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Oscillations of a Fluid in a Channel

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Contents

Introduction

In this work we investigate the motion of a viscous, incompressible fluid contained in an uncovered three-dimensional rectangular channel. The upper surface changes with the motion of the fluid, so we deal with a free boundary problem. We consider small perturbations of a uniform flow with a flat free surface. We include the effect of the surface tension; the external forces are gravity, and the wind force which acts on the free boundary (in the Section 2.3).

The motion of the fluid in the channel is governed by the Navier-Stokes equations. The variables are, as usual, the velocity and the pressure of the fluid in the interior of the domain and a function parameterizing the free boundary. The pressure can be expressed in terms of the other two variables, which are coupled as follows: the fluid velocity at the free boundary prescribes the speed of the boundary, and the mean curvature of the free surface creates a pressure jump via the surface tension.

We consider the system to be periodic in the direction of the length of the channel. Technically, we identify the inflow boundary with the outflow boundary of the channel and then we consider the second spatial variable belonging to the circle S^1. In order to obtain a well-posed model, we have to prescribe the value of the dynamic contact angle between the walls and the free boundary (see [Schw2], [Re]) and we choose it to be $\frac{\pi}{2}$. As boundary conditions, we consider that the walls are impenetrable together with a perfect slip condition, and a no slip condition for the bottom.

The main aim of this paper is to analyse the qualitative behavior of the flow (oscillations of periodic solutions) using tools of bifurcation theory. In order to do this we need fundamental facts of existence and regularity of solutions, spectral analysis of the linear system connected with the free boundary value problem taking into account the underlying symmetries, and techniques of equivariant Hopf bifurcation theorem.

J. T. Beale studied the problem of the motion of a viscous incompressible fluid in a semi-infinite domain, bounded below by a solid floor and above by an atmosphere of constant pressure, either with ([Be1]) or without ([Be2]) surface tension. In [Be1] he used the Fourier transformation to prove resolvent estimates. These estimates combined with the Laplace transformation in time were used to prove the solvability of the time-dependent problem. He transformed the free boundary value problem to an initial boundary value problem on a fixed domain in a special way. This method is crucial in his existence proof and was also adapted and used by [Re], [Schw1], [Schw2]. We will apply it also in this paper.

B. Schweizer treated in [Schw1] the case of a liquid drop (with viscosity and surface tension) in a free space, so a full free boundary problem. With the help of semigroup methods, he studied linearized equations and get also existence results for the nonlinear problem. He computed the spectrum of the generator of the semigroup. Nonreal eigenvalues appeared for large values of the surface tension. An additional exterior linear force proportional to the normal velocity and acting on the free surface leaded to a Hopf bifurcation with $O(2)$-symmetry.

As soon as contact between a fixed boundary and a free boundary arises, the analytic investigations are getting more complicated. Already in case of a flow in a domain with non smooth fixed boundary, the regularity of the solutions is restricted (see e.g. [Dau]). The problem how to prescribe conditions for the contact is still in discussion. There exists a huge number of publications dealing with the solvability of free boundary problems with contact points and lines and therefore only some of the works and authors can be mentioned.

V. A. Solonnikov proved existence results for free boundary problems for the Navier-Stokes equations for both static or dynamic contact points and lines. He proved estimates for stationary problem for limiting values of contact angle 0 or π, in weighted Hölder spaces (see [Sol1], [Sol2] and the references presented there). For the solvability of stationary free boundary problems with a Navier type slip condition on the rigid walls see [Kr] and [Soc]. This condition can be applied in the case of a domain with rough boundaries by replacing the rough boundary with a smooth one where the Navier condition is fulfilled.

M. Renardy ([Re]) proved existence and uniqueness results for a two dimensional free surface flow problem with open boundaries. Both steady and initial value problems are investigated. He considered the case where velocity boundary conditions are prescribed on both the inflow and the outflow boundary. The smoothness of the solution is limited by the singularity at the corner between the free surface and the inflow (or outflow) boundary.

In [Schw2], B. Schweizer discussed conditions for the dynamic contact angle and well-posedness of the equations for a flow in a two dimensional domain. For the case of $\frac{\pi}{2}$ contact angle and slip boundary conditions he proved resolvent estimates which, using techniques developed in [Re], yielded an existence result for the nonlinear initial boundary value problem.

The studies of the oscillatory behavior of a fluid in a channel is continuing the research of B. Schweizer who analyzed the oscillation of a liquid drop [Schw1]. Due to the solid boundary in our problem, the techniques in this paper have to be changed due to difficulties arising from the additional boundary conditions. We are able to obtain results for the channel similar to those B. Schweizer obtained for the oscillating drop.

The present work is divided into two chapters. The first chapter treats the existence of solutions for the nonstationary linear and nonlinear problem. The boundary conditions chosen for the walls and the $\frac{\pi}{2}$ dynamic contact angle allow us to avoid the problems which might appear in dealing with the regularity of the solution, because the domain is not smooth. We can construct symmetric extensions of the solution through the walls obtaining functions in the extended domain, which will satisfy the same equations as the initial ones. The problem becomes equivalent to one of a fluid in a container with periodic lateral boundary conditions.

In order to study the spectral behavior of the linearized problem, we write the corresponding system in the form $\partial_t x + \mathcal{L}x = 0$, where x contains two of the variables: the velocity field and the position of the free boundary. The third unknown, the pressure, can be taken out from the Navier-Stokes equations as follows: using a harmonic extension operator (see equations (1.2.12) and (1.2.13)), we can express the pressure as a map depending on the velocity and the position of the free boundary. In an appropriate Hilbert space X^r (see Definition 1.2.1), $\mathcal{L}$ has a compact resolvent and its spectrum is contained in a sector of the complex plane (see Proposition 1.2.6 and Theorem 1.2.9). $\mathcal{L}$ and the nonlinearity in the full nonlinear system define maps from $\tilde{X}^{r+2}$ to X^r, but the operator $\mathcal{L}$ does not have the usual regularization property: the inverse does not map X^r to $\tilde{X}^{r+2}$ (see Remark 1.2.7(c)). We will use the fact that the right hand side of the nonlinear equation is always contained in a subspace of the form $(F, 0) \in X^r$ (see the equation (1.3.9)). Both, the optimal regularization property and a resolvent estimate, hold on such a subspace (see Theorem 1.2.11 and its consequence formulated in Theorem 1.2.13). Using the inverse of the Laplace transformation, the resolvent estimates gives us a unique solution of the time dependent linear problem (see Theorem 1.2.15).

In order to solve the nonlinear problem we follow the method presented in [Be1]: we transform the nonlinear problem defined on the unknown domain into one on the equilibrium domain (which has a flat surface on the top), by stretching or compressing on the vertical line segments (see Section 1.3). The nonlinearity has the optimal properties we have already mentioned. We treat it as the right hand side of the linear equation. Then, the implicit function theorem gives us, for small enough initial values, a solution of the time-dependent nonlinear problem (see Theorem 1.3.2).

The second chapter follows essentially the ideas presented in [Schw1] and contains the main result of this work, a Hopf bifurcation theorem with $\mathbb{Z}_k$-symmetry for this Navier-Stokes system (see Theorem 2.3.6). For general tools in bifurcation theory, especially for abstract results about the Hopf bifurcation, see e.g. [GSS], [Cr,Ra] and [Ma,Mc]. The group of symmetries in our model is determined by the shape of the domain and the boundary conditions, so our problem provides an $O(2)$-equivariance. Using the eigenfunctions of the Laplace operator in a rectangle, we can find an $\mathcal{L}$-invariant decomposition of the spaces $X^r = \oplus X^r_{n,k}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ (see Proposition 2.1.1 and Proposition 2.1.2). The isotropy subgroup of the position of the boundary function in $X^r_{n,k}$ is isomorphic to the cyclic group $\mathbb{Z}_k$ (see Proposition 2.1.3). We investigate the eigenvalues of $\mathcal{L}$ in such a space $X^r_{n,k}$ with $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ fixed.

In Section 2.2, we obtain a detailed picture of the position of the eigenvalues of $\mathcal{L}|_{\tilde{X}^r_{n,k}}$ depending on gravity and the surface tension which, together, we denoted by α (see Theorem 2.2.6 and Figure 2). For $\alpha = 0$ the spectrum consists of Stokes eigenvalues together with zero. With increasing α, the eigenvalues can become complex. For α greater than a certain α_0, the first two merge and leave the real axis. Between every two consecutive Stokes eigenvalues we can find at least one real eigenvalue of $\mathcal{L}$, and only one for $\alpha \to \infty$, which approaches the next lower Stokes eigenvalue. For $\alpha \to \infty$, the modulus of the nonreal eigenvalues is not bounded.

For a fixed $\alpha > \alpha_0$, a similar picture can be drawn if an additional exterior linear force of strength ξ acts on the free surface (see Theorem 2.3.3 and Figure 3). The operator $\mathcal{L}_\xi$ has similar invertibility properties and the solution satisfies similar resolvent estimates like in the case of $\mathcal{L}$ (see Proposition 2.3.4). The behavior of the eigenvalues of $\mathcal{L}_\xi$ depending on ξ presents two important differences compared with the behavior of the eigenvalues of $\mathcal{L}$ depending on α: eigenvalues with negative real part will appear and the modulus of nonreal eigenvalues is bounded independent of ξ (see Theorem 2.3.1 and Proposition 2.3.2). For $|\xi| \to +\infty$, all the eigenvalues of $\mathcal{L}_\xi$ are real and interspersed with the eigenvalues of the Stokes operator. Following the eigenvalues between $\xi \to -\infty$ and $\xi \to +\infty$, we prove the existence of a pair of nonreal eigenvalues for $\xi \in (\xi_1, \xi_2)$ which crosses the imaginary axis transversally for a value ξ^* of ξ (see Theorem 2.3.3). They are simple in every space $X^r_{n,k}$, up to the symmetry $\mathbb{Z}_k$. We formulate a generalized nonresonance condition (see Definition 2.3.5) and we assume that the pair of purely imaginary eigenvalues of $\mathcal{L}_{\xi^*}$ satisfies this generalized nonresonance condition. Then we can prove an equivariant version of the Hopf bifurcation, and thus the existence of a branch of $\mathbb{Z}_k$-symmetric and periodically oscillating solutions of the Navier-Stokes system (see Theorem 2.3.6).

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Chapter 1

The Existence Theory

1.1 Formulation of the Problem

We want to model the nonstationary motion of a viscous, incompressible fluid contained in an uncovered rectangular channel. The upper surface changes with the motion of the fluid, so we deal with a free boundary problem. The unknown functions are not only the velocity field u and the pressure $\bar{p}$, but also the domain Ω. The effect of the surface tension on the upper free boundary is included. The external forces are the gravity and the wind force which acts on the free boundary and in fact generates the motion of the flow.

We consider the channel of width b and length $l = 2\pi$ to be deep enough such that the fluid will never overflow it. We impose a periodicity condition in the direction of the length of the channel (for all unknown functions). We write the equations using the euclidian coordinates (x_1, x_2, x_3); the components of the velocity field are then denoted by (u_1, u_2, u_3). In describing the equations of motion we will assume that all variables are nondimensionalized in the usual way.

Let $(0, b) \times (0, 2\pi) \times (-h, +\infty)$, $b, h > 0$ be the channel and Ω the domain occupied by the fluid with the free boundary denoted by Γ and fixed boundary Σ composed from the walls Σ_1, Σ_2 and the bottom Σ_{-h}. Let C_1, C_2 be the intersection curves between the free boundary and the walls. The periodicity in x_2 is technically incorporated by considering the independent variable x_2 belonging to the circle S^1. So, we have identified (and actually eliminated as boundaries) the surfaces $(0, b) \times \{0\} \times (-h, +\infty)$ and $(0, b) \times \{2\pi\} \times (-h, +\infty)$. The channel $(0, b) \times S^1 \times (-h, +\infty)$ is now considered "without curvature in the x_2-direction", i.e. the equations will not be transformed (this is not a domain transformation, it is only an identification).

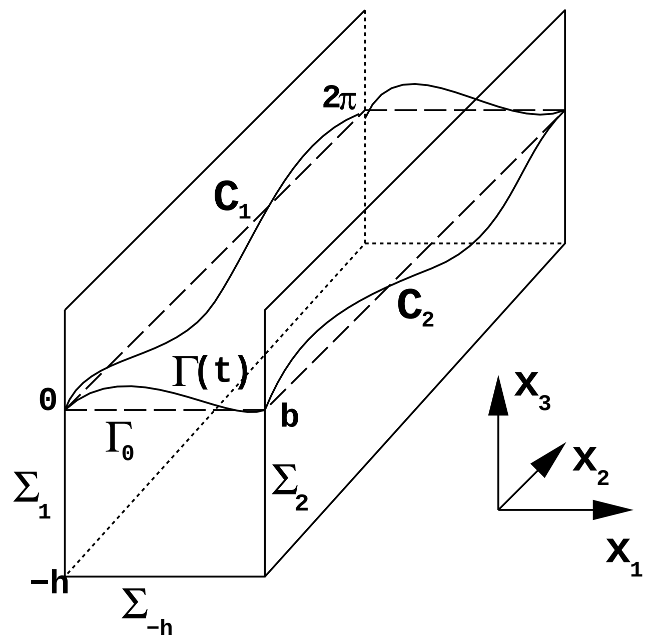

Figure 1:

We take the domain of the fluid at equilibrium to be

$$\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < b,\, x_2 \in S^1, -h < x_3 < 0\},$$

with the upper boundary Γ_0

$$\Gamma_0 = (0, b) \times S^1 \times \{0\},$$

and the fixed boundary composed from the walls $\Sigma_{1,0}, \Sigma_{2,0}$ and the bottom Σ_{-h}. The contact curves between the free boundary and the walls are denoted by $C_{1,0}, C_{2,0}$. Where no confusion can appear, we will omit the index 0 from the notation for the walls and contact lines of the equilibrium domain. When we want to refer to the walls together, we will denote them by $\Sigma_{1,2}$ and the same for contact lines $C_{1,2}$.

To describe the free surface of the fluid, we assume small perturbations of the equilibrium surface Γ_0 and parametrize the free boundary of the liquid with a function $\eta(t, \cdot) : \Gamma_0 \longrightarrow \mathbb{R}$. Thus the height of the free surface is a function of horizontal coordinates: $x_3 = \eta(t, x_1, x_2)$, $(x_1, x_2) \in \Gamma_0$ and the graph of η gives the shape of Γ. The domain occupied by the fluid is

$$\Omega = \Omega(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < b, x_2 \in S^1, -h < x_3 < \eta(t, x_1, x_2)\}.$$

The velocity field is a function $u(t, \cdot) : \Omega(t) \longrightarrow \mathbb{R}^3$.

As usual, we introduce the deformation tensor S_u with the components

$$(S_u)_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and the stress tensor σ with the components

$$\sigma_{ij} = -\bar{p}\delta_{ij} + 2\nu(S_u)_{ij}.$$

The motion of the fluid in the interior is governed by the Navier-Stokes equations for an incompressible fluid with viscosity ν:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla\bar{p} + g\nabla x_3 = 0 \tag{1.1.1}$$

$$\nabla \cdot u = 0 \tag{1.1.2}$$

where g is the acceleration of gravity. It is natural to substract the hydrostatic presure from $\bar{p}$, so we set

$$p := \bar{p} - P_0 + gx_3$$

where P_0 is the atmospheric pressure above the liquid. The density does not appear because of the nondimensionalization. After substitution, the gravity term in (1.1.1) is eliminated.

On the free surface we have the kinematic boundary condition which states that the fluid particles do not cross the free surface (which is equivalent with the geometric condition that η always parametrizes the free surface):

$$\partial_t\eta = u_3 - (\partial_1\eta)u_1 - (\partial_2\eta)u_2 \quad \text{on } \Gamma. \tag{1.1.3}$$

If we neglected the surface tension, the remaining boundary condition on Γ would be the continuity of the stress across the free surface, so $-\sum_{j=1}^{3}\sigma_{ij}n_j = P_0\, n_i + f_i n_i$ for $i = 1, 2, 3$, where $n = (n_1, n_2, n_3)$ is the outward normal to Γ and $f = (f_1, f_2, f_3)$ is the exterior force (for example the wind force). The effect of surface tension is to introduce a discontinuity in the normal stress, proportional to the mean curvature $H(\eta)$ of the free surface Γ. Our boundary condition on Γ is therefore (using $p := \bar{p} - P_0 + gx_3$ and $x_3 = \eta$ on Γ)

$$p\, n_i - \nu\sum_{j=1}^{3}(\partial_i u_j + \partial_j u_i)n_j = (g\eta + \beta H(\eta) + f_i)n_i \quad i = 1, 2, 3 \tag{1.1.4}$$

where $\beta > 0$ is the nondimensionalized coefficient of the surface tension and the mean curvature of the surface Γ is given by

$$H(\eta) = -\overline{\nabla} \cdot \frac{\overline{\nabla}\eta}{\sqrt{1 + |\overline{\nabla}\eta|^2}}. \tag{1.1.5}$$

We have denoted here by $\overline{\nabla}$ the gradient with respect to the first two variables x_1, x_2; then let $\underline{\Delta} := \overline{\nabla} \cdot \overline{\nabla}$.

If nothing else is specified, in the following, we denote by n the outward normal and by τ_i, $i = 1, 2$, the two tangential directions to the surface.

From a physical point of view, the usual boundary condition $u = 0$ on Σ can not be considered here because of the unknown contact between the free surface and the walls (we can not assume that it is not moving at all on the walls, so we can not "stick" the free surface on the fixed boundary); but it is natural to consider the no-slip condition on the bottom:

$$u\big|_{\Sigma_{-h}} = 0 \tag{1.1.6}$$

and the velocity vanishing in the normal direction of the walls

$$u \cdot n\big|_{\Sigma_1 \cup \Sigma_2} := u_n\big|_{\Sigma_{1,2}} = u_1\big|_{\Sigma_{1,2}} = 0 \tag{1.1.7}$$

together with a perfect slip condition

$$n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} = 0. \tag{1.1.8}$$

We need also to prescribe the contact angle between the free surface and the fixed boundary. We shall choose it to be $\frac{\pi}{2}$. So, the free surface is moving on the walls, but the value of the contact angle should remain constant. This condition can be writen as:

$$\overline{\nabla}\eta \cdot n^{\Sigma_1} = \overline{\nabla}\eta \cdot n^{\Sigma_2} = \partial_1 \eta = 0 \quad \text{on } C_1 \cup C_2. \tag{1.1.9}$$

For similar problems with contact angle 0 or π see [Sol1], [Sol2] and the references presented there.

The unknown functions u, p, η are periodic in the x_2 direction of the channel, so

$$(u, p, \eta)(t, x_1, x_2, x_3) = (u, p, \eta)(t, x_1, x_2 + 2\pi, x_3). \tag{1.1.10}$$

The initial condition is

$$(u, \eta)\big|_{t=0} = (u_0, \eta_0). \tag{1.1.11}$$

The equations (1.1.1)$-$(1.1.11) are the evolutionary nonlinear equations describing the oscillations of a fluid in an uncovered channel.

1.2 The Linear Equations and Estimates

The linear problem for which we derive estimates is the one obtained by linearizing equations (1.1.1)−(1.1.11) about equilibrium, replacing the initial data by zero and introducing a right hand side. We note that the linearization of the mean curvature in Γ_0 is $-\underline{\Delta}\eta$, where $\underline{\Delta}$ is the Laplacian with respect to the "horizontal" variables x_1, x_2. Because $\Gamma_0 = \{x_3 = 0\}$, we have $n_i = \delta_{i3}$, $i = 1, 2, 3$, in the equation (1.1.4).

For the begining we consider the exterior force to be zero. The influence of a nonzero exterior force (for example the wind force) will be considered for the study of the Hopf bifurcation in Section 2.3.

We observe that the equation (1.2.5) is equivalent to the condition on the vanishing of the tangential stress on Γ_0, so it can be writen also in the form $n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} = 0$ We also use the notations

$$S_u : S_v := \sum_{i,j=1}^{3} (S_u)_{ij}(S_v)_{ij}$$

$$S_u^n := n \cdot S_u \cdot n \qquad S_u^{\tau_i} := n \cdot S_u \cdot \tau_i.$$

Our linear problem becomes

$$u(t,\cdot) : \Omega_0 \longrightarrow \mathbb{R}^3, \quad p(t,\cdot) : \Omega_0 \longrightarrow \mathbb{R}, \quad \eta(t,\cdot) : \Gamma_0 \longrightarrow \mathbb{R},$$

$$\begin{aligned}
\partial_t u - \nu\Delta u + \nabla p &= 0 && (1.2.1)\\
\nabla \cdot u &= 0 && (1.2.2)\\
\partial_t \eta &= u_3\big|_{\Gamma_0} = u_n\big|_{\Gamma_0} && (1.2.3)\\
(p - 2\nu\partial_3 u_3)\big|_{\Gamma_0} = (p - 2\nu S_u^n)\big|_{\Gamma_0} &= g\eta - \beta\underline{\Delta}\eta && (1.2.4)\\
(\partial_3 u_i + \partial_i u_3)\Big|_{\Gamma_0} = n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} &= 0 \quad (i = 1, 2) && (1.2.5)\\
u\big|_{\Sigma_{-h}} &= 0 && (1.2.6)\\
u_1\big|_{\Sigma_{1,2}} = u_n\big|_{\Sigma_{1,2}} &= 0 && (1.2.7)\\
\partial_1 u_i\big|_{\Sigma_{1,2}} \overset{(1.2.7)}{=} n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} &= 0 \quad (i = 2, 3) && (1.2.8)\\
\partial_1 \eta\big|_{x_1 \in \{0,b\}} &= 0 && (1.2.9)\\
(u, p, \eta)(t, x_1, x_2, x_3) &= (u, p, \eta)(t, x_1, x_2 + 2\pi, x_3) && (1.2.10)\\
(u, \eta)\big|_{t=0} &= (0, 0) && (1.2.11)
\end{aligned}$$

We want to write the linear equations in the form $\partial_t x + \mathcal{L}x = 0$ and to satisfy the boundary conditions by the choice of appropriate function spaces. To estimate solutions of this equation, we use the Laplace transform in time.

Following [Be1] and [Schw1], we use a harmonic extension operator and replace the pressure term from the equation (1.2.1) by a gradient term which is determined by the other unknowns (u and η). In order to solve the equation $\Delta p = 0$ in Ω_0, we have to find appropriate boundary conditions for p on $\Sigma_{1,2}$ and Σ_{-h} (the boundary condition on Γ_0 is the equation (1.2.4)).

$$\text{on } \Sigma_{1,2}\text{: } \partial_n p = \partial_1 p = \nu\Delta u_1 - \partial_t u_1 \overset{(1.2.7)}{=} \nu\partial_1^2 u_1 \overset{(1.2.2)}{=} -\nu\partial_1(\partial_2 u_2 + \partial_3 u_3)$$
$$= -\nu\partial_2(\partial_1 u_2) - \nu\partial_3(\partial_1 u_3) \overset{(1.2.8)}{=} 0$$

$$\text{on } \Sigma_{-h}\text{: } \partial_n p = \partial_3 p = \nu\Delta u_3 - \partial_t u_3 \overset{(1.2.8)}{=} \nu\,\partial_3^2 u_3 = \nu(\partial_n S_u^n)\big|_{\Sigma_{-h}}$$

It seams to be to restrictive to impose this condition for the pressure on the bottom because it is not well-understood that we have enough regularity for u (this condition requires $u \in H^r(\Omega_0)^3$ with $r \geq 2$). At least locally, this will become clear after we will symmetrize the equations and eliminate the walls (see equations (1.2.34) and the Definition 1.2.12).

The harmonic extension function is defined as the unique solution of the problem

$$\begin{aligned} \Delta p &= 0 \quad \text{in } \Omega_0 && (a)\\ p\big|_{\Gamma_0} &= 2\nu S_u^n\big|_{\Gamma_0} + g\eta - \beta\underline{\Delta}\eta && (b)\\ \partial_n p\big|_{\Sigma_{1,2}} &= 0 && (c)\\ \partial_n p\big|_{\Sigma_{-h}} &= \nu(\partial_n S_u^n)\big|_{\Sigma_{-h}} && (d) \end{aligned} \tag{1.2.12}$$

So, define the linear operator

$$\tilde{\mathcal{H}} : H^{r-1/2}(\Gamma_0) \times H^{r-3/2}(\Sigma_{-h}) \longrightarrow H^r(\Omega_0)^3$$

which essentially maps a function defined on Γ_0 to its harmonic extension in Ω_0. The order r of the Sobolev space will be established later. We can consider p as a harmonic function defined on the whole domain,

$$\begin{aligned} p &= \tilde{\mathcal{H}}(2\nu S_u^n\big|_{\Gamma_0} + g\eta - \beta\underline{\Delta}\eta, \nu(\partial_n S_u^n)\big|_{\Sigma_{-h}})\\ &= \tilde{\mathcal{H}}(2\nu S_u^n\big|_{\Gamma_0}, \nu\partial_n S_u^n\big|_{\Sigma_{-h}}) + \tilde{\mathcal{H}}(g\eta - \beta\underline{\Delta}\eta, 0)\\ &:= \mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \mathcal{H}(g\eta - \beta\underline{\Delta}\eta). \end{aligned} \tag{1.2.13}$$

In the last equality of (1.2.13), we have only simplified the notation for the operator $\tilde{\mathcal{H}}$ (i.e. we have not included the condition on the bottom Σ_{-h}), because generally we are more interested to solve the problem near the free surface. Anytime when we refer to $\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0})$ we have to understand the condition (1.2.12)(d) to be satisfied too, and when we refer to $\mathcal{H}(g\eta - \beta\underline{\Delta}\eta)$ we have to understand the condition (1.2.12)(d) with zero right hand side, i.e. $\partial_n p\big|_{\Sigma_{-h}} = 0$

In the following we will consider complex valued functions and denote with $\bar{u}$ the complex conjugate of u. We use the following notations for the norms: $\forall r \in \mathbb{R}$ ($r = 0$ denotes the L^2-norm)

$$\begin{aligned} \|u\|_{H^r(\Omega_0)^3} &:= \|u\|_{r,\Omega_0} \\ \|\eta\|_{H^r(\Gamma_0)} &:= \|\eta\|_{r,\Gamma_0}\,. \end{aligned}$$

Definition 1.2.1 *Define the Hilbert spaces (over $\mathbb{C}$):*

$$\begin{aligned} X^r &:= \{(u,\eta) \in H^r(\Omega_0)^3 \times H^{r+1/2}(\Gamma_0) \bigm| \nabla \cdot u = 0,\ u_n\big|_{\Sigma_{1,2,-h}} = 0\} \\ \tilde{X}^r &:= \{(u,\eta) \in X^r \bigm| n \cdot S_u \cdot \tau_i\big|_{\Gamma_0 \cup \Sigma_{1,2}} = 0,\ u_{\tau_i}\big|_{\Sigma_{-h}} = 0,\ \partial_1 \eta\big|_{x_1 \in \{0,b\}} = 0\} \end{aligned}$$

with the natural norm inherited from the product space, i.e.

$$\|(u,\eta)\|_{X^r} := \|u\|_{r,\Omega_0} + \|\eta\|_{r+1/2,\Gamma_0}$$

and the operator

$$\mathcal{L} : \tilde{X}^{r+2} \longrightarrow X^r,$$

by

$$\mathcal{L}\begin{pmatrix} u \\ \eta \end{pmatrix} := \begin{pmatrix} -\nu\Delta u + \nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \nabla\mathcal{H}(g\eta - \beta\underline{\Delta}\eta) \\ -u_n\big|_{\Gamma_0} \end{pmatrix}.$$

Remark: The fact that $\mathcal{L}$ maps to X^r follows after a similar calculation we have done to find the condition for $\partial_n p\big|_{\Sigma_{1,2,-h}}$.

Lemma 1.2.2 *For smooth functions $u, v : \Omega_0 \to \mathbb{C}^3$ with $\nabla \cdot u = 0$ there holds*

$$2\int_{\Omega_0} S_u : S_{\bar{v}} = -\int_{\Omega_0} \Delta u \cdot \bar{v} + 2\int_{\partial\Omega_0} n \cdot S_u \cdot \bar{v}.$$

In the case $\nabla \cdot v = 0$, $v\big|_{\Sigma_{-h}} = 0$, $v_n\big|_{\Sigma_{1,2}} = 0$, and $n \cdot S_u \cdot \tau_i\big|_{\Gamma_0 \cup \Sigma_{1,2}} = 0$ (where τ_i is any tangent vector and n the normal vector corresponding to Γ_0, Σ_1 or Σ_2 respectively), we obtain the identity

$$2\int_{\Omega_0} S_u : S_{\bar{v}} = \int_{\Omega_0} [-\Delta u + \nabla\mathcal{H}(2S_u^n\big|_{\Gamma_0})] \cdot \bar{v}$$

Proof:

$$
\begin{aligned}
I &:= 2\int_{\Omega_0} S_u : S_{\bar v} \\
&= \frac{1}{2}\int_{\Omega_0} \sum_{i,j=1}^{3} (\partial_i u_j + \partial_j u_i)(\partial_i \bar v_j + \partial_j \bar v_i) \\
&= \int_{\Omega_0} \sum_{i,j=1}^{3} \frac{1}{2}(\partial_i u_j \partial_i \bar v_j + \partial_j u_i \partial_j \bar v_i) + \int_{\Omega_0} \sum_{i,j=1}^{3} \frac{1}{2}(\partial_i u_j \partial_j \bar v_i + \partial_j u_i \partial_i \bar v_j) \\
&= \int_{\Omega_0} \sum_{i,j=1}^{3} (\partial_i u_j \partial_i \bar v_j + \partial_j u_i \partial_i \bar v_j) \\
&=: \sum_{j=1}^{3} I_j .
\end{aligned}
$$

Let j be fixed. Integration by parts gives:

$$
\begin{aligned}
I_j &= -\int_{\Omega_0} \sum_{i=1}^{3} (\partial_i^2 u_j + \partial_j \partial_i u_i)\bar v_j + \int_{\partial\Omega_0} \sum_{i=1}^{3} (\partial_i u_j) n_i \bar v_j + \int_{\partial\Omega_0} \sum_{i=1}^{3} (\partial_j u_i) n_i \bar v_j \\
&= -\int_{\Omega_0} \Delta u_j \bar v_j + \int_{\partial\Omega_0} \sum_{i=1}^{3} (\partial_i u_j + \partial_j u_i) n_i \bar v_j \\
I &= \sum_{j=1}^{3} I_j = -\int_{\Omega_0} \Delta u \cdot \bar v + 2\int_{\partial\Omega_0} n \cdot S_u \cdot \bar v
\end{aligned}
$$

If additionaly $n \cdot S_u \cdot \tau_i\big|_{\Gamma_0 \cup \Sigma_{1,2}} = 0$, the tangent components of the vector $n \cdot S_u\big|_{\Gamma_0 \cup \Sigma_{1,2}}$ are zero; so together with the conditions for v $(v_n\big|_{\Sigma_{1,2}} = v\big|_{\Sigma_{-h}} = 0)$, we can write:

$$
n \cdot S_u \cdot \bar v\big|_{\partial\Omega_0} = n \cdot S_u \cdot \bar v\big|_{\Gamma_0 \cup \Sigma_{1,2}} = (n \cdot S_u \cdot n)(\bar v \cdot n)\big|_{\Gamma_0 \cup \Sigma_{1,2}} = S_u^n\big|_{\Gamma_0} \cdot \bar v_n\big|_{\Gamma_0}.
$$

Using again integration by parts we obtain:

$$
\begin{aligned}
2\int_{\Omega_0} S_u : S_{\bar v} &= -\int_{\Omega_0} \Delta u \cdot \bar v + \int_{\Gamma_0} 2S_u^n \cdot \bar v_n \\
&= -\int_{\Omega_0} \Delta u \cdot \bar v + \int_{\Omega_0} \nabla \mathcal{H}(2S_u^n\big|_{\Gamma_0}) \cdot \bar v + \mathcal{H}(2S_u^n\big|_{\Gamma_0})\, \nabla \cdot \bar v \\
&= \int_{\Omega_0} [-\Delta u + \nabla \mathcal{H}(2S_u^n\big|_{\Gamma_0})] \cdot \bar v
\end{aligned}
$$

□

We will use the results of Lemma 1.2.2 especially in the particular case when u and v satisfy the same conditions. We state this identities in the next Corollary; the proof follows immediately.

Corollary 1.2.3 *For function (u,p) and (v,q) satisfying the conditions*

$$\begin{aligned} \nabla\cdot u &= \nabla\cdot v = 0 \\ u_n\big|_{\Sigma_{1,2}} &= v_n\big|_{\Sigma_{1,2}} = 0 \\ u\big|_{\Sigma_{-h}} &= v\big|_{\Sigma_{-h}} = 0 \\ n\cdot S_u\cdot\tau_i\big|_{\Gamma_0\cup\Sigma_{1,2}} &= n\cdot S_v\cdot\tau_i\big|_{\Gamma_0\cup\Sigma_{1,2}} = 0, \end{aligned}$$

the following identities hold:

$$\begin{aligned} 2\int_{\Omega_0} S_u : S_{\bar v} &= -\int_{\Omega_0}\Delta u\cdot\bar v + 2\int_{\Gamma_0} S_u^n\cdot\bar v_n && (1.2.14)\\ &= -\int_{\Omega_0}\Delta\bar v\cdot u + 2\int_{\Gamma_0} S_{\bar v}^n\cdot u_n \\ 2\nu\int_{\Omega_0} S_u : S_{\bar v} &= \int_{\Omega_0}[-\nu\Delta u + \nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0})]\cdot\bar v && (1.2.15) \end{aligned}$$

$$\int_{\Omega_0}[-\nu\Delta u+\nabla p]\bar v - \int_{\Gamma_0}[p-2\nu S_u^n]\bar v_n = \int_{\Omega_0}[-\nu\Delta\bar v+\nabla\bar q]u - \int_{\Gamma_0}[\bar q-2\nu S_{\bar v}^n]u_n \qquad (1.2.16)$$

Definition 1.2.4 (Energy-norms)
For functions $u,v:\Omega_0\to\mathbb{C}^3$, $\eta,\sigma:\Gamma_0\to\mathbb{C}$ we define the scalar products:

$$\begin{aligned} \langle u,v\rangle_{E,\Omega_0} &:= \int_{\Omega_0} u\cdot\bar v \\ \langle \eta,\sigma\rangle_{E,\Gamma_0} &:= \int_{\Gamma_0}\eta\cdot(g\bar\sigma - \beta\underline{\Delta}\bar\sigma) \\ \left\langle \begin{pmatrix} u\\ \eta\end{pmatrix}, \begin{pmatrix} u\\ \eta\end{pmatrix}\right\rangle_E &:= \langle u,v\rangle_{E,\Omega_0} + \langle\eta,\sigma\rangle_{E,\Gamma_0} \end{aligned}$$

The corresponding norms are denoted by $\|\cdot\|_{E,\Omega_0}$, $\|\cdot\|_{E,\Gamma_0}$ and $\|\cdot\|_E$.

Remark: For $(u,\eta),(v,\sigma)\in\tilde{X}^r$, we have

$$\langle\eta,\sigma\rangle_{E,\Gamma_0}=g\int_{\Gamma_0}\eta\cdot\bar{\sigma}+\beta\int_{\Gamma_0}\overline{\nabla}\eta\cdot\overline{\nabla}\bar{\sigma},$$

so

$$\|\eta\|^2_{E,\Gamma_0}=g\|\eta\|^2_{0,\Gamma_0}+\beta\|\overline{\nabla}\eta\|^2_{0,\Gamma_0}$$

and because g and β are positive constants, we obtain immediately the norm equivalence

$$\|\eta\|_{E,\Gamma_0}\approx\|\eta\|_{1,\Gamma_0}.$$

For u we have $\|u\|_{E,\Omega_0}=\|u\|_{0,\Omega_0}$.

Theorem 1.2.5 (Position of eigenvalues of $\mathcal{L}$ w.r.t. $\|\cdot\|_E$)
Let $(u,\eta)\in\tilde{X}^r$ be an eigenfunction (considered complex) of $\mathcal{L}$ with eigenvalue λ. Then

$$\begin{aligned}\mathrm{Re}\lambda\left\|\begin{pmatrix}u\\ \eta\end{pmatrix}\right\|^2_E &= 2\nu\int_{\Omega_0}|S_u|^2 && (1.2.17)\\ \mathrm{Im}\lambda\left\|\begin{pmatrix}u\\ \eta\end{pmatrix}\right\|^2_E &= 2\mathrm{Im}\int_{\Gamma_0}(-u_n|_{\Gamma_0})(g\bar{\eta}-\beta\underline{\Delta}\bar{\eta}). && (1.2.18)\end{aligned}$$

In the case of $\mathrm{Im}\lambda\neq 0$*, the energy equality holds:*

$$\|u\|^2_{E,\Omega_0}=\|\eta\|^2_{E,\Gamma_0}=\frac{1}{2}\left\|\begin{pmatrix}u\\ \eta\end{pmatrix}\right\|^2_E. \qquad (1.2.19)$$

Proof:

$$\begin{aligned}\lambda\left\|\begin{pmatrix}u\\ \eta\end{pmatrix}\right\|^2_E &= \left\langle\mathcal{L}\begin{pmatrix}u\\ \eta\end{pmatrix},\begin{pmatrix}u\\ \eta\end{pmatrix}\right\rangle_E\\ &= \left\langle\begin{pmatrix}-\nu\Delta u+\nabla\mathcal{H}(2\nu S_u^n|_{\Gamma_0})+\nabla\mathcal{H}(g\eta-\beta\underline{\Delta}\eta)\\ -u_n|_{\Gamma_0}\end{pmatrix},\begin{pmatrix}u\\ \eta\end{pmatrix}\right\rangle_E\\ &= \int_{\Omega_0}[-\nu\Delta u+\nabla\mathcal{H}(2\nu S_u^n|_{\Gamma_0})]\cdot\bar{u}+\int_{\Omega_0}\nabla\mathcal{H}(g\eta-\beta\underline{\Delta}\eta)\cdot\bar{u}\\ &\quad+\int_{\Gamma_0}(-u_n|_{\Gamma_0})(g\bar{\eta}-\beta\underline{\Delta}\bar{\eta})\\ &= 2\nu\int_{\Omega_0}|S_u|^2+\int_{\Gamma_0}\bar{u}_n|_{\Gamma_0}(g\eta-\beta\underline{\Delta}\eta)-u_n|_{\Gamma_0}(g\bar{\eta}-\beta\underline{\Delta}\bar{\eta})\end{aligned}$$

Looking at the last equality, the first term is real, the second is imaginary and this implies the assertion on the real and imaginary part of λ.

To prove the energy equality we use the second part of the eigenvalue equation, $-u_n\big|_{\Gamma_0} = \lambda\eta$:

$$\mathrm{Im}\lambda \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|_E^2 = 2\mathrm{Im} \int_{\Gamma_0} \lambda\eta(g\bar{\eta} - \beta\underline{\Delta}\bar{\eta}) = 2\,\mathrm{Im}\lambda\,\|\eta\|^2_{E,\Gamma_0}.$$

□

Proposition 1.2.6 *The operator $\mathcal{L}^{-1} : X^r \to \tilde{X}^{r+1}$, $r \geq 1$, is bounded.*

Proof: We want to solve $\mathcal{L}(u,\eta) = (f,h) \in X^r$ for (u,η). Let (u,p) be the solution of the Stokes system:

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \\ \nabla\cdot u &= 0 \\ -u_n\big|_{\Gamma_0} &= h \\ u_n\big|_{\Sigma_{1,2}} &= 0 \\ n\cdot S_u\cdot\tau_i\big|_{\Gamma_0\cup\Sigma_{1,2}} &= 0 \\ u\big|_{\Sigma_{-h}} &= 0, \end{aligned}$$

with the usual bounds for the solution of the Stokes problem:

$$\|u\|_{r+1,\Omega_0} + \|\nabla p\|_{r-1,\Omega_0} \leq c_1\{\|f\|_{r-1,\Omega_0} + \|h\|_{r+1/2,\Gamma_0}\}.$$

For these estimates we observe at first that the Stokes system is elliptic and the considered boundary conditions satisfy the complementary conditions from [ADN]. In order to obtain a domain with smooth boundary, we can perform a reflection at the walls as in the equations (1.2.34) and in the Definition 1.2.12.

The pressure p yields η because $g - \beta\underline{\Delta}$ is invertible in our function spaces. The first part of the pressure can be estimated by

$$\begin{aligned} \|\nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0})\|_{r-1,\Omega_0} &\leq c_2\|u\|_{r+1,\Omega_0} \\ &\leq c_1c_2\{\|f\|_{r-1,\Omega_0} + \|h\|_{r+1/2,\Gamma_0}\} \end{aligned}$$

and therefore it follows for the second part of the pressure:

$$\begin{aligned} \|\nabla\mathcal{H}(g\eta - \beta\underline{\Delta}\eta)\|_{r-1,\Omega_0} &\leq \|\nabla p\|_{r-1,\Omega_0} + \|\nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0})\|_{r-1,\Omega_0} \\ &\leq c_1(1+c_2)\{\|f\|_{r-1,\Omega_0} + \|h\|_{r+1/2,\Gamma_0}\}. \end{aligned}$$

This implies $\eta \in H^{r+3/2}(\Gamma_0)$, so we obtain a bound for $(u,\eta) \in \tilde{X}^{r+1}$. □

Remark 1.2.7

(a) *By Proposition 1.2.6,* $\mathcal{L}^{-1} : X^r \to X^r$ *is compact, because the embeding* $H^{r+1} \hookrightarrow H^r$ *is compact. So* $\mathcal{L}$ *has a pure point spectrum and the eigenvalues have no finite accumulation point*

(b) $-\mathcal{L}$ *is dissipative by the calculation in the proof of Theorem 1.2.5. Together with Proposition 1.2.6, it follows that* $-\mathcal{L}$ *is an operator with compact resolvent and the resolvent set of* $-\mathcal{L}$ *(which is an open set) contains 0. This implies that*

- $\exists \mu > 0$ *contained in the resolvent set of* $-\mathcal{L}$*;*
- *the resolvent* $(\mu + \mathcal{L})^{-1}$ *exists and is compact* $\forall \mu$ *in the resolvent set of* $-\mathcal{L}$*.*

(c) *We point out that* $\mathcal{L}^{-1} : X^0 \to \tilde{X}^2$ *is not bounded: let* (u, η) *solve* $\mathcal{L}(u, \eta) = (0, h)$*. A bound for* $\|u\|_{2,\Omega_0}$ *would imply* $h = u_n\big|_{\Gamma_0} \in H^{3/2}(\Gamma_0)$*; but apriori only* $h \in H^{1/2}(\Gamma_0)$ *holds. We will formulate later (see Theorem 1.2.17) a result similar with Proposition 1.2.6 where a better regularity for* h *is assumed.*

In the next Proposition we remember some well-known inequalities we will need in order to obtain the resolvent estimates. For the proof (in a Lipschitz domain, where the function is zero only on a part of the boundary) see [Ad], [Ci] or [Gi,Ra].

Proposition 1.2.8

For $u \in H^1(\Omega_0)^3$ *with* $u\big|_{\Sigma_{-h}} = 0$*, the following inequalities hold (with positive constants* C_K*,* C_P*,* C_T *and* C_I*):*
Korn inequality:

$$\frac{1}{C_K}\|u\|_{1,\Omega_0} \le \|S_u\|_{0,\Omega_0} \le C_K\|u\|_{1,\Omega_0}; \tag{1.2.20}$$

Poincaré inequality:

$$\|u\|_{0,\Omega_0} \le C_P\|\nabla u\|_{0,\Omega_0}; \tag{1.2.21}$$

Trace inequality:

$$\|u_n\|_{1/2,\Gamma_0} \le C_T\|u\|_{1,\Omega_0}; \tag{1.2.22}$$

Interpolation inequality (for which we need $u \in H^2(\Omega_0)^3$*)*

$$\|u\|_{1,\Omega_0} \le C_I\|u\|_{0,\Omega_0}^{1/2}\|u\|_{2,\Omega_0}^{1/2}. \tag{1.2.23}$$

Theorem 1.2.9 (Position of the spectrum of $\mathcal{L}$)
The spectrum of $\mathcal{L}$ *consists only of eigenvalues and is contained in a sector*

$$S_C = \{\lambda \in \mathbb{C} \big| \, |\mathrm{Im}\lambda| \le C\mathrm{Re}\lambda\}.$$

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{L}$ with eigenvector (u, η). If $\mathrm{Im}\lambda = 0$ then λ is contained in any sector, so we assume $\mathrm{Im}\lambda \neq 0$.

We can apply the operator $\overline{\nabla} = (\partial_1, \partial_2, 0)$ to the eigenvalue equation for $\mathcal{L}$ and obtain that

$$\mathcal{L}\begin{pmatrix} \partial_i u \\ \partial_i \eta \end{pmatrix} = \lambda \begin{pmatrix} \partial_i u \\ \partial_i \eta \end{pmatrix} \text{ for } i = 1, 2,$$

so

$$\mathcal{L}\begin{pmatrix} \overline{\nabla} u \\ \overline{\nabla} \eta \end{pmatrix} = \lambda \begin{pmatrix} \overline{\nabla} u \\ \overline{\nabla} \eta \end{pmatrix}.$$

But we can not say that $(\partial_i u, \partial_i \eta)$, $i = 1, 2$ is an eigenvector of $\mathcal{L}$ because some of the boundary conditions are not satisfied (in the sense required for $\tilde{X}^r$). On the other hand we can do similar calculations to that of Theorem 1.2.5 and obtain the same results for $(\overline{\nabla} u, \overline{\nabla} \eta)$. In particular, for $\mathrm{Im}\lambda \neq 0$ the energy equality holds:

$$\|\overline{\nabla} u\|^2_{E,\Omega_0} = \|\overline{\nabla} \eta\|^2_{E,\Gamma_0}. \tag{1.2.24}$$

In the following calculations, we will use repeatedly the identities from Theorem 1.2.5 and the inequalities from Proposition 1.2.8:

$$\begin{aligned} |\mathrm{Im}\lambda| \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|^2_E &= 2 \left| \mathrm{Im} \int_{\Gamma_0} (-u_n|_{\Gamma_0})(g\bar{\eta} - \beta \underline{\Delta} \bar{\eta}) \right| \\ &\leq \int_{\Gamma_0} |u_n|_{\Gamma_0}|^2 + \int_{\Gamma_0} |g\bar{\eta} - \beta \underline{\Delta} \bar{\eta}|^2. \end{aligned}$$

We can estimate the first term by:

$$\begin{aligned} \int_{\Gamma_0} |u_n|_{\Gamma_0}|^2 &\leq C_T \|u\|^2_{1,\Omega_0} \\ &\leq C_T C_K \int_{\Omega_0} |S_u|^2 = \frac{C_T C_K}{2\nu} \mathrm{Re}\lambda \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|^2_E \end{aligned}$$

and the second by:

$$\begin{aligned} \int_{\Gamma_0} |g\bar{\eta} - \beta \underline{\Delta} \bar{\eta}|^2 &= g\|\eta\|^2_{E,\Gamma_0} - \beta \int_{\Gamma_0} (\underline{\Delta} \eta)(g\bar{\eta} - \beta \underline{\Delta} \bar{\eta}) \\ &= g\|\eta\|^2_{E,\Gamma_0} + \beta \|\overline{\nabla} \eta\|^2_{E,\Gamma_0} \\ &\overset{(1.2.24)}{=} g\|\eta\|^2_{E,\Gamma_0} + \beta \|\overline{\nabla} u\|^2_{E,\Omega_0} \\ &\leq g\|\eta\|^2_{E,\Gamma_0} + \beta c_1 \, 2\nu \int_{\Omega_0} |S_u|^2 \\ &= \frac{g}{2} \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|^2_E + \beta c_1 \mathrm{Re}\lambda \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|^2_E, \end{aligned}$$

so we obtain a sector of the form

$$|\mathrm{Im}\lambda| \leq \left(\frac{C_T C_K}{2\nu} + \beta c_1\right) \mathrm{Re}\lambda + \frac{g}{2}.$$

But we know from Theorem 1.2.5 that all the eigenvalues of $\mathcal{L}$ have positive real part and using Remark 1.2.7(a) we can say

$$\exists \delta > 0 \text{ such that } \forall \lambda \text{ eigenvalue of } \mathcal{L}: \; \mathrm{Re}\lambda \geq \delta > 0,$$

and than find a positive constant C such that all eigenvalues of $\mathcal{L}$ are contained in a sector

$$S_C = \{\lambda \in \mathbb{C} \big| \, |\mathrm{Im}\lambda| \leq C\mathrm{Re}\lambda\}.$$

$\square$

We apply the Laplace transform in time to our linear equations and prove an estimate for the resolvent of $-\mathcal{L}$, first on a subspace of the form $\{(f,0)\big| f \in L^2(\Omega_0)^3\}$. We denote the transformed functions by $(\hat{u}, \hat{\eta})$, but in the following, where no confusion can appear, we will omit to write the $\hat{}$ (especially in the proofs). So, let us investigate the solutions (u, η) of the equation

$$(\lambda + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} := \begin{pmatrix} \lambda u - \nu \Delta u + \nabla \mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \nabla \mathcal{H}(g\eta - \beta \underline{\Delta} \eta) \\ \lambda \eta - u_n\big|_{\Gamma_0} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1.2.25}$$

The following Lemma gives us useful estimates we will need in order to obtain the resolvent estimate.

Lemma 1.2.10 *For $\lambda \in \mathbb{C} \setminus (-S_C)$, solutions of (1.2.25) satisfy:*

$$\int_{\Omega_0} |\overline{\nabla} S_u|^2 \;\leq\; C_2\big(|\lambda| \, \|u\|_{0,\Omega_0} + \|f\|_{0,\Omega_0}\big)\|u\|_{2,\Omega_0} \tag{1.2.26}$$

$$|\lambda|^2 \|u\|^2_{0,\Omega_0} \;\leq\; C_3\big(\|u\|^{1/2}_{0,\Omega_0}\|u\|^{3/2}_{2,\Omega_0} + \|f\|^2_{0,\Omega_0}\big) \tag{1.2.27}$$

Proof: We carry out the estimates on the region of the complex plane where λ is not an eigenvalue of $-\mathcal{L}$, this is

$$\lambda \in \mathbb{C} \setminus (-S_C) = \{\lambda \in \mathbb{C} \big| \mathrm{Re}\lambda \leq 0, |\mathrm{Im}\lambda| \geq C|\mathrm{Re}\lambda|\} \cup \{\lambda \in \mathbb{C} \big| \mathrm{Re}\lambda > 0\}.$$

(a) Let $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \leq 0$, $|\mathrm{Im}\lambda| \geq C|\mathrm{Re}\lambda|$; we have:

$$|\mathrm{Im}\lambda|^2 \leq |\lambda|^2 = |\mathrm{Im}\lambda|^2 + |\mathrm{Re}\lambda|^2 \leq (1 + \frac{1}{C^2})|\mathrm{Im}\lambda|^2 = C_1^2 |\mathrm{Im}\lambda|^2.$$

We substitute the second equation of (1.2.25) in the first and obtain:

$$\lambda u - \nu\Delta u + \nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \frac{1}{\lambda}\nabla\mathcal{H}\big((gu_n - \beta\underline{\Delta}u_n)\big|_{\Gamma_0}\big) = f\,. \tag{1.2.28}$$

Multiplying this equation by $\underline{\Delta}\bar{u}$, integrating over Ω_0 and using Lemma (1.2.2), we obtain:

$$-\lambda\int_{\Omega_0}|\overline{\nabla}u|^2 - 2\nu\int_{\Omega_0}|\overline{\nabla}S_u|^2 - \frac{1}{\lambda}\int_{\Gamma_0}(g|\overline{\nabla}u_n|^2 + \beta|\underline{\Delta}u_n|^2) = \int_{\Omega_0} f\underline{\Delta}\bar{u} \tag{1.2.29}$$

Taking the imaginary part of (1.2.29) and then the absolute value, we obtain:

$$\frac{|\mathrm{Im}\lambda|}{|\lambda|^2}\int_{\Gamma_0}(g|\overline{\nabla}u_n|^2 + \beta|\underline{\Delta}u_n|^2) \le |\mathrm{Im}\lambda|\int_{\Omega_0}|\overline{\nabla}u|^2 + \|f\|_{0,\Omega_0}\|u\|_{2,\Omega_0}$$

and this multiplied by $\frac{|\lambda|}{|\mathrm{Im}\lambda|} \le C_1$ gives

$$\frac{1}{|\lambda|}\int_{\Gamma_0}(g|\overline{\nabla}u_n|^2 + \beta|\underline{\Delta}u_n|^2) \le |\lambda|\ \|\overline{\nabla}u\|_{0,\Omega_0}^2 + C_1\|f\|_{0,\Omega_0}\|u\|_{2,\Omega_0} \tag{1.2.30}$$

Taking the real part of (1.2.29), then the absolute value, using (1.2.30) and (1.2.23), we obtain:

$$\begin{aligned}
2\nu\int_{\Omega_0}|\overline{\nabla}S_u|^2 &\le |\lambda|\int_{\Omega_0}|\overline{\nabla}u|^2 + \frac{1}{|\lambda|}\int_{\Gamma_0}(g|\overline{\nabla}u_n|^2 + \beta|\underline{\Delta}u_n|^2) + \|f\|_{0,\Omega_0}\|u\|_{2,\Omega_0}\\
&\overset{(1.2.30)}{\le} 2|\lambda|\ \|u\|_{1,\Omega_0}^2 + (1+C_1)\|f\|_{0,\Omega_0}\|u\|_{2,\Omega_0}\\
&\overset{(1.2.23)}{\le} \big(2C_I|\lambda|\ \|u\|_{0,\Omega_0} + (1+C_1)\|f\|_{0,\Omega_0}\big)\|u\|_{2,\Omega_0}
\end{aligned}$$

and this proves (1.2.26) with $C_2 = \max\{\frac{C_I}{\nu}, \frac{1+C_1}{2\nu}\}$.

Multiplying the equation (1.2.28) by $\bar{u}$, integrating over Ω_0 and using Lemma (1.2.2), we obtain:

$$\lambda\int_{\Omega_0}|u|^2 + 2\nu\int_{\Omega_0}|S_u|^2 + \frac{1}{\lambda}\int_{\Gamma_0}(g|u_n|^2 + \beta|\overline{\nabla}u_n|^2) = \int_{\Omega_0} f\bar{u}. \tag{1.2.31}$$

Taking the imaginary part, then the absolute value, multiplying by $\frac{|\lambda|^2}{|\mathrm{Im}\lambda|} \le C_1|\lambda|$ and using the inequalities from Proposition 1.2.8, we obtain

$$\begin{aligned}
|\lambda|^2\|u\|_{0,\Omega_0}^2 &\le g\|u_n\|_{0,\Gamma_0}^2 + \beta\int_{\Gamma_0} u_n\,\underline{\Delta}u_n + C_1\|f\|_{0,\Omega_0}\,|\lambda|\ \|u\|_{0,\Omega_0}\\
&\le gC_T\|u\|_{1,\Omega_0}^2 + \beta C_T\|u\|_{1,\Omega_0}\|u\|_{2,\Omega_0} + \frac{C_1^2}{2}\|f\|_{0,\Omega_0}^2 + \frac{|\lambda|^2}{2}\|u\|_{0,\Omega_0}^2\\
&\le C_TC_I(g+\beta)\|u\|_{0,\Omega_0}^{1/2}\|u\|_{2,\Omega_0}^{3/2} + \frac{C_1^2}{2}\|f\|_{0,\Omega_0}^2 + \frac{|\lambda|^2}{2}\|u\|_{0,\Omega_0}^2
\end{aligned}$$

We can absorb the last term in the left hand side and obtain (1.2.27) with $C_3 = \max\{2C_T C_I(g+\beta), C_1^2\}$.

(b) Let now $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > 0$. In this case, the estimates are much easier. Taking the real part of (1.2.29), we observe that all terms on the left hand side are negative, so taking the absolute value, we obtain:

$$\int_{\Omega_0} |\overline{\nabla} S_u|^2 \leq \frac{1}{2\nu}\|f\|_{0,\Omega_0}\|u\|_{2,\Omega_0}$$

and in particular (1.2.26) holds. In a similar way we can prove (1.2.27), too. □

Theorem 1.2.11 (The resolvent $(\lambda+\mathcal{L})^{-1}$ in the case $(f,0) \in X^0$)
There exist constants C_R and c such that solutions (u,η) of (1.2.25) with $\lambda \in \mathbb{C} \setminus (-S_C)$ satisfy the regularity

$$\|(u,\eta)\|_{X^2} \leq c\|(f,0)\|_{X^0} \tag{1.2.32}$$

and for $|\lambda|$ large enough, the resolvent estimate

$$\|(u,\eta)\|_{X^0} \leq \frac{C_R}{|\lambda|}\|(f,0)\|_{X^0}\,. \tag{1.2.33}$$

Proof:

Looking at the first equation in (1.2.25), we can interpret u as the solution of a Stokes system in Ω_0 with right hand side $f - \lambda u$ and with prescribed boundary data $u_n\big|_{\Gamma_0}$. To complete the boundary conditions we consider the equations (1.2.5)-(1.2.8) to be satisfied too. These imply the estimate (with the positive constant C_S):

$$\|u\|_{2,\Omega_0}^2 \leq C_S(\|u_n\|_{3/2,\Gamma_0}^2 + |\lambda|^2\|u\|_{0,\Omega_0}^2 + \|f\|_{0,\Omega_0}^2).$$

Using the trace and the Korn inequality for $\overline{\nabla} u_3$, and the inequalities of Lemma 1.2.10, we can calculate further:

$$\begin{aligned}
\|u\|_{2,\Omega_0}^2 &\leq C_S C_{T,K}\int_{\Omega_0}|\overline{\nabla} S_u|^2 + C_S(|\lambda|^2\|u\|_{0,\Omega_0}^2 + \|f\|_{0,\Omega_0}^2) \\
&\overset{(1.2.26)}{\leq} C_S C_{T,K} C_2\|u\|_{2,\Omega_0}(|\lambda|\|u\|_{0,\Omega_0} + \|f\|_{0,\Omega_0}) + C_S(|\lambda|^2\|u\|_{0,\Omega_0}^2 + \|f\|_{0,\Omega_0}^2) \\
&\leq \frac{1}{2}\|u\|_{2,\Omega_0}^2 + \Big(\frac{C_S^2 C_{T,K}^2 C_2^2}{2} + C_S\Big)|\lambda|^2\|u\|_{0,\Omega_0}^2 + \Big(\frac{C_S^2 C_{T,K}^2 C_2^2}{2} + C_S\Big)\|f\|_{0,\Omega_0}^2 \\
&\overset{(1.2.27)}{\leq} \frac{1}{2}\|u\|_{2,\Omega_0}^2 + C_3\Big(\frac{C_S^2 C_{T,K}^2 C_2^2}{2} + C_S\Big)\|u\|_{0,\Omega_0}^{1/2}\|u\|_{2,\Omega_0}^{3/2} \\
&\quad + \Big(\frac{C_S^2 C_{T,K}^2 C_2^2}{2} + C_S + C_3\Big)\|f\|_{0,\Omega_0}^2\,.
\end{aligned}$$

We can absorb the first term of the last inequality in the left hand side and obtain:

$$\|u\|^2_{2,\Omega_0} \le C_4\big(\|u\|^{1/2}_{0,\Omega_0}\|u\|^{3/2}_{2,\Omega_0} + \|f\|^2_{0,\Omega_0}\big),$$

with $C_4 = \max\{C_S^2 C_{T,K}^2 C_2^2 C_3 + 2C_S C_3, C_S^2 C_{T,K}^2 C_2^2 + 2C_S + 2C_3\}$; this implies a bound of the form $\|u\|_{2,\Omega_0} < c(\|u\|_{0,\Omega_0} + \|f\|_{0,\Omega_0})$. More explicitly, for small $\epsilon > 0$ we have:

$$\begin{aligned}\|u\|^2_{2,\Omega_0} &\le \epsilon\|u\|^2_{2,\Omega_0} + \frac{C_4^2}{\epsilon}\|u\|_{0,\Omega_0}\|u\|_{2,\Omega_0} + C_4\|f\|^2_{0,\Omega_0}\\ &\le \epsilon\|u\|^2_{2,\Omega_0} + \frac{1}{2}\|u\|^2_{2,\Omega_0} + \frac{C_4^4}{2\epsilon^2}\|u\|^2_{0,\Omega_0} + C_4\|f\|^2_{0,\Omega_0}\end{aligned}$$

and absorbing the first two terms of the right hand side in the left hand side, we obtain the desired estimate.

The first equation of (1.2.25) connects second derivatives of η in Ω_0 with traces of functions bounded in $H^1(\Omega_0)$, so we obtain a bound for $\|\eta\|_{5/2,\Gamma_0}$.

Using again the inequality (1.2.27), we obtain an estimate of the form

$$|\lambda|\|u\|_{0,\Omega_0} \le c(\|u\|_{0,\Omega_0} + \|f\|_{0,\Omega_0}),$$

so, in the case of large $|\lambda|$, we get the estimate for $\|u\|_{0,\Omega_0}$.

Using the second equation of (1.2.25), we have

$$|\lambda|\ \|\eta\|_{1/2,\Gamma_0} = \|u_n\|_{1/2,\Gamma_0}$$

which can be bounded by $\|u\|_{2,\Omega_0}$ and therefore (for $|\lambda|$ large enough) by $\|f\|_{0,\Omega_0}$.

$\square$

In the following we are going to derive estimates for the higher derivatives which are needed for the existence theory and for the nonlinear problem. In order to avoid difficulties with the corners we will perform a reflection across the walls. Without loss of generality we may restrict to one of the sides, let $x_1 = 0$. Our boundary conditions on the walls $\Sigma_{1,2}$ allow us to define symmetric extensions of (u, η, p) across Σ_1. We denote them by $(\tilde{u}, \tilde{\eta}, \tilde{p})$. These functions will be periodic in the x_1-direction in the domain $\tilde{\Omega}_0 = (-b, b) \times S^1 \times (-h, 0)$ with the upper boundary $\tilde{\Gamma}_0 = (-b, b) \times S^1 \times \{0\}$. The symmetries are as follows:

$$\left.\begin{aligned}\tilde{u}_1(t, -x_1, x_2, x_3) &= -u_1(t, x_1, x_2, x_3)\\ \tilde{u}_2(t, -x_1, x_2, x_3) &= u_2(t, x_1, x_2, x_3)\\ \tilde{u}_3(t, -x_1, x_2, x_3) &= u_3(t, x_1, x_2, x_3)\\ \tilde{p}(t, -x_1, x_2, x_3) &= p(t, x_1, x_2, x_3)\\ \tilde{\eta}(t, -x_1, x_2) &= \eta(t, x_1, x_2)\end{aligned}\right\} \tag{1.2.34}$$

and consistently we define $\tilde{f}_1$ to be odd and $\tilde{f}_2, \tilde{f}_3$ to be even with respect to the first variable (considered as functions of (x_1, x_2, x_3)). It is easy to see that these new functions satisfies the same equations in $\tilde{\Omega}_0$ as the old one in Ω_0.

Definition 1.2.12 *We define $(\tilde{u}, \tilde{\eta}, \tilde{p})$ to be the solution of the following problem in $\tilde{\Omega}_0$ periodic in x_1- and x_2-direction:*

$$\left.\begin{array}{rcll} \partial_t\tilde{u} - \nu\Delta\tilde{u} + \nabla\tilde{p} & = & \tilde{f} & \text{in } \tilde{\Omega}_0 \\ \nabla\cdot\tilde{u} & = & 0 & \text{in } \tilde{\Omega}_0 \\ \partial_t\tilde{\eta} & = & \tilde{u}_3 & \text{on } \tilde{\Gamma}_0 \\ \frac{\partial\tilde{u}_i}{\partial x_3} + \frac{\partial\tilde{u}_3}{\partial x_i} & = & 0 & \text{on } \tilde{\Gamma}_0 \quad (i=1,2) \\ \tilde{p} - 2\nu\frac{\partial\tilde{u}_3}{\partial x_3} - (g\tilde{\eta} - \beta\underline{\Delta}\tilde{\eta}) & = & 0 & \text{on } \tilde{\Gamma}_0 \\ \tilde{u} & = & 0 & \text{on } \tilde{\Sigma}_{-h} \end{array}\right\} \tag{1.2.35}$$

In a similar way like Theorem 1.2.11, we can prove estimates for the solution of the problem (1.2.35) in higher Sobolev spaces, using well-known techniques: we differentiate the equations (1.2.35) with respect to the variable x_1 and x_2, then the corresponding derivatives of u satisfy the same equations with the differentiated right hand side. The estimates of the derivatives with respect to x_3 can be obtained from the first equation. Using the same methods as before, we obtain estimates similar to (1.2.32) and (1.2.33) for the derivatives of $(\tilde{u}, \tilde{\eta})$ in $\tilde{\Omega}_0$. We formulate now the analog of Theorem 1.2.11 in higher Sobolev norms, for the restricted solution (u, η) in Ω_0:

Theorem 1.2.13 (The resolvent $(\lambda + \mathcal{L})^{-1}$ in the case $(f, 0) \in X^r$)
There exist constants C_R and c such that solutions (u, η) of (1.2.25) with $\lambda \in \mathbb{C} \setminus (-S_C)$ satisfy for $(f, 0) \in X^r$, with $r \geq 0$, the regularity

$$\|(u, \eta)\|_{X^{r+2}} \leq c\|(f, 0)\|_{X^r} \tag{1.2.36}$$

and for $|\lambda|$ large enough, the resolvent estimate

$$\|(u, \eta)\|_{X^r} \leq \frac{C_R}{|\lambda|}\|(f, 0)\|_{X^r}\,. \tag{1.2.37}$$

Corollary 1.2.14 (The resolvent $(\lambda + \mathcal{L})^{-1}$ for $(f, h) \in X^r$ with $h \neq 0$)
Let (u, η) be a solution of the equation

$$(\lambda + \mathcal{L})\begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}, \tag{1.2.38}$$

with $(f, h) \in X^{r+2}$, $r \geq 0$. Then there exists a constant $M > 0$ such that for all $\lambda \in \mathbb{C} \setminus (-S_C)$, $|\lambda|$ large enough, there holds:

$$\|(u, \eta)\|_{X^{r+2}} \leq \frac{M}{|\lambda|}\|(f, h)\|_{X^{r+2}} \tag{1.2.39}$$

Proof: Let (u_1, η_1) be a solution of the equation

$$(\lambda + \mathcal{L}) \begin{pmatrix} u_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Define

$$\begin{aligned} u_2 &:= u - u_1 \\ \eta_2 &:= \eta - \eta_1 - \frac{1}{\lambda} h. \end{aligned}$$

Then the pair (u_2, η_2) satisfies the equation

$$(\lambda + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\lambda} \nabla \mathcal{H}(gh - \beta \underline{\Delta} h) \\ 0 \end{pmatrix}.$$

We apply now the Theorem 1.2.13, i.e. the inequality (1.2.37) for (u_1, η_1) and the inequality (1.2.36) for (u_2, η_2):

$$\begin{aligned} \|(u_1, \eta_1)\|_{X^{r+2}} &\leq \frac{C_R}{|\lambda|} \|(f, 0)\|_{X^{r+2}} \\ \|(u_2, \eta_2)\|_{X^{r+2}} &\leq c \|(-\frac{1}{\lambda} \nabla \mathcal{H}(gh - \beta \underline{\Delta} h), 0)\|_{X^r} \\ &\leq \frac{c*}{|\lambda|} \|h\|_{r+3-1/2, \Gamma_0} \\ &= \frac{c*}{|\lambda|} \|(0, h)\|_{X^{r+2}}. \end{aligned}$$

Using the triangle inequality we obtain the desired estimate for (u, η). □

We can now apply the inverse of the Laplace transformation and formulate our existence result for the linear problem.

Theorem 1.2.15 (Linear existence result for $(f, 0)$)
We consider $\mathcal{L} : \tilde{X}^{r+2} \to X^r$, $r \geq 1$ and $(f, 0) \in L^2([0, T], X^r)$, $T > 0$. Then the problem

$$(\partial_t + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

with initial conditions $(u, \eta)\big|_{t=0} = (u_0, \eta_0) \in \tilde{X}^{r+2}$ has a unique solution

$$(u, \eta) \in H^1([0, T], \tilde{X}^r) \cap L^2([0, T], \tilde{X}^{r+2}).$$

Proof: We substract the initial conditions from the solution, so define the pair

$$\begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} u \\ \eta \end{pmatrix} - \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix}$$

which solves the problem

$$(\partial_t + \mathcal{L}) \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ h \end{pmatrix} := \begin{pmatrix} f \\ 0 \end{pmatrix} - \mathcal{L} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix}$$

and has zero initial conditions. Considering the Laplace transform in time,

$$\hat{v}(\lambda, \cdot) = \int_0^\infty e^{-\lambda t} v(t, \cdot) dt,$$

we obtain the equation

$$(\lambda + \mathcal{L}) \begin{pmatrix} \hat{v} \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \hat{\tilde{f}} \\ \hat{h} \end{pmatrix} \tag{1.2.40}$$

which (for all $\lambda \in \mathbb{C}$ with $|\lambda|$ large enough) has a solution $(\hat{v}, \hat{\sigma})$ satisfying the estimates (1.2.39), so $(\hat{v}, \hat{\sigma}) \in \tilde{X}^r$, $r \geq 2$. The second component of the right hand side of the equation (1.2.40) is not zero, but Theorem 1.2.13 gives us actually $(\hat{v}, \hat{\sigma}) \in \tilde{X}^{r+2}$ because $\hat{h} = \hat{u}_{0n}|_{\Gamma_0} \in H^{r+3/2}(\Gamma_0)$ is regular enough. See the proof of Proposition 1.2.6 (where h has now a special form and a better regularity) and also the next two Theorems 1.2.17 and 1.2.18. For the same reason that $\hat{h}$ is more regular than the space X^r required, we obtain the properties of the solution for $r \geq 1$ (see the proof of Corollary 1.2.14).

We apply the inverse of the Laplace transformation

$$\begin{pmatrix} v \\ \sigma \end{pmatrix} (t, \cdot) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\lambda t} \begin{pmatrix} \hat{v} \\ \hat{\sigma} \end{pmatrix} (\lambda, \cdot)\, d\lambda$$

where $\lambda = \alpha + is$ ($\alpha = \mathrm{Re}\lambda$ is large enough in order to have the resolvent estimates) and obtain a solution $(v, \sigma) \in L^2([0, T], \tilde{X}^{r+2})$. Using the isometry of the Laplace transformation, we can calculate (with a generic constant C):

$$\begin{aligned} \int_{\alpha - i\infty}^{\alpha + i\infty} \left\| \begin{pmatrix} \widehat{\partial_t v} \\ \widehat{\partial_t \sigma} \end{pmatrix} (\lambda, \cdot) \right\|_{X^r}^2 ds &= \int_{\alpha - i\infty}^{\alpha + i\infty} |\lambda|^2 \left\| \begin{pmatrix} \hat{v} \\ \hat{\sigma} \end{pmatrix} (\lambda, \cdot) \right\|_{X^r}^2 ds \\ &\overset{(1.2.39)}{\leq} C \int_{\alpha - i\infty}^{\alpha + i\infty} \left\| \begin{pmatrix} \hat{\tilde{f}} \\ \hat{h} \end{pmatrix} (\lambda, \cdot) \right\|_{X^r}^2 ds \\ &= C \int_0^\infty \left\| \begin{pmatrix} \tilde{f} \\ h \end{pmatrix} (t, \cdot) \right\|_{X^r}^2 e^{-2\alpha t} dt \end{aligned} \tag{1.2.41}$$

which proves that $(v,\sigma) \in H^1([0,T],\tilde{X}^r)$ and

$$\int_0^\infty \left\| \partial_t \begin{pmatrix} v \\ \sigma \end{pmatrix} (t,\cdot) \right\|_{X^r}^2 dt \le C \int_0^\infty \left\| \begin{pmatrix} \tilde{f} \\ h \end{pmatrix} (t,\cdot) \right\|_{X^r}^2 dt.$$

As a consequence we obtain immediatly that $(v,\sigma) \in C^{1/2}([0,T],\tilde{X}^r)$.

Then we obtain $(u,\eta) \in H^1([0,T],\tilde{X}^r) \cap L^2([0,T],\tilde{X}^{r+2})$ with a bound depending on $\|f\|_{L^2([0,T],H^r(\Omega_0)^3)}$ and $\|(u_0,\eta_0)\|_{\tilde{X}^{r+2}}$. □

Remark 1.2.16 *Differentiating once more w.r.t. time and cutting off the solution at $t=0$, one can obtain after calculations similar to (1.2.41) that $(u,\eta) \in C^{1,\gamma}((0,T],\tilde{X}^r)$ with $\gamma \le \frac{1}{2}$.*

The next theorem is a consequence of the Proposition 1.2.6 and Theorem 1.2.13. It states that we can generalize the regularity estimate (1.2.36) and obtain it also for a nonzero second component of the right hand side, if this is more regular than the space X^r required. This means we have to introduce a new space

$$X^r_{3/2} := \{(f,h) \in H^r(\Omega_0)^3 \times H^{r+3/2}(\Gamma_0) \,\big|\, \nabla \cdot f = 0, f_n\big|_{\Sigma_{1,2,-h}} = 0\} \qquad (1.2.42)$$

with the natural norm inherited from the product space. Using this notation, our X^r spaces coincide with the $X^r_{1/2}$ spaces, but we will keep the old notation for X^r.

Theorem 1.2.17 (Properties of $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$)
The operator $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \ge 0$, is invertible, the inverse is bounded and we have the regularity estimate

$$\|(u,\eta)\|_{X^{r+2}} \le c\|(f,h)\|_{X^r_{3/2}}. \qquad (1.2.43)$$

The same result holds for the operator $\lambda + \mathcal{L}$, too, when $-\lambda$ is not an eigenvalue of $\mathcal{L}$.

We can immediatly formulate the analog of the linear existence Theorem 1.2.15 for this special form of the right hand side:

Theorem 1.2.18 (Linear existence result for (f,h) with $h \neq 0$)
We consider $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \ge 1$, and $(f,h) \in L^2([0,T],X^r_{3/2})$, $T > 0$. Then the problem

$$(\partial_t + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

with initial conditions $(u,\eta)\big|_{t=0} = (u_0,\eta_0) \in \tilde{X}^{r+2}$ has a unique solution

$$(u,\eta) \in H^1([0,T],\tilde{X}^r_{3/2}) \cap L^2([0,T],\tilde{X}^{r+2}).$$

1.3 Transformation to the Fixed Domain and the Nonlinear Problem

Following [Be1], we convert our (initial) nonlinear problem (1.1.1) - (1.1.11) defined on the unknown domain Ω to one on the equilibrium domain Ω_0 by stretching or compressing on the vertical line segments. In this section, we denote the variables and functions in Ω_0 by capital letters, so $X_1 = x_1$, $X_2 = x_2$, X_3 will be specified later.

For every time t, given a small $\eta : \mathbb{R}^+ \times \Gamma_0 \to \mathbb{R}$ with $\frac{\partial \eta}{\partial x_1}\big|_{x_1 \in \{0,b\}} = 0$, we can choose $\tilde{\eta}(t, X_1, X_2, \cdot)$ close to the identity which transforms the interval $[-h, 0]$ to the interval $[-h, \eta(t, x_1, x_2)]$. We can choose the extension $\tilde{\eta}$ to have maximal regularity as given by the trace theorem and such that $\forall t$, $\tilde{\eta}(t)$ depends only on $\eta(t)$, the contact line condition for η is prelonged on the whole $\Sigma_{1,2}$ and $\tilde{\eta}$ satisfies also the boundary condition on Σ_{-h}. So we define $\tilde{\eta}$ such that:

$$
\begin{array}{rcll}
\tilde{\eta}(t, X_1, X_2, 0) & = & \eta(t, x_1, x_2) & (a) \\
\dfrac{\partial \tilde{\eta}}{\partial X_1}\Big|_{X_1 \in \{0,b\}} & = & 0 & (b) \\
\tilde{\eta}\big|_{X_3 = -h} & = & 0 & (c)
\end{array}
\tag{1.3.1}
$$

For each t we define the transformation $\Theta(t, \cdot) : \Omega_0 \to \Omega$,

$$
(x_1, x_2, x_3) = \Theta(t, X_1, X_2, X_3) := (X_1, X_2, X_3 + (1 + \tfrac{X_3}{h})\tilde{\eta}) \tag{1.3.2}
$$

and calculate:

$$
\begin{aligned}
D\Theta = \left(\frac{\partial x_i}{\partial X_j}\right)_{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_1} & (1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_2} & 1 + \frac{\tilde{\eta}}{h} + (1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_3} \end{pmatrix} \\
J := \det D\Theta &= 1 + \frac{\tilde{\eta}}{h} + (1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_3} \\
\left(\frac{\partial X_i}{\partial x_j}\right)_{ij}\Bigg|_{x = \Theta(X)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{J}(1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_1} & -\frac{1}{J}(1 + \frac{X_3}{h})\frac{\partial \tilde{\eta}}{\partial X_2} & \frac{1}{J} \end{pmatrix}.
\end{aligned}
$$

We could transform the velocity field only by composition, but then the divergence free condition would be lost. Instead, for U in Ω_0, we define u in $\Omega = \Theta(\Omega_0)$ by:

$$
u_i = \frac{1}{J}\frac{\partial x_i}{\partial X_j} U_j \tag{1.3.3}
$$

where repeated indices are summed. It is understood here that for $(x_1, x_2, x_3) \in \Omega$, the right hand side is evaluated at $\Theta^{-1}(x_1, x_2, x_3) = (X_1, X_2, X_3)$. With this definition, U has divergence zero in Ω_0 iff u has the same property in Ω.

There is a further advantage to this transformation of the velocity field: the right hand side of (1.1.3) is replaced simply by U_3. More explicitly, on the upper surface Γ_0 we have $X_3 = 0$ and $\tilde{\eta} = \eta$, so we can calculate:

$$\begin{aligned} (u_1, u_2, u_3) &= \frac{1}{J}(U_1, U_2, U_1 \frac{\partial \eta}{\partial x_1} + U_2 \frac{\partial \eta}{\partial x_2} + JU_3) \\ \frac{\partial \eta}{\partial t} &= \frac{1}{J} U_1 \frac{\partial \eta}{\partial x_1} + \frac{1}{J} U_2 \frac{\partial \eta}{\partial x_2} + U_3 - \frac{1}{J} U_1 \frac{\partial \eta}{\partial x_1} - \frac{1}{J} U_2 \frac{\partial \eta}{\partial x_2} \\ &= U_3. \end{aligned} \tag{1.3.4}$$

The derivatives of u are:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial X_l}{\partial x_j} \frac{\partial}{\partial X_l} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_k} U_k \right).$$

In rewriting $\frac{\partial u_i}{\partial t}$ we have terms arising from the fact that Θ depends on t:

$$\frac{\partial u_i}{\partial t} = \frac{1}{J} \frac{\partial x_i}{\partial X_j} \frac{\partial U_j}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_j} \right) U_j + \frac{\partial}{\partial X_3} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_j} U_j \right) \frac{\partial (\Theta^{-1})_3}{\partial t}.$$

Let $p \circ \Theta = P$. The other three terms in the Navier-Stokes equations can be writen as:

$$\begin{aligned} (u \cdot \nabla u)_i &= \left(\frac{1}{J} \frac{\partial x_j}{\partial X_m} U_m \right) \frac{\partial X_l}{\partial x_j} \frac{\partial}{\partial X_l} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_k} U_k \right) \\ \Delta u_i &= \sum_{j=1}^{3} \left[\frac{\partial^2 X_l}{\partial x_j^2} \frac{\partial}{\partial X_l} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_k} U_k \right) + \frac{\partial X_l}{\partial x_j} \frac{\partial X_m}{\partial x_j} \frac{\partial^2}{\partial X_l \partial X_m} \left(\frac{1}{J} \frac{\partial x_i}{\partial X_k} U_k \right) \right] \\ (\nabla p)_i &= \frac{\partial X_k}{\partial x_i} \frac{\partial P}{\partial X_k} \end{aligned}$$

Finally, multiplying by $\left(J \frac{\partial X_i}{\partial x_j} \right)$, expressing the time derivatives of $\frac{1}{J} \frac{\partial x_i}{\partial X_j}$ and Θ by time derivatives of η and using (1.3.4), we can write the Navier-Stokes equations for (U, P):

$$\begin{aligned} \partial_t U - \nu \Delta U + \nabla P &= F_0(U, \eta, \nabla P) \\ \nabla \cdot U &= 0 \end{aligned} \tag{1.3.5}$$

The condition on the free boundary

$$pn_i - \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)n_j = \left(g\eta - \beta\overline{\nabla}\cdot\frac{\overline{\nabla}\eta}{\sqrt{1+|\overline{\nabla}\eta|^2}}\right)n_i$$

can be writen in terms of the new variables as

$$PN_i - \nu\left(\frac{\partial X_l}{\partial x_j}\frac{\partial}{\partial X_l}\left(\frac{1}{J}\frac{\partial x_i}{\partial X_k}U_k\right) + \frac{\partial X_l}{\partial x_i}\frac{\partial}{\partial X_l}\left(\frac{1}{J}\frac{\partial x_j}{\partial X_k}U_k\right)\right)N_j =$$

$$= \left(g\eta - \beta\overline{\nabla}\cdot\frac{\overline{\nabla}\eta}{\sqrt{1+|\overline{\nabla}\eta|^2}}\right)N_i$$

where $N = n \circ \Theta$. It is convenient to replace this vector equation with components tangential and normal to the physical surface. Let $T_1 = (1, 0, \frac{\partial\eta}{\partial x_1})$, $T_2 = (0, 1, \frac{\partial\eta}{\partial x_2})$ be two tangent vectors and $N_3 = (-\frac{\partial\eta}{\partial x_1}, -\frac{\partial\eta}{\partial x_2}, 1)$ be the normal to Γ in the point $(x_1, x_2, \eta(x_1, x_2))$. Projecting the equation on this three directions we obtain equations of the form:

$$\begin{aligned} \frac{\partial U_i}{\partial X_3} + \frac{\partial U_3}{\partial X_i} &= G_i(U,\eta) \quad i = 1,2 && (1.3.6)\\ P - 2\nu\frac{\partial U_3}{\partial X_3} - (g\eta - \beta\underline{\Delta}\eta) &= G_3(U,\eta)\,. && (1.3.7)\end{aligned}$$

The boundary conditions on the fixed boundary are preserved , so

$$\left.\begin{aligned} U\big|_{\Sigma_{-h}} &= 0 \\ U_n\big|_{\Sigma_{1,2}} &= 0 \\ n\cdot S_U\cdot\tau_i\big|_{\Sigma_{1,2}} &= 0 \quad i = 1,2 \end{aligned}\right\} \qquad (1.3.8)$$

This is easy to see for every particular form of these conditions, doing direct calculations and using the boundary conditions $(1.3.1)(b,c)$ we required for the extension $\tilde{\eta}$.

With the help of (1.3.7) we can solve a problem similar to (1.2.12) and, again, take out the pressure (as an unknown) from the equation (1.3.5). We can write now our full nonlinear problem in terms of the operator $\mathcal{L}$, so

$$\partial_t\begin{pmatrix} U \\ \eta \end{pmatrix} + \mathcal{L}\begin{pmatrix} U \\ \eta \end{pmatrix} = \begin{pmatrix} F(U,\eta) \\ 0 \end{pmatrix} \qquad (1.3.9)$$

with the boundary conditions (1.3.8) for the fixed boundary and (1.3.6) for the free boundary. For F and G we have the properties for $r \geq 1$ (see [Be1], [Schw1] and [Ta], Ch.13):

$$F : X^{r+2} \to H^r(\Omega_0)^3,\ F(0,0) = 0,\ DF \text{ exists and } DF(0,0) = 0,$$

$$G : X^{r+2} \to H^{r+1/2}(\Gamma_0)^2,\ G(0,0) = 0,\ DG \text{ exists and } DG(0,0) = 0.$$

We have to be careful because the condition of vanishing tangential stress on the free boundary is not fulfilled. We correct this by a function Φ.

Definition 1.3.1 *For a function $g = (g_1, g_2) \in H^{r+1/2}(\Gamma_0)^2$, we define the vector field $\Phi(g) : \Omega_0 \to \mathbb{R}^3$ which has the correct boundary values: let A be the Stokes operator, i.e. $Au := A(u,p) = -\nu\Delta u + \nabla p$ and $\nabla \cdot u = 0$. We define $\Phi(g)$ with the help of A to be the unique solution (we are not interested in the corresponding pressure for $\Phi(g)$) of:*

$$\begin{aligned} A\Phi(g) &= 0 \quad \text{in } \Omega_0 \\ \Phi(g)\big|_{\Sigma_{-h}} &= 0 \\ \Phi_n(g)\big|_{\Gamma_0\cup\Sigma_{1,2}} &= 0 \\ n \cdot S_{\Phi(g)} \cdot \tau_i\big|_{\Sigma_{1,2}} &= 0 \\ n \cdot S_{\Phi(g)} \cdot \tau_i\big|_{\Gamma_0} &= g_i\,. \end{aligned}$$

As the solution of the Stokes operator A with these boundary conditions, we have the following regularity estimates for Φ (see [ADN] and [Schw1]): $\forall r \geq 0$

$$\|\Phi(g)\|_{r+2,\Omega_0} \leq C\,\|g\|_{r+1/2,\Gamma_0}\,.$$

In the following we denote our variables together, so we want to find solutions $x := (U, \eta)$ of

$$(\partial_t + \mathcal{L})x = \begin{pmatrix} F(x) \\ 0 \end{pmatrix} \tag{1.3.10}$$

$$n \cdot S_U \cdot \tau_i\big|_{\Gamma_0} = G_i(x). \tag{1.3.11}$$

We consider now new variables, namely

$$\tilde{x} := x - \begin{pmatrix} \Phi \circ G(x) \\ 0 \end{pmatrix}. \tag{1.3.12}$$

If $x \in X^r$ satisfies the boundary condition (1.3.6), then $\tilde{x} \in \tilde{X}^r$, so $\tilde{x}$ has the correct (in the sense $\tilde{X}$) boundary conditions of vanishing tangential stress on the free boundary and we have not lost regularity through $\Phi \circ G$. Because $DG(0,0) = 0$, the map $\tilde{x} \mapsto x$ is close to identity, so we can locally solve (1.3.12) by $x = \varphi(\tilde{x})$. In the $\tilde{x}$ variable, the equation (1.3.9) becomes:

$$(\partial_t + \mathcal{L})\tilde{x} = \begin{pmatrix} \tilde{F}(\tilde{x}) \\ 0 \end{pmatrix} := \begin{pmatrix} F \circ \varphi(\tilde{x}) \\ 0 \end{pmatrix} - (\partial_t + \mathcal{L}) \begin{pmatrix} \Phi \circ G \circ \varphi(\tilde{x}) \\ 0 \end{pmatrix} \tag{1.3.13}$$

$$\tilde{x}(0) = x(0) - \begin{pmatrix} \Phi \circ G(x(0)) \\ 0 \end{pmatrix}. \tag{1.3.14}$$

We observe that we have a vanishing second component on the right hand side of (1.3.13) because of the property of Φ, $\Phi_n(g)\big|_{\Gamma_0} = 0$. $\tilde{F}$ keeps the properties of F, so

$$\tilde{F} : \tilde{X}^{r+2} \to H^r(\Omega_0)^3, \quad \tilde{F}(0) = 0, \quad D\tilde{F} \text{ exists and } D\tilde{F}(0) = 0.$$

Define the following operator ($r \geq 1$):

$$\mathcal{M} : H^1([0,T], \tilde{X}^r) \cap L^2([0,T], \tilde{X}^{r+2}) \cap \{z | z(0) \in \tilde{X}^{r+2}\} \to L^2([0,T], X^r) \times \tilde{X}^{r+2}$$

$$z \longmapsto \left((\partial_t + \mathcal{L})z - \begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix}(z), z(0) \right)$$

$\mathcal{M}$ has the following properties (using the properties of $\tilde{F}$):

$$D\mathcal{M}\Big|_{z=0} : y \longmapsto ((\partial_t + \mathcal{L})y, y(0))$$

It was shown in Theorem 1.2.15 that the problem

$$(\partial_t + \mathcal{L})y = \begin{pmatrix} f \\ 0 \end{pmatrix} \in L^2([0,T], X^r)$$

$$y(0) = y_0 \in \tilde{X}^{r+2}$$

has an unique solution, so $D\mathcal{M}\big|_{z=0}$ is an isomorphism between the spaces where $\mathcal{M}$ is defined. Then the implicit function theorem proves the existence of a unique solution of the nonlinear problem $\mathcal{M}(z) = ((f,0), z_0)$ for small enough $(f,0) \in L^2([0,T], X^r)$ and small enough initial values $z_0 \in \tilde{X}^{r+2}$. We can state now our nonlinear existence result:

Theorem 1.3.2 (Nonlinear existence result for $\tilde{X}$-spaces)
For $r \geq 1$, small enough $(f,0) \in L^2([0,T], X^r)$ and small enough initial values $z_0 \in \tilde{X}^{r+2}$, there exists a unique solution $z \in H^1([0,T], \tilde{X}^r) \cap L^2([0,T], \tilde{X}^{r+2})$ of the nonlinear problem $\mathcal{M}(z) = ((f,0), z_0)$.

Remark: A similar nonlinear existence result holds also in the spaces $X^r_{3/2}$ defined in (1.2.42). We observe that the transformation Θ we have done produced no term in the second component of the right hand side of the equation (1.3.10) (see also equation (1.3.4)). Moreover, if we consider from the beginning a nonzero second component of the right hand side, its regularity will be kept through Θ. The result is not needed for our Hopf bifurcation analysis, but for the seek of completeness we will formulate it here:

Theorem 1.3.3 (Nonlinear existence result for $\tilde{X}_{3/2}$-spaces)
Define the operator

$$\mathcal{N} : H^1([0,T], \tilde{X}^r_{3/2}) \cap L^2([0,T], \tilde{X}^{r+2}) \cap \{z | z(0) \in \tilde{X}^{r+2}\} \to L^2([0,T], X^r_{3/2}) \times \tilde{X}^{r+2}$$

$$z \longmapsto \left((\partial_t + \mathcal{L})z - \begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix} (z), z(0) \right),$$

where $\tilde{F}$ is defined in (1.3.13). Then, for r ≥ 1, small enough $(f,h) \in L^2([0,T], X^r_{3/2})$ and small enough initial values $z_0 \in \tilde{X}^{r+2}$, there exists a unique solution $z \in H^1([0,T], \tilde{X}^r_{3/2}) \cap L^2([0,T], \tilde{X}^{r+2})$ of the nonlinear problem $\mathcal{N}(z) = ((f,h), z_0)$.

Chapter 2

The Bifurcation Theory

2.1 The $\mathcal{L}$-invariant Decomposition

We want to split X^r and $\tilde{X}^r$ into a direct sum of $\mathcal{L}$-invariant subspaces $(X_i^r)_{i\in I}$.

The normed eigenvectors of $-\underline{\Delta}$ on Γ_0, with Neumann boundary conditions in the x_1-direction of the channel, form an orthonormal basis for $L^2(\Gamma_0)$. In order to find this basis explicitly, we solve the eigenvalue problem

$$\begin{aligned} -\underline{\Delta}\eta(x_1, x_2) &= \lambda\eta(x_1, x_2) \\ \partial_1\eta\big|_{x_1\in\{0,b\}} &= 0 \end{aligned}$$

using the method of separation of variables.

It is well-known (see e.g.[Da,Li], Ch.VIII, Th.8 and the applications presented here) that this problem has a countable number of eigenvalues $\lambda^{n,k}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ which are real, positive and simple. The eigenfunctions are

$$\eta^{n,k}(x_1, x_2) = c^{n,k} \cos\left(\frac{\pi}{b} n x_1\right) e^{ikx_2},$$

the constants $c^{n,k}$ being chosen in such a way that

$$\int_{\Gamma_0} |\eta^{n,k}|^2 dx_1\, dx_2 = 1.$$

So, $L^2(\Gamma_0)$ can be decomposed into a direct Hilbert sum

$$L^2(\Gamma_0) = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} l^2_{n,k}$$

where

$$l^2_{n,k}(\Gamma_0) = span\{\eta^{n,k}(x_1, x_2)\}.$$

Using the basis we found for $L^2(\Gamma_0)$, we want to construct a basis for $L^2(\Omega_0)^3$. Let $\vec{e}_3 = (0,0,1)$ be the normal vector on Γ_0, $\overline{\nabla} = \vec{e}_1\frac{\partial}{\partial x_1} + \vec{e}_2\frac{\partial}{\partial x_2}$ and $\overline{\nabla}^\perp = \vec{e}_1\frac{\partial}{\partial x_2} - \vec{e}_2\frac{\partial}{\partial x_1}$, where $\vec{e}_1 = (1,0,0)$ and $\vec{e}_2 = (0,1,0)$ are two tangent vectors to Γ_0.

Proposition 2.1.1 *The set*

$$\mathcal{B} = \{\eta^{n,k}(x_1, x_2)\vec{e}_3,\ \overline{\nabla}\eta^{n,k}(x_1, x_2)\,,\ \overline{\nabla}^\perp\eta^{n,k}(x_1, x_2)\}$$

is a basis for $L^2(\Gamma_0)^3$.

Proof: Because these vectors are orthogonal, they are linear independent. It remains to show that they span $L^2(\Gamma_0)^3$.

Let $u : \Gamma_0 \longrightarrow \mathbb{R}^3$ be a function orthogonal to every element in $\mathcal{B}$. Because it is orthogonal to $\vec{e}_3$, it is a tangent vector, so the third component of u is zero and we refer to u as $u \in L^2(\Gamma_0)^2$.

We prove that $u = 0$. Let C_0 be a smooth cut in Γ_0 such that $\Gamma_0 \setminus C_0$ is simply connected. Following [Te], we use the decomposition

$$L^2(\Gamma_0)^2 = H_0 \oplus \ker(\overline{\nabla}^\perp) =: H_0 \oplus H_1 \oplus H_2 \oplus H_c$$

where

$$\begin{aligned}
H_0 &= \{u \in L^2(\Gamma_0)^2 \,|\, \overline{\nabla}\cdot u = 0,\ u_n\big|_{C_{1,2}} = 0,\ \int_{C_0} u_n\, dC_0 = 0\} \\
H_1 &= \{u \in L^2(\Gamma_0)^2 \,|\, u = \overline{\nabla}q,\ \underline{\Delta}q = 0,\ q \in H^1(\Gamma_0)\} \\
H_2 &= \{u \in L^2(\Gamma_0)^2 \,|\, u = \overline{\nabla}q,\ q \in H^1_0(\Gamma_0)\} \\
H_c &= \{u \in L^2(\Gamma_0)^2 \,|\, u = \overline{\nabla}q,\ \underline{\Delta}q = 0 \text{ in } \Gamma_0 \setminus C_0,\ q \in H^1(\Gamma_0), \\
&\qquad [q]_{C_0} = \text{const},\ \left[\frac{\partial q}{\partial n}\right]_{C_0} = 0,\ \frac{\partial q}{\partial n}\Big|_{C_{1,2}} = 0\}
\end{aligned}$$

where $[q]_{C_0}$ denotes the jump of q on C_0. We know that the dimension of H_c is equal to the number of cuts which we need to make in order to obtain a simply connected domain, so is one.

Corresponding to this decomposition, we can split $u = u_0 + u_1 + u_2 + u_c$. Because of the direct sum, it follows that every u_0, u_1, u_2, u_c is orthogonal to every $\overline{\nabla}\eta^{n,k}$ and

$\overline{\nabla}^{\perp}\eta^{n,k}$. Then,

$$\begin{aligned}
0 &= \int_{\Gamma_0} u_1 \cdot \overline{\nabla}\eta^{n,k} \\
&= \int_{\Gamma_0} \overline{\nabla} q_1 \cdot \overline{\nabla}\eta^{n,k} \\
&= -\int_{\Gamma_0} q_1 \,\underline{\Delta}\eta^{n,k} + \int_{\partial\Gamma_0} q_1 \big(\overline{\nabla}\eta^{n,k} \cdot n\big) \\
&= \int_{\Gamma_0} q_1 \,\lambda^{n,k}\,\eta^{n,k}
\end{aligned}$$

and this implies

$$q_1 = 0 \quad\text{and}\quad u_1 = 0$$

because $\{\eta^{n,k}\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ is a basis for $L^2(\Gamma_0)$. In an analog way we obtain also $u_2 = u_c = 0$.

For $u_0 \in H_0$ we can calculate

$$0 = \int_{\Gamma_0} u_0 \cdot \overline{\nabla}^{\perp}\eta^{n,k} = -\int_{\Gamma_0} (\overline{\nabla}^{\perp}\, u_0)\eta^{n,k} + \int_{\partial\Gamma_0} u_{0_n}\,\eta^{n,k}$$

and then $\overline{\nabla}^{\perp}u_0 = 0$. But $u_0 \in H_0$ which is the ortogonal complement of $\ker(\overline{\nabla}^{\perp})$ in $L^2(\Gamma_0)^2$, so $u_0 = 0$ and the proof is complete. $\square$

Using the basis $\mathcal{B}$ for $L^2(\Gamma_0)^3$, we can decompose a function $u(x_1, x_2, x_3) \in L^2(\Omega_0)^3$:

$$\begin{aligned}
u(x_1,x_2,x_3) &= \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} U_1^{n,k}(x_3)\overline{\nabla}\eta^{n,k}(x_1,x_2) + U_2^{n,k}(x_3)\overline{\nabla}^{\perp}\eta^{n,k}(x_1,x_2) \\
&\qquad + U_3^{n,k}(x_3)\eta^{n,k}(x_1,x_2)\vec{e}_3 \\
&=: \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} u^{n,k}(x_1,x_2,x_3)
\end{aligned}$$

where $U_{1,2,3}^{n,k}$ are arbitrary real functions depending only on x_3, not all of them identically zero. Then,

$$L^2(\Omega_0)^3 = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} L^2_{n,k},$$

where $L^2_{n,k}$ is the corresponding space in the decomposition of $L^2(\Omega_0)^3$, for n, k fixed.

In order to find a $\mathcal{L}$-invariant decomposition for X^r we will see now how the divergence free condition and the boundary conditions are carried over. We fixe $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

$\vec{u}^{n,k}(x_1, x_2, x_3) =$

$$\begin{aligned}
&= \Big[-n\frac{\pi}{b}U_1^{n,k}(x_3)\sin\left(\frac{\pi}{b}nx_1\right) + ik\frac{2\pi}{l}U_2^{n,k}(x_3)\cos\left(\frac{\pi}{b}nx_1\right)\Big]e^{ikx_2}\vec{e}_1 \\
&\quad +\Big[ik\frac{2\pi}{l}U_1^{n,k}(x_3)\cos\left(\frac{\pi}{b}nx_1\right) + n\frac{\pi}{b}U_2^{n,k}(x_3)\sin\left(\frac{\pi}{b}nx_1\right)\Big]e^{ikx_2}\vec{e}_2 \\
&\quad +U_3^{n,k}(x_3)\cos\left(\frac{\pi}{b}nx_1\right)e^{ikx_2}\vec{e}_3 \\
&=: u_1^{n,k}\vec{e}_1 + u_2^{n,k}\vec{e}_2 + u_3^{n,k}\vec{e}_3
\end{aligned}$$

The divergence-free condition:

$$\nabla\cdot\vec{u}^{n,k} = 0 \Leftrightarrow (U_3^{n,k})'(x_3) = \lambda^{n,k}U_1^{n,k}(x_3) \quad \text{for } x_3 \in (-h, 0)$$

For X^r, we have to satisfy also the condition $\vec{u}\cdot n\big|_{\Sigma_{1,2,-h}} = 0$, which means:

on $\Sigma_{1,2}$: $n = \pm\vec{e}_1$

$$u_1^{n,k}\big|_{x_1\in\{0,l\}} = 0 \quad \Leftrightarrow \quad U_2^{n,k}(x_3) = 0 \quad \forall x_3 \in (-h, 0)$$

on Σ_{-h}: $n = -\vec{e}_3$

$$u_3^{n,k}\big|_{x_3=-h} = 0 \quad \Leftrightarrow \quad U_3^{n,k}(-h) = 0$$

For $\tilde{X}^r$ we have to satisfy additionally the conditions for the zero tangential stress on the free boundary and the walls, and zero tangential velocity on the bottom. We observe that the conditions on the walls $\Sigma_{1,2}$ are automatically satisfied.

on Σ_{-h}: $\vec{e}_1$ and $\vec{e}_2$ are tangential directions:

$$u_1^{n,k}\big|_{\Sigma_{-h}} = u_2^{n,k}\big|_{\Sigma_{-h}} = 0 \quad \Leftrightarrow \quad U_1^{n,k}(-h) = 0$$

on Γ_0: $n = \vec{e}_3$, $\vec{e}_1$ and $\vec{e}_2$ are tangential directions:

$$(\partial_3 u_i^{n,k} + \partial_i u_3^{n,k})\big|_{\Gamma_0} = 0, \ i = 1, 2 \quad \Leftrightarrow \quad U_3^{n,k}(0) + (U_1^{n,k})'(0) = 0.$$

Proposition 2.1.2 *The $\mathcal{L}$-invariant decompositions of the spaces X^r and $\tilde{X}^r$ are:*

$$X^r = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} X_{n,k}^r \qquad \tilde{X}^r = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} \tilde{X}_{n,k}^r$$

with

$$\begin{aligned}
X^r_{n,k} \;=\; \{(u^{n,k},\eta^{n,k}) \in H^r(\Omega_0)^3 \times H^{r+1/2}(\Gamma_0) \;\big|\; & \\
\eta^{n,k}(x_1,x_2) &= c^{n,k}\cos\left(\tfrac{\pi}{b}nx_1\right)e^{ikx_2},\\
\vec{u}^{n,k}(x_1,x_2,x_3) &= U_1^{n,k}(x_3)\overline{\nabla}\eta^{n,k}(x_1,x_2) + U_3^{n,k}(x_3)\eta^{n,k}(x_1,x_2)\vec{e}_3,\\
(U_3^{n,k})'(x_3) &= \lambda^{n,k}U_1^{n,k}(x_3),\ \ x_3\in(-h,0),\\
U_3^{n,k}(-h) &= 0\}
\end{aligned}$$

$$\begin{aligned}
\tilde{X}^r_{n,k} \;=\; \{(u^{n,k},\eta^{n,k}) \in X^r_{n,k} \;\big|\; & U_1^{n,k}(-h)=0\\
& U_3^{n,k}(0)+(U_1^{n,k})'(0)=0\}.
\end{aligned}$$

Proof:

It remains to prove that $\mathcal{L}$ defined on $\tilde{X}_{n,k}$ maps to $X_{n,k}$. Let $(u^{n,k},\eta^{n,k}) \in \tilde{X}_{n,k}$. Then

$$\mathcal{L}\begin{pmatrix} u^{n,k} \\ \eta^{n,k} \end{pmatrix} = \begin{pmatrix} -\nu\Delta u^{n,k} + \nabla\mathcal{H}(p\big|_{\Gamma_0}) \\ -u_n^{n,k}\big|_{\Gamma_0} \end{pmatrix},$$

where

$$\begin{aligned}
p\big|_{\Gamma_0} &= 2\nu S^n_{u^{n,k}}\big|_{\Gamma_0} + g\eta^{n,k} - \beta\underline{\Delta}\eta^{n,k}\\
&= [2\nu(U_3^{n,k})'(0) + g + \beta\lambda^{n,k}]\eta^{n,k}(x_1,x_2)
\end{aligned}$$

Because the solution of the problem (1.2.12) is unique and $\partial_1\eta^{n,k}\big|_{x_1\in\{0,b\}} = 0$, the harmonic extension of the pressure has the form

$$\mathcal{H}(p\big|_{\Gamma_0}) = P(x_3)\eta^{n,k}(x_1,x_2)$$

where $P(x_3)$ can be found explicitly as the solution of the problem

$$\begin{aligned}
P''(x_3) &= \lambda^{n,k}P(x_3) \qquad \text{for } x_3\in(-h,0)\\
P(0) &= 2\nu(U_3^{n,k})'(0) + g + \beta\lambda^{n,k}\\
P'(-h) &= \nu\,(U_3^{n,k})''(-h).
\end{aligned}$$

We have $P(x_3) = c_1e^{\sqrt{\lambda^{n,k}}x_3} + c_2e^{-\sqrt{\lambda^{n,k}}x_3}$, where c_1 and c_2 can be determined from the boundary conditions for $P(0)$ and $P'(-h)$.

After some simple calculations using the special form of $u^{n,k}$, we obtain:

$$\begin{aligned}
-\nu\Delta u^{n,k} + \nabla\mathcal{H}(p\big|_{\Gamma_0}) &= (\nu\lambda^{n,k}U_1^{n,k} - \nu(U_1^{n,k})'' + P)(x_3)\overline{\nabla}\eta^{n,k}(x_1,x_2)\\
&\quad +(\nu\lambda^{n,k}U_3^{n,k} - \nu(U_3^{n,k})'' + P')(x_3)\eta^{n,k}(x_1,x_2)\vec{e}_3\\
&=: f_1(x_3)\overline{\nabla}\eta^{n,k}(x_1,x_2) + f_3(x_3)\eta^{n,k}(x_1,x_2)\vec{e}_3
\end{aligned}$$

$$-u_n^{n,k}\big|_{\Gamma_0} = -U_3^{n,k}(0)\eta^{n,k}(x_1,x_2)$$

where f_1 and f_3 satisfy the conditions required in the $X_{n,k}$-space (because $(u^{n,k}, \eta^{n,k}) \in \tilde{X}_{n,k}$). The proof is complete. $\square$

Since we study the eigenvalue problem for $\mathcal{L}$, we can restrict ourself to such a space $X^r_{n,k}$ and make all considerations there. This is proved in the next proposition:

Proposition 2.1.3 *Let λ be an arbitrary eigenvalue of $\mathcal{L}$. Then there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that λ is an eigenvalue for $\mathcal{L}\big|_{\tilde{X}^r_{n,k}}$.*

Proof:

Let $\lambda \in \mathbb{C}$ be an arbitrary eigenvalue of $\mathcal{L}$, so $\exists\,(0,0) \not\equiv (u,\eta) \in \tilde{X}^r$ such that $\mathcal{L}(u,\eta) = \lambda(u,\eta)$. Decompose in a unique way

$$(u,\eta) = (\sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} u^{n,k}, \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} \eta^{n,k}) = \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} (u^{n,k}, \eta^{n,k}),$$

with $(u^{n,k}, \eta^{n,k}) \in \tilde{X}^r_{n,k}$.

We have $\lambda(u^{n,k}, \eta^{n,k}) \in \tilde{X}^r_{n,k}$ and the following equalities hold in the weak sense:

$$\begin{aligned} \sum \lambda(u^{n,k}, \eta^{n,k}) &= \mathcal{L}\big(\sum (u^{n,k}, \eta^{n,k})\big) \\ &= \sum \mathcal{L}(u^{n,k}, \eta^{n,k}). \end{aligned}$$

The decomposition is invariant under $\mathcal{L}$ and because of the direct sum, it follows: $\exists\, n \in \mathbb{N}$ and $\exists\, k \in \mathbb{Z}$ with

$$\lambda(u^{n,k}, \eta^{n,k}) = \mathcal{L}(u^{n,k}, \eta^{n,k})$$

with $\eta^{n,k} \not\equiv 0$, so $\lambda^{n,k}$ is an eigenvalue of $\mathcal{L}\big|_{\tilde{X}^r_{n,k}}$. $\square$

So we can restrict our considerations on such a space $X^r_{n,k}$ (actually we fixe $\eta^{n,k}$) and we will denote the functions there without indicies.

2.2 A Bifurcation Picture w.r.t α

Since the Navier-Stokes equations are invariant under the Euclidian group E_3 of all translations, rotations and reflections of space, the group of symmetries of a given model is a subgroup of E_3 determined by the shape of the domain and the boundary conditions. In our problem, we consider the symmetries obtained by translations along x_2 and reflections through a plane perpendicular to the x_2-axis.

The assumption on periodic boundary conditions in the x_2-direction allows us to identify these translations with the action of a circle group. These lead to an $O(2)$ symmetry, so our problem provides an $O(2)$-equivariance.

Remark: A reflection through the plane $\{x_1 = \frac{b}{2}\}$ is also a symmetry for our model. We did not consider it because it does not increase the dimension of the kernel spaces in the bifurcation theorem. This will become clear from the form of the function $\eta^{n,k}$.

$O(2)$ is generated by $SO(2)$ together with the flip $\varkappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $SO(2)$ consists of planar rotations $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. We refer to elements of $O(2)$ as 3×3 matrices, adding the third line and the third column $(0,0,1)$. We define the action of an element $\gamma \in O(2)$ on X^r by

$$\left.\begin{array}{rcl} \gamma * u & := & u \circ \gamma^{-1} \\ \gamma * \eta & := & \eta \circ \gamma^{-1} \\ \gamma * \begin{pmatrix} u \\ \eta \end{pmatrix} & := & \begin{pmatrix} \gamma * u \\ \gamma * \eta \end{pmatrix} . \end{array}\right\} \tag{2.2.1}$$

$SO(2)$ may be identified with the circle group S^1, the identification being $R_\theta \mapsto \theta$. Using this identification, we describe the action of $O(2) = \{se^{i\theta} : \theta \in \mathbb{R}, s \in \{id, \varkappa\}\}$ on X^r as follows: if $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$ is the velocity field,

$$\left.\begin{array}{rcl} \theta * \vec{u}(x_1,x_2,x_3) & := & u_1(x_1,x_2-\theta,x_3)\vec{e}_1 + u_2(x_1,x_2-\theta,x_3)\vec{e}_2 \\ & & +u_3(x_1,x_2-\theta,x_3)\vec{e}_3 \\ \\ \varkappa * \vec{u}(x_1,x_2,x_3) & := & u_1(x_1,-x_2,x_3)\vec{e}_1 - u_2(x_1,-x_2,x_3)\vec{e}_2 \\ & & +u_3(x_1,-x_2,x_3)\vec{e}_3 \\ \\ \theta * \eta(x_1,x_2) & := & \eta(x_1,x_2-\theta) \\ \\ \varkappa * \eta(x_1,x_2) & := & \eta(x_1,-x_2) \ . \end{array}\right\} \tag{2.2.2}$$

It is easy to see that $\mathcal{L}$ is $O(2)$-equivariant w.r.t. this action, i.e.

$$\gamma * \mathcal{L}\begin{pmatrix} u \\ \eta \end{pmatrix} = \mathcal{L}\left(\gamma * \begin{pmatrix} u \\ \eta \end{pmatrix}\right).$$

Lemma 2.2.1
The function $\eta^{n,k}$ has an isotropy subgroup $\Sigma_{\eta^{n,k}}$ of $O(2)$ isomorphic to $\mathbb{Z}_k$.

Proof: The actions of $e^{i\theta}$ and $\varkappa$ on $\eta^{n,k}$ are:

$$\begin{aligned}\theta * \eta^{n,k}(x_1,x_2) &= c^{n,k}\cos\left(\frac{\pi}{b}nx_1\right)e^{ik(x_2-\theta)}\,,\\ \varkappa * \eta^{n,k}(x_1,x_2) &= c^{n,k}\cos\left(\frac{\pi}{b}nx_1\right)e^{-ikx_2}\,.\end{aligned}$$

Imposing the isotropy condition we obtain

$$\theta * \eta^{n,k} = \eta^{n,k} \quad\Leftrightarrow\quad k\theta = 2m\pi,\quad m\in\mathbb{Z}$$

$$\gamma \in \Sigma_{\eta^{n,k}} \quad\Leftrightarrow\quad \gamma \in \{e^{i\frac{2\pi m}{k}}\,\big|\, m\in\mathbb{Z}\} \,\approx\, \mathbb{Z}_k.$$

□

We are now able to study the position of the eigenvalues of $\mathcal{L}$ depending on the gravity g and on the surface tension β. The position can be calculated explicitly for $g=\beta=0$ and for $g,\beta\to+\infty$. It is not of interest to study the problem for g and β separately. Anyway, these parameters are physical measures and they are fixed for a given liquid, but the "formal" analysis we are presenting here gives us useful ideas for the study of Hopf bifurcation in the next section. Then

$$(g-\beta\underline{\Delta})\eta^{n,k} = (g+\beta\lambda^{n,k})\eta^{n,k} =: \alpha\,\eta^{n,k},$$

with $\alpha := g+\beta\lambda^{n,k}\in[0,\infty)$.

Remark: In this section, n and k are fixed, so $\lambda^{n,k}$ is fixed, and varying α in the Theorem 2.2.6 means actually to vary g and β. This is also the reason for which we do not introduce n and k in the notation α for $g+\beta\lambda^{n,k}$.

Let $A:(u,p)\mapsto -\nu\Delta u+\nabla p$ together with the following conditions:

$$\left.\begin{aligned}\text{in }\Omega_0:\ \nabla\cdot u &= 0\\ n\cdot S_u\cdot\tau_i\big|_{\Gamma_0\cup\Sigma_{1,2}} &= 0\\ u_n\big|_{\Sigma_{1,2}} &= 0\\ u\big|_{\Sigma_{-h}} &= 0\end{aligned}\right\}\tag{2.2.3}$$

be the Stokes operator. In order to study eigenvalue problems for A, we have to impose one boundary condition more, i.e. one for the normal velocity on the free boundary Γ_0. We have two possibilities, to prescribe the normal velocity on Γ_0 (and obtain than a "Dirichlet" problem for the Stokes operator) or to prescribe the normal stress on the free boundary (and obtain than a "Neumann" problem for the Stokes operator). As soon as we have imposed a condition for $u_n\big|_{\Gamma_0}$ or for $(p-2\nu S_u^n)\big|_{\Gamma_0}$, we can calculate the value of the other one. Because we are in $X^r_{n,k}$, both of them should be multiple of $\eta^{n,k}$. Also, for fixed $\eta^{n,k}$, the pressure p is known as a function of u and $\eta^{n,k}$ (see (1.2.13)). Therefore, when we don't need to write the pressure explicitly, we will simplify the notation:

$$A(u,p) = -\nu\Delta u + \nabla p =: Au.$$

Definition 2.2.2 (The Stokes operators A_D and A_N)

Denote by A_D the Stokes operator A on $\tilde{X}^r$ together with the boundary condition of a vanishing normal component of the velocity at the free boundary. It is known that its eigenvalues are countable, real, positive and simple; we denote them by $\{\kappa_j\}_{j\in\mathbb{N}}$. The corresponding eigenfunctions with symmetry $\mathbb{Z}_k$ are unique up to a multiplicative constant. Let $\{u_j\}_{j\in\mathbb{N}}$ be the normed eigenfunctions with symmetry $\mathbb{Z}_k$ and $\{p_j\}_{j\in\mathbb{N}}$ be the pressure functions such that $(p_j - 2\nu S^n_{u_j})\big|_{\Gamma_0} = \eta^{n,k}$.

Denote by A_N the Stokes operator A on $\tilde{X}^r$ together with the boundary condition of a vanishing normal stress on the free boundary. It is known that its eigenvalues are countable, real, positive and simple; we denote them by $\{\rho_j\}_{j\in\mathbb{N}}$.

The Stokes operators A_D and A_N are elliptic in the sense of Agmon, Douglis and Nirenberg (see [ADN], and also [Be1], [Schw1]).

Following [Schw1], we define for every $\mu \in \mathbb{C}\setminus\{\kappa_j \mid j\in\mathbb{N}\}$, $(\tilde{u}(\mu), \tilde{p}(\mu))$ to be the unique solution of the problem ($\tilde{p}(\mu)$ is here unique up to an additive constant):

$$(\mu - A)\tilde{u}(\mu) \;=\; 0 \tag{2.2.4}$$
$$\tilde{u}_n(\mu)\big|_{\Gamma_0} \;=\; -\mu\eta^{n,k} \tag{2.2.5}$$

We know from the perturbation theory for linear operators (see [Ka], and also [Schw1]) that $(\tilde{u}(\mu), \tilde{p}(\mu))$ is an analytic family of functions for $\mu \in \mathbb{C}\setminus\{\kappa_j \mid j\in\mathbb{N}\}$.

One verifies easily that $X^r_{n,k}$ are invariant subspaces also for A_D and A_N. Therefore the (unique) solution of (2.2.4)-(2.2.5) must be in $X^r_{n,k}$. In particular $(\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\big|_{\Gamma_0}$ is a multiple of $\eta^{n,k}$. We define $\tilde{r}(\mu)\in\mathbb{C}$ by

$$(\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\big|_{\Gamma_0} =: \tilde{r}(\mu)\eta^{n,k}. \tag{2.2.6}$$

Of course, every $\mu \neq \kappa_j$ eigenvalue of $\mathcal{L}$ together with the corresponding eigenfunction satisfy the problem (2.2.4)-(2.2.5). Reciprocally, a $\mu\in\mathbb{C}$ is an eigenvalue on $\mathcal{L}$ with eigenfunction $(\tilde{u}(\mu), \eta^{n,k})$, if and only if

$$\tilde{r}(\mu) = \alpha.$$

Lemma 2.2.3 *We have: $\mu\in\mathbb{R}$ implies $\tilde{r}(\mu)\in\mathbb{R}$.*

Proof: Testing the eigenvalue equation (2.2.4) with $\bar{\tilde{u}}$ and using Corollary 1.2.3, we obtain:

$$\begin{aligned}
\mu \int_{\Omega_0} |\tilde{u}(\mu)|^2 &= \int_{\Omega_0} [-\nu\Delta\tilde{u}(\mu) + \nabla\mathcal{H}(2\nu S^n_{\tilde{u}(\mu)}\big|_{\Gamma_0})]\,\bar{\tilde{u}}(\mu) \\
&\quad + \int_{\Omega_0} [\nabla\tilde{p}(\mu) - \nabla\mathcal{H}(2\nu S^n_{\tilde{u}(\mu)}\big|_{\Gamma_0})]\,\bar{\tilde{u}}(\mu) \\
&= 2\nu \int_{\Omega_0} S_{\tilde{u}(\mu)} : S_{\bar{\tilde{u}}(\mu)} + \int_{\Gamma_0} (\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\,\bar{\tilde{u}}_n(\mu) \\
&= 2\nu \int_{\Omega_0} |S_{\tilde{u}(\mu)}|^2 + \int_{\Gamma_0} \tilde{r}(\mu)\eta^{n,k}(-\bar{\mu}\bar{\eta}^{n,k}) \\
&= 2\nu \int_{\Omega_0} |S_{\tilde{u}(\mu)}|^2 - \bar{\mu}\int_{\Gamma_0} \tilde{r}(\mu)\,|\eta^{n,k}|^2
\end{aligned}$$

and the lemma is proved.

□

In the following we abbreviate by $\|\cdot\|$ (without indicies) the $L^2(\Omega_0)^3$-norm or the $L^2(\Gamma_0)$-norm.

Proposition 2.2.4 (Properties of $\tilde{u}(\mu)$)
(a) In κ_j there holds

$$\|\tilde{u}(\mu)\| \to +\infty \quad for \quad \mu \to \kappa_j. \tag{2.2.7}$$

(b) The rescaled functions approximate the eigenfunctions of A_D, so

$$u_j := \lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|} = -\lim_{\mathbb{R}\ni\mu\searrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|}. \tag{2.2.8}$$

(c)

$$\|\tilde{u}(\mu)\| \to +\infty \quad for \quad |\mu| \to +\infty. \tag{2.2.9}$$

Proof: We define the familly of functions $(u(\mu), p(\mu))$ which depends smooth on μ in a neighborhood of κ_j to be the unique (nonzero) solution of the following problem for the Stokes operator:

$$\begin{aligned}
(\mu - A)u(\mu) &= 0 && (2.2.10)\\
(p(\mu) - 2\nu S^n_{u(\mu)})\big|_{\Gamma_0} &= \eta^{n,k}. && (2.2.11)
\end{aligned}$$

Denote

$$u_n(\mu)\big|_{\Gamma_0} =: s(\mu)\eta^{n,k}, \tag{2.2.12}$$

we have the properties: $s(\cdot)$ is differentiable and $s(\kappa_j) = 0$ (because the eigenvalues of A_D are simple), so for $\mu = \kappa_j$, $u(\kappa_j)$ is a multiple of u_j

$$u(\kappa_j) = \mathrm{const}_1\, u_j \not\equiv 0.$$

(a) Comparing the problem (2.2.4),(2.2.5) with the problem (2.2.10),(2.2.12) we obtain:

$$\tilde{u}(\mu) = \frac{-\mu}{s(\mu)} u(\mu),$$

so

$$\|\tilde{u}(\mu)\| = \left|\frac{-\mu}{s(\mu)}\right| \, |\mathrm{const}_1| \cdot 1$$

and

$$\|\tilde{u}(\mu)\| \to +\infty \quad \text{for} \quad \mu \to \kappa_j.$$

(b) Then

$$\lim_{\mathbb{R}\ni\mu\to\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|} = -\mathrm{sign}\; s(\kappa_j) \cdot \mathrm{sign}(\mathrm{const}_1) \cdot u_j$$

and the sign of the limit will be established by showing that the function $s(\cdot)\big|_{\mathbb{R}}$ changes sign in κ_j.

Assume this is not true, so $\partial_\mu s(\kappa_j) = 0$. Defining the functions (v, q), $v := \partial_\mu u(\kappa_j)$ and $q := \partial_\mu p(\kappa_j)$, they satisfy the following equations:

$$\begin{aligned} (\kappa_j - A)v &= -u(\kappa_j) \\ v_n\big|_{\Gamma_0} = \partial_\mu s(\kappa_j)\eta^{n,k} &= 0 \\ (q - 2\nu S^n_v)\big|_{\Gamma_0} &= 0. \end{aligned} \tag{2.2.13}$$

Testing the equation (2.2.13) with $\bar{u}(\kappa_j)$, integrating by parts and using the equality (1.2.16) together with the boundary conditions of the equation (2.2.13), yields:

$$\begin{aligned} 0 \neq -\|u(\kappa_j)\|^2 &= \int_{\Omega_0} [\kappa_j v + \nu\Delta v - \nabla q]\, \bar{u}(\kappa_j) \\ &= \int_{\Omega_0} [\kappa_j \bar{u}(\kappa_j) + \nu\Delta\bar{u}(\kappa_j) - \nabla\bar{p}(\kappa_j)]\, v \\ &= 0, \end{aligned}$$

a contradiction, so $s(\cdot)\big|_{\mathbb{R}}$ changes sign in κ_j, i.e.

$$\lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|} = -\lim_{\mathbb{R}\ni\mu\searrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|},$$

and we choose

$$u_j := \lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|}.$$

(c) As the solution of the Stokes system (2.2.4)$-$(2.2.5), $\tilde{u}(\mu)$ is sufficiently smooth and satisfies the estimate ($C_S > 0$ is a constant):

$$\|\tilde{u}(\mu)\|_{2,\Omega_0} \leq C_S\big(|\mu|\ \|\tilde{u}(\mu)\|_{0,\Omega_0} + |\mu|\ \|\eta^{n,k}\|_{3/2,\Gamma_0}\big).$$

We use now $\tilde{u}_n(\mu)\big|_{\Gamma_0} = -\mu\eta^{n,k}$, a trace formula (with constant $C_T > 0$) and an interpolation (with constant $C_I > 0$) (see also Proposition 1.2.8) to calculate:

$$\begin{aligned}
|\mu|^2\|\eta^{n,k}\|^2_{0,\Gamma_0} &= \|\tilde{u}_n(\mu)\|^2_{0,\Gamma_0} \\
&\leq C_T\|\tilde{u}(\mu)\|^2_{1,\Omega_0} \\
&\leq C_T\, C_I\|\tilde{u}(\mu)\|_{0,\Omega_0}\|\tilde{u}(\mu)\|_{2,\Omega_0} \\
&\leq C_T\, C_I\, C_S\|\tilde{u}(\mu)\|_{0,\Omega_0}\big(|\mu|\ \|\tilde{u}(\mu)\|_{0,\Omega_0} + |\mu|\ \|\eta^{n,k}\|_{3/2,\Gamma_0}\big),
\end{aligned}$$

and then

$$|\mu|\|\eta^{n,k}\|^2_{0,\Gamma_0} \leq C_T\, C_I\, C_S\|\tilde{u}(\mu)\|_{0,\Omega_0}\big(\|\tilde{u}(\mu)\|_{0,\Omega_0} + \|\eta^{n,k}\|_{3/2,\Gamma_0}\big)$$

which imply $\|\tilde{u}(\mu)\|_{0,\Omega_0} \to +\infty$ for $|\mu| \to +\infty$.

$\square$

Proposition 2.2.5 (Properties of $\tilde{r}(\mu)$)
The function $\tilde{r}(\mu)$ satisfies:
(a)

$$\lim_{\mathbb{R}\ni\mu\searrow 0} \tilde{r}(\mu) = 0\,; \tag{2.2.14}$$

(b)

$$\lim_{\mathbb{R}\ni\mu\searrow\kappa_j} \tilde{r}(\mu) = -\lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \tilde{r}(\mu) = +\infty\,; \tag{2.2.15}$$

(c) $\tilde{r}(\mu)$ is positive for small $\mu > 0$ and $\partial_\mu\tilde{r}(\mu)\big|_{\mu=0} > 0$;
(d) it has exactly one turning point on each interval (κ_j, κ_{j+1}), $j \in \mathbb{N}$;
it does not have turning points on the interval $(-\infty, \kappa_0)$;
(e) critical values of $\tilde{r}(\mu)$ are positive.

Proof:

(a) Putting $\mu = 0$ in the problem (2.2.4)$-$(2.2.5) we obtain that $\tilde{u}(0) \equiv 0$ (because 0 is not an eigenvalue of A_D) and then $\tilde{r}(0) = 0$.

(b) The function

$$w(\mu) := \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|} = \tilde{u}(\mu)\left|\frac{s(\mu)\mathrm{const}_1}{\mu}\right|$$

(together with the corresponding pressure $\frac{\tilde{p}(\mu)}{\|\tilde{u}(\mu)\|}$) satifies (2.2.4) and the boundary conditions on Γ_0:

$$\begin{aligned} w_n(\mu)\big|_{\Gamma_0} &= -\mu\,\eta^{n,k}\cdot\left|\frac{s(\mu)\mathrm{const}_1}{\mu}\right| \\ \left(\frac{\tilde{p}(\mu)}{\|\tilde{u}(\mu)\|} - 2\nu S^n_{w(\mu)}\right)\Big|_{\Gamma_0} &= \frac{\tilde{r}(\mu)}{\|\tilde{u}(\mu)\|}\eta^{n,k}. \end{aligned}$$

Because $s(\kappa_j) = 0$ and $\|w(\mu)\| = 1$, using Proposition 2.2.4 it follows

$$\begin{aligned} w(\mu) &\to \pm u_j \quad \text{for} \quad \mu \to \kappa_j \\ \frac{\tilde{p}(\mu)}{\|\tilde{u}(\mu)\|} &\to \pm\mathrm{const}_2 p_j \quad \text{for} \quad \mu \to \kappa_j, \end{aligned}$$

so

$$\frac{\tilde{r}(\mu)}{\|\tilde{u}(\mu)\|} \to \pm\mathrm{const}_3 \quad \text{for} \quad \mu \to \kappa_j.$$

Because of (2.2.7), $\tilde{r}(\mu)$ cannot stay finite for $\mu \to \kappa_j$ and $\tilde{r}(\cdot)\big|_{\mathbb{R}}$ changes sign in κ_j like $s(\cdot)\big|_{\mathbb{R}}$ does.

(c, d, e) Consider the functions $\tilde{v}(\mu) := \partial_\mu \tilde{u}(\mu)$ and $\tilde{q}(\mu) := \partial_\mu \tilde{p}(\mu)$ which satisfy:

$$\begin{aligned} (\mu - A)\tilde{v}(\mu) &= -\tilde{u}(\mu) \\ \tilde{v}_n(\mu)\big|_{\Gamma_0} &= -\eta^{n,k} \\ (\tilde{q}(\mu) - 2\nu S^n_{\tilde{v}(\mu)})\big|_{\Gamma_0} &= \partial_\mu \tilde{r}(\mu)\eta^{n,k}. \end{aligned} \tag{2.2.16}$$

For $\mu = 0$, so $(\tilde{v}(0), \tilde{q}(0))$, the right hand side of the equation (2.2.16) becomes zero and testing it with $\bar{\tilde{v}}(0)$ yields

$$\int_{\Omega_0} |S_{\tilde{v}(0)}|^2 = \int_{\Gamma_0} (\tilde{q}(0) - 2\nu S^n_{\tilde{v}(0)})\tilde{v}_n(0) = \partial_\mu \tilde{r}(\mu)\big|_{\mu=0} \cdot \|\eta^{n,k}\|^2$$

which implies $\partial_\mu \tilde{r}(\mu)\big|_{\mu=0} > 0$ $\big($we have $\partial_\mu \tilde{r}(\mu)\big|_{\mu=0} \neq 0$ because 0 is not an eigenvalue of $A_N\big)$; this proves (c) .

Testing the equation (2.2.16) with $\bar{\tilde{u}}(\mu)$ and using the identity (1.2.16), we obtain

$$\begin{aligned} \|\tilde{u}(\mu)\|^2 &= \int_{\Omega_0} (\mu\tilde{v}(\mu) + \nu\Delta\tilde{v}(\mu) - \nabla\tilde{q}(\mu))\bar{\tilde{u}}(\mu) \\ &= \int_{\Omega_0} (\mu\bar{\tilde{u}}(\mu) + \nu\Delta\bar{\tilde{u}}(\mu) - \nabla\bar{\tilde{p}}(\mu))\tilde{v}(\mu) \\ &\quad + \int_{\Gamma_0} (\bar{\tilde{p}}(\mu) - 2\nu S^n_{\bar{\tilde{u}}(\mu)})\tilde{v}_n(\mu) - \int_{\Gamma_0} (\tilde{q}(\mu) - 2\nu S^n_{\tilde{v}(\mu)})\bar{\tilde{u}}_n(\mu)\ . \end{aligned}$$

This yields, together with (2.2.5), (2.2.6) and the boundary conditions from the problem (2.2.16):

$$\|\tilde{u}(\mu)\|^2 - \tilde{r}(\mu)\|\eta^{n,k}\|^2 + \mu\partial_\mu\tilde{r}(\mu)\|\eta^{n,k}\|^2 = 0 \tag{2.2.17}$$

which implies for any critical point $\mu_{crit} \in \mathbb{R}\setminus\{0, \kappa_j | j \in \mathbb{N}\}$ of $\tilde{r}$ (so $\partial_\mu\tilde{r}(\mu)\big|_{\mu=\mu_{crit}} = 0$), that $\tilde{r}(\mu_{crit}) > 0$ $\big(\|\tilde{u}(\mu_{crit})\|^2 = 0$ would imply $\tilde{u}(\mu_{crit}) \equiv 0$ which is in contradiction with the boundary condition (2.2.5)$\big)$; this proves (e).

Differentiating (2.2.17) w.r.t. μ we obtain

$$\partial_\mu\left(\|\tilde{u}(\mu)\|^2\right) + \mu\partial_\mu^2\tilde{r}(\mu)\|\eta^{n,k}\|^2 = 0, \tag{2.2.18}$$

which implies

$$\partial_\mu\left(\|\tilde{u}(\mu)\|^2\right) = 0 \quad \overset{\mu\neq 0}{\Longleftrightarrow} \quad \partial_\mu^2\tilde{r}(\mu) = 0,$$

so the turning points of $\tilde{r}(\mu)$, $\mu \in \mathbb{R}\setminus\{0, \kappa_j | j \in \mathbb{N}\}$, coincide with the critical points of $\|\tilde{u}(\mu)\|^2$, $\mu \in \mathbb{R}\setminus\{0, \kappa_j | j \in \mathbb{N}\}$.

We can calculate further

$$\partial_\mu^2\left(\|\tilde{u}(\mu)\|^2\right) = 2\|\partial_\mu\tilde{u}(\mu)\|^2 + 2\langle\tilde{u}(\mu), \partial_\mu^2\tilde{u}(\mu)\rangle \tag{2.2.19}$$

and we are looking for an expresion for $\langle\tilde{u}(\mu), \partial_\mu^2\tilde{u}(\mu)\rangle$.

Define the functions $\tilde{w}(\mu) := \partial_\mu^2\tilde{u}(\mu)$, $\tilde{t}(\mu) := \partial_\mu^2\tilde{p}(\mu)$ which satisfy

$$\begin{aligned} (\mu - A)\tilde{w}(\mu) &= -2\tilde{v}(\mu) \\ \tilde{w}_n(\mu)\big|_{\Gamma_0} &= 0 \\ (\tilde{t}(\mu) - 2\nu S^n_{\tilde{w}(\mu)})\big|_{\Gamma_0} &= \partial_\mu^2\tilde{r}(\mu)\eta^{n,k}. \end{aligned} \tag{2.2.20}$$

Testing the equation (2.2.20) with $\bar{\tilde{v}}(\mu)$ and using the identity (1.2.16) we obtain:

$$\begin{aligned} -2\|\tilde{v}(\mu)\|^2 &= \int_{\Omega_0}(\mu\tilde{w}(\mu) + \nu\Delta\tilde{w}(\mu) - \nabla\tilde{t}(\mu))\bar{\tilde{v}}(\mu) \\ &= \int_{\Omega_0}(\mu\bar{\tilde{v}}(\mu) + \nu\Delta\bar{\tilde{v}}(\mu) - \nabla\bar{\tilde{q}}(\mu))\tilde{w}(\mu) \\ &\quad + \int_{\Gamma_0}(\bar{\tilde{q}}(\mu) - 2\nu S^n_{\bar{\tilde{v}}(\mu)})\tilde{w}_n(\mu) - \int_{\Gamma_0}(\tilde{t}(\mu) - 2\nu S^n_{\tilde{w}(\mu)})\bar{\tilde{v}}_n(\mu) \end{aligned}$$

and using the problems (2.2.16) and (2.2.20), this yields the equation:

$$2\|\tilde{v}(\mu)\|^2 - \langle\tilde{u}(\mu), \tilde{w}(\mu)\rangle + \partial_\mu^2\tilde{r}(\mu)\|\eta^{n,k}\|^2 = 0 \tag{2.2.21}$$

$\big($where we have denoted by $\langle\cdot,\cdot\rangle$ the usual scalar product in $L^2(\Omega_0)^3\big)$. Together with (2.2.19) we obtain:

$$\partial_\mu^2\left(\|\tilde{u}(\mu)\|^2\right) = 6\|\tilde{v}(\mu)\|^2 + 2\partial_\mu^2\tilde{r}(\mu)\|\eta^{n,k}\|^2 . \tag{2.2.22}$$

Let $\mu_{turn} \in \mathbb{R}\setminus\{0,\kappa_j|j\in\mathbb{N}\}$ be a critical point of $\|\tilde{u}(\mu)\|^2$. It is also a turning point of $\tilde{r}(\mu)$ and using (2.2.22) for $\mu=\mu_{turn}$, we can calculate:

$$\partial_\mu^2\left(\|\tilde{u}(\mu)\|^2\right)\big|_{\mu=\mu_{turn}} = 6\|\tilde{v}(\mu_{turn})\|^2 > 0 .$$

$\Big(\|\tilde{v}(\mu_{turn})\|^2 = 0$ would imply $\tilde{v}(\mu_{turn}) \equiv 0$ which is a contradiction because $\tilde{v}_n(\mu)\big|_{\Gamma_0} = -\eta^{n,k}\ \forall\mu$ (see the problem (2.2.16))$\Big)$, so all the critical points of $\|\tilde{u}(\mu)\|^2$ in $\mathbb{R}\setminus\{0,\kappa_j|j\in\mathbb{N}\}$ are points of local minimum. We collect now the properties of the function $\|\tilde{u}(\mu)\|^2$:

$\|\tilde{u}(\mu)\|^2 \geq 0$	$\forall\mu\in\mathcal{D}(\|\tilde{u}(\mu)\|^2) = \mathbb{R}\setminus\{\kappa_j\|j\in\mathbb{N}\}$;
Every critical point	$\mu_{turn}\neq 0$ is a local minimum of $\|\tilde{u}(\mu)\|^2$;
$\|\tilde{u}(\mu)\| \to +\infty$	for $\|\mu\|\to+\infty$ or $\mu\to\kappa_j(j\in\mathbb{N})$ (see (2.2.7) and (2.2.9)) ;
$\|\tilde{u}(0)\|^2 = 0$	because 0 is not an eigenvalue of A_D .

Then we can conclude that the function $\|\tilde{u}(\mu)\|^2$ has exactly one critical point (and this is a minimum) on each interval $(-\infty,\kappa_0)$, (κ_j,κ_{j+1}) $(j\in\mathbb{N})$. For every interval (κ_j,κ_{j+1}), this is equivalent to say that the function $\tilde{r}(\mu)$ has exactly one turning point on each (κ_j,κ_{j+1}).

For the interval $(-\infty,\kappa_0)$, we know that the unique critical point of the function $\|\tilde{u}(\mu)\|^2$ is the point $\mu=0$, but we can not say that this is also a turning point for $\tilde{r}$ (see the equation (2.2.18)). Moreover, we will show that it is not a turning point of $\tilde{r}$, so $\partial_\mu^2\tilde{r}(\mu)\big|_{\mu=0}\neq 0$.

Suppose that $\partial_\mu^2\tilde{r}(\mu)\big|_{\mu=0} = 0$. For $\mu=0$ we know $\tilde{u}(0)\equiv 0$ (because 0 is not an eigenvalue of A_D) and using (2.2.21) we obtain $\tilde{v}(0)\equiv 0$ which is a contradiction because $\tilde{v}_n(0)\big|_{\Gamma_0} = -\eta^{n,k}$ (see the problem (2.2.16)).

So, for the interval $(-\infty,\kappa_0)$ we can conclude that $\tilde{r}$ does not have turning points; (d) is also proved.

□

We can draw now the graph of $\tilde{r}$ for $\mu\in\mathbb{R}$ (see Figure 2). On $(0,\kappa_0)$ we know exactly how it looks like, on (κ_j,κ_{j+1}) we have two possibilities: $\tilde{r}$ is monoton descending or has a local maximum and a local minimum, both positive. We have drawn the graph of $\tilde{r}$ also for negative μ (because we need it for the next section).

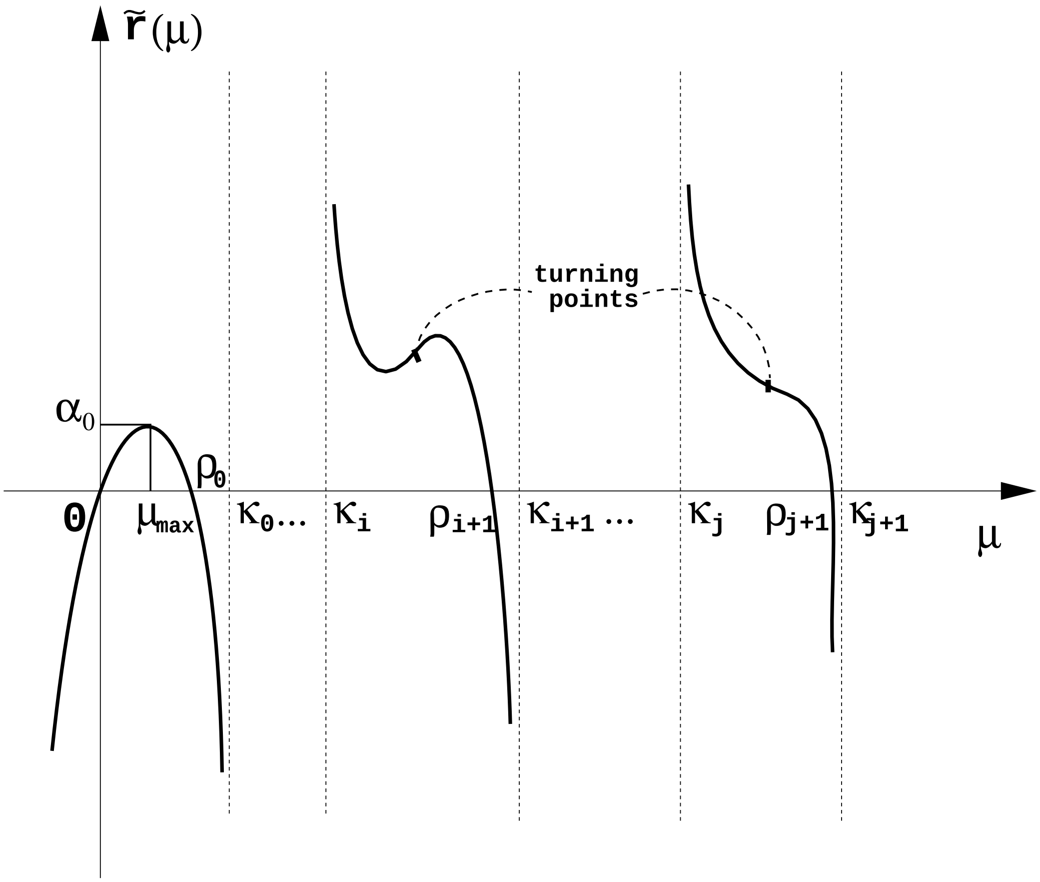

Figure 2: The Graph of $\tilde{r}(\mu)$

We know that $\tilde{r}$ has no negative zeros, and no turning points on $(-\infty, \kappa_0)$, so it should looks like a "parabola" on this interval.

The numbers $\rho_j > 0$ are zeros of the $\tilde{r}(\mu)$, so the shape of $\tilde{r}$ implies

$$\rho_j < \kappa_j < \rho_{j+1} \quad \forall j \in \mathbb{N}.$$

Theorem 2.2.6 (The global bifurcation picture in α)
For $\alpha = 0$ all the eigenvalues of $\mathcal{L}_\alpha\big|_{\tilde{X}^r_{n,k}}$ are real. Denoting them by $\{\mu_j\}_{j\in\mathbb{N}}$, it holds

$$\mu_0 = 0, \;\; \mu_{j+1} = \rho_j \quad \forall j \in \mathbb{N}.$$

For some $\alpha_0 > 0$ the first two eigenvalues merge and leave the real axis.
Given a number $\omega \in \mathbb{R}$, there exists $\alpha_\omega > 0$ such that for $\alpha > \alpha_\omega$ every interval (κ_j, κ_{j+1}) with $\kappa_{j+1} < \omega$ contains one and only one eigenvalue $\mu(\alpha)$ of $\mathcal{L}_\alpha$ (which

is the unique real solution of the equation $\tilde{r}(\mu) = \alpha$ on this interval) and this real eigenvalue satisfies

$$\mu^{\mathbb{R}}(\alpha) \searrow \kappa_j \quad for \quad \alpha \to +\infty.$$

For the nonreal eigenvalues it holds

$$|\mu^{\mathbb{C}}(\alpha)| \to +\infty \quad for \quad \alpha \to +\infty.$$

Proof: The statements for the real eigenvalues of $\mathcal{L}_\alpha$ are clear from the graph of $\tilde{r}$.

For $\alpha = 0$ we can compute a complete set of eigenfunctions in $X^r_{n,k}$:

$$\begin{aligned} \mu_0 &= 0 \quad \text{with eigenfunction} \quad (0, \eta^{n,k}) \\ \mu_{j+1} &= \rho_j \quad \text{with eigenfunction} \quad (\tilde{u}(\rho_j), \eta^{n,k}) \end{aligned}$$

Let $\mu_{\max}$ be the critical point of $\tilde{r}$ on $(0, \kappa_0)$. Then $\alpha_0 := \tilde{r}(\mu_{\max})$ and from the shape of $\tilde{r}$ we see that for $\alpha \geq \alpha_0$ the first two eigenvalues merge and leave the real axis.

Let $\omega \in \mathbb{R}$ be given, then there exists $i \in \mathbb{N}$ such that $0 < \kappa_0 < \ldots < \kappa_i < \omega$ and define α_ω to be the biggest local maximum of $\tilde{r}(\mu)$ on (κ_j, κ_{j+1}) for all j, $j < i$. The rest is clear from the shape of $\tilde{r}$.

It remains now to prove only the assertion on the nonreal eigenvalues. We suppose we have a sequence of nonreal eigenvalues $\mu(\alpha)$ of $\mathcal{L}_\alpha$ which are bounded independent of α, so suppose:

$$\mu(\alpha) \to \mu_\infty \in \mathbb{C} \quad \text{for a sequence } \alpha \to +\infty.$$

Denoting the corresponding eigenfunctions of $\mathcal{L}_\alpha$ with $(\tilde{u}(\mu(\alpha)), \eta^{n,k})$, they satisfy the energy equality $\forall \alpha \in \mathbb{R}$:

$$\|\tilde{u}(\mu(\alpha))\|^2 = \alpha \|\eta^{n,k}\|^2$$

and the condition for the normal stress on the free boundary:

$$(\tilde{p}(\mu(\alpha)) - 2\nu S^n_{\tilde{u}(\mu(\alpha))})\big|_{\Gamma_0} = \alpha \eta^{n,k}.$$

where $\tilde{p}(\mu(\alpha)$ is the corresponding pressure function. Because $\mu(\alpha)$ is nonreal $\forall \alpha$, it never meets κ_j and the pair $(\tilde{u}(\mu(\alpha)), \tilde{p}(\mu(\alpha)))$ is also a nonzero solution of the problem $(2.2.4) - (2.2.5)$.

Then the pair $(v(\mu(\alpha)), q(\mu(\alpha)))$,

$$v(\mu(\alpha)) := \frac{\tilde{u}(\mu(\alpha))}{\alpha} \quad \text{and} \quad q(\mu(\alpha)) := \frac{\tilde{p}(\mu(\alpha))}{\alpha},$$

satisfies the equations:

$$\begin{aligned}(\mu(\alpha) - A)v(\mu(\alpha)) &= 0\\ v_n(\mu(\alpha))\big|_{\Gamma_0} &= -\frac{\mu(\alpha)}{\alpha}\eta^{n,k}\\ (q(\mu(\alpha)) - 2\nu S^n_{v(\mu(\alpha))})\big|_{\Gamma_0} &= \eta^{n,k}.\end{aligned}$$

Passing to the limit $\alpha \to +\infty$ in all these equations, using our hypothesis $\mu(\alpha) \to \mu_\infty \in \mathbb{C}$ and continuity w.r.t. μ of the functions v and q, the pair of the limit functions $(v(\mu_\infty), q(\mu_\infty))$

$$v(\mu_\infty) := \lim_{\alpha\to+\infty} v(\mu(\alpha)) \quad \text{and} \quad q(\mu_\infty) := \lim_{\alpha\to+\infty} q(\mu(\alpha))$$

satisfies the following equations:

$$\begin{aligned}(\mu_\infty - A)v(\mu_\infty) &= 0\\ v_n(\mu_\infty)\big|_{\Gamma_0} &= 0\\ (q(\mu_\infty) - 2\nu S^n_{v(\mu_\infty)})\big|_{\Gamma_0} &= \eta^{n,k}.\end{aligned}$$

and because the normal stress on the free boundary is $\eta^{n,k}$, the solution $v(\mu_\infty) \not\equiv 0$.

On the other hand, using the energy equality we can calculate:

$$\begin{aligned}0 \neq \|v(\mu_\infty)\|^2 &= \lim_{\alpha\to+\infty}\left\|\frac{\tilde{u}(\mu(\alpha))}{\alpha}\right\|^2\\ &= \lim_{\alpha\to+\infty}\frac{\alpha\|\eta^{n,k}\|^2}{\alpha^2}\\ &= 0,\end{aligned}$$

a contradiction, so for nonreal eigenvalues, $|\mu(\alpha)| \to +\infty$ for $\alpha \to +\infty$.

□

Proposition 2.2.7
(a) Eigenvalues of $\mathcal{L}_\alpha$ leave the real axis with an infinite speed (w.r.t. α).
(b) The qualitative shape of $\tilde{r}_\nu(\mu)$ is independent of the viscosity ν:

$$\tilde{r}_{\epsilon\nu}(\epsilon\mu) = \epsilon^2\tilde{r}_\nu(\mu).$$

Proof:
(a) The eigenvalues of $\mathcal{L}_\alpha$ leave the real axis in a critical point of $\tilde{r}$ and we denote it by $\mu_{crit}(\alpha) \in \mathbb{R}$. Because $\tilde{r}$ is an analytic function in $\mathbb{C} \setminus \{\kappa_j \mid j \in \mathbb{N}\}$, we have:

$$\frac{\partial \mathrm{Re}(\tilde{r}(\mu(\alpha)))}{\partial\mu}\Big|_{\mu=\mu_{crit}} = \frac{\partial \mathrm{Im}(\tilde{r}(\mu(\alpha)))}{\partial\mu}\Big|_{\mu=\mu_{crit}} = 0.$$

Because $\tilde{r}(\mu_{crit}(\alpha)) = \alpha$, we can calculate

$$1 = \frac{\partial \mathrm{Re}(\tilde{r}(\mu(\alpha)))}{\partial \alpha}\Big|_{\mu=\mu_{crit}} = \frac{\partial \mathrm{Re}(\tilde{r}(\mu(\alpha)))}{\partial \mu}\Big|_{\mu=\mu_{crit}} \cdot \frac{\partial \mu}{\partial \alpha} = 0 \cdot \frac{\partial \mu}{\partial \alpha},$$

so the speed of $\mu(\alpha)$ gets infinite.

(b) Multiplying the equation (2.2.4) with ϵ^2, we obtain the system

$$\begin{aligned}
(\epsilon\mu)(\epsilon\tilde{u}(\mu)) + (\epsilon\nu)\Delta(\epsilon\tilde{u}(\mu)) - \nabla(\epsilon^2\tilde{p}(\mu)) &= 0 \\
\nabla \cdot (\epsilon\tilde{u}(\mu)) &= 0 \\
(\epsilon\tilde{u})\big|_{\Gamma_{-1}} &= 0 \\
\tau \cdot S_{(\epsilon\tilde{u}(\mu))} \cdot n\big|_{\Gamma_0 \cup \Sigma_{1,2}} &= 0 \\
(\epsilon\tilde{u})_n(\mu)\big|_{\Sigma_{1,2}} &= 0 \\
(\epsilon\tilde{u})_n(\mu)\big|_{\Gamma_0} &= -(\epsilon\mu)\eta^{n,k}
\end{aligned}$$

together with the condition for the normal stress on the free boundary:

$$(\epsilon^2\tilde{p}(\mu) - 2(\epsilon\nu)S^n_{\epsilon\tilde{u}})\big|_{\Gamma_0} = \epsilon^2\tilde{r}_\nu(\mu)\eta^{n,k}.$$

By definition of $\tilde{r}$ the last line coincide with $\tilde{r}_{\epsilon\nu}(\epsilon\mu)\eta^{n,k}$ and (b) is also proved.

□

2.3 Hopf Bifurcation with Symmetry

The Hopf bifurcation refers to a phenomenon in which a steady state of an evolution equation evolves into a periodic orbit as a bifurcation parameter is varied. When the symmetry appears, the problem becomes more complicated because the symmetry can lead to multiple eigenvalues. In order to state an equivariant Hopf bifurcation theorem we have to prove the existence of a pair of purely imaginary eigenvalues of $\mathcal{L}$ which are $\mathbb{Z}_k$-simple together with the transversality condition that these eigenvalues cross the imaginary axis with a nonzero speed, when the bifurcation parameter is varied.

In this section we consider the influence of an exterior force (e.g. the wind force) acting on the free surface of the fluid. In general such a force will depend on the position and the velocity of the free surface and result in an increase or decrease of the pressure at the free boundary. With a parameter ξ for the strength we write

$$(p - 2\nu S^n_u)\big|_{\Gamma_0} = g\eta - \beta\underline{\Delta}\eta + \xi F(\eta, u_n\big|_{\Gamma_0}).$$

Linearizing F in 0 we notice that $D_1 F \cdot \eta$ acts like an additional surface tension, the effect of which we know in any subspace $X^r_{n,k}$ (Section 2.2). So we will concentrate on a linear force of the form

$$F(\eta, u_n\big|_{\Gamma_0}) = D_2 F \cdot u_n\big|_{\Gamma_0}.$$

This force can be written in terms of the representation $X^r = \oplus X^r_{n,k}$. We assume that the decomposition remains invariant and study the force $D_2 F = -id$ in $X^r_{n,k}$ which has the structure of a negative damping. We are interested in the position of eigenvalues and restrict all the calculations to $X^r_{n,k}$. The linearized equation are the same like that one in Chapter 1, except the equation (2.3.4) where the term $\xi u_n\big|_{\Gamma_0}$ appears additionaly:

$$\begin{aligned}
\partial_t u - \nu\Delta u + \nabla p &= 0 && (2.3.1)\\
\nabla \cdot u &= 0 && (2.3.2)\\
\partial_t \eta^{n,k} &= u_n\big|_{\Gamma_0} && (2.3.3)\\
(p - 2\nu S^n_u)\big|_{\Gamma_0} &= g\eta^{n,k} - \beta\underline{\Delta}\eta^{n,k} - \xi u_n\big|_{\Gamma_0} && (2.3.4)\\
n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} &= 0, \;\; i = 1,2 && (2.3.5)\\
u\big|_{\Sigma_{-h}} &= 0 && (2.3.6)\\
u_n\big|_{\Sigma_{1,2}} &= 0 && (2.3.7)\\
n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} &= 0 && (2.3.8)\\
\partial_1 \eta^{n,k}\big|_{x_1 \in \{0,b\}} &= 0 && (2.3.9)\\
(u, p, \eta^{n,k})(t, x_1, x_2, x_3) &= (u, p, \eta^{n,k})(t, x_1, x_2 + 2\pi, x_3) && (2.3.10)
\end{aligned}$$

Because we are working in the space $\tilde{X}^r_{n,k}$ or $X^r_{n,k}$, so we have a special form for $\eta^{n,k}$, some of the conditions (2.3.1)-(2.3.10) are automatically satisfied; however, for the seek of completeness we wrote the whole Stokes problem.

In analogy with the previous sections we define the operator

$$\mathcal{L}_\xi \begin{pmatrix} u \\ \eta^{n,k} \end{pmatrix} := \begin{pmatrix} -\nu\Delta u + \nabla\mathcal{H}(2\nu S^n_u\big|_{\Gamma_0}) + \nabla\mathcal{H}(g\eta^{n,k} - \beta\underline{\Delta}\eta^{n,k}) - \nabla\mathcal{H}(\xi u_n\big|_{\Gamma_0}) \\ -u_n\big|_{\Gamma_0} \end{pmatrix}, \tag{2.3.11}$$

where $\mathcal{H}(\xi u_n\big|_{\Gamma_0}) := \tilde{\mathcal{H}}(\xi u_n\big|_{\Gamma_0}, 0)$. We denote by $\mathcal{L}_\xi u$ the first component in the definition (2.3.11).

We prove how the Theorem 1.2.5 carries over. We observe that the next Theorem is true also in the whole space $\tilde{X}^r$ (i.e. for an eigenvector $(u, \eta) \in \tilde{X}^r$ of $\mathcal{L}_\xi$).

Theorem 2.3.1 (Position of eigenvalues of $\mathcal{L}_\xi$ w.r.t. $\|\cdot\|_E$)
Let $\begin{pmatrix} u \\ \eta^{n,k} \end{pmatrix} \in \tilde{X}^r_{n,k}$ *be an eigenfunction (considered complex) of* $\mathcal{L}_\xi$ *with eigenvalue* μ. *Then*

$$\mathrm{Re}\mu \left\| \begin{pmatrix} u \\ \eta^{n,k} \end{pmatrix} \right\|_E^2 = 2\nu \int_{\Omega_0} |S_u|^2 - \xi|\mu|^2 \|\eta^{n,k}\|^2_{0,\Gamma_0} \tag{2.3.12}$$

$$\mathrm{Im}\mu \left\| \begin{pmatrix} u \\ \eta^{n,k} \end{pmatrix} \right\|_E^2 = 2\mathrm{Im} \int_{\Gamma_0} (-u_n\big|_{\Gamma_0})(g\bar{\eta}^{n,k} - \beta\underline{\Delta}\bar{\eta}^{n,k}). \tag{2.3.13}$$

In the case of $\mathrm{Im}\mu \neq 0$ *the energy equality holds :*

$$\|u\|^2_{0,\Omega_0} = \|u\|^2_{E,\Omega_0} = \|\eta^{n,k}\|^2_{E,\Gamma_0} = \frac{1}{2} \left\| \begin{pmatrix} u \\ \eta^{n,k} \end{pmatrix} \right\|_E^2 = \alpha \|\eta^{n,k}\|^2_{0,\Gamma_0} \,. \tag{2.3.14}$$

Proof: Following the proof of Theorem 1.2.5 this Theorem can be proved without difficulties. The only difference which appear is the expresion (2.3.4) for the normal stress on the free boundary.

□

We abbreviate again by $\|\cdot\|$ without indicies the $L^2(\Omega_0)^3$-norm (or $L^2(\Gamma_0)$-norm) and by $\langle\cdot,\cdot\rangle$ the L^2-scalar product.

We want to get a global picture of the position of eigenvalues as in the previous section, but now depending on the parameter ξ. Looking at the results of Theorem 2.3.1, we see that two important differences will appear:

(*a*) *The eigenvalues may have a negative real part;*
(*b*) *The energy equality for eigenvectors* $(u, \eta^{n,k})$ *remains unchanged and does not depend on the bifurcation parameter* ξ*; we will exploit this to prove that for* $|\xi| \to +\infty$*, the nonreal eigenvalues are bounded.*

Proposition 2.3.2
(a) The modulus of nonreal eigenvalues is bounded independent of ξ.
(b) For $|\xi| \to +\infty$ *all eigenvalues of* $\mathcal{L}_\xi$ *are real.*

Proof:
(*a*) We suppose that for $|\xi| \to +\infty$ we can find a sequence of nonreal eigenvalues $\mu(\xi) \in \mathbb{C}\setminus\mathbb{R}$ of $\mathcal{L}_\xi$ with $|\mu(\xi)| \to +\infty$. For every such complex eigenvalue with the eigenfunction $u(\mu(\xi))$, we know from the energy equality (2.3.14):

$$\|u(\mu(\xi))\|^2 = \alpha\|\eta^{n,k}\|^2 \text{ is bounded independent of } \xi.$$

The function $u(\mu(\xi))$ satisfies the problem (2.2.4)$-$(2.2.5) (together with the corresponding pressure function). We can use the result of Proposition 2.2.4(c) from the previous section, because its proof did not exploit $\xi = 0$, and we conclude:

$$\|u(\mu(\xi))\| \to +\infty, \quad \text{for} \quad |\mu(\xi)| \to +\infty,$$

a contradiction.

(b) For the second part we treat separately the cases $\xi \to -\infty$ and $\xi \to +\infty$.

(i) $\xi \to -\infty$

The equation (2.3.12) implies

$$\mathrm{Re}\mu(\xi) \to +\infty \quad \text{for} \quad \xi \to -\infty$$

which implies $\mu(\xi) \in \mathbb{R}$ (because the nonreal eigenvalues are bounded).

(ii) $\xi \to +\infty$

We suppose that for any ξ arbitrary large, we can find a nonreal eigenvalue $\mu(\xi)$ of $\mathcal{L}_\xi$, so we can construct a sequence of nonreal eigenvalues (which are bounded) and consider $\mu(\xi) \to \mu_\infty$. Let $(\tilde{u}(\mu(\xi)), \eta^{n,k})$ be an eigenfunction of $\mathcal{L}_\xi$ corresponding to $\mu(\xi)$ and $\tilde{p}(\mu(\xi))$ be the corresponding pressure function. Because $\mu(\xi) \in \mathbb{C} \setminus \mathbb{R}$, it never meets κ_j, so $(\tilde{u}(\mu(\xi)), \tilde{p}(\mu(\xi)))$ is a nonzero solution of (2.2.4)$-$(2.2.5) (for $\mu(\xi)$).

We distinguish two cases:

(1) $\mu_\infty = 0$

Letting $\xi \to +\infty$, the limit function $(\tilde{u}(0), \tilde{p}(0))$ is a solution of the problem (2.2.4)$-$(2.2.5) for $\mu = 0$, so $\tilde{u}(0)$ is identically zero. On the other hand, every $\tilde{u}(\mu(\xi))$ satisfies the energy equality (2.3.14) and passing to the limit, we obtain

$$0 = \|\tilde{u}(0)\|^2 = \lim_{\xi \to +\infty} \|\tilde{u}(\mu(\xi))\|^2 = \alpha \|\eta^{n,k}\|^2 \neq 0$$

a contradiction.

(2) $\mu_\infty \neq 0$

Then the pair $(v(\mu(\xi)), q(\mu(\xi)))$,

$$v(\mu(\xi)) := \frac{\tilde{u}(\mu(\xi))}{\xi} \quad \text{and} \quad q(\mu(\xi)) := \frac{\tilde{p}(\mu(\xi))}{\xi}$$

satisfies the equations:

$$\begin{aligned} (\mu(\xi) - A) v(\mu(\xi)) &= 0 \\ v_n(\mu(\xi))\big|_{\Gamma_0} &= -\frac{\mu(\xi)}{\xi} \eta^{n,k} \\ \left(q(\mu(\xi)) - 2\nu S^n_{v(\mu(\xi))}\right)\big|_{\Gamma_0} &= \frac{\alpha}{\xi} \eta^{n,k} + \mu(\xi) \eta^{n,k}. \end{aligned}$$

Passing to the limit $\xi \to +\infty$, using our hypothesis $\mu(\xi) \to \mu_\infty \in \mathbb{C}$ and continuity w.r.t μ of the functions v and q, the pair of the limit functions $(v(\mu_\infty), q(\mu_\infty))$

$$v(\mu_\infty) := \lim_{\xi \to +\infty} v(\mu(\xi)) \quad \text{and} \quad q(\mu_\infty) := \lim_{\xi \to +\infty} q(\mu(\xi))$$

satisfies the following equations:

$$\begin{aligned}
(\mu_\infty - A)v(\mu_\infty) &= 0 \\
v_n(\mu_\infty)\big|_{\Gamma_0} &= 0 \\
(q(\mu_\infty) - 2\nu S^n_{v(\mu_\infty)})\big|_{\Gamma_0} &= \mu_\infty \eta^{n,k}.
\end{aligned}$$

and $v(\mu_\infty) \not\equiv 0$ because the normal stress on the free boundary is still nonzero.

On the other hand, using the energy equality (2.3.14) we have:

$$\begin{aligned}
0 \neq \|v(\mu_\infty)\|^2 &= \lim_{\xi \to +\infty} \left\| \frac{\tilde{u}(\mu(\xi))}{\xi} \right\|^2 \\
&= \lim_{\xi \to +\infty} \frac{\alpha \|\eta^{n,k}\|^2}{\xi^2} \\
&= 0,
\end{aligned}$$

a contradiction.

So, $\exists \xi_0 > 0$ such that for $|\xi| > \xi_0$ all eigenvalues of $\mathcal{L}_\xi$ are real.

□

We resume now two useful results from the previous Section 2.2. First, for $\xi = 0$ we know:

the first two eigenvalues of $\mathcal{L}_{0,\alpha}$ become nonreal when α exceeds α_0

and for the analysis in this section we fixed such an α (and omit it from the notation of $\mathcal{L}_{\xi,\alpha}$). Second, for $\mu \in \mathbb{C}\backslash\{\kappa_j : j \in \mathbb{N}\}$ we have defined $\tilde{u}(\mu)$ as the unique solution of the problem (2.2.4)–(2.2.5), and $\tilde{r}(\mu)$. We know that $\mu(\alpha)$ is an eigenvalue of $\mathcal{L}_{0,\alpha} \Leftrightarrow \tilde{r}(\mu) = \alpha$. With the exterior force acting through ξ, we have:

$$\begin{aligned}
\mu(\xi) \in \mathbb{C} \textit{ is an eigenvalue of } \mathcal{L}_\xi \iff \tilde{r}(\mu)\eta^{n,k} &= \alpha\eta^{n,k} - \xi \tilde{u}_n\big|_{\Gamma_0} \\
&= \alpha\eta^{n,k} + \xi\mu\eta^{n,k} \\
\iff \tilde{r}(\mu) &= \alpha + \xi\mu,
\end{aligned}$$

so, we find the real eigenvalues of $\mathcal{L}_\xi$ at the intersection of the graph of the function $\tilde{r}(\mu) - \alpha$ (which is already known) with the line $y = \xi\mu$ (see Figure 3).

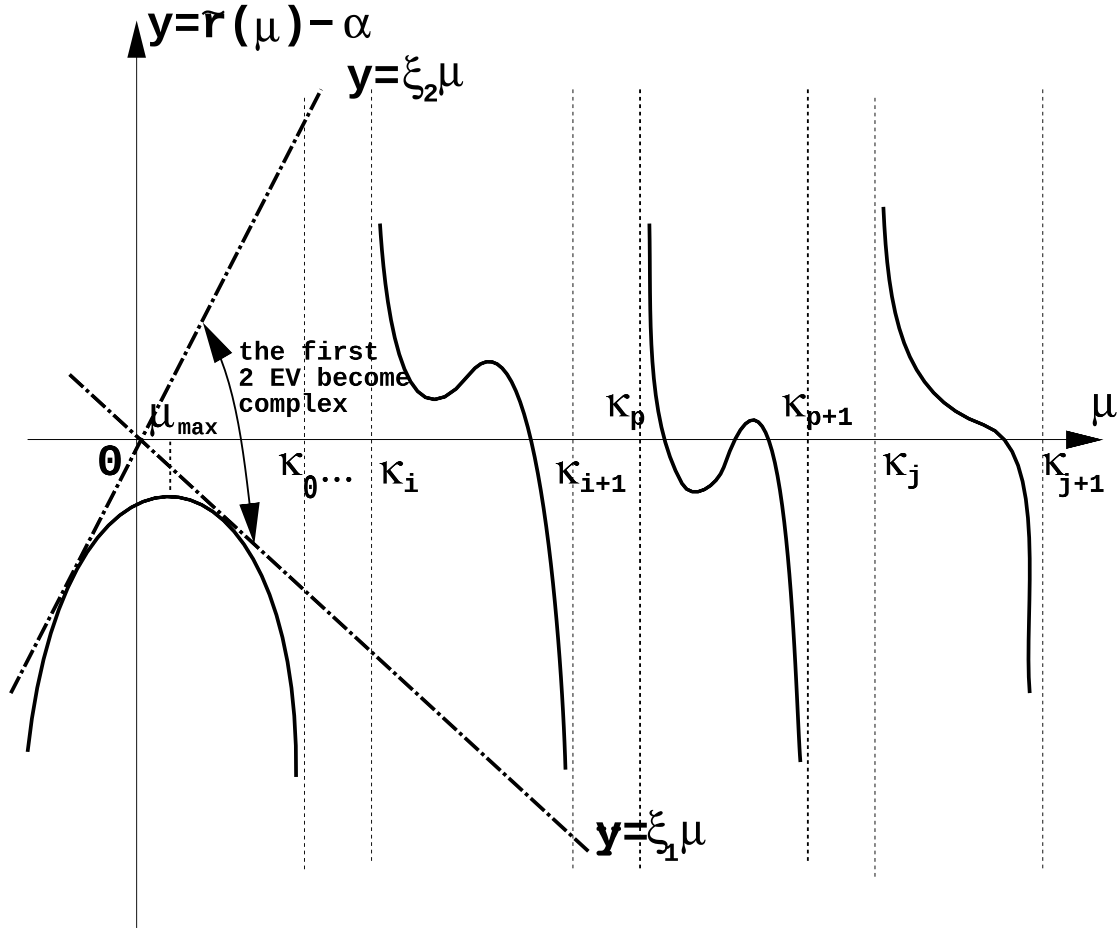

Figure 3: The intersection of the graph of $\tilde{r}(\mu) - \alpha$ with the line $y = \xi\mu$

We observe:

- For any $\xi \in \mathbb{R}$, the line $y = \xi\mu$ intersects the graph of $\tilde{r}(\mu) - \alpha$ on each interval (κ_j, κ_{j+1}), $j \in \mathbb{N}$, at least once, so $\mathcal{L}_\xi$ has at least one real eigenvalue lying on each interval (κ_j, κ_{j+1}).

- There exists the values $\xi_1 < 0$ and $\xi_2 > 0$ such that the lines $y = \xi_1\mu$ and $y = \xi_2\mu$ are tangent to the graph of $\tilde{r}(\mu) - \alpha$ on the interval $(0, \kappa_0)$ and $(-\infty, 0)$ respectively. For $\xi \in (\xi_1, \xi_2)$ the line $y = \xi\mu$ does not intersect the graph of $\tilde{r}(\mu) - \alpha$ for $\mu \in (-\infty, \kappa_0)$. Because of the analyticity of $\tilde{r}$ (the number of zeros of $\tilde{r}$, each counted with its multiplicity, is locally constant), a pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ appears for $\xi = \xi_1 + \epsilon$ and $\xi = \xi_2 - \epsilon$ ($\epsilon > 0$ small). Denote them by $\mu_0(\xi)$ and $\mu_1(\xi)$ with $\mu_0(\xi) = \bar{\mu}_1(\xi)$.

• For $\xi \in (-\infty, \xi_1)$, the line $y = \xi\mu$ intersects the graph of $\tilde{r}(\mu) - \alpha$ twice for $\mu \in (0, \kappa_0)$, so the first two eigenvalues are real and positive.

• For $\xi \in (\xi_2, +\infty)$, the line $y = \xi\mu$ intersects the "first part" of the graph of $\tilde{r}(\mu) - \alpha$ twice, but for $\mu \in (-\infty, 0)$, so the first two eigenvalues are real and negative.

We denote the first two eigenvalues of $\mathcal{L}_\xi$ with $\mu_0(\xi)$, $\mu_1(\xi)$ and the ordered sequence of the (rest) real eigenvalues with $\{\mu_j(\xi)\}_{j\in\mathbb{N}, j\geq 2}$.

Theorem 2.3.3 (The global bifurcation picture in ξ)

For $\xi \in (-\infty, \xi_1)$ the first two eigenvalues of $\mathcal{L}_\xi$ are real and positive:

$$0 < \mu_0(\xi) < \mu_1(\xi) < \kappa_0.$$

For $\xi \to -\infty$ all eigenvalues of $\mathcal{L}_\xi$ are real, every interval (κ_j, κ_{j+1}) contains one real eigenvalue μ_{j+2} of $\mathcal{L}_\xi$ and $\mu_0(\xi) \searrow 0$, $\mu_{j+2} \nearrow \kappa_{j+1}$, $j \in \mathbb{N} \cup \{-1\}$.

For $\xi \in (\xi_2, +\infty)$ the first two eigenvalues of $\mathcal{L}_\xi$ are real and negative:

$$\mu_0(\xi), \mu_1(\xi) < 0.$$

For $\xi \to +\infty$ all eigenvalues of $\mathcal{L}_\xi$ are real, every interval (κ_j, κ_{j+1}) contains one real eigenvalue μ_{j+2} of $\mathcal{L}_\xi$ and $\mu_{j+2} \searrow \kappa_{j+1}$, $j \in \mathbb{N}$.

There exists a point $\xi^* \in (\xi_1, \xi_2)$ where a pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ crosses the imaginary axis transversally. The imaginary axis can be crossed only with negative real part of the velocity.

Proof: During this proof we have to keep in mind that each of μ, u, p depends on ξ, but we will not write this explicitly.

The first two statements are clear from Proposition 2.3.2(b) and Figure 3, which also implies (because $\tilde{r}$ is an analytic function): for small $\epsilon > 0$,

• for $\xi = \xi_1 + \epsilon$ the pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ has a positive real part;

• for $\xi = \xi_2 - \epsilon$ the pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ has a negative real part.

The eigenvalues of $\mathcal{L}_\xi$ depend continuously on ξ and together with Proposition 2.3.2(a) we can conclude: there exists $\xi^* \in (\xi_1, \xi_2)$ such that $\mu(\xi^*)$ is purely imaginary, $\mathrm{Re}\mu(\xi^*) = 0$.

The eigenvalues $\mu(\xi) \neq \kappa_j$ of $\mathcal{L}_\xi$ are geometrically simple (in every $\tilde{X}^r_{n,k}$ and up to the $\mathbb{Z}_k$-symmetry) because for every eigenfunction $(u(\mu(\xi)), \eta^{n,k})$, $u(\mu(\xi))$ satisfies also the problem (2.2.4)$-$(2.2.5) which has unique solution. The eigenvalues have the same geometric and algebraic multiplicity for $|\xi| \to +\infty$; for the proof see [Schw1].

We have to prove now the transversality (for $\xi = \xi^*$, $\partial_\xi(\mathrm{Re}\mu) \neq 0$) and the direction of crossing (for $\xi = \xi^*$, $\partial_\xi(\mathrm{Re}\mu) < 0$).

From the energy equality (2.3.14) we see that the norm of the eigenfunction u does not depend on ξ, and we can calculate:

$$0 = \partial_\xi \|u\|^2 = \partial_\xi \int_{\Omega_0} u \cdot \bar{u} = \int_{\Omega_0} u \cdot \partial_\xi \bar{u} + \overline{u \cdot \partial_\xi \bar{u}} = 2\mathrm{Re}\langle u, \partial_\xi u\rangle\,. \tag{2.3.15}$$

We make first some further calculations, (2.3.16) and (2.3.17), for $\partial_\xi u \neq 0$. Multiplying the first component of the eigenvalue equation for $\mathcal{L}_\xi$ with $\partial_\xi \bar{u}$, integrating over Ω_0 and using Corollary 1.2.3 and Theorem 2.3.1, we obtain:

$$\begin{aligned}
\langle \mu u, \partial_\xi u\rangle &= \langle \mathcal{L}_\xi u, \partial_\xi u\rangle \\
&= 2\nu \int_{\Omega_0} S_u : S_{\partial_\xi \bar{u}} + \int_{\Gamma_0} (\alpha \eta^{n,k} - \xi u_n\big|_{\Gamma_0}) \cdot \partial_\xi \bar{u}_n\big|_{\Gamma_0} \\
&= \frac{1}{2}\partial_\xi \left(2\nu \int_{\Omega_0} S_u : S_{\bar{u}}\right) + \int_{\Gamma_0} (\alpha + \xi\mu)\eta^{n,k}(-\partial_\xi \bar{\mu})\bar{\eta}^{n,k} \\
&= \frac{1}{2}\partial_\xi \left(\mathrm{Re}\mu \cdot 2\alpha \|\eta^{n,k}\|^2 + \xi|\mu|^2 \|\eta^{n,k}\|^2\right) - \partial_\xi \bar{\mu} \cdot \alpha \|\eta^{n,k}\|^2 - \xi\mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2 \\
&= \underbrace{\partial_\xi(\mathrm{Re}\mu)\alpha \|\eta^{n,k}\|^2 + \frac{1}{2}|\mu|^2 \|\eta^{n,k}\|^2 + \frac{1}{2}\xi(\partial_\xi |\mu|^2)\|\eta^{n,k}\|^2}_{\in \mathbb{R}} \\
&\quad -\partial_\xi \bar{\mu} \cdot \alpha \|\eta^{n,k}\|^2 - \xi\mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2
\end{aligned} \tag{2.3.16}$$

Differentiating the first component of the eigenvalue equation for $\mathcal{L}_\xi$ w.r.t. ξ, multiplying with $\partial_\xi \bar{u}$, integrating over Ω_0 and using Corollary 1.2.3 and Theorem 2.3.1, we obtain:

$$\begin{aligned}
0 &= \langle \partial_\xi(\mu u), \partial_\xi u\rangle - \langle \partial_\xi(\mathcal{L}_\xi u), \partial_\xi u\rangle \\
&= \partial_\xi \mu \langle u, \partial_\xi u\rangle + \mu \|\partial_\xi u\|^2 \\
&\quad + \langle \nu \Delta \partial_\xi u - \nabla \mathcal{H}(2\nu S^n_{\partial_\xi u}\big|_{\Gamma_0}), \partial_\xi u\rangle - \langle \nabla \mathcal{H}(\partial_\xi(-\xi u_n\big|_{\Gamma_0})), \partial_\xi u\rangle \\
&= \partial_\xi \mu \langle u, \partial_\xi u\rangle + \mu \|\partial_\xi u\|^2 - 2\nu \int_{\Omega_0} S_{\partial_\xi u} : S_{\partial_\xi \bar{u}} - \int_{\Gamma_0} \partial_\xi(\xi\mu)\eta^{n,k}(-\partial_\xi \bar{\mu})\bar{\eta}^{n,k} \\
&= \partial_\xi \mu \underbrace{\langle u, \partial_\xi u\rangle}_{\in \mathbb{C}\setminus\mathbb{R}} + \mu \underbrace{\|\partial_\xi u\|^2}_{\in \mathbb{R}} - \underbrace{2\nu \|S_{\partial_\xi u}\|^2 + \xi |\partial_\xi \mu|^2 \|\eta^{n,k}\|^2}_{\in \mathbb{R}} \\
&\quad + \mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2
\end{aligned} \tag{2.3.17}$$

We prove now that the speed of nonreal eigenvalues never vanishes. Let μ be a nonreal eigenvalue of $\mathcal{L}_\xi$ and suppose $\partial_\xi \mu = 0$, so $\partial_\xi(\mathrm{Re}\mu) = \partial_\xi(\mathrm{Im}\mu) = 0$. We prove first that this implies also $\partial_\xi u = 0$. Suppose $\partial_\xi u \neq 0$, so $\partial_\xi \bar{u} \neq 0$, too. Introducing this in the equation (2.3.17) we obtain

$$\mu \|\partial_\xi u\|^2 = 2\nu \|S_{\partial_\xi u}\|^2$$

which implies $\mu \in \mathbb{R}$, a contradiction. So

$$\partial_\xi \mu = 0 \implies \partial_\xi u = \partial_\xi \bar{u} = 0 \implies \partial_\xi S_u = 0.$$

Differentiating the equation (2.3.12) w.r.t. ξ we obtain

$$\begin{aligned} 0 &= \partial_\xi(\mathrm{Re}\mu) 2\alpha \|\eta^{n,k}\|^2 \\ &= 2\nu \partial_\xi \Big(\int_{\Omega_0} |S_u|^2\Big) - \xi \partial_\xi |\mu|^2 \|\eta^{n,k}\|^2 - |\mu|^2 \|\eta^{n,k}\|^2 \\ &= -|\mu|^2 \|\eta^{n,k}\|^2, \end{aligned}$$

a contradiction. Therefore we know for nonreal eigenvalues: $\partial_\xi \mu \neq 0$, $\forall \xi$.

In order to prove the transversality condition for $\xi = \xi^*$ and the direction of crossing of the imaginary axis, we take the real part of (2.3.16) together with (2.3.15) to obtain:

$$-\mathrm{Im}\mu \cdot \mathrm{Im}\langle u, \partial_\xi u\rangle \overset{(2.3.15)}{=} \mathrm{Re}\langle \mu u, \partial_\xi u\rangle \overset{(2.3.16)}{=} \frac{1}{2}|\mu|^2 \|\eta^{n,k}\|^2$$

and the imaginary part of (2.3.17) to obtain

$$0 = \partial_\xi(\mathrm{Re}\mu)\mathrm{Im}\langle u, \partial_\xi u\rangle + \mathrm{Im}\mu \|\partial_\xi u\|^2 + \mathrm{Im}(\mu \partial_\xi \bar{\mu}) \|\eta^{n,k}\|^2.$$

Multiplying the last equation with $2\mathrm{Im}\mu \neq 0$ and using the previous equation, we obtain:

$$\partial_\xi(\mathrm{Re}\mu)|\mu|^2 \|\eta^{n,k}\|^2 = 2\mathrm{Im}^2\mu \|\partial_\xi u\|^2 + 2\mathrm{Im}\mu \cdot \mathrm{Im}(\mu \partial_\xi \bar{\mu}) \|\eta^{n,k}\|^2.$$

For $\xi = \xi^*$, we are on the imaginary axis, so we have

$$\begin{aligned} \mathrm{Re}\mu = 0 \;\Rightarrow\; & \mathrm{Im}(\mu \partial_\xi \bar{\mu}) = \mathrm{Im}\mu \cdot \partial_\xi(\mathrm{Re}\mu) \\ & |\mu|^2 = \mathrm{Im}^2 \mu \neq 0 \end{aligned}$$

and then

$$-\partial_\xi(\mathrm{Re}\mu) \|\eta^{n,k}\|^2 = 2\|\partial_\xi u\|^2 > 0.$$

□

We can formulate results similar to Proposition 1.2.6, Theorem 1.2.13 and Theorem 1.2.17 for $\mathcal{L}_\xi$ $(\forall \xi)$. The proofs follow immediatly because only the value of $p\big|_{\Gamma_0}$ is modified with $\xi u_n\big|_{\Gamma_0}$ and we can estimate $\|\nabla \mathcal{H}(\xi u_n|_{\Gamma_0})\|_{r,\Omega_0} \leq c\|u\|_{r+1,\Omega_0}$:

Proposition 2.3.4 (Properties of $\mathcal{L}_\xi$)
(a) The operator $\mathcal{L}_\xi^{-1} : X^r \to \tilde{X}^{r+1}$, $r \geq 1$ is bounded $\forall \xi$.

(b) The solution (u, η) of the equation $\mathcal{L}_\xi(u, \eta) = (f, 0) \in X^r$ satisfies the regularity:

$$\|(u, \eta)\|_{X^{r+2}} \leq c\|(f, 0)\|_{X^r}.$$

(c) The operator $\mathcal{L}_\xi : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \geq 0$, is invertible and the inverse is bounded $\forall \xi$. The same result holds for $\lambda + \mathcal{L}_\xi$, too, when $-\lambda$ is not an eigenvalue of $\mathcal{L}_\xi$.

(d) Linear existence results, similar to Theorem 1.2.15 and Theorem 1.2.18, hold for $\mathcal{L}_\xi$, $\forall \xi$, too.

Definition 2.3.5 (Generalized nonresonance condition)

We say that the pair $\mu^\pm$ of pure imaginary eigenvalues of $\mathcal{L}_{\xi^*}$ satisfies the generalised nonresonance condition, when the following two requirements are fullfiled:
(a) the usual nonresonance condition: $\forall a \in \mathbb{Z} \setminus \{\pm 1\}$, $a\mu^+$ is not an eigenvalue of $\mathcal{L}_{\xi^*}$;
(b) a simplicity condition: for the fixed value ξ^* of the bifurcation parameter (for which we have proved the transversality condition), the eigenvalues $\mu^\pm$ of $\mathcal{L}_{\xi^*}$ are eigenvalues of $\mathcal{L}_{\xi^*}\big|_{\tilde{X}_{n,k}}$ only for one $n \in \mathbb{N}$ and for one $k \in \mathbb{Z}$.

We are now in position to formulate a Hopf bifurcation theorem for the full nonlinear problem. We can consider we have written it in the form (after similar transformations we have done in Section 1.3):

$$(\partial_t + \mathcal{L}_\xi) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} F(u, \eta) \\ 0 \end{pmatrix} \tag{2.3.18}$$

where F contains all the nonlinearities and correction terms. We recall that F has the following properties: for $r \geq 1$, $F : X^{r+2} \to H^r(\Omega_0)^3$, $F(0, 0) = 0$, DF exists and $DF(0, 0) = 0$.

Theorem 2.3.6 (Hopf bifurcation theorem)
For every space $X_{n,k}$ there exists a critical value ξ^* of the bifurcation parameter ξ such that $\mathcal{L}_{\xi^*}$ has a pair $\mu^\pm$ of purely imaginary eigenvalues and the transversality condition is fullfiled. We assume that this pair of eigenvalues satisfies the generalized nonresonance condition of Definition 2.3.5.
Then a Hopf bifurcation occurs and there exists a branch of $\mathbb{Z}_k$-symmetric, periodic solutions of the nonlinear equation.

Proof: We are looking for periodic solution in t of period p and with prescribed spatial symmetry $\mathbb{Z}_k$ for the equation (2.3.18). We can rescale the time through $t \mapsto 2\pi t/p$, so we look for periodic solutions of (2.3.18) of period 2π and introduce the unknown period as a parameter. We define the spaces $\tilde{X}^r_{\mathbb{Z}_k}$ and $\tilde{X}^r_{3/2,\mathbb{Z}_k}$ which contain functions from $\tilde{X}^r$ and $\tilde{X}^r_{3/2}$ respectively, which have spatial symmetry $\mathbb{Z}_k$, i.e.:

$$\begin{aligned}\tilde{X}^r_{\mathbb{Z}_k} &= \{x(t,\cdot) \in \tilde{X}^r \,|\, \gamma * x(t,\cdot) = x(t,\cdot), \forall \gamma \in \mathbb{Z}_k,\ \forall t\}\\ \tilde{X}^r_{3/2,\mathbb{Z}_k} &= \{x(t,\cdot) \in \tilde{X}^r_{3/2} \,|\, \gamma * x(t,\cdot) = x(t,\cdot), \forall \gamma \in \mathbb{Z}_k,\ \forall t\}\end{aligned}$$

where the composition $*$ is defined in (2.2.1) and (2.2.2) ($*$ acts only on the second spatial variable x_2). The definition is similar for the spaces without $\tilde{}$, too.

We want to solve the equation

$$\phi((u,\eta),p,\xi) := \frac{2\pi}{p}\partial_t \begin{pmatrix} u \\ \eta \end{pmatrix} + \mathcal{L}_\xi \begin{pmatrix} u \\ \eta \end{pmatrix} - \begin{pmatrix} F(u,\eta) \\ 0 \end{pmatrix} = 0 \tag{2.3.19}$$

where

$$\phi : H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k}) \times \mathbb{R} \times \mathbb{R} \to L^2_{per}([0,2\pi], X^r_{3/2,\mathbb{Z}_k}),$$

(see Proposition 2.3.4(d)).

Applying $\mathcal{L}_\xi^{-1}$ we obtain an equation equivalent with (2.3.19):

$$\psi((u,\eta),p,\xi) := \frac{2\pi}{p}\mathcal{L}_\xi^{-1}\partial_t \begin{pmatrix} u \\ \eta \end{pmatrix} + \begin{pmatrix} u \\ \eta \end{pmatrix} - \mathcal{L}_\xi^{-1} \begin{pmatrix} F(u,\eta) \\ 0 \end{pmatrix} = 0 \tag{2.3.20}$$

and for $(u,\eta) \in H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k})$, we know from the regularity theory of Chapter 1 (see also Proposition 2.3.4):

$$\begin{aligned} F(u,\eta) \in H^r_{\mathbb{Z}_k}(\Omega_0) \quad &\Rightarrow \quad \mathcal{L}_\xi^{-1} \begin{pmatrix} F(u,\eta) \\ 0 \end{pmatrix} \in \tilde{X}^{r+2}_{\mathbb{Z}_k} \\ \partial_t \begin{pmatrix} u \\ \eta \end{pmatrix} \in \tilde{X}^r_{3/2,\mathbb{Z}_k} \quad &\Rightarrow \quad \mathcal{L}_\xi^{-1}\partial_t \begin{pmatrix} u \\ \eta \end{pmatrix} \in \tilde{X}^{r+2}_{\mathbb{Z}_k}, \end{aligned}$$

so

$$\psi : H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k}) \times \mathbb{R} \times \mathbb{R} \to L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k}).$$

Let ξ^* be the critical value of ξ we have found in Theorem 2.3.3. Then consider μ^+ has $\mathrm{Im}\mu^+ > 0$ and let $p^* := \frac{2\pi}{\mathrm{Im}\mu^+}$. We have to show that the operator $P := D_1\psi((0,0),p^*,\xi^*)$ is Fredholm of index zero with a two dimensional kernel, where

$$P : H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k}) \to L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k})$$

$$Pv := D_1\psi((0,0),p^*,\xi^*)\langle v\rangle = \frac{2\pi}{p^*}\partial_t \mathcal{L}_{\xi^*}^{-1}v + v\,,$$

where $v = (u,\eta)$. We apply the Fourier expansion in time setting $\forall m \in \mathbb{Z}$

$$v_m(\cdot) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} v(t,\cdot)e^{imt}dt\,.$$

Then

$$P = \bigoplus_{m\in\mathbb{Z}} P_m$$

with

$$\begin{aligned} P_m &: \quad \tilde{X}^{r+2}_{\mathbb{Z}_k} \to \tilde{X}^{r+2}_{\mathbb{Z}_k} \\ P_m &:= \quad -im\frac{2\pi}{p^*}\mathcal{L}_{\xi^*}^{-1} + id\,. \end{aligned}$$

P_m is a Fredholm operator of index 0 $\forall m \in \mathbb{Z}\setminus\{0\}$, because the operator $\mathcal{L}_{\xi^*}^{-1}$: $\tilde{X}^{r+2}_{\mathbb{Z}_k} \to \tilde{X}^{r+3}_{\mathbb{Z}_k} \hookrightarrow \tilde{X}^{r+2}_{\mathbb{Z}_k}$ is compact; $P_0 = id$. We know from Theorem 2.3.3 that $\mathcal{L}_{\xi^*}$ has a pair of purely imaginary complex conjugate eigenvalues $\mu^\pm$ which are simple up to the $\mathbb{Z}_k$-symmetry in every space $\tilde{X}^{r+2}_{n,k}$ ($n \in \mathbb{N}$, $k \in \mathbb{Z}$) for which $\mu^\pm$ are eigenvalues of $\mathcal{L}_{\xi^*}\big|_{\tilde{X}^{r+2}_{n,k}}$. (We observe that the number of such $\tilde{X}^{r+2}_{n,k}$ is finite, because $\mathcal{L}_{\xi^*}^{-1}$ is compact, so the eigenvalues of $\mathcal{L}_{\xi^*}$ have also finite multiplicity). We have assumed that the generalized nonresonance condition of Definition 2.3.5 is fullfiled, so the eigenvalues $\mu^\pm$ of $\mathcal{L}_{\xi^*}$ are $\mathbb{Z}_k$-simple in $\tilde{X}^{r+2}$ and

$$\begin{aligned} &P_m \text{ is invertible} \quad \forall m \in \mathbb{Z}\setminus\{\pm1\} \\ &\ker P_1 = \ker P_{-1} \\ &\dim_{\mathbb{R}} \ker P_m = 2 \quad \text{for } m = \pm 1\,. \end{aligned}$$

For the kernel of P, we calculate:

$$\begin{aligned} Pv = 0 \quad &\Leftrightarrow \quad P\left(\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}} v_m e^{-imt}\right) = 0 \\ &\Leftrightarrow \quad P_m v_m = 0 \;\; \forall m \in \mathbb{Z} \\ &\Leftrightarrow \quad \begin{cases} v_m = 0 & \forall m \neq \pm 1 \\ v_m \in \ker P_m & \text{for } m = \pm 1 \end{cases} . \end{aligned}$$

Then $\dim_{\mathbb{R}} \ker P = \dim_{\mathbb{R}} \ker P_{\pm1} = 2$. To complete the proof that P is a Fredholm operator of index zero, it remains to show that

$$\begin{aligned} \tilde{P} &:= \bigoplus_{m\in\mathbb{Z}\setminus\{\pm1\}} P_m \\ \tilde{P} &: \quad H^1_{per}([0,2\pi],\tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi],\tilde{X}^{r+2}_{\mathbb{Z}_k}) \to L^2_{per}([0,2\pi],\tilde{X}^{r+2}_{\mathbb{Z}_k}) \end{aligned}$$

is surjective. Let

$$g \in L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k}), \quad g = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} g_m e^{-imt} \text{ with } g_1 = g_{-1} = 0\,.$$

For every $g_m \in \tilde{X}^{r+2}_{\mathbb{Z}_k}$ $(m \neq \pm 1)$ we can find $v_m \in \tilde{X}^{r+2}_{\mathbb{Z}_k}$ such that $P_m v_m = g_m$ (by the Fredholm property of P_m).

Denote by

$$g_M = \frac{1}{\sqrt{2\pi}} \sum_{m=-M}^{M} g_m e^{-imt} \ , \quad v_M = \frac{1}{\sqrt{2\pi}} \sum_{m=-M}^{M} v_m e^{-imt}.$$

Then v_M satisfies the equation

$$\frac{2\pi}{p^*}\partial_t \mathcal{L}^{-1}_{\xi^*} v_M + v_M = g_M \quad \Leftrightarrow \quad \frac{2\pi}{p^*}\partial_t v_M + \mathcal{L}_{\xi^*} v_M = \mathcal{L}_{\xi^*} g_M$$

and

$$\begin{aligned} \|v_M\|_{H^1_{per}([0,2\pi],\tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi],\tilde{X}^{r+2}_{\mathbb{Z}_k})} &\leq C_1 \|\mathcal{L}_{\xi^*} g_M\|_{L^2_{per}([0,2\pi],X^r_{3/2,\mathbb{Z}_k})} \\ &\leq C_2 \|g_M\|_{L^2_{per}([0,2\pi],\tilde{X}^{r+2}_{\mathbb{Z}_k})} \\ &\leq C_3 \|g\|_{L^2_{per}([0,2\pi],\tilde{X}^{r+2}_{\mathbb{Z}_k})}\,. \end{aligned}$$

We find v such that $v_M \rightharpoonup v$ weakly in $H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k})$ and v solves $\tilde{P}v = g$. The regularity theory of Chapter 1 (see also Proposition 2.3.4) implies now $v \in H^1_{per}([0,2\pi], \tilde{X}^r_{3/2,\mathbb{Z}_k}) \cap L^2_{per}([0,2\pi], \tilde{X}^{r+2}_{\mathbb{Z}_k})$. Then P is a Fredholm operator of index 0 (as a direct sum of two Fredholm operators of index zero $P_{\pm 1}$ with an invertible operator $\tilde{P}$) and $\dim_{\mathbb{R}} \ker P = 2$.

Using Ljapunov-Schmidt techniques, we reduce the problem of finding periodic solutions of the equation $\psi = 0$ (which is equivalent to $\phi = 0$) to one in two dimensions. We can apply now standard technical arguments used in the proof of the Hopf bifurcation theorem (see [GSS] and [Cr,Ra]) and the result follows.

$\square$

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