline{u} = (B+k)\overline{v}^{-1}\chi_{\Omega_1}$ for some measurable set $\Omega_1 \subset \Omega$ and $\overline{v} > 0$ satisfies a quadratic equation. Instability of non-homogeneous steady states was discussed in [69, Remark C.2] via $\sigma(A_{11}) = \{\pm(B+k)\}$ for $B, k > 0$ and $|\Omega_1| > 0$ since the shadow operator $\mathbf{L} \in \mathcal{L}(L^\infty(\Omega) \times \mathbb{R})$ is given by

$$
\mathbf{L}(\overline{u}(x), \overline{v}) = \begin{pmatrix} -(B+k) + 2(B+k)\chi_{\Omega_1}(x) & \overline{u}^2(x) \\ -2(B+k)\langle \chi_{\Omega_1} \cdot \rangle_\Omega & -B - \langle \overline{u}^2 \rangle_\Omega \end{pmatrix}.
$$

Note that the constant steady state $(0, 1)$ satisfies $\sigma(\mathbf{L}) = \{-(B+k), -B\}$ and is locally asymptotically stable [14, Chapter VII, Theorem 2.1].

Finally, I will discuss linearized stability of steady states of the shadow limit for a quite popular activator-inhibitor system.

Example 5.10. Consider a shadow system of Gierer-Meinhardt type with a possibly inhomogeneous coefficient, compare [110, Chapter 7].

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\mu(x)u + u^p v^{-q} \qquad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad && u(\cdot,0) = u^0 \quad \text{in} \quad \Omega,\\
\tau\frac{\mathrm{d}v}{\mathrm{d}t} &= -v + \langle u^r \rangle_\Omega v^{-s} \qquad \text{in} \quad \mathbb{R}_{>0}, && v(0) = \langle v^0 \rangle_\Omega.
\end{aligned}
$$

Here, we have real parameters $p > 1$, $q, r, \tau > 0, s \geq 0$ and $u^0 \geq 0, \langle v^0 \rangle_\Omega > 0$. The coefficient $\mu \in L^\infty(\Omega)$ satisfies $\mu(x) \geq \mu_0 > 0$ for a.e. $x \in \Omega$. Non-negative steady states are given by $(\overline{u}, \overline{v}) \in L^\infty(\Omega) \times \mathbb{R}$ with $\overline{u} = (\mu^{-1}\overline{v}^q)^{1/(p-1)}\chi_{\Omega_1}$ for some measurable set $\Omega_1 \subset \Omega$ with $|\Omega_1| > 0$ and a real number $\overline{v} > 0$ determined by an integral over $\mu^{-1/(p-1)}$. The corresponding linearized shadow operator is given by

$\mathbf{L} \in \mathcal{L}(L^\infty(\Omega) \times \mathbb{R})$ with

$$\mathbf{L}(\overline{u}(x), \overline{v}) = \begin{pmatrix} -\mu(x) + p(\mu^{-1}\chi_{\Omega_1})(x) & -q\overline{u}^p(x)\overline{v}^{-(q+1)} \\ r\langle \overline{u}^{r-1}\overline{v}^{-s} \cdot \rangle_\Omega & -1 - s\langle \overline{u}^r \overline{v}^{-(s+1)} \rangle_\Omega \end{pmatrix}.$$

By Proposition C.1, the spectrum $\sigma(A_{11})$ of the multiplication operator is given by the essential range of $-\mu + p\mu^{-1}\chi_{\Omega_1}$ in Ω. Depending on μ, a positive spectral radius $s(A_{11}) > 0$ would already lead to instability by Proposition 5.7 and [14, Chapter VII, Theorem 2.3].
In the case $\mu \equiv 1$, instability of steady states $(\overline{u}, \overline{v})$ results from $\sigma(A_{11}) = \{-1, p-1\}$ since $p > 1$. Hence, the constant steady state $(1, 1)$ as well as inhomogeneous steady states are all unstable. Recall the works [61, Corollary 1.3], [85] which consider linearized stability of steady states to the classical shadow system with a diffusion $D^u > 0$ for the u-component and $\mu \equiv 1$. In contrast to the above shadow problem, [85, Theorem B] reveals stable configurations for some parameter sets in the classical shadow case.

5.2.2 Steady states of the diffusive system

Let us study the spectrum of the diffusive counterpart of the linearized shadow operator $\mathbf{L}$. Consider a bounded steady state of system (1.1)–(1.3), i.e., a solution $(\overline{\mathbf{u}}_D, \overline{\mathbf{v}}_D) \in L^\infty(\Omega)^{m+k}$ of the problem

$$\begin{aligned} -\mathbf{D}^u \Delta \mathbf{u}_D &= \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x) \quad \text{in} \quad \Omega, & \frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} &= \mathbf{0} \quad \text{on} \quad \partial\Omega, \\ -\mathbf{D}^v \Delta \mathbf{v}_D &= \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x) \quad \text{in} \quad \Omega, & \frac{\partial \mathbf{v}_D}{\partial \mathbf{n}} &= \mathbf{0} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

According to Assumption A4, $\mathbf{f}, \mathbf{g}$ are once continuously differentiable with respect to the unknown variables $\mathbf{u}, \mathbf{v}$. The linearization is induced by the operator

$$\mathbf{L}_D \xi(x) = \mathbf{D}\Delta\xi(x) + \mathbf{J}_D(x)\xi(x)$$

where we used the Jacobian $\mathbf{J}_D$ of $(\mathbf{f}, \mathbf{g})$ evaluated at the steady state $(\overline{\mathbf{u}}_D, \overline{\mathbf{v}}_D)$.
As already mentioned for shadow systems, stability or instability properties may often be verified using linearization, see references at the beginning of the last section. Hence, knowing its spectrum becomes important for many applications.
Following the same ideas leading to the definition of $(\mathbf{S}(t))_{t \in \mathbb{R}_{\geq 0}}$ in (2.5) yields a strongly continuous contraction semigroup on $L^p(\Omega)^{m+k}$ for each $1 \leq p < \infty$ gener-

ated formally by $\mathbf{D}\Delta$. The corresponding generator of $(\mathbf{S}(t))_{t\in\mathbb{R}_{\geq 0}}$,

$$\mathbf{H}_p : \mathcal{D}(\mathbf{H}_p) \subset L^p(\Omega)^{m+k} \to L^p(\Omega)^{m+k},$$

is a densely defined, closed, linear operator [23, Chapter II, Theorem 1.4]. Let us once again split $\mathbf{D}^u = \operatorname{diag}(\mathbf{0}, \mathbf{D}_+)$ in its zero and positive part, for $\mathbf{0} \in \mathbb{R}_{\geq 0}^{\tilde{m}\times\tilde{m}}$ and $\mathbf{D}_+ \in \mathbb{R}_{>0}^{\ell\times\ell}$ (note that $m = \tilde{m} + \ell$). This yields $\mathcal{D}(\mathbf{H}_p) = L^p(\Omega)^{\tilde{m}} \times \mathcal{D}(H_p)^{\ell+k}$ as a domain for the unbounded operator $\mathbf{L}_D$ and boundedness of the steady state results in a bounded multiplication operator $\mathbf{J}_D$ induced by the Jacobian evaluated at $(\overline{\mathbf{u}}_D, \overline{\mathbf{v}}_D)$. Subsequently, let us take $L^p(\Omega)^{m+k}$ as the underlying function space to get an impression of the method in use. Using perturbation theory, the operator $\mathbf{L}_D$ can be defined by

$$\mathbf{L}_D = \mathbf{H}_p + \mathbf{J}_D : \mathcal{D}(\mathbf{H}_p) \subset L^p(\Omega)^{m+k} \to L^p(\Omega)^{m+k} \tag{5.10}$$

for each $1 \leq p < \infty$. The operator $\mathbf{L}_D$ is still unbounded, closed, linear, and densely defined [23, Chapter III, Theorem 1.3]. It even generates an analytic semigroup by [23, Chapter III, Proposition 1.12] and the spectral mapping theorem is valid for $1 < p < \infty$ [23, Chapter IV, Corollary 3.12].

Similar to the shadow case, the spectrum $\sigma(\mathbf{L}_D)$ differs in its structure depending on zero entries of $\mathbf{D}^u \in \mathbb{R}_{\geq 0}^{m\times m}$. Let us begin with the case of no zero entries.

Proposition 5.11. *Let $\mathbf{D}^u \in \mathbb{R}_{>0}^{m\times m}$ and let $\mathbf{L}_D$ be the diffusive operator defined in (5.10) on $\mathcal{D}(H_p)^{m+k} \subset L^p(\Omega)^{m+k}$ for a bounded coefficient matrix $\mathbf{J}_D$ and a finite $1 < p < \infty$. Then $\sigma(\mathbf{L}_D)$ is a discrete set of eigenvalues of $\mathbf{L}_D$. The same holds for the particular case $p = 1 = n$.*

Proof. Let $\mathbf{D}^u \in \mathbb{R}_{>0}^{m\times m}$, then it follows in a straight forward manner from [50, Chapter III, Theorem 6.29] that $\sigma(\mathbf{L}_D)$ is a discrete set of eigenvalues. To see this, take some real number $\lambda > \|\mathbf{J}_D\|_{L^\infty(\Omega)^{m\times m}}$ and consider the resolvent $(\lambda I - \mathbf{L}_D)^{-1}$ which is the solution operator of the elliptic problem

$$(\lambda I - \mathbf{L}_D)\xi = \psi \in L^p(\Omega)^{m+k}$$

endowed with zero flux boundary conditions. Existence and uniqueness for large λ follows from Lemma A.3. By Lemma B.5, a weak solution $\xi \in W^{1,p}(\Omega)^{m+k}$ is an element of $\mathcal{D}(\mathbf{H}_p) \subset W^{1,p}(\Omega)^{m+k}$. The compact embedding $W^{1,p}(\Omega) \hookrightarrow^c L^p(\Omega)$

from [1, Theorem 6.3] finally implies compactness of the resolvent for $\lambda \in \rho(\mathbf{L}_D)$ [23, Chapter II, Proposition 4.25]. $\square$

Now let us assume that $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$ possesses $\tilde{m}$ zeroes. Rewrite the linear operator $\mathbf{L}_D$ similarly to the shadow case (5.8) in the form

$$\mathbf{L}_D \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} \mathbf{0} \\ \mathbf{D}^+\Delta\xi_2(x) \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{11}(x)\xi_1(x) + \tilde{\mathbf{B}}(x)\xi_2(x) \\ \tilde{\mathbf{C}}(x)\xi_1(x) + \tilde{\mathbf{D}}(x)\xi_2(x) \end{pmatrix} \tag{5.11}$$

where $\xi_1 \in L^p(\Omega)^{\tilde{m}}, \xi_2 \in \mathcal{D}(H_p)^{\ell+k}$, and $\mathbf{D}^+ \in \mathbb{R}^{(\ell+k)\times(\ell+k)}_{>0}$ comprises all positive entries of $\mathbf{D}$. Recall that $\mathbf{A}_{11}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$, and $\tilde{\mathbf{D}}$ are assumed to be bounded matrices. Let us start from the problem

$$(\lambda I - \mathbf{L}_D)\xi = \psi \quad \Leftrightarrow \quad \begin{cases} (\lambda I - \mathbf{A}_{11})\xi_1 - \tilde{\mathbf{B}}\xi_2 = \psi_1, \\ -\tilde{\mathbf{C}}\xi_1 + (\lambda I - \mathbf{D}^+\Delta - \tilde{\mathbf{D}})\xi_2 = \psi_2 \end{cases}$$

for some $\lambda \in \mathbb{C}$. According to [104, Proposition 2.2.14], we denote the multiplication operator induced by $\mathbf{A}_{11}$ on $L^p(\Omega)^{\tilde{m}}$ still by $\mathbf{A}_{11}$. Then the first equation can be solved for $\lambda \in \rho(\mathbf{A}_{11})$ with respect to

$$\xi_1 = (\lambda I - \mathbf{A}_{11})^{-1}(\psi_1 + \tilde{\mathbf{B}}\xi_2).$$

Similarly to the case $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{>0}$ one shows

Lemma 5.12. *Let $\mathbf{D}^u \in \mathbb{R}^{m\times m}_{\geq 0}$ have at least one diagonal entry which is zero and let $\mathbf{D}^+ \in \mathbb{R}^{(\ell+k)\times(\ell+k)}_{>0}$ be the positive part of $\mathbf{D}$ for some $0 \leq \ell < m$. Let $\mathbf{L}_D$ be the partly diffusive operator defined in (5.11) on $L^p(\Omega)^{\tilde{m}} \times \mathcal{D}(H_p)^{\ell+k} \subset L^p(\Omega)^{m+k}$ for bounded coefficient matrices and a finite $1 < p < \infty$. Then*

$$\Sigma_D := \sigma(\mathbf{L}_D) \cap \rho(\mathbf{A}_{11}) \subset \sigma_p(\mathbf{L}_D)$$

is a discrete (probably empty) set of eigenvalues of $\mathbf{L}_D$. Moreover, there holds

$$\sigma(\mathbf{L}_D) \subset \sigma(\mathbf{A}_{11}) \,\dot{\cup}\, \Sigma_D.$$

The same is true for the particular case $p = 1 = n$.

Proof. For $\lambda \in \rho(\mathbf{A}_{11})$ invertibility of $\lambda I - \mathbf{L}_D$ is equivalent to solving

$$(\lambda I - \mathbf{D}^+\Delta - \mathbf{M}(\lambda))\xi_2 = \eta_2 \tag{5.12}$$

with respect to $\xi_2 \in \mathcal{D}(H_p)^{\ell+k}$ for

$$\mathbf{M}(\lambda) = \tilde{\mathbf{D}} + \tilde{\mathbf{C}}(\lambda I - \mathbf{A}_{11})^{-1}\tilde{\mathbf{B}} \qquad \text{and} \qquad \eta_2 = \psi_2 + \tilde{\mathbf{C}}(\lambda I - \mathbf{A}_{11})^{-1}\psi_1.$$

Let us first show that this equation is solvable for some $\lambda \in \rho(\mathbf{A}_{11})$. Since the bounded multiplication operator $\mathbf{A}_{11}$ induces a uniformly continuous semigroup, we obtain a resolvent estimate of $(\lambda I - \mathbf{A}_{11})^{-1}$ for all large λ [23, Chapter II, Theorem 3.8]. Hence, $\mathbf{M}(\lambda)$ is uniformly bounded for all large λ and the last equation (5.12) can be solved with respect to $\xi_2 \in \mathcal{D}(H_p)^{\ell+k}$ as an elliptic problem.
We apply an analytic Fredholm theorem [33, Theorem 4.34] to prove that equation (5.12) is not uniquely solvable with respect to $\xi_2 \in \mathcal{D}(H_p)^{\ell+k}$ only if $\lambda \in \rho(\mathbf{A}_{11})$ is contained in a discrete set Σ_D of eigenvalues of $\mathbf{L}_D$. To do this, let us note that solving problem (5.12) is equivalent to solving equation

$$\left[I + (lI - \mathbf{D}^+\Delta)^{-1}\left((\lambda - l)I - \mathbf{M}(\lambda)\right)\right]\xi_2 = \tilde{\eta}_2$$

for some sufficiently large $l > 0$ where $\tilde{\eta}_2 = (lI - \mathbf{D}^+\Delta)^{-1}\eta_2$. Calculations from Proposition 5.11 apply to show compactness of the resolvent $(lI - \mathbf{D}^+\Delta)^{-1}$. Hence,

$$\mathbf{k}(\lambda) := -(lI - \mathbf{D}^+\Delta)^{-1}\left((\lambda - l)I - \mathbf{M}(\lambda)\right)$$

is compact for each $\lambda \in \rho(\mathbf{A}_{11})$ and $\mathbf{k}$ is analytic in λ since the resolvent mapping of $\mathbf{A}_{11}$ is. An application of the Fredholm theorem to $[I - \mathbf{k}(\lambda)]\xi_2 = \tilde{\eta}_2$ yields the claim $\Sigma_D \subset \sigma_p(\mathbf{L}_D)$ since equation (5.12) is solvable for some $\lambda > 0$ large enough. □

We infer that Σ_D need not to be closed but accumulation points are included in $\sigma(\mathbf{A}_{11})$ by the same argument as for Σ following on Lemma 5.6. Let us remind ourselves of the characterization of the spectrum $\sigma(\mathbf{A}_{11})$ of the multiplication operator in $L^p(\Omega)^{\tilde{m}}$ in Proposition C.1. The following result is the analogon of Proposition 5.7 and is inspired by the partly diffusive case considered in [71, Sections 4.3, 4.4]. The latter work concerns a system of two equations with $\tilde{m} = 1$. The subsequent proposition generalizes the results from [71] to systems with compartments of arbitrary size and to less regular boundaries $\partial\Omega \in C^{0,1}$. Along the same lines of the proof of Proposition 5.7 we can show

Proposition 5.13. *Let $\mathbf{A}_{11}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}$ be matrix-valued functions which have entries in $L^\infty(\Omega)$ according to the linear operator $\mathbf{L}_D$ defined by* (5.11) *on $L^p(\Omega)^{\tilde{m}+(\ell+k)}$ for some $1 < p < \infty$. Then, using the same notation for the multiplication operator* *induced by* $\mathbf{A}_{11}$ *on* $L^p(\Omega)^{\tilde{m}}$*, we have*

$$\sigma(\mathbf{L}_D) = \sigma(\mathbf{A}_{11}) \,\dot{\cup}\, \Sigma_D$$

where $\Sigma_D \subset \sigma_p(\mathbf{L}_D)$ *is the discrete (possibly empty) set defined in Lemma 5.12. The same is true for the particular case* $p = 1 = n$*.*

Proof. From the above considerations we already know the relation

$$\rho(\mathbf{A}_{11}) \cap \Sigma_D \subset \sigma(\mathbf{L}_D) \subset \sigma(\mathbf{A}_{11}) \cup \Sigma_D.$$

It is sufficient to show $\sigma_{\text{ess}}(\mathbf{A}_{11}) \subset \sigma(\mathbf{L}_D)$ since from Proposition C.1 we infer $\sigma(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{A}_{11})$. We apply [5, Theorem 2.2] to show $\sigma_{\text{ess}}(\mathbf{A}_{11}) = \sigma_{\text{ess}}(\mathbf{L}_D)$. For an application of [5, Theorem 2.2] we have to resort above operators. Using the notation of their paper, we take the bounded multiplication operator $D := \mathbf{A}_{11}$ in $X_2 := L^p(\Omega)^{\tilde{m}}$ and A is given by the $\ell + k$ equations of operator (5.11) induced by the reaction-diffusion terms on $X_1 := L^p(\Omega)^{\ell+k}$. In view of Proposition 5.11 and previous discussions, A is a densely defined, closed operator with non-empty resolvent set and compact resolvent. The remaining operators B, C in notation of [5] consist of bounded multiplication operators. Note that in assumption (e) of their paper $S(\mu)$ for $\mu \in \rho(A)$ is given by

$$S(\mu) = D - C(A - \mu I)^{-1}B,$$

where we can choose $S_0 = D$ and compactness of $M(\mu) = S(\mu) - S_0$ holds by [50, Chapter III, Theorem 4.8]. Then $\sigma_{\text{ess}}(\mathbf{L}_D) = \sigma_{\text{ess}}(S_0) = \sigma_{\text{ess}}(\mathbf{A}_{11})$ is a consequence of [5, Theorem 2.2] since $\mathbf{L}_D$ is a closed operator. $\square$

Following discussions after [35, Proposition 4.4], there holds $\sigma_p(\mathbf{A}_{11}) \not\subset \sigma_p(\mathbf{L}_D)$ in general. Moreover, Σ_D in the above splitting of the spectrum might not contain all eigenvalues of $\mathbf{L}_D$ since $\sigma_{\text{ess}}(\mathbf{L}_D)$ possibly also contains eigenvalues.

Let us conclude this chapter with some examples discussing stability of steady states of some particular diffusive problems. Returning to Example 3.6, we will see that, similar to the shadow case, instability from the linearization $\mathbf{L}_D$ is inherited from the unstable subsystem with $a - \lambda_j D^u > 0$.

Example 5.14. Steady states of the linear system in Example 3.6 satisfy

$$
\begin{aligned}
-D^u\Delta\overline{u}_D &= a\overline{u}_D + b\overline{v}_D \quad \text{in} \quad \Omega, & \frac{\partial \overline{u}_D}{\partial \mathbf{n}} &= 0 \quad \text{on} \quad \partial\Omega,\\
-D\Delta\overline{v}_D &= c\overline{u}_D + d\overline{v}_D \quad \text{in} \quad \Omega, & \frac{\partial \overline{v}_D}{\partial \mathbf{n}} &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
$$

The trivial steady state is $\mathbf{0}$ and integration over Ω yields $\langle\overline{u}_D\rangle_\Omega = \langle\overline{v}_D\rangle_\Omega = 0$ for any other stationary solution, since $\mathbf{M}$ is invertible with $\det(\mathbf{M}) > 0$ and $\operatorname{tr}(\mathbf{M}) < 0$. There exists a non-trivial steady state if and only if $0 \in \sigma_p(\mathbf{L}_D)$ and it remains to compute $\sigma(\mathbf{L}_D)$.

If $D^u = 0$, Proposition 5.13 implies $\sigma(\mathbf{L}_D) = \{a\} \cup \Sigma_D$ where $\lambda \in \Sigma_D$ is given by those $\lambda \neq a$ for which equation (5.12) is not uniquely solvable. The latter is the case if and only if $(\lambda - d - c(\lambda - a)^{-1}b) = -D\lambda_j$ for some $j \in \mathbb{N}_0$. These values are actually all eigenvalues of the shifted matrix $\mathbf{M}_{D,j} = \mathbf{M} - \lambda_j\mathbf{D}$. We have $\operatorname{tr}(\mathbf{M}_{D,j}) < 0$ for all $D > 0, j \in \mathbb{N}_0$ but

$$
\det(\mathbf{M}_{D,j}) = \det(\mathbf{M}) - a\lambda_j D
$$

changes its sign for $a > 0$ and all $j \in \mathbb{N}$ for growing D. Apart from $a > 0$, additional instabilities arise due to some $\lambda \in \Sigma_D$ with positive real part. Since $a > 0$ and $\det(\mathbf{M}) > 0$, we conclude that $0 \in \sigma_p(\mathbf{L}_D)$ if and only if $0 \in \sigma(\mathbf{M}_{D,j})$ for some $j \in \mathbb{N}$. The last condition is satisfied if and only if $0 = \det(\mathbf{M}_{D,j})$. Hence, non-trivial steady states exist of the form $(\overline{u}_{D,j}, \overline{v}_{D,j}) = w_j\mathbf{z}_{D,j}$ for several small diffusion D with some eigenvector $\mathbf{z}_{D,j}$ of the matrix $\mathbf{M}_{D,j}$. They disappear, however, for growing D and $\mathbf{0}$ is the only steady state of the diffusive system. From $a > 0$ and

$$
\sigma(\mathbf{L}_D) = \{a\} \cup \Sigma_D = \{a\} \cup \bigcup_{j\in\mathbb{N}_0} \sigma(\mathbf{M} - \lambda_j\mathbf{D})
$$

it is clear that all steady states are unstable if $D^u = 0$.

If $D^u > 0$, we know from Proposition 5.11 that $\sigma(\mathbf{L}_D) = \sigma_p(\mathbf{L}_D)$ and we search for possible eigenfunctions. For each eigenvector $\mathbf{z}_{D,j}$ of the shifted matrix $\mathbf{M}_{D,j}$ we can choose an eigenfunction $\xi = \mathbf{z}_{D,j}w_j$ for the operator $\mathbf{L}_D$ where we used the eigenfunction w_j of $-\Delta$ corresponding to λ_j. This shows one inclusion of the equality

$$
\bigcup_{j\in\mathbb{N}_0} \sigma(\mathbf{M} - \lambda_j\mathbf{D}) = \sigma_p(\mathbf{L}_D). \tag{5.13}
$$

Indeed, an eigenfunction ξ of $\mathbf{L}_D$ corresponding to eigenvalue $\lambda \in \mathbb{C}$ is a non-trivial weak solution $\xi = (\xi_1, \xi_2) \in \mathcal{D}(H_p)^2$ of

$$(\lambda I - \mathbf{L}_D)\xi = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} (\lambda - D^u\Delta - a)\xi_1 - b\xi_2 = 0, \\ -c\xi_1 + (\lambda - D\Delta - d)\xi_2 = 0. \end{cases}$$

Testing with eigenfunctions $w_j \in W^{1,q}(\Omega)$ for some $j \in \mathbb{N}_0$ in each component yields

$$(\lambda - \mathbf{M}_{D,j})\overline{\xi}_j = \mathbf{0} \qquad \text{for} \qquad \overline{\xi}_j = \begin{pmatrix} \int_\Omega \xi_1 w_j \, \mathrm{d}x \\ \int_\Omega \xi_2 w_j \, \mathrm{d}x \end{pmatrix}$$

where we used the weak formulation of w_j from Proposition A.1, see Corollary A.2 for regularity. Since $(w_j)_{j\in\mathbb{N}_0}$ forms a spectral basis of $L^2(\Omega)$, $\overline{\xi}_j$ cannot vanish for all j if $p \geq 2$. Thus, there is some index j for which $\overline{\xi}_j \neq \mathbf{0}$ is an eigenvector to $\mathbf{M}_{D,j}$. If $p < 2$, density of the span of eigenfunctions in $L^2(\Omega) \subset L^p(\Omega)$ carries over to $L^p(\Omega)$ and the same argument applies. Using identity (5.13), the spectrum on $\mathbf{L}_D$ can be studied via the matrices $\mathbf{M}_{D,j}$. It is not difficult to show $\operatorname{tr}(\mathbf{M}_{D,j}) < 0$ provided $\operatorname{tr}(\mathbf{M}) < 0$ and

$$\det(\mathbf{M}_{D,j}) = \det(\mathbf{M}) - d\lambda_j D^u + \lambda_j D(\lambda_j D^u - a).$$

There are again non-trivial steady states for several values of diffusions (D^u, D) if and only if $\det(\mathbf{M}_{D,j}) = 0$, which are stable since $\operatorname{tr}(\mathbf{M}_{D,j}) < 0$. As $D \to \infty$, almost all eigenvalues stabilize due to $\lambda_j D^u - a > 0$ but some indices satisfy $\det(\mathbf{M}_{D,j}) < 0$ and we have instability (recall $a > \lambda_{j_0} D^u$ for some $j_0 \in \mathbb{N}$ in Example 3.6). Also if $\lambda_j D^u = a$ for some $j \in \mathbb{N}$, then $\det(\mathbf{M}_{D,j}) = -bc \neq 0$. In any case, $\mathbf{0}$ is the only remaining steady state as $D \to \infty$, which is unstable.

The example above also applies to results obtained in [106, Theorem 4.2] for general linear diffusive systems with constant coefficients. We refer to the literature given in [106] for further investigations concerning diffusion-driven instability. Let us briefly discuss similarity of the shadow spectrum $\sigma(\mathbf{L})$ with its diffusive counterpart $\sigma(\mathbf{L}_D)$ for the zero solution faced with in Example 5.14.

- Let $D^u = 0$. Stability of the shadow solution via $\operatorname{Re}(\lambda) < 0$ for all spectral values $\lambda \in \sigma(\mathbf{L})$ implies stability of the diffusive solution by [106, Theorem

4.2] using the inclusion

$$\{a\} \cup \sigma(\mathbf{M}) = \sigma(\mathbf{L}) \subset \sigma(\mathbf{L}_D) = \{a\} \cup \bigcup_{j \in \mathbb{N}_0} \sigma(\mathbf{M} - \lambda_j \mathbf{D}).$$

Clearly, instability of the shadow solution via $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \sigma(\mathbf{L})$ yields instability for the diffusive system.

- Let $D^u > 0$. Stability of the shadow solution via $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(\mathbf{L})$ implies stability of the diffusive solution by [106, Theorem 4.2],

$$\sigma(\mathbf{L}) = \sigma(\mathbf{M}) \cup \bigcup_{j \in \mathbb{N}} \{a - \lambda_j D^u\} \quad \text{and} \quad \sigma(\mathbf{L}_D) = \bigcup_{j \in \mathbb{N}_0} \sigma(\mathbf{M} - \lambda_j \mathbf{D}).$$

We infer from Example 5.14 that instability via $\mathrm{Re}(\lambda) > 0$ of the shadow solution yields instability for the diffusive system, either due to $\sigma(\mathbf{M}) \subset \sigma(\mathbf{L}_D)$ or $a - \lambda_{j_0} D^u > 0$ for some $j_0 \in \mathbb{N}_0$.

- Let $D^u \geq 0$ and $a \leq 0$. If the shadow solution is stable while some spectral value satisfies $\mathrm{Re}(\lambda) = 0$, stability for the diffusive problem can be shown via resolvent estimates [21, Theorem 1.11]. We refer to [80, Lemma 2.3] for a representation of the resolvent operator of $\mathbf{L}_D$. For instance, Example 4.2 corresponds to this degenerated case.

In the nonlinear case, there are mainly two ways showing instability via linearization. On the one hand, superlinear decay of the nonlinear reaction term in combination with a non-empty intersection of the spectrum with the complex right half-plane is sufficient for instability, see [97, Theorem 1] or [14, Chapter VII, Theorem 2.3] for bounded operators. On the other hand, a growth estimate of the nonlinear reaction term with respect to two different norms and a spectral gap close to the imaginary axis lead to instability, see [26, Theorem 2.1] or [14, Chapter VII, Theorem 2.2] for bounded operators. The latter method is used to provide the instability result [71, Theorem 2.11] which can be generalized to systems and discontinuous, bounded steady states in view of Proposition 5.13. Let us conclude with the diffusive counterparts of Examples 5.9 and 5.10 applying to the latter instability result in [71].

Example 5.15. Non-negative steady states $(\overline{u}_D, \overline{v}_D)$ of the partly diffusive Gray-Scott model

$$\frac{\partial u_D}{\partial t} = -(B + k)u_D + u_D^2 v_D \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad u_D(\cdot, 0) = u^0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial v_D}{\partial t} - D\Delta v_D = B(1-v_D) - u_D^2 v_D \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad v_D(\cdot,0) = v^0 \quad \text{in} \quad \Omega,$$

endowed with zero flux boundary conditions for v_D satisfy $\overline{u}_D\overline{v}_D = B+k$ or $\overline{u}_D = 0$, i.e., $\overline{u}_D = (B+k)\overline{v}_D^{-1}\chi_{\Omega_1}$ for some measurable set $\Omega_1 \subset \Omega$ and $\overline{v}_D$ satisfies a corresponding nonlinear elliptic problem. Uniqueness of the corresponding linear elliptic problem yields $\overline{v}_D = 1$ if $\Omega_1 = \emptyset$, see Lemma A.3. The linearization around the constant steady state $(0,1)$ satisfies

$$\sigma(\mathbf{L}_D) = \{-(B+k)\} \cup \bigcup_{j\in\mathbb{N}_0} \{-B - \lambda_j D\}.$$

Hence, $(0,1)$ is locally asymptotically stable [106, Theorem 4.2]. All other steady states (also non-homogeneous if they exist) are unstable as shown in [71, Section 3.1] via $\sigma(A_{11}) = \{B+k\}$ for $B,k>0$.

The corresponding diffusive system of activator-inhibitor type from Example 5.10 can be treated in a similar way. Additional model examples satisfying $D^u = 0$ can be found in [71, 105] where instability is investigated, too.

Example 5.16. Non-negative, bounded steady states (if they exist apart from constant ones) of the partly diffusive system

$$\begin{aligned} \frac{\partial u_D}{\partial t} &= -u_D + u_D^p v_D^{-q} && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad u_D(\cdot,0) = u^0 \quad \text{in} \quad \Omega, \\ \tau\frac{\partial v_D}{\partial t} - D\Delta v_D &= -v_D + u_D^r v_D^{-s} && \text{in} \quad \Omega \times \mathbb{R}_{>0}, \qquad v_D(\cdot,0) = v^0 \quad \text{in} \quad \Omega \end{aligned}$$

are of the form $(\overline{u}_D, \overline{v}_D) \in L^\infty(\Omega)^2$ with

$$\overline{u}_D(x) = \overline{v}_D^{q/(p-1)}(x)\chi_{\Omega_1}(x)$$

for some measurable (not necessarily connected) subset $\Omega_1 \subset \Omega$. Here, $\overline{v}_D > 0$ satisfies an elliptic problem with zero Neumann boundary conditions. There holds $|\Omega_1| > 0$ because $\overline{u}_D = 0$ and uniqueness of the elliptic problem would imply $\overline{v}_D = 0$, see Lemma A.3, which yields no reasonable steady state of the above system.
Similar to Example 5.10, linearization yields $\sigma(A_{11}) = \{-1, p-1\}$. Since $p>1$ and elements of the discrete set Σ_D possibly accumulate only in $\sigma(A_{11})$, this implies a spectral gap of $\sigma(\mathbf{L}_D)$ near the imaginary axis and [71, Theorem 4.8] applies to show instability of these steady states [71, Theorem 2.11]. Recall that even the constant steady state $(1,1)$ is unstable no matter what relation holds between the parameters

$q, r, \tau > 0$ and $s \geq 0$; although there are stable constellations for the ODE case. For the more complex case $D^u > 0$, we refer for instance to the survey [109, Section 5.4] in which non-homogeneous steady states are constructed and their stability is investigated. Actually, there exist stable, so called spike patterns in contrast to the case of $D^u = 0$ discussed here.

6 Model examples

There is a huge amount of applications arising from diverse fields of natural sciences, see e.g. [27, 60, 82, 83, 94], subject to sufficiently large diffusion components. Since the classical shadow approximation is well studied [32, 53, 78, 79, 86, 109], this chapter is mainly devoted to reaction-diffusion-ODE models. I refer to Examples 5.9 and 5.10 based on models of Gray-Scott type [29] and Gierer-Meinhardt type [28]. These examples as well as Model 6.1 below include classical shadow problems and reaction-diffusion-ODE type problems. The following model examples in this chapter rather illustrate various dynamics of solutions of both the shadow limit and the diffusive problem than show differences to classical shadow systems (see Introduction). The aim of this chapter is to exemplify the results of this thesis in showing accuracy of the shadow approximation on different time scales for distinct models.

Before studying each model in detail, let us briefly depict the general approach to applications of the above main theorems 4.10 and 5.2. Model assumptions A1–A4 are not difficult to check. Usually nonlinearities are continuously differentiable with respect to the solution variables $(\mathbf{u}, \mathbf{v})$, hence they satisfy the local Lipschitz condition A1 on page 12. Boundedness of initial conditions stated in Assumption A2 on page 12 depends on the considered model. The differentiability assumption A4 on page 52 is satisfied by a nonlinearity which is of class C^2 with respect to the unknown solution and which possesses uniformly bounded gradients. Note that in many cases the nonlinearities are time-independent, hence the decay estimate (4.13) for the mean value correction in Assumption A3 on page 44 is trivially satisfied. Assumption B on page 51 concerning global existence and uniform boundedness of the shadow limit is more delicate and has to be checked for each model in particular. As can be seen in the following examples, it is often useful to consider the corresponding differential equation for the masses $(\langle \mathbf{u} \rangle_\Omega, \mathbf{v})$ in order to show some properties of the shadow limit $(\mathbf{u}, \mathbf{v})$ itself. There are several techniques to show global existence and uniform bounds of solutions of which I just mention a few: almost linear growth of nonlinearities, feedback arguments, maximum principles and

invariant rectangles for ordinary differential equations as well as reaction-diffusion equations, see [12] and the books [14, Chapter VII] or [94].
Concerning linearization, it is quite standard to check Assumption L0 on the ODE subsystem using Bohl exponents as in [14, Chapter III, §3]. A Bohl exponent determines the exponential growth of the evolution operators from which we can derive stability of the evolution system. This concept is a generalization of the growth bound of a semigroup to time-dependent evolution systems. Alternatively, to show Assumption L0 stated on page 49, one can use the dissipativity condition Dp on page 68 in its equivalent form given in Lemma 4.14. Nevertheless, Assumption L1p stated on page 43 is crucial for an application of Theorem 4.10 resp. Theorem 5.2 and an analytical verification is not easy. Stability of Bohl exponents for evolution systems under time-dependent perturbations, which tend to zero as $t \to \infty$, often allows reducing considerations to some known dynamics, e.g., stationary or periodic structures. In such a situation, the linearized evolution equation is asymptotically comparable to the linearized equation of the stationary pattern. This stability property is well known for bounded operators [14, Corollary 4.2] but is also applicable for perturbed unbounded operators, see [16, Theorem 5], [96, Corollary 4.2] or [40, §7.1].

Now let us turn to three model examples to see the methodology described above in more detail. The following models have been selected to present various dynamics of solutions of both the shadow limit and the diffusive problem. At first, a Lotka-Volterra-type system from ecology is considered which has only constant steady states [82]. While this model features global convergence results, the following models show that accuracy of the shadow approximation highly depends on the time scale. The well-known Lengyel-Epstein model from [60] exhibiting various patterns demonstrates convergence results on different time scales depending on the pattern. Finally, following [27], a system modeling stem-cell dynamics highlights the discrepancy between the diffusive system and its shadow limit for asymptotic time scales. Although the shadow is a valuable approximation for intermediate times according to Theorem 4.10, the dynamics of the two problems differs drastically as time $t \to \infty$.

6.1 Predator-prey model

Consider a closed system describing predator-prey dynamics, with a predator denoted by u_D and a mobile prey v_D. In fresh-water ecology, a biological example can be given by Hydra and Daphnia where the predator Hydra is sedentary, i.e., $D^u = 0$ [82, Example (b)]. The following model adapted from [82] includes both cases $D^u = 0$ and $D^u > 0$. The corresponding (partly) diffusive system reads

$$\frac{\partial u_D}{\partial t} - D^u \Delta u_D = -pu_D + bv_D \quad \text{in} \quad \Omega_T, \qquad u_D(\cdot, 0) = u^0 \quad \text{in} \quad \Omega, \tag{6.1}$$

$$\frac{\partial v_D}{\partial t} - D \Delta v_D = (d - au_D - cv_D)v_D \quad \text{in} \quad \Omega_T, \qquad v_D(\cdot, 0) = v^0 \quad \text{in} \quad \Omega, \tag{6.2}$$

$$\frac{\partial u_D}{\partial \mathbf{n}} = 0, \qquad \frac{\partial v_D}{\partial \mathbf{n}} = 0 \qquad \text{on} \quad \partial\Omega \times (0, T). \tag{6.3}$$

Here, $a, b, c, d, p > 0$ are constants and the initial values $u^0, v^0 \geq 0$ satisfy Assumption A2 as well as non-negativity almost everywhere in Ω with $\langle v^0 \rangle_\Omega > 0$. The corresponding shadow limit is given by

$$\frac{\partial u}{\partial t} - D^u \Delta u = -pu + bv \quad \text{in} \quad \Omega_T, \qquad u(\cdot, 0) = u^0 \quad \text{in} \quad \Omega, \tag{6.4}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (d - a\langle u \rangle_\Omega - cv)v \quad \text{in} \quad (0, T), \qquad v(0) = \langle v^0 \rangle_\Omega \tag{6.5}$$

where u is endowed with a zero flux boundary condition if $D^u > 0$. If we integrate the mild solution u of differential equation (6.4) over Ω, compare Proposition 2.6 and Proposition B.3, we obtain an ODE system for the masses $(\langle u \rangle_\Omega, v)$. This system admits the global attractor $(\overline{u}, \overline{v})$ where

$$\overline{u} = \frac{dp}{cp + ab} \qquad \text{and} \qquad b\overline{v} = p\overline{u}.$$

Convergence to the equilibrium is a consequence of the radially unbounded Lyapunov functional

$$L(\langle u \rangle_\Omega, v) = \frac{a}{2}(\langle u \rangle_\Omega - \overline{u})^2 + b(v - \overline{v} - \overline{v}\log(v/\overline{v}))$$

adapted from [82], where L is dissipative on trajectories, i.e.,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -ap(\langle u \rangle_\Omega - \overline{u})^2 - bc(v - \overline{v})^2 \leq 0.$$

Thus, we obtain the asymptotics $(\langle u \rangle_\Omega, v) \to (\overline{u}, \overline{v})$ as $t \to \infty$ as well as $u \to \overline{u}$, since

$$u(\cdot, t) - \langle u \rangle_\Omega(t) = (S^u(t)u^0 - \langle u^0 \rangle_\Omega)\mathrm{e}^{-pt} \to 0 \qquad \text{for} \quad t \to \infty.$$

Hence, all assumptions A1–A4, B are satisfied. For an application of Theorem 5.2, it remains to compute the linearization

$$\mathbf{J}(x,t) = \begin{pmatrix} -p & b \\ -av(t) & d - 2cv(t) - au(x,t) \end{pmatrix}$$

around the shadow limit (u, v). The corresponding shadow evolution system $\mathcal{W}$ defined in (4.39) is induced by the operator

$$\begin{aligned} &\mathbf{D}^S\Delta + \mathbf{L}_0(t) : L^p(\Omega) \times \mathbb{R} \to L^p(\Omega) \times \mathbb{R}, \\ &\mathbf{L}_0(t)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} -p\xi_1(x) + b\xi_2 \\ -av(t)\langle \xi_1 \rangle_\Omega + (d - 2cv(t) - a\langle u(\cdot,t)\rangle_\Omega)\xi_2 \end{pmatrix} \end{aligned}$$

where $\mathbf{D}^S = \operatorname{diag}(D^u, 0) \in \mathbb{R}^{2\times 2}_{\geq 0}$ is a diagonal matrix.

Lemma 6.1. *Let (u, v) be a shadow solution of system (6.4)–(6.5) for bounded initial conditions $u^0, v^0 \geq 0$ satisfying $\langle v^0 \rangle_\Omega > 0$. Then assumptions L0 and L1p are satisfied for $p = \infty$. Moreover, the corresponding evolution system is uniformly exponentially stable for the exponent $\eta = p > 0$ and some $\sigma > 0$, respectively.*

Proof. Assumption L0 is satisfied since $(U(t))_{t\in\mathbb{R}_{\geq 0}}$ with $U(t) = S^u(t)\mathrm{e}^{-pt}$ is uniformly exponentially stable with exponent $\eta = p > 0$. The semigroup $(S^u(t))_{t\in\mathbb{R}_{\geq 0}}$ generated by $D^u\Delta$ on $L^2(\Omega)$ is given by $S^u(t) = S_\Delta(D^u t)$, see the definition in (2.5). Concerning the evolution system $\mathcal{W}$ defined for Assumption L1p on page 56, let us split the shadow operators $\mathbf{L}_0(t) = \mathbf{L}_\infty + \mathbf{B}(t)$ for operator matrices

$$\begin{aligned} &\mathbf{L}_\infty, \mathbf{B}(t) : L^p(\Omega) \times \mathbb{R} \to L^p(\Omega) \times \mathbb{R}, \\ &\mathbf{L}_\infty \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} -p\xi_1(x) + b\xi_2 \\ -a\overline{v}\langle \xi_1 \rangle_\Omega - c\overline{v}\xi_2 \end{pmatrix} =: \begin{pmatrix} A_{11}\xi_1(x) + B_*\xi_2 \\ C_*\langle \xi_1 \rangle_\Omega + D_*\xi_2 \end{pmatrix}, \\ &\mathbf{B}(t)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} 0 \\ -a(v(t) - \overline{v})\langle \xi_1 \rangle_\Omega + \left[-2c(v(t) - \overline{v}) - a(\langle u(\cdot,t)\rangle_\Omega - \overline{u})\right]\xi_2 \end{pmatrix}. \end{aligned}$$

Since $\lim_{t\to\infty} \mathbf{B}(t) = \mathbf{0}$ with respect to the operator norm on $L^\infty(\Omega) \times \mathbb{R}$, evolution systems induced by $\mathbf{D}^S\Delta + \mathbf{L}_0(t)$ and $\mathbf{D}^S\Delta + \mathbf{L}_\infty$ are asymptotically comparable. We will show that it remains to consider the latter semigroup for exponential stability of the former evolution system. To recognize this, we start from the definition of $\mathcal{W}$ in condition L1p. This evolution system is given by evolution operators $\mathbf{W}(t, s)$ for

$t, s \in \mathbb{R}_{\geq 0}, s \leq t$, defined by

$$\xi(\cdot, t) = \mathbf{W}(t, s)\xi(\cdot, s), \qquad \xi(\cdot, 0) = \xi^0 \in L^p(\Omega) \times \mathbb{R},$$

where $\xi \in C(\mathbb{R}_{\geq 0}; L^p(\Omega)^m \times \mathbb{R}^k)$ is the unique solution of the shadow problem

$$\frac{\partial \xi}{\partial t} - \mathbf{D}^S \Delta \xi = \mathbf{L}_0(t)\xi = \mathbf{L}_\infty \xi + \mathbf{B}(t)\xi \quad \text{in} \quad \Omega \times \mathbb{R}_{>0}$$

endowed with zero Neumann boundary conditions for ξ_1 if necessary. We split the full operator into a time-independent, possibly unbounded part $\mathbf{L} = \mathbf{D}^S\Delta + \mathbf{L}_\infty$ and the bounded time-varying operator family $(\mathbf{B}(t))_{t \in \mathbb{R}_{\geq 0}}$. We are able to compare both evolution systems, the system $\mathcal{W}_\infty$ induced by a semigroup $(\mathbf{W}_\infty(t))_{t \in \mathbb{R}_{\geq 0}}$ which is generated by the operator $\mathbf{L}$ and the full evolution system $\mathcal{W}$, using the integral representation

$$\mathbf{W}(t, s)\xi^0 = \mathbf{W}_\infty(t - s)\xi^0 + \int_s^t \mathbf{W}_\infty(t - \tau)\mathbf{B}(\tau)\mathbf{W}(\tau, s)\xi^0 \, \mathrm{d}\tau \qquad \forall\, 0 \leq s \leq t$$

from [23, Chapter VI, Theorem 9.19]. Once we have shown uniform exponential stability for $(\mathbf{W}_\infty(t))_{t \in \mathbb{R}_{\geq 0}}$, estimations in $L^p(\Omega) \times \mathbb{R}$ for $1 \leq p \leq \infty$ yield

$$\begin{aligned}\|\mathbf{W}(t, s)\xi^0\|_{L^p(\Omega)\times\mathbb{R}} \leq{}& C\mathrm{e}^{-\sigma_\infty(t-s)}\|\xi^0\|_{L^p(\Omega)\times\mathbb{R}} \\ &+ \int_s^t C\mathrm{e}^{-\sigma_\infty(t-\tau)}\|\mathbf{B}(\tau)\|_{L^\infty(\Omega)\times\mathbb{R}}\|\mathbf{W}(\tau, s)\xi^0\|_{L^p(\Omega)\times\mathbb{R}} \, \mathrm{d}\tau.\end{aligned}$$

Gronwall's inequality results in the estimate

$$\|\mathbf{W}(t, s)\xi^0\|_{L^p(\Omega)\times\mathbb{R}} \leq C\mathrm{e}^{-\sigma_\infty(t-s)} \exp\left(\int_s^t C\|\mathbf{B}(\tau)\|_{L^\infty(\Omega)\times\mathbb{R}} \, \mathrm{d}\tau\right) \|\xi^0\|_{L^p(\Omega)\times\mathbb{R}}.$$

Although the theory of Bohl exponents was established for bounded operators in [14, Chapter III, pp. 118], the same estimates used to prove [14, Corollary 4.2] apply to the above estimate in the context of semigroup theory, see further [96, Corollary 4.2], [16, Theorem 5] or [40, §7.1, p. 195]. More precisely, since $\lim_{t\to\infty} \mathbf{B}(t) = \mathbf{0}$, for each $\gamma \in (0, 1)$ there is a $t_0 > 0$ such that $C\|B(t)\|_{L^\infty(\Omega)\times\mathbb{R}} \leq \gamma\sigma_\infty$ for all $t \geq t_0$. This implies the following estimate

$$\|\mathbf{W}(t, s)\xi^0\|_{L^p(\Omega)\times\mathbb{R}} \leq \tilde{C}\mathrm{e}^{-\gamma\sigma_\infty(t-s)}\|\xi^0\|_{L^p(\Omega)\times\mathbb{R}}$$

for $\tilde{C} = C \exp(\int_0^{t_0} C\|B(\tau)\|_{L^\infty(\Omega)\times\mathbb{R}} \,\mathrm{d}\tau)$. Hence, uniform exponential stability carries over from the evolution system $\mathcal{W}_\infty$ to the full evolution system $\mathcal{W}$ on $L^p(\Omega) \times \mathbb{R}$, provided $\lim_{t\to\infty} \mathbf{B}(t) = \mathbf{0}$.
Using the spectral mapping theorem [23, Chapter IV, Corollary 3.12] for analytical semigroups, it is well known that uniform exponential stability of the semigroup $(\mathbf{W}_\infty(t))_{t\in\mathbb{R}_{\geq 0}}$ can be verified via the spectrum of its generator $\mathbf{L} = \mathbf{D}^S\Delta + \mathbf{L}_\infty$. It remains to show $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(\mathbf{D}^S\Delta + \mathbf{L}_\infty)$ for uniform exponential stability of the evolution system $\mathcal{W}_\infty$ resp. $\mathcal{W}$ [23, Chapter V, Theorem 1.10]. We infer from Proposition 5.7 that in case of $D^u = 0$

$$\sigma(\mathbf{L}) = \sigma(\mathbf{L}_\infty) = \{A_{11}\} \cup \Sigma,$$

where Σ consists of all eigenvalues of the constant coefficient matrix

$$\mathbf{J}_\infty = \begin{pmatrix} A_{11} & B_* \\ C_* & D_* \end{pmatrix}.$$

Note that $A_{11} = -p < 0$ and both eigenvalues of $\mathbf{J}_\infty$ have negative real parts since $\mathrm{tr}(\mathbf{J}_\infty) = -p - c\overline{v} < 0$ and $\det(\mathbf{J}_\infty) = (pc + ab)\overline{v} > 0$.
From Proposition 5.5 we know that $\sigma(\mathbf{D}^S\Delta + \mathbf{L}_\infty)$ is a discrete set for $D^u > 0$ and one could follow the strategy of [80, Appendix]. However, since the evolution of the linear shadow limit is quite simple, we apply a different approach. The semigroup $(\mathbf{W}_\infty(t))_{t\in\mathbb{R}_{\geq 0}}$ is defined by the solution $\xi = (w, z)$ of $\partial_t\xi - \mathbf{D}^S\Delta\xi = \mathbf{L}_\infty\xi$. While $w(\cdot, t) - \langle w\rangle_\Omega(t) = (S^u(t)w^0 - \langle w^0\rangle_\Omega)\mathrm{e}^{-pt}$, integration yields

$$\langle\xi\rangle_\Omega = \begin{pmatrix} \langle w\rangle_\Omega \\ z \end{pmatrix}(t) = \exp(\mathbf{J}_\infty t)\begin{pmatrix} \langle w^0\rangle_\Omega \\ \langle z^0\rangle_\Omega \end{pmatrix}.$$

It is well-known from the theory of ODEs that $\langle\xi\rangle_\Omega$ decays exponentially to zero since $\mathbf{J}_\infty$ is a stable matrix [14, Chapter I, Theorem 4.1]. Choosing $\sigma \in \mathbb{R}_{>0}$ such that $\sigma < \min\{p, \min_{\lambda\in\sigma(\mathbf{J}_\infty)} |\mathrm{Re}\lambda|\}$ yields an estimation of both expressions. This results in

$$\|\mathbf{W}_\infty(t)\xi^0\|_{L^\infty(\Omega)\times\mathbb{R}} = \|\xi(\cdot, t)\|_{L^\infty(\Omega)\times\mathbb{R}} \leq C_\sigma \mathrm{e}^{-\sigma t}\|\xi^0\|_{L^\infty(\Omega)\times\mathbb{R}}$$

for some $C_\sigma > 0$. Thus, Assumption L1p is satisfied for $p = \infty$ and each $D^u \geq 0$ with a uniformly exponentially stable evolution system $\mathcal{W}$. □

In summary, Theorem 5.2 yields global estimates

$$\|u_D - u\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})} + \|v_D - v - \psi_D\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})} \leq CD^{-1}.$$

Note that the results on Lyapunov functions in [39, Proposition 2.1] is also applicable to partly diffusing systems. The same Lyapunov function which is known from the theory of ODEs can be extended to the reaction-diffusion case. Consequently, $(\overline{u}, \overline{v})$ is the only positive attractor for the diffusive system (6.1)–(6.3) and $(\overline{u}, \overline{v})$ is globally (for positive initial data) asymptotically stable by Lyapunov's direct method.

6.2 Lengyel-Epstein model

Consider the partially diffusive Lengyel-Epstein model in [36, Section 5.7.2] originated from [60]. For constants $a, b > 0$, it is given by the following system

$$\frac{\partial u_D}{\partial t} = a - \left(1 + \frac{4v_D}{1+u_D^2}\right)u_D \quad \text{in} \quad \Omega_T, \quad u_D(\cdot,0) = u^0 \quad \text{in} \quad \Omega, \tag{6.6}$$

$$\frac{\partial v_D}{\partial t} - D\Delta v_D = b\left(1 - \frac{v_D}{1+u_D^2}\right)u_D \quad \text{in} \quad \Omega_T, \quad v_D(\cdot,0) = v^0 \quad \text{in} \quad \Omega, \tag{6.7}$$

$$\frac{\partial v_D}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{6.8}$$

Those equations originally model the Chlorite-Iodide-Malonic Acid (CIMA) reaction in [60], wherein u_D describes an activator which is set to be immobile in system (6.6)–(6.8) and v_D is a diffusing inhibitor. The corresponding shadow limit reduction of equations (6.6)–(6.8) yields the following system of integro-differential equations

$$\frac{\partial u}{\partial t} = a - \left(1 + \frac{4v}{1+u^2}\right)u \quad \text{in} \quad \Omega_T, \qquad u(\cdot,0) = u^0 \quad \text{in} \quad \Omega, \tag{6.9}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = b\left\langle\left(1 - \frac{v}{1+u^2}\right)u\right\rangle_\Omega \quad \text{in} \quad (0,T), \qquad v(0) = \langle v^0\rangle_\Omega. \tag{6.10}$$

Fundamental properties of the shadow solution are summarized in the following proposition.

Proposition 6.2. *Let $u^0, v^0 \geq 0$ a.e. in Ω satisfy Assumption A2. Then there is a unique solution $(u,v) \in C^1(\mathbb{R}_{\geq 0}; L^\infty(\Omega) \times \mathbb{R})$ of system (6.9)–(6.10) which is uniformly bounded and each component is non-negative. Moreover, assumptions A1–A4 and B are fulfilled.*

Proof. Existence and uniqueness of a local-in-time mild solution is provided by Proposition 2.6. For non-negative initial data $u^0, v^0 \geq 0$ the shadow limit component u is non-negative due to

$$u(\cdot,t) = \hat{U}(\cdot,t)u^0 + \int_0^t \hat{U}(\cdot,t)\hat{U}^{-1}(\cdot,\tau)a \, \mathrm{d}\tau$$

for

$$\hat{U}(\cdot,t) = \exp\left(-\int_0^t 1 + \frac{4v(\tau)}{1+u^2(\cdot,\tau)} \, \mathrm{d}\tau\right).$$

Similarly, $v \geq 0$ since

$$v(t) = V(t)\langle v^0\rangle_\Omega + \int_0^t V(t)V^{-1}(\tau)b\langle u(\cdot,\tau)\rangle_\Omega \, \mathrm{d}\tau$$

for

$$V(t) = \exp\left(-b\int_0^t \left\langle \frac{u(\cdot,\tau)}{1+u^2(\cdot,\tau)}\right\rangle_\Omega \mathrm{d}\tau\right) \leq 1.$$

The first component u is bounded uniformly by $\max\{a, \|u^0\|_{L^\infty(\Omega)}\}$ due to exponential decay of $(\hat{U}(x,t))_{t\in\mathbb{R}_{\geq 0}}$. If v were unbounded, i.e., $v(t) \geq 1 + \|u\|^2_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})}$ for some $t \geq t_0$, then v would decrease and we obtain boundedness, to the contrary. Since the nonlinearities in system (6.9)–(6.10) are of class C^2 with respect to (u, v), also their derivatives evaluated at the bounded shadow limit are uniformly bounded. In summary, assumptions A1–A4 and B are fulfilled. □

To check stability assumption L1p, we consider the non-symmetric Jacobian

$$\mathbf{J}(x,t) = \begin{pmatrix} -1 - 4v(t)\frac{1-u^2(x,t)}{(1+u^2(x,t))^2} & -4\frac{u(x,t)}{1+u^2(x,t)} \\ b - bv(t)\frac{1-u^2(x,t)}{(1+u^2(x,t))^2} & -b\frac{u(x,t)}{1+u^2(x,t)} \end{pmatrix} = \begin{pmatrix} A_*(x,t) & B_*(x,t) \\ C_*(x,t) & D_*(x,t) \end{pmatrix} \quad (6.11)$$

evaluated at the shadow limit (u, v) with shadow operator

$$\mathbf{L}_0(t) : L^p(\Omega)\times\mathbb{R} \to L^p(\Omega)\times\mathbb{R},$$
$$\mathbf{L}_0(t)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}(x) = \begin{pmatrix} A_*(x,t)\xi_1(x) + B_*(x,t)\xi_2 \\ \langle C_*(\cdot,t)\xi_1\rangle_\Omega + \langle D_*(\cdot,t)\xi_2\rangle_\Omega \end{pmatrix}.$$

Let us consider the following two examples for an application of Theorems 4.10 and 5.2. A shadow solution close to the homogeneous stationary solution and a shadow limit close to a non-homogeneous stationary solution of problem (6.9)–(6.10).

Example 6.3. Let $(u, v) \to (\overline{u}, \overline{v})$ converge uniformly in $L^\infty(\Omega)$ as $t \to \infty$, where the unique constant steady state is given by $\overline{u} = a/5$ and $\overline{v} = 1 + \overline{u}^2$. Decompose

$$\mathbf{L}_0(t) = \mathbf{L}_\infty + (\mathbf{L}_0(t) - \mathbf{L}_\infty) \quad \text{with} \quad \lim_{t\to\infty} \|\mathbf{L}_0(t) - \mathbf{L}_\infty\|_{\mathcal{L}(L^\infty(\Omega)\times\mathbb{R}))} = 0$$

for the steady state shadow operator

$$\begin{aligned} &\mathbf{L}_\infty : L^p(\Omega) \times \mathbb{R} \to L^p(\Omega) \times \mathbb{R}, \\ &\mathbf{L}_\infty \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \frac{1}{\overline{v}} \begin{pmatrix} (3\overline{u}^2 - 5)\xi_1(x) - 4\overline{u}\xi_2 \\ 2b\overline{u}^2\langle\xi_1\rangle_\Omega - b\overline{u}\xi_2 \end{pmatrix} =: \begin{pmatrix} \overline{A}\xi_1(x) + \overline{B}\xi_2 \\ \overline{C}\langle\xi_1\rangle_\Omega + \overline{D}\xi_2 \end{pmatrix}. \end{aligned}$$

We infer from [14, Corollary 4.2] that uniform exponential stability of the full evolution system $\mathcal{W}$ induced by $\mathbf{L}_0$ on $L^\infty(\Omega) \times \mathbb{R}$ is inherited from the semigroup $(\exp(\mathbf{L}_\infty t))_{t\in\mathbb{R}_{\geq 0}}$ generated by the shadow operator $\mathbf{L}_\infty$ at the steady state. By virtue of the spectral mapping theorem [23, Chapter I, Lemma 3.13] for uniformly continuous semigroups, it remains to check the spectrum of the shadow operator $\mathbf{L}_\infty$ at the steady state. We conclude from Proposition 5.7 that

$$\sigma(\mathbf{L}_\infty) = \{\overline{A}\} \cup \Sigma = \{\overline{A}\} \cup \sigma(\mathbf{J}_\infty)$$

where the Jacobian $\mathbf{J}_\infty$ at the steady state is given by

$$\mathbf{J}_\infty = \frac{1}{\overline{v}} \begin{pmatrix} 3\overline{u}^2 - 5 & -4\overline{u} \\ 2b\overline{u}^2 & -b\overline{u} \end{pmatrix}.$$

Since $\det(\mathbf{J}_\infty) = ab/\overline{v} > 0$, it remains to check $\overline{A}, \operatorname{tr}(\mathbf{J}_\infty) < 0$ for exponential stability of the semigroup generated by $\mathbf{L}_\infty$. From the calculations

$$\overline{A} < 0 \quad \Leftrightarrow \quad 3\overline{u}^2 < 5 \quad \Leftrightarrow \quad a^2 < \frac{125}{3}$$

we infer (also in the case of $\overline{A} \leq 0$) that

$$\operatorname{tr}(\mathbf{J}_\infty) = \overline{v}^{-1}[(3\overline{u}^2 - 5) - b\overline{u}] \leq -b\overline{v}^{-1}\overline{u} < 0.$$

This shows that Assumption L1p is satisfied for all $1 \leq p \leq \infty$ with an exponential stability for some $\sigma > 0$ if $a^2 < 125/3$, and Theorem 5.2 applies for estimates.
If however $\overline{A} = 0$, i.e., $3a^2 = 125$, let us consider $(u, v) = (\overline{u}, \overline{v})$, and we can check boundedness of the semigroup using a characterization via the resolvent of $\mathbf{L}_\infty$ [21,

Chapter III, Theorem 1.11]. It is not difficult to invert $\lambda I - \mathbf{L}_\infty$ explicitly for each $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$,

$$\lambda(\lambda I - \mathbf{L}_\infty)^{-1}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(x) = \begin{pmatrix}\psi_1(x) + \frac{\overline{BC}}{\lambda(\lambda-\overline{D})-\overline{BC}}\langle\psi_1\rangle_\Omega + \frac{\lambda\overline{B}}{\lambda(\lambda-\overline{D})-\overline{BC}}\psi_2\\ \frac{\lambda\overline{C}}{\lambda(\lambda-\overline{D})-\overline{BC}}\langle\psi_1\rangle_\Omega + \frac{\lambda^2}{\lambda(\lambda-\overline{D})-\overline{BC}}\psi_2\end{pmatrix}.$$

Using $|\langle\psi_1\rangle_\Omega| \le \max\{1, |\Omega|^{-1}\}\|\psi_1\|_{L^p(\Omega)}$ for each finite $1 \le p < \infty$, we reach at the resolvent estimate $\|(\lambda I - \mathbf{L}_\infty)^{-2}\| \le C|\lambda|^{-2}$ in $L^p(\Omega)\times\mathbb{R}$ where C is independent of $p < \infty$. Then condition (b) of Theorem 1.11 in [21, Chapter III] is satisfied and its proof yields uniform boundedness of the semigroup with a bound independent of p for all large $p < \infty$. Continuity of the L^p norm yields the same bound in $L^\infty(\Omega)\times\mathbb{R}$ ([1, Theorem 2.14] applies since $|\Omega| < \infty$). Hence, Theorem 4.10 is applicable for $a^2 \le 125/3$ and $p = \infty$ where $(u, v) \equiv (\overline{u}, \overline{v})$ is constant.

The next example shows that even discontinuous, non-monotone patterns of the Lengyel-Epstein model (6.6)–(6.8) can be approximated globally by its discontinuous shadow limit.

Example 6.4. Consider space-dependent steady states $(\overline{u}(x), \overline{v})$ of the shadow system (6.9)–(6.10). Since $\overline{v}$ is constant, the first equation yields

$$(a - \overline{u})(1 + \overline{u}^2) = 4\overline{uv}$$

with at most three constant solutions $0 < \overline{u}_i = \overline{u}_i(\overline{v}) < a$ depending on $\overline{v} > 0$. Thus, a steady state may be written in the form $(\overline{u}, \overline{v}) \in L^\infty(\Omega)\times\mathbb{R}$ where

$$\overline{u}(x) = \sum_{i=1}^{3}\overline{u}_i\chi_{\Omega_i}(x) \tag{6.12}$$

is a step function for some measurable, disjoint sets $\Omega_i \subset \Omega$ with $|\Omega_i| \ge 0$. The second equation (6.10) yields a relation between sets Ω_i and solutions $\overline{u}_i$:

$$\bigcup_{i=1}^{3}\Omega_i = \Omega, \qquad \sum_{i=1}^{3}|\Omega_i|(5\overline{u}_i - a) = 0 \tag{6.13}$$

Notice that the sets Ω_i are not necessarily connected and $\overline{u}$ may have infinitely many jumps in Ω, hence it is not monotone in general. Such a construction is done for instance in [36, Section 5.4].

We check the spectrum of the linearized shadow operator $\mathbf{L}$ which depends on $\overline{v} > 0$

and the model parameters $a, b > 0$. It consists of

$$\sigma(A_{11}) = \bigcup_{i=1}^{3} \left\{ -1 - \frac{4\overline{v}(1-\overline{u}_i^2)}{(1+\overline{u}_i^2)^2} \right\}$$

and the discrete set Σ which is given by the (at most four) solutions $\lambda \in \rho(A_{11})$ of the complex equation

$$H(\lambda) = \lambda - \langle D_* \rangle_\Omega - \langle C_*(\lambda - A_{11})^{-1} B_* \rangle_\Omega = 0. \tag{6.14}$$

The space-dependent coefficients A_{11}, B_*, C_*, D_* result from the linearization (6.11) around the steady state $(\overline{u}, \overline{v})$. Certainly, there are a lot of unstable patterns. However, let us give an example inspired from [36, Section 5.7.2] to which our convergence results apply.

Take the parameter value $a = 6$ and consider $4\overline{v} = 10$, then there exist three constant solutions $\overline{u}_i = i$ for $i = 1, 2, 3$ of the first equation (6.9). It is easy to compute the Jacobians $\mathbf{J}_i$ evaluated at $(\overline{u}_i, \overline{v})$ for each case:

$$\mathbf{J}_1 = \begin{pmatrix} -1 & -2 \\ b & -0.5b \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} 0.2 & -1.6 \\ 1.3b & -0.4b \end{pmatrix}, \quad \mathbf{J}_3 = \begin{pmatrix} -0.2 & -1.2 \\ 1.2b & -0.3b \end{pmatrix}$$

We infer that a combination of only two values $\overline{u}_1, \overline{u}_3$ by (6.12) and (6.13) is successful with $|\Omega_1| = 9|\Omega_3| > 0$ and $\Omega_2 = \emptyset$. Since the Jacobian around $(\overline{u}, \overline{v})$ is given by

$$\mathbf{J}(x) = \mathbf{J}_1 \chi_{\Omega_1}(x) + \mathbf{J}_3 \chi_{\Omega_3}(x),$$

we immediately see that $\sigma(A_{11}) = \{-1, -0.2\} \subset \mathbb{R}_{<0}$ does not induce any instability, and it remains to check Σ for an application of Theorem 5.2 (Note that a combination including $\overline{u}_2$ implies instability). We will verify

$$\Sigma \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \leq -\sigma\} =: S_\sigma$$

for some $\sigma > 0$ following the proof of [36, Corollary 3.10]. To do so, it remains to show that the complex number $H(\lambda)$ defined in equation (6.14) is non-zero for each $\lambda \in \mathbb{C} \setminus S_\sigma$. From the above specific form of the Jacobian $\mathbf{J}$ we infer that

$$A_{11} = -0.2 - 0.8\chi_{\Omega_1}, \quad B_* = -1.2 - 0.8\chi_{\Omega_1},$$

$$C_* = b + 0.2b\chi_{\Omega_3}, \quad \langle D_* \rangle_\Omega = -4.8b|\Omega_3|/|\Omega|.$$

Consider the real part of $H(\lambda)$, then for $\lambda_1 := \operatorname{Re}\lambda$ and $B_*C_* < 0$ in Ω

$$\operatorname{Re} H(\lambda) = \lambda_1 - \langle D_* \rangle_\Omega - \left\langle \frac{B_*C_*(\lambda_1 - A_{11})}{|\lambda - A_{11}|^2} \right\rangle_\Omega > 0$$

for $\lambda_1 > -\sigma$ where $\sigma := \min\{\inf_{x\in\Omega} |A_{11}(x)|, |\langle D_* \rangle_\Omega|\} > 0$. In the latter estimate we used the fact that $\langle D_* \rangle_\Omega, A_{11}, B_*C_*$ are bounded from above by a negative number since $b > 0$. Considering the steady state $(\overline{u}, \overline{v})$, or shadow limits (u, v) converging uniformly to the steady state, Theorem 5.2 applies and we obtain global convergence

$$\|u_D - u\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})} + \|v_D - v - \psi_D\|_{L^\infty(\Omega\times\mathbb{R}_{\geq 0})} \leq CD^{-1}.$$

Remark that the shadow solution $(\overline{u}, \overline{v})$ is no steady state of the diffusive problem (6.6)–(6.8) although we have a global estimate. Concerning the diffusive problem, I want to mention a result of the authors of [71] due to private communication which has not yet been published: They found a method to show existence of steady states close to the above non-degenerated constant steady states $(\overline{u}_i, \overline{v})$ in a suitable sense. Moreover, their work allows to study stability properties of the constructed steady states with spectral methods similar to those used in this thesis, see Section 5.2.

Recall that, in case of classical shadow limits with another fixed diffusion in equation (6.6), all non-monotone, stationary patterns of Example 6.4 are unstable, compare [87, Theorem 4.1] or [84] for $\Omega = (0, 1) \subset \mathbb{R}$.

6.3 Stem cell model

Consider a model of stem cell dynamics in [27] consisting of two compartments denoted by stem cells u_D and mature cells v_D. The self-renewal rate $s(v_D)$ which depends on v_D is given here by a diffusing component w_D. We refer to [10] for a different infinite-dimensional extension of the model of ordinary differential equations in [27] and for further literature. Let us consider the system

$$\frac{\partial u_D}{\partial t} = (2aw_D - 1)\, pu_D \quad \text{in} \quad \Omega_T, \quad u_D(\cdot, 0) = u^0 \quad \text{in} \quad \Omega, \tag{6.15}$$

$$\frac{\partial v_D}{\partial t} = 2\,(1 - aw_D)\, pu_D - dv_D \quad \text{in} \quad \Omega_T, \quad v_D(\cdot, 0) = v^0 \quad \text{in} \quad \Omega, \tag{6.16}$$

$$\frac{\partial w_D}{\partial t} - D\Delta w_D = 1 - kv_D w_D - w_D \qquad \text{in} \quad \Omega_T, \;\; w_D(\cdot,0) = w^0 \quad \text{in} \quad \Omega, \tag{6.17}$$

$$\frac{\partial w_D}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{6.18}$$

for constant parameters $a, k, d, p > 0$ with $a < 1$. The corresponding shadow limit of equations (6.15)–(6.18) is given by the following system of integro-differential equations

$$\frac{\partial u}{\partial t} = (2aw - 1)\, pu \qquad \text{in} \quad \Omega_T, \qquad u(\cdot,0) = u^0 \quad \text{in} \quad \Omega, \tag{6.19}$$

$$\frac{\partial v}{\partial t} = 2\,(1 - aw)\, pu - dv \qquad \text{in} \quad \Omega_T, \qquad v(\cdot,0) = v^0 \quad \text{in} \quad \Omega, \tag{6.20}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 1 - k\langle v\rangle_\Omega w - w \qquad \text{in} \quad (0,T), \qquad w(0) = \langle w^0\rangle_\Omega. \tag{6.21}$$

Fundamental properties of the shadow solution are summarized in the following result.

Proposition 6.5. *Let $u^0, v^0, w^0 \geq 0$ a.e. in Ω satisfy Assumption A2 and let $\langle w^0\rangle_\Omega < 1/a$. Then there is a unique solution $(u,v,w) \in C^1(\mathbb{R}_{\geq 0}; L^\infty(\Omega)^2 \times \mathbb{R})$ of the shadow system* (6.19)*–*(6.21) *which is uniformly bounded and each component is non-negative. Hence, assumptions A1–A4 and B are satisfied.*

Proof. Proposition 2.6 yields local-in-time solutions for which regularity results from the following reformulations of the implicit integral equations (2.6)–(2.7). In view of ordinary differential equation (6.19), u is given by the formula

$$u(\cdot,t) = u^0 \exp\left(p \int_0^t 2aw(\tau) - 1 \,\mathrm{d}\tau\right) \geq 0. \tag{6.22}$$

Continuity of the local solution w implies differentiability of u. Similarly, w is positive since

$$w(t) = h(t)\langle w^0\rangle_\Omega + \int_0^t h(t)h^{-1}(s)\,\mathrm{d}s \tag{6.23}$$

for the exponential function

$$h(t) := \exp\left(-\int_0^t 1 + k\langle v\rangle_\Omega(\tau)\,\mathrm{d}\tau\right).$$

In order to show non-negativity of v, let us assume $v^0 > 0$ a.e. in Ω. The general case $v^0 \geq 0$ can be shown by approximation from above with $v^{0,\varepsilon} := v^0 + \varepsilon > 0$ and the fact that problem (6.19)–(6.21) depends continuously on its initial condition. Consider the function $z := u/v \geq 0$ which is well-defined locally in time. The quotient z satisfies the generalized logistic equation

$$\frac{\partial z(\cdot,t)}{\partial t} = z(\cdot,t)\left[(d + (2aw(t)-1)p) - 2p(1-aw(t))z(\cdot,t)\right] =: z[a(t) - b(t)z],$$

for some $a, b \in C^1([0,T])$, some $T > 0$ and $z(\cdot,0) \geq 0$. Clearly, if $u^0(x) = 0$ for some $x \in \Omega$, then $u(x,t) = 0$ for all times. Integrating equation (6.20) yields

$$v(x,t) = \mathrm{e}^{-d(t-t_0)}v(x,t_0) + \int_{t_0}^{t} \mathrm{e}^{-d(t-\tau)}b(\tau)u(x,\tau)\,\mathrm{d}\tau \geq 0. \tag{6.24}$$

Thus, to show non-negativity of v, it remains to consider $x \in \Omega$ for which $z(x,0) > 0$. On the one hand, we conclude $z \geq 0$ from representation (6.24) if $b \geq 0$. On the other hand, if b changes its sign and becomes negative at some $t_0 \geq 0$, we infer

$$z(\cdot,t) = A(t,t_0)z(\cdot,t_0) - \int_{t_0}^{t} A(t,\tau)b(\tau)z^2(\cdot,\tau)\,\mathrm{d}\tau \geq 0$$

with evolution operators $A(t,s) = \exp\left(\int_s^t a(\tau)\,\mathrm{d}\tau\right)$. If b changes again its sign and becomes positive at some $t_0 \geq 0$, we make use of formula (6.24). Hence, we obtain $z \geq 0$ in any case.

In order to verify Assumption B, let us first note that w is uniformly bounded by $\max\{1, \langle w^0\rangle_\Omega\}$ due to $v \geq 0$ and the representation (6.23). This implies global existence since u and v do not blow-up in finite time $T > 0$. The latter follows from an estimation of equations (6.22) and (6.24), respectively. To show uniform boundedness of the shadow limit, let us integrate the shadow limit equations to obtain a system of ordinary differential equations for the masses $(\langle u\rangle_\Omega, \langle v\rangle_\Omega, w)$

$$\frac{\mathrm{d}\langle u\rangle_\Omega}{\mathrm{d}t} = (2aw-1)\,p\langle u\rangle_\Omega \quad \text{in} \quad (0,T), \quad \langle u\rangle_\Omega(0) = \langle u^0\rangle_\Omega, \tag{6.25}$$

$$\frac{\mathrm{d}\langle v\rangle_\Omega}{\mathrm{d}t} = 2\,(1-aw)\,p\langle u\rangle_\Omega - d\langle v\rangle_\Omega \quad \text{in} \quad (0,T), \quad \langle v\rangle_\Omega(0) = \langle v^0\rangle_\Omega, \tag{6.26}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 1 - k\langle v\rangle_\Omega w - w \quad \text{in} \quad (0,T), \quad w(0) = \langle w^0\rangle_\Omega. \tag{6.27}$$

We first show uniform boundedness of the solution to system (6.25)–(6.27) using techniques of ordinary differential equations. We will show that the uniform bound on w does imply a uniform bound for $\langle v\rangle_\Omega$. If $\langle v\rangle_\Omega$ were unbounded as $t \to \infty$, we would have

$$\langle v(\cdot, t)\rangle_\Omega \geq \frac{4a-1}{k} \qquad \forall\, t \geq t^*$$

for some $t^* \geq 0$. Using the equation (6.23) for w, this implies

$$w(t) \leq \mathrm{e}^{-4at}\langle w^0\rangle_\Omega + \int_0^t \mathrm{e}^{-4a(t-\tau)}\,\mathrm{d}\tau \leq \frac{1}{2a}$$

for all sufficiently large t. Consequently, $\langle u\rangle_\Omega$ is uniformly bounded by formula (6.22). This, in turn, would imply uniform boundedness of $\langle v\rangle_\Omega$ by equation (6.24), what yields a contradiction. Hence, w and $\langle v\rangle_\Omega$ are uniformly bounded. Next, we make use of the assumption $\langle w^0\rangle_\Omega < 1/a$ to show uniform boundedness of $\langle u\rangle_\Omega$. Let us assume $\langle v^0\rangle_\Omega > 0$ for the moment. Applying the same idea as above, let us consider the logistic equation for the quotient $\overline{z} = \langle u\rangle_\Omega/\langle v\rangle_\Omega \geq 0$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{z} = \overline{z}(t)[a(t) - b(t)\overline{z}(t)].$$

Due to the above assumptions and $w \leq \max\{\langle w^0\rangle_\Omega, 1\}$, we have $b(t) \geq b_0 > 0$ as well as $a(t) \leq a_0 < \infty$ for all $t \in \mathbb{R}_{\geq 0}$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{z} \leq \overline{z}(t)[a_0 - b_0\overline{z}(t)].$$

A standard comparison principle for ordinary differential equations yields

$$\frac{\langle u\rangle_\Omega}{\langle v\rangle_\Omega} = \overline{z} \leq \max\left\{\frac{a_0}{b_0}, \frac{\langle u^0\rangle_\Omega}{\langle v^0\rangle_\Omega}\right\}$$

and $\langle u\rangle_\Omega$ is uniformly bounded, too. If $\langle v^0\rangle_\Omega = 0$, formula (6.24) shows that v is either identical to zero if $u \equiv 0$ or is positive on (t_0, ∞) if u becomes positive at $t_0 \geq 0$. Hence, the above method applies to the quotient $\overline{z}$ on $[t_0 + 1, \infty)$ and shows boundedness of $\langle u\rangle_\Omega$.

The latter result implies uniform boundedness of the shadow component u since the function u has the same growth factor as $\langle u\rangle_\Omega$, see representation (6.22). An estimation of v given by formula (6.24) implies validity of Assumption B. Since the smooth nonlinearities do not depend explicitly on space or time, assumptions A1–A4 are fulfilled. □

Let us consider a steady state $(\overline{u}, \overline{v}, \overline{w})$ in $L^\infty(\Omega)^2 \times \mathbb{R}$ of the shadow system for an application of Theorem 4.10. Stationary solutions of both the diffusive system and the shadow system are specified in the next result.

Proposition 6.6. *The diffusive system* (6.15)–(6.18) *admits only constant steady states in* $L^\infty(\Omega)^3$ *given by*

$$S^0 = (0,0,1) \qquad \text{and} \qquad S^* = (u^*, v^*, w^*) = \left(\frac{d}{p}\frac{2a-1}{k}, \frac{2a-1}{k}, \frac{1}{2a}\right).$$

The shadow problem (6.19)–(6.21) *possesses the constant steady states* S^0 *and* S^* *as well as non-homogeneous steady states*

$$S = (\overline{u}, \overline{v}, \overline{w}) \in L^\infty(\Omega)^2 \times \mathbb{R}_{>0} \qquad \text{with} \quad p\overline{u} = d\overline{v},$$

where spatial means $\langle \overline{u} \rangle_\Omega = u^*$, $\langle \overline{v} \rangle_\Omega = v^*$, *and* $\overline{w} = w^*$ *are prescribed.*

Proof. From the first two steady state equations of system (6.15)–(6.18) we obtain $p\overline{u}_D = d\overline{v}_D$ and inserting $2a\overline{v}_D\overline{w}_D = \overline{v}_D$ in the elliptic equation for $\overline{w}_D$ yields

$$-D\Delta(\overline{w}_D - 1) + (\overline{w}_D - 1) = -\frac{k}{2a}\overline{v}_D \in L^\infty(\Omega). \tag{6.28}$$

A solution $\overline{w}_D - 1$ resp. $\overline{w}_D$ of problem (6.28) endowed with zero flux boundary conditions is an element of $W^{1,2}(\Omega)$. As the right-hand side is in $L^p(\Omega)$ for $p > n$, this solution is necessarily continuous as a bootstrap argument similar to the proof of Corollary A.2 shows. Consider the measurable sets

$$\Omega_0 := \{x \in \Omega \mid \overline{w}_D(x) \neq 1/(2a)\} \qquad \text{and} \qquad \Omega_1 = \Omega \setminus \Omega_0,$$

of which Ω_0 is open and $\Omega_1 \subset \Omega$ is compact in $\mathbb{R}^n$ since $\overline{w}_D$ is continuous. We infer from the first two equations (6.15)–(6.16) that $\overline{u}_D = 0 = \overline{v}_D$ in Ω_0. Consequently, testing the above elliptic problem (6.28) with $\varphi \in C_c^\infty(\Omega_0)$ yields the following weak Dirichlet problem

$$\int_{\Omega_0} D\nabla(\overline{w}_D - 1)\nabla\varphi + (\overline{w}_D - 1)\varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^\infty(\Omega_0).$$

An application of the Theorem of Lax-Milgram from [8, Theorem 9.21] on $W_0^{1,2}(\Omega_0)$ yields $\overline{w}_D \equiv 1$ on Ω_0. Note that we need no boundary regularity or connectivity of Ω_0 for Poincaré's inequality [8, Corollary 9.19]. Since $\overline{w}_D$ is a continuous function,

which is identical 1 on Ω_0 and $1/(2a)$ on the complement Ω_1, we have shown that either $|\Omega_0| = 0$ or $|\Omega_1| = 0$, which results in either S^0 or S^*.
Concerning bounded steady states $(\overline{u}, \overline{v}, \overline{w}) \in L^\infty(\Omega)^2 \times \mathbb{R}$ of the shadow problem (6.19)–(6.21), the analysis is easier since $\overline{w}$ is a real number. The case $\overline{w} \neq 1/(2a)$ implies the steady state S^0 as above. In case of $\overline{w} = 1/(2a)$ only spatial mean values of $\overline{u}$ and $\overline{v}$ are prescribed, having $p\overline{u} = d\overline{v}$ in mind. □

Let us consider a shadow limit reduction around stationary shadow solutions. For an application of Theorem 4.10, we compute the linearization of the shadow problem to verify Assumption L0 and L1p stated on page 49 and page 56, respectively. Linearization of the system (6.19)–(6.21) at the semitrivial point $S^0 = (0, 0, 1)$ of the shadow system yields the following Jacobian

$$\mathbf{J}(0,0,1) = \begin{pmatrix} 2a-1 & 0 & 0 \\ 2(1-a)p & -d & 0 \\ 0 & -k & -1 \end{pmatrix}.$$

We do not consider this semitrivial steady state in much detail but we mention that, due to [14, Chapter VII, Theorem 2.3], S^0 is unstable for the shadow system for parameters $2a > 1$, since

$$\sigma(\mathbf{L}) = \sigma_p(\mathbf{L}) = \{2a-1, -d, -1\}.$$

If $2a < 1$, we have local asymptotic stability by [14, Chapter VII, Theorem 2.1].
Considering non-trivial steady states given by S (including S^*), we focus on positive masses for $2a > 1$, a condition which is usually satisfied in applications. To verify Assumption L1p, we consider the non-symmetric Jacobian

$$\mathbf{J}(\overline{u}(x), \overline{v}(x), \overline{w}) = \begin{pmatrix} 0 & 0 & 2ap\overline{u}(x) \\ p & -d & -2ap\overline{u}(x) \\ 0 & -kw^* & -(1+k\overline{v}(x)) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{B}_*(x) \\ \mathbf{C}_* & D_*(x) \end{pmatrix}$$

evaluated at the shadow limit $S = (\overline{u}, \overline{v}, \overline{w})$ with shadow operator

$$\mathbf{L} : L^\infty(\Omega)^2 \times \mathbb{R} \to L^\infty(\Omega)^2 \times \mathbb{R},$$
$$\mathbf{L}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}(x) = \begin{pmatrix} \mathbf{A}_{11}\xi_1(x) + \mathbf{B}_*(x)\xi_2 \\ \mathbf{C}_*\langle \xi_1 \rangle_\Omega + \langle D_* \rangle_\Omega \xi_2 \end{pmatrix}.$$

Clearly, $\langle D_* \rangle_\Omega = -2a$ and $\sigma(\mathbf{A}_{11}) = \{-d, 0\}$ by Proposition C.1. Since in the definition of $\mathbf{L}$ only $\mathbf{B}_*$ is space-dependent, the discrete set Σ from Proposition 5.7 actually is the set of eigenvalues of the constant matrix

$$\mathbf{J}(u^*, v^*, w^*) = \begin{pmatrix} 0 & 0 & 2apu^* \\ p & -d & -2apu^* \\ 0 & -kw^* & -2a \end{pmatrix}.$$

Using the Routh-Hurwitz criterion for stability, we infer that all $\lambda \in \Sigma$ satisfying

$$\lambda^3 + (d + 2a)\lambda^2 + d\lambda + p^2 k u^* = 0$$

have negative real parts if and only if

$$d(d + 2a) - p^2 k u^* > 0 \qquad \Leftrightarrow \qquad (d + 2a) > p(2a - 1). \tag{6.29}$$

Since $a < 1$, the last inequality is satisfied if for instance $p \leq d+2$. Due to $0 \in \sigma(\mathbf{L})$, an immediate application of Theorem 4.10 is not obvious in the case of assumption (6.29). To show boundedness of the semigroup generated by the shadow operator $\mathbf{L}$, we will apply [21, Chapter III, Theorem 1.11]. Inverting $\lambda I - \mathbf{L}$ explicitly yields

$$(\lambda I - \mathbf{L})^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x) = \begin{pmatrix} \xi_1(x) \\ \xi_2 \end{pmatrix}$$

with components

$$\begin{aligned} \xi_1(x) &= (\lambda I - \mathbf{A}_{11})^{-1}(\psi_1(x) + \mathbf{B}_*(x)\xi_2), \\ \xi_2 &= \frac{\psi_2 + \mathbf{C}_*(\lambda I - \mathbf{A}_{11})^{-1}\langle \psi_1 \rangle_\Omega}{\lambda - \langle D_* \rangle_\Omega - \mathbf{C}_*(\lambda I - \mathbf{A}_{11})^{-1}\langle \mathbf{B}_* \rangle_\Omega} \end{aligned}$$

for each $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$. Independently of $1 \leq p < \infty$, we obtain the estimate $|\langle \psi_1 \rangle_\Omega| \leq \max\{1, |\Omega|^{-1}\}\|\psi_1\|_{L^p(\Omega)}$. Since $\lambda(\lambda I - \mathbf{A}_{11})^{-1}$ is uniformly bounded in λ, we reach at the following resolvent estimate in $L^p(\Omega)^2 \times \mathbb{R}$

$$\|(\lambda I - \mathbf{L})^{-2}\| \leq C|\lambda|^{-2}$$

where C is independent of $p < \infty$. Then condition (b) of Theorem 1.11 in [21, Chapter III] is satisfied and its proof yields uniform boundedness of the semigroup with a bound independent of p for all large $p < \infty$. Continuity of the L^p norm implies the same bound in $L^\infty(\Omega)^2 \times \mathbb{R}$, as [1, Theorem 2.14] applies for $|\Omega| < \infty$. Hence, Theorem 4.10 is applicable with $p = \infty$.

Above calculations in combination with the result of Theorem 4.10 show that the shadow approximation around S is in fact useful on intermediate time scales T of the order $T \sim D^\ell$ for some $0 < \ell < 1$, provided stability assumption (6.29) holds. However, the different stationary solutions in Proposition 6.6 already indicate a mismatch of asymptotic behavior between solutions of the diffusive problem (6.15)–(6.18) and the shadow problem (6.19)–(6.21). A stability consideration of the above non-homogeneous steady states of the shadow problem will subsequently confirm this. Let us start from a linearized stability analysis for the system (6.25)–(6.27) of masses provided that a, d, p satisfy inequality (6.29) and $1/2 < a < 1$. The latter conditions imply local asymptotic stability of the constant steady state S^* as a solution of the system (6.25)–(6.27) of ordinary differential equations [14, Chapter VII, Theorem 2.1]. Concerning the shadow system around the steady state S, the standard linearized stability analysis does not apply since $0 \in \sigma(\mathbf{L})$. For this reason nonlinear stability of S for the shadow system has to be considered.

Proposition 6.7. *Let $1/2 < a < 1$ and d, p satisfy inequality* (6.29). *Then steady states S with positive entries and S^* are nonlinearly stable (for non-negative initial data) in $L^\infty(\Omega)^2 \times \mathbb{R}$, but not asymptotically stable. More precisely, the shadow limit from Proposition 6.5 locally converges to*

$$\left(\frac{u^0}{\langle u^0 \rangle_\Omega} u^*, \frac{u^0}{\langle u^0 \rangle_\Omega} v^*, w^* \right)$$

if non-negative initial conditions satisfy $\|(u^0 - \overline{u}, v^0 - \overline{v}, \langle w^0 \rangle_\Omega - \overline{w})\|_{L^\infty(\Omega)^2 \times \mathbb{R}} \le \delta$ for sufficiently small $\delta > 0$.

Proof. Starting close to a steady state with positive entries implies that initial data is non-negative and the assumptions of Proposition 6.5 are fulfilled. The initial conditions for the masses $(\langle u \rangle_\Omega, \langle v \rangle_\Omega, w)$ satisfy the estimate

$$|(\langle u^0 \rangle_\Omega - u^*, \langle v^0 \rangle_\Omega - v^*, \langle w^0 \rangle_\Omega - w^*)| \le \|(u^0 - \overline{u}, v^0 - \overline{v}, \langle w^0 \rangle_\Omega - \overline{w})\|_{L^\infty(\Omega)^2 \times \mathbb{R}} \le \delta.$$

Hence, for sufficiently small $\delta > 0$, the local asymptotic stability of S^* for the system (6.25)–(6.27) of ordinary differential equations implies local convergence

$$\langle u(\cdot, t) \rangle_\Omega \to u^*, \quad \langle v(\cdot, t) \rangle_\Omega \to v^*, \quad w(t) \to w^* \qquad \text{as} \quad t \to \infty.$$

As mentioned above, the growth factor of u and $\langle u\rangle_\Omega$ equals in the sense that

$$u(\cdot,t) = u^0 g(t), \quad \langle u(\cdot,t)\rangle_\Omega = \langle u^0\rangle_\Omega g(t) \qquad \text{for} \qquad g(t) := \exp\left(p\int_0^t 2aw(\tau) - 1 \, \mathrm{d}\tau\right).$$

This implies local convergence in $L^\infty(\Omega)$ of the shadow component u as $t \to \infty$;

$$u(\cdot,t) \to \frac{u^0}{\langle u^0\rangle_\Omega} u^*.$$

Although the spatial mean of this limit equals $\langle \overline{u}\rangle_\Omega = u^*$, the limit differs in general from $\overline{u}$ and no asymptotic stability is possible. We infer local convergence of v from representation (6.24) and the fact that $w(t)$ and $b(t) = 2p(1 - aw(t))$ converge as $t \to \infty$. Actually, we have $w(t) \to w^*$,

$$v(\cdot,t) \to \frac{u^0}{\langle u^0\rangle_\Omega} v^*$$

and the shape of u^0 is inherited by v. In order to show nonlinear stability of the shadow system, we use triangle inequality twice to obtain an estimate for u:

$$\begin{aligned}\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} \le{}& \left\|u(\cdot,t) - \frac{u^0}{\langle u^0\rangle_\Omega} u^*\right\|_{L^\infty(\Omega)} \\ &+ \frac{1}{|\langle u^0\rangle_\Omega|} \left\|-u^0(\langle u^0\rangle_\Omega - u^*) + \langle u^0\rangle_\Omega (u^0 - \overline{u})\right\|_{L^\infty(\Omega)},\end{aligned}$$

while

$$u(\cdot,t) - \frac{u^0}{\langle u^0\rangle_\Omega} u^* = \frac{u^0}{\langle u^0\rangle_\Omega}(\langle u(\cdot,t)\rangle_\Omega - u^*).$$

To estimate v, let us write

$$v(\cdot,t) - \overline{v} = v(\cdot,t) - \frac{p}{d} u(\cdot,t) + \frac{p}{d}(u(\cdot,t) - \overline{u}).$$

The sum $\partial_t(u+v) = -d(u+v) + (d+p)u$ implies

$$\begin{aligned}v(\cdot,t) - \frac{p}{d}u(\cdot,t) ={}& \mathrm{e}^{-dt}\left(v^0 - \frac{p}{d}u^0\right) \\ &+ \left(1 + \frac{p}{d}\right)\left[\mathrm{e}^{-dt}u^0 - u(\cdot,t) + \int_0^t d\mathrm{e}^{-d(t-\tau)} u(\cdot,\tau)\,\mathrm{d}\tau\right] \\ ={}& \mathrm{e}^{-dt}\left(v^0 - \overline{v} - \frac{p}{d}(u^0 - \overline{u})\right) - \left(1 + \frac{p}{d}\right)\int_0^t \mathrm{e}^{-d(t-\tau)}\partial_\tau u(\cdot,\tau)\,\mathrm{d}\tau\end{aligned}$$

owing to partial integration and $p\overline{u} = d\overline{v}$. Since $|\langle u^0\rangle_\Omega| \geq |u^*| - \delta \geq 1/2|u^*|$ for sufficiently small $\delta > 0$, we obtain the estimates

$$\begin{aligned}
\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} &\leq C_{\overline{u}} \left(|\langle u(\cdot,t)\rangle_\Omega - u^*| + |\langle u^0\rangle_\Omega - u^*| + \|u^0 - \overline{u}\|_{L^\infty(\Omega)} \right), \\
\|v(\cdot,t) - \overline{v}\|_{L^\infty(\Omega)} &\leq C_{a,p,d}\Big(\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} + \|u^0 - \overline{u}\|_{L^\infty(\Omega)} + \|v^0 - \overline{v}\|_{L^\infty(\Omega)} \\
&\qquad + \int_0^t \mathrm{e}^{-d(t-\tau)} \left(\|\overline{u}\|_{L^\infty(\Omega)} + \|u(\cdot,\tau) - \overline{u}\|_{L^\infty(\Omega)} \right) |w(\tau) - w^*| \,\mathrm{d}\tau \Big).
\end{aligned}$$

Herein, we used $\partial_t u = 2apu(w - w^*)$ under the integral and accumulated all constants of the system in $C_{a,p,d} > 0$. These estimates together with local asymptotic stability of the masses $(\langle u\rangle_\Omega, \langle v\rangle_\Omega, w)$ imply stability of the shadow system in $L^\infty(\Omega)^2 \times \mathbb{R}$. $\square$

It is remarkable that the initial shape of the shadow limit component u persists as $t \to \infty$. The shape of the component u and is even inherited by the component v modulo some factor. Let us for example choose initial conditions oscillating around the constant steady state S^*:

$$(u^0, v^0, w^0) = (u^* + \delta_1 w_{j_1}, v^* + \delta_2 w_{j_2}, w^* + \delta_3 w_{j_3})$$

for some $\delta_i \in \mathbb{R}, i = 1,2,3$, and eigenfunctions $w_{j_i}, j_i \in \mathbb{N}$, from a spectral basis of $-\Delta$ in Proposition A.1. Since in this case the masses are time-independent and given by S^*, we obtain the shadow solution

$$u \equiv u^0, \quad v(\cdot,t) = \frac{p}{d}u^0 + \mathrm{e}^{-dt}\left(v^0 - \frac{p}{d}u^0\right), \quad w \equiv w^*.$$

The behavior of the shadow limit and its diffusive approximants is illustrated by simulations on the domain $\Omega = (0,1) \subset \mathbb{R}$ using eigenfunctions $w_{j_i}(x) = \sqrt{2}\cos(j_i\pi x)$, $j_i = 4i$, parameter values

$$a = 2/3, p = 1, d = 2, k = 1/30 \tag{6.30}$$

and initial conditions

$$\begin{aligned}
u^0(x) &= 20 + 4\cos(4\pi x), \\
v^0(x) &= 10 + 4\cos(8\pi x), \\
w^0(x) &= 0.75 + 1.5\cos(12\pi x).
\end{aligned} \tag{6.31}$$

Whereas the shadow components u and w are time-independent and given by

$$u(x,t) = u^0(x) = 20 + 4\cos(4\pi x) \qquad \text{and} \qquad w(t) = \langle w^0 \rangle_\Omega = 0.75,$$

the component v converges to $0.5u^0$ as $t \to \infty$ as shown in Figure 6.1. The shadow components are obtained via an implicit Euler scheme of spatial mesh size $h = 10^{-3}$ and temporal mesh size $k = 10^{-3}$. The spatial mean value in equation (6.21) is handled as a mean value of the discretized solution in the latter algorithm.

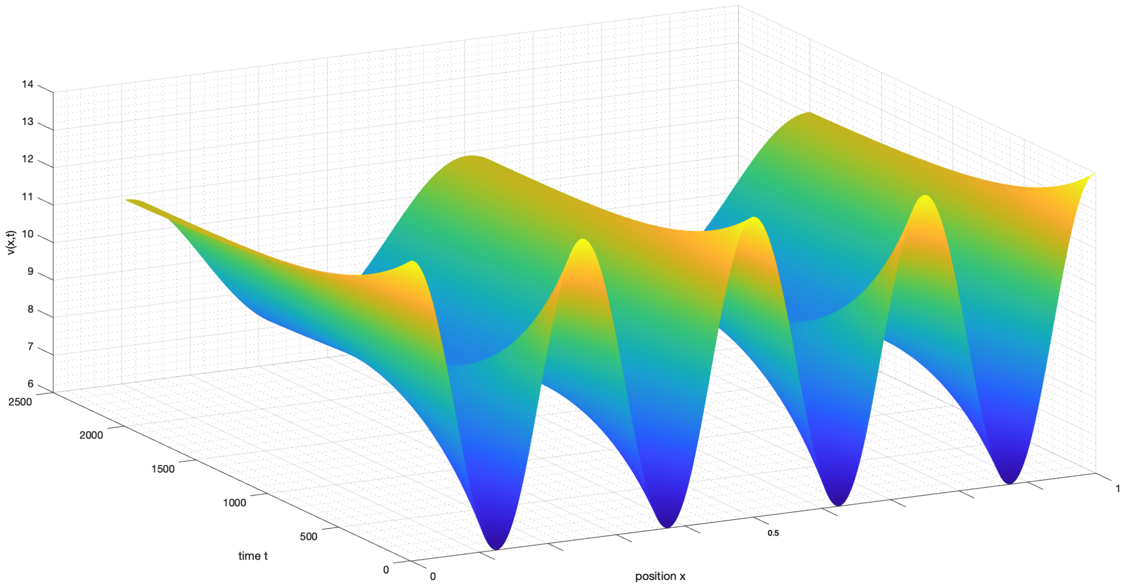

Figure 6.1: Shadow component v of system (6.19)–(6.21) for parameter set (6.30) and initial conditions from (6.31).

In contrast to this behavior of the shadow system, the diffusive problem (6.15)–(6.18) only has constant steady states for each diffusion $D > 0$ as shown in Proposition 6.6. Simulations for the same parameter setting (6.30) and initial conditions (6.31) are shown in Figure 6.2. They are obtained with the `pdepe` solver of MATLAB for a spatial mesh size $h = 4 \cdot 10^{-3}$ and temporal mesh size $k = 1$. These simulations indicate that $(u_D, v_D, w_D) \to S^*$ as $t \to \infty$ for big diffusion parameters D such that no global convergence result as in Theorem 5.2 is possible. Hence, the dynamics of the two problems differ drastically as time tends to infinity, although the shadow limit is a valuable approximation for intermediate times as an application of Theorem 4.10 shows above. Since numerical simulations take place on finite time scales, one might run the risk of misinterpreting the corresponding visualization of the approximate shadow solution. As this example shows, a usage of the shadow limit might yield no information on the asymptotic behavior of the approximated diffusive solution.

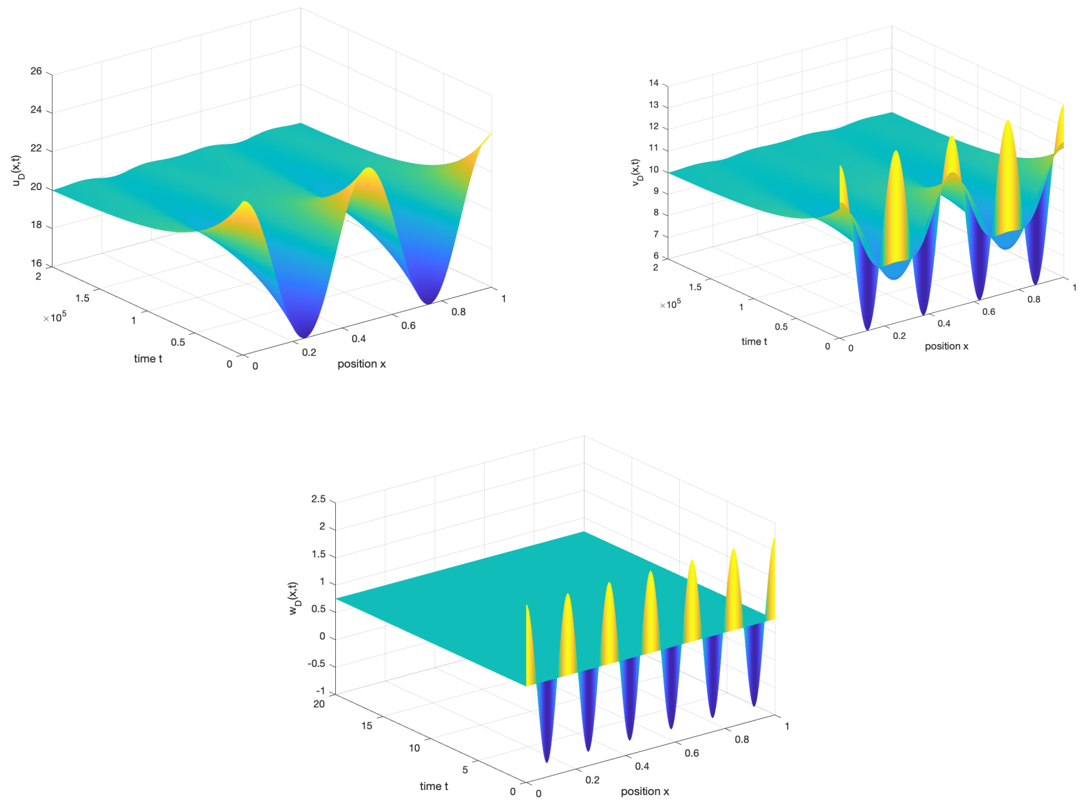

Figure 6.2: Diffusive components (u_D, v_D, w_D) of system (6.15)–(6.18) for parameter set (6.30), initial conditions (6.31) and for a diffusion coefficient $D = 100$. The steady state is given by $S^* = (20, 10, 0.75)$.

Inspired from the behavior of the solution (u_D, v_D, w_D) in simulations similar to Figure 6.2, a natural question concerning stability of the steady state S^* arises. Unfortunately, linearization is not helpful to determine stability of the steady state S^* of the diffusive system (6.15)–(6.18) either. A linearization around S^* yields

$$\mathbf{L}_D = \begin{pmatrix} 0 & 0 & 2apu^* \\ p & -d & -2apu^* \\ 0 & -kw^* & -2a + D\Delta \end{pmatrix}.$$

The subsystem $\mathbf{A}_{11}$ corresponding to the ODE subsystem satisfies $\sigma(\mathbf{A}_{11}) = \{0, -d\}$. Hence, Proposition 5.13 yields $0 \in \sigma(\mathbf{L}_D) = \sigma(\mathbf{A}_{11}) \dot\cup \Sigma_D$. Following the proof of

Lemma 5.12, we infer that $\Sigma_D = \sigma(\mathbf{L}_D) \cap \rho(\mathbf{A}_{11})$ is given by

$$\Sigma_D = \bigcup_{j\in\mathbb{N}_0} \sigma_p(\mathbf{M}_{D,j}) \qquad \text{for} \qquad \mathbf{M}_{D,j} = \begin{pmatrix} 0 & 0 & 2apu^* \\ p & -d & -2apu^* \\ 0 & -kw^* & -2a - D\lambda_j \end{pmatrix},$$

where λ_j are the eigenvalues associated to $-\Delta$ from Proposition A.1. Since $\lambda_0 = 0$, we again assume that stability inequality (6.29) holds. For sufficiently large $D > 0$ and $j \in \mathbb{N}$ the matrices $\mathbf{M}_{D,j}$ have only real, negative roots by Cardano's formula. One of these eigenvalues diverges to $-\infty$ as $j \to \infty$ and, since their product is $-dp(2a-1) < 0$ independent of $\lambda_j D$, one eigenvalue converges to 0 from below as $j \to \infty$. Hence, $0 \in \sigma(\mathbf{L}_D) \setminus \sigma_p(\mathbf{L}_D)$ is an approximate eigenvalue and there is no spectral gap such that methods from center manifold theory are not applicable for nonlinear stability analysis either. Using resolvent estimates [21, Theorem 1.1] for the diffusive linearized operator similar to above calculations for the shadow operator, one can show linearized stability of the problem. Nevertheless, nonlinear stability of the equilibrium S^* remains undetermined.

Let us conclude this example with one more comment on the current work of the authors of the scientific paper [71]. They establish a method to construct non-homogeneous steady states to the diffusive system (1.1)–(1.3) close to constant steady states. The authors assume that the latter steady states are non-degenerated in the sense that the linearization around the steady state implies an invertible matrix for which the submatrix corresponding to the ODE subsystem is invertible as well. For instance, Example 6.4 applies to this setting with two non-degenerated constant steady states. However, the current model 6.3 possesses a non-degenerated steady state S^0 and a degenerated steady state S^* arising from the explicit form of $\mathbf{L}_D$ above. Proposition 6.6 shows that the method of the authors of [71] does not apply to this degenerated case.

7 Conclusion and Outlook

This research aimed to investigate the relation between a reaction-diffusion-ODE system for large diffusive components and its shadow limit on long-time scales. There have been only few qualitative attempts including [62, 75] which analyze such a long-time behavior. This thesis extended the uniform convergence results [7, 75] which already exist on short-time intervals to large time intervals including uniform error estimates.

Motivated by applications such as [36, 69], only low regularity assumptions on the solutions, the initial datum of the system and on the domain Ω were made, see next section *Basics*. Solutions of the partly diffusive system and its shadow limit were compared using uniform error estimates with respect to the $L^\infty(\Omega \times (0,T))$ norm. The latter estimates, which depend explicitly on an upper bound of the time T, were obtained by linearization of the original system around the shadow solution. Additional stability assumptions on the linearized shadow system and the ODE subsystem allowed to show convergence results either for times scaling with the least positive diffusion D tending to infinity, i.e., $T \sim D^\ell$ for some $\ell \geq 0$, or for the asymptotic time scale $T = \infty$. Moreover, the errors were estimated by a bound proportional to a power of the inverse D^{-1} as $D \to \infty$. Compare the following sections *The linear case* and *The nonlinear case* for details.

Such estimates provide information on the long-term dynamics of reaction-diffusion-type models from results obtained for their associated shadow limit. To obtain a comprehensive picture of this limit process, the thesis included a critical discussion of the made assumptions with various examples and applications from natural sciences. This detailed study showed that a shadow limit reduction has characteristic time scales on which a uniform approximation result can be achieved under additional stability assumptions on a shadow system. Details can be found in the last concluding section.

Apart from further details on the most relevant results including the main steps of the proofs and challenges of this work, possible directions of future work are depicted in the following sections.

Basics

The semigroup framework in this thesis allows to study the uniform shadow limit approximation for classical reaction-diffusion systems and reaction-diffusion-ODE systems. With a minor modification, one may also include the case $m = 0$. This corresponds to [100, Theorem 14.17] in which all diffusion coefficients tend to infinity. Although the used techniques are well known, the theory of existence and uniqueness of mild solutions for both the diffusive system and its shadow system defined on a domain with a Lipschitz boundary could not be found in standard textbooks. Chapter 2 reviews fundamental results for bounded, mild solutions of both systems under low boundary regularity. In order to obtain error estimates, we followed the approach from [75]. The latter considers a linearization of the diffusive system around the shadow solution to obtain error estimates. In general, a shadow solution is space- and time-dependent and probably of low regularity in space [69]. This motivated the low regularity of solutions in this thesis.
It was crucial to properly define the heat semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ within this low-regular setting of [18] and to recombine this with the notion of mild solutions similar to [94]. The heat semigroup for zero flux boundary conditions is hypercontractive and satisfies a uniform decay estimate similar to [112],

$$\|S_\Delta(\tau)(z^0 - \langle z^0\rangle_\Omega)\|_{L^\infty(\Omega)} \leq \overline{C}\mathrm{e}^{-\lambda_1\tau}\|z^0 - \langle z^0\rangle_\Omega\|_{L^\infty(\Omega)} \qquad \forall\, \tau \in \mathbb{R}_{\geq 0}, z^0 \in L^\infty(\Omega),$$

which was essential for the entire work. It has to be emphasized that such an estimate for some $\lambda_1 > 0$ can also be verified for different boundary conditions and even for more general parabolic differential operators, see [94, Part I, Lemma 3] and references following on Lemma B.4 in this work. Hence, the general framework in this thesis is not restricted to zero flux boundary conditions but rather applies to a wide class of problems that satisfy the above decay estimate.
For instance, a shadow limit can be performed for boundary conditions of Robin type [32, Equations (1.6)–(1.8)]. More generally, instead of considering (1.1)–(1.2) endowed with zero flux boundary conditions, the reaction-diffusion-type system

$$\begin{aligned}
\frac{\partial \mathbf{u}_D}{\partial t} - \mathbf{D}^u\Delta\mathbf{u}_D &= \mathbf{f}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{u}_D(\cdot, 0) = \mathbf{u}^0 \quad \text{in} \quad \Omega,\\
\frac{\partial \mathbf{v}_D}{\partial t} - \mathbf{D}^v\Delta\mathbf{v}_D &= \mathbf{g}(\mathbf{u}_D, \mathbf{v}_D, x, t) \quad \text{in} \quad \Omega_T, \qquad \mathbf{v}_D(\cdot, 0) = \mathbf{v}^0 \quad \text{in} \quad \Omega,\\
\mathbf{D}^u\frac{\partial \mathbf{u}_D}{\partial \mathbf{n}} + \mathbf{M}^u(x)\mathbf{u}_D = \mathbf{0}, &\quad \mathbf{D}^v\frac{\partial \mathbf{v}_D}{\partial \mathbf{n}} + \mathbf{M}^v(x)\mathbf{v}_D = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0, T)
\end{aligned}$$

can be studied for some diagonal matrices $\mathbf{M}^u, \mathbf{M}^v$ with entries in $L^\infty(\Omega)$. The shadow system as $D = \min_{j=1,\dots,k} D_j^v \to \infty$ is then given by

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}^u \Delta \mathbf{u} &= \mathbf{f}(\mathbf{u}, \mathbf{v}, x, t) && \text{in} \quad \Omega_T, \qquad \mathbf{u}(\cdot, 0) = \mathbf{u}^0 \quad \text{in} \quad \Omega, \\
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \langle \mathbf{g}(\mathbf{u}(\cdot, t), \mathbf{v}(t), \cdot, t) \rangle_\Omega + \overline{\mathbf{M}^v}\mathbf{v} && \text{in} \quad (0, T), \qquad \mathbf{v}(0) = \langle \mathbf{v}^0 \rangle_\Omega.
\end{aligned}$$

The shadow component $\mathbf{u}$ still satisfies the boundary condition $\mathbf{D}^u \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \mathbf{M}^u(x)\mathbf{u} = \mathbf{0}$ on $\partial\Omega \times (0, T)$ and the additional linear term in the second equation arises from the boundary integral

$$\overline{\mathbf{M}^v} = -\frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{M}^v(x) \, \mathrm{d}x.$$

Such a shadow system is considered in [67] for Robin boundary conditions. Concerning future research, it would be interesting to adapt the current work to different, even time-dependent differential operators and other boundary conditions such as the above ones. A further step could be to consider additional advection terms such as models including self- or cross-diffusion [45, 57, 107]. All these possible directions should be compared to the more abstract but also more regular setting in [7, Section 3.1] satisfying an exponential decay estimate.

The linear case

Starting from a shadow solution, a direct comparison of solutions of the diffusive system and its singular limit bares its own difficulties. In order to obtain estimates up to the initial time $t = 0$, a suitable mean value correction was incorporated similar to [75]. The latter correction term ψ_D decays as time grows and is negligible in error estimates on long-time scales, compare decay estimate (4.13).

The linear case already includes all main aspects of extending the uniform error estimates obtained in [75] to long-time intervals. Hence, understanding this case in detail allows a deeper insight in the limit process of the shadow approximation for general nonlinear systems.

Using a stability condition for the evolution of the linear ODE subsystem as in [55], we could show uniform convergence results in Theorem 4.1 on time scales of the order $T \sim D^\ell$ for some $\ell \in [0, 1)$, where $D \to \infty$. In Theorem 4.5, we found that the space-dependence of coefficients required a further stability condition on the entire shadow system for long-time error estimates. Both stability conditions are uniform in time and space, i.e., we assumed uniform stability of the subsystem of ordinary differential equations in $L^\infty(\Omega)^{\tilde{m}}$ and of the shadow system in $L^\infty(\Omega)^m \times \mathbb{R}^k$. Finally, global error estimates up to $T = \infty$ were derived under the stronger condition of uniform exponential stability of the linear evolutionary system. Using a bootstrap argument for parabolic equations, the stability condition on the linear shadow system could be relaxed to uniform stability in $L^p(\Omega)^m \times \mathbb{R}^k$ for sufficiently large $p \geq 1$. Additional examples illustrated that the stability conditions are optimal in the sense that they are required to obtain convergence results.

The nonlinear case

In approaching general semilinear problems – with nonlinearities that are space-dependent as well as time-dependent –, the main part of this dissertation made use of the powerful tool of linearization. It turned out that the stability of the evolution system induced by the linearization around the shadow limit yields a natural condition for solutions of the diffusive system to stay nearby the shadow solution for all large diffusions. Using the truncation method from [75], the remaining nonlinear part was cut off and uniform estimates for the localized error functions were derived. The stability conditions implied that the localized errors are small with respect to the $L^\infty(\Omega \times (0,T))$ norm. This in turn made the truncation redundant as diffusion D grows and estimates for the original errors were obtained in Theorem 4.10 and Theorem 5.2. Such estimates allow to deduce long-time behavior of the solution to the reaction-diffusion system (1.1)–(1.3) solely from the corresponding behavior of its shadow limit (1.4)–(1.6).
Example 5.4 showed that, in general, instability of the evolution system induced by the linearization around the shadow solution does not imply divergence of the error functions. The stability assumptions are sufficient conditions for convergence results on long-time scales and are far from being necessary in the nonlinear case. Classical works for the shadow limit suggest that there could be a similar result implying instability of the diffusive solution close to an unstable shadow solution. It would be interesting to address such a scientific issue in future studies.

Verification of stability conditions

The last parts of this thesis were devoted to possible analytical ways of verifying the two stability assumptions described in the last two sections. Nevertheless, a verification using numerical simulations is also reasonable and inevitable in many cases [53, 87].

Dissipativity conditions for the evolution system induced by the linearization were discussed in Section 4.3 with respect to different L^p spaces, following the approach in [66] for time-dependent evolution systems. It was shown that this is a much stronger stability assumption compared to the one we required in Theorem 4.10. However, the notion of dissipativity is useful in particular if diffusion is involved in the corresponding shadow system: we only had to impose conditions on the linearization without diffusion to obtain a stable shadow evolution system.
In many applications, structures are considered which converge as $t \to \infty$ to a stationary solution, i.e., a time-independent solution of the shadow problem. Concerning a linearization around such a structure, it was shown that the stability behavior of the corresponding linearized evolution system can be deduced solely from its linearization at the steady state. Using the relation between the spectrum of the linear steady state operator and its corresponding semigroup resulted in a second possibility of how to verify stability assumptions of the established theorems. The spectrum of the corresponding linear, stationary shadow operator was characterized for reaction-diffusion-ODE systems. As a consequence, stability properties of stationary patterns could be derived not only for an application of the proven theorems but also for a linearized stability analysis of the nonlinear shadow problem. This characterization of the spectrum showed that the instability result in [69, Appendix B] is not restricted to a pure point spectrum.

Partly diffusive system vs. shadow system

It is well known that the shadow limit is a valuable approximation of reaction-diffusion-type equations for short-time intervals, i.e., $T = \mathcal{O}(1)$ as $D \to \infty$. In particular, solutions of both systems can be compared and error estimates can be derived with respect to the $L^\infty(\Omega \times (0,T))$ norm [32, 75]. This dissertation showed that an extension of the uniform error estimates for long-time intervals may fail without additional stability assumptions as described above. It turned out that different time scales have to be considered in showing accuracy of the shadow approximation. We distinguished estimates on long-time intervals $[0,T]$ with $T \sim D^\ell$ scaling with diffusion for $0 < \ell < 1$ and global estimates on $[0,\infty)$ including asymptotic behavior. Examples in Chapter 6 highlighted that validity of the approximation on short- or long-time intervals does not extend to an approximation result on a larger time scale and, in particular, on the global time scale. Consequently, in using the shadow solution as a simplification of the solution to the reaction-diffusion-type system, nu-

merical simulations on finite time scales have to be interpreted with caution.
Asymptotic behavior of solutions to both systems, the partly diffusive system and its shadow limit, have been studied extensively for the classical shadow limit [32, 78, 87, 109]. It is remarkable that most proofs in [32], revealing a relation between compact attractors of the reaction-diffusion system and its shadow limit, do not apply to reaction-diffusion-ODE systems. This is due to a loss of compactness of the solution map induced by the ODE subsystem. Reaction-diffusion-ODE systems thus can be seen as a partially degenerated reaction-diffusion system.
However, linearized stability analysis in this thesis indicated that the behavior of solutions to both systems is similar around stable stationary patterns of the shadow limit. This similarity was reflected in the fact that the same method of proof which lead to a characterization of the spectrum of a linear shadow operator was applicable to characterize the spectrum of a linear partly diffusive operator. Such a spectral decomposition can be used to verify stability of steady states to the partly diffusive system (1.1)–(1.3); for instance, for steady states which are constructed from stationary shadow solutions as in [78, 103, 109] or [38]. Concerning future work, the spectral characterization of Proposition 5.13 could be a basis for a generalization of the instability result [71, Theorem 2.11] for reaction-diffusion-ODE systems: instead of regular steady states, one might be able to consider bounded, discontinuous steady states.

Appendices

A Spectral theory for the Laplacian

Solving the heat equation (2.1) on the Hilbert space $L^2(\Omega)$ is essentially based on a Galerkin approximation which makes use of a spectral basis of the Laplace operator [24, Section 7.1]. Following the idea of [100, Chapter 11, §A], it is shown that a bounded domain with Lipschitz boundary $\partial\Omega$ is sufficient for existence of such an orthonormal basis in $L^2(\Omega)$.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Then there exists a spectral basis $(\lambda_j, w_j)_{j\in\mathbb{N}_0}$ such that*

$$\begin{cases} -\Delta w_j = \lambda_j w_j & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \mathbf{n}} w_j = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is satisfied in the weak sense, i.e.,

$$\int_\Omega \nabla w_j \nabla \varphi \, \mathrm{d}x = \lambda_j \int_\Omega w_j \varphi \, \mathrm{d}x \qquad \forall \, \varphi \in H^1(\Omega).$$

This sequence has the following properties:

(1) $(\lambda_j)_{j\in\mathbb{N}_0}$ is a non-decreasing sequence of non-negative eigenvalues satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \quad (j \to \infty).$$

Each eigenspace is finite-dimensional.

(2) The set of eigenfunctions $(w_j)_{j\in\mathbb{N}_0} \subset H^1(\Omega)$ form an orthonormal basis for $L^2(\Omega)$ and an orthogonal basis for $H^1(\Omega) = W^{1,2}(\Omega)$.

(3) The normed eigenfunction for the principal eigenvalue $\lambda_0 = 0$ is $w_0 = |\Omega|^{-1/2}$ and the corresponding eigenfunctions satisfy $\langle w_j \rangle_\Omega = 0$ for each $j \geq 1$.

Proof. We use a spectral decomposition theorem for compact, self-adjoint operators on separable Hilbert spaces such as $L^2(\Omega)$ [8, Theorem 6.11]. For this purpose, it suffices to show that the inverse of the negative Laplacian induces a compact and self-adjoint operator on some separable Hilbert subspace of $L^2(\Omega)$. Define the closed, separable subspaces

$$L := \{f \in L^2(\Omega) \mid \langle f \rangle_\Omega = 0\} \subset L^2(\Omega) \qquad \text{and} \qquad H := L \cap H^1(\Omega) \subset H^1(\Omega)$$

with inherited norms, where $\langle f \rangle_\Omega$ denotes the spatial mean value of $f \in L^2(\Omega)$. Consider the weak formulation of Poisson's equation $-\Delta u = f$ for some arbitrary $f \in L$, i.e., search for $u \in H$ satisfying

$$\int_\Omega \nabla u \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \qquad \forall\, \varphi \in H^1(\Omega). \tag{A.1}$$

The choice of L guarantees the existence of a solution $u \in H^1(\Omega)$. Indeed, by considering $\varphi - \langle \varphi \rangle_\Omega$, one can replace $\varphi \in H^1(\Omega)$ by $\varphi \in H$ in the variational formulation above to apply the theorem of Lax-Milgram [8, Corollary 5.8]. To show coercivity of the bilinear form, recall that Poincaré's inequality holds on H [24, §5.8.1, Theorem 1]. The restriction to solutions $u \in H$ is crucial for uniqueness since a solution is only unique up to additive constants on the connected set Ω. Having the theorem of Lax-Milgram in mind, we define the solution operator

$$T : L \to H \subset L, \quad f \mapsto u$$

on the Hilbert space L. This operator assigns each right-hand side $f \in L$ to the corresponding weak solution $u \in H$ of the elliptic boundary value problem (A.1). The theorem of Lax-Milgram shows continuity of T due to the estimate

$$\|u\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)}$$

for some constant $C > 0$. The compact embedding $H^1(\Omega) \hookrightarrow^c L^2(\Omega)$ of Rellich-Kondrachov's theorem in [8, Theorem 9.16] implies compactness of T [23, Chapter II, Proposition 4.25]. Thus, it remains to show that the inverse of the negative Laplace operator, denoted by T, is a self-adjoint operator. Using the variational formulation (A.1), we obtain

$$(Tf, g)_L = \int_\Omega g T f \, \mathrm{d}x = \int_\Omega \nabla(Tg) \nabla(Tf) \, \mathrm{d}x = \int_\Omega f T g \, \mathrm{d}x = (f, Tg)_L$$

since for all $f, g \in L$ we have test functions $Tf, Tg \in H$. This computation also shows that the spectrum of T only consists of non-negative real eigenvalues [8, Proposition 6.9].

Now we are able to apply a spectral decomposition to the space L with respect to T [8, Theorem 6.11]. This results in a sequence $(\mu_j, w_j)_{j\in\mathbb{N}}$ of eigenvalues $\mu_j \geq 0$ and eigenfunctions $w_j \in H$ of the operator T which form an orthonormal basis of L. In view of the variational equation (A.1), the functions $(w_j)_{j\in\mathbb{N}}$ are also pairwise orthogonal in H. In order to obtain a spectral basis of the negative Laplacian, we consider the reciprocals $\lambda_j = \mu_j^{-1}$ for $j \in \mathbb{N}$ and we receive

$$\int_\Omega \nabla w_j \nabla \varphi \,\mathrm{d}x = \lambda_j \int_\Omega w_j \varphi \,\mathrm{d}x \qquad \forall\, \varphi \in H. \tag{A.2}$$

Recall that every $\varphi \in H^1(\Omega)$ can be written as a sum of an element in H and its spatial mean value

$$\varphi = \varphi - \langle\varphi\rangle_\Omega + \langle\varphi\rangle_\Omega.$$

Thus, the variational equation (A.2) holds for all $\varphi \in H^1(\Omega)$ since $w_j \in H$, i.e., $\langle w_j\rangle_\Omega = 0$. To provide an orthonormal basis of $L^2(\Omega)$ itself, we add constants by choosing $w_0 = |\Omega|^{-1/2} \in H^1(\Omega)$ and consider the sequence of eigenfunctions $(w_j)_{j\in\mathbb{N}_0}$ of the operator $-\Delta$. We observe that all eigenvalues of $-\Delta$ must be non-negative and that we obtained all positive ones, yet including their eigenspaces. Additionally, since Ω is connected, the corresponding eigenspace to the eigenvalue $\lambda_0 = 0$ is spanned by the normed eigenfunction $w_0 = |\Omega|^{-1/2}$. We infer from formulation (A.2) that $(w_j)_{j\in\mathbb{N}_0}$ forms an orthogonal basis of $H^1(\Omega)$ as well. Consequently, for each $j \in \mathbb{N}$, the eigenfunctions w_j satisfy $\langle w_j\rangle_\Omega = 0$ and change their sign on Ω.

It remains to prove assertion (1) of the above listing. Compactness of T implies $0 \in \sigma(T)$ [8, Theorem 6.8]. If 0 was an eigenvalue of the operator T, its eigenspace would consist of all $f \in L$ such that

$$\int_\Omega f\varphi \,\mathrm{d}x = \int_\Omega \nabla(Tf)\nabla\varphi \,\mathrm{d}x = 0 \quad \forall\, \varphi \in H.$$

The fundamental lemma of calculus of variations yields $f = 0$ since the equation actually holds for each $\varphi \in H^1(\Omega)$ – a contradiction to a non-zero eigenspace. Hence, all eigenvalues μ_j of T are positive and have finite-dimensional eigenspaces [8, Theorem 6.11]. The latter implies an infinite sequence $(\mu_j)_{j\in\mathbb{N}}$ of eigenvalues of T converging to 0 and, consequently, 0 an approximate eigenvalue of T. The sequence of reciprocals shows that $(\lambda_j)_{j\in\mathbb{N}}$ diverges to infinity. All in all, assertions (1)–(3) are proven, if we arrange the eigenvalues monotonically increasing. $\square$

In the same manner one can proof a result for the eigenvalue problem corresponding to a general uniformly elliptic differential operator of second order with lower order terms. The operator can be endowed with a possibly different boundary condition such as a Robin boundary condition [100, Chapter 11, §A].
Certainly, the regularity of the eigenfunctions depends on the regularity of the boundary $\partial\Omega$ as the following Corollary shows.

Corollary A.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Then a spectral basis $(\lambda_j, w_j)_{j\in\mathbb{N}_0}$ of $-\Delta$ from Proposition A.1 satisfies $w_j \in W^{1,p}(\Omega)$ for each finite $1 \le p < \infty$, and consequently $w_j \in C(\overline{\Omega})$. Moreover, a boundary $\partial\Omega$ of class $\partial\Omega \in C^{\ell-1,1}$ for some $\ell \in \mathbb{N}$ yields orthogonal eigenfunctions $w_j \in H^\ell(\Omega)$, and similarly $w_j \in C^{\ell-1}(\overline{\Omega})$.*

Proof. The first result is a bootstrap argument in combination with Sobolev embeddings. The right-hand side $\lambda_j w_j \in W^{1,2}(\Omega) = H^1(\Omega)$ of the eigenvalue problem can be embedded in $L^q(\Omega)$ for some $q > 2$ [1, Theorem 4.12]. We infer from elliptic regularity theory that the solution in fact satisfies $w_j \in W^{1,q}(\Omega)$, see Lemma A.3 below. Applying this bootstrap procedure $k \ge n/2$ times leads to $w_j \in W^{1,n}(\Omega)$. Sobolev embedding results in $w_j \in L^p(\Omega)$ for all $1 \le p < \infty$. Finally, elliptic regularity yields $w_j \in W^{1,p}(\Omega)$ for all $1 \le p < \infty$ and the eigenfunction w_j has a representative which is Hölder continuous [1, Theorem 4.12]. The second result is due to higher elliptic regularity theory for weak solutions [30, Theorem 2.2.2.5]; compare to [8, Theorem 9.26] for $\partial\Omega \in C^\ell$. Classical solutions are again recovered by the above bootstrap argument which implies $w_j \in W^{\ell,p}(\Omega)$ for all $1 \le p < \infty$. $\square$

The following result on existence and uniqueness of solutions to the Poisson equation (A.1) on $L^p(\Omega)$ already implied higher regularity of the eigenfunctions in the last statement. This is similar to the theorem of Lax-Milgram for the Hilbertian case $p = 2$ [8, Corollary 5.8]. Moreover, this result allows a characterization of the abstract domain of the heat semigroup on $L^p(\Omega)$ in Lemma B.5.

Lemma A.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $f \in L^p(\Omega)$ for some $1 < p < \infty$ and $c \in L^\infty(\Omega)$. Consider the elliptic boundary value problem*

$$\begin{cases} -\Delta w + c(x)w = f & in \quad \Omega, \\ \frac{\partial}{\partial \mathbf{n}} w = 0 & on \quad \partial\Omega \end{cases}$$

in the weak sense, i.e., search for a solution $w \in W^{1,p}(\Omega)$ such that it holds

$$\int_\Omega \nabla w \nabla \varphi \, \mathrm{d}x = \int_\Omega (f - c(x)w)\varphi \, \mathrm{d}x \qquad \forall\, \varphi \in W^{1,q}(\Omega) \tag{A.3}$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Then there exits a unique weak solution $w \in W^{1,p}(\Omega)$ provided $c \geq 0$ a.e. in Ω is not identically zero. Moreover, we have the estimate

$$\|w\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{A.4}$$

where $C > 0$ does not depend on f or w. The same is true for the case $n = 1 = p$. If $c \equiv 0$, we have existence of a solution $w \in W^{1,p}(\Omega)$ which is unique up to constants if and only if $\langle f \rangle_\Omega = 0$. In this case there holds

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Proof. This is essentially a result from elliptic regularity theory. For $n = 1$, Ω is just a bounded interval and the result is clear by integration of the corresponding ordinary differential equation. For $n \geq 2$, [19, Theorem 5] provides a similar result for an elliptic operator with oscillating principle part but a Lipschitz boundary with small Lipschitz constant. However, since the principal part of $-\Delta$ has only constant coefficients and is symmetric, we can adapt the proof to the situation above. The assumption of leading coefficients with small BMO norm is trivially fulfilled and Theorem 1–3 in [19] remain valid in our situation. Moreover, Lemmata 6 and 9 which are used in the proof of [19, Theorem 5] still applies. Since the principal part of $-\Delta$ is constant, we have $N_0 = 0$ in estimate (47) and the crucial inequality (48) in their paper is valid independent of the Lipschitz constant of the boundary. Finally, the result of [19, Theorem 5] remains valid. □

Recall that the spectrum of $-\Delta$ considered on $L^p(\Omega)$ is independent of $1 \leq p < \infty$ by [2, Example 1.2]. The same independence holds for uniformly elliptic operators, see [2, Example 5.2]. We deduce from Corollary A.2 and the weak formulation (A.3) in $W^{1,p}(\Omega)$ that $(w_j)_{j\in\mathbb{N}_0} \subset W^{1,p}(\Omega)$ are also corresponding eigenfunctions in $L^p(\Omega)$ to the eigenvalues λ_j of $-\Delta$.

Basic examples for Lipschitz domains are given by hyperrectangles. In this case, the Fourier theory yields

Example A.4. For an interval $\Omega = (a, b) \subset \mathbb{R}$ for some $a < b$ the functions

$$w_j(x) := \begin{cases} \frac{1}{\sqrt{b-a}} & \text{for} \quad j = 0, \\ \sqrt{\frac{2}{b-a}} \cos\left(j\pi \frac{x-a}{b-a}\right) & \text{for} \quad j \in \mathbb{N} \end{cases}$$

serve as a spectral basis for $-\Delta$ endowed with zero flux boundary conditions which has the eigenvalues $(\lambda_j)_{j \in \mathbb{N}_0}$ given by

$$\lambda_j = \left(\frac{j\pi}{b-a}\right)^2.$$

If $\Omega = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ for some $a_i < b_i$, then a product ansatz yields for each multi-index $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$

$$w_{\mathbf{k}}(x_1, \ldots, x_n) = \prod_{i=1}^n w_{k_i}(x_i) \qquad \text{and} \qquad \lambda_{\mathbf{k}} = \sum_{i=1}^n \lambda_{k_i},$$

where w_{k_i} are given by the one-dimensional case above for each $i = 1, \ldots, n$.

B Parabolic regularity theory

For solving the heat equation (2.1) with less boundary regularity than used for the standard semigroup theory in [8, 92] or [94], I recall the existence theory on the Hilbert space $L^2(\Omega)$. The latter is essentially based on semigroup theory and Fourier techniques which make use of a spectral basis of the Laplacian deduced in Proposition A.1 [24, Section 7.1]. Subsequently, we define the heat semigroup $(S_\Delta(t))_{t\in\mathbb{R}_{\geq 0}}$ according to [18] and show its hypercontractivity stated in Lemma 2.1 which is crucial for the entire work. Additionally to the Hilbertian case, $L^\infty(\Omega_T)$ estimates are derived for solutions of the inhomogeneous heat equation with explicit dependence on time T.

B.1 The heat semigroup

Let us recall some basics from semigroup theory. An introduction to the topic and further results on semigroup theory can be found for example in [8, 17, 23, 40, 92] or [94]. This semigroup approach is motivated by the fact that the solution of the homogeneous heat equation can be represented in a short way by a semigroup acting on the initial value.

Definition B.1. Let $(B, \|\cdot\|)$ be a Banach space and let $(S(t))_{t\in\mathbb{R}_{\geq 0}} \subset \mathcal{L}(B)$ be a family of linear, bounded operators satisfying the conditions of a semigroup, i.e.,

$$\begin{cases} S(t+s) = S(t)S(s) \quad \forall\, t, s \in \mathbb{R}_{\geq 0}, \\ \quad S(0) = I. \end{cases}$$

The semigroup $(S(t))_{t\in\mathbb{R}_{\geq 0}}$ is called *strongly continuous* if for all $z \in B$ the so called *orbit*

$$\xi_z : \mathbb{R}_{\geq 0} \to B, \quad t \mapsto S(t)z$$

is continuous. A strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_{\geq 0}}$ is called *contractive* if

$$\|S(t)z\| \leq \|z\| \qquad \forall\, z \in B, t \in \mathbb{R}_{\geq 0}$$

is satisfied. The *generator* A of a strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_{\geq 0}}$ is defined on the domain

$$\mathcal{D}(A) := \left\{ z \in B \;\middle|\; \lim_{t\searrow 0} \frac{S(t)z - z}{t} \text{ exists in } B \right\}$$

and is uniquely given by the limit

$$Az := \lim_{t\searrow 0} \frac{S(t)z - z}{t} \qquad \forall\, z \in \mathcal{D}(A).$$

Many applications of semigroups arise from the consideration of evolution equations such as initial value problems on some Banach space B, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) \qquad \text{for} \quad t > 0, \qquad u(0) = u^0 \in B.$$

Let us assume that $(A, \mathcal{D}(A))$ is the generator of a strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_{\geq 0}}$ and $u^0 \in \mathcal{D}(A)$. Then the function $t \mapsto u(t) := S(t)u^0$ is differentiable, hence, it is the unique solution of the above initial value problem [23, Chapter II, Proposition 6.2]. The operator $A : \mathcal{D}(A) \subset B \to B$ is usually unbounded but linear, closed and densely defined on B [23, Chapter II, Lemma 1.3, Theorem 1.4]. In applications, a differential operator is given on some Banach space and the aim is to find a solution of the above initial value problem via its semigroup. However, the identification of the generator of the semigroup and its abstract domain which is defined by the semigroup is not trivial, and the latter might even differ from the given differential operator. In the case of the Laplacian $A = -\Delta$ with a certain domain, we use the fact that $-\Delta$ induces a quadratic form on the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$.

Proposition B.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $z^0 \in L^2(\Omega)$. Then the homogeneous heat equation*

$$\begin{aligned} \frac{\partial z}{\partial t} - \Delta z = 0 \quad &\textit{in} \quad \Omega \times \mathbb{R}_{>0}, \\ \frac{\partial z}{\partial \mathbf{n}} = 0 \quad &\textit{on} \quad \partial\Omega \times \mathbb{R}_{>0}, \qquad z(\cdot, 0) = z^0 \quad \textit{in} \quad \Omega \end{aligned}$$

has a unique variational solution $z \in C(\mathbb{R}_{\geq 0}; L^2(\Omega)) \cap C^1(\mathbb{R}_{>0}; L^\infty(\Omega))$ *satisfying* $z \in L^2(0,T; H^1(\Omega))$ *for each* $T > 0$. *The solution is given by the Fourier expansion*

$$z(x,t) = (S_\Delta(t)z^0)(x) = \sum_{j\in\mathbb{N}_0} \mathrm{e}^{-\lambda_j t}(z^0, w_j)_{L^2(\Omega)} w_j(x)$$

where $(\lambda_j, w_j)_{j\in\mathbb{N}_0}$ *is a spectral basis of* $-\Delta$ *from Proposition A.1.*

Proof. We will apply [8, Theorem 7.7] to get a unique solution given via the semigroup defined in [18, Theorem 1.3.9]. The latter author considers the closed, non-negative quadratic form

$$Q : H^1(\Omega) \to \mathbb{R}_{\geq 0}, \qquad f \mapsto \tilde{Q}(f,f) \quad \text{with} \quad \tilde{Q}(f,g) = \int_\Omega \nabla f \nabla g \,\mathrm{d}x$$

with dense domain $C^\infty(\Omega) \cap H^1(\Omega)$. Thus, by [17, Theorem 4.12], Q is induced by a non-negative self-adjoint operator $H \geq 0$ on $L^2(\Omega)$ via

$$Q(f) = (H^{1/2}f, H^{1/2}f)_{L^2(\Omega)} \qquad \text{for} \quad f \in \mathcal{D}(H^{1/2}) = H^1(\Omega).$$

Due to [18, Theorem 1.2.10], the abstract domain $\mathcal{D}(H) \subset H^1(\Omega)$ is given by the following characterization:

$$\begin{aligned} f \in \mathcal{D}(H) \quad \iff \quad & f \in H^1(\Omega) \text{ and there exists a function } g \in L^2(\Omega) \text{ such that} \\ & \tilde{Q}(f,\varphi) = (H^{1/2}f, H^{1/2}\varphi)_{L^2(\Omega)} = (g,\varphi)_{L^2(\Omega)} \qquad \forall\, \varphi \in H^1(\Omega). \end{aligned}$$

As usual in the weak sense, we identify $Hf = g$ and obtain the estimate

$$\|f\|^2_{H^1(\Omega)} = \|f\|^2_{L^2(\Omega)} + (g,f)_{L^2(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega)} + \|Hf\|^2_{L^2(\Omega)}\right)$$

for all $f \in \mathcal{D}(H)$. Obviously, H is an extension of $-\Delta$ for smooth functions and H is densely defined since $C_c^\infty(\Omega) \subset \mathcal{D}(H)$. As a self-adjoint operator, H is necessarily closed in $L^2(\Omega)$. Writing

$$\|(\lambda I + H)z\|^2_{L^2(\Omega)} = \lambda^2\|z\|^2_{L^2(\Omega)} + 2\lambda(Hz,z)_{L^2(\Omega)} + \|Hz\|^2_{L^2(\Omega)}$$

for $\lambda \in \mathbb{R}, z \in \mathcal{D}(H)$, it is not difficult to infer from $H \geq 0$ that the resolvent of $-H$ exists for each $\lambda > 0$ with the estimate

$$\|(\lambda I + H)^{-1}\| \leq \frac{1}{\lambda}.$$

Due to the well-known characterization of contraction semigroups by Hille and Yosida in [17, Corollary 2.22], $-H$ with domain $\mathcal{D}(H)$ generates a strongly continuous semigroup of contractions $(S_\Delta(t))_{t\in\mathbb{R}_{\geq 0}}$ on $L^2(\Omega)$. The semigroup may then be defined by the formula used in [18, Theorem 1.3.2], see [23, Corollary 5.5, Chapter III]. Since $-H$ is densely defined, we extend the unique solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) + Hz(t) = 0 \qquad \text{for} \quad t > 0, \qquad z(0) = z^0 \in \mathcal{D}(H)$$

given by $z(t) = S_\Delta(t)z^0$ to all $z^0 \in L^2(\Omega)$. The solution $z \in C(\mathbb{R}_{\geq 0}; L^2(\Omega))$ is still continuous due to strong continuity of the semigroup. However, it may lose its differentiability at $t = 0$ and we get a possibly singular time derivative. To see this and, furthermore, to obtain higher regularity for $t > 0$, we apply [8, Theorem 7.7] to the self-adjoint, maximal monotone operator H on $L^2(\Omega)$ in the sense of [8, Definition in §7.1]. The latter theorem yields $z \in C^k(\mathbb{R}_{>0}; \mathcal{D}(H^\ell))$ for all integers $k, \ell \in \mathbb{N}_0$ where $\mathcal{D}(H^0) := L^2(\Omega)$. Let us show how this yields higher regularity in space using the above characterization of the domain $\mathcal{D}(H)$. It suffices to prove $z \in C^k(\mathbb{R}_{>0}; W^{1,p}(\Omega))$ for all $k \in \mathbb{N}_0, 1 \leq p < \infty$ since Sobolev's embedding yields $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ for $p > n$. According to [8, Section 7.3] the domain $\mathcal{D}(H^\ell)$ is given by

$$\mathcal{D}(H^{\ell+1}) = \{z \in \mathcal{D}(H^\ell) \mid Hz \in \mathcal{D}(H^\ell)\} \qquad \forall\, \ell \in \mathbb{N}.$$

Using the characterization of $\mathcal{D}(H)$, we infer that $\mathcal{D}(H) \subset W^{1,2}(\Omega)$. Sobolev's embedding implies $\mathcal{D}(H) \subset L^{p_2}(\Omega)$ for some $p_2 > 2$ [1, Theorem 4.12]. Elliptic regularity theory from Lemma A.3 in combination with this embedding yields

$$\mathcal{D}(H^\ell) \subset W^{1,p_\ell}(\Omega) \qquad \text{for} \quad \frac{1}{p_\ell} = \frac{1}{2} - \frac{\ell - 1}{n}.$$

Applying this bootstrap procedure $\ell \geq n/2$ times leads to $\mathcal{D}(H^\ell) \subset W^{1,n}(\Omega)$. The Sobolev embedding results in $W^{1,n}(\Omega) \subset L^p(\Omega)$ for all $1 \leq p < \infty$. Finally, elliptic regularity yields $\mathcal{D}(H^{\ell+1}) \subset W^{1,p}(\Omega)$ for all $1 \leq p < \infty$ and the solution $z(\cdot, t)$ has a representative which is Hölder continuous for each $t > 0$ [1, Theorem 4.12]. Hence, the boundary condition is satisfied in the sense of distributions by the trace operator $W^{1,p}(\Omega) \hookrightarrow W^{-1/p,p}(\partial\Omega)$ from [30, Theorem 1.5.1.2].

To determine the Fourier coefficients, we recall that the unique solution z solves the following variational equation for all $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(x,t)\varphi(x)\,\mathrm{d}x + \int_\Omega \nabla z(x,t)\nabla\varphi(x)\,\mathrm{d}x = 0 \qquad \forall\,\varphi \in H^1(\Omega).$$

Since $z(\cdot,t) \in L^2(\Omega)$, we can expand this function in a Fourier series using Proposition A.1 and get for all $t \geq 0$

$$z(\cdot,t) = \sum_{j\in\mathbb{N}_0} b_j(t)w_j \qquad \text{with} \quad b_j(t) = \int_\Omega z(x,t)w_j(x)\,\mathrm{d}x.$$

Inserting the series in the variational formula and choosing $\varphi = w_i$ for $i \in \mathbb{N}_0$ yields an ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}b_i(t) + \lambda_i b_i(t) = 0 \qquad \text{for} \quad t > 0.$$

Solving this with respect to the initial condition leads to the series representation. Concerning the regularity $z \in L^2(0,T;H^1(\Omega))$, we note that the partial sums

$$z_m(\cdot,t) = \sum_{j=0}^{m} \mathrm{e}^{-\lambda_j t}(z^0,w_j)_{L^2(\Omega)}w_j$$

of the Fourier series form a Cauchy sequence in this space. Indeed, for $\ell > m \geq 0$

$$\begin{aligned}\|z_m - z_\ell\|^2_{L^2(0,T;H^1(\Omega))} &= \sum_{j=m+1}^{\ell}\int_0^T (1+\lambda_j)\mathrm{e}^{-2\lambda_j t}\,\mathrm{d}t\,|(z^0,w_j)_{L^2(\Omega)}|^2 \\ &\leq C\sum_{j=m+1}^{\ell}|(z^0,w_j)_{L^2(\Omega)}|^2 \longrightarrow 0 \qquad (\text{as } m,\ell\to\infty)\end{aligned}$$

due to Parseval's equality for $z^0 \in L^2(\Omega)$ and $C = 1/2 + 1/(2\lambda_1)$. □

We remark that the set of more regular functions

$$H^2_N(\Omega) := \{u \in H^2(\Omega) \mid \partial_{\mathbf{n}}u = 0 \text{ a.e. in } \Omega\}$$

is included in $\mathcal{D}(H)$ defined in the above proof. In general, elliptic regularity is restricted for low-regular boundaries such as $\partial\Omega \in C^{0,1}$ [30, Section 4.4.3]. By the characterization of contraction semigroups via maximal dissipative generators in [17, Theorem 6.4], we obtain $H^2_N(\Omega) \subsetneq \mathcal{D}(H)$ for a general Lipschitz boundary $\partial\Omega \in C^{0,1}$. As shown in [113, Theorem 2.15], there holds $\mathcal{D}(H) = H^2_N(\Omega)$ in case of

$\partial\Omega \in C^{1,1}$ since [30, Theorem 2.4.1.3] is used for this boundary regularity.

As a simple consequence of Duhamel's formula, we obtain the following result for inhomogeneous initial value problems.

Proposition B.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $z^0 \in L^2(\Omega)$ and $r \in L^2(\Omega_T)$ be given. Then the inhomogeneous heat equation*

$$\frac{\partial z}{\partial t} - \Delta z = r(x,t) \quad \text{in} \quad \Omega_T, \tag{B.1}$$

$$\frac{\partial z}{\partial \mathbf{n}} = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \qquad z(\cdot,0) = z^0 \quad \text{in} \quad \Omega \tag{B.2}$$

has a unique mild solution $z \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, given by the separation of variables formula

$$\begin{aligned} z(x,t) &= S_\Delta(t)z^0 + \int_0^t S_\Delta(t-s)r(\cdot,s)\,\mathrm{d}s \qquad\qquad (\text{B.3}) \\ &= \sum_{j\in\mathbb{N}_0} \mathrm{e}^{-\lambda_j t}(z^0,w_j)_{L^2(\Omega)}w_j(x) + \int_0^t \sum_{j\in\mathbb{N}_0} \mathrm{e}^{-\lambda_j(t-s)}(r(\cdot\ ,s),w_j)_{L^2(\Omega)}w_j(x)\,\mathrm{d}s. \end{aligned}$$

In addition, $z^0 \in H^1(\Omega)$ implies a weak solution $z \in L^\infty(0,T;H^1(\Omega))$ which has a weak derivative $\partial_t z \in L^2(\Omega_T)$, and the weak formulation

$$(\partial_t z(\cdot,t),\varphi)_{L^2(\Omega)} + (\nabla z(\cdot,t),\nabla\varphi)_{L^2(\Omega)} = (r(\cdot,t),\varphi)_{L^2(\Omega)} \qquad \forall\,\varphi \in H^1(\Omega)$$

holds for a.e. $t \in (0,T)$.

Proof. It is well known from [92, Section 4.2] that there exists a unique mild solution $z \in C([0,T];L^2(\Omega))$ of problem (B.1)–(B.2). This is given by the integral formula

$$z(\cdot,t) = S_\Delta(t)z^0 + \int_0^t S_\Delta(t-s)r(\cdot,s)\,\mathrm{d}s$$

where the contraction semigroup $(S_\Delta(t))_{t\geq 0}$ on $L^2(\Omega)$ is defined in Proposition B.2. In order to show that z is an element of $L^2(0,T;H^1(\Omega))$ resp. $L^\infty(0,T;H^1(\Omega))$, it is again sufficient to prove the Cauchy property of the corresponding partial sums given by expression (B.3). For $\ell > m \geq 0$ we obtain by Hölder's inequality

$$\|(z_m - z_\ell)(\cdot,t)\|^2_{H^1(\Omega)} \leq \sum_{j=m+1}^{\ell} (1+\lambda_j)\mathrm{e}^{-2\lambda_j t}\,|(z^0,w_j)_{L^2(\Omega)}|^2$$

$$
\begin{aligned}
&+ \sum_{j=m+1}^{\ell} (1+\lambda_j) \left(\int_0^t \mathrm{e}^{-\lambda_j (t-s)} (r(\cdot,s), w_j)_{L^2(\Omega)} \, \mathrm{d}s \right)^2 \\
\leq &\sum_{j=m+1}^{\ell} (1+\lambda_j) |(z^0, w_j)_{L^2(\Omega)}|^2 \\
&+ (T+1) \int_0^t \sum_{j=m+1}^{\ell} |(r(\cdot,s), w_j)_{L^2(\Omega)}|^2 \, \mathrm{d}s
\end{aligned}
$$

and the same reasoning as in the proof of Proposition B.2 applies to obtain a Cauchy sequence in the corresponding Banach space.

Concerning the weak formulation, we use Galerkin's approximation. Following this classical approach, one might establish a notion of weak solutions for $z^0 \in L^2(\Omega)$ using the dual space of $H^1(\Omega)$ [24, Section 7.1]. Since we will use this result for more regular initial data $z^0 \in H^1(\Omega)$, a proof for this case shall be given here for completeness. Using the previous ansatz given by formula (B.3), the partial sums can be written as

$$
z_m(\cdot,t) = \sum_{j=0}^{m} c_j(t) w_j = \sum_{j=0}^{m} \left(\mathrm{e}^{-\lambda_j t} (z^0, w_j)_{L^2(\Omega)} + \int_0^t \mathrm{e}^{-\lambda_j (t-s)} (r(\cdot\ ,s), w_j)_{L^2(\Omega)} \, \mathrm{d}s \right) w_j
$$

and converge to the mild solution z in $L^2(0,T;H^1(\Omega))$. By construction, each partial sum satisfies an equation of type (B.1)–(B.2) in the weak sense. Testing this weak formulation with the orthogonal spectral basis $(w_j)_{j\in\mathbb{N}_0}$ of $H^1(\Omega)$ from Proposition A.1 yields

$$
(\partial_t z_m(\cdot,t), w_j)_{L^2(\Omega)} + (\nabla z_m(\cdot,t), \nabla w_j)_{L^2(\Omega)} = (r(\cdot,t), w_j)_{L^2(\Omega)} \tag{B.4}
$$

for each $j = 0, \ldots, m$ and a.e. $t \in (0,T)$.

Our aim is to show weak convergence of a subsequence of $(\partial_t z_m)_{m\in\mathbb{N}_0}$ in $L^2(\Omega_T)$. This will then lead to a weak solution satisfying $\partial_t z \in L^2(\Omega_T)$. Since $L^2(\Omega_T)$ is reflexive, it remains to show boundedness of $(\partial_t z_m)_{m\in\mathbb{N}_0}$ in $L^2(\Omega_T)$. We multiply the variational formulation (B.4) with time derivatives c_j' of the Fourier coefficients for each $j = 0, \ldots, m$ and sum up to get

$$
(\partial_t z_m, \partial_t z_m)_{L^2(\Omega)} + (\nabla z_m, \nabla(\partial_t z_m))_{L^2(\Omega)} = (r, \partial_t z_m)_{L^2(\Omega)} \qquad \forall\, m \in \mathbb{N}_0.
$$

Reformulation and Young's inequality imply an estimate of the form

$$
\|\partial_t z_m\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla z_m\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|r\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t z_m\|^2_{L^2(\Omega)}.
$$

Integration over $(0,T)$ yields

$$\frac{1}{2}\|\partial_t z_m\|^2_{L^2(\Omega_T)} + \frac{1}{2}\left(\|\nabla z_m(\cdot,T)\|^2_{L^2(\Omega)} - \|\nabla z_m(\cdot,0)\|^2_{L^2(\Omega)}\right) \leq \frac{1}{2}\|r\|^2_{L^2(\Omega_T)}.$$

In view of the Fourier coefficients, we get $c_j(0) = (z^0, w_j)_{L^2(\Omega)}$ and may further estimate

$$\|\partial_t z_m\|^2_{L^2(\Omega_T)} \leq \|r\|^2_{L^2(\Omega_T)} + \|\nabla z^0\|^2_{L^2(\Omega)}.$$

The uniform bound then implies a weakly convergent subsequence – which will still be denoted by $(\partial_t z_m)_{m\in\mathbb{N}_0}$ – with a limit $v \in L^2(\Omega_T)$. Actually, we will see that v equals the weak derivative $\partial_t z$ satisfying the usual partial integration

$$\int_0^T \partial_t z(\cdot,t)\varphi(t)\,\mathrm{d}t = -\int_0^T z(\cdot,t)\varphi'(t)\,\mathrm{d}t \qquad \forall\,\varphi \in C_c^\infty((0,T)).$$

To recognize this, let $w \in L^2(\Omega)$ be arbitrary and consider $\varphi w \in L^2(\Omega_T)$ for some test function $\varphi \in C_c^\infty((0,T))$. On the one hand, we obtain by Fubini's rule and weak convergence

$$\begin{aligned}\int_0^T \Big(\partial_t z_m(\cdot,t), \varphi(t) w\Big)_{L^2(\Omega)}\,\mathrm{d}t &= \int_{\Omega_T} \partial_t z_m(x,t)\varphi(t)w(x)\,\mathrm{d}(x,t)\\ &\longrightarrow \left(\int_0^T v(\cdot,t)\varphi(t)\,\mathrm{d}t, w\right)_{L^2(\Omega)}\end{aligned}$$

as $m\to\infty$. On the other hand, since $\partial_t z_m$ are weak derivatives with coefficients $c_j' \in L^2((0,T))$, we have

$$\begin{aligned}\left(\int_0^T \partial_t z_m(\cdot,t)\varphi(t)\,\mathrm{d}t, w\right)_{L^2(\Omega)} &= \left(-\int_0^T z_m(\cdot,t)\varphi'(t)\,\mathrm{d}t, w\right)_{L^2(\Omega)}\\ &= -\int_{\Omega_T} z_m(x,t)\varphi'(t)w(x)\,\mathrm{d}(x,t)\\ &\longrightarrow \left(-\int_0^T z(\cdot,t)\varphi'(t)\,\mathrm{d}t, w\right)_{L^2(\Omega)}\end{aligned}$$

with similar arguments. As $w \in L^2(\Omega)$ is arbitrary, this implies

$$\int_0^T v(\cdot,t)\varphi(t)\,\mathrm{d}t = -\int_0^T z(\cdot,t)\varphi'(t)\,\mathrm{d}t,$$

and uniqueness of weak derivatives yields $\partial_t z = v \in L^2(\Omega_T)$.
It remains to deduce the weak formulation for a.e. $t\in(0,T)$. For this purpose, let

us express an arbitrary $v \in H^1(\Omega)$ by its Fourier series with partial sums

$$v_\ell = \sum_{j=0}^{\ell} d_j w_j \qquad \text{for} \quad d_j = (v, w_j)_{L^2(\Omega)}.$$

Multiplying the weak formulation (B.4) with d_j and $\varphi \in C_c^\infty((0,T))$, summing up and integrating over $(0,T)$ yields for all $m \geq \ell$

$$\int_0^T (\partial_t z_m, \varphi v_\ell)_{L^2(\Omega)} \,\mathrm{d}t + \int_0^T (\nabla z_m, \varphi \nabla v_\ell)_{L^2(\Omega)} \,\mathrm{d}t = \int_0^T (r, \varphi v_\ell)_{L^2(\Omega)} \,\mathrm{d}t.$$

We pass to the weak limit z as $m \to \infty$ to obtain for all $\ell \in \mathbb{N}_0$

$$\int_0^T (\partial_t z, v_\ell)_{L^2(\Omega)} \varphi \,\mathrm{d}t + \int_0^T (\nabla z, \nabla v_\ell)_{L^2(\Omega)} \varphi \,\mathrm{d}t = \int_0^T (r, v_\ell)_{L^2(\Omega)} \varphi \,\mathrm{d}t.$$

For $\ell \to \infty$ we can substitute v_ℓ by the arbitrary function $v \in H^1(\Omega)$. Since the test function $\varphi \in C_c^\infty(\Omega)$ was arbitrary as well, the fundamental lemma of calculus of variations implies the weak formulation for a.e. time $t \in (0,T)$. □

Remark that above propositions may be shown for different parabolic differential operators as well as other boundary conditions [15, Theorem 2.4].
Now, we are capable of proving Lemma 2.1 which we recall for a comprehensible proof, and write simply z instead of z^0.

Lemma B.4 (Lemma 2.1)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{0,1}$ and $\lambda_1 > 0$ the first non-zero eigenvalue of $-\Delta$ endowed with zero Neumann boundary conditions. Then there exists a constant $C > 0$, merely depending on Ω, such that for all $1 \leq q \leq p \leq \infty$*

$$\|S_\Delta(\tau) z\|_{L^p(\Omega)} \leq C m(\tau)^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \mathrm{e}^{-\lambda_1 \tau} \|z\|_{L^q(\Omega)} \qquad \forall\, \tau \in \mathbb{R}_{>0} \tag{B.5}$$

holds for all $z \in L^q(\Omega)$ satisfying $\langle z \rangle_\Omega = 0$. Here, we denote $m(\tau) = \min\{1, \tau\}$. Moreover, $(S_\Delta(\tau))_{\tau \in \mathbb{R}_{\geq 0}}$ is a contraction semigroup on $L^p(\Omega)$ for each $1 \leq p \leq \infty$, which is strongly continuous for $1 \leq p < \infty$ and analytic for $1 < p < \infty$.

Proof. The first part of this lemma has been proven with similar arguments in [94] or [112] assuming stronger boundary regularity $\partial\Omega \in C^{2,\alpha}$. For completeness, a proof along the same lines of [112, Lemma 1.3] is given here but estimates for the heat kernel from [18, Theorem 3.2.9] for a Lipschitz boundary are used instead. The second part of this lemma is a consequence of [18, Theorems 1.3.9, 1.3.3] and

analyticity follows from [18, Theorems 1.4.1, 1.4.2].
Finally, let us derive decay estimate (B.5). Using spectral theory developed in Proposition B.2, we can express the semigroup action by a Fourier series

$$(S_\Delta(\tau)z)(x) = \sum_{j\in\mathbb{N}_0} \mathrm{e}^{-\lambda_j\tau}(z, w_j)_{L^2(\Omega)} w_j(x)$$

On the one hand, for $z \in L^2(\Omega)$ with $\langle z\rangle_\Omega = 0$, we can use that $\lambda_0 = 0$ and $w_0 = |\Omega|^{-1/2}$ is constant and obtain

$$\|S_\Delta(\tau)z\|_{L^2(\Omega)} \le \mathrm{e}^{-\lambda_1\tau}\|z\|_{L^2(\Omega)} \qquad \forall\, \tau \in \mathbb{R}_{\ge 0}. \tag{B.6}$$

On the other hand, [18, Theorem 2.4.4] shows that $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\ge0}}$ is hypercontractive and maps $L^2(\Omega)$ to $L^\infty(\Omega)$ for each $\tau > 0$. Thus for each $\tau > 0$, $x \in \Omega$, the map $z \mapsto S_\Delta(\tau)z(x) \in L^2(\Omega)^*$ can be represented by a function $K(\tau, x, \cdot) \in L^2(\Omega)$, the so called heat kernel, such that

$$(S_\Delta(\tau)z)(x) = \int_\Omega K(\tau, x, y)z(y)\,\mathrm{d}y.$$

We infer an estimation of the heat kernel from [18, Theorem 3.2.9], more precisely,

$$0 \le K(\tau, x, y) \le C_1 m(\tau)^{-n/2} \exp\left(\frac{-(x-y)^2}{C_2\tau}\right) =: f_\tau(x-y),$$

where $m(\tau) = \min\{1, \tau\}$ as in [94, Part I, Lemma 3]. Using Young's inequality, we get for $r \ge 1$ with $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$

$$\begin{aligned}
\|S_\Delta(\tau)z\|_{L^p(\Omega)} &\le \left(\int_\Omega \left(\int_\Omega K(\tau, x, y)|z(y)|\,\mathrm{d}y\right)^p \mathrm{d}x\right)^{1/p} \\
&\le \|f_\tau * (|z|\chi_\Omega)\|_{L^p(\mathbb{R}^n)} \\
&\le \|f_\tau\|_{L^r(\mathbb{R}^n)}\|z\|_{L^q(\Omega)}.
\end{aligned}$$

The evaluation of the Gauss integral yields a constant $C_3 = C_1(C_2\pi)^{n/2} > 0$ only depending on Ω such that for all $\tau > 0$ and $r \ge 1$

$$\|S_\Delta(\tau)z\|_{L^p(\Omega)} \le C_3 m(\tau)^{-n/2}\tau^{n/(2r)}\|z\|_{L^q(\Omega)} \le C_3 m(\tau)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|z\|_{L^q(\Omega)}. \tag{B.7}$$

In view of the latter inequality (B.7), estimate (B.5) holds for small $\tau \in (0,2)$. Thus, it remains to consider $\tau \geq 2$, i.e., $m(\tau) = 1$. In the following argument, we distinguish the cases $p < 2$ resp. $p \geq 2$.

- If $1 \leq p < 2$, then Hölder's inequality and the semigroup properties imply

$$\|S_\Delta(\tau)z\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{2}}\|S_\Delta(\tau)z\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}}\|S_\Delta(\tau-1)S_\Delta(1)z\|_{L^2(\Omega)},$$

  which can be further estimated using inequalities (B.6), (B.7). Since we assume $1 \leq q \leq p \leq 2$, we obtain

$$\|S_\Delta(\tau)z\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{2}}\mathrm{e}^{-\lambda_1(\tau-1)}\|S_\Delta(1)z\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}}\mathrm{e}^{-\lambda_1(\tau-1)}C_3\|z\|_{L^q(\Omega)}.$$

- If $p \geq 2$, we proceed in a similar way using semigroup properties and estimates (B.7), (B.6) to reach at

$$\begin{aligned}\|S_\Delta(\tau)z\|_{L^p(\Omega)} &= \|S_\Delta(1)S_\Delta(\tau-1)z\|_{L^p(\Omega)} \leq C_3\|S_\Delta(\tau-1)z\|_{L^2(\Omega)}\\ &= C_3\|S_\Delta(\tau-2)S_\Delta(1)z\|_{L^2(\Omega)} \leq C_3\mathrm{e}^{-\lambda_1(\tau-2)}\|S_\Delta(1)z\|_{L^2(\Omega)}.\end{aligned}$$

  Applying once more inequality (B.7) for $q \leq 2$ resp. Hölder's inequality for $q > 2$ yields

$$\|S_\Delta(\tau)z\|_{L^p(\Omega)} \leq C_3\mathrm{e}^{-\lambda_1(\tau-2)}\max\{C_3, |\Omega|^{\frac{1}{2}}\}\|z\|_{L^q(\Omega)}.$$

All in all, estimate (B.5) holds for all $\tau > 0$ where C is independent of p, q. □

Remark that different uniformly elliptic differential operators of second order with possibly different boundary conditions induce similar semigroup estimates, see [4, Theorem 4.9] and subsequent remark therein for Robin boundary conditions. Even for space- and time-dependent parabolic coefficients, there are general works on Gaussian bounds by [15] and [58] including estimates of the evolution operators.

The following final result characterizes the abstract domain of the generator of the heat semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ for low boundary regularity $\partial\Omega \in C^{0,1}$. In this general case, the Laplace operator is not defined on a domain in $W^{2,p}(\Omega)$.

Lemma B.5. *Let $1 < p < \infty$ and the semigroup $(S_\Delta(\tau))_{\tau\in\mathbb{R}_{\geq 0}}$ be defined as in Lemma B.4 with generator $H_p : \mathcal{D}(H_p) \subset L^p(\Omega) \to L^p(\Omega)$. Then the domain $\mathcal{D}(H_p)$*

is characterized by $\mathcal{D}(H_p) = W^{1,p}_N(\Omega)$ *where*

$$w \in W^{1,p}_N(\Omega) \quad :\Leftrightarrow \quad w \in W^{1,p}(\Omega) \text{ and there exists a function } f \in L^p(\Omega) \text{ such that}$$
$$(\nabla w, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \qquad \forall\, \varphi \in W^{1,q}(\Omega).$$

As usual in the weak sense, we identify $H_p w = f$ *and obtain the estimate*

$$\|w\|_{W^{1,p}(\Omega)} \leq C \left(\|w\|_{L^p(\Omega)} + \|H_p w\|_{L^p(\Omega)} \right)$$

for all $w \in \mathcal{D}(H_p)$*. Additionally, if* $\langle w \rangle_\Omega = 0$*, we obtain*

$$\|w\|_{W^{1,p}(\Omega)} \leq C \|\nabla w\|_{L^p(\Omega)} \leq C \|H_p w\|_{L^p(\Omega)}.$$

Proof. This characterization of the domain $\mathcal{D}(H_p)$ is shown similar to [113, Theorem 2.15]. Starting from the quadratic form Q on $H^1(\Omega)$ as in the proof of Proposition B.2, we have shown in Lemma B.4 that the latter induces sectorial operators on $L^p(\Omega)$ for each finite $1 < p < \infty$ [23, Chapter II, Theorem 4.6]. However, this was just an abstract result without knowledge of the domain. The same can be done analog to [113, Theorem 2.12] while we note that the preceding results from [113, Propositions 2.1, 2.3] hold for each $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re} \lambda \leq 0$. The operators which we gain by this procedure are sectorial in the sense of [23, Chapter II, Definition 4.1] and generate analytic semigroups [23, Chapter II, Proposition 4.3]. The semigroups coincide on the dense subset $L^p(\Omega) \cap L^2(\Omega)$ of $L^p(\Omega)$ with the ones defined above by [18, Theorem 1.4.1]. As a consequence, each bounded semigroup operator coincides on $L^p(\Omega)$ and hence, the unique generator H_p of the semigroup can be characterized using the method of [113, Section 2.4].
First of all, we have $W^{1,p}_N(\Omega) \subset \mathcal{D}(H_p)$ since we can identify $H_p w = f \in L^p(\Omega)$ in the weak sense for each $w \in W^{1,p}_N(\Omega)$, compare to Proposition B.2. Conversely, let $w \in \mathcal{D}(H_p)$, i.e., there exists some $f \in L^p(\Omega)$ with $f = H_p w$ and $w \in L^p(\Omega)$. Let us consider some constant $c > 0$ for which we obtain a unique solution $\tilde{w} \in W^{1,p}(\Omega)$ of the elliptic problem

$$-\Delta \tilde{w} + c\tilde{w} = f + cw$$

in its weak formulation (A.3). Actually, $\tilde{w} \in W^{1,p}_N(\Omega)$ since

$$-\Delta \tilde{w} = f + c(w - \tilde{w}) \in L^p(\Omega).$$

Since $H_p w = f$, this can be rewritten as $(H_p + cI)(\tilde{w} - w) = 0$ and considering the weak formulation (A.3) for this problem yields $\tilde{w} = w$ by uniqueness. All in all, $\mathcal{D}(H_p) = W_N^{1,p}(\Omega)$ and we obtain above estimates which hold on $W_N^{1,p}(\Omega)$ by estimate (A.4) and Poincaré's inequality. $\square$

B.2 A bootstrap argument

Let us consider a bounded weak solution $z \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ which solves the following linear parabolic equation

$$\frac{\partial z}{\partial t} - d\Delta z = R(x,t) \quad \text{in} \quad \Omega_T, \tag{B.8}$$

$$\frac{\partial z}{\partial \mathbf{n}} = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \qquad z(\cdot,0) = 0 \quad \text{in} \quad \Omega \tag{B.9}$$

with right-hand side $R \in L_{p,r}(\Omega_T)$ and diffusion d, see Proposition B.3. The aim of this section is to show $L^\infty(\Omega_T)$ estimates for the solution z which depend explicitly on time T and the mixed norm $\|R\|_{p,r}$. This is based on a bootstrap method developed by Ladyzenskaja in [59, Chapter III, §7] which essentially uses parabolic $L_{p,r}(\Omega_T)$ estimates in combination with the well-known truncation method of Stampacchia. Within this procedure, the exponent p is restricted due to Sobolev embeddings by $p \geq 1 = n$ and $p > n/2$ for $n \geq 2$. The parameter $1 \leq r \leq \infty$ is chosen according to [59, Chapter III, §7], see definition (B.14) below.
The Lebesgue space $L_{p,r}(\Omega_T)$ is given by all measurable functions ψ on Ω_T with finite mixed norm

$$\|\psi\|_{p,r} := \left(\int_0^T \left(\int_\Omega |\psi(x,t)|^p \, \mathrm{d}x \right)^{r/p} \mathrm{d}t \right)^{1/r} \qquad \text{for} \quad 1 \leq p, r < \infty$$

and an obvious modification for $r = \infty$ [59, Chapters I, II, §1 in both cases]. It is well known that $L_{p,r}(\Omega_T) = L^r(0,T;L^p(\Omega))$ for $p, r < \infty$ since simple functions are dense in both spaces with the same norm [3, Section 1.1]. If we do not specify the region of integration within the notation $\|\cdot\|_{p,r}$, we assume to integrate over Ω_T.

For simplicity, we normalize the right-hand side R of equation (B.8) by dividing with $\|R\|_{p,r}$. Due to linearity of the latter parabolic equation, the rescaled solution $\tilde{z}$ equals the former solution z but divided by the norm $\|R\|_{p,r}$. Our goal is to prove that $\|\tilde{z}\|_{L^\infty(\Omega_T)}$ can be estimated by a power of T. In achieving this goal, we estimate the positive part of $\tilde{z} - h$ for $h \in \mathbb{R}_{\geq 0}$, more precisely, the measurable function

$$(\tilde{z} - h)_+ := \max\{\tilde{z} - h, 0\}.$$

Let us define its positivity set

$$A_h(t) = \{x \in \Omega \mid \tilde{z}(x,t) > h\} \qquad \text{for} \quad t > 0. \tag{B.10}$$

In order to find an estimate for $\tilde{z}$ in $L^\infty(\Omega_T)$, we will show existence of $h > 0$ such that $|A_h(t)| = 0$ for a.e. $t > 0$. In the end, rescaling establishes a corresponding bound for the original solution z in terms of T and $\|R\|_{p,r}$. I follow the strategy of the proof of Theorem 7.1 in [59, Chapter III, §7] and adapt calculations to our situation. For simplicity, we write

$$B_h := (\tilde{z} - h)_+ \qquad \text{and} \qquad \overline{B}_h := \langle(\tilde{z} - h)_+\rangle_\Omega.$$

First of all, we derive an estimate for $\|B_h - \overline{B}_h\|_{\hat{p},\hat{r}}$ for some $\hat{p}, \hat{r}$ to be determined. Since for almost every time $t \in [0,T]$ we have $B_h(\cdot,t) \in H^1(\Omega)$, the well-known Gagliardo-Nirenberg interpolation inequality holds (see [1, Theorem 5.8] for $n \geq 2$ or [59, Chapter II, Theorem 2.2] for $n \geq 1$). The latter inequality yields in combination with Poincaré's inequality [24, §5.8.1, Theorem 1]

$$\|B_h(\cdot,t) - \overline{B}_h(t)\|_{L^{\hat{p}}(\Omega)} \leq \varrho \|\nabla_x B_h(\cdot,t)\|^{\omega}_{L^2(\Omega)} \|B_h(\cdot,t) - \overline{B}_h(t)\|^{1-\omega}_{L^2(\Omega)}, \tag{B.11}$$

with $\omega = n/2 - n/\hat{p}$ and ϱ depending on Ω, n, and $2 \leq \hat{p} \leq \infty$. We observe ranges $0 \leq \omega \leq 1/2$ for $n = 1$, $0 \leq \omega < 1$ for $n = 2$. For $n \geq 3$ we use the Sobolev conjugate 2^* of 2 which restricts $\hat{p}$,

$$2 \leq \hat{p} \leq 2^* := \frac{2n}{n-2},$$

and implies $0 \leq \omega \leq 1$. Following the method of Ladyzenskaja, we choose a parameter $\hat{r}$ such that we are able to make use of our differential equation (B.8), and especially of the bound $\|\tilde{R}\|_{p,r} = 1$. Using the weak formulation of equation (B.8) in $L^2(\Omega_T)$, an $L_{\hat{p},\hat{r}}(\Omega_T)$ estimate can be deduced from inequality (B.11) by choosing $\omega\hat{r} = 2$ [59, Chapter II, §3]. We reach at the following estimate.

Lemma B.6. *Let $h \in \mathbb{R}_{\geq 0}$, $\omega\hat{r} = 2$, and consequently $2 < \hat{p} \leq \infty$ in estimate* (B.11). *Then there exists a constant $C > 0$ which depends only on $\varrho, \hat{r}$, and $\hat{p}$ such* *that*

$$\|B_h - \overline{B}_h\|_{\hat{p},\hat{r}}^2 \le C d^{-2/\hat{r}} \left(\frac{1}{2} \|B_h\|_{2,\infty}^2 + d \|\nabla_x B_h\|_{2,2}^2 \right). \tag{B.12}$$

Proof. Let us apply inequality (B.11) to the case $\omega\hat{r} = 2$ to deduce

$$\begin{aligned}
\|B_h - \overline{B}_h\|_{\hat{p},\hat{r}} &= \left(\int_0^T \|B_h(\cdot,\tau) - \overline{B}_h(\tau)\|_{L^{\hat{p}}(\Omega)}^{\hat{r}} \,\mathrm{d}\tau \right)^{1/\hat{r}} \\
&\le \varrho \left(\int_0^T \|\nabla_x B_h(\cdot,\tau)\|_{L^2(\Omega)}^{\omega\hat{r}} \|B_h(\cdot,\tau) - \overline{B}_h(\tau)\|_{L^2(\Omega)}^{(1-\omega)\hat{r}} \,\mathrm{d}\tau \right)^{1/\hat{r}} \\
&\le \varrho \left(\int_0^T \|\nabla_x B_h(\cdot,\tau)\|_{L^2(\Omega)}^{2} \|B_h(\cdot,\tau)\|_{L^2(\Omega)}^{\hat{r}-2} \,\mathrm{d}\tau \right)^{1/\hat{r}}
\end{aligned}$$

from the fact that the mean $\overline{B}_h$ is the minimum of the parabola $s \mapsto \|B_h - s\|_{L^2(\Omega)}^2$. The spatial gradient can be connected to the weak formulation by estimating

$$\begin{aligned}
\|B_h - \overline{B}_h\|_{\hat{p},\hat{r}}^2 &\le \varrho^2 d^{-2/\hat{r}} \left(d \int_0^T \|\nabla_x B_h(\cdot,\tau)\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau \right)^{2/\hat{r}} \|B_h\|_{2,\infty}^{2(1-2/\hat{r})} \\
&\le \varrho^2 d^{-2/\hat{r}} \left(\frac{2}{\hat{r}} d \|\nabla_x B_h\|_{2,2}^2 + \left(1 - \frac{2}{\hat{r}}\right) \|B_h\|_{2,\infty}^2 \right)
\end{aligned}$$

where we used Young's inequality for the exponent $\hat{r}/2 \ge 1$. □

In order to obtain an estimate for the expression $\|B_h\|_{\hat{p},\hat{r}}$, we aim for a similar estimate of the spatial mean $\overline{B}_h$ to compare $\overline{B}_h$ to $\tilde{R}$ as well.

Lemma B.7. *For each $h \in \mathbb{R}_{\ge 0}$ there holds the integral inequality*

$$\overline{B}_h(t) \le |\Omega|^{-1} \int_0^t \int_{A_h(\tau)} |\tilde{R}(x,\tau)| \,\mathrm{d}x \,\mathrm{d}\tau. \tag{B.13}$$

Proof. For convenience, we omit the tilde in the following proof. In order to obtain inequality (B.13), we formally test the corresponding weak formulation of equation (B.8) for z with right-hand side R by $B_h^{\ell-1} = (z-h)_+^{\ell-1}$ for $\ell > 1$. To do this mathematically rigorously, we consider bounded test functions

$$\varphi_\delta(\cdot,t) := (z(\cdot,t) - h)_+^{\ell-1} \chi_{A_{h+\delta}(t)} \in H^1(\Omega)$$

for arbitrary $\delta > 0$, a.e. $t \in (0,T)$ and obtain

$$\int_{A_{h+\delta}(t)} \partial_t z (z-h)_+^{\ell-1} \,\mathrm{d}x + d(\ell-1) \int_{A_{h+\delta}(t)} |\nabla z|^2 (z-h)_+^{\ell-2} \,\mathrm{d}x$$

$$= \int_{A_{h+\delta}(t)} R(x,t)(z-h)_+^{\ell-1}\,\mathrm{d}x$$

in view of Proposition B.3. Notice that the second term on the left-hand side is non-negative. Moreover, the bounded function $(z-h)^\ell$ satisfies $(z-h)^\ell \in H^1(0,T)$ for a.e. $x \in \Omega$ which implies $(z-h)_+^\ell \in H^1(0,T)$ by [1, Lemma 8.34] and

$$\frac{1}{\ell}\int_{A_{h+\delta}(t)} \partial_t (z-h)_+^\ell\,\mathrm{d}x \le \int_{A_{h+\delta}(t)} R(x,t)(z-h)_+^{\ell-1}\,\mathrm{d}x.$$

By dominated convergence, we conclude that $A_{h+\delta}(t)$ can be substituted as $\delta \to 0$ by $A_h(t)$ or simply by Ω as domain of integration. Integration over $(0,t)$ yields

$$\frac{1}{\ell}\int_\Omega (z(x,t)-h)_+^\ell\,\mathrm{d}x \le \int_0^t \int_{A_h(\tau)} R(x,\tau)(z-h)_+^{\ell-1}\,\mathrm{d}x\,\mathrm{d}\tau$$

in view of the initial condition $z(\cdot,0) = 0$. Finally, the theorem of dominated convergence implies estimate (B.13) in the limit $\ell \to 1$. $\square$

Following the current method of Ladyzenskaja, we choose parameters r, κ_1 according to [59, Chapter III, §7], i.e., we choose

$$\frac{1}{r} + \frac{n}{2p} = 1 - \kappa_1 \in (0,1) \tag{B.14}$$

for given $1 \le p \le \infty$ with slight modifications for $n = 1$

$$\begin{aligned} &\frac{1}{1-\kappa_1} \le r \le \frac{2}{1-2\kappa_1} \quad \text{and} \quad \kappa_1 \in (0,1/2) && \text{for} \quad n = 1,\\ &\frac{1}{1-\kappa_1} \le r \le \infty \quad \text{and} \quad p \ge \frac{n}{2(1-\kappa_1)} > \frac{n}{2} && \text{for} \quad n \ge 2. \end{aligned}$$

Furthermore, we define parameters

$$\hat{p} := (1+\kappa)\frac{2p}{p-1}, \quad \hat{r} := (1+\kappa)\frac{2r}{r-1} \qquad \text{for} \quad \kappa := 2\kappa_1/n < 1 \tag{B.15}$$

with ranges

$$\begin{aligned} &2(1+\kappa) \le \hat{p} \le \infty, \quad 4 \le \hat{r} \le \frac{4(1+\kappa)}{\kappa} && \text{for} \quad n = 1,\\ &2(1+\kappa) \le \hat{p} \le \frac{2n(1+\kappa)}{n-2(1-\kappa_1)}, \quad 2(1+\kappa) \le \hat{r} \le \frac{2(1+\kappa)}{\kappa_1} && \text{for} \quad n \ge 2. \end{aligned}$$

In particular, $\hat{p}$ satisfies the requirements of inequality (B.11) and above Lemma B.6 is applicable since $\omega\hat{r} = 2$. For each $h \in \mathbb{R}_{\geq 0}$ and $p > 1$ consider the integral

$$\mu(h) = \int_0^T |A_h(\tau)|^{\hat{r}/\hat{p}} \, \mathrm{d}\tau = \int_0^T |A_h(\tau)|^{\gamma} \, \mathrm{d}\tau \tag{B.16}$$

for which we will show $\mu(h) = 0$ for some $h \in \mathbb{R}_{\geq 0}$ to establish an upper bound for the solution $\tilde{z}$. Recall that $\gamma = (1 - 1/p)/(1 - 1/r) > 0$ in this case and A_h defined in (B.10) is the positivity set of $\tilde{z} - h$. For the case $p = 1 = n$, a modification of μ is required since definition (B.16) does not yield any information on the measure of the positivity set $A_h(t)$. Instead, Ladyzenskaja proposes in [59, footnote on p. 185] to use

$$\mu(h) := |P_h| \qquad \text{where} \quad P_h = \{t \in [0, T] \mid |A_h(t)| > 0\} \tag{B.17}$$

for $p = 1 = n$. Having these definitions in mind, the following is a consequence of Lemma B.7.

Corollary B.8. *Let $h \in \mathbb{R}_{\geq 0}$ and $1 \leq p, r \leq \infty$ be chosen according to the parameter setting* (B.14)*. Then there holds*

$$\|\overline{B}_h\|_{\hat{p},\hat{r}} \leq T^{1/\hat{r}} |\Omega|^{1/\hat{p}-1} \mu(h)^{1-1/r} \tag{B.18}$$

provided definitions (B.16) *and* (B.17) *for the function μ and $\hat{p}, \hat{r}$ given by* (B.15)*.*

Proof. Let $\tilde{R}$ be normalized in the $L_{p,r}(\Omega_T)$-norm. A further estimation of inequality (B.13) using Hölder's inequality results in

$$|\Omega| \left|\overline{B}_h(t)\right| \leq \int_0^T \|\tilde{R}(\cdot, \tau)\|_{L^p(\Omega)} |A_h(\tau)|^{1-1/p} \, \mathrm{d}\tau \leq \left(\int_0^T |A_h(\tau)|^{\gamma} \, \mathrm{d}\tau \right)^{1-1/r}$$

where $\gamma = (1 - 1/p)/(1 - 1/r) > 0$ for $p > 1$. With regard to the mixed norm, we obtain

$$\begin{aligned} \|\overline{B}_h\|_{\hat{p},\hat{r}} &= \left(\int_0^T \left(\int_\Omega |\overline{B}_h(\tau)|^{\hat{p}} \, \mathrm{d}x \right)^{\hat{r}/\hat{p}} \mathrm{d}\tau \right)^{1/\hat{r}} \leq \sup_{\tau \in [0,T]} |\overline{B}_h(\tau)| |\Omega|^{1/\hat{p}} T^{1/\hat{r}} \\ &\leq T^{1/\hat{r}} |\Omega|^{1/\hat{p}-1} \left(\int_0^T |A_h(\tau)|^{\gamma} \, \mathrm{d}\tau \right)^{1-1/r} . \end{aligned}$$

For the case $p = 1$, we start again from inequality (B.13) and further estimate

$$\overline{B}_h(t) \leq |\Omega|^{-1} \int_0^T \int_{A_h(\tau)} |\tilde{R}(x, \tau)| \, \mathrm{d}x \, \mathrm{d}\tau$$

$$\leq |\Omega|^{-1} \int_0^T \chi_{P_h}(\tau) \|\tilde{R}(\cdot,\tau)\|_{L^1(\Omega)} \, \mathrm{d}\tau.$$

Applying Hölder's inequality and $\|\tilde{R}\|_{1,r} = 1$ yield $\overline{B}_h(t) \leq |\Omega|^{-1}\mu(h)^{1-1/r}$ where we use definition (B.17) of μ. As shown above, this implies

$$\|\overline{B}_h\|_{\hat{p},\hat{r}} \leq T^{1/\hat{r}} |\Omega|^{-1} \mu(h)^{1-1/r}.$$

Recall that $\hat{p} = \infty, \hat{r} = 4$ for $p = 1 = n$ while $1 < r < \infty$. □

Combining the above results, we gain an estimate for $\|B_h\|_{\hat{p},\hat{r}}$ which still depends on the spatial gradient of B_h. However, the embedding

$$L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \subset L_{\hat{p},\hat{r}}(\Omega_T)$$

is a consequence of the choice of the exponents and allows to dispose of the gradient.

Lemma B.9. *Let p, r satisfy the setting* (B.14) *and let $\kappa, \hat{p}$ and $\hat{r}$ be defined as in* (B.15)*. Then, for arbitrary $h \in \mathbb{R}_{\geq 0}, T \geq 1$, the function $B_h = (\tilde{z} - h)_+$ can be estimated by*

$$\|B_h\|_{\hat{p},\hat{r}} \leq C T^{2/\hat{r}} \mu^{(2\kappa+1)/\hat{r}}(h) \tag{B.19}$$

where the constant $C > 0$ does only depend on Ω, n, the parameters p, r and a lower bound of the diffusion d.

Proof. Let us once again omit the tilde. Above lemmata imply the inequality

$$\|B_h\|_{\hat{p},\hat{r}}^2 \leq C_1 d^{-2/\hat{r}} \left(\frac{1}{2} \|B_h\|_{2,\infty}^2 + d \|\nabla_x B_h\|_{2,2}^2 \right) + C_2 T^{2/\hat{r}} \mu^{4(1+\kappa)/\hat{r}}(h). \tag{B.20}$$

For a further estimation, we test the scaled version of the parabolic equation (B.8) with $B_h = (z - h)_+$ to reach at

$$\frac{1}{2} \|B_h\|_{2,\infty}^2 + d \|\nabla_x B_h\|_{2,2}^2 \leq \int_0^T \int_{A_h(\tau)} R(z-h)_+ \, \mathrm{d}x \, \mathrm{d}\tau. \tag{B.21}$$

Let us first consider $p > 1$. We use twice Hölder's inequality for $p, r > 1$ to gain

$$\int_0^T \int_{A_h(\tau)} R(z-h)_+ \, \mathrm{d}x \, \mathrm{d}\tau \leq \|(z-h)\chi_{\Omega_T(h)}\|_{p/(p-1),r/(r-1)}$$

where we used $\Omega_T(h) := \{(x,t) \in \Omega_T \mid z(x,t) - h > 0\}$. Applying the parabolic Hölder inequality [59, Chapter II, §1] to the characteristic function on $\Omega_T(h)$ implies

$$\|(z-h)\chi_{\Omega_T(h)}\|_{p/(p-1),r/(r-1)} \le \|(z-h)_+\|_{\hat{p},\hat{r}}\mu^{(2\kappa+1)/\hat{r}}(h), \tag{B.22}$$

where $\hat{p}, \hat{r}$ are defined as in (B.15). Hence, we may apply estimate (B.20) to the function $B_h = (z-h)_+$ and obtain by Cauchy's inequality for some $\delta > 0$

$$\begin{aligned}\|(z-h)\chi_{\Omega_T(h)}\|_{p/(p-1),r/(r-1)} \le \delta \Big(C_1 d^{-2/\hat{r}} \left(\frac{1}{2}\|B_h\|_{2,\infty}^2 + d\|\nabla_x B_h\|_{2,2}^2 \right) \\ + C_2 T^{2/\hat{r}} \mu^{4(1+\kappa)/\hat{r}}(h) \Big) + (4\delta)^{-1}\mu^{2(2\kappa+1)/\hat{r}}(h).\end{aligned}$$

Inserting the latter estimate into inequality (B.21) and absorbing terms on the left-hand side (for small $\delta > 0$ in case of a lower bound $d \ge d_0$), this yields

$$\frac{1}{2}\|B_h\|_{2,\infty}^2 + d\|\nabla_x B_h\|_{2,2}^2 \le C \left(T^{2/\hat{r}} \mu^{4(1+\kappa)/\hat{r}}(h) + \mu^{2(2\kappa+1)/\hat{r}}(h) \right).$$

Since $\mu(h) \le |\Omega|^{\hat{r}/\hat{p}} T$ and we consider very long time intervals $[0,T]$, the assertion follows from estimate (B.20).
A similar argument which involves the positivity set P_h is used to verify the case $p = 1 = n$. In doing so, only estimate (B.22) has to be verified for the modified definition of μ since Lemma B.6 remains valid. The former is still a consequence of Hölder's inequality applied to the integral on the left-hand side of estimate (B.22). Apart from this modification, the assertion follows along the same lines using the estimate $\mu(h) \le T$. □

Next, we use estimate (B.19) of Lemma B.9 to obtain decay estimates for the function μ. A well-known truncation lemma due to Stampacchia then yields an estimate for the diffusive component z after rescaling with $\|R\|_{p,r}$.

Proposition B.10. *Let $z \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ be a bounded weak solution of the initial boundary value problem* (B.8)–(B.9) *with $R \in L_{p,r}(\Omega_T)$ and parameter values p, r given by* (B.14). *Then there exists a constant $C > 0$ which only depends on Ω, n, p, r and a lower bound of the diffusion d such that the diffusive component z satisfies the estimate*

$$\|z\|_{L^\infty(\Omega_T)} \le C T^{1-1/r} \|R\|_{p,r}. \tag{B.23}$$

Proof. Due to inequality (B.19) there holds

$$\|(\tilde{z}-h)_+\|_{\hat{p},\hat{r}} \leq CT^{2/\hat{r}}\mu^{(2\kappa+1)/\hat{r}}(h).$$

To derive decay estimates for μ, we estimate the left-hand side from below. Therefore let $\tilde{h} > h \geq 0$ which implies $z - h > (\tilde{h} - h)$ on $A_{\tilde{h}}(t)$. An integration over $A_{\tilde{h}}(t) \subset A_h(t)$ yields

$$|A_{\tilde{h}}(t)|^{1/\hat{p}}(\tilde{h}-h) \leq \|(\tilde{z}-h)_+\|_{L^{\hat{p}}(A_h(t))}$$

for $p > 1$ resp. $\hat{p} < \infty$. For $p = 1$, we may take the supremum over x in the inequality $(\tilde{h}-h)\chi_{P_{\tilde{h}}}(t) \leq (\tilde{z}-h)_+\chi_{P_{\tilde{h}}}(t)$ which holds on $A_{\tilde{h}}(t)$. Integrating over t yields the lower bound

$$\mu(\tilde{h})(\tilde{h}-h)^{\hat{r}} \leq \|(\tilde{z}-h)_+\|^{\hat{r}}_{\hat{p},\hat{r}}.$$

Both bounds for the decreasing function μ imply the relation

$$\mu(\tilde{h}) \leq \frac{(CT^{2/\hat{r}})^{\hat{r}}}{(\tilde{h}-h)^{\hat{r}}}\mu^{2\kappa+1}(h) \qquad \forall\, \tilde{h} > h \geq 0.$$

Then a technical lemma of Stampacchia [101, Lemma 4.1] applies (with the same constant C) to the function μ and yields extinction:

$$\mu(h) = 0 \qquad \text{for all} \quad h \geq h_0 = 2^{1+1/(2\kappa)}CT^{2/\hat{r}}\mu(0)^{2\kappa/\hat{r}}$$

The definition of μ leads to $\tilde{z} \leq h_0$ a.e. in Ω_T. Since $-\tilde{z}$ satisfies the same type of equation as $\tilde{z}$ with $-\tilde{R}$ instead of $\tilde{R}$, we obtain the estimate $\|\tilde{z}\|_{L^\infty(\Omega_T)} \leq h_0$. Accordingly rescaled, this means

$$\|z\|_{L^\infty(\Omega_T)} \leq h_0\|R\|_{p,r} \leq CT^{1-1/r}\|R\|_{p,r}$$

where we used again $|\mu(h)| \leq |\Omega|^{\hat{r}/\hat{p}}T$ and the parameter setting (B.15). $\square$

C Multiplication operators

Each matrix $\mathbf{A} \in L^\infty(\Omega)^{m \times m}$ induces a corresponding multiplication operator

$$\mathbf{M_A} : L^p(\Omega)^m \to L^p(\Omega)^m, \qquad \mathbf{z} \mapsto \mathbf{Az}$$

where $(\mathbf{Az})(x) := \mathbf{A}(x)\mathbf{z}(x)$ for each $\mathbf{z} \in L^p(\Omega)^m$. Since $\|\mathbf{M_A}\| \leq \|\mathbf{A}\|_\infty$, this is a bounded, linear operator for each $1 \leq p \leq \infty$ [104, Proposition 2.2.14]. Let us simply write $\mathbf{A}$ instead of $\mathbf{M_A}$ in the following. The knowledge of the spectrum of a multiplication operator $\mathbf{A}$ allows us to characterize the spectrum of the shadow operator $\mathbf{L}$ defined in (5.8) and the partly diffusive operator $\mathbf{L}_D$ defined in (5.10). We refer to [22, Chapter IX] and [35, Sections 1-3] for several characterizations of the spectrum $\sigma(\mathbf{A})$ of the multiplication operator $\mathbf{A}$ on $L^p(\Omega)^m$ for $1 \leq p < \infty$. The following result concerning the essential spectrum is known for the case $p = 2$ by [35, Proposition 3.2, Corollary 3.4] and for the scalar case by [102, Proposition 3]. A generalization to arbitrary exponents $1 \leq p \leq \infty$ is given next.

Proposition C.1. *Let* $\mathbf{A} \in L^\infty(\Omega)^{m \times m}, m \in \mathbb{N}$, *and let* $\mathbf{A}$ *denote its corresponding multiplication operator on* $L^p(\Omega)^m$ *for some* $1 \leq p \leq \infty$. *Then there exists a null set* $N \subset \Omega$ *such that*

$$\sigma(\mathbf{A}) = \overline{\bigcup_{x \in \Omega \setminus N} \sigma(\mathbf{A}(x))}. \tag{C.1}$$

Moreover, the whole spectrum is essential in the sense of Wolf, i.e.,

$$\sigma(\mathbf{A}) = \sigma_{\mathrm{ess}}(\mathbf{A}) := \{\lambda \in \mathbb{C} \mid \lambda I - \mathbf{A} \textit{ is not a Fredholm operator}\}.$$

Proof. Boundedness of the multiplication operator leads to a non-empty resolvent set $\rho(\mathbf{A}) \neq \emptyset$. For $1 \leq p < \infty$, [22, Chapter IX, Theorem 2.4] states

$$\sigma(\mathbf{A}) = \{\lambda \in \mathbb{C} \mid |N_{\lambda,\varepsilon}| > 0 \quad \forall\, \varepsilon > 0\} =: \text{ess}-\sigma(\mathbf{A}(\Omega)),$$

for measurable sets

$$N_{\lambda,\varepsilon} := \{x \in \Omega \mid \operatorname{dist}(\lambda, \sigma(\mathbf{A}(x))) < \varepsilon\}.$$

On the one hand, along the same lines of that proof, $\sigma(\mathbf{A}) \subset \text{ess}-\sigma(\mathbf{A}(\Omega))$ also holds for $p = \infty$. On the other hand, the proof of $\sigma(\mathbf{A}) \supset \text{ess}-\sigma(\mathbf{A}(\Omega))$ given in [22, Chapter IX, Theorem 2.4] does not apply for $p = \infty$. In order to prove the above representation (C.1) of the spectrum, it remains to show the inclusion $\text{ess}-\sigma(\mathbf{A}(\Omega)) \subset \sigma(\mathbf{A})$ and [22, Chapter IX, Remark 2.3] yields the result.
Using the idea of [35, Theorem 3.3], we show that each $\lambda \in \text{ess}-\sigma(\mathbf{A}(\Omega))$ is in the spectrum of $\mathbf{A}$. As the characteristic polynomial of the matrix $\mathbf{A}(x)$ factorizes with eigenvalues $\lambda_i(x) \in \mathbb{C}$, we obtain

$$|\det(\lambda I - \mathbf{A}(x))| = \prod_{i=1}^{m} |\lambda - \lambda_i(x)| \geq \operatorname{dist}(\lambda, \sigma(\mathbf{A}(x)))^m \tag{C.2}$$

for a.e. $x \in \Omega$. This estimate yields the inclusion

$$\Gamma_{\lambda,\varepsilon} := \{x \in \Omega \mid |\det(\lambda I - \mathbf{A}(x))| < \varepsilon^m\} \subset N_{\lambda,\varepsilon}$$

and we conclude that

$$\Gamma_\lambda := \{x \in \Omega \mid \det(\lambda I - \mathbf{A}(x)) = 0\} \tag{C.3}$$

satisfies $0 \leq |\Gamma_\lambda| \leq \lim_{\varepsilon\to 0} |N_{\lambda,\varepsilon}|$ as the limit of the above subsets of $N_{\lambda,\varepsilon}$ as $\varepsilon \to 0$. The sequence $(|\Gamma_{\lambda,\varepsilon}|)_{\varepsilon>0}$ of non-negative numbers is non-increasing as $\varepsilon \to 0$ with a limit which is either positive or zero. In the former case, we conclude that Γ_λ defined in (C.3) has positive measure which is equivalent to $\lambda \in \sigma_p(\mathbf{A})$ using [43, Theorem 2.1] or [35, Theorem 2.5]. In the latter case, $|\Gamma_\lambda| = \lim_{\varepsilon\to 0} |\Gamma_{\lambda,\varepsilon}| = 0$, we show that the injective operator $\lambda I - \mathbf{A}$ is not bounded from below, hence $\lambda \in \sigma(\mathbf{A})$. Although we know from $\lambda \in \text{ess}-\sigma(\mathbf{A}(\Omega))$ that $|N_{\lambda,\varepsilon}| > 0$ for all $\varepsilon > 0$, there are still two possibilities for the zero sequence $(|\Gamma_{\lambda,\varepsilon}|)_{\varepsilon>0}$: either $|\Gamma_{\lambda,\varepsilon}| > 0$ for all $\varepsilon > 0$ or the sequence becomes stationary in the sense that $|\Gamma_{\lambda,\varepsilon}| = 0$ for all $0 < \varepsilon \leq \varepsilon_0$ and some $\varepsilon_0 > 0$. In both cases we construct a sequence $(\mathbf{f}_j)_{j\in\mathbb{N}} \subset L^p(\Omega)^m$ with $\|\mathbf{f}_j\|_{L^p(\Omega)^m} = 1$ for which $\|(\lambda I - \mathbf{A})\mathbf{f}_j\|_{L^p(\Omega)^m} \to 0$ as $j \to \infty$, hence $\lambda I - \mathbf{A}$ can not be bounded from below.

- Let $|\Gamma_{\lambda,\varepsilon}| > 0$ for all $\varepsilon > 0$. Thus, we are able to extract a decreasing subsequence $(\Gamma_{\lambda,\varepsilon_j})_{j\in\mathbb{N}}$ with $\varepsilon_j \to 0$ as $j \to \infty$ such that

$$|\Gamma_{\lambda,\varepsilon_j}| > 0, \qquad \Gamma_{\lambda,\varepsilon_{j+1}} \subset \Gamma_{\lambda,\varepsilon_j} \qquad \text{and} \qquad \left|\Gamma_{\lambda,\varepsilon_j} \setminus \Gamma_{\lambda,\varepsilon_{j+1}}\right| > 0.$$

By choosing measurable sets $M_j \subset \Gamma_{\lambda,\varepsilon_j} \setminus \Gamma_{\lambda,\varepsilon_{j+1}}$ with $|M_j| > 0$ for all $j \in \mathbb{N}$ we obtain the estimate

$$\varepsilon_{j+1}^m \leq |\det(\lambda I - \mathbf{A}(x))| < \varepsilon_j^m \qquad \forall\, x \in M_j. \tag{C.4}$$

This enables us to apply [35, Lemma 3.1] to the matrix $(\lambda - \mathbf{A}(x))^{-1}$. Consequently, we find measurable vector-valued functions $\mathbf{v}_j : M_j \to \mathbb{C}^m$ satisfying

$$|\mathbf{v}_j(x)|_2 = 1 \qquad \text{and} \qquad \left|(\lambda I - \mathbf{A}(x))^{-1}\mathbf{v}_j(x)\right|_2 = \left|(\lambda I - \mathbf{A}(x))^{-1}\right|_2$$

for all $x \in M_j$, where we used the Euclidean norm $|\cdot|_2$ on $\mathbb{C}^m$ and for the induced matrix norm $|\cdot|_2$. Define $\mathbf{u}_j(x) = (\lambda I - \mathbf{A}(x))^{-1}\mathbf{v}_j(x)$ as well as functions $\mathbf{f}_j \in L^p(\Omega)^m$ by

$$\mathbf{f}_j(x) = c_p(j)\frac{\mathbf{u}_j(x)}{|\mathbf{u}_j(x)|_2}\chi_{M_j}(x)$$

where $c_p(j) = |M_j|^{-1/p}$ for $p < \infty$ and $c_p(j) = 1$ for $p = \infty$. Here, we fix $|\cdot|_2$ as the vector norm on $\mathbb{C}^m$. This implies

$$\|\mathbf{f}_j\|_{L^p(\Omega)^m}^p = \int_\Omega |\mathbf{f}_j(x)|_2^p \,\mathrm{d}x = 1$$

with an obvious modification for $p = \infty$. Applying $\lambda I - \mathbf{A}$ to $\mathbf{f}_j$ yields

$$(\lambda I - \mathbf{A}(x))\mathbf{f}_j(x) = c_p(j)\chi_{M_j}(x)\mathbf{v}_j(x)\left|(\lambda I - \mathbf{A}(x))^{-1}\right|_2^{-1}.$$

From the invertibility condition (C.4) we infer

$$\left|(\lambda I - \mathbf{A}(x))^{-1}\right|_2^{-1} \leq \operatorname{dist}(\lambda, \sigma(\mathbf{A}(x))) \qquad \forall\, x \in M_j$$

where we used [23, Chapter IV, Corollary 1.14]. A combination of estimates (C.2) and (C.4) yields

$$\operatorname{dist}(\lambda, \sigma(\mathbf{A}(x))) < \varepsilon_j \qquad \forall\, x \in M_j,$$

which implies
$$\|(\lambda I - \mathbf{A})\mathbf{f}_j\|_{L^p(\Omega)^m} \leq \varepsilon_j.$$

Since $\varepsilon_j \to 0$, λ is an approximate eigenvalue of $\mathbf{A}$, i.e., $\lambda \in \sigma(\mathbf{A})$.

- Let $|\Gamma_{\lambda,\varepsilon}| = 0$ for all $0 < \varepsilon \leq \varepsilon_0$. The definition of $\Gamma_{\lambda,\varepsilon}$ yields the pointwise invertibility condition

$$|\det(\lambda I - \mathbf{A}(x))| \geq \varepsilon_0^m > 0 \qquad \text{for a.e. } x \in \Omega.$$

Taking $M_j := N_{\lambda,\varepsilon_j} \subset \Omega$ with $|M_j| > 0$ for any zero sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, we find, similar to the above reasoning, a sequence $(\mathbf{f}_j)_{j\in\mathbb{N}} \subset L^p(\Omega)^m$ satisfying

$$\|(\lambda I - \mathbf{A})\mathbf{f}_j\|_{L^p(\Omega)^m} \leq \varepsilon_j.$$

Since $\varepsilon_j \to 0$, λ is an approximate eigenvalue of $\mathbf{A}$, i.e., $\lambda \in \sigma(\mathbf{A})$. Note that in this case, $N_{\lambda,\varepsilon_j}$ cannot become stationary since then M_j and $\mathbf{f}_j$ would become stationary which implies $(\lambda I - \mathbf{A})\mathbf{f}_j = \mathbf{0}$ – a contradiction to $\mathbf{f}_j \neq \mathbf{0}$.

It remains to show that $\lambda I - \mathbf{A}$ is not Fredholm for all $\lambda \in \sigma(\mathbf{A})$. To do so, we prove that for each $\lambda \in \sigma(\mathbf{A})$ either $\lambda I - \mathbf{A}$ has no closed range or an infinite-dimensional kernel. This implies that $\lambda I - \mathbf{A}$ is not Fredholm.
If $\lambda \in \sigma_p(\mathbf{A})$, notice that the results [35, Lemma 2.4, Theorem 2.5] hold independently of $1 \leq p \leq \infty$. Hence, the first part of the proof of [35, Proposition 3.2] is still applicable: we infer $\sigma_p(\mathbf{A}) \subset \sigma_{\text{ess}}(\mathbf{A})$ from an infinite-dimensional kernel of $\lambda I - \mathbf{A}$ containing a subspace isomorphic to $L^p(\Gamma_\lambda)$ due to [35, Corollary 2.6].
If $\lambda \in \sigma(\mathbf{A}) \setminus \sigma_p(\mathbf{A})$, we necessarily have $|\Gamma_\lambda| = \lim_{\varepsilon\to 0} |\Gamma_{\lambda,\varepsilon}| = 0$. From above reasoning we know that $\lambda I - \mathbf{A}$ is not bounded from below. Thus, the injective operator $\lambda I - \mathbf{A}$ cannot have closed range by [8, Theorem 2.19, Remark 18] and $\lambda I - \mathbf{A}$ is not a Fredholm operator. Recall that the constructed sequence $(\mathbf{f}_j)_{j\in\mathbb{N}} \subset L^p(\Omega)^m$ is in fact singular, see [20, Chapter 9, Definition 1.2], subject to a similar choice of disjoint sets M_j in the second case above.

□

Remark that the above proof may be shortened extremely for the cases $1 \leq p < \infty$. One can essentially use the same method of proof from [35, Proposition 3.2] for the case $p = 2$ having the characterization from [22, Chapter IX, Proposition 1.4] for the dual multiplication operator in mind.

References

[1] R.A. Adams, J.F. Fournier, Sobolev spaces. Academic Press Amsterdam, 2003.

[2] W. Arendt, Gaussian estimates and interpolation of the spectrum in L^p. Diff. Int. Equ., Vol. 7, No. 5 (1994), pp. 1153-1168.

[3] W. Arendt, C. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems. Monographs in Mathematics, Vol. 96, 2nd Edition, Springer-Verlag Basel, 2011.

[4] W. Arendt, A.F.M. ter Elst, Gaussian estimates for second order elliptic operators with boundary conditions. J. Operator Theory, Vol. 38, No. 1 (1997), pp. 87-130.

[5] F.V. Atkinson, H. Langer, R. Mennicken, A.A. Shkalikov, The essential spectrum of some matrix operators. Math. Nachr., Vol. 167, No. 1 (1994), pp. 5-20.

[6] A. Bobrowski, Singular perturbations involving fast diffusion. J. Math. Anal. Appl., Vol. 427, No. 2 (2015), pp. 1004-1026.

[7] A. Bobrowski, M. Kunze, Irregular convergence of mild solutions of semilinear equations. J. Math. Anal. Appl., Vol. 472, No. 2 (2019), pp. 1401-1419.

[8] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. Springer-Verlag New York, 2010.

[9] C. Buşe, A.D.R. Choudary, S.S. Dragomir, M.S. Prajea, On uniform exponential stability of exponentially bounded evolution families. Int. Equ. Oper. Theory, Vol. 61 (2008), pp. 325-340.

[10] J.-E. Busse, P. Gwiazda, A. Marciniak-Czochra, Mass concentration in a nonlocal model of clonal selection. J. Math. Biol., Vol. 73 (2016), pp. 1001-1033.

[11] J. Carr, Applications of centre manifold theory. Springer-Verlag New York, 1981.

[12] K.N. Chueh, C.C. Conley, J.A. Smoller, Positively invariant regions for systems of nonlinear diffusion equations. Indiana Univ. Math. J., Vol. 26 (1977), pp. 373-392.

[13] J.A. Clarkson, Uniformly convex spaces. Trans. Amer. Math. Soc., Vol. 40, No. 3 (1936), pp. 396-414.

[14] Ju.L. Daleckiĭ, M.G. Kreĭn, Stability of solutions of differential equations in Banach spaces. Translations in Math. Monographs, Vol. 43, Amer. Math. Soc. Providence, 1974.

[15] D. Daners, Heat kernel estimates for operators with boundary conditions. Math. Nachr., Vol. 217, No. 1 (2000), pp. 13-41.

[16] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space. SIAM J. Math. Anal., Vol. 3, No. 3 (1972), pp. 428-445.

[17] E.B. Davies, One-parameter semigroups. LMS Monographs No. 15, Academic Press London, 1980.

[18] E.B. Davies, Heat kernels and spectral theory. Cambridge Tracts in Math., Vol. 92, Cambridge Univ. Press Cambridge, 1989.

[19] H. Dong, D. Kim, Elliptic equations in divergence form with partially BMO coefficients. Arch. Rational Mech. Anal., Vol. 196 (2010), pp. 25-70.

[20] D.E. Edmunds, W.D. Evans, Spectral theory and differential operators. 2nd Edition, Oxford Mathematical Monographs, Clarendon Press Oxford, 2018.

[21] T. Eisner, Stability of operators and operator semigroups. Operator Theory: Advances and Applications, Vol. 209, Birkhäuser-Verlag Basel, 2010.

[22] K.-J. Engel, Operator matrices and systems of evolution equations. Habilitationsschrift, Universität Tübingen, 1995.

[23] K.-J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations. Springer-Verlag New York, 2000.

[24] L.C. Evans, Partial differential equations. 2nd Edition. Amer. Math. Soc. Providence, 2010.

[25] M. Fila, H. Ninomiya, Reaction versus diffusion: blow-up induced and inhibited by diffusivity. Russian Math. Surveys, Vol. 60, No. 6 (2005), pp. 1217-1235.

[26] S. Friedlander, W. Strauss, M. Vishik, Nonlinear instability in an ideal fluid. Ann. Inst. H. Poincaré, Anal. Non Linéaire, Vol. 14, No. 2 (1997), pp. 187-209.

[27] P. Getto, A. Marciniak-Czochra, Y. Nakata, M. Vivanco, Global dynamics of two-compartment models for cell production systems with regulatory mechanisms. Math. Biosciences, Vol. 245, No. 2 (2013), pp. 258-268.

[28] A. Gierer, H. Meinhardt, A theory of biological pattern formation. Kybernetik, Vol. 12 (1972), pp. 30-39.

[29] P. Gray, S.K. Scott, Autocatalytic reactions in the isothermal continuous stirred tank reactor: Isolas and other forms of multistability. Chem. Eng. Sci., Vol. 38, No. 1 (1983), pp. 29-43.

[30] P. Grisvard, Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, Vol. 24, Pitman Press Boston, 1985.

[31] P.V. Hai, Polynomial stability of evolution cocycles and Banach function spaces. Bull. Belg. Math. Soc. Simon Stevin, Vol. 26, No. 2 (2019), pp. 299-314.

[32] J.K. Hale, K. Sakamoto, Shadow systems and attractors in reaction-diffusion equations. Appl. Anal., Vol. 32, No. 3-4 (1989), pp. 287-303.

[33] G.W. Hanson, A.B. Yakovlev, Operator theory for electromagnetics: an introduction. Springer-Verlag New York, 2002.

[34] M. Haragus, G. Iooss: Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems. Springer-Verlag London, 2011.

[35] V. Hardt, E. Wagenführer, Spectral properties of a multiplication operator. Math. Nachr., Vol. 178, No. 1 (1996), pp. 135-156.

[36] S. Härting, Reaction-diffusion-ODE systems: de-novo formation of irregular patterns and model reduction. Dissertation, Heidelberg University, 2016.

[37] S. Härting, A. Marciniak-Czochra, Spike patterns in a reaction-diffusion-ODE model with Turing instability. Math. Meth. Appl. Sci., Vol. 37, No. 9 (2014), pp. 1377-1391.

[38] S. Härting, A. Marciniak-Czochra, I. Takagi, Stable patterns with jump discontinuity in systems with Turing instability and hysteresis. Disc. Cont. Dyn. Syst. - A, Vol. 37, No. 2 (2017), pp. 757-800.

[39] K. Hattaf, N. Yousfi, Global stability for reaction-diffusion equations in biology. Comp. Math. Appl., Vol. 66, No. 8 (2013), pp. 1488-1497.

[40] D. Henry, Geometric theory of semilinear parabolic equations. Lecture Notes in Math., Vol. 840, Springer-Verlag Berlin, 1981.

[41] V. Hernández-Santamaría, K. Le Balc'h, Local-controllability of the one-dimensional nonlocal Gray-Scott model with moving controls (2020). Preprint available on arXiv.

[42] V. Hernández-Santamaría, E. Zuazua, Controllability of shadow reaction-diffusion systems. J. Diff. Eq., Vol. 268, No. 7 (2019), pp. 3781-3818.

[43] R. Heymann, Eigenvalues and stability properties of multiplication operators and multiplication semigroups. Math. Nachr., Vol. 287, No. 5-6 (2014), pp. 574-584.

[44] S. Hock, Y. Ng, J. Hasenauer, D. Wittmann, D. Lutter, D. Trümbach, W. Wurst, N. Prakash, F.J. Theis, Sharpening of expression domains induced by transcription and microRNA regulation within a spatio-temporal model of mid-hindbrain boundary formation. BMC Syst. Biol., Vol. 7, No. 48 (2013).

[45] J. Hu, Q. Wang, J. Yang, L. Zhang, Global existence and steady states of a two competing species Keller-Segel chemotaxis model. Kinetic & Related Models, Vol. 8, No.4 (2015), pp. 777-807

[46] K. Ishige, H. Yagisita, Blow-up problems for a semilinear heat equation with large diffusion. J. Diff. Equ., Vol. 212, No. 1 (2005), pp. 114-128.

[47] A. Jeribi, N. Moalla, I. Walha, Spectra of some block operator matrices and application to transport operators. J. Math. Anal. Appl., Vol. 351, No. 1 (2009), pp. 315-325.

[48] G. Karch, K. Suzuki, J. Zienkiewicz, Finite-time blowup of solutions to some activator-inhibitor systems. Discr. Cont. Dyn. Syst., Vol. 36, No. 9 (2016), pp. 4997-5010.

[49] T. Kato, Integration of the equation of evolution in a Banach space. J. Math. Soc. Japan, Vol. 5, No. 2 (1953), pp. 208-234.

[50] T. Kato, Perturbation theory for linear operators. Springer-Verlag Berlin, 1995.

[51] J.P. Keener, Activators and inhibitors in pattern formation. Studies in Appl. Math., Vol. 59, No. 1 (1978).

[52] V. Klika, R.E. Baker, D. Headon, E.A. Gaffney, The influence of receptor-mediated interactions on reaction-diffusion mechanisms of cellular self-organization. Bull. Math. Biol., Vol. 74 (2012), pp. 935-957.

[53] S. Kondo, M. Mimura, A reaction-diffusion system and its shadow system describing harmful algal blooms. Tamkang J. Math., Vol. 47, No. 1 (2016), pp. 71-92.

[54] A. Köthe, A. Marciniak-Czochra, I. Takagi, Hysteresis-driven pattern formation in reaction-diffusion-ODE systems. Disc. Cont. Dyn. Syst. - A, Vol. 40, No. 6 (2020), pp. 3595-3627.

[55] C. Kowall, A. Marciniak-Czochra, A. Mikelić, Long-time shadow limit for a reaction-diffusion-ODE system. Appl. Math. Letters, Vol. 112 (2021) 106790.

[56] G.I. Kresin, V.G. Maz'ya, Criteria for validity of the maximum modulus principle for solutions of linear parabolic systems. Ark. Mat., Vol. 32, No. 1 (1994), pp. 121-155.

[57] K. Kuto, T. Tsujikawa, Limiting structure of steady-states to the Lotka-Volterra competition model with large diffusion and advection. J. Diff. Equ., Vol. 258, No. 5 (2015), pp. 1801-1858.

[58] H. Laasri, D. Mugnolo, Ultracontractivity and Gaussian bounds for evolution families associated with nonautonomous forms. Math. Meth. Appl. Sci., Vol. 43, No. 3 (2020), pp. 1409-1436.

[59] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and quasi-linear equations of parabolic type. Translations of Mathematical Monographs Reprint, Vol. 23, Amer. Math. Soc. Providence, 1968.

[60] I. Lengyel, I.R. Epstein, A chemical approach to designing Turing patterns in reaction-diffusion systems. Proc. Natl. Acad. Sci. USA, Vol. 89, No. 9 (1992), pp. 3977-3979.

[61] F. Li, K. Nakashima, W.-M. Ni, Stability from the point of view of diffusion, relaxation and spatial inhomogeneity. Discrete Contin. Dyn. Syst., Vol. 20, No. 2 (2008), pp. 259-274.

[62] F. Li, W.-M. Ni, On the global existence and finite time blow-up of shadow systems. J. Diff. Equ., Vol. 247, No. 6 (2009), pp. 1762-1776.

[63] F. Li, N.K. Yip, Finite time blow-up of parabolic systems with nonlocal terms. Indiana Univ. Math. J., Vol. 63, No. 3 (2014), pp. 783-829.

[64] G.M. Lieberman, Second order parabolic differential equations. World Scientific Singapore, 1996.

[65] C.-P. Lo, A blow-up problem of reaction diffusion equation related to the diffusion induced blow-up phenomenon (2003). arXiv: math.AP/0307397.

[66] D.L. Lovelady, R.H. Martin, Jr., A global existence theorem for a nonautonomous differential equation in a Banach space. Proc. Amer. Math. Soc., Vol. 35, No. 2 (1972), pp. 445-449.

[67] P.K. Maini, J. Wei, M. Winter, Stability of spikes in the shadow Gierer-Meinhardt system with Robin boundary conditions. Chaos, Vol. 17, No. 3 (2007), 037106.

[68] A. Marasco, A. Iuorio, F. Cartení, G. Bonanomi, D.M. Tartakovsky, S. Mazzoleni, F. Giannino, Vegetation pattern formation due to interactions between water availability and toxicity in plant-soil feedback. Bull. Math. Biol., Vol. 76 (2014), pp. 2866-2883.

[69] A. Marciniak-Czochra, S. Härting, G. Karch, K. Suzuki, Dynamical spike solutions in a nonlocal model of pattern formation. Nonlinearity, Vol. 31, No. 5 (2018).

[70] A. Marciniak-Czochra, G. Karch, K. Suzuki, Unstable patterns in reaction-diffusion model of early carcinogenesis. J. Math. Pures Appl., Vol. 99, No. 5 (2013), pp. 509-543.

[71] A. Marciniak-Czochra, G. Karch, K. Suzuki, Instability of Turing patterns in reaction-diffusion-ODE systems. J. Math. Biol., Vol. 74 (2017), pp. 583-618.

[72] A. Marciniak-Czochra, G. Karch, K. Suzuki, J. Zienkiewicz, Diffusion-driven blowup of nonnegative solutions to reaction-diffusion-ODE systems. Diff. Int. Equ., Vol. 29, No. 7-8 (2016), pp. 715-730. arXiv: math.AP/1511.02510.

[73] A. Marciniak-Czochra, M. Kimmel, Modelling of early lung cancer progression: influence of growth factor production and cooperation between partially transformed cells. Math. Models Methods Appl. Sci., Vol. 17, suppl. (2007), pp. 1693-1719.

[74] A. Marciniak-Czochra, M. Kimmel, Reaction-diffusion model of early carcinogenesis: the effects of influx of mutated cells. Math. Model. Nat. Phenom., Vol. 3, No. 7 (2008), pp. 90-114.

[75] A. Marciniak-Czochra, A. Mikelić, Shadow limit using renormalization group method and center manifold method. Vietnam J. Math., Vol. 45 (2017), pp. 103-125.

[76] R.H. Martin, Jr., Nonlinear operators and differential equations in Banach spaces. Pure Appl. Math., Wiley Interscience Publication New York, 1976.

[77] L. Mattner, Complex differentiation under the integral. Nieuw Arch. Wiskd., Vol. 5, No. 2 (2001), pp. 32-35.

[78] Y. Miyamoto, Stability of a boundary spike layer for the Gierer-Meinhardt system. Euro. J. Appl. Math., Vol. 16, No. 4 (2005).

[79] Y. Miyamoto, Upper semicontinuity of the global attractor for the Gierer-Meinhardt model. J. Diff. Equ., Vol. 223, No. 1 (2006), pp. 185-207.

[80] Y. Miyamoto, On stable patterns for reaction-diffusion equations and systems. RIMS Kôkyûroku Bessatsu B3 (2007), pp. 59-82.

[81] J. Morgan, On the question of blow-up for semilinear parabolic systems. Diff. Int. Equ., Vol. 3, No. 3 (1990), pp. 937-978.

[82] P. De Mottoni, F. Rothe, Convergence to homogeneous equilibrium state for generalized Volterra-Lotka systems with diffusion. SIAM J. Appl. Math., Vol. 37, No. 3 (1979), pp. 648-663.

[83] J.D. Murray, Mathematical biology II. Spatial models and biomedical applications. 3rd Edition. Interdisc. Appl. Math., Vol. 18, Springer-Verlag New York, 2003.

[84] W.-M. Ni, P. Poláčik, E. Yanagida, Monotonicity of stable solutions in shadow systems. Trans. Amer. Math. Soc., Vol. 353, No. 12 (2001), pp. 5057-5069.

[85] W.-M. Ni, I. Takagi, E. Yanagida, Stability of least energy patterns of the shadow system for an activator-inhibitor model. Japan J. Indust. Appl. Math., Vol. 18, No. 259 (2001), pp. 259-272.

[86] Y. Nishiura, Global structure of bifurcating solutions of some reaction-diffusion systems. SIAM J. Math. Anal., Vol. 13, No. 4 (1982), pp. 555-593.

[87] Y. Nishiura, Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit. Dynamics Reported, Vol. 3 (1994), pp. 25-103.

[88] E.M. Ouhabaz, L^p contraction semigroups for vector valued functions. Positivity, Vol. 3 (1999), pp. 83-93.

[89] E.M. Ouhabaz, Analysis of heat equations on domains. Princton Univ. Press Princton, 2005.

[90] N.H. Pavel, Semilinear equations with dissipative time-dependent domain perturbations. Israel J. Math., Vol. 46, No. 1-2 (1983), pp. 103-122.

[91] N.H. Pavel, Nonlinear evolution operators and semigroups. Applications to partial differential equations. Lecture Notes in Math., Vol. 1260, Springer-Verlag Berlin, 1987.

[92] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer-Verlag New York, 1983.

[93] E.J. Putzer, Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients. Amer. Math. Monthly, Vol. 73, No. 1 (1966), pp. 2-7.

[94] F. Rothe, Global solutions of reaction-diffusion systems. Lecture Notes in Math., Vol. 1072, Springer-Verlag Berlin, 1984.

[95] N. Rouche, P. Habets, M. Laloy, Stability theory by Lyapunov's direct method. Appl. Math. Sci., Vol. 22. Springer-Verlag New York, 1977.

[96] R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations. Forum Math., Vol. 11, No. 5 (1999), pp. 543-566.

[97] J. Shatah, W. Strauss, Spectral condition for instability. Contemp. Math., Vol. 255, Amer. Math. Soc. Providence, 2000, pp. 189-198.

[98] R.E. Showalter, Monotone operators in Banach spaces and nonlinear partial differential equations. Math. Surv. Monogr., Vol. 49, Amer. Math. Soc. Providence, 1997.

[99] S. Smith, N. Dalchau, Model reduction enables Turing instability analysis of large reaction-diffusion models. J. Roy. Soc. Interf., Vol. 15, No. 140 (2018).

[100] J. Smoller, Shock waves and reaction-diffusion equations. Grundlehren der mathematischen Wissenschaften, Vol. 258, Springer-Verlag New York, 1994.

[101] G. Stampacchia, Équations elliptiques du second ordre a coefficients discontinus. Presses de l'Université de Montréal, Canada, 1965.

[102] H. Takagi, Fredholm weighted composition operators. Int. Equ. Oper. Theory, Vol. 16 (1993), pp. 267-276.

[103] I. Takagi, Point condensation for a reaction-diffusion system. J. Diff. Equ., Vol. 61, No. 2 (1986), pp. 208-249.

[104] S. Thomaschewski, Form methods for autonomous and nonautonomous Cauchy problems, Dissertation, Ulm University, 2003.

[105] J. Wang, The stability of equilibria for a reaction-diffusion-ODE system on convex domains. Appl. Math. Letters, Vol. 93 (2019), pp. 147-152.

[106] L. Wang, M.Y. Li, Diffusion-driven instability in reaction-diffusion system. J. Math. Anal. Appl., Vol. 254, No. 1 (2001), pp. 138-153.

[107] L. Wang, Y. Wu, Q. Xu, Instability of spiky steady states for S–K–T biological competing model with cross-diffusion. Nonlinear Analysis, Vol. 159 (2017), pp. 424-457.

[108] G.F. Webb, Theory of nonlinear age-dependent population dynamics. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 89, Marcel Dekker, New York, 1985.

[109] J. Wei, Existence and stability of spikes for the Gierer-Meinhardt system. Handbook of Differential Equations: Stationary Partial Differential Equations V, North-Holland Amsterdam, 2008, pp. 487-585.

[110] J. Wei, M. Winter, Mathematical aspects of pattern formation in biological systems. Springer-Verlag London, 2014.

[111] K.A.J. White, C.A. Gilligan, Spatial heterogeneity in three species, plant-parasite-hyperparasite, systems. Phil. Trans. R. Soc. London B, Vol. 353, No. 1368 (1998), pp. 543-557.

[112] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. J. Diff. Equ., Vol. 248, No. 12 (2010), pp. 2889-2905.

[113] A. Yagi, Abstract parabolic evolution equations and their applications. Springer Monographs in Mathematics, Springer-Verlag Berlin, 2010.