Inaugural-Dissertation

zur

Erlangung der Doktorwürde

der

Naturwissenschaftlich-Mathematischen Gesamtfakultät

der

Ruprecht-Karl-Universität

Heidelberg

vorgelegt von

Diplom-Mathematikerin

Kathrin Bringmann

aus: Münster

Tag der mündlichen Prüfung: 23.07.04

Thema

Applications of Poincaré series on Jacobi groups

Gutachter: Prof. Dr. Winfried Kohnen
Prof. Dr. Eberhard Freitag

Contents

Abstract

In meiner Arbeit betrachte ich Anwendungen von Poincaréreihen zur verallgemeinerten Jacobigruppe.
Im ersten Teil schätze ich Fourierkoeffizienten Siegelscher Spitzenformen ab.
Zunächst betrachte ich den Fall Siegelscher Spitzenformen kleiner Gewichte zur vollen Siegelschen Modulgruppe Γ_g; anschließend untersuche ich eine Untergruppe $\Gamma_{0,g}(N)$ von Γ_g.
Im zweiten Teil der Arbeit geht es um Liftungs-Abbildungen von einem Vektorraum verallgemeinerter Jacobi Spitzenformen in einen Teilraum elliptischer Spitzenformen.

In our work we regard applications of Poincaré series for generalized Jacobi groups. The first part deals with estimates of Fourier coefficients of Siegel cusp forms. First we consider the case of Siegel modular forms for the full modular group Γ_g having small weight. Afterwards the case of a certain subgroup $\Gamma_{0,g}(N)$ of Γ_g is regarded.
In the second part we construct lifting maps from a vector space of generalized Jacobi cusp forms to a subspace of elliptic modular forms.

Chapter 1

Introduction

In this thesis we discuss applications of Poincaré series on certain (higher- dimensional) Jacobi groups.
In the first part (Chapter 3) we derive estimates of Fourier coefficients of Siegel cusp forms. In the second part (Chapter 4) we construct certain lifting maps from a vector space of Jacobi cusp forms to a certain subspace of elliptic modular forms. Chapter 2 contains preliminary facts on Siegel and Jacobi modular forms.

Let F be a cusp form of integral weight k with respect to the Siegel modular group $\Gamma_g = Sp_g(\mathbb{Z}) \subset GL_{2g}(\mathbb{Z})$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix. Then a conjecture of Resnikoff and Saldaña (cf. [RS]) says that

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-(g+1)/4+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F. For $g = 1$ this conjecture is true (Ramanujan-Petersson conjecture, proved by Deligne for $k \geq 2$ [DE], and by Deligne and Serre for $k = 1$ [DS]). This estimate is the best possible one, because due to Rankin we have (for $f \neq 0$)

$$\limsup_{T\to\infty} \left|A(T)T^{(1-k)/2}\right| = \infty.$$

([Ra]).
For arbitary g there are known counter examples for the conjecture of Resnikoff and Saldaña as the following theorem (cf. [K5]) shows

Theorem 1.1 *Let $g \equiv 1 \pmod 4$, $g \equiv k \pmod 2$ and $F \in S_{k+g}(\Gamma_{2g})$ be a Hecke eigenform that is the Ikeda lift of a normalized Hecke eigenform $f \in S_{2k}(\Gamma_1)$. Then the conjecture of Resnikoff and Saldaña is not true.*

For $g \geq 2$ the estimate is at least known on average (cf. [K7]).

From the classical Hecke argument the following bound

$$a(T) \ll_F (\det T)^{k/2}$$

follows readily, where the constant implied in $\ll_F$ only depends on F.
For $k > g+1$, the at present best estimate is

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-c_g+\epsilon} \qquad (\epsilon > 0), \tag{1.1}$$

where

$$c_g := \begin{cases} \frac{13}{36} & \text{if } g = 2 \quad ([K1]), \\ \frac{1}{4} & \text{if } g = 3 \quad ([Br]), \\ \frac{1}{2g} + (1-\frac{1}{g})\alpha_g & \text{if } g > 3 \quad ([BK]), \end{cases}$$

where

$$\alpha_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}. \tag{1.2}$$

Clearly

$$\alpha_g \to 0 \qquad \text{for } g \to \infty,$$

i.e., one is far away from the conjecture of Resnikoff and Saldaña. To be more precise, in [BK] it is proved that

$$a(T) \ll_{\epsilon,F} (m_{g-1}(T))^{1/2-\alpha_g+\epsilon} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0), \tag{1.3}$$

where

$$m_{g-1} := min \left\{ T[U]_{g-1} | \, U \in GL_g(\mathbb{Z}) \right\},$$

where $T[U] := U^t T U$ (U^t = transpose of U) and $T[U]_{g-1}$ denotes the $(g-1) \times (g-1)$ minor of $T[U]$. From (1.3) the estimate (1.1) follows directly if one uses the bound $m_{g-1}(T) \ll_g (\det T)^{1-1/g}$, which in turn follows readily from reduction theory.

The method in [BK] is the following (for details cf. Chapter 2, Section 2 or [BK]): If we write $Z \in \mathbb{H}_g$ as $Z = \begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix}$, where $\tau \in \mathbb{H}$, $z \in \mathbb{C}^{(1,g-1)}$, and $\tau' \in \mathbb{H}_{g-1}$, we see that the function $F(Z) \in S_k(\Gamma_g)$ has a so-called Fourier-Jacobi expansion of the kind

$$F(Z) = \sum_{m>0} \phi_m(\tau, z) e^{2\pi i \mathrm{tr}\,(m\tau')} \qquad (\tau' \in \mathbb{H}_{g-1}),$$

where m runs through all positive definite symmetric half-integral $(g-1)\times(g-1)$ matrices, and where the coefficients $\phi_m(\tau, z)$ are Jacobi cusp forms (the definition of a Jacobi cusp form is given in Chapter 2). In [BK] the Fourier coefficients of

Jacobi cusp forms are estimated by developing a Petersson coefficient formula for Jacobi cusp forms and by estimating the Fourier coefficients of certain Jacobi-Poincaré series $P_{k,m;(n,r)}$ uniformly in $\det m$. The restriction $k > g+1$ is needed for the absolute convergence of $P_{k,m;(n,r)}$. Then the Petersson norm of a Jacobi cusp form is estimated for the particular case where the Jacobi cusp form arises from a Fourier-Jacobi expansion. This can be done by using certain Dirichlet series of Rankin-Selberg type which have a meromorphic continuation to the whole complex plane with finitely many poles and satisfy a certain functional equation (cf. [KS]). Then one can use a version of the Theorem of Sato and Shintani.
The method in [Br] is quite the same, he only uses a different splitting of $Z \in \mathbb{H}_3$, with $\tau \in \mathbb{H}_2$, $z \in \mathbb{C}^{(2,1)}$, and $\tau' \in \mathbb{H}$. Breulmann tried to generalize his result for arbitrary g by using in the general case the splitting $\tau \in \mathbb{H}_2$, $z \in \mathbb{C}^{(2,g-2)}$, and $\tau' \in \mathbb{H}_{g-2}$. Unfortunately for $g > 3$ his estimate is worse than the one of [BK]. In the case $g \geq 7$ it is even worse than the trivial Hecke bound.

In Chapter 3 we prove generalizations of the estimates of [BK] in various directions. More precisely we regard the limiting case $k = g+1$, and the case where Γ_g is replaced by the subgroup

$$\Gamma_{g,0}(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C \equiv 0 \pmod{N} \right\}$$

of Γ_g.

In the case $k = g+1$ the Poincaré series for Jacobi forms cannot be defined as before, because one can show that these series are not absolutely convergent (cf. Lemma 2.27). Therefore we use the so-called Hecke trick and multiply every summand of the Poincaré series with a factor depending on a complex variable s such that the new series $P_{k,m;(n,r),s}$ is again absolutely convergent for $\sigma = \mathrm{Re}(s)$ sufficiently large. Moreover this factor is chosen such that the new series is again invariant under the slash operation of the Jacobi group. For $g = 2$ the use of the Hecke trick is suggested in [GKZ] but not carried out. Now the method is the following one: we compute the Fourier expansion of $P_{k,m;(n,r),s}$ (cf. Theorem 3.4) and show that it is even absolutely and locally uniformly convergent in a larger domain of $\mathbb{C}$ that contains the point $s = 0$ if $k = g+1$. For this we have to estimate certain integrals by changing the path of integration in a suitable way (cf. Corollary 3.7) and we have to estimate generalized Kloosterman sums, using some formulas for Gauß sums and some knowledge about quadratic forms over the p-adic numbers $\mathbb{Z}_p$ (cf. Lemma 3.9). Combining these results, we can show that the series that is defined through the above Fourier expansion is absolutely and locally uniformly convergent in s in the larger domain (cf. Lemma 3.10). Therefore we have a definition for the Poincaré series for $k = g+1$. What is left to show is that these series are Jacobi cusp forms and that the Petersson coefficient formula is still valid. The first claim follows quite easily from what we already know; the second claim is more difficult. The problem is that the scalar product $\langle \phi, P_{g+1,m;(n,r)} \rangle$ cannot be computed directly but only via the limit $\lim_{\sigma\to 0} \langle \phi, P_{g+1,m;(n,r)\sigma} \rangle$. Therefore we have to show that these scalar products are absolutely convergent (cf. Lemma 3.12), compute it explicitely (cf. Lemma 3.13) and show that we may change limit and integration (cf. Lemma 3.14). Thus we obtain (cf. Theorem 3.20)

Theorem 1.2 *Let $g \geq 2$ and suppose that $k \geq g+1$. Then we have*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/(2g)-(1-1/g)\alpha_g+\epsilon} \qquad (\epsilon > 0),$$

where α_g is defined in (1.2), and where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F, i.e., the estimate of [BK] is still valid in the border case $k = g+1$.

In the second section we generalize the estimates for Fourier coefficients to Siegel cusp forms on $\Gamma_{g,0}(N)$.
Thus in what follows we let F be a cusp form of integral weight k with respect to $\Gamma_{g,0}(N)$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix.
For $g = 2, 3$ we obtain estimates of the same quality as in the case of the full modular group; for $g > 3$ we obtain a slightly weaker estimtate. For the proof we define a vector space of certain Jacobi forms for subgroups using the same techniques as in [BK]. We estimate the Fourier coefficients of these Jacobi cusp forms (again using for $k = g+1$ the Hecke-trick). The difficulty lies in the estimates of the Petersson norm of the Fourier-Jacobi coefficients since it is not obvious how to define similar Dirichlet series of Rankin-Selberg type with a simple functional equation. Thus we instead use the classical Hecke argument to obtain a slightly weaker estimate for the Petersson norm which leads to (cf. Theorem 3.74 and Corollary 3.75)

Theorem 1.3 *Let $g \geq 2$, $k \geq g+1$. Then*

$$a(T) \ll_{\epsilon,F} (m_{g-1}(T))^{1/2} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where $m_{g-1}(T) := min\{T[U]|_{g-1} | \, U \in GL_g(\mathbb{Z})\}$, where $T[U]_{g-1}$ denotes the $(g-1) \times (g-1)$ minor of $T[U]$, and where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 1.4 *Let $g \geq 2$, $k \geq g+1$. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/(2g)+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

We now proceed using a different splitting in the Fourier-Jacobi expansion (in the case $g = 3$ this splitting coincides with the splitting used in [Br]). Doing so, we again can define certain Dirichlet series of Rankin-Selberg type (cf. Definition 3.69) which have a meromorphic continuation to the whole complex plane and satisfy a certain quite complicated functional equation that sets this Dirichlet series in connection to other Dirichlet series (cf. Theorem 3.72). Thus we obtain, again using a modified version of the Theorem of Sato and Shintani (cf. Theorem 3.73) the following improvements for $g = 2$ and $g = 3$ (cf. Theorem 3.76, Corollary 3.77, Theorem 3.78, and Corollary 3.79)

Theorem 1.5 *Let $g = 2$ and $k \geq 3$. Then*

$$a(T) \ll_{\epsilon,F} (min(T))^{5/18+\epsilon} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where $min(T)$ is the least positive integer that is represented by T, and where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 1.6 *Let $g = 2$ and $k \geq 3$. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-13/36+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Theorem 1.7 *Let $g = 3$ and $k \geq 8$ an even integer. Then*

$$a(T) \ll_{\epsilon,F} (min(T))^{1-3/13+\epsilon} \cdot m_2(T)^{-1/2} \cdot (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 1.8 *Let $g = 3$ and $k \geq 8$ an even integer. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Thus in the cases $g = 2$ and $g = 3$ the estimates for the full Siegel modular group obtained in [K1] and [Br], respectively, are also valid for $\Gamma_{g,0}(N)$. Moreover we have, similar as in [K8], the following estimate on average for $g = 2$ (cf. Corollary 3.80)

Corollary 1.9 *We have*

$$\sum_{\{T>0, tr(T)=N\}} |a(T)|^2 \ll_{\epsilon,F} N^{k-1/2+\epsilon}.$$

In the second part of our work we generalize (under certain restrictions) the construction of lifting maps from the vector space of Jacobi cusp forms to a certain subspace of elliptic modular forms to arbitrary g. The case $g = 1$ is treated in [GKZ].
Let $n, k, g \in \mathbb{N}$, where $g \equiv 1 \pmod 8$, and $k \geq \frac{g+3}{2}$. Moreover let m be a positive definite symmetric half-integral $g \times g$ matrix (this implies in particular that $\frac{1}{2}\det(2m)$ is an integer), $r \in \mathbb{Z}^{(1,g)}$, $D_0 := -\det 2\begin{pmatrix} n_0 & \frac{r_0}{2} \\ \frac{r_0^t}{2} & m \end{pmatrix} < 0$ such that D_0 is a square modulo $\frac{1}{2}\det(2m)$, $\frac{1}{2}\det(2m)$ is odd, coprime to D_0 and D_0 is a fundamental discriminant. Let us put $\Gamma^J_{1,g} := \Gamma_1 \ltimes \left(\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{g,1}\right)$ and denote by $J^{cusp}_{k,m}(\Gamma^J_{1,g})$ the vector space of Jacobi cusp forms with respect to $\Gamma^J_{1,g}$. Let $S_k(\frac{1}{2}\det(2m))^-$ be the subspace of elliptic cusp forms of weight k with respect to $\Gamma_0(\frac{1}{2}\det(2m))$ that have eigenvalue -1 under the Fricke involution. Then we can give the following

Definition 1.10 *For $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}(\Gamma^J_{1,g})$ we define*

$$\mathcal{S}_{D_0,r_0}(\phi)(w) := 2^{1-g} \cdot \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{D_0}{d}\right) d^{k-1} c_\phi\left(\frac{n^2}{d^2}n_0, \frac{n}{d}r_0\right)\right) e^{2\pi i n w} \qquad (w \in \mathbb{H}),$$

where $c_\phi(n, r)$ is the $(n, r)-$th Fourier coefficient of ϕ.

Definition 1.11 *For $f \in S_{2k}(\frac{1}{2}\det(2m))^-$ we define for $(\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}$*

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau, z) := \left(\frac{i}{\det(2m)}\right)^{k-1} \cdot \sum_{\substack{n\in\mathbb{Z} \\ r\in\mathbb{Z}^{(1,g)} \\ 4n > m^{-1}[r^t]}} r_{k,\frac{1}{2}\det(2m),D_0 D,r_0(2m)^* r^t,D_0}(f) e^{2\pi i(n\tau + rz)},$$

where $D := -\det 2\begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix}$, and where $r_{k,\frac{1}{2}\det(2m),D_0 D,r_0(2m)^* r^t,D_0}(f)$ is a certain cycle integral (cf. Definition 4.13).

We prove the following version (cf. Theorem 4.19)

Theorem 1.12 *If ϕ is an element of $J^{cusp}_{k+\frac{g+1}{2},m}(\Gamma^J_{1,g})$, then the function $\mathcal{S}_{D_0,r_0}(\phi)(w)$ is an element of $S_{2k}(\frac{1}{2}\det(2m))^-$.*
If $f \in S_{2k}(\frac{1}{2}\det(2m))^-$, then the function $\mathcal{S}^*_{D_0,r_0}(f)(\tau, z)$ is an element of $J^{cusp}_{k+\frac{g+1}{2},m}(\Gamma^J_{1,g})$.
The maps

$$\mathcal{S}_{D_0,r_0} : J^{cusp}_{k+\frac{g+1}{2},m} \to S_{2k}(\frac{1}{2}\det(2m))^-$$

$$\mathcal{S}^*_{D_0,r_0} : S_{2k}(\frac{1}{2}\det(2m))^- \to J^{cusp}_{k+\frac{g+1}{2},m}$$

are adjoint maps with respect to the Petersson scalar products.

For the proof we follow the same method as in [GKZ] and define a function $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ that can easily be shown to be a holomorphic kernel function for the map $\mathcal{S}^*_{D_0,r_0}$. To prove the theorem we have to show that $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is also a holomorphic kernel function for the map $\mathcal{S}_{D_0,r_0}$. Using the Petersson coefficient formula for Jacobi cusp forms, we have to show that $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ has a Fourier expansion where the Fourier coefficients are certain linear combinations of Jacobi-Poincaré series. Therefore we have to manipulate certain higher dimensional congruences and compute sums of multi-variable Kloosterman sums.

I wish to thank Prof. Dr. W. Kohnen for suggesting the issue of this work and for giving helpful advice. Moreover I want to thank my family and all my friends for having a lot of appreciation. In particular I want to thank my father for correcting English phrases and Anke for helping with computer problems.

Chapter 2

Preliminaries

In this chapter we want to fix some notations and recall some basic definitions of Siegel and Jacobi cusp forms. Furthermore we want to introduce Poincaré series for the Jacobi group. For details we refer the reader to [EZ], [Fr], [Zi], and [BK].

2.1 Basic facts about Siegel and Jacobi cusp forms

For an integer g and a commutative ring R let us denote by $Sym_g(R)$ the set of symmetric matrices of size g with entries from R.
Moreover, if R is a commutative ring with 1, then $GL_g(R)$ denotes the group of $g \times g$ matrices with entries in R that are invertible in R (i.e., the inverse matrix has also entries in R).
Moreover let

$$Sp_g(\mathbb{R}) := \{M \in GL_g(\mathbb{R}) |\, I_g[M] = I_g\}$$

be the real symplectic group of genus g. Here we have used the abbreviations $I_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$, where E_g ist the identity matrix of size g and $A[B] := B^t AB$ for $A \in R^{(n,n)}$, $B \in R^{(n,m)}$ $(n, m \in \mathbb{N})$.
Additionally, we define

$$\Gamma_g := Sp_g(\mathbb{R}) \cap GL_g(\mathbb{Z}),$$

the full Siegel modular group of genus g, and

$$\Gamma_{g,0}(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C \equiv 0 \pmod{N} \right\},$$

where $C \equiv 0 \pmod{N}$ means that every entry of C is divisible by N.

Remark 2.1 *The group* Γ_g *is generated by the matrices* $\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$ *and* $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$ $(S = S^t)$.

Remark 2.2 $\Gamma_{g,0}(N)$ *is a subgroup of* Γ_g *of finite index.*

The set of positive definite $g \times g$ matrices with entries from $\mathbb{R}$ will be denoted by $\mathbb{P}_g$.

Definition 2.3 *A matrix* $Y \in \mathbb{P}_g$ *is called Minkowski-reduced if the following conditions are satisfied:*

$$\begin{array}{ll} (i) & Y[h] \geq Y[e_k] \quad \forall\, h = (h_1, \ldots, h_k)^t \in \mathbb{Z}^{(g,1)} \text{ with } (h_k, \ldots, h_g) = 1, 1 \leq k \leq g, \\ (ii) & e_i^t Y e_{i+1} \geq 0, \quad 1 \leq i < g, \end{array}$$

where e_i *denotes the* i*-th unit vector.*

Lemma 2.4 *For all* $Y \in \mathbb{P}_g$ *there exists a unimodular matrix* $U \in GL_g(\mathbb{Z})$ *such that* $Y[U]$ *is Minkowski-reduced.*

Let us denote by

$$\mathbb{H}_g := \{Z = X + iY \in Sym_g(\mathbb{C}) |\, Y > 0\}$$

the Siegel upper half space of degree g ($g \in \mathbb{N}$), where $Y > 0$ means that $Y \in \mathbb{P}_g$. For $\mathbb{H}_1$ we also use the abbreviation $\mathbb{H}$.

Definition 2.5 *An element* $Z \in \mathbb{H}_g$ *is called Siegel-reduced if the following conditions are satisfied:*

1. $|\det(CZ + D)| \geq 1$ *for all matrices* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$,
2. $Y = Im(Z)$ *is Minkowski-reduced,*
3. $X = Re(Z)$ *is reduced modulo* 1.

We will abbreviate the set of all Siegel-reduced matrices by $\mathbb{F}_g$.

Lemma 2.6 *For every element* $Z \in \mathbb{H}_g$ *there exists a matrix* $M \in \Gamma_g$ *such that* $M \circ Z$ *is Siegel-reduced.*

Remark 2.7 *If* $g = 1$, *then the set of all Siegel-reduces matrices is given by*

$$\left\{ z \in \mathbb{H} \,\middle|\, -\frac{1}{2} \leq x \leq \frac{1}{2}, x^2 + y^2 \geq 1 \right\},$$

that is the classical fundamental domain of the operation for $SL_2(\mathbb{Z})$. *In particular we have* $y \geq \frac{\sqrt{3}}{2}$.

In the following let us abbreviate

$$m_{g-1}(T) := min\{T[U]|_{g-1}|\, U \in GL_g(\mathbb{Z})\}, \tag{2.1}$$

where $T[U]_{g-1}$ denotes the $(g-1)\times(g-1)$ minor of $T[U]$. Then from reduction theory follows

Remark 2.8 *We have*

$$m_{g-1}(T) \ll_g (\det T)^{1-1/g},$$

where the constant implied in $\ll_g$ only depends on g.

Remark 2.9 *The group $Sp_g(\mathbb{R})$ acts on $\mathbb{H}_g$ in the usual way by*

$$M \circ Z := (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Definition 2.10 *Let $f : \mathbb{H}_g \to \mathbb{C}$. Then we define for $k \in \mathbb{Z}$*

$$f|_k M = \det(CZ+D)^{-k} f(M\circ Z) \qquad \left(\forall\, M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{R}), Z \in \mathbb{H}_g\right).$$

Let $\Gamma \subset \Gamma_g$ be a subgroup of Γ_g of finite index.
A function $f : \mathbb{H}_g \to \mathbb{C}$ is called a Siegel cusp form of weight k and degree g with respect to Γ if the following conditions are satisfied:
(i) f is holomorphic,
(ii) $f|_k M(Z) = f(Z) \qquad (\forall\, M \in \Gamma,\, Z \in \mathbb{H}_g)$,
(iii) for all $M \in \Gamma_g$, there exists a positive integer l such that the function $f|_k M$ has a Fourier expansion of the form

$$f|_k M(Z) = \sum_{T>0} a(T) e^{\frac{2\pi i}{l} tr\,(TZ)},$$

where the summation extends over all positive definite symmetric half-integral $g\times g$-matrices.
Let us denote by $S_k(\Gamma)$ the vector space of Siegel cusp forms with respect to Γ. If $g=1$ and $\Gamma = \Gamma_0(N)$, then we also use the abbreviation $S_k(N) := S_k(\Gamma_0(N))$.

Remark 2.11 *(i) In Definition 2.10 it is sufficient to claim condition (iii) for a set of representatives of $\Gamma\backslash\Gamma_g$.*
(ii) If F is a Siegel cusp form with respect to Γ_g or $\Gamma_{g,0}(N)$, then F has a Fourier expansion

$$F(Z) = \sum_{T>0} a(T) e^{2\pi i\; tr(TZ)},$$

where the summation extends over all positive definite symmetric half-integral $g \times g$-matrices.
Morever: if F is a Siegel cusp form with respect to $\Gamma_{g,0}(N)$, then $F|_k\gamma$ ($\gamma \in \Gamma_g$) has a Fourier expansion

$$F(Z) = \sum_{T>0} a(T) e^{\frac{2\pi i}{l}\ tr(TZ)},$$

where the summation extends over all positive definite symmetric half-integral $g \times g$-matrices.

Remark 2.12 *Let $f \in S_k(\Gamma)$ and let $U \in GL_g(\mathbb{Z})$ with $\begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \in \Gamma$. Then*

$$a(T[U]) = (\det U)^k a(T).$$

Lemma 2.13 *If $f \in S_k(\Gamma)$, then the function*

$$h(Z) := |f(Z)|(\det Y)^{k/2},$$

where Y =Im(Z), is invariant under the action of Γ and has a maximum in $\mathbb{H}_g$. If $Z \in \mathbb{F}_g$, then

$$|h(Z)| \ll_h e^{-a\sum_{i=1}^g y_i},$$

where $y_i = Y[e_i]$ $(1 \leq i \leq g)$, where a is a constant only depending on h, and where the constant implied in $\ll_h$ also only depends on h.

Definition 2.14 *For $f, g \in S_k(\Gamma)$ we define*

$$< f, g > := \frac{1}{[\Gamma_g : \Gamma]} \cdot \int_{\mathbb{F}} f(Z) \cdot \overline{g(Z)} \cdot (\det Y)^k \, dV_g, \tag{2.2}$$

where $\mathbb{F}$ is a fundamental domain of the action of Γ on $\mathbb{H}_g$, where $dV_g = (\det Y)^{-(g+1)} dX dY$ is the symplectic volume element, and where we have written Z as $Z = X + iY$.

Remark 2.15
1. *The fundamental domain of the group Γ has a finite symplectic volume, i.e.,*
$$\int_{\mathbb{F}} dV_g < \infty.$$
2. *The integral in (2.2) is absolutely convergent and independent of the choice of the fundamental domain.*
3. *$< \cdot, \cdot >$ defines a scalar product on $S_k(\Gamma)$, the so-called Petersson scalar product.*

4. *One can also form $< f, g >$ for arbitrary complex-valued functions f and g that are defined on $\mathbb{H}_g$, and that are invariant under the slash operation of Γ, if the integral is absolutely convergent with respect to one fundamental domain (and therefore with respect to all fundamental domains).*

5. *If we want to emphasize with respect to which subgroup Γ of Γ_g we consider the scalar product, we also write $< \cdot, \cdot >_\Gamma$ instead of $< \cdot, \cdot >$.*

Next we want to give the definition of a Jacobi cusp form. Therefore we recall what the Jacobi-group is and how it acts.

Definition 2.16 *Let $l, n \in \mathbb{N}$ and let us abbreviate*

$$\begin{aligned} \Gamma^J_{l,n} &:= \Gamma_l \ltimes \left(\mathbb{Z}^{(n,l)} \times \mathbb{Z}^{(n,l)}\right), \\ \Gamma^J_{l,n,0}(N) &:= \Gamma_{l,0}(N) \ltimes \left(\mathbb{Z}^{(n,l)} \times \mathbb{Z}^{(n,l)}\right). \end{aligned}$$

Then $\Gamma^J_{l,n}$ acts on $\mathbb{H}_l \times \mathbb{C}^{(n,l)}$ in the usual way by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z) := \left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right).$$

Let k be an integer, m be a positive definite symmetric half-integral $g \times g$ matrix. Moreover let $\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J_{l,n}$, and $\phi : \mathbb{H}_l \times \mathbb{C}^{(n,l)} \to \mathbb{C}$. Then we define the following action

$$\begin{aligned} \phi|_{k,m}\gamma(\tau, z) \;:=\; & \det(C\tau + D)^{-k} \cdot e\left(-tr\left(m(C\tau + D)^{-1}Cm[(z + \lambda\tau + \mu)^t]\right.\right. \\ & \left.\left. + m\tau[\lambda^t] + 2m\lambda z^t\right)\right) \cdot \phi(\gamma \circ (\tau, z)), \end{aligned}$$

where

$$e(x) := e^{2\pi i x} \qquad (\forall\, x \in \mathbb{R}).$$

Remark 2.17 *Later we will only need the case $(l, n) = (1, g - 1)$ and $(l, n) = (2, 1)$.*

Now we can give the definition of a Jacobi cusp form.

Definition 2.18 *Let m be a positive definite symmetric half-integral $g \times g$ matrix, and let Γ be a subgroup of $\Gamma^J_{l,n}$ of finite index. A function $\phi : \mathbb{H}_l \times \mathbb{C}^{(n,l)} \to \mathbb{C}$ is called a Jacobi cusp form of weight k and index m with respect to Γ, if the following conditions are satisfied:*
(i) ϕ is holomorphic,
(ii) $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z) \qquad (\forall\, \gamma \in \Gamma,\ (\tau, z) \in \mathbb{H}_l \times \mathbb{C}^{(n,l)})$,

(iii) for all $\gamma \in \Gamma^J_{n,l}$ there exists a positive integer M such that the function $\phi|_{k,m}\gamma$ has a Fourier expansion of the form

$$\phi|_{k,m}\gamma(\tau, z) = \sum_{\substack{n\in\mathbb{Z}^{(l,l)} \\ r\in\mathbb{Z}^{(l,n)} \\ \frac{4n}{N} > m^{-1}[r^t]}} c(n,r)e\left(\frac{1}{M}tr\,(n\tau) + tr\,(rz)\right).$$

Let us denote by $J^{cusp}_{k,m}(\Gamma^J_{l,n})$ and $J^{cusp}_{k,m}(\Gamma^J_{l,n,0}(N))$ the vector spaces of Jacobi cusp forms with respect to $\Gamma^J_{l,n}$ and $\Gamma^J_{l,n,0}(N)$, respectively. If there is no room for confusion we merely write $J^{cusp}_{k,m}$ and $J^{cusp}_{k,m}(N)$, respectively.

Remark 2.19 *Let T be a real-valued positive definite symmetric $g \times g$ matrix. Then we can write T in the form*

$$\begin{aligned} T \; &=: \begin{pmatrix} n & r \\ r^t & m \end{pmatrix} = \begin{pmatrix} n - m^{-1}[r^t] & 0 \\ 0 & m \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ m^{-1}r^t & E_{g-1} \end{pmatrix}\right] \\ &= \begin{pmatrix} n & 0 \\ 0 & m - n^{-1}[r] \end{pmatrix} \left[\begin{pmatrix} 1 & n^{-1}r \\ 0 & E_{g-1} \end{pmatrix}\right], \end{aligned}$$

where $n \in \mathbb{P}_l$, $r \in \mathbb{C}^{(l,n-l)}$, and $m \in \mathbb{P}_{n-l}$. This decomposition is called Jacobi decomposition.
From this decomposition one can directly see that the following conditions are equivalent:

1. $T > 0$,
2. $n - m^{-1}[r^t] > 0$, *and* $m > 0$,
3. $n > 0$, *and* $m - n^{-1}[r^t] > 0$.

One can embed the Jacobi group into the Siegel modular group, i.e., we have the following

Remark 2.20 *For $\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J_{l,n}$ we define*

$$\gamma^{\uparrow} := \begin{pmatrix} A & 0 & B & \tilde{\mu} \\ \lambda & E & \mu & 0 \\ C & 0 & D & -\tilde{\lambda} \\ 0 & 0 & 0 & E \end{pmatrix},$$

where $(\tilde{\lambda}^t, \tilde{\mu}^t) := (\lambda, \mu)\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$. Then $\gamma^{\uparrow}$ is an element of Γ_{l+n}. If additionally $\gamma \in \Gamma^J_{l,n,0}(N)$, then $\gamma^{\uparrow} \in \Gamma_{l+n}(N)$. Let us denote by $(\Gamma^J_{l,n})^{\uparrow}$ the set of all these $\gamma^{\uparrow}$.

Remark 2.21 *The map $\gamma \mapsto \gamma^{\uparrow}$ is not a group morphism. If one substitutes the Jacobi group by the Heisenberg group, then one obtains a group morphism, but this property is not needed in our work.*

The following remark shows why Jacobi cusp forms play such an important role for deriving estimates of Fourier coefficients of Siegel cusp forms.

Remark 2.22 1. *Let $F \in S_k(\Gamma)$, $\Gamma \in \{\Gamma_g, \Gamma_{g,0}(N)\}$, $n, l \in \mathbb{N}$ with $n + l = g$. Let us decompose $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix} \in \mathbb{H}_g$ ($\tau \in \mathbb{H}_l$, $z \in \mathbb{C}^{(n,l)}$, and $\tau' \in \mathbb{H}_n$). Then the Fourier expansion of F can be written in the form*

$$F(Z) = \sum_{m>0} \phi_m(\tau, z) e^{2\pi i \, tr(m\tau')}, \tag{2.3}$$

were the summation extends over all positive definite symmetric half-integral $n \times n$ matrices. Then the coefficients $\phi_m(\tau, z)$ are Jacobi cusp forms with respect to $\Gamma^J_{l,n}$ if $\Gamma = \Gamma_g$ and with respect to $\Gamma^J_{l,n,0}(N)$ if $\Gamma = \Gamma_{g,0}(N)$. The developement in (2.3) is called Fourier Jacobi developement.

2. *If $M \in \Gamma_g^{\uparrow}$, $F \in S_k(\Gamma_{g,0}(N))$, then $F|_k M$ has a Fourier expansion*

$$F|_k M(Z) = \sum_{m>0} \tilde{\phi}_m(\tau, z) e^{2\pi i \, tr(m\tau')},$$

were the summation extends over all positive definite symmetric half-integral $n \times n$ matrices. The coefficients $\tilde{\phi}_m(\tau, z)$ have a Fourier expansion of the form

$$\tilde{\phi}_m(\tau, z) = \sum_{\substack{n \in \mathbb{Z}^{(l,l)} \\ r \in \mathbb{Z}^{(l,n)} \\ \frac{4n}{N} > m^{-1}[r^t]}} c(n, r) e\left(\frac{1}{N} \; tr(n\tau) + \; tr(rz) \right).$$

Proof. Remark 2.22 follows directly from Remark 2.20 applying the transformation law of F, and using that $F|_k M \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$ has period 1 in z and τ', and period N in τ. $\square$

Lemma 2.23 *If ϕ is a Jacobi cusp form with respect to Γ, where Γ is a subgroup of $\Gamma^J_{l,n}$ of finite index, then the funtion*

$$h(\tau, z) := |\phi(\tau, z)| \cdot (\det v)^{k/2} \cdot e^{-2\pi tr(mv^{-1}[y])},$$

where v =Im(τ) and y =Im(z), is invariant under the action of Γ and has a maximum in $\mathbb{H}_l \times \mathbb{H}^{(n,l)}$.

Definition 2.24 *For Jacobi cusp forms ϕ and ψ with respect to Γ, where Γ is a subgroup of $\Gamma^J_{l,n}$ of finite index, we define*

$$< \phi, \psi > := \frac{1}{[\Gamma^J_{l,n} : \Gamma]} \int_{\mathbb{F}} \phi(\tau, z) \overline{\psi(\tau, z)} (\det v)^k \exp\left(-4\pi tr \ \left(mv^{-1}[y]\right)\right) dV^J_n, \quad (2.4)$$

where $dV^J_n = (\det v)^{-n-2} dudvdxdy$ denotes the invariant volume element of the Jacobi group, where we have written $\tau = u + iv$, $z = x + iy$, and where $\mathbb{F}$ denotes a fundamental domain of the action of Γ on $\mathbb{H}_l \ltimes \left(\mathbb{C}^{(n,l)} \times \mathbb{C}^{(n,l)}\right)$.

Remark 2.25

1. *As a fundamental domain for the action of $\Gamma \ltimes \left(\mathbb{Z}^{(n,l)} \times \mathbb{Z}^{(n,l)}\right)$ on $\mathbb{H}_l \times \mathbb{C}^{(n,l)}$, where Γ is a subgroup of Γ_l of finite index, we can choose the set*

$$\left\{(\tau, z) \in \mathbb{H}_l \times \mathbb{C}^{(n,l)} \,\middle|\, \tau \in \mathcal{F}_\Gamma,\ 0 \le x_\nu \le 1\,(1 \le \nu \le g),\ 0 \le (yv^{-1})_{\nu\mu} \le 1 \right.$$
$$\left. (1 \le \nu, \mu \le g)\right\} / \{(\tau, z) \mapsto (\tau, -z)\},$$

where we have written τ and z as $\tau = u + iv$ and $z = x + iy$, respectively, where x_ν denotes the ν-th component of x, and where $(yv^{-1})_{\nu\mu}$ denotes the (ν, μ)-th entry of the matrix yv^{-1}. Moreover $\mathcal{F}_\Gamma$ denotes a fundamental domain of the action of Γ on $\mathbb{H}_l$.

2. *The fundamental domain of the group Γ, where Γ is a subgroup of $\Gamma^J_{l,n}$ of finite index, has a finite volume, i.e.,*

$$\int_{\mathbb{F}} dV^J_n < \infty.$$

3. *The integral in (2.4) is absolutely convergent and independent of the choice of the fundamental domain.*

2.2 Poincaré series

Let us recall the following formal

Definition 2.26 *Let $n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)}$, and let m be a positive definite symmetric half-integral $g \times g$ matrix such that $4n > m^{-1}[r^t]$. Then we define a Poincaré series of exponential type by*

$$P_{k,m;(n,r)}(\tau, z) := \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g}} e^{n,r}|_{k,m}\gamma(\tau, z) \qquad \left(\tau \in \mathbb{H},\ z \in \mathbb{C}^{(g,1)}\right),$$

where

$$e^{n,r}(\tau, z) := e(n\tau + rz) := e^{2\pi i(n\tau + rz)} \qquad \left(\tau \in \mathbb{H}, z \in \mathbb{C}^{(g,1)}\right),$$

and where

$$(\Gamma_{1,g}^J)_\infty := \left\{ \left(\left(\begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right), (0,\mu) \right) \middle| \, n \in \mathbb{Z},\, \mu \in \mathbb{Z}^{(g,1)} \right\}$$

is the stabilizer group of the function $e^{n,r}$.

Then we have the following

Lemma 2.27 *The series* $P_{k,m;(n,r)}(\tau,z)$ *is absolutely and locally uniformly convergent on* $\mathbb{H} \times \mathbb{C}^{(g,1)}$ *if* $k > g+2$.
For $k \leq g+2$ *the series is not absolutely convergent at the point* $(i,0)$.

Remark 2.28 *The first part of Lemma 2.27 is already stated in [BK], but no proof is given.*

Proof. For clarity we subdivide the proof into two steps.
I. In the first step we show that in the case $k > g+2$ the series is absolutely and locally uniformly convergent on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ if it is absolutely convergent at the point $(i,0)$.
Therefore we let K be an arbitrary compact subset of $\mathbb{H} \times \mathbb{C}^{(g,1)}$ and write τ and z as $\tau = u + iv$ and $z = x + iy$, respectively. We introduce for $(\tau, z) \in K$ an auxiliary variable $\tau' = u' + iv' \in \mathbb{H}_g$ such that

$$Z_{(\tau,z,\tau')} := \left(\begin{array}{cc} \tau & z^t \\ z & \tau' \end{array} \right) \in \mathbb{H}_{g+1}.$$

From the Jacobi decomposition (cf. Remark 2.19) it follows that we can choose for example $\tau' = u' + iv'$ with $v' = i\left(\frac{yy^t}{v} + \epsilon E_g\right)$, where $\epsilon > 0$ is chosen arbitrarily. Moreover for $(\tau, z) = (i, 0)$ we can choose $\tau' \in \mathbb{H}_g$ arbitrarily. Now let $K' \subset \mathbb{H}_{g+1}$ be a compact set that contains $Z_{(\tau,z,\tau')}$ and $Z_{(i,0,\tau')}$ for all elements (τ, z) from K (where we have chosen one τ' for every element (τ, z)).
As described in Remark 2.20, if $\gamma = \left(\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), (\lambda, \mu) \right)$ is an element of $\Gamma_{1,g}^J$, then the matrix $\gamma^\uparrow := \left(\begin{array}{cccc} a & 0 & b & \tilde{\mu} \\ \lambda & E & \mu & 0 \\ c & 0 & d & -\tilde{\lambda} \\ 0 & 0 & 0 & E \end{array} \right)$ is an element of Γ_{g+1}.
Put $T := \left(\begin{array}{cc} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{array} \right) > 0$ and $e^T(Z) := e^{2\pi i\, tr(TZ)}$ $(\forall\, Z \in \mathbb{H}_{g+1})$.
Then one can easily show that $\forall\, \gamma \in \Gamma_{1,g}^J, (\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}$ we have

$$\left(e^{n,r}|_{k,m}\gamma\right)(\tau,z) = e^{-2\pi i \mathrm{tr}\,(m\tau')} \cdot (e^T|_k \gamma^\uparrow)(Z_{(\tau,z,\tau')}).$$

Thus

$$|e^{n,r}|_{k,m}\gamma(\tau,z)| = |c\tau+d|^{-k}\cdot e^{2\pi \mathrm{tr}\,(mv')}\cdot e^{-2\pi\mathrm{tr}\,(T\,\mathrm{Im}(\gamma^{\uparrow}\circ Z_{(\tau,z,\tau')}))}.$$

It is known that there exists a positive constant c_1 such that

$$\mathrm{Im}(M\circ\tilde{Z}) \le c_1\cdot\mathrm{Im}(M\circ Z) \quad (\forall\, M\in\Gamma_{g+1},\,\forall\, Z,\,\tilde{Z}\in\tilde{K})$$

(cf. [Ch]). Here $A\le B$ for real quadratic matrices means that the matrix $B-A$ is positive semi-definite (in particular we have $\mathrm{tr}(A)\le\mathrm{tr}(B)$). Thus

$$\mathrm{Im}(\gamma^{\uparrow}\circ Z_{(i,0,\tau')}) \ll_K c_1\cdot\mathrm{Im}(\gamma^{\uparrow}\circ Z_{(\tau,z,\tau')}) \qquad \left(\forall\,\gamma\in\Gamma^J_{1,g},(\tau,z)\in K\right),$$

where the constant implied in $\ll_K$ only depends on K. Clearly we may choose $c_1\in\mathbb{N}$.
Moreover there exists a constant $\epsilon>0$ such that

$$|c\tau+d|^2\ge\epsilon(c^2+d^2) \quad \begin{array}{l}\forall\, c,d\in\mathbb{Z} \text{ with } (c,d)=1,\\ \forall\,\tau\in\mathbb{H} \text{ s.t. } (\tau,z)\in K,\end{array}$$

where the constant ϵ only depends on K (cf. [FB] p. 313).
Therefore we get, using that $\mathrm{tr}(AB)\ge 0$ if $A,B\ge 0$,

$$\begin{aligned}|e^{n,r}|_{k,m}\gamma(\tau,z)| &\ll_K (c^2+d^2)^{-k/2}\cdot e^{-2\pi c_1\mathrm{tr}\,\left(T\,\mathrm{Im}\left(\gamma^{\uparrow}\circ Z_{(i,0,\tau')}\right)\right)}\cdot e^{2\pi\mathrm{tr}\,(c_1 mv')}\\ &= |e^{c_1 n,c_1 r}|_{k,c_1 m}\gamma(i,0)|,\end{aligned}$$

where the constant implied in $\ll_K$ only depends on K.
Thus the claim in I. follows.

II. In the second step we want to show the absolute convergence at the point $(i,0)$ in the case $k>g+2$. Therefore let us take as a set of representatives of $(\Gamma^J_{1,g})_\infty\backslash\Gamma^J_{1,g}$ the elements

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(a\lambda,b\lambda)\right), \tag{2.5}$$

where $c,d\in\mathbb{Z}$ with $(c,d)=1$, $\lambda\in\mathbb{Z}^{(g,1)}$, and where for each pair (c,d) we have chosen $a,b\in\mathbb{Z}$ such that $ad-bc=1$.
Then the series of the absolute values of terms of the Poincaré series at the point $(i,0)$ is given by

$$\sum_{(c,d)=1}(c^2+d^2)^{-k/2}\cdot e^{-\frac{2\pi n}{(c^2+d^2)}}\sum_{\lambda\in\mathbb{Z}^{(g,1)}} e^{-\frac{2\pi}{c^2+d^2}(m[\lambda]+r\lambda)}. \tag{2.6}$$

Completing the square, we get that the value of the inner sum in (2.6) equals

$$e^{\frac{\pi}{2(c^2+d^2)}m^{-1}[r^t]}\cdot\sum_{\lambda\in\mathbb{Z}^{(g,1)}} e^{-\frac{2\pi}{(c^2+d^2)}m[\lambda+\frac{1}{2}m^{-1}r^t]}. \tag{2.7}$$

Since m is a positive definite matrix there exists a positive constant α such that $\alpha E_g \leq m$. Thus (2.6) can be estimated against a constant times

$$\sum_{(c,d)=1} \left(c^2+d^2\right)^{-k/2} \cdot e^{\frac{\pi\left(m^{-1}[r^t]-4n\right)}{2(c^2+d^2)}} \cdot \prod_{i=1}^{g} \sum_{\lambda_i \in \mathbb{Z}} e^{-\frac{2\pi}{(c^2+d^2)}\alpha\left(\lambda_i+\left(\frac{1}{2}m^{-1}r^t\right)_i\right)^2}, \qquad (2.8)$$

where $\left(\frac{1}{2}m^{-1}r^t\right)_i$ denotes the $i-$th entry of the vector $\frac{1}{2}m^{-1}r^t$.
Next we want to estimate the inner sums in (2.8). Therefore we show that for $l \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ we have

$$\sum_{\lambda \in \mathbb{Z}} e^{-l(\lambda+\mu)^2} \ll_\mu \left(1+l^{-1/2}\right), \qquad (2.9)$$

where the constant implied in $\ll_\mu$ only depends on μ modulo $\mathbb{Z}$. First the following estimate holds:

$$\sum_{\lambda \in \mathbb{Z}} e^{-l(\lambda+\mu)^2} \leq 1 + 2\sum_{\lambda \in \mathbb{N}} e^{-l\lambda^2}. \qquad (2.10)$$

Indeed: inequality (2.10) is clear for $\mu \in \mathbb{Z}$, because in this case $\lambda+\mu$ runs through $\mathbb{Z}$ if λ does. If $\mu \notin \mathbb{Z}$, inequality (2.10) follows directly from the estimates

$$-(\lambda+\mu)^2 \leq \begin{cases} -(\lambda+[\mu]+1)^2 & \text{for } \lambda \leq -[\mu]-1 \\ -(\lambda+[\mu])^2 & \text{for } \lambda \geq -[\mu], \end{cases}$$

where $[\mu]$ denotes the Gauß bracket.
Since $e^{-l\lambda^2} > 0$ and this function is decreasing it is known from elementary analysis that we can compare the sum with an integral which leads to the estimate

$$\sum_{\lambda \in \mathbb{N}} e^{-l\lambda^2} \leq \int_0^\infty e^{-lx^2} dx.$$

Thus we get the desired estimate if we use

$$\int_0^\infty e^{-lx^2} dx = \frac{1}{2\sqrt{l}} \cdot \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{2} \cdot \sqrt{\frac{\pi}{l}}.$$

Now we can use (2.9) with $l = \frac{2\pi\alpha}{c^2+d^2}$ and $\mu = \left(\frac{1}{2}m^{-1}r^t\right)_i$ in (2.8). Due to $c^2+d^2 \geq 1$ we can estimate the sum in (2.8) against a constant times

$$\sum_{(c,d)=1} \left(c^2+d^2\right)^{(g-k)/2} \cdot e^{\frac{\pi\left(m^{-1}[r^t]-4n\right)}{2(c^2+d^2)}}. \qquad (2.11)$$

In view of $4n - m^{-1}[r^t] > 0$, (2.11) can be estimated against a constant times

$$\sum_{(c,d)=1} (c^2+d^2)^{(g-k)/2}.$$

Therefore the series (2.6) converges if $k > g+2$ because it is well known that the series $\sum_{(c,d)=1}(c^2+d^2)^{-l}$, where $l \in \mathbb{R}^+$, is convergent if (and only if) $l > 1$.

The divergence at the point $(i, 0)$ in the case $k \leq g+2$ can be shown similarly if we use the existence of a constant β such that $m \leq \beta E_g$ and use that $e^{-\frac{\pi}{2}\left(4n-m^{-1}[r^t]\right)} \leq e^{-\frac{\pi}{2(c^2+d^2)}\left(4n-m^{-1}[r^t]\right)}$, since $c^2+d^2 \geq 1$ and $4n - m^{-1}[r^t] > 0$. □

From now on to the end of Subsection 2.2 we assume that $k > g+2$. We have the following

Theorem 2.29 *The function $P_{k,m;(n,r)}$ is an element of $J^{cusp}_{k,m}$.*
It has the Fourier expansion

$$P_{k,m;(n,r)}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)} \\ 4n' > m^{-1}[r'^t]}} g^{\pm}_{k,m;(n,r)}(n', r') e(n'\tau + r'z),$$

where $\pm = (\pm 1)^k$, where

$$g^{\pm}_{k,m;(n,r)}(n', r') := g_{k,m;(n,r)}(n', r') + (-1)^k g_{k,m;(n,r)}(n', -r'),$$

where

$$g_{k,m;(n,r)}(n', r') := \delta_m(n, r, n', r') + 2\pi i^k \cdot (\det(2m))^{-1/2} \cdot (D'/D)^{k/2-g/4-1/2}$$
$$\times \sum_{c \geq 1} e_{2c}(r' m^{-1} r^t) \cdot H_{m,c}(n, r, n', r') \cdot J_{k-g/2-1}\left(\frac{2\pi\sqrt{D'D}}{\det(2m) \cdot c}\right) \cdot c^{-g/2-1},$$

and where $D := -\det 2 \begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix}$, $D' := -\det 2 \begin{pmatrix} n' & \frac{r'}{2} \\ \frac{r'^t}{2} & m \end{pmatrix}$.
Furthermore,

$$\delta_m(n, r, n', r') := \begin{cases} 1 & \text{if} \quad D' = D,\ r' \equiv r \pmod{\mathbb{Z}^{(1,g)} \cdot 2m} \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{m,c}(n, r, n', r') := \sum_{\substack{x(c) \\ y(c)^*}} e_c((m[x] + rx + n)\bar{y} + n'y + r'x),$$

where x and y run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$, respectively, where $\bar{y}$ denotes an inverse of $y \pmod{c}$, and where $J_{k-g/2-1}$ is the Bessel function of order $k - g/2 - 1$.

Proof. For a proof, cf. [BK]. □

Remark 2.30 *We have used a notation slightly different from the one in [BK].*

Moreover we have

Theorem 2.31 *(Petersson coefficient formula)*
One has

$$\left\langle \phi, P_{k,m;(n,r)} \right\rangle = \lambda_{k,m,D} \cdot c_\phi(n,r) \qquad (\forall\, \phi \in J_{k,m}^{cusp}), \tag{2.12}$$

where $c_\phi(n,r)$ denotes the (n,r)-th Fourier coefficient of ϕ and

$$\lambda_{k,m,D} := 2^{(g-1)(k-g/2-1)-g} \cdot \Gamma\left(k-g/2-1\right) \cdot \pi^{-k+g/2+1} \cdot (\det m)^{k-(g+3)/2} \cdot |D|^{-k+g/2+1}.$$

Proof. For a proof, cf. [BK]. □

From Theorem 2.31 we directly obtain

Corollary 2.32 *As a unitary vector space with respect to the Petersson scalar product, $J_{k,m}^{cusp}$ is generated by the Poincaré series*

$$\left\{ P_{k,m;(n,r)} \,|\, n \in \mathbb{Z},\, r \in \mathbb{Z}^{(1,g)};\, 4n > m^{-1}[r^t] \right\}.$$

Chapter 3

Estimates for Fourier coefficients of Siegel cusp forms

3.1 The full Siegel modular group

Our aim in this chapter is to estimate the Fourier coefficients of Siegel cusp forms of weight $k = g+1$. For this we construct certain Poincaré series as a generating system of $J^{cusp}_{g+2,m}$. Remember that the case $k > g+2$ has been treated in Chapter 2.

3.1.1 Poincaré series of small weight

In Lemma 2.27 we have seen that $P_{k,m;(n,r)}(\tau, z)$ is not absolutely convergent at the point $(i, 0)$ if $k \leq g+2$. Thus the Poincaré series of weight $k = g+2$ has to be defined in a different way. We use the so-called Hecke trick (cf. [He]) and multiply $e^{n,r}|_{k,m}\gamma(\tau, z)$ in Definition 2.26 with a factor depending on a complex variable s such that the new series is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large (we will be more precise in Lemma 3.3) and can thus be analytically continued to $s = 0$. In his work Hecke uses this trick in order to define Eisenstein series of weight 1 and 2 in real quadratic fields.
In our case one may think of choosing the factor $|c\tau + d|^{-2s}$. However, later on we need scalar products of the Poincaré series with Jacobi cusp forms. Therefore it is better to adapt this factor such that the new series is again invariant under the slash operation of the Jacobi group. In the following, we write $\tau = u + iv$.

Definition 3.1 *Let $n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)}$, and m be a positive definite symmetric half-integral $g \times g$ matrix such that $4n > m^{-1}[r^t]$, $s \in \mathbb{C}$. Then we define a* *formal Poincaré series of exponential type by*

$$P_{k,m;(n,r),s}(\tau,z) := \sum_{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}} \left(\frac{v}{|c\tau+d|^2}\right)^s \cdot e^{n,r}|_{k,m}\gamma(\tau,z) \qquad \left((\tau,z)\in\mathbb{H}\times\mathbb{C}^{(g,1)}\right),$$

where $\left(\Gamma^J_{1,g}\right)_\infty$ *and* $e^{n,r}$ *are given in Definition 2.26.*

Remark 3.2 *The use of Hecke's trick for* $g=1$ *is also suggested in [GKZ].*

Lemma 3.3 *The series* $P_{k,m;(n,r),s}(\tau,z)$ *is absolutely and locally uniformly convergent on* $\mathbb{H}\times\mathbb{C}^{(g,1)}$ *if* $\sigma := Re(s) > \frac{1}{2}(g-k+2)$.
In this case it satisfies the transformation law

$$P_{k,m;(n,r),s}|_{k,m}\gamma(\tau,z) = P_{k,m;(n,r),s}(\tau,z) \qquad (\forall\,\gamma\in\Gamma^J_{1,g}\,,(\tau,z)\in\mathbb{H}\times\mathbb{C}^{(g,1)}).$$

Proof. The proof of the absolute and local uniform convergence goes exactly as the proof of Lemma 2.27. The transformation law follows directly from the absolute convergence. □

Now our aim is to compute the Fourier expansion of the Poincaré series and show that the Fourier series is even absolutely and locally uniformly convergent in s for $\sigma > \frac{1}{2}\left(g/2-k+2\right)$; for these values the series is holomorphic in s and can be taken as a definition for $P_{k,m;(n,r),s}$. In particular the series is holomorphic at $s=0$ for $k > g/2+2$.

Theorem 3.4 *Suppose that* $\sigma > \frac{1}{2}(g-k+2)$. *Then the Poincaré series has the Fourier expansion*

$$P_{k,m;(n,r),s}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v}(n',r')e(n'\tau+r'z),$$

where $\pm = (\pm1)^k$, *where*

$$g^{\pm}_{k,m;(n,r);s,v}(n',r') := g_{k,m;(n,r);s,v}(n',r') + (-1)^k g_{k,m;(n,r);s,v}(n',-r').$$

Here

$$g_{k,m;(n,r);s,v}(n',r') := v^s\cdot\delta_m(n,r,n',r') + \sum_{c\geq1} H_{m,c}(n,r,n',r')\cdot\Phi_{k,m,c,v}(n',r',s)\cdot c^{-k-2s},$$

with $D, D', \delta_m(n,r,n',r')$ *and* $H_{m,c}(n,r,n',r')$ *defined as in Theorem 2.29. Moreover*

$$\Phi_{k,m,c,v}(n',r',s) := (\det(2m))^{-1/2}\cdot i^{-g/2}\cdot v^{g/2-k-s+1}\cdot e_{2c}(r'm^{-1}r^t)$$
$$\times\int_{-\infty}^{\infty}(u+i)^{g/2-k-s}\cdot(u-i)^{-s}\cdot e\left((2\det(2m))^{-1}\left(D'v(u+i)+\frac{D}{vc^2(u+i)}\right)\right)du.$$

Proof. To compute the Fourier expansion of $P_{k,m;(n,r),s}(\tau,z)$ we proceed as in [BK]. Thus, taking the set of representatives given in (2.5), we obtain

$$P_{k,m;(n,r),s}(\tau,z) = \sum_{\substack{(c,d)=1\\ \lambda\in\mathbb{Z}^{(g,1)}}} (c\tau+d)^{-k}\cdot|c\tau+d|^{-2s}\cdot v^s\cdot e\Bigg(-m[z]\,\frac{c}{c\tau+d}$$
$$+m[\lambda]\,\frac{a\tau+b}{c\tau+d}+2\lambda^t mz\,\frac{1}{c\tau+d}+n\,\frac{a\tau+b}{c\tau+d}+rz\,\frac{1}{c\tau+d}+r\lambda\,\frac{a\tau+b}{c\tau+d}\Bigg).$$

We now split the sum into the terms with $c=0$ and the terms with $c\neq 0$.
If $c=0$, then $a=d=\pm 1$. Thus these terms give the contribution

$$v^s\sum_{\lambda\in\mathbb{Z}^{(g,1)}} e((m[\lambda]+r\lambda+n)\tau)[e((r+2\lambda^t m)z)+(-1)^k e((-r-2\lambda^t m)z)]$$
$$=v^s\sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}\\ 4n'>m^{-1}[r'^t]}} \left(\delta_m(n,r,n',r')\cdot e(n'\tau+r'z)+(-1)^k\delta_m(n,r,n',-r')e(n'\tau-r'z)\right).$$

The terms for $c<0$ are obtained from those with $c>0$ by multiplying with $(-1)^k$ and replacing z by $-z$, thus it sufficies to consider the terms with $c>0$. Using the identities

$$\frac{a\tau+b}{c\tau+d}=\frac{a}{c}-\frac{1}{c(c\tau+d)},$$
$$\frac{1}{c\tau+d}z+\frac{a\tau+b}{c\tau+d}\lambda=\frac{1}{c\tau+d}\left(z-\frac{1}{c}\lambda\right)+\frac{a}{c}\lambda,$$
$$\frac{a\tau+b}{c\tau+d}m[\lambda]+\frac{2}{c\tau+d}\lambda^t mz-\frac{c}{c\tau+d}m[z]=-\frac{c}{c\tau+d}m\left[z-\frac{1}{c}\lambda\right]+\frac{a}{c}m[\lambda],$$

we get, replacing d by $d+\alpha c$ and λ by $\lambda+\beta c$, with d running $\pmod{c}^*$, $\lambda \pmod{c}$, $\alpha\in\mathbb{Z}$, $\beta\in\mathbb{Z}^{(g,1)}$, that the terms for $c>0$ give the contribution

$$\sum_{c>0} c^{-k-2s}\sum_{\substack{d(c)^*\\ \lambda(c)}} e_c((m[\lambda]+r\lambda+n)\bar{d})\cdot\mathcal{F}_{k,m,c;(n,r),s}\left(\tau+d/c,z-\lambda/c\right),$$

where

$$\mathcal{F}_{k,m,c;(n,r),s}(\tau,z):=v^s\cdot\sum_{\substack{\alpha\in\mathbb{Z}\\ \beta\in\mathbb{Z}^{(g,1)}}}(\tau+\alpha)^{-k}\cdot|\tau+\alpha|^{-2s}\cdot e\Bigg(-\frac{1}{\tau+\alpha}m[z-\beta]$$
$$-\frac{n}{c^2(\tau+\alpha)}+\frac{1}{c(\tau+\alpha)}r(z-\beta)\Bigg)\left(\tau\in\mathbb{H},z\in\mathbb{C}^{(g,1)}\right).$$

The function $\mathcal{F}_{k,m,c;(n,r),s}(\tau,z)$ has period 1 in τ and z. Therefore it has a Fourier expansion of the form

$$\mathcal{F}_{k,m,c;(n,r),s}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} \Phi_{k,m,c,v}(n',r',s)e(n'u+r'z),$$

where

$$\Phi_{k,m,c,v}(n',r',s) = v^s\cdot e^{-2\pi vn'}\cdot \int_{-\infty}^{\infty}(u+iv)^{-k-s}\cdot(u-iv)^{-s}\cdot e^{-2\pi in'(u+iv)}$$
$$\int_{ic_2-\infty}^{ic_2+\infty}\cdots\int_{ic_2-\infty}^{ic_2+\infty} e\left(-\frac{1}{u+iv}m[z]-\frac{n}{c^2(u+iv)}+\frac{1}{c(u+iv)}rz-r'z\right)dudz \tag{3.1}$$

($c_2\in\mathbb{R}$). Here we have used the Poisson summation formula, which can be applied since for example the following three conditions are satisfied:
(i) $\mathcal{F}_{k,m,c;(n,r),s}(\tau,z)$ is holomorphic in z.
(ii) The series

$$\sum_{\substack{\alpha\in\mathbb{Z}\\ \beta\in\mathbb{Z}^{(g,1)}}}(\tau+\alpha)^{-k}\cdot|\tau+\alpha|^{-2s}\cdot e\left(-\frac{1}{\tau+\alpha}m[z-\beta]-\frac{n}{c^2(\tau+\alpha)}+\frac{1}{c(\tau+\alpha)}r(z-\beta)\right)$$

is uniformly convergent in u. This is a consequence of the compact convergence in τ already shown before (cf. Lemma 2.27), because we can choose u between 0 and 1 due to the periodicity 1 as a function of u.
(iii) The Fourier series

$$\sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} \Phi_{k,m,c,v}(n',r',s)e(n'u+r'z)$$

is absolutely convergent (which will be shown in Theorem 3.10).
Thus we get, using the computations of the inner integral of [BK] for (3.1) and substituting $u\mapsto \frac{u}{v}$,

$$\Phi_{k,m,c,v}(n',r',s) = (\det(2m))^{-1/2}\cdot i^{-g/2}\cdot v^{g/2-k-s+1}\cdot e_{2c}(r'm^{-1}r^t)$$
$$\int_{-\infty}^{\infty}(u+i)^{g/2-k-s}\cdot(u-i)^{-s}\cdot e\left((2\det(2m))^{-1}\left(D'v(u+i)+\frac{D}{vc^2(u+i)}\right)\right)du,$$

which shows the Theorem. $\square$

In order to show that the Fourier series is absolutely and locally uniformly convergent for $\sigma>\frac{1}{2}(g/2-k+2)$, the main difficulty to overcome is to estimate the integrals $\Phi_{k,m,c,v}(n',r',s)$. For this we need the following

Lemma 3.5 *Let $c_1, c_2, c_3 \in \mathbb{R}$, $c_3 > 0$. Then the integral*

$$f_s(c_1, c_2, c_3) := \int_{-\infty}^{\infty} (u+i)^{-c_1-s} \cdot (u-i)^{-s} \cdot e\left(-c_2 u - \frac{c_3}{u+i}\right) du$$

is locally uniformly convergent in s for $\sigma > \frac{1}{2}(1-c_1)$ and thus defines a holomorphic function in s.
Let K be an arbitrary compact set of the domain $\sigma > \frac{1}{2}(1-c_1)$. Then for all $s \in K$ we have the estimates

$$|f_s(c_1, c_2, c_3)| \ll_{K,c_1} \begin{cases} e^{2\pi c_2 v_1} & \text{if } c_2 < 0 \\ e^{-2\pi c_2 v_1} \cdot e^{\frac{\pi c_3}{A}} & \text{if } c_2 > 0 \\ 1 & \text{if } c_2 = 0 \end{cases}, \tag{3.2}$$

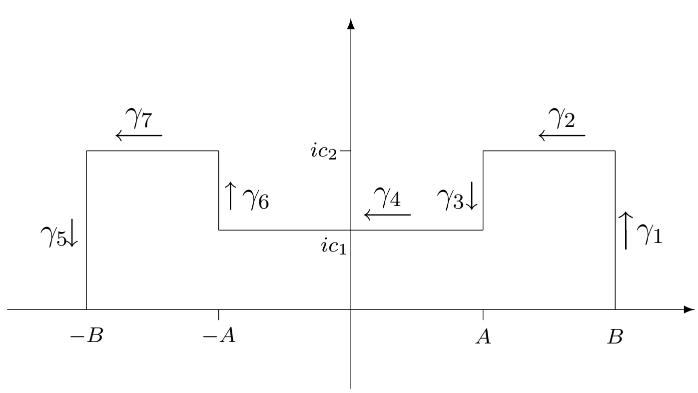
where the constant implied in $\ll_{K,c_1}$ only depends on K and c_1, and where v_1 and A are positive constants.

Proof. For the proof let us abbreviate the integrand with $g(u, s)$ and consider the three cases $c_2 = 0$, $c_2 < 0$, and $c_2 > 0$.
Let us start with the simplest case $c_2 = 0$. In view of $c_3 > 0$ we obtain the trivial estimate

$$|f_s(c_1, 0, c_3)| \leq \int_{-\infty}^{\infty} (u^2+1)^{-(c_1/2+\sigma)} \cdot e^{-\frac{2\pi c_3}{(u^2+1)}} du \leq 2 \int_{0}^{\infty} (u^2+1)^{-(c_1/2+\sigma)} du.$$

If $s \in K$, then the integral has a convergent majorant independent of s. Thus assertion (3.2) follows for $c_2 = 0$.

In the case $c_2 < 0$ we want to consider the following path of integration:



with

$$\gamma_j(t) = \begin{cases} B+it & 0 \le t \le v_2 \quad B>0,\ v_2>1 \quad \text{for } j=1 \\ (iv_2+t)^{-1} & A \le t \le B \quad 1<A<B \quad \text{for } j=2 \\ (A+it)^{-1} & v_1 \le t \le v_2 \quad 0<v_1<1 \quad \text{for } j=3 \\ (t+iv_1)^{-1} & -A \le t \le A \quad \text{for } j=4 \\ -A+it & v_1 \le t \le v_2 \quad \text{for } j=5 \\ (t+iv_2)^{-1} & -B \le t \le -A \quad \text{for } j=6 \\ (-B+it)^{-1} & 0 \le t \le v_2 \quad \text{for } j=7 \end{cases},$$

where A, B, v_1, and v_2 are positive constants with $v_1 < 1 < v_2, A$. Applying the residue theorem we find

$$\int_{-B}^{B} g(u,s)du = -\sum_{j=1}^{7} \int_{\gamma_j} g(u,s)\, du. \tag{3.3}$$

First we want to estimate the integrals along the paths γ_1 and γ_7. Since $c_2 < 0 < c_3$, the values of both integrals are less or equal than

$$\int_0^{v_2} (B^2+(1+t)^2)^{-\frac{c_1+\sigma}{2}} \cdot (B^2+(1-t)^2)^{-\frac{\sigma}{2}} \cdot e^{2\pi c_2 t - \frac{2\pi c_3(t+1)}{B^2+(t+1)^2}}\, dt$$
$$\le \int_0^{v_2} (B^2+(1+t)^2)^{-\frac{c_1+\sigma}{2}} \cdot (B^2+(1-t)^2)^{-\frac{\sigma}{2}}\, dt.$$

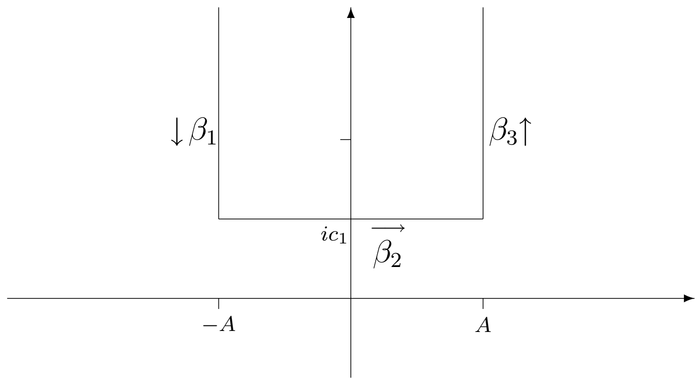
If $\sigma > -\frac{c_1}{2}$, then this integral tends to 0 if B tends to infinity.
Next we estimate the integrals along the paths γ_2 and γ_6. They are less or equal than

$$\int_A^B (t^2+(1+v_2)^2)^{-\frac{c_1+\sigma}{2}} \cdot (t^2+(1-v_2)^2)^{-\frac{\sigma}{2}} \cdot e^{2\pi c_2 v_2 - \frac{2\pi c_3(1+v_2)}{t^2+(v_2+1)^2}}\, dt.$$

Due to $c_2 < 0$ this integral tends to 0 for $v_2 \to \infty$.
Therefore we obtain from (3.3), letting B and v_2 both tend to infinity,

$$\int_{-\infty}^{\infty} g(u,s)\, du = \sum_{i=1}^{3} \int_{\beta_i} g(u,s)\, du,$$

where the paths β_j are given as



$$\beta_j = \begin{cases} (-A+it)^{-1} & v_1 \le t < \infty \quad \text{for } j=1 \\ t+iv_1 (=\gamma_4^{-1}) & -A \le t \le A \quad \text{for } j=2 \\ A+it & v_1 \le t < \infty \quad \text{for } j=3 \end{cases} .$$

Now we estimate the integrals along the paths β_1 and β_3.
Their values are less or equal than

$$\int_{v_1}^{\infty} (A^2+(1+t)^2)^{-\frac{c_1+\sigma}{2}} \cdot (A^2+(1-t)^2)^{-\frac{\sigma}{2}} \cdot e^{2\pi c_2 t - \frac{2\pi c_3(t+1)}{A^2+(t+1)^2}}\, dt.$$

Making the substitution $t \mapsto t+v_1$ we get, using $c_2 < 0 < c_3$, that the values of the integrals are less or equal than

$$e^{2\pi c_2 v_1} \cdot \int_0^{\infty} (A^2+(1+t+v_1)^2)^{-\frac{c_1+\sigma}{2}} \cdot (A^2+(1-t-v_1)^2)^{-\frac{\sigma}{2}}\, dt.$$

If $s \in K$, then the integral has a convergent majorant independent of s.
Finally we estimate the integral along the path β_2. In view of $0 < c_3$, we infer that the value of this integral is less or equal than

$$\int_{-A}^{A} (t^2+(1+v_1)^2)^{-\frac{c_1+\sigma}{2}} \cdot (t^2+(1-v_1)^2)^{-\frac{\sigma}{2}} \cdot e^{2\pi c_2 v_1 - \frac{2\pi c_3(v_1+1)}{t^2+(v_1+1)^2}}\, dt$$
$$\le e^{2\pi c_2 v_1} \cdot \int_{-A}^{A} (t^2+(1+v_1)^2)^{-\frac{c_1+\sigma}{2}} \cdot (t^2+(1-v_1)^2)^{-\frac{\sigma}{2}}\, dt.$$

If again $s \in K$, then the integral has a convergent majorant independent of s. Thus (3.2) follows for $c_2 < 0$.

In the case $c_2 > 0$ we reflect the paths from the case $c_2 < 0$ along the real line. The notations are kept the same, now taken for the new paths.
The difference we have to observe is that the exponential factor including the term with c_3 now cannot be estimated against 1. This does not influence the estimates of the integrals along the paths $\gamma_1, \gamma_2, \gamma_6$, and γ_7, because in this case the exponential factor including the term with c_3 tends to 1 if B and v_2 both

tend to infinity.
Due to $v_1 < 1$, the exponential factor including c_3 in the integral along the path β_2 is negative, and can thus be estimated against 1.
Therefore we are left with the estimates of the integrals along the paths β_1 and β_3. These are both less or equal than

$$\int_{v_1}^{\infty} (A^2 + (1-t)^2)^{-\frac{c_1+\sigma}{2}} \cdot (A^2 + (1+t)^2)^{-\frac{\sigma}{2}} \cdot e^{-2\pi c_2 t + \frac{2\pi c_3 (t-1)}{A^2+(t-1)^2}} \, dt. \tag{3.4}$$

To estimate the integrand we look for the maximum of the function $g(t) := e^{\frac{2\pi c_3(t-1)}{A^2+(t-1)^2}}$ in the interval $[v_1, \infty)$. We have

$$g'(t) = 2\pi c_3 \cdot \frac{A^2 - (t-1)^2}{(A^2+(t-1)^2)^2} \cdot e^{\frac{2\pi c_3(t-1)}{A^2+(t-1)^2}},$$

which is 0 if and only if $A^2 = (t-1)^2$. Due to $A > 1$, therefore the only solution of $g'(t) = 0$ in the interval $[v_1, \infty)$ is at the point $t = A+1$, which is directly seen to be a maximum of the function g. In this case $g(t)$ has the value $e^{\frac{2\pi c_3}{A}}$.
Thus we obtain, using that $\lim_{t\to\infty} e^{\frac{2\pi c_3(t-1)}{A^2+(t-1)^2}} = 1$, that the expression in (3.4) is less or equal than

$$e^{\frac{\pi c_3}{A}} \cdot \int_{v_1}^{\infty} (A^2 + (1-t)^2)^{-\frac{c_1+\sigma}{2}} \cdot (A^2 + (1+t)^2)^{-\frac{\sigma}{2}} \cdot e^{-2\pi c_2 t} \, dt.$$

Therefore, applying the substitution $t \mapsto t + v_1$, and using that $c_2 > 0$, we obtain that the integrals along the paths β_1 and β_3 are both less or equal than

$$e^{\frac{\pi c_3}{A}} \cdot e^{-2\pi c_2 v_1} \cdot \int_0^{\infty} (A^2 + (1-t-v_1)^2)^{-\frac{c_1+\sigma}{2}} \cdot (A^2 + (1+t+v_1)^2)^{-\frac{\sigma}{2}} \, dt.$$

If $s \in K$, then the integral has a convergent majorant independent of s. Thus assertion (3.2) also follows for $c_2 > 0$.

Therefore the holomorphicity follows in all three cases as a consequence of the independence of s. □

Remark 3.6 *The above described path of integration is suggested from Hecke's work. Indeed: he cuts the complex plane starting from i up to infinity and shifts the path of integration such that the cut is surrounded (cf. [He]).*

Corollary 3.7 *The coefficients $\Phi_{k,m,c,v}(n', r', s)$, that are defined in Theorem 3.4, are holomorphic functions in s with $\sigma > \frac{1}{2}(1 + g/2 - k)$.*
In particular they are holomorphic at $s = 0$ if $k > g/2 + 1$.
If K is a compact subset of the domain $\sigma > \frac{1}{2}(1 + \frac{g}{2} - k)$ with $s \in K$, then they satisfy the following estimate

$$\Phi_{k,m,c,v}(n', r', s) \ll_K v^{g/2-k-\sigma+1} \cdot e^{\frac{-D}{Av}} \cdot e^{\frac{-\pi D' v}{\det(2m)}(1+sign(D')v_1)},$$

where v_1 and A are positive constants, and where the constant implied in $\ll_K$ only depends on K.

Proof. The proof follows directly from Theorem 3.4 and Lemma 3.5 with $c_1 = k-g/2$, $c_2 = \frac{-D'v}{2\det(2m)}$, and $c_3 = \frac{-D}{c^2 v\cdot 2\det(2m)} > 0$, using that $1 \le e^{\frac{-\pi D}{2Ac^2\det(2m)v}} = e^{\frac{\pi c_3}{A}} \le e^{\frac{-\pi D}{Av}}$. $\square$

The next step is to estimate the Kloosterman sums occuring in the Forier coefficients of the Poincaré series (cf. Theorem 3.4). Therefore we require some well known formulas for Gauß sums.

Lemma 3.8 *Let $a, b \in \mathbb{Z}$, $\nu \in \mathbb{N}_0$, and let p be a prime number. Define*

$$G(a,b,p^\nu) := \sum_{x(p^\nu)} e_{p^\nu}(ax^2+bx).$$

Let $\alpha := \nu_p(a)$, where $a = p^\alpha a'$, $(a',p)=1$.

1. *For $\alpha \ge \nu$ we have*

$$G(a,b,p^\nu) = \begin{cases} p^\nu & \text{if } b \equiv 0 \pmod{p^\nu} \\ 0 & \text{otherwise} \end{cases}.$$

2. *For $0 \le \alpha < \nu$ and $b \not\equiv 0 \pmod{p^\alpha}$ we have*

$$G(a,b,p^\nu) = 0.$$

3. *If $p \ne 2$ and $b \equiv 0 \pmod{p^\alpha}$, $0 \le \alpha < \nu$, we have*

$$G(a,b,p^\nu) = p^{\frac{\alpha+\nu}{2}} \cdot \epsilon(p^{\nu-\alpha}) \cdot \left(\frac{a/p^\nu}{p^{\nu-\alpha}}\right) \cdot e_{p^{\nu+\alpha}}\left(-b^2\overline{\frac{4a}{p^\alpha}}\right),$$

where $\overline{\frac{4a}{p^\alpha}}$ is an inverse of $\frac{4a}{p^\alpha} \pmod{p^{\nu+\alpha}}$, and where $\epsilon(x) = 1$ or i according as $x \equiv 1$ or $3 \pmod 4$.

4. *The sum $G(a,b,2^\nu)$ is equal to 2^ν if $\nu-\alpha = 1$ and $b \not\equiv 0 \pmod 2$, has the value*

$$2^{\frac{\nu+\alpha}{2}} \cdot (i+1) \cdot \left(\frac{-2^{\nu+\alpha}}{a/2^\alpha}\right) \cdot \epsilon(a/2^\alpha) \cdot e_{2^{\nu+\alpha}}\left(-\frac{\overline{a}}{2^\alpha}\frac{b^2}{4}\right)$$

if $\nu-\alpha > 1$ and $b \equiv 0 \pmod{2^{\alpha+1}}$, and is 0 otherwise. Here $\overline{a/2^\alpha}$ is an inverse of $a/2^\alpha \pmod{p^{\nu+\alpha+2}}$.

Proof. The claim of Lemma 3.8 and its proof can be found in [Br]. However there is a misprint in the case $p = 2$ and $\nu - \alpha > 1$. In [Br] it is stated that the formula is only valid if $\nu - \alpha$ is an even integer and otherwise the sum $G(a, b, 2^\nu)$ has value 0. His misprint is based on a wrong citation of [La]. $\square$

Lemma 3.9 *Let $H_{m,c}(n, r, n', r')$ be defined as in Theorem 2.29. Then we have*

$$|H_{m,c}(n, r, n', r')| \ll_{D,m,\epsilon} c^{g/2+1+\epsilon}, \tag{3.5}$$

where the constant implied in $\ll_{D,m,\epsilon}$ only depends on D, m and ϵ.

Proof. Since both sides of inequality (3.5) are multiplicative in c it is sufficient to show that for all primes p and all $\nu \in \mathbb{N}$

$$|H_{m,p^\nu}(n, r, n', r')| \leq \begin{cases} p^{\frac{g}{2} \cdot v_p(2\det(2m)D)} \cdot (p^\nu)^{g/2+1} & \text{if } p \text{ is odd} \\ 2^{\frac{g}{2}} \cdot 2^{\frac{g}{2} \cdot v_2(2\det(2m)D)} \cdot (2^\nu)^{g/2+1} & \text{if } p = 2. \end{cases} \tag{3.6}$$

Counting the number of summands of $H_{m,p^\nu}(n, r, n', r')$ we obtain

$$|H_{m,p^\nu}(n, r, n', r')| \leq (p^\nu)^{1+g}.$$

Therefore (3.6) follows trivially if p^ν divides D. Thus we may assume $p^\nu \nmid D$. Let $U \in GL_g(\mathbb{Z}\ /p^\nu\mathbb{Z})$. Then

$$\begin{aligned} H_{m,p^\nu}(n, r, n', r') &= \sum_{\substack{x(p^\nu) \\ y(p^\nu)^*}} e_{p^\nu}((m[U][\bar{U}x] + rU \cdot \bar{U}x + n)\bar{y} + n'y + r'U \cdot \bar{U}x) \\ &= \sum_{\substack{x(p^\nu) \\ y(p^\nu)^*}} e_{p^\nu}((m[U][x] + rUx + n)\bar{y} + n'y + r'Ux), \end{aligned}$$

where $\bar{U}$ is an inverse of $U \pmod{c}$. For the last identity we used that $\bar{U}x$ runs $(\bmod\ \mathbb{Z}^{(g,1)} \cdot c)$ if x does. Thus the left-hand side of (3.6) remains unchanged if we replace T by $T\left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\right]$ (which is the same as changing m, r, and r' into $m[U]$, rU, and $r'U$, respectively). Moreover $\det(2m)$ is replaced by $(\det U)^2 \cdot \det(2m)$ and D_0 by $(\det U)^2 \cdot D_0$. Thus $\nu_p(4\det(2m) \cdot D_0)$ is not changed, because $\det U$ and p are coprime.
Let us now distinguish the two cases $p \neq 2$ and $p = 2$.
Since a non-degenerate integral quadratic form over $\mathbb{Z}_p$ is diagonisable over $\mathbb{Z}_p$ if $p \neq 2$ is a prime (cf. [Ca]) we may assume that m is a diagonal matrix with diagonal elements $m_1, \ldots, m_g$. Let $\mu_i := \nu_p(m_i)$ $(1 \leq i \leq g)$.
In case $\mu_i \geq \nu$ for at least one m_i $(1 \leq i \leq g)$ the claim follows trivially since in this case we have the estimate

$$|H_{m,p^\nu}(n, r, n', r')| \leq (p^\nu)^{g/2+1} \cdot (p^{\mu_i})^{g/2} \leq p^{g/2 \cdot \nu_p(\det(2m))} \cdot (p^\nu)^{g/2+1}.$$

Thus we may assume $\mu_i < \nu\,(1 \le i \le g)$.
Using Lemma 3.8 we find

$$|H_{m,p^\nu}(n,r,n',r')| \le p^{\nu(g/2+1)} \cdot \left(p^{\sum_{i=1}^{g}\mu_i}\right)^{1/2} \le p^{\nu(g/2+1)} \cdot p^{\frac{1}{2}\nu_p(\det(2m))}.$$

Since in the case $p = 2$ a non-degenerate integral quadratic form over $\mathbb{Z}_2$ is equivalent to a sum of forms $2^l\epsilon x^2$, $2^l xy$, $2^l(x^2+xy+y^2)$, where $l \in \mathbb{Z}$, and $\epsilon \in \{1,3,5,7\}$ (cf. [Ca]) we may assume that the quadratic form corresponding to m is a sum of forms of the above types. Let the first type occur g_1, the second g_2, and the third g_3 times, i.e., $g = g_1 + 2g_2 + 2g_3$. Then clearly

$$\det(2m) = 2^{g_1+\sum_{i=1}^{g_1+g_2+g_3} l_i} \cdot \left(\prod_{i=1}^{g_1}\epsilon_i\right) \cdot (-1)^{g_2} \cdot 3^{g_3}.$$

Let

$$l := max\{l_i|\, 1 \le i \le g_1+g_2+g_3\}.$$

If $l \ge \nu - 1$, then we have

$$|H_{m,2^\nu}(n,r,n',r')| \le (2^\nu)^{g+1} \le (2^\nu)^{g/2+1} \cdot (2^{l+1})^{g/2} \le 2 \cdot (2^\nu)^{g/2+1} \cdot 2^{\frac{g}{2}\cdot\nu_2(\det(2m))},$$

from which (3.6) follows directly. Therefore we may assume that $l < \nu - 1$.
We now estimate the three types of sums that can occur. Therefore we write $r = (r_1, \dots, r_g)$ and $r' = (r'_1, \dots, r'_g)$.
Using Lemma 3.8 we directly see that the first type of sum

$$\sum_{x(2^\nu)} e_{2^\nu}\left(2^{l_i}\epsilon_i\bar{y}x^2 + x\left(r_i\bar{y} + r'_i\right)\right) \qquad (\epsilon_i \in \{1,3,5,7\})$$

has an absolute value that is less or equal than $2^{\frac{v+l}{2}+1}$.
From Lemma 3.8 it furthermore follows that the second type of sum

$$\sum_{x_{i+1}(2^\nu)} e_{2^\nu}\left(\left(r_{i+1}\bar{y} + r'_{i+1}\right)x_{i+1}\right) \sum_{x_i(2^\nu)} e_{2^\nu}\left(x_i\left(2^{l_i}x_{i+1}\bar{y} + r_i\bar{y} + r'_i\right)\right)$$

has an absolute value that is less or equal than

$$2^\nu \sum_{\substack{x_{i+1}(2^\nu)\\ 2^{l_i}x_{i+1}\bar{y}+(r_i\bar{y}+r'_i)\equiv 0(2^\nu)}} |e_{2^\nu}((r_{i+1}\bar{y}+r'_{i+1})x_{i+1})| \le 2^{\nu+l_i},$$

because the congruence $2^{l_i}x_{i+1}\bar{y} + (r_i\bar{y}+r'_i) \equiv 0 \pmod{2^\nu}$ has at most l_i solutions $\pmod{2^\nu}$ since $\bar{y}$ is coprime to 2.
From Lemma 3.8 it moreover follows that the third type of sum

$$\sum_{x_{i+1}(2^\nu)} e_{2^\nu}\left(2^{l_i}x_{i+1}^2\bar{y} + \left(r_{i+1}\bar{y} + r'_{i+1}\right)x_{i+1}\right) \sum_{x_i(2^\nu)} e_{2^\nu}\left(2^{l_i}x_i^2\bar{y} + \left(2^{l_i}x_{i+1}\bar{y} + r_i\bar{y} + r'_i\right)x_i\right)$$

has an absolute value less or equal than

$$2^{1+\frac{l_i+\nu}{2}}\left|\sum_{\substack{x_{i+1}(2^\nu)\\ x_{i+1}\equiv -\frac{r_i+r_i'y}{2^{l_i}}(2)}} e_{2^\nu}\left(2^{l_i-2}3\bar{y}x_{i+1}^2+\frac{x_{i+1}}{2}\left(2r_{i+1}\bar{y}+2r'_{i+1}-r_i\bar{y}-r'_i\right)\right)\right|,$$

if $r_i\bar{y}+ri'\equiv 0 \pmod{2^{l_i}}$ and is 0 otherwise. Next we replace x_{i+1} by $2x_{i+1}-\frac{r_i+r_i'y}{2^{l_i}}$ with the new x_{i+1} running $\pmod{2^{\nu-1}}$. Then we obtain, again using Lemma 3.8, that this sum is less or equal than

$$\begin{aligned}
&2^{1+\frac{l_i+\nu}{2}}\left|\sum_{x_{i+1}(2^{\nu-1})} e_{2^\nu}\left(2^{l_i-2}3\bar{y}\left(2x_{i+1}-\frac{r_i+r_i'y}{2^{l_i}}\right)^2+\right.\right.\\
&\qquad\left.\left.\frac{1}{2}\left(2x_{i+1}-\frac{r_i+r_i'y}{2^{l_i}}\right)\left(2r_{i+1}\bar{y}+2r'_{i+1}-r_i\bar{y}-r'_i\right)\right)\right|\\
&\le 2^{\frac{l_i+\nu}{2}}\left|\sum_{x_{i+1}(2^\nu)} e_{2^\nu}\left(2^{l_i}3\bar{y}x_{i+1}^2+2x_{i+1}((r_{i+1}\bar{y}+r'_{i+1})-2(r_i\bar{y}+r'_i))\right)\right|\le 2^{l_i+\nu+1}.
\end{aligned}$$

Thus we have the estimate

$$|H_{m,2^\nu}(n,r,n',r')|\le 2^{\sum_{i=1}^{g_1+g_2+g_3}l_i}\cdot(2^\nu)^{g/2}\cdot 2^g\cdot 2^\nu\le 2^{g/2}\cdot 2^{g/2\cdot\nu_2(2\det(2m))}\cdot(2^\nu)^{g/2+1},$$

which proves the assertion. □

Now we can define the Poincaré series $P_{k,m;(n,r),s}$ in the larger domain $\sigma>\frac{1}{2}(g/2+2-k)$ by just taking the Fourier expansion as a definition.

Theorem 3.10 *Let $g^{\pm}_{k,m;(n,r);s,v}(n',r')$, $H_{m,c}(n,r,n',r')$ and $\Phi_{k,m,c,v}(n',r',s)$ be defined as in Theorem 2.29 and in Theorem 3.4, respectively, where $\pm=(\pm1)^k$. Then the Fourier series*

$$P_{k,m;(n,r),s}(\tau,z):=\sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v}(n',r')e(n'\tau+r'z)\qquad\left((\tau,z)\in\mathbb{H}\times\mathbb{C}^{(g,1)}\right)$$

is absolutely and locally uniformly convergent in s and defines a holomorphic function in s for $\sigma>\frac{1}{2}(g/2+2-k)$. In particular the series $P_{k,m;(n,r),0}(\tau,z)$ is absolutely convergent if $k>g/2+2$.
Let us define

$$\begin{aligned}
g^{(1)}(n',r') &:= \textstyle\sum_{c\ge1}H_{m,c}(n,r,n',r')\cdot\Phi_{k,m,c,v}(n',r',s)\cdot c^{-k-2s},\\
g^{(1)}(n',r')^{\pm} &:= g^{(1)}(n',r')+(-1)^k g^{(1)}(n',-r').
\end{aligned}$$

Let $k = g+2$, $(\tau, z) \in \mathbb{F}$, where $\mathbb{F}$ is the standard fundamental domain for the action of the Jacobi group on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ (cf. Remark 2.25), and suppose that s varies in a compact set K such that $0 < \sigma < 1$. Then we have the estimate

$$\left| \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{(1)}(n', r')^{\pm} e(n'\tau + r'z) \right| \ll_K v^{-g/2-1},$$

where the constant implied in $\ll_K$ is independent of τ and z.

Proof. To show that the series is absolutely and locally uniformly convergent in s, it is clearly sufficient to estimate the series

$$\sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{(1)}(n', r') e(n'\tau + r'z).$$

First we want to estimate $g^{(1)}(n', r')$.
Due to Corollary 3.7 and Lemma 3.9, there exist positive constants v_1, A such that

$$|H_{m,c}(n, r, n', r') \cdot \Phi_{k,m,c,v}(n', r', s)| \ll e^{\frac{-\pi D' v(1+sign(D')v_1)}{\det(2m)}} \cdot v^{g/2-k-\sigma+1} \cdot e^{\frac{-D}{Av}} \cdot c^{g/2+1+\epsilon}.$$

Using that $\sum_{c>0} c^{-l}$ converges for $l > 1$ and taking $\epsilon > 0$ small enough, we find

$$\left| \sum_{c>0} c^{-k-2s} \cdot H_{m,c}(n, r, n', r') \cdot \Phi_{k,m,c,v}(n', r', s) \right|$$

$$\begin{aligned} &\ll \quad e^{\frac{-D}{Av}} \cdot \sum_{c>0} c^{g/2+1-k-2\sigma+\epsilon} \cdot e^{\frac{-\pi D' v(1+sign(D')v_1)}{\det(2m)}} \cdot v^{g/2-k-\sigma+1} \\ &\ll \quad e^{\frac{-D}{Av}} \cdot e^{\frac{-\pi D' v(1+sign(D')v_1)}{\det(2m)}} \cdot v^{g/2-k-\sigma+1}. \end{aligned}$$

Thus we obtain, using that $D' = \frac{1}{2}\det(2m) \cdot (-4n' + m^{-1}[r'^t])$ (which follows directly from the Jacobi decomposition, see Remark 2.19),

$$\left| \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{(1)}(n', r') e(n'\tau + r'z) \right|$$

$$
\begin{aligned}
&\ll_K \quad v^{g/2-k-\sigma+1} \cdot e^{\frac{-D}{Av}} \cdot \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} e^{\frac{-\pi D' v}{\det(2m)}(1+sign(D')v_1) - 2\pi n' v - 2\pi r' y} \\
&\ll_K \quad v^{g/2-k-\sigma+1} \cdot e^{\frac{-D}{Av}} \cdot \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} e^{\frac{-\pi sign(D') D' v_1}{\det(2m)} v - \frac{1}{2}\pi m^{-1}[r'^t] v - 2\pi r' y} \\
&\ll_K \quad e^{\frac{-D}{Av}} \cdot v^{g/2-k-\sigma+1} \cdot \sum_{\substack{D' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} e^{-a \cdot sign(D') D' v - \frac{1}{2}\pi m^{-1}[r'^t] v - 2\pi r' y} \\
&\ll_K \quad e^{\frac{-D}{Av}} \cdot v^{g/2-k-\sigma+1} \cdot \left(\sum_{r' \in \mathbb{Z}^{(1,g)}} e^{-\frac{1}{2}\pi m^{-1}[r'^t] v - 2\pi r' y} \right) \left(1 + 2 \sum_{D' > 0} e^{-aD'v} \right),
\end{aligned}
$$

where a is a positive constant independent of D', r', s, τ, and z.
Therefore the absolute and local uniform convergence in s follows because the first sum is a special value of a Jacobi theta series and the second sum is a geometric sum. This establishes the holomorphicity in the variable s.
If $k = g + 2$ and $(\tau, z) \in \mathbb{F}$, then there exists a positive constant b, independent of (τ, z) such that

$$
\begin{aligned}
\sum_{r' \in \mathbb{Z}^{(1,g)}} e^{-\frac{1}{2}\pi m^{-1}[r'^t] v - 2\pi r' y} \quad &\ll \quad \left(\sum_{r' \in \mathbb{N}_0} e^{-br'^2 v + 2\pi r' v} \right)^g \ll \left(\sum_{r' \in \mathbb{N}_0} e^{-br'^2 v} \right)^g \\
&\le \quad \left(\sum_{r' \in \mathbb{N}_0} e^{-br'^2 \frac{\sqrt{3}}{2}} \right)^g \ll 1.
\end{aligned}
$$

Moreover, we have for $(\tau, z) \in \mathbb{F}$

$$
\sum_{D' > 0} e^{-aD'v} \le \sum_{D' > 0} e^{\frac{-aD'\sqrt{3}}{2}} \ll 1.
$$

Thus we get the desired estimate using that $v^{-\sigma} \ll 1$ for $(\tau, z) \in \mathbb{F}$, and under the given conditions on σ, and that $e^{\frac{-D}{Av}} \ll 1$. $\square$

In the remainder of the section we restrict ourselves to the case $k = g + 2$.

Lemma 3.11 *The function $P_{g+2,m;(n,r)}(\tau, z) := P_{g+2,m;(n,r),0}(\tau, z)$ is an element of $J^{cusp}_{g+2,m}$.*

Proof. The transformation law is clear since we have for all $(\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}$ and for all $\gamma \in \Gamma^J_{1,g}$

$$
\begin{aligned}
P_{g+2,m;(n,r)}|_{g+2,m}\gamma(\tau, z) \quad &= \quad \lim_{\sigma \to 0^+} P_{g+2,m;(n,r),\sigma}|_{g+2,m}\gamma(\tau, z) \\
&= \quad \lim_{\sigma \to 0^+} P_{g+2,m;(n,r),\sigma}(\tau, z) \\
&= \quad P_{k,m;(n,r)}(\tau, z).
\end{aligned}
$$

The Fourier expansion of $P_{g+2,m;(n,r)}$ is known per definition (cf. Theorem 3.10). Thus it is left to show that the Fourier coefficients of $P_{g+2,m;(n,r),0}(\tau,z)$ are constant functions of $v = \mathrm{Im}(\tau)$. Checking the definitions, it is therefore enough to show that the integral

$$\int_{iv-\infty}^{iv+\infty} \tau^{-g/2-2} \cdot e\left((2\det(2m))^{-1}\left(D'\tau + \frac{D}{c^2\tau}\right)\right) d\tau \tag{3.7}$$

is independent of v.
For $D' < 0$ in (3.7) we make the substitution $\tau = \frac{i}{c}\cdot(D/D')^{1/2}\cdot s$ to obtain

$$\alpha \cdot \int_{v'-i\infty}^{v'+i\infty} s^{-g/2-2}\cdot \exp\left(\frac{2\pi}{c\cdot\det(2m)}\cdot(DD')^{1/2}(s-s^{-1})\right) ds, \tag{3.8}$$

where α is a constant independent of v, and where v' depends on v. For $\mu > 0, \kappa > 0$ the functions $t \mapsto (t/\kappa)^{(\mu-1)/2}\cdot J_{\mu-1}(2\sqrt{\kappa t}) \quad (t \geq 0)$ and $s \mapsto s^{-\mu}\cdot e^{-\kappa/s}$ $(\mathrm{Re}(s) > 0)$ are inverse to each other w.r.t. the Laplace transform ([AS] 29.3.80). Therefore we see, taking $t = \kappa = \frac{2\pi}{c\cdot\det(2m)}\cdot(D'D)^{1/2}$ and $\mu = g/2+2$, that the integral in (3.8) has the value

$$2\pi i \cdot J_{g/2+1}\left(\frac{2\pi}{\det(2m)\cdot c}\cdot(D'D)^{1/2}\right),$$

which is independent of v' (and so of v).
If $D' = 0$, then we can use the same Laplace tranform with $t = 0$, $\kappa = \frac{-D}{c^2\cdot 2\det(2m)}$. Thus we get that the value of (3.7) is zero.
If $D' > 0$, then in (3.7) we make the substitution $\tau = \frac{i}{c}\cdot(-D/D')^{1/2}\cdot s$ to obtain

$$\beta \cdot \int_{v'-i\infty}^{v'+i\infty} s^{-g/2-2}\cdot \exp\left(\frac{\pi}{c\cdot\det(2m)}\cdot(-DD')^{1/2}(s+s^{-1})\right) ds, \tag{3.9}$$

where β is a constant independent of v, and where v' depends on v.
For $\mu > 0, \kappa > 0$ the functions $t \mapsto (t/\kappa)^{(\mu-1)/2} I_{\mu-1}(2\sqrt{\kappa t}) \quad (t \geq 0)$, where $I_{\mu-1}(x)$ $(x \in \mathbb{R})$ denotes the I-Bessel function of order $\mu - 1$, and $s \mapsto s^{-\mu}e^{\kappa/s}$ $(\mathrm{Re}(s) > 0)$ are inverse to each other w. r. t. the Laplace transform (cf. [AS] 29.3.80). Therefore we get, taking $t = \kappa = \frac{\pi}{c\cdot\det(2m)}\cdot(-D'D)^{1/2}$ and $\mu = g/2+2$, that the integral in (3.9) has the value

$$2\pi i \cdot I_{g/2+1}\left(\frac{2\pi}{\det(2m)\cdot c}\cdot(-D'D)^{\frac{1}{2}}\right),$$

which is independent of v' (and so of v).
Thus the integral in (3.7) is independent of v in all three cases, i.e., the holomorphicity of $P_{g+2,m;(n,r)}(\tau,z)$ follows.

The vanishing of the Fourier coefficients for $D' \geq 0$ can be established if we deform the path of integration up to infinity. $\square$

Now it is left to show that in case $k = g + 2$ the Petersson coefficient formula is still valid. The difficulty is that the scalar product cannot be calculated directly. It can only be computed by means of the scalar products

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle ,$$

where $\sigma > 0$. Therefore we first have to show that these scalar products are absolutely convergent (cf. Lemma 3.12); note that $P_{g+2,m;(n,r),\sigma}$ is not necessarily a cusp form (cf. Theorem 3.4). Afterwards we calculate the scalar products (cf. Lemma 3.13). Both is done by means of the usual unfolding argument.
Next - and this is the main difficulty - we have to show that we are allowed to interchange limit and integration (cf. Lemma 3.14), i.e.,

$$\lim_{\sigma \to 0} \left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle = \left\langle \phi, P_{g+2,m;(n,r)} \right\rangle . \tag{3.10}$$

Then we get the desired value of the scalar product $\left\langle \phi, P_{g+2,m;(n,r)} \right\rangle$ by taking the limit of the values of the scalar products $\left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle$ (cf. Theorem 3.15).

Lemma 3.12 *Let $\sigma > 0$ and $\phi \in J_{k,m}^{cusp}$. Then the scalar product*

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle$$

is absolutely convergent.

Proof. The well-definedness of the scalar product is clear because ϕ is a Jacobi cusp form and $P_{g+2,m;(n,r),\sigma}$ is invariant under the slash operation of the Jacobi group (cf. Lemma 3.3). To show the absolute convergence we let V be a fixed set of representatives of $(\Gamma_{1,g}^J)_\infty \backslash \Gamma_{1,g}^J$. Then, using the usual unfolding argument and Levi's Theorem, we obtain in the sense of formal agreement

$$\begin{aligned}
&\int_{\mathbb{F}} \left| \phi(\tau, z) \cdot \overline{P_{g+2,m;(n,r),s}(\tau, z)} \cdot \exp(-4\pi m[y] v^{-1}) \right| dudvdxdy \\
\leq &\int_{\mathbb{F}} |\phi(\tau, z)| \cdot \exp(-4\pi m[y] v^{-1}) \cdot \sum_{\gamma \in V} v^\sigma \cdot |c\tau + d|^{-2s} \cdot |e^{n,r}|_{k,m} \gamma(\tau, z)| \, dudvdxdy \\
&= \int_{\cup_{\gamma \in V} \gamma \mathbb{F}} v^\sigma \cdot \exp(-4\pi m[y] v^{-1})| \cdot \phi(\tau, z)| \cdot |e^{n,r}(\tau, z)| \, dudvdxdy.
\end{aligned} \tag{3.11}$$

Since $\cup_{\gamma \in V} \gamma \mathbb{F}$ is a fundamental domain of the action of $(\Gamma_{1,g}^J)_\infty$ on $\mathbb{H} \times \mathbb{C}^{(g,1)}$, and since the integrand of the previous integral is invariant under this action, we can choose an arbitrary fundamental domain of this action, if the integral converges with respect to this fundamental domain. For example we may choose

$$\tilde{F} = \left\{(\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)} \middle| \, 0 \leq u \leq 1;\ v > 0;\ 0 \leq x_\nu \leq 1 \text{ for } \nu = 1, \ldots g,\ y \in \mathbb{R}^{(g,1)} \right\}. \tag{3.12}$$

Here $\tau = u+iv$ as before , $z = x+iy$, and x_ν are the components of x $(1 \leq \nu \leq g)$. Thus

$$\begin{aligned} &\int_{\tilde{F}} v^\sigma \cdot |\phi(\tau, z)| \cdot \exp(-4\pi m[y]v^{-1}) \cdot |e^{n,r}(\tau, z)| \, dudvdxdy \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^g} |\phi(\tau, z)| \cdot e^{-2\pi nv - 2\pi ry} \cdot v^\sigma \cdot \exp(-4\pi m[y]v^{-1}) \, dydv. \end{aligned} \tag{3.13}$$

Using the boundness condition of Lemma 2.23, we infer that the integral in (3.13) is less or equal than

$$\int_0^\infty v^{\sigma - g/2 - 1} \cdot e^{-2\pi nv} \cdot \int_{\mathbb{R}^g} e^{-2\pi ry - 2\pi m[y]v^{-1}} \, dydv. \tag{3.14}$$

Completing the square we obtain that the value of the inner integral in (3.14) equals

$$2^{-g/2} \cdot (\det m)^{-1/2} \cdot v^{g/2} \cdot e^{\frac{\pi}{2} vm^{-1}[r^t]}.$$

Thus

$$\begin{aligned} &\int_{\tilde{F}} v^\sigma \cdot \exp(-4\pi m[y]v^{-1}) \cdot |\phi(\tau, z)| \cdot |e^{n,r}(\tau, z)| \, dudvdxdy \\ &\qquad \ll \int_0^\infty v^{-1+\sigma} \cdot e^{-\frac{\pi v}{2}(4n - m^{-1}[r^t])} \, dv < \infty \end{aligned}$$

since $\sigma > 0$, which proves the assertion. □

Lemma 3.13 *For $\sigma > 0$ and $\phi \in J^{cusp}_{g+2,m}$ one has*

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle = \lambda_{g+2,m,D,\sigma} \cdot c_\phi(n, r), \tag{3.15}$$

where $c_\phi(n, r)$ denotes the (n, r)-th Fourier coefficient of ϕ and

$$\begin{aligned} \lambda_{g+2,m,D,\sigma} \ := \ & 2^{(g-1)(g/2+\sigma+1)-g} \cdot \Gamma\left(g/2 + \sigma + 1\right) \cdot \pi^{-g/2-\sigma-1} \cdot (\det m)^{g/2+\sigma+1/2} \\ & \cdot |D|^{-g/2-\sigma-1}. \end{aligned}$$

Proof. Due to Lemma 3.12 the scalar product on the left-hand side of (3.15) is well defined and the integral is absolutely convergent. Thus we get, using the usual unfolding argument,

$$\begin{aligned}
&\left\langle \phi \, , \, P_{g+2,m;(n,r),\sigma} \right\rangle \\
&\quad = \int_{\Gamma^J_{1,g}\backslash \mathbb{H}\times\mathbb{C}^{(g,1)}} \phi(\tau,z)\cdot \overline{P_{g+2,m;(n,r),\sigma}(\tau,z)}\cdot \exp(-4\pi m[y]v^{-1})\, dudvdxdy \\
&\quad = \int_{(\Gamma^J_{1,g})_\infty\backslash \mathbb{H}\times\mathbb{C}^{(g,1)}} v^\sigma\cdot\phi(\tau,z)\cdot\overline{e^{n,r}(\tau,z)}\exp(-4\pi m[y]v^{-1})\, dudvdxdy.
\end{aligned}$$

Using the fundamental domain given in (3.12) and inserting the Fourier expansion of ϕ

$$\phi(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}\\ 4n'>m^{-1}[r'^t]}} c(n',r')e(n'\tau+r'z),$$

we obtain

$$\begin{aligned}
\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle = &\sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}\\ 4n'>m^{-1}[r'^t]}} c(n',r')\int_0^\infty e^{-2\pi(n+n')v}\cdot v^\sigma \\
&\int_{\mathbb{R}^g} e^{-4\pi m[y]v^{-1}-2\pi y(r'+r)}\int_0^1 e^{2\pi i(n'-n)u}\int_{[0,1]^g} e^{2\pi i(r'-r)x}\, dvdydudx.
\end{aligned}$$

Here we have used the absolute convergence of the Fourier expansion of ϕ in order to be allowed to interchange limit and integration.
The integrals over x and u clearly vanish unless $r=r'$ and $n=n'$. In this case both integrals have the value 1. Thus we obtain

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle = c(n,r)\cdot\int_0^\infty e^{-4\pi nv}\cdot v^\sigma\cdot\int_{\mathbb{R}^g} e^{-4\pi m[y]v^{-1}-4\pi yr}\, dvdy. \quad (3.16)$$

Completing the square we get that the inner integral in (3.16) has the value

$$2^{-g}\cdot(\det m)^{-1/2}\cdot v^{g/2}\cdot e^{\pi v m^{-1}[r^t]}.$$

Thus we obtain, using $-4n+m^{-1}[r^t] = 2^{1-g}\cdot(\det m)^{-1}\cdot D$,

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle = c(n,r)\cdot 2^{-g}\cdot(\det m)^{-1/2}\cdot\int_0^\infty v^{g/2+\sigma}\cdot e^{-\pi v(4n-m^{-1}[r^t])}\, dv$$

$$\begin{aligned}
&= c(n,r)\cdot 2^{-g}\cdot(\det m)^{-1/2}\cdot\pi^{-g/2-\sigma-1}\cdot\left(4n-m^{-1}[r^t]\right)^{-g/2-\sigma-1}\cdot\Gamma\left(g/2+\sigma+1\right) \\
&= c(n,r)\cdot\lambda_{k,m,D,\sigma}.
\end{aligned}$$

□

Lemma 3.14 *For $0<\sigma<1$ and $\phi \in J^{cusp}_{g+2,m}$ we have*

$$\lim_{\sigma\to 0} \left\langle \phi, P_{g+2,m;(n,r),\sigma} \right\rangle = \left\langle \phi, P_{g+2,m;(n,r)} \right\rangle . \tag{3.17}$$

Proof. The existence of the limit on the left-hand side of (3.17) follows directly from Lemma 3.13. Thus it is left to show that we may interchange limit and integration. For this we use the fundamental domain given in Remark 2.25. We use Lebesgue's Theorem of bounded convergence and construct a majorant $g(\tau,z)>0$ on $\mathbb{H}\times\mathbb{C}^{(g,1)}$ such that the following two conditions are satisfied:

$$\begin{aligned} &(i) \quad |P_{g+2,m;(n,r),\sigma}(\tau,z)| \ll g(\tau,z) \qquad (\forall\, \sigma>0,\, (\tau,z)\in\mathbb{F})\,,\\ &(ii) \quad \int_{\mathbb{F}} |\phi(\tau,z)|\cdot g(\tau,z) e^{-4\pi m[y]v^{-1}}\, dudvdxdy < \infty. \end{aligned}$$

In the rest of the proof we show that for $g(\tau,z)$ we may take the function

$$\sum_{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}} \frac{v}{|c\tau+d|^2}\cdot |e^{n,r}|_{g+2,m}\gamma(\tau,z)| \\ + v^{-g/2-1} + \chi_{\{(\tau,z)\in\mathbb{F}|\,\frac{\sqrt{3}}{2}<v<1\}} \cdot \sum_{\substack{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}\\ c=0}} |e^{n,r}|_{g+2,m}\gamma(\tau,z)|\,,$$

where χ_M denotes the characteristic function of a set $M\subset\mathbb{H}\times\mathbb{C}^{(g,1)}$, and $c=0$ means that if $\gamma=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda a,\lambda b)\right)\in\Gamma^J_{1,g}$, then $c=0$. Here we only have to show the convergence of the third term since the convergence of the first term is shown in Lemma 3.12 and the convergence of the second term is trivial. In the following we abbreviate the terms of $g(\tau,z)$ by $g_1(\tau,z)$, $g_2(\tau,z)$, and $g_3(\tau,z)$ in an obvious sense. To show (i) we separate for a fixed but arbitary $\sigma>0$ the series $P_{g+2,m;(n,r),\sigma}(\tau,z)$ into two parts according to $c=0$ or $c\neq 0$. The part with $c=0$ is given by

$$v^\sigma\cdot \sum_{\substack{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}\\ c=0}} e^{n,r}|_{g+2,m}\gamma(\tau,z).$$

For $(\tau,z)\in\mathbb{F}$ this has an absolute value less or equal than

$$\chi_{\{(\tau,z)\in\mathbb{F}|\,v\geq 1\}}\cdot \sum_{\substack{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}\\ c=0}} v\cdot |e^{n,r}|_{g+2,m}\gamma(\tau,z)| \\ + \chi_{\{(\tau,z)\in\mathbb{F}|\,\frac{\sqrt{3}}{2}<v<1\}}\cdot \sum_{\substack{\gamma\in\left(\Gamma^J_{1,g}\right)_\infty\backslash\Gamma^J_{1,g}\\ c=0}} |e^{n,r}|_{g+2,m}\gamma(\tau,z)|$$

$$\leq \sum_{\gamma\in\left(\Gamma_{1,g}^J\right)_\infty\backslash\Gamma_{1,g}^J} \frac{v}{|c\tau+d|^2}\cdot |e^{n,r}|_{g+2,m}\gamma(\tau,z)|$$

$$+\chi_{\{(\tau,z)\in\mathbb{F}|\ \frac{\sqrt{3}}{2}<v<1\}}\cdot \sum_{\substack{\gamma\in\left(\Gamma_{1,g}^J\right)_\infty\backslash\Gamma_{1,g}^J\\ c=0}} |e^{n,r}|_{g+2,m}\gamma(\tau,z)|\,,$$

which coincides with the first two terms of the function $g(\tau,z)$. Here we have enlarged the first summand that much because we want to use Lemma 3.12 and Lemma 3.13 with $s=1$ to get the absolute convergence of the series and the value of the integral.
We still have to show the absolute convergence of the series $g_2(\tau,z)$. Therefore we calculate it explicitly, using that in this case $a=d=\pm 1$. Thus we have for $\frac{\sqrt{3}}{2}<v<1$:

$$\begin{aligned}
g_2(\tau,z) &= \sum_{\lambda\in\mathbb{Z}^{(g,1)}} \left|e(m[\lambda]+r\lambda+n)\tau)\cdot e(\pm(r+2\lambda^t m)z)\right| \\
&= \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}\\ 4n'>m^{-1}[r'^t]}} |(\delta_m(n,r,n',\pm r')\cdot e(n'\tau\pm r'z)| = 2\sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}}{}^{*}\ e^{-2\pi n'v-2\pi r'y},
\end{aligned}$$

where $\sum^*$ is used as an abbreviation for the condition $4n'>m^{-1}[r'^t]$. This sum coincides with the part of the absolute values of $P_{k,m;(n,r)}(\tau,z)$ that belongs to $c=0$ for k arbitrary but sufficiently large, and is therefore convergent. In particular every subseries of this series is convergent.
Now we estimate the part belonging to $c\neq 0$. Using Theorem 3.10, this part can be estimated against $v^{-g/2-1}$, which is the third term of the function $g(\tau,z)$. Therefore $g(\tau,z)$ is a majorant of $P_{g+2,m;(n,r),\sigma}(\tau,z)$.
The rest of the proof is devoted to the claim (ii), i.e.,

$$\int_{\mathbb{F}} g(\tau,z)\cdot|\phi(\tau,z)|\cdot\exp(-4\pi m[y]v^{-1})\,dudvdxdy<\infty.$$

It is sufficient to prove the claim separately for $g_1(\tau,z), g_2(\tau,z)$, and $g_3(\tau,z)$.
The convergence of

$$\int_{\mathbb{F}} g_1(\tau,z)\cdot|\phi(\tau,z)|\cdot e^{-4\pi m[y]v^{-1}}\,dvdudxdy$$

has already been shown in Lemma 3.12.
To show the convergence of

$$\int_{\mathbb{F}} g_2(\tau,z)\cdot|\phi(\tau,z)|\cdot\exp(-4\pi m[y]v^{-1})\,dudvdxdy,$$

it is sufficient to regard the 3^g subseries of g_2 with $sign(r_i')(1 \leq i \leq g)$ fixed. We define for the components $r_i'\,(1 \leq i \leq g)$ of r'

$$\epsilon(r_i') = \begin{cases} 1 & r_i' < 0 \\ 0 & r_i' \geq 0 \end{cases},$$

and regard one fixed but arbitrary of the 3^g subseries, which we denote by $\sum_{n',r'}^{**}$. Then we have, using Lemma 2.23 and Lemma 2.25,

$$\int_{\mathbb{F}} |\phi(\tau,z)| \cdot \sum_{n',r'}^{**} e^{-2\pi n'v - 2\pi r'y} \cdot e^{-4\pi m[y]v^{-1}}\, dudvdxdy$$

$$\begin{aligned}
&\ll \int_{\frac{\sqrt{3}}{2}}^{1} \int_{[0,v]^g} v^{-g/2-1} \cdot e^{-2\pi m[y]v^{-1}} \sum_{n',r'}^{**} e^{-2\pi n'v - 2\pi r'y}\, dvdy \\
&\leq \sum_{n',r'}^{**} \int_{\frac{\sqrt{3}}{2}}^{1} v^{-g/2-1} \cdot e^{-2\pi n'v} \int_{[0,v]^g} e^{-2\pi m[y]v^{-1}} \cdot e^{-2\pi\epsilon(r_i')r_i'v}\, dydv \\
&\leq \sum_{n',r'}^{**} \int_{\frac{\sqrt{3}}{2}}^{1} v^{g/2-1} \cdot e^{-2\pi n'\frac{\sqrt{3}}{2}} \cdot e^{-2\pi\epsilon(r_i')r_i'v}\, dv \\
&= \sum_{n',r'}^{**} e^{-2\pi n'\frac{\sqrt{3}}{2} - 2\pi\epsilon(r_i')r_i'} \int_{\frac{\sqrt{3}}{2}}^{1} v^{-g/2-1}\, dv,
\end{aligned}$$

which is clearly finite since the integral is finite and $\sum_{n',r'}^{**} e^{-2\pi n'\frac{\sqrt{3}}{2} - 2\pi\epsilon(r_i')r_i'}$ is a special value of a subseries of the convergent series $\sum_{n',r'}^{**} e^{2\pi i(n'\tau + r'z)}$.
Finally we have to show the convergence of

$$\int_{\mathbb{F}} g_3(\tau,z) \cdot |\phi(\tau,z)| \cdot \exp(-4\pi m[y]v^{-1})\, dudvdxdy.$$

Again using Lemma 2.23 and Theorem 3.10, we obtain that this integral is less or equal than

$$\int_1^\infty v^{-g-2} \int_{[0,v]^g} e^{-2\pi m[y]v^{-1}}\, dydv \ll \int_1^\infty v^{-2}\, dv < \infty.$$

The claim then follows directly. □

Thus we have shown, combining Lemmas 3.12, 3.13, and 3.14

Theorem 3.15 *For $\phi \in J_{g+2,m}^{cusp}$ we have*

$$\left\langle \phi, P_{g+2,m;(n,r)} \right\rangle = \lambda_{g+2,m,D} \cdot c_\phi(n,r),$$

where $c_\phi(n,r)$ and $\lambda_{g+2,m,D}$ are defined as in Theorem 2.31, i.e., the Petersson coefficient formula is still valid in the limiting case $k = g+2$.

Corollary 3.16 *As a unitary vector space with respect to the Petersson scalar product, $J_{g+2,m}^{cusp}$ is generated by the Poincaré series*

$$\{P_{g+2,m;(n,r)} |\, n \in \mathbb{Z},\, r \in \mathbb{Z}^{(1,g)};\, 4n > m^{-1}[r^t]\}.$$

3.1.2 The final estimates

We have the following

Theorem 3.17 *Suppose that $k \geq g+2$. Let $\phi \in J_{k,m}^{cusp}$ with Fourier coefficients $c(n,r)$. Then we have*

$$c(n,r) \ll_{\epsilon,k} \left(1 + \frac{|D|^{g/2+\epsilon}}{(\det m)^{(g+1)/2}}\right)^{1/2} \cdot \frac{|D|^{k/2-g/4-1/2}}{(\det m)^{k/2-(g+3)/4}} \cdot \parallel \phi \parallel \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,k}$ only depends on ϵ and k .

Proof. The Cauchy-Schwarz inequality and the Petersson coefficient formula (cf. Theorem 2.31 and Theorem 3.15) we find

$$\begin{aligned} |c(n,r)|^2 = \lambda_{k,m,D}^{-2} \cdot \left|\left\langle \phi, P_{k,m;(n,r)} \right\rangle\right|^2 &\leq \lambda_{k,m,D}^{-2} \cdot \parallel \phi \parallel^2 \cdot \left\langle P_{k,m;(n,r)}, P_{k,m;(n,r)} \right\rangle \\ &= \lambda_{k,m,D}^{-1} \cdot b_{n,r}(P_{k,m;(n,r)}) \cdot \parallel \phi \parallel^2, \end{aligned}$$

where $b_{n,r}(P_{k,m;(n,r)})$ denotes the (n,r)-th Fourier coefficient of the Poincaré series $P_{k,m;(n,r)}$. In order to prove Theorem 3.17, we therefore only need to estimate the Fourier coefficients of the Poincaré series. Since these are of the same type as in the case $k > g+2$ we can proceed as in [BK] . $\square$

Corollary 3.18 *Suppose that $k \geq g+2$. Let $\phi \in J_{k,m}^{cusp}$ with Fourier coefficients $c(n,r)$. Then we have*

$$c(n,r) \ll_{\epsilon,\phi} |D|^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,\phi}$ only depends on ϵ and ϕ.

Remark 3.19 *Of course Corollary 3.18 is not useful for the estimates of Fourier coefficients of Siegel cusp forms because this estimate is not uniform in m.*

Moreover we can prove, and this is the main result of Section 3.1,

Theorem 3.20 *Let $g \geq 2$ and suppose that $k \geq g+1$. Let $F \in S_k(\Gamma_g)$ with Fourier coefficients $a(T)$. Then we have*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/(2g)-(1-1/g)\alpha_g+\epsilon} \qquad (\epsilon > 0),$$

where $\alpha_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}$, and where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Proof. The proof for $k > g+1$ is already given in [BK], using certain Dirichlet series of Rankin-Selberg type, which have a meromorphic continuation to the whole complex plane with finitely many poles and satisfy a certain functional equation. Then a version of the Theorem of Sato and Shintani can be used.
The restriction $k > g+1$ is only needed for the estimates of $c(n,r)$. □

3.2 The subgroup $\Gamma_{g,0}(N)$

In this section we want to estimate the Fourier coefficients of Siegel cusp forms with respect to the subgroup $\Gamma_{g,0}(N)$ of Γ_g defined in Chapter 2.
In the first two sections we define Poincaré series for Jacobi cusp forms on certain subgroups and estimate their Fourier coefficients. In the third section we estimate certain Petersson norms.

3.2.1 Poincaré series for Jacobi cusp forms

i) The case $\Gamma^J_{1,g,0}(N)$

Recall that

$$\Gamma^J_{1,g,0}(N) := \Gamma_0(N) \ltimes \left(\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}\right)$$

(cf. Definition 2.16). We proceed as in Section 2.2.

Definition 3.21 *Let n, r, and m be given as in Definition 2.26. Then we define a Poincaré series of exponential type for $\Gamma^J_{1,g,0}(N)$ by*

$$P^N_{k,m;(n,r)}(\tau,z) := \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g,0}(N)} e^{n,r}|_{k,m}\gamma(\tau,z) \qquad \left(\tau \in \mathbb{H}, z \in \mathbb{C}^{(g,1)}\right),$$

where the notations are the same as in Definition 2.26.

Then we have the following

Lemma 3.22 *The series $P^N_{k,m;(n,r)}(\tau,z)$ is absolutely and locally uniformly convergent on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ if $k > g+2$.*
If $k \le g+2$ it is not absolutely convergent at the point $(i,0)$.
It satisfies the transformation law

$$P^N_{k,m;(n,r)}|_{k,m}\gamma(\tau,z) = P^N_{k,m;(n,r)}(\tau,z) \qquad \left(\forall\, (\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)},\ \gamma \in \Gamma^J_{1,g,0}(N)\right).$$

Proof. Most of Lemma 3.22 follows directly from Lemma 2.27 because $P^N_{k,m;(n,r)}$ occurs as a subseries of $P_{k,m;(n,r)}$. What remains to show is that the series of absolute values is divergent at the point $(i,0) \in \mathbb{H}\times\mathbb{C}^{(g,1)}$. This can be done similarly as in the case of the full Jacobi group , using that $\sum_{\substack{(c,d)=1\\ c\equiv 0(N)}}(c^2+d^2)^{-1}$ is divergent. $\square$

To show that $P^N_{k,m;(n,r)}$ is an element of $J^{cusp}_{k,m}(N)$ we need for all matrices $\gamma \in \Gamma^J_{1,g,0}(N)$ the Fourier expansion of $P^N_{k,m;(n,r)}|_{k,m}\gamma$. First we regard the case $\gamma = E_g$.

Theorem 3.23 *Let $k > g+2$. Then the function $P^N_{k,m;(n,r)}(\tau,z)$ has the Fourier expansion*

$$P^N_{k,m;(n,r)}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}\\ 4n'>m^{-1}[r'^t]}} g^{\pm}_{k,m;(n,r),N}(n',r')e(n'\tau+r'z),$$

where

$$g^{\pm}_{k,m;(n,r),N}(n',r') := g_{k,m;(n,r),N}(n',r') + (-1)^k g_{k,m;(n,r),N}(n',-r'),$$

where $\pm = (\pm 1)^k$, and where

$$\begin{aligned} g_{k,m;(n,r),N}(n',r') &:= \delta_m(n,r,n',r') + 2\pi i^k \cdot (\det(2m))^{-1/2}\cdot (D'/D)^{k/2-g/4-1/2} \\ &\times \sum_{\substack{c\geq 1\\ N|c}} e_{2c}(r'm^{-1}r^t)\cdot H_{m,c}(n,r,n',r')\cdot J_{k-g/2-1}\left(\frac{2\pi\sqrt{D'D}}{\det(2m)\cdot c}\right)\cdot c^{-g/2-1}, \end{aligned}$$

and where D, D', $\delta_m(n,r,n',r')$, $H_{m,c}(n,r,n',r')$ and $J_{k-g/2-1}$ are defined as in Theorem 2.29.

Proof. The proof is very similar to the proof of Theorem 2.29, using as a set of representatives of $(\Gamma^J_{1,g})_\infty\backslash\Gamma^J_{1,g,0}(N)$ the elements $\left\{\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, (a\lambda, b\lambda)\right)\right\}$, where $c,d\in\mathbb{Z}$ with $(c,d)=1$, $c\equiv 0 \pmod N$, $\lambda\in\mathbb{Z}^{(g,1)}$, and where for each pair (c,d) we have chosen $a,b\in\mathbb{Z}$ such that $ad-bc=1$. $\square$

Next we want to compute $P^N_{k,m;(n,r)}|_{k,m}\gamma(\tau,z)$, where $\gamma = (M,(0,0))\in\Gamma^J_{1,g}$, with $M\notin\Gamma^J_{1,g,0}(N)$, since it is enough to show the cusp condition in Definition 2.18 for a set of representatives of $\Gamma^J_{1,g,0}(N)\backslash\Gamma^J_{1,g}$. The following remark restricts the computation to certain matrices

Remark 3.24 *It is sufficient to choose $\gamma = (M,(0,0))$, where M runs through a set of representatives of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$.*

Lemma 3.25 *Let $M \in SL_2(\mathbb{Z}), M \notin \Gamma_0(N)$. Then one has*

$$\Gamma_0(N)M = \dot{\bigcup_{\substack{c \neq 0 \\ d(N|c|)}}} \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty^0(N), \tag{3.18}$$

where $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$, $\Gamma_\infty^0(N) := \left\{ \begin{pmatrix} 1 & nN \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$, and where in the union we have chosen for each pair $(c,d) \in \mathbb{Z}^2$ a fixed pair (a,b) such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)M$.

Proof. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)M$. Since $M \notin \Gamma_0(N)$ we have $\gamma \neq 0$. Write $\delta = nN\gamma + r$, with $n \in \mathbb{Z}$ and $r \in \mathbb{Z}$, $0 \leq r < N\gamma$. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta - Nn\alpha \\ \gamma & r \end{pmatrix} \begin{pmatrix} 1 & Nn \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta - Nn\alpha \\ \gamma & r \end{pmatrix} = \begin{pmatrix} 1 & -Nn \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_\infty \Gamma_0(N)M,$$

which shows that the left-hand side of (3.18) is contained in the right-hand side. Next we show that the right-hand side of (3.18) is contained in the left-hand side. For this it is sufficient to show that for all $n \in \mathbb{Z}$

$$M \begin{pmatrix} 1 & Nn \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)M,$$

or equivalently

$$M \begin{pmatrix} 1 & Nn \\ 0 & 1 \end{pmatrix} M^{-1} \in \Gamma_0(N). \tag{3.19}$$

But (3.19) follows directly since

$$M \begin{pmatrix} 1 & Nn \\ 0 & 1 \end{pmatrix} M^{-1} \equiv E \pmod{N}.$$

Finally to show the disjointness of the union in (3.18), let us assume that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty^0(N),$$

i.e, there exist $n, m \in \mathbb{Z}$ such that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & Nm \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ** & * \\ c & d + Nmc \end{pmatrix}.$$

Then we obtain $c = c'$ and $d = d'$, since d and d' run over a set of representatives of $\mathbb{Z}/(Nc)\,\mathbb{Z}$. This proves the Lemma $\square$

Using Lemma 3.25 we directly obtain

Lemma 3.26 *Let $M \in SL_2(\mathbb{Z})$, $M \notin \Gamma_0(N)$. Then we can choose as a set of representatives of $(\Gamma^J_{1,g})_\infty \backslash \Gamma^J_{1,g,0}(N)M$, the elements*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & nN \\ 0 & 1 \end{pmatrix}, (a\lambda, (b + nNa)\lambda) \right\},$$

where (c,d) runs through the elements in Lemma 3.25.

Now we can prove

Theorem 3.27 *Let $k > g + 2$, $M \in SL_2(\mathbb{Z}), M \notin \Gamma_0(N)$.*
Then $P^N_{k,m;(n,r)}|_{k,m}(M,(0,0))$ has the Fourier expansion

$$P^N_{k,m;(n,r)}|_{k,m}(M,(0,0))(\tau,z) = \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)} \\ \frac{4n'}{N} > m^{-1}[r'^t]}} g_{k,m;(n,r),N}(n',r') e(n'/N\tau + r'z),$$

where

$$g_{k,m;(n,r),N}(n',r') := 2\pi i^k \cdot (\det(2m))^{-1/2} \cdot \left(\tilde{D}/D\right)^{k/2-g/4-1/2} \cdot \sum_{(c,d)} c^{-g/2-1}$$

$$\times \sum_{\lambda(|c|)} e_{|c|}(sign(c)\,(m[\lambda] + r\lambda + n)\,\bar{d} + dn'/N + \lambda r') \cdot e_{2|c|}(sign(c) r' m^{-1} r^t)$$

$$\times\, J_{k-g/2-1}\left(\frac{2\pi\sqrt{\tilde{D}D}}{\det(2m)\cdot c} \right).$$

Here (c,d) runs through the same elements as in Lemma 3.25, and
$\tilde{D} := -\det 2 \begin{pmatrix} \frac{n'}{N} & \frac{r'}{2} \\ \frac{r'^t}{2} & m \end{pmatrix}$. In particular $P^N_{k,m;(n,r)}$ is an element of $J^{cusp}_{k,m}(N)$.

Proof. We have

$$\begin{aligned} P^N_{k,m;(n,r)}|_{k,m}(M,(0,0))(\tau,z) &= \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g,0}(N)} e^{n,r}|_{k,m}\gamma|_{k,m}(M,(0,0))(\tau,z) \\ &= \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g,0}(N)(M,(0,0))} e^{n,r}|_{k,m}\gamma(\tau,z). \end{aligned}$$

Thus we get, using Lemma 3.26,

$$P^N_{k,m;(n,r)}|_{k,m}(M,(0,0))(\tau,z) = \sum_{\substack{(c,d)\\ \lambda\in\mathbb{Z}^{(g,1)}\\ \alpha\in\mathbb{Z}}} (c\tau+d+N\alpha c)^{-k}\cdot e\Bigg(-m[z]\frac{c}{c\tau+d+N\alpha c}+$$
$$m[\lambda]\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c}+2\lambda^t mz\frac{1}{c\tau+d+N\alpha c}+n\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c}$$
$$+rz\frac{1}{c\tau+d+N\alpha c}+r\lambda\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c}\Bigg),$$

where (c,d) runs over the same set of elements as in Lemma 3.25; in particular we have $c\neq 0$. Thus we obtain, using the identities

$$\begin{aligned}\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c} &= \frac{a}{c}-\frac{1}{c(c\tau+d+N\alpha c)},\\ \frac{1}{c\tau+d+N\alpha c}z+\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c}\lambda &= \frac{1}{c\tau+d+N\alpha c}\left(z-\frac{1}{c}\lambda\right)+\frac{a}{c}\lambda,\end{aligned}$$

$$\frac{a\tau+b+N\alpha a}{c\tau+d+N\alpha c}m[\lambda]+\frac{2}{c\tau+d+N\alpha c}\lambda^t mz-\frac{c}{c\tau+d+N\alpha c}m[z]$$
$$=-\frac{c}{c\tau+d+N\alpha c}m\left[z-\frac{1}{c}\lambda\right]+\frac{a}{c}m[\lambda],$$

and replacing λ by $\lambda+\beta c$, with the new λ running $\pmod c$ and $\beta\in\mathbb{Z}^{(g,1)}$,

$$\sum_{(c,d)} c^{-k}\sum_{\lambda(|c|)} e_{|c|}(sign(c)(m[\lambda]+r\lambda+n)\bar{d})\cdot\mathcal{F}_{k,m,c;(n,r),N}\left(\tau+d/c,z-\lambda/c\right),$$

where

$$\mathcal{F}_{k,m,c;(n,r),N}(\tau,z) := \sum_{\substack{\alpha\in\mathbb{Z}\\ \beta\in\mathbb{Z}^{(g,1)}}} (\tau+N\alpha)^{-k}\cdot e\Bigg(-\frac{1}{\tau+N\alpha}m[z-\beta]$$
$$-\frac{n}{c^2(\tau+N\alpha)}+\frac{1}{c(\tau+N\alpha)}r(z-\beta)\Bigg)\ (\tau\in\mathbb{H},z\in\mathbb{C}^{(g,1)}).$$

The function $\mathcal{F}_{k,m,c;(n,r),N}(\tau,z)$ has period N in τ and period 1 in z and therefore a Fourier expansion

$$\mathcal{F}_{k,m,c;(n,r),N}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(g,1)}}} \gamma_N(n',r')e(n'/N\tau+r'z),$$

where

$$\gamma_N(n',r') = \frac{1}{N} \cdot \int_{ic_1+[0,N]} \int_{ic_2+[0,1]^g} \mathcal{F}_{k,m,c;(n,r),N}(\tau,z) \cdot e(-n'/N\tau - r'z)\, d\tau dz$$

($c_1 > 0$, $c_2 \in \mathbb{R}^g$). This integral can be computed exactly as in the proof of Theorem 2.29 and gives the desired value. $\square$

Theorem 3.28 *(Petersson coefficient formula.)*
One has

$$\left\langle \phi, P^N_{k,m;(n,r)} \right\rangle = \frac{1}{\left[\Gamma^J_{1,g} : \Gamma^J_{1,g,0}(N)\right]} \cdot \lambda_{k,m,D} \cdot c_\phi(n,r) \qquad (\forall\, \phi \in J^{cusp}_{k,m}(N)),$$

where $c_\phi(n,r)$ denotes the (n,r)-th Fourier coefficient of ϕ and $\lambda_{k,m,D}$ is defined as in Theorem 2.31.

From this we get

Corollary 3.29 *As a unitary vector space with respect to the Petersson scalar product, $J^{cusp}_{k,m}(N)$ is generated by the Poincaré series*

$$\{P^N_{k,m;(n,r)} |\, n \in \mathbb{Z},\, r \in \mathbb{Z}^{(1,g)};\, 4n > m^{-1}[r^t]\}.$$

Next, as in the case of the full Jacobi group, we want to construct Poincaré series for $k = g+2$, using Hecke's trick.

Definition 3.30 *For $s \in \mathbb{C}$ let us define for $(\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}$ a formal series by*

$$P^N_{k,m;(n,r),s}(\tau,z) := \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g,0}(N)} \left(\frac{v}{|c\tau+d|^2}\right)^s \cdot e^{n,r}|_{k,m}\gamma(\tau,z).$$

We now need the Fourier expansion of $P^N_{k,m;(n,r),s}|_{k,m}\gamma(\tau,z)$, where $\gamma = (M,(0,0)) \in \Gamma^J_{1,g}$. Let us start with the case $M = E_g$.

Theorem 3.31 *The series $P^N_{k,m;(n,r),s}(\tau,z)$ is absolutely convergent in $\mathbb{H}\times\mathbb{C}^{(g,1)}$ if $\sigma > \frac{1}{2}(g-k+2)$. It satisfies the transformation law*

$$P^N_{k,m;(n,r),s}|_{k,m}\gamma(\tau,z) = P^N_{k,m;(n,r),s}(\tau,z) \qquad \left(\forall (\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)},\ \gamma \in \Gamma^J_{1,g,0}(N)\right),$$

and has the Fourier expansion

$$P^N_{k,m;(n,r),s}(\tau,z) = \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v,N}(n',r')e(n'\tau + r'z),$$

where $\tau = u + iv$, $\pm = (\pm 1)^k$, *and where*

$$g^{\pm}_{k,m;(n,r);s,v,N}(n',r') := g_{k,m;(n,r);s,v,N}(n',r') + (-1)^k g_{k,m;(n,r);s,v,N}(n',-r').$$

Moreover

$$g_{k,m;(n,r);s,v,N}(n',r') := v^s \cdot \delta_m(n,r,n',r') + \sum_{\substack{c\geq 1\\ N|c}} H_{m,c}(n,r,n',r') \cdot \Phi_{k,m,c,v}(n',r',s) \cdot c^{-k-2s},$$

where $D, D', \delta_m(n,r,n',r')$ *and* $H_{m,c}(n,r,n',r')$ *are defined as in Theorem 2.29 and* $\Phi_{k,m,c,v}(n',r',s)$ *as in Theorem 3.4.*

Proof. The proof of Theorem 3.31 is very similar to the proof of Theorem 3.4 and can therefore be left to the reader. $\square$

Theorem 3.32 *For* $\gamma = (M,(0,0)) \in \Gamma^J_{g,1}$*, where* $M \notin \Gamma_0(N)$*, and* $(\tau, z) \in \mathbb{H}\times\mathbb{C}^{(g,1)}$ *the series* $P^N_{k,m;(n,r),s}|_{k,m}\gamma$ *has a Fourier expansion*

$$P^N_{k,m;(n,r),s}|_{k,m}\gamma(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} g^N_{k,m;(n,r);s,v}(n',r')e(n'/N\tau + r'z),$$

where $\tau = u + iv$, *and where*

$$g^N_{k,m;(n,r);s,v}(n',r') := \sum_{(c,d)} H^N_{m,c}(n,r,n',r') \cdot \Phi^N_{k,m,c,v}(n',r',s) \cdot c^{-k-2s},$$

where (c,d) *runs through the set of elements of Lemma 3.25, and where*

$$H^N_{m,c}(n,r,n',r') := \sum_{\substack{x(|c|)\\ y(|c|)^*}} e_{|c|}(sign(c)(m[x] + rx + n)\bar{y} + n'/Ny + r'x),$$

and

$$\Phi^N_{k,m,c,v}(n',r',s) = \frac{1}{N} \cdot (\det(2m))^{-1/2} \cdot i^{-g/2} \cdot v^{g/2-k-s+1} \cdot e_{2|c|}(sign(c) r' m^{-1} r^t)$$
$$\times \int_{-\infty}^{\infty} (u+i)^{g/2-k-s} \cdot (u-i)^{-s} \cdot e\left((2\det(2m))^{-1} \left(\tilde{D}v(u+i) + \frac{D}{vc^2(u+i)} \right) \right) du,$$

where D *is defined as in Theorem 2.29 and* $\tilde{D}$ *in Theorem 3.27*

Proof. The proof is very similar to the proofs of the Theorems 2.29 and 3.4, and can therefore be omitted. $\square$

Now we have to estimate the integrals $\Phi_{k,m,c,v}(n',r',s)$ and $\Phi^N_{k,m,c,v}(n',r',s)$ and the Kloosterman sums $H_{m,c}(n,r,n',r')$ and $H^N_{m,c}(n,r,n',r')$. We have already estimated $\Phi_{k,m,c,v}(n',r',s)$ and $H_{m,c}(n,r,n',r')$ in Corallary 3.7 and Lemma 3.9, respectively. Thus it is left to consider $\Phi^N_{k,m,c,v}(n',r',s)$ and $H^N_{m,c}(n,r,n',r')$.

Lemma 3.33 *The coefficients $\Phi^N_{k,m,c,v}(n',r',s)$, defined as in Theorem 3.32, are holomorphic functions in s with $\sigma > \frac{1}{2}(1+g/2-k)$. In particular they are holomorphic at $s=0$ if $k > g/2+1$.*
If K is a compact set of the domain $\sigma > \frac{1}{2}(1+g/2-k)$, with $s \in K$, then they satisfy the estimate

$$\Phi^N_{k,m,c,v}(n',r',s) \ll_K v^{g/2-k-\sigma+1} \cdot e^{\frac{-D}{Av}} \cdot e^{\frac{-\pi\tilde{D}v}{\det(2m)}(1+sign(\tilde{D})v_1)},$$

where v_1 and A are positive constants with $0 < v_1 < 1$, and where the constant implied in $\ll_K$ only depends on the set K.

Proof. The lemma follows directly from Lemma 3.5 with $c_1 = k - g/2$, $c_2 = \frac{-\tilde{D}v}{2\det(2m)}$, and $c_3 = \frac{-D}{c^2 \cdot v \cdot 2\det(2m)}$. □

Lemma 3.34 *Let $H^N_{m,c}(n,r,n',r')$ be defined as in Theorem 3.32. Then we have*

$$|H^N_{m,c}(n,r,n',r')| \ll_{D,m,\epsilon} c^{g/2+1+\epsilon},$$

where the constant implied in $\ll_{D,m,\epsilon}$ only depends on D, m, and ϵ.

Proof. The proof is very similar to the proof of Lemma 3.9 and will therefore be omitted. □

Theorem 3.35 *Let $g^{\pm}_{k,m;(n,r);s,v,N}(n',r')$, $H_{m,c}(n,r,n',r')$, $\Phi_{k,m,c,v}(n',r',s)$, $H^N_{m,c}(n,r,n',r')$, and $\Phi^N_{k,m,c,v}(n',r',s)$ be defined as in the Theorems 2.29, 3.4, and 3.31, respectively, where $\pm = (\pm 1)^k$.*
Then the Fourier series

$$P_{k,m;(n,r),s}(\tau,z) := \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v,N}(n',r') e(n'\tau + r'z) \qquad ((\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)})$$

is absolutely and locally uniformly convergent in s and defines a holomorphic function in s for $\sigma > \frac{1}{2}(g/2+2-k)$. In particular $P^N_{k,m,(n,r),0}(\tau,z)$ is absolutely convergent if $k > g/2+2$.
Let us define

$$\begin{aligned} g^{\pm}_N(n',r') &:= g_N(n',r') + (-1)^k g_N(n',-r'), \\ g_N(n',r') &:= \textstyle\sum_{\substack{c \geq 1 \\ N|c}} H_{m,c}(n,r,n',r') \cdot \Phi_{k,m,c,v}(n',r',s) \cdot c^{-k-2s}, \\ h_{N,M}(n',r') &:= \textstyle\sum_{(c,d)} H^N_{m,c}(n,r,n',r') \cdot \Phi^N_{k,m,c,v}(n',r',s) \cdot c^{-k-2s}, \end{aligned}$$

where (c,d) runs through the elements of Lemma 3.25 and where $\pm = (\pm 1)^k$.
Let $k = g+2, (\tau,z) \in \mathbb{F}$, where $\mathbb{F}$ is the standard fundamental domain for the

action of the Jacobi group on $\mathbb{H} \times \mathbb{C}^{(g,1)}$, given in Remark 2.25, and let s with $0 < \sigma < 1$ be an element from a compact set K. Then we have the estimates

$$\begin{aligned} \left| \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g_N^{\pm}(n',r') e(n'\tau + r'z) \right| &\ll_K \quad v^{-g/2-1}, \\ \left| \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} h_{N,M}(n',r') e(n'\tau + r'z) \right| &\ll_K \quad v^{-g/2-1}, \end{aligned}$$

where the constants implied in $\ll_K$ are independent of τ and z.

Proof. The proof is very similar to the proof of Lemma 3.10 and can therefore be left to the reader. □

Lemma 3.36 *The function $P^N_{g+2,m;(n,r)}(\tau,z) := P^N_{g+2,m;(n,r),0}(\tau,z)$ is an element of $J^{cusp}_{g+2,m}(N)$.*

Proof. The proofs of the transformation law and the Fourier expansion can be given as in the proof of Lemma 3.11. Moreover the Fourier expansion in an arbitrary cusp follows from Theorem 3.32 by taking the limit. □

It is left to show that the Petersson coefficient formula is still valid. This is proved in the following three lemmas.

Lemma 3.37 *Let $\sigma > 0$ and $\phi \in J^{cusp}_{k,m}(N)$. Then the scalar product*

$$\left\langle \phi, P^N_{g+2,m;(n,r),\sigma} \right\rangle$$

is absolutely convergent.

Proof. We proceed as in the case of the full Jacobi group. Thus we let V be a fixed set of representatives of $(\Gamma^J_{1,g})_\infty \backslash \Gamma^J_{1,g,0}(N)$. Then, using the usual unfolding argument and Levi‘s Theorem, we have in the sense of formal agreement

$$\begin{aligned} &\int_{\mathbb{F}_N} \left| \phi(\tau,z) \cdot \overline{P^N_{g+2,m;(n,r),s}(\tau,z)} \cdot \exp(-4\pi m[y]v^{-1}) \right| dudvdxdy \\ \leq &\int_{\mathbb{F}_N} |\phi(\tau,z)| \cdot \exp(-4\pi m[y]v^{-1}) \sum_{\gamma \in V} v^{\sigma} \cdot |c\tau + d|^{-2s} \cdot |e^{n,r}|_{k,m}\gamma(\tau,z)| \, dudvdxdy \\ &\qquad = \int_{\cup_{\gamma \in V} \gamma \mathbb{F}_N} v^{\sigma} \cdot \exp(-4\pi m[y]v^{-1}) \cdot |\phi(\tau,z)| \cdot |e^{n,r}(\tau,z)| \, dudvdxdy, \end{aligned}$$

where $\mathbb{F}_N$ is a fundamental domain for the action of $\Gamma^J_{1,g,0}(N)$ on $\mathbb{H} \times \mathbb{C}^{(g,1)}$. Since $\cup_{\gamma \in V} \gamma \mathbb{F}_N$ is a fundamental domain for the action of $(\Gamma^J_{1,g})_\infty$ on $\mathbb{H} \times \mathbb{C}^{(g,1)}$, the integral can be estimated as in the case of the full Jacobi group (using Lemma 3.12). □

Lemma 3.38 *For $\sigma > 0$ and $\phi \in J^{cusp}_{g+2,m}(N)$ we have*

$$\left\langle \phi, P^N_{g+2,m;(n,r),\sigma} \right\rangle = \frac{1}{\left[\Gamma^J_{1,g} : \Gamma^J_{1,g,0}(N)\right]} \cdot \lambda_{g+2,m,D,\sigma} \cdot c_\phi(n,r),$$

where $c_\phi(n,r)$ denotes the (n,r)-th Fourier coefficient of ϕ and $\lambda_{g+2,m,D,\sigma}$ is defined as in Lemma 2.24.

Proof. The proof is similar to the proof of Lemma 3.13 and can therefore be omitted. $\square$

Lemma 3.39 *For $0 < \sigma < 1$ and $\phi \in J^{cusp}_{g+2,m}(N)$ we have*

$$\lim_{\sigma \to 0} \left\langle \phi, P^N_{g+2,m;(n,r),\sigma} \right\rangle = \left\langle \phi, P^N_{g+2,m;(n,r)} \right\rangle. \tag{3.20}$$

Proof. The existence of the limit on the right-hand side of (3.20) follows directly from Lemma 3.38. Thus it is left to show that we may interchange limit and integration.
As a fundamental domain for the action of $\Gamma^J_{1,g,0}(N)$ on $\mathbb{H}\times\mathbb{C}^{(g,1)}$ we can choose $\cup_{\gamma_i \in V} \gamma_i \mathbb{F}$, where $\mathbb{F}$ is a fundamental domain for the action of $\Gamma^J_{1,g}$ on $\mathbb{H}\times\mathbb{C}^{(g,1)}$, and where V is a set of representatives of $\Gamma^J_{1,g,0}(N)\backslash\Gamma^J_{1,g}$.
Since $[\Gamma^J_{1,g} : \Gamma^J_{1,g,0}(N)] < \infty$ it is sufficient to show for $\gamma_i \in V$

$$\begin{aligned}\lim_{\sigma \to 0} \int_{\gamma_i \mathbb{F}} \phi(\tau,z) \cdot \overline{P^N_{g+2,m;(n,r),\sigma}(\tau,z)} \cdot \exp\left(-4\pi m[y]v^{-1}\right) dudvdxdy \\ = \int_{\gamma_i \mathbb{F}} \phi(\tau,z) \cdot \overline{P^N_{g+2,m;(n,r)}(\tau,z)} \cdot \exp\left(-4\pi m[y]v^{-1}\right) dudvdxdy,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\lim_{\sigma \to 0} \int_{\mathbb{F}} \phi|_{g+2,m}\gamma_i^{-1}(\tau,z) \cdot \overline{P^N_{g+2,m;(n,r),\sigma}|_{g+2,m}\gamma_i^{-1}(\tau,z)} \cdot \exp\left(-4\pi m[y]v^{-1}\right) dudvdxdy \\ = \int_{\mathbb{F}} \phi|_{g+2,m}\gamma_i^{-1}(\tau,z) \cdot \overline{P^N_{g+2,m;(n,r)}|_{g+2,m}\gamma_i^{-1}(\tau,z)} \cdot \exp\left(-4\pi m[y]v^{-1}\right) dudvdxdy.\end{aligned}$$

Now we can proceed as in Lemma 3.14 and construct a Lebesgue majorant using the estimates in Lemma 2.23 which is possible since $\phi|_{g+2,m}\gamma_i^{-1}(\tau,z)$ is a Jacobi cusp form on $\gamma_i \Gamma^J_{1,g,0}(N)\gamma_i^{-1}$ (which has finite index in $\Gamma^J_{1,g}$). $\square$

Lemma 3.40 *As a unitary vector space with respect to the Petersson scalar product, $J^{cusp}_{g+2,m}(N)$ is generated by the Poincaré series*

$$\left\{ P^N_{g+2,m;(n,r)} \mid n \in \mathbb{Z},\, r \in \mathbb{Z}^{(1,g)};\, 4n > m^{-1}[r^t] \right\}.$$

Now we want to estimate the Fourier coefficients of Jacobi cusp forms with respect to $\Gamma^J_{1,g,0}(N)$.

Theorem 3.41 *Suppose that $k \geq g+2$. Let $\phi \in J^{cusp}_{k,m}(N)$ with Fourier coefficients $c(n,r)$. Then we have*

$$c(n,r) \ll_{\epsilon,k} \left(1 + \frac{|D|^{g/2+\epsilon}}{(\det m)^{(g+1)/2}}\right)^{1/2} \cdot \frac{|D|^{k/2-g/4-1/2}}{(\det m)^{k/2-(g+3)/4}} \cdot \parallel \phi \parallel \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,k}$ only depends on ϵ and k .

Proof. As in the case of the full Jacobi group we get, using the Cauchy-Schwarz inequality and the Petersson coefficient formula,

$$|c(n,r)|^2 \leq \lambda^{-1}_{k,m,D} \cdot b_{n,r}(P^N_{k,m;(n,r)}) \cdot \parallel \phi \parallel^2,$$

where $b_{n,r}(P^N_{k,m;(n,r)})$ is the (n,r)-th Fourier coefficient of $P^N_{k,m;(n,r)}$. Thus to prove Theorem 3.41 we only have to estimate the Fourier coefficients of the Poincaré series $P^N_{k,m;(n,r)}$. Therefore it is sufficient to estimate

$$\sum_{\substack{c \geq 1 \\ N|c}} \left| H_{m,c}(n,r,n,\pm r) \cdot J_{k-g/2-1}\left(\frac{-2\pi \cdot D}{\det(2m) \cdot c}\right)\right|,$$

which is trivially less or equal than

$$\sum_{c \geq 1} \left| H_{m,c}(n,r,n,\pm r) \cdot J_{k-g/2-1}\left(\frac{-2\pi \cdot D}{\det(2m) \cdot c}\right)\right|.$$

This sum has already been estimated in [BK] . $\square$

Corollary 3.42 *Suppose that $k \geq g+2$. Let $\phi \in J^{cusp}_{k,m}(N)$ with Fourier coefficients $c(n,r)$. Then we have*

$$c(n,r) \ll_{\epsilon,\phi} |D|^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,\phi}$ only depends on ϵ and ϕ.

ii) The case $\Gamma^J_{2,1,0}(N)$

Recall that

$$\Gamma^J_{2,1,0}(N) \quad := \quad \Gamma_{2,0}(N) \ltimes \left(\mathbb{Z}^{(1,2)} \times \mathbb{Z}^{(1,2)}\right)$$

(cf. Definition 2.16).

Definition 3.43 *Let n be a positive definite symmetric half-integral 2×2 matrix, $r \in \mathbb{C}^{(1,2)}$, and $m \in \mathbb{N}$ such that $n - \frac{rr^t}{4m} > 0$. Let us define a Poincaré series of exponential type for $\Gamma^J_{2,1,0}(N)$ by*

$$\mathcal{P}^N_{k,m;(n,r)}(\tau,z) := \sum_{\gamma \in \left(\Gamma^J_{2,1}\right)_\infty \backslash \Gamma^J_{2,1,0}(N)} e^{n,r}|_{k,m}\gamma(\tau,z) \qquad \left((\tau,z)\in \mathbb{H}\times\mathbb{C}^{(1,2)}\right),$$

where $e^{n,r}(\tau,z) := e(tr\,(n\tau + rz))$, and where
$(\Gamma^J_{2,1})_\infty := \left\{ \left(\begin{pmatrix} E_2 & S \\ 0 & E_2 \end{pmatrix}, (0,\mu) \right) \in \Gamma^J_{2,1} \middle| S \in Sym_2(\mathbb{Z}) \right\}$.

Then we have the following

Lemma 3.44 *Let $k \geq 8$ be an even integer. Then the series $\mathcal{P}^N_{k,m;(n,r)}(\tau,z)$ is absolutely and locally uniformly convergent on $\mathbb{H}_2 \times \mathbb{C}^{(1,2)}$. It defines a holomorphic function and satisfies the transformation law*

$$\mathcal{P}^N_{k,m;(n,r)}|_{k,m}\gamma(\tau,z) = \mathcal{P}^N_{k,m;(n,r)}(\tau,z) \qquad \left(\forall(\tau,z)\in\mathbb{H}_2\times\mathbb{C}^{(1,2)},\ \gamma\in\Gamma^J_{2,1,0}(N)\right).$$

Proof. The absolute and local uniform convergence follows from the absolute and local uniform convergence of the Poincaré series $\mathcal{P}_{k,m;(n,r)}$ for the full Jacobi group (cf. [Br] p. 8). Here the restriction $k \geq 8$ even is needed since $e(m\tau')\cdot\mathcal{P}_{k,m;(n,r)}$ is a subseries of another Poincaré series for the Siegel modular group which converges absolutely in the case $k \geq 8$ even. Thus also the holomorphicity and the transformation law are clear. □

In order to compute the Fourier expansion we first need

Lemma 3.45 *Let $\tilde{R}$ be a complete set of representatives of $(\Gamma_2)_\infty\backslash\Gamma_{2,0}(N)$. Then*

$$R := \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda A, \lambda B) \right) \in \Gamma^J_{2,1} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{R},\ \lambda \in \mathbb{Z}^{(1,2)} \right\}$$

is a complete set of representatives of $(\Gamma^J_{2,1})_\infty\backslash\Gamma^J_{2,1,0}(N)$.

Proof. The proof is an easy and straightforward calculation and can be left to the reader. □

In the computation of the Fourier series of $\mathcal{P}^N_{k,m;(n,r)}$ we proceed as in [Br].

Corollary 3.46 *Let $\tilde{R}$ be a complete set of representatives of $(\Gamma_2)_\infty\backslash\Gamma_{2,0}(N)$. Then we have for $(\tau,z)\in\mathbb{H}\times\mathbb{C}^{(1,2)}$*

$$\mathcal{P}^N_{k,m;(n,r)}(\tau,z) = \sum_{M=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\tilde{R}} \ \sum_{\lambda\in\mathbb{Z}^{(1,2)}} (\det(C\tau+D))^{-k}$$
$$\times\, e(-m((C\tau+D)^{-1}C)[z^t] + tr\,(T(A\tau+B)(C\tau+D)^{-1}) + z(C\tau+D)^{-1}r$$
$$+\lambda(A\tau+B)(C\tau+D)^{-1}r + 2mz(C\tau+D)^{-1}\lambda^t + m(A\tau+B)(C\tau+D)^{-1}[\lambda^t]).$$

Proof. The claim follows directly from Lemma 3.45, the definition of the slash operator, and the elementary identities

$$\begin{aligned} A-(A\tau+B)(C\tau+D)^{-1}C &= (C\tau+D)^{-t}, \\ (A\tau+B)(C\tau+D)^{-1}C(A\tau+B)^t &= A(A\tau+B)^t-(A\tau+B)(C\tau+D)^{-1}.\square \end{aligned}$$

To compute the Fourier expansion of the Poincaré series $\mathcal{P}_{k,m;(n,r)}(\tau,z)$ we need the following

Lemma 3.47 *Let H be a complete set of representatives of $(\Gamma_2)_\infty\backslash\Gamma_{2,0}(N)/(\Gamma_2)_\infty$.*
Then we have

1. $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in H$ *is parametrized by C with $C\equiv 0 \pmod N$ and $D \pmod{C\Lambda}$, where $\Lambda := Sym_2(\mathbb{Z})$.*

2. *For $M\in\Gamma_{2,0}(N)$ we have*

$$(\Gamma_2)_\infty M(\Gamma_2)_\infty = \dot{\bigcup_{S\in\Lambda/\Theta(M)}} (\Gamma_2)_\infty M\begin{pmatrix} I & S \\ 0 & I\end{pmatrix},$$

where

$$\Theta(M) := \left\{ S\in\Lambda \,\middle|\, M\begin{pmatrix} I & S \\ 0 & I\end{pmatrix}M^{-1}\in(\Gamma_2)_\infty\right\}$$

is an additive subgroup of Λ.

Proof. The proof follows in the same way as in the case of the full modular group (cf. [Ki], p. 158). $\square$

Thus we obtain

Corollary 3.48 *For $(\tau,z)\in\mathbb{H}_2\times\mathbb{C}^{(1,2)}$ and H as in Lemma 3.47 we have*

$$\begin{aligned} \mathcal{P}^N_{k,m;(n,r)}(\tau,z) = & \sum_{M=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in H}\ \sum_{\substack{S\in\Lambda/\Theta(M)\\ \lambda\in\mathbb{Z}^{(1,2)}}} (\det(C(\tau+S)+D))^{-k} \\ & \times e(-m((C(\tau+S)+D)^{-1}C)[z^t] + tr\,(T(A(\tau+S)+B)(C(\tau+S)+D)^{-1}) \\ & \quad + z(C(\tau+S)+D)^{-1}r + \lambda(A(\tau+S)+B)(C(\tau+S)+D)^{-1}r \\ & \quad + 2mz(C(\tau+S)+D)^{-1}\lambda^t + m(A(\tau+S)+B)(C(\tau+S)+D)[\lambda^t]). \end{aligned}$$

Proof. The proof follows directly from Lemma 3.47, using the absolute convergence of the Poincaré series. $\square$

We now subdivide the Poincaré series into three parts according as $rk(C)=0,1,$ and 2, respectively and denote these by $P^1(\tau,z)$, $P^2(\tau,z)$, and $P^3(\tau,z)$, respectively.
First we want to compute $P^1(\tau,z)$. We have the following

Lemma 3.49 *1. As a set of representatives* $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H, rk(C) = 0\right\}$, *we can choose*

$$H^{(0)} := \left\{ \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \middle| U \in GL_2(\mathbb{Z}) \right\}.$$

We have for $M \in H^{(0)}$:

$$\Theta(M) = \Lambda.$$

2. We have for $(\tau, z) \in \mathbb{H}_2 \times \mathbb{C}^{(1,2)}$

$$P^{(1)}(\tau, z) = \sum_{\substack{n' \in \Lambda^* \\ r' \in \mathbb{Z}^{(1,2)} \\ \eta' > 0}} A(n', r') e(tr\,(n'\tau + r'z)),$$

where $\Lambda^* := \{S \in Sym_2(\mathbb{Q}),\ S \text{ half-integral}\}$, *and where*

$$A(n', r') \quad = \quad \sharp \left\{ U \in GL_2(\mathbb{Z}) \,\middle|\, \eta[U^t] = \eta',\ U^{-1}n' - r \in \mathbb{Z}^{(1,2)} \cdot 2m \right\},$$

where $\eta := n - \frac{rr^t}{4m}$, *and where* $\eta' := n' - \frac{r'r'^t}{4m}$.

Proof. The proof can be taken from [Br] since we can take as a set of representatives the same set as in the case of Γ_2. □

Next we want to compute $P^{(2)}(\tau, z)$. We have

Lemma 3.50 *1. As a set of representatives* $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H, rk(C) = 1\right\}$, *we can choose*

$$H^{(1)} := \left\{ M = \begin{pmatrix} * & * \\ U^{-1}\left(\begin{smallmatrix} c_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) V^t & U^{-1}\left(\begin{smallmatrix} d_1 & d_2 \\ 0 & d_4 \end{smallmatrix}\right) V^{-1} \end{pmatrix} \in \Gamma_2 \middle| \right.$$
$$U \in \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2(\mathbb{Z}) \right\} \Big\backslash GL_2(\mathbb{Z}), V \in GL_2(\mathbb{Z}) \Big/ \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}) \right\}$$
$$\left. c_1 \in \mathbb{N},\, c_1 \equiv 0 \pmod N, d_4 = \pm 1, d_1, d_2 \pmod{c_1}, (c_1, d_1) = 1 \right\}.$$

For $M \in H^{(1)}$ *we have:*

$$\Theta(M) = \left\{ S \in \Lambda \middle| S[V] = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\}$$

with V *as above.*

2. For $(\tau, z) \in \mathbb{H}_2 \times \mathbb{C}^{(1,2)}$ *we have*

$$P^{(1)}(\tau, z) = \sum_{\substack{n' \in \Lambda^* \\ r' \in \mathbb{Z}^{(1,2)} \\ \eta' > 0}} B(n', r') e(tr\,(n'\tau + r'z)),$$

where

$$B(n', r') = \sum_{\substack{U \in \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in GL_2(\mathbb{Z})\right\} \backslash GL_2(\mathbb{Z}) \\ V \in GL_2(\mathbb{Z}) / \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in GL_2(\mathbb{Z})\right\}}} \sum_{\substack{c_1 \in \mathbb{N} \\ c_1 \equiv 0 (N) \\ d_4 = \pm 1}} \sum_{\substack{d_1 (c_1)^* \\ d_2 (c_1)}} \delta^{(2m)}_{d_4 r_2, r'_2} \cdot \delta_{F,F'}$$

$$\times\, e_{c_1}\Big(\overline{d_1} n_1 - \overline{d_1} d_2 d_4 n_2 + \overline{d_1} d_2^2 n_4 - r_1 r'_1/(2m) + d_2 d_4 r_2 r'_1/(2m) + d_1 n'_1$$
$$+ d_2\left(n'_2 - r'_1 r'_2/(2m)\right) - \frac{d_4}{2F}\left(n'_2 - r_1 r'_2/(2m)\right)\left(n_2 - r_1 r'_1/(2m)\right)\Big)$$
$$\times \sum_{\lambda_1 (c_1)} e_{c_1}\left(\overline{d_1} m \lambda_1^2 + \overline{d_1} r_1 \lambda_1 - \overline{d_1} d_2 d_4 r_2 \lambda_1 - r'_1 \lambda\right) \cdot c_1^{-2} \pi i^{-k} \cdot (Fm)^{-1/2}$$
$$\times\, (E'/E)^{k/2-1} \cdot J_{k-2}\left(\frac{4\pi}{c_1 F}\sqrt{E'E}\right),$$

where

$$\delta^{(n)}_{x,y} := \begin{cases} 1 & \text{if } x \equiv y \pmod{n} \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix}\left[\begin{pmatrix} U^t & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} n_1 & n_2/2 & r_1/2 \\ n_2/2 & n_4 & r_2/2 \\ r_1/2 & r_2/2 & m \end{pmatrix},$$

$$\begin{pmatrix} n' & r'/2 \\ r'^t/2 & m \end{pmatrix}\left[\begin{pmatrix} U^t & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} n'_1 & n'_2/2 & r'_1/2 \\ n'_2/2 & n'_4 & r'_2/2 \\ r'_1/2 & r'_2/2 & m \end{pmatrix},$$

where $\overline{d_1}$ *is an inverse of* $d_1 \pmod{c_1}$, $\eta[U^t] = \left(\begin{smallmatrix} * & * \\ * & F \end{smallmatrix}\right)$, $\eta'[V^{-t}] = \left(\begin{smallmatrix} * & * \\ * & F' \end{smallmatrix}\right)$, $E := \det \eta$, $E' := \det \eta'$, *and* J_{k-2} *denotes the Bessel function of order* $k-2$.

Proof. 1. can be obtained by intersecting the set of representatives of $(\Gamma_2)_\infty \backslash \Gamma_2 / (\Gamma_2)_\infty$ chosen in [Ki], p. 159 with $\Gamma_{2,0}(N)$.

2. For a fixed $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H^{(1)}$ the inner sum in Corollary 3.48 can be computed as in [Br] (pp. 17-26) to

$$\delta^{2m}_{r_2, r'_2} \cdot \delta_{F,F'} \cdot \pi i^{-k} \cdot (Fm)^{-1/2} \cdot c^{-2} \cdot (E'/E)^{k/2-1}$$
$$\times\, e_c\left(-r_1 r'_1/(2m) + a t_1 + n'_1 d\right) \cdot e\left(\frac{n_2 r'_1 r'_2}{4cFm} + \frac{n'_2 r_1 r_2}{4cFm} - \frac{n_2 n'_2}{2cF} - \frac{r_1 r_2 r'_1 r'_2}{8cFm^2}\right)$$
$$\times \sum_{\lambda_1 (c)} e_c\left(am\lambda_1^2 + a r_1 \lambda_1 - r'_1 \lambda_1\right) \cdot J_{k-2}\left(\frac{4\pi}{cF}\sqrt{E'E}\right).$$

Using 1., we get the claim after an easy straightforward calculation. $\square$

Finally we need the Fourier expansion of $P^{(2)}(\tau, z)$.

Lemma 3.51 *1. As a set of representatives for* $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H, rk(C) = 2\right\}$, *we can take*

$$H^{(2)} := \left\{ \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_{2,0}(N) \middle| \det C \neq 0\,, D \pmod{C\Lambda} \right\}.$$

We have for $M \in H^{(2)}$

$$\Theta(M) = \{0\}.$$

2. We have for $(\tau, z) \in \mathbb{H}_2 \times \mathbb{C}^{(1,2)}$

$$P^{(2)}(\tau, z) = \sum_{\substack{n' \in \Lambda^* \\ r' \in \mathbb{Z}^{(1,2)} \\ \eta' > 0}} C(n', r') e(tr\ (n'\tau + r'z)),$$

where

$$C(n', r') = \frac{1}{2m} \cdot \sum_{\substack{C \in M_2(\mathbb{Z}) \\ \det C \neq 0}} e\left(\frac{1}{2m} n^t C^{-1} r\right) \sum_{\substack{D (C\Lambda) \\ \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{2,0}(N)}} \sum_{\lambda \in \mathbb{Z}^{(1,2)}/\mathbb{Z}^{(1,2)} C^t}$$
$$\times\, e(tr\ (n' C^{-1} D - r' \lambda C^{-t}) + m A C^{-1}[\lambda^t] + \lambda A C^{-1} r + tr\ (n A C^{-1}))$$
$$\times\, (E'/E)^{k/2-1} \cdot (\det C)^{-2} \cdot \tilde{J}(Q),$$

where A is chosen arbitrarily such that $\begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \Gamma_{2,0}(N)$, *and where* $\tilde{J}(\cdot)$ *is a certain matrix-argument Bessel function defined for* $Sym_2(\mathbb{R})$ *by*

$$\tilde{J}(R) := \int_{X \in Sym_2(\mathbb{R})} (\det \tau)^{1/2-k} \cdot e(-tr\ (R(\tau + \tau^{-1})))dX$$

and $Q = \sqrt{\epsilon' \left[\sqrt{\epsilon[C^{-t}]}\right]}$.

Remark 3.52 *Of course one has to take a suitable choice of square root in Lemma 3.49. This can be done as in [Br].*

Proof. Lemma 3.51 can be proved exactly as in the case of the full Jacobi group. The restriction $C \equiv 0 \pmod N$ does not change the calculations (cf. [Br], pp. 28-30). $\square$

Next we want to compute the Fourier expansion of $\mathcal{P}_{k,m;(n,r),N}|_{k,m}\gamma$, where $\gamma = (M, (0,0)) \in \Gamma^J_{2,1}$. For this we need the following

Lemma 3.53 *Let $M \in \Gamma_2$, $M \notin \Gamma_{2,0}(N)$. Then we have*

$$\Gamma_{2,0}(N)M = \dot{\bigcup_{\substack{rgC>0 \\ \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{2,0}(N)M \\ NS \in \Lambda/\Theta(M)}}} (\Gamma_2)_\infty \begin{pmatrix} * & * \\ C & D \end{pmatrix} \begin{pmatrix} I & NS \\ 0 & I \end{pmatrix}. \qquad (3.21)$$

Proof. That the right-hand side of (3.21) is contained in the left-hand side follows from

$$M \begin{pmatrix} I & NS \\ 0 & I \end{pmatrix} M^{-1} \equiv E \pmod N.$$

That the left-hand side of (3.21) is contained in the right-hand side and the disjointness of the union follows directly from the definition of $\Theta(M)$. □

Now we are able to prove the following

Theorem 3.54 *Let $M \in \Gamma_2$, $M \notin \Gamma_{2,0}(N)$. Then $\mathcal{P}_{k,m;(n,r),N}|_{k,m}(M,0)(\tau,z)$ for all $(\tau,z) \in \mathbb{H}_2 \times \mathbb{C}^{(1,2)}$ has a Fourier expansion*

$$\mathcal{P}_{k,m;(n,r),N}|_{k,m}(M,0)(\tau,z) = \sum_{\substack{n' \in \Lambda^* \\ r' \in \mathbb{Z}^{(1,2)} \\ \frac{4n'}{N} > \frac{r'r'^t}{m}}} A(n',r')e(tr\,(n'\tau + r'z)),$$

where $A(n',r') \in \mathbb{C}$.

Proof. We only give a sketch of proof, because the computations are similar to the ones for $M = E_g$. We have

$$\begin{aligned} \sum_{\gamma \in \left(\Gamma^J_{2,1}\right)_\infty \backslash \Gamma^J_{2,1,0}(N)} e^{n,r}|_{k,m}(M,0)\gamma(\tau,z) &= \sum_{\gamma \in (\Gamma^J_{2,1})^J_\infty \backslash (M,0)\Gamma^J_{2,1,0}(N)} e^{n,r}|_{k,m}\gamma(\tau,z) \\ &= \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{2,0}(N)M} \sum_{\substack{S \in N\Gamma/\Theta(M) \\ \lambda \in \mathbb{Z}^{(1,2)}}} (\det(c(\tau+S)+D)^{-k}) \\ \times\, e(-m((C(\tau+S)+D)^{-1}C)[z^t] &+ \mathrm{tr}\,(T(A(\tau+S)+B)(C(\tau+S)+D)^{-1}) \\ &+ z(C(\tau+S)+D)^{-1}r + \lambda(A(\tau+S)+B)(C(\tau+S)+D)^{-1}r \\ &+ 2mz(C(\tau+S)+D)^{-1}\lambda^t + m(A(\tau+S)+B)(C(\tau+S)+D)^{-1}[\lambda^t]). \end{aligned}$$

Now we can split the sum into two parts acccording as $rk(C) = 1$ or 2 (the case $C = 0$ can obviously not occur).
In the case $rk(C) = 1$ one can easily show that one may assume that every $\tilde{M} \in \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_{2,0}(N)M$ has the form

$$\tilde{M} = \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) & \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} c & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{pmatrix} \begin{pmatrix} V^t & 0 \\ 0 & V^{-1} \end{pmatrix},$$

where $a,b,c,d \in \mathbb{Z}$, $c \geq 1$, $ad-bc=1$, $U,V \in GL_2(\mathbb{Z})$. We now fix an $\tilde{M}$ and develop the inner sum into a Fourier series with respect to τ (using that the sum has period N in τ). Then we can show that the Fourier coefficients have period 1 in z and therefore have a Fourier expansion in z. Now we can show that the Fourier coefficients of the Fourier expansion with respect to (τ, z) vanish unless $\frac{4n'}{N} > \frac{r'r'^t}{m}$.
In the case $rk(C) = 2$ we replace λ by $\lambda + \mu C^t$ with $\mu \in \mathbb{Z}^{(1,2)}$, and with the new λ running $\pmod{\mathbb{Z}^{(1,2)} \cdot C}$. Using $\Theta(M) = \{0\}$, the claim follows easily for this case. □

Thus we have shown

Theorem 3.55 *The series $\mathcal{P}_{k,m;(n,r),N}(\tau, z)$ is a Jacobi cusp form with respect to $\Gamma^J_{2,1,0}(N)$.*

Lemma 3.56 *We have*

$$\left\langle \phi, \mathcal{P}_{k,m;(n,r),N} \right\rangle = \lambda_{k,m,\eta} \cdot c(n,r) \qquad (\forall\, \phi \in J^{cusp}_{k,m}(N)),$$

where

$$\lambda_{k,m,\eta} := 2^{6-4k} \cdot \pi^{9/2-2k} \cdot \Gamma(k-2) \cdot \Gamma(k-5/2) \cdot m^{-1} \cdot (\det \eta)^{2-k},$$

with η defined as in Lemma 3.51.

Proof. Lemma 3.56 follows in a similar way as in the case of the full Jacobi group.
□

Thus we can estimate the Fourier coefficients of Jacobi cusp forms with respect to $\Gamma^J_{2,1,0}(N)$. We have

Theorem 3.57 *Let $k \geq 8$ be an even integer and let $\phi \in J^{cusp}_{k,m}(N)$ with Fourier coefficients $c(n,r)$. Then for a Minkowski-reduced matrix $\begin{pmatrix} m & \frac{r^t}{2} \\ \frac{r}{2} & n \end{pmatrix}$ we have*

$$c(n,r) \ll_{\epsilon,k} \left(1 + m^{-1/2+\epsilon} \cdot (\det \eta)^{1+\epsilon} + m^{-1/2+\epsilon} \cdot (min(\eta))^{-1} \cdot (\det \eta)^{3/2+\epsilon}\right) \cdot \parallel \phi \parallel,$$

where η is defined as in Lemma 3.49, where $min(\eta) := min\{\eta[x] | 0 \neq x \in \mathbb{Z}^{(2,1)}\}$, and where the constant implied in $\ll_{\epsilon,k}$ only depends on ϵ and k.

Proof. The estimate can be proved similarly as in the case of the full Jacobi group (cf. [Br] pages 59-78). □

3.2.2 Estimates of the Petersson norm of Fourier-Jacobi coefficients

Let $F \in S_k(\Gamma_{g,0}(N))$. As described in Remark 2.22, F has the following two Fourier-Jacobi expansions

$$F(Z) = \sum_{\tilde{m}>0} \psi_{\tilde{m}}(\tilde{\tau}, \tilde{z}) e^{2\pi i \mathrm{tr}\ (\tilde{m}\tilde{\tau}')} = \sum_{m \geq 1} \phi_m(\tau, z) e^{2\pi i m \tau'}. \tag{3.22}$$

Here in the first sum the summation extends over all positive definite symmetric half-integral $(g-1) \times (g-1)$ matrices, and $Z \in \mathbb{H}_g$ is written as $Z = \begin{pmatrix} \tilde{\tau} & \tilde{z} \\ \tilde{z}^t & \tilde{\tau}' \end{pmatrix}$, with $\tilde{\tau} \in \mathbb{H}$, $\tilde{z} \in \mathbb{C}^{(1,g-1)}$, and $\tilde{\tau}' \in \mathbb{H}_{g-1}$.
In the second sum the summation extends over all positive integers, and $Z \in \mathbb{H}_g$ is written as $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$, with $\tau \in \mathbb{H}_{g-1}, z \in \mathbb{C}^{(1,g-1)}$, and $\tau' \in \mathbb{H}$.
Moreover $\psi_{\tilde{m}}$ and ϕ_m are Jacobi cusp forms with respect to $\Gamma^J_{1,g-1,0}(N)$ and $\Gamma^J_{g-1,1,0}(N)$, respectively. In this chapter we estimate the Petersson norms of the coefficients $\psi_{\tilde{m}}(\tilde{\tau}, \tilde{z})$ and $\phi_m(\tau, z)$. If $g = 2$ both cases coincide. Comparing the estimates gives that the second one is slightly better.

The estimate we obtain in the following Lemma uses the classical Hecke argument.

Lemma 3.58 *We have the estimates*

$$\begin{aligned} \parallel \psi_{\tilde{m}} \parallel &\ll_F (\det \tilde{m})^{k/2}, \\ \parallel \phi_m \parallel &\ll_F m^{k/2}, \end{aligned}$$

where the constants implied in $\ll_F$ only depend on F.

Remark 3.59 *For $N = 1$ the proof is given in [KS] for $g = 2$ and in [Kr] for ϕ_m for arbitrary g.*

Proof. We only give the proof for $\psi_{\tilde{m}}$, the estimate for $\parallel \phi_m \parallel$ is obtained similarly. Using the first Fourier-Jacobi expansion in (3.22), we obtain

$$\psi_{\tilde{m}}(\tilde{\tau}, \tilde{z}) = \int_{iC}^{iC+E_{g-1}} F(Z) e(-\mathrm{tr}\ (\tilde{m}\tilde{\tau}')) d\tilde{\tau}',$$

where $C > 0$ depends on $\tilde{v}$ and $\tilde{y}$ and satisfies the condition $Y = \begin{pmatrix} \tilde{v} & \tilde{y}^t \\ \tilde{y} & C \end{pmatrix} > 0$, where $\tilde{v} :=\mathrm{Im}(\tilde{\tau})$ and $\tilde{y} :=\mathrm{Im}(\tilde{z})$.

From the Jacobi decomposition (cf. Remark 2.19) it follows directly that we can choose
$$C = \tilde{m}^{-1} + \frac{\tilde{y}\tilde{y}^t}{\tilde{v}}.$$
In this case we have
$$\det Y = (\det \tilde{m})^{-1}\tilde{v}.$$
Thus we obtain, using that $(\det Y)^{k/2}F(Z)$ is bounded on $\mathbb{H}_g$ (cf. Lemma 2.13),
$$\begin{aligned} |\psi_{\tilde{m}}(\tilde{\tau},\tilde{z})| &\leq \int_{iC}^{iC+E_{g-1}} |F(Z)| \cdot e^{2\pi \mathrm{tr}\,(\tilde{m}C)}\, d\tilde{\tau} \\ &\ll_F (\det \tilde{m})^{k/2} \cdot \tilde{v}^{-k/2} \cdot e^{2\pi \mathrm{tr}\,(\tilde{m}[\tilde{y}^t]\tilde{v}^{-1})}. \end{aligned}$$
Therefore we get
$$\begin{aligned} \| \psi_{\tilde{m}} \|^2 &\leq \frac{1}{\left[\Gamma^J_{1,g-1} : \Gamma^J_{1,g-1,0}(N)\right]} \int_{\mathbb{F}} \tilde{v}^k \cdot |\psi_{\tilde{m}}(\tilde{\tau},\tilde{z})|^2 \cdot e^{-4\pi \mathrm{tr}\,(\tilde{m}[\tilde{y}^t]\tilde{v}^{-1})}\, dV^J_{g-1} \\ &\ll_F (\det \tilde{m})^k \cdot \int_{\mathbb{F}} dV^J_{g-1}, \end{aligned}$$
where $\mathbb{F}$ is a fundamental domain for the action of $\Gamma^J_{1,g-1,0}(N)$ on $\mathbb{H} \times \mathbb{C}^{(g-1,1)}$. Thus the claim follows if we use that $\int_{\mathbb{F}} dV^J_{g-1} < \infty$ (cf. Remark 2.25). □

Next we want to improve the estimate for the Petersson norm $\| \phi_m \|$. The idea is to use the Rankin-Selberg method and prove for a Dirichlet series with essential $\| \phi_m \|^2$ as general coefficient a meromorphic continuation and functional equation and then estimate $\| \phi_m \|^2$ by using a modified version of the Theorem of Sato and Shintani. In the following we more generally replace $\| \phi_m \|^2$ by $\langle \phi_m, \psi_m \rangle$, where ϕ_m and ψ_m are the Fourier-Jacobi coefficients of two Siegel cusp forms with respect to $\Gamma_{g,0}(N)$. We want to prove the following

Theorem 3.60 *We have*
$$\| \phi_m \| \ll_{\epsilon,F} m^{k/2-g/(4g+1)+\epsilon} \qquad (\epsilon > 0),$$
where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

For the proof we need some properties about certain non-holomorphic Eisenstein series. First we want to show the following technical

Lemma 3.61 *Let $(c,d) \in \mathbb{Z}^{(1,2g)}$ be a primitive vector (i.e., $ggT(c,d) = 1$) with $c \equiv 0 \pmod{N}$.*
Then there exists a matrix $M \in \Gamma_{g,0}(N)$ with (c,d) as last row.

Remark 3.62 *For $N = 1$ the lemma is well known (cf. for example [Ma]).*

Proof. For clarity the proof is subdivided into seven steps.

I. In the first step we want to prove the claim for $c = 0$.
In this case d is a primitive vector in $\mathbb{Z}^{(1,g)}$. Due to the Lemma of Gauß there exists a matrix $U \in GL_g(\mathbb{Z})$ with d as last row. Thus the matrix

$$\begin{pmatrix} U^{-t} & 0 \\ 0 & U \end{pmatrix} \text{ lies in } \Gamma_{g,0}(N),$$

and has (c, d) as last row.

II. We may assume that $|c_g| \neq 0$ is minimal under all entries c_i $(1 \leq i \leq g)$ coming from $(c, d)M$ with $M \in \Gamma_{g,0}(N)$.
Indeed: Due to I. we may assume that $c \neq 0$. Let $c_i \in \mathbb{Z}\backslash\{0\}$ with $|c_i|$ minimal under the c_j $(1 \leq j \leq g)$ coming from $(c, d)M$ with $M \in \Gamma_{g,0}(N)$, and let V be the following $g \times g$ permutation matrix

$$V := \begin{pmatrix} e_1^t \\ \vdots \\ e_g^t \\ \vdots \\ e_i^t \end{pmatrix}.$$

Then

$$\begin{pmatrix} V & 0 \\ 0 & V^{-t} \end{pmatrix} \in \Gamma_{g,0}(N).$$

Thus we obtain, setting $(\tilde{d}_1, \ldots, \tilde{d}_g) := (d_1, \ldots, d_g)V^{-t}$,

$$(c_1, \ldots, c_g, d_1 \ldots, d_g) \begin{pmatrix} V & 0 \\ 0 & V^{-t} \end{pmatrix} = (c_1, \ldots, c_g, c_{i+1}, \ldots, c_i, \tilde{d}_1 \ldots, \tilde{d}_g),$$

i.e., we may assume $|c_g| \neq 0$ is minimal under all entries c_i $(1 \leq i \leq g)$ coming from $(c, d)M$ with $M \in \Gamma_{g,0}(N)$. For $1 \leq i < g$ let us write $c_i = \lambda_i c_g + r_i$ with $0 \leq r_i < c_g$ and $\lambda_i \in \mathbb{Z}$. Then clearly

$$V := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ -\lambda_1 & \ldots & -\lambda_{g-1} & 1 \end{pmatrix} \in SL_g(\mathbb{Z}),$$

and

$$\begin{pmatrix} V & 0 \\ 0 & V^{-t} \end{pmatrix} \in \Gamma_{g,0}(N).$$

Thus, setting $(\tilde{d}_1, \ldots, \tilde{d}_g) := (d_1, \ldots, d_g)V^{-t}$, we find

$$(c_1, \ldots, c_g, d_1 \ldots, d_g) \begin{pmatrix} V & 0 \\ 0 & V^{-t} \end{pmatrix} = (r_1, \ldots, r_{g-1}, c_g, \tilde{d}_1 \ldots, \tilde{d}_g).$$

From the minimality of $|c_g|$, it follows that $r_i = 0$ $(1 \leq i \leq g-1)$.

III. Due to II. we may assume $(c, d) = (0 \ldots, 0, c_g, d_1, \ldots d_g)$, $c_g \equiv 0 \pmod N$. If $(d_1, \ldots, d_{g-1}) = 0$, then c_g and d_g are coprime since c and d are coprime.
In this case we set

$$\begin{aligned} C &:= \begin{pmatrix} 0 & 0 \\ 0 & c_g \end{pmatrix}, \\ D &:= \begin{pmatrix} E & 0 \\ 0 & d_g \end{pmatrix}. \end{aligned}$$

Then clearly the matrix (C, D) is primitive (i.e., there exists a matrix $U \in GL_{2g}(\mathbb{Z})$ with (C, D) as the lower $g \times (2g)$ block) and satisfies $CD^t = DC^t$.
Thus the pair (C, D) can be completed to a matrix belonging to Γ_g (cf. [Fr]), which is then clearly in $\Gamma_{g,0}(N)$ and has (c, d) as last row.
If $d_i \neq 0$ for at least one d_i $(1 \leq i \leq g-1)$, we let V be the following $(g-1) \times (g-1)$ permutation matrix

$$V := \begin{pmatrix} e_1^t \\ \vdots \\ e_{g-1}^t \\ e_{i+1}^t \\ \vdots \\ e_i^t \end{pmatrix}.$$

Then

$$(0, \ldots, 0, c_g, d_1, \ldots, d_{g-1}, d_g) \begin{pmatrix} V^{-t} & & & \\ & 1 & & \\ & & V & \\ & & & 1 \end{pmatrix}$$
$$= (0, \ldots, 0, c_g, d_1 \ldots, d_{g-1}, d_{i+1}, \ldots, d_i, d_g).$$

For the same reasons as in I. we may assume that
$(c, d) = (0, \ldots, 0, c_g, 0, \ldots, 0, d_{g-1}, d_g)$, with $d_{g-1} \neq 0$ and $c_g \equiv 0 \pmod N$.

IV. According to [KK], page 116 there exists a $\lambda \in \mathbb{Z}$ with $(d_{g-1}, d_g + \lambda c) = 1$.
Let

$$S := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \lambda \end{pmatrix}.$$

Then clearly $S^t = S$ and therefore

$$\begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \in \Gamma_{g,0}(N).$$

Thus

$$(0, \ldots, 0, c_g, 0, \ldots 0, d_{g-1}, d_g) \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} = (0, \ldots, 0, c_g, 0, \ldots, d_{g-1}, d_g + \lambda c_g).$$

Therefore we may assume that $(c, d) = (0, \ldots, 0, c_g, 0, \ldots, 0, d_{g-1}, d_g)$, with $c_g \equiv 0 \pmod N$ and $(d_{g-1}, d_g) = 1$.

V. Since $\mathbb{Z}$ is euclidean, there exist $d_1, d_2 \in \mathbb{Z}$ such that

$$\begin{pmatrix} d_1 & d_2 \\ d_{g-1} & d_g \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{3.23}$$

Let

$$\begin{aligned} C &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 & c_2 \\ 0 & 0 & c_g \end{pmatrix}, \\ D &:= \begin{pmatrix} E & 0 & 0 \\ 0 & d_1 & d_2 \\ 0 & d_{g-1} & d_g \end{pmatrix}, \end{aligned}$$

where $c_1, c_2 \in \mathbb{Z}$. Then clearly the matrix (C, D) is primitive.
We now want to choose c_1, c_2 such that $c_1, c_2 \equiv 0 \pmod N$ and $CD^t = DC^t$, because in this case the pair (C, D) can be completed to a matrix in $\Gamma_{g,0}(N)$ (cf. [Fr]). These conditions are satisfied for

$$\begin{aligned} c_1 &:= -c_g d_2^2, \\ c_2 &:= c_g d_2 d_1. \end{aligned}$$

Indeed: Due to $c_g \equiv 0 \pmod N$ we clearly have $c_1, c_2 \equiv 0 \pmod N$. Thus it is left to show that the matrix CD^t is symmetric. We have

$$CD^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 & c_2 \\ 0 & 0 & c_g \end{pmatrix} \begin{pmatrix} E & 0 & 0 \\ 0 & d_1 & d_{g-1} \\ 0 & d_2 & d_g \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 d_1 + c_2 d_2 & c_1 d_{g-1} + c_2 d_g \\ 0 & c_g d_2 & c_g d_g \end{pmatrix}.$$

Thus we have to show

$$c_1 d_{g-1} + c_2 d_g = d_2 c_g.$$

Inserting the definitions of c_1 and c_2 and using (3.23) we directly see that this condition is satisfied. □

We now define non-holomorphic Eisenstein series as in [KS] or [Kr].

Definition 3.63 *We formally define the following non-holomorphic Eisenstein series*

$$\begin{aligned} E_{s,N}(Z) &:= \sum_{M\in\mathcal{C}_N\backslash\Gamma_{g,0}(N)} \left(\frac{\det(\,Im(M<Z>))}{\det(\,Im(M<Z>)_1)}\right)^s, \\ E_s(Z) &:= E_{s,1}(Z), \end{aligned}$$

where $\mathcal{C}_N$ denotes the subgroup of $\Gamma_{g,0}(N)$ consisting of all those matrices with $(0,\dots 0,1)$ as last row, and $Im(M<Z>)_1$ means the upper $(g-1)\times(g-1)$ block of the matrix $Im(M<Z>)$.
Moreover let

$$E^*_{s,N}(Z) := \pi^{-s}\cdot\Gamma(s)\cdot\zeta_N(2s)\cdot E_{s,N}(Z),$$

where

$$\zeta_N(s) := \sum_{\substack{n\in\mathbb{N}\\(n,N)=1}} n^{-s}.$$

Then we have the following

Lemma 3.64 *The series $E_{s,N}(Z)$ is well-defined, converges absolutely and locally uniformly for $Re(s)>g$, and is invariant under $\Gamma_{g,0}(N)$.*

Proof. The lemma is well known for $N=1$ (cf. [Kr]). For $N>1$ it is proved in exactly the same way. □

Remark 3.65 *We have*

$$E_{s,N}(Z) = \sum_{\substack{(c,d)\in\mathbb{Z}^{(2g,1)}\\(c,d)=1\\c\equiv 0(N)}} \left(P_Z\left[\begin{pmatrix} c\\ d\end{pmatrix}\right]\right)^{-s} \qquad (\forall\, Z\in\mathbb{H}_g),$$

where $P_Z := \begin{pmatrix} Y & 0\\ 0 & Y^{-1}\end{pmatrix}\left[\begin{pmatrix} I & 0\\ X & I\end{pmatrix}\right] > 0.$

Proof. The proof follows directly if we use Lemma 3.61 and the identity

$$\frac{\det(\,\mathrm{Im}(M<Z>))}{\det(\,\mathrm{Im}(M<Z>)_1)} = P_Z[\lambda]^{-1},$$

where λ^t denotes the last row of M. □

Lemma 3.66 *The function $E^*_{s,N}(Z)$ has a meromorphic continuation to the whole complex s-plane.*
If $N=1$ it is holomorphic except for two simple poles at $s=0$ and at $s=g$ with residues -1 and 1, respectively.
In this case it satisfies the functional equation

$$E^*_s(Z) = E^*_{g-s}(Z).$$

If $N \neq 1$ the only singularity is a simple pole at $s = g$ with residue

$$N^{-g} \cdot \sum_{l|N} \mu(l) l^{-g},$$

where $\mu(l)$ denotes the Moebius function.
In this case it satisfies the functional equation

$$E^*_{g-s,N}(Z) = N^{2s-2g} \cdot \sum_{l_1|N} \mu\left(\frac{N}{l_1}\right) \cdot l_1^{g-2s} \cdot \sum_{l_2|l_1} l_2^{2s} \cdot E^*_{s,l_2}(Z).$$

Moreover we have the following identity between the Eisenstein series

$$E^*_{s,N}(Z) = \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot E^*_s\left(\frac{N}{l} Z\right).$$

Proof. Since the case $N = 1$ is treated in [KS] for $g = 2$ and in [Kr] for arbitrary g, we may assume $N > 1$.
Write $\lambda = \begin{pmatrix} c \\ d \end{pmatrix}$. We use Remark 3.65, the directly verified identity

$$P_Z[\lambda] = Y[c] + Y^{-1}[d + Xc] \qquad \left(\forall\, c, d \in \mathbb{Z}^{(g,1)}\right), \tag{3.24}$$

and the well-known property of the Moebius function

$$\sum_{l|n} \mu(l) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}, \tag{3.25}$$

to deduce

$$\begin{aligned}
\zeta_N(2s) E_{s,N}(Z) &= \sum_{\substack{n\in\mathbb{N} \\ (N,n)=1}} n^{-2s} \sum_{\substack{(c,d)\in(N\mathbb{Z})^{(g,1)}\times\mathbb{Z}^{(g,1)} \\ (c,d)=1}} \left(Y[c] + Y^{-1}[Xc + d]\right)^{-s} \\
&= \sum_{\substack{n\in\mathbb{N} \\ (N,n)=1 \\ (c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)} \\ (Nc,d)=1}} \left(Y[Nnc] + Y^{-1}[NnXc + dn]\right)^{-s} \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)} \\ (N,d)=1}} \left(Y[Nc] + Y^{-1}[NXc + d]\right)^{-s} \\
&= \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)}} \left(Y[Nc] + Y^{-1}[NXc + d]\right)^{-s} \sum_{l|(N,d)} \mu(l) \\
&= \sum_{l|N} \mu(l) \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)}} \left(Y[Nc] + Y^{-1}[NXc + ld]\right)^{-s}
\end{aligned}$$

$$= \sum_{l|N}(Nl)^{-s}\cdot\mu(l) \sum_{(c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)}}{}' \left(\left(\frac{N}{l}Y\right)[c]+\left(\frac{N}{l}Y\right)^{-1}\left[\left(\frac{N}{l}X\right)c+d\right]\right)^{-s}$$

$$= \sum_{n\in\mathbb{N}}\sum_{l|N}(Nl)^{-s}\cdot\mu(l) \sum_{\substack{(c,d)\in\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)}\\(c,d)=1}}{}' \left(\left(\frac{N}{l}Y\right)[nc]+\left(\frac{N}{l}Y\right)^{-1}\left[\left(\frac{N}{l}X\right)nc+nd\right]\right)^{-s}$$

$$= \zeta(2s)\cdot\sum_{l|N}\mu(l)\cdot(Nl)^{-s}\cdot E_s\left(\frac{N}{l}Z\right).$$

Here $\sum'$ means as usual that the vector 0 is omitted in the summation.
Hence

$$E^*_{s,N}(Z) = \sum_{l|N}\mu(l)\cdot(Nl)^{-s}\cdot E^*_s\left(\frac{N}{l}Z\right) = N^{-2s}\cdot\sum_{l|N}\mu\left(\frac{N}{l}\right)\cdot l^s\cdot E^*_s(lZ) \tag{3.26}$$

Applying the Moebius inversion formula to (3.26), we get

$$E^*_s(NZ) = N^{-s}\cdot\sum_{l|N} l^{2s}\cdot E^*_{s,l}(Z). \tag{3.27}$$

From (3.26), the meromorphicity of $E^*_{s,N}(Z)$ as a function of s follows from the well-known meromorphicity of $E^*_s(Z)$. The only possible poles are those of $E^*_s(Z)$, i.e., $s=0$ and $s=g$ and they are at most of first order.

Using elementary rules for calculating residues, the fact $N\neq 1$, the formulas (3.25) and (3.26), the fact that $E^*_s(Z)$ has a pole of first order with residue -1 at $s=0$, we find

$$\operatorname{Res}\left(E^*_{s,N}(Z), s=0\right) = \lim_{s\to 0} sE^*_{s,N}(Z) = \sum_{l|N}\mu(l)\cdot\lim_{s\to 0} sE^*_s\left(\frac{N}{l}Z\right) = -\sum_{l|N}\mu(l) = 0.$$

Thus $E^*_{s,N}(z)$ is holomorphic at $s=0$.
At $s=g$ we can compute the residue with the same arguments as in the case $s=0$. Using the fact that $E_s(Z)^*$ has a pole of first order with residue 1 at $s=g$, we derive

$$\begin{aligned}\operatorname{Res}\left(E^*_{s,N}(Z), s=g\right) &= \lim_{s\to g}(s-g)E^*_{s,N}(Z)\\ &= N^{-g}\cdot\sum_{l|N}\mu(l)\cdot l^{-g}\cdot\lim_{s\to g}(s-g)E^*_s\left(\frac{N}{l}Z\right) = N^{-g}\cdot\sum_{l|N} l^{-g}\cdot\mu(l)\neq 0.\end{aligned}$$

Thus $E^*_{s,N}(z)$ has a simple pole with the residue $N^{-g}\cdot\sum_{l|N} l^{-g}\cdot\mu(l)$ at $s=g$.

It is left to show the functional equation for $E^*_{s,N}(Z)$. Using the functional equation of $E^*_s(Z)$ and (3.26), we find

$$E^*_{g-s,N}(Z) = N^{2s-2g} \cdot \sum_{l|N} \mu\left(\frac{N}{l}\right) \cdot l^{g-s} \cdot E^*_{g-s}(lZ) = N^{2s-2g} \cdot \sum_{l|N} \mu\left(\frac{N}{l}\right) \cdot l^{g-s} \cdot E^*_s(lZ).$$

Thus we obtain by using (3.27)

$$E^*_{g-s,N}(Z) = N^{2s-2g} \cdot \sum_{l|N} \mu\left(\frac{N}{l}\right) \cdot l^{g-2s} \sum_{\tilde{l}|l} \tilde{l}^{2g} \cdot E^*_{s,\tilde{l}}(Z).$$

□

In order to prove a later claim on the meromorphicity of $\langle FE_{s,N}, G\rangle$ as a function of s, where $F, G \in S_k(\Gamma_{g,0}(N))$, we need knowledge about the growth behaviour of the Eisenstein series $E_{s,N}$ for $\mathrm{tr}(Y) \to \infty$. More exactly speaking we want to prove the following

Lemma 3.67 *Fix $s \in \mathbb{C}$. Then for every $Z \in \mathbb{F}_g$, $\gamma \in \Gamma_g$, there exists a real constant α such that*

$$E^*_{s,N}(\gamma \circ Z) - \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot \left(\frac{1}{s} + \frac{1}{g-s}\right) \ll_{g,N,\gamma} (tr\, Y)^{\alpha},$$

where $\mathbb{F}_g$ denotes the standard fundamental domain for the action of Γ_g on $\mathbb{H}_g$, and where the constant implied in $\ll_{g,N,\gamma}$ only depends on g, N, and γ.

Proof. Due to Lemma 3.66 it is sufficient to prove Lemma 3.67 for $E_s(l(\gamma \circ Z))$, where l is a positive divisor of N. For this we need an integral representation of the Eisenstein series.

Lemma 3.68 *For all $s \in \mathbb{C}$, we have the integral representation*

$$E^*_s(Z) - \left(\frac{1}{s} + \frac{1}{g-s}\right) = \int_1^{\infty} \sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t P_Z[\lambda]} \cdot (t^s + t^{g-s}) \frac{dt}{t} \qquad (\forall\, Z \in \mathbb{H}_g),$$

where P_Z is defined as in Remark 3.65.

Proof. For the proof we need some well known properties of the Epstein zeta-function (cf. [Te] pp. 58). As in the proof of Lemma 3.66 we need the identity

$$\zeta(2s) \cdot E_s(Z) = \sum_{\lambda \in \mathbb{Z}^{(2g,1)}} (P_Z[\lambda])^{-s}.$$

Now for a positive definite symmetric real-valued $(2g) \times (2g)$ matrix A we define a theta series by

$$\Theta(t, A) := \sum_{\lambda \in \mathbb{Z}^{(2g,1)}} e^{-\pi t A[\lambda]} \qquad \left(\forall\, t \in \mathbb{R}^+\right).$$

In particular we set

$$\Theta_t(Z) := \Theta(t, P_Z) \qquad (\forall\, Z \in \mathbb{H}_g).$$

Then it is shown in [Te] that

$$\int_0^\infty (\Theta_t(Z) - 1) \cdot t^s \,\frac{dt}{t} = \pi^{-s} \cdot \Gamma(s) \cdot \zeta(2s) \cdot E_s(Z) = E_s^*(Z) \qquad (\mathrm{Re}(s) > g),$$

and

$$\Theta_{1/t}(Z) = \Theta(t^{-1}, P_Z) = (\det P_Z)^{-1/2} \cdot t^g \cdot \Theta(t, P_Z^{-1}).$$

Due to

$$\det P_Z = 1,$$

and

$$\begin{aligned} P_Z^{-1}\left[\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right] &= \begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix}\left[\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\right]\left[\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right] \\ &= Y[\lambda_2] + Y^{-1}[\lambda_1 - X\lambda_2] = P_Z\left[\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix}\right], \end{aligned}$$

we have

$$\Theta_{1/t}(Z) = t^g \cdot \Theta_t(Z) \qquad (\forall\, t > 0).$$

Therefore we have for all s with $\sigma > 0$ the following integral representation

$$\begin{aligned} E_s^*(Z) &= \int_1^\infty (\Theta_t(Z) - 1) \cdot t^s \,\frac{dt}{t} + \int_1^\infty (t^g \Theta_t(Z) - 1) \cdot t^{-s} \,\frac{dt}{t} \\ &= \int_1^\infty (\Theta_t(Z) - 1)\,(t^s + t^{g-s}) \frac{dt}{t} - \left(\frac{1}{s} + \frac{1}{g-s}\right). \end{aligned}$$

Thus the lemma follows for all $s \in \mathbb{C}$ using meromorphic continuation. $\square$

Now we can prove Lemma 3.67, using the integral representation computed in Lemma 3.68. First we want to estimate $\sum'_\lambda e^{-\pi t P_{l_\gamma \circ Z}[\lambda]}$. Therefore we show that for all integers r_1 and r_2 there exist integers r_1' and r_2' such that

$$\sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' \; e^{-\pi t l^{r_1} P_{l^{r_2}\gamma \circ Z}[\lambda]} \leq \sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' \; e^{-\pi t l^{r_1'} P_{l^{r_2'} Z}[\lambda]}. \tag{3.28}$$

Since the matrix γ is an element of Γ_g, it can be written as a product up to $\pm E$ of matrices of the form $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, $(S^t = S)$ and $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. We now want to prove (3.28) per induction on the minimal number n of matrices of these two types that are needed. Since in the case $n = 0$ there is nothing to show we may assume that $n > 0$. Then we can write $\gamma = \gamma_1\gamma_2$, where $\gamma_1 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ or $\gamma_1 = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, and $\gamma_2 \in \Gamma_g$.

In the following we need the identity (cf. [Kr])

$$P_{M<Z>} = P_Z[M^t] \qquad (\forall\, Z \in \mathbb{H}_g, M \in Sp_g(\mathbb{R}))\,. \tag{3.29}$$

Let us first assume $\gamma_1 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and set $\tilde{Z} := \gamma_2 \circ Z$. Then we have

$$l^{r_2}\left(\gamma_1 \circ \tilde{Z}\right) = l^{r_2}\left(-\frac{1}{\tilde{Z}}\right) = -\frac{1}{l^{-r_2}\tilde{Z}} = \gamma_1 \circ \left(l^{-r_2}\tilde{Z}\right).$$

Using identity (3.29) we find

$$P_{l^{r_2}(\gamma_1\circ\tilde{Z})} = P_{\gamma_1\circ\left(l^{-r_2}\tilde{Z}\right)} = P_{l^{-r_2}\tilde{Z}}[\gamma_1^t].$$

Thus we obtain, using that $\gamma_1^t\lambda$ runs over $\mathbb{Z}^{(2g,1)} \setminus \{0\}$ if λ does,

$$\sum_{\lambda\in\mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{r_2}\gamma_1\gamma_2\circ Z}[\lambda]} = \sum_{\lambda\in\mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{-r_2}(\gamma_2\circ Z)}[\gamma_1^t\lambda]} = \sum_{\lambda\in\mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{-r_2}(\gamma_2\circ Z)}[\lambda]}.$$

If $\gamma_1 = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, we have

$$l^{r_2}(\gamma_1 \circ \tilde{Z}) = l^{r_2}\tilde{Z} + l^{r_2}S = \begin{pmatrix} E & l^{r_2}S \\ 0 & E \end{pmatrix} \circ (l^{r_2}\tilde{Z}).$$

We now distinguish the cases $r_2 \geq 0$ and $r_2 < 0$.

If $r_2 \geq 0$, then $\begin{pmatrix} E & l^{r_2}S \\ 0 & E \end{pmatrix}^t \lambda$ runs over $\mathbb{Z}^{(2g,1)} \setminus \{0\}$ if λ does. Thus we get with the same arguments as before

$$\sum_{\lambda\in\mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{r_2}(\gamma_1\gamma_2\circ Z)}[\lambda]} = \sum_{\lambda\in\mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{r_2}(\gamma_2\circ Z)}[\lambda]}.$$

If $r_2 < 0$, we replace r_2 by $-r_2$ with the new $r_2 > 0$. Then we have

$$P_{l^{-r_2}(\gamma_1\circ\tilde{Z})}\left[\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right]$$

$$\begin{aligned}
&= P_{l^{-r_2}\tilde{Z}}\left[\begin{pmatrix} E & 0 \\ l^{-r_2}S & E \end{pmatrix}\right]\left[\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right] \\
&= \begin{pmatrix} \left(l^{-r_2}\tilde{Y}\right) & 0 \\ 0 & \left(l^{-r_2}\tilde{Y}\right)^{-1} \end{pmatrix}\left[\begin{pmatrix} E & 0 \\ l^{-r_2}\tilde{X} & E \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 + l^{-r_2}S\lambda_1 \end{pmatrix}\right] \\
&= \left(l^{-r_2}\tilde{Y}\right)[\lambda_1] + \left(l^{-r_2}\tilde{Y}\right)^{-1}\left[\left(l^{-r_2}\tilde{X}\right)\lambda_1 + \lambda_2 + l^{-r_2}S\lambda_1\right] \\
&= l^{-2r_2}\left(\left(l^{-r_2}\tilde{Y}\right)[l^{r_2}\lambda_1] + \left(l^{-r_2}\tilde{Y}\right)^{-1}\left[\left(l^{-r_2}\tilde{X}\right)l^{r_2}\lambda_1 + l^{r_2}\lambda_2 + S\lambda_1\right]\right).
\end{aligned}$$

Moreover for a fixed $\lambda_1 \neq 0$ and $\lambda_1 = 0$, $l^{r_2}\lambda_2 + S\lambda_1$ runs through a subset of $\mathbb{Z}^{(g,1)} \setminus \{0\}$ and $\mathbb{Z}^{(g,1)}$, respectively if λ_2 runs through $\mathbb{Z}^{(g,1)} \setminus \{0\}$ and $\mathbb{Z}^{(g,1)}$, respectively. Thus we have

$$\sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{r_2}(\gamma_1\gamma_2\circ Z)}[\lambda]} \leq \sum_{\lambda_1,\lambda_2 \in \mathbb{Z}^{(g,1)}}{}' e^{-\pi t l^{r_1-2r_2}\left(\left(l^{-r_2}\tilde{Y}\right)[l^{r_2}\lambda_1] + \left(l^{-r_2}\tilde{Y}\right)^{-1}\left[\left(l^{-r_2}\tilde{X}\right)l^{r_2}\lambda_1 + \lambda_2\right]\right)}.$$

Moreover $l^{r_2}\lambda_1$ runs through a subset of $\mathbb{Z}^{(2g,1)}$ and $\mathbb{Z}^{(2g,1)} \setminus \{0\}$ if λ_2 runs through $\mathbb{Z}^{(g,1)}$ and $\mathbb{Z}^{(g,1)} \setminus \{0\}$, respectively. Thus we have

$$\sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1} P_{l^{r_2}(\gamma_1\gamma_2\circ Z)}[\lambda]} \leq \sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t l^{r_1-2r_2} P_{l^{-r_2}(\gamma_2\circ Z)}[\lambda]}.$$

Thus we can use the induction step in both cases and get the claim.
Now we can show as in [Ku] (using that the largest and the smallest eigenvalue of Y can be estimated against $e_{min} \geq e_1$ and $e_{max} \leq e_2 \cdot \mathrm{tr}\ (Y)$, respectively, with constants e_1, e_2 only depending on g) that there exists a positive constant α such that

$$\sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t P_{l\gamma\circ Z}[\lambda]} \ll_{g,l,\gamma} (\mathrm{tr}\ Y)^{\alpha} \cdot e^{-\pi t a}.$$

Using Lemma 3.68 this leads to the estimate

$$E_s^*(l\gamma \circ Z) - \left(\frac{1}{s} + \frac{1}{g-s}\right) \ll_{g,l,\gamma} (\mathrm{tr}\ Y)^{\alpha} \cdot \int_1^{\infty} e^{-\pi t a} \cdot (t^{\sigma} + t^{g-\sigma})\, dt.$$

As in [Ku] we get, distinguishing the cases $a > 1$ and $a \leq 1$, that the integral is bounded by a power of tr (Y). $\square$

Definition 3.69 *Let us now formally define the Dirchlet series*

$$\begin{aligned}
D_{F,G,N}(s) &:= \tfrac{k_N}{i_N} \cdot \zeta_N(2s - 2k + 2g) \cdot \textstyle\sum_{m \geq 1} \langle \phi_m, \psi_m \rangle\, m^{-s}, \\
D^*_{F,G,N}(s) &:= (4\pi)^{-s} \cdot \Gamma(s) \cdot \Gamma(s - k + g) \cdot D_{F,G,N}(s),
\end{aligned}$$

where $F, G \in S_k(\Gamma_{g,0}(N))$, ϕ_m and ψ_m are the m-th Fourier-Jacobi coefficients of F and G, respectively, $i_N := [\Gamma_g : \Gamma_{g,0}(N)]$, and $k_N := [\Gamma_1 : \Gamma_0(N)]$.
Moreover we define for a positive divisor l of N the Dirichlet series

$$D_{F,G,N,l}(s) := \zeta_l(2s-2k+2g) \cdot \sum_{m \geq 1} b_l(m) m^{-s},$$

where

$$b_l(m) := \frac{1}{i_l} \cdot \sum_j \int_{\mathbb{F}_l} \phi_{m,\gamma_j}(\tau, z) \cdot \overline{\psi_{m,\gamma_j}(\tau, z)} \cdot (\det v)^{k-(g+1)} \cdot e^{-4\pi m v^{-1}[y]} \, dudvdxdy,$$

and

$$D^*_{F,G,N,l}(s) := (4\pi)^{-s} \cdot \Gamma(s) \cdot \Gamma(s-k+g) \cdot D_{F,G,N,l}(s),$$

where $\mathbb{F}_l$ is a fixed fundamental domain of the action of $\Gamma^J_{g-1,1}(l)$ on $\mathbb{H}_{g-1} \times \mathbb{C}^{(1,g-1)}$. Moreover $\phi_{m,\gamma_j}(\tau, z)$ and $\psi_{m,\gamma_j}(\tau, z)$ denote the Fourier-Jacobi coefficient of $F|\gamma_j$ and $G|\gamma_j$, respectively, where γ_j runs through a set of representatives of $\Gamma_{g,0}(N)\backslash\Gamma_{g,0}(l)$.

Remark 3.70 *The proof of Theorem 3.72 will show that the fundamental domain $\mathbb{F}_l$ in Definition 3.69 can be chosen arbitrarily.*

Lemma 3.71 *The coefficients $\langle \phi_m, \psi_m \rangle$ and $b(m,l)$ of $D_{F,G,N}(s)$ and $D_{F,G,N,l}(s)$, respectively, satisfy the estimates*

$$\left. \begin{matrix} \langle \phi_m, \psi_m \rangle \\ b_l(m) \end{matrix} \right\} \ll_{F,G} m^k,$$

where the constants implied in $\ll_{F,G}$ only depend on F and G.
Thus the series $D_{F,G,N}(s)$ and $D_{F,G,N,l}(s)$ are absolutely and locally uniformly convergent for $Re(s) > k+1$ and therefore holomorphic.

Proof. The proof is very similar to the proof of Lemma 3.58, and therefore can be left to the reader. □

Theorem 3.72 *The functions $D_{F,G,N}(s)$ and $D_{F,G,N,l}(s)$ have meromorphic continuations to the whole complex plane with only finitely many poles.*
The function $D_{F,G,N}(s)$ is entire if $\langle F, G \rangle = 0$, and otherwise at $s = k$ has a simple pole of residue

$$\frac{(4\pi)^k}{(k-1)!(g-1!)} \cdot \pi^{g-k} \cdot \langle F, G \rangle \cdot N^{-g} \cdot \sum_{l|N} \mu(l) \cdot l^{-g}.$$

It satisfies the functional equation

$$D^*_{F,G,N}(2k-g-s) = N^{2s-2k} \cdot \sum_{l_1|N} \mu\left(\frac{N}{l_1}\right) \cdot l_1^{-2s+2k-g} \sum_{l_2|l_1} l_2^{2(s-k+g)} \cdot D^*_{F,G,N,l_2}(s).$$

Proof. Since the proof for $N=1$ is given in [KS] for $g=2$ and in [Kr] for arbitrary g we may assume $N>1$. Let $F,G\in S_k(\Gamma_{g,0}(N))$. Then clearly $\langle FE_{s,N},G\rangle$ is well-defined. Thus we obtain, using the usual unfolding argument,

$$\begin{aligned}\langle FE_{s,N},G\rangle &= i_N^{-1}\cdot\int_{\Gamma_{g,0}(N)\backslash\mathbb{H}_g} F(Z)\cdot E_{s,N}(Z)\cdot\overline{G(Z)}\cdot(\det Y)^{k-g-1}\,dXdY\\ &= i_N^{-1}\cdot\int_{\mathcal{C}_N\backslash\mathbb{H}_g} F(Z)\cdot\overline{G(Z)}\cdot(\det Y)^{k-g-1+s}\cdot(\det v)^{-s}\,dXdY,\end{aligned}$$

where as before $\mathcal{C}_N$ denotes the subgroup of $\Gamma_{g,0}(N)$ consisting of all those matrices with $(0,\dots 0,1)$ as last row. As a fundamental domain for the action of $\mathcal{C}_N$ on $\mathbb{H}_g$, we can choose

$$\left\{\left(\begin{array}{cc}\tau & z^t\\ z & \tau'\end{array}\right)\middle|\,(\tau,z)\in\mathcal{F}_N,\,v'>v^{-1}[y],\,0\le u'\le 1\right\},$$

where $\mathcal{F}_N$ is a fundamental domain for the action of $\Gamma^J_{g-1,1}(N)$ on $\mathbb{H}_{g-1}\times\mathbb{C}^{(1,g-1)}$, and where we have written τ, z, and τ' as $\tau=u+iv$, $z=x+iy$, and $\tau'=u'+iv'$, respectively.

Inserting the Fourier-Jacobi expansions of F and G

$$\begin{aligned}F(Z) &= \textstyle\sum_{m\ge1}\psi_m(\tau,z)e^{2\pi im\tau'},\\ G(Z) &= \textstyle\sum_{m\ge1}\phi_m(\tau,z)e^{2\pi im\tau'},\end{aligned}$$

we obtain for $\mathrm{Re}(s)>g+1$, making similar calculations as in [Kr] or [KS],

$$\langle FE_{s,N},G\rangle=(4\pi)^{-(s+k-g)}\cdot\Gamma(s+k-g)\cdot\frac{k_N}{i_N}\cdot\sum_{m\ge1}\langle\phi_m,\psi_m\rangle\,m^{-(s+k-g)}.$$

Hence we have the identity

$$\pi^{g-k}\cdot\left\langle E^*_{s-k+g,N}F,G\right\rangle=D^*_{F,G,N}(s).$$

Using that $F,G\in S_k(\Gamma_{g,0}(N))$, Lemma 2.13 and Lemma 3.67, we get that $D^*_{F,G,N}(s)$ has a meromorphic continuation to $\mathbb{C}$ having at most simple poles at $s=k$ and $s=k-g$.

Since the Gamma function is a meromorphic function with no zeros we can conclude that $D_{F,G,N}(s)$ has a meromorphic continuation to the whole complex plane and is holomorphic exept for possible simple poles at $s=k$ and at $s=k-g$.

Using that $E^*_{s,N}(Z)$ has a simple pole with residue $N^{-g}\cdot\sum_{l|N}\mu(l)\cdot l^{-g}$ at $s=g$ and Lemma 3.66 we find

$$\begin{aligned}
\operatorname{Res}(D_{F,G,N}(s), s=k) &= \lim_{s\to k}(s-k)D_{F,G,N}(s) \\
&= \lim_{s\to k}(s-k)\frac{D^*_{F,G,N}(s)\cdot(4\pi)^s}{\Gamma(s)\cdot\Gamma(s-k+g)} \\
&= \frac{(4\pi)^k}{(k-1)!(g-1)!}\cdot\pi^{g-k}\cdot\left\langle \lim_{s\to k}(s-k)E^*_{s-k+g,N}F, G\right\rangle \\
&= \frac{(4\pi)^k\cdot\pi^{g-k}}{(k-1)!(g-1)!}\cdot\langle F,G\rangle\cdot N^{-g}\cdot\sum_{l|N}\mu(l)\cdot l^{-g}.
\end{aligned}$$

Since $D_{F,G,N}(s)$ has at most a simple pole at the point $s=k-g$ and since the Gamma function has a simple pole at zero and no zeros we obtain

$$\begin{aligned}
\operatorname{Res}(D_{F,G,N}(s), s=k-g) &= \lim_{s\to k-g}(s-(k-g))D_{F,G,N}(s) \\
&= \lim_{s\to k-g}(s-(k-g))\frac{D^*_{F,G,N}(s)\cdot(4\pi)^s}{\Gamma(s)\cdot\Gamma(s-k+g)} = 0,
\end{aligned}$$

i.e., $D_{F,G,N}(s)$ is holomorphic at $s=k-g$.
Now it remains to show the functional equation for $D_{F,G,N}(s)$.
Using Lemma 3.66 (with $g-k+s$ instead of s) we find

$$\begin{aligned}
&D^*_{F,G,N}(2k-g-s) = \pi^{g-k}\cdot\left\langle E^*_{g-(g-k+s),N}F, G\right\rangle \\
&\quad = N^{2s-2k}\cdot\sum_{l_1|N}\mu\left(\frac{N}{l_1}\right)\cdot l_1^{-2s+2k-g}\sum_{l_2|l_1} l_2^{2(s-k+g)}\cdot\pi^{g-k}\cdot\left\langle\iota\left(E^*_{s-k+g,l_2}\right)F, G\right\rangle,
\end{aligned}$$

where ι denotes the inclusion map. To prove the functional equation we thus have to show that

$$\pi^{g-k}\cdot\langle\iota(E_{s-k+g,l})F, G\rangle = D_{F,G,N,l}(s).$$

Therefore we have to calculate scalar products of the type

$$\langle\iota(E_{s,l})F, G\rangle \qquad (l|N),$$

which are clearly well-defined.
Denoting by ι^* the adjoint of the map ι we find

$$\langle\iota(E_{s,l})F, G\rangle_{\Gamma_{g,0}(N)} = \left\langle\iota(E_{s,l}), \bar{F}G\right\rangle_{\Gamma_{g,0}(N)} = \left\langle E_{s,l}, \iota^*(\bar{F}G)\right\rangle_{\Gamma_{g,0}(l)}.$$

It is well known that

$$\iota^*(\bar{F}G) = \sum_j (\bar{F}G)|\gamma_j,$$

where γ_j runs through a set of representatives of $\Gamma_{g,0}(N)\backslash\Gamma_{g,0}(l)$. Thus we have

$$\langle \iota(E_{s,l})F, G\rangle = i_l^{-1}\cdot \sum_j \int_{\Gamma_{g,0}(l)/\mathbb{H}_g} E_{s,l}(Z)\cdot F|\gamma_j(Z)\cdot \overline{G|\gamma_j(Z)}\cdot (\det Y)^{k-(g+1)}\, dXdY.$$

Using the usual unfolding argument we find

$$\langle \iota(E_{s,l})F, G\rangle = i_l^{-1}\cdot \sum_j \int_{\mathcal{C}_l\backslash\mathbb{H}_g} F|\gamma_j(Z)\cdot \overline{G|\gamma_j(Z)}\cdot (\det v)^{-s}\cdot (\det Y)^{k-(g+1)+s}\, dXdY.$$

As a fundamental domain for the action of $\mathcal{C}_l$ on $\mathbb{H}_g$, we may choose

$$\left\{ \left(\begin{array}{cc} \tau & z^t \\ z & \tau' \end{array}\right) \middle| \, (\tau, z)\in\mathcal{F}_l,\, v' > v^{-1}[y],\, 0\le u'\le 1, \right\},$$

where $\mathcal{F}_l$ is given in Definition 3.69, and where we have written $\tau = u+iv$, $z = x+iy$, and $\tau' = u'+iv'$, respectively. .
Inserting the Fourier-Jacobi expansions of $F|\gamma_j(z)$ and $G|\gamma_j(z)$

$$F|\gamma_j(z) = \sum_{m\ge 1}\phi_{m,\gamma_j}(\tau,z)e^{2\pi i m\tau'},$$

$$G|\gamma_j(z) = \sum_{m\ge 1}\psi_{m,\gamma_j}(\tau,z)e^{2\pi i m\tau'},$$

we find

$$\langle \iota(E_{s,l})F, G\rangle = i_l^{-1}\cdot \int_{\mathcal{F}_l}\int_{\substack{v'>v^{-1}[y]\\ 0\le u'\le 1}} \sum_{m,n\ge 1}\phi_{m,\gamma_j}(\tau,z)\cdot\overline{\psi_{n,\gamma_j}(\tau,z)}$$

$$\times e^{-2\pi(m+n)v'}\cdot e^{2\pi i(m-n)u'}\cdot (\det v)^{k-(g+1)}\cdot\left(\det\left(v'-v^{-1}[y]\right)\right)^{k-(g+1)+s}\; du'dv'dudvdxdy.$$

The integrals over v' and u' can be evaluated exactly as before and give the value

$$\frac{1}{i_l}\cdot(4\pi)^{-(s+k-g)}\cdot\Gamma(s+k-g)\cdot\sum_{m\ge 1}\left(\sum_j\int_{\mathcal{F}_l}\phi_{m,\gamma_j}(\tau,z)\cdot\overline{\psi_{m,\gamma_j}(\tau,z)}\cdot(\det v)^{k-(g+1)}\right.$$
$$\left.\times e^{-4\pi m v^{-1}[y^t]}\, dudvdxdy\right) m^{-(s+k-g)}$$

for $\mathrm{Re}(s) > g+1$.
Thus we have the identity

$$D^*_{F,G,N,l}(s) = \pi^{g-k}\cdot\left\langle\iota(E^*_{s-k+g,l})F, G\right\rangle.$$

Now one can show with the same arguments as before that $D^*_{F,G,N,l}(s)$ has a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s=k$. Moreover the functional equation follows directly. $\square$

To prove Theorem 3.60 we only need the case $F = G$. Clearly $D_{F,F}(s)$ and $D_{F,F,N,l}(s)$ have non-negative coefficients. Thus a classical Theorem of Landau says that they must have the first real singularity at their abscissa of convergence. Thus they converge for $\mathrm{Re}(s) > k$.
We now need the following modified version of Landau's Hauptsatz (cf [SS])

Theorem 3.73 *Suppose $Z(s) = \sum_{n\geq 1} c(n)n^{-s}$ and $\eta_i(s) = \sum_{n\geq 1} b_i(n)n^{-s}$, $1 \leq i \leq l$ ($l \in \mathbb{N}$) are Dirichlet series with non-negative coefficients which converge for Re$(s) > \sigma_0$, have a meromorphic continuation to $\mathbb{C}$ with finitely many poles and satisfy a functional equation*

$$Z^*(\delta - s) = \sum_{i=1}^{l} \pm\eta_i^*(s),$$

where

$$\begin{aligned} Z^*(s) &= A^{-s} \cdot \textstyle\prod_{j=1}^{J} \Gamma(a_j s + b_j) \cdot Z(s), \\ \eta_i^*(s) &= A_i^{-s} \cdot \textstyle\prod_{j=1}^{J} \Gamma(a_j s + b_j) \cdot \eta_i(s), \end{aligned}$$

where A , $A_i > 0$, $J \in \mathbb{N}$, $a_j > 0$, $b_j \in \mathbb{R}$. Suppose that

$$\kappa := (2\sigma_0 - \delta) \cdot \sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$

Then

$$\sum_{n\leq x} c(n) = \sum_{\textit{all poles}} Res\left(\frac{\zeta(s)}{s} x^s\right) + O_\eta(x^\eta),$$

for any $\eta > \eta_0 := (\delta + \sigma_0(\kappa - 1))/(\kappa + 1)$.

Proof. As in [K2] we write the functional equation asymmetrically as

$$Z(\delta - s) = A^{\delta-s} \cdot \prod_{j=1}^{J} \Gamma(a_j s + b_j)/\Gamma(-a_j s + a_j\delta + b_j) \sum_{i=1}^{l} \pm A_i^{-s} \cdot \eta_i(s) \qquad (\forall\, s \in \mathbb{C})\,.$$

If we use that $\Gamma(x) \cdot \Gamma(1 - x) = \pi \cdot \sin(\pi x)^{-1}$, then Theorem 3.73 follows directly from the version of Sato and Shintani (cf. [SS]). $\square$

Now we want to use Theorem 3.73 with

$$Z(s) = D_{F,F,N}(s), \quad \eta_i(s) = D_{F,F,N,p(i)}(s) \cdot N^{-2k} \cdot (p(i))^{2k-g} \cdot (q(i))^{-2k+2g},$$

where $(p(i), q(i))$ runs through the elements (l_1, l_2) with $l_1|N$, $l_2|l_1$ and $\mu\left(\frac{N}{l_1}\right) \neq 0$. Moreover $\sigma_0 = k$, $\delta = 2k - g$, $A = 4\pi$, $A_i = 4\pi(p(i))^2(q(i))^{-2}N^{-2}$, $J = 2$, $a_1 =$

$a_2 = 1$, $b_1 = 0$, $b_2 = g - k$.
Hence $\kappa = 2g - \frac{1}{2} > 0$, $\eta_0 = k - \frac{2g}{4g+1}$. Thus all conditions of Theorem 3.73 are satisfied. Therefore we have

$$\sum_{n \le x} c(n) = cx^k + O_\eta\left(x^{k - \frac{2g}{4g+1} + \epsilon}\right) \qquad (\forall\, \epsilon > 0)\,,$$

where $c = Res_{s=k} \frac{D_{F,F}(s)}{s}$.
Therefore we get, taking $x = m$ and $x = m-1$ and substracting, that

$$c(m) \ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon}.$$

Writing $\zeta_N(s)^{-1} := \sum_{n \ge 1} \mu_N(n) n^{-s}$ and using that the coefficients of $\zeta_N(s)^{-1}$ are bounded by 1, we find

$$\begin{aligned} \| \phi_m \|^2 &= \sum_{d^2 | m} \mu_N(d) \cdot d^{2k-2g} \cdot c\left(\frac{m}{d^2}\right) \ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon} \cdot \sum_{d \ge 1} d^{-2g(1 - \frac{2}{4g+1})} \\ &\ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon}. \end{aligned}$$

□

3.2.3 Final estimates

In this section we want to collect our results and give the final estimates. Therefore we let $F \in S_k(\Gamma_{g,0}(N))$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix ($g \in \mathbb{N}$, $g \ge 2$). First we have

Theorem 3.74 *Let $g \ge 2$, $k \ge g+1$. Then*

$$a(T) \ll_{\epsilon,F} (m_{g-1}(T))^{1/2} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where $m_{g-1}(T)$ is defined as in (2.1), and where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 3.75 *Let $g \ge 2$, $k \ge g+1$. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2 - 1/(2g) + \epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Proof. Let us write T as $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix}$. Then clearly $a(T)$ is the (n,r)-th Fourier coefficient of the m-th Fourier-Jacobi coefficient of F. Thus we get, using Theorem 3.41 with $g-1$ instead of g, and Lemma 3.58

$$\begin{aligned} a(T) &\ll_{\epsilon,F} \left(1 + \frac{(\det T)^{(g-1)/2+\epsilon}}{(\det m)^{g/2}}\right)^{1/2} \cdot \frac{(\det T)^{k/2 - g/4 - 1/4}}{(\det m)^{k/2 - g/4 - 1/2}} \cdot (\det m)^{k/2} \\ &= \left((\det m)^{g/2} + (\det T)^{(g-1)/2+\epsilon}\right)^{1/2} \cdot (\det T)^{k/2 - g/4 - 1/4} \cdot (\det m)^{1/2}. \end{aligned}$$

Now we may assume that $\det m = m_{g-1}(T)$.
Indeed, otherwise replace T by $T[U]$, with $U \in GL_g(\mathbb{Z})$ such that $T[U]|_{g-1} = m_{g-1}(T)$, which changes neither the left- nor the right-hand side of the estimate in Theorem 3.74 (here we use that $\Gamma_{g,0}(N)$ contains all matrices of the form $\begin{pmatrix} U & 0 \\ 0 & U^{-t} \end{pmatrix}$ $\forall U \in GL_g(\mathbb{Z})$, and therefore $|a(T[U])| = |a(T)|$ due to Lemma 2.12). Therefore we obtain

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/2+\epsilon} \cdot (m_{g-1}(T))^{1/2},$$

which proves Theorem 3.74. Corollary 3.75 follows directly from Theorem 3.74, if we use the well known estimate $m_{g-1}(T) \ll_g (\det T)^{1-1/g}$ (cf. Remark 2.8). □

For $g = 2$ and $g = 3$, we can obtain the following improvements

Theorem 3.76 *Let $g = 2$ and $k \geq 3$. Then*

$$a(T) \ll_{\epsilon,F} (min(T))^{5/18+\epsilon} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 3.77 *Let $g = 2$ and $k \geq 3$. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-13/36+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Proof. The proof follows with the same arguments as in the proof of Theorem 3.74, using Theorem 3.41 and Theorem 3.60. □

Theorem 3.78 *Let $g = 3$ and let $k \geq 8$ be an even integer. Then*

$$a(T) \ll_{\epsilon,F} (min(T))^{-3/13+\epsilon} \cdot (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0).$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Corollary 3.79 *Let $g = 3$ and let $k \geq 8$ be an even integer. Then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Proof. The proof follows directly from Theorem 3.57 and Theorem 3.60 similar as in the case of the full Siegel modular group (cf. [Br], pp. 79-85). □

Moreover, for $g = 2$ we obtain in the same way as in the case of the full Siegel modular group (cf. [K7]) the following estimate on average

Corollary 3.80 *Let $g = 2$ and let $k \geq 3$ be an even integer. Then*

$$\sum_{\{T>0, tr\,(T)=N\}} |a(T)|^2 \ll_{\epsilon,F} N^{k-1/2+\epsilon} \quad (\epsilon > 0),$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

3.3 Subgroups of finites index and open questions

In this section we want to sketch how the results of the above sections can be generalized to subgroups Γ of Γ_g of finite index that contain all matrices of the form $\begin{pmatrix} U & 0 \\ 0 & U^{-t} \end{pmatrix}$, where $U \in GL_g(\mathbb{Z})$. This restriction is only needed for the final estimates in order to be allowed to replace T by $T[U]$. One again starts with estimates of the Fourier coefficients of Jacobi forms, which are again defined in a way such that the Fourier coefficients of Siegel cusp forms belong to this class of cusp forms. The method is the same as before: One first constructs Jacobi-Poincaré series, which can be shown to be cusp forms if $k > g+2$, and estimates their Fourier coefficients. The calculations (in particular the estimates of certain generalized Kloosterman sums) are more complicated than in the case of $\Gamma_{1,g,0}(N)^J$ but nevertheless straighforward. Afterwards one can regard the case $k = g+2$, again using Hecke's trick. Proceeding as in Section 3.1 and Section 3.2, one gets an analogous result as in Theorem 3.74, using the same estimate for the Petersson norm $\| \phi_m \|$ as in Lemma 3.58 and 3.75.

Open questions

- How is it possible to extend the range of the estimate in Theorem 3.74 regarding the weight k, i.e, enlarge the range of the estimate in Theorem? 3.41.
- How is it possible to obtain an estimate of the coefficients of cusp forms of the quality of Theorem 3.74 for general g? Probably for this one needs the general Langland's theory of Eisenstein series on symplectic groups.

Chapter 4

Lifting maps

In this chapter we want to construct a lifting map from the vector space $J^{cusp}_{k+\frac{g+1}{2},m}$ to a certain subspace of $S_{2k}\left(\frac{1}{2}\det(2m)\right)$ and vice versa under certain conditions on D_0, k, and g such that both mappings are adjoint with respect to the Petersson scalar products.

4.1 The generalized genus character and geodesic cycle integrals

First of all we want to recall some basic facts about quadratic forms. For details we refer the reader to [GKZ]. We have the following

Definition 4.1 *For $a, b, c \in \mathbb{Z}$ let us define the integral binary quadratic form*

$$[a, b, c](x, y) := ax^2 + bxy + cy^2.$$

The group $SL_2(\mathbb{Z})$ acts on these forms by

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) := [a, b, c](\alpha x + \beta y, \gamma x + \delta y) \quad (x,\, y \in \mathbb{Z}).$$

Let $\Delta \in \mathbb{Z}$ be a discriminant (i.e., $\Delta \neq 0$, $\Delta \equiv 0, 1 \pmod 4$). Let us denote by $\mathcal{D}_\Delta$ the set of integral binary quadratic forms with discriminant Δ. Then $SL_2(\mathbb{Z})$ acts on $\mathcal{D}_\Delta$. Furthermore let us denote for $l \in \mathbb{N}$ by $\mathcal{D}_{l,\Delta} \subset \mathcal{D}_\Delta$ the set of all quadratic forms with the additional condition that $a \equiv 0 \pmod l$. Moreover let us define for integers $\rho \pmod{2l}$ with $\Delta \equiv \rho^2 \pmod{4l}$ the set

$$\mathcal{D}_{l,\Delta,\rho} := \left\{[a, b, c] \in \mathcal{D}_\Delta \,|\, a \equiv 0 \pmod l, b \equiv \rho \pmod{2l}\right\}.$$

Remark 4.2 *The sets* $\mathcal{D}_{l,\Delta}$ *and* $\mathcal{D}_{l,\Delta,\rho}$ *are* $\Gamma_0(l)$ *invariant.*

Remark 4.3 *We have the decomposition*

$$\mathcal{D}_{l,\Delta} = \bigcup_{\substack{\rho(2l) \\ \Delta\equiv\rho^2(4l)}} \mathcal{D}_{l,\Delta,\rho}.$$

Now we can define a generalized genus character.

Definition 4.4 *Let* l *be a positive integer,* D_0 *be a fundamental discriminant and* Δ *be a discriminant that divides* D_0 *such that both* D_0 *and* Δ/D_0 *are squares* $\pmod{4l}$. *Then we define for* $Q = [al, b, c] \in \mathcal{D}_{l,\Delta}$ *:*

$$\chi_{D_0}(Q) := \begin{cases} \left(\frac{D_0}{n}\right) & \text{if } (a,b,c,D) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\left(\frac{D_0}{n}\right)$ *denotes the Kronecker symbol. Here* n *is an integer coprime to* D_0 *represented by the form* $[al_1, b, cl_2]$ *for some decomposition* $l = l_1 l_2$ *,* $l_i > 0$ $(i = 1, 2)$.

Remark 4.5 *Such an* n *always exists and the value of* $\left(\frac{D_0}{n}\right)$ *is independent of the choice of* l_1, l_2, *and* n.

We have the following

Theorem 4.6 *The function* χ_{D_0} *is* $\Gamma_0(l)-$*invariant and has the following properties:*

P1 (Multiplicativity):

$$\chi_{D_0}([al, b, c]) = \chi_{D_0}([a_1 l, b, ca_2])\chi_{D_0}([a_2 l, b, ca_1]) \qquad \text{if } a = a_1 a_2, (a_1, a_2) = 1.$$

P2 (Invariance under the Fricke involution):

$$\chi_{D_0}([al, b, c]) = \chi_{D_0}([cl, -b, a]).$$

P3 (Explicit formula):

$$\chi_{D_0}([al, b, c]) = \left(\frac{D_1}{l_1 a}\right)\left(\frac{D_2}{l_2 c}\right)$$

for any splitting $D_0 = D_1 D_2$ *of* D_0 *into coprime fundamental discriminants and* $l = l_1 l_2$ *of* l *into positive factors such that* $(D_1, l_1 a) = (D_1, l_2 c) = 1$, $\chi_{D_0}([al, b, c]) = 0$ *if no such splitting exists.*

Definition 4.7 *Let $l \in \mathbb{N}$, $\rho \in \mathbb{Z}/2l\mathbb{Z}$, $\Delta > 0$ be a discriminant satisfying $\Delta \equiv \rho^2 \pmod{4l}$. Let D_0 be a fundamental discriminant dividing Δ such that both D_0 and Δ/D_0 are squares $\pmod{4l}$. Let k be an integer > 1. Then we define*

$$f_{k,l,\Delta,\rho,D_0}(z) := \sum_{Q \in \mathcal{D}_{l,\Delta,\rho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \qquad (z \in \mathbb{H}).$$

Definition 4.8 *We define by $S_{2k}(l)^- \subset S_{2k}(l)$ the space of cusp forms on $\Gamma_0(l)$ with eigenvalue -1 under the Fricke involution*

$$f(z) \mapsto (-lz^2)^{-k} f(-1/(lz)).$$

We know from [GKZ]

Lemma 4.9 *The series $f_{k,l,\Delta,\rho,D_0}(z)$ is absolutely and locally uniformly convergent for $k > 1$ and is an element of $S_{2k}(l)^-$.*

For the following, we need the Fourier expansion of $f_{k,l,\Delta,\rho,D_0}(z)$.

Lemma 4.10 *The Fourier expansion of $f_{k,l,\Delta,\rho,D_0}(z)$ $(k \geq 1)$ is given by*

$$f_{k,l,\Delta,\rho,D_0}(z) = \sum_{m=1}^{\infty} c^{\pm}_{k,l}(m,\Delta,\rho,D_0) e^{2\pi i m z},$$

where

$$c^{\pm}_{k,l}(m,\Delta,\rho,D_0) := c_{k,l}(m,\Delta,\rho,D_0) + (-1)^{k+1} c_{k,l}(m,\Delta,-\rho,D_0),$$

where $\pm = (\pm 1)^{k+1}$, where

$$c_{k,l}(m,\Delta,\rho,D_0) = i^k \cdot (-1)^{-\frac{1}{2}} \cdot \frac{(2\pi)^k}{(k-1)!} \cdot (m^2/\Delta)^{\frac{k-1}{2}} \cdot \Big[|D_0|^{-\frac{1}{2}} \cdot \epsilon_l(m,\Delta,\rho,D_0)$$
$$+ i^{k+1} \cdot \pi \cdot \sqrt{2} \cdot \left(m^2/\Delta\right)^{\frac{1}{4}} \cdot \sum_{a \geq 1} (la)^{-\frac{1}{2}} \cdot S_{la}(m,\Delta,\rho,D_0) \cdot J_{k-1/2}\left(\frac{\pi m \sqrt{\Delta}}{la}\right) \Big].$$

Here

$$\epsilon_l(m,\Delta,\rho,D_0) := \begin{cases} \left(\frac{D_0}{m/f}\right) & \text{if } \Delta = D_0^2 \cdot f^2 \ (f > 0),\ f|m,\ D_0 f \equiv \rho \pmod{2l} \\ 0 & \text{otherwise} \end{cases},$$

$$S_{la}(m,\Delta,\rho,D_0) = \sum_{\substack{b(2la) \\ b \equiv \rho(2l) \\ b^2 \equiv \Delta(4la)}} \chi_{D_0}\left(\left[al, b, \frac{b^2-\Delta}{4la}\right]\right) \cdot e_{2la}(mb),$$

and $J_{k-1/2}(t)$ is the Bessel function of order $k - 1/2$.

Moreover we need the relation of these functions to cycle integrals of modular forms:

Definition 4.11 *For $f \in S_{2k}(l)$ and $Q = [a,b,c] \in \mathcal{D}_{l,\Delta,\rho}$ we set*

$$r_{k,l,Q}(f) := \int_{\gamma_Q} f(z) \cdot Q(z,1)^{k-1}\, dz,$$

where γ_Q is the image in $\Gamma_0(l)\backslash\mathbb{H}$ of the semicircle $a|z|^2 + bx + c = 0$ ($x = Re(z)$), orientated from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$ if $a \neq 0$ or if $a = 0$ of the vertical line $bx + c = 0$, orientated from $-\frac{c}{b}$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-\frac{c}{b}$ if $b < 0$.

Lemma 4.12 *The above given definition of $r_{k,l,Q}(f)$ makes sense, i.e., the integral is invariant with respect to the subgroup of $\Gamma_0(l)$ perserving Q, and depends only on the $\Gamma_0(l)$ equivalence class of Q.*

Definition 4.13 *Define*

$$r_{k,l,\Delta,\rho,D_0}(f) := \sum_{Q \in \mathcal{D}_{l,\Delta,\rho}/\Gamma_0(l)} \chi_{D_0}(Q) \cdot r_{k,l,Q}(f).$$

Then the following holds

Theorem 4.14 *For $f \in S_{2k}(l)^-$ we have*

$$\langle f, f_{k,l,\Delta,\rho,D_0} \rangle = \pi \cdot \binom{2k-2}{k-1} \cdot 2^{-2k+2} \cdot \Delta^{-k+1/2} \cdot r_{k,l,\Delta,\rho,D_0}(f).$$

4.2 Construction of lifting maps

Let $n_0 \in \mathbb{Z}$, $r_0 \in \mathbb{Z}^{(1,g)}$, m be a positive definite symmetric half-integral $g \times g$ matrix, and $D_0 := -\det 2 \begin{pmatrix} n_0 & \frac{r_0}{2} \\ \frac{r_0^t}{2} & m \end{pmatrix} < 0$. For this section let us make the following

Assumptions:

1. $g \equiv 1 \pmod 8$,
2. $k \geq \frac{g+3}{2}$,
3. D_0 is fundamental, $\frac{1}{2}\det(2m)$ is odd, and $(\det(2m), D_0) = 1$,
4. D_0 is a square $\pmod{\frac{1}{2}\det(2m)}$.

Remark 4.15 *1. If $g = 1$, claim 4. follows automatically. Moreover in this case instead of condition 3 it suffices to assume that D_0 is a fundamental discriminant.*

2. The integer $\det(2m)$ *is even, thus* $\frac{1}{2}\det(2m)$ *is an integer.*

3. We have $D_0 \equiv 1 \pmod 4$*.*

4. The integer D_0 *is a square* $\pmod{2\det(2m)}$*.*

Proof. 1. If $g = 1$ we have the identity $D_0 = r_0^2 - 4n_0 m$, which is clearly a square $\pmod m$. Moreover in this case the statements of this chapter are proved in [GKZ]. Here assumption 4. and the last two claims of assumption 3. are not needed.
2. The claim follows directly from the Laplace development if we use that g is odd and m is symmetric and half-integral.
3. From the Laplace development it follows readily that D_0 is congruent $0, 1 \pmod 4$. Thus the claim follows directly from 2., since $(D_0, \det(2m)) = 1$.
4. Since $(D_0, \det(2m)) = 1$ and $\frac{1}{2}\det(2m)$ is odd the claim follows from 3. and assumption 4. □

Let us show that quadratic forms with the above conditions exist. For this we have to show

Remark 4.16 *For every* $g \equiv 1 \pmod 8$ *a positive definite half-integral* $g \times g$ *matrix* m *exists, such that* $\frac{1}{2}\det(2m)$ *is odd,* $(D_0, \det(2m)) = 1$*,* D_0 *is fundamental and* D_0 *is a square* $\pmod{\frac{1}{2}\det(2m)}$*.*

Proof. We may for example choose

$$2m := E_8^{\oplus \frac{g-1}{8}} \oplus 2N,$$

where N is an odd integer, where

$$E_8 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

and where

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

for square integral matrices.
Then we get, using that $\det E_8 = 1$ (cf. [KKo]).

$$\det(2m) = 2N.$$

Then clearly $\frac{1}{2}\det(2m) = N$ is odd per assumption.
Moreover, writing $r_0 = (r'_1, \ldots, r'_g)$, we obtain from the Jacobi decomposition (cf. Remark 2.19)

$$D_0 = -2\det(2m)\cdot n_0 + (2m)^*[r_0^t] \equiv r_g'^2 \pmod N,$$

where $(2m)^*$ denotes the adjoint of the matrix $2m$. Now the claim follows if we choose r'_g such that $(r'_g, N) = 1$. $\square$

Now we want to define the desired maps (see also Definitions 1.10 and 1.11 in the Introduction). Let us start with the map $\mathcal{S}_{D_0,r_0}(\phi)(w)$ $(w \in \mathbb{H})$ that is defined for a Jacobi cusp form ϕ in terms of a Fourier expansion in $e^{2\pi i w}$, where the Fourier coefficients are certain sums of special values of the Fourier coefficients of ϕ.

Definition 4.17 *For $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}$ we define*

$$\mathcal{S}_{D_0,r_0}(\phi)(w) := 2^{1-g}\cdot\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D_0}{d}\right)\cdot d^{k-1}\cdot c_\phi\left(\frac{n^2}{d^2}n_0, \frac{n}{d}r_0\right)\right)e^{2\pi i n w} \qquad (w \in \mathbb{H}),$$

where $c_\phi(n,r)$ is the $(n,r)-$th Fourier coefficient of ϕ.

Next we define for $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$ the map $\mathcal{S}^*_{D_0,r_0}(f)(\tau,z)$ $((\tau,z) \in \mathbb{H}\times\mathbb{C}^{(g,1)})$ as a Fourier expansion in $e^{2\pi i\tau}$ and $e^{2\pi i z}$, where the Fourier coefficients are certain cycle integrals.

Definition 4.18 *For $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$ we define*

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z) := \left(\frac{i}{\det(2m)}\right)^{k-1}\cdot \sum_{\substack{n\in\mathbb{Z}\\ r\in\mathbb{Z}^{(1,g)}\\ 4n>m^{-1}[r^t]}} r_{k,\frac{1}{2}\det(2m),D_0D,r_0(2m)^*r^t,D_0}(f)e^{2\pi i(n\tau+rz)},$$

where $(\tau,z)\in\mathbb{H}\times\mathbb{C}^{(g,1)}$, and where $D := -\det 2\begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m\end{pmatrix}$.

Later on (cf. Lemma 4.20) we show that this definition is allowed. But first we want to state the main Theorem of this chapter.

Theorem 4.19 *If $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}$, then the function $\mathcal{S}_{D_0,r_0}(\phi)(w)$ is an element of $S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$.*

If $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$, then the function $\mathcal{S}^*_{D_0,r_0}(f)(\tau,z)$ is an element of $J^{cusp}_{k+\frac{g+1}{2},m}$. The maps

$$\begin{aligned}\mathcal{S}_{D_0,r_0}: &\quad J^{cusp}_{k+\frac{g+1}{2},m} \to S_{2k}(\tfrac{1}{2}\det(2m))^- \\ \mathcal{S}^*_{D_0,r_0}: &\quad S_{2k}(\tfrac{1}{2}\det(2m))^- \to J^{cusp}_{k+\frac{g+1}{2},m}\end{aligned}$$

are adjoint maps with respect to the Petersson scalar products, i.e., we have for all $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$ and for all $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}$

$$\left\langle \mathcal{S}_{D_0,r_0}(\phi), f\right\rangle = \left\langle \phi, \mathcal{S}^*_{D_0,r_0}(f)\right\rangle .$$

Before we can prove Theorem 4.19 we have to show

Lemma 4.20 *Definition 4.18 is allowed, i.e., $DD_0 > 0$ is a discriminant, D_0 is a fundamental discriminant, D and D_0 are both squares $\pmod{2\det(2m)}$, and*

$$DD_0 \equiv \left(r_0(2m)^* r^t\right)^2 \pmod{2\det(2m)}. \tag{4.1}$$

Proof. Since D_0 and D are negative, D_0 is a square $\pmod{2\det(2m)}$ and a fundamental discriminant, $D \equiv 0, 1 \pmod 4$ (which follows from the Laplace development) and $(2\det(2m), D_0) = 1$ it is enough to show congruence (4.1). For this we need some knowledge about quadratic forms over the p-adic ring $\mathbb{Z}_p$.

Lemma 4.21 *Let p be an arbitrary prime.*

1. *If $p \neq 2$, then there exists a matrix $U \in GL_g(\mathbb{Z}_p)$ and there exist $m_1, \ldots, m_g \in \mathbb{Z}_p$ such that*

$$(2m)[U] = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_g \end{pmatrix}$$

   and

$$(2m[U])^* = \begin{pmatrix} \prod_{\substack{i=1\\ i\neq 1}}^g m_i & & \\ & \ddots & \\ & & \prod_{\substack{i=1\\ i\neq g}}^g m_i \end{pmatrix}.$$

2. *If $p = 2$, then there exists a matrix $U \in GL_g(\mathbb{Z}_2)$ and there exist $M_1, \ldots, M_r \in \left\{2^\nu l, \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\}$, where $\nu \in \mathbb{N}$, $\nu \geq 1$, $l \in \{1, 3, 5, 7\}$ and $r \in \mathbb{N}$ such that*

$$(2m)[U] = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_r \end{pmatrix}$$

and

$$(2m[U])^* = \det(2m)\begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_r \end{pmatrix}.$$

Here for $1 \le i \le r$ *we define*

$$N_i := \begin{cases} \frac{1}{2l} & \text{if } M_i = 2l \\ \frac{1}{3}\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) & \text{if } M_i = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) & \text{if } M_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \end{cases}.$$

Proof. Lemma 4.21 follows directly from the theory of quadratic forms over $\mathbb{Z}_p$ (see [Ca]). □

Remark 4.22 *In Lemma 4.21 1.* $(\prod_{i=1}^g m_i, p) = 1$ *if* $p \nmid \det(2m)$ *and* $p|m_i$ *for exactly one* m_i $(1 \le i \le \nu)$ *if* $p|\det(2m)$.
Moreover in 2., there occurs exactly one block $2^\nu l$ *and* $\nu = 1$.

Proof. Let us first assume $p \neq 2$.
Clearly $p \nmid \prod_{i=1}^g m_i$ if $p \nmid \det(2m)$, since $(\det U, p) = 1$.
If $p|\det(2m)$, then $p|\prod_{i=1}^g m_i$, thus $p|m_i$ for at least on m_i, without loss of generality we may assume $p|m_1$.
Setting

$$\begin{aligned} \begin{pmatrix} \tilde{n_0} & \frac{\tilde{r_0}}{2} \\ \frac{\tilde{r_0}^t}{2} & m \end{pmatrix} &:= \begin{pmatrix} n_0 & \frac{r_0}{2} \\ \frac{r_0^t}{2} & m \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\right], \\ \tilde{r_0} &:= (r_1', \ldots, r_g'), \end{aligned}$$

we see from the Jacobi decomposition that $\tilde{D_0} := \det 2\begin{pmatrix} \tilde{n_0} & \frac{\tilde{r_0}}{2} \\ \frac{\tilde{r_0}^t}{2} & m \end{pmatrix}$ equals

$$\sum_{j=1}^g \prod_{\substack{i=1 \\ i\neq j}}^g m_i\, {r_j'}^2 - 4n_0 \prod_{i=1}^g m_i \equiv \left(\prod_{i=2}^g m_i\right) {r_1'}^2 \pmod p.$$

Since $(D_0, \det(2m)) = 1$ and $\tilde{D_0} = (\det U)^2 \cdot D_0$ with $(\det U, p) = 1$, we have $p \nmid \tilde{D_0}$. Thus $p \nmid \prod_{i=2}^g m_i$ as claimed.
For the same reasons we can conclude that exactly one block $2^\nu l$ occurs if $p = 2$ and that $\nu = 1$.

Proof of Lemma 4.20. Since $\frac{1}{2}\det(2m)$ is odd and $DD_0 \equiv 0$ or $1 \pmod 4$, which are the only squares $\pmod 4$, we have to show

$$DD_0 \equiv \left(r(2m)^* r_0^t\right)^2 \pmod{p^\nu}, \tag{4.2}$$

where p is a prime that divides $\det(2m)$ of $\nu-$th order. Now our aim is to use Lemma 4.21 and replace $2m$ by a diagonal matrix and a block-diagonal matrix if $p \neq 2$ and $p = 2$, respectively.
Let $U \in GL_g\ (\mathbb{Z}_p)$. We set

$$\begin{aligned} 2\tilde{m} &= 2m[U], \\ \tilde{r}_0 &= r_0 U, \\ \tilde{r} &= rU. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{D}_0 &= \det 2 \begin{pmatrix} n_0 & \frac{r_0}{2} \\ \frac{r_0^t}{2} & m \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \right] = (\det U)^2 \cdot D_0, \\ \tilde{D} &= \det 2 \begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \right] = (\det U)^2 \cdot D, \\ (2\tilde{m})^* &= (\det U)^2 \cdot U^{-1} (2m)^* U^{-t}, \\ \det(2\tilde{m}) &= (\det U)^2 \cdot \det(2m). \end{aligned}$$

Therefore we have, using that p and $\det U$ are coprime since $U \in GL_g(\mathbb{Z}_p)$, that congruence (4.2) is equivalent to

$$\begin{aligned} \tilde{D}_0 \cdot \tilde{D} &= (\det U)^4 \cdot D \cdot D_0 \equiv (\det U)^4 \cdot (rU \cdot U^{-1} (2m)^* U^{-t} \cdot U^t r_0^t)^2 \\ &\equiv (\tilde{r} (2\tilde{m})^* \tilde{r}_0^t)^2 \pmod{p^\nu}. \end{aligned}$$

In the following we may replace m, r_0, and r by $\tilde{m}$, $\tilde{r}_0$, and $\tilde{r}$, respectively. In particular we replace D and D_0 by $\tilde{D}$ and $\tilde{D}_0$, respectively. This is possible since in the proof none of the restrictions given on D_0 and m at the beginning of section 4.2 is needed. We distinguish the cases $p \neq 2$ and $p = 2$.
If $p \neq 2$, due to Lemma 4.21, we may assume that $2m$ is a diagonal matrix of the form

$$2m = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_g \end{pmatrix},$$

where $m_i \in \mathbb{Z}_p$ and where p divides exactly one m_i $(1 \leq i \leq g)$ of $\nu-$th order. Without loss of generality we may assume that p divides m_1. Thus, setting $r_0 := (r_1', \ldots, r_g')$ and $r := (r_1, \ldots, r_g)$, we obtain from the Jacobi decomposition (cf. Remark 2.19) that the left-hand side of (4.2) is congruent to

$$\left(-2\det(2m)\cdot n + (r_1,\dots,r_g)\begin{pmatrix}\prod_{\substack{i=1\\i\neq 1}}^g m_i & & \\ & \ddots & \\ & & \prod_{\substack{i=1\\i\neq g}}^g m_i\end{pmatrix}\begin{pmatrix}r_1\\ \vdots\\ r_g\end{pmatrix}\right)$$
$$\times\left(-2\det(2m)\cdot n_0 + (r_1',\dots,r_g')\begin{pmatrix}\prod_{\substack{i=1\\i\neq 1}}^g m_i & & \\ & \ddots & \\ & & \prod_{\substack{i=1\\i\neq g}}^g m_i\end{pmatrix}\begin{pmatrix}r_1'\\ \vdots\\ r_g'\end{pmatrix}\right)$$
$$\equiv\left(r_1r_1'\prod_{i=2}^g m_i\right)^2 \pmod{p^\nu},$$

whereas the right-hand side of (4.2) is congruent to

$$\left((r_1,\dots,r_g)\begin{pmatrix}\prod_{\substack{i=1\\i\neq 1}}^g m_i & & \\ & \ddots & \\ & & \prod_{\substack{i=1\\i\neq g}}^g m_i\end{pmatrix}\begin{pmatrix}r_1'\\ \vdots\\ r_g'\end{pmatrix}\right)^2 \equiv \left(r_1r_1'\prod_{i=2}^g m_i\right)^2 \pmod{p^\nu}.$$

Thus (4.2) is proved for $p\neq 2$.

If $p=2$ we may assume, again using Lemma 4.21, that $2m$ is a block-diagonal matrix with blocks from the set $\left\{2l, \begin{pmatrix}2&1\\1&2\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$, where l is odd and where the type $2l$ occurs exactly once, without loss of generality at the first position. Thus, using that $\frac{1}{2}\det(2m)$ and l are odd integers, we obtain that the left-hand side of (4.2) is congruent to

$$(r_1,\dots,r_g)\begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}\begin{pmatrix}r_1\\ \vdots\\ r_g\end{pmatrix}\cdot(r_1',\dots,r_g')\begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}\begin{pmatrix}r_1'\\ \vdots\\ r_g'\end{pmatrix}$$
$$\equiv r_1'r_1 \pmod{2},$$

whereas the right-hand side of (4.2) is congruent to

$$\left((r_1,\dots,r_g)\begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}\begin{pmatrix}r_1'\\ \vdots\\ r_g'\end{pmatrix}\right)^2$$
$$\equiv r_1'r_1 \pmod{2},$$

which proves (4.2) for $p=2$. $\square$

Now we want to prove Theorem 4.19. The proof is seperated into several theorems and lemmas. First of all we define a function $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ which is a holomorphic kernel function for the map $\mathcal{S}^*_{D_0,r_0}$. That is we give the following

Definition 4.23 *Let*

$$\Omega_{k,m,D_0,r_0}(w;\tau,z) := c_{k,m,D_0} \cdot \sum_{\substack{n\in\mathbb{Z} \\ r\in\mathbb{Z}^{(1,g)} \\ 4n>m^{-1}[r^t]}} |D|^{k-1/2} \cdot f_{k,\frac{1}{2}\det(2m),D_0D,r_0(2m)^*r^t,D_0}(w) \cdot e^{2\pi i(n\tau+rz)},$$

where

$$c_{k,m,D_0} := \frac{(-2i)^{k-1}\cdot|D_0|^{k-1/2}}{\left(\frac{1}{2}\det(2m)\right)^{k-1}\cdot\pi\cdot\binom{2k-2}{k-1}}.$$

Then we can prove

Lemma 4.24 *The series $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is absolutely convergent. As a function of w it is an element of $S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$.*
Moreover we have the identity

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z) = \langle f, \Omega_{k,m,D_0,r_0}(\cdot;-\bar{\tau},-\bar{z})\rangle \qquad \left(\forall\, f\in S_k\left(\frac{1}{2}\det(2m)\right)^-\right).$$

Proof. The absolute convergence of Ω_{k,m,D_0,r_0} can easily be shown, using the Fourier expansion of $f_{k,\frac{1}{2}\det(2m),D_0D,r_0(2m)^*r^t,D_0}(w)$ given in Lemma 4.10.
Clearly $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is an element of $S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$ as a function of w (due to the absolute convergence and Lemma 4.9).
Moreover we have, using Theorem 4.14 for $f\in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$,

$$\begin{aligned}
&\langle f, \Omega_{k,m,D_0,r_0}(\cdot;-\bar{\tau},-\bar{z})\rangle \\
&= \overline{c_{k,m,D_0}} \cdot \sum_{\substack{n\in\mathbb{Z} \\ r\in\mathbb{Z}^{(1,g)} \\ 4n>m^{-1}[r^t]}} |D|^{k-1/2}\cdot\left\langle f, f_{k,\frac{1}{2}\det(2m),D_0D,r_0(2m)^*r^t,D_0}(w)\right\rangle e^{2\pi i(n\tau+rz)} \\
&= \frac{(2i)^{k-1}\cdot|D_0|^{k-1/2}}{\left(\frac{1}{2}\det(2m)\right)^{k-1}\cdot\pi\cdot\binom{2k-2}{k-1}}\cdot\pi\cdot\binom{2k-2}{k-1}\cdot 2^{-2k+2}\cdot|D_0|^{-k+1/2} \\
&\quad\times \sum_{\substack{n\in\mathbb{Z} \\ r\in\mathbb{Z}^{(1,g)} \\ 4n>m^{-1}[r^t]}} r_{k,\frac{1}{2}\det(2m),D_0D,r_0(2m)^*,D_0}(f)e^{2\pi i(n\tau+rz)} \\
&= \mathcal{S}^*_{D_0,r_0}(f)(\tau,z).\Box
\end{aligned}$$

Now our aim is to show that the function Ω_{k,m,D_0,r_0} is also a holomorphic kernel function for the map $\mathcal{S}_{D_0,r_0}$, i.e.,

$$\mathcal{S}_{D_0,r_0}(\phi)(\omega) = \langle \phi, \Omega_{k,m,D_0,r_0}(-\bar{\omega}, \cdot, \cdot)\rangle \quad \left(\forall\, \phi \in J^{cusp}_{k+\frac{g+1}{2},m}\right).$$

Since the Fourier coefficients of $\mathcal{S}_{D_0,r_0}$ are given by certain linear combinations of the Fourier coefficients of ϕ, due to the Petersson coefficient formula it is sufficient to show that $\mathcal{S}_{D_0,r_0}$ is a suitable linear combination of Poincaré series in $J^{cusp}_{k+\frac{g+1}{2},m}$. More precisely, since

$$\Omega_{k,m,D_0,r_0}(-\bar{\tau}, -\bar{z}; w) = \overline{\Omega_{k,m,D_0,r_0}(\tau, z; -\bar{w})}$$

we have to prove

Theorem 4.25 *The identity*

$$\Omega_{k,m,D_0,r_0}(w;\tau,z) = c_{k,m,D_0} \cdot \frac{i^{k-1}\cdot(2\pi)^k}{(k-1)!} \times \sum_{l\geq 1} l^{k-1}\left(\sum_{dd'=l}\left(\frac{D_0}{d}\right)\cdot d'^k \cdot P_{k+\frac{g+1}{2},m,(n_0 d'^2, r_0 d')}(\tau,z)\right) e^{2\pi i l w}, \quad (4.3)$$

holds.

Proof. The definition of the right-hand of equation (4.3) is formally allowed because

$$\frac{1}{2}\det(2m)\cdot(-4n_0 d'^2 + m^{-1}[r_0^t d]) = d'^2 \cdot D_0 < 0,$$

since $D_0 < 0$. Moreover it can easily be shown, using the Fourier expansion from Theorem 2.29 and Theorem 3.10 that the right-hand side of (4.3) is absolutely convergent.
Now the idea is to expand both sides of (4.3) in double Fourier series and compare Fourier coefficients. Using the Fourier expansion from Theorem 2.29, Theorem 3.10 and Lemma 4.10, we have to show since $g \equiv 1 \pmod 8$

$$i^{k-1}\cdot\frac{(2\pi)^k}{(k-1)!}\cdot|D|^{k-1/2}\cdot(l^2/D_0 D)^{(k-1)/2}\cdot\Bigg(|D_0|^{-\frac{1}{2}}\cdot\epsilon_{\frac{1}{2}\det(2m)}(l, D_0 D, r(2m)^* r_0^t, D_0)$$
$$+ i^{k+1}\cdot\pi\cdot\sqrt{2}\cdot(l^2/(DD_0))^{1/4}\cdot\sum_{a\geq 1}\left(\frac{1}{2}\det(2m)\right)^{-1/2}\cdot a^{-1/2}$$
$$\times S_{\frac{1}{2}\det(2m)a}(l, D_0 D, r(2m)^* r_0^t, D_0)\cdot J_{k-1/2}\left(\frac{2\pi\cdot l}{\det(2m)\cdot a}\cdot\sqrt{D_0 D}\right)\Bigg)$$

$$= i^{k-1} \cdot \frac{(2\pi)^k}{(k-1)!} \cdot l^{k-1} \cdot \sum_{d|l} \left(\frac{D_0}{d}\right) \cdot (l/d)^k \cdot \left(\delta_m \left(\left(\frac{l}{d}\right)^2 n_0, \frac{l}{d} r_0, n, r \right) \right.$$
$$+ 2\pi \cdot (\det(2m))^{-1/2} \cdot i^{k+\frac{g+1}{2}} \cdot \left(\frac{D}{l^2/d^2 D_0}\right)^{k/2-1/4} \cdot \sum_{c\geq 1} e_{2c}(rm^{-1} r_0^t l/d)$$
$$\left. \times H_{m,c}\left(\frac{l^2}{d^2} n_0, \left(\frac{l}{d}\right) r_0, n, r\right) \cdot J_{k-1/2}\left(\frac{2\pi \cdot l}{\det(2m) \cdot c \cdot d} \cdot \sqrt{D_0 D}\right) \cdot c^{-g/2-1} \right).$$

This is equivalent to saying that

$$l^{k-1} \cdot (D/D_0)^{k/2} \cdot \epsilon_{\frac{1}{2}\det(2m)}(l, D_0 D, r(2m)^* r_0^t, D_0) + i^{k+1} \cdot (D/D_0)^{k/2-1/4} \cdot l^{k-1/2} \cdot \pi$$
$$\times 2(\det(2m))^{-1/2} \cdot \sum_{a\geq 1} a^{-1/2} \cdot S_{\frac{1}{2}\det(2m)a}(l, D_0 D, r(2m)^* r_0^t, D_0)$$
$$\cdot J_{k-1/2}\left(\frac{2\pi \cdot l}{\det(2m) \cdot a} \cdot \sqrt{D_0 D}\right)$$
$$= l^{k-1} \cdot \sum_{d|l} \left(\frac{D_0}{d}\right) \cdot (l/d)^k \cdot \delta_m \left(\left(\frac{l}{d}\right)^2 n_0, \frac{l}{d} r_0, n, r \right) + 2 \cdot i^{k+(g+1)/2}$$
$$\times (D/D_0)^{k/2-1/4} \cdot l^{k-1/2} \cdot \pi \cdot (\det 2m)^{-1/2} \cdot \sum_{d|l} \left(\frac{D_0}{d}\right) \cdot d^{-1/2} \sum_{c\geq 1} e_{2c}(rm^{-1} r_0^t l/d)$$
$$\times H_{m,c}\left(\frac{l^2}{d^2} n_0, \left(\frac{l}{d}\right) r_0, n, r\right) \cdot J_{k-1/2}\left(\frac{2\pi \cdot l}{\det(2m) \cdot c \cdot d} \cdot \sqrt{D_0 D}\right) \cdot c^{-g/2-1}. \quad (4.4)$$

We first want to show that the first terms of (4.4) agree with each other. For this we to show the following

Lemma 4.26 *We have*

$$(D/D_0)^{k/2} \cdot \epsilon_{\frac{1}{2}\det(2m)}(l, D_0 D, r(2m)^* r_0^t, D_0)$$
$$= \sum_{d|l} \left(\frac{D_0}{d}\right) \cdot (l/d)^k \cdot \delta_m \left(\left(\frac{l}{d}\right)^2 n_0, \frac{l}{d} r_0, n, r \right). \quad (4.5)$$

Proof. Inserting the definition of $\epsilon_{\frac{1}{2}\det(2m)}$ from Lemma 4.10 we see that the left-hand side of (4.5) is zero unless $D = D_0 f^2$ for some $f \in \mathbb{N}$ with $f|l$ and

$$D_0 f \equiv r(2m)^* r_0^t \pmod{\det(2m)}. \quad (4.6)$$

Using

$$D_0 = r_0(2m)^* r_0^t - 2n_0 \det(2m)$$

we see that (4.6) is equivalent to

$$r_0(2m)^* r_0^t f \equiv r(2m)^* r_0^t \pmod{\det(2m)}.$$

In this case the left-hand side of (4.5) is equal to $\left(\frac{D_0}{l/f}\right) \cdot f^k$.
Inserting the definition of δ_m from Theorem 2.29, we see that right-hand side of (4.5) is zero unless $D = D_0(l/d)^2$ and $r \equiv r_0 l/d \pmod{\mathbb{Z}^{(1,g)} \cdot 2m}$. Setting $f = l/d$ we see that in this case it has the value $\left(\frac{D_0}{l/f}\right) \cdot f^k$.
Thus we have to show

Lemma 4.27 *Under the assumptions on m and D_0 given in the beginning of this section, the following congruences are equivalent:*

$$(r - r_0 f)(2m)^* r_0^t \equiv 0 \pmod{\det(2m)}, \tag{4.7}$$

$$r - r_0 f \equiv 0 \pmod{\mathbb{Z}^{(1,g)} \cdot 2m}. \tag{4.8}$$

Remark 4.28 *Lemma 4.27 does not hold for arbitrary m and D_0. As an example, choose $g > 1$, $m = E_g$, $r_0 = (1, \dots, 1)$, and $f = 1$.*
Then we see, writing $r = (r_1, \dots, r_g)$, that (4.7) is equivalent to

$$\sum_{i=1}^{g} r_i \equiv 1 \pmod 2, \tag{4.9}$$

and (4.8) is equivalent to

$$r_i \equiv 1 \pmod 2 \qquad (1 \leq i \leq g). \tag{4.10}$$

Thus (4.9) has more solutions than (4.10).

The proof of Lemma 4.27 is subdivided into several lemmas.

Lemma 4.29 *If r is a solution of (4.8), then r is also a solution of (4.7).*

Proof. Let r be a solution of congruence (4.8). Then there exists a $\lambda \in \mathbb{Z}^{(1,g)}$ with

$$r - r_0 f = \lambda \cdot 2m.$$

Multiplying both sides with $(2m)^* r_0^t$ from the right gives that r is a solution of the congruence (4.7). □

Lemma 4.30 *For $r \in \mathbb{Z}^{(1,g)}$, the following conditions are equivalent:*

$$(i)\, r \equiv 0 \pmod{\mathbb{Z}^{(1,g)} \cdot 2m} \tag{4.11}$$

$$(ii) \text{ The congruence } \lambda \cdot 2m \equiv r \pmod{\mathbb{Z}^{(1,g)} \cdot \det(2m)} \text{ is solvable.} \tag{4.12}$$

Proof. Suppose that (i) is satisfied, i.e., $r = \mu \cdot 2m$, with $\mu \in \mathbb{Z}^{(1,g)}$. Then in (ii) we can take $\lambda = \mu$.
Conversely, suppose that (ii) holds, i.e., there exists $\lambda, \mu \in \mathbb{Z}^{(1,g)}$ such that $r - \lambda \cdot 2m = \mu \cdot \det(2m)$. Observing that

$$\mu \cdot \det(2m) = \mu \cdot \det(2m) \cdot (2m)^{-1} \cdot 2m = \mu \cdot (2m)^* \cdot 2m,$$

we then see that (i) is satisfied. □

Now we let p be a fixed prime such that p divides $\det(2m)$ of order ν. Due to Lemma 4.30 it is sufficient to consider the congruences:

$$(r - r_0 f)(2m)^* r_0^t \equiv 0 \pmod{p^\nu}, \tag{4.13}$$

$$\lambda \cdot 2m \equiv r - r_0 f \pmod{\mathbb{Z}^{(1,g)} \cdot p^\nu} \tag{4.14}$$

and show that every solution r of (4.13) gives a solution λ of (4.14). For this let us consider for $U \in GL_g(\mathbb{Z}_p)$ the following system of congruences for $\tilde{r} \in \mathbb{Z}^{(1,g)}$:

$$(\tilde{r} - r_0 U f)(2m[U])^* (r_0 U)^t \equiv 0 \pmod{p^\nu}, \tag{4.15}$$

$$\lambda \cdot 2m[U] \equiv \tilde{r} - r_0 U f \pmod{\mathbb{Z}^{(1,g)} \cdot p^\nu} \text{ is solvable.} \tag{4.16}$$

The following lemma shows how this system corresponds to the congruences (4.13) and (4.14).

Lemma 4.31 *1. If r is a solution of (4.13), then $\tilde{r} := rU$ is a solution of (4.15).*
Conversely: If $\tilde{r}$ is a solution of (4.15), then $r := \tilde{r}\bar{U}$ is a solution of (4.13), where $\bar{U}$ is an inverse of U in $GL_g(\mathbb{Z}_p)$.

2. If r is a solution of (4.14), then $\tilde{r} := rU$ is a solution of (4.16).
Conversely: If $\tilde{r}$ is a solution of (4.16), then $r := \tilde{r}\bar{U}$ is a solution of (4.14), where $\bar{U}$ is defined in 1..

Proof. 1. Due to $(\det U, p) = 1$ and $(2m[U])^* = (\det U)^2 \cdot \det(2m) \cdot U^{-1} \cdot (2m[U])^{-1}$, congruence (4.13) can be written as

$$\det(2m) \cdot (rU - r_0 U f) \cdot (2m[U])^{-1} \cdot U^t r_0^t \equiv 0 \pmod{p^\nu}.$$

Thus the claim follows if we multiply both side with $(\det U)^2$.
2. Due to $U \in GL_g(\mathbb{Z}_p)$, the congruence

$$\lambda \cdot 2m \equiv r - r_0 f \pmod{p^\nu} \tag{4.17}$$

is solvable for $r \in \mathbb{Z}^{(1,g)}$ if and only if the congruence

$$\lambda \cdot \bar{U}^t \cdot U^t 2mU \equiv rU - r_0 Uf \pmod{p^\nu}$$

is solvable. Thus the claim follows directly. $\square$

Lemma 4.32 *Let $U \in GL_g(\mathbb{Z}_p)$ be given as in Lemma 4.21. If $\tilde{r}$ is a solution of (4.15), then $\tilde{r}$ is also a solution of (4.16).*

Proof. Let us abbreviate

$$\begin{aligned} s &:= (s_1, \dots, s_g)^t := \tilde{r} - r_0 Uf, \\ r_0 U &:= (r'_1, \dots, r'_g)^t, \\ \lambda &= (\lambda_1, \dots, \lambda_g)^t. \end{aligned}$$

For the proof we treat the cases $p \neq 2$ and $p = 2$ separately.
In the case $p \neq 2$ we conclude by Lemma 4.21 that (4.15) is equivalent to

$$\sum_{i=1}^{g} \left(\prod_{j \neq i} m_j \right) s_i r'_i \equiv 0 \pmod{p^\nu} \tag{4.18}$$

and that (4.16) is equivalent to the solvability of the congruences

$$\lambda_i \cdot m_i \equiv s_i \pmod{p^\nu} \qquad (1 \leq i \leq g). \tag{4.19}$$

Moreover we know that p divides exactly one m_i of order ν $(1 \leq i \leq g)$. We may without loss of generality assume that p divides m_1. Thus (4.18) has the form

$$s_1 r'_1 \prod_{j=2}^{g} m_j \equiv 0 \pmod{p^\nu}. \tag{4.20}$$

As shown before $\left(\prod_{j=2}^{g} m_j, p \right) = 1$.
Moreover we have that changing T into $T[U]$ changes D_0 into

$$(\det U)^2 \cdot D_0 \equiv r_1'^2 \prod_{j=2}^{g} m_j \pmod{p^\nu}.$$

Thus $(r'_1, p) = 1$ follows from $((\det U)^2 D_0, p) = 1$.
Therefore (4.20) is equivalent to

$$s_1 \equiv 0 \pmod{p^\nu}.$$

It is therefore left to show that the congruences

$$\begin{aligned} \lambda_2 \cdot m_2 &\equiv s_2 \pmod{p^\nu}, \\ &\vdots \\ \lambda_g \cdot m_g &\equiv s_g \pmod{p^\nu}, \end{aligned}$$

are solvable, which is trivially satisfied, because the numbers $m_2, \ldots, m_g$ are coprime to p as shown above.
In the case $p = 2$ we get with the same abbreviations as in the case $p \neq 2$, using Lemma 4.21 (and without loss of generality assuming that the block $2l$ occurs at the first position) that (4.15) has the form

$$s_1 r_1' \equiv 0 \pmod 2, \tag{4.21}$$

and (4.16) is equivalent to the solvability of the congruences

$$\begin{aligned} s_1 &\equiv 0 \pmod 2, \\ \lambda_3 &\equiv s_2 \pmod 2, \\ \lambda_2 &\equiv s_3 \pmod 2, \\ &\vdots \\ \lambda_{g-1} &\equiv s_g \pmod 2. \end{aligned}$$

Clearly the last $g-1$ congruences are solvable. Moreover we obtain as in the case $p \neq 2$, using that $2 \nmid \tilde{D}_0$, that $2 \nmid r_1'$. Thus (4.21) is equivalent to

$$s_1 \equiv 0 \pmod 2.$$

Therefore we have proved Lemma 4.27 and therefore Lemma 4.26, too. $\square$
Thus the first terms in (4.4) agree. Next we have to show that the second terms in (4.4) agree. In the second term on the right-hand side of (4.4) we substitute $cd = a$ to get, using $g \equiv 1 \pmod 8$,

$$i^{k+1} \cdot (D/D_0)^{k/2-1/4} \cdot l^{k-1/2} \cdot 2\pi \cdot (\det(2m))^{-1/2} \cdot \sum_{a \geq 1} \sum_{d|(a,l)} \left(\frac{D_0}{d} \right) \cdot d^{-1/2} \cdot e_{2a}(rm^{-1}r_0^t)$$
$$\times H^{\pm}_{m,a/d} \left(\frac{l^2}{d^2} n_0, \frac{l}{d} r_0, n, r \right) \cdot J_{k-1/2} \left(\frac{2\pi \cdot l}{\det(2m) \cdot a} \cdot \sqrt{D_0 D} \right) \cdot (a/d)^{-g/2-1}.$$

Thus it is sufficient to show

Lemma 4.33 *For $l \geq 1, n \geq 0, r \in \mathbb{Z}^{(1,g)}$ we have*

$$S_{\frac{1}{2}\det(2m)a}(l, DD_0, r_0(2m)^* r^t, D_0)$$
$$= \sum_{d|(a,l)} \left(\frac{D_0}{d} \right) \cdot (a/d)^{g/2} \cdot e_{2a/d}(rm^{-1}r_0^t) \cdot H_{m,a/d} \left(\frac{l^2}{d^2} n_0, \frac{l}{d} r_0, n, r \right).$$

Proof. If we insert the definitions of $S_{\frac{1}{2}\det(2m)a}$ and $H_{m,a/d}$ and multiply both sides with $e_{2a}(-r_0 m^{-1} r^t)$, then we see that we have to show

$$\sum_{\substack{b(a\det(2m))\\ b\equiv r_0(2m)^* r^t(\det(2m))\\ b^2\equiv DD_0(2\det(2m)a)}} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right)\cdot e_a\left(\frac{b-r_0(2m)^{-1}r}{\det(2m)} l\right)$$
$$=\sum_{d|(a,l)}\left(\frac{D_0}{d}\right)\cdot (a/d)^{-(g+1)/2}\cdot \sum_{\substack{\rho(a/d)^*\\ \lambda(a/d)}} e_{a/d}\left(\left(m[\lambda]+\frac{l}{d}r_0\lambda+\frac{l^2}{d^2}n_0\right)\bar{\rho}+n\rho+r\lambda\right).$$

Since both sides are periodic in l with period a, it is sufficient to show that their Fourier transforms are equal, i.e., we have to show that for every $h' \in \mathbb{Z}/a\mathbb{Z}$ we have

$$\frac{1}{a}\cdot \sum_{\substack{b(a\det(2m))\\ b\equiv r_0(2m)^* r^t(\det 2m)\\ b^2\equiv DD_0(2\det(2m)a)}} \sum_{l(a)} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right)$$
$$\times e_a\left(\left(\frac{b-r_0(2m)^{-1}r^t}{\det(2m)}-h'\right)l\right)$$

$$=\frac{1}{a}\cdot\sum_{l(a)}\sum_{d|(a,l)}\left(\frac{D_0}{d}\right)\cdot (a/d)^{-(g+1)/2}\cdot \sum_{\substack{\rho(a/d)^*\\ \lambda(a/d)}}$$
$$\times e_{a/d}\left(\left(m[\lambda]+\frac{l}{d}r_0\lambda+\frac{l^2}{d^2}n_0\right)\bar{\rho}+n\rho+r\lambda-h'\frac{l}{d}\right). \quad (4.22)$$

Setting $h=\det(2m)h'+r_0(2m)^* r^t$ we see the left-hand side of (4.22) is equal to

$$\frac{1}{a}\cdot \sum_{\substack{b(a\det(2m))\\ b\equiv r_0(2m)^* r^t(\det(2m))\\ b^2\equiv DD_0(2\det(2m)a)}} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right)\cdot\sum_{l(a)} e_a\left(\frac{l}{\det(2m)}(b-h)\right)$$

$$=\begin{cases}\chi_{D_0}\left(\left[a\frac{1}{2}\det(2m), h, \frac{h^2-DD_0}{2\det(2m)a}\right]\right) & \text{if } h^2\equiv DD_0 \pmod{2a\det(2m)}\\ 0 & \text{otherwise}\end{cases}.$$

For the right-hand side of (4.22) we obtain, after replacing l by ld and then (λ, l) by $(\rho\lambda, \rho l)$,

$$\frac{1}{a}\cdot\sum_{d|a}\left(\frac{D_0}{d}\right)\cdot (d/a)^{(g+1)/2}\cdot \sum_{\substack{\rho(a/d)^*\\ \lambda,l(a/d)}} e_{a/d}\left(\rho\left(m[\lambda]+r_0 l\lambda+n_0 l^2+r\lambda-h'l+n\right)\right).$$

Thus it is left to prove the following

Lemma 4.34 *Suppose that* $b \equiv r(2m)^* r_0^t \pmod{\det(2m)}$. *Let*

$$F(x,y) := m[x] + r_0 xy + n_0 y^2 + rx + sy + n, \qquad (\, x \in \mathbb{Z}^{(g,1)}, y \in \mathbb{Z})$$

where

$$s = r(2m)^{-1} r_0^t - \frac{b}{\det(2m)},$$

and

$$F_c(m, r_0, n_0, r, s, n) := F_c := c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} \sum_{x,y(c)} e_c(\lambda F(x,y)).$$

Then we have for any $a \geq 1$

$$\frac{1}{a} \cdot \sum_{d|a} \left(\frac{D_0}{d} \right) \cdot F_{a/d} = \begin{cases} \chi_{D_0}\left(\left[\frac{a}{2} \det(2m), b, \frac{b^2 - DD_0}{2\det(2m)a} \right] \right) & \textit{if } a | \frac{b^2-DD_0}{2\det(2m)} \\ 0 & \textit{otherwise} \end{cases}. \tag{4.23}$$

For the proof we need the well known

Lemma 4.35 *Let* $p \neq 2$ *be a prime,* c *be a* p*-power,* $A \in \mathbb{Z}$ *with* $p|A$. *Then we have*

$$\sum_{\lambda(c)} \left(\frac{\lambda}{p} \right) \cdot e_{pc}(\lambda A) = \begin{cases} \epsilon(p) \cdot \frac{c}{\sqrt{p}} \cdot \left(\frac{A/c}{p} \right) & \textit{if } c|A \\ 0 & \textit{otherwise} \end{cases}.$$

Recall that $\epsilon(x) = 1$ *or* i *according as* $x \equiv 1$ *or* $3 \pmod 4$.

Proof of Lemma 4.34. For the proof we set

$$C := \frac{b^2 - DD_0}{2\det(2m)}.$$

Since both sides of (4.23) are multiplicative functions in a we may assume that a is a prime-power.
Moreover it is sufficient to show that

$$\frac{1}{a} \cdot \sum_{d|a} \left(\frac{D_0}{d} \right) \cdot F_{a/d} = \begin{cases} 0 & \text{if } a \nmid C \\ \left(\frac{D_0}{a} \right) & \text{if } a|C \textit{ and } p \nmid D_0 \\ \left(\frac{D_0/p^*}{a} \right) \left(\frac{\frac{1}{2}\det(2m)C/a}{p} \right) & \text{if } a|C \text{ and } p|D_0. \end{cases} \tag{4.24}$$

Indeed, if $a|C$ and $p \nmid D_0$, then we can take as a splitting in Theorem 4.6, P3,

$$D_1 = D_0, \qquad D_2 = 1, \qquad l_1 = 1, \qquad l_2 = \det(2m)/2.$$

Then clearly we have a splitting with coprime fundamental discriminants and with

$$(D_0, a) = (1, \det(2m)C/(2a)) = 1,$$

and

$$\chi_{D_0}\left(\left[a\det(2m)/2, b, \frac{C}{a}\right]\right) = \left(\frac{D_0}{a}\right).$$

If $a|C$ and $p|D_0$, then clearly $p \nmid \det(2m)$ because $2|\det(2m)$ (in particular $p \neq 2$) and $(p, D_0/p) = 1$ (because D_0 is fundamental), i.e., we can take as a splitting

$$D_1 = D_0/p^*, \qquad D_2 = p^*, \qquad l_1 = 1, \qquad l_2 = \det(2m)/2,$$

where

$$p^* := \left(\frac{-1}{p}\right) p \equiv 1 \pmod 4.$$

Then D_1 and D_2 are fundamental and coprime. We may assume $p \nmid (C/a)$, because otherwise both $\left(\frac{D_0/p^*}{a}\right)\left(\frac{\frac{1}{2}\det(2m)\cdot C/a}{p}\right)$ and the generalized genus character vanish.
In this case

$$(D_0/p^*, a) = (p^*, \det(2m)\cdot C/(2a)) = 1,$$

and we have

$$\chi_{D_0}\left(\left[a\cdot\det(2m)/2, b, \frac{C}{a}\right]\right) = \left(\frac{D_0/p^*}{a}\right)\left(\frac{p^*}{\det(2m)C/(2a)}\right).$$

The Jacobi symbol $\left(\frac{p^*}{\det(2m)C/(2a)}\right)$ is obtained by using the quadratic residue law, distinguishing the cases whether $\det(2m)C/(2a)$ is even or odd and using that the only squares $\pmod 8$ are 0 and 1. The proof can be left out here since it is a straightforward calculation and can be done exactly as in the case $g = 1$.
Thus we have proved that is is sufficient to show identity (4.24).

Remark 4.36 *If $p \neq 2$ is a prime that divides* $\det(2m)$*, then* $\left(\frac{D_0}{p}\right) = 1$*, because in this case D_0 is a square* $\pmod p$*.*

Now let $U \in GL_g(\mathbb{Z}_p)$. We want to replace m by $m[U]$, choosing U as in Lemma 4.21, since in this case the sums F_c are easier to compute (in fact we take the integers that are congruent this $p-$adic numbers $\pmod{p^\nu}$, we just don't mention this all the time).
For this let $\tilde{D}, \tilde{D}_0, \tilde{N}, \tilde{r}$, and $\tilde{r}_0$ be defined as in the proof of Lemma 4.20. Moreover let

$$\begin{aligned} \tilde{b} &:= \tilde{r}(2\tilde{m})^*\tilde{r}_0^t - \det(2\tilde{m})s = (\det U)^2\cdot b, \\ \tilde{C} &:= \tfrac{\tilde{b}^2-\tilde{D}\tilde{D}_0}{2\det(2\tilde{m})} = (\det U)^2\cdot C. \end{aligned}$$

We now have to show that by changing D, D_0, b, and C into $\tilde{D}, \tilde{D}_0, \tilde{b}$, and $\tilde{C}$, respectively, we neither change the left- nor the right-hand side of (4.24). Let us start with the left-hand side. We have

$$F_c = c^{-(g+1)/2} \sum_{\lambda(c)^*} e_c(\lambda n) \sum_{y(c)} e_c\left(\lambda\left(n_0 y^2 + sy\right)\right) \sum_{x(c)} e_c\left(\lambda\left(m[x] + r_0 xy + rx\right)\right)$$
$$= c^{-(g+1)/2} \sum_{\lambda(c)^*} e_c(\lambda n) \sum_{y(c)} e_c\left(\lambda\left(n_0 y^2 + sy\right)\right) \sum_{x(c)} e_c(\lambda(m[U][\bar{U}x]+r_0 U\bar{U}xy+rU\bar{U}x))$$
$$= F_c\left(\tilde{m}, \tilde{r}_0, n_0, \tilde{r}, s, n\right),$$

where $\bar{U}$ is an inverse of U in $GL_g(\mathbb{Z}_p)$. For the previous identity we used that $\bar{U}x$ runs $\pmod c$ if x does since $U \in GL_g(\mathbb{Z}_p)$.
Moreover we obtain, using that $(\det U, d) = 1$, that for all positive divisors d of a

$$\left(\frac{D_0}{d}\right) = \left(\frac{D_0 \cdot (\det U)^2}{d}\right) = \left(\frac{\tilde{D}_0}{d}\right).$$

Thus the left-hand side of (4.24) is equal to

$$\frac{1}{a} \cdot \sum_{d|a} \left(\frac{\tilde{D}_0}{d}\right) \cdot F_{a/d}(\tilde{m}, \tilde{r}_0, n_0, \tilde{r}, s, n).$$

We now show that the right-hand side of (4.24) remains unchanged.
Due to $(a, \det U) = 1$, we have

$$\begin{array}{lcl} a|C & \Leftrightarrow & a|\tilde{C} = (\det U)^2 \cdot C, \\ p|D_0 & \Leftrightarrow & p|\tilde{D}_0 = (\det U)^2 \cdot D_0. \end{array}$$

If $a|C$ and $p \nmid D_0$, we have due to $(\det U, a) = 1$

$$\left(\frac{\tilde{D}_0}{a}\right) = \left(\frac{(\det U)^2 \cdot D_0}{a}\right) = \left(\frac{D_0}{a}\right).$$

If $a|C$ and $p|D_0$ we have, again using that $(\det U, a) = 1$,

$$\begin{aligned} \left(\frac{\tilde{D}_0/p^*}{a}\right)\left(\frac{\det(2\tilde{m})\tilde{C}/(2a)}{p}\right) &= \left(\frac{(\det U)^2 D_0/p^*}{a}\right)\left(\frac{(\det U)^4 \det(2m)C/(2a)}{p}\right) \\ &= \left(\frac{D_0/p^*}{a}\right)\left(\frac{\det(2m)C/(2a)}{p}\right). \end{aligned}$$

We now have to be carefull, since the restrictions on D_0 and $\det(2m)$ given at the beginning of this section are now changed (e.g. $(\det(2\tilde{m}), \tilde{D}_0) = (\det U)^2$, which is not necessarily 1). But for the proof of identity (4.24) we only need that p must not divide both $\tilde{D}_0$ and $\det(2\tilde{m})$, which remains since $(p, \det U) = 1$, and

$(\det(2m), D_0) = 1$.
Thus in the following we may replace m, r, and r_0 by $\tilde{m}$, $\tilde{r}$, and $\tilde{r}_0$, respectively. Let us abbreviate

$$\begin{aligned} \tilde{r} &=: (r_1, \ldots, r_g), \\ \tilde{r}_0 &=: (r_1', \ldots, r_g'). \end{aligned}$$

First we treat the case $p \neq 2$. Due to Lemma 4.21 we may assume that m has the form

$$m = \begin{pmatrix} \bar{2}\tilde{m}_1 & & \\ & \ddots & \\ & & \bar{2}\tilde{m}_g \end{pmatrix} =: \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_g \end{pmatrix},$$

where $\bar{2}$ denotes an inverse of 2 (mod c). We have $(m_i, p) = 1$ if $p \nmid \det(2m)$ and p divides m_i for exactly one m_i of $\nu-$th order $(1 \leq i \leq g)$ if $p | \det(2m)$.
Thus we have

$$\begin{aligned} D &= \det(2m)/2 \cdot \left(\textstyle\sum_{i=1}^g r_i^2/m_i - 4n\right), \\ D_0 &= \det(2m)/2 \cdot \left(\textstyle\sum_{i=1}^g r_i'^2/m_i - 4n_0\right), \\ b &= \det(2m)/2 \cdot \left(\textstyle\sum_{i=1}^g r_i r_i'/m_i - 2s\right). \end{aligned} \tag{4.25}$$

Moreover in this case the sum F_c has the form

$$F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) \sum_{y(c)} e_c\left(\left(n_0 y^2 + sy\right)\lambda\right) \\ \times \prod_{i=1}^g \sum_{x_i(c)} e_c\left(\left(m_i x_i^2 + \left(y r_i' + r_i\right) x_i\right)\lambda\right). \tag{4.26}$$

Let us first assume $p \nmid \det(2m)$, i.e., $(p, m_i) = 1$ for all m_i $(1 \leq i \leq g)$. Applying Lemma 3.8 (with $\alpha = 0$) leads to

$$F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) \sum_{y(c)} e_c\left(\left(n_0 y^2 + sy\right)\lambda\right) \\ \times \prod_{i=1}^g \sqrt{c} \cdot \epsilon(c) \cdot \left(\frac{\lambda m_i}{c}\right) \cdot e_c\left(-\lambda \left(y r_i' + r_i\right)^2 / (4 m_i)\right).$$

Since $g \equiv 1 \pmod 4$ and since c is odd we obtain

$$\begin{aligned} \epsilon(c)^g &= \epsilon(c), \\ \textstyle\prod_{i=1}^g \left(\frac{m_i}{c}\right) &= \left(\frac{2^{g-1}\prod_{i=1}^g m_i}{c}\right) = \left(\frac{\frac{1}{2}\det(2m)}{c}\right), \\ \left(\frac{\lambda}{c}\right)^g &= \left(\frac{\lambda}{c}\right). \end{aligned} \tag{4.27}$$

Thus we get

$$F_c = c^{-1/2} \cdot \epsilon(c) \cdot \left(\frac{\frac{1}{2}\det(2m)}{c} \right) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c \left(\lambda \left(n - \sum_{i=1}^{g} r_i^2/(4m_i) \right) \right)$$
$$\sum_{y(c)} e_c \left(\left(\lambda \left(n_0 - \sum_{i=1}^{g} r_i'^2/(4m_i) \right) y^2 + \left(s - \sum_{i=1}^{g} r_i' r_i/(2m_i) \right) y \right) \right).$$

Due to $(c, \det(2m)) = 1$, we can replace λ by $2\det(2m) \cdot \lambda$ and use (4.25), which leads to

$$F_c = c^{-1/2} \cdot \epsilon(c) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c(-\lambda D) \sum_{y(c)} e_c(-D_0 \lambda y^2 - 2b\lambda y). \tag{4.28}$$

We now proceed, treating the cases $p \nmid D_0$ and $p|D_0$, separately. As described previously, $p^2 \nmid D_0$ in both cases.
In the case $p \nmid D_0$ we have, again using Lemma 3.8,

$$\begin{aligned} F_c &= c^{-1/2} \cdot \epsilon(c) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c(-\lambda D) c^{1/2} \cdot \epsilon(c) \cdot \left(\frac{-D_0 \lambda}{c} \right) \cdot e_c \left(\lambda b^2 / D_0 \right) \\ &= \left(\frac{D_0}{c} \right) \cdot \sum_{\lambda(c)^*} e_c \left(\lambda (b^2 - D D_0)/D_0 \right). \end{aligned}$$

Due to $(c, D_0 \cdot 2\det(2m)) = 1$ we can replace λ by $\lambda \cdot D_0 \cdot \overline{2\det(2m)}$, where $\overline{2\det(2m)}$ is an inverse of $2\det(2m) \pmod{c}$. This leads to

$$F_c = \left(\frac{D_0}{c} \right) \cdot \sum_{\lambda(c)^*} e_c(\lambda C).$$

Thus we obtain

$$\sum_{d|a} \left(\frac{D_0}{d} \right) \cdot F_{a/d} = \left(\frac{D_0}{a} \right) \sum_{\lambda(a)} e_a(\lambda C) = \begin{cases} \left(\frac{D_0}{a} \right) \cdot a & \text{if } a|C \\ 0 & \text{otherwise} \end{cases}.$$

In the case $p|D_0$ we obtain, using Lemma 3.8 for the sum in (4.28), that F_c vanishes if $p \nmid b$ and otherwise has the value

$$F_c = c^{-1/2} \cdot \epsilon(c) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c(-D\lambda) \cdot \sqrt{pc} \cdot \epsilon(c/p) \cdot \left(\frac{-\lambda D_0/p}{c/p} \right) \cdot e_{c/p} \left(\frac{(2b/p)^2}{4D_0/p} \right).$$

Since $p|D_0$ and $p \nmid \det(2m)$, the condition $p|b$ is satisfied if and only if $p|C$. Thus we get, using

$$\begin{aligned} \epsilon(c/p) \cdot \epsilon(c) &= \epsilon(p), \\ \left(\tfrac{-D_0/p}{c/p} \right) &= \left(\tfrac{D_0/p^*}{c/p} \right), \end{aligned} \tag{4.29}$$

that

$$F_c = p^{1/2} \cdot \epsilon(p) \cdot \left(\frac{D_0/p^*}{c/p}\right) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{p}\right) \cdot e_c\left(\lambda\left(b^2 - DD_0\right)/D_0\right).$$

Due to $(p, 2\det(2m) \cdot D_0/p) = 1$ we can replace λ by $\overline{2\det(2m)} \cdot D_0/p \cdot \lambda$, where $\overline{2\det(2m)}$ is an inverse of $2\det(2m) \pmod{c}$, which leads to

$$F_c = p^{1/2} \cdot \epsilon(p) \cdot \left(\frac{D_0/p^*}{c/p}\right)\left(\frac{\frac{1}{2}\det(2m)}{p}\right)\left(\frac{D_0/p}{p}\right)\sum_{\lambda(c)^*}\left(\frac{\lambda}{p}\right) \cdot e_c(\lambda C/p). \quad (4.30)$$

Thus we obtain in case $c|C$, using Lemma 4.35,

$$F_c = c \cdot \left(\frac{D_0/p^*}{c}\right) \cdot \left(\frac{\frac{1}{2}\det(2m)C/c}{p}\right). \quad (4.31)$$

Othervice the sum in (4.24) vanishes. Clearly expression (4.31) is zero if $p|(C/c)$. Therefore the sum on the left-hand side of (4.24) is reduced to a single term F_a.

Now let us assume that $p|\det(2m)$. As shown in Remark 4.22 p divides exactly one m_i $(1 \le i \le g)$ of order $\nu = \nu_p(\det 2m)$. Without loss of generality we may assume $p|m_g$.
We now distinguish the case whether $p^\nu|c$ and $p^\nu \nmid c$. Let us first assume $p^\nu|c$. Then we have, again using Lemma 3.8 for the sum in (4.26),

$$\begin{aligned} F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) & \sum_{\substack{y(c) \\ yr'_g + r_g \equiv 0(p^\nu)}} e_c((n_0 y^2 + sy)\lambda) \\ & \times \prod_{i=1}^{g-1} \sqrt{c} \cdot \epsilon(c) \cdot \left(\frac{\lambda m_i}{c}\right) \cdot e_c\left(-\lambda(yr'_i + r_i)^2/(4m_i)\right) \\ & \quad \times \sqrt{p^\nu c} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{\lambda m_g/p^\nu}{c/p^\nu}\right) \cdot e_{c/p^\nu}\left(-\lambda \frac{((yr'_g + r_g)/p^\nu)^2}{4m_g/p^\nu}\right). \end{aligned}$$

Thus since $g \equiv 1 \pmod{4}$, we obtain

$$F_c = c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{\prod_{i=1}^{g-1} m_i}{c}\right) \cdot \left(\frac{m_g/p^\nu}{c/p^\nu}\right) \cdot \sum_{\lambda(c)^*} e_c\left(\left(n - \sum_{i=1}^{g} r_i^2/(4m_i)\right)\lambda\right)$$

$$\left(\frac{\lambda}{c/p^\nu}\right) \cdot \sum_{\substack{y(c) \\ yr'_g + r_g \equiv 0(p^\nu)}} e_c\left(\lambda\left(y^2\left(n_0 - \sum_{i=1}^{g} r_i'^2/(4m_i)\right) + y\left(s - \sum_{i=1}^{g} r_i r'_i/(2m_i)\right)\right)\right).$$

Due to $\left(\frac{2\det(2m)}{p}, p\right) = 1$, we may replace λ by $\frac{2\det(2m)}{p} \cdot \lambda$. Using (4.25) we obtain

$$F_c = c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{\prod_{i=1}^{g-1} m_i}{c}\right) \cdot \left(\frac{m_g/p^\nu}{c/p^\nu}\right) \cdot \left(\frac{2\det(2m)/p^\nu}{c/p^\nu}\right)$$
$$\times \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c(-\lambda D/p^\nu) \sum_{\substack{y(c) \\ yr'_g + r_g \equiv 0 (p^\nu)}} e_c\left(\lambda\left(-y^2 D_0/p^\nu - 2b/p^\nu y\right)\right).$$

Since $p \nmid D_0$ we can apply the same arguments as before to conclude that $(r'_g, p) = 1$. Thus we can replace y by $-r_g\overline{r'_g} + p^\nu y$, where $\overline{r'_g}$ is an inverse of $r'_g \pmod c$ and where the new y runs $\pmod{c/p^\nu}$. Using (4.25) we obtain

$$\left(\prod_{i=1}^{g-1} m_i\right) D_0 \equiv \left(\prod_{i=1}^{g-1} m_i r'_g 2^{\frac{g-1}{2}}\right)^2 \pmod{p^\nu},$$

i.e.,

$$\left(\frac{\prod_{i=1}^{g-1} m_i D_0}{p^\nu}\right) = 1,$$

i.e., we have, due to Remark 4.36,

$$\left(\frac{\prod_{i=1}^{g-1} m_i}{p}\right) = \left(\frac{D_0}{p}\right) = 1.$$

Thus we obtain

$$F_c = c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c(-\lambda D/p^\nu)$$
$$\times \sum_{y(c/p^\nu)} e_c\left(\lambda\left(-\left(p^\nu y - r_g\overline{r'_g}\right)^2 D_0/p^\nu - 2b/p^\nu \cdot \left(p^\nu y - r_g\overline{r'_g}\right)\right)\right)$$
$$= c^{-1/2} \cdot p^\nu \cdot \epsilon(c/p) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c\left(\lambda\left(-D/p^\nu - D_0/p^\nu \cdot \left(r_g\overline{r'_g}\right)^2 + 2b/p^\nu \left(r_g\overline{r'_g}\right)\right)\right)$$
$$\times \sum_{y(c/p^\nu)} e_c\left(\lambda\left(-p^\nu y^2 D_0 + 2y\left(D_0 r_g\overline{r'_g} - b\right)\right)\right).$$

Due to (4.25) we have

$$D_0 r_g\overline{r'_g} - b \equiv 0 \pmod{p^\nu},$$

i.e., we get, using Lemma 3.8 and $p \nmid D_0$,

$$F_c = c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c \left(\lambda \left(-D/p^\nu - D_0/p^\nu \left(r_g \overline{r'_g}\right)^2 + 2b/p^\nu \left(r_g \overline{r'_g}\right)\right)\right)$$

$$\times (c/p^\nu)^{1/2} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{-D_0 \lambda}{c/p^\nu}\right) \cdot e_{c/p^\nu} \left(\frac{\left((D_0 r_g \overline{r'_g} - b)/p^\nu\right)^2}{D_0}\right).$$

Thus we have, again using Remark 4.36,

$$F_c = \sum_{\lambda(c)^*} e_c \left(\frac{\lambda(b^2 - DD_0)}{p^\nu D_0}\right).$$

Therefore we deduce, changing λ into $D_0 \cdot \overline{2\det(2m)/p^\nu} \cdot \lambda$, where $\overline{2\det(2m)/p^\nu}$ denotes an inverse of $2\det(2m)/p^\nu \pmod{c}$,

$$F_c = \sum_{\lambda(c)^*} e_c(C\lambda). \tag{4.32}$$

Now let us assume $p^\nu \nmid c$. Then we have, again using Lemma 3.8 for the sum in (4.26),

$$F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) \cdot c \sum_{\substack{y(c) \\ y r'_g + r_g \equiv 0 (c)}} e_c((n_0 y^2 + sy)\lambda)$$

$$\times \prod_{i=1}^{g-1} \sqrt{c} \cdot \epsilon(c) \cdot \left(\frac{\lambda m_i}{c}\right) \cdot e_c\left(-\lambda (y r'_i + r_i)^2/(4m_i)\right).$$

Thus since $g \equiv 1 \pmod 4$, we obtain

$$F_c = \left(\frac{\prod_{i=1}^{g-1} m_i}{c}\right) \cdot \sum_{\lambda(c)^*} e_c \left(\left(n - \sum_{i=1}^{g-1} r_i^2/(4m_i)\right)\lambda\right) \cdot \left(\frac{\lambda}{c/p^\nu}\right)$$

$$\times e_c\left(\lambda \left(\left(r_g \overline{r'_g}\right)^2 y^2 \left(n_0 - \sum_{i=1}^{g-1} r_i'^2/(4m_i)\right) + r_g \overline{r'_g} y \left(s - \sum_{i=1}^{g-1} r_i r'_i/(2m_i)\right)\right)\right),$$

where we have again used that $p \nmid r'_g$.
As before we have $\left(\frac{\prod_{i=1}^{g-1} m_i}{p}\right) = 1$. Moreover we have, using that $p^\nu | m_g$ and $c | p^\nu$

$$
\begin{aligned}
C &= \frac{1}{2\det(2m)} \cdot (b^2 - D \cdot D_0) \\
&= \frac{\det(2m)}{2} \cdot \left(\left(\sum_{i=1}^{g} \frac{r_i r_i'}{4m_i} - \frac{s}{2} \right)^2 - \left(\sum_{i=1}^{g} \frac{r_i^2}{4m_i} - n \right) \left(\sum_{i=1}^{g} \frac{{r_i'}^2}{4m_i} - n_0 \right) \right) \\
&\quad = \frac{\det(2m)}{2} \cdot \left(\frac{r_g r_g'}{2m_g} \left(\sum_{i=1}^{g-1} \frac{r_i r_i'}{4m_i} - \frac{s}{2} \right) - \frac{r_g^2}{4m_g} \left(\sum_{i=1}^{g-1} \frac{{r_i'}^2}{4m_i} - n \right) \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. - \frac{{r_g}^2}{4m_g} \left(\sum_{i=1}^{g-1} \frac{{r_i'}^2}{4m_i} - n_0 \right) . \right)
\end{aligned}
$$

Thus we obtain, replacing λ by $-\frac{\det(2m)}{m_g} \cdot \bar{4} \cdot \lambda$, where $\bar{4}$ denotes an inverse of 4 (mod c),

$$
F_c = \sum_{\lambda(c)^*} e_c(C\lambda). \tag{4.33}
$$

Combing (4.32) and (4.33) and using $\left(\frac{D_0}{d}\right) = 1$, we infer that

$$
\sum_{d|a} \left(\frac{D_0}{d}\right) \cdot F_{a/d} = \sum_{\lambda(c)} e_a(\lambda C) = \begin{cases} a & \text{if } a|C \\ 0 & \text{otherwise} \end{cases} .
$$

Thus formula (4.24) is proved in the case $p \neq 2$.
Now let us assume $p = 2$.
Using Lemma 4.21 we may, without loss of generality, assume that m has the form

$$
m = \begin{pmatrix} l & & & \\ & \begin{smallmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{smallmatrix} \end{pmatrix} .
$$

Let the type $\left(\begin{smallmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix}\right)$ occur g_1 times and the type $\left(\begin{smallmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{smallmatrix}\right)$ occur g_2 times, i.e., $g = 1 + 2g_1 + 2g_2$ (we may without loss of generality assume that the blocks occur in this order, otherwise we change the numeration).
Let us set

$$
\begin{aligned}
I &:= \{2, 4, \ldots, 2g_1\} \subset \mathbb{N}, \\
J &:= \{2g_1 + 2, 2g_1 + 4, \ldots, g - 1\} \subset \mathbb{N}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
D &= \frac{1}{2}\det(2m)\cdot\left(-4n+\frac{r_1^2}{l}+\frac{4}{3}\cdot\sum_{i\in I}\left(r_i^2-r_ir_{i+1}+r_{i+1}^2\right)+4\cdot\sum_{i\in J}r_ir_{i+1}\right),\\
D_0 &= \frac{1}{2}\det(2m)\cdot\left(-4n_0+\frac{r_1'^2}{l}+\frac{4}{3}\cdot\sum_{i\in I}\left(r_i'^2-r_i'r_{i+1}'+r_{i+1}'^2\right)+4\cdot\sum_{i\in J}r_i'r_{i+1}'\right),\\
b &= \frac{1}{2}\det(2m)\cdot\left(-2s+\frac{r_1r_1'}{l}+\frac{2}{3}\cdot\sum_{i\in I}\left(2r_ir_i'+2r_{i+1}r_{i+1}'-r_i'r_{i+1}-r_{i+1}'r_i\right)\right.\\
&\qquad\qquad\left.+2\cdot\sum_{i\in J}\left(r_i'r_{i+1}+r_ir_{i+1}'\right)\right) \qquad (4.34)
\end{aligned}
$$

and

$$
\begin{aligned}
C = \frac{1}{2}\det(2m)\cdot&\left(\left(-s+\frac{r_1r_1'}{2l}+\frac{1}{3}\cdot\sum_{i\in I}\left(2r_i'r_i-r_i'r_{i+1}-r_{i+1}'r_i+2r_{i+1}'r_{i+1}\right)\right.\right.\\
&\qquad\left.+\sum_{i\in J}\left(r_i'r_{i+1}+r_{i+1}'r_i\right)\right)^2\\
&-\left(-2n+\frac{r_1^2}{2l}+\frac{2}{3}\cdot\sum_{i\in I}(r_i^2-r_ir_{i+1}+r_{i+1}^2)+2\cdot\sum_{i\in J}r_ir_{i+1}\right)\\
&\left.\times\left(-2n_0+\frac{r_1^2}{2l}+\frac{2}{3}\cdot\sum_{i\in I}\left(r_i'^2-r_i'r_{i+1}'+r_{i+1}'^2\right)+2\cdot\sum_{i\in J}r_i'r_{i+1}'\right)\right). \qquad (4.35)
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
F_c = c^{-(g+1)/2}\cdot&\sum_{\lambda(c)^*}e_c(n\lambda)\sum_{y(c)}e_c\left(\left(n_0y^2+sy\right)\lambda\right)\sum_{x_1(c)}e_c\left(\left(lx_1^2+\left(yr_1'+r_1\right)x_1\right)\lambda\right)\\
&\times\prod_{i\in I}\sum_{\substack{x_i(c)\\x_{i+1}(c)}}e_c\left(\lambda\left(x_i^2+x_ix_{i+1}+x_{i+1}^2+\left(r_i'y+r_i\right)x_i+\left(r_{i+1}'y+r_{i+1}\right)x_{i+1}\right)\right)\\
&\times\prod_{i\in J}\sum_{\substack{x_i(c)\\x_{i+1}(c)}}e_c\left(\lambda\left(x_ix_{i+1}+\left(r_i'y+r_i\right)x_i+\left(r_{i+1}'y+r_{i+1}\right)x_{i+1}\right)\right). \qquad (4.36)
\end{aligned}
$$

We now want to determine the different types of sums that can appear. For this let us first assume $c\neq 2$. If $r_1'y+r_1\equiv 0 \pmod 2$ we obtain, using Lemma 3.8

and $(l,2)=1$,

$$\sum_{x_1(c)} e_c\left(\left(lx_1^2+\left(yr_1'+r_1\right)x_1\right)\lambda\right)$$
$$= c^{1/2}\cdot\left(\frac{-c}{\lambda l}\right)\cdot\epsilon(l\lambda)\cdot(1+i)\cdot e_c\left(-\lambda\left(r_1'y+r_1\right)^2/(4l)\right).$$

Otherwise the sum has the value 0. Moreover we have, again using Lemma 3.8,

$$\sum_{x_{i+1}(c)} e_c\left(\lambda\left(x_{i+1}^2+\left(r_{i+1}'y+r_{i+1}\right)x_{i+1}\right)\right)\sum_{x_i(c)} e_c\left(\lambda\left(x_i^2+\left(x_{i+1}+r_i'y+r_i\right)x_i\right)\right)$$
$$=\sum_{\substack{x_{i+1}(c)\\ x_{i+1}+r_i'y+r_i\equiv 0(2)}} e_c\left(\lambda\left(x_{i+1}^2+\left(r_{i+1}'y+r_{i+1}\right)x_{i+1}\right)\right)\cdot c^{1/2}\left(\frac{-c}{\lambda}\right)\cdot\epsilon(\lambda)\cdot(1+i)$$
$$\times e_c\left(-\lambda\frac{\left(x_{i+1}+r_i'y+r_i\right)^2}{4}\right).$$

Now we can replace x_{i+1} by $-r_i'y-r_i+2x_{i+1}$ with the new x_{i+1} running (mod $c/2$), which leads to

$$c^{1/2}\cdot\left(\frac{-c}{\lambda}\right)\cdot\epsilon(\lambda)\cdot(1+i)$$
$$\sum_{x_{i+1}(c/2)} e_c\left(\lambda\left(\left(2x_{i+1}-\left(r_i+r_i'y\right)\right)^2+\left(r_{i+1}'y+r_{i+1}\right)\left(2x_{i+1}-\left(r_i+r_i'y\right)\right)-x_{i+1}^2\right)\right)$$
$$=c^{1/2}\cdot\left(\frac{-c}{\lambda}\right)\cdot\epsilon(\lambda)\cdot(1+i)\cdot e_c\left(\lambda\left(\left(r_i+r_i'y\right)^2-\left(r_{i+1}'y+r_{i+1}\right)\left(r_i'y+r_i\right)\right)\right)$$
$$\times\frac{1}{2}\cdot\sum_{x_{i+1}(c)} e_c\left(\lambda\left(\left(3x_{i+1}^2+x_{i+1}\left(2\left(r_{i+1}'y+r_{i+1}\right)-4\left(r_i'y+r_i\right)\right)\right)\right)\right)$$
$$=c^{1/2}\cdot\left(\frac{-c}{\lambda}\right)\cdot\epsilon(\lambda)\cdot(1+i)\cdot e_c\left(\lambda\left(\left(r_i+r_i'y\right)^2-\left(r_{i+1}'y+r_{i+1}\right)\left(r_i'y+r_i\right)\right)\right)$$
$$\times\frac{1}{2}\cdot c^{1/2}\cdot\left(\frac{-c}{3\lambda}\right)\cdot\epsilon(3\lambda)\cdot(1+i)\cdot e_c\left(-\frac{\lambda\left(\left(2r_i-r_{i+1}\right)+\left(2r_i'-r_{i+1}'\right)y\right)^2}{3}\right).$$

We have $\epsilon(\lambda)\cdot\epsilon(3\lambda)=i$. Thus the last expression equals

$$c\cdot\left(\frac{-c}{3}\right)\cdot(-1)\cdot e_c\left(\bar{3}\lambda\left(r_ir_{i+1}-r_i^2-r_{i+1}^2\right)+y^2\left(r_i'r_{i+1}'-r_i'^2-r_{i+1}'^2\right)\right.$$
$$\left.+y\left(r_i'r_{i+1}+r_{i+1}'r_i-2r_i'r_i-2r_{i+1}'r_{i+1}\right)\right).$$

Moreover we have, again using Lemma 3.8, that the third type of sum is equal to

$$\sum_{x_{i+1}(c)} e_c\left(\lambda\left(r_{i+1}'y+r_{i+1}\right)x_{i+1}\right)\sum_{x_i(c)} e_c\left(\lambda\left(x_{i+1}+r_i'y+r_i\right)x_i\right)$$

$$= c \cdot \sum_{\substack{x_{i+1}(c) \\ x_{i+1}+r'_i y+r_i \equiv 0(c)}} e_c\left(\lambda\left(\left(r'_{i+1}y + r_{i+1}\right)x_{i+1}\right)\right)$$
$$= c \cdot e_c\left(-\lambda\left(r'_{i+1}y + r_{i+1}\right)\left(r'_i y + r_i\right)\right).$$

Thus we get, using that $\left(\frac{-c}{-1}\right) = -1$, that the third type of sum equals

$$c \cdot (-1) \cdot \left(\frac{-c}{-1}\right) \cdot e_c\left(-\lambda\left(y^2 r'_i r'_{i+1} + y\left(r'_i r_{i+1} + r'_{i+1} r_i\right) + r_i r_{i+1}\right)\right).$$

Thus the sum in (4.36) equals

$$\begin{aligned}
F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) \sum_{\substack{y(c) \\ r'_1 y + r_1 \equiv 0(2)}} e_c\left(\left(n_0 y^2 + sy\right)\lambda\right) \cdot c^{1/2}\left(\frac{-c}{\lambda l}\right) \cdot \epsilon(l\lambda) \cdot (1+i) \\
\times e_c\left(-\lambda\left(r_1'^2 y^2/(4l) + r_1 r'_1 y/(2l) + r_1^2/(4l)\right)\right) \\
\times \prod_{i\in I} (-1) \cdot c\left(\frac{-c}{3}\right) \cdot e_c\left(\bar{3}\lambda\left(\left(r_i r_{i+1} - r_i^2 - r_{i+1}^2\right) + \right.\right. \\
\left.\left.\left(r'_i r'_{i+1} - r_i'^2 - r_{i+1}'^2\right) y^2 + \left(r'_i r_{i+1} + r'_{i+1} r_i - 2r'_i r_i - 2r_{i+1} r'_{i+1}\right) y\right)\right) \\
\times \prod_{i\in J} c \cdot (-1) \cdot \left(\frac{-c}{-1}\right) \cdot e_c\left(-\lambda\left(y^2 r'_i r'_{i+1} + y\left(r'_i r_{i+1} + r'_{i+1} r_i\right) + r_i r_{i+1}\right)\right) \\
= (-1)^{(g-1)/2} \cdot c^{1/2}\left(\frac{-c}{\frac{1}{2}\det(2m)}\right) \cdot (1+i) \cdot \sum_{\lambda(c)^*}\left(\frac{-c}{\lambda}\right) \cdot \epsilon(l\lambda) \\
e_c\left(\lambda\left(n - r_1^2/(4l) + \bar{3}\sum_{i\in I}\left(r_i r_{i+1} - r_i^2 - r_{i+1}^2\right) - \sum_{i\in J} r_i r_{i+1}\right)\right) \\
\sum_{\substack{y(c) \\ r'_1 y + r_1 \equiv 0(2)}} e_c\left(\lambda\left(y^2\left(n_0 - r_1'^2/(4l) + \sum_{i\in I}\left(r'_i r'_{i+1} - r_i'^2 - r_{i+1}'^2\right) - \sum_{i\in J} r'_i r'_{i+1}\right)\right.\right. \\
+ y\left(s - r_1 r'_1/(2l) + \bar{3}\sum_{i\in I}\left(r'_i r_{i+1} + r'_{i+1} r_i - 2r'_i r_i - 2r'_{i+1} r_{i+1}\right)\right. \\
\left.\left.\left. - \sum_{i\in J}\left(r'_i r_{i+1} + r_i r'_{i+1}\right)\right)\right)\right).
\end{aligned}$$

Thus we obtain by changing λ into $\frac{1}{2}\det(2m) \cdot \lambda$ and by using that $g \equiv 1 \pmod 4$ and (4.34),

$$F_c = c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon\left(l\frac{1}{2}\det(2m)\lambda\right) \cdot e_c\left(-\lambda D/4\right)$$
$$\times \sum_{\substack{y(c)\\ r_1' y + r_1 \equiv 0(2)}} e_c(\lambda(-D_0/4y^2 - b/2y)).$$

Since D_0 is odd, we get with the same arguments as used before that r_1' has to be odd. Thus we can replace y by $2y - r_1\overline{r_1'}$, where $\overline{r_1'}$ is an inverse of $r_1 \pmod c$ and where the new y runs $\pmod{c/2}$. Thus we get, using that l is odd and $g \equiv 1 \pmod 4$,

$$\frac{l}{2} \cdot \det(2m) = l^2 \cdot 3^{g_1} \cdot (-1)^{g_2} \equiv (-1)^{\frac{g-1}{2}} \equiv 1 \pmod 4.$$

Thus

$$\epsilon\left(\frac{l}{2}\det(2m)\lambda\right) = \epsilon(\lambda).$$

Therefore we have

$$\begin{aligned}
F_c &= c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon(\lambda) \cdot e_c\left(-\lambda D/4\right) \\
&\qquad \times \sum_{y(c/2)} e_c\left(\lambda\left(-D_0/4\left(2y - r_1\overline{r_1'}\right)^2 - b/2\left(2y - r_1\overline{r_1'}\right)\right)\right) \\
&= c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon(\lambda) \cdot e_c\left(\lambda\left(-D/4 - D_0/4(r_1\bar{r}_1)^2 + b/2r_1\overline{r_1'}\right)\right) \\
&\qquad \times \sum_{y(c/2)} e_c\left(\lambda\left(-D_0y^2 + \left(D_0r_1\overline{r_1'} - b\right)y\right)\right).
\end{aligned}$$

Moreover we get, using that r_1' and $\frac{1}{2}\det(2m)$ are odd and (4.34),

$$D_0 \cdot r_1 \cdot \overline{r_1'} \equiv r_1 \cdot \overline{r_1'} \cdot \det(2m)/(2l) \cdot r_1'^2 \equiv r_1 \pmod 2,$$

and

$$b \equiv r_1 \cdot \det(2m)/(2l) \cdot r_1' \equiv r_1 \pmod 2.$$

Thus

$$D_0 \cdot r_1 \cdot \overline{r_1'} - b \equiv 0 \pmod 2.$$

Therefore we obtain, using Lemma 3.8,

$$\begin{aligned}
F_c &= c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon(\lambda) \cdot e_c\left(\lambda\left(-D/4 - D_0/4\left(r_1\bar{r}_1\right)^2 + b/2r_1\overline{r_1'}\right)\right) \\
&\quad \times \frac{1}{2} \cdot (1+i) \cdot c^{1/2} \cdot \left(\frac{-c}{-\lambda D_0}\right) \cdot \epsilon(-D_0\lambda) \cdot e_c\left(\lambda\left(D_0r_1\overline{r_1'} - b\right)^2/(4D_0)\right).
\end{aligned}$$

Since $D_0 \equiv 1 \pmod 4$, we have

$$\epsilon(\lambda) \cdot \epsilon(-D_0\lambda) = i,$$

and

$$\left(\frac{-c}{-D_0}\right) = -\left(\frac{-1}{D_0}\right)\left(\frac{c}{D_0}\right) = -\left(\frac{c}{D_0}\right).$$

Thus

$$F_c = \left(\frac{c}{D_0}\right) \cdot \sum_{\lambda(c)^*} e_c\left(\lambda\left(b^2/(4D_0) - D/4\right)\right) = \left(\frac{c}{D_0}\right) \cdot \sum_{\lambda(c)^*} e_c(\lambda C),$$

where we have changed λ into $\overline{\det(2m)/2} \cdot D_0 \cdot \lambda$ in the previous equality, where $\overline{\det 2m/2}$ is an inverse of $\det 2m/2 \pmod c$.
Next let us assume that $c = 2$. Then clearly $\lambda = 1$. We again want to compute the three types of sums in (4.36). We have, using that l and r_1' are odd and $x^2 \equiv x \pmod 2$ for all integers x,

$$\begin{aligned}\sum_{x_1(2)} e_2\left(\left(lx_1^2 + (yr_1' + r_1)\, x_1\right)\right) &= \sum_{x_1(2)} e_c\left(x_1\left(1 + y + r_1\right)\right) \\ &= \begin{cases} 2 & \text{if } y \equiv 1 + r_1 \pmod 2 \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

Furthermore

$$\begin{aligned}&\sum_{x_{i+1}(2)} e_2\left(x_{i+1}^2 + \left(r_{i+1}'y + r_{i+1}\right) x_{i+1}\right) \sum_{x_i(2)} e_2\left(x_i^2 + \left(x_{i+1} + r_i'y + r_i\right) x_i\right) \\ = {} & \sum_{x_{i+1}(2)} e_2\left(x_{i+1}\left(1 + r_{i+1}'y + r_{i+1}\right)\right) \sum_{x_i(2)} e_2\left(x_i\left(1 + x_{i+1} + r_i'y + r_i\right)\right) \\ = {} & 2 \cdot \sum_{\substack{x_{i+1}(2) \\ x_{i+1} + r_i'y + r_i + 1 \equiv 0(2)}} e_2\left(x_{i+1}\left(1 + r_{i+1}'y + r_{i+1}\right)\right) \\ = {} & 2 \cdot e_2\left(\left(1 + r_i'y + r_i\right)\left(1 + r_{i+1}'y + r_{i+1}\right)\right) \\ = {} & -2 \cdot e_2\left(y\left(r_{i+1}' + r_i' + r_i'r_{i+1}' + r_i'r_{i+1} + r_ir_{i+1}'\right) + \left(r_i + r_{i+1} + r_ir_{i+1}\right)\right).\end{aligned}$$

Moreover we have

$$\begin{aligned}&\sum_{x_{i+1}(2)} e_2\left(\left(r_{i+1}'y + r_{i+1}\right) x_{i+1}\right) \sum_{x_i(2)} e_2\left(\left(x_{i+1} + r_i'y + r_i\right) x_i\right) \\ = {} & 2 \cdot \sum_{\substack{x_{i+1}(2) \\ x_{i+1} \equiv r_i'y + r_i(2)}} e_2\left(\left(r_{i+1}'y + r_{i+1}\right) x_{i+1}\right) = 2 \cdot e_2\left((r_i'y + r_i)(r_{i+1}'y + r_{i+1})\right) \\ = {} & 2 \cdot e_2(y(r_i'r_{i+1}' + r_i'r_{i+1} + r_ir_{i+1}') + r_ir_{i+1}).\end{aligned}$$

Inserting this into (4.36) leads to

$$F_2 = e_2(n) \cdot \sum_{\substack{y(2)\\ y\equiv 1+r_1(2)}} e_2((n_0+s)y)$$
$$\times \prod_{i\in I}(-1)\cdot e_2\left(y\left(r'_{i+1}+r'_i+r'_ir'_{i+1}+r'_ir_{i+1}+r_ir'_{i+1}\right)+\left(r_i+r_{i+1}+r_ir_{i+1}\right)\right)$$
$$\times \prod_{i\in J} e_2\left(y\left(r'_ir'_{i+1}+r'_ir_{i+1}+r_ir'_{i+1}\right)+r_ir_{i+1}\right).$$

As a solution of the congruence $y \equiv 1+r_1 \pmod 2$ we can choose $y = 1+r_1$. Thus we get

$$F_2 = e_2\Bigg(n+g_1+\sum_{i\in I}(r_i+r_{i+1}+r_ir_{i+1})+\sum_{i\in J}r_ir_{i+1}+(1+r_1)\Bigg(n_0+s+\sum_{i\in I}$$
$$\left(r'_i+r'_{i+1}+r'_ir'_{i+1}+r'_ir_{i+1}+r'_{i+1}r_i\right)+\sum_{i\in J}\left(r'_ir'_{i+1}+r'_{i+1}r_i+r'_ir_{i+1}\right)\Bigg)\Bigg). \quad (4.37)$$

Moreover we obtain using (4.35) , that $\frac{1}{2}\det(2m)$ and r'_1 are odd, and that $x^2 \equiv x \pmod 2$ for all integers x

$$C \equiv \Bigg(r_1r'_1\Bigg(\sum_{i\in I\cup J}\left(r'_ir_{i+1}+r'_{i+1}r_i\right)+s\Bigg)+\sum_{i\in I\cup J}\left(r'_ir_{i+1}+r'_{i+1}r_i\right)+s\Bigg)$$
$$+r_1^2\Bigg(n_0+\sum_{i\in I}\left(r_i'^2+r'_ir'_{i+1}+r_{i+1}'^2\right)+\sum_{i\in J}r'_ir'_{i+1}\Bigg)$$
$$+r_1'^2\Bigg(n+\sum(r_i^2+r_ir_{i+1}+r_{i+1}^2)+\sum_{i\in J}r_ir_{i+1}\Bigg)$$
$$\equiv \Bigg(r_1\Bigg(\sum_{i\in I\cup J}\left(r'_ir_{i+1}+r'_{i+1}r_i\right)+s\Bigg)+\sum_{i\in I\cup J}\left(r'_ir_{i+1}+r'_{i+1}r_i\right)+s\Bigg)$$
$$+r_1\Bigg(n_0+\sum_{i\in I}\left(r'_i+r'_ir'_{i+1}+r'_{i+1}\right)+\sum_{i\in J}r'_ir'_{i+1}\Bigg)$$
$$+\Bigg(n+\sum_{i\in I}(r_i+r_ir_{i+1}+r_{i+1})+\sum_{i\in J}r_ir_{i+1}\Bigg) \pmod 2. \quad (4.38)$$

Therefore we have, using (4.37) and (4.38) and that $e_2(x) = (-1)^x$ ($\forall x \in \mathbb{Z}$),

$$F_2 = (-1)^{n_0+\sum_{i\in I}\left(r'_i+r'_{i+1}+r'_ir'_{i+1}\right)+\sum_{i\in J}r'_ir'_{i+1}+g_1}\cdot e_2(C).$$

Next we want to show that

$$\left(\frac{2}{D_0}\right) = (-1)^{n_0+\sum_{i\in I}\left(r'_i+r'_{i+1}+r'_ir'_{i+1}\right)+\sum_{i\in J} r'_ir'_{i+1}+g_1}.$$

We obtain by using $D_0 \equiv 1 \pmod 4$,

$$\left(\frac{2}{D_0}\right) = (-1)^{1/8(D_0^2-1)} = \begin{cases} 1 & \text{if } D_0 \equiv 1 \pmod 8 \\ -1 & \text{if } D_0 \equiv 5 \pmod 8 \end{cases}.$$

Thus we have to check the values of $D_0 \pmod 8$.
From (4.34) we know

$$D_0 = \det(2m)/(2l)\, {r'_1}^2 + 2\det(2m)\left(\frac{1}{3}\sum_{i\in I}\left(r_i^{2'} - r'_ir'_{i+1} + r_{i+1}^{'2}\right) + \sum_{i\in J} r'_ir'_{i+1} - n_0\right). \quad (4.39)$$

We obtain by using that r'_1 is odd and $g \equiv 1 \pmod 4$,

$$\begin{aligned}\det(2m)/(2l)\cdot r_1^{'2} &\equiv 3^{g_1}\cdot(-1)^{g_2} \equiv 5^{g_1}\cdot(-1)^{g_1+g_2} \equiv 5^{g_1}\cdot(-1)^{\frac{g-1}{2}} \\ &\equiv 5^{g_1} \equiv \begin{cases} 1 \pmod 8 & \text{if } g_1 \text{ is even} \\ 5 \pmod 8 & \text{if } g_1 \text{ is odd} \end{cases}. \end{aligned} \quad (4.40)$$

Moreover

$$\begin{aligned} 2\det(2m)\cdot&\left(\frac{1}{3}\sum_{i\in I}\left(r_i^{'2} - r'_ir'_{i+1} + r_{i+1}^{'2}\right) + \sum_{i\in J} r'_ir'_{i+1} - n_0\right) \\ &\equiv 2\det(2m)\cdot\left(\sum_{i\in I}\left(r'_i + r'_ir'_{i+1} + r'_{i+1}\right) + \sum_{i\in J} r'_ir'_{i+1} + n_0\right) \\ \equiv &\begin{cases} 0 \pmod 8 & \text{if } \sum_{i\in I}\left(r'_i + r'_ir'_{i+1} + r'_{i+1}\right) + \sum_{i\in J} r'_ir'_{i+1} + n_0 \text{ is even} \\ 4 \pmod 8 & \text{otherwise} \end{cases}. \end{aligned} \quad (4.41)$$

Inserting (4.40) and (4.41) into (4.39) we obtain

$$D_0 \equiv \begin{cases} 1 \pmod 8 & \text{if } g_1 + \sum_{i\in I}\left(r'_i + r'_ir'_{i+1} + r_{i+1} + r'_{i+1}\right) + \sum_{i\in J} r'_ir'_{i+1} + n_0 \\ & \text{is even} \\ 5 \pmod 8 & \text{otherwise} \end{cases}.$$

Thus we have

$$\left(\frac{2}{D_0}\right) = (-1)^{\frac{1}{8}(D_0^2-1)} = (-1)^{g_1+n_0+\sum_{i\in I}\left(r'_i+r_ir'_{i+1}+r_{i+1}\right)+\sum_{i\in J} r_ir_{i+1}},$$

i.e., we have shown for an arbitrary 2-power

$$F_c = \left(\frac{c}{D_0}\right) \cdot \sum_{\lambda(c)^*} e_c(\lambda C).$$

Therefore (4.24) follows very similarly as in the case $p \neq 2$. $\square$
From this we obtain Theorem 4.19. $\square$

Remark 4.37 *We can also weaken the assumptions, made at the beginning of this section, in a way that in case $g = 1$ we have the same restrictions as in [GKZ]. That is we can skip the condition that $\frac{1}{2}\det(2m)$ is odd and furthermore replace $(\det(2m), D_0) = 1$ by the following conditions: If p divides both, $\det(2m)$ and D_0, p^2 must not divide $\det(2m)$ if $p \neq 2$, p^3 must not divide $\det(2m)$ if $p = 2$ and $\frac{D_0}{4}$ is odd, and p^4 must not divide $\det(2m)$ is $p = 2$ and $\frac{D_0}{4}$ is even. Moreover if $p \neq 2$, $\prod_{\substack{i=1 \\ i\neq j}} m_i$ has to be assumed to be a square $\pmod p$, where the m_i are choosen such that $\exists\, U \in GL_g(\mathbb{Z}/p\mathbb{Z})$ with $(2m)[U] \equiv \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_g \end{pmatrix} \pmod p$, $p|m_j$.*

We don't want to prove the claims of this chapter under the restrictions of the above Lemma because this would make calculations still more complicated since more cases have to be distinguished. We just want to mention a few words about what has to be changed. In order to prove that we are allowed to skip the condition that $\frac{1}{2}\det(2m)$ is odd we first show that in Lemma 4.21 only one block of the form $2^\nu l$ can occur. The proof that the first terms of (4.4) coincide can be adopted with little modifications. What is more difficult to show is that the second terms of (4.4) coincide. For this one has to take more cases into account. In order to show that one can also change the second condition, again the biggest difficulty lies in showing that the second terms of (4.4) coincide. Here the restriction $\prod_{\substack{i=1 \\ i\neq j}} m_i$ is a square $\pmod p$ is needed in order to obtain $\left(\frac{\prod_{\substack{i=1 \\ i\neq j}} m_i}{p}\right) = 1$ (which can be deduced in case the $p|\frac{1}{2}\det(2m)$ and $p \nmid \det(2m)$).

Appendix: List of often used symbols

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	set of positive integers, integers, rational, real and complex numbers
tr, det	trace and determinant of a matrix
Im, Re	imaginary and real part of matrices
$Sym_g(R)$	set of symmetric matrices of size g with entries from a commutative ring R
R^*	units of a (commutative) ring R with 1
$M_g(R)$	ring of $g \times g$ matrices with entries in R
$GL_g(R)$	group of invertible matrices from $M_g(R)$
E_g	identity matrix of size g
I_g	$p.$ 11
A^t	transpose of the matrix A
$A[B]$	B^tAB
$A > B, A \geq B$	$A - B$ is positive (semi-) definite
$\mathbb{P}_g$	set of positive definite matrices of size g with entries from $\mathbb{R}$
$Sp_g(\mathbb{R})$	real symplectic group of genus g
Γ_g	Siegel modular group of genus g
$SL_g(\mathbb{Z})$	special linear group of genus g
$\Gamma_{g,0}(N)$	subgroup of Γ_g $p.$ 11
$\Gamma_0(N)$	$\Gamma_{1,0}(N)$
$a \equiv b \pmod{c}$	c divides $a - b$
e_i	i-th unit vector
$\mathbb{H}_g$	Siegel upper half space of genus g
$\mathbb{H}$	$\mathbb{H}_1$
$\mathbb{F}_g$	set of Siegel reduced matrices $p.$ 12
$m_{g-1}(T)$	$p.$ 13
T_{g-1}	$(g-1) \times (g-1)$ minor of T
$M \circ Z$, $N \circ (\tau, z)$	operation of the symplectic and the Jacobi group $p.$ 13, $p.$15
$\vert_k$	slash operation of the Siegel modular and the Jacobi group $p.$ 13, 15
$S_k(\Gamma)$	vector space of Siegel cusp forms with respect to Γ
$S_k(N)$	$S_k(\Gamma_0(N))$
$S_k(N)^-$	$p.$ 86
$< \cdot, \cdot >$, $< \cdot, \cdot >_\Gamma$	Petersson scalar product for Siegel and Jacobi cusp forms $p.$ 14, $p.$ 18
dV_g, dV_g^J	symplectic volume for the Siegel modular and Jacobi group $p.$ 14, $p.$ 18
$\Gamma_{l,n}^J, \Gamma_{l,n,0}^J(N)$,	Jacobi groups 15
Λ	$Sym_2(\mathbb{Z})$
Λ^*	$\{S \in Sym_2(\mathbb{Q}) \vert\, S \text{ half-integral}\}$
$e(z)$	$\exp(2\pi i z)$
$e_c(z)$	$e(\frac{z}{c})$
$e^{n,r}(\tau, z)$	$e(n\tau + rz)$
$\left(\Gamma_{1,g}^J\right)_\infty$	stabilizer group of the function $e^{n,r}$ $p.$ 19
$J_{k,m}^{cusp}$, $J_{k,m}^{cusp}(N)$	vector spaces of Jacobi cusp forms

$(\Gamma^J_{l,n})^\uparrow$	embedding of the Jacobi group in the Siegel modular group
$P_{k,m;(n,r)}$, $P^N_{k,m;(n,r)}$, $\mathcal{P}_{k,m;(n,r)}$	Poincaré series $p.\,18$, $p.\,46$, $p.\,57$
$P_{k,m;(n,r),s}$, $P^N_{k,m;(n,r),s}$	non-holomorphic Poincaré series $p.\,24$, $p.\,24$
δ_m	$p.\,22$
$H_{m,c}$, $H^N_{m,c}$	generalized Kloosterman sums $p.\,22$, $p.\,53$
J_n	Bessel function of order n
I_n	I-Bessel function of order n
$\tilde{J}$	matrix-argument Bessel function $p.\,61$
$\lambda_{k,m,D}$	$p.\,23$
$\Gamma(\cdot)$	gamma function
$\Phi_{k,m,c,v}$, $\Phi^N_{k,m,c,v}$	certain integrals $p.\,25$
$G(a,b,c)$	Gauss sums $p.\,32$
$\nu_p(\cdot)$	p-order
$\epsilon(\cdot)$	$p.\,32$
$\mathbb{Z}_p$	ring of p-adic numbers
χ_M	characteristic function of the set M
α_g	$p.\,45$
Γ_∞	$p.\,48$
$\left(\Gamma^J_{2,1}\right)_\infty$	$p.\,57$
$\Gamma^0_\infty(N)$	$p.\,48$
$\Theta(M)$	$p.\,58$
rk	rang of a quadratic matrix
$\delta^{(n)}_{x,y}$	$p.\,60$
$E_{s,N}(Z)$, $E_s(Z)$	non-holomorphic Eisenstein series $p.\,69$
$E^*_{s,N}(Z)$, $E^*_s(Z)$	$p.\,69$
$\zeta(s)$	zeta-function
$\zeta_N(s)$	$p.\,69$
P_Z	$p.\,69$
$\mu(\cdot)$	Moebius function
$D_{F,G,N}(s)$, $D^*_{F,G,N}(s)$	Dirichlet series $p.\,75$
ι	inclusion map
$\mathcal{D}_\Delta$, $\mathcal{D}_{l,\Delta}$	sets of quadratic forms $p.\,84$
$\chi_{D_0}(\cdot)$	generalized genus character $p.\,85$
f_{k,l,Δ,ρ,D_0}	$p.86$
$\epsilon_l(m,\Delta,\rho,D_0)$	$p.\,86$
S_{la}	$p.\,86$
$r_{k,l,Q}$, r_{k,l,Δ,ρ,D_0}	cycle integrals $p.\,87$, $p.\,87$
$\mathcal{S}_{D_0,r_0}$, $\mathcal{S}^*_{D_0,r_0}$	lifting maps $p.\,89$, $p.\,89$
M^*	adjoint of a matrix
Ω_{k,m,D_0,r_0}	$p.\,94$

Bibliography

[AS] Abramowitz, N., Stegun, I.: Handbook of mathematical functions. New York: Dover (1965).

[BK] Böcherer, S., Kohnen, W.: Estimates for Fourier coefficients of Siegel cusp forms. Math. Ann. **297**, 499-517 (1993).

[Br] Breulmann, S.: Abschätzungen für Fourierkoeffizienten Siegelscher Spitzenformen vom Geschlecht 3. Doktorarbeit, Universität Heidelberg (1996).

[Ca] Cassels, J.W.S.: Rational quadratic forms. L.M.S. Monographs, No.13. London, New York, San Francisco: Academic Press (1978).

[Ch] Christian, U.: Über gewisse Poincarésche Reihen zu elliptischen Modulgruppen. Math. Gottingensis, Schriftenr. Sonderforschungsbereichs Geom. Anal. **17** (1987).

[De] Deligne, P.: La conjecture de Weil. I. Publ. Math., Inst. Hautes tud. Sci. **43**, 273-307 (1973).

[DS] Deligne, P., Serre, J.-P.: Formes modulaires de poids 1. Ann. Sci. Éc. Norm. Supér., IV. Sér. 7, 507-530 (1974).

[EZ] Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in Mathematics, **Vol. 55**. Boston-Basel-Stuttgart: Birkhuser (1985).

[GKZ] Gross, B.; Kohnen, W.; Zagier, D.: Heegner points and derivatives of L-series. II. Math. Ann. **278**, 497-562 (1987).

[FB] Freitag,E., Busam, R.: Funktionentheorie 1. Berlin Heidelberg New York: Springer (2000) (3. Aufl.).

[Fr] Freitag, E.: Siegelsche Modulfunktionen. Berlin Heidelberg New York: Springer (1983).

[He] Hecke, E.: Mathematische Werke. Hrsg. im Auftr. der Akademie der Wissenschaften in Göttingen. 3., durchges. Aufl. Göttingen: Vandenhoeck & Ruprecht (1983).

[KK] Koecher, Krieg: Elliptische Funktionen und Modulformen. Berlin: Springer (1998).

[Ki] Kitaoka, Y.: Fourier coefficients of Siegel cusp forms of degree two. Proc. Japan Acad., **Ser. A 58**, 41-43 (1982).

[K1] Kohnen, W. : Estimates of Fourier coefficients of Siegel cusp forms of degree two. Compos. Math. **87**, 231- 240 (1993).

[K2] Kohnen, W. : Estimates for Fourier coefficients of Siegel cusp forms of degree two II. Nagoya Math. J. **128** 171-176 (1992).

[K3] Kohnen, W.: On the uniform convergence of Poincaré series of exponential type on Jacobi groups. Abh. Math. Semin. Univ. Hamb. **66**, 131-134 (1996).

[KKo] Kohnen, W., Kojima, H.: A Maass space in higher genus. To appear in Compos. Math.

[K4] Kohnen, W.: On the average growth of Fourier coefficients of Siegel cusp forms of genus 2. Pac. J. Math. **179**, No.1, 119-121 (1997).

[K5] Kohnen, W.: Non-holomorphic Poincaré-type series on Jacobi groups. J. Number Theory **46**, No.1, 70-99 (1994).

[K6] Kohnen, W.: Class numbers, Jacobi forms and Siegel-Eisenstein series of weight 2 on $Sp_2(\mathbb{Z})$. Math. Z. **213**, No.1, 75-95 (1993).

[K7] Kohnen, W.: On the average growth of Fourier coefficients of Siegel cusp forms of genus 2. Pac. J. Math. **179**, No.1, 119-121 (1997).

[KS] Kohnen, W., Skoruppa, N.-P.: A certain Dirichlet series attached to Siegel modular forms of degree two. Invent Math. **95**, No 3, 541-558 (1989).

[Kl] Klingen, H.: Bemerkungen zur Konvergenz von Poincareschen Reihen. Nachr. Akad. Wiss. Göttingen, II. Math.-Phys. Kl. 1966, 1-9 (1966).

[Kr] Krieg, A.: A Dirichlet series for modular forms of degree n. Acta Arith. **59**, No.3, 243-259 (1991).

[Ku] Kuss, M.: Die Funktionalgleichung der getwisteten Spinorzetafunktion und dieBöcherer Vermutung. Heidelberg: Univ. Heidelberg, Naturwissenschaftlich-Mathematische Gesamtfakultät (2000).

[La] Lang, S.: Algebraic number theory. 2nd ed. Graduate Texts in Mathematics. **110**. New York: Springer-Verlag (1994).

[Ma] Maass, H.: Siegel's modular forms and Dirichlet series. Course given at the University of Maryland, 1969-1970. Lecture Notes in Mathematics. 216. Berlin Heidelberg New York: Springer (1971).

[Mi] Miyake, T.: Modular forms. Berlin etc.: Springer-Verlag (1989).

[Ra] Rankin, R.A.: An Ω-result for the coefficients of cusp forms. Math. Ann. **203**, 239-250 (1973).

[RS] Resnikoff, H.L.; Saldana, R.L.: Some properties of Fourier coefficients of Eisenstein series of degree two. J. Reine Angew. Math. **265**, 90-109 (1974).

[SS] Sato, M.; Shintani, T. On zeta functions associated with prehomogeneous vector spaces. Ann. Math. (2) **100**, 131-170 (1974).

[Te] Terras, A.: Harmonic analysis on symmetric spaces and applications. I. New York etc.: Springer (1985).

[Z2] Zagier, D.: Modular forms associated to real quadratic fields. Invent. Math. **30**, 1-46 (1975).

[Zi] Ziegler, C.: Jacobi forms of higher degree. Abh. Math. Semin. Univ. Hamb. **59**, 191-224 (1989).