Countable state Markov shifts with automorphism groups being a direct sum

MICHAEL SCHRAUDNER

Institut für Angewandte Mathematik, Universität Heidelberg
Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
(e-mail: schraudner@math.uni-heidelberg.de)

ABSTRACT. In this short note we prove the existence of a class of transitive, locally compact, countable state Markov shifts whose automorphism groups split into a direct sum of two groups; one being the infinite cyclic group generated by the shift map, the other being a countably infinite, centerless group H, which contains all automorphisms that act on the orbit-complement of certain finite sets of symbols like the identity. Such a decomposition is well known from the automorphism groups of coded systems, in which case one can explicitly construct example subshifts with $\mathrm{Aut}(\sigma) = \langle\sigma\rangle \oplus H$ to a variety of abstract groups H. A similar result for SFTs is yet only established for full p-shifts (p prime), where H equals the set of inert automorphisms. For general SFTs no direct sum representation is known so far. Thus our result may help to distinguish between the countable automorphism groups of SFTs and countable state Markov shifts.

KEYWORDS: countable state Markov shift, automorphism group, SFT, coded system, **(FMDP)**-graph, thinned-out graph

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1. PRELIMINARIES

We give a short introduction to the notions and definitions used in this paper. For a general overview in symbolic dynamics and some background information the author favors the monographs by D. Lind and B. Marcus [**LM**] or by B. Kitchens [**Kit**]. A more detailed reference on questions concerning notation can be found in the first section of [**Sch2**].

Let $\mathcal{A}^{\mathbb{Z}}$ denote the product space of all bi-infinite sequences over a countably infinite alphabet $\mathcal{A}$ endowed with the product topology of the discrete topology on $\mathcal{A}$. $\mathcal{A}^{\mathbb{Z}}$ is a non-compact, totally disconnected, perfect metric space. The (left-)shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $\sigma\big((x_i)_{i\in\mathbb{Z}}\big) := (x_{i+1})_{i\in\mathbb{Z}}$ is a homeomorphism inducing some dynamics on $\mathcal{A}^{\mathbb{Z}}$.

Any shift-invariant subset X of $\mathcal{A}^{\mathbb{Z}}$ endowed with the induced subspace topology, which is generated by the countable set of clopen cylinders $_n[a_0 \ldots a_m] := \{(x_i)_{i\in\mathbb{Z}} \in X \mid \forall\, 0 \le i \le m : x_{n+i} = a_i\}$ ($n \in \mathbb{Z}$, $m \in \mathbb{N}_0$) yields a subshift (X, σ).

Two subshifts (X_1, σ_1) and (X_2, σ_2) are (topologically) conjugate, if there is a shiftcommuting homeomorphism $\gamma : X_1 \to X_2$. We denote by $\mathrm{Pres}(X)$ the set of presentations of the subshift (X, σ), i.e. the set of all subshifts conjugate to (X, σ). A subshift (X, σ) is called countable state Markov shift, if its set of presentations contains an edge shift (X_G, σ) on some directed graph $G = (V, E)$ with countably infinite edge set E and σ acting on the set of bi-infinite walks $X_G := \big\{ (x_i)_{i \in \mathbb{Z}} \in E^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : \mathfrak{t}(x_i) = \mathfrak{i}(x_{i+1}) \big\}$ along the edges of G.
A subshift (X, σ) is called locally compact, if X is locally compact. For countable state Markov shifts this is equivalent to the compactness of any (every) cylinder set and to the local finiteness of G for any (every) graph presentation (X_G, σ).
A subshift (X, σ) is called (topologically) transitive, if X is irreducible. An edge shift (X_G, σ) is transitive, iff G is strongly connected.

Let (X, σ) be some subshift. A map $\varphi : X \to X$ is called an automorphism, if φ is a shiftcommuting homeomorphism from X onto itself. Obviously the set of automorphisms forms a group $\mathrm{Aut}(\sigma)$ under composition which is a conjugacy-invariant reflecting the inner structure and symmetries of the subshift.
The investigation of $\mathrm{Aut}(\sigma)$ for topological countable state Markov shifts has been initiated in [**Sch2**] where the algebraic properties and the subgroup structure of the automorphism group played the dominant role. In particular the similarities between SFTs and countable state Markov shifts with $\mathrm{Aut}(\sigma)$ countable have been emphasized and the question how to distinguish between their automorphism groups has been asked. In this paper we exhibit a subclass of locally compact, countable state Markov shifts whose countably infinite automorphism groups decompose into a direct sum.
For SFTs we do not know of any such decomposition except in the case of full-shifts on an alphabet of prime cardinality p, where the automorphism group splits due to the dimension group representation $\delta : \mathrm{Aut}(\sigma_p) \to \mathrm{Aut}(s_p) \cong \mathbb{Z}$ into the cyclic group generated by the shift map being the image of δ and the centerless, normal subgroup of inert automorphisms being the kernel of δ (see e.g. [**Wag**] or [**KRW**]). Unfortunately even this partial result is rather non-constructive due to the fact that the set of inert automorphisms is not fully understood for general SFTs. Therefore further investigations in this direction may give rise to the demanded difference and thus solve the open problem in [**Sch2**].

2. Main results

The research carried out and published by D. Fiebig and U.-R. Fiebig in [**FF2**] proves that the automorphism groups of coded systems often split into a direct sum of the cyclic group generated by the shift map and a second group which can vary – depending on the coded system – in a large set of abstract groups. Moreover their method is highly constructive.
We will show the same phenomenon for the class of transitive, locally compact, countable state Markov shifts that can be presented as edge shifts on thinned-out graphs:

DEFINITION 2.1. *A strongly connected, locally finite directed graph $G = (V, E)$ with $|E| = \aleph_0$ is a thinned-out graph, iff it contains a vertex $v \in V$ such that the set*

$L := \{l_n \mid n \in \mathbb{N}\}$ *of first-return-loops at v satisfies:*

$$\forall\, M \in \mathbb{N}_0 \;\exists\, N \in \mathbb{N} : \; \forall\, n \geq N : \; |l_{n+1}| - |l_n| > M \qquad\qquad \text{(GC)}$$

REMARK: The term 'thinned-out' has been chosen, since the gaps in the sequence of lengths of the first-return-loops at v grow unbounded (the farer from v, the thinner the structure of G). For any given bound M there are at most finitely many first-return-loops at v with length-difference less or equal to M. In particular the growth condition (GC) for $M = 0$ implies the existence of at most finitely many pairs of first-return-loops at v having a common length.

The following two propositions expose some general properties of thinned-out graphs. Proposition 2.2 proves v to be already a (one-element) vertex-ROME for G, whereas proposition 2.3 shows that the thinned-out graphs form a proper subclass of the **(FMDP)**-graphs as defined in [**Sch2**]:

A strongly connected, directed graph has the property **(FMDP)**, iff it contains at most Finitely Many pairwise edge-disjoint DoublePaths.

Recall that a doublepath in a graph G is a pair of two distinct paths of equal length connecting a common initial with a common terminal vertex. A set of doublepaths is pairwise edge-disjoint, if no edge from G is part of more than one doublepath in this set.
As usual a path/loop is called simple, if it has no proper closed subpath.

PROPOSITION 2.2. *Let $G = (V, E)$ be some thinned-out graph. The set of first-return-loops at a vertex $v \in V$ satisfies (GC), iff v is part of any bi-infinite walk along the edges of G. All the first-return-loops at such a vertex v are simple and v shows up in every non-simple path in G.*
Neither does G contain two vertex-disjoint loops nor two vertex-disjoint, bi-infinite walks.

PROOF: "$\Longrightarrow$": Suppose for every vertex $v \in V$ there is a loop $l_v := e_1\, e_2 \ldots e_{|l_v|}$ $(e_i \in E)$ avoiding v, i.e. $\mathfrak{t}(e_{|l_v|}) = \mathfrak{i}(e_1) \neq v$ and $\forall\, 1 \leq i < |l_v| : \; \mathfrak{t}(e_i) = \mathfrak{i}(e_{i+1}) \neq v$. Since G is strongly connected one can choose a shortest path p from v to $\mathfrak{i}(e_1)$ and a shortest path q from $\mathfrak{t}(e_{|l_v|})$ back to v. The subset $\left\{ p\, l_v{}^i\, q \mid i \in \mathbb{N}_0 \right\}$ of first-return-loops at v contradicts the growth condition (GC) for $M := |l_v|$. As G is thinned-out, it contains at least one vertex that is part of every loop.
Now suppose there is a bi-infinite, simple walk $w := \ldots w_{-3}\, w_{-2}\, w_{-1} \cdot w_0\, w_1\, w_2 \ldots$ in G ($w_i \in E$, $\mathfrak{t}(w_i) = \mathfrak{i}(w_{i+1}) \neq v$ $\forall\, i \in \mathbb{Z}$) avoiding v. Let p_1 be some shortest path connecting v to $\mathfrak{i}(w_0)$. Fix a shortest path p_2 from v to $\mathfrak{i}(w_{-|p_1|})$ and choose an infinite sequence $(q_i)_{i \in \mathbb{N}}$ of shortest paths connecting $\mathfrak{t}(w_{n_i})$ with $n_i := \sum_{j=1}^{i-1} |q_j|$ back to v. All these paths are non-empty and distinct. Once more they yield an infinite number of pairs $p_1\, w_0 \ldots w_{n_i}\, q_i$, $p_2\, w_{-|p_1|} \ldots w_0 \ldots w_{n_i}\, q_i$ $(i \in \mathbb{N})$ of first-return-loops at v, that violate (GC) for $M := |p_2|$. Therefore any vertex that is not part of every bi-infinite walk in G does not fulfill (GC).

"$\Longleftarrow$": Let $v \in V$ be contained in any bi-infinite walk along the edges of G and choose $w \in V$ such that the first-return-loops at w satisfy (GC). For $w = v$ the statement is obviously true. Assume $w \neq v$. Using the previous part of the proof w shows up in every bi-infinite walk as well. In particular w is part of any loop in
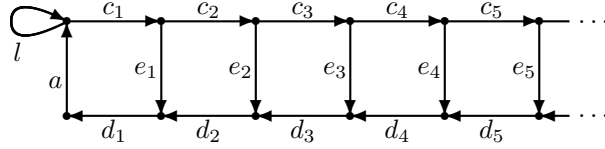
FIGURE 1: Graph presentation of a transitive, locally compact, countable state Markov shift with **(FMDP)**. The vertex $\mathfrak{t}(a)$ is a ROME; every bi-infinite walk contains at least one of the edges labeled a or l.

G and the sequence of lengths of first-return-loops at w and at v agree with each other. Actually there is either only one simple path leading from w to v or only one simple path leading from v to w. The first-return-loops at v are cyclically permuted first-return-loops at w and vice versa.

The remaining statements follow immediately. □

PROPOSITION 2.3. *Every thinned-out graph is a* **(FMDP)**-*graph.*

PROOF: Let $G = (V, E)$ be a thinned-out graph. Assume the set $L := \{l_n \mid n \in \mathbb{N}\}$ of first-return-loops at $v \in V$ satisfies the growth condition (GC). As pointed out in the remark following definition 2.1 the sequence $(|l_n|)_{n \geq N}$ is strictly increasing after some bound $N \in \mathbb{N}$. Obviously the finite subset $F := \{e \in E \mid \mathfrak{i}(e) = v\} \cup \{e \in E \mid \exists n \leq N : e \in l_n\}$ of edges covers all doublepaths in G. According to proposition 3.1 in [**Sch2**] this proves G to be a **(FMDP)**-graph. □

Figure 1 shows an example of a **(FMDP)**-graph that is not thinned-out. The lengths of the first-return-loops at $\mathfrak{t}(a)$ are $1, 4, 6, 8, 10, \ldots$, whereas for any other vertex this sequence comprises all large enough natural numbers.

Obviously definition 2.1 is a priori not invariant under (topological) conjugacy in the sense that given a transitive, locally compact, countable state Markov shift (X_G, σ) defined on a thinned-out graph G we can – using a finite number of state splittings – easily construct another graph presentation that contains vertex-disjoint loops and is therefore no longer thinned-out.
We overcome this technicality by calling a transitive, locally compact, countable state Markov shift (X, σ) thinned-out, iff its set of presentations $\mathrm{Pres}(X)$ contains an edge shift on some thinned-out graph. Via this little detour we define a conjugacy invariant subclass of all countable state Markov shifts.

The reason for studying this class is the rigid structure of a thinned-out graph G forcing each automorphism to map the set of σ-orbits corresponding to bi-infinite, simple walks along the edges of G onto itself. Usually this need not be the case even for transitive, countable state Markov shifts with **(FMDP)**. We illustrate this for the edge shift (X, σ) on the **(FMDP)**-graph displayed in figure 1. There is an order 2 automorphism $\varphi : X \to X$ that scanning a point $x \in X$ replaces every block $a\,l\,l\,c_1 \ldots c_n\,e_n$ with $a\,c_1 \ldots c_{n+1}\,e_{n+1}\,d_{n+1}$ ($\forall n \in \mathbb{N}$) and vice versa. By continuity φ maps the point $y := \ldots d_3\,d_2\,d_1\,.\,a\,c_1\,c_2\,c_3 \ldots \in X$ that corresponds to a bi-infinite, simple walk into $\varphi(y) = \ldots d_3\,d_2\,d_1\,.\,a\,l\,l\,c_1\,c_2\,c_3 \ldots \in X$, a point that

does not correspond to any bi-infinite, simple walk in G.

Automorphisms of topological Markov shifts seem not to distinguish between bi-infinite walks and bi-infinite, simple walks as long as the minimal differences between the lengths of first-return-loops remain bounded. Whereas for thinned-out graphs (unbounded length-differences) they have to respect simple walks, as we will see below. This property can be used to show that any automorphism acts on all points avoiding a certain finite set of edges like a power of the shift map. Moreover this set can always be chosen from the complement of all bi-infinite, simple walks. Therefore one can factor out the cyclic group generated by σ and represent any automorphism as a composition of a power of the shift with an automorphism which is the identity on all points corresponding to bi-infinite, simple walks. This gives the desired direct sum decomposition of $\mathrm{Aut}(\sigma)$ for edge shifts on thinned-out graphs. As the existence of such a decomposition is a purely group theoretical property of the conjugacy invariant automorphism group, the result holds – independently of the chosen presentation – for all thinned-out Markov shifts.

THEOREM 2.4. *Let (X, σ) be some thinned-out Markov shift. Any automorphism acts on the set of σ-orbits corresponding to bi-infinite, simple walks on any graph presentation on some thinned-out graph like a power of the shift map.*

The automorphism group $\mathrm{Aut}(\sigma)$ splits into the direct sum of the cyclic group generated by σ and another countably infinite, centerless group. Is (X, σ) presented on a thinned-out graph $G = (V, E)$ one gets:

$$\mathrm{Aut}(\sigma) \cong \langle \sigma \rangle \oplus \big\{ \varphi \in \mathrm{Aut}(\sigma) \,\big|\, \exists\, K_\varphi \subsetneq E \text{ finite: } K_\varphi \text{ does not contain any}$$
$$\text{edge from a bi-infinite, simple walk on } G \ \wedge \ \varphi|_{\mathrm{Orb}(K_\varphi)^{\complement}} = \mathrm{Id}_{\mathrm{Orb}(K_\varphi)^{\complement}} \big\}$$

where $\mathrm{Orb}(K_\varphi)^{\complement} := X \setminus \bigcup_{n \in \mathbb{Z}} \sigma^n \big(\bigcup_{k \in K_\varphi} {}_0[k] \big)$ denotes the orbit-complement of K_φ.

We postpone the proof of theorem 2.4 to the last section of this paper. Instead to round off our work we describe some thinned-out graph presentations. Theorem 2.4 can be applied directly to the corresponding subshifts giving examples of transitive, locally compact, countable state Markov shifts with countably infinite automorphism groups being a direct sum.

First look at the graph G displayed in figure 1. Let $S : \mathbb{N} \to \mathbb{N}$ be some superlinear function, i.e. S is monotone and grows eventually faster than any linear function (like $S(n) := n^2$, $S(n) := 2^n$ etc.). Deleting any edge e_i with $i \notin S(\mathbb{N})$ from G gives a thinned-out graph.

Figure 2 shows two strongly connected, locally finite, countable graphs. Once more let $S : \mathbb{N} \to \mathbb{N}$ be a superlinear function. Remove all edges e_i, e'_j with $i \notin S(2\mathbb{N})$ and $j \notin S(2\mathbb{N}+1)$ to get thinned-out graphs with more than one bi-infinite, simple walk (figure 2, top graph) or even with a canonical boundary (for an exact definition of this term see [**FF1**]) consisting of more than one orbit (figure 2, bottom graph). More complicated thinned-out graphs can be constructed by identifying the vertex-ROMEs of two or more appropriate thinned-out graphs with each others: For instance merge two copies $G = (V, E)$ and $G' = (V', E')$ of the graph from figure 1 by identifying the vertices $\mathfrak{t}(a) \in V$ and $\mathfrak{t}(a') \in V'$ and remove all edges $e_i \in E$ with $i \notin S(2\mathbb{N})$ as well as $e'_j \in E'$ with $j \notin S(2\mathbb{N}+1)$ for $S : \mathbb{N} \to \mathbb{N}$ superlinear.
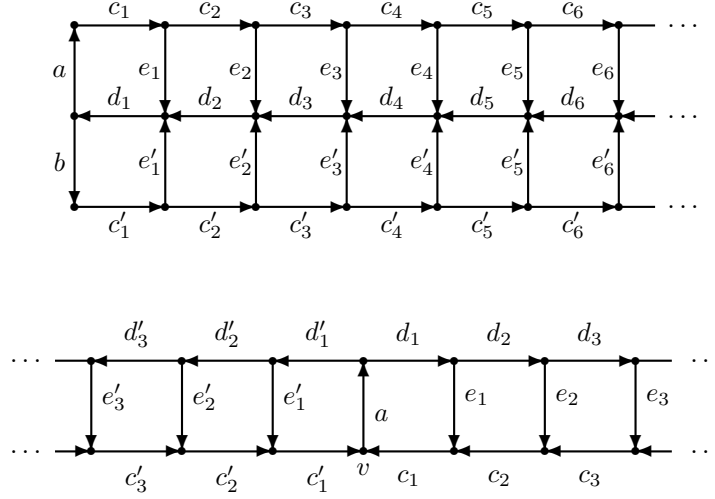
FIGURE 2: Graph presentations of two transitive, locally compact, countable state Markov shifts. The top graph has two bi-infinite, simple walks whereas the bottom graph contains four bi-infinite, simple walks and the canonical boundary of its Markov shift consists of two orbits.

3. PROOF OF THEOREM 2.4

This final section is primarily dedicated to the proof of theorem 2.4, even so it comprises two more propositions (3.2 and 3.3) on the structure of strongly connected graphs with an one-element vertex-ROME, which may be of general interest.

At first recall that every topological conjugacy $\gamma : \ X \to Y$ (in particular every automorphism) between two locally compact, countable state Markov shifts (X, σ_X), (Y, σ_Y) has some kind of local coding-lengths:

$$\forall\, C \subsetneq Y \text{ cylinder set } \ \exists\, s_C \leq t_C \in \mathbb{Z}:$$
$$(x_i)_{i \in \mathbb{Z}} \in \gamma^{-1}(C) \iff {}_{s_C}[x_{s_C} \dots x_{t_C}] \subseteq \gamma^{-1}(C) \qquad \text{(CL)}$$

In other words $\gamma^{-1}(C) = \bigcup_{x \in \gamma^{-1}(C)} {}_{s_C}[x_{s_C} \dots x_{t_C}]$ and is thus presentable as a finite union of cylinder sets. The easy argument for this is that $C \subsetneq Y$ compact-open immediately forces $\gamma^{-1}(C)$ to be compact-open.

We start the proof of theorem 2.4 with a lemma pointing out that any automorphism not only respects the σ-orbits of bi-infinite, simple walks on a thinned-out graph but that all corresponding points are just shifted by an uniform amount. This is due to the unbounded growth of the length differences in the sequence of first-return-loops at an one-element vertex-ROME. The structure of the graph far away from this single vertex has thus a large influence on the (non-)existence of certain automorphisms.

LEMMA 3.1. *Let (X_G, σ) be an edge shift on some thinned-out graph $G = (V, E)$. Any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ induces some permutation on the bi-infinite, simple walks along the edges of G.*

Moreover φ acts on corresponding points like a fixed power of the shift map and so the permutation induced by φ is actually the identity.

PROOF: We build up a whole string of arguments based on the local compactness of X_G and the validity of the growth condition (GC) for the set L of first-return-loops at some vertex $v \in V$. Let $F := \{e \in E \mid \mathfrak{i}(e) = v\}$ the finite set of out-going edges at v, then F constitutes a finite edge-ROME in G.

In the following we do not distinguish between bi-infinite walks on G and σ-orbits of corresponding points in X_G. As φ is a shiftcommuting bijection it induces an injective mapping $\tilde{\varphi} : \mathrm{Orb}(X_G) \to \mathrm{Orb}(X_G)$, $\mathrm{Orb}(x) \mapsto \mathrm{Orb}(\varphi(x))$ on the σ-orbits/walks in the obvious way. To prove the first statement of lemma 3.1 one has to show that – by choice of F – any automorphism φ maps all representatives $x := \ldots x_{-3}\, x_{-2}\, x_{-1}\, . \, x_0\, x_1\, x_2 \ldots \in \mathrm{Orb}(x)$ of bi-infinite, simple walks with $x_0 \in F$ to representatives of simple walks on G. This is done in the next two claims.

Thereafter surjectivity and thus bijectivity of $\tilde{\varphi}$ restricted to the simple walks in G immediately follows from the existence of the inverse $\tilde{\varphi}^{-1}$ defined via $\tilde{\varphi}^{-1}\big(\mathrm{Orb}(x)\big) := \mathrm{Orb}\big(\varphi^{-1}(x)\big)$, which itself acts on the bi-infinite, simple walks.

CLAIM 1. *Let $K \subsetneq E$ be any finite subset of edges. The distance from an edge in K to the nearest edge in F is uniformly bounded for all points in X_G, i.e. there exists some bound $I \in \mathbb{N}$ independent of $y \in \bigcup_{k \in K} {}_0[k] \subsetneq X_G$ such that the block $y_{[-I,I]}$ does contain an element from F.*

PROOF: As long as $B := \big\{ y_{[-i,i]} \mid y \in \bigcup_{k \in K} {}_0[k] \, \wedge \, i \in \mathbb{N} \, \wedge \, \forall\, |j| \leq i : \, y_j \notin F \big\}$ is finite (or empty) the statement is trivial. Otherwise at least one $B_k := \big\{ y_{[-i,i]} \mid y \in {}_0[k] \, \wedge \, i \in \mathbb{N} \, \wedge \, \forall\, |j| \leq i : \, y_j \notin F \big\} \subseteq B$ for some $k \in K$ has to be infinite. In which case using the local finiteness of G one could inductively choose a sequence of admissible blocks $\big(b^{(n)}\big)_{n \in \mathbb{N}_0}$ with $b^{(0)} := k$, $b^{(n+1)} := a_n\, b^{(n)}\, c_n$ $(a_n, c_n \in E)$ such that $\big(B_{b^{(n)}} := \big\{ y_{[-i,i]} \mid y \in {}_{-n}[b^{(n)}] \, \wedge \, i > n \, \wedge \, \forall\, |j| \leq i : \, y_j \notin F \big\} \big)_{n \in \mathbb{N}_0}$ is a decreasing nested sequence of infinite sets. Since X_G is closed this would imply the existence of a point $x \in X_G$ with $x_{[-i,i]} \in B_{b^{(i)}} \; \forall\, i \in \mathbb{N}_0$. x completely avoids all edges in F and thus contradicts v being a vertex-ROME for G. $\square$

Since X_G is locally compact the finite union of all zero-cylinders ${}_0[f]$ with $f \in F$ is compact-open. So for any $\varphi \in \mathrm{Aut}(\sigma)$ one can find a finite subset $K \subsetneq E$ of edges (depending on φ) with

$$F \subseteq K \quad \text{and} \quad \varphi^{-1}\big(\textstyle\bigcup_{f \in F} {}_0[f]\big) \subseteq \textstyle\bigcup_{k \in K} {}_0[k] \tag{FP}$$

Using claim 1 for a set $K \subsetneq E$ as assumed in (FP) shows that all representatives $x \in X_G$ of simple walks with $x_0 \in F$ do not contain any edges from K outside the central block $x_{[-I,I]}$. So neither the left-infinite ray $\big(\varphi(x)\big)_{(-\infty,-I)}$ nor the right-infinite ray $\big(\varphi(x)\big)_{(I,\infty)}$ of the image of x under φ contains an edge from F.

As a vertex-ROME v shows up particularly in every loop in G. So the simple walks are characterized by the property of containing precisely one edge starting at v. Therefore φ acts on the representatives of bi-infinite, simple walks in G iff

CLAIM 2. *The central block $\big(\varphi(x)\big)_{[-I,I]}$ contains exactly one edge from F.*

PROOF: According to (CL) the finite block $\big(\varphi(x)\big)_{[-I,I]}$ has bounded coding-length, that is there exists $J \geq I$ such that $\big(\varphi(y)\big)_{[-I,I]} = \big(\varphi(x)\big)_{[-I,I]}$ for all $y \in {}_{-J}[x_{-J} \ldots x_J]$.

Construct a special point $y := x_{(-\infty,-1]} \cdot x_{[0,J]} \underbrace{??\,?}\, x_{[-J,-1]}\, x_{[0,\infty)} \in X_G$ where the central block $x_{[0,J]} \underbrace{??\,?}\, x_{[-J,-1]}$ consists of precisely two first-return-loops $l_m, l_n \in L$ at v chosen by the following procedure:

W.l.o.g. let the elements in L be sorted according to their length, i.e. $|l_i| \leq |l_j|$ whenever $i \leq j$. The validity of (GC) for L implies the existence of $N_1 \in \mathbb{N}$ such that $|l_{N_1}| > 2J$ and $\big||l_i| - |l_j|\big| > 2I$ for all $i \geq N_1$ and $j \neq i$. Choose $l_m \in L$ to be the shortest first-return-loop starting with $x_{[0,|l_{N_1}|)}$. Let $N_2 \in \mathbb{N}$ with $|l_{N_2}| > |l_m|$ and $\big||l_i| - |l_j|\big| > |l_m| + 2I$ for all $i \geq N_2$ and $j \neq i$. Finally take $l_n \in L$ minimal ending in $x_{[-|l_{N_2}|,-1]}$ and set $x_{[0,J]} \underbrace{??\,?}\, x_{[-J,-1]} := l_m\, l_n$.

As we have shown above $\big(\varphi(x)\big)_{[-I,I]}$ has to contain at least one edge from the set F (v is a vertex-ROME; there is no edge from F in $\big(\varphi(x)\big)_{(-\infty,-I)}$ or $\big(\varphi(x)\big)_{(I,\infty)}$). Therefore define $a := \max\big\{i \in \mathbb{Z} \;\big|\; -I \leq i \leq I \,\wedge\, \big(\varphi(x)\big)_i \in F\big\}$ and $b := \min\big\{i \in \mathbb{Z} \;\big|\; -I \leq i \leq I \,\wedge\, \big(\varphi(x)\big)_i \in F\big\}$.

Since $y_{(-\infty,J]} = x_{(-\infty,J]}$, $y_{[|l_m\, l_n|-J,\infty)} = x_{[-J,\infty)}$ and $2J+1$ is a coding-length for $\big(\varphi(x)\big)_{[-I,I]}$ one gets $\big(\varphi(y)\big)_{[-I,I]} = \big(\varphi(y)\big)_{[|l_m\, l_n|-I,|l_m\, l_n|+I]} = \big(\varphi(x)\big)_{[-I,I]}$ and the block $B := \big(\varphi(y)\big)_{[a,|l_m\, l_n|+b]}$ consists of a concatenation of elements from L.

Suppose $B = l_j \in L$ is a single first-return-loop, to get an immediate contradiction:

$$\big||l_n| - |l_j|\big| = \big||l_n| - |l_m\, l_n| - b + a\big| = \big||l_m| + b - a\big| \leq |l_m| + |b - a| \leq |l_m| + 2I$$

By choice of l_n this estimate would imply $l_j = l_n$, but $|l_m| > 2J \geq 2I$ and $-2I \leq b - a \leq 0$ results in $|l_j| = |l_m| + |l_n| + b - a > |l_n|$.

The equivalence $y_i \in F \iff i \in \{0, |l_m|, |l_m\, l_n|\}$ together with claim 1 yields:

$$\forall\, I < i < |l_m| - I:\; y_i \notin K \quad \text{as well as} \quad \forall\, |l_m| + I < i < |l_m\, l_n| - I:\; y_i \notin K$$

Consequently the appropriate restrictions on the image point are:

$$\forall\, a < i < |l_m| - I:\; \big(\varphi(y)\big)_i \notin F \quad \text{and} \quad \forall\, |l_m| + I < i < |l_m\, l_n| + b:\; \big(\varphi(y)\big)_i \notin F$$

Now if B is assumed to start with a loop $l_i \in L$ and end in $l_j \in L$ the above gives: $|l_m| - I - a \leq |l_i| \leq |l_m| + I - a$, which can be transformed into $\big||l_m| - |l_i|\big| \leq 2I$, forcing $l_i = l_m$. In the same manner $|l_m\, l_n| + b - |l_m| - I \leq |l_j| \leq |l_m\, l_n| + b - |l_m| + I$ can be manipulated into $\big||l_n| - |l_j|\big| \leq 2I$, forcing $l_j = l_n$. Finally use $b \leq a$ to establish an upper bound on the length of B: $|B| = |l_m\, l_n| + b - a \leq |l_m| + |l_n|$. This shows $B = l_m\, l_n$ and $a = b$. So $\big(\varphi(x)\big)_{[-I,I]}$ contains precisely one edge from F and φ induces some permutation on the set of bi-infinite, simple walks.   □

We slightly generalize the idea from the proof of claim 2 to show the remaining statements:

CLAIM 3. *The unique edge from F inside $\big(\varphi(x)\big)_{[-I,I]}$ is located at a common coordinate $-M_\varphi$ $(-I \leq M_\varphi \leq I)$ for all representatives $x \in X_G$ of bi-infinite, simple walks with $x_0 \in F$.*

PROOF: Let $x^{(1)}, x^{(2)} \in X_G$ be representatives of two distinct bi-infinite, simple walks on G with $x_0^{(1)}, x_0^{(2)} \in F$ and denote by $a_1, a_2 \in \{-I, -I+1, \ldots, I\}$ the

coordinates of the unique edge from F in $\big(\varphi(x^{(1)})\big)_{[-I,I]}$, $\big(\varphi(x^{(2)})\big)_{[-I,I]}$ respectively. W.l.o.g. let $a_1 \geq a_2$.

As before construct a point $y := x^{(1)}_{(-\infty,-1]} \cdot l_m \, l_n \, x^{(2)}_{[0,\infty)} \in X_G$ with $l_m, l_n \in L$ such that l_m starts with $x^{(1)}_{[0,|l_{N_1}|)}$ and l_n ends in $x^{(2)}_{[-|l_{N_2}|,-1]}$, where N_1, $N_2 \in \mathbb{N}$ are chosen as in the previous proof (Here $J \geq I$ is a common coding-length for $\big(\varphi(x^{(1)})\big)_{[-I,I]}$ and $\big(\varphi(x^{(2)})\big)_{[-I,I]}$). Substituting $a := a_1$ and $b := a_2$ the remaining proof carries over directely from claim 2 and $a_1 = a_2$. $\qquad\square$

Since φ commutes with the shift map one instantly gets the equivalence $x_i \in F$ iff $\big(\varphi(x)\big)_{i-M_\varphi} \in F$ for any $i \in \mathbb{Z}$ and any representative $x \in X_G$ of some bi-infinite, simple walk.

CLAIM 4. *φ maps any representative $x \in X_G$ of some bi-infinite, simple walk on G into $\mathrm{Orb}(x)$, i.e. $\tilde{\varphi}$ is the identity.*

PROOF: Let $J \geq I$ be a coding-length for $\big(\varphi(x)\big)_{[-I,I]}$ and define $N_1 \in \mathbb{N}$ as before. Construct a sequence of points $\big(y^{(k)} := x_{(-\infty,-1]} \, l_m^{(k)} \, l_n^{(k)} \cdot l_m^{(k)} \, l_n^{(k)} \, x_{[0,\infty)} \in X_G\big)_{k\in\mathbb{N}}$ converging to x: Choose shortest first-return-loops $l_m^{(k)}$ starting with $x_{[0,|l_{N_1}|+k)}$. For every $k \in \mathbb{N}$ fix $N_2^{(k)} \in \mathbb{N}$ such that $\big|l_{N_2^{(k)}}\big| > \big|l_m^{(k)}\big|$ and $\big||l_i| - |l_j|\big| > \big|l_m^{(k)}\big| + 2I$ for all $i \geq N_2^{(k)}$ and $j \neq i$ exactly as above. $l_n^{(k)}$ be the shortest element in L ending in $x_{[-|l_{N_2^{(k)}}|,-1]}$. Since $y^{(k)}_{[-|l_{N_2^{(k)}}|,|l_{N_1}|+k]} = x_{[-|l_{N_2^{(k)}}|,|l_{N_1}|+k]}$ and $\big|l_{N_2^{(k)}}\big| > \big|l_{N_1}\big| + k$ this procedure forces the convergence $y^{(k)} \xrightarrow{k\to\infty} x$.

The image points $\varphi(y^{(k)})$ satisfy:

$$\forall\, k \in \mathbb{N}: \quad \big(\varphi(y^{(k)})\big)_{[-I-|l_m^{(k)} \, l_n^{(k)}|,\, I-|l_m^{(k)} \, l_n^{(k)}|]} = \big(\varphi(y^{(k)})\big)_{[-I,I]} =$$
$$= \big(\varphi(y^{(k)})\big)_{[-I+|l_m^{(k)} \, l_n^{(k)}|,\, I+|l_m^{(k)} \, l_n^{(k)}|]} = \big(\varphi(x)\big)_{[-I,I]}$$

so the blocks $\big(\varphi(y^{(k)})\big)_{[-M_\varphi-|l_m^{(k)} \, l_n^{(k)}|,\, -M_\varphi)}$ and $\big(\varphi(y^{(k)})\big)_{[-M_\varphi,\, -M_\varphi+|l_m^{(k)} \, l_n^{(k)}|)}$ consist of concatenations of first-return-loops from L. Now prove that these are both equal to $l_m^{(k)} \, l_n^{(k)}$ as above to establish:

$$\Big(\sigma^{-M_\varphi}\big(\varphi(y^{(k)})\big)\Big)_{[-|l_m^{(k)} \, l_n^{(k)}|,\, |l_m^{(k)} \, l_n^{(k)}|)} = \big(\varphi(y^{(k)})\big)_{[-M_\varphi-|l_m^{(k)} \, l_n^{(k)}|,\, -M_\varphi+|l_m^{(k)} \, l_n^{(k)}|)} =$$
$$= l_m^{(k)} \, l_n^{(k)} \, l_m^{(k)} \, l_n^{(k)} = \big(y^{(k)}\big)_{[-|l_m^{(k)} \, l_n^{(k)}|,\, |l_m^{(k)} \, l_n^{(k)}|)}$$

For k increasing $\big(\sigma^{-M_\varphi} \circ \varphi\big)(y^{(k)})$ and $y^{(k)}$ coincide on longer and longer blocks symmetric to the zero-coordinate: $\big((\sigma^{-M_\varphi} \circ \varphi)(y^{(k)})\big)_{[-k,k]} = y^{(k)}_{[-k,k]}$. As φ is continuous the convergence $y^{(k)} \xrightarrow{k\to\infty} x$ guarantees $\varphi(y^{(k)}) \xrightarrow{k\to\infty} \varphi(x)$ and in the limit one gets the demanded result:

$$\varphi(x) = \lim_{k\to\infty} \Big(\big(\varphi \circ \sigma^{-M_\varphi}\big)\big(\sigma^{M_\varphi}(y^{(k)})\big)\Big)_{[-k,k]} = \lim_{k\to\infty} \big(\sigma^{M_\varphi}(y^{(k)})\big)_{[-k,k]} = \sigma^{M_\varphi}(x)$$

$\qquad\square$

Combining claims 3 and 4 φ acts on the set of points corresponding to bi-infinite, simple walks like the M_φ-th power of the shift map. $\qquad\square$

PROPOSITION 3.2. *If a strongly connected directed graph $G = (V, E)$ has an one-element vertex-ROME $v \in V$, then every loop in G contains an edge which does not show up in any bi-infinite, simple walk.*

PROOF: Suppose there is a loop $l := e_1 e_2 \ldots e_{|l|}$ ($e_i \in E$) incompatible with the statement. W.l.o.g. l is simple with $|l| \geq 2$ and $\mathfrak{i}(e_1) = \mathfrak{t}(e_{|l|}) = v$. As $e_{|l|}$ is part of some bi-infinite, simple walk there is a left-infinite, simple walk $x_- := \ldots x_{-2} x_{-1}$ avoiding v and ending at the earliest possible vertex in l, i.e. $\mathfrak{t}(x_{-1}) = \mathfrak{t}(e_m) \neq v$ with $1 \leq m < |l|$ minimal. In the same way denote by $x_+ := x_1 x_2 \ldots$ a right-infinite, simple walk which never visits v and leaves l at the latest possible vertex $\mathfrak{i}(x_1) = \mathfrak{i}(e_n) \neq v$ with $1 < n \leq |l|$ maximal. Such x_+ exists, since e_1 shows up in some bi-infinite, simple walk in G.
If $m < n$, then one would have a bi-infinite walk $x_- e_{m+1} \ldots e_{n-1} x_+$ completely avoiding the vertex-ROME v. Therefore assume $m \geq n$: By choice of m, n every bi-infinite walk containing an edge e_i ($n \leq i \leq m$) would visit v at least two times – once before e_i, once afterwards. So the edges $e_n, e_{n+1}, \ldots, e_m$ can never show up in any bi-infinite, simple walk contradicting the assertion on l. $\square$

PROPOSITION 3.3. *Let $G = (V, E)$ be a strongly connected, locally finite directed graph (with $|E| = \aleph_0$) having an one-element vertex-ROME $v \in V$. Every simple path p in G which is part of infinitely many first-return-loops at v is already contained in some bi-infinite, simple walk.*

PROOF: Denote by L the set of first-return-loops at v and by (X_G, σ) the Markov shift given on G. Inductively construct a representative $x \in X_G$ of some bi-infinite, simple walk: For $b^{(0)} := p$ define the infinite set $L_{b^{(0)}} := \{p\,t\,r \mid \exists l \in L, r, t \text{ paths}: l = r\,p\,t\} \cup \{p\,s \mid \exists l \in L, r, s, t \text{ paths}: l = r\,s\,t \ \wedge \ t\,r = p\}$ of all (cyclically permuted) first-return-loops starting with p. Choose $a_n, c_n \in E$ ($n \in \mathbb{N}$) such that $b^{(n)} := a_n b^{(n-1)} c_n$ is a valid path in G and the set $L_{b^{(n)}} := \left\{ l \in L_{b^{(0)}} \,\middle|\, |l| \geq 2n + |p| \ \wedge \ \exists q \text{ path}: l = p\,q \ \wedge \ q\,p\,q \text{ contains } b^{(n)} \right\}$ remains infinite. This yields a nested sequence $_0[b^{(0)}] \supseteq _{-1}[b^{(1)}] \supseteq _{-2}[b^{(2)}] \supseteq \ldots$ of non-empty cylinders converging to a point $x \in \bigcap_{n \in \mathbb{N}} {}_{-n}[b^{(n)}]$.
Since elements in $L_{b^{(n)}}$ have at least length $2n + |p|$ and contain exactly one edge from $F := \{e \in E \mid \mathfrak{i}(e) = v\}$, there are at most two edges from F in $q\,p\,q$ separated by a block of length $|l| - 1 \geq 2n + |p| - 1$. Therefore the block $b^{(n)}$ is too short to comprise more than one edge from F. As F is an edge-ROME, x has precisely one edge in F and is thus a representative of some bi-infinite, simple walk with $x_{[0, |p|-1]} = p$. $\square$

LEMMA 3.4. *Let (X_G, σ) be an edge shift on some thinned-out graph $G = (V, E)$, $v \in V$ a vertex-ROME and L the set of first-return-loops at v. For any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ there is an integer $M_\varphi \in \mathbb{Z}$ and a finite set $K_\varphi \subsetneq E$ of edges lying in the complement of all bi-infinite, simple walks in G such that $\varphi|_{\mathrm{Orb}(K_\varphi)^\complement} = \sigma^{M_\varphi}|_{\mathrm{Orb}(K_\varphi)^\complement}$.*
Moreover the condition that none of the edges in K_φ is contained in some bi-infinite, simple walk is equivalent to K_φ marking only finitely many elements in L.

PROOF: Denote by $M_\varphi \in \mathbb{Z}$ the integer found in lemma 3.1, i.e. φ acts like σ^{M_φ} on all representatives of bi-infinite, simple walks in G. To prove the main statement

one has to construct a finite set $K_\varphi \subsetneq E$ such that the back-shifted automorphism $\varphi \circ \sigma^{-M_\varphi}$ induces the identity on $\mathrm{Orb}(K_\varphi)^{\complement}$. (The notation K_φ instead of $K_{\varphi \circ \sigma^{-M_\varphi}}$ is justified, since $\mathrm{Orb}(K_\varphi)^{\complement}$ is shiftinvariant and thus K_φ is instantly valid for all automorphisms $\varphi \circ \sigma^i$ ($i \in \mathbb{Z}$).)

Be $F := \{e \in E \mid \mathfrak{i}(e) = v\}$ the set of out-going edges at v and $I \in \mathbb{N}$ a global bound for the distance from any edge in $K \subsetneq E$ given as in (FP) to the nearest edge from F. Since $\bigcup_{f \in F} {}_0[f]$ is compact-open there is a common coding-length $J \geq I$ for all edges in F such that for all representatives $x \in X_G$ of bi-infinite, simple walks with $x_0 \in F$ and $y \in {}_{-J}[x_{-J} \ldots x_J]$ the zero-coordinates of the image points coincide: $\left(\varphi \circ \sigma^{-M_\varphi}(y)\right)_0 = \left(\varphi \circ \sigma^{-M_\varphi}(x)\right)_0 = x_0 \in F$. Once more the validity of (GC) allows one to choose $N \in \mathbb{N}$ such that $|l_N| > 2J$ and $\big||l_i| - |l_j|\big| > 2I$ for all $i \geq N$ and $j \neq i$.

As there exist only finitely many paths of length J ($J+1$) ending (starting) at v, almost all elements in L start with some block from $B_+ := \big\{y_{[0,J]} \mid y \in \bigcup_{f \in F} {}_0[f] \wedge y$ represents some bi-infinite, simple walk$\big\}$ and end in a block from $B_- := \big\{y_{[-J,-1]} \mid y \in \bigcup_{f \in F} {}_0[f] \wedge y$ represents some bi-infinite, simple walk$\big\}$. To show this, define finite sets $A_+ := \big\{y_{[0,J]} \mid y \in \bigcup_{f \in F} {}_0[f] \wedge y_{[0,J]}$ a simple path$\big\}$ and $A_- := \big\{y_{[-J,-1]} \mid y \in \bigcup_{f \in F} {}_0[f] \wedge y_{[-J,-1]}$ a simple path$\big\}$. A_+ (A_-) comprises the prefixes (suffixes) of all first-return-loops at v of length greater than $J+2$. According to proposition 3.3 any $a \in A_+$ (or $a \in A_-$) being part of infinitely many first-return-loops at v is already an element in B_+ (or B_-). Thus there is only a finite set of exceptional elements in L.

Define $L_\varphi := \{l_i \in L \mid i \geq N \wedge \exists\, u \in B_+, w \in B_-, b \text{ path}: l_i = u\,b\,w\}$. Its complement $L \setminus L_\varphi$ is finite. Using proposition 3.2 one can build up K_φ taking from every element in $L \setminus L_\varphi$ a single edge which is not part in any bi-infinite, simple walk.

Obviously every point in $\mathrm{Orb}(K_\varphi)^{\complement}$ can be approximated by a convergent sequence of points being infinite concatenations of elements from L_φ. For such points $x \in L_\varphi{}^\infty$ the equality $\varphi \circ \sigma^{-M_\varphi}(x) = x$ can be established using the same arguments as in the proof of lemma 3.1. Continuity of $\varphi \circ \sigma^{-M_\varphi}$ then shows $\varphi \circ \sigma^{-M_\varphi}\big|_{\mathrm{Orb}(K_\varphi)^{\complement}} = \mathrm{Id}_{\mathrm{Orb}(K_\varphi)^{\complement}}$.

Finally it remains to show the demanded equivalence for K_φ:

"$\Longrightarrow$": Suppose K_φ marks infinitely many elements in L then this is already true for some $k \in K_\varphi$ and this implies $\mathfrak{i}(k) \neq \mathfrak{t}(k)$ (self loops cannot show up in several first-return-loops). Following from proposition 3.3 the simple path $p := k$ would be part of some bi-infinite, simple walk.

"$\Longleftarrow$": Assume $k \in K_\varphi$ shows up in a representative $x := (x_i)_{i \in \mathbb{Z}} \in X_G$ of some bi-infinite, simple walk with $\mathfrak{i}(x_0) = v$, i.e. $x_N = k$ for some $N \in \mathbb{Z}$. For $n \geq N \geq 0$ choose some minimal path $p^{(n)}$ from $\mathfrak{t}(x_n)$ back to v. This gives an infinite subset $\big\{x_{[0,n]}\, p^{(n)} \mid n \geq N\big\} \subseteq L$ of first-return-loops containing k. For $N < 0$ the infinite subset $\big\{q^{(n)}\, x_{[-n,-1]} \mid n \geq |N|\big\} \subseteq L$ with $q^{(n)}$ a shortest path from v to $\mathfrak{i}(x_{-n})$ ($n \geq |N|$) forces the same contradiction. $\qquad\square$

After these preparations we can easily finish the proof of theorem 2.4. Most of the work is already done: The first statement is essentially lemma 3.1; the existence of K_φ is shown in lemma 3.4. What remains is the decomposition of $\mathrm{Aut}(\sigma)$:

PROOF: It is easy to see that the second part of the direct sum representation

$H := \big\{ \varphi \in \mathrm{Aut}(\sigma) \mid \exists\, K_\varphi \subsetneq E \text{ finite, as in lemma 3.4} \wedge \varphi|_{\mathrm{Orb}(K_\varphi)^\complement} = \mathrm{Id}_{\mathrm{Orb}(K_\varphi)^\complement} \big\}$ is actually a subgroup of $\mathrm{Aut}(\sigma)$: Let $\varphi, \phi \in H$ with corresponding finite sets $K_\varphi, K_\phi \subsetneq E$, then $K_{\varphi\phi} := K_\varphi \cup K_\phi \subsetneq E$ is still finite and does not contain any edge from bi-infinite, simple walks. Moreover $\mathrm{Orb}(K_{\varphi\phi})^\complement \subseteq \mathrm{Orb}(K_\varphi)^\complement \cap \mathrm{Orb}(K_\phi)^\complement$ such that $\big(\varphi \circ \phi\big)|_{\mathrm{Orb}(K_{\varphi\phi})^\complement} = \mathrm{Id}_{\mathrm{Orb}(K_{\varphi\phi})^\complement}$. Of course $\mathrm{Id} \in H$ (choose $K_{\mathrm{Id}} := \emptyset$) and $\varphi \in H$ implies $\varphi^{-1} \in H$ by means of $K_{\varphi^{-1}} := K_\varphi$.

The map $\alpha: \mathrm{Aut}(\sigma) \to \langle \sigma \rangle$, $\varphi \mapsto \sigma^{M_\varphi}$ with $M_\varphi \in \mathbb{Z}$ as in lemma 3.1 is a well-defined additive homomorphism with $M_{\varphi\phi} := M_\varphi + M_\phi$ for $\varphi, \phi \in \mathrm{Aut}(\sigma)$, i.e. $\alpha(\varphi \circ \phi) = \sigma^{M_{\varphi\phi}} = \sigma^{M_\varphi + M_\phi} = \sigma^{M_\varphi} \circ \sigma^{M_\phi} = \alpha(\varphi) \circ \alpha(\phi)$ and $K_{\varphi\phi} = K_\varphi \cup K_\phi$ as above gives $\big(\varphi \circ \phi\big)|_{\mathrm{Orb}(K_{\varphi\phi})^\complement} = \sigma^{M_{\varphi\phi}}|_{\mathrm{Orb}(K_{\varphi\phi})^\complement}$.

The kernel of α is H (see lemma 3.4); so there are two normal subgroups $\langle \sigma \rangle, \ker(\alpha) \trianglelefteq \mathrm{Aut}(\sigma)$ with $\langle \sigma \rangle \cdot \ker(\alpha) = \{ \sigma^n \circ \varphi \mid n \in \mathbb{Z} \wedge \varphi \in \ker(\alpha) \} = \mathrm{Aut}(\sigma)$ and $\langle \sigma \rangle \cap \ker(\alpha) = \{ \mathrm{Id} \}$ and so $\mathrm{Aut}(\sigma)$ decomposes into a direct sum.

Since thinned-out Markov shifts have countable automorphism groups (see proposition 2.3 above and theorem 2.3 in [**Sch2**]), H is countably infinite. Furthermore using theorem 5.4 in [**Sch2**] the center of $\mathrm{Aut}(\sigma)$ has to be isomorphic to $\mathbb{Z}$ for non-trivial Markov shifts. This proves H centerless. $\square$

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