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Resolution of Stratifolds and Connection to Mather's Abstract Pre-Stratified Spaces

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Preface

Many spaces which naturally occur in topology and algebraic geometry are not manifolds, but have a decomposition as a disjoint union of manifolds. Examples include algebraic varieties, orbit spaces of proper smooth group actions on manifolds and mappings cylinders of maps between manifolds.

In 1998 M. Kreck began to develop a concept of stratified spaces such that their bordism theory leads to a homology theory, which has the same coefficients as the singular homology. He presented geometrical proofs of various classical results such as the Künneth-Formula or the Hurewicz-Theorem. At the same time one began to develop the interest in the objects themselves. After more than 4 years of developments, M. Kreck introduced the category of stratifolds, stratified spaces, together with an algebra of real valued functions satisfying some subtle conditions. In this thesis we will focus on two subclasses of stratifolds with more geometrical structure, namely p-stratifolds and cornered p-stratifolds.

Roughly speaking, in the case of p-stratifolds, topological spaces are constructed by attaching manifolds with boundary by a map to the space, which is already inductively constructed. The other way to think of p-stratifolds is to think of the generalization of CW-complexes, where one uses arbitrary manifolds with boundaries instead of unit balls D^n. The attaching map has to fulfil certain properties and, in contrast to CW-complexes, is a part of the data.

This thesis is divided into three parts. In the first chapter we give definitions of different kinds of stratifolds: stratifolds, locally trivial stratifolds, parametrized stratifolds and cornered p-stratifolds. We give the general definitions and work out an alternative geometrical description, which will be used in the following chapters. The second and third chapters deal with two different problems and can be read separately.

The second chapter investigates the resolution of p-stratifolds. The definition of the resolution is modelled on the definition from algebraic geometry, see [Hi].

In contrast to algebraic varieties, resolutions of p-stratifolds in general do not exist, even for isolated singularities, where the topological space is obtained from a manifold with boundary by collapsing boundary components to single points. But in this case M. Kreck stated a simple necessary and sufficient condition (Theorem 2.2.1). In this thesis, we investigate the case of isolated singularities in more detail, find necessary and sufficient conditions for the existence of so-called optimal resolutions (Theorem 2.2.5), then go on with the classification of optimal resolutions in the even-dimensional case and state our classification result in Theorem 2.2.8, which is valid from dimension 6 and up.

We also consider algebraic varieties and show that every algebraic variety with isolated singularities admits a canonical structure of a p-stratifold. We further conclude, that a hypersurface with isolated singularities always admits an optimal resolution.

The special case of 4-dimensional p-stratifolds with isolated singularities will be treated in §2.2.3.

In the next section we apply the methods developed by dealing with isolated singularities to the general situation of p-stratifolds. We obtain results for the existence of (optimal) resolutions up to deformation of the attaching map of the highest dimensional manifold (Theorems 2.3.2 & 2.3.3).

Section 2.4 is concerned with a special case of a p-stratifold, which is built out of a sphere by attaching a higher-dimensional manifold via a differential fibre bundle on the boundary. In this situation, we give an inductive construction of a resolution and show that the sufficient condition for the existence of the resolution derived in the previous section is also necessary in this special situation (Corollary 2.4.2). P-stratifolds of this type occur for example if one takes a Witten space, 7-dimensional closed Riemannian manifold W with $\mathrm{iso}(W) \cong SU(3) \times SU(2) \times U(1)$, where $SU(3) \times SU(2) \times U(1)$ acts transitively, and divides out an $S^1 \in \mathrm{iso}(W)$. We investigate optimal spin resolutions of 6-dimensional p-stratifolds built out of a 2-sphere (the quotient W/S^1 described above is precisely of this type), see Theorem 2.4.3. We further state a classification result for optimal resolution in Theorem 2.4.10. Then we leave the world of resolutions and address the connections of p-stratifolds to Mather's abstract pre-stratified spaces.

In the last chapter we establish the main theorem, which states that every abstract pre-stratified space in the sense of Mather [Ma] is a cornered p-stratifold. We first give an introduction to abstract pre-stratified spaces. Secondly we define, analogously to stratifolds with boundary, abstract pre-stratified spaces with boundary and extend some results concerning controlled vector fields to the bounded objects in §3.3.1. In addition we show that by using the algebra of controlled real valued functions on an abstract pre-stratified space, we obtain a stratifold (Lemma 3.3.7). In §3.4 we aim to construct a cornered parametrization on an abstract pre-stratified space (Theorem 3.4.2). Since various classes of singular spaces, such as Whitney stratified spaces, algebraic varieties and orbit spaces of proper smooth group actions on manifolds admit a structure of an abstract pre-stratified space, the last theorem implies that they admit a structure of a cornered p-stratifold as well.

The research described in this thesis took place in the years 1999-2002, during which period I was resident in the University of Heidelberg. Most of all, I wish to thank my supervisor Professor Matthias Kreck for the invaluable assistance he has offered me and the helpful comments and suggestions he has made. I owe my knowledge of this subject to his teaching and I was able to write this thesis only with the help of his constant encouragement and interest. My thanks also go to Christian Ewald, Augusto Minatta, Markus Ulke and Julia Weber for many mathematical and non-mathematical discussions.

Contents

Chapter 1

Introduction to Stratifolds

In this chapter we give the basic definition of stratifolds, the objects we are going to work with. Stratifolds are topological spaces, together with an algebra of real valued maps satisfying some subtle conditions. As the name suggests a stratifold admits a decomposition into smooth manifolds, and is therefore a stratified space. The concept was developed by M. Kreck in [Kr2] and [Kr3], where you can find more information about and results regarding stratifolds.

We start with the general definition of stratifolds and then go on to describe two special classes of stratifolds, namely p-stratifolds and cornered p-stratifolds.

1.1 Stratifolds

In this section we give a brief introduction to stratifolds. For various examples, explicit arguments and applications we refer to [Kr2].

We use the language of differential spaces introduced by Sikorski [Si]. First of all, however, we have to introduce some notation.

Definition: An algebra $\mathcal{C} \subset \mathcal{C}^0(X)$ is called *locally detectable* if a continuous function $f : X \to \mathbb{R}$ is in $\mathcal{C}$ if and only if for each $x \in X$ there is an open neighbourhood V of x and an element $g \in \mathcal{C}$ such that $g|_V = f|_V$.

To obtain a feeling for the use of locally detectable algebras we introduce the following notation. If $\mathcal{C} \subset \mathcal{C}^0(X)$ is an algebra and U an open subspace of X, we consider the algebra $\mathcal{C}(U)$ which is defined as the continuous maps $f : U \to \mathbb{R}$ such that for each $x \in U$ there is an open neighbourhood $V \subset U$

of x and $g \in \mathcal{C}$ such that $g|_V = f|_V$. Obviously, if $f \in \mathcal{C}$, then $f|_U$ is in $\mathcal{C}(U)$. Using this language, an algebra $\mathcal{C}$ of functions on X is locally detectable if and only if $\mathcal{C}(X) = \mathcal{C}$.

Now, if $\mathcal{C}$ is a locally detectable algebra, we can construct functions in $\mathcal{C}$ in the following way, which is commonly known and used for the construction of continuous or smooth functions. Suppose that we have a covering of X by open subsets U_i, i.e. $X = \cup U_i$, and that we have elements $f_i \in \mathcal{C}(U_i)$ such that for all i and j we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Then there is a unique function $f \in \mathcal{C}$ such that for all i we have $f|_{U_i} = f_i$. Clearly there is a unique continuous function f with this property and we only have to show that $f \in \mathcal{C}$. For this we use that $\mathcal{C}$ is locally detectable. For each $x \in X$ there is an i such that $x \in U_i$ and, since $f|_{U_i} \in \mathcal{C}(U_i)$, there is an open neighbourhood V of x in U_i and $g \in \mathcal{C}$ such that $g|_V = f_V$ implying $f \in \mathcal{C}$, if $\mathcal{C}$ is locally detectable.

Definition: A *differential space* is a pair $(X, \mathcal{C})$, where X is a topological space and $\mathcal{C} \subset \mathcal{C}^0(X)$ an algebra of continuous functions such that

1. $\mathcal{C}$ is locally detectable,
2. for all $f_1, \ldots, f_k \in \mathcal{C}$ and $g : \mathbb{R}^k \longrightarrow \mathbb{R}$, a smooth function, the function $x \mapsto g(f_1(x), \ldots, f_k(x))$ is in $\mathcal{C}$.

We have discussed the use of the first condition above already. The second condition is obviously desirable to construct new elements of $\mathcal{C}$ by composition with smooth maps.

The considerations above show that if M is a k-dimensional smooth manifold then $(M, \mathcal{C}^\infty(M))$ is a differential space. This is the fundamental class of examples which is the model for our generalization to stratifolds in the following discussion. We further note if a differential space $(X, \mathcal{C})$ admits a smooth structure such that $\mathcal{C}^\infty(X) = \mathcal{C}$, then this smooth structure is uniquely determined.

Definition: Let $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ be differential spaces. Then we define $\mathcal{C}(X, X')$ as the continuous maps $f : X \longrightarrow X'$ such that for all $\rho \in \mathcal{C}'$ we have $\rho f \in \mathcal{C}$. We call such a map a *morphism* from $(X, \mathcal{C})$ to $(X', \mathcal{C}')$.

We define a stratifold as a differential space with certain properties. The main consequence of these properties will be that the space is decomposed into a disjoint union of smooth manifolds of various dimensions. If only a single dimension occurs we will get a smooth manifold. Thus there should be a natural decomposition of a differential space into subspaces (which should be smooth manifolds in the end). We begin with a definition of this decomposition.

Let $(X, \mathcal{C})$ be a differential space. For a point $x \in X$ we define the *germ* of function near x as the equivalence classes of $f \in \mathcal{C}(U)$ for an open neighbourhood U of X, where $f : U \longrightarrow \mathbb{R}$ is equivalent to $f' : U' \longrightarrow \mathbb{R}$ if there is an open neighbourhood $V \subset U \cap U'$ of x such that $f|_V = f'|_V$.

Definition: Let $(X, \mathcal{C})$ be a differential space and $x \in X$. The vector space of derivations at x is called the *tangent space* of X at x and denoted by T_xX.

We define the subspace

$$X^i := \{x \in X \,\big|\, \dim T_xX = i\}.$$

We impose conditions on $\mathcal{C}$ which make X^i an i-dimensional smooth manifold.

For a subset Y in a differential space $(X, \mathcal{C})$ we consider the *germ* of smooth functions near Y. This is an equivalence class of functions in $\mathcal{C}(V)$ for some open neighbourhood V of Y in X, where two such functions $f : V \to \mathbb{R}$ and $f' : V' \to \mathbb{R}$ are equivalent if there is an open neighbourhood $V'' \subset V \cap V'$ with $g|_{V''} = g'|_{V''}$. We denote the set of germs by $\Gamma(Y; X)$. For example if $Y = x$ is a point we obtain the previously defined germ $\Gamma(x) = \Gamma(x; X)$. The restriction to Y gives a map $i^* : \Gamma(Y; X) \to \mathcal{C}^0(Y)$.

Now we are able to formulate the condition:

- There is a smooth structure on X^i such that for each x in X^i there is an open neighbourhood W of x in $\mathscr{S}$ such that

$$i^* : \Gamma(W \cap X^i; W) \to \mathcal{C}^\infty(W \cap X^i)$$

  is an isomorphism.

Note that this smooth structure is unique.

As a consequence of the above condition we see that for each $x \in X^i$ the germ of smooth functions on X^i is isomorphic to the germ $\Gamma(x)$ of functions in $\mathcal{C}(U)$ under the restriction map for some open neighbourhood of x in X. In particular we see that the dimension of X^i is i.

We call X^i the *i-stratum* of X. In other concepts of spaces which are decomposed as smooth manifolds the connected components of X^i are called the strata but we prefer to collect the i-dimensional strata into a single stratum. We call $\cup_{i \leq r} X^i =: \Sigma^r$ the *r-skeleton* of X.

Definition: A k-dimensional *stratifold* is a differential space $(\mathscr{S}, \mathcal{C}_{\mathbf{S}}(X))$, where $\mathscr{S}$ is a locally compact Hausdorff space with countable basis, the skeleta Σ^i are closed subspaces and for each $j > i$ we require $\overline{\mathscr{S}^i} \cap \mathscr{S}^j = \emptyset$. In addition we assume:

1. for all $i \leq k$ there is a smooth structure on $\mathscr{S}^i$ such that for each x in $\mathscr{S}^i$ there is an open neighbourhood W of x in $\mathscr{S}$ such that
$$i^* : \Gamma(W \cap \mathscr{S}^i; W) \to \mathcal{C}^\infty(W \cap \mathscr{S}^i)$$
is an isomorphism,

2. $dim T_x \mathscr{S} \leq k$ for all $x \in \mathscr{S}$, i. e. all tangent spaces have dimension $\leq k$,

3. for each $x \in \mathscr{S}$ and open neighbourhood $U \in \mathscr{S}$, there is a function $\rho \in \mathcal{C}_{\mathbf{S}}(X)$ such that $\rho(x) \neq 0$ and supp $\rho \subseteq U$.

The last condition implies, that every open covering of a stratifold $\mathscr{S}$ admits a subordinated partition of unity out of elements of $\mathcal{C}_{\mathbf{S}}(\mathscr{S})$. One says, $\mathscr{S}$ is *paracompact with respect to $\mathcal{C}_{\mathbf{S}}(\mathscr{S})$*.

We denote with $\Gamma(\mathscr{S}^i, M; \mathscr{S})$ the set of germs of morphisms from an open neighbourhood of $\mathscr{S}^i$ in $\mathscr{S}$ to a smooth manifold M, considered as stratifold together with differentiable maps. As the next consequence we note:

Proposition 1.1.1. *Let M be a smooth manifold. The inclusion induces an isomorphism*
$$\Gamma(\mathscr{S}^i, M; \mathscr{S})|_{\mathscr{S}^i} \cong \mathcal{C}^\infty(\mathscr{S}^i, M),$$
between the germs of maps near $\mathscr{S}^i$, which are in $\Gamma(\mathscr{S}^i, M; X)$, and the smooth maps from $\mathscr{S}^i$ to M.

See [Kr3] for a proof. An important consequence is that for all strata we have a *canonical germ of retractions*

$$\pi_i : U_i \longrightarrow \mathscr{S}^i,$$

where U_i is an open neighbourhood of $\mathscr{S}^i$ in $\mathscr{S}$. We have to apply Proposition 1.1.1 to the identity map from $\mathscr{S}^i$ to $\mathscr{S}^i$.

The following is a simple consequence of the definition of a stratifold:

Proposition 1.1.2. *There are representatives of the canonical germs of retractions*

$$\pi_i : U_i \longrightarrow \mathscr{S}^i,$$

such that for all $j > i$

$$\pi_i \pi_j(x) = \pi_i(x)$$

whenever both sides are defined.

This proposition allows an alternative description of a stratifold. Namely, a k-dimensional stratifold $\mathscr{S}$ gives a decomposition into the strata $\mathscr{S}^i$ for $0 \le i \le k$ together with germs of retractions $\pi_i : U_i \longrightarrow \mathscr{S}^i$ such that for all $j > i$

$$\pi_i \pi_j(x) = \pi_i(x)$$

whenever both sides are defined. These maps are smooth on all strata. In addition, for each map $f \in \mathcal{C}_{\mathbf{S}}(\mathscr{S})$ there are representatives of the canonical germs of retractions $\pi_i : U_i \longrightarrow \mathscr{S}^i$, such that $f|_{U_i}$ *commutes* with all π_j, i.e.

$$f\pi_j(x) = f(x),$$

whenever defined.

In turn, suppose we have a space $\mathscr{S}$, together with a decomposition into smooth manifolds $\mathscr{S}^i$ of dimension $\le k$, the strata, and germ of retractions $\pi_i : U_i \longrightarrow \mathscr{S}^i$, which are smooth on all strata, such that for all $j > i$ we have $\pi_i \pi_j(x) = \pi_i(x)$ whenever both sides are defined. Then we can define an algebra $\mathcal{C}_{\{\pi_i\}}(\mathscr{S})$ consisting of continuous functions $f : \mathscr{S} \longrightarrow \mathbb{R}$, which are smooth on all strata and, such that there exist representatives of the canonical germs of retractions $\pi_i : U_i \longrightarrow \mathscr{S}^i$, such that $f|_{U_i}$ commutes with all π_j. The tuple $(\{\mathscr{S}^i\}, [\{\pi_i : U_i \longrightarrow \mathscr{S}^i\}])$ is called *controlled stratification* on $\mathscr{S}$.

If we require that $\mathscr{S}$ is locally compact with countable basis and paracompact with respect to $\mathcal{C}_{\{\pi_i\}}(\mathscr{S})$ and if the condition

- for all $i < j$ we have $\overline{\mathscr{S}^i} \cap \mathscr{S}^j = \emptyset$

holds, then

$$(\mathscr{S}, \mathcal{C}_{\{\pi_i\}}(\mathscr{S}))$$

is a k-dimensional stratifold. Thus we have shown:

Proposition 1.1.3. *There is a bijection between stratifolds and spaces $\mathscr{S}$ with controlled stratification such that $\mathscr{S}$ is paracompact with respect to the algebra $\mathcal{C}_{\{\pi_i\}}(\mathscr{S})$, and for all $i < j$ we have $\overline{\mathscr{S}^i} \cap \mathscr{S}^j = \emptyset$.*

In analogy with maps from a smooth manifold to a smooth manifold we call the morphisms f from a stratifold $\mathscr{S}$ to a smooth manifold *smooth maps.*

1.2 Locally trivial stratifolds

In this section we present a special class of stratifolds, which have "nice" neighbourhood retractions.

Definition: Let $\mathscr{S}$ be a stratifold and M a smooth manifold. A smooth map $p : \mathscr{S} \longrightarrow M$ is called a *stratifold bundle* if for each $x \in M$ there is an open neighbourhood V of x in M, a stratifold F and an isomorphism of stratifolds φ, making the following diagram commutative:

$$\begin{tikzcd}
\varphi : p^{-1}(V) \arrow[rr] \arrow[dr, "p"'] & & V \times F \arrow[dl, "\mathrm{pr}_1"] \\
& V &
\end{tikzcd}$$

Now we consider a stratifold $\mathscr{S}$, a stratum $\mathscr{S}^i$ and a canonical germ of retractions $[\pi_i : U_i \longrightarrow \mathscr{S}^i]$, introduced in the last section.

One can require π_i to be a stratifold bundle, but this condition in general depends on the representative π_i. To avoid this difficulty we make the following definition.

Definition: A stratifold $\mathscr{S}$ is called *locally trivial* if for each i and for each representative $\pi_i : U_i \longrightarrow \mathscr{S}^i$ of the germ of retractions, there is an open neighbourhood U_i' of $\mathscr{S}^i$ in U_i, such that $\pi_i|_{U_i'}$ is a stratifold bundle.

Amongst the locally trivial stratifolds one can distinguish subclasses by imposing conditions on the fibre. For example one can require that the germ of all fibres of π_i is the germ of some cone (*locally coned*) or of the products of some cones (*locally trivial product coned*).

In the following sections we are going to introduce another large class of stratifolds.

1.3 c-manifolds

In this thesis we are studying a special class of stratifolds, namely stratifolds with (cornered) strats, which come with much more structure. To give a definition of these objects we first need a notion of c-manifolds. A c-manifold (collared manifold) is a manifold with boundary, together with a representative of the collar, where we allow more general collars than in the classical theory.

First we have to introduce some notation. A pair of topological spaces is a space X, together with a subspace S. We denote a pair by (X, S).

Furthermore, let U be a neighbourhood of S in X, together with the following continuous functions:

- a retraction $\pi : U \longrightarrow S$ and

- $\rho : U \longrightarrow [0, \infty)$.

Given a continuous map $\delta : S \longrightarrow (0, \infty)$, we introduce the following subsets of U.

$$\begin{array}{rcl} U^{\delta}_{(\pi,\rho)} & := & \{x \in U \big| \rho(x) = \delta(\pi(x))\} \\ U^{<\delta}_{(\pi,\rho)} & := & \{x \in U \big| \rho(x) < \delta(\pi(x))\} \\ U^{\leq\delta}_{(\pi,\rho)} & := & \{x \in U \big| \rho(x) \leq \delta(\pi(x))\} \end{array}$$

In many cases in this thesis, the space S is a smooth manifold such that $\rho^{-1}(0) = S$ and δ is a smooth map. The sets $U^{<\delta}_{(\pi,\rho)}$ $U^{\leq\delta}_{(\pi,\rho)}$ are then neighbourhoods of S in contrast to $U^{\delta}_{(\pi,\rho)}$.

Example:
Let S be a smooth manifold. Consider $X := S \times [0, \infty)$ and identify S with $S \times \{0\} \subset X$. The space X itself is an open neighbourhood of S and the maps π and ρ are given by pr_1 and pr_2 respectively. Let δ be a constant map

with value $c > 0$. Then

$$\begin{aligned}
(S\times[0,\infty))^{\delta}_{(\pi,\rho)} &= S\times\{c\},\\
(S\times[0,\infty))^{<\delta}_{(\pi,\rho)} &= S\times[0,c) \text{ and}\\
(S\times[0,\infty))^{\leq\delta}_{(\pi,\rho)} &= S\times[0,c].
\end{aligned}$$

If the maps π and ρ are the obvious ones as in the case of $S\times[0,\infty)$, we omit the subscript (π,ρ) and just write $(S\times[0,\infty))^{\delta}$ for a given map δ.

We further define

$$(S\times(0,\infty))^{<\delta}_{(\pi,\rho)} := (S\times[0,\infty))^{<\delta}_{(\pi,\rho)} \cap S\times(0,\infty),$$

analogously $(S\times(0,\infty))^{\delta}_{(\pi,\rho)}$ and $(S\times(0,\infty))^{\leq\delta}_{(\pi,\rho)}$.

Definition: Let $(W,\partial W)$ be a pair of topological spaces, such that ∂W and $W-\partial W$ are smooth manifolds and ∂W is closed in W. A *collar* is a homeomorphism

$$\mathbf{c}: (\partial W\times[0,\infty))^{<\delta} \longrightarrow V,$$

where $\delta:\partial W\longrightarrow(0,\infty)$ is a continuous map and V is an open neighbourhood of ∂W in W, such that $\mathbf{c}|_{\partial W\times\{0\}}$ is the identity map to ∂W and $\mathbf{c}|_{(\partial W\times(0,\infty))^{<\delta}}$ is a diffeomorphism onto $V-\partial W$.

If δ is a constant map, we obtain a classical definition of the collar, see [BJ, Def. 13.5].

The *germ* is an equivalence class of collars, where two collars $\mathbf{c}:(\partial W\times[0,\infty))^{<\delta}\longrightarrow W$ and $\mathbf{c}':(\partial W\times[0,\infty))^{<\delta'}\longrightarrow W$ are called equivalent if there is a map δ'', such that $\mathbf{c}|_{(\partial W\times[0,\infty))^{<\delta''}}\equiv\mathbf{c}'|_{(\partial W\times[0,\infty))^{<\delta''}}$. As usual, when we consider equivalence classes, we denote the germ represented by a collar $\mathbf{c}$ by $[\mathbf{c}]$.

Passing to the germ allows us to choose our neighbourhood $(\partial W\times[0,\infty))^{<\delta}$ of ∂W of a particular nice form. For example if ∂W is compact then we can always choose δ to be a constant map with the image $\{\varepsilon\}$, leading to a neighbourhood $\partial W\times[0,\varepsilon)$.

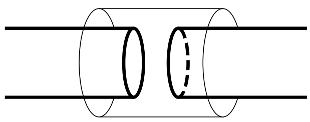
Definition: An n-dimensional *c-manifold* W is a pair of topological spaces $(W,\partial W)$, where $\mathring{W} := W-\partial W$ is a smooth n-dimensional manifold and ∂W is a closed subspace and an $(n-1)$-dimensional manifold, together with a germ of collars $[\mathbf{c}]$. We call ∂W the *boundary* of W.

REMARK: According to the Collar-Theorem (compare [BJ, Thm. 13.6]), each manifold admits a collar of a constant length, that means $(\partial M \times [0,\infty))^{<\delta} = \partial M \times [0,\varepsilon)$ for an $\varepsilon > 0$. On the other hand, a collar in our sense leads to a collar of constant length by a simple reparametrization, which in general changes the equivalence class.

We allow that ∂W is empty. Then, of course, a c-manifold is nothing but a smooth manifold (without boundary, or better with an empty boundary). Thus the smooth manifolds are incorporated into the world of c-manifolds as those c-manifolds W with $\partial W = \emptyset$.

One of the main reasons for introducing c-manifolds is that a collar is what we need to glue manifolds.

Let W and W' be c-manifolds with $\partial W = \partial W'$. As a topological space the glued manifold is obtained from the disjoint union of W and W' by identifying points in the boundary. To describe the smooth structure we construct this space differently.



We choose representatives of the collars $\mathbf{c} : (\partial W \times [0,\infty))^{<\delta} \longrightarrow W$ and $\mathbf{c}' : (\partial W \times [0,\infty))^{<\delta'} \longrightarrow W'$ and introduce the reflection s on $\partial W \times [0,\infty)$ which maps (x,t) to $(x,-t)$. Then we obtain a topological space from the disjoint union of $\mathring{W}$, $\mathring{W}'$ and $(\partial W \times [0,\infty))^{<\delta} \cup s((\partial W \times [0,\infty))^{<\delta'}) \subset \partial W \times (-\infty,\infty)$ by passing to the quotient space identifying $(x,t) \in (\partial W \times (0,\infty))^{<\delta}$ with $\mathbf{c}(x,t)$ and $(x,t) \in s((\partial W \times (0,\infty))^{<\delta'})$ with $\mathbf{c}'(x,-t)$. The resulting space is a Hausdorff space. It has a countable basis and the smooth structure on this space is characterized by the property that the canonical projections (identification maps) on $\mathring{W}$, $\mathring{W}'$ and $(\partial W \times [0,\infty))^{<\delta} \cup s((\partial W \times [0,\infty))^{<\delta'})$ are diffeomorphisms on their images. Thus we have constructed a smooth manifold denoted

$$W \cup_{\partial W = \partial W'} W',$$

whose underlying topological space is of course the space described above obtained from W and W' by identifying points in the boundary. This manifold only depends on the germ of the collars $\mathbf{c}$ and $\mathbf{c}'$.

Definition: Let W be a c-manifold and N a smooth manifold. A *c-map* from W to N is a continuous map $f : W \to N$ such that $f|_{\mathring{W}}$ is smooth and there is a representative of the germ of collars $\mathbf{c} : (\partial W \times [0,\infty))^{<\delta} \longrightarrow V \subset W$ of W, such that for all $(x,t) \in (\partial W \times [0,\infty))^{<\delta}$ we have

$$f(\mathbf{c}(x,t)) = f(x)$$

Thus, a c-map is a smooth map, which is constant in the direction of the collar in a small neighbourhood of the boundary.

We also introduce the concept of manifolds with corners, as usually equipped with collars.

Definition: A smooth k-dimensional *manifold with corners* is a topological manifold W with boundary, together with a maximal smooth atlas $\varphi_i : U_i \longrightarrow V_i$, where U_i is an open subset of W and V_i an open subset of $\mathbb{R}^p \times (\mathbb{R}_{\geq 0})^{k-p}$. The *p-face* of W is the set

$$\partial^p(W) := \{x \in W | \exists \text{ chart } \varphi : U \longrightarrow V \subset \mathbb{R}^p \times (\mathbb{R}_{\geq 0})^{k-p} \text{ with } \varphi(x) \in \mathbb{R}^p \times \{0\}\}.$$

This definition is independent of the choice of charts, see [Kr3]. The boundary of W is the disjoint union of the faces $\partial W = \sqcup_{0 \leq p \leq k-1} \partial^p(W)$.

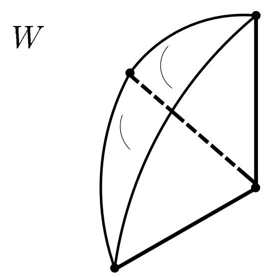


Figure 1.1: Manifold with corners.

In an obvious way one defines a *collar* of W to be a sequence of diffeomorphisms $\mathbf{c}_p : U_p \longrightarrow V_p$ for $0 \leq p \leq k-1$, where U_p is an open neighbourhood of $\partial^p(W) \times \{0\}$ in $\partial^p(W) \times (\mathbb{R}_{\geq 0})^{k-p}$ and V_p is an open neighbourhood of $\partial^p(W)$ in W, such that $\mathbf{c}_p|_{\partial^p W \times \{0\}} \equiv \mathrm{id}$. In addition a compatibility condition is required, as described in the following.

Let $(t_1, \dots, t_r)$ be a tuple of real numbers, and let $I = (i_1, \dots, i_s) \in \mathbb{N}^s$ be a multi-index such that $1 \leq i_1 < i_2 < \cdots < i_s \leq r$. Set

$$t_I := (t_{i_1}, \dots, t_{i_s}) \in \mathbb{R}^s$$

and

$$\hat{t}_I \in \mathbb{R}^r \quad \text{by} \quad (\hat{t}_I)_j := \begin{cases} t_j & \text{for } j \notin I \\ 0 & \text{for } j \in I \end{cases}$$

We require the compatibility condition for the collars

$$\mathbf{c}_p(x, (t_1, \ldots, t_{k-p})) = \mathbf{c}_{p+s}(\mathbf{c}_p(x, \hat{t}_I), t_I),$$

for all $I \in \mathbb{N}^{k-p-s}$ with $1 \leq i_1 < i_2 < \cdots < i_{k-p-s} \leq k-p$, whenever both sides make sense.

For example, for a 2-dimensional manifold with corners, one requires the compatibility conditions

$$\begin{array}{rcll} \mathbf{c}_0(x, (t_1, t_2)) & = & \mathbf{c}_1(\mathbf{c}_0(x, (t_1, 0)), t_2) & \text{and} \\ \mathbf{c}_0(x, (t_1, t_2)) & = & \mathbf{c}_1(\mathbf{c}_0(x, (0, t_2)), t_1). & \end{array}$$

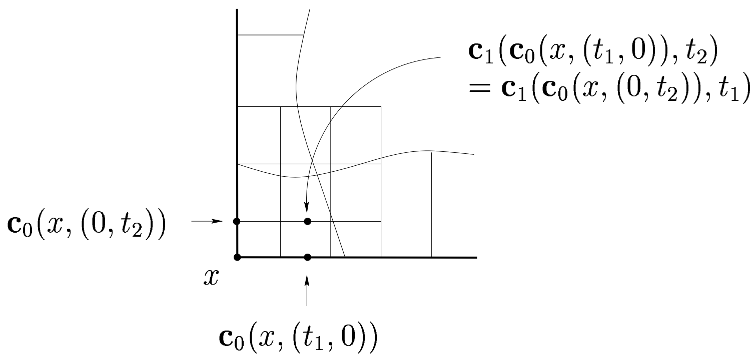


Figure 1.2: Compatibility of the collars.

As in the case of c-manifolds, one goes over to *germs* of collars.

Definition: A *c-manifold with corners* is a smooth manifold with corners, together with a germ of collars.

REMARK: After passing to possibly smaller U_p's and V_p's we can always achieve that $y \in \operatorname{im} \mathbf{c}_p \cap \operatorname{im} \mathbf{c}_{p+s}$ if and only if

$$y = \mathbf{c}_p(x, (t_1, \ldots, t_{k-p})) = \mathbf{c}_{p+s}(\mathbf{c}_p(x, \hat{t}_I, t_I)$$

for suitable $(x, (t_1, \ldots, t_{k-p}))$ and multi-index I. We often make use of this property without explicit mention.

The c-manifolds clearly form a subclass of c-manifolds with corners.

To define a category of c-manifolds with corners, we need a notion of morphisms. In order to do this, we first introduce two classes of maps between c-manifolds with corners.

Let $f : W \longrightarrow W'$ be a smooth map between c-manifolds with corners and $\partial^p W_i$ be a connected component of $\partial^p W$. We say that f is *retracting along* $\partial^p W_i$ if there is a representative of the germs of collars $\mathbf{c}_p : U_p \longrightarrow V_p$ around $\partial^p W_i$ such that for all $(x, (t_1, \ldots, t_{k-p})) \in U_p$ we have

$$f(\mathbf{c}_p(x, t_1 \ldots, t_{k-p})) = f(x).$$

We say that f is *commuting with the collars along* $\partial^p W_i$ if there is a component $\partial^p W_i'$ of $\partial^p W'$ and representatives of germs of collars $\mathbf{c}_p : U_p \longrightarrow V_p$ of $\partial^p W_i'$ and $\mathbf{c}_p' : U'_p \longrightarrow V'_p$ of $\partial^p W_i'$ such that $f(\partial^p W_i) \subset \partial^p W_i'$ and

$$f(\mathbf{c}_p(x, (t_1, \ldots, t_{k-p}))) = \mathbf{c}_p'(f(x), (t_1, \ldots, t_{k-p})).$$

Definition: A *morphism* from a c-manifold W with corners to a c-manifold W' with corners is a smooth map $f : W \longrightarrow W'$ such that for each component $\partial^p W_i$ of $\partial^p W$ either f retracts along $\partial^p W_i$ or commutes with the collars along $\partial^p W_i$. The class of morphisms is denoted by $\mathcal{C}_{\mathbf{CMC}}(W, W')$.

Remark: If W is a c-manifold without corners, then the morphisms $\mathcal{C}_{\mathbf{CMC}}(W, \mathbb{R})$ are precisely the c-maps.

One reason for introducing c-manifolds with corners is that the product of two c-manifolds $\tilde{W} := W_1 \times W_2$ is a c-manifold with corners. Given two representatives of the collars $\mathbf{c}_1$ and $\mathbf{c}_2$ of W_1 and W_2 respectively, it is possible to construct a representative of the collar of $\tilde{W}$, but the construction in general depends on the choice of the representatives, see [Kr2].

Nevertheless we note the following observation:

Proposition 1.3.1. *Every c-manifold with corners admits a smooth structure of a manifold with boundary.*

To see this, we define a map

$$\begin{array}{rccc} h_2 : & [0,\infty) \times [0,\infty) & \longrightarrow & \mathbb{R} \times [0,\infty) \\ & r\exp(i\varphi) & \longmapsto & r\exp(i2\varphi) \end{array}$$

where we identify $\mathbb{R}^2$ with $\mathbb{C}$.

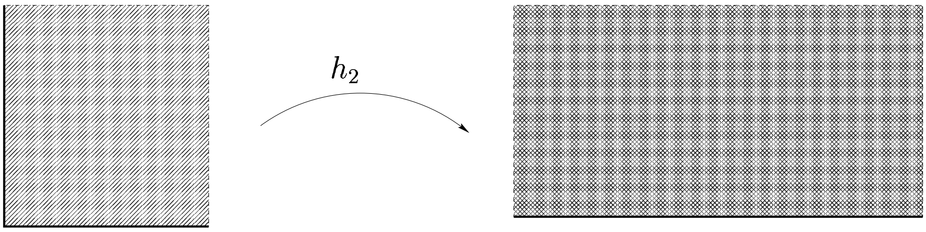


Figure 1.3: Straightening the corners.

The map is a homeomorphism and a diffeomorphism outside of $\{0\}$. Iterating the procedure we obtain a homeomorphism $h_k : (\mathbb{R}_{\geq 0})^k \longrightarrow \mathbb{R}^{k-1} \times [0, \infty)$. Thus, every m-dimensional c-manifold with corners is locally diffeomorphic to $\mathbb{R}^{m-1} \times [0, \infty)$, which implies the proposition. According to collar theorem ([BJ, Thm. 13.7]), every manifold with boundary is equipped with a collar, and two different collars are isotopic.

Finally, we establish a connection between manifolds with corners and stratifolds.

Lemma 1.3.2. *Let W be a c-manifold with corners. The pair $(W, \mathcal{C}_{\mathbf{CMC}}(W, \mathbb{R}))$ is a stratifold.*

Proof. We make use of the Proposition 1.1.3 and give a decomposition of W into smooth manifolds together with retractions. The decomposition is given by $\{\partial^p W\}_p$ and $\mathring{W}$. Since $\mathring{W}$ is open, the neighbourhood of $\mathring{W}$ in W is given by $\mathring{W}$ itself, and the retraction is the identity map. A neighbourhood around $\partial^p W$ is given by $U_p := V_p$ and $\pi_p(\mathbf{c}_p(x, (t_1, \ldots, t_{k-p}))) := x$, where $\mathbf{c}_p : U_p \longrightarrow V_p$ is a representative of the collars. The compatibility relation of the collars implies the compatibility of the retractions $\{\pi_p\}$. It is easy to verify, that W is paracompact with respect to the algebra $\mathcal{C}_{\{\pi_p\}}(W)$. The fact that the map f is in $\mathcal{C}_{\mathbf{CMC}}(W)$ if and only if f is an element of $\mathcal{C}_{\{\pi_p\}}(W)$ finishes the proof. $\square$

1.4 Parametrized stratifolds

In this section we will define parametrized stratifolds. These are stratifolds which are constructed by attaching manifolds with boundary by a map to the already inductively constructed space. The attaching map has to fulfil some subtle properties. For more detailed information regarding parametrized stratifolds and various applications see [Kr2, Kr3].

Definition: A *k-dimensional strat* of a topological space X is a proper continuous map $f : W \longrightarrow X$ such that $f|_{\mathring{W}} : \mathring{W} \longrightarrow f(\mathring{W})$ is a homeomorphism, where W is a k-dimensional c-manifold.

Given a family of strats $f_i : W^i \longrightarrow X$, we call a continuous map $\rho : X \longrightarrow \mathbb{R}$ *smooth with respect to* $\{f_i\}$ if all compositions $\rho f_i : W^i \longrightarrow \mathbb{R}$ are smooth c-maps. For a smooth manifold M, a map $g : M \longrightarrow X$ is called *smooth with resp. to* f_i if for all smooth maps $\rho : X \longrightarrow \mathbb{R}$ with resp. to f_i the composition ρg is again smooth.

EXAMPLE:
Let W be a smooth compact n-dimensional c-manifold and define $X := W \cup_\partial x$, where the boundary is collapsed to a single point. Take as strat the canonical projection $f_n : W \longrightarrow X$. It is an immediate consequence of the definition of c-maps that a map $\rho : X \longrightarrow \mathbb{R}$ is smooth if and only if ρ is continuous, the restriction $\rho|_{\mathring{W}} : \mathring{W} \longrightarrow \mathbb{R}$ is smooth and ρ is constant on a small neighbourhood of $x \in X$.

It is also not hard to verify that a map $g : M \longrightarrow X$ from a smooth manifold M to X is smooth if and only if the restriction $g|_{g^{-1}(\mathring{W})}$ is smooth.

Definition: A *parametrization* on a stratifold $(\mathscr{S}, \mathcal{C}_{\mathbf{S}}(\mathscr{S}))$ is a family of strats $\{f_i : W^i \longrightarrow \mathscr{S}\}$ such that

- $\sqcup_i f_i(\mathring{W}^i) = \mathscr{S}$,
- $\dim W^i = i$,
- f_i is a morphism of stratifolds for all i,
- $f_i|_{\mathring{W}^i} : W^i \longrightarrow f_i(W^i)$ is an isomorphism for all i,
- a subset $U \subset X$ is open if and only if for all i the set $f_i^{-1}(U)$ is open in W^i.

Two parametrisations $\{f_i : W^i \longrightarrow \mathscr{S}\}$ and $\{\tilde{f}_i : \tilde{W}^i \longrightarrow \mathscr{S}\}$ are called *isomorphic* or *equivalent* if there are isomorphisms $\varphi_i : W^i \longrightarrow \tilde{W}^i$ of c-manifolds, making the following diagram commutative:

$$\begin{tikzcd} W^i \arrow[d, "\varphi_i"'] \arrow[dr, "f_i"] & \\ \tilde{W}^i \arrow[r, "\tilde{f}_i"'] & \mathscr{S} \end{tikzcd}$$

Note that a map $f : \mathscr{S} \longrightarrow \mathscr{S}$ is smooth with respect to $\{f_i : W^i \longrightarrow \mathscr{S}\}$ if and only if it is smooth with respect to $\{\tilde{f}_i : \tilde{W}^i \longrightarrow \mathscr{S}\}$. This allows us to make the following definition.

Definition: An m-dimensional *parametrised stratifold* (short *p-stratifold*) is a stratifold $(\mathscr{S}, \mathcal{C}_{\mathbf{S}}(\mathscr{S}))$, together with an equivalence class of parametrizations $[\{f_i : W^i \longrightarrow \mathscr{S}\}_{\{1\leq i\leq m\}}]$ such that

$$\mathcal{C}_{\mathbf{S}}(\mathscr{S}) = \{f : \mathscr{S} \longrightarrow \mathbb{R} \big| f \text{ is smooth with respect to parametrisation}\}.$$

The class of p-stratifolds will be denoted by **PS**.

The manifolds W^i are not supposed to be non-empty, in particular the dimension of a p-stratifold can not be derived from the underlying topological space. To avoid misunderstandings, we assume throughout this thesis, that the top stratum is non-empty.

The condition $\overline{\mathscr{S}^i} \cap \mathscr{S}^j = \emptyset$ for $i < j$ implies that $f_i(\partial W^i) \subseteq \cup_{j\leq i-1} f_j(W^j)$. In the case of all strata below the top dimension being empty, we obtain a common manifold without boundary, together with its diffeomorphism class.

Recall, that the union $\Sigma^k := \cup_{i=1}^k f_i(W^i)$ is called the *k-skeleton of* $\mathscr{S}$. The codimension 1 skeleton Σ^{m-1} is denoted simply by Σ. Using the collar of W^m the $(m-1)$-skeleton Σ is equipped with the germ of neighbourhoods $[U\Sigma]$ by taking

$$U\Sigma := (f_m \cup \mathrm{id}\,)(\mathbf{c}((\partial W^m \times [0,\infty))^{<\delta}) \cup_{f_m \mathrm{pr}_1|_{\partial W^m \times \{0\}}} \Sigma),$$

where $\mathbf{c} : (\partial W^m \times [0,\infty))^{<\delta} \longrightarrow W^m$ is a representative of the germ of the collar. The collar also gives us a retraction $r : U\Sigma \longrightarrow \Sigma$.

If we want to point out the dependency on the representative $\mathbf{c}$, we use the notation $U\Sigma_{\mathbf{c}}$.

Define further the germ of closed neighbourhoods, after choosing a smooth map $\delta : \partial W^m \longrightarrow (0, \infty)$, by setting

$$\overline{U}\Sigma := (f_m \cup \mathrm{id}\ )(\mathbf{c}((\partial W^m \times [0,\infty))^{\leq \delta/2}) \cup_{f_m \mathrm{pr}_1|_{\partial W^m \times \{0\}}} \Sigma).$$

In the case of a singularity Σ^{m-1}, decomposed into compact connected components Σ_i, the preimage $L_i := (f_m)^{-1}(\Sigma_i)$ is a compact collection of boundary components of W^m, called the *link of the singularity* Σ_i. Hence there always exists a representative of the germ of collars with constant length along L_i. In this case, the germ of closed neighbourhoods $[\overline{U}\Sigma]$ has a representative $\sqcup_i \overline{U}\Sigma_i$ by setting $\overline{U}\Sigma_i := (f_m \cup \mathrm{id}\ )(\mathbf{c}_i(L_i \times [0, \varepsilon_i/2]) \cup_{f_m \mathrm{pr}_1|_{L_i \times \{0\}}} \Sigma_i)$, where $\mathbf{c}_i : L_i \times [0, \varepsilon_i) \longrightarrow W^m$ is a representative of the germ of collars around L_i.

The following result by M.Kreck [Kr2] is a constructive description of stratifolds.

Lemma 1.4.1. *Let X be a topological space and $f_i : W^i \longrightarrow X$ for $0 \leq i \leq m$ be strats satisfying the following conditions:*

- $\sqcup_i f_i(\mathring{W}^i) = \mathscr{S}$,
- $\dim W^i = i$,
- $f_i(\partial W^i) \subseteq \cup_{j \leq i-1} f_j(W^j)$,
- *the restriction $f_i|_{\partial W^i}$ is smooth with resp. to f_j for $j \leq i-1$,*
- *a subset $U \subset X$ is open if and only if for all j the set $f_j^{-1}(U)$ is open in W^j.*

Then the pair $(X, \mathcal{C}_{\{f_i\}})$ is an m-dimensional p-stratifold with parametrization $[\{f_i : W^i \longrightarrow \mathscr{S}\}]$, where

$$\mathcal{C}_{\{f_i\}}(\mathscr{S}) = \{f : \mathscr{S} \longrightarrow \mathbb{R} \big| f \text{ is smooth with respect to parametrisation}\}.$$

1.5 Cornered p-stratifolds

In this section we present another class of stratifolds, of which the p-stratifolds we studied before are a special case. The idea is to replace c-manifolds as objects of which the p-stratifolds are built, with c-manifolds with corners.

Using c-manifolds W^i with corners instead of c-manifolds we define *cornered strats* in the same way as strats in the last paragraph. In the same way using cornered strats, one defines *cornered parametrisation* in exactly the same way as introduced in §1.4. The 3rd condition

- $f_i : W^i \longrightarrow \mathscr{S}$ is a morphism of stratifolds

makes sense using Lemma 1.3.2.

Given a family of cornered strats $f_i : W^i \longrightarrow X$, we call a continuous map $\rho : X \longrightarrow \mathbb{R}$ *smooth with respect to* $\{f_i\}$ if all compositions $\rho f_i : W^i \longrightarrow \mathbb{R}$ are morphisms of c-manifolds with corners. The equivalence relation on cornered parametrisations is done in the similar way using isomorphisms of c-manifolds with corners. Again one notes, the the condition "being smooth with respect to cornered parametrisation" is independent on the choice of the representative. For an explicit definition see [Kr3].

Definition: An m-dimensional *cornered p-stratifold* is a stratifold $(\mathscr{S}, \mathcal{C}_{\mathbf{S}}(\mathscr{S}))$, together with an equivalence class of cornered parametrizations $[\{f_i : W^i \longrightarrow \mathscr{S}\}_{\{1 \leq i \leq m\}}]$ such that

$$\mathcal{C}_{\mathbf{S}}(\mathscr{S}) = \{f : \mathscr{S} \longrightarrow \mathbb{R} \big| f \text{ is smooth with respect to cornered parametrisation}\}.$$

The class of cornered p-stratifolds will be denoted by **CPS**.

As in the case of p-stratifolds there is a pure geometrical (constructive) description of cornered p-stratifolds. Recall from 1.3.2 that a c-manifold with corners together with an algebra of morphism to $\mathbb{R}$ is a stratifold, in particular the boundary of W, the codimension 1 skeleton, is again a stratifold.

Lemma 1.5.1. *Let X be a topological space and $f_i : W^i \longrightarrow X$ for $0 \leq i \leq m$ be cornered strats satisfying the following conditions:*

- $\sqcup_i f_i(\mathring{W}^i) = \mathscr{S}$,
- $\dim W^i = i$,
- $f_i(\partial W^i) \subseteq \cup_{j \leq i-1} f_j(W^j)$,
- *for all* $\rho : (\cup_{j \leq i-1} f_j(W^j)) \longrightarrow \mathbb{R}$, *such that* $\rho f_j : W^j \longrightarrow \mathbb{R}$ *is a morphism for all* $j \leq i-1$, *the map* $\rho f_i|_{\partial W^i}$ *is in* $\mathcal{C}_{\mathbf{S}}(\partial W^i)$,
- *a subset* $U \subset X$ *is open if and only if for all* j *the set* $f_j^{-1}(U)$ *is open in* W^j.

Then the pair $(X, \mathcal{C}_{\{f_i\}})$ is an m-dimensional cornered p-stratifold with cornered parametrization $[\{f_i : W^i \longrightarrow \mathscr{S}\}]$, where

$$\mathcal{C}_{\{f_i\}}(\mathscr{S}) = \{f : \mathscr{S} \longrightarrow \mathbb{R} \big| f \text{ is smooth with respect to parametrisation}\}.$$

We begin with some preparations and assume in the following that we have a space X fulfilling the conditions of the last lemma. The requirement on the "attaching maps" $f|_{\partial W^i}$ in the definition of cornered p-stratifolds is essential. It allows us to do inductive constructions on X using the collars of ∂W^j to proceed from the $(j-1)$st stratum to the higher dimensional one. (Here we call the manifold $X^i := f_i(\mathring{W}^i)$ the i-strata and set $\Sigma^i := \sqcup_{j \leq i} X^i$ the i-skeleton.) To see this, consider an open subset D of $f_i(\mathring{W}^i)$ and construct open neighbourhoods D^j of D in Σ^j for $j \geq i$. We do this inductively on j, where the first step is done by setting $D^i := D$. Assume we have already constructed D^{k-1} for $k-1 \geq i$. Let f_k^p denote the restriction of f_k on the p-dimensional face $\partial^p(W^k)$ and let $\mathbf{c}_p^k : U_p^k \longrightarrow W^k$ be a representative of the collar around $\partial^p(W^k)$. Define first

$$D_p^k := (f_k \cup \mathrm{id}\,)\big(\mathbf{c}_p^i(((f_k^p)^{-1}(D^{k-1}) \times [0,\infty)^{k-p}) \cap U_p^k) \cup_{f_k^p \times \{0\}^{k-p}} D^{k-1}\big)$$

where $\cup_{f_k^p \times \{0\}^{k-p}}$ denotes the following gluing prescription: $f_k^p \times \{0\}^{k-p}$: $(f_k^p)^{-1}(D^{k-1}) \times \{0\}^{k-p} \to D^{k-1}$, $(x, 0) \mapsto f_k^p(x)$. Then set $D^k := \cup_{0 \leq p \leq k-1} D_p^k$. This is clearly an open neighbourhood of D in Σ^k.

Every open subset U of X is again provided with cornered strats satisfying the conditions of the lemma by setting $W_U^i := f_i^{-1}(U)$ and $f_i^U := f_i|_{W_U^i}$: $W_U^i \longrightarrow U$. Thus it makes sense to speak of smooth maps (with respect to strats) on open subsets of X.

Now we want to construct smooth retractions π_k from D^k to D. We again argue inductively. Set $\pi_i = \mathrm{id}$ and assume that we have already defined π_{k-1}. On every D_p^k there are canonical retractions $\pi_k^p : D_p^k \longrightarrow D$ given by :

$$x \mapsto \begin{cases} \pi_{k-1} f_k^p(y) & \text{for } x = f_k(\mathbf{c}_p(y, t_1, \ldots t_{k-p})) \\ \pi_{k-1}(x) & \text{for } x \in D^{k-1} \end{cases}$$

Since D^k is attached via f_k^p to D^{k-1}, the assignment above defines a well defined map, which is smooth since π_{k-1} is smooth. Thus, according to the 4th condition of the lemma, $\pi_{k-1} f_k^p$ is smooth as well.

Proposition 1.5.2. *The map $\pi_k : D^k \longrightarrow D$ given by $\cup_{0 \leq p \leq k-1} \pi_k^p$ is well-defined and is a smooth retraction.*

Proof. One only has to show that the assignment is well-defined. First observe that since $\pi_{k-1}f_k^p = \pi_{k-1}f_k|_{\partial^p(W^k)}$ and since $\pi_{k-1}f_k$ is smooth, the map $\pi_{k-1}f_k^p$ retracts along every collar $\mathbf{c}_r^k$ for $0 \leq r \leq k-1$. Let now $y := (f_k|_{\mathring{W}^k})^{-1}(x) \in \mathrm{im}\,\mathbf{c}_p^k \cap \mathrm{im}\,\mathbf{c}_{p+s}^k$, then

$$y = \mathbf{c}_p^k(z,(t_1,\dots,t_{k-p})) = \mathbf{c}_{p+s}^k(\mathbf{c}_p^k(z,(\hat{t}_I),t_I)$$

for suitable $(z,(t_1,\dots,t_{k-p}))$ and I. Then

$$\begin{array}{lclcl} \pi_k^{p+s}(x) & = & \pi_{k-1}f_k^{p+s}(\mathbf{c}_p(z,(\hat{t}_I))) & \overset{(*)}{=} & \\ \pi_{k-1}f_k^{p+s}(z) & = & \pi_{k-1}f_k^p(z) & = & \pi_k^p(x), \end{array}$$

where the equation $(*)$ is true because $\pi_{k-1}f_k^{p+s}$ retracts along the collars. Thus the map is well defined and the proposition is proved.

□

Summarizing the discussion above, we have shown:

Lemma 1.5.3. *Every stratum $X^i = f_i(\mathring{W}^i)$ possesses an open neighbourhood $U(X^i)$ and a retraction $\pi_i : U(X^i) \longrightarrow X^i$, which is smooth with respect to $\{f_i^{U(X^i)}\}$.*

REMARK: If we change the representatives of the germs of collars the neighbourhoods will also change, but their germ is still well defined.

The last lemma shows that $(\{f_i(\mathring{W}^i)\}, [(U(\mathring{W}^i),\pi_i)])$ is a controlled stratification on X. Furthermore from the definition of π_i's we conclude:

Lemma 1.5.4. *A map $f : X \longrightarrow \mathbb{R}$ is smooth with respect to $\{f_i\}$ if and only if $f \in \mathcal{C}_{\{\pi_i\}}(X)$.*

Let $\pi_i : U(X^i) \longrightarrow X^i$ be representatives of the germs of neighbourhood retractions, defined above. Define the set of germs of functions around X^i by

$$\Gamma_{\{\pi_i\}}(X^i) := \{g \in \mathcal{C}_{\{\pi_i\}}(U) \big| U \text{ is an open neighbourhood of } X^i \text{ in } X\}/\sim$$

where $(g : U \longrightarrow \mathbb{R}) \sim (g' : U' \longrightarrow \mathbb{R})$ if there is an open neighbourhood U'' of X^i in $U \cap U'$ such that $g|_{U''} \equiv g'|_{U''}$.
The restriction gives us a well defined map:

$$\begin{array}{rccc} \tau_i : & \Gamma_{\{\pi_i\}}(X^i) & \longrightarrow & \mathcal{C}^\infty(X^i) \\ & [f : U \longrightarrow \mathbb{R}] & \longmapsto & f|_{X^i} : X^i \longrightarrow \mathbb{R} \end{array}$$

Lemma 1.5.5. *The map τ_i described above is an isomorphism of algebras.*

Proof. The map is injective, for given $g : U' \longrightarrow \mathbb{R}$ and $g' : U \longrightarrow \mathbb{R}$ there is a representative of the neighbourhood germ $\pi_{X^i} : U(X^i) \longrightarrow X^i$ such that $g\pi_{X^i}(x) = g(x)$ and $g'\pi_{X^i}(x) = g'(x)$ for all $x \in U(X^i)$. From $g|_{X^i} \equiv g'|_{X^i}$ it follows that $g(x) = g\pi_{X^i}(x) = g'\pi_{X^i}(x) = g'(x)$ for all $x \in U(X^i)$, hence $g = g' \in \Gamma_{\{\pi_i\}}(X^i)$. For surjectivity, take a map $h \in \mathcal{C}^\infty(X^i)$ and consider a representative $\pi_{X^i} : U(X^i) \longrightarrow X^i$. The composition $h\pi_{X^i}$ gives a smooth extension of h to a neighbourhood of X^i, and the lemma is proved.

□

We see that every smooth map defined on a stratum $f_i(\mathring{W}^i)$ has a unique smooth extension to the space X. But the condition on the attaching maps in the Lemma 1.5.1 allows even more.

Lemma 1.5.6. *Let X be as above and $f : \Sigma^i \longrightarrow \mathbb{R}$ a smooth map with respect to $\{f_j\}_{j \leq i}$ on the i-skeleton of X. Then there exists a smooth map $g : X \longrightarrow \mathbb{R}$ such that $g|_{\Sigma^i} \equiv f$.*

Proof. It is enough to show the statement in the case of $i+1$ strats on X, the general case will then follow inductively. Consider $ff_{i+1} : \partial W^{i+1} \longrightarrow \mathbb{R}$. According to the 4th condition of the Lemma 1.5.1 this is a morphism, an element of $\mathcal{C}_{\mathbf{S}}(\partial W^{i+1})$. Let $\{\mathbf{c}_p : U_p \longrightarrow V_p\}$ be a representative of the collars of W^{i+1}. Set $U := \cup_p V_p$ and define $h_U : U \longrightarrow \mathbb{R}$ by $h_U(\mathbf{c}_p(x, (t_1, \ldots, t_{i+1-p}))) := ff_{i+1}(x)$. First we show that the map is well defined. Let

$$y = \mathbf{c}_p(x, (t_1, \ldots, t_{i+1-p})) = \mathbf{c}_{p+s}(\mathbf{c}_p(x, (\hat{t}_I), t_I)$$

Then

$$\begin{aligned} &h_U(\mathbf{c}_{p+s}(\mathbf{c}_p(x, (\hat{t}_I)), t_I)) &&= \\ &ff_{i+1}(\mathbf{c}_p(x, (\hat{t}_I))) &&\overset{(*)}{=} \\ &ff_{i+1}(x) = h_U(\mathbf{c}_p(x, (t_1, \ldots, t_{i+1-p}))), \end{aligned}$$

where the equality (*) is true because of the definition of the retractions on the boundary ∂W^{i+1} and the fact that ff_{i+1} commutes with the retractions. Let now A be a closed neighbourhood of ∂W^{i+1}, contained in U. Then we can use a smooth partition of unity on $X^{i+1} := f_{i+1}(\mathring{W}^{i+1})$ to obtain a map $h : X^{i+1} \longrightarrow \mathbb{R}$ such that $h|_A \equiv (h_U)|_A$. Define finally $g := h \sqcup f : X = X^{i+1} \sqcup \Sigma^i \longrightarrow \mathbb{R}$, which is a smooth map on X according to the construction.

□

Using similar arguments one obtains:

Lemma 1.5.7. *Every X with properties as above possesses a smooth partition of unity.*

Summarizing the discussion we conclude that the pair $(X, \mathcal{C}_{\{f_i\}}(X)) = (X, \mathcal{C}_{\{\pi_i\}}(X))$ is a stratifold, as stated in Lemma 1.5.1.

One advantage of cornered p-stratifolds over those without corners is that the product of stratifolds (with or without corners) is in the canonical way an object with corners. Although it is possible to define a product of two p-stratifolds without corners again as an object without corners using the straightening of the corners, the construction in general depends on the choice of representatives of the germ of collars. For an explicit argument, see [Kr3].

Using Proposition 1.3.1, we conclude

Proposition 1.5.8. *Every stratifold with cornered strat structure admits a structure of a p-stratifold, which is unique up to isotopy.*

Still in the category of cornered p-stratifolds, we give an example similar to the product of stratifolds, which will play an important role in §3.4.

EXAMPLES:

1. Let W be a c-manifold with corners and $\delta : W \longrightarrow (0, \infty)$ a smooth map. Consider $\tilde{W} := (W \times [0, \infty))^{\leq \delta}$. Then $\tilde{W}$ is again a smooth c-manifold with corners. Observe first that since $(W \times [0, \infty))^{\delta} = \operatorname{graph}(\delta) \cong W$, the space $\tilde{W}$ is a topological manifold. The faces of $\tilde{W}$ are given by

$$\{(\partial^p W \times (0, \infty))^{<\delta},\ (\partial^p W \times (0, \infty))^{\delta},\ \partial^p W \times \{0\}\}_p,\ \mathring{W} \times \{0\} \text{ and } (\mathring{W} \times (0, \infty))^{\delta}.$$

Next we give the collars of the faces. Let $\{\mathbf{c}_p : U_p \longrightarrow V_p\}$ be a representative of the collar of W, then we define the following collars of $\tilde{W}$:

$$\begin{array}{llcrll}
\tilde{\mathbf{c}}_{p+1} & := & \mathbf{c}_p \times \mathrm{id} & : & (U_p \times (0, \infty))^{<\delta \mathbf{c}_p} & \longrightarrow & \tilde{W} \\
\tilde{\mathbf{c}}_p & & & : & (U_p \times [0, \infty))^{<\delta \mathbf{c}_p} & \longrightarrow & \tilde{W} \\
 & & & & (u, t) & \longmapsto & (\mathbf{c}_p(u), \delta(\mathrm{pr}_1(u)) - t) \\
\tilde{\mathbf{c}}_p & := & \mathbf{c}_p \times \mathrm{id} & : & (U_p \times [0, \infty))^{<\delta \mathbf{c}_p} & \longrightarrow & \tilde{W} \\
\tilde{\mathbf{c}}_1 & := & \mathrm{id} \times \mathrm{id} & : & (\mathring{W} \times [0, \infty))^{<\delta} & \longrightarrow & \tilde{W} \\
\tilde{\mathbf{c}}_1 & & & : & (\mathring{W} \times [0, \infty))^{<\delta} & \longrightarrow & \tilde{W} \\
 & & & & (x, t) & \longmapsto & (x, \delta(x) - t)
\end{array}$$

This gives $\tilde{W}$ the structure of a c-manifold with corners.

2. Let now $\mathscr{S}$ be a cornered p-stratifold and $\delta : \mathscr{S} \longrightarrow (0, \infty)$ a smooth map. We will give the topological space $\tilde{\mathscr{S}} := (\mathscr{S} \times [0, \infty))^{\leq \delta}$ a canonical structure of a cornered p-stratifold. Set $\tilde{W}^k := (W^{k-1} \times [0, \infty))^{\leq \delta}$. According

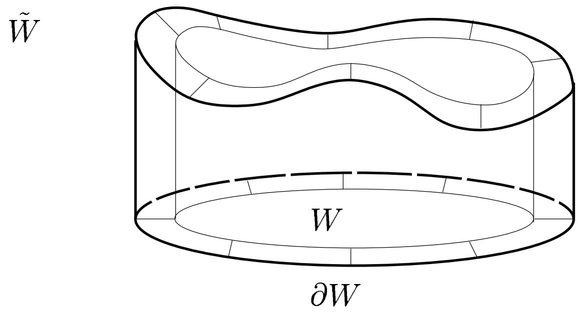


Figure 1.4: New manifold with corners $\tilde{W}$.

to the previous discussion, this is again a c-manifold with corners. Define strats $\tilde{f}_k := f_{k-1} \times \mathrm{id} \; : \tilde{W}^k \longrightarrow \tilde{\mathscr{S}}$. We now have to verify that the strats satisfy the conditions of Lemma 1.5.1. All conditions except the 4th one are obviously fulfilled. And for the 4th condition, one observes that for every morphism $\rho : \tilde{\Sigma}^{k-1} \longrightarrow \mathbb{R}$ such that $\rho(f_{k-1} \times \mathrm{id}\;)$ is again a morphism, the last map retracts near

$$\partial W^{k-1} \times \{0\} \text{ and } (\partial W^{k-1} \times [0,\infty))^{\delta}$$

and the boundary of W^k is given by

$$\partial(\tilde{W}^k) =$$
$$(\partial W^{k-1} \times [0,\infty))^{\leq\delta} \cup_{\partial W^{k-1}\times\{0\}\sqcup((\partial W^{k-1}\times[0,\infty))^{\delta}} (W^{k-1} \times \{0\} \sqcup W^{k-1} \times [0,\infty))^{\delta}).$$

Thus we have shown, that $\tilde{\mathscr{S}}$ is a cornered p-stratifold.

1.6 Stratifolds with boundary

Once one knows how to build products with $\mathbb{R}$ in the category of stratifolds, one can introduce stratifolds with boundary in the same way as c-manifolds, see [Kr3] for details. We omit the construction of products and objects with boundary in the category of stratifolds and refer to [Kr2] and [Kr3]. In the case of (cornered) p-stratifolds the construction is very simple. One takes $f_i \times \mathrm{id} \; : W^i \times \mathbb{R} \longrightarrow \mathscr{S} \times \mathbb{R}$ as strats, where $W^i \times \mathbb{R}$ has an obvious structure as c-manifold (with corners).

To introduce a collar in the case of p-stratifolds we first recall:

Definition: An *isomorphism of cornered p-stratifolds* $\varphi : \mathscr{S} \longrightarrow \tilde{\mathscr{S}}$ between cornered p-stratifolds is a homeomorphism such that there are isomorphisms of c-manifolds with corners $\varphi_j : W^j \longrightarrow \tilde{W}^j$ making the following diagram commutative for every j:

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\varphi"] & \tilde{\mathscr{S}} \\ W^j \arrow[u, "f_j"] \arrow[r, "\varphi_j"] & \tilde{W}^j \arrow[u, "\tilde{f}_j"'] \end{tikzcd}$$

Remark: An isomorphism of cornered p-stratifolds is certainly an isomorphism of stratifolds, but the converse is in general false, since an isomorphism of stratifolds does not control the velocity of the collar. Consider, for example, the half open interval $I := [0,1)$ together with two collars the inclusion $\mathbf{c} : [0,1/2) \hookrightarrow I$ and the map $\tilde{\mathbf{c}} : [0,1/2) \longrightarrow I$ with $\tilde{\mathbf{c}}(x) := 2x$. The identity map id $: I \longrightarrow I$ is certainly an isomorphism of stratifolds since the retractions induced by the collars are equivalent. The collars themselves are admittedly not equivalent, hence the identity map is not an isomorphism of p-stratifolds.

A *cornered p-stratifold with boundary* is then a pair $(\mathscr{S}, \partial\mathscr{S})$, where $\mathring{\mathscr{S}} := \mathscr{S} - \partial\mathscr{S}$ as well as $\partial\mathscr{S}$ are cornered p-stratifolds and $\partial\mathscr{S}$ is closed in $\mathscr{S}$, together with a germ of collars $[\mathbf{c} : (\partial\mathscr{S} \times [0,\infty))^{<\delta} \longrightarrow \mathscr{S}]$, such that $\mathbf{c}|_{\partial\mathscr{S}\times\{0\}}$ is the identity map and $\mathbf{c}|_{(\partial\mathscr{S}\times(0,\infty))^{<\delta}}$ is an isomorphism of cornered p-stratifolds onto its image, where $(\mathrm{im}\,\mathbf{c})$ is open in $\mathscr{S}$.

One also introduces the gluing of stratifolds along a common boundary in exactly the same way as for c-manifolds. A detailed construction can be found in [Kr2].

Chapter 2

Resolution of singularities

We have seen that stratifolds are objects with singularities. In this chapter, we introduce the notion of a resolution of a stratifold and study the existence and uniqueness of resolutions in some special cases in the category of p-stratifolds. The existence of strats, i.e. of attaching maps, is essential in the following discussion.

2.1 Resolution of stratifolds

A stratifold can be seen as a space with singularities. A natural question which occurs in this context is to try to resolve the singularities, leaving the smooth top stratum untouched.

Definition: Let $\mathscr{S}$ be an m-dimensional stratifold. A *resolution* of $\mathscr{S}$ is a map $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ such that

- $\hat{\mathscr{S}}$ is a smooth manifold;

- p is a proper morphism;

- the restriction of p to $p^{-1}(\mathscr{S}^m)$ is a diffeomorphism onto $\mathscr{S}^m$;

- $p^{-1}(\mathscr{S}^m)$ is dense in $\hat{\mathscr{S}}$.

As already mentioned, the only results we have take place in the category of p-stratifolds. The existence of the attaching maps allows us to construct resolving manifolds under certain conditions. Since a p-stratifold is a stratifold with additional structure, we also impose an additional requirement on the resolution.

Definition: Let $\mathscr{S}$ be an m-dimensional p-stratifold. A *resolution* of $\mathscr{S}$ as a p-stratifold is a resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ such that additionally

- the inclusion $\hat{\Sigma} := p^{-1}(\Sigma) \hookrightarrow U\hat{\Sigma} := p^{-1}(U\Sigma)$ is a homotopy equivalence for a representative of the neighbourhood $U\Sigma$ of Σ;

A resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called *optimal*, if $p|_{\hat{\Sigma}} : \hat{\Sigma} \longrightarrow \Sigma$ is an $[n/2]$-equivalence.

If V is an algebraic variety, Hironaka has shown [Hi] that there is a resolution of singularities in the sense of algebraic geometry. The above topological definition is modelled on the one from algebraic geometry. All conditions are analogous except the new one, which is always fulfilled in the context of algebraic geometry. As explained in [BR], a neighbourhood U of the singular set Σ of an algebraic variety V such that the inclusion $\Sigma \hookrightarrow U$ is a homotopy equivalence can be obtained from a proper algebraic map $\rho : V \longrightarrow \mathbb{R}$ with $\Sigma = \rho^{-1}(0)$ by taking $U = \rho^{-1}[0, r)$, provided $r > 0$ is small enough. Thus for a resolution $p : \hat{V} \longrightarrow V$ the preimage $\hat{U} := p^{-1}(U)$ is a neighbourhood of $\hat{\Sigma} := p^{-1}(\Sigma)$ in $\hat{V}$ obtained from $\hat{\rho} := \rho p$, hence the inclusion $\hat{\Sigma} \hookrightarrow \hat{U}$ is a homotopy equivalence.

If one representative of the top strat $f_n : W^n \longrightarrow \mathscr{S}$ comes from a manifold with more structure, e.g. oriented or spin, the isomorphism class of domains of top strats are equipped with the induced structure. In such cases we introduce corresponding resolutions, which have more structure.

Definition: Let $\mathscr{S}$ be a p-stratifold with oriented W^n. A resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called an *oriented resolution*, if $\mathscr{S}$ is oriented and $p|_{p^{-1}(\mathscr{S}^n)}$ is orientation preserving. Analogously, if W^n is spin, then $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is called a *spin resolution* if $\mathscr{S}$ is spin and $p|_{p^{-1}(\mathscr{S}^n)}$ preserves the spin structure.

As mentioned in the introduction, we are particularly interested in the classification of resolutions. Thus we have to decide when we are going to consider two resolutions as equivalent. We can restrict our attention to the resolving manifolds and introduce a relation on them, e.g. diffeomorphism, but in this case we completely ignore an important part of the resolution data, namely the resolving map. Hence, one can ask for diffeomorphisms between the resolving manifolds commuting with the resolving maps. This relation is very strong and, therefore, very hard to control. In the following definition, we combine these two ideas.

Definition: Let $\mathscr{S}$ be a p-stratifold and $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $p' : \hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ two resolutions of $\mathscr{S}$. We call the resolutions *equivalent*, if, for every representative of the neighbourhood germ $\overline{U}\Sigma_{\mathbf{c}}$, there is a diffeomorphism $\varphi_{\mathbf{c}} : \hat{\mathscr{S}} \longrightarrow \hat{\mathscr{S}}'$ such that the following holds:

- $p'\varphi_{\mathbf{c}} = p$ on $\hat{\mathscr{S}} - \overline{U}\hat{\Sigma}_{\mathbf{c}}$ and
- $rp'\varphi_{\mathbf{c}} = rp$ on $\overline{U}\hat{\Sigma}_{\mathbf{c}}$, where $r : \overline{U}\Sigma_{\mathbf{c}} \longrightarrow \Sigma$ is the neighbourhood's retraction.

This means outside of an arbitrary small neighbourhood of the singularity, the diffeomorphism commutes with the resolving maps and near Σ it only commutes after the composition with the retraction.

Observe that $\varphi_{\mathbf{c}}$ gives a diffeomorphism $\partial\overline{U}\hat{\Sigma}_{\mathbf{c}} \longrightarrow \partial\overline{U}\hat{\Sigma}'_{\mathbf{c}}$.

2.2 Isolated singularities

The situation simplifies very much if we consider only p-stratifolds with isolated singularities, where the construction of the strats is done in two steps only. The first step is the choice of a countable number of points $\{x_i\}_{i\in I\subseteq\mathbb{N}}$, which become the isolated singularities. The second step is the choice of a smooth manifold W of dimension m, together with a proper map $g : \partial W \longrightarrow \{x_i\}_i$, where $\{x_i\}_i$ is considered as a 0-dimensional manifold, and the collection of boundary components $f^{-1}(x_i)$ is equipped with a germ of collars. The stratifold is obtained by forming

$$\mathscr{S} = W \cup_g \{x_i\}_i.$$

We reformulate this in a slightly different way.

Definition: An m-dimensional p-stratifold $\mathscr{S}$ is said to have *isolated singularities* if $\mathscr{S}^i = \emptyset$ for all $i \in \{1, \ldots, m-1\}$.

The zero dimensional stratum $\mathscr{S}^0$ is a countable set of points $\{x_i\}_{i\in I\subseteq\mathbb{N}}$ and the topological space $\mathscr{S}$ in this case is homeomorphic to $W^m \cup_{f_m|_{\partial W^m}} \{x_i\}_i$. Thus, $\mathscr{S}$ depends on the diffeomorphism type of W^m and $L_i := (f_m)^{-1}(x_i)$, the collection of boundary components mapped to a singular point x_i.

It is not hard to verify that, in the special case of p-stratifolds with isolated singularities, the map $g : M \longrightarrow \mathscr{S}$ from a smooth manifold to $\mathscr{S}$ is a morphism if and only if the restriction $g|_{g^{-1}(\mathscr{S}-\Sigma)}$ is smooth.

The most important examples of such stratifolds are algebraic varieties with isolated singularities.

Example (**Algebraic varieties with isolated singularities**):
Consider an algebraic variety $V \subset \mathbb{R}^n$ with isolated singularities, i.e. the singular set Σ is zero-dimensional. Let $s_i \in \Sigma$ be a singular point. There is nothing to do if s_i is open in V. Otherwise consider the distance function ρ_i on $\mathbb{R}^n$ given by $\rho_i(x) := ||x - s_i||^2$. It is well known that there is an $\varepsilon_i > 0$ such that on $V_{\varepsilon_i}(s_i) := V \cap D_{\varepsilon_i}(s_i)$, the restriction $\rho_i|_{V_{\varepsilon_i} - \{s_i\}}$ has no critical values. Here $D_{\varepsilon_i}(s_i)$ denotes the closed ball in $\mathbb{R}^n$ of radius ε_i centred at s_i. Set $\partial V_{\varepsilon_i}(s_i) := V_{\varepsilon_i}(s_i) \cap \partial D_{\varepsilon_i}(s_i)$.

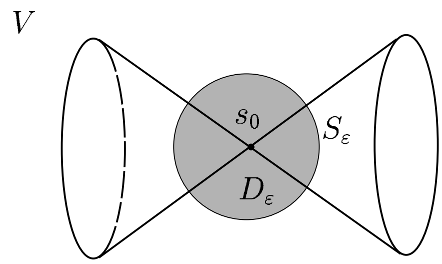


Figure 2.1: An algebraic variety V with an isolated singularity at s_0.

By following the integral curves of the gradient vector field of $\rho_i|_{V_{\varepsilon_i} - \{s_i\}}$, we obtain a diffeomorphism

$$\begin{tikzcd} h : \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i) \arrow[rr] \arrow[dr, "\mathrm{pr}_2"'] & & V_{\varepsilon_i} - \{s_i\} \arrow[dl, "\varepsilon_i - \rho_i"] \\ & {[0, \varepsilon_i)} & \end{tikzcd}$$

being the identity on $\partial V_{\varepsilon_i}(s_i) \times \{0\}$, see [H, §6.2]. We extend this map to a continuous map

$$\bar{h} : \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i] \longrightarrow V_{\varepsilon_i}.$$

Finally, we define a c-manifold W (with obvious collar) by setting

$$W := V - (\sqcup_i \mathring{D}_{\varepsilon_i}(s_i)) \cup_{\mathrm{id}} \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i].$$

The map $f = \mathrm{id} \cup \bar{h} : W \longrightarrow V$ gives V the structure of a p-stratifold with isolated singularities, compare Figure 2.2.

Since every complex algebraic variety is in particular a real one, we obtain the same result for a complex algebraic variety with isolated singularities.

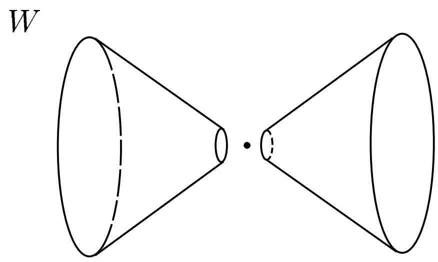


Figure 2.2: Stratification of V.

CONCLUSION: *Every real or complex algebraic variety with isolated singularities admits a canonical structure of a p-stratifold.*

2.2.1 Resolution of isolated singularities

In contrast to algebraic varieties, resolutions of stratifolds in general do not exist, not even for isolated singularities. But in this case there is a simple necessary and sufficient condition, see [Kr3].

Theorem 2.2.1. *An n-dimensional p-stratifold with isolated singularities admits a resolution if and only if each link of the singularity L_i vanishes in the bordism group Ω_{n-1}.*

EXAMPLE: The p-stratifold $\mathscr{S} = \mathbb{C}P^2 \times I \cup_f \{x_0, x_1\}$ with the obvious strats, such that $f(\mathbb{C}P^2 \times \{0\}) = x_0$ and $f(\mathbb{C}P^2 \times \{1\}) = x_1$, does not admit a resolution.

Although the proof of the theorem can be found in [Kr3], it is useful to understand its nature for the succeeding results.

One of the basic tools for constructing a resolving map is the following lemma, which can be proved with the help of Morse theory, see [Kr3].

Lemma 2.2.2. *Let M be a smooth compact manifold with boundary. There is a thin compact subspace X of M and a continuous map $\partial M \longrightarrow X$, such that M is homeomorphic to $\partial M \times [0,1] \cup X$, where on $\partial M \times [0,1)$ the homeomorphism can be chosen to be a diffeomorphism.*

In other words, every smooth manifold with boundary arises from its collar by attaching a thin set. The notation *thin* stands for the complement of a dense set. With this information, we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Let $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ be a resolution. Set $M_i := \overline{U\hat{\Sigma}_i}$.

Since M_i is a compact manifold with boundary L_i, we obtain $[L_i] = 0$ in Ω_{n-1}.

On the other hand let M_i be a compact manifold bounded by L_i and let $f(\mathring{W})$ be the top stratum of $\mathscr{S}$. Set $\hat{\mathscr{S}} := W \cup_\partial (\sqcup_i M_i)$ and construct the following resolving map with the help of the last Lemma:

$$\hat{\mathscr{S}} \cong W \cup_i (L_i \times [0, \varepsilon_i] \cup_{f_i} X_i) \xrightarrow{p} W \cup_i (L_i \times [0, \varepsilon_i] \cup \{x_i\}) = \mathscr{S}\,.$$

id

id

□

We have shown above that every algebraic variety with isolated singularities admits a structure of a p-stratifold. One may ask the converse question. When does a p-stratifold with isolated singularities admit an algebraic structure? The following Theorem of Akbulut and King [AK, Thm. 4.1] clarifies the situation in the case of a complex algebraic structure.

Theorem 2.2.3. *A topological space X is homeomorphic to an algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold M with boundary $\partial M = \cup_{i=1}^r L_i$, where each L_i bounds, then crushing some L_i's to points and deleting the remaining L_i's.*

Combining this result with Theorem 2.2.1 we immediately obtain:

Corollary 2.2.4. *A p-stratifold $\mathscr{S}$ with isolated singularities is homeomorphic to an algebraic set with isolated singularities if and only if $\mathscr{S}$ admits a resolution.*

The question of optimal resolutions simplifies in the situation of isolated singularities. It is not hard to verify that a resolution $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is optimal if and only if the manifolds $\overline{U\hat{\Sigma}_i}$ are $([n/2] - 1)$-connected.

Before proceeding with the existence of an optimal resolution, we have to explain the notation. For a topological space X, let $X\langle k\rangle$ be the k-connected cover of X, which always comes with a fibration $p : X\langle k\rangle \longrightarrow X$. For further information, see, for example, [Ba]. We take X to be the classifying space BO and denote by $\Omega_n^{BO\langle k\rangle}$ the bordism group of closed n-dimensional manifolds, together with a lift of the normal Gauss map (compare [St, Chap. I]).

Theorem 2.2.5. *An n-dimensional p-stratifold with isolated singularities admits an optimal resolution if and only if the normal Gauss map $\nu_j : L_j \longrightarrow BO$ admits a lift over $BO\langle[n/2]-1\rangle$ such that $[L_j, \bar{\nu}_j]$ vanishes in $\Omega_{n-1}^{BO}\langle[n/2]-1\rangle$ for every link of the singularity L_j.*

For the proof, see [Gr]. As an example, we obtain the following corollary.

Corollary 2.2.6. *Let $\mathscr{S}$ be a p-stratifold with parallelizable links of isolated singularities L_i. Assume every L_i is bounded by a parallelizable manifold, then $\mathscr{S}$ admits an optimal resolution.*

Example (**Resolutions of hypersurfaces with isolated singularities**): Let $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a polynomial with isolated singularities $\{s_i\}_i$, i.e. $s_i \in V := p^{-1}(0)$ and s_i is an isolated critical point of p. Assume further that the points s_i are not open. According to a previous example, the hypersurface V admits a canonical structure of a p-stratifold. We have to investigate the link of the singularity, which is given by $\partial V_{\varepsilon_i}(s_i)$.

Choose a $\delta > 0$ such that all c with $|c| \leq \delta$ are regular values of p and take c such that $p^{-1}(c) \neq \emptyset$. Then $p^{-1}(c)$ is a smooth manifold with trivial normal bundle. With the help of a gradient vector field we see that $p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i)$ is diffeomorphic to $p^{-1}(0) \cap S^n_{\varepsilon_i}(s_i) = \partial V_{\varepsilon_i}(s_i)$. Thus, $\partial V_{\varepsilon_i}(s_i) \cong p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i) \cong \partial(p^{-1}(c) \cap D^{n+1}_{\varepsilon_i}(s_i))$. We see that a resolution always exists, and since the bounding manifolds are automatically parallelizable, we even obtain an optimal resolution after choosing an appropriate bordism (compare with Figure 2.3).

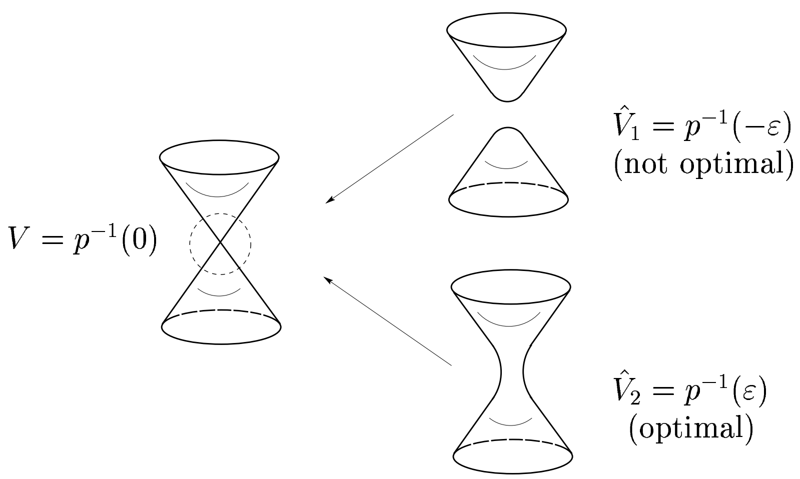


Figure 2.3: $p(x, y, z) = x^2 + y^2 - z^2$.

In the case of a complex polynomial $p : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ $(n > 0)$, every deformation $p^{-1}(c)$ gives us an optimal resolution, provided $||c||$ is small enough. This follows from a result of Milnor [Mi2, Thm. 6.5] which states that $M := p^{-1}(c) \cap D_\varepsilon^{2n+2}(s_i)$ is homotopy equivalent to a wedge of $\mu \geq 1$ copies of S^n and thus $(n-1)$-connected.

CONCLUSION: *Let $\mathbb{K}$ denote the field of real or complex numbers and let $n > 1$. Each hypersurface $V = p^{-1}(0)$ in $\mathbb{K}^n$ with isolated singularities admits an optimal resolution. In the case of $\mathbb{K} = \mathbb{C}$, the deformation $p^{-1}(c)$ is itself the total space of an optimal resolution, provided $||c||$ is small enough.*

Consider another interesting class of p-stratifolds with isolated singularities, namely those arising from a smooth group action.

Definition: A smooth S^1-action on a smooth manifold M is called *semi-free* if the action is free outside of the fixed point set, i.e. if $gx = x$ for a $g \in S^1, g \neq 1$ and $x \in M$, then $hx = x$ for all $h \in S^1$.

Lemma 2.2.7. *Let M be a closed oriented manifold with semi-free S^1-action with only isolated fixed points. Then M/S^1 admits an oriented resolution if and only if* $\dim M \equiv 0(mod\, 4)$.

Proof. Let $\dim M = n$. There is nothing to show if the action is free. Thus let $x \in M$ be a fixed point. The differential of the action gives a representation of S^1 on T_xM and there is an equivariant local diffeomorphism from T_xM onto a neighbourhood of x in M. According to [BT, Prop. (II.8.1)], every irreducible representation of S^1 on $\mathbb{R}^n$ is equivalent to:

$$\begin{pmatrix} z^{n_1} & 0 & \dots & \dots & 0 \\ 0 & \cdot & \ddots & & \vdots \\ \vdots & \ddots & \cdot & \ddots & \vdots \\ \vdots & & \ddots & z^{n_{[n/2]}} & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} z^{n_1} & 0 & \dots & \dots & 0 \\ 0 & \cdot & \ddots & & \vdots \\ \vdots & \ddots & \cdot & \ddots & \vdots \\ \vdots & & \ddots & \cdot & 0 \\ 0 & \dots & \dots & 0 & z^{n_{n/2}} \end{pmatrix}$$

considered as a representation on $\mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{R}$, if n is odd

considered as a representation on $\mathbb{C} \times \cdots \times \mathbb{C}$, if n is even

Since the action is semi-free and x an isolated fixed point we conclude that $\dim M$ is even. We can further assume $n_i = 1$ for all $i \in \{1, \dots, n/2\}$. Let $\dim M = 2m$ and let $\{x_1, \dots, x_k\}$ be the set of fixed points. Choose equivariant disks D_{x_i} around x_i. In this situation we have

$$M/S^1 = (M - \sqcup_i \mathring{D}_{x_i})/S^1 \cup \{x_1, \dots, x_k\}.$$

The domain of the top strat is then given by $W^n := (M - \sqcup_i \mathring{D}_{x_i})/S^1$ and the singular set is $\Sigma := \{x_1, \ldots, x_k\}$. The links of singularities are given by $L_i \cong S^{2m-1}/S^1 = \mathbb{C}P^{m-1}$. Using Theorem 2.2.1 we conclude that the resolution exists if and only if $[\mathbb{C}P^{m-1}]$ vanishes in Ω^{SO}_{2m-2}. For $m = 2l+1$ the signature of $\mathbb{C}P^{m-1}$ is equal to 1, hence $\mathbb{C}P^{m-1}$ does not bound. In the case of an even $m = 2l+2$ we have $\mathbb{C}P^{2l+1} = S^{4l+3}/S^1 = S(\mathbb{H}^{l+1})/S^1$, where $S(\mathbb{H}^{l+1})/S^1$ is the sphere bundle

$$\begin{tikzcd} S^2 = S^3/S^1 \arrow[r, hook] & S(\mathbb{H}^{l+1})/S^1 \arrow[d] \\ & S(\mathbb{H}^{l+1})/S^3 = \mathbb{H}P^l \end{tikzcd}$$

and the associated disk bundle bounds.

□

The classification of optimal resolutions is quite a difficult problem. For if $\hat{\mathscr{S}}$ is an optimal resolution of $\mathscr{S}$, then $\hat{\mathscr{S}}\sharp\mathcal{S}$ is again optimal for an arbitrary homotopy sphere $\mathcal{S}$. In particular, consider the sphere S^n stratified as $D^n \cup pt$, then every homotopy sphere $\mathcal{S}^n$ gives us a resolution of S^n. Thus, we weaken the problem and ask for the equivalence up to a homotopy sphere.

Definition: Two resolutions $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ are called *almost equivalent* if $\hat{\mathscr{S}}\sharp\mathcal{S}$ is equivalent to $\hat{\mathscr{S}}'$ for a homotopy sphere $\mathcal{S}$.

Before formulating the classification result, we need to introduce some notation.

Let B be a fibration over BO, a *normal B-structure* on a manifold M is a lift $\bar{\nu}$ of the normal Gauss map $\nu : M \longrightarrow BO$ to B.

Definition: Let B be a fibration over BO.

1. A normal B-structure $\bar{\nu} : M \longrightarrow B$ of a manifold M in B is a *normal k-smoothing* if it is a $(k+1)$-equivalence.

2. We say that B is *k-universal* if the fibre of the map $B \longrightarrow BO$ is connected and its homotopy groups vanish in dimension $\geq k+1$.

Obstruction theory implies that if B and B' are both k-universal and admit a normal k-smoothing of the same manifold M, then the two fibrations are fibre homotopy equivalent. Furthermore, the theory of Moore-Postnikov decompositions implies that for each manifold M there is a k-universal fibration

B^k over BO admitting a normal k-smoothing (compare [Ba, §5.2]). Thus, the fibre homotopy type of the fibration B^k over BO is an invariant of the manifold M and we call it the *normal k-type of M*.

There is an obvious bordism relation on closed n-dimensional manifolds with normal B-structures and the corresponding bordism group is denoted $\Omega_n^{(p:B\to BO)}$.

Theorem 2.2.8. *For $n > 2$, let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two optimal resolutions of a $2n$-dimensional p-stratifold $\mathscr{S}$ with isolated singularities $\{x_i\}_{i\in I}$, such that each link L_i is $(n-2)$-connected. Assume further that for a suitable representative $\overline{U}\Sigma$ the following conditions hold for all $i \in I$:*

- *$e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}_i')$;*
- *there exist normal $(n-1)$-smoothings of $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}_i'$ in a fibration $B \longrightarrow BO$, compatible on the boundaries;*
- *$\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}_i'$ is elementary.*

If n is odd, then $\hat{\mathscr{S}}$ is almost equivalent to $\hat{\mathscr{S}}'$.
If n is even, then $\hat{\mathscr{S}}\sharp k(S^n \times S^n)$ is almost equivalent to $\hat{\mathscr{S}}'\sharp k(S^n \times S^n)$ for a $k \in \{0, 1\}$.

We have to explain the last condition in the theorem.

Let M be a $2n$-dimensional $(n-1)$-connected manifold. According to the Hurewicz-Theorem every element of $H_n(M)$ is represented by a map $S^n \longrightarrow M$. Using a result of Haefliger [Hae] we can without loss of generality assume that the map $S^n \longrightarrow M$ is an embedding, and two embeddings which correspond to the same homology class are regular homotopic. Thus, assigning its normal bundle to an embedded sphere gives us a well defined map $\nu_* : H_n(M) \longrightarrow \pi_{n-1}(SO(n))$.

Definition: An $(n-1)$-connected closed $2n$-dimensional manifold M is called *elementary* if $H_n(M)$ admits a Lagrangian $\mathcal{L}$ with respect to the self intersection form such that $(\nu)_*|_{\mathcal{L}} \equiv 0$.

The proof of the Theorem is based on surgery and is carried out in [Gr]. In the next paragraph, we are going to find an algebraic description of the last condition in the theorem.

2.2.2 Algebraic invariants

In this section we will find algebraic invariants, which allow us to decide whether an $(n-1)$-connected closed $2n$-dimensional manifold is elementary or not $(n > 2)$.

Recall the algebraic data corresponding to such a manifold M. We have a triple (H, Λ, ν_*), where $H = H_n(M)$ is a free $\mathbb{Z}$-module, $\Lambda : H \times H \longrightarrow \mathbb{Z}$ is the intersection product and $\nu_* : H \longrightarrow \pi_{n-1}(SO_n)$ is a normal bundle map, described in the previous section. The map ν_* is not a homomorphism, but satisfies the following equation:

$$\nu_*(x+y) = \nu_*(x) + \nu_*(y) + \partial\Lambda(x,y), \tag{$*$}$$

where $\partial : \mathbb{Z} \cong \pi_n(S^n) \longrightarrow \pi_{n-1}(SO_n)$ is the boundary map from the long exact homotopy sequence of the fibration $SO(n) \hookrightarrow SO(n+1) \longrightarrow S^n$, see [W1].

Thus, we obtain an algebraic object, the set $\mathcal{T}_n$ of triples (H, Λ, ν_*), where H is a free $\mathbb{Z}$-module, $\Lambda : H \times H \longrightarrow \mathbb{Z}$ is an $(-1)^n$-symmetric unimodular quadratic form and $\nu_* : H \longrightarrow \pi_{n-1}(SO_n)$ is a map satisfying $(*)$. We want to investigate the assumptions under which an element $(H, \Lambda, \nu_*) \in \mathcal{T}_n$ is elementary, i. e. when H possesses a Lagrangian $\mathcal{L}$ with respect to Λ such that $\nu_*|_{\mathcal{L}} \equiv 0$.

We begin with an observation that for a $4k$-dimensional manifold the normal bundle, information can be replaced by the stable normal bundle map.

Lemma 2.2.9. *Let n be even and let $S^n \hookrightarrow M^{2n}$ be an embedding. The normal bundle $\nu(S^n)$ of S^n in M is trivial if and only if $\nu \oplus \mathbb{R}$ is trivial and the Euler class of $\nu(S^n)$ vanishes.*

Thus, instead of considering $(H, \Lambda, \nu_*) \in \mathcal{T}_n$ we can go over to $(H, \Lambda, s\nu_*)$, where $s\nu_* : H \longrightarrow \pi_{n-1}(SO)$ corresponds to the stable normal bundle map. Since the Euler class of an embedded sphere representing $x \in H$ can be identified with the self intersection class we conclude:

Lemma 2.2.10. *Let n be even. Then $(H, \Lambda, \nu_*) \in \mathcal{T}_n$ is elementary if and only if $(H, \Lambda, s\nu_*)$ is elementary.*

Let $\mathcal{T}_n^s$ denote the set of triples $(H, \Lambda, s\nu_*)$, with H and Λ as above and $s\nu_* : H \longrightarrow \pi_{n-1}(SO)$ a homomorphism. According to the different possibilities for $\pi_{n-1}(SO)$ we distinguish 3 cases.

(1) $\pi_{n-1}(SO) = 0$.
CLAIM: $(H, \Lambda, s\nu_*) \in \mathcal{T}_n^s$ is elementary if and only if $\mathrm{sign}(\Lambda) = 0$, where sign denotes the signature of a quadratic form.

(2) $\pi_{n-1}(SO) = \mathbb{Z}$. Since Λ is unimodular it induces an isomorphism $H \xrightarrow{\cong} H^*$, which we also denote by Λ. The map $s\nu_*$ gives an element of H^* and we consider $\kappa_{s\nu_*} := \Lambda^{-1}(s\nu_*) \in H$.
CLAIM: $(H, \Lambda, s\nu_*) \in \mathcal{T}_n^s$ is elementary if and only if $\mathrm{sign}(\Lambda) = 0$ and $\kappa_{s\nu_*}^2 = 0$.

(3) $\pi_{n-1}(SO) = \mathbb{Z}_2$. Let $(H, \Lambda, s\nu_*)$ be an element of $\mathcal{T}_n^s$ with vanishing signature and suppose Λ is of type II, i.e. $\Lambda(x, x) = 0 \pmod 2$ for all $x \in H$. Note that since $n \neq 8$ in this case, an elementary element corresponding to a manifold always has a type II quadratic form. Thus, the dimension of H is even and according to [Mi1, Lem. 9] we can choose a basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ satisfying

$$\Lambda(\lambda_i, \lambda_j) = 0, \quad \Lambda(\mu_i, \mu_j) = 0 \text{ and } \Lambda(\lambda_i, \mu_j) = \delta_{ij}.$$

Consider the set of all elements $x \in H$ with $\Lambda(x, x) = 0$ and denote its image under canonical projection on $H \otimes \mathbb{Z}_2$ by H^0. This class $\Phi(H, \Lambda, s\nu_*) := \sum_{i=1}^k s\nu_*(\lambda_i) s\nu_*(\mu_i) \in \mathbb{Z}_2$ is well-defined and is equal to the value $s\nu_*$ takes most frequently on the finite set H^0, the class is called Arf invariant.
CLAIM: An element $(H, \Lambda, s\nu_*) \in \mathcal{T}_n^s$ with type II form Λ is elementary if and only if $\mathrm{sign}(\Lambda) = 0$ and $\Phi(H, \Lambda, s\nu_*) = 0$.

Consider now the case of an odd n. The quadratic form is now skew symmetric. Depending on the values of ν_* there are again three different cases (compare [Ke]), which were studied in [W1].

(4) $\pi_{n-1}(SO_n) = 0$. In this case every element of $\mathcal{T}_n$ is elementary.

(5) $\pi_{n-1}(SO_n) = \mathbb{Z}_2$. As in (3), we can define the Arf invariant $\Phi(H, \Lambda, \nu_*) = \sum_{i=1}^k \nu_*(\lambda_i)\nu_*(\mu_i) \in \mathbb{Z}_2$, using a symplectic basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ of H.
CLAIM: An element $(H, \Lambda, \nu_*) \in \mathcal{T}_n$ is elementary if and only if $\Phi(H, \Lambda, \nu_*) = 0$.

(6) $\pi_{n-1}(SO_n) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We consider again the stable normal bundle map $s\nu_* : H \longrightarrow \mathbb{Z}_2$, the projection on the first component. As in (2) using Λ, we obtain an element κ (determined mod $2H$) with $s\nu_*(x) = \Lambda(\kappa, x) \pmod 2$ for all $x \in H$.

Claim: An element $(H, \Lambda, \nu_*) \in \mathcal{T}_n$ is elementary if and only if $\Phi(H, \Lambda, \nu_*) = 0$ and $\mathrm{pr}_2\nu_*(\kappa) = 0$, where pr_2 denotes the projection on the second component.

Proof. ad (2): Let $(H, \Lambda, s\nu_*)$ be elementary and $\mathcal{L} = \langle\lambda_1, \dots \lambda_k\rangle$ a Lagrangian with $s\nu_*|_{\mathcal{L}} \equiv 0$. Thus $\mathrm{sign}(\Lambda) = 0$ and $0 = s\nu_*(\lambda_i) = \Lambda\langle\kappa_{s\nu_*}, \lambda_i\rangle$ for all $1 \leq i \leq k$. Since $\mathcal{L}$ is maximal it follows that $\kappa_{s\nu_*} \in \mathcal{L}$ and therefore $s\nu_*(\kappa_{s\nu_*}) = 0$.

On the other hand let $\mathrm{sign}(\Lambda) = 0$ and $s\nu_*(\kappa_{s\nu_*}) = 0$. Choose a basis $\{\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k\}$ of H such that $\Lambda(\lambda_i, \lambda_j) = 0$ and $\Lambda(\lambda_i, \mu_j) = \delta_{ij}$.

There is nothing to show if $\kappa_{s\nu_*} = 0$. Otherwise we can without loss of generality assume that $s\nu_*(\lambda_i) = 0$ for all $i > 1$, since $s\nu_*$ is a homomorphism. Recall the equality $s\nu_*(v) = \Lambda(\kappa_{s\nu_*}, v)\ \ \forall v \in H$. Since $\kappa_{s\nu_*} \in H$ there are $a_i, b_i \in \mathbb{Z}$ such that $\kappa_{s\nu_*} = \sum_{i=1}^k (a_i\lambda_i + b_i\mu_i)$. Consider a sub-Lagrangian $\mathcal{L}' := \langle\lambda_2, \dots, \lambda_k\rangle$. If $\kappa_{s\nu_*} \notin \mathcal{L}'$, build $\mathcal{L} := \langle\lambda_2, \dots, \lambda_k, \tilde{\kappa}_{s\nu_*}\rangle$, where $\tilde{\kappa}_{s\nu_*}$ is a primitive element of H with $\kappa_{s\nu_*} \in \langle\tilde{\kappa}_{s\nu_*}\rangle$. This is a Lagrangian, satisfying $s\nu_*|_{\mathcal{L}} \equiv 0$. In the case of $\kappa_{s\nu_*} \in \mathcal{L}'$, the coefficients $a_1, b_1, \dots, b_k$ have to be zero, thus $\mathcal{L} = \langle\lambda_1, \dots, \lambda_k\rangle$ is a Lagrangian with the desired property.

ad (3): The conditions are obviously necessary. To see that they are also sufficient choose a symplectic basis $\{\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k\}$ of H. Sort the generators in the following way

$$\begin{aligned} s\nu_*(\lambda_i) &= s\nu_*(\mu_i) = 1 && \text{for } i \leq s, \\ s\nu_*(\lambda_i) &= 0 && \text{for } i > s, \end{aligned}$$

where s is an integer between 0 and k. The assumption $\Phi(H) = \sum_{i=1}^k s\nu_*(\lambda_i)s\nu_*(\mu_i) = 0$ implies that $s \equiv 0 (\mathrm{mod}\ 2)$. Construct a new basis $\{\lambda_1', \dots, \mu_k'\}$ for H by the substitution

$$\begin{aligned} \lambda_{2i-1}' &= \lambda_{2i-1} + \lambda_{2i}, &\qquad \lambda_{2i}' &= \mu_{2i-1} - \mu_{2i}, \\ \mu_{2i-1}' &= \mu_{2i} & \mu_{2i}' &= \lambda_{2i} \end{aligned}$$

for $2i \leq s$, and

$$\lambda_i' = \lambda_i \qquad \mu_i' = \mu_i$$

for $i > s$. This new basis is again symplectic and satisfies the condition

$$s\nu_*(\lambda_1') = \cdots = s\nu_*(\lambda_k') = 0.$$

□

Knowing the algebraic description of elementary manifolds, we formulate a special case of Theorem 2.2.8.

Corollary 2.2.11. *Let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two resolutions of a $2n$-dimensional p-stratifold $\mathscr{S}$ having $(n-2)$-connected links of isolated singularities. Assume that $n \equiv 6 \pmod 8$ and that $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}'_i$ are parallelizable with compatible parallelization on the boundary. Let further $e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}'_i)$ and $sign(\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}'_i) = 0$. Then there is a $k \in \{0,1\}$ such that $\hat{\mathscr{S}}\sharp k(S^n \times S^n)$ is almost equivalent to $\hat{\mathscr{S}}'\sharp k(S^n \times S^n)$.*

2.2.3 4-dimensional results

In this section we consider the exceptional case of a 4-dimensional p-stratifold and give a similar classification result in that situation.

For a 4-dimensional stratifold $\mathscr{S}$, every link of the singularity L_i is a 3-dimensional manifold. According to the computation of Ω_* by Thom [Th2] we immediately obtain from Theorem 2.2.1:

Corollary 2.2.12. *A four-dimensional p-stratifold with isolated singularities always admits a resolution.*

If we further assume the links to be oriented we can use the following result:

Proposition 2.2.13. *Every orientable 3-manifold is parallelizable, hence in particular spin.*

Proof. Let M be an orientable 3-manifold. The tangent bundle TM is trivial if and only if the classifying map $g : M \longrightarrow BO_3$ is null-homotopic. According to Whitehead's theorem this is the case if and only if the induced map in homotopy $g_* : \pi_i(M) \longrightarrow \pi_i(BO_3)$ is trivial for $i = 0, \ldots, 3$. The last statement is clearly true if all Stiefel-Whitney classes $w_i(M)$ vanish. Since M is orientable $w_1(M) = 0$. For the third Stiefel-Whitney class we have the equality $w_3(TM) = e(M) \bmod 2$, and the right side vanishes by Poincaré duality. By the Wu formula we have $w_1(M) = v_1$ and $w_2(M) = v_1^2 + v_2$. From the definition of the Wu classes we see that $v_2 = 0$ if $Sq^2 = 0$, and the triviality of the 2nd Stiefel-Whitney class is clear for dimension reasons.

□

The normal 1-type of a simply connected 4-dimensional spin-manifold is given by BSpin. Since $\Omega_3^{\mathrm{spin}} = 0$ (see [Mi3, Lem. 9]) we obtain the following corollary from Theorem 2.2.5.

Corollary 2.2.14. *A four-dimensional p-stratifold with isolated singularities admits an optimal resolution if and only if all links of singularities are orientable.*

Let us concentrate on resolutions by spin manifolds.

We have to develop some notation in the topological category. Use BTOP to denote the classifying space of topological vector bundles and let BTOPSpin be the 2-connected cover over BTOP. Let M be a simply connected 4-manifold. Using the Wu-Formula we can explain the Stiefel-Whitney-classes of M. We call the topological manifold M *spin* if $w_2(M)$ vanishes. One can show that the topological Gauss map of M lifts to BTOPSpin if and only if M is spin. Note further that if such a lift exists, it is unique.

Using [Kr1, Thm. 2] and the h-cobordism-Theorem in dimension 4 [Fr, Thm. 10.3] we formulate:

Theorem 2.2.15. *Let M_1 and M_2 be compact 4-dimensional topological spin manifolds with $e(M_1) = e(M_2)$ and let $g : \partial M_1 \longrightarrow \partial M_2$ be a homeomorphism compatible with the induced spin-strucures on the boundaries. If $M_1 \cup_g M_2$ vanishes in $\Omega_4^{BTOPSpin}$, then g can be extended to a homeomorphism $G : M_1 \sharp k(S^2 \times S^2) \longrightarrow M_2 \sharp k(S^2 \times S^2)$ for $k \in \{0, 1\}$.*

We call two resolutions *topologically equivalent* if the diffeomorphism $\varphi_{\mathbf{c}}$ in the definition of equivalent resolutions in §2.1 is replaced by a homeomorphism. Using this notation, we obtain the following classification result in dimension four:

Theorem 2.2.16. *Let $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $\hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two optimal resolutions of a 4-dimensional p-stratifold $\mathscr{S}$ with isolated singularities $\{x_i\}_{i \in I}$, such that each link L_i is connected. Assume that both $\hat{\mathscr{S}}$ and $\hat{\mathscr{S}}'$ are spin and that for a suitable representative $\overline{U}\Sigma$ of the neighbourhood germ, the following conditions hold for all $i \in I$:*

- *$e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}'_i)$,*
- *the spin-structures of $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}'_i$ coincide on the boundary,*
- *$sign(\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}'_i) = 0$.*

Then $\hat{\mathscr{S}} \sharp k(S^2 \times S^2)$ is topologically equivalent to $\hat{\mathscr{S}}' \sharp k(S^2 \times S^2)$ for a $k \in \{0, 1\}$.

Proof. As in the proof of Theorem 2.2.8 we conclude that it is enough to show that the diffeomorphism on the boundary $\partial \overline{U}\hat{\Sigma}_i \longrightarrow \partial \overline{U}\hat{\Sigma}'_i$ can be extended to a homeomorphism on $\overline{U}\hat{\Sigma}_i \sharp k(S^2 \times S^2) \longrightarrow \overline{U}\hat{\Sigma}'_i \sharp k(S^2 \times S^2)$. Since the resolutions are optimal, the manifolds $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}'_i$ are 1-connected, hence

$M := \overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}'_i$ is again 1-connected. In order to apply Theorem 2.2.15 we have to show that the closed 4-dimensional spin manifold with vanishing signature is bordant to a homotopy sphere. Then we apply the topological 4-dimensional Poincaré conjecture proved by Freedman [Fr, Thm. 1.6] and obtain the desired statement.

We want to use surgery to prove that M is bordant to a homotopy sphere. Since M is spin and $\operatorname{sign}(M) = 0$, there is a basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ of $H_2(M)$ satisfying

$$\Lambda(\lambda_i, \lambda_j) = 0 \quad \Lambda(\mu_i, \mu_j) = 0 \quad \Lambda(\lambda_i, \mu_j) = \delta_{ij}.$$

We can not use the Haefliger's embedding theorem in dimension 4, but according to [Fr, Thm. 3.1, 1.1] every generator λ_i is represented by a topological embedding $S^2 \times D^2 \hookrightarrow M$. Knowing this we can proceed in exactly the same way as in the proof of Theorem 2.2.8, see [Gr].

It remains to show that the conditions of the theorem are true for every representative of the neighbourhood germs $[\overline{U}\Sigma_i]$, once we have checked them on one representative. If $\mathbf{c}_i : L_i \times [0, \varepsilon_i/2] \longrightarrow \mathscr{S}$ and $\tilde{\mathbf{c}}_i : L_i \times [0, \varepsilon'_i/2] \longrightarrow \mathscr{S}$ with $\varepsilon_i < \varepsilon'_i$ are two representatives of the germ of collars around L_i, then there exists a positive $\varepsilon''_i < \varepsilon_i/2$ such that $\mathbf{c}_i$ coincides with $\tilde{\mathbf{c}}_i$ on $L_i \times [0, \varepsilon'']$. We choose a diffeomorphism $\eta_i : [0, \varepsilon_i/2] \longrightarrow [0, \varepsilon'_i/2]$ with $\eta \equiv \mathrm{id}$ on $[0, \varepsilon''_i]$. The map induces an isomorphism

$$\begin{array}{ccc} \overline{U}\Sigma_{i\mathbf{c}_i} & \longrightarrow & \overline{U}\Sigma_{i\tilde{\mathbf{c}}_i} \\ f(\mathbf{c}_i(x,t)) & \longmapsto & f(\tilde{\mathbf{c}}_i(x, \eta_i(t))) \end{array}$$

being the identity on a small neighbourhood of $\Sigma_i \subset \mathscr{S}$. This gives us a diffeomorphism between $\overline{U}\hat{\Sigma}_{i\mathbf{c}_i}$ and $\overline{U}\hat{\Sigma}_{i\tilde{\mathbf{c}}_i}$ making the following diagram commutative:

$$\begin{tikzcd} \partial\overline{U}\hat{\Sigma}_{i\mathbf{c}_i} \arrow[r, hook] \arrow[d, "\cong"] & \overline{U}\hat{\Sigma}_{i\mathbf{c}_i} \arrow[r, "\cong"] & \overline{U}\hat{\Sigma}_{i\tilde{\mathbf{c}}_i} & \partial\overline{U}\hat{\Sigma}_{i\tilde{\mathbf{c}}_i} \arrow[l, hook'] \arrow[d, "\cong"] \\ \partial\overline{U}\hat{\Sigma}'_{i\mathbf{c}_i} \arrow[r, hook] & \overline{U}\hat{\Sigma}'_{i\mathbf{c}_i} \arrow[r, "\cong"] & \overline{U}\hat{\Sigma}'_{i\tilde{\mathbf{c}}_i} & \partial\overline{U}\hat{\Sigma}'_{i\tilde{\mathbf{c}}_i} \arrow[l, hook'] \end{tikzcd}$$

This completes the proof.

$\square$

2.3 First approach to the resolution of non isolated singularities

In this section we apply the methods obtained by dealing with isolated singularities to the general situation of a p-stratifold. We start with an n-dimensional p-stratifold $\mathscr{S}$ with strats $f_i : W^i \longrightarrow \mathscr{S}$. Denote as usual the $(n-1)$-skeleton by Σ. Observe that we can always assume $\mathscr{S}$ to be $W^n \cup_{f_n|_{\partial W^n}} \Sigma$. Since, given a representative $f_n : W^n \longrightarrow \mathscr{S}$ of the top strat, we can give the space $f_n(\mathring{W}^n) \sqcup \partial W^n$ a structure of a c-manifold using the bijection $f_n|_{\mathring{W}^n} \sqcup \mathrm{id}|_{\partial W^n} W^n \longrightarrow f_n(\mathring{W}^n) \sqcup \partial W^n$. Thus, the strat $\mathrm{id}|_{f_n(\mathring{W}^n)} \sqcup f|_{\partial W^n} : f_n(\mathring{W}^n) \sqcup \partial W^n \longrightarrow \mathscr{S}$ is equivalent to $f_n : W^n \longrightarrow \mathscr{S}$.

In this section we always assume that $\mathscr{S} = W^n \cup_{f_n|_{\partial W^n}} \Sigma$. We further assume throughout this section that Σ has a decomposition $\{\Sigma_i\}_{i \in I \subseteq \mathbb{N}}$ into compact connected components. Set $g := f_n|_{\partial W^n} : \partial W^n \longrightarrow \Sigma$ and consider the links of the singularities $L_i := g^{-1}(\Sigma_i)$. Further denote the restriction of g to L_i by g_i. Since the manifolds L_i are compact, we can always choose a representative of the germ of collars $\mathbf{c} : (\partial W \times [0,\infty))^{<\delta} \longrightarrow W$ such that $\delta|_{L_i}(x) = 2\varepsilon_i$ for all $x \in L_i$. In the following we will always use this representative and denote its restriction to $(L_i \times [0,\infty))^{<\delta}$ by $\mathbf{c}_i$.

In the same way as in the case of isolated singularities we conclude:

Proposition 2.3.1. *If a p-stratifold $\mathscr{S}$ admits a resolution then $[L_i, g_i]$ vanishes in the bordism group $\Omega_{n-1}(\Sigma_i)$ for all $i \in J$.*

Now assume the link $[L_i, g_i]$ is zero bordant in $\Omega_{n-1}(\Sigma_i)$. Thus there is a manifold M_i together with a map $G_i : M_i \longrightarrow \Sigma_i$ with $\partial M_i = L_i$ and $G_i|_{L_i} = g_i$. We apply Lemma 2.2.2 and obtain a closed thin subset X_i of M_i and a map $h_i : \partial M_i \longrightarrow X_i$ such that $\partial M_i \times [0,\varepsilon_i] \cup_{h_i \mathrm{pr}_1|_{\partial M_i \times \{\varepsilon_i\}}} X_i$ is homeomorphic to M_i. Further, the homeomorphism can be chosen in such a way that it is the identity on the boundary and a diffeomorphism outside of X_i. We are in a similar situation as in the case of isolated singularities. We are finished here if there is a map $p_i : X_i \longrightarrow \Sigma_i$ such that $g_i = p_i h_i$, for this would give us a resolution

$$\begin{array}{c}
\hat{\mathscr{S}} := W^n \cup_{\partial W^n = \partial W^n \times \{0\}} \left(\cup_i (L_i \times [0,\varepsilon_i] \cup_{h_i \mathrm{pr}_1 : L_i \times \{\varepsilon_i\} \to X_i} X_i)\right) \\
\Big\downarrow {\scriptstyle \mathrm{id} \cup (\cup_i (\mathrm{id} \cup p_i))} \\
W^n \cup_{\partial W^n = \partial W^n \times \{0\}} \left(\cup_i (L_i \times [0,\varepsilon_i] \cup_{g_i \mathrm{pr}_1 : L_i \times \{\varepsilon_i\} \to X_i} \Sigma_i)\right) = \mathscr{S}.
\end{array}$$

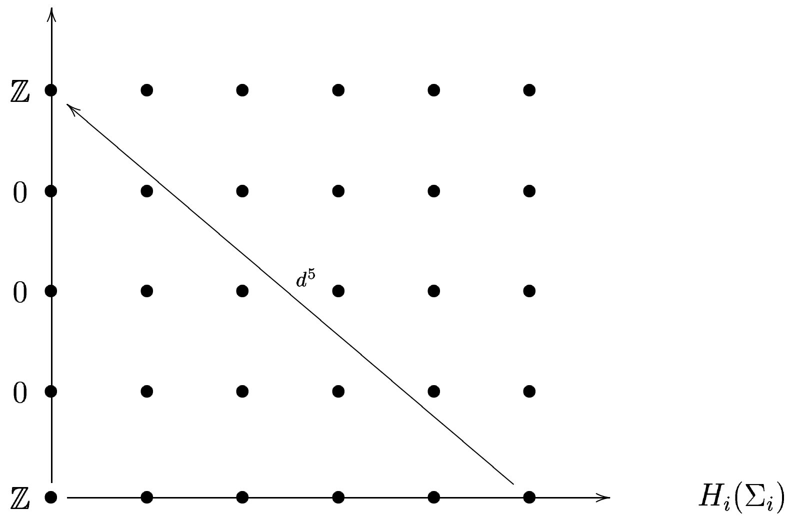
Consider the map $\tilde{p}_i := G_i|_{X_i} : X_i \longrightarrow \Sigma_i$. Unfortunately we only know that its composition with h_i is homotopy equivalent to g_i. The homotopy is given by the collar:

$$\begin{tikzcd} H_t : L_i \arrow[r] \arrow[drr] & L_i \times \{t\} \subset M_i \arrow[d, "G_i|_{L_i \times \{t\}}"] \\ & & \Sigma_i \end{tikzcd}$$

We summarize the discussion in the following theorem.

Theorem 2.3.2. *Let $\mathscr{S} = W^n \cup_g \Sigma$ be a smooth n-dimensional p-stratifold. Then g is homotopic to g' such that $\mathscr{S}' = W^n \cup_{g'} \Sigma$ admits a resolution if and only if $[L_i, g_i]$ vanishes in $\Omega_{n-1}(\Sigma_i)$ for all $i \in J$.*

Example: Let $\mathscr{S} = W^4 \cup_g \Sigma$ be a 4-dimensional p-stratifold with oriented domain W^4 of a representative of the top strat and singular set Σ, decomposed into compact connected components $\{\Sigma_i\}_i$. We are interested in the existence of an oriented resolution up to deformation of g. Denote as usual the links of singularities by $L_i := g^{-1}(\Sigma_i)$. According to Theorem 2.3.2, the obstructions to finding such a resolution are given by $[L_i, g_i] \in \Omega_3^{SO}(\Sigma_i)$, where g_i denotes the restriction of g to L_i. Apply the Atiyah-Hirzebruch spectral sequence to $\Omega_*^{SO}(\Sigma_i)$. The E^2-term is given by $H_i(\Sigma_i, \Omega_j^{SO}(\mathrm{pt}))$.



Thus, we obtain an isomorphism $\Omega_3^{SO}(\Sigma_i) \longrightarrow H_3(\Sigma_i)$ given by $[N, h] \mapsto h_*([N])$. In summary, we obtain:

Conclusion: *$\mathscr{S}$ admits an oriented resolution up to deformation of g if and only if $(g_i)_*[L_i] = 0$ in $H_3(\Sigma_i)$ for all $i \in J$.*

Now we want to collect necessary conditions for the existence of an optimal resolution.

We start again with an n-dimensional p-stratifold $\mathscr{S} = W^n \cup_g \Sigma$ with $(n-1)$-skeleton Σ. Let $\{\Sigma_i\}_{i \in J}$ be the decomposition of Σ into compact connected components. Suppose $\mathscr{S}$ admits an optimal resolution $r : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$. Set $M_i := r^{-1}(\overline{U}\Sigma^i)$. Thus, M_i is a non-empty connected manifold with boundary, and the boundary is isomorphic to $L_i := g^{-1}(\Sigma_i)$, the link of the singularity. The neighbourhood's retraction $\overline{U}\Sigma_i \longrightarrow \Sigma_i$ gives us a map $G_i : M_i \longrightarrow \Sigma_i$ extending the given map on the boundary. In the same way as for isolated singularities, we consider the map

$$G_i \times \nu_i : M_i \longrightarrow \Sigma_i \times BO,$$

where ν_i denotes the normal Gauss map. We look at the Moore-Postnikov tower of this map, the $[n/2]$-floor gives us a fibration $p_i : B_i \longrightarrow \Sigma_i \times BO$ with fibre F_i having the properties:

- $\pi_i(F_i) = 0$ for $i \geq [n/2]$ and
- $G_i \times \nu_i : M_i \longrightarrow \Sigma_i$ admits a lift over B_i by an $[n/2]$-equivalence:

$$\begin{tikzcd} & B_i \arrow[d, "p_i"] \\ M_i \arrow[ur, "\overline{G_i \times \nu_i}"] \arrow[r, "G_i \times \nu_i"'] & \Sigma_i \times BO \end{tikzcd}$$

We know that both G_i and $\overline{G_i \times \nu_i}$ are $[n/2]$-equivalences. So by considering the following commutative diagram

$$\begin{tikzcd} & \pi_i(B_i) \arrow[d, "\mathrm{pr}_1\, p_i"] \\ \pi_i(M) \arrow[ur, "(\overline{G_i \times \nu_i})_*"] \arrow[r, "G_{i*}"'] & \pi_i(\Sigma_i) \end{tikzcd}$$

we conclude that $\mathrm{pr}_1\, p_i$ is also an $[n/2]$-equivalence. We summarize necessary conditions for the existence of an optimal resolution: For all $i \in J$ there should exist a fibration $p_i : B_i \longrightarrow \Sigma_i \times BO$ such that

1. $\pi_j(F_i) = 0$ for $j \geq [n/2]$, where F_i is the fibre of p_i,
2. $\mathrm{pr}_1 p_i$ is an $[n/2]$-equivalence,
3. $g_i \times \nu_i : L_i \longrightarrow \Sigma_i \times BO$ admits a lift $\overline{g_i \times \nu_i}$ over B_i,

4. $[L_i, \mathrm{pr}_2\, \overline{g_i \times \nu_i}]$ vanishes in $\Omega_{n-1}^{B_i \to BO}$, where the fibration over BO is given by $\mathrm{pr}_2\, p_i$.

These conditions are also sufficient if we again slightly deform the attaching map.

Theorem 2.3.3. *Let $\mathscr{S} = W^n \cup_g \Sigma$ be an n-dimensional p-stratifold. Then g is homotopic to g' such that $\mathscr{S}' = W^n \cup_{g'} \Sigma$ admits an optimal resolution if and only if there are fibrations $p_i : B_i \longrightarrow \Sigma_i \times BO$ fulfilling the above conditions 1 to 4.*

Proof. We only have to verify that our conditions are sufficient. So suppose they are fulfilled. Then we can choose an n-dimensional manifold M_i bounding L_i together with a lift $\bar{\nu}_i : M_i \longrightarrow B_i$ of the normal Gauss-map making the following diagram commutative:

$$\begin{tikzcd} & B_i \arrow[d, "\mathrm{pr}_2 p_i"] \\ M_i \arrow[ur, "\bar{\nu}_i"] \arrow[r, "\nu_i"'] & BO \end{tikzcd}$$

such that the restriction of the lift to the boundary L_i coincides with the given map $\mathrm{pr}_2\, \overline{g_i \times \nu_i}$.

Using Proposition 4 from [Kr1] we can assume $\bar{\nu}_i$ to be an $[n/2]$-equivalence, so we conclude that the map

$$G_i : M_i \xrightarrow{\bar{\nu}_i} B_i \xrightarrow{p_i} \Sigma_i \times BO \xrightarrow{\mathrm{pr}_1} \Sigma_i$$

is an $[n/2]$-equivalence as well. According to Lemma 2.2.2 we can write M_i as $L_i \times [0, \varepsilon_i] \cup_{h_i} X_i$ and obtain an optimal resolution

$$\begin{tikzcd} \hat{\mathscr{S}} := W^n \cup_{\partial W^n = \partial W^n \times \{0\}} (\cup_i (L_i \times [0, \varepsilon_i] \cup_{h_i \mathrm{pr}_1 : L_i \times \{\varepsilon_i\} \to X_i} X_i)) \arrow[d, "\mathrm{id} \cup (\cup_i (\mathrm{id} \cup G_i|_X)"] \\ W^n \cup_{\partial W^n = \partial W^n \times \{0\}} (\cup_i (L_i \times [0, \varepsilon_i] \cup_{g_i' \mathrm{pr}_1 : L_i \times \{\varepsilon_i\} \to \Sigma_i} \Sigma_i)) = \mathscr{S}' \end{tikzcd}$$

where $g_i' = G_i|_{X_i} h_i$.

$\square$

Example: As in the previous example, consider a 4-dimensional stratifold $\mathscr{S} = W^4 \cup_g \Sigma$ built from an oriented manifold W^4 and the singularity Σ, decomposed into compact connected components Σ_i. Let us discuss the existence of an oriented optimal resolution up to deformation of g. According to Theorem 2.3.3, we first have to find a fibration $F_i \hookrightarrow B_i \xrightarrow{p_i} \Sigma_i \times BSO$ with $\pi_j(F) = 0$ for $j \geq 2$ and $B_i \longrightarrow \Sigma_i \times BSO \longrightarrow \Sigma_i$ a 2-equivalence, and a lift $\overline{g_i \times \nu_i}$ of $g_i \times \nu_i : L_i \longrightarrow \Sigma_i \times BSO$ over B_i such that $[L_i, \overline{g_i \times \nu_i}]$ vanishes in $\Omega_3^{B_i \to BSO}$.

Given such a fibration we obtain the following exact homotopy sequence:

$$\begin{tikzcd}
 & & \pi_2(BSO) = \mathbb{Z}_2 \arrow[d] & & & \\
0 \arrow[r] & \pi_2(B_i) \arrow[r] \arrow[dr, two heads] & \pi_2(\Sigma_i \times BSO) \arrow[r] \arrow[d] & \pi_1(F_i) \arrow[r] & \pi_1(B_i) \arrow[r] \arrow[dr, two heads, "\cong"'] & \pi_1(\Sigma_i \times BSO) \arrow[d, equal] \\
 & & \pi_2(\Sigma_i) & & & \pi_1(\Sigma_i)
\end{tikzcd}$$

Thus the fibre F_i is the Eilenberg-MacLane space $K(G,1)$, where $G = \operatorname{coker}((p_i)_* : \pi_2(B_i) \longrightarrow \pi_2(\Sigma_i \times BSO))$. Since $\pi_2(\Sigma_i \times BSO) = \pi_2(\Sigma_i) \oplus \mathbb{Z}/2$ and $\pi_2(\Sigma_i) \subset \text{im } (p_i)*$, there are only two possibilities for G, namely $G = 0$ or $G = \mathbb{Z}/2$.

In the first case, we obtain $B_i := \Sigma_i \times BSO$ and p_i is given by the identity map.

Using the the Atiyah-Hirzebruch spectral sequence from the last example we see that all differentials below d^5 are trivial, thus we conclude: $\Omega_3^{B_i \to BSO}$ is isomorphic to $H_3(B_i) = H_3(BSO) \oplus H_3(\Sigma_i) \oplus H_1(\Sigma_i) \otimes \mathbb{Z}/2$, where the isomorphism is given by $[N, \bar{\nu}_N] \longmapsto (\bar{\nu}_N)_*([N])$.

In the second case, we additionally assume that Σ_i is 1-connected. Then, according to a result by Baues [Ba, §5.2], the fibration $p_i : B_i \longrightarrow \Sigma_i \times BSO$ is a pull-back of the path-fibration over $K(\mathbb{Z}/2, 2)$ via a map $\theta_i : \Sigma_i \times BSO \longrightarrow K(\mathbb{Z}/2, 2)$. We can consider θ_i as an element of $\operatorname{Hom}(\pi_2(\Sigma_i) \times \mathbb{Z}/2, \mathbb{Z}/2)$. Considering again the above homotopy sequence, we see that θ_i is surjective and $\pi_2(\Sigma_i)$ lies in the kernel of θ_i. Thus, there is only one possibility for θ_i, namely

$$\begin{array}{ccccc}
\pi_2(\Sigma_i) \oplus \pi_2(BSO) & = & \pi_2(\Sigma_i) \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \\
 & & (x, y) & \longmapsto & y
\end{array},$$

leading to $B_i = \Sigma_i \times B\mathrm{Spin}$. The map p_i is given by id $\times p$, where p denotes the standard fibration $p : B\mathrm{Spin} \longrightarrow BO$.

Conclusion: *The stratifold $W^4 \cup_g \Sigma$ with a simply connected singularity Σ admits an optimal resolution up to deformation of g if and only if $(g_i)_*[L_i] = 0 \in H_3(\Sigma_i)$ and $(\nu_i)_*([L_i]) = 0 \in H_3(BSO)$ for all $i \in J$.*

As in the case of isolated singularities, we obtain the following classification result using Corollary 3 from [Kr1].

Theorem 2.3.4. *Let $\mathscr{S}$ be a $2n$-dimensional stratifold with compact connected components $\{\Sigma_i\}_{i\in J}$ of Σ. Two optimal resolutions $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $p' : \hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ of $\mathscr{S}$ are equivalent up to connected sum with $r(S^n \times S^n)$ if and only if for every $i \in J$:*

- *$e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}'_i)$,*
- *there is a fibration $B_i \longrightarrow \Sigma_i \times BO$ and lifts of $p_i \times \nu_i$ and $p'_i \times \nu'_i$ over B_i by $(n-1)$-equivalences, compatible with the diffeomorphism on the boundary such that*
- *$[\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}'_i]$ vanishes in $\Omega_{2n}^{B_i \to BO}$.*

Here p_i denotes the composition of $p|_{\overline{U}\hat{\Sigma}_i}$ with the retraction to Σ_i, and p'_i is the corresponding map $\overline{U}\hat{\Sigma}'_i \longrightarrow \Sigma'_i$.

2.4 Differential fibre bundles over spheres

In this section we consider another special case of p-stratifolds. Just as in the case of isolated singularities, we assume that there is only one non empty stratum aside from the top one, i.e., we consider an m-dimensional p-stratifold $\mathscr{S}$ with strats $f_m : W^m \longrightarrow \mathscr{S}$ and $f_n : W^n \longrightarrow \mathscr{S}$, with $\mathscr{S}^m \neq \emptyset \neq \mathscr{S}^n$ and $W^j = \emptyset$ for all $j < m, j \neq n$.

The manifold W^n is a smooth manifold without boundary, thus we can identify W^n with its image in $\mathscr{S}$. Using the discussion at the beginning of the last section, we can choose a representative of the parametrization such that $\mathscr{S} = W^m \cup_g W^n$, where g denotes the restriction of f_m to ∂W^m. To simplify the notation we will work with this parametrization throughout this section.

Assume further that the manifold W^n is the n-sphere and that the attaching map g is a differential fibre bundle. Set $W := W^m$ and denote the fibre of $g : \partial W \longrightarrow S^n$ with F. Since ∂W is compact, there is a representative of the germ of collars $\mathbf{c} : \partial W \times [0, 4\varepsilon) \longrightarrow W$.

2.4.1 Construction of a resolution

From the last section we know that one necessary condition for the existence of a resolution of $\mathscr{S}$ is a zero-bordism of $L := \partial W$ in $\Omega_{m-1}(S^n)$. With the help of the Theorem of Saard, we see that the fibre F also has to be zero-bordant. Thus, assume there is a manifold T such that $\partial T = F$.

First we consider the easiest case where the fibre bundle $g : L \longrightarrow S^n$ is trivial. In this situation we know that L is diffeomorphic to the product space $F \times S^n$. Now we can write down an obvious resolution, namely $W \cup_{\partial W} T \times S^n$. To specify the resolving map, we again make use of the Lemma 2.2.2 and decompose T as $F \times [0, \varepsilon] \cup_h X$. The resolving map is then given by:

$$\begin{array}{rcccc}
W \cup T \times S^n = & W & \cup_{L=L\times\{0\}} F \times S^n \times [0,\varepsilon] & \cup_{(x,y,\varepsilon)\mapsto(h(x),g(x,y))} & X \times S^n \\
 & \Big\downarrow \mathrm{id} & \Big\downarrow \mathrm{id} & & \Big\downarrow \mathrm{pr}_2 \\
W \cup S^n = & W & \cup_{L=L\times\{0\}} F \times S^n \times [0,\varepsilon] & \cup_{g\mathrm{pr}_1 : L\times\{\varepsilon\}\to S^n} & S^n
\end{array}$$

If the fibre bundle is not trivial, we consider the sphere as $D^n_+ \cup D^n_-$, two disks identified along the common boundary. We obtain two trivial fibre bundles by restricting the given one to each disk. Let

$$\alpha : S^{n-1} \longrightarrow \mathrm{Diff}(F)$$

contain the gluing information, such that the gluing map of the fibre bundle, which we also denote by α, is given by

$$\begin{array}{rccc}
\alpha : & S^{n-1} \times F & \longrightarrow & S^{n-1} \times F \\
 & (x, y) & \longmapsto & (x, \alpha(x)y)
\end{array}$$

The idea is to resolve the stratifold $\mathscr{S}$ step by step, where in each step the dimension of the singularity, i.e. the dimension of the attached sphere, is reduced until we find ourselves in the familiar case of isolated singularities.

Since the bundle restricted to a disk is trivial with unique trivialization, we can resolve the restriction over D^n_+ by building $D^n_+ \times T$ and the one over D^n_- by $D^n_- \times T'$, where T and T' are zero-bordisms of F. Apply Lemma 2.2.2 on T and T' and write $T = F \times [0, 4\varepsilon] \cup_{F\times\{4\varepsilon\}} X$ and $T' = F \times [0, 4\varepsilon] \cup_{F\times\{4\varepsilon\}} X'$. Let $\tilde{\mathbf{c}} : F \times [0, 4\varepsilon) \hookrightarrow T$ and $\tilde{\mathbf{c}}' : F \times [0, 4\varepsilon) \hookrightarrow T'$ be the collars of T and T' given by the identity. Now consider the manifold

$$W_1 := W \cup_{\partial W} (D^n_+ \times T \cup_{\mu_1} D^n_- \times T'),$$

where

$$\begin{array}{rccc} \mu_1 : & S^{n-1} \times \tilde{\mathbf{c}}(F \times [0, 3\varepsilon]) & \longrightarrow & S^{n-1} \times \tilde{\mathbf{c}}'(F \times [0, 3\varepsilon]) \\ & (x, \tilde{\mathbf{c}}(y, t)) & \longmapsto & (x, \tilde{\mathbf{c}}'(\alpha_x(y), t)) \,. \end{array}$$

The boundary of W_1 is given by

$$S^{n-1} \times (F \times [3\varepsilon, 4\varepsilon] \cup_{F\times\{4\varepsilon\}} X) \bigcup_{\alpha\times\mathrm{id}\,|_{S^{n-1}\times F\times\{3\varepsilon\}}} S^{n-1} \times (F \times [3\varepsilon, 4\varepsilon] \cup_{F\times\{4\varepsilon\}} X')$$

and we specify a collar $\mathbf{d} : \partial W_1 \times [0, 3\varepsilon) \longrightarrow W_1$ using the decomposition of T and T' to give the collar on $S^{n-1} \times F \times \{3\varepsilon\} \cup_{\alpha\times\mathrm{id}} S^{n-1} \times F \times \{3\varepsilon\}$ and then using a collar of length $[0, 3\varepsilon)$ of S^{n-1} in D^n_+ and D^n_-.

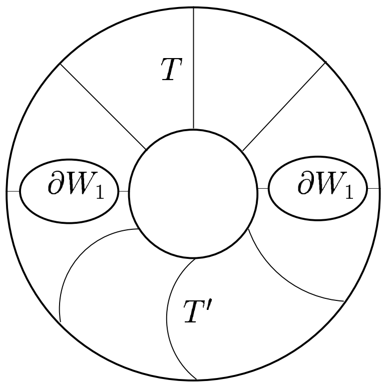


Figure 2.4: The space $D^n_+ \times T \cup_{\mu_1} D^n_- \times T'$.

We have a map from ∂W_1 to S^{n-1} given by the projection to the first factor. This map is obviously a submersion and thus, according to the Theorem of Ehresmann, a fibration. Later we will give the explicit description of this new bundle. The fibre over a point $x \in S^{n-1}$ is diffeomorphic to $T \cup_{\alpha(x)} T'$ and we can without loss of generality assume that this manifold is again zero-bordant. For if this is not the case, we can pass to the manifold $T' \sharp (T \cup_{\alpha(x)} T')$ instead of T'. Remember that we assumed the link L to be zero-bordant in $\Omega_{m-1}(S^n)$.

Set $g_1 := \mathrm{pr}_1 : \partial W_1 \longrightarrow S^{n-1}$, and consider a new stratifold $\mathscr{S}_1 := W_1 \cup_{g_1} S^{n-1}$ with obvious strats.

The following theorem allows an inductive approach indicated at the beginning of the section.

Theorem 2.4.1. *If $\mathscr{S}_1$ admits a resolution so does $\mathscr{S}$.*

Proof. Let $\mathbf{c} : \partial W \times [0, 4\varepsilon) \hookrightarrow W$ be a representative of the germ of collars and consider the neighbourhood $\overline{U}\Sigma = \mathbf{c}(L \times [0, 3\varepsilon]) \cup_{g:L\to S^n} S^n$ of the singularity.

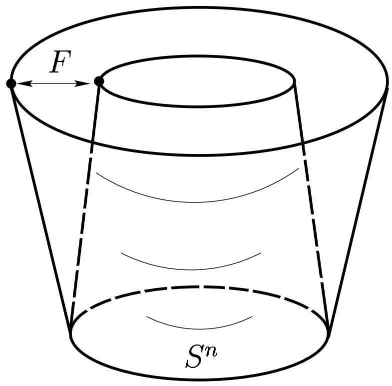


Figure 2.5: The neighbourhood $\overline{U}\Sigma$.

Within this neighbourhood look at the following neighbourhood of $S^{n-1} \subset S^n$:

$$A_1 := (S^{n-1} \times C_{2\varepsilon}F \cup_{\lambda_1} S^{n-1} \times C_{2\varepsilon}F') \times [0, \varepsilon) \cup_{\lambda_2} S^{n-1},$$

where

- $C_{2\varepsilon}F$ denotes the cone of length 2ε over F, i.e.

$$\begin{array}{rcl} C_{2\varepsilon}F & = & \tilde{\mathbf{c}}(F \times [0, 2\varepsilon))/\tilde{\mathbf{c}}(F \times \{0\}) \quad \text{and} \\ C_{2\varepsilon}F' & = & \tilde{\mathbf{c}}'(F \times [0, 2\varepsilon))/\tilde{\mathbf{c}}'(F \times \{0\}); \end{array}$$

- the additional product with $[0, \varepsilon)$ is given using a small collar of S^{n-1} in D^n_- and D^n_+ and identifying $S^{n-1} \times [0, \varepsilon)$ with its image, and

- the gluing maps λ_i are the following:

$$\begin{array}{rrcl} \lambda_1 : & S^{n-1} \times \tilde{\mathbf{c}}(F \times [\varepsilon, 2\varepsilon)) & \longrightarrow & S^{n-1} \times \tilde{\mathbf{c}}'(F \times [\varepsilon, 2\varepsilon)) \\ & (x, \tilde{\mathbf{c}}(y, t)) & \longmapsto & (x, \tilde{\mathbf{c}}'(\alpha_x(y), t)) \end{array}$$

and

$$\lambda_2 := \mathrm{pr}_1 : \quad (S^{n-1} \times \tilde{\mathbf{c}}(F \times [0, \varepsilon]) \cup_{\lambda_1} S^{n-1} \times \tilde{\mathbf{c}}'(F \times [0, \varepsilon])) \times \{0\} \quad \longrightarrow \quad S^{n-1}.$$

The complement of this set in $\overline{U}\Sigma$ is diffeomorphic to

$$A_2 := D_+^n \times C_{3\varepsilon}F \cup_{\lambda_3} D_-^n \times C_{3\varepsilon}F',$$

where the sets are glued together via:

$$\begin{array}{rccc} \lambda_3 : & S^{n-1} \times \tilde{\mathbf{c}}(F \times (\varepsilon, 3\varepsilon]) & \longrightarrow & S^{n-1} \times \tilde{\mathbf{c}}'(F \times (\varepsilon, 3\varepsilon]) \\ & (x, \tilde{\mathbf{c}}(y,t)) & \longmapsto & (x, \tilde{\mathbf{c}}'(\alpha_x(y), t)) \end{array}$$

Using the collar $\mathbf{d}$ of ∂W_1 in W_1 of length 3ε we build the following neighbourhood of the singularity:

$$B_1 := \mathbf{d}(\partial W_1 \times [0, \varepsilon]) \cup_{\mu_2} S^{n-1}$$

where $\mu_2 := g_1 \mathrm{pr}_1|_{\partial W_1 \times \{0\}}$. Now consider the complement of this set in the subset of $\mathscr{S}_1$ given by

$$\mathbf{d}(\partial W_1 \times [0, 2\varepsilon]) \cup_{\mu_1} S^{n-1}.$$

We see that the resulting space is diffeomorphic to

$$B_2 := D_+^n \times (F \times [\varepsilon, 4\varepsilon] \cup_{F \times \{4\varepsilon\}} X) \bigcup_{\mu_3} D_-^n \times (F \times [\varepsilon, 4\varepsilon] \cup_{F \times \{4\varepsilon\}} X')$$

where the map $\mu_3 : S^{n-1} \times \tilde{\mathbf{c}}(F \times (\varepsilon, 3\varepsilon]) \longrightarrow S^{n-1} \times \tilde{\mathbf{c}}'(F \times (\varepsilon, 3\varepsilon])$ is given by $\mu_3(x, \tilde{\mathbf{c}}(y,t)) := (x, \tilde{\mathbf{c}}'(\alpha_x(y), t))$.

After all these preparations we can write down a continuous map $\varphi : \mathscr{S}_1 \longrightarrow \mathscr{S}$ satisfying the following conditions:

1. $\varphi^{-1}(W)$ is dense in $\mathscr{S}_1$, where W will be considered as $W \cup_\partial \mathbf{c}(\partial W \times [0, 3\varepsilon])$.

2. $\varphi|_{\varphi^{-1}(\mathring{W})} \longrightarrow \mathring{W}$ is a diffeomorphism.

$$\begin{array}{ccccc} W & \cup_\partial & B_2 & \cup & B_1 \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow \\ W & \cup_\partial & A_2 & \cup & A_1 \end{array}$$

where the maps on B_1 and B_1 are identities (more precisely id $\times (4\varepsilon -$ id $))$ outside of X and X', which are mapped to the corresponding cone points.

Now we can complete the proof. Let $p : \hat{\mathscr{S}_1} \longrightarrow \mathscr{S}_1$ be a resolution of $\mathscr{S}_1$, then one easily verifies that $\varphi p : \hat{\mathscr{S}_1} \longrightarrow \mathscr{S}$ is a resolution of $\mathscr{S}$.

□

Corollary 2.4.2. *Let $\mathscr{S}$ be an m-dimensional stratifold having only two non-empty strats $f_m : W^m \longrightarrow \mathscr{S}$ and $f_n : W^n \longrightarrow \mathscr{S}$, such that*

- *$W^n = \sqcup_{i \in J} S_i^n$ (a countable union of spheres) and*
- *$g_i := f_m|_{L_i} : L_i \longrightarrow S_i^n$ is a differential fibre bundle.*

Assume that every link of the singularity S_i^n given by $L_i := (f_m|_{\partial W^m})^{-1}(S_i^n)$ bounds in $\Omega_{m-1}(S^n)$. Then $\mathscr{S}$ admits a resolution.

REMARK: (**Special case of a resolution.**) After introducing the general approach to reach a resolution, we see that there are a lot of choices one has to make in each step. One observes that we can take T' to be T (in the case of oriented resolution T with opposite orientation). In the next step we can take the zero bordism $T \times I$, bounding $T \cup_\partial T$ and so on, until we get to the last induction step. Thus, we obtain a resolution depending on fewer indeterminates.

2.4.2 Bundle description

In this section, we want to understand the bundles we get in the induction step described above. To do this, we give an explicit description of the trivialization and the gluing map.

We recover that we start with a bundle $F \hookrightarrow L \longrightarrow S^n$ where the fibre F is zero-bordant. Let the gluing information be stored in $\alpha : S^{n-1} \longrightarrow \mathrm{Diff}(F)$. Next, we choose bounding manifolds T and T', i.e. $\partial T = F = \partial T'$ and build a new bundle $S^{n-1} \times T \cup_\alpha S^{n-1} \times T' \longrightarrow S^{n-1}$.

Set $\alpha_+ := \alpha|_{D_+^{n-1}}$ and $\alpha_- := \alpha|_{D_-^{n-1}}$. Without loss of generality we can assume the equality $\alpha_+(0) = \mathrm{id} = \alpha_-(0)$. To describe the trivialization we first consider the following map:

$$\begin{array}{rcl} \gamma : D_+^{n-1} \times \partial T \times I & \longrightarrow & D_+^{n-1} \times \partial T \times I \\ (x, y, t) & \longmapsto & (x, \alpha_+(tx)y, t) \end{array}$$

where I denotes the unite interval $[0,1]$. Now we define the trivialization φ of the new vector bundle by setting:

$$\begin{tikzcd} D_+^{n-1} \times T \cup_{\mathrm{id}\,: D_+^{n-1} \times T \to D_+^{n-1} \times T \times \{0\}} \arrow[d, "\mathrm{id}"] & D_+^{n-1} \times \partial T \times I \cup_{\alpha \mathrm{pr}_1|_{(D_+^{n-1} \times \partial T) \times \{1\}}} \arrow[d, "\gamma"] & D_+^{n-1} \times T' \arrow[d, "\mathrm{id}"] \\ D_+^{n-1} \times T \cup_{\mathrm{id}\,: D_+^{n-1} \times T \to D_+^{n-1} \times T \times \{0\}} & D_+^{n-1} \times \partial T \times I \cup_{\mathrm{id}\ \mathrm{pr}_1|_{(D_+^{n-1} \times \partial T) \times \{1\}}} & D_+^{n-1} \times T' \end{tikzcd}$$

The map φ is well-defined. To see this, let $(x, y, 0) \in D_+^{n-1} \times \partial T \times I$, then we compute

$$\gamma(x, y, 0) = (x, \alpha_+(0)y, 0) = (x, y, 0) \sim (x, y) = \mathrm{id}\ (x, y),$$

and for $(x, y, 1) \in D_+^{n-1} \times \partial T \times I$

$$\gamma(x, y, 1) = (x, \alpha_+(x)y, 1) \sim (x, \alpha_+(x)y) = \mathrm{id}\ (x, \alpha_+(x)y).$$

In the same way, one obtains a trivialization on $D_-^{n-1} \times T \cup_\alpha D_-^{n-1} \times T'$ using the map α_- instead of α_+. The gluing map's information is concentrated in

$$\begin{array}{ccc} S^{n-2} & \longrightarrow & \mathrm{Diff}(\partial T \times I)_{\mathrm{rel}\partial} \\ x & \longmapsto & (y, t) \mapsto (\alpha_-(xt)\alpha_+^{-1}(xt)y, t), \end{array}$$

and can be extended with identity to

$$\beta : S^{n-2} \longrightarrow \mathrm{Diff}((\partial T \times I \cup T) \cup_{\partial T \times \{0\}} T \cup_{\partial T \times \{1\}} T').$$

This describes the new fibre bundle over S^{n-1}.

2.4.3 Optimal spin resolutions of fibrations over S^2

In this section, we will find necessary conditions under which the construction introduced in 2.4.1 leads to an optimal resolution. We use the special case of construction where one uses the same bordism above each disk to construct a resolution.

We only study 6-dimensional p-stratifolds with spin top stratum attached to a 2-sphere.

The interest in such spaces partially comes from physics and was seized by M. Kreck (see [Kr3]).

There are two classes of spaces which are of particular interest: Witten spaces and Calabi-Yau-manifolds. A *Witten space* is a 7-dimensional closed Riemannian manifold W with $\mathrm{iso}(W) \cong SU(3) \times SU(2) \times U(1)$, where $SU(3) \times SU(2) \times U(1)$ acts transitively. A *Calabi-Yau-manifold* is a closed Kähler manifold M with $c_1(M) = 0$ and $\dim_{\mathbb{C}} M = 3$. One asks: Is there a geometric connection between Witten spaces and Calabi-Yau-manifolds? The first naive idea, finding $S^1 \in \mathrm{iso}(W)$ such that W/S^1 is Calabi-Yau, unfortunately does not work. M. Kreck showed that in the case where W/S^1

is smooth, it is not spin. Thus, one has to work with a space W/S^1 having singularities. M. Kreck studied a special case $X := S^5 \times S^3/S^1$, where the action is given by $z(x,t) := (z^2x, z^3y)$. This space has a p-stratifold structure of the type we consider in this section. For this special X, M. Kreck constructed an optimal resolution by a Calabi-Yau.

Let $S = W \cup_g S^2$ be a 6-dimensional p-stratifold and $f : \partial W \longrightarrow S^2$ a smooth fibre bundle with fibre F. The construction of the resolution takes place in three steps. See §2.4.1 for details. First we choose a zero bordism T of F and build

$$R_1 := \partial W \times I \cup_{\partial R \times \{0\}} (D^2_+ \times T \cup_{\alpha \times \mathrm{id}} D^2_- \times T).$$

This is again a six dimensional manifold with boundary

$$\partial R_1 = \partial R + S^1 \times T \cup_\alpha S^1 \times T.$$

The second boundary component admits a fibration over S^1 using the projection onto the first factor. The fibre is given by $T \cup_\partial T$.

In the next step, we take as a zero-bordism of $T \cup_\partial T$ the cylinder $M := T \times I$ and obtain

$$R_2 := R_1 \cup_{S^1 \times T \cup_\alpha S^1 \times T} (D^1 \times M \cup_{\beta \times \mathrm{id}} D^1 \times M),$$

where $\beta : S^0 \times \partial M \longrightarrow S^0 \times \partial M$ is the gluing map. We have

$$\partial R_2 = \partial W + S^0 \times M \cup_\beta S^0 \times M.$$

Finally, we choose a zero-bordism Z (consisting of two connected components) of the second component of the boundary and obtain

$$R_3 := R_2 \cup_{\partial Z} Z.$$

Since we start with a spin manifold W, we are interested in spin resolutions.

Theorem 2.4.3. *Let $\mathscr{S} = W \cup_g S^2$ be a 6-dimensional stratifold with spin manifold W and simply connected fibre F. Then $\mathscr{S}$ admits an optimal spin resolution.*

Proof. Since $\Omega_3^{\mathrm{spin}} = \Omega_5^{\mathrm{spin}} = 0$, observe that there are spin-bordisms T and $Z = Z_1 + Z_2$ of F and $S^0 \times M \cup_\beta S^0 \times M$ respectively. Using [Kr1, Prop. 4] we can without loss of generality assume that T, as well as, Z_i is 1-connected.

According to the Theorem of Whitehead, the resolution $\hat{\mathscr{S}} \longrightarrow \mathscr{S}$ is optimal if and only if $\overline{U}\hat{S}^2 = R_3$ is 1-connected and $H_2(\overline{U}\hat{S}^2)$ maps bijectively onto $H_2(S^2) = \mathbb{Z}$. Thus, we have to control the generator of $H_2(S^2)$ during the construction.

Let us first see that using T and Z as above the manifold R_3 is simply connected. Since we assumed T to be 1-connected, the manifold R_1 is again simply connected according to the Seifert-van Kampen-Theorem. To see that R_2 is simply connected, note that every map $S^1 \longrightarrow D^1 \times M \sqcup_{\beta\times\mathrm{id}} D^1 \times M$ can be deformed to ∂R_1, hence R_2 is 1-connected. Finally R_3 is simply connected as well using again the Seifert-van Kampen-Theorem.

Now we investigate $H_2(R_3)$. The manifold R_1 is obviously homotopy equivalent to $D^2_+ \times T \cup D^2_- \times T$. Use the Mayer-Vietoris sequence and obtain the following exact sequence:

$$\begin{array}{c} H_2(D^2_+ \times T) \oplus H_2(D^2_- \times T) \\ \downarrow \\ H_2(D^2_+ \times T \cup D^2_- \cup T) \\ \downarrow \\ H_1(S^1 \times F) = H_1(S^1) \\ \downarrow \\ 0 \end{array}$$

Hence, the relevant generator of $H_2(S^2) = H_1(S^1)$ survives in R_1 and maps bijectively onto $H_1(S^1)$.

We proceed with the next step in our construction and consider the Mayer-Vietoris sequence for R_2 leading to the following commutative diagram:

$$
\begin{tikzcd}
H_2(T) \oplus H_2(T) \arrow[r] \arrow[r, bend left, "\cong"] \arrow[rr, bend left=40, "\cong"] \arrow[d] & H_2(T) \oplus H_2(T) \arrow[d] & H_2(M) \oplus H_2(M) \arrow[d] \\
H_2(S^1 \times T \cup_\alpha S^1 \times T) \arrow[r] \arrow[d] & H_2(D^2 \times T \cup D^2 \times T) \arrow[d] & \oplus\, H_2(D^1 \times M \cup D^1 \times M) \\
H_1(S^1) \oplus H_1(F) \arrow[d, "\mathrm{id}\quad 0"'] & H_1(S^1) \oplus H_1(F) \arrow[d] & \\
2(H_1(S^1) \oplus H_1(T)) & 2(H_1(D^2) \oplus H_1(T)) &
\end{tikzcd}
$$

From this diagram, we conclude that the relevant free factor $\mathbb{Z} \cong H_1(S_1) \subseteq H_2(D^2 \times T \cup D^2 \times T) = H_2(R_1)$ does not lie in the image of $H_2(S^1 \times T \cup_\alpha S^1 \times T)$ and, therefore, maps injectively into $H_2(R_2)$.

The last step concerns the manifold R_3, which is built from R_2 after choosing a zero-bordism Z of $S^0 \times M \cup_\beta S^0 \times M$. A part of the Mayer-Vietoris sequence is the following.

$$H_2(\partial R_2) \longrightarrow H_2(R_2) \oplus H_2(Z) \longrightarrow H_2(R_3) \longrightarrow 0$$

Adding the corresponding sequences for ∂R_2 and R_2 we obtain:

$$
\begin{tikzcd}
H_2(M) \oplus H_2(M) \arrow[r, "\mathrm{id}"] \arrow[d] & (0 \arrow[d] & \oplus & H_2(M) \oplus H_2(M)) \arrow[d] \\
2(H_2(M \cup_\beta M)) \arrow[d] & H_2(D^2 \times T \cup D^2 \times T) & \oplus \arrow[d] & H_2(D^1 \times M \cup D^1 \times M) \\
H_2(\partial R_2) \arrow[rr] & & H_2(R_2) &
\end{tikzcd}
$$

Thus our $\mathbb{Z} \subset H_2(R_2)$ does not lie in the image of $H_2(\partial R_2)$ and will, therefore, be injectively mapped to $H_2(R_3)$.

Summarizing the discussion, we conclude that the map $R_3 \longrightarrow S_2$ is 1-connected. To make the map 3-connected as desired, we have to eliminate the remaining classes except the mentioned $\mathbb{Z}$, which maps bijectively onto $H_2(S^2)$.

From the Mayer-Vietoris sequence for R_3 we see that every homology class $x \in H_2(R_3)$ comes from a class $x' \in H_2(R_2) \oplus H_2(Z)$. We have seen that since $H_1(F) = 0$, the relevant homology classes of $H_2(R_2)$ only come from $H_2(T) \cong H_2(M)$, but these classes can be represented as classes of $H_2(Z)$ since $\partial Z = S^0 \times M \cup_\beta S^0 \times M$.

First note that every homology class of Z is represented by an embedding $S^2 \hookrightarrow Z$, see [Hae]. Since Z is spin, it admits a framing of the 2-skeleton, hence, every embedded 2-sphere has trivial normal bundle. Thus, we can perform surgery (see [Kr1, Lemma 2]) to eliminate these classes and the theorem is proved.

□

2.4.4 Equivalent resolutions

In this section, we investigate the classification of optimal resolutions in the 6-dimensional case treated in the last section. Thus, we consider stratifolds $\mathscr{S} = W \cup_g S^2$, where W is a 6-dimensional spin manifold with boundary $\partial R =: L$ and the attaching map $g : L \longrightarrow S^2$ is a differential fibre bundle. The classification of the resolutions is a rather complicated intension, even in the case of isolated singularities, see, for example, [Gr]. We weaken the requirements a little bit by introducing the following definition.

Definition: Let $\mathscr{S}$ be a stratifold, and $p : \hat{\mathscr{S}} \longrightarrow \mathscr{S}$ and $p' : \hat{\mathscr{S}}' \longrightarrow \mathscr{S}$ be two resolutions of $\mathscr{S}$. We call the resolutions *weakly equivalent* if, for every representative of the neighbourhood germ $\overline{U}\Sigma_{\mathbf{c}}$, there is a diffeomorphism $\varphi_{\mathbf{c}} : \hat{\mathscr{S}} \longrightarrow \hat{\mathscr{S}}'$ such that $p'\varphi_{\mathbf{c}} = p$ on $\hat{\mathscr{S}} - \overline{U}\hat{\Sigma}_{\mathbf{c}}$.

The classification is based on the following theorem of M. Kreck, which is proved in [Kr1].

Theorem 2.4.4. *Let M and M' be two simply connected $2(2q+1)$-dimensional manifolds with normal $2q$-smoothings in a fibration $B \longrightarrow BO$. Let $\varphi : \partial M \longrightarrow \partial M'$ be a diffeomorphism compatible with the normal $2q$-smoothings ν and ν'. Let further*

- $e(W) = e(W')$ *and*
- $[W \cup_\varphi (-W'), \bar{\nu} \cup \bar{\nu}'] = 0 \in \Omega^B_{2(2q+1)}$.

Then the diffeomorphism on the boundary can be extended to a diffeomorphism $\Phi : M \longrightarrow M'$.

Now we are going to adapt the theorem to our special situation. Let $\hat{\mathscr{S}}$ and $\hat{\mathscr{S}}'$ be two optimal resolutions of the stratifold $\mathscr{S} = W \cup \Sigma$. Set $M :=$

$\overline{U}\hat{\Sigma}$ and $M' := \overline{U}\hat{\Sigma}'$, then M and M' are smooth manifolds with boundary diffeomorphic to ∂W. Assume that M and M' are spin with compatible spin structure on the boundary. The normal 2-type of M and M' is $\mathbb{C}P^\infty \times B\mathrm{Spin}$. Thus, we obtain the following corollary.

Corollary 2.4.5. *Let $\hat{\mathscr{S}}$ and $\hat{\mathscr{S}}'$ be two optimal resolutions of the 6-dimensional stratifold $\mathscr{S} = W \cup \Sigma$, with simply connected singularity Σ (decomposed into compact connected components Σ_i) and spin manifold W. Assume that $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}'_i$ are spin with compatible spin structure on the boundary. Then $\hat{\mathscr{S}}$ is weakly equivalent to $\hat{\mathscr{S}}$ if the following conditions are fulfilled:*

- $e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}_i)$ *and*
- $[\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}'_i] = 0$ *in* $\Omega_6^{spin}(\mathbb{C}P^\infty)$.

Proof. We see that the conditions of Theorem 2.4.4 are fulfilled for the manifolds $\overline{U}\hat{\Sigma}_i$ and $\overline{U}\hat{\Sigma}'_i$. Thus, the diffeomorphism on the boundary can be extended to a diffeomorphism $\overline{U}\hat{\Sigma}_i \longrightarrow \overline{U}\hat{\Sigma}'_i$. From the definition of the resolutions and from the construction of the diffeomorphism on the boundary, we see that the obtained diffeomorphism extends in the obvious way to a diffeomorphism $\hat{\mathscr{S}} \longrightarrow \hat{\mathscr{S}}'$ having the desired properties.

As carried out in the proof of Theorem 2.2.16, it is enough to consider one representative of the neighbourhood germ.

□

We will investigate the second condition in the corollary and compute $\Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty)$. In order to do this, we first need some information concerning complete intersections.

Complete intersections

In this section, we recover some facts about complete intersections. Let $f_1, \dots f_r \in \mathbb{C}[x_0, \dots, x_{n+r}]$ be homogeneous polynomials with degree $f_i = d_i$, such that the gradients are all linearly independent.

Definition and Remark: The space

$$X(f_1, ..., f_r) := \{x \in \mathbb{C}P^{n+1} | f_1(x) = \dots = f_r(x) = 0\} \subset \mathbb{C}P^{n+r}$$

is called the *complete intersection* of complex dimension n of the polynomials $f_1, \dots, f_r$.

Since the gradients of the polynomials are linearly independent, $X(f_1, ..., f_r)$ is a complex submanifold of dimension n in $\mathbb{C}P^{n+r}$.

A natural question is: Given another polynomials $f_1', \dots f_r'$ with degreef_i = degreef_i', are the manifolds $X(f_1, \dots, f_r)$ and $X(f_1', \dots, f_r')$ diffeomorphic? This question was answered by Thom in the early fifties.

Theorem 2.4.6. *Let $f_1, \dots, f_r$ be polynomials as above. The complete intersection $X(f_1, \dots, f_r)$ depends up to diffeomorphism only on the degrees $d_1, \dots, d_r$.*

From now on we write $X(d_1, \dots, d_r)$ for $X(f_1, \dots, f_r)$ or shorter $X(d)$, where $d = (d_1, \dots, d_r)$ is the multi-degree. Set $|d| := \prod_{i=1}^r d_i$, the total degree of $X(d)$, which will turn out to be a diffeomorphism invariant if $n \geq 3$.

The homology of $X(d)$ can be almost completely computed with the help of the following theorem of Lefschetz.

Theorem 2.4.7. *For $n \geq 2$ the manifold $X(d)$ is simply connected and the inclusion $i : X(d) \hookrightarrow \mathbb{C}P^{n+r}$ is an n-equivalence, i.e. $i_* : \pi_j(X(d)) \longrightarrow \pi_j(\mathbb{C}P^{n+r})$ is an isomorphism for $j < n$ an an epimorphism for $j = n$.*

The next interesting question concerns the characteristic classes. The following results can be proved using the basic techniques of differential topology.

Let H be the dual of the Hopf bundle. First of all, we have to compute the tangent bundle of $X(d)$. Since it is a submanifold of $\mathbb{C}P^{n+r}$, we obtain the following bundle equality $T\mathbb{C}P^{n+r}|_{X(d)} = TX(d) \oplus \nu(X(d); \mathbb{C}P^{n+r})$. The tangent bundle of $\mathbb{C}P^{n+r}$ is stably well known, namely $T\mathbb{C}P^{n+r} \oplus \mathbb{R} = (n+r+1)H$. We now compute the normal bundle of $X(d)$ in $\mathbb{C}P^{n+r}$. Let $\gamma := i^*H$ be the pull back of the Hopf bundle to our complete intersection $X(d)$. Denote with γ^d the d-fold tensor product of γ. It is easy to prove:

$$\nu(X(d); \mathbb{C}P^{n+r}) = \gamma^{d_1} \oplus \dots \oplus \gamma^{d_r}.$$

Combining the facts together we see that $TX(d)$ is stably isomorphic to $(n+r+1)\gamma - (\gamma^{d_1} \oplus \dots \oplus \gamma^{d_r})$. From this we conclude:

Lemma 2.4.8. *The total Chern class of $X(d)$ is given by*

$$c(TX(d)) = (1+x)^{n+r+1} \prod_{j=1}^{r} (1+d_j x)^{-1},$$

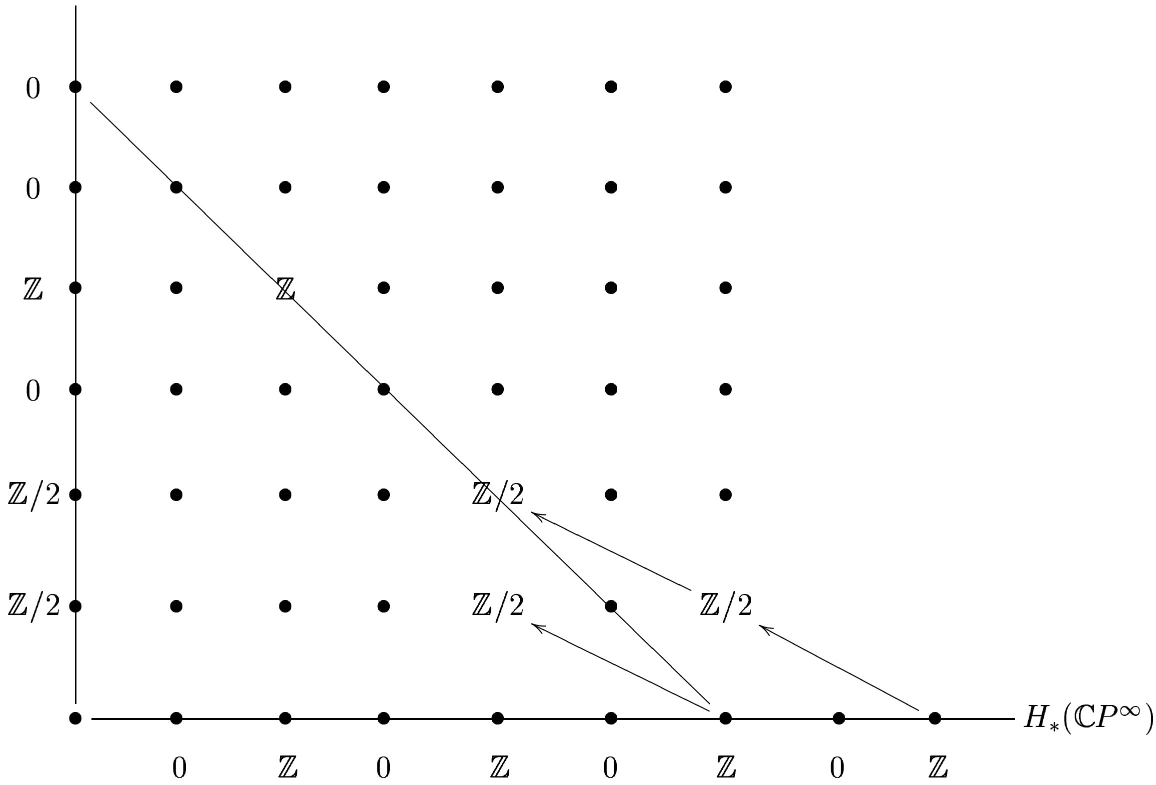
where $x := i^* c_1(H)$. Thus, for the total Pontrjagin class we have

$$p(TX(d)) = (1+x^2)^{n+r+1} \prod_{j=1}^{r} (1+d_j^2 x^2)^{-1}.$$

With this information, we are ready to compute $\Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty)$.

The bordism group $\Omega_6^{\mathrm{spin}}\mathbb{C}P^\infty$

We apply the Atiyah-Hirzebruch-Spectral Sequence to compute the bordism group $\Omega_6^{\mathrm{spin}}\mathbb{C}P^\infty$. Consider the E^2-term, given by $H_p(\mathbb{C}P^\infty;\Omega_q^{\mathrm{spin}})$, which by the universal coefficient theorem is equal to $H_p(\mathbb{C}P^\infty)\otimes\Omega_q^{\mathrm{spin}}$.



The three interesting d^2 differentials are well known. The two maps $H_6(\mathbb{C}P^\infty)\longrightarrow H_4(\mathbb{C}P^\infty;\mathbb{Z}/2)$ and $H_8(\mathbb{C}P^\infty)\longrightarrow H_6(\mathbb{C}P^\infty;\mathbb{Z}/2)$ are given by the composition of the reduction modulo 2 with the dual of the second Steenrod square.

$$H_6(\mathbb{C}P^\infty)\xrightarrow{\otimes\mathbb{Z}/2}H_6(\mathbb{C}P^\infty;\mathbb{Z}/2)\xrightarrow{(Sq^2)^*}H_4(\mathbb{C}P^\infty;\mathbb{Z}/2)$$

$$H_8(\mathbb{C}P^\infty)\xrightarrow{\otimes\mathbb{Z}/2}H_8(\mathbb{C}P^\infty;\mathbb{Z}/2)\xrightarrow{(Sq^2)^*}H_6(\mathbb{C}P^\infty;\mathbb{Z}/2)$$

The third map $H_6(\mathbb{C}P^\infty;\mathbb{Z}/2)\longrightarrow H_4(\mathbb{C}P^\infty;\mathbb{Z}/2)$ is given just by the dual of Sq^2.

Let $x \in H^2(\mathbb{C}P^\infty;\mathbb{Z}/2)$ be a generator of the cohomology ring of $\mathbb{C}P^\infty$, then x^2 generates $H^4(\mathbb{C}P^\infty;\mathbb{Z}/2)$ and x^3 generates $H^6(\mathbb{C}P^\infty;\mathbb{Z}/2)$. According to the Cartan-formula, we obtain

$$Sq^2(x^2) = Sq^2(x)x + Sq^2(x)x + Sq^1(x)Sq^1(x) = 2x^3 = 0 \in H^6(\mathbb{C}P^\infty;\mathbb{Z}/2)$$

and using this computation

$$Sq^2(x^3) = Sq^2(x)x^2 = x^4.$$

Thus, both differentials $H_6(\mathbb{C}P^\infty;\mathbb{Z}/2) \longrightarrow H_4(\mathbb{C}P^\infty;\mathbb{Z}/2)$ and $H_6(\mathbb{C}P^\infty) \longrightarrow H_4(\mathbb{C}P^\infty;\mathbb{Z}/2)$ are zero and the differential $H_8(\mathbb{C}P^\infty) \longrightarrow H_6(\mathbb{C}P^\infty;\mathbb{Z}/2)$ is surjective. Since there are no more relevant differentials, we obtain the following exact sequence

$$0 \longrightarrow F \longrightarrow \Omega_6^{\text{spin}}(\mathbb{C}P^\infty) \longrightarrow H_6(\mathbb{C}P^\infty) \longrightarrow 0$$

where the space F comes from the exact sequence

$$0 \longrightarrow \Omega_4^{\text{spin}} \longrightarrow F \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Here we identified the $E^\infty_{2,4}$ with Ω_4^{spin}. We now have to decide whether the last sequence splits or not. For this we need additional information about the occurring bordism groups. The generator of Ω_4^{spin} is the Kummer surface K, the complete intersection in $\mathbb{C}P^3$ given by the homogeneous polynomial $x^4+y^4+z^4$. There is also a map from Ω_4^{spin} to $\Omega_6^{\text{spin}}(\mathbb{C}P^\infty)$ given by $K \longmapsto K \times S^2$, where the map to $\mathbb{C}P^\infty$ is given by $K \times S^2 \longrightarrow S^2 \longrightarrow \mathbb{C}P^\infty$. Further there is a map from $\Omega_6^{\text{spin}}(\mathbb{C}P^\infty)$ to $\mathbb{Z}$ mapping $[M,f]$ to $\langle f^*x \cup p_1(M)/2, [M]\rangle$. Observe that since M is spin, the first Pontrjagin class is always even. The fact is that the composition

$$\Omega_4^{\text{spin}} \longrightarrow \Omega_6^{\text{spin}}(\mathbb{C}P^\infty) \longrightarrow \mathbb{Z}$$

is injective. This allows us to compute $\Omega_6^{\text{spin}}(\mathbb{C}P^\infty)$. Summarizing the previous considerations, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{Z} & & \\
0 \arrow[r] & F \arrow[r] & \Omega_6^{\text{spin}}(\mathbb{C}P^\infty) \arrow[u, "{[M,f]\mapsto\langle f^*x\cup p_1(M)/2,[M]\rangle}"'] \arrow[r, "{[M,f]\mapsto f_*([M])}"'] & H_6(\mathbb{C}P^\infty) \arrow[r] & 0 \\
0 \arrow[r] & \Omega_4^{\text{spin}} \arrow[uur, hook] \arrow[ur, "K\mapsto K\times S^2"'] \arrow[r] & F \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

The question of whether the lower sequence splits or not is now equivalent to the existence of $[X, f] \in \Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty)$, such that $f^*([X]) = 0$ and $\langle f^*x \cup p_1(X)/2, [X]\rangle = -12$.

For, if such an X exists, then by exactness of the first sequence we know that X comes from F. We now consider $2X \in F$ which obviously maps to 0 in $\mathbb{Z}/2$ and thus, there is an $\tilde{X} \in \Omega_4^{\mathrm{spin}}$ with $\tilde{X} \times S^2 = 2X$. Considering now $Y := K \times S^2 + 2X$, we see that $\langle f^*x \cup p_1(Y)/2, [Y]\rangle = \langle f^*x \cup p_1(K \times S^2)/2, [K \times S^2]\rangle + 2\langle f^*x \cup p_1(X)/2, [X]\rangle = 24 - 2 \cdot 12 = 0$. Thus, K maps to $2X$ in the lower sequence, which implies that it does not split.

We are going to construct such a manifold X. Consider the complete intersection $X' := \{[z_1, z_2, z_3, z_4] \in \mathbb{C}P^4 | z_i^3 = 0\}$. Using Lemma 2.4.8, we compute the Chern and thus the Pontrjagin classes of X'.

$$\begin{aligned} c_1(X') &= (5-3)x \\ p_1(X') &= (5-3^2)x^2 \end{aligned}$$

From the formula for the Chern classes, we see $w_2(X') = c_1(X') \mathrm{mod}\ (2) = 0$, thus X' is spin. From the second equation, we obtain the first Pontrjagin class $p_1(X') = -4x^2$.

Now we consider $X := X' - 3[\mathbb{C}P^3, i]$ and conclude $\langle x^3, f_*[X]\rangle = \langle (f^*x)^3, [X]\rangle = \langle (f^*x)^3, [X']\rangle - 3\langle (i^*x)^3, \mathbb{C}P^3\rangle = 3 - 3 = 0$, which implies $f_*[X] = 0$.

Using the fact that $f^*(x)$ generates X' (Theorem 2.4.7) and the equality $\langle (f^*x)^3, [X']\rangle = 3$ already used in the previous consideration, we compute

$$\begin{aligned} \langle f^*x \cup p_1(X)/2, [X]\rangle &= \langle f^*x \cup p_1(X')/2, [X']\rangle - 3\langle i^*x \cup p_1(\mathbb{C}P^3)/2, [\mathbb{C}P^3]\rangle \\ &= \langle -2(f^*x)^3, [X']\rangle - 3\langle 2x^3, [\mathbb{C}P^3]\rangle = -6 - 6 = -12. \end{aligned}$$

To summarize we obtain:

Theorem 2.4.9. *There is an isomorphism*

$$\begin{array}{ccc} \Omega_6^{spin}(\mathbb{C}P^\infty) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ [M, f] & \longmapsto & (\langle x^3, f_*[M]\rangle, \frac{1}{12}\langle f^*x \cup p_1(X)/2, [X]\rangle) \end{array}$$

Knowing the relevant bordism group $\Omega_6^{\mathrm{spin}}(\mathbb{C}P^\infty)$, we obtain the following classification result:

Theorem 2.4.10. *Let $\hat{\mathscr{S}}$ and $\hat{\mathscr{S}}'$ be two optimal resolutions of the 6-dimensional stratifold $\mathscr{S} = W \cup_g S^2$, with spin W. Assume that the spin structure on $M := \overline{U}\hat{S}^2$ and $M' := \overline{U}\hat{S}^{2'}$ is compatible with the diffeomorphism on the boundary $\varphi : \partial M \longrightarrow \partial M'$ for a representative of the neighbourhood's germ $[US^2]$ of S^2. Let $\hat{r} : M \longrightarrow S^2$ and $\hat{r}' : M' \longrightarrow S^2$ be the maps given by the composition of the resolving maps with the neighbourhood's retraction on S^2. Assume further that*

- *the following diagram is commutative:*

$$\begin{tikzcd} & H^2(S^2) \cong \mathbb{Z} \arrow[dl, "\hat{r}^*"', "\cong"] \arrow[dr, "(\hat{r}')^*", "\cong"'] & \\ H^2(M) \arrow[d] & & H^2(M') \arrow[d] \\ H^2(\partial M) \arrow[rr, "\varphi^*"] & & H^2(\partial M') \end{tikzcd}$$

- *$e(M) = e(M')$ and*

- *$p_1(M \cup M') \cup u^* x = 0$ for all $x \in H^6(\mathbb{C}P^\infty)$, where $u : M \cup M' \longrightarrow \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ is the map $u_M \cup_\partial u_{M'}$, where u_M corresponds to $\hat{r}^*(1)$ and $u_{M'}$ corresponds to $(\hat{r}')^*(1)$.*

Then $\hat{\mathscr{S}}$ is weakly equivalent to $\hat{\mathscr{S}}'$.

Proof. According to Corollary 2.4.5 and Theorem 2.4.9, we have to verify that the class $u_*[M \cup M']$ vanishes in $H_6(\mathbb{C}P^\infty)$. The cohomology ring of $\mathbb{C}P^\infty$ is well known to be a polynomial ring in one generator of dimension two, again denoted by x.

$$u_*[M \cup M'] = 0 \Leftrightarrow \langle x^3, u_*[M \cup M'] \rangle = 0 \Leftrightarrow \langle u^* x^3, [M \cup M'] \rangle = 0$$

Consider the exact cohomology sequence. Since u_M and $u_{M'}$ coincide on the boundary, the inclusion $j^*(u)$ vanishes in $H^2(\partial M)$. Hence, there are cohomology classes v_M and $v_{M'}$ fitting into the diagram:

$$\begin{tikzcd} H^2(M, \partial M) \oplus H^2(M', \partial M') \arrow[r] & H^2(M \cup M') \arrow[r] & H^2(\partial M) \\ (v_M, v_{M'}) \arrow[r, mapsto, "i_1{}^* - i_2{}^*"] & u \arrow[r, mapsto, "j^*"] & 0 \end{tikzcd}$$

Thus, we conclude

$$((v_M)^* x)^3 = ((u_{M'})^* x)^3 \Rightarrow u_*[M \cup M'] = 0$$

The condition on the left side is always fulfilled, since the triple products on $H^2(\hat{\mathscr{S}})$ and on $H^2(\hat{\mathscr{S}}')$ are completely determined by the corresponding products on $H^2(\mathscr{S})$.

As in the proof of Theorem 2.2.16, we see that it is enough to verify the conditions for one representative of the neighbourhood germ.

$\square$

Chapter 3

Connections to other stratified spaces

One natural question arises when dealing with p-stratifolds. What is the connection of our new objects to the old concepts of stratification? First of all, one can think of the stratifolds as objects with singularities. As already mentioned in §2.1, our definition of resolutions is the same as from algebraic geometry, adapted to our category. We now ask about the similarities of these two sorts of singular spaces. In the case of isolated singularities, we gave an answer to this question in an example from §2.2. On the other hand a stratifold is a stratified space, so we are interested in its connection to the stratified spaces studied earlier, for example Whitney stratified spaces.

Consider a real algebraic set $V \subseteq \mathbb{R}^n$, which is the common locus of finitely many polynomials. The singular set ΣV of all points where V fails to be a smooth manifold is again an algebraic set (of strictly lower dimension). Thus, one obtains a finite filtration $V = V^m \supseteq V^{m-1} \supseteq \cdots \supseteq V^0 \supseteq V^{-1} = \emptyset$ by defining V^{i-1} to be ΣV^i if $\dim V^i = i$, and to be V^i if $\dim V^i < i$. The space $V^i - V^{i-1}$ is a smooth manifold of dimension i (perhaps empty). This decomposition makes V a stratified space.

Apart from the observation made in §2.2, which states that every algebraic variety with isolated singularities admits a structure of a p-stratifold, there is another result [BCR, Prop. 9.4.4] establishing a connection between algebraic varieties and p-stratifolds.

Theorem 3.0.11. *Let $Z \subset S$ be two closed and bounded semi-algebraic sets. Let f be a non-negative continuous semi-algebraic function on S such that $f^{-1}(0) = Z$. Then there are $\delta > 0$ and a continuous semi-algebraic mapping $h : f^{-1}(\delta) \times [0, \delta] \longrightarrow f^{-1}([0, \delta])$, such that $f(h(x, t)) = t$ for every*

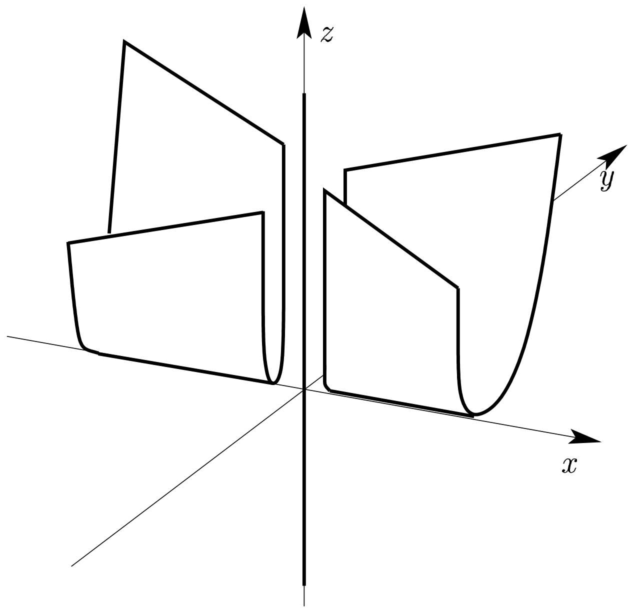


Figure 3.1: Naive stratification of the Whitney umbrella.

$(x,t) \in f^{-1}(\delta) \times [0,\delta]$, $h(x,\delta) = x$ for every $x \in f^{-1}(\delta)$, and $h|_{f^{-1}(\delta)\times]0,\delta]}$ is a homeomorphism onto $f^{-1}(]0,\delta])$.

The proof of this theorem uses a triangulation, which is not canonical. It is also not clear if one can make the attaching map smooth in our sense. Nevertheless, the theorem implies the following result.

Corollary 3.0.12. *Every algebraic variety admits a structure of a topological p-stratifold.*

One can try to adapt the idea used for isolated singularities and "slide" along the gradient vector field towards the singular set. But it is generally not clear whether the gradient flow converges, and, if this is the case, whether the resulting map is smooth. We will not follow up this idea in this thesis, but will investigate other concepts of stratification.

3.1 Stratified spaces

We start with a very general notion of stratified spaces.

Definition: Let X be a locally compact topological space with countable basis. A *stratification* of X is a partition $\mathcal{X} = \{X_i\}$ of X into locally finite, pairwise disjoint, locally closed subsets of X, such that each X_i is a smooth manifold. We call the pair $(X, \mathcal{X})$ a *stratified space* and the manifolds $X_i \in \mathcal{X}$ the *strata of* X. The class of stratified spaces is denoted by $\mathbf{SF}$.

Definition: An isomorphism of stratified spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ is a homeomorphism $\varphi : X \longrightarrow Y$, such that for all $X_i \in \mathcal{X}$ there exists $Y_i \in \mathcal{Y}$, such that $\varphi(X_i) = Y_i$ and $\varphi|_{X_i}$ is a diffeomorphism. Denote the class of isomorphisms between stratified spaces with $\mathbf{ISO_{SF}}$.

As *smooth maps* on $(X, \mathcal{X})$, we take

$$\mathcal{C}_{\mathbf{SF}}(X) := \{f \in \mathcal{C}^0(X) \big| f|_{X_i} \text{ is smooth } \forall X_i \in \mathcal{X}\},$$

where $\mathcal{C}^0(X)$ denotes the continuous maps on X.

3.2 Whitney stratified spaces

We now concentrate our attention on Whitney stratified spaces. In two fundamental papers [Wh1, Wh2], Whitney developed some ideas on stratifications and introduced his conditions (A) and (B).

To motivate these conditions, consider again an algebraic variety V, with stratification as explained before. With this construction, the strata need not have geometrically "well-behaved" neighbourhoods, this means that the local topological type need not be locally constant along the strata. The simple illustration of this fact is provided by the Whitney umbrella, an algebraic set in $\mathbb{R}^3$, which is the locus of $y^2 - zx^2$. The construction above gives us a stratification consisting of a surface with two connected components and a line, see Figure 3.1.

If we take a point on the z-axis and sketch the intersection of a small ball centred at that point with V, we obtain the neighbourhoods illustrated in Figure 3.2.

Clearly, something rather special happens at the origin, where the local topological type changes. If we now split the z-axis into $z < 0$, $z = 0$ and $z > 0$, we obtain a second stratification of the Whitney umbrella, indicated in the Figure 3.3.

To avoid such situations as in Figure 3.1, the following condition was introduced.

Definition: If X and Y are smooth submanifolds of $\mathbb{R}^n$, then X is *Whitney regular over* Y if, whenever $(x_i)_i \subset X$ and $(y_i)_i \subset Y$ are sequences of points

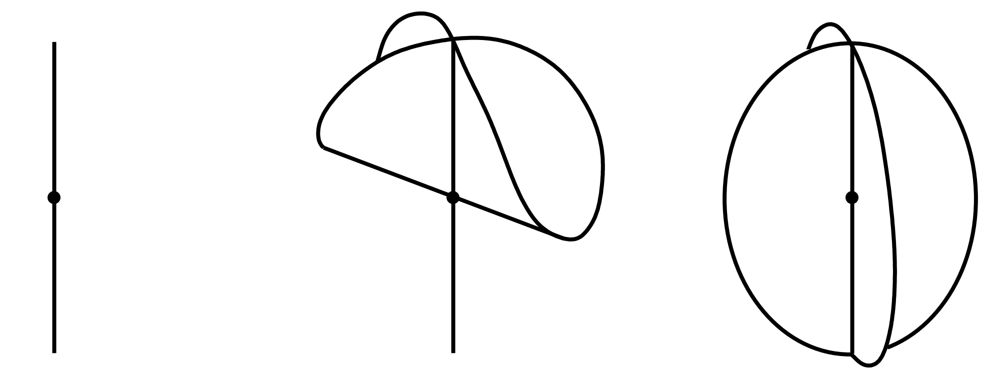


Figure 3.2: Three different neighbourhoods.

both converging to some $y \in Y$, such that the lines $l_i = \overline{x_i y_i}$ converge to a line l (in the Grassmannian of 1 - dimensional subspaces of $\mathbb{R}^n$) and the tangent spaces $T_{x_i}X$ converge to a space τ (in the Grassmannian of $(\dim X)$-dimensional subspaces of $\mathbb{R}^n$), then
(A) $T_yY \subseteq \tau$ and
(B) $l \subseteq \tau$.

Definition: A stratification $\mathcal{X}$ on a subset X of $\mathbb{R}^n$ is called *Whitney stratification* if it satisfies the following conditions:

- Every $X_i \in \mathcal{X}$ is a smooth submanifold of $\mathbb{R}^n$.
- Whitney regularity condition: Any stratum $Y \in \mathcal{X}$ is regular over any other stratum $X \in \mathcal{X}$.
- Frontier condition: Let X, Y be strata with $X \cap \overline{Y} \neq \emptyset$, then $X \subseteq \overline{Y}$ (i.e. the frontier of a stratum is a union of strata).

A pair $(X, \mathcal{X})$ is called a *Whitney stratified space.*

It is easy to see that the stratification of the Whitney umbrella in Fig. 3.1 is not a Whitney stratification. The surface fails to be Whitney-regular over the z-axis at the origin. On the other hand, the stratification of Fig. 3.3 is a Whitney stratification.

Various theories have been developed around Whitney stratified spaces. In [GM] the theory of Morse-Functions for Whitney stratified spaces was studied. Stratified Morse theory is the natural extension of Morse theory to

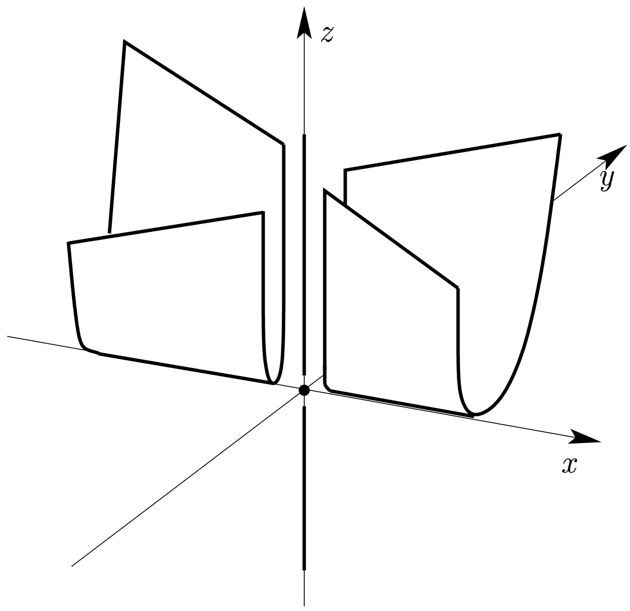


Figure 3.3: Another stratification of the Whitney umbrella.

include singular spaces. One of the the fundamental results is the following Theorem [GM, p. 6], which states a similarity between singular and non-singular theory.

Theorem 3.2.1. *Let $X \subset \mathbb{R}^n$ be a compact Whitney stratified space, and $f : X \longrightarrow \mathbb{R}$ a restriction of a smooth function on $\mathbb{R}^n$ to X. As c varies within the open interval between two adjacent critical values, the topological type of $X_{\leq c}$ remains constant, where critical value means critical value of f, restricted to some stratum.*

In [Go], M. Goresky introduced "geometric" chains and cochains and developed homology and cohomology in the context of Whitney stratifications.

Much research concerning Whitney stratified spaces makes use of an additional structure on the spaces, which one obtains using the conditions (A) and (B). We will study this structure in the next section.

3.3 Abstract pre-stratified spaces

In this section, we introduce the concept of abstract pre-stratified spaces developed by J. Mather in [Ma]. We additionally define abstract pre-stratified spaces with boundary and extend some results of Mather to bounded objects.

First we need the notion of tubular neighbourhoods.

Definition: Let $(V, \mathcal{S})$ be a stratified space. A system of tubular neighbourhoods of V is a set $\{N_X\}_{X \in \mathcal{S}}$ of triples $N_X = (N_X, \pi_X, \rho_X)$ where for each $X \in \mathcal{S}$

- N_X is an open neighbourhood of X in V,
- $\pi_X : N_X \longrightarrow X$ is a continuous retraction and
- $\rho_X : N_X \longrightarrow [0, \infty)$ is a continuous function such that
- $X = \rho_X^{-1}(0)$.

We call N_X the tubular neighbourhood of X, π_X the local retraction and ρ_X the tubular function on X.

Remark: Let W be a Whitney stratified space. In particular W consists of manifolds locally closed embedded into $\mathbb{R}^n$. In this situation, we obtain a canonical system of tubular neighbourhoods of W from the Riemannian normal bundle of each stratum X.

For any two strata X and Y we set

$$N_{X,Y} = N_X \cap Y, \quad \pi_{X,Y} = \pi_X|_{N_{X,Y}}, \quad \rho_{X,Y} = \rho_X|_{N_{X,Y}}.$$

Of course $N_{X,Y}$ may be empty, in which case the two mappings above are also empty.

Definition: A system of tubular neighbourhoods $\{(N_X, \pi_X, \rho_X)\}$ is said to be *controlled* if the following conditions are satisfied:

- $(\pi_{X,Y}, \rho_{X,Y}) : N_{X,Y} \longrightarrow X \times (0, \infty)$ is a smooth submersion.
- For any strata X, Y and Z we have

$$\begin{aligned} \pi_{X,Y}\pi_{Y,Z}(v) &= \pi_{X,Z}(v) \\ \rho_{X,Y}\pi_{Y,Z}(v) &= \rho_{X,Z}(v) \end{aligned}$$

  whenever both sides make sense.

Two controlled systems of tubular neighbourhoods $\{(N_X, \pi_X, \rho_X)\}$ and $\{(N'_X, \pi'_X, \rho'_X)\}$ are called *equivalent* if, for all $X \in \mathcal{S}$, there is a neighbourhood U of X in $N_X \cap N'_X$, such that $\pi_X|_U \equiv \pi'_X|_U$ and $\rho_X|_U \equiv \rho'_X|_U$. The equivalence class is called the germ of controlled systems of tubular neighbourhoods or simply the *controlled neighbourhood structure.*

Definition: An *abstract pre-stratified space* is a triple $(V, \mathcal{S}, \mathcal{N})$ such that:

1. V is a Hausdorff, locally compact topological space with a countable basis for its topology.
2. $\mathcal{S}$ is a family of locally closed subsets of V, such that V is the disjoint union of the members of $\mathcal{S}$. The members of $\mathcal{S}$ will be called *strata* of V.
3. Each stratum of V is a topological manifold (in the induced topology), provided with a smooth structure.
4. The family $\mathcal{S}$ is locally finite, i.e. every point of V has a neighbourhood which meets at most finitely many strata.
5. The family $\mathcal{S}$ satisfies the axiom of the frontier: if $X, Y \in \mathcal{S}$ and $Y \cap \overline{X} \neq \emptyset$, then $Y \subseteq \overline{X}$.
6. $\mathcal{N} = [\{N_X = (N_X, \pi_X, \rho_X)\}]$ is a controlled neighbourhood structure.

Denote by **APS** the class of abstract pre-stratified spaces.

REMARK: Taking the equivalence class of tubular neighbourhood systems in the above definition corresponds to passing to equivalence classes of abstract pre-stratified spaces as intended by Mather, see [Ma, §8].

If $Y \subset \overline{X}$ and $Y \neq X$, we write $Y < X$. The relation is obviously transitive, i.e $Z < Y$ and $Y < X$ imply $Z < X$.

Definition: The length of the longest chain $X_1 < X_2 < \cdots < X_k$ of strats of an abstract pre-stratified space V is called the *depth* of V and is denoted by depth(V).

From the normality of arbitrary subsets of an abstract pre-stratified space, it follows that we can always choose representatives of the neighbourhood structure satisfying:

- If X, Y are strata and $N_{X,Y} \neq \emptyset$, then $X < Y$.

- If X, Y are strata and $N_X \cap N_Y \neq \emptyset$, then X and Y are comparable, i.e., one of the following holds: $X < Y$, $Y < X$, or $X = Y$.

We often make use of this property without explicit indication.

EXAMPLES:

1. Every smooth manifold is an abstract pre-stratified space with only one stratum.

2. A manifold M with boundary admits a structure of an abstract pre-stratified space with two strata, namely $\mathring{M}$ and ∂M. The existence of a tubular neighbourhood follows from the Collar Theorem.

3. Let V be a pre-stratified space and U an open subset of V. Then U has a canonical structure of a pre-stratified space by setting $\mathcal{S}^U := \{X \cap U | X \in \mathcal{S}\}$ and $\mathcal{N}^U := [\{(N_X^U, \pi_X|_{N_X^U}, \rho_X|_{N_X^U})\}]$, where $N_X^U := U \cap \pi_X^{-1}(N_X \cap U)$. In particular, if S a closed stratum of V, then $V - S$ has a canonical structure of a pre-stratified space.

4. Let V be a pre-stratified space and M a smooth manifold without boundary. Then $V \times M$ has a structure of a pre-stratified space, taking $X \times M$ to be the strata for every stratum X of V, the neighbourhoods $N_{X \times M} := N_X \times M$ with mappings $\pi_{X \times M} := \pi_X \times \mathrm{id}$ and $\rho_{X \times M} := \rho_X \circ \mathrm{pr}_1$.

5. The open cone $\mathring{C}V = V \times (0, 1]/V \times \{1\}$ over a pre-stratified space V admits a structure as a pre-stratified set, setting as strata $\{X \times (0, 1) | X \in \mathcal{S}\} \cup \{\mathrm{pt}\}$ and taking for $X \times (0, 1)$ the neighbourhood structure defined in the previous example and, around the top point, the space $\mathring{C}V$ itself with canonical projection onto $\{\mathrm{pt}\}$ and the tubular function $\rho_{\mathrm{pt}} = (1 - \mathrm{pr}_2)$.

The following theorem completes the list of examples of abstract pre-stratified spaces.

Theorem 3.3.1. *Every Whitney stratified space admits a structure of a pre-stratified space.*

One has to show the last condition, i.e., find a controlled neighbourhood structure. This was done by Mather [Ma, Lemma 7.1], where the structure is unique up to isotopy, see [Ma, Proposition 6.1].

Definition: Let $(V, \mathcal{S}, \mathcal{N})$ be an abstract pre-stratified space and M a smooth manifold. Denote with $\mathcal{C}_{\mathbf{APS}}(V, M)$ the class of continuous functions $f : V \longrightarrow M$, satisfying:

- $f|_X$ is a smooth map for all $X \in \mathcal{S}$,
- for every stratum X, there is a neighbourhood N'_X of X in N_X such that $f(x) = f\pi_X(x)$ for all $x \in N'_X$.

The maps from $\mathcal{C}_{\mathbf{APS}}(V, M)$ are called *smooth* or *controlled.* The map f is called a *controlled* or *smooth submersion* if $f|_X$ is a submersion for each stratum X of V.

Let $(V', \mathcal{S}, \mathcal{N})$ be another abstract pre-stratified space. A continuous map $\varphi : V \longrightarrow V'$ is called an *isomorphism of abstract pre-stratified spaces* if $\varphi \in \mathbf{ISO_{SF}}$ and, for all $X \in \mathcal{S}$, there are representatives N_X and $N'_{\varphi(X)}$ of the neighbourhood structure of X and $\varphi(X)$ respectively, such that

$$\varphi\pi_X(x) = \pi'_{\varphi(X)}(\varphi(x)) \quad \text{and} \quad \rho'_{\varphi(X)}(\varphi(x)) = \rho_X(x).$$

Definition: Let $(V, \mathcal{S}, \mathcal{N})$ be an abstract pre-stratified space , $X \in \mathcal{S}$ a stratum of V and $N_X = (N_X, \pi_X, \rho_X)$ a representative of the neighbourhood structure. Further let $\delta : X \longrightarrow (0, \infty)$ be a continuous map. Recall the definition from §1.3 and set

$$N_X^\delta := N^\delta_{X(\pi_X, \rho_X)}$$

Analogously we define $N_X^{<\delta}$ and $N_X^{\leq\delta}$.

One should keep in mind that $N_X^{<\delta}$ not only depends on the map δ, but also on the chosen neighbourhood N_X. If, for example, N'_X is another neighbourhood of X with $N'_X \subset N_X$, then $N'^{<\delta}_X \neq N_X^{<\delta}$ in general.

One useful property of abstract pre-stratified spaces is the following:

Lemma 3.3.2. *Let $(V, \mathcal{S}, \mathcal{N})$ be an abstract pre-stratified space and $X \in \mathcal{S}$ a stratum of V. For every representative N'_X of the neighbourhood germ of X there exists a representative $N_X \subset N'_X$ such that for every open neighbourhood U of X in N_X, there exists a smooth map $\delta : X \longrightarrow (0, \infty)$, such that $N_X^{<\delta} \subset U$ and $(\pi_X, \rho_X) : N_X^{<\delta} \longrightarrow (X \times (0, \infty))^{<\delta}$ is proper and surjective.*

Proof. We start with a representative N'_X of the neighbourhood germ of X, equipped with tubular functions π'_X and ρ'_X. For an arbitrary $x \in X$, we choose a compact neighbourhood A^c_x of x in N'_X.

Claim: There is an open neighbourhood B of x in X and an $n \in \mathbb{N}$, such that $(\pi'_X)^{-1}(B) \cap (\rho'_X)^{-1}(0, 1/n) \cap A_x^c \subset U$.

Assume that this claim is false. Let φ be a manifold chart around x with $\varphi(x) = 0 \in \mathbb{R}^k$. Set $B_n := \varphi^{-1}(\{y \in \mathbb{R}^k | \, ||y|| < 1/n\})$, then $B_{n+1} \subset B_n$ and every open neighbourhood of x in X contains the open set B_n for a big enough $n \in \mathbb{N}$. According to the assumption, for every $n \in \mathbb{N}$ there is an $y_n \in (\pi'_X)^{-1}(B_n) \cap (\rho'_X)^{-1}(0, 1/n) \cap A_x^c - U$. Since the sequence y_n lies in the compact set A_x^c we can without loss of generality assume y_n converges to a point $y \in A_x^c$. If this is not the case, we can go over to a convergent subsequence. From this construction, we obtain $\pi'_X(y) = \lim \pi'_X(y_n) = x$ and $\rho'_X(y) = \lim \rho'_X(y_n) = 0$, hence we get $\lim y_n = x$, which is a contradiction to the choice $y_n \notin U$.

Set $N_X := N'_X \cap (\cup_x \mathring{A}_x^c)$ and define $\pi_X := \pi'_X|_{N_X}$ and $\rho_X := \rho'_X|_{N_X}$. Then for every x, there is an open set B_x of x in X, and an $n_x \in \mathbb{N}$, such that $\pi_X^{-1}(B_x) \cap \rho_X^{-1}[0, 1/n_x) \subset U \cap N_X$. Define $\delta_x : \mathring{A}_x^c \longrightarrow (0, \infty)$ by setting $\delta_x := 1/n_x$, where n_x is chosen as above. Use a smooth partition of unity on X to combine these maps in a smooth map $\delta : X \longrightarrow (0, \infty)$, satisfying $N_X^{<\delta} \subset U \cap N_X$.

For the second statement, we observe that we can choose δ, such that $((\pi_X|_{N_X^{<\delta}}), (\rho_X|_{N_X^{<\delta}}))^{-1}(x, t) = ((\pi_X|_{A_x^c}), (\rho_X|_{A_x^c}))^{-1}(x, t)$. This implies that the preimage of every point is compact. Since a submersion is always an open mapping, this implies the properness.

□

From now on we assume that every tubular neighbourhood considered has the property stated in the last lemma.

According to the last lemma, for every stratum X of an abstract prestratified space V, there is a smooth map $\delta_X : X \longrightarrow (0, \infty)$, such that $(\pi_X, \rho_X) : N_X^{<\delta_X} \longrightarrow (X \times [0, \infty))^{<\delta_X}$ is proper and a smooth submersion restricted on every stratum $Y \neq X$. Assume δ_X takes the minimum ε on X. In this case we can replace δ_X with the constant function $\varepsilon : X \longrightarrow (0, \infty)$, $x \mapsto \varepsilon$ and obtain a proper submersion $(\pi_X, \rho_X) : N_X^{<\varepsilon} \longrightarrow X \times [0, \varepsilon)$. If this is not the case, we can replace ρ_X with $\rho'_X(y) := \rho_X(y)/\delta_X(\pi_X(y))$. The map $(\pi_X, \rho'_X) : N_X^{<\delta_X} \longrightarrow X \times [0, 1)$ is then proper and a smooth submersion restricted to any stratum $Y \neq X$. Further, ρ'_X satisfies the compatibility relation:

$$\rho'_{X,Y}\pi_{Y,Z}(v) = \frac{\rho_{X,Y}(\pi_{Y,Z}(v))}{\delta_X(\pi_{X,Y}\pi_{Y,Z}(v))} = \frac{\rho_{X,Z}(v)}{\delta_X(\pi_{X,Z}(v))} = \rho'_{X,Z}(v).$$

Nevertheless, one should keep in mind that such a replacement of ρ in general changes the isomorphism class.

According to [Ma, Cor. 10.2], we conclude:

Lemma 3.3.3. *For every stratum X of an abstract pre-stratified space V there is a representative of the neighbourhood germ N_X, and a smooth map $\delta : X \longrightarrow (0,\infty)$, such that $(\pi_X, \rho_X) : N_X^{<\delta} - X \longrightarrow (X \times (0,\infty))^{<\delta}$ is a locally trivial fibration, where the trivialization is an isomorphism of abstract pre-stratified spaces.*

Once we know that open subsets preserve the structure, how to build the product with an open interval $(0,1)$, and what the isomorphisms are, we can define objects with boundary.

In the same way as in the case of c-manifolds or p-stratifolds, we first define collars.

Definition: Let $(P, \partial P)$ be a pair of topological spaces such that $\mathring{P} := P - \partial P$ and ∂P are abstract pre-stratified spaces and ∂P is closed in P. A *collar* is a homeomorphism

$$\mathbf{c} : (\partial P \times [0,\infty))^{<\delta} \longrightarrow U,$$

where $\delta : \partial P \longrightarrow (0,\infty)$ is a continuous map and U an open neighbourhood of ∂P in P such that $\mathbf{c}|_{\partial P \times \{0\}}$ is the identity map and $\mathbf{c}|_{(\partial P \times (0,\infty))^{<\delta}}$ is an isomorphism of abstract pre-stratified spaces onto $U - \partial P$.

As before we go over to *germs of collars*. This is an equivalence class of collars, where two collars are equivalent if they coincide on $(\partial P \times [0,\infty))^{<\tilde{\delta}}$ for an appropriate map $\tilde{\delta} : \partial P \longrightarrow (0,\infty)$.

Definition: An *abstract pre-stratified space P with boundary* is a pair $(P, \partial P)$ consisting of abstract pre-stratified spaces $\mathring{P} := P - \partial P$ and ∂P, together with a germ of collars $[\mathbf{c}]$. The abstract pre-stratified space ∂P is called the *boundary of P*, and $\mathring{P} := P - \partial P$ the *interior of P*.

The class of abstract pre-stratified spaces with boundary is denoted by $\mathbf{APS}_\partial$.

EXAMPLES:

1. The class of c-manifolds provides examples of abstract pre-stratified spaces with boundaries by setting $P = (P, \partial P)$ and taking the germ of collars of ∂P in P.

2. The closed cone $CV := V \times [0,1]/V \times \{1\}$ over an abstract pre-stratified space V is an abstract pre-stratified space with boundary $V \times \{0\}$. The

interior $CV - \partial CV$ is the open cone, which has a canonical structure as an abstract pre-stratified space as indicated in a previous example. A collar is an obvious inclusion $V \times [0,1) \longrightarrow CV$.

Definition: Let $(P, \partial P)$ be an abstract pre-stratified space with boundary and T a c-manifold. A continuous map $g : P \longrightarrow T$ is called *smooth* if $g|_{\mathring{P}} \in \mathcal{C}_{\mathbf{APS}}(\mathring{P}, \mathring{T})$, $g|_{\partial P} \in \mathcal{C}_{\mathbf{APS}}(\partial P, \partial T)$, satisfying the compatibility condition

$$g(\mathbf{c}(x,t)) = \tilde{\mathbf{c}}(g(x), t)$$

for suitable representatives $\mathbf{c}$ and $\tilde{\mathbf{c}}$ of the germ of collars of P and T respectively. A map $h : P \longrightarrow M$ to a smooth manifold is called *smooth* if $h(c(x,t)) = h(x)$ for a suitable representative $\mathbf{c}$.

Definition: Let $f \in \mathcal{C}_{\mathbf{APS}}(V, M)$ be a smooth map. A point $t \in M$ is called a *regular value of* f if, for all $x \in f^{-1}(t)$, the restriction $f|_X$ to the stratum X containing x has x as a regular point. The point x is called *regular point of* f. As usual, one denotes the complement of the regular values and the regular points *critical values* resp. *critical points.*

According to Sard's theorem, the set of critical values has measure zero in M. Another classical result regarding regular points is the following.

Lemma 3.3.4. *The set of regular points of an abstract pre-stratified space is open.*

Proof. Let $f : V \longrightarrow M$ be a smooth map and $x \in V$ a regular point of f, i.e., x is a regular point of f restricted to the stratum X containing x. After passing to a possible smaller neighbourhood N_X, we may assume that f commutes with π_X on N_X. Since the set of regular points is open on every smooth manifold, there is an open subset U of x in X, such that f has no critical points on U. Consider now $\tilde{U} := \pi_X^{-1}(U)$. Suppose there exists a critical point $y \in \tilde{U}$, lying in a stratum Y, then $D_y f$ is not surjective. Using the commutativity of f with π_X, we conclude

$$D_y f = D_y(f\pi_X) = D_{\pi_X(y)} f D_y \pi_X$$

Since π_X is a submersion, it follows that $D_{\pi(y)} f$ is not surjective as well, leading to a contradiction to the choice of $\tilde{U}$.

□

As in the case of smooth manifolds, we conclude.

Lemma 3.3.5. *Let $f : V \longrightarrow M$ be a smooth map and $t \in M$ a regular value of f. Then $f^{-1}(t)$ has a canonical structure as an abstract pre-stratified space.*

Analogously for a smooth map $f : P \longrightarrow M$ from an abstract pre-stratified space $(P, \partial P)$ with boundary to a manifold M and for a $t \in M$ being a regular value both for $f|_{\mathring{P}}$ and $f|_{\partial P}$, the preimage $f^{-1}(t)$ is again an abstract pre-stratified space with boundary $(f|_{\partial P})^{-1}(t)$.

Proof. As strata of $f^{-1}(t)$ we take $X^{f^{-1}(t)} := (f|_X)^{-1}(t)$ for all $X \in \mathcal{S}$. Since t was assumed to be regular, all strata are smooth manifolds. After passing to a possibly smaller neighbourhood N_X, we can assume that f commutes with π_X on N_X for all $X \in \mathcal{S}$. Now we set

$$\begin{array}{rcl} N_{X^{f^{-1}(t)}} & := & N_X|_{X^{f^{-1}(t)}} \\ \pi_{X^{f^{-1}(t)}} & := & \pi_X|_{X^{f^{-1}(t)}} \\ \rho_{X^{f^{-1}(t)}} & := & \rho_X|_{X^{f^{-1}(t)}} \end{array}$$

We only have to show that the maps are well-defined, the required relations will then follow from the corresponding statements for π_X and ρ_X. But this is a consequence of the compatibility of f with π_X. Let $x \in N_{X^{f^{-1}(t)}}$, then $f(x) = f(\pi_X(x))$, hence $\pi_X(x) \in f^{-1}(y)$.

The case of an abstract pre-stratified space with boundary follows analogously using the corresponding result for manifolds with boundary and the compatibility with the collar.

□

Since every abstract pre-stratified space V is a Hausdorff, locally compact topological space having a countable basis of the topology, it follows, as in the case of stratifolds, that V is metrizable and paracompact. The next lemma shows that V even has a smooth partition of unity.

Lemma 3.3.6. *Let V be an abstract pre-stratified space and U an open covering of V. There exists a subordinated smooth partition of unity.*

Proof. We prove the following assertion, from which the statement of the lemma follows using classical arguments.

Claim: Let $x \in V$ and let U be an open neighbourhood of x. There exists a smooth map $\lambda : V \longrightarrow [0, \infty)$ with compact support supp $\lambda \subset U$ and $\lambda(x) \neq 0$.

Let X be the stratum containing x. According to Lemma 3.3.2, after possible reduction of N_X there is an open neighbourhood D of x in X and an $n \in \mathbb{N}$ such that $\pi_X^{-1}(D) \cap \rho_X^{-1}[0, 1/n) \subset U \cap N_X$. Choose a function $\lambda_X : X \longrightarrow [0, \infty)$ satisfying the desired properties for D and a smooth map

$\eta : [0,\infty) \longrightarrow [0,\infty)$ such that $\eta(t) = 1$ for $t \in [0, 1/4n)$ and $\eta(t) = 0$ for $t > 1/2n$. Now define the map $\lambda : \pi_X^{-1}(\overset{\circ}{D}) \cap \rho_X^{-1}(0, 1/n) \longrightarrow [0,\infty)$ by setting $\lambda(x) := \eta(\rho_X(x))\lambda_X(\pi_X(x))$. This map can be extended by zero to V and has the desired properties. Note that since $\eta(t) = 1$ near zero the map λ is indeed smooth.

□

One does not require the strata of an abstract pre-stratified space $(V, \mathcal{S}, \mathcal{N})$ to have different dimensions. But since strata of the same dimension can not be incident, i.e., $\overline{X} \cap Y = \emptyset$ for $X, Y \in \mathcal{S}$ with $\dim X = \dim Y$, we can always find representatives of the neighbourhood germs N_X and N_Y with $N_X \cap N_Y = \emptyset$. Thus, after collecting the strata of the same dimensions to a single stratum we can assume without loss of generality that all strata of V have different dimensions. Let Σ^k denote the collection of strata up to dimension k, i.e. $\Sigma^k := \{X | \dim X \leq k\}$. An abstract pre-stratified space V is called *finite-dimensional* if there exists $k \in \mathbb{N}$ such that $V = \Sigma^k$. The class of finite-dimensional abstract pre-stratified spaces (with strata having different dimensions) will be denoted by $\mathbf{APS^f}$.

Summarizing the discussion we conclude.

Lemma 3.3.7. *Let $(V, \mathcal{S}, \mathcal{N})$ be an element of $\mathbf{APS^f}$. The pair $(V, \mathcal{C}_{\mathbf{APS}}(V))$ is a locally trivial stratifold.*

In the last section of this chapter, we see that V even admits a structure of a cornered p-stratifold.

3.3.1 Vector fields and flows

Definition: Let $(V, \mathcal{S}, \mathcal{N})$ be a pre-stratified space. A *stratified vector field* η on V is a collection of smooth vector fields $\{\eta_X | X \in \mathcal{S}\}$, where η_X is a smooth vector field on X.

Let $\mathcal{N} = [\{N_X = (N_X, \pi_X, \rho_X)\}]$ be the controlled neighbourhood structure on V.

Definition: A stratified vector field η on V is said to be *controlled* if the following conditions are satisfied: For any stratum Y there exists a neighbourhood N'_Y of Y in N_Y such that for any second stratum $X > Y$ and any $v \in N'_Y \cap X$, we have

$$\begin{aligned} D\rho_{Y,X}\ \eta_X(v) &= 0 \\ D\pi_{Y,X}\ \eta_X(v) &= \eta_Y\ \pi_{Y,X}(v) \end{aligned}$$

One very useful tool in constructing controlled vector fields is the following lemma [Ma, Prop. 9.1].

Lemma 3.3.8. *Let $f : V \longrightarrow M$ be a controlled submersion and ξ a smooth vector field on M. Then there exists a controlled vector field η on V making the following diagram commutative for every stratum X*

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "f"'] & TX \arrow[d, "Df|_X"] \\ M \arrow[r, "\xi"] & TM \end{tikzcd}$$

When considering abstract pre-stratified spaces with boundary it is useful to have a relative version of the lemma above.

Definition: Let $(V, \partial V)$ be an abstract pre-stratified space with boundary. A *controlled vector field* on $(V, \partial V)$ is a pair (η, η^∂) such that η is a controlled vector field on $\mathring{V}$, and η^∂ is a controlled vector field on ∂V, satisfying the compatibility condition

$$\eta(\mathbf{c}(x,t)) = D\mathbf{c}(x,t)\eta^\partial(x)$$

for a representative $\mathbf{c}$ of the germ of collars.

Lemma 3.3.9. *Let $(V, \partial V)$ be an abstract pre-stratified space with boundary, $(T, \partial T)$ a c-manifold and $\varphi : V \longrightarrow T$ a smooth submersion. For any smooth vector field (ξ, ξ^∂) on T, there is a controlled vector field (η, η^∂) on $(V, \partial V)$, such that $D\varphi\eta(v) = \xi(\varphi(v))$ and $D\varphi\eta^\partial(w) = \xi^\partial(\varphi(w))$ for all $v \in V$ and $w \in \partial V$.*

Proof. Studying the proof of Lemma 3.3.8 in [Ma], we see that the proof is carried out inductively on the dimension of the abstract pre-stratified space. In the inductive step, one takes local vector fields on the higher dimensional stratum satisfying the desired conditions, and puts them together with a partition of unity to form a global vector field. Hence, we only have to check that the vector field $\tilde{\eta}$ on $\mathbf{c}(\partial V \times (0,\infty))^{<\delta}$ defined by $\tilde{\eta}(\mathbf{c}(x,t)) := D\mathbf{c}(x,t)\eta^\partial(x)$ satisfies the desired properties for a representative $\mathbf{c}$ of the germs of collars, where η^∂ is a vector field on ∂V obtained with the help of Lemma 3.3.8.

Choose collars $\mathbf{c} : (\partial V \times [0,\infty))^{<\delta} \hookrightarrow V)$ of ∂V and $\mathbf{d} : (\partial T \times [0,\infty))^{<\delta'} \hookrightarrow T$ satisfying all compatibility relations. We first show that $\tilde{\eta}$ is controlled.

For this we compute:

$$\begin{array}{lclc}
D\pi_X(\tilde{\eta}(\mathbf{c}(x,t))) & = & D\pi_X(D\mathbf{c}(x,t)\eta^\partial(x)) & = \\
D(\pi_X\mathbf{c}(x,t))\eta^\partial(x) & = & D(\mathbf{c}(\pi_{\partial X}(x),t))\eta^\partial(x) & = \\
D\mathbf{c}(\pi_{\partial X}(x),t)D\pi_{\partial X}\eta^\partial(x) & = & D\mathbf{c}(\pi_{\partial X}(x),t)\eta^\partial(\pi_{\partial X}(x)) & = \\
\tilde{\eta}(\mathbf{c}(\pi_{\partial X}(x),t)) & = & \tilde{\eta}(\pi_X(\mathbf{c}(x,t))). &
\end{array}$$

Analogously, one computes: $\tilde{\eta}\rho_X(\mathbf{c}(x,t)) = 0$. To show the connection to ξ we compute:

$$\begin{array}{lclc}
D\varphi\tilde{\eta}(\mathbf{c}(x,t)) & = & D\varphi(D\mathbf{c}(x,t))\eta^\partial(x) & = \\
D(\varphi\mathbf{c}(x,t))\eta^\partial(x) & = & D(\mathbf{d}(\varphi(x),t))\eta^\partial(x) & = \\
D\mathbf{d}(\varphi(x),t)D\varphi(x)\eta^\partial(x) & = & D\mathbf{d}(\varphi(x),t)\xi^\partial(\varphi(v)) & = \\
\xi(\mathbf{d}(\varphi(x),t)) & = & \xi(\varphi(\mathbf{c}(x,t))). &
\end{array}$$

$\square$

Let V be a topological space. A *flow* on V is a continuous map $\alpha : \mathbb{R}\times V \longrightarrow V$, such that $\alpha_{t+s}(v) = \alpha_t\alpha_s(v)$ for all $t,s\in\mathbb{R}$ and for all $v\in V$.

Now suppose V is an abstract pre-stratified space $(V,\mathcal{S},\mathcal{N})$, and α is stratum preserving. Further let η be a stratified vector field on V. We say that η *generates* α if the following condition is satisfied: For any $v\in V$, the mapping $t \longrightarrow \alpha_t(v)$ of $\mathbb{R}$ into V is smooth as a mapping into the stratum which contains v and

$$\frac{d}{dt}(\alpha_t(v))\big|_{t=0} = \eta(v).$$

Let $(\eta,\partial\eta)$ be a controlled vector field on an abstract pre-stratified space $(V,\partial V)$ with boundary. A *flow* on $(V,\partial V)$ is a pair of flows (α,α^∂), such that α is a flow on $\mathring{V}$ and α^∂ is a flow on ∂V satisfying

$$\alpha(\mathbf{c}(x,t),s) = \mathbf{c}(\alpha^\partial(x,s),t)$$

for a representative $\mathbf{c}$ of the germ of collars.

In analogy to smooth manifolds, one defines *local flows* and explains the notion of vector fields generating local flows. For details, see [Ma, §9], where the proof of the following lemma can also be found.

Lemma 3.3.10. *Let η be a controlled vector field on an abstract pre-stratified space V. Then η generates a unique maximal local flow (J,α).*

REMARK: It follows from the control condition of η that

$$\rho_X(\alpha_y(s)) = \rho_X(y) \quad \text{and } \pi_X(\alpha_y(s)) = \alpha_{\pi_X(y)}(s)$$

for appropriate representatives of the neighbourhood germs N_X.

We again formulate a relative version of the lemma.

Lemma 3.3.11. *Let (η, η^∂) be a controlled vector field on an abstract pre-stratified space $(V, \partial V)$ with boundary. Then (η, η^∂) generates a unique maximal local flow $((J, \alpha), (J^\partial, \alpha^\partial))$.*

Proof. Let (J, α) be the maximal local flow on $\mathring{V}$ and $(J^\partial, \alpha^\partial)$ be the maximal local flow on ∂V and let $\mathbf{c} : (\partial V \times [0, \infty))^{<\delta} \hookrightarrow V$ be a representative of the germ of collars for ∂V, such that (η, η^∂) is compatible with $\mathbf{c}$. First of all, observe that $\{(\mathbf{c}(x,t), s) | (x,t) \in (\partial V \times (0, \infty))^{<\delta}, (x,s) \in J^\partial\} = J \cap (\mathbf{c}(\partial V \times (0,\infty))^{<\delta} \times \mathbb{R}) =: \tilde{J}$. Define the following map:

$$\begin{array}{cccc} \tilde{\alpha} : & \tilde{J} & \longrightarrow & V \\ & (\mathbf{c}(x,t), s) & \longmapsto & \mathbf{c}(\alpha^\partial(x,s), t). \end{array}$$

Then we have

$$\begin{array}{lcll} \frac{d}{ds}\tilde{\alpha}(\mathbf{c}(x,t), s)\big|_{s=0} & = & \frac{d}{ds}\mathbf{c}(\alpha^\partial(x,s), t) & = \\ D\mathbf{c}(\alpha^\partial(x,s), t)\frac{d}{ds}\alpha^\partial(x,s)\big|_{s=0} & = & D\mathbf{c}(\alpha^\partial(x,0), t)\eta^\partial(x) & = \\ D\mathbf{c}(x,t)\eta^\partial(x) & = & \eta(\mathbf{c}(x,t)). & \end{array}$$

Hence $\tilde{\alpha}$ solves the differential equation, furthermore

$$\tilde{\alpha}(\mathbf{c}(x,t), 0) = \mathbf{c}(\alpha^\partial(x,0), t) = \mathbf{c}(x,t) = \alpha(\mathbf{c}(x,t), 0).$$

Since the solutions are unique, $\tilde{\alpha}$ has to coincide with α on $\tilde{J}$, thus $(\alpha, \alpha^\partial)$ satisfies the flow condition.

□

EXAMPLES:
1. Let $(V, \mathcal{S}, \mathcal{N})$ be an abstract pre-stratified space and $X \in \mathcal{X}$ a stratum of V. The subspace $N_X - X$ inherits a canonical structure of an abstract pre-stratified space from V as explained previously (remembering that X is closed in N_X).
According to Lemma 3.3.2, there exists a smooth map $\delta' : X \longrightarrow (0, \infty)$ and a representative of the neighbourhood germ $N_X = (N_X, \pi_X, \rho_X)$, such that

$$(\pi_X, \rho_X) : N_X - X \longrightarrow (X \times (0, \infty))^{<\delta'}$$

is a proper surjective submersion on each stratum. The compatibility conditions for π_X and ρ_X, restricted to each stratum, imply that the above map is a controlled submersion. By Lemma 3.3.8 there exists a controlled vector field η on $N_X - X$ with the following conditions:

$$\begin{aligned} D\pi_X\,\eta &= 0 \\ D\rho_X\,\eta &= \tfrac{d}{dt} \end{aligned}$$

Let $\alpha : J' \longrightarrow N_X - X$ be the maximal local flow generated by η. Consider the real valued function $\rho_X\alpha_x : (a_x, b_x) \longrightarrow \mathbb{R}$ for a fixed $x \in X$ and compute its derivative:

$$\frac{d}{dt}\rho_X\alpha_x(t) = D\rho_X \circ \dot{\alpha}_x(t) = (D\rho_X\eta)(\alpha_x(t)) = \frac{d}{dt}$$

In the first equation, we used the chain rule, then the fact that η generates α and finally, the second condition on η.

Thus, after the canonical identification $\frac{d}{dt} \mapsto \langle \frac{d}{dt}, dt\rangle$ we obtain

$$\frac{d}{dt}\rho_X\,\alpha_x(t) = 1$$

In view of the initial condition, it follows that

$$\rho_X\,\alpha_x(t) = t + \rho_X(x). \tag{$*$}$$

With this information, it is possible to determine the range of definition of the flow α, namely

$$J' = \{(x,t) | -\rho_X(x) < t < \delta'(\pi_X(x)) - \rho_X(x)\}.$$

The first condition of the flow leads to the equality

$$\pi_X\,\alpha_x(t) = \pi_X(x). \tag{$**$}$$

Combining equality (*) with (**), we see that by setting $\alpha_x(-\rho_X(x)) := \pi_X(x)$, the flow α can be continuously extended to

$$J = \{(x,t) | -\rho_X(x) \le t < \delta'(\pi_X(x)) - \rho_X(x)\}.$$

2. Let $(V, \partial V)$ be an abstract pre-stratified space with boundary, X a stratum of $\mathring{V}$ and ∂X a stratum of ∂V, such that $\mathbf{c}(\partial X \times (0,\infty))^{<\lambda} \subset X$ for a representative $\mathbf{c}$ of the germ of collars. Let $N_X = (N_X, \pi_X, \rho_X)$ and $N_{\partial X} = (N_{\partial X}, \pi_{\partial X}, \rho_{\partial X})$ be representatives of the neighbourhood structures of X resp. ∂X, such that $\mathbf{c}(N_{\partial X} \times (0,\infty))^{<\lambda} \subset N_X$. Shrinking the collar $\mathbf{c}$ on

the manifold $(\partial X \times [0,\infty))^{<\lambda}$, we see for $\overline{X} := (\partial X \times [0,\infty))^{<\lambda} \cup_{\mathbf{c}|_{(\partial X \times (0,\infty))^{<\lambda}}} X$ that the pair $(\overline{X}, \partial X)$ is a c-manifold. Let $\delta' : \overline{X} \longrightarrow (0,\infty)$ be a smooth map, such that

$$\begin{array}{llll} (\pi_X, \rho_X): & N_X - X & \longrightarrow & (X \times (0,\infty))^{<\delta'} \quad \text{and} \\ (\pi_{\partial X}, \rho_{\partial X}): & N_{\partial X} - \partial X & \longrightarrow & (\partial X \times (0,\infty))^{<\delta'} \end{array}$$

are proper surjective submersions. Such a map δ' always exists, since we can choose the appropriate map on ∂X, first extend it with the help of the collar to $\mathbf{c}(\partial X \times (0,\infty))^{<\lambda}$ and use a smooth partition of unity to extend it to a smooth map on $\overline{X}$ with the desired properties. Consider

$$\begin{array}{lll} T & := & (\partial X \times [0,\infty) \times (0,\infty))^{\{<\lambda, <\delta'\}} \cup_{\mathbf{c}|_{(\partial X \times (0,\infty))^{<\lambda}} \times \mathrm{id}} (X \times (0,\infty))^{<\delta'}, \\ \partial T & := & (\partial X \times (0,\infty))^{<\delta'}. \end{array}$$

Then $(T, \partial T)$ is a c-manifold with the interior $\mathring{T} = (X \times (0,\infty))^{<\delta'}$ and the germ of collars $[\mathbf{c}^T]$, where $\mathbf{c}^T : (\partial T \times (0,\infty))^{\lambda \mathrm{pr}_1} \longrightarrow T$ is given by $\mathbf{c}^T((x,t),s) := (\mathbf{c}(x,s),t)$.

We build $(P, \partial P)$ in the same way by setting

$$\begin{array}{lll} P & := & ((N_{\partial X} - \partial X) \times [0,\infty))^{<\lambda)} \cup_{\mathbf{c}|_{((N_{\partial X} - \partial X) \times (0,\infty))^{<\lambda}}} (N_X - X) \quad \text{and} \\ \partial P & := & N_{\partial X} - \partial X. \end{array}$$

We obtain an abstract pre-stratified space $(P, \partial P)$ with boundary with the germ of collars $[\mathbf{c}^P]$, where $\mathbf{c}^P := \mathbf{c}|_{((N_{\partial X} - \partial X) \times [0,\infty))^{<\lambda}}$. The interior of P is given by $\mathring{P} = N_X - X$.
Define a map $\varphi : P \longrightarrow T$ by the assignment

$$\begin{array}{lll} \varphi|_{\mathring{P}} & := & (\pi_X, \rho_X) : N_X - X \longrightarrow (X \times (0,\infty))^{<\delta'} \qquad \text{and} \\ \varphi|_{\partial P} & := & (\pi_{\partial X}, \rho_{\partial X}) : N_{\partial X} - \partial X \longrightarrow (\partial X \times (0,\infty))^{<\delta'}. \end{array}$$

According to the definition of $(P, \partial P)$ and $(T, \partial T)$, the map φ is continuous, furthermore, both $\varphi|_{\mathring{P}}$ and $\varphi|_{\partial P}$ are controlled submersions. We verify the compatibility condition:

$$\begin{array}{lll} \varphi(\mathbf{c}^P(x,t)) & = & (\pi_X(\mathbf{c}(x,t)), \rho_X(\mathbf{c}(x,t))) \quad = \\ (\mathbf{c}(\pi_{\partial X}(x),t), \rho_{\partial X}(x)) & = & \mathbf{c}^T(\varphi(x),t). \end{array}$$

Thus, $\varphi : P \longrightarrow T$ is a smooth surjective submersion.

Let (η, η^∂) be a controlled vector field on $(P, \partial P)$ satisfying

$$\begin{aligned} D\varphi\eta &= (0, \tfrac{d}{dt}) \quad \text{and} \\ D\varphi\eta^\partial &= (0, \tfrac{d}{dt}). \end{aligned}$$

We can now combine Lemma 3.3.9 and Lemma 3.3.11 with the first part of the example and obtain the maximal local flow $((J', \alpha), (J'^\partial, \alpha^\partial))$ with

$$J' = \{(x,t) | -\rho_X(x) < t < \delta'(\pi_X(x)) - \rho_X(x)\}$$

and

$$J'^\partial = \{(y,s) | -\rho_{\partial X}(y) < s < \delta'(\pi_{\partial X}(y)) - \rho_{\partial X}(y)\}.$$

Setting $\alpha_x(-\rho_X(x)) := \pi_X(x)$ and $\alpha_y^\partial(-\rho_{\partial X}(y)) := \pi_{\partial X}(y)$, the flow can be continuously extended to (J, J^∂) with

$$J = \{(x,t) | -\rho_X(x) \le t < \delta'(\pi_X(x)) - \rho_X(x)\}$$

and

$$J^\partial = \{(y,s) | -\rho_{\partial X}(y) \le s < \delta'(\pi_{\partial X}(y)) - \rho_{\partial X}(y)\}.$$

3.4 Main Theorem

Finally, we are going to prove that every abstract pre-stratified space admits a structure of a cornered p-stratifold.

First of all, we need to make some technical preparations.

Proposition 3.4.1. *Let $\mathscr{S}$ be a cornered p-stratifold having only strats of dimension $\geq m$, and let S be an m-dimensional manifold. Let further $g : \mathscr{S} \longrightarrow S$ be a smooth map and $\delta : S \longrightarrow (0,\infty)$ a smooth map on S. Then*

$$\tilde{\mathscr{S}} := (\mathscr{S} \times [0,\infty))^{\le \delta g} \cup_{gpr_1 : (\mathscr{S}\times[0,\infty))^{\delta g} \to S} S$$

admits a structure of a cornered p-stratifold with boundary $\partial\tilde{\mathscr{S}} = \mathscr{S} \times \{0\}$.

Proof. Let $f_i : W^i \longrightarrow \mathscr{S}$ be the strats of $\mathscr{S}$. We want to apply Lemma 1.5.1 and construct cornered strats for $\mathring{\tilde{\mathscr{S}}}$. Define for $k > m$ manifolds $\tilde{W}^k := (W^{k-1} \times (0,\infty))^{\le \delta g f_{k-1}}$. In a canonical way this is a c-manifold with corners. Moreover, the space $(\mathscr{S} \times (0,\infty))^{\le \delta g}$ has a canonical stratifold structure as described in an example in §1.5. In dimension m, we define $\tilde{W}^m := S$. For $k \neq m$, the strats of $\mathring{\tilde{\mathscr{S}}}$ are given by $\tilde{f}_k : \tilde{W}^k \longrightarrow \tilde{\mathscr{S}}$, the composition of strats of $(\mathscr{S} \times (0,\infty))^{\leq \delta g}$ with the canonical projection $(\mathscr{S} \times (0,\infty))^{\leq \delta g} \longrightarrow \mathring{\tilde{\mathscr{S}}}$ and $\tilde{f}_m : \tilde{W}^m \longrightarrow \mathring{\tilde{\mathscr{S}}}$ is the inclusion.

The maps $\tilde{f}_k$ are proper continuous maps, being homeomorphisms on the interiors of $\tilde{W}^k$. We now have to show that $\tilde{f}_k(\partial \tilde{W}^k) \subset \tilde{\Sigma}^{k-1}$. There is nothing to show for $k = m$. Otherwise

$$\partial \tilde{W}^k = (\partial W^{k-1} \times (0,\infty))^{\leq \delta g f_{k-1}} \cup_{(\partial\partial W^{k-1} \times (0,\infty))^{\delta g f_{k-1}}} (W^{k-1} \times (0,\infty))^{\delta g f_{k-1}}.$$

We compute $\tilde{f}_k((\partial W^{k-1} \times (0,\infty))^{\leq \delta g f_{k-1}}) = (f_{k-1}(\partial W^{k-1}) \times (0,\infty))^{\leq \delta g} \subset (\Sigma^{k-2} \times (0,\infty))^{\leq \delta g} \sqcup S = \tilde{\Sigma}^{k-1}$ and $\tilde{f}_k((W^{k-1} \times (0,\infty))^{\delta g f_{k-1}}) \subset S \subset \tilde{\Sigma}^{k-1}$. Finally, we observe that the maps are smooth because $(\mathscr{S} \times [0,\infty))^{\leq \delta g}$ is a stratifold and $g : \mathscr{S} \longrightarrow S$ was assumed to be a smooth map. Thus, $\mathring{\tilde{\mathscr{S}}}$ is a stratifold with cornered strats. The collar of $\mathscr{S} \times \{0\}$ in $\tilde{\mathscr{S}}$ is obvious.

□

In the following discussion, we assume all abstract pre-stratified space to have strata of different dimensions. We further denote the class of finite dimensional abstract pre-stratified spaces with boundary by $\mathbf{APS}^{\mathbf{f}}_{\partial}$. Next we need a concept which allows us to compare abstract pre-stratified spaces with stratifolds. This can be done using the weak definition of stratification introduced in §3.1.

Given a stratified space $(X, \mathcal{X})$ and a map $\delta : X \longrightarrow (0,\infty)$, the product space $(X \times (0,\infty))^{<\delta}$ is again a stratified space, where the stratification is given by $\{(X_i \times (0,\infty))^{<\delta}\}$. Knowing this, we can define objects with boundary, as explained for abstract pre-stratified spaces with boundary.

Definition: Let $(X, \partial X)$ and $(Y, \partial Y)$ be two stratified spaces with boundary. A map $\varphi : X \longrightarrow Y$ is called an *isomorphism of stratified spaces with boundary* if $(\varphi|_{\mathring{X}} : \mathring{X} \longrightarrow \mathring{Y}) \in \mathbf{ISO_{SF}}$, $(\varphi|_{\partial X} : \partial X \longrightarrow \partial Y) \in \mathbf{ISO_{SF}}$, and there are representatives of the germs of the collars $\mathbf{c} : (\partial X \times [0,\infty))^{<\delta} \hookrightarrow X$ and $\mathbf{d} : (\partial Y \times [0,\infty))^{<\delta'} \hookrightarrow Y$ such that $\varphi(\mathbf{c}(x,t)) = \mathbf{d}(\varphi(x),t)$ for all $(x,t) \in (\partial X \times [0,\infty))^{<\delta}$.

We are now going to establish a connection between abstract pre-stratified spaces and cornered p-stratifolds and prove the following statement:

Let $(V, \mathcal{S}, \mathcal{N})$ be a finite dimensional abstract pre-stratified space. There is a cornered p-stratifold $\mathscr{S}$ and an isomorphism of stratified spaces $\varphi : V \longrightarrow \mathscr{S}$.

Since the actual proof is rather technical we are going to give an overview of the construction.

We prove the theorem inductively on the depth of V. Let $X = X_1 < X_2 < \cdots < X_k$ be the longest chain of V.
We shall see that the construction can be carried out separately for each chain. Thus, we always assume that the space V has only one chain of length k and that X is connected.

There is nothing to show for $k = 1$. To familiarize ourselves with the arguments, we also provide a proof of the next induction step. Thus, assume $k = 2$.

Choose $\delta' : X \longrightarrow (0, \infty)$ such that

$$(\pi_X, \rho_X) : N_X - X \longrightarrow (X \times (0, \infty))^{<\delta'}$$

is a proper surjective submersion for a suitable representative of the neighbourhood germ of X. Set $F_X := (\pi_X, \rho_X) : N_X \longrightarrow (X \times [0, \infty))^{<\delta'}$ and let $\delta := \delta'/4$.
Define a smooth manifold $P = F_X^{-1}((X \times (0, \infty))^{\delta})$ and consider $(P \times [0, \infty))^{\leq \delta\pi_X}$. According to Example 1 from §1.5, this is a manifold with boundary $M \sqcup N$, where $M = P \times \{0\}$ and $N = P \times [0, \infty)^{\delta\pi_X}$. Set

$$\mathscr{S} := (P \times [0, \infty))^{\leq \delta\pi_X} \cup_{\pi_X \mathrm{pr}_1 : N \to X} X.$$

According to Proposition 3.4.1, the space $\mathscr{S}$ is a p-stratifold with boundary $\partial\mathscr{S} = M$.
Use the vector field η from Example 1 in §3.3.1 together with the corresponding local flow α and construct the following map

$$\begin{array}{rrcl} \varphi_P : & \mathscr{S} & \longrightarrow & F_X^{-1}(X \times [0, \infty))^{\leq \delta} \\ & (P \times [0, \infty))^{\leq \delta\pi_X} \ni (x, t) & \longmapsto & \alpha_x(-t) \\ & X \ni y & \longmapsto & y \end{array}$$

For $(x, t) = (x, \delta\pi_X(x)) \in (P \times [0, \infty))^{\delta\pi_X})$ we have $\rho_X(x) = \delta\pi_X(x)$ and, therefore, $\alpha_x(-\delta\pi_X(x)) = \alpha_x(-\rho(x)) = \pi_X(x)$. Hence the map is well defined, and the flow properties imply that the map is an isomorphism of stratified spaces.

We now consider the space $T := X_2 - F_X^{-1}(X \times (0, \infty))^{<\delta}$. This is a smooth manifold with boundary $\partial T := F_X^{-1}(X \times (0, \infty))^{\delta}$, where the collar is given by

$$\begin{array}{rrcl} \mathbf{d} : & \partial T \times [0, \infty)^{<\delta\pi_X} & \hookrightarrow & T \\ & (x, t) & \longmapsto & \alpha_x(t). \end{array}$$

Recall that $\rho_X(\alpha_x(t)) = t + \rho_X(x)$, therefore, for $x \in \partial T$, we obtain $\rho_X(\alpha_x(t)) = t + \delta(\pi_X(x)) < 2\delta$.
Now glue T and $\mathscr{S}$ together along $\partial\mathscr{S} = \partial T$ and obtain the desired map

$$\varphi := \mathrm{id} \cup \varphi_P : T \cup_\partial \mathscr{S} \longrightarrow V$$

The properties of the flow guarantee the requirements on φ.

Figure 3.4 illustrates that the construction can be carried out separately for each connected component of the "deepest" stratum of V. The figure shows an abstract pre-stratified space with one-dimensional stratum consisting of three connected components, namely X, X' and X'', and a two-dimensional stratum having X^1 and X^2 as connected components.

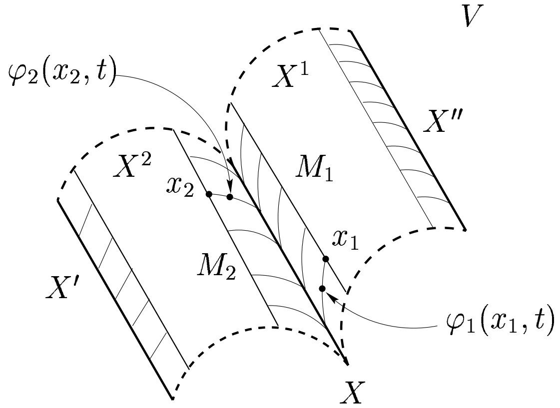


Figure 3.4: Abstract pre-stratified space of depth 2.

In the inductive step we assume $k > 2$. Before proceeding with the detailed proof, let us explain some basic ideas of the construction. Again we consider the smallest stratum X in the chain and define three spaces: $\mathring{B} := \rho_X^{-1}[0,\varepsilon)$, $A := \rho_X^{-1}(\varepsilon)$ and $\mathring{D} := V - \rho_X^{-1}[0,\varepsilon]$.

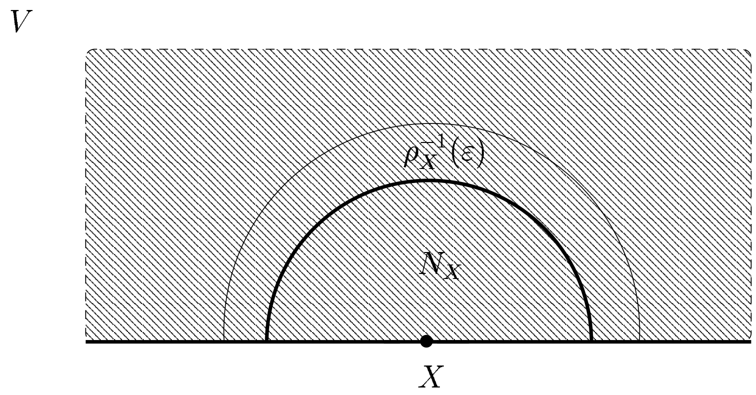


Figure 3.5: Decomposition of V

Space A is again an abstract pre-stratified space according to Lemma 3.3.5, but has smaller depth than V. Thus, according to the inductive assumption, the space A admits a structure of a cornered p-stratifold. The same is true for the space $\mathring{D}$, where we again assume that V has only one chain of depth k, since the construction can be carried out for each of the "deepest" strata X separately. Observe that we may assume $\mathring{D}$ to be open in V, since X is closed. To see that $\mathring{B}$ also admits a structure of a p-stratifold, we again make use of the flow constructed in the example from §3.3.1 to obtain a homeomorphism between $\mathring{B}$ and $(A \times (0,\infty))^{\leq \delta\pi_X} \cup_{\pi_X \mathrm{pr}_1|_{(A\times(0,\infty))^{\delta\pi_X}}} X$ being a diffeomorphism on every stratum. Thus, we can apply Proposition 3.4.1 to obtain a structure of a cornered p-stratifold on $\mathring{B}$. Therefore, we have 3 spaces admitting a cornered p-stratifold structure.

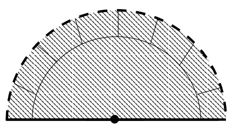


Figure 3.6: Subspace $\mathring{B}$.

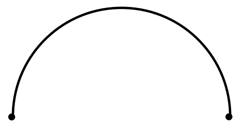


Figure 3.7: Subspace A.

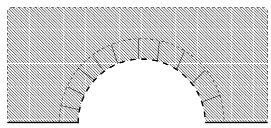


Figure 3.8: Subspace $\mathring{D}$.

Our next task is to put the pieces together to obtain a stratification of V. This can be done using the "collar of A" in $\mathring{B}$ as well as in $\mathring{D}$ as indicated in Figures 3.6 and 3.8.

So we start with the actual proof of the theorem. Since the construction should respect the collar in order to perform the last gluing step, we need more control.

Theorem 3.4.2. *Let $(V, \partial V)$ be a finite dimensional abstract pre-stratified space with boundary, then $(V, \mathcal{C}_{\mathbf{APS}}(V))$ admits a parametrization, which gives $(V, \partial V)$ a structure of a cornered p-stratifold with boundary.*

The proof is based on the following lemma. First of all, we recall that for every stratum ∂X of ∂V, we can build a c-manifold $X \sqcup \partial X$ with boundary ∂X, where X is the corresponding stratum in $\mathring{V}$, see the example after Lemma 3.3.11. If the closure of a stratum Y of $\mathring{V}$ in V does not meet ∂V, we simply set $\overline{Y} = Y$. Using Lemma 3.3.2 we can choose a c-map $\delta_X : \overline{X} \longrightarrow (0,\infty)$, such that

$$\begin{array}{llll} (\pi_X, \rho_X): & N_X^{<\delta_X|_X} - X & \longrightarrow & (X \times (0,\infty))^{<\delta_X|_X} \quad \text{and} \\ (\pi_{\partial X}, \rho_{\partial X}): & N_{\partial X}^{<\delta_X|_{\partial X}} - \partial X & \longrightarrow & (\partial X \times (0,\infty))^{<\delta_X|_{\partial X}} \end{array}$$

are proper surjective submersions for representatives (N_X, π_X, ρ_X) and $(N_{\partial X}, \pi_{\partial X}, \rho_{\partial X})$ of the neighbourhood germs of X and ∂X respectively. We may also assume that $\mathbf{c}(N_{\partial X} \times (0,\infty))^{<\lambda} \subset N_X$ for a representative of the collar $\mathbf{c} : (\partial V \times [0,\infty))^{<\lambda} \hookrightarrow V$, and that for an n-dimensional stratum X that space $\overset{\circ}{V} - \Sigma^n(\overset{\circ}{V}) - N_X^{\leq \delta_X/2}$ is open in $\overset{\circ}{V}$, and $\partial V - \Sigma^{n-1}(\partial V) - N_{\partial X}^{\leq \delta_X/2}$ is open in ∂V. Recall that Σ^k denotes the k-skeleton, the collection of strata up to dimension k.

In the following lemma $\{\delta_X\}$ and $\{N_X,\ N_{\partial X}\}$ are assumed to satisfy the above conditions.

Lemma 3.4.3. *There is a map*

$$\begin{array}{c} \Phi : \{(V, \{\delta_X\}, \{N_X,\ N_{\partial X}\}) \big| V \in \mathbf{APS}^{\mathbf{f}}_{\partial}\} \\ \Big\downarrow \\ \{(\mathscr{S}, \varphi) \big| \mathscr{S} \in \mathbf{CPS}_{\partial}, (\varphi : \mathscr{S} \longrightarrow V) \in \mathbf{ISO}_{\mathbf{SF}_\partial} \textit{ for a } V \in \mathbf{SF}_\partial\} \end{array}$$

such that for $\Phi(V, \{\delta_X\}, \{N_X,\ N_{\partial X}\}) = (\mathscr{S}, \varphi)$ the following conditions hold:

1. *$\varphi : \mathscr{S} \longrightarrow V$.*

2. *For every stratum $\mathscr{S}^k$ of $\overset{\circ}{\mathscr{S}}$ there are representatives $U(\mathscr{S}^k)$ and N_X of the neighbourhood structures of $\mathscr{S}^k$ and $X := \varphi(\mathscr{S}^k)$, making the following diagram commutative:*

$$\begin{array}{ccc} U(\mathscr{S}^k) & \xrightarrow{\varphi} & N_X \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ \mathscr{S}^k & \xrightarrow{\varphi} & X \end{array}$$

The corresponding statement for $\partial\mathscr{S}$ and ∂V is also true.

Proof. Let $(V, \partial V)$ be an abstract pre-stratified space with boundary. We argue inductively on the depth of V. There is nothing to show for $\mathrm{depth}(V) = 1$. Hence let $\mathrm{depth}(V) = \mathrm{depth}(\overset{\circ}{V}) = k \geq 2$, and let $X = X_1 < \cdots < X_k$ be the longest chain in $\overset{\circ}{V}$.

Figure 3.9 shows an abstract pre-stratified space with boundary. The interior consists of a 1-dimensional stratum X, a 2-dimensional stratum decomposed in two connected components, and a 3-dimensional stratum “filling the box”. The boundary of V has ∂X as its smallest stratum. The “tunnel” indicates a δ-neighbourhood around X. Observe that the neighbourhoods in V and ∂V can be chosen as compatible.

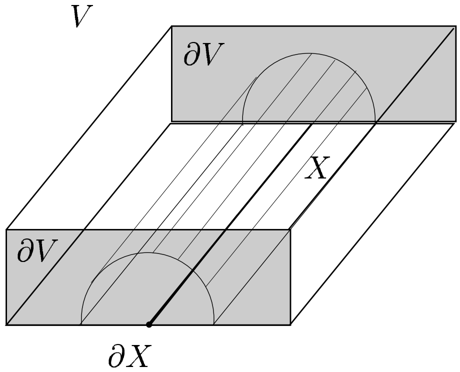


Figure 3.9: Abstract pre-stratified space with boundary.

Step 1: Preparation of data.

Let $\mathbf{c} : (\partial V \times [0,\infty))^{<\lambda} \hookrightarrow V$ be a representative of the germ of collars, and let ∂X be the stratum of ∂V such that $\mathbf{c}(N_{\partial X} \times (0,\infty))^{<\lambda} \subset N_X$. Observe that since X is a "deepest" stratum of X, it is closed in $\mathring{V}$. Similarly the stratum ∂X is closed in ∂V. This brings us to the situation of the second example of the last section. We define

$$\overline{X} := (\partial X \times [0,\infty))^{<\lambda} \cup_{\mathbf{c}|_{(\partial X \times (0,\infty))^{<\lambda}}} X$$

and conclude that $(\overline{X}, \partial X)$ is a c-manifold with a representative of the neighbourhood germ $\mathbf{c}^{\overline{X}} := \mathbf{c}|_{(\partial X \times [0,\infty))^{<\lambda}} : (\partial X \times [0,\infty))^{<\lambda} \hookrightarrow \overline{X}$. In addition, for the map $\delta' := \delta_X : \overline{X} \longrightarrow (0,\infty)$ the functions

$$\begin{array}{llll} (\pi_X, \rho_X): & N_X - X & \longrightarrow & (X \times (0,\infty))^{<\delta'} \quad \text{and} \\ (\pi_{\partial X}, \rho_{\partial X}): & N_{\partial X} - \partial X & \longrightarrow & (\partial X \times (0,\infty))^{<\delta'} \end{array}$$

are proper surjective submersions. Let $((J,\alpha),(J^\partial,\alpha^\partial))$ be the flow from the example of the last section, corresponding to the vector field $(0, \frac{d}{dt})$.

We denote with F_X the map $(\pi_X, \rho_X) : N_X \longrightarrow X \times [0,\infty)$ and let $F_{\partial X}$ be the map $(\pi_{\partial X}, \rho_{\partial X}) : N_{\partial X} \longrightarrow \partial X \times [0,\infty)$. We also define

$$\begin{array}{lll} \overline{N}_X & := & (N_{\partial X} \times [0,\infty))^{<\lambda} \cup_{\mathbf{c}|_{(N_{\partial X} \times (0,\infty))^{<\lambda}}} N_X \quad \text{and} \\ \partial \overline{N}_X & := & N_{\partial X} \end{array}$$

and obtain an abstract pre-stratified space $(\overline{N}_X, \partial \overline{N}_X)$ with boundary with a representative of the germ of collars $\mathbf{c}^{\overline{N}_X} := \mathbf{c}|_{(N_{\partial X} \times [0,\infty))^{<\lambda}} : (N_{\partial X} \times [0,\infty))^{<\lambda} \hookrightarrow \overline{N}_X$. The compatibility of π_X with $\pi_{\partial X}$ as well as ρ_X with $\rho_{\partial X}$

implies that $\overline{F}_X := F_{\partial X} \sqcup F_X$ becomes a smooth submersion from $\overline{N}_X - \overline{X}$ onto $(\overline{X} \times (0,\infty))^{<\delta'}$.

Step 2: Construction of Φ on some subspaces of V.

After the preparations we set $\delta := 1/4\delta'$ and define

$$\begin{array}{rcl} B & := & F_X^{-1}(X \times [0,\infty))^{\leq\delta} \\ \partial B & := & F_X^{-1}(X \times (0,\infty))^{\delta} \end{array}$$

The space $B - \partial B$ is an open subset of V and, therefore, an abstract pre-stratified space. The space ∂B is also an element of **APS** according to Lemma 3.3.5. We make the pair $(B, \partial B)$ into an abstract pre-stratified space with boundary by specifying the collar

$$\begin{array}{rrcl} \mathbf{d}^B : & (\partial B \times [0,\infty))^{<\delta\pi_X} & \longrightarrow & B - X \\ & (x,t) & \longmapsto & \alpha_x(-t). \end{array}$$

The properties of the flow guarantee the requirements on $\mathbf{d}^B$. We also define the corresponding sets on the boundary of V.

$$\begin{array}{rcl} \tilde{B} & := & F_{\partial X}^{-1}(\partial X \times [0,\infty))^{\leq\delta} \\ \partial\tilde{B} & := & F_{\partial X}^{-1}(\partial X \times (0,\infty))^{\delta} \end{array}$$

The collar is given by:

$$\begin{array}{rrcl} \mathbf{d}^{\tilde{B}} : & (\partial\tilde{B} \times [0,\infty))^{\delta\pi_{\partial X}} & \longrightarrow & \tilde{B} - \partial X \\ & (x,t) & \longmapsto & \alpha_x^{\partial}(-t). \end{array}$$

We are going to construct the map Φ on $(B, \partial B)$, as well as on $(\tilde{B}, \partial\tilde{B})$, where one takes induced neighbourhoods, and the corresponding maps δ_Y are given by restriction of the given ones. To make the construction compatible, note that the restriction of the collar of V to $\partial\tilde{B}$ gives us a map

$$\begin{array}{rrcl} \mathbf{c}^{\partial B} : & (\partial\tilde{B} \times (0,\infty))^{<\lambda} & \longrightarrow & \partial B \\ & (x,t) & \longmapsto & \mathbf{c}(x,t) \end{array}$$

leading to the abstract pre-stratified space with boundary $(\overline{\partial B}, \partial\tilde{B})$, where

$$\overline{\partial B} := (\partial\tilde{B} \times [0,\infty))^{<\lambda} \cup_{\mathbf{c}^{\partial B}:(\partial\tilde{B}\times(0,\infty))^{\lambda}\to\partial B} \partial B.$$

Since we assumed that V has only one chain of length k, the abstract pre-stratified space $\overline{\partial B}$ has a strictly smaller depth than V. Using the inductive assumption, we may therefore assume that there is a cornered p-stratifold $\mathscr{S}_{\overline{\partial B}}$ with boundary and an isomorphism of stratified spaces with boundaries $\varphi_{\overline{\partial B}} : \mathscr{S}_{\overline{\partial B}} \longrightarrow \overline{\partial B}$ fulfilling the conditions of the theorem. Set $\mathscr{S}_{\partial B} := \mathscr{S}_{\overline{\partial B}} - \partial\mathscr{S}_{\overline{\partial B}}$ and $\mathscr{S}_{\partial\tilde{B}} := \partial\mathscr{S}_{\overline{\partial B}}$. Let $\varphi|_{\partial B}$ denote the restriction of $\varphi_{\overline{\partial B}}$ on the interior $\varphi|_{\mathscr{S}_{\partial B}} : \mathscr{S}_{\partial B} \longrightarrow \partial B$, and respectively $\varphi_{\partial\tilde{B}} := \varphi|_{\mathscr{S}_{\partial\tilde{B}}} : \mathscr{S}_{\partial\tilde{B}} \longrightarrow \partial\tilde{B}$.

We have a controlled map $\pi_X \sqcup \pi_{\partial X} : \overline{\partial B} \longrightarrow \overline{X}$, thus, according to the inductive assumption, the map $(\pi_X \sqcup \pi_{\partial X})\varphi_{\overline{\partial B}}$ is a smooth map from $\mathscr{S}_{\overline{\partial B}}$ to $\overline{X}$. Set $g_{\partial B} := \pi_X\varphi_{\partial B}\mathrm{pr}_1 : (\mathscr{S}_{\partial B} \times [0,\infty))^{\delta\pi_X\varphi_{\partial B}} \longrightarrow X$ and $g_{\partial\tilde{B}} := \pi_{\partial X}\varphi_{\partial\tilde{B}}\mathrm{pr}_1 : (\mathscr{S}_{\partial\tilde{B}} \times [0,\infty))^{\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}} \longrightarrow \partial X$.

We again make use of our flow α to construct the following map.

$$\begin{array}{rrccc}
\varphi_B : & (\mathscr{S}_{\partial B} \times [0,\infty))^{\leq\delta\pi_X\varphi_{\partial B}} \cup_{g_{\partial B}} X & & \stackrel{\alpha\cup id}{\longrightarrow} & B \\
& (\mathscr{S}_{\partial B} \times [0,\infty))^{\leq\delta\pi_X\varphi_{\partial B}} \ni & (x,t) & \longmapsto & \alpha_{\varphi_{\partial B}(x)}(-t) \\
& X \ni & y & \longmapsto & y
\end{array}$$

For $(x,t) \in \mathscr{S}_{\partial B} \times [0,\infty))^{\delta\pi_X\varphi_{\partial B}}$ we have $\varphi_{\partial B}(x) \in \partial B$ and $t = \delta(\pi_X(\varphi(x)))$, therefore, $\rho_X(\varphi_{\partial B}(x)) = \delta(\pi_X(\varphi_{\partial B}(x)))$. Hence

$$\alpha_{\varphi_{\partial\tilde{B}}(x)}(-t) = \alpha_{\varphi_{\partial\tilde{B}}(x)}(-\rho_X\varphi_{\partial B}(x)) = \pi_X(\varphi_{\partial\tilde{B}}(x)) = g_{\partial B}(x,t),$$

which implies that $\varphi_{\partial B}$ is well-defined.

Analogously using α^∂, we obtain a well defined map $\varphi_{\tilde{B}}$, defined as follows.

$$\begin{array}{rrccc}
\varphi_{\tilde{B}} : & (\mathscr{S}_{\partial\tilde{B}} \times [0,\infty))^{\leq\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}} \cup_{g_{\partial\tilde{B}}} \partial X & & \stackrel{\alpha^\partial\cup id}{\longrightarrow} & \tilde{B} \\
& (\mathscr{S}_{\partial\tilde{B}} \times [0,\infty))^{\leq\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}} \ni & (x,t) & \longmapsto & \alpha^\partial_{\varphi_{\partial\tilde{B}}(x)}(-t) \\
& \partial X \ni & y & \longmapsto & y
\end{array}$$

According to Proposition 3.4.1, the spaces

$$\begin{array}{rcl}
\mathscr{S}_B & := & (\mathscr{S}_{\partial B} \times [0,\infty))^{\leq\delta\pi_X\varphi_{\partial B}} \cup_{g_{\partial B}} X \ \text{ and} \\
\mathscr{S}_{\tilde{B}} & := & (\mathscr{S}_{\partial\tilde{B}} \times [0,\infty))^{\leq\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}} \cup_{g_{\partial\tilde{B}}} \partial X
\end{array}$$

are cornered p-stratifolds, thus we define $\Phi(B) := (\mathscr{S}_B, \varphi_B)$ and $\Phi(\tilde{B}) := (\mathscr{S}_{\tilde{B}}, \varphi_{\tilde{B}})$ using the corresponding neighbourhoods and δ's.

Step 3: Verifying the properties of Φ.

We verify the requirements on $(\mathscr{S}_B, \varphi_B)$. The properties of the flow, along with the definition of the collars of $\mathscr{S}_B$ and φ_B, imply that $\pi_B : \mathscr{S}_B \longrightarrow B$ is an isomorphism of stratified spaces with boundary. It remains to show the compatibility with neighbourhoods and retractions. The strata of $\mathring{\mathscr{S}}_B$ are given by $\mathring{\mathscr{S}}_B^{k+1} := (\mathscr{S}_{\partial B}^k \times (0,\infty))^{<\delta\pi_X\varphi_{\partial B}}$ and X. The neighbourhoods around $\mathring{\mathscr{S}}_B^{k+1}$ are

$$U(\mathring{\mathscr{S}}_B^{k+1}) = (U(\mathscr{S}_{\partial B}^k) \times (0,\infty))^{\delta\pi_X\varphi_{\partial B}\pi_{\partial B}^k} \cap \mathring{\mathscr{S}}_B$$

and the retractions π_{k+1}^B are given by

$$\begin{array}{rccc} \pi_{k+1}^B: & U(\mathring{\mathscr{S}}_B^{k+1}) & \longrightarrow & \mathring{\mathscr{S}}_B^{k+1} \\ & (u,t) & \longmapsto & (\pi_k^{\partial B}(u), t) \end{array}$$

According to the inductive assumption, after passing to a possibly smaller $U(\mathscr{S}_{\partial B}^k)$, we can assume that there exists a representative $\mathcal{N}_Y^{\partial B}$ of the tubular neighbourhood of $Y := \varphi_{\partial B}(\mathscr{S}_{\partial B}^k)$, making the following diagram commutative.

$$\begin{tikzcd} U(\mathscr{S}_{\partial B}^k) \arrow[r, "\varphi_{\partial B}"] \arrow[d, "\pi_k^{\partial B}"'] & N_Y^{\partial B} \arrow[d, "\pi_Y^{\partial B}"] \\ \mathscr{S}_{\partial B}^k \arrow[r, "\varphi_{\partial B}"] & Y \end{tikzcd}$$

Let Z be a stratum of $\mathring{B}$ such that $\varphi_B(\mathring{\mathscr{S}}_B^{k+1}) = Z$, then $Z = \mathbf{d}^B(Y \times (0,\infty))^{<\delta\pi_X}$. Thus, we have a commutative diagram

$$\begin{tikzcd} (U(\mathscr{S}_{\partial B}^k) \times (0,\infty))^{\delta\pi_X\varphi_{\partial B}\pi_k^{\partial B}} \arrow[r, "\pi_{\partial B}\times\mathrm{id}"] \arrow[d, "\pi_{k+1}^B"'] & (N_Y^{\partial B} \times (0,\infty))^{\delta\pi_X} \arrow[r, "\mathbf{c}^B"] \arrow[d, "\pi_Y^{\partial B}\times\mathrm{id}"] & N_Z^{\mathring{B}} \arrow[d, "\pi_Z^{\mathring{B}}"] \\ \mathring{\mathscr{S}}_B^{k+1} \arrow[r, "(\mathbf{d})^{-1}(\varphi_{\partial B}\times\mathrm{id})"] & (Y \times (0,\infty))^{\delta\pi_X} \arrow[r, "\mathbf{c}^B"] & Z \end{tikzcd}$$

where the first square commutes according to the definition of π_{k+1}^B and the second square, since $\mathbf{c}$ is an isomorphism of abstract pre-stratified spaces. Furthermore, one notes that the composition $\mathbf{c}^B\pi_{\partial B} \times \mathrm{id}$ is nothing but our map φ_B, thus, we have shown the compatibility around $\mathscr{S}_{\partial B}^k$. We still have to consider the stratum X. According to the construction of $\mathscr{S}_B$, one neighbourhood of X is given by $(\mathscr{S}_{\partial B} \times (0,\infty))^{\leq\delta\pi_X\varphi_{\partial B}} \cup X$ and the retraction is $\pi := \pi_X\varphi_{\partial B}\mathrm{pr}_1 \cup \mathrm{id}$, thus, we immediately obtain the desired commutative diagram:

$$\begin{tikzcd} (\mathscr{S}_{\partial B} \times (0,\infty))^{\leq\delta\pi_X\varphi_{\partial B}} \cup X \arrow[r, "\varphi_B"] \arrow[d, "\pi"'] & \mathring{B} \arrow[d, "\pi_X"] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd}$$

Thus, the second condition of the theorem is fulfilled. In summary, we have shown that $(\mathscr{S}_B, \varphi_B)$ fulfils the requirements of the theorem. The arguments in the case of $(\mathscr{S}_{\tilde{B}}, \varphi_{\tilde{B}})$ are exactly the same.

Step 4: Finishing the construction.

Finally, we consider the remaining part of V. Let

$$\begin{array}{rcl} C & := & V - \overline{F}_X^{-1}((\overline{X} \times [0,\infty))^{\leq \delta}) \ \text{ and} \\ \partial C & := & \partial V - \overline{F}_{\partial X}^{-1}((\partial X \times [0,\infty))^{\leq \delta}). \end{array}$$

The pair $(C, \partial C)$ is an abstract pre-stratified space with boundary, where the collar is $\mathbf{c}^C := \mathbf{c}|_{(\partial C \times [0,\infty))^{<\lambda}}$. According to the inductive assumption, there is a stratifold $\mathscr{S}_C$ with cornered strats and an isomorphism of stratified spaces with boundaries $\varphi_C : \mathscr{S}_C \longrightarrow C$. Consider further

$$\begin{array}{rcl} D & := & \mathring{V} - F_X^{-1}((X \times [0,\infty))^{<\delta}) \ \text{ and} \\ \partial D & := & F_X^{-1}((X \times [0,\infty))^{\delta}). \end{array}$$

The pair $(D, \partial D)$ is again an abstract pre-stratified space with boundary, where the collar is given by

$$\begin{array}{rrcl} \mathbf{c}^D : & (\partial D \times [0,\infty))^{\delta \pi_X} & \longrightarrow & D \\ & (x,t) & \longmapsto & \alpha_x(t). \end{array}$$

Thus, there exists a cornered p-stratifold $\mathscr{S}_D$ with boundary and an isomorphism $\varphi_D : \mathscr{S}_D \longrightarrow D$ of stratified spaces with boundary. Observe further that $\mathring{D} = \mathring{C}$, hence $\mathring{\mathscr{S}}_D = \mathring{\mathscr{S}}_C$ and $\varphi_{\mathring{\mathscr{S}}_D} = \varphi_{\mathring{\mathscr{S}}_C}$.

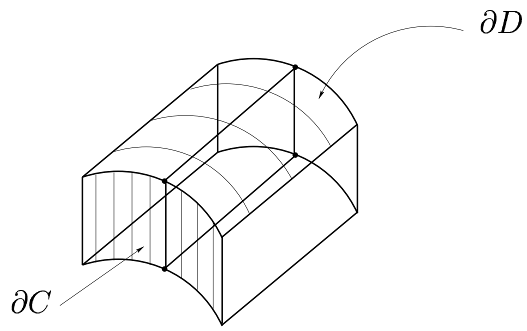


Figure 3.10: Stratified space "with two boundaries".

Figure 3.10 demonstrates the situation. The interior of the "box" is $\mathring{D} = \mathring{C}$, where ∂C is the front and the back part, whereas ∂D consists of the bottom and the top parts of the "box".

We further define

$$\begin{array}{rcl}\tilde{D} & := & \partial V - F_{\partial X}^{-1}((\partial X \times [0,\infty))^{<\delta}) \text{ and} \\ \partial\tilde{D} & := & F_{\partial X}^{-1}((\partial X \times [0,\infty))^{\delta}),\end{array}$$

where the collar of the pair $(\tilde{D}, \partial\tilde{D})$ is given by

$$\begin{array}{rccc}\mathbf{c}^{\tilde{D}}: & (\partial\tilde{D} \times [0,\infty))^{\delta\pi_{\partial X}} & \longrightarrow & \tilde{D} \\ & (x,t) & \longmapsto & \alpha_x^{\partial}(t).\end{array}$$

Let $\Phi(\tilde{D}) = (\mathscr{S}_{\tilde{D}}, \varphi_{\tilde{D}})$, now glue $\mathscr{S}_D$ with $\mathscr{S}_B$ together to form a stratifold $\mathring{\mathscr{S}}$ and let $\partial\mathscr{S} := \mathscr{S}_{\tilde{D}} \cup_\partial \mathscr{S}_{\tilde{B}}$ be the corresponding stratifold on the boundary. First of all, note that we have a map $\varphi_{\partial\mathscr{S}} := \varphi_{\tilde{D}} \cup \varphi_{\tilde{B}} : \partial\mathscr{S} \longrightarrow \partial V$, which, by the construction an isomorphism of stratified spaces, satisfies the properties of the theorem, and a map $\varphi_{\mathring{\mathscr{S}}} := \varphi_D \cup \varphi_B : \mathring{\mathscr{S}} \longrightarrow \mathring{V}$, which is also an isomorphism of stratified spaces.

We now specify a collar of $\partial\mathscr{S}$ in $\mathring{\mathscr{S}}$, the map $\mathbf{e} : (\partial\mathscr{S} \times (0,\infty))^{<\lambda\varphi_{\partial\mathscr{S}}} \longrightarrow \mathring{\mathscr{S}}$, by the assignment

$$(x,t) \mapsto \begin{cases} \mathbf{e}^C(x,t) & \text{for } x \in \partial\mathscr{S}_C \\ \mathbf{e}^{\tilde{B}}(x,t) & \text{for } x \in \partial\mathscr{S}_{\tilde{B}} \\ (\mathbf{e}^{\tilde{B}}(y,t), s) & \text{for } x = (y,s) \in (\partial\mathscr{S}_{\tilde{B}} \times (0,\infty))^{<\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}} \\ \mathbf{c}^{\partial X}(x,t) & \text{for } x \in \partial X \end{cases}$$

where $\mathbf{e}^C$ denotes the collar of $\mathscr{S}_C$ and $\mathbf{e}^{\tilde{B}}$ denotes the collar of $\mathscr{S}_{\overline{\partial B}}$. The construction implies that the map satisfies the properties of the collar.

To see that $\mathbf{e}$ is compatible with $\varphi_{\partial\mathscr{S}}$ and $\varphi_{\mathring{\mathscr{S}}}$, we only have to show the compatibility on $\mathscr{S}_B$. For $x = (y,s) \in (\mathscr{S}_{\tilde{B}} \times (0,\infty))^{<\delta\pi_{\partial X}\varphi_{\partial\tilde{B}}}$ we compute

$$\begin{array}{lclcll}\varphi(\mathbf{e}(x,t)) & = & \varphi_B(\mathbf{e}^{\tilde{B}}(y,t), s) & = & \alpha_{\varphi_{\tilde{B}}(\mathbf{e}^{\tilde{B}}(y,t))}(s) & = \\ \alpha_{\mathbf{c}^{\tilde{B}}(\varphi_{\partial\tilde{B}}(y),t)}(s) & = & \mathbf{c}^{\tilde{B}}(\alpha^{\partial}_{\varphi_{\partial\tilde{B}}(y)}(s), t) & = & \mathbf{c}(\varphi(x), t). & \end{array}$$

Thus, we set $\mathscr{S} := \mathring{\mathscr{S}} \cup \partial\mathscr{S}$ using the collar constructed above and define $\varphi := \varphi_{\mathring{\mathscr{S}}} \cup \varphi_{\partial\mathscr{S}}$. The discussion above implies that $\varphi : \mathscr{S} \longrightarrow V$ is an isomorphism of stratified spaces with boundaries having the desired properties.

□

Now we finish the proof of the theorem.

Proof of Theorem 3.4.2. Let $(V, \partial V)$ be a finite dimensional abstract pre-stratified space with boundary. Choose a system of mappings δ_X and the representatives of the neighbourhoods structure N_X and $N_{\partial X}$ for every stratum X of $\mathring{V}$ satisfying the usual conditions. Using Lemma 3.4.3 we obtain a cornered p-stratifold $\mathscr{S}$ and an isomorphism of stratified spaces $\varphi : \mathscr{S} \longrightarrow V$. Let $[\{f_i : W^i \longrightarrow \mathscr{S}\}]$ be the parametrization of $\mathscr{S}$, then the composition $[\{\varphi f_i : W^i \longrightarrow V\}]$ equips V with a parametrization. According to the second property of φ, the parametrization makes $((\mathring{V}, \mathcal{C}_{\mathbf{APS}}(\mathring{V})), (\partial V, \mathcal{C}_{\mathbf{APS}}(\partial V)))$ into a cornered p-stratifold with boundary.

Since the tubular maps π_X of $\mathring{V}$ as well as $\pi_{\partial X}$ are submersions, restricted to small enough neighbourhoods, we immediately see that the resulting stratifold is locally trivial.

□

Let us make some remarks concerning the behaviour of the map Φ. Let V and V' be two abstract pre-stratified spaces with $\Phi(V) = \Phi(V') = (\mathscr{S}, \varphi)$. The second condition on Φ implies the compatibility of π_X and π'_X. The maps ρ_X are also reconstructible. To see this, observe that given a stratifold $\mathscr{S}$ together with distance mappings $\rho_i : U(\mathscr{S}^i) \longrightarrow [0, \infty)$, such that $(\rho_i)|_{\mathscr{S}^j}$ is smooth for all j and $(\rho_i)^{-1}(0) = \mathscr{S}^i$, the stratifold $\tilde{\mathscr{S}} := (\mathscr{S} \times [0,\infty))^{\leq \delta g} \cup_{g\mathrm{pr}_1 : (\mathscr{S} \times [0,\infty))^{\delta g} \to X} X$ from Proposition 3.4.1 again has this property. The strata of $\tilde{\mathscr{S}}$ are given by $(\mathscr{S}_i \times (0,\infty))^{<\delta g}$ and X. A representative of the neighbourhood germ of $(\mathscr{S}_i \times (0,\infty))^{<\delta g}$ is given by $(U(\mathscr{S}_i) \times (0,\infty))^{<\delta g}$ and we define $\tilde{\rho}_i := \rho_i \mathrm{pr}_1$. The neighbourhood around X is just $(\mathscr{S} \times (0,\infty))^{\leq \delta g} \cup_{g\mathrm{pr}_1 : (\mathscr{S} \times [0,\infty))^{\delta g} \to X} X$ and we set:

$$\tilde{\rho}_X(z) := \begin{cases} \delta g(x) - t & \text{for } z = (x,t) \in (\mathscr{S} \times (0,\infty))^{\leq \delta g} \\ 0 & \text{for } z \in X \end{cases}$$

In the special case of our construction, the map g is given by $\pi_X \varphi$, where $\varphi : \mathscr{S} \longrightarrow A$ is an isomorphism of stratified spaces, where the space A is a subset of an abstract pre-stratified space V with $A = \{x \in V | \rho_X(x) = \delta(\pi_X(x))\}$. We conclude that $\tilde{\rho}_X(x,t) = \delta \pi_X(\varphi(x)) - t = \rho_X(\varphi(x)) - t$.

In general, it is not clear how to construct such a distance map on a cornered p-stratifold, hence we do not see that every locally trivial cornered p-stratifolds comes form an abstract pre-stratified space.

Remark: Straightening the corners in every step of the construction would assign a p-stratifold (without corners) to every abstract pre-stratified space and a map δ, but, as already mentioned in §1.3, the construction would in general depend on δ and would not be canonical. Furthermore, the algebra of maps obtained from the parametrization will in general differ from the one of controlled maps. Using Proposition 1.5.8 we nevertheless obtain:

Corollary 3.4.4. *Let V be an abstract pre-stratified space. There exists a p-stratifold $(\mathscr{S}, \mathcal{C}(\mathscr{S}), [\{f_i\}])$ such that V is isomorphic to $\mathscr{S}$ as stratified space.*

Theorem 3.4.2 allows us to embed various stratified spaces into the category of stratifolds with (cornered) strat structure and to give answers on questions addressed at the beginning of the chapter.

The first conclusion is a direct consequence of Theorem 3.3.1.

Corollary 3.4.5. *Every Whitney stratified space admits a structure as a cornered p-stratifold.*

Using Corollary 3.4.4, we see that the last observation extends the result of Thom [Th2, Thm. 2.B.1] to non-compact Whitney stratified spaces.

One should be careful with the notion of smooth maps. If $X \subset \mathbb{R}^N$ is a Whitney stratified space, a continuous map $f : X \longrightarrow \mathbb{R}$ is called smooth if there is a smooth map $F : \mathbb{R}^N \longrightarrow \mathbb{R}$, such that $f \equiv F|_X$. Let $V(X) = (X, \mathcal{S}, \mathcal{N})$ be an abstract pre-stratified space and $\Phi(V(X)) = (\mathscr{S}, \varphi)$. Then $f\varphi$ is in general not smooth on $\mathscr{S}$. Vice versa for a smooth map $g : \mathscr{S} \longrightarrow \mathbb{R}$ the composition $g\varphi^{-1}$ does not generally extend to a smooth map on $\mathbb{R}^N$. For examples, see [GWPL, Ex. 2.1].

Since any semi-algebraic set $V \in \mathbb{R}^n$ admits a canonical Whitney stratification $\mathcal{X}$ having finitely many semi-algebraic strata [GWPL, Theorem (I.2.7)], we conclude using Theorem 3.3.1:

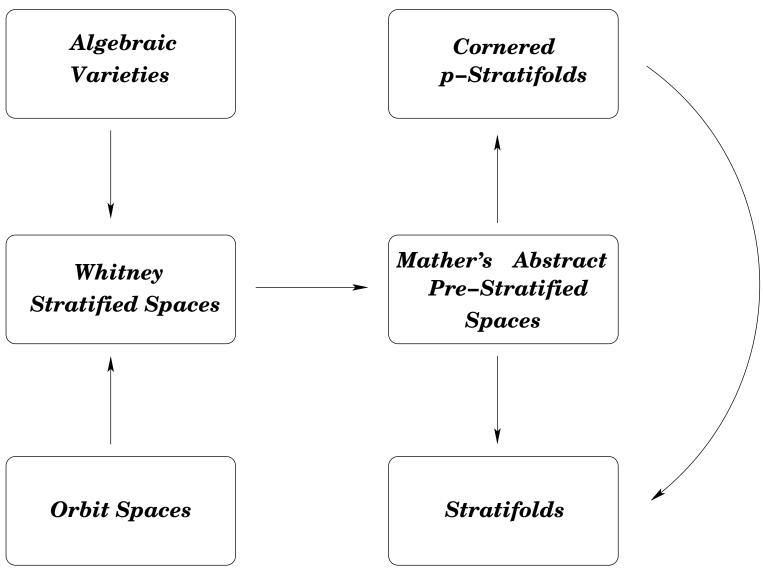
Corollary 3.4.6. *Any semi-algebraic set V admits a structure as a cornered p-stratifold.*

Note that the stratification $\mathcal{X}$ of V is not the one given by regular respectively singular points. We already saw in §3.2, in the example of the Whitney umbrella that we had to refine this first stratification. However, Whitney showed that the set of points where the Whitney conditions are not satisfied is again semi-algebraic, so the refinement is canonical.

The orbit space M/G of a smooth group action of a compact Lie group G on a smooth manifold M is an abstract pre-stratified space (even Whitney stratified), as carried out in [Fe], hence we obtain another large class of objects mapping into the category of cornered p-stratifolds. The strata are obtained by taking the connected components of the subsets of points whose isotropy groups are conjugated.

Corollary 3.4.7. *The orbit space M/G of a smooth group action of a compact Lie group G on a smooth manifold M admits a structure as a cornered p-stratifold.*

In conclusion, we provide a diagram, which shows the connection of various classes of spaces to the stratifolds.



Bibliography

[AK] S. Akbulut, H. King, *Topology of Real Algebraic Sets.* L'Enseignement Mathématique, 29 (1983), 221-261.

[Ba] H. H. Baues, *Obstruction Theory, on Homotopy Classification of Maps.* Springer Lectures Notes 628, 1977.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry.* Springer-Verlag, 1998.

[BJ] Th. Bröcker and K. Jänich, *Einführung in die Differentialtopologie.* Springer-Verlag, 1973.

[BR] R. Benedetti and Jean-Jacques Risler, *Real Algebraic and Semi-Algebraic Sets.* Hermann, Éditeurs des science et des arts, 1990.

[BT] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups.* Springer-Verlag, 1985.

[Fe] M. Ferrarotti, *G-Manifolds and Stratifications.* Rendiconti dell'Istituto di Matematica dell'Universita di Trieste, Vol XXVI (1994), Edizione Elettronica.

[Fr] M. H. Freedman, *The Topology of Four-Dimensional Manifolds.* J. Differ. Geom. 17 (1982), 357-453.

[GM] M. Goresky and R. MacPherson, *Stratified Morse Theory.* Springer-Verlag, 1988.

[Go] M. Goresky, *Whitney Chains and Cochains.* Trans. Am. Math. Soc. Vol 267 (1981), 175-196.

[Gr] A. Grinberg, *Auflösungen von differentialtopologischen Varietäten mit isolierten Singularitäten.* Diplomarbeit Universität Mainz, 1999.

[GWPL] C. G. Gibson, K. Wirthmüller, A. A. du Plessis E. J. N. Looijenga *Topological Stability of Smooth Mappings.* Lecture Notes in Math. 552, Springer-Verlag, 1976.

[H] M. W. Hirsch, *Differential Topology.* Springer-Verlag, 1976.

[Hae] A. Haefliger, *Differentiable Imbeddings.* Bull. Amer. Soc. 67 (1961), 109-112.

[Hi] H. Hironaka, *Resolutions of Singularities of an Algebraic Variety over a Field of Characteristic Zero: I,II.* Ann. of Math. 79 (1964), 109 - 326.

[Ke] M. A. Kervaire, *Some Non-Stable Homotopy Groups of Lie Groups.* Illinois J. Math. Vol 4 (1960), 161-169.

[Kr1] M. Kreck, *Surgery and Duality.* Annals of Mathematics Vol. 149 (1999), 704-754.

[Kr2] M. Kreck, *Differential Algebraic Topology.* Preprint.

[Kr3] M. Kreck, *Stratifolds.* Preprint.

[Ma] J. Mather, *Notes on Topological Stability.* Harvard University Press, 1970.

[Mi1] J. Milnor, *A Procedure for Killing the Homotopy Groups of Differentiable Manifolds.* Symposia in Pure Math. A.M.S., Vol. III (1961), 39-55.

[Mi2] J. Milnor, *Singular Points of Complex Hypersurfaces.* Princeton University Press, 1968.

[Mi3] J. Milnor, *Remarks Concerning Spin Manifolds.* Differ. and Combinat. Topology, Sympos. Marston Morse, Princeton, 1965, 55-62.

[Sch] H. Schubert, *Topologie.* B.G. Teubner, 1969.

[Si] R. Sikorski, *Differential Modules.* Colloq. Math. 24 (1971), 45-79.

[St] R. E. Stong, *Notes on Cobordism Theory.* Mathematical Notes, Princeton University Press, 1968.

[Th1] R. Thom, *Quelques Properiétés Globales des Varietétés Differentiables.* Comm. Mathe. Helv. Vol 28, 1954, 17-86.

[Th2] R. Thom, *Ensembles et Morphismes Stratifiés*. Bull. Amer. Math. Soc. 75 (1969), 240-284.

[W1] C. T. C. Wall, *Classification of* $(n-1)$*-Connected* $2n$*-Manifolds*. Annals of Mathematics Vol. 75, 1962, 163-189.

[Wh1] H. Whitney, *Local Properties of Ananlytic Varieties*. Differ. and Combinat. Topology, Princeton University Press, 1965, 205-244.

[Wh2] H. Whitney, *Tangents to an Analytic Variety*. Ann. of Math. 81 (1965), 496-549.