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EXISTENCE OF SMOOTH SHOCK PROFILES FOR HYPERBOLIC SYSTEMS WITH RELAXATION

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Finally, I dedicate this thesis to my parents.

Abstract: The aim of this thesis is the proof of the existence of relaxation shock profiles. The existence results apply if the reduced system is strictly hyperbolic, satisfies the strict entropy condition, and if the underlying hyperbolic system with relaxation fulfills easy-to-check structural conditions. In general, the ODE system for the relaxation shock profile has a singular right-hand-side. The structural conditions allow the construction of a locally invariant manifold $\tilde{\Gamma}$, where the vector field to this ODE system has a smooth extension from a dense subset of $\tilde{\Gamma}$ throughout $\tilde{\Gamma}$ and the classical center manifold theorem applies. We apply our results to exponentially based moment closure systems.

Zusammenfassung: Ziel dieser Arbeit ist der Beweis der Existenz von Relaxationsschockprofilen. Die Existenzresultate finden Anwendung, wenn das reduzierte System strikt hyperbolisch ist, die strikte Entropiebedingung erfüllt und das zugrunde liegende Relaxationssystem leicht nachzuprüfende Strukturbedingungen erfüllt. Im Allgemeinen hat das gewöhnliche Differentialgleichungssystem für das Schockprofil eine singuläre rechte Seite. Die Strukturbedingungen erlauben die Konstruktion einer lokal invarianten Mannigfaltigkeit $\tilde{\Gamma}$, auf der das Vektorfeld zu diesem gewöhnlichen Differentialgleichungssystem eine glatte Fortsetzung von einer dichten Teilmenge von $\tilde{\Gamma}$ auf ganz $\tilde{\Gamma}$ besitzt und das klassische Zentrumsmannigfaltigkeitstheorem Anwendung findet. Wir wenden unsere Ergebnisse auf exponentiell basierte Momentenabschluss-Systeme an.

Contents

Introduction **3**
Organization of this work 5

1 Structural conditions and main results **7**
1.1 Structural conditions 8
1.2 Main theorem 12
1.3 Traveling waves for simple shocks 14

2 Proof of main theorem **21**
2.1 Preliminaries 22
2.2 Checking of weakened structural condition 31
2.3 Algebraic structure 35
2.3.1 Basic proposition 36
2.3.2 Generalized kernel of linearization 41
2.3.3 Signature of real parts of nonvanishing eigenvalues . . . 46
2.4 Center manifold reduction 57

3 Moment closure systems **66**
3.1 Derivation 66
3.2 Structural conditions on moments fulfilling the Galilean invariance property 68
3.3 Smooth extension 69
3.4 The modified moment closure system 71
3.5 The entropy condition 74
3.6 Existence of smooth shock profiles 74
3.6.1 Checking of weakened structural condition 74
3.6.2 Checking of simplicity condition for a special case . . . 81
3.7 Appendix 84

3.7.1 Strict entropy condition 84
3.7.2 Conclusions from Galilean Invariance 88

Basic notation **92**

Bibliography **94**

Introduction

A large number of physical phenomena involving nonequilibrium processes is modelled by first-order partial differential equations of the form

$$U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = Q(U)/\epsilon, \tag{1}$$

where $\epsilon > 0$ is a small parameter, U is an unknown n-vector valued function of $(x, t) \equiv (x_1, \cdots, x_d, t) \in \mathbb{R}^d \times [0, \infty)$ and Q, F_j are given smooth n-vector valued functions.

Furthermore, we assume that we have *conservation structure*, i.e. the first $n - r$ components of $Q(U)$ vanish.

Important examples occur in inviscid gas dynamics with relaxation, magnetohydrodynamics, kinetic theories, extended thermodynamics, nonlinear optics, numerics of conservation laws, and so on. Typical examples and further references are given in [Y3].

In order to have a well-defined limit for $\epsilon \to 0$ (also called *relaxation limit*) Yong introduced in [Y1] the so-called *second stability condition* which from now on is referred to as *stability condition*.

This condition consists of the decomposition of $Q(U)$ into a conservation and a relaxation part, the symmetrizability of system (1) and a condition on the coupling of the symmetrizer with the relaxation term (see (1.1.4)).

In the context of shock structure problems, the existence of traveling waves, i.e. solutions of the form $\phi(\xi)$, $\xi = (-st + \omega_j x_j)/\epsilon$ ($\omega = (\omega_1, \cdots \omega_d) \in \mathbb{S}^{d-1}$) solving

$$(2) \qquad -s\phi_\xi + \sum_{j=1}^{d} \omega_j F_j(\phi)_\xi = Q(\phi),$$

is of independent physical interest ([WE]).

From the conservation structure it follows that for each solution $\phi(\cdot)$ to (2) with trajectory starting at a point U_- there exist $n-r$ constant quantities $c_1, \cdots, c_{n-r}$:

$$c_1(\phi(\cdot)) \equiv c_1(U_-), \cdots, c_{n-r}(\phi(\cdot)) \equiv c_{n-r}(U_-).$$

We say that the trajectory is contained in the *conservation manifold* through point U_-.

If $(-sI_n + \sum_{j=1}^{d} \omega_j F_{jU}(\phi))^{-1}$ does not become singular and if the stability condition together with generic coupling conditions is fulfilled, then the existence of traveling wave solutions has been proven (see [Y-Z]).

Here, the center manifold reduction works as well as in the existence proof for viscous profiles by Majda and Pego (see [M-P]): System (2) can be brought into the form

$$(3) \qquad \phi_\xi = \mathcal{F}(s, \phi),$$

where $\mathcal{F}$ is smooth.

Application of the the center manifold theorem gives the existence of a slow invariant manifold of system (3) containing the fixed points of this system.

The intersection of this slow manifold with the conservation manifold is a curve.

The orbit for the profile is bounded by two neighboring fixed points on the curve and is contained in the stable manifold at one of them and in the unstable manifold at the other one.

If $(-sI_n + \sum_{j=1}^{d} \omega_j F_{jU}(\phi))^{-1}$ becomes singular, the right-hand-side $\mathcal{F}$ becomes singular, and the application of the center manifold theorem is not straightforward.

For this case we develop a modification of the center manifold reduction method applied in [Y-Z].

This is possible, if, in addition to the stability condition, the *dissipativity condition* (also called *Kawashima's condition*) formulated by Shizuta and Kawashima in [S-K], is fulfilled and if the *simplicity condition* is fulfilled, i.e. zero is an eigenvalue of $-sI_n + \sum_{j=1}^{d} \omega_j F_{jU}(\phi)$ with multiplicity less than or equal to one.

We construct a manifold Γ where the ODE system (2) defines a smooth vector field and then apply the center manifold theorem in order to obtain a slow invariant submanifold of Γ containing the fixed points.

The existence theorem for this slow manifold (see (1.2.1)) is referred to as main theorem.

Furthermore, the main theorem gives information about the dimension of the stable and unstable manifolds at fixed points.

If the shock profile represents a *simple shock* (see chapter 1, section 3), the trajectory of the shock profile is localized in the one-dimensional intersection of the slow manifold of Theorem (1.2.1) with the conservation manifold and is bounded by two neighboring fixed points (see Theorem (1.3.1)).

The methods developed in this thesis allow the construction of smooth shock profiles connecting two states in a small enough neighborhood.

The existence of smooth profiles to general shocks remains to be an unresolved problem.

Organization of this work

This work is organized as follows:

In chapter 1 we introduce the *stability condition*, the *dissipativity condition* and the *weakened structural condition* and state the main theorems.

For the construction of the slow invariant submanifold we need the simplicity condition and the weakened structural condition which is implied by the stability and the dissipativity condition.

In chapter 2 we prove the main results.

We will show that the stability and dissipativity condition in fact implies the weakened structural condition and we will construct the slow invariant manifold for the traveling wave equations.

Furthermore, we will analyze the qualitative behavior on the slow manifold.

In chapter 3 we apply our results to exponentially based moment closure systems.

The relaxation limit of these equations are the Euler equations for compressible fluids.

If two states connected by a simple shock solving the Euler equations are close enough to each other and the simplicity condition is fulfilled then they can be represented by relaxation shock profiles which are solutions of moment closure systems.

Chapter 1

Structural conditions and main results

We consider first-order hyperbolic PDE with source term of the form

$$U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = \frac{1}{\epsilon} Q(U), \tag{1.1}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{U} \subset \mathbb{R}^n, u \in \mathbb{R}^{n-r}, v \in \mathbb{R}^r$, $\mathcal{U}$ is open, ϵ is a positive real parameter, U is an n-vector valued function of $(x,t) \equiv (x_1, \cdots, x_d, t) \in \mathbb{R}^d \times [0, \infty)$.

The n-vector valued function $Q(U)$ and the n-vector valued functions $F_j(U)$ are assumed to be of class $\mathbf{C}^5$.

Set $B(U) = Q_U(U)$ and $A_j(U) = F_{jU}(U)$ for $j = 1, \cdots, d$.

Furthermore, we assume that $Q(U)$ has the form

$$Q(U) = Q(u,v) = \begin{pmatrix} 0_{n-r} \\ q(u,v) \end{pmatrix}. \tag{1.2}$$

The first $n-r$ zero entries of Q correspond to $n-r$ conserved quantities.

In many physically relevant systems these conserved quantities are mass, momentum and energy.

1.1 Structural conditions

We motivate the structural conditions for linear systems with constant coefficients of the form

$$U_t + \sum_{j=1}^{d} A_j U_{x_j} = BU/\epsilon, \tag{1.3}$$

which can be viewed as linearization of system (1.1) in a constant state in equilibrium.

In order to have a well-defined limit $\epsilon \to 0$ for x-independent solutions

$$U^\epsilon(x,t) = exp(tB/\epsilon)U^\epsilon(0,0)$$

we introduce the following condition:

Condition 1.1.1 *For each $U \in \mathcal{E} := \{U \in \mathcal{U} \mid Q(U) = 0\}$ the Jacobian $B(U) = Q_U(U)$ has no nonzero purely imaginary eigenvalues.*

Furthermore, we propose an analogous condition for the x_j- direction:

Condition 1.1.2 *If $U \in \mathcal{E}$ and $A_j(U)$ is invertible, then $A_j^{-1}(U)B(U)$ has no nonzero purely imaginary eigenvalues.*

The Fourier transform of the solution $U^\epsilon(x,t)$ of (1.3) with respect to x is

$$\hat{U}^\epsilon(\xi,t) = exp(tH_r(1/\epsilon,\xi))\hat{U}_0(\xi),$$

where $\hat{U}_0$ is the Fourier transform of the initial value U_0 and

$$H_r(\eta,\xi) = \eta B + i\sum_{j=1}^{d} \xi_j A_j.$$

In order to have well-posedness for the initial value problem of the linear system we assume that the hyperblicity assumption is fulfilled, that is, there exists a positive constant C such that

$$|exp(H_r(0,\xi))| \le C$$

for all $\xi \in \mathbb{R}^d$, where $|\cdot|$ denotes some norm for matrices.

Assume that the hyperbolicity assumption is fulfilled and that

$$\sup_{\eta \ge 0, \xi \in \mathbb{R}^d} |exp(H_r(\eta,\xi))| = \infty.$$

Under these conditions it is proven in [Y3] that for any $t > 0$, there exists $U_0 \in L^2$ such that the unique global solution $U^\epsilon(x,t)$ to (1.3) with initial data U_0 satisfies

$$\limsup_{\epsilon \to 0} \|U^\epsilon(\cdot,t)\|_{L^2} = \infty.$$

This fact implies a necessary criterion for a well-defined limiting behavior for $\epsilon \to 0$, the so-called *stability criterion*:

> There is $C(U) > 0$ such that
>
> $$\left| exp\left(\eta B(U) + i\sum_{j=1}^{d} \xi_j A_j(U)\right)\right| \le C(U) \qquad (1.4)$$
>
> for $Q(U) = 0$, $\eta \ge 0$ and $\xi \in \mathbb{R}^d$ where $|\cdot|$ denotes some norm for matrices.

In order to give a characterization of the stability criterion in terms of verifiable conditions Yong proved the following fundamental lemma by applying the Kreiss matrix theorem (for further references see [KR], [K-L] and [Y3]):

Lemma 1.1.3 *The stability criterion is equivalent to there being a positive constant C and a Hermitian matrix $A_0(\xi)$, defined for $\xi \in \mathbb{R}^d$ with $|\xi| \le 1$, such that*

$$C^{-1}I_n \le A_0(\xi) \le CI_n, \quad A_0(\frac{\xi}{\eta+|\xi|})H_r(\eta,\xi) + H_r^*(\eta,\xi)A_0(\frac{\xi}{\eta+|\xi|}) \le 0$$

for all (η,ξ) with $\eta \ge 0$ and $\xi \in \mathbb{R}^d$.

Many physically relevant relaxation systems admit the existence of a positive definite Hermitian matrix $A_0(\xi)$ independent of ξ such that the condition of the lemma is fulfilled.

Such systems fulfill the conditions (1.1.1), (1.1.2) and the stability criterion, if the so-called *stability condition* is fulfilled (see [Y1], [Y2]):

Condition 1.1.4 (stability condition) *There is an invertible $n \times n$-matrix $P(U)$ and an invertible $r \times r$-matrix ($0 < r \leq n$) $S(U)$ defined on $\mathcal{E} = \{U \in \mathcal{U} \mid Q(U) = 0\}$ such that*

$$P(U)B(U) = \begin{pmatrix} 0 & 0 \\ 0 & S(U) \end{pmatrix} P(U). \tag{1.5}$$

As a hyperbolic system, (1) is symmetrizable, that means: For each $U \in \mathcal{U}$ there exists a positive definite Hermitian matrix $A_0(U)$ with

$$A_0(U)A_j(U) = A_j^*(U)A_0(U) \ \ for \ \ all \ \ j. \tag{1.6}$$

The hyperbolic part and the source term are coupled in the following sense:

$$A_0(U)B(U) + B^*(U)A_0(U) \leq -P^*(U) \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} P(U) \ \ \forall U \in \mathcal{E}. \tag{1.7}$$

For the solution $U^1(\cdot, t)$ of (1.3) to $\epsilon = 1$ the time-asymptotic limit requirement

$$\lim_{t \to \infty} \|U^1(\cdot, t)\|_{L^2} = 0$$

is fulfilled, if $H_r(1, \xi)$ is stable, i.e., all of its eigenvalues have negative real parts.

If the stability condition is satisfied, the matrix $H_r(1, \xi)$ is stable, if $\sum_{j=1}^d \xi_j A_j$ has no eigenvectors in $ker(B)$ (see [Y3], p. 279f).

Hence, we introduce the *dissipativity condition*:

Condition 1.1.5 (dissipativity condition) *For each $U \in \mathcal{E}$, $\omega \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$ it holds*

$$ker(\sum_{j=1}^{d}\omega_j A_j(U) - sI_n) \cap ker(Q_U(U)) = \{0\}.$$

Set $A(U) = \sum_{j=1}^{d} \omega_j A_j(U)$ and introduce for $U \in \mathcal{E}$ the block decomposition corresponding to $P(U)$

$$P(U)A(U)P^{-1}(U) = \begin{pmatrix} A_{11}(U) & A_{12}(U) \\ A_{21}(U) & A_{22}(U) \end{pmatrix}. \tag{1.8}$$

From now on, we assume that the origin is contained in $\mathcal{E}$, and non-mentioning the variable means evaluation at the origin.

Let Π denote the projector onto $ker(A)$ commuting with A and define

$$\tilde{A} = \frac{1}{2\pi i}\int_{\Gamma} \zeta^{-1}(\zeta - A)^{-1} d\zeta, \tag{1.9}$$

where Γ encircles the nonvanishing eigenvalues of A in counterclockwise direction and does not enclose the origin.

Furthermore, set

$$\tilde{\Lambda} = \frac{1}{2\pi i}\int_{C} (\zeta - \Pi B)^{-1} d\zeta,$$

where C is a curve which surrounds the origin in counterclockwise direction and does not enclose the non-vanishing eigenvalues of ΠB.

The operator $\tilde{\Lambda}$ commutes with ΠB and projects onto the generalized eigenspace of ΠB to eigenvalue zero.

Choose bases $(r_1, \cdots, r_m)$ of $ker(A_{11})$ and $(l_1^*, \cdots, l_m^*)$ of $ker(A_{11}^*)$, such that $l_i r_j = \delta_{ij}$, and, for $r = \sum_{i=1}^{m} \alpha_i r_i$, set

$$l(r) = \sum_{i=1}^{m} \alpha_i l_i. \tag{1.10}$$

The stability and dissipativity condition have already been checked for a lot of relaxation systems, and we will show that they imply the weakened structural condition which reads as follows:

Condition 1.1.6 (weakened structural condition) *With P fulfilling relation (1.5), and with the block decomposition (1.8) induced by P there holds:*

PAP^{-1} *has a positive definite block diagonal symmetrizer of the form* $diag(A_{01}, A_{02})$ *corresponding to the partition (1.8). Zero is a semisimple eigenvalue of* ΠB *and* $\Pi diag(0_{n-r}, I_r)$*, each nonzero eigenvalue of* $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ *and* ΠB *has nonzero real part. Furthermore, there holds*

$$Re(l(r)A_{12}S^{-1}A_{21}r) < 0 \ \ \forall\, r \in ker(A_{11}) \setminus \{0\}, \tag{1.11}$$

where $l(r)$ is defined in (1.10).

After eventual linear transformation by P and multiplication of $P^{-*}A_0P^{-1}$ (with A_0 being the symmetrizer of A) from the left we can assume that system (1.1) has *normal form*, i.e. $P = A_0 = I_n$.

In the next chapter we will show the following proposition:

Proposition 1.1.7 *Assume that system (1.1) has normal form and that the stability and dissipativity condition are fulfilled. Then the weakened structural condition (1.1.6) is satisfied.*

1.2 Main theorem

The main task of this work is to treat the ODE system for traveling waves $\phi(\xi)$ under the scaling

$$\xi = (-st + \sum_{j=1}^{d} \omega_j x_j)/\epsilon = (-(\sigma + \lambda_p)t + \sum_{j=1}^{d} \omega_j x_j)/\epsilon, \tag{1.12}$$

which has the form

$$-s\phi_\xi + \sum_{j=1}^{d} \omega_j F_j(\phi)_\xi = \tag{1.13}$$

$$= -(\sigma + \lambda_p)\phi_\xi + \sum_{j=1}^{d} \omega_j F_j(\phi)_\xi = Q(\phi) = \begin{pmatrix} 0 \\ q(\phi) \end{pmatrix},$$

where λ_p is the p-th eigenvalue of $A_{11} = A_{11}(0)$.

We remark that after the scaling (1.12) the parameter ϵ does not occur any more in equation (1.13): If traveling waves exist then the thickness of their profile is of order $O(\epsilon)$.

The reason for writing the shock speed s in the form

$$s = \sigma + \lambda_p$$

is the fact that σ is a critical parameter, i.e. the qualitative behavior of solutions of (1.13) changes under change of the sign of σ.

Traveling waves occur, if there exists an intersection of the stable manifold at one limit state and the unstable manifold at the other one.

The first step in the construction of an intersection of a stable manifold and an unstable manifold to system (1.13) consists of answering the question, how the dimensions of the stable and unstable manifolds change under variation of σ near $\sigma = 0$.

The following theorem answers this question and gives information about the tangent space of the slow manifold where this change takes place.

In particular, the theorem controls the change of the dimensions of the stable and unstable submanifolds of the slow invariant manifold (called $\mathcal{M}_\sigma$) under variation of σ.

Remembering that $\mathcal{E} = \{\phi \in \mathcal{U} \mid Q(\phi) = 0\}$ and writing

$$B = B(0), \;\; S = S(0), \;\; A_{ij} = A_{ij}(0), \;\; \tilde{\Lambda} = \tilde{\Lambda}(0), \;\; A = A(0), \;\; \tilde{A} = \tilde{A}(0)$$

this theorem reads as follows:

Theorem 1.2.1 *Assume that the weakened structural condition (1.1.6) is fulfilled.*

Furthermore, assume that $A - \lambda_p I_n$ *is invertible or zero is a simple eigenvalue of* $A - \lambda_p I_n$.

Then there exists a $\mathbf{C}^1$ manifold $\mathcal{M} \subset \mathbb{R}^{n+1}$ and a real number $\delta > 0$ such that it holds: For any $|\sigma| < \delta$ it holds for the section $\mathcal{M}_\sigma = \mathcal{M} \cap \{(\bar\sigma, \phi) \mid \bar\sigma = \sigma\}$: The set $\{\phi \in \mathcal{U} \mid (\sigma, \phi) \in \mathcal{M}_\sigma\}$ is locally invariant for $\mathbf{C}^1$ solutions $\phi(\cdot)$ of (1.13). If $(\sigma, \phi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ then $(\sigma, \phi) \in \mathcal{M}_\sigma$. Furthermore, the following claims are true:

1. *The tangent space $T_0\mathcal{M}$ is equal to $\mathbb{R} \times V$, where V is the generalized eigenspace of $\tilde\Lambda \tilde A B|_{\tilde\Lambda \mathbb{R}^n}$ to eigenvalue zero, and V has the form*

$$V = P^{-1}\left(\mathbb{R}^{n-r} \times \left(S^{-1} A_{21}\left(ker(A_{11})\right)\right)\right). \tag{1.14}$$

2. *Let $\hat{\mathcal{P}}$ be the projector onto V commuting with $\tilde\Lambda \tilde A B \tilde\Lambda$, set $\psi = \hat{\mathcal{P}}\phi$, let $\psi \mapsto \phi(\sigma, \psi)$ be a $\mathbf{C}^1$ parametrization of $\{\phi \in \mathcal{U} \mid (\sigma, \phi) \in \mathcal{M}_\sigma\}$. Then, system (1.13) induces an autonomous ODE system having the form*

$$\psi_\xi = \mathcal{V}(\sigma, \psi), \tag{1.15}$$

where $\mathcal{V}(\cdot, \cdot)$ is of class $\mathbf{C}^1$ and for $\sigma > 0$ (resp. $\sigma < 0$) the dimension of the stable manifold (resp. unstable manifold) to system (1.15) at the origin is equal to $dim(ker(A_{11}))$ as long as $A_{11} + \bar\sigma I$ is invertible for each $\bar\sigma \in (0, \sigma]$ (resp. for each $\bar\sigma \in [\sigma, 0)$).

The form of V given in (1.14) has the following interpretation:

Writing $V = P^{-1}(\mathbf{V}_1 \times \mathbf{V}_2)$, where $\mathbf{V}_1 = \mathbb{R}^{n-r}$ and $\mathbf{V}_2 = S^{-1} A_{21}\left(ker(A_{11})\right)$, the component $\mathbf{V}_1$ is the trivial part corresponding to the $n - r$ conserved quantities, the component $\mathbf{V}_2$ is the nontrivial part corresponding to the instability near $\sigma = 0$.

1.3 Traveling waves for simple shocks

In our PDE system

$$U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = \frac{1}{\epsilon} \begin{pmatrix} 0_{n-r} \\ q(U) \end{pmatrix} \tag{1.16}$$

the first $n-r$ entries of F_j are denoted by f_j.

Taking into account that the equilibrium set $\mathcal{E}$ has the representation

$$\mathcal{E} = \{U = \begin{pmatrix} u \\ v \end{pmatrix} \mid v = h(u)\} \tag{1.17}$$

for a smooth function h, we get in the limit $\epsilon \to 0$ the reduced system

$$u_t + \sum_{j=1}^{d} f_j(u, h(u))_{x_j} = 0. \tag{1.18}$$

We consider the limit from (1.16) to (1.18) for weak solutions of the form

$$u(x,t) = \begin{cases} u_- & if \;\; \omega \cdot x < st, \\ u_+ & if \;\; \omega \cdot x > st, \end{cases}$$

where u_-, u_+, s and $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{S}^{d-1}$ are given and satisfy the Rankine-Hugoniot relation

$$s(u_+ - u_-) = f(\omega, u_+, h(u_+)) - f(\omega, u_-, h(u_-)) \tag{1.19}$$

for $f(\omega, U) = \sum_{j=1}^{d} \omega_j f_j(U)$.

Remembering the decomposition (1.8) we assume that λ_p is a simple eigenvalue of the constant matrix

$$A_{11}(u_-, h(u_-)) = \left. \frac{\partial f(\omega, u, h(u))}{\partial u} \right|_{u=u_-}.$$

Denote by r_p the corresponding eigenvector.

We make the assumption that for a real number $\bar{\delta} > 0$ the p^{th} Hugoniot curve exists, represented by a function

$$[0, \bar{\delta}) \longrightarrow \mathbb{R} \times \mathbb{R}^{n-r}, \;\; \rho \mapsto (s(\rho), u(\rho))$$

with the properties

i.) $s(0) = \lambda_p(u_-)$, $u(0) = u_-$, ii.) $s(\rho)(u(\rho) - u_-) = f(\omega, u(\rho), h(u(\rho)) - f(\omega, u_-, h(u_-)))\ \ \forall \rho \in [0, \bar{\delta})$,

iii.) $\partial_\rho u(\rho)|_{\rho=0} = r_p(u_-)$,

iv.) $s(\cdot)$ is strictly monotone,

and we assume that the strict entropy condition

$$\lambda_p(u(\rho)) < s(\rho) < \lambda_p(u_-)\ \ \forall \rho \in (0, \bar{\delta}) \tag{1.20}$$

or the reversed strict entropy condition

$$\lambda_p(u_-) < s(\rho) < \lambda_p(u(\rho))\ \ \forall \rho \in (0, \bar{\delta}) \tag{1.21}$$

is fulfilled.

For any $v \in \mathbb{R}^n$ denote by v^I the vector with the first $n - r$ entries of v and by v^{II} the vector with the last r entries of v.

If we have proven Theorem (1.2.1), it is not hard to show the following theorem about the existence of smooth profiles to simple shocks:

Theorem 1.3.1 *Assume that $Q(U_-) = 0$, the weakened structural condition (1.1.6) is fulfilled, $\lambda_p(u_-)$ is a simple eigenvalue of $\partial_u f(\omega, u, h(u))|_{u=u_-}$ and that the strict entropy condition is fulfilled.*

Furthermore, assume that $A - \lambda_p I_n$ is invertible or zero is a simple eigenvalue of $A - \lambda_p I_n$.

Then there exists a real number $\delta > 0$ such that the following statements are true:

1. *If there exists a solution $\phi \in \mathbf{C}^1(\mathbb{R}, \mathcal{U})$ of (1.13) with $\phi(-\infty) = U_-$, $\phi(\infty) = U_+$, if $|U_+ - U_-| < \delta$ and if $\lambda_p(U_+^I) < s < \lambda_p(U_-^I)$ (resp. $\lambda_p(U_-^I) < s < \lambda_p(U_+^I)$) then it holds: U_+^I is contained in the p^{th} Hugoniot curve of U_-^I, i.e. $(U_+^I, s) = (u(\rho), s(\rho))$ for a parameter value $\rho \in I$.*

2. *If $U_-, U_+ \in \mathcal{E}$, $|U_+ - U_-| < \delta$, the strict entropy condition (1.20) (resp. the reversed strict entropy condition (1.21)) holds and U_+^I is*

contained in the p^{th} Hugoniot curve of U_-^I then there exists a solution $\phi \in \mathbf{C}^1(\mathbb{R},\mathcal{U})$ of (1.13) such that $\phi(-\infty) = U_-$ and $\phi(\infty) = U_+$ (resp. $\phi(-\infty) = U_+$ and $\phi(\infty) = U_-$).

Proof:

From now on, we assume $\lambda_p(U_-^I) = 0$ and $U_- = 0$ for notational simplicity, and we can write $s(\cdot) = \sigma(\cdot)$.

Assume there exists a solution $\phi \in \mathbf{C}^1(\mathbb{R},\mathcal{U})$ with $\phi(-\infty) = 0$ and $\phi(\infty) = U_+$.

$Q^I \equiv 0$ implies

$$\left(\sum_{i=1}^{d} \omega_i f_i(\phi(\xi)) - \sigma \phi^I(\xi) \right)_\xi = 0 \quad \forall \xi \in \mathbb{R}. \tag{1.22}$$

Integrating (1.22) from $-R$ to R (for $R > 0$) leads to the relation

$$f(\omega, \phi(R)) - f(\omega, \phi(-R)) = \sigma \phi^I(R) - \sigma \phi^I(-R).$$

In the limit $R \to \infty$ we get the Rankine-Hugoniot relation

$$f(\omega, u_+, h(u_+)) - f(\omega, u_-, h(u_-)) \tag{1.23}$$

$$= f(\omega, u_+, h(u_+)) - f(\omega, 0, 0)) = \sigma u_+ - \sigma u_- = \sigma u_+$$

for $u_- = U_-^I = 0$ and $u_+ = U_+^I$.

Due to (1.20) (resp. (1.21)), the only solutions of (1.23) are contained in the p^{th} Rankine-Hugoniot curve for $|U_+|$ small enough, i.e. $\exists \rho > 0$ such that $(u_+, \sigma) = (u(\rho), \sigma(\rho))$, and claim (1) follows.

Define the conservation manifold

$$\mathcal{N}_\sigma = \{\phi \in \mathcal{U} \subset \mathbb{R}^n \ | \ f(\omega, \phi) - f(\omega, 0) = \sigma \phi^I\}.$$

For a solution $\phi(\cdot)$ of system (1.13) with $\phi(-\infty) = 0$ (resp. $\phi(\infty) = 0$) it holds $\phi(\xi) \in \mathcal{N}_\sigma \ \forall \xi \in \mathbb{R}$.

For the tangent space $T_0\mathcal{N}_0$ of $\mathcal{N}_0$ at the origin it holds

$$T_0\mathcal{N}_0 = ker((A_{11} \;\; A_{12})P). \tag{1.24}$$

From claim (1) of the main theorem and $ker(A_{11}) = \mathbb{R}r_p$ it follows

$$T_0\mathcal{M}_0 = \{0\} \times V, \tag{1.25}$$

where $V = P^{-1}\left(\mathbb{R}^{n-r} \times \mathbb{R}S^{-1}A_{21}r_p\right)$.

If $w \in V \cap T_0\mathcal{N}_0$ then w has the form

$$w = P^{-1}\begin{pmatrix} v \\ cS^{-1}A_{21}r_p \end{pmatrix},$$

and it holds after multiplication of $(A_{11} \;\; A_{12})P$ from the left:

$$0 = A_{11}v + cA_{12}S^{-1}A_{21}r_p. \tag{1.26}$$

After multiplication of $l(r_p)$ (with $l(\cdot)$ defined in (1.10)) from the left we get due to $l(r_p)A_{11} = 0$:

$$0 = l(r_p)A_{11}v + cl(r_p)A_{12}S^{-1}A_{21}r_p = cl(r_p)A_{12}S^{-1}A_{21}r_p.$$

From (1.11) it follows $c = 0$ so that relation (1.26) is fulfilled if and only if $v \in \mathbb{R}r_p$.

We get

$$V \cap T_0\mathcal{N}_0 = \mathbb{R}P^{-1}\begin{pmatrix} r_p \\ 0 \end{pmatrix}. \tag{1.27}$$

In other words, the invariant set $\mathcal{C}_\sigma = \{\phi \in \mathcal{U} \;\mid\; (\sigma, \phi) \in \mathcal{M}_\sigma\} \cap \mathcal{N}_\sigma$ is a curve which approximately tangent to $\mathbb{R}P^{-1}\begin{pmatrix} r_p \\ 0 \end{pmatrix}$ for σ small.

Assume that $U_+ \in \mathcal{E}$, that $|U_+|$ is small enough and that $u_+ = U_+^I$ fulfills the Rankine-Hugoniot condition (1.23) for $u_- = U_-^I = 0$ and $\sigma = \sigma_+ = \sigma(\rho_+) \neq 0$.

Obviously, $0, U_+$ are contained in $\mathcal{N}_{\sigma_+}$, and, due to the main theorem, $(\sigma_+, 0)$ and (σ_+, U_+) are contained in $\mathcal{M}_{\sigma_+}$.

Hence, $0, U_+$ are contained in $\mathcal{C}_{\sigma_+}$.

Due to the strict entropy condition (1.20) (resp. the reversed strict entropy condition (1.21)), it holds

$$\sigma = \sigma_+ < \lambda(U_-^I) = 0, \quad (resp.\ 0 = \lambda(U_-^I) < \sigma = \sigma_+).$$

Due to claim (2) of the main theorem, there exists a one dimensional unstable (resp. stable) manifold at $U_- = 0$ (being contained in $\mathcal{M}_\sigma$ for $\sigma = \sigma_+$) for the ODE system induced on $\bar{\mathcal{M}}_{\sigma_+} = \{\phi \in \mathcal{U} \mid (\sigma_+, \phi) \in \mathcal{M}_{\sigma_+}\}$.

As this unstable (resp. stable) manifold is also contained in $\mathcal{N}_{\sigma_+}$ it is contained $\mathcal{C}_{\sigma_+}$.

Assume that the trajectory of a solution $\phi(\cdot)$ of the traveling wave system corresponds to a solution of the ODE system (in claim (2) of the main theorem) which is induced on $\mathcal{M}_\sigma$ by the traveling wave system (for $\sigma = \sigma_+$).

Furthermore, assume that this trajectory has a non-void intersection with the aforementioned unstable (resp. stable) manifold and a non-void intersection with the part of the line $\mathcal{C}_{\sigma_+}$ which connects the points 0 and U_+.

Then, $0 = lim_{\xi \to -\infty} \phi(\xi)$ (resp. $0 = lim_{\xi \to \infty} \phi(\xi)$), and there exists $0 \neq U_* \in \mathcal{C}_{\sigma_+}$ such that $U_* = lim_{\xi \to \infty} \phi(\xi)$ (resp. $U_* = lim_{\xi \to -\infty} \phi(\xi)$).

As the stationary points 0 and U_+ are contained in the curve $\mathcal{C}_{\sigma_+}$, such a point $U_* \neq 0$ being contained in $\mathcal{C}_{\sigma_+}$ exists, and U_* is either equal to U_+ or is located between 0 and U_+ on the invariant curve $\mathcal{C}_{\sigma_+}$.

Now, we show that $U_* = U_+$.

Due to (1), U_*^I is contained in the Rankine-Hugoniot curve through $U_-^I = 0$ for $\sigma = \sigma_+$.

For $|U_+^I|$ and $\sigma = \sigma_+$ small enough, $u_+ = U_+^I$ is the only solution of the Rankine-Hugoniot relation (1.23).

Due to $U_+, U_* \in \mathcal{E}$ it holds $U_+^{II} = h(U_+^I)$, $U_*^{II} = h(U_*^I)$ (remembering (1.17).

Hence, we have $U_+ = U_*$, i.e. there exists a solution ϕ with asymptotic states $U_- = 0$ and U_+, and (2) has been proven. $\square$

Chapter 2

Proof of main theorem

Before we go into the details we give the main structure of this chapter:

In section 2.1 we will cite the center manifold theorem and some basic facts from perturbation theory of linear operators in $\mathbb{R}^n$.

In section 2.2 we will prove Proposition (1.1.7) which says that the stability and dissipativity condition imply the weakened structural condition.

In section 2.3 we will prove the main theorem under the assumptions that the weakened structural condition is fulfilled and that a locally invariant manifold $\tilde{\Gamma}$ with the following property exists: The vector field to the traveling wave ODE system has a smooth extension from a dense set in $\tilde{\Gamma}$ throughout $\tilde{\Gamma}$.

Section 2.3 consists of three subsections:

> In subsection 2.3.1 we will prove the basic proposition about the existence of the invariant manifold $\mathcal{M}$ of the main theorem.
>
> In subsection 2.3.2 ("Generalized kernel of the linearization") we will prove claim (1) of the main theorem about the tangent space of the invariant manifold $\mathcal{M}$ at the origin.
>
> In subsection 2.3.3 we will prove claim (2) of the main theorem about the signature of real parts of nonvanishing eigenvalues of

the linearization of the ODE system induced on $\mathcal{M}$ by the traveling wave ODE system.

In section 2.4 we will prove the existence of the invariant manifold $\tilde{\Gamma}$ whose existence is assumed in section 2.3, and we are ready with the proof of the main theorem.

2.1 Preliminaries

For an ODE system of the form

$$y_\xi = Ay + F(y,z), \quad z_\xi = Bz + G(y,z), \tag{2.1}$$

where the eigenvalues of A have zero real parts, the eigenvalues of B have nonzero real parts, F, G belong to $\mathbf{C}^k$, $k \geq 2$ and vanish along with their first derivative at $(y,z) = 0$, the center manifold theorem says:

Theorem 2.1.1 *([KE]) With the variables (y,z), the matrices A, B and functions F, G defined in (2.1) there exists a locally invariant manifold*

$$\mathcal{M} = \{(y,z) \mid |y| < \delta, \; z^* = z(y)\}$$

where z^* is a $\mathbf{C}^{k-1}$ function defined for $|y| < \delta$ for some δ sufficiently small and vanishes along with its first derivative at $y = 0$. In other words: The tangent space of $\mathcal{M}$ at $(y,z) = (0,0)$ is the linear space corresponding to the eigenvalues of $diag(A,B)$ with zero real parts. Moreover, any fixed point y with $|y| < \delta$ is contained in $\mathcal{M}$.

Let's repeat the well-known notation of the Dunford-Taylor integral of linear operators on $\mathbb{R}^n$:

For a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, f holomorphic on a neighborhood of Ω in the complex plane and a closed simple curve C contained in Ω which does not intersect the eigenvalues of T, we define

$$f(T) = \frac{1}{2\pi i} \oint_C f(\zeta)(\zeta - T)^{-1} d\zeta, \tag{2.2}$$

where the integral path has to be taken counterclockwise.

If f and g are holomorphic on a neighborhood of Ω then the well-known identity $f(T)g(T) = (fg)(T)$ ([KAT]) implies for $M_C = \frac{1}{2\pi i} \oint_C \frac{1}{\zeta}(\zeta - T)^{-1} d\zeta$

$$TM_C = M_C T = \Pi_C = \frac{1}{2\pi i} \oint_C (\zeta - T)^{-1} d\zeta,$$

i.e. TM_C is a projector onto the subspace corresponding to the eigenvalues of T which are encircled by C.

Now we will cite important theorems about the dependence of eigenvalues on analytic perturbations.

We start from a given power series for $T(\sigma)$:

$$T(\sigma) = T + \sigma T^{(1)} + \sigma^2 T^{(1)} + \ldots.$$

Let λ be one of the eigenvalues of the unperturbed operator $T = T(0)$ with algebraic multiplicity m, and let P and D be the associated projection and eigennilpotent. Thus

$$TP = PT = PTP = \lambda P + D,\ rank(P) = m,\ D^m = 0,\ PD = DP = D.$$

The following theorem is about the dependence of eigenvalues on analytic perturbations:

Theorem 2.1.2 *([KAT]) If $T(\sigma)$ is analytic in σ there exist $p_1, \cdots, p_q \in \mathbb{N}$ such that for $\omega_{p_j} = exp(\frac{2\pi i}{p_j})$ there exist $\alpha_\nu^j \in \mathbb{C},\ \ \nu \in \mathbb{N}$ and a strictly positive real number δ such that the Puiseux series*

$$\lambda_k^j(\sigma) = \lambda + \sum_{\nu=1}^{\infty} \alpha_\nu^j \omega_{p_j}^{\nu k} \sigma^{\frac{\nu}{p_j}},\ \ k = 0, \cdots, p_j - 1$$

converges for $|\sigma| < \delta$.

In particular, $\lambda_k^j(\cdot)$ $(j = 1, \cdots, q,\ k = 0, \cdots, p_j - 1)$ are continuous functions on $(-\delta, \delta)$.

We call $\{\lambda_0^j(\cdot), \cdots, \lambda_{p-1}^j(\cdot)\}$ the j^{th} λ-group.

If $P(\sigma)$ is the commuting projector of $T(\sigma)$ with $P(0) = P$ and if $D = 0$ then we can write $(T(\sigma) - \lambda)P(\sigma)$ in the form (see [KAT])

$$(T(\sigma) - \lambda)P(\sigma) = \sum_{n=1}^{\infty} \sigma^m \tilde{T}^{(n)},$$

and the following theorem holds:

Theorem 2.1.3 *([KAT], p. 82) If λ is a semisimple eigenvalue of the unperturbed operator $T = T(0)$ (i.e. $D = 0$) and $T(\sigma)$ is analytic in σ then each element of the λ-group is continuously differentiable near $\sigma = 0$. Furthermore, $\tilde{T}(\sigma) = \frac{1}{\sigma}(T(\sigma) - \lambda)P(\sigma)$ is analytic in σ and*

$$\tilde{T}(\sigma) = P(0)\left(T^{(1)} - \lambda\right)P(0) + O(\sigma) \tag{2.3}$$

where $P(\sigma)$ denotes the projector commuting with $T(\sigma)$ corresponding to its λ-group.

The following lemma is a straightforward consequence of the last theorem

Lemma 2.1.4 *If A is noninvertible, zero is a semisimple eigenvalue of A, $\Pi = \frac{1}{2\pi i}\oint_\Gamma (\zeta - A)^{-1} d\zeta$, (where Γ encircles the origin in counterclockwise direction, but does not enclose the nonvanishing eigenvalues of A) and zero is a semisimple eigenvalue of ΠB then*

$$(A - \sigma I_n)^{-1} B \tilde{\Lambda}(\sigma) = \tilde{\Lambda}\tilde{A}B\tilde{\Lambda} + O(\sigma),$$

where $\tilde{A} = \frac{1}{2\pi i}\oint_C \zeta^{-1}(\zeta - A)^{-1} d\zeta$ (where C encircles the nonzero eigenvalues of A in counterclockwise direction, but does not encircle the origin), $\tilde{\Lambda} = \tilde{\Lambda}(0)$ is the projector onto $ker(\Pi B)$ commuting with ΠB and $\tilde{\Lambda}(\sigma)$ is the projector commuting with $\sigma(A - \sigma I_n)^{-1}B$ corresponding to the eigenvalues $\lambda(\sigma)$ of $\sigma(A - \sigma I_n)^{-1}B$ with the following property: For each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|\lambda(\sigma)| < \epsilon$ for $|\sigma| < \delta(\epsilon)$.

Proof:

Set $\Lambda = I_n - \Pi$ and define $\tilde{A}(\sigma)$ by

$$\tilde{A}(\sigma) = \frac{1}{2\pi i} \oint_C \zeta^{-1}(\zeta - (A - \sigma\Lambda))^{-1} d\zeta,$$

where C encircles the nonzero eigenvalues of A in counterclockwise direction, but does not encircle the origin.

For σ small enough it holds

$$\tilde{A}(\sigma)(A - \sigma\Lambda) = (A - \sigma\Lambda)\tilde{A}(\sigma) = \Lambda$$

and

$$\Pi\tilde{A}(\sigma) = \tilde{A}(\sigma)\Pi = 0,$$

so that

$$(\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi)(A - \sigma I_n) = (\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi)(A - \sigma\Lambda - \sigma\Pi) = I_n =$$

$$= (A - \sigma I_n)(\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi).$$

Hence, there holds (taking $\tilde{A}(0) = \tilde{A}$ into account):

$$\sigma(A - \sigma I_n)^{-1} B\tilde{\Lambda}(\sigma) = (\sigma\tilde{A}(\sigma) - \Pi)B\tilde{\Lambda}(\sigma) = \sigma\tilde{\Lambda}\tilde{A}B\tilde{\Lambda} + O(\sigma^2).$$

The last identity follows due to the semisimplicity of eigenvalue zero of ΠB and formula (2.3) with $T(\sigma) = (\sigma\tilde{A}(\sigma) - \Pi)B$ and $P(\sigma) = \tilde{\Lambda}(\sigma)$ plugged into the claim of Theorem (2.1.3). □

Define $A(\sigma, \phi) := -(\sigma + \lambda_p)I_n + \sum_{j=1}^{d} \omega_j A_j(\phi)$ and consider the function

$$(\sigma, \phi) \mapsto A^{-1}(\sigma, \phi)Q(\phi). \qquad (2.4)$$

Denote by $\tilde{\Lambda}_\sigma$ the family of projectors commuting with $A^{-1}(\sigma, 0)B$ corresponding to the eigenvalues $\lambda(\sigma)$ of $A^{-1}(\sigma, 0)B$ with the following property:

There exist $K > 0$ and $\delta > 0$ such that $|\lambda(\sigma)| < K$ for each $\sigma \in (-\delta, \delta)$.

Define $\tilde{\Pi}_\sigma = I - \tilde{\Lambda}_\sigma$ and set $\tilde{\Lambda} = \tilde{\Lambda}_0$, $\tilde{\Pi} = \tilde{\Pi}_0$.

We will need the following proposition about the linearization of function (2.4) at the origin:

Proposition 2.1.5 *If* $v(\cdot) : \mathbb{R} \to \mathbb{R}^n$ *is a parametrization with* $\lim_{\sigma\to 0} v(\sigma) = v$ *then relation*

$$\limsup_{\sigma\to 0} |A^{-1}(\sigma,0)Bv(\sigma)| < \infty$$

implies $v \in \tilde{\Lambda}\mathbb{R}^n$. *Furthermore,* $\tilde{\Lambda}$ *is the commuting projector onto the (generalized) kernel of* ΠB.

Proof:

Let Π be the projector onto $ker(A(0,0))$ commuting with $A(0,0)$ and set $\Lambda = I_n - \Pi$.

Define $\tilde{A}(\sigma)$ by

$$\tilde{A}(\sigma) = \frac{1}{2\pi i}\oint_C \zeta^{-1}(\zeta - (A(0,0) - \sigma\Lambda))^{-1} d\zeta,$$

where C encircles the nonzero eigenvalues of $A(0,0)$ in counterclockwise direction, but does not encircle the origin.

For σ small enough it holds

$$\tilde{A}(\sigma)(A(0,0) - \sigma\Lambda) = (A(0,0) - \sigma\Lambda)\tilde{A}(\sigma) = \Lambda$$

and

$$\tilde{A}(\sigma)\Pi = \Pi\tilde{A}(\sigma) = 0.$$

Hence, it holds $\sigma \neq 0$ small enough

$$(\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi)A(\sigma,0) = (\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi)(A(0,0) - \sigma\Lambda - \sigma\Pi) = I_n =$$

$$= A(\sigma,0)(\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi).$$

We can write

$$A^{-1}(\sigma,0)B = (\tilde{A}(\sigma) - \frac{1}{\sigma}\Pi)B = \tilde{A}(\sigma)B - \frac{1}{\sigma}\Pi B.$$

Hence, from

$$\limsup_{\sigma\to 0} |A^{-1}(\sigma,0)Bv(\sigma)| < \infty$$

it follows

$$\limsup_{\sigma\to 0} \left|\frac{1}{\sigma}\Pi Bv(\sigma)\right| < \infty.$$

This implies

$$0 = \lim_{\sigma\to 0} \Pi Bv(\sigma) = \Pi Bv,$$

so that

$$v \in ker(\Pi B). \tag{2.5}$$

Due to the weakened structural condition, zero is a semisimple eigenvalue of

$$\lim_{\sigma\to 0} \sigma A^{-1}(\sigma,0)B = -\Pi B.$$

Due to Theorem (2.1.3), each eigenvalue $\lambda(\sigma)$ in any 0 - group (due to the notation of Theorem (2.1.3)) of $\sigma A^{-1}(\sigma,0)B$ is continuously differentiable in σ.

As $\tilde{\Lambda}_\sigma$ is the projector commuting with $A^{-1}(\sigma,0)B$ corresponding to the eigenvalues which are uniformly bounded with respect to σ for σ small, it is the projector commuting with $\sigma A^{-1}(\sigma,0)B$ corresponding to the eigenvalues of order $O(\sigma)$.

As each eigenvalue $\lambda(\sigma)$ in any 0 - group of $\sigma A^{-1}(\sigma,0)B$ is continuously differentiable in σ, each eigenvalue $\lambda(\sigma)$ of $\sigma A^{-1}(\sigma,0)B$ is of order $O(\sigma)$ if and only if it is in some 0 - group of $\sigma A^{-1}(\sigma,0)B$.

Hence, $\tilde{\Lambda} = \tilde{\Lambda}_0$ is the commuting projector onto the kernel of

$$\lim_{\sigma\to 0} \sigma A^{-1}(\sigma,0)B = -\Pi B.$$

As $\tilde{\Lambda}$ is the projector onto the kernel of ΠB commuting with ΠB, the claim of the proposition follows. □

Set

$$\mathcal{S} = \{(\sigma, \phi) \mid \exists A^{-1}(\sigma, \phi)\}.$$

If Proposition (2.1.5) has been shown it is not hard to prove the following proposition:

Proposition 2.1.6 *Assume that $\tilde{\Gamma}$ is a $\mathbf{C}^2$ manifold of dimension $rank(\tilde{\Lambda})+1$ such that the section $\tilde{\Gamma}_\sigma = \{(\bar{\sigma}, \phi) \in \tilde{\Gamma} \mid \bar{\sigma} = \sigma\}$ is a $\mathbf{C}^2$ manifold of dimension $rank(\tilde{\Lambda})$ containing $(0, \sigma)$ for σ small enough and that the restriction of $(\sigma, \phi) \mapsto A^{-1}(\sigma, \phi)Q(\phi)$ on $\mathcal{S} \cap \tilde{\Gamma}$ has an extension of class $\mathbf{C}^2$ throughout $\tilde{\Gamma}$.*

Then the tangent space of $\tilde{\Gamma}$ at the origin is equal to $\mathbb{R} \times \tilde{\Lambda}\mathbb{R}^n$.

Proof:

As the restriction of $(\sigma, \phi) \mapsto (0, A^{-1}(\sigma, \phi)Q(\phi))$ on $\mathcal{S} \cap \tilde{\Gamma}$ has a smooth extension of class $\mathbf{C}^2$ throughout $\tilde{\Gamma}$, for any family of curves $(\sigma, \tau) \mapsto \gamma_\sigma(\tau)$ with $\gamma_\sigma(0) = 0$, $(\sigma, \tau) \mapsto \partial_\tau \gamma_\sigma(\tau)$ smooth in (σ, τ) and $(\sigma, \gamma_\sigma(\tau)) \in \tilde{\Gamma}_\sigma$ for $(\sigma, \tau) \in I \times I$ (where I is an open interval containing 0) the derivative

$$\partial_\tau [A^{-1}(\sigma, \gamma_\sigma(\tau))Q(\gamma_\sigma(\tau))]|_{\tau=0} = A^{-1}(\sigma, 0)B\partial_\tau \gamma_\sigma(0)$$

exists for small $\sigma \neq 0$ and is uniformly bounded with respect to σ for σ small.

As $(\sigma, \tau) \mapsto \partial_\tau \gamma_\sigma(\tau)$ is smooth the limit $\lim_{\sigma \neq 0, \sigma \to 0} \partial_\tau \gamma_\sigma(0)$ exists.

From Proposition (2.1.5) it follows

$$\partial_\tau \gamma_0(0) \in \tilde{\Lambda}\mathbb{R}^n. \tag{2.6}$$

As $\tilde{\Gamma}_0$ is a manifold of dimension $rank(\tilde{\Lambda})$ and (2.6) is true for any smooth curve $\tau \mapsto \gamma_0(\tau)$ with $\gamma_0(0) = 0$ and $\gamma_0(\tau) \in \tilde{\Gamma}_0$, it follows

$$T_{(0,0)}\tilde{\Gamma}_0 = \{0\} \times \tilde{\Lambda}\mathbb{R}^n.$$

As $(\sigma, 0) \in \tilde{\Gamma}$ for σ small enough, it holds

$$T_{(0,0)}\tilde{\Gamma} = \mathbb{R} \times \tilde{\Lambda}\mathbb{R}^n. \square$$

We will need the following lemma about matrices with semisimple eigenvalue zero:

Lemma 2.1.7 *Assume that zero is a semisimple eigenvalue of* $M \in \mathbb{C}^{n\times n}$*. Then zero is a simple singularity of* $\zeta \mapsto (\zeta - M)^{-1}$*, i.e. there exists* $C > 0$ *such that* $|(\zeta - M)^{-1}| \leq C\frac{1}{|\zeta|}$ *(where* $|\cdot|$ *denotes some matrix norm).*

Proof:

As zero is a simple eigenvalue of M we can choose an invertible matrix S such that SMS^{-1} has block form

$$SMS^{-1} = diag(M_1, 0_r),$$

where $rank(M_1) = n - r$.

Then zero is a simple singularity of $\zeta \mapsto (\zeta - M)^{-1}$ if and only if zero is a simple singularity of

$$\zeta \mapsto S(\zeta - M)^{-1}S^{-1} = (\zeta - SMS^{-1})^{-1} = diag((\zeta - M_1)^{-1}, \zeta^{-1}I_r).$$

Obviously, the first block of the block matrix in the last expression is analytic in ζ for ζ small.

The second block has a simple singularity at $\zeta = 0$. □

Furthermore, we will need the following lemma:

Lemma 2.1.8 *Let* $\mathcal{U} \subset \mathbb{R}^n$ *be an open neighborhood containing zero and* r *an integer with* $0 < r < n$*. Assume that* $\bar{w}_1(\cdot), \cdots \bar{w}_r(\cdot) \in \mathbf{C}^k(\mathcal{U}, \mathbb{R}^n)$ *and that for each* $\phi \in \mathcal{U}$ *the vectors* $\bar{w}_1(\phi), \cdots, \bar{w}_r(\phi)$ *are linearly independent. Set*

$$V = span(\bar{w}_1(0), \cdots, \bar{w}_r(0)).$$

Furthermore, assume that $\tilde{\Pi} : \mathbb{R}^n \to \mathbb{R}^n$ *is a projector with* $ker(\tilde{\Pi}) = V$*.*

Then there exists a projector valued function $\hat{\Pi}(\cdot) : \mathcal{O} \to \mathbb{R}^{n\times n}$ *of class* $\mathbf{C}^k$ *for an open neighborhood containing zero such that* $\hat{\Pi}(0) = \tilde{\Pi}$ *and that for each* $\phi \in \mathcal{U}$ *it holds*

$$ker(\hat{\Pi}(\phi)) = span(\bar{w}_1(\phi), \cdots, \bar{w}_r(\phi)).$$

Proof:

Before we go into the details, we remark that we will use the well-known Gram-Schmidt orthonormalization for this proof.

There exists an invertible matrix S such that $\tilde{\Pi} = S^{-1}(\sum_{i=1}^{n-r} e_i e_i^T)S$, where e_i is the i - th canonical base vector.

Obviously it holds

$$e_i^T S\bar{w}_j(0) = 0 \;\; for \;\; i = 1, \cdots, n-r, \;\; j = 1, \cdots, r. \tag{2.7}$$

Set $\bar{v}_i(\phi) := S\bar{w}_i(\phi)$ for $i = 1, \cdots, r$ and

$$\tilde{v}_1(\phi) := \frac{\bar{v}_1(\phi)}{|\bar{v}_1(\phi)|},$$

and define inductively for $j = 2, \cdots, r$

$$\tilde{v}_j(\phi) = \frac{\bar{v}_j(\phi) - \sum_{i=1}^{j-1} \tilde{v}_i^T(\phi)\bar{v}_j(\phi)\tilde{v}_i(\phi)}{|\bar{v}_j(\phi) - \sum_{i=1}^{j-1} \tilde{v}_i^T(\phi)\bar{v}_j(\phi)\tilde{v}_i(\phi)|}.$$

Then, $\tilde{v}_1(\cdot), \cdots, \tilde{v}_r(\cdot) \in \mathbf{C}^k(\mathcal{U}, \mathbb{R}^n)$ and $\tilde{v}_i^T(\phi)\tilde{v}_j(\phi) = \delta_{ij}$.

For $\tilde{w}_i(\phi) = S^{-1}\tilde{v}_i(\phi)$ it holds

$$span(\tilde{w}_1(\phi), \cdots, \tilde{w}_r(\phi)) = S^{-1} span(\tilde{v}_1(\phi), \cdots, \tilde{v}_r(\phi)) =$$

$$= S^{-1} span(\bar{v}_1(\phi), \cdots, \bar{v}_r(\phi)) = span(\bar{w}_1(\phi), \cdots, \bar{w}_r(\phi))$$

Note that (2.7) implies that $\tilde{v}_j^T(0)e_i = 0$ for $i = 1, \cdots, n-r, \;\; j = 1, \cdots, r$.

Set

$$\bar{e}_1(\phi) = \frac{e_1 - \sum_{i=1}^{r} \tilde{v}_i^T(\phi)e_1\tilde{v}_i(\phi)}{|e_1 - \sum_{i=1}^{r} \tilde{v}_i^T(\phi)e_1\tilde{v}_i(\phi)|}$$

and define inductively for $j = 2, \cdots, n-r$

$$\bar{e}_j(\phi) = \frac{e_j - \sum_{i=1}^{r} \tilde{v}_i^T(\phi) e_j \tilde{v}_i(\phi) - \sum_{i=1}^{j-1} \bar{e}_i^T(\phi) e_j \bar{e}_i(\phi)}{|e_j - \sum_{i=1}^{r} \tilde{v}_i^T(\phi) e_j \tilde{v}_i(\phi) - \sum_{i=1}^{j-1} \bar{e}_i^T(\phi) e_j \bar{e}_i(\phi)|}.$$

Then, $\bar{e}_1(\cdot), \cdots, \bar{e}_{n-r}(\cdot) \in \mathbf{C}^k(\mathcal{O}, \mathbb{R}^n)$ for an open neighborhood $\mathcal{O}$ containing zero, $\bar{e}_i^T(\phi)\bar{e}_j(\phi) = \delta_{ij}$, $\bar{e}_i(0) = e_i$ and $\bar{e}_i^T(\phi)\tilde{v}_j(\phi) = 0$ for $i = 1, \cdots, n-r$ and $j = 1, \cdots, r$.

Set

$$\hat{\Pi}(\phi) = S^{-1}(\sum_{i=1}^{n-r} \bar{e}_i(\phi)\bar{e}_i^T(\phi))S.$$

Due to $\bar{e}_i^T(\phi)\bar{e}_j(\phi) = \delta_{ij}$, the matrix $\sum_{i=1}^{n-r} \bar{e}_i(\phi)\bar{e}_i^T(\phi)$ is an orthogonal projector. Hence, $\hat{\Pi}(\phi)$ is a projector of rank $n-r$.

As $0 = \bar{e}_i^T(\phi)\tilde{v}_j(\phi) = \bar{e}_i^T(\phi)S\tilde{w}_j(\phi)$ for $i = 1, \cdots, n-r$ and $j = 1, \cdots, r$, it holds

$$ker(\hat{\Pi}(\phi)) = span(\tilde{w}_1(\phi), \cdots, \tilde{w}_r(\phi)) = span(\bar{w}_1(\phi), \cdots, \bar{w}_r(\phi)).\square$$

2.2 Checking of weakened structural condition

In this section we will show that the stability and dissipativity condition imply the weakened structural condition.

The analysis of the algebraic structure of the linearization at the bifurcation point uses the following lemma about the existence of a block-diagonal symmetrizer of PAP^{-1}:

Lemma 2.2.1 *([Y2]) Assume system (1.1) satisfies (1.5), (1.6) and*

$$A_0(U)Q_U(U) + Q_U(U)^* A_0(U) \leq 0.$$

Then $P^{-*}(U)A_0(U)P^{-1}(U)$ is a block-diagonal matrix corresponding to partition (1.8).

Now we turn to the proof of Proposition (1.1.7).

Due to the block symmetrizer Lemma (2.2.1), the matrix PAP^{-1} has a positive definite block-diagonal symmetrizer corresponding to partition (1.8).

Hence, it remains to show:

(i). $Re(l(r)A_{12}S^{-1}A_{21}r) < 0 \;\; \forall\, r \in ker(A_{11}) \setminus \{0\}$, where $l(r)$ is given by (1.10).

(ii). Zero is a semisimple eigenvalue of ΠB and $\Pi diag(0_{n-r}, I_r)$.

(iii). The eigenvalues of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ which are different from zero have a nonvanishing real part.

(iv). The eigenvalues of ΠB which are different from zero have a nonvanishing real part.

Remember that we assume that (after eventual linear transformation by $P = P(0)$ and multiplication of $P^{-*}A_0P^{-1} = P^{-*}A_0(0)P^{-1}$ from the left) system (1.1) has normal form, i.e. $P = A_0 = I_n$ and that A and A_{11} are already symmetric.

As A_{11} is symmetric, it holds $l(r) = r^*$.

ad (i):

It holds $A_{21}r \neq 0$ for each $r \in ker(A_{11}) \setminus \{0\}$. Otherwise,

$$P^{-1}\begin{pmatrix} r \\ 0 \end{pmatrix} \in ker(B) \cap ker(A) = ker(B) \cap ker \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

a contradiction to the dissipativity condition (1.1.5).

As the matrix A is symmetric it follows

$$A_{12} = A_{21}^*, \;\; A_{11} = A_{11}^*. \tag{2.8}$$

The latter implies $A_{11}^*r = 0$ and therefore $r^*A_{11} = 0$.

Having made the above preparations, we get

$$Re\left[r^* A_{12} S^{-1} A_{21} r\right] = Re\left[r^* A_{21}^* S^{-1} A_{21} r\right] < 0,$$

because $S + S^* \leq -I_r$ (due to (1.7) with $A_0(0) = I_n$), and (i) follows.

ad (ii):

If $z = \Pi B y \neq 0,\ \ \Pi B z = 0$ then $z \in range(\Pi) \setminus \{0\}$ or

$$z \in ker(A) \setminus \{0\}. \tag{2.9}$$

Assume it holds $z^{II} = 0$, then we get $z^I \in ker(A_{11})$ and $0 = (Az)^{II} = A_{21} z^I$, and (i) implies $z^I = 0$, a contradiction to (2.9).

Hence, relation (2.9) implies $z^{II} \neq 0$, and we conclude from (1.7) (with $P = A_0(0) = I_n$):

$$z^* \left(B + B^*\right) z \leq \tag{2.10}$$

$$\leq -z^* diag(0_{n-r}, I_r) z = -|z^{II}|^2 < 0.$$

On the other hand, it follows from $\Pi B z = 0$

$$z^* \Pi^* B \Pi z = 0, \ \ z^* \Pi^* B^* \Pi z = 0 \tag{2.11}$$

so that $z = \Pi z$ and adding the two relations in (2.11) imply

$$z^* \left(B + B^*\right) z = 0, \tag{2.12}$$

a contradiction to (2.10), and the first part of (ii) follows.

If $z = \Pi diag(0_{n-r}, I_r) y \neq 0,\ \ \Pi diag(0_{n-r}, I_r) z = 0$ then $z \in range(\Pi) \setminus \{0\}$ or

$$z \in ker(A) \setminus \{0\}. \tag{2.13}$$

Assume it holds $z^{II} = 0$, then we get $z^I \in ker(A_{11})$ and $0 = (Az)^{II} = A_{21} z^I$, and (i) implies $z^I = 0$, a contradiction to (2.13).

Hence, relation (2.13) implies $z^{II} \neq 0$, and we conclude:

$$z^* diag(0_{n-r}, I_r) z = -|z^{II}|^2 < 0. \tag{2.14}$$

On the other hand, it follows from $\Pi diag(0_{n-r}, I_r)z = 0$

$$z^* \Pi^* diag(0_{n-r}, I_r) \Pi z = 0 \tag{2.15}$$

so that $z = \Pi z$ implies

$$z^* diag(0_{n-r}, I_r) z = 0, \tag{2.16}$$

a contradiction to (2.14), and the second part of (ii) follows.

ad (iii):

Assume there exists $y \neq 0$ with $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}y = i\kappa y$, $0 \neq \kappa$ real.

As zero is a semisimple eigenvalue of ΠB and due to identity (2.3) in Theorem (2.1.3) it holds for $A(\sigma) = A + \sigma\Pi$, where $\tilde{\Lambda}(\sigma)$ denotes the projector commuting with $(A(\sigma))^{-1}B$ corresponding to the eigenvalues which are uniformly bounded for σ small:

$$(A(\sigma))^{-1}B\tilde{\Lambda}(\sigma) = (\tilde{A} + \frac{1}{\sigma}\Pi)B\tilde{\Lambda}(\sigma) = \tilde{\Lambda}\tilde{A}B\tilde{\Lambda} + O(\sigma),$$

and we get for $y(\sigma) = \tilde{\Lambda}(\sigma)y$

$$(A(\sigma))^{-1}By(\sigma) = i\kappa y + r(\sigma), \tag{2.17}$$

where $\lim_{\sigma \to 0} r(\sigma) = 0$.

Multiplication of $A(\sigma)$ resp. $y^*A(\sigma)$ to (2.17) from the left and performing the limit $\sigma \longrightarrow 0$ give

$$By = i\kappa Ay, \quad y^*By = i\kappa y^*Ay. \tag{2.18}$$

A being symmetric, it follows from the second relation in (2.18) that $Re(y^*By) = 0$, or in other words:

$$y^* \left[B + B^*\right] y = 0.$$

The last r components of y are zero due to (1.7).

Return to the first relation in (2.18).

Obtain $i\kappa Ay = By = diag(0, S)y = 0$, i.e. $Ay = 0$.

As the last r components of y are zero, it follows $A_{11}y^I = 0$ and $A_{21}y^I = 0$, and we conclude from (i) that $y^I = 0$, a contradiction to $y \neq 0$, and (iii) follows.

ad (iv):

Assume there exists $y \neq 0$ with $\Pi By = i\kappa y$, $\kappa \neq 0$.

Then it holds:

$$y^*\Pi By = i\kappa y^* y. \tag{2.19}$$

From (2.19) it follows $Re(y^*\Pi By) = 0$, or in other words:

$$y^*\left[\Pi B + B^*\Pi\right]y = y^*\left[B + B^*\right]y = 0.$$

The last r components of y are zero due to (1.7).

y being an eigenvector of ΠB and $\Pi By \neq 0$ imply that y is contained in $ker(A) \setminus \{0\}$.

As the last r components of y are zero, it follows $A_{11}y^I = 0$ and $A_{21}y^I = 0$, and we conclude from (i) that $y^I = 0$, a contradiction to $y \neq 0$, and (iv) follows.

2.3 Algebraic structure

In order to prove the main theorem we analyze the linearization of the traveling wave system.

The basic proposition in section 2.3.1 gives the connection between the algebraic structure of the linearization and the invariant manifolds.

In section 2.3.2 and 2.3.3 the algebraic structure will be analyzed in terms of the generalized kernel and the signature of the nonvanishing eigenvalues.

2.3.1 Basic proposition

For $A(\sigma,\phi) := -(\sigma+\lambda_p)I + \sum_{j=1}^{d}\omega_j A_j(\phi)$ the traveling wave system augmented by the equation for σ has the form

$$\sigma_\xi = 0, \quad A(\sigma,\phi)\phi_\xi = Q(\phi). \tag{2.20}$$

Denote by $\tilde{\Lambda}_\sigma$ (resp. $\hat{\mathcal{P}}(\sigma)$) the projector commuting with $A^{-1}(\sigma,0)B$ corresponding to the eigenvalues remaining uniformly bounded in σ for σ small (resp. to the eigenvalues being zero in the limit $\sigma \to 0$) and set $\tilde{\Pi}_\sigma = I - \tilde{\Lambda}_\sigma$.

Set $\tilde{\Lambda} = \tilde{\Lambda}_0$ and $\tilde{\Pi} = \tilde{\Pi}_0$.

The projector $\tilde{\Lambda}$ is the commuting projector onto the (generalized) kernel of ΠB; set $\tilde{\Pi} = \tilde{\Pi}_0$.

Remember the definition (1.9) of $\tilde{A}$, remember

$$\mathcal{E} = \{Q(\phi) = 0\}, \quad B = DQ(0)$$

and set

$$\mathcal{S} = \{(\sigma,\phi) \mid \exists A^{-1}(\sigma,\phi)\}.$$

We will prove the following proposition about invariant manifolds for system (2.20):

Proposition 2.3.1 *Assume that the weakened structural condition is fulfilled, that $\tilde{\Gamma} \subset \mathbb{R}\times\mathcal{U}$ is a locally invariant $\mathbf{C}^2$ manifold of dimension rank$(\tilde{\Lambda})+1$ for system (2.20), that the section $\tilde{\Gamma}_\sigma = \tilde{\Gamma}\cap\{(\bar{\sigma},\phi) \mid \bar{\sigma}=\sigma\}$ is a locally invariant $\mathbf{C}^2$ manifold of dimension rank$(\tilde{\Lambda})$ with $(\sigma,0)\in\tilde{\Gamma}_\sigma$ for σ small enough, that $\mathcal{S}\cap\tilde{\Gamma}$ is dense in $\tilde{\Gamma}$ and that the restriction of $(\sigma,\phi)\mapsto A^{-1}(\sigma,\phi)Q(\phi)$ on $\mathcal{S}\cap\tilde{\Gamma}$ has a $\mathbf{C}^2$ extension throughout $\tilde{\Gamma}$. Furthermore, assume there exists $\delta>0$ with the following property: If $(\sigma,\phi)\in B_\delta(0)\cap(\mathbb{R}\times\mathcal{E})$*

then $(\sigma, \phi) \in \tilde{\Gamma}$. *Then* $\tilde{\Gamma}$ *contains a locally invariant* $\mathbf{C}^1$ *submanifold* $\mathcal{M}$ *such that the section* $\mathcal{M}_\sigma = \mathcal{M} \cap \tilde{\Gamma}_\sigma$ *is invariant and that the following conditions are fulfilled:*

1. *There exists* $\delta > 0$ *with the following property: If* $(\sigma, \phi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ *then* $(\sigma, \phi) \in \mathcal{M}$.

2. *The tangent space* $T_{(0,0)}\mathcal{M}$ *has the form* $\mathbb{R} \times W$, *where* W *is the generalized kernel of* $\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}$.

3. *The dimension of the intersection of* $\mathcal{M}_\sigma$ *with the stable (resp. unstable) manifold for system (2.20) at the origin is equal to the sum of the algebraic multiplicities of eigenvalues of*

$$\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)B|_{\hat{\mathcal{P}}(\sigma)\mathbb{R}^n}$$

with strictly negative (resp. strictly positive) real part.

Remark 2.3.2 *In section 2.4 we will prove the existence of a manifold* $\tilde{\Gamma}$ *fulfilling the assumptions of Proposition (2.3.1).*

If $A = A(0,0)$ is invertible then $\Pi = \Pi B = 0$ and $\tilde{\Lambda} = I_n$.

Hence, we can make the following remark

Remark 2.3.3 *If* A *is invertible then* $\tilde{\Gamma}$ *is an open subset of* $\mathbb{R} \times \mathcal{U}$.

Proof of Proposition (2.3.1):

Due to Proposition (2.1.6), the tangent space of $\tilde{\Gamma}$ at the origin is equal to $\mathbb{R} \times \tilde{\Lambda}\mathbb{R}^n$, and due to assumption, it holds $(\sigma, 0) \in \tilde{\Gamma}_\sigma$ for σ small enough.

Hence, there exist an open interval containing zero and a $\mathbf{C}^2$ function $g(\cdot, \cdot) : I \times \tilde{\Lambda}\mathbb{R}^n \to \tilde{\Pi}\mathbb{R}^n$, with $g(\sigma, 0) = 0$ for $\sigma \in I$ and $D_{(\sigma,\phi_1)}g(\sigma, \phi_1)|_{(\sigma,\phi_1)=0} = 0$ such that (σ, ϕ^σ) is contained in $\tilde{\Gamma}$ if and only if

$$\phi^\sigma = \phi_1 + g(\sigma, \phi_1) \tag{2.21}$$

for some $\phi_1 \in \mathcal{O} \subset \tilde{\Lambda}\mathbb{R}^n$, where $\mathcal{O}$ is an open neighborhood of 0 in the topology induced on $\tilde{\Lambda}\mathbb{R}^n$.

Let $V : \tilde{\Gamma} \to \mathbb{R}^n$ be the $\mathbf{C}^2$ extension of the restriction of $(\sigma, \phi) \mapsto A^{-1}(\sigma, \phi)Q(\phi)$ on $\mathcal{S} \cap \tilde{\Gamma}$.

This extension exists due to the assumptions of the theorem.

As $(0, A^{-1}(\sigma, \phi)Q(\phi)) \in T_{(\sigma,\phi)}\tilde{\Gamma}$ for each $(\sigma, \phi) \in \tilde{\Gamma} \cap \mathcal{S}$ (due to the invariance of $\tilde{\Gamma}$) and $\tilde{\Gamma} \cap \mathcal{S}$ is dense in $\tilde{\Gamma}$, it holds $(0, V(\sigma, \phi)) \in T_{(\sigma,\phi)}\tilde{\Gamma}$ for each $(\sigma, \phi) \in \tilde{\Gamma}$.

As $(0, V(\sigma, \phi)) \in T_{(\sigma,\phi)}\tilde{\Gamma}$ for each $(\sigma, \phi) \in \tilde{\Gamma}$, the set $\tilde{\Gamma}$ is a locally invariant manifold for system

$$\sigma_\xi = 0, \quad \phi_\xi = V(\sigma, \phi). \tag{2.22}$$

Plugging (2.21) into the second equation in (2.22) and applying $\tilde{\Lambda}\tilde{\Lambda}_\sigma$ from the left give

$$\tilde{\Lambda}\tilde{\Lambda}_\sigma(\tilde{\Lambda} + D_{\phi_1} g(\sigma, \phi_1))\phi_{1\xi} = \tilde{\Lambda}\tilde{\Lambda}_\sigma V(\sigma, \phi_1 + g(\sigma, \phi_1)). \tag{2.23}$$

Set

$$\hat{\Lambda}(\sigma, \phi_1) = \frac{1}{2\pi i} \int_\Gamma \zeta^{-1}(\zeta - \tilde{\Lambda}\tilde{\Lambda}_\sigma(\tilde{\Lambda} + D_{\phi_1} g(\sigma, \phi_1))\tilde{\Lambda})^{-1} d\zeta,$$

where Γ is a curve which encloses the eigenvalue 1 of the projector

$$\tilde{\Lambda} = \tilde{\Lambda}\tilde{\Lambda}_\sigma(\tilde{\Lambda} + D_{\phi_1} g(\sigma, \phi_1))\tilde{\Lambda}|_{(\sigma,\phi_1)=(0,0)}$$

in counterclockwise direction and does not enclose the origin.

After multiplication of $\hat{\Lambda}(\sigma, \phi_1)$ from the left to (2.23), noting that

$$\hat{\Lambda}(\sigma, \phi_1)\tilde{\Lambda}\tilde{\Lambda}_\sigma(\tilde{\Lambda} + D_{\phi_1} g(\sigma, \phi_1))\tilde{\Lambda} = \tilde{\Lambda}$$

and $\tilde{\Lambda}\phi_{1\xi} = \phi_{1\xi}$ and augmenting by the equation for σ we get

$$\sigma_\xi = 0, \quad \phi_{1\xi} = \hat{\Lambda}(\sigma, \phi_1)\tilde{\Lambda}\tilde{\Lambda}_\sigma V(\sigma, \phi_1 + g(\sigma, \phi_1)). \tag{2.24}$$

Obviously, the map

$$(\sigma, \phi_1) \mapsto \hat{\Lambda}(\sigma, \phi_1)\tilde{\Lambda}\tilde{\Lambda}_\sigma V(\sigma, \phi_1 + g(\sigma, \phi_1))$$

is of class $\mathbf{C}^2$.

From (2.24) we get after augmenting by the equation for σ and noting that $\lim_{(\sigma,\phi_1)\to 0} \hat{\Lambda}(\sigma, \phi_1) = \tilde{\Lambda}$ and that, due to Lemma (2.1.4), $\lim_{\sigma\to 0} \tilde{\Lambda}_\sigma A^{-1}(\sigma, 0)B = \tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$:

$$\sigma_\xi = 0, \quad \phi_{1\xi} = \tilde{\Lambda}\tilde{A}B\tilde{\Lambda}\phi_1 + O(\sigma^2 + |\phi_1|^2).$$

As, due to the weakened structural condition, each nonzero eigenvalue of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ has nonzero real part, we conclude from the center manifold theorem that there exists a locally invariant $\mathbf{C}^1$ - manifold $\bar{\mathcal{M}}$ for system (2.24) such that the following claims are true:

(i). For (σ, ϕ_1) small enough it holds: If $\hat{\Lambda}(\sigma, \phi_1)\tilde{\Lambda}\tilde{\Lambda}_\sigma V(\sigma, \phi_1 + g(\sigma, \phi_1)) = 0$ then $(\sigma, \phi_1) \in \bar{\mathcal{M}}$.

(ii). The tangent space $T_{(0,0)}\bar{\mathcal{M}}$ has the form $\mathbb{R} \times W$, where W is the generalized kernel of $\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}$.

If $(\sigma, \phi_1(\cdot))$ is a solution of system (2.24) then

$$(\sigma, \phi(\cdot)) = (\sigma, \phi_1(\cdot) + g(\sigma, \phi_1(\cdot)))$$

is a solution of system (2.22).

Hence, the set

$$\mathcal{M} = \{(\sigma, \phi) \mid \exists \phi_1 \in \bar{\mathcal{M}} \ \ with \ \ \phi = \phi_1 + g(\sigma, \phi_1)\} \tag{2.25}$$

is a locally invariant manifold for system

$$\sigma_\xi = 0, \quad \phi_\xi = V(\sigma, \phi).$$

If $(\sigma, \phi_1 + g(\sigma, \phi_1)) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ for δ small enough then

$$\hat{\Lambda}(\sigma, \phi_1)\tilde{\Lambda}\tilde{\Lambda}_\sigma V(\sigma, \phi_1 + g(\sigma, \phi_1)) = 0,$$

and it follows $(\sigma, \phi_1) \in \bar{\mathcal{M}}$ (due to (i)).

On the other hand, we conclude: If $(\sigma, \phi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ then $(\sigma, \phi) \in \tilde{\Gamma}$, i.e. there exists ϕ_1 with

$$(\sigma, \phi) = (\sigma, \phi_1 + g(\sigma, \phi_1)).$$

Hence, for δ small enough, $(\sigma, \phi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ implies

$$(\sigma, \phi) = (\sigma, \phi_1 + g(\sigma, \phi_1)),$$

where $(\sigma, \phi_1) \in \bar{\mathcal{M}}$.

Therefore, $(\sigma, \phi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ implies $(\sigma, \phi) \in \mathcal{M}$.

Hence, claim (1) of the theorem is true for the manifold $\mathcal{M}$.

Recall definition (2.25) of $\mathcal{M}$.

As $D_{(\sigma,\phi_1)} g(\sigma, \phi_1)|_{(\sigma,\phi_1)=0} = 0$, the tangent space $T_{(0,0)}\mathcal{M}$ is equal to $T_{(0,0)}\bar{\mathcal{M}}$, i.e. it has the form $\mathbb{R} \times W$, where W is the generalized kernel of $\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}$ (due to (ii)).

In other words: Claim (2) of the theorem is true for the manifold $\mathcal{M}$.

If $\phi_\xi(\xi) = V(\sigma, \phi(\xi))$ and $(\sigma, \phi(\xi)) \in \mathcal{M}$ then it holds

$$A(\sigma, \phi(\xi))\phi_\xi(\xi) = A(\sigma, \phi(\xi))V(\sigma, \phi(\xi)) =$$

$$= \lim_{(\bar{\sigma},\bar{\phi}) \in \mathcal{S} \cap \tilde{\Gamma}, (\bar{\sigma},\bar{\phi}) \to (\sigma, \phi(\xi))} A(\bar{\sigma}, \bar{\phi}) A^{-1}(\bar{\sigma}, \bar{\phi}) Q(\bar{\phi}) = Q(\phi(\xi)).$$

Hence, the fact that $\mathcal{M}$ is a locally invariant manifold for system

$$\sigma_\xi = 0, \quad \phi_\xi = V(\sigma, \phi)$$

implies that $\mathcal{M}$ is a locally invariant manifold for system (2.20), and due to $\sigma_\xi = 0$, the section $\mathcal{M}_\sigma = \mathcal{M} \cap \tilde{\Gamma}_\sigma$ is invariant.

Note that $T_0\mathcal{M}_0 = \{0\} \times \hat{\mathcal{P}}(0)\mathbb{R}^n$.

There exists a $\mathbf{C}^1$ - function $\eta^\sigma : \hat{\mathcal{P}}(\sigma)\mathbb{R}^n \to (I - \hat{\mathcal{P}}(\sigma))\mathbb{R}^n$ such that for each ϕ with $(\sigma, \phi) \in \mathcal{M}_\sigma$ it holds

$$\phi = \psi^\sigma + \eta^\sigma(\psi^\sigma), \tag{2.26}$$

where $\eta^\sigma(0) = 0$ and $D\eta^0(0) = 0$.

Note that $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)B(I - \hat{\mathcal{P}}(\sigma)) = A^{-1}(\sigma, 0)B\hat{\mathcal{P}}(\sigma)(I - \hat{\mathcal{P}}(\sigma)) = 0$ and

$$D\eta^\sigma(\cdot) \equiv (I - \hat{\mathcal{P}}(\sigma))D\eta^\sigma(\cdot)$$

because of $(I - \hat{\mathcal{P}}(\sigma))\eta^\sigma \equiv \eta^\sigma$.

Hence, $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)BD\eta^\sigma(0) = \hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)B(I - \hat{\mathcal{P}}(\sigma))D\eta^\sigma(0) = 0$ and $\hat{\mathcal{P}}(\sigma)D\eta^\sigma(\cdot) = \hat{\mathcal{P}}(\sigma)(I - \hat{\mathcal{P}}(\sigma))D\eta^\sigma(\cdot) \equiv 0$.

After plugging (2.26) into the second equation in (2.20), multiplication of $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, \psi^\sigma + \eta^\sigma(\psi^\sigma)$ from the left and taking into account that $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)BD\eta^\sigma(0) = 0$ and $\hat{\mathcal{P}}(\sigma)D\eta^\sigma(\cdot) \equiv 0$ we get:

$$\psi^\sigma_\xi = \hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)B\psi^\sigma + O(|\psi^\sigma|^2) \tag{2.27}$$

Claim (3) of the theorem for the manifold $\mathcal{M}$ follows from taking into account the signature of $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma, 0)B|_{\hat{\mathcal{P}}(\sigma)\mathbb{R}^n}$. □

2.3.2 Generalized kernel of linearization

Recall the definiton of $\tilde{\Lambda}$ in the last section.

Define $\tilde{A}$ to be the generalized inverse of $-\lambda_p I_n + \sum_{j=1}^d A_j(0)$ corresponding to its nonvanishing eigenvalues.

Due to Proposition (2.3.1), there exists a locally invariant manifold with tangent space $\{0\} \times V$, where V is the generalized kernel of $\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}$, if the invariant manifold $\tilde{\Gamma}$ in Proposition (2.3.1) exists.

We will prove the existence of such a locally invariant manifold $\tilde{\Gamma}$ in section 2.4.

In this section we will analyze the generalized kernel of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$, i.e. we will prove the following theorem

Theorem 2.3.4 *If the weakened structural condition holds, then the following claims are true:*

1. *It holds* $ker(\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}) = P^{-1}(\mathbb{R}^{n-r} \times \{0\})$ *and*
$$range(\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}) \cap ker(\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}) = P^{-1}\left(ker(A_{11}) \times \{0\}\right).$$
2. *The generalized kernel of* $\tilde{\Lambda}\tilde{A}B|_{\tilde{\Lambda}\mathbb{R}^n}$ *is the linear space*
$$P^{-1}\left(\mathbb{R}^{n-r} \times \left(S^{-1}A_{21}\left(ker(A_{11})\right)\right)\right).$$
3. *The normal form of* $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ *has no Jordan block to eigenvalue zero of order larger than two.*

If the invariant manifold $\tilde{\Gamma}$ in Proposition (2.3.1) exists, then, due to claim (2) in Proposition (2.3.1), the tangent space $T_0\mathcal{M}_0$ is the generalized kernel of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$, and we can make the following remark:

Remark 2.3.5 *If the invariant manifold* $\tilde{\Gamma}$ *in Proposition (2.3.1) exists then claim (1) of the main theorem is a direct consequence of Theorem (2.3.4).*

Proof of Theorem (2.3.4):

We can assume that $P = I_n$, i.e. $B = diag(0, S)$.

For general P our claims follow via linear transformation by P.

Repeat from the weakened structural condition that

$$l(r)A_{12}S^{-1}A_{21}r \neq 0 \;\; \forall r \in ker(A_{11}) \setminus \{0\} \tag{2.28}$$

with the block decomposition (1.8) for A, where $l(r)$ is defined in (1.10).

Let $\tilde{\Lambda}(\sigma)$ be the projector commuting with $\frac{1}{\sigma}\left(\Pi + \sigma\tilde{A}\right)B$ corresponding to its eigenvalues which are uniformly bounded with respect to σ for σ small.

Define $\mathcal{A}(\sigma) := \frac{1}{\sigma}\left(\Pi B + \sigma\tilde{A}B\right)\tilde{\Lambda}(\sigma),\ \mathcal{A} := \lim_{\sigma\to 0,\sigma\neq 0}\mathcal{A}(\sigma) = \tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$.

As zero is a semisimple eigenvalue of ΠB the last relation follows due to relation (2.3) in Theorem (2.1.3).

Due to Theorem (2.1.3), the matrices $\mathcal{A}(\sigma)$ and $\tilde{\Lambda}(\sigma)$ have an analytic extension up to $\sigma = 0$.

This implies that, if $y \in ker(\mathcal{A}|_{\tilde{\Lambda}\mathbb{R}^n})$, then it holds for $y(\sigma) = \tilde{\Lambda}(\sigma)y$

$$\lim_{\sigma\to 0,\sigma\neq 0}\mathcal{A}(\sigma)y(\sigma) = \mathcal{A}y = 0,$$

and it follows for $A(\sigma) = A + \sigma\Pi$ the relation

$$0 = \lim_{\sigma\to 0,\sigma\neq 0} A(\sigma)\mathcal{A}(\sigma)y(\sigma) = \lim_{\sigma\to 0} By(\sigma) = By.$$

As $B = diag(0, S)$ and $S \in GL(\mathbb{R}, r)$ this implies $y^{II} = 0$.

We can conclude:

$$y \in ker(\mathcal{A}|_{\tilde{\Lambda}\mathbb{R}^n}) \ \ \Rightarrow \ \ y^{II} = 0. \tag{2.29}$$

On the other hand, if $y^{II} = 0$ then $\frac{1}{\sigma}\left(\Pi + \sigma\tilde{A}\right)By = 0$ for each $\sigma \neq 0$, i.e. y is in the kernel of $\frac{1}{\sigma}\left(\Pi + \sigma\tilde{A}\right)B$ which implies that $\tilde{\Lambda}(\sigma)y = y$ for each $\sigma \neq 0$.

Hence, we can conclude that

$$y^{II} = 0 \ \ \Rightarrow \ \ \lim_{\sigma\to 0,\sigma\neq 0}\frac{1}{\sigma}\left(\Pi + \sigma\tilde{A}\right)B\tilde{\Lambda}(\sigma)y = \mathcal{A}y = 0 \ \ and \ \ \tilde{\Lambda}y = y.$$

From this conclusion and (2.29) it follows

$$y \in ker(\mathcal{A}|_{\tilde{\Lambda}\mathbb{R}^n}) \ \ \Leftrightarrow \ \ y^{II} = 0. \tag{2.30}$$

This is the first part of claim (1) in the theorem for $P = I_n$.

Let $y \in V = \tilde{\Lambda}\mathbb{R}^n$ be such that $z = \mathcal{A}y \neq 0,\ \ \mathcal{A}^2 y = 0$, set $y(\sigma) = \tilde{\Lambda}(\sigma)y$ and $z(\sigma) = \mathcal{A}(\sigma)y(\sigma)$.

Obviously, it holds $y = \lim_{\sigma\to 0,\sigma\neq 0} y(\sigma),\ \ \ z = \lim_{\sigma\to 0,\sigma\neq 0} z(\sigma)$.

After multiplication of $A(\sigma) = A + \sigma\Pi$ from the left to $\mathcal{A}(\sigma)y(\sigma) = z(\sigma)$ and $\mathcal{A}(\sigma)z(\sigma)$ for $\sigma \neq 0$ (using $A(\sigma)\mathcal{A}(\sigma) = B$) and taking into account that $\lim_{\sigma\to 0,\sigma\neq 0} \mathcal{A}(\sigma)z(\sigma) = \mathcal{A}z = \mathcal{A}^2 y = 0$ we get:

$$By = \lim_{\sigma\to 0,\sigma\neq 0} By(\sigma) = \lim_{\sigma\to 0,\sigma\neq 0} A(\sigma)z(\sigma) = Az,\ \ \mathcal{A}z = 0.$$

These relations imply (recalling the notation in (1.8)):

$$A_{11}z^I + A_{12}z^{II} = 0,\ \ A_{21}z^I + A_{22}z^{II} = Sy^{II},\ \ \mathcal{A}z = 0. \tag{2.31}$$

From the third equation in (2.31) and (2.30) we see that $z^{II} = 0$. Because of the first equation it is $z^I \in ker(A_{11})$, and we can conclude that

$$range(\mathcal{A}|_V) \cap ker(\mathcal{A}|_V) \subset \{z \in V \mid z^I \in ker(A_{11}),\ z^{II} = 0\}. \tag{2.32}$$

As $z \neq 0$ and $z^{II} = 0$ it follows $z^I \in ker(A_{11}) \setminus \{0\}$.

Due to (2.28), it follows $A_{21}z^I \neq 0$.

The second equation in (2.31), $z^{II} = 0$ and $A_{21}z^I \neq 0$ imply

$$y^{II} = S^{-1}A_{21}z^I \neq 0. \tag{2.33}$$

Taking (2.30) into account it follows for $P = I_n$

$$ker(\mathcal{A}^2|_V) \subset \mathbb{R}^{n-r} \times \left(S^{-1}A_{21}\left(ker(A_{11})\right)\right). \tag{2.34}$$

On the other hand assume there is a vector $\tilde{y} \in V$ such that $\mathcal{A}\tilde{y} \neq 0$, $\mathcal{A}^2\tilde{y} \neq 0$ and $\mathcal{A}^3\tilde{y} = 0$. Set $y = \mathcal{A}\tilde{y}$ and $z = \mathcal{A}y$. The above argument shows

$$y^{II} = S^{-1}A_{21}z^I \neq 0\ \ with\ \ z^I \in ker(A_{11}). \tag{2.35}$$

We obtain

$$\mathcal{A}(\sigma)\tilde{y} = \begin{pmatrix} y^I \\ S^{-1}A_{21}z^I \end{pmatrix} + O(\sigma).$$

Multiplication of $A(\sigma)$ to the last relation from the left for $\sigma \neq 0$ and performing the limit $\sigma \to 0$ give $B\tilde{y} = diag(0, S)\tilde{y} = Ay$.

We get $(Ay)^I = 0$, or in other words (using relation (2.35)):

$$A_{11}y^I + A_{12}S^{-1}A_{21}z^I = 0, \;\; z^I \in ker(A_{11}).$$

From $\bar{z}^I A_{11} = 0$ for $\bar{z}^I = l(z^I)$ (with $l(\cdot)$ defined in (1.10)) we conclude:

$$\bar{z}^I A_{12}S^{-1}A_{21}z^I = 0. \tag{2.36}$$

Because of (2.28) it holds $l(r)A_{12}S^{-1}A_{21}r \neq 0 \;\; \forall r \in ker(A_{11}) \setminus \{0\}$, where $l(r)$ is defined in (1.10) .

Hence, relation (2.36) implies $z^I = 0$.

On the other hand, it is $z^I \neq 0$ because of (2.35). This is a contradiction, and we conclude: The normal form of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ has no Jordan block to eigenvalue zero of order larger than two.

This is claim (3) of the theorem.

For $z \in \mathbb{R}^n$ with $z^I \in ker(A_{11})$, $z^{II} = 0$ set y with $y^I = 0$ and $y^{II} = S^{-1}A_{21}z^I$.

Then the relations in (2.31) hold, i.e. $Az = By$, $\mathcal{A}z = 0$.

We conclude that $By = \lim_{\sigma\neq 0, \sigma\to 0} A(\sigma)z$, and it follows

$$\lim_{\sigma\neq 0, \sigma\to 0} A^{-1}(\sigma)By = \lim_{\sigma\neq 0, \sigma\to 0} (\tilde{A} + \frac{1}{\sigma}\Pi)By = z, \;\; \mathcal{A}z = 0.$$

Hence, $\lim_{\sigma\neq 0, \sigma\to 0} \sigma A^{-1}(\sigma)By = \Pi By = 0$, and it follows $y \in ker(\Pi B)$, i.e. $y = \tilde{\Lambda}y$, and we conclude that

$$\lim_{\sigma\neq 0, \sigma\to 0} B\tilde{\Lambda}(\sigma)y = B\tilde{\Lambda}y = By = \lim_{\sigma\neq 0, \sigma\to 0} A(\sigma)z.$$

Hence,

$$z = \lim_{\sigma\neq 0, \sigma\to 0} A^{-1}(\sigma)B\tilde{\Lambda}(\sigma)y = \lim_{\sigma\neq 0, \sigma\to 0} \frac{1}{\sigma}(\Pi + \sigma\tilde{A})B\tilde{\Lambda}(\sigma)y = \mathcal{A}y. \tag{2.37}$$

We obtain $z = \mathcal{A}y$, $\mathcal{A}z = 0$, where $y = \tilde{\Lambda}y$.

Together with (2.32), this implies

$$\{z^I \in ker(A_{11}), \;\; z^{II} = 0\} = range(\mathcal{A}|_{\tilde{\Lambda}\mathbb{R}^n}) \cap ker(\mathcal{A}|_{\tilde{\Lambda}\mathbb{R}^n}).$$

This is the second claim of claim (1) in the theorem for $P = I_n$.

Furthermore, it follows

$$\mathbb{R}^{n-r} \times \left(S^{-1}A_{21}\left(ker(A_{11})\right)\right) \subset ker(\mathcal{A}^2|_V). \tag{2.38}$$

As the normal form of $\tilde{\Lambda}\tilde{A}B\tilde{\Lambda}$ has no Jordan block of order larger than two, claim (2) of the theorem follows from (2.34) and (2.38) for $P = I_n$. □

2.3.3 Signature of real parts of nonvanishing eigenvalues

Remember the notation introduced in chapter 1, where $l_1^*, \cdots l_m^* \in ker(A_{11}^*)$ and $r_1, \cdots r_m \in ker(A_{11})$ are chosen such that $l_i r_j = \delta_{ij}$.

First, we assume that $A \in GL(n, \mathbb{R})$.

For $\sigma \in \mathbb{R}$ and

$$R_i = \begin{pmatrix} r_i l_i & 0 \\ 0 & 0 \end{pmatrix}, \;\; R = \sum_{i=1}^{m} R_i$$

define

$$A(\sigma) := A + \sigma P^{-1}RP. \tag{2.39}$$

Let $\hat{\mathcal{P}}(\sigma)$ be the projector commuting with $A^{-1}(\sigma)B$ corresponding to the eigenvalues of $A^{-1}(\sigma)B$ which tend to zero for $\sigma \to 0$.

We will prove the following theorem:

Theorem 2.3.6 *If the weakened structural condition is fulfilled and $A \in GL(n, \mathbb{R})$ then it holds:*

If $\sigma > 0$ (resp. $\sigma < 0$) then the sum of the algebraic multiplicities of eigenvalues of $\hat{\mathcal{P}}(\sigma)A^{-1}(\sigma)B$ with strictly negative (resp. strictly positive) real part is equal to $dim(ker(A_{11}))$ for σ small enough.

Before we prove this theorem we tell how we obtain claim (2) of the main theorem for invertible A.

Consider

$$\mathcal{A}(\sigma, \tilde{\sigma}) = (A + \sigma P^{-1}RP + \tilde{\sigma}(I_n - P^{-1}RP))^{-1}B.$$

Theorem (2.3.6) says that the change of the sum of algebraic multiplicities of the eigenvalues of $\hat{\mathcal{P}}(\sigma)\mathcal{A}(\sigma, 0)$ with strictly negative (resp. strictly positive) real part under change of the sign of σ is equal to $dim(ker(A_{11}))$.

We will see that we can repeat the proof of the theorem for $A+\tilde{\sigma}(I_n-P^{-1}RP)$ instead of A as long as $\tilde{\sigma}$ is small enough.

Hence, the change of the sum of algebraic multiplicities of the eigenvalues of $\mathcal{A}(\sigma, \tilde{\sigma})$ with strictly negative (resp. strictly positive) real part under change of the sign of σ is equal to $dim(ker(A_{11}))$ for $\tilde{\sigma}$ small enough.

On the other hand, the sum of algebraic multiplicities of the eigenvalues of $\mathcal{A}(0, \tilde{\sigma})$ with strictly negative (resp. strictly positive) real part does not change under variation of $\tilde{\sigma}$ as long as $\tilde{\sigma}$ is small enough. In fact, the dimension of the generalized kernel of $\mathcal{A}(0, \tilde{\sigma})$ does not change under variation of $\tilde{\sigma}$: Analogously as in the proof of Theorem (2.3.4) we can show: The generalized kernel of $\mathcal{A}(0, \tilde{\sigma})$ is equal to

$$P^{-1}(\mathbb{R}^{n-r} \times (S^{-1}A_{21}(\tilde{\sigma})(ker(A_{11}(\tilde{\sigma})))))$$

with the well-known notation for the blocks $A_{ij}(\tilde{\sigma})$.

We conclude that the change of the sum of algebraic multiplicities of the eigenvalues of $\mathcal{A}(\sigma, \sigma) = (A + \sigma I_n)^{-1}B$ with strictly negative (resp. strictly positive) real part under change of the sign of σ is equal to $dim(ker(A_{11}))$.

Hence, we can make the following remark:

Remark 2.3.7 *If the invariant manifold $\tilde{\Gamma}$ in Proposition (2.3.1) exists then, for invertible A, claim (2) of the main theorem is a direct consequence of Theorem (2.3.6).*

We will treat the case of non-invertible A in this section later.

Proof of Theorem (2.3.6):

Because of

$$A^{-1}B = P^{-1}PA^{-1}P^{-1}PBP^{-1}P = P^{-1}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} diag(0,S)P$$

the matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} diag(0,S)$ is similar to $A^{-1}B$, and we can assume $P = I_n$.

Set for each σ where $A(\sigma) \in GL(\mathbb{R}, n)$

$$T(\sigma) := \left(A(\sigma)^{-1}B\right)_{22}. \tag{2.40}$$

Denote by $\bar{P}(\sigma)$ the projector commuting with $T(\sigma)$ corresponding to its eigenvalues tending to zero for $\sigma \to 0$ and set $\bar{P} = \bar{P}(0)$.

We will prove the following claims:

(i) $ker(T(0)) = S^{-1}A_{21}\left(ker(A_{11})\right)$, $ker(T^*(0)) = A_{12}^*\left(ker(A_{11}^*)\right)$

(ii) Zero is a semisimple eigenvalue of $T(0)$.

(iii) $rank(\bar{P}\left(\partial_\sigma T(\sigma)\right)|_{\sigma=0}\bar{P}) = m$

(iv) For $0 \neq r = \sum_{j=1}^m \alpha_j r_j$ and $l = \sum_{j=1}^m \alpha_j l_j$ it holds: If $0 \neq \lambda \in spec\left(\bar{P}\partial_\sigma T(0)\bar{P}\right)$, then $sign(Re(\lambda)) = -sign(Re(lA_{12}S^{-1}A_{21}r)) \neq 0$.

(v) The changes of the sum of the algebraic multiplicities of eigenvalues with strictly negative (strictly positive) real part of $A(\sigma)^{-1}B$ and $T(\sigma)$ under variation of σ are equal.

Assume the claims above are true.

Due to (ii), zero is a semisimple eigenvalue of $T(0)$, and it follows due to Theorem (2.1.3) (after multiplication of σ to relation (2.3)):

$$T(\sigma)\bar{P}(\sigma) = \sigma\bar{P}T^{(1)}\bar{P} + o(\sigma) \;\; for \;\; T^{(1)} = (\partial_\sigma T(\sigma))\,|_{\sigma=0}.$$

Claims (iii) and (iv) imply that, with σ changing from a strictly negative value to a strictly positive value, m eigenvalues of $T(\sigma)$ (i.e. whose sum of multiplicities is equal to m) change from a strictly positive value (resp. strictly negative value) to a strictly negative value (resp. a strictly positive value) if $Re(lA_{12}S^{-1}A_{21}r)$ is strictly negative (resp. strictly positive), and the claim of the theorem follows from (v).

It remains to prove claims (i) - (v):

ad (i):

$$\left(A^{-1}B\right)_{22} v = 0 \Leftrightarrow \exists y \in \mathbb{R}^{n-r} \;\; with \;\; A^{-1}B\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \exists y \in \mathbb{R}^{n-r} \;\; with \;\; \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \exists y \in \mathbb{R}^{n-r} \;\; with \;\; A_{11}y = 0, \;\; Sv = A_{21}y$$

$$\Leftrightarrow \exists y \in ker(A_{11}) \;\; with \;\; v = S^{-1}A_{21}y.$$

Hence, it follows

$$ker(T(0)) = S^{-1}A_{21}(ker(A_{11})).$$

If $y^*A_{11} = 0$ then it follows from

$$\left(A_{11}(A^{-1})_{12} + A_{12}(A^{-1})_{22}\right)S = \left(AA^{-1}\right)_{12}S = (I_n)_{12}S = 0$$

the relation

$$y^* A_{12} T(0) = y^* A_{12} \left(A^{-1}B\right)_{22} = y^* A_{12} (A^{-1})_{22} S = 0.$$

Hence, it follows

$$A_{12}^*(ker(A_{11}^*)) \subset ker(T^*(0)).$$

As

$$dim(ker(T(0))) = dim(ker(T^*(0)))$$

and

$$dim(A_{12}^*(ker(A_{11}^*))) = dim(S^{-1}A_{21}(ker(A_{11})))$$

claim (i) follows.

ad (ii):

Assume there exists $w \in \mathbb{R}^r$ such that

$$\left(A^{-1}B\right)_{22} w \neq 0, \quad \left(A^{-1}B\right)_{22}^2 w = 0. \tag{2.41}$$

From (i) it follows that there exists $v \in \mathbb{R}^r \setminus \{0\}$ of the form

$$v = S^{-1}A_{21} r \neq 0, \quad r \in ker(A_{11}) \tag{2.42}$$

such that

$$\exists y \in \mathbb{R}^{n-r} \;\; with \;\; A^{-1}B \begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} y \\ v \end{pmatrix}. \tag{2.43}$$

Choose $\alpha_1, \cdots, \alpha_m$ such that $r = \sum_{j=1}^m \alpha_j r_j$ and set $l = \sum_{j=1}^m \alpha_j l_j$.

Multiplication with $\begin{pmatrix} l^* \\ 0 \end{pmatrix}^* A$ from the left to (2.43) gives

$$\begin{pmatrix} l^* \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} l^* \\ 0 \end{pmatrix}^* \begin{pmatrix} A_{11}y + A_{12}v \\ A_{21}y + A_{22}v \end{pmatrix}.$$

Using $lA_{11} = 0$ gives for the first entry

$$0 = lA_{12}v = lA_{12}S^{-1}A_{21}r = 0.$$

Hence, it follows $r = 0$, a contradiction to (2.42).

ad (iii) and (iv):

From

$$(\partial_\sigma T(\sigma))|_{\sigma=0} = \left((\partial_\sigma(A+\sigma R)^{-1})|_{\sigma=0}B\right)_{22} = -\left(A^{-1}RA^{-1}B\right)_{22},$$

and $B = diag(0, S)$ we get

$$(\partial_\sigma T(\sigma))|_{\sigma=0} = -(A^{-1})_{21}R_{11}(A^{-1})_{12}S. \tag{2.44}$$

Set

$$\tilde{T} = S^{-1}A_{21}R_{11}A_{12} = \sum_{i=1}^{m} S^{-1}A_{21}r_i l_i A_{12}. \tag{2.45}$$

Assume that λ is a nonvanishing eigenvalue of $\tilde{T}$.

After an appropriate choice of $\{r_1, \cdots, r_m\}$ we can assume that $S^{-1}A_{21}r_1$ is an eigenvector of $\tilde{T}$ to eigenvalue λ, i.e. $l_j A_{12}S^{-1}A_{21}r_1 = 0$ for $j \neq 1$ and $\lambda = l_1 A_{12}S^{-1}A_{21}r_1$, and we get $sign(Re(\lambda)) = sign(Re(l_1 A_{12}S^{-1}A_{21}r_1))$.

As $S^{-1}A_{21}r_1$ can be assumed to be the multiple of an arbitrary eigenvector of $\tilde{T}$ we conclude that the signs of the real parts of the nonvanishing eigenvalues of $\tilde{T}$ are equal to the sign in (1.11).

Remember that $\bar{P}$ is the projector onto $ker(T(0))$ commuting with $T(0)$.

Due to (i), $r_i \in ker(A_{11})$ and $l_i^* \in ker(A_{11}^*)$ it holds for $i = 1, \cdots, m$

$$\bar{P}S^{-1}A_{21}r_i = S^{-1}A_{21}r_i \;\; and \;\; l_i A_{12}\bar{P} = l_i A_{12}.$$

Hence, it follows

$$\bar{P}\tilde{T} = \tilde{T}\bar{P} = \tilde{T}. \tag{2.46}$$

Obviously it holds

$$R_{11}A_{12}(A^{-1})_{21} = R_{11}\left(A_{11}(A^{-1})_{11} + A_{12}(A^{-1})_{21}\right) = \tag{2.47}$$

$$= R_{11}\left(AA^{-1}\right)_{11} = R_{11}$$

and

$$(A^{-1})_{12}A_{21}R_{11} = \left((A^{-1})_{11}A_{11} + (A^{-1})_{12}A_{21}\right)R_{11} = \tag{2.48}$$

$$= \left(A^{-1}A\right)_{11}R_{11} = R_{11}.$$

Using these identities we get

$$m \geq rank(\tilde{T}) \geq rank((A^{-1})_{12}S\tilde{T}(A^{-1})_{21}) = rank(\sum_{i=1}^{m} r_i l_i) = m.$$

Taking into account relation (2.46) and using the identities (2.44), (2.45), (2.47) and (2.48) it is not hard to verify the relation

$$\bar{P}\tilde{T}\bar{P}\partial_\sigma T(0)\bar{P}\tilde{T}\bar{P} = \tilde{T}\partial_\sigma T(0)\tilde{T} = -S^{-1}A_{21}R_{11}A_{12} = \tag{2.49}$$

$$= -\tilde{T} = -\bar{P}\tilde{T}\bar{P}.$$

Multiplication of the generalized inverse of $-\bar{P}\tilde{T}\bar{P}$ (corresponding to its non-vanishing eigenvalues) to (2.49) from the right gives

$$-\bar{P}\tilde{T}\bar{P}\partial_\sigma T(0)\bar{P} = -\bar{P}\partial_\sigma T(0)\bar{P}\tilde{T}\bar{P} = \bar{P}. \tag{2.50}$$

Relation (2.50) implies that $\bar{P}\tilde{T}\bar{P} = \tilde{T}$ is the generalized inverse of $-\bar{P}(\partial_\sigma T(\sigma))|_{\sigma=0}\bar{P}$ corresponding to the nonvanishing eigenvalues of $\bar{P}\partial_\sigma T(0)\bar{P}$, and we can write

$$-\bar{P}\partial_\sigma T(0)\bar{P} = \frac{1}{2\pi i}\int_\Gamma \zeta^{-1}(\zeta - \tilde{T})^{-1}d\zeta,$$

where the curve Γ encircles the nonvanishing eigenvalues of $\tilde{T}$ in counter-clockwise direction and does not enclose the origin.

The real part of each nonvanishing eigenvalue of $\tilde{T}$ is strictly negative (resp. strictly positive) if and only if each nonvanishing eigenvalue of $\bar{P}\partial_\sigma T(0)\bar{P}$ is strictly positive (resp. strictly negative).

Remembering that $rank(\tilde{T}) = m$ and that the signs of the real parts of the nonvanishing eigenvalues of $\tilde{T}$ are equal to the sign in (1.11), claims (iii) and (iv) follow.

ad (v):

If $\bar{T}(\sigma)$ is a transformation such that $\bar{T}^{-1}(\sigma)\,(A(\sigma)^{-1}B)_{22}\,\bar{T}(\sigma)$ has Jordan normal form then it holds for $\bar{\mathbf{T}}(\sigma) = diag(I_{n-r}, \bar{T}(\sigma))$

$$\bar{\mathbf{T}}(\sigma)^{-1}A(\sigma)^{-1}B\bar{\mathbf{T}}(\sigma) = \begin{pmatrix} I_{n-r} & 0 \\ 0 & \bar{T}(\sigma)^{-1} \end{pmatrix} \begin{pmatrix} 0 & (A^{-1}(\sigma)B)_{12} \\ 0 & (A^{-1}(\sigma)B)_{22} \end{pmatrix} \begin{pmatrix} I_{n-r} & 0 \\ 0 & \bar{T}(\sigma) \end{pmatrix} =$$

$$= \begin{pmatrix} I_{n-r} & 0 \\ 0 & \bar{T}^{-1}(\sigma) \end{pmatrix} \begin{pmatrix} 0 & (A(\sigma)^{-1}B)_{12}\,\bar{T}(\sigma) \\ 0 & (A(\sigma)^{-1}B)_{22}\,\bar{T}(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & (A(\sigma)^{-1}B)_{12}\,\bar{T}(\sigma) \\ 0 & \bar{T}(\sigma)^{-1}\,(A(\sigma)^{-1}B)_{22}\,\bar{T}(\sigma) \end{pmatrix}.$$

Hence, the changes of the sum of algebraic multiplicities of eigenvalues with strictly negative and strictly positive real part of $(A(\sigma)^{-1}B)_{22}$ and $A(\sigma)^{-1}B$ under variation of σ are equal and we are ready with the case of invertible A. $\square$

Now we treat the case of noninvertible A. We can assume that $P = I_n$.

Set $A(\delta, \sigma) = A + \delta diag(0_{n-r}, I_r) + \sigma R$.

We use the following lemma:

Lemma 2.3.8 *There exists a real number $C > 0$ such that for $0 < |\delta| < C$ the inverse $A(\delta, 0)^{-1}$ exists.*

Before we prove this lemma, we tell, how we conclude for the case of noninvertible A.

As for $\delta \neq 0$ small enough the matrix $A(\delta, 0)$ is invertible we can repeat the proof above for $A(\delta, 0)$ instead of A.

Remember relation

$$-\bar{P}\tilde{T}\bar{P}((\partial_\sigma A(\sigma)^{-1}B)_{22})|_{\sigma=0}\bar{P} = -\bar{P}((\partial_\sigma A(\sigma)^{-1}B)_{22})|_{\sigma=0}\bar{P}\tilde{T}\bar{P} = \bar{P}$$

for the case that A is invertible, where $\bar{P}$ is the commuting projector onto $ker(T(0))$ and $\tilde{T}$ is given in (2.45).

This relation together with $\bar{P}\tilde{T}\bar{P} = \tilde{T}$ implies

$$-\bar{P}((\partial_\sigma A(\sigma)^{-1}B)_{22})|_{\sigma=0}\bar{P} = \frac{1}{2\pi i}\int_\Gamma \zeta^{-1}(\zeta - \tilde{T})^{-1}d\zeta,$$

where Γ encircles the nonvanishing eigenvalues of $\tilde{T}$ in counterclockwise direction and does not enclose the origin.

If A is not invertible, we can conclude in an analogous way for invertible $A(\delta, 0)$:

$$-\bar{P}((\partial_\sigma A(\delta,\sigma)^{-1}B)_{22})|_{\sigma=0}\bar{P} = \frac{1}{2\pi i}\int_\Gamma \zeta^{-1}(\zeta - \tilde{T})^{-1}d\zeta.$$

Remembering $B = diag(0_{n-r}, S)$ and taking into account that $\tilde{T}$ does not change under variation of A_{22} (due to representation (2.45)), we conclude that the matrix $\bar{P}((\partial_\sigma A(\delta,\sigma)^{-1}B)|_{\sigma=0})_{22}\bar{P}$ is independent of δ.

In other words: The change of the signature of $T(\delta,\sigma) = (A(\delta,\sigma)^{-1}B)_{22}$ under change of σ is independent of δ as well as the change of the signature of $A(\delta,\sigma)^{-1}B$.

Hence, it remains to prove Lemma (2.3.8):

We have to show that $A(\delta)v = 0$ implies $v = 0$ for $\delta \neq 0$ small enough.

Relation $A(\delta)v = 0, \;\; \delta \neq 0$ implies

$$\Lambda A(\delta)\Lambda v^I + \delta\Lambda diag(0_{n-r}, I_r)\Pi v^{II} = 0, \tag{2.51}$$

$$\Pi diag(0_{n-r}, I_r)\Lambda v^I + \Pi diag(0_{n-r}, I_r)\Pi v^{II} = 0. \tag{2.52}$$

We will show later that

(2.53) $$rank(\Pi) = rank(\Pi diag(0_{n-r}, I_r)\Pi).$$

Hence, relation

(2.54) $$v^I = -\delta\ M_1 v^{II},\ \ v^{II} = -M_2 v^I$$

is equivalent to (2.51) and (2.52) for

$$v^I = \Lambda v,\ v^{II} = \Pi v,\ M_1 = \frac{1}{2\pi i}\oint_\Gamma \zeta^{-1}\left(\zeta - \Lambda A(\delta)\Lambda\right)^{-1} d\zeta\ \Lambda diag(0_{n-r}, I_r)\Pi$$

and

$$M_2 = \frac{1}{2\pi i}\oint_\Gamma \zeta^{-1}\left(\zeta - \Pi diag(0_{n-r}, I_r)\Pi\right)^{-1} d\zeta\ \Pi diag(0_{n-r}, I_r)\Lambda$$

where Γ is a simple closed curve which does not enclose the origin and surrounds the nonzero eigenvalues of $\Lambda A(\delta)\Lambda$ and $\Pi diag(0_{n-r}, I_r)\Pi$ in counterclockwise direction.

Plugging the second equality for v^{II} in (2.54) into the first one yields

(2.55) $$v^I = \delta M_1 M_2 v^I.$$

If $|\delta M_1 M_2| < 1$ then $v^I = 0$, the second equality in (2.54) implies $v^{II} = 0$ and Lemma (2.3.8) has been proven, and it remains to show relation (2.53).

As zero is a semisimple eigenvalue of $\Pi diag(0_{n-r}, I_r)$ (due to the weakened structural condition), there holds

(2.56) $$rank\left((\Pi diag(0_{n-r}, I_r))^2\right) = rank(\Pi diag(0_{n-r}, I_r)).$$

Hence, relation (2.56) together with

$$rank\left((\Pi diag(0_{n-r}, I_r))^2\right) = rank\left(\Pi diag(0_{n-r}, I_r)\Pi diag(0_{n-r}, I_r)\right) \leq$$

$$\leq rank\left(\Pi diag(0_{n-r}, I_r)\Pi\right) \leq rank\left(\Pi diag(0_{n-r}, I_r)\right)$$

implies

$$(2.57)\qquad rank(\Pi diag(0_{n-r}, I_r)) = rank\left(\Pi diag(0_{n-r}, I_r)\Pi\right).$$

Assume that

$$rank(\Pi) > rank(\Pi diag(0_{n-r}, I_r)).$$

Hence, for $v_1, \cdots, v_n$ being the n row vectors of Π, there exists $v \in span(v_1, \cdots, v_n)$ with

$$(2.58)\qquad v^I \neq 0 \;\; and \;\; v^{II} = 0,$$

where v^I denotes the row vector with the first $n-r$ entries of v and v^{II} denotes the row vector with the last r entries of v.

On the other hand, due to $vA = 0$, it holds: $(vA)^I = v^I A_{11} = 0$ and $(vA)^{II} = v^I A_{12} = 0$.

Remembering the notation in (1.11), let r be the element $ker(A_{11})$ with $l(r) = v^I$.

Due to (1.11), $v^I A_{11} = 0$ implies $v^I = 0$, a contradiction to (2.58).

Hence, we conclude that

$$(2.59)\qquad rank(\Pi) = rank(\Pi diag(0_{n-r}, I_r)),$$

and, together with (2.57), relation (2.53) follows.

2.4 Center manifold reduction

Before we turn to the details, we introduce the following convention for this section: 'smooth' means 'of class $\mathbf{C}^2$'.

In this section we will prove the existence of the invariant manifold $\tilde{\Gamma}$ of Proposition (2.3.1).

Recall that the traveling wave system reads as

$$(2.60) \qquad (-(\sigma+\lambda_p)\phi + \sum_{j=1}^{d} \omega_j F_j(\phi))_\xi = Q(\phi) = \begin{pmatrix} 0 \\ q(\phi) \end{pmatrix},$$

where F_j and Q are of class $\mathbf{C}^5$.

Without loss of generality we can assume for the matrix $P(U)$ in (1.5) that $P(0) = I_n$, i.e.

$$Q_\phi(0) = B(0) = B = \begin{pmatrix} 0 & 0 \\ 0 & S(0) \end{pmatrix},$$

where $S(0)$ is invertible.

Define the nonlinear transformation

$$(2.61) \qquad \psi(\phi) = \begin{pmatrix} \phi^I \\ S^{-1}(0)q(\phi) \end{pmatrix}.$$

As $q_{\phi^{II}}(0) = S(0)$ is invertible the transformation $\psi = \psi(\phi)$ has an inverse $\phi = \phi(\psi)$ for ϕ close to 0.

For $\bar{A}_j(\psi) = \phi_\psi^{-1}(\psi)F_{j\phi}(\phi(\psi))\phi_\psi(\psi)$, $\bar{B}(\psi) = diag(0_{n-r}, S^{-1}(0)q_{\phi^{II}}(\phi(\psi))S(0))$ and

$$(2.62) \qquad \bar{A}(\sigma,\psi) = -(\sigma+\lambda_p)I_n + \sum_{j=1}^{d} \omega_j \bar{A}_j(\psi)$$

system (2.60) augmented by the equation for σ can be written in the form

$$(2.63) \qquad \sigma_\xi = 0, \;\; \bar{A}(\sigma,\psi)\psi_\xi = \bar{B}(\psi)\psi.$$

We remark that $\bar{A}(\cdot,\cdot)$ and $\bar{B}(\cdot)$ are matrix valued functions of class $\mathbf{C}^4$.

Obviously it holds $\bar{B}(0) = B = Q_\phi(0)$, and, due to $\phi(0) = 0$ and $\phi_\psi(0) = I_n$, we conclude

$$\bar{A}(0,0) = -\lambda_p I_n + \sum_{j=1}^{d} \omega_j DF_j(0) = A. \tag{2.64}$$

If A is invertible, the existence of the locally invariant manifold $\tilde{\Gamma}$ for system (2.60) in Proposition (2.3.1) is straightforward: Due to Remark (2.3.3), $\tilde{\Gamma}$ is an open subset of $\mathbb{R}^{n+1}$.

In this section we want to prove the existence of the invariant manifold $\tilde{\Gamma}$ for the case that zero is an eigenvalue of A with algebraic multiplicity one.

Obviously, the invariant manifold $\tilde{\Gamma}$ exists, if we have proven the existence of a locally invariant manifold for system (2.63) fulfilling the conditions in Proposition (2.3.1).

As long as no misunderstanding will be caused we denote this locally invariant manifold by $\tilde{\Gamma}$, too.

If zero is a simple eigenvalue of $A = \bar{A}(0,0)$, then the eigenvalue $\lambda(\sigma,\psi)$ of $\bar{A}(\sigma,\psi)$ with $\lambda(0,0) = 0$ is a function of class $\mathbf{C}^4$ in a neighborhood of $(0,0)$.

Remembering (2.62) we can write

$$\lambda(\sigma,\psi) = -\sigma + \mu(\psi), \tag{2.65}$$

where μ is of class $\mathbf{C}^4$ for ψ close to $(0,0)$.

Set

$$\bar{\Pi}(\sigma,\psi) = \frac{1}{2\pi i}\int_\Gamma \left(\zeta - \bar{A}(\sigma,\psi)\right)^{-1} d\zeta,$$

where Γ is a curve which surrounds the origin in counterclockwise direction and does not enclose the nonzero eigenvalues of A. For (σ,ψ) close to $(0,0)$, the operator $\bar{\Pi}(\sigma,\psi)$ is the projector commuting with $\bar{A}(\sigma,\psi)$ corresponding to the eigenvalue $\lambda(\sigma,\psi)$.

Set

$$\tilde{A}(\sigma,\psi) = \frac{1}{2\pi i}\int_{\bar{\Gamma}} \zeta^{-1}\left(\zeta - \bar{A}(\sigma,\psi)\right)^{-1} d\zeta, \tag{2.66}$$

where $\bar{\Gamma}$ is a curve which surrounds the nonvanishing eigenvalues of $A = \bar{A}(0,0)$ in counterclockwise direction and does not enclose the origin.

Define

$$\mathcal{S} = \{(\sigma,\psi) \mid \exists \bar{A}^{-1}(\sigma,\psi)\}.$$

As long as $(\sigma,\psi) \neq (0,0)$ is contained in $\mathcal{S}$, we can write

$$\bar{A}^{-1}(\sigma,\psi) = \tilde{A}(\sigma,\psi) + \lambda^{-1}(\sigma,\psi)\bar{\Pi}(\sigma,\psi) = \tilde{A}(\sigma,\psi) + \frac{1}{\mu(\psi)-\sigma}\bar{\Pi}(\sigma,\psi).$$

We will prove that $\mathcal{S}\cap\tilde{\Gamma}$ is dense in $\tilde{\Gamma}$.

After scaling

$$\lambda(\sigma,\psi)d\tau = d\xi$$

we get from (2.63)

$$\sigma_\tau = 0, \quad \psi_\tau = \left(\lambda(\sigma,\psi)\tilde{A}(\sigma,\psi) + \bar{\Pi}(\sigma,\psi)\right)\bar{B}(\psi)\psi. \tag{2.67}$$

The right-hand-side of system (2.67) is of class $\mathbf{C}^4$.

Set

$$\tilde{\Lambda}(\sigma,\psi) = \frac{1}{2\pi i}\int_C \left(\zeta - \bar{\Pi}(\sigma,\psi)\bar{B}(\psi)\right)^{-1} d\zeta, \tag{2.68}$$

$$\tilde{\Pi}(\sigma,\psi) = I_n - \tilde{\Lambda}(\sigma,\psi), \tag{2.69}$$

where C is a curve which surrounds the origin in counterclockwise direction and does not enclose the nonvanishing eigenvalue of $\Pi B = \bar{\Pi}(0,0)\bar{B}(0)$. It holds

$$\tilde{\Lambda}(\sigma,\psi)\bar{\Pi}(\sigma,\psi)\bar{B}(\psi) = \frac{1}{2\pi i}\int_C \zeta\left(\zeta - \bar{\Pi}(\sigma,\psi)\bar{B}(\psi)\right)^{-1} d\zeta. \tag{2.70}$$

As assumed in the the weakened structural condition, each nonvanishing eigenvalue of $\Pi B = \bar{\Pi}(0,0)\bar{B}(0)$ has nonvanishing real part and zero is a semisimple eigenvalue of ΠB. Moreover, it holds

$$rank(\Pi B) = rank(\Pi diag(0_{n-r}, S(0))) = rank(\Pi diag(0_{n-r}, S(0)) diag(I_{n-r}, S^{-1}(0))) =$$

$$= rank(\Pi diag(0_{n-r}, I_r)) = rank(\Pi) = 1.$$

The fourth identity follows from (2.59).

Therefore ΠB has only one eigenvalue with nonvanishing real part.

Hence, for (σ,ψ) close to $(0,0)$,

$$rank(\bar{\Pi}(\sigma,\psi)\bar{B}(\psi)) = rank(\bar{\Pi}(\sigma,\psi)) = 1,$$

and $\bar{\Pi}(\sigma,\psi)\bar{B}(\psi)$ has only one eigenvalue with nonvanishing real part.

These facts imply that, for (σ,ψ) close to $(0,0)$, zero is a semisimple eigenvalue of $\bar{\Pi}(\sigma,\psi)\bar{B}(\psi)$.

From Lemma (2.1.7) it follows that, for (σ,ψ) close to $(0,0)$, zero is a simple singularity of

$$\zeta \mapsto \left(\zeta - \bar{\Pi}(\sigma,\psi)\bar{B}(\psi)\right)^{-1}.$$

We conclude that $\zeta \mapsto \zeta\left(\zeta - \bar{\Pi}(\sigma,\psi)\bar{B}(\psi)\right)^{-1}$ has an analytic extension throughout the region encircled by C, and it follows for (σ,ψ) small enough:

$$\frac{1}{2\pi i}\int_C \zeta\left(\zeta - \bar{\Pi}(\sigma,\psi)\bar{B}(\psi)\right)^{-1} d\zeta = 0.$$

Hence, relation (2.70) implies for (σ,ψ) small enough:

$$\tilde{\Lambda}(\sigma,\psi)\bar{\Pi}(\sigma,\psi)\bar{B}(\psi) = 0. \tag{2.71}$$

Set $\mathcal{E} = \{\bar{B}(\psi)\psi = 0\}$.

We are ready to prove the following theorem about the existence of a locally invariant manifold $\tilde{\Gamma}$ for system (2.63):

Theorem 2.4.1 *Assume that $\lambda(0,0)$ is a simple eigenvalue of $\bar{A}(0,0)$ and that the weakened structural condition is fulfilled.*

Then there exists a locally invariant $\mathbf{C}^3$ - manifold $\tilde{\Gamma} \subset \mathbb{R} \times \mathbb{R}^n$ for system (2.63) of dimension n such that for each σ small enough the section $\tilde{\Gamma}_\sigma = \tilde{\Gamma} \cap \{(\bar{\sigma}, \psi) \mid \bar{\sigma} = \sigma\}$ is a locally invariant $\mathbf{C}^3$ - manifold of dimension $n-1$ containing $(0,\sigma)$. Furthermore, $\mathcal{S} \cap \tilde{\Gamma}$ is dense in $\tilde{\Gamma}$ and $\bar{A}^{-1}(\sigma,\psi)\bar{B}(\psi)\psi$ defined for $(\sigma,\psi) \in \mathcal{S} \cap \tilde{\Gamma}$ has a $\mathbf{C}^2$ extension throughout $\tilde{\Gamma}$. There exists $\delta > 0$ with the following property: If $(\sigma,\psi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ then $(\sigma,\psi) \in \tilde{\Gamma}$.

Proof:

Write system (2.67) in the form

$$\sigma_\tau = 0, \quad \psi_\tau = \bar{\Pi}(0,0)\bar{B}(0)\psi + R(\sigma,\psi), \tag{2.72}$$

where $D_{(\sigma,\psi)}R(0,0) = 0$.

Note that the right-hand-side of system (2.67) is of class $\mathbf{C}^4$.

Remember that, due the weakened structural condition, it holds for the matrix $\Pi B = \bar{\Pi}(0,0)\bar{B}(0)$: Zero is a semisimple eigenvalue and each nonvanishing eigenvalue has nonvanishing real part.

Furthermore, we know that $rank(\Pi B) = 1$ and that $\tilde{\Lambda}(0,0)$ commutes with ΠB and maps onto $ker(\Pi B)$.

Application of the center manifold theorem gives the existence of an n - dimensional locally invariant $\mathbf{C}^3$ - manifold $\tilde{\Gamma} \subset \mathbb{R} \times \mathcal{U}$ for system (2.67) with $T_{(0,0)}\tilde{\Gamma} = \mathbb{R} \times \tilde{\Lambda}(0,0)\mathbb{R}^n$.

Furthermore, we conclude from the center manifold theorem that there exists $\delta > 0$ such that the following claim is true: If $(\sigma,\psi) \in B_\delta(0) \cap (\mathbb{R} \times \mathcal{E})$ then

$(\sigma, \psi) \in \tilde{\Gamma}$.

In particular, it follows $(\sigma, 0) \in \tilde{\Gamma}$ for σ small enough.

Let $e_1 \in \mathbb{R}^{n+1}$ be the vector whose first entry is equal to one and whose other entries are zero.

From (2.65) it follows that $\nabla_{(\sigma,\psi)}\lambda(0,0) \cdot e_1 = -1 \neq 0$.

Hence, due to $e_1 \in T_{(0,0)}\tilde{\Gamma}$, the set

$$\mathcal{S} \cap \tilde{\Gamma} = \{(\sigma, \psi) \in \tilde{\Gamma} \ \mid \ \lambda(\sigma, \psi) \neq 0\}$$

is dense in $\tilde{\Gamma}$.

It follows for each $(\sigma, \psi) \in \mathcal{S} \cap \tilde{\Gamma}$: The vector

$$(0, (\lambda(\sigma, \psi)\tilde{A}(\sigma, \psi) + \bar{\Pi}(\sigma, \psi))\bar{B}(\psi)\psi) = (0, \lambda(\sigma, \psi)\bar{A}^{-1}(\sigma, \psi)\bar{B}(\psi)\psi)$$

is contained in the tangent space of $\tilde{\Gamma}$ at point (σ, ψ) if and only if the vector $(0, \bar{A}^{-1}(\sigma, \psi)\bar{B}(\psi)\psi)$ is contained in the tangent space of $\tilde{\Gamma}$ at point (σ, ψ).

The manifold $\tilde{\Gamma}$ is invariant with respect to system (2.67), i.e.

$$(0, (\lambda(\sigma, \psi)\tilde{A}(\sigma, \psi)+\bar{\Pi}(\sigma, \psi))\bar{B}(\psi)\psi) = (0, \lambda(\sigma, \psi)\bar{A}^{-1}(\sigma, \psi)\bar{B}(\psi)\psi) \in T_{(\sigma,\psi)}\tilde{\Gamma}$$

for each $(\sigma, \psi) \in \mathcal{S} \cap \tilde{\Gamma}$.

Hence, it follows for $V(\sigma, \psi) = \bar{A}^{-1}(\sigma, \psi)\bar{B}(\psi)\psi$:

$$(\sigma, \psi) \in \mathcal{S} \cap \tilde{\Gamma} \Rightarrow (0, V(\sigma, \psi)) \in T_{(\sigma,\psi)}\tilde{\Gamma}. \tag{2.73}$$

Now we show: The map $(\sigma, \psi) \mapsto V(\sigma, \psi)$ has a smooth extension from $\mathcal{S} \cap \tilde{\Gamma}$ throughout $\tilde{\Gamma}$.

For $(\sigma, \psi) \in \mathcal{S} \cap \tilde{\Gamma}$ write $V(\sigma, \psi)$ in the form

$$V(\sigma, \psi) = \left(\tilde{A}(\sigma, \psi) + \lambda^{-1}(\sigma, \psi)\bar{\Pi}(\sigma, \psi)\right) \bar{B}(\psi)\psi, \tag{2.74}$$

where $\tilde{A}(\sigma,\psi)$ is defined in (2.66) and $\bar{\Pi}(\sigma,\psi)$ is the projector commuting with $\bar{A}(\sigma,\psi)$ corresponding to eigenvalue $\lambda(\sigma,\psi)$.

Multiplication of $\tilde{\Lambda}(\sigma,\psi)$ to (2.74) from the left and taking into account relation (2.71) give

$$\tilde{\Lambda}(\sigma,\psi)V(\sigma,\psi) = \tilde{\Lambda}(\sigma,\psi)\tilde{A}(\sigma,\psi)\bar{B}(\psi)\psi \tag{2.75}$$

for each $(\sigma,\psi) \in \mathcal{S} \cap \tilde{\Gamma}$.

From $T_{(0,0)}\tilde{\Gamma} = \mathbb{R} \times \tilde{\Lambda}(0,0)\mathbb{R}^n$ it follows for $W = \{(\sigma,v) \in \mathbb{R} \times \mathbb{R}^n \;\mid\; \sigma = 0\}$:

$$W \cap T_{(0,0)}\tilde{\Gamma} = \{0\} \times \tilde{\Lambda}(0,0)\mathbb{R}^n.$$

The spaces $T_{(0,0)}\tilde{\Gamma}$ and W are transversal to each other and the dimension of their intersection is equal to $n-1$.

Let $w_1(\sigma,\psi),\cdots,w_{n-1}(\sigma,\psi)$ be $(n-1)$ families of vectors being smooth in (σ,ψ) such that $\{w_1(\sigma,\psi),\cdots,w_{n-1}(\sigma,\psi)\}$ is a base of $W \cap T_{(\sigma,\psi)}\tilde{\Gamma}$ for $(\sigma,\psi) \in \tilde{\Gamma}$ small enough.

Obviously it holds for $i \in \{1,\cdots,n-1\}$ and $(\sigma,\psi) \in \tilde{\Gamma}$ small enough:

$$w_i(\sigma,\psi) = (0,\bar{w}_i(\sigma,\psi)), \;\; where \;\; \bar{w}_i(\sigma,\psi) \in \mathbb{R}^n. \tag{2.76}$$

The vectors $\bar{w}_1(\sigma,\psi),\cdots,\bar{w}_{n-1}(\sigma,\psi)$ are linearly independent, and, for $i \in \{1,\cdots,n-1\}$, the map $(\sigma,\psi) \mapsto \bar{w}_i(\sigma,\psi)$ is smooth for (σ,ψ) small enough.

There exists a smooth family of projectors $(\sigma,\psi) \mapsto \hat{\Pi}(\sigma,\psi)$ such that

$$\hat{\Pi}(0,0) = \tilde{\Pi}(0,0) \tag{2.77}$$

and that for any $(\sigma,\psi) \in \tilde{\Gamma}$, (σ,ψ) small enough the following claim is true:

$$(0,v) \in T_{(\sigma,\psi)}\tilde{\Gamma} \Leftrightarrow \hat{\Pi}(\sigma,\psi)v = 0. \tag{2.78}$$

Namely it holds

$$span(\bar{w}_1(0,0),\cdots,\bar{w}_{n-1}(0,0)) = \tilde{\Lambda}(0,0)\mathbb{R}^n = ker(\tilde{\Pi}(0,0)),$$

and we conclude

$$(0,v) \in T_{(\sigma,\psi)}\tilde{\Gamma} \Leftrightarrow (0,v) \in W \cap T_{(\sigma,\psi)}\tilde{\Gamma} = span(w_1(\sigma,\psi), \cdots, w_{n-1}(\sigma,\psi)) =$$

$$= \{0\} \times span(\bar{w}_1(\sigma,\psi), \cdots, \bar{w}_{n-1}(\sigma,\psi)) \Leftrightarrow v \in span(\bar{w}_1(\sigma,\psi), \cdots, \bar{w}_{n-1}(\sigma,\psi)).$$

As $(\sigma,\psi) \mapsto \bar{w}_i(\sigma,\psi)$ is smooth for $i \in \{1, \cdots, n-1\}$, it holds due to Lemma (2.1.8):

There exists a family of projectors $(\sigma,\psi) \mapsto \hat{\Pi}(\sigma,\psi)$ being smooth in (σ,ψ) for (σ,ψ) small enough such that

$$\hat{\Pi}(0,0) = \tilde{\Pi}(0,0) \;\; and \;\; ker(\hat{\Pi}(\sigma,\psi)) = span(\bar{w}_1(\sigma,\psi), \cdots, \bar{w}_{n-1}(\sigma,\psi)).$$

Hence, taking into account that

$$(0,v) \in T_{(\sigma,\psi)}\tilde{\Gamma} \Leftrightarrow v \in span(\bar{w}_1(\sigma,\psi), \cdots, \bar{w}_{n-1}(\sigma,\psi)),$$

a smooth family of projectors $(\sigma,\psi) \mapsto \hat{\Pi}(\sigma,\psi)$ fulfilling relations (2.77) and (2.78) exists.

Due to (2.73) and (2.78), it follows for each $(\sigma,\psi) \in \mathcal{S} \cap \tilde{\Gamma}$:

$$\hat{\Pi}(\sigma,\psi)V(\sigma,\psi) = 0. \tag{2.79}$$

The matrix $T(\sigma,\psi) = \tilde{\Lambda}(\sigma,\psi) + \hat{\Pi}(\sigma,\psi)$ is invertible for (σ,ψ) small enough because of $T(0,0) = I_n$ (remember $\hat{\Pi}(0,0) = \tilde{\Pi}(0,0)$ and (2.69)).

Adding (2.75) and (2.79) and multiplication of $T^{-1}(\sigma,\psi)$ from the left give

$$V(\sigma,\psi) = T^{-1}(\sigma,\psi)\tilde{\Lambda}(\sigma,\psi)\tilde{A}(\sigma,\psi)\bar{B}(\psi)\psi, \tag{2.80}$$

if $(\sigma,\psi) \in \mathcal{S} \cap \tilde{\Gamma}$.

As $(\sigma,\psi) \mapsto T^{-1}(\sigma,\psi)\tilde{\Lambda}(\sigma,\psi)\tilde{A}(\sigma,\psi)\bar{B}(\psi)\psi$ has a smooth extension from $\mathcal{S} \cap \tilde{\Gamma}$ throughout $\tilde{\Gamma}$, it follows from (2.80) that the map $(\sigma,\psi) \mapsto V(\sigma,\psi) = \bar{A}^{-1}(\sigma,\psi)\bar{B}(\psi)\psi$ has a smooth extension from $\mathcal{S} \cap \tilde{\Gamma}$ throughout $\tilde{\Gamma}$.

Hence, the claim of the theorem about the smooth extension has been proven.

In the following, denote the smooth extension of $V(\cdot,\cdot)$ by $V(\cdot,\cdot)$, too.

Relation $(0, V(\sigma,\psi)) \in T_{(\sigma,\psi)}\tilde{\Gamma}$ for each $(\sigma,\psi) \in \mathcal{S} \cap \tilde{\Gamma}$ implies that $(0, V(\sigma,\psi)) \in T_{(\sigma,\psi)}\tilde{\Gamma}$ for each $(\sigma,\psi) \in \tilde{\Gamma}$.

Hence, $\tilde{\Gamma}$ is a locally invariant manifold for system

$$\sigma_\xi = 0, \quad \psi_\xi = V(\sigma,\psi).$$

If $\psi_\xi(\xi) = V(\sigma,\psi(\xi))$ and $(\sigma,\psi(\xi)) \in \tilde{\Gamma}$ then it holds

$$\bar{A}(\sigma,\psi(\xi))\psi_\xi(\xi) = \bar{A}(\sigma,\psi(\xi))V(\sigma,\psi) =$$

$$= \lim_{(\bar{\sigma},\bar{\psi}) \in \mathcal{S}\cap\tilde{\Gamma}, (\bar{\sigma},\bar{\psi}) \to (\sigma,\psi(\xi))} \bar{A}(\bar{\sigma},\bar{\psi})\bar{A}^{-1}(\bar{\sigma},\bar{\psi})\bar{B}(\bar{\psi})\bar{\psi} = \bar{B}(\psi(\xi))\psi(\xi),$$

i.e.

$$\bar{A}(\sigma,\psi(\xi))\psi_\xi(\xi) = \bar{B}(\psi(\xi))\psi(\xi).$$

Hence, $\tilde{\Gamma}$ is a locally invariant manifold for system (2.63), and, due to $\sigma_\xi = 0$, the section $\tilde{\Gamma}_\sigma$ is a locally invariant manifold for system (2.63). □

Chapter 3

Moment closure systems

3.1 Derivation

A standard mathematical model describing the kinetic particle density $f(x,t,v)$ of rarefied gases at the position-time-velocity point $(x,t,v) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$ is the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}(f), \tag{3.1}$$

where $\mathcal{C}(f) = \frac{1}{\epsilon} \int_{(\omega,v') \in \mathbb{S}^{d-1} \times \mathbb{R}^d} (f_* f'_* - f f') B(\omega, v, v') d\omega dv'$, ϵ is proportional to the mean free path, $f_* = f(x,t,v_*)$, $f'_* = f(x,t,v'_*)$, $f' = f(x,t,v')$ with

$$v_* = v - \omega \cdot (v - v')\omega, \quad v'_* = v' + \omega \cdot (v - v')\omega$$

and $d\omega$ denotes the normalized measure on the unit sphere $\mathbb{S}^{d-1}$.

The collision kernel $B = B(\omega, v, v')$ is positive almost everywhere and fulfills the symmetry properties

$$B(\omega, v, v') = B(\omega, v', v) = B(\omega, v_*, v'_*). \tag{3.2}$$

First, we recall the identity (see [C])

$$4 \int \phi(v)(f_* f'_* - f f') B d\omega dv' dv \tag{3.3}$$

$$= \int (\phi + \phi' - \phi_* - \phi'_*)(f_* f'_* - f f') B d\omega dv' dv$$

for any continuous function $\phi = \phi(v)$.

Set

$$\mathbb{E} = span\{1, v_1, \cdots, v_d, |v|^2\}.$$

Mass, momentum and energy are the only collision invariants: For continuous ϕ it holds (see [C]):

$$\phi + \phi' = \phi_* + \phi'_* \quad \Leftrightarrow \quad \phi(v) \in \mathbb{E}. \tag{3.4}$$

Define the linear space

$$\mathbb{M} = span\{c_1, \cdots, c_n\}, \tag{3.5}$$

where $c_1, \cdots, c_n \in \mathbf{L}^1_{loc}(\mathbb{R}^d)$ and set $c = (c_1, \cdots, c_n)$.

Furthermore, assume

$$c_1 \equiv 1, \;\; c_2(v) = v_1, \cdots, c_{d+1}(v) = v_d, \;\; c_{d+2}(v) = |v|^2.$$

By $\mathbb{M}_k$ we denote the space of polynomials up to k - th order.

Multiplying (3.1) with $c_k(v)$ and integrating with respect to $v \in \mathbb{R}^d$ leads to n equations

$$\partial_t \int c_k f dv + \nabla_x \cdot \int v c_k f dv = \int c_k (f_* f'_* - f f') B d\omega dv' dv. \tag{3.6}$$

Maximization of entropy

$$H(f)(x,t) = -\int_{\mathbb{R}^d} f(v,x,t)\left(ln(f(v,x,t)) - 1\right) dv$$

under the constraints

$$\mathbf{I}_i(f)(x,t) = \int_{\mathbb{R}^d} f(x,t,v) c_i(v) dv, \;\; i = 1, \cdots, n$$

leads to the necessary condition

$$\int_{\mathbb{R}^d} \left[-ln(f(x,t,v)) + \sum_{i=1}^{n} \alpha_i(x,t) c_i(v) \right] \phi(v) dv = 0 \;\; \forall \phi \in \mathbf{C}_0^\infty(\mathbb{R}^d)$$

where $\alpha_i(x,t)$ $(i = 1,\cdots,n)$ is the Lagrange multiplier to constraint $\mathbf{I}_i(f)(x,t)$.

We get the exponentially based distribution

$$f = f(x,t,v) = exp\left(c(v)\cdot\alpha(x,t)\right), \tag{3.7}$$

where $c(v)\cdot\alpha(x,t)$ denotes $\sum_{k=1}^{n} c_k(v)\alpha_k(x,t)$.

Define $\bar{\eta}, q_j, \mathcal{Q}$ by

$$\bar{\eta}(\alpha) = \int exp\left(c(v)\cdot\alpha\right)dv, \quad q_j(\alpha) = \bar{\eta}_{\alpha_{j+1}}(\alpha) = \int v_j exp\left(c(v)\cdot\alpha\right)dv,$$

and the $n-$vector $\mathcal{Q}(\alpha) = (\mathcal{Q}_1(\alpha),\cdots,\mathcal{Q}_n(\alpha))^T$ with entries

$$\mathcal{Q}_i(\alpha) = \int c_i(v)\left[e^{c_*(v)\cdot\alpha + c'_*(v)\cdot\alpha} - e^{c(v)\cdot\alpha + c'(v)\cdot\alpha}\right] B d\omega dv' dv.$$

Then plugging (3.7) into (3.6) results in a first-order PDE system for the $n-$vector $\alpha = \alpha(x,t)$:

$$\frac{\partial\bar{\eta}_\alpha(\alpha)}{\partial t} + \sum_{j=1}^{d}\frac{\partial q_{j\alpha}(\alpha)}{\partial x_j} = \mathcal{Q}(\alpha). \tag{3.8}$$

3.2 Structural conditions on moments fulfilling the Galilean invariance property

For strictly positive γ define

$$\mathcal{M}_\gamma = \{\alpha\in\mathbb{R}^n \;\mid\; \bar{\eta}(\alpha)\le\gamma\} \tag{3.9}$$

We formulate the conditions on moment closure systems fulfilling the Galilean invariance property:

Condition 3.2.1 *The functions $c_1,\cdots,c_n$ are linearly independent and contained in $\mathbf{L}^1_{loc}(\mathbb{R}^d)$, and it holds*

$$c_1\equiv 1, \;\; c_2(v) = v_1,\ldots,c_{d+1}(v) = v_d, \;\; c_{d+2}(v) = |v|^2. \tag{3.10}$$

For each $\gamma > 0$, the interior $int(\mathcal{M}_\gamma)$ is nonvoid, and $\mathcal{Q} \in \mathbf{C}^\infty(clos(\mathcal{M}_\gamma), \mathbb{R}^n)$ (i.e. each derivative of $\mathcal{Q}$ has a continuous continuation up to the boundary of $\mathcal{M}_\gamma$). Furthermore, the Galilean invariance requirement is fulfilled, i.e.

$$p \in \mathbb{M} \Rightarrow p(\mathbf{O} \cdot + \tau) \in \mathbb{M} \;\; \forall\, \tau \in \mathbb{R}^d, \;\; \mathbf{O} \in SO(n). \tag{3.11}$$

As we will show in the appendix, the Galilean invariance property implies that $c_1, \cdots, c_n$ are polynomials.

It holds $\mathcal{Q} \in \mathbf{C}^\infty(clos(\mathcal{M}_\gamma), \mathbb{R}^n)$ for the following classes of intermolecular forces:

1. $B \in \mathbf{L}^1_{loc}(S^{d-1}, \mathbb{R}^d, \mathbb{R}^d)$ fulfilling the following growth condition:

$$B(\omega, v, v') \leq C_1 + C_2 |v - v'|^\gamma, \tag{3.12}$$

2. B resulting from an inverse k^{th}-power force for $k > d$ (see the proof in [T-M], p. 315 f. for the case $d = 3$ and prove analogously for general d).

Relation (3.12) holds in the classical case of the hard-spheres model, where

$$B(\omega, v, v') = |(v - v') \cdot \omega|.$$

3.3 Smooth extension

If the highest-order polynomial of $\mathbb{M}$ is of order larger than two then the set of Maxwellian states

$$\mathbf{E} = \{\alpha \in \mathbb{R}^n \;\; | \;\; c(v) \cdot \alpha \in \mathbb{E}, \;\; \alpha_{d+2} < 0\}$$

is contained in the boundary of the domain of definition of $\bar{\eta}$.

If we bring system (3.8) into the form of a relaxation system with certain F_j and Q having the form given at the beginning of chapter "Structural conditions and main results", this means that the equilibrium manifold $\mathcal{E}$ of this relaxation system is contained in the boundary of the domain of definition

$\mathcal{U}$ of F_j and Q.

On the other hand, it is assumed that the equilibrium states are contained in the interior of $\mathcal{U}$.

We will resolve this problem by means of Whitney's Extension Theorem ([W2], [W1], [A-R]):

Theorem 3.3.1 *Let W be a Banach space, $M \subset \mathbb{R}^n$ a closed subset, and $f : M \to W$. By $L_s^k(\mathbb{R}^n, W)$ denote the Banach space of symmetric k - linear maps from $\mathbb{R}^n$ to W. Then the following claims are true:*

1. *f extends to a $\mathbf{C}^r$ $(r \geq 0)$ function $F : \mathbb{R}^n \to W$ provided that there exist $f_0, \cdots, f_r$ with $f_0 = f$, $f_k : M \to L_s^k(\mathbb{R}^n, W)$, $(k = 0, \cdots, r)$, and for $k = 0, \cdots, r$, the following condition is satisfied: If $R_k : M \times M \to L_s^k(\mathbb{R}^n, W)$ is defined by*

   $$f_k(y) = \sum_{i \leq r-k} \frac{f_{k+i}(x)}{i!}(y-x)^i + R_k(x,y) \tag{3.13}$$

   for $x, y \in M$, then for each $x_0 \in M$,

   $$\frac{|R_k(x_1,x_2)|}{|x_1 - x_2|^{r-k}} \to 0 \tag{3.14}$$

   as $x_1, x_2 \to x_0$ in M, i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in M$,

   $$|R_k(x_1,x_2)| < \epsilon |x_1 - x_2|^{r-k} \tag{3.15}$$

   whenever $|x_1 - x_0|, |x_2 - x_0| < \delta$.

2. *The extension F of f may be chosen so that $D^k F|_M = f_k$ for all appropriate k.*

For $\alpha_1, \alpha_2 \in \mathcal{M}_\gamma$ and $t_1 \in [0,1]$ we obtain from *exp* being convex

$$\bar{\eta}(t_1\alpha_1 + (1-t_1)\alpha_2) = \int_{\mathbb{R}^d} exp\left((t_1\alpha_1 + (1-t_1)\alpha_2) \cdot c(v)\right) dv \leq$$

$$\leq \int_{\mathbb{R}^d} t_1 exp\left(\alpha_1 \cdot c(v)\right) + (1-t_1) exp\left(\alpha_2 \cdot c(v)\right) dv =$$

$$= t_1 \bar{\eta}(\alpha_1) + (1-t_1)\bar{\eta}(\alpha_2) \leq \gamma,$$

i.e. $\mathcal{M}_\gamma$ is convex.

Due to the structural conditions (3.2.1) the interior $int(\mathcal{M}_\gamma)$ is non-void.

For any $\alpha \in int(\mathcal{M}_\gamma)$ and multi-index β the derivative has the form

$$D^\beta \bar{\eta}(\alpha) = \int p_\beta(v) exp(c(v) \cdot \alpha) dv,$$

where p_β is a polynomial.

For any $R \in (0, \infty)$ and any multi-index β, the derivative $D^\beta \bar{\eta}(\cdot)$ is uniformly bounded on $int(\mathcal{M}_\gamma) \cap B_R(0)$.

Hence, for any multi-index β, the derivative $D^\beta \bar{\eta}(\cdot)$ has smooth extension throughout $\mathcal{M}_\gamma$.

In particular, $\bar{\eta}|_{\mathcal{M}_\gamma}$ is continuous.

Hence, if $\lim_{n\to\infty} \alpha_n = \alpha$ and $\alpha_n \in \mathcal{M}_\gamma\ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} = \bar{\eta}(\alpha_n) = \bar{\eta}(\alpha) \leq \gamma$, i.e. $\alpha \in \mathcal{M}_\gamma$, and we conclude that $\mathcal{M}_\gamma$ is closed.

From Theorem (3.3.1) it follows:

Corollary 3.3.2 *For each $m \in \mathbb{N}$ and $\gamma \in (0, \infty)$ the function $\bar{\eta}$ may be extended from $\mathcal{M}_\gamma$ throughout $\mathbb{R}^d$ as a function of class $\mathbf{C}^m$.*

3.4 The modified moment closure system

In this section we modify system (3.8) by means of Whitney's extension theorem and bring it into the form of a relaxation system, where the equilibrium set is in the interior of the domain of definition of F_j and Q.

We assume that the structural conditions (3.2.1) are fulfilled.

Let $\tilde{\eta}$ be a $\mathbf{C}^7$ extension of $\bar{\eta}$ from $\mathcal{M}_\gamma$ throughout $\mathbb{R}^n$ which exists according to Corollary (3.3.2).

For $\alpha_e \in \mathbf{E} \cap \mathcal{M}_\gamma$ let $\mathcal{G}$ be an open neighborhood of α_e such that $\alpha \mapsto U = \tilde{\eta}_\alpha(\alpha)$ defines a $\mathbf{C}^5$ bijection from $\mathcal{G}$ to an open set G.

As $\tilde{\eta}_{\alpha\alpha}(\alpha_e) = \bar{\eta}_{\alpha\alpha}(\alpha_e)$ is positive definite (because $c_1, \cdots, c_n$ are linearly independent), this neighborhood exists.

Furthermore, we can assume that

$$\gamma = 2\bar{\eta}(\alpha_e). \tag{3.16}$$

As $\tilde{\eta}_\alpha(\alpha_e) \neq 0$, we can assume that

$$\tilde{\eta}(\alpha) \leq \gamma \quad \Rightarrow \quad \tilde{\eta}(\alpha) = \bar{\eta}(\alpha) \tag{3.17}$$

for each $\alpha \in \mathcal{G}$, if we choose $\mathcal{G}$ small enough.

For $U \in G$ set

$$\eta(U) = \alpha(U) \cdot U - \tilde{\eta}(\alpha(U)).$$

We see that the inverse function $\alpha(U)$ is equal to $\eta_U(U)$.

Set

$$F_j(U) := \tilde{\eta}_{\alpha_{j+1}\alpha}(\eta_U(U)) \;\; \textit{and} \;\; Q(U) := \tilde{\mathcal{Q}}(\eta_U(U)),$$

where $\tilde{\mathcal{Q}}$ is a $\mathbf{C}^5$ extension of $\mathcal{Q}$ from $\mathcal{M}_\gamma$ throughout $\mathbb{R}^n$, where the first $d+2$ entries of $\tilde{\mathcal{Q}}$ vanish identically.

As $\tilde{\eta}$ is of class $\mathbf{C}^7$, F_j is of class $\mathbf{C}^5$ as assumed at the beginning of the chapter "Structural conditions and main results".

We consider the following first-order PDE system

$$\frac{\partial \tilde{\eta}_\alpha(\alpha)}{\partial t} + \sum_{j=1}^{d} \frac{\partial \tilde{\eta}_{\alpha_{j+1}\alpha}(\alpha)}{\partial x_j} = \tilde{\mathcal{Q}}(\alpha), \tag{3.18}$$

which is equivalent to the following system of balance laws:

$$U_t + \sum_{j=1}^{d} F_j(U)_{x_j} = Q(U). \tag{3.19}$$

Note that system (3.18) is equivalent to system (3.8) as long as $\tilde{\eta}(\alpha(x,t)) = \bar{\eta}(\alpha(x,t))$.

For each $\alpha_e \in \mathbf{E}$, the kinetic particle density $f(v) = exp(c(v) \cdot \alpha_e)$ can be expressed in terms of the density ρ, the macroscopic velocity $\bar{u}$ and the temperature θ via

$$exp(c(v) \cdot \alpha_e) = \frac{\rho}{(2\pi\theta^{d/2})} exp\left(-\frac{|v-\bar{u}|^2}{2\theta}\right). \tag{3.20}$$

Consider the reduced system to (3.19) (remembering the notation in section "Traveling waves for simple shocks"):

$$u_t + \sum_{j=1}^{d} f_j(u, h(u))_{x_j} = 0. \tag{3.21}$$

We can write $u = u(\rho, \bar{u}, \theta) = u(\rho, \bar{u}_1, \cdots, \bar{u}_d, \theta)$.

As we will show in the appendix, the $d+2$ eigenvalues of

$$\hat{A}_i = \partial_u f_i(u(\rho, \bar{u}, \theta), h(u(\rho, \bar{u}, \theta)))$$

are

$$\lambda_1^i = \bar{u}_i - \bar{a}(\theta), \;\; \lambda_2^i = \cdots = \lambda_{d+1}^i = \bar{u}_i, \;\; \lambda_{d+2}^i = \bar{u}_i + \bar{a}(\theta), \tag{3.22}$$

where $\bar{a}(\theta) = \left(\frac{2+d}{d}\right)^{1/2} \theta^{1/2}$.

Furthermore, we will show in the appendix the existence of a shock curve such that the strict entropy condition is fulfilled.

3.5 The entropy condition

In order to introduce an easy-to-check sufficient condition for the stability and dissipativity condition for system (3.19), we introduce the so-called entropy condition introduced by Yong in [Y4].

Remembering the notation of the chapter "Structural conditions and main results" this condition reads as follows:

Condition 3.5.1 *(i). $q_v(U)$ is invertible.*

(ii). There is a strictly convex smooth function $\eta(U)$, defined in a convex compact neighborhood G of $U_e \in \mathcal{E}$ such that $\eta_{UU}(U)F_{kU}(U)$ is symmetric for all $U \in G$ and all k.

(iii). There is a positive constant c_G, depending on the compact neighborhood G, such that for all $U \in G$,

$$[\eta_U(U) - \eta_U(U_e)]\, Q(U) \leq -c_G|Q(U)|^2.$$

(iv). The kernel of the Jacobian $Q_U(U_e)$ contains no eigenvectors of $\sum_{k=1}^{d} \omega_k F_{kU}(U_e)$for each $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{S}^{d-1}$.

If the entropy condition is fulfilled, then the stability condition and the dissipativity condition are fulfilled (see [Y3], Prop. 5.6 and [Y4], Theorem 2.1).

Due to Proposition (1.1.7), we can make the following remark

Remark 3.5.2 *If the entropy condition holds, then the weakened structural condition (1.1.6) is fulfilled.*

3.6 Existence of smooth shock profiles

3.6.1 Checking of weakened structural condition

In this section we prove that for system (3.19) the weakened structural condition is fulfilled and that traveling wave solutions represent solutions of the Boltzmann equation.

In order to do this we first consider the relaxation system (3.19) and prove that this system fulfills the entropy condition (3.5.1).

As for the reduced system the strict entropy condition is fulfilled, it follows due to Remark (3.5.2) that the assumptions of Theorem (1.3.1) are fulfilled.

In the last step we will prove that the shock profiles, which exist due to Theorem (1.3.1), represent solutions of the Boltzmann equation.

For system (3.19) repeat the notation for A and A_{11} given in the section "Structural conditions" in the chapter "Structural conditions and main results".

Furthermore, let λ_p be one of the simple eigenvalues of A_{11} for which the strict entropy condition (shown in the appendix) is fulfilled.

The purpose of this section is the proof of the following theorem:

Theorem 3.6.1 *If the structural conditions (3.2.1) are fulfilled and* $\mathbb{M}_3 \subset \mathbb{M}$*, then for system (3.19) the entropy condition (3.5.1) is fulfilled. In particular, the weakened structural condition is fulfilled. Furthermore, if* $A - \lambda_p I_n$ *is invertible or zero is a simple eigenvalue of* $A - \lambda_p I_n$*, then there exist smooth shock profiles which represent solutions of the Boltzmann equation.*

Proof:

First we will check the conditions (i) - (iv) of the entropy condition (3.5.1).

In order to check condition (i) remember that we assume that the distribution f is exponentially based, i.e.

$$f(x,t,v) = exp(c(v) \cdot \alpha(x,t)).$$

It follows that

$$\frac{\partial \mathcal{Q}_k}{\partial \alpha_j}|_{\mathcal{M}_\gamma} =$$

$$= \frac{1}{4} \int \left(c_k + c'_k - c_{k\,*} - c'_{k\,*}\right) \left(\left(c_{j\,*} + c'_{j\,*}\right)(f_* f'_*)\,|_{\mathcal{M}_\gamma} - \left(c_j + c'_j\right)(f f')\,|_{\mathcal{M}_\gamma}\right) B d\omega dv' dv.$$

Remember that

$$\mathbf{E} = \{\alpha \in \mathbb{R}^n \ \mid \ c(v) \cdot \alpha \in \mathbb{E}, \ \ \alpha_{d+2} < 0\}.$$

From $(\mathbf{E} \cap \mathcal{M}_\gamma) \subset \partial \mathcal{M}_\gamma$ and $(f_* f'_*)|_{\mathbf{E}} = (ff')|_{\mathbf{E}}$ we conclude that

$$\frac{\partial \mathcal{Q}_k}{\partial \alpha_j}|_{\mathbf{E} \cap \mathcal{M}_\gamma} =$$

$$= -\frac{1}{4} \int (c_k + c'_k - c_{k\,*} - c'_{k\,*}) \left(c_j + c'_j - c_{j\,*} - c'_{j\,*}\right) (ff') \,|_{\mathbf{E} \cap \mathcal{M}_\gamma} B d\omega dv' dv.$$

Hence, we can write $\mathcal{Q}_\alpha|_{\mathbf{E} \cap \mathcal{M}_\gamma}$ in the form

$$\mathcal{Q}_\alpha|_{\mathbf{E} \cap \mathcal{M}_\gamma} = -\frac{1}{4} \int (c + c' - c_* - c'_*) \otimes (c + c' - c_* - c'_*) \, (ff') \,|_{\mathbf{E} \cap \mathcal{M}_\gamma} B d\omega dv' dv$$

and see that $\mathcal{Q}_\alpha|_{\mathbf{E} \cap \mathcal{M}_\gamma}$ is symmetric.

In addition, let $y \in \mathbb{R}^n$. Then it is easy to see that

$$-4y^* \mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = \int |(c + c' - c_* - c'_*) \cdot y|^2 (ff')|_{\mathbf{E}} B d\omega dv' dv \geq 0.$$

Hence, $\mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}}$ is nonpositive.

Furthermore, $y^* \mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = 0$ if and only if $(c + c' - c_* - c'_*) \cdot y = 0$ almost everywhere.

Due to (3.4) it follows that $y^* \mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = 0$ if and only if $(c+c'-c_*-c'_*) \cdot y \in \mathbb{E}$.

It follows that $y^* \mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = 0$ if and only if $y \in span(\mathbf{E})$.

On the other hand, since $\mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}}$ is symmetric nonpositive, it is elementary that $y^* \mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = 0$ if and only if $\mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}} y = 0$. Hence,

$$ker(\mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}}) = span(\mathbf{E}) \tag{3.23}$$

and $rank(\mathcal{Q}_\alpha(\alpha)|_{\mathbf{E}}) = n - dim(span(\mathbf{E})) = n - d - 2$.

As $\mathcal{Q}_\alpha(\alpha_e)$ is symmetric nonpositive and the first $d+2$ entries of $\mathcal{Q}$ are identically equal to zero, it follows for $\alpha_e \in \mathbf{E}$ that $\mathcal{Q}_\alpha(\alpha_e)$ has the form $\mathcal{Q}_\alpha(\alpha_e) = diag(0_{d+2}, \tilde{H}(\alpha_e))$, where $\tilde{H}(\alpha_e)$ is negative definite.

We calculate for $\alpha_e = \eta_U(U_e)$

$$Q_U(U_e) = \mathcal{Q}_\alpha(\eta_U(U_e))\eta_{UU}(U_e) = diag(0_{d+2}, \tilde{H}(\alpha_e))\eta_{UU}(U_e) =$$

$$= \begin{pmatrix} 0 & 0 \\ * & \tilde{H}(\alpha_e)\eta_{vv}(U_e) \end{pmatrix}.$$

As $\tilde{H}(\alpha_e)$ is negative definite and $\eta_{vv}(U_e)$ is positive definite, the matrix

$$q_v(U_e) = \tilde{H}(\alpha_e)\eta_{vv}(U_e)$$

is invertible (remembering the notation for v at the beginning of chapter "Structural conditions and main results", where $r = n - d - 2$).

Hence, there exists an open neighborhood G of U_e such that $q_v(U)$ is invertible if U is contained in G, and (i) follows.

(ii) is a direct consequence of

$$\eta_{UU}(U)F_{k\,U}(U) = \eta_{UU}(U)\tilde{\eta}_{\alpha_{k+1}\,\alpha\alpha}(\eta_U(U))\eta_{UU}(U) = F_{k\,U}^T(U)\eta_{UU}(U).$$

ad (iii):

For $\alpha_e \in \mathbf{E}$ the matrix $\mathcal{Q}_\alpha(\alpha_e)$ is symmetric and nonpositive, so that for U_e with $\eta_U(U_e) = \alpha_e$ it follows

$$(3.24)\left(\left[\eta_U - \eta_U(U_e)\right] Q\right)_{UU}(U_e) = \eta_{UU}(U_e)Q_U(U_e) + Q_U(U_e)^*\eta_{UU}(U_e) =$$

$$= 2\eta_{UU}(U_e)\mathcal{Q}_\alpha(\eta_U(U_e))\eta_{UU}(U_e) \le 0.$$

The last $(n-d-2)$ entries of $\eta_U(U_e)$ and the first $(d+2)$ entries of Q are zero, so that

$$\eta_U(U_e)Q(U) = 0 \;\; \forall (U_e, U) \in \mathcal{E} \times G. \tag{3.25}$$

Due to Theorem 2.1 and Proposition 2.2 in [Y4], (iii) is implied by (i), (ii), (3.24) and (3.25).

Condition (iv) is fulfilled if

$$ker\left(-s\bar{\eta}_{\alpha\alpha}(\alpha_e) + \sum_{j=1}^{d} \omega_j \bar{\eta}_{\alpha_{j+1}\alpha\alpha}(\alpha_e)\right) \cap ker(\mathcal{Q}_\alpha(\alpha_e)) = \{0\}$$

for each $\alpha_e \in \mathbf{E}$ and $(s, \omega) \in \mathbb{R} \times \mathbb{S}^{d-1}$.

As $ker(\mathcal{Q}_\alpha(\alpha_e)) = \mathbf{E}$ this is equivalent to the following condition:

For each $\alpha_e \in \mathbf{E}$ and $(s, \omega, \bar{\alpha}) \in \mathbb{R} \times \mathbb{S}^{d-1} \times (\mathbf{E} \setminus \{0\})$ there exists $\tilde{\alpha} \in \mathbb{R}^n$, such that

$$\tilde{\alpha}^T \left(-s\bar{\eta}_{\alpha\alpha}(\alpha_e) + \sum_{j=1}^{d} \omega_j \bar{\eta}_{\alpha_{j+1}\alpha\alpha}(\alpha_e)\right) \bar{\alpha} =$$

$$= \tilde{\alpha}^T \int_{\mathbb{R}^d} (\omega \cdot v - s) \;\; c(v) \otimes c(v) \; exp(\alpha_e \cdot c(v)) dv \; \bar{\alpha} \neq 0.$$

Hence, the dissipativity condition is equivalent to that for each $\alpha_e \in \mathbf{E}$ and $(s, \omega, \bar{\mathfrak{p}}) \in \mathbb{R} \times \mathbb{S}^{d-1} \times (\mathbb{E} \setminus \{0\})$ there exists $\tilde{\mathfrak{p}} \in \mathbb{M}$ such that

$$\int_{\mathbb{R}^d} (\omega \cdot v - s) \;\; \tilde{\mathfrak{p}}(v) \; \bar{\mathfrak{p}}(v) \; exp(\alpha_e \cdot c(v)) dv \neq 0.$$

After eventual rotation and translation we can assume $\omega \cdot v = v_d$ and

$$exp(\alpha_e \cdot c(v)) = const \; exp(-C|v|^2),$$

where $const \neq 0, \;\; C > 0$.

Define on $\mathbb{M}$ the scalar product

$$\langle p, q \rangle := \int_{\mathbb{R}^d} p(v) q(v) exp(-C|v|^2) dv \tag{3.26}$$

denoting the induced norm with $\|\cdot\|$.

Then the dissipativity condition is equivalent to that for each $(s,\bar{\mathfrak{p}}) \in \mathbb{R} \times (\mathbb{E}\setminus\{0\})$ there exists $\tilde{\mathfrak{p}} \in \mathbb{M}$ such that $\langle \mathfrak{l}_s\tilde{\mathfrak{p}}, \bar{\mathfrak{p}}\rangle \neq 0$, where $\mathfrak{l}_s(v) = v_d - s$.

In other words: We have to show:

$$\mathbb{M}_3 \subset \mathbb{M} \;\Rightarrow\; (for\;\; each\;\; (s,\bar{\mathfrak{p}}) \in \mathbb{R}\times(\mathbb{E}\setminus\{0\}) \;\; \exists\tilde{\mathfrak{p}} \in \mathbb{M} : \langle \mathfrak{l}_s\,\bar{\mathfrak{p}}, \tilde{\mathfrak{p}}\rangle \neq 0)\,.$$

There holds $\mathfrak{l}_s\bar{\mathfrak{p}} \in \mathbb{M}_3 \subset \mathbb{M}$ for each $(s,\bar{\mathfrak{p}}) \in \mathbb{R}\times\mathbb{E}$.

For $\tilde{\mathfrak{p}} = \mathfrak{l}_s\bar{\mathfrak{p}}$ we get $\langle \mathfrak{l}_s\,\bar{\mathfrak{p}}, \tilde{\mathfrak{p}}\rangle = \|\mathfrak{l}_s\bar{\mathfrak{p}}\|^2 \neq 0$, and condition (iv) has been shown.

We have shown that the entropy condition is fulfilled.

Due to Remark (3.5.2), the weakened structural condition is fulfilled.

As the weakened structural condition is fulfilled and the first and $(d+2)$ - th eigenvalue are simple and fulfill the strict entropy condition, Theorem (1.3.1) applies, if $A - \lambda_p I_n$ is invertible or zero is a simple eigenvalue of $A - \lambda_p I_n$.

Then, for $\xi = -st + \sum_{j=1}^n \omega_j x_j$ there exists a solution $\alpha = \alpha(\xi)$ of

$$-s\frac{\partial\tilde{\eta}_\alpha(\alpha(\xi))}{\partial\xi} + \sum_{j=1}^{d}\omega_j\frac{\partial\tilde{\eta}_{\alpha_{j+1}\alpha}(\alpha(\xi))}{\partial\xi} = \mathcal{Q}(\alpha(\xi)) \tag{3.27}$$

with $\alpha(-\infty) = \alpha_- = \alpha_e$ and $\alpha(\infty) = \alpha_+$, if $|\alpha_- - \alpha_+|$ is small enough.

We remark that α_+ corresponds to a point of the Rankine-Hugoniot curve through α_- corresponding to shock speed s.

It remains to show that α represents a solution of the Boltzmann equation, if $|\alpha_+ - \alpha_-|$ is small enough.

In order to do this we show that there exists a real number $\gamma > 0$ such that

$$\tilde{\eta}(\alpha(\xi)) \leq \gamma \;\; \forall\xi \in \mathbb{R}, \tag{3.28}$$

if $|\alpha_+ - \alpha_-|$ is small enough.

Remember that, if condition (3.28) is fulfilled, then, due to (3.17), it holds

$$\tilde{\eta}(\alpha(\xi)) = \bar{\eta}(\alpha(\xi)) \ \ \forall \xi \in \mathbb{R},$$

and α represents a solution of the Boltzmann equation.

Due to rotational invariance, we can assume that system (3.27) has the form

$$\frac{\partial \tilde{\eta}_{\alpha_2 \alpha}(\alpha(\xi))}{\partial \xi} - s \frac{\partial \tilde{\eta}_{\alpha}(\alpha(\xi))}{\partial \xi} = \mathcal{Q}(\alpha(\xi)). \tag{3.29}$$

Remember that $c(v) = (1, v_1, \cdots, v_d, |v|^2, c_{d+3}(v), \cdots, c_n(v))$.

Due to translational invariance, we can assume that in (3.22) it is $u_1 = 0$, so that (remembering $\alpha_- = \alpha_e$)

$$\tilde{\eta}^1_{\alpha_2 \alpha}(\alpha_-) = \int_{\mathbb{R}^d} v_1 exp\left(c(v) \cdot \alpha_-\right) dv = 0, \tag{3.30}$$

where superscript 1 means taking the first entry.

Note that the first $d + 2$ entries of $\mathcal{Q}$ are identically equal to zero.

After integration of (3.29) from $-\infty$ to ξ, taking (3.30) into account and noting that $\bar{\eta}^1_{\alpha}(\alpha) = \bar{\eta}(\alpha)$, we get

$$\tilde{\eta}^1_{\alpha_2 \alpha}(\alpha(\xi)) - s\tilde{\eta}(\alpha(\xi)) = -s\bar{\eta}(\alpha_-) \ \ \forall \xi \in \mathbb{R}. \tag{3.31}$$

There exists a real number $R > 0$ such that

$$|\tilde{\eta}^1_{\alpha_2 \alpha}(\alpha)| \leq \frac{|s|}{2} \tilde{\eta}(\alpha) \ \ \forall \alpha \in B_R(\alpha_-), \tag{3.32}$$

if $|s - \lambda^1_1(\alpha_-)|$ or $|s - \lambda^1_{d+2}(\alpha_-)|$ is small enough, where λ^1_1 and λ^1_{d+2} are given in (3.22).

Such an R exists due to the continuity of $\tilde{\eta}^1_{\alpha_2 \alpha}$, due to (3.30) and $\lambda^1_{d+2}(\alpha_-) \neq 0$, because we have $u_1 = 0$ and $a(\theta) \neq 0$ in (3.22).

Furthermore, the values $|s - \lambda^1_1(\alpha_-)|$ (resp. $|s - \lambda^1_{d+2}(\alpha_-)|$) are small enough, if the point α_+ on the Rankine-Hugoniot curve through α_- is close enough to α_-.

Due to (3.31) and (3.32), we obtain:

$$\tilde{\eta}(\alpha) \leq 2\bar{\eta}(\alpha_-) \ \ \forall \alpha \in B_R(\alpha_-),$$

and due to assumption (3.16) (remembering that $\alpha_- = \alpha_e$) we conclude that

$$\alpha(\xi) \in B_R(\alpha_-) \,\forall \xi \in \mathbb{R} \ \ \Rightarrow \ \ \tilde{\eta}(\alpha(\xi)) \leq \gamma \ \ \forall \xi \in \mathbb{R}.$$

Due to (3.17), we conclude that

$$\alpha(\xi) \in B_R(\alpha_-) \,\forall \xi \in \mathbb{R} \ \ \Rightarrow \ \ \tilde{\eta}(\alpha(\xi)) = \bar{\eta}(\alpha(\xi)) \ \ \forall \xi \in \mathbb{R}.$$

As for $|\alpha_+ - \alpha_-|$ small enough it holds $\alpha(\xi) \in B_R(\alpha_-)$ $\forall \xi \in \mathbb{R}$, the last claim of the theorem has been proven. □

3.6.2 Checking of simplicity condition for a special case

It remains to check that for system (3.19) zero is an eigenvalue of

$$-\lambda_p I_n + A = -\lambda_p I_n + \sum_{i=1}^{d} \omega_i F_{iU}(U_e)$$

with multiplicity less than or equal to one for $U_e \in \mathcal{E}$ if for $\lambda_p(\cdot)$ the strict entropy condition is fulfilled (remembering the notations and definitions of section "Traveling waves for simple shocks").

Writing $\lambda_p = \lambda_p(\alpha_e)$ and taking into account that system (3.18) is equivalent to system (3.19) we have to show that for α_e with $\tilde{\eta}_\alpha(\alpha_e) = U_e$ zero is an eigenvalue of

$$-\lambda_p(\alpha_e)\tilde{\eta}_{\alpha\alpha}(\alpha_e) + \sum_{i=1}^{d} \omega_i \tilde{\eta}_{\alpha_{i+1}\alpha\alpha}(\alpha_e) =$$

$$= -\lambda_p(\alpha_e) \int c(v) \otimes c(v) exp(\alpha_e \cdot c(v)) dv + \sum_{i=1}^{d} \omega_i \int v_i c(v) \otimes c(v) exp(\alpha_e \cdot c(v)) dv$$

with multiplicity less than or equal to one.

Due to rotational invariance we can assume that $\omega = (1, 0, \cdots, 0)$.

For each $\alpha_e \in \mathbf{E}$, the kinetic particle density $f(v) = exp(c(v) \cdot \alpha_e)$ can be expressed in terms of the density ρ, the macroscopic velocity u and the temperature θ via

$$exp(c(v) \cdot \alpha_e) = \frac{\rho}{(2\pi\theta^{d/2})} exp\left(-\frac{|v-u|^2}{2\theta}\right). \tag{3.33}$$

Note that can write $\rho = \rho(\alpha_e)$, $u = u(\alpha_e)$ and $\theta = \theta(\alpha_e)$.

Due to translational invariance (in the velocity space) we can assume that $u = 0$.

For $\tau = (2\theta)^{1/2}$ substitute $v = \tau\bar{v}$. Then it holds $exp(c(v) \cdot \alpha_e) = exp(-|\bar{v}|^2)$.

There exists an invertible matrix $T(\tau)$ such that

$$c(v) \otimes c(v) = c(\tau\bar{v}) \otimes c(\tau\bar{v}) = T^t(\tau) c(\bar{v}) \otimes c(\bar{v}) T(\tau). \tag{3.34}$$

Furthermore, from (3.22) it easily follows that for $\bar{\alpha}_e$ with

$$-\frac{|v|^2}{2\theta} = c(v) \cdot \alpha_e = c(\bar{v}) \cdot \bar{\alpha}_e = -|\bar{v}|^2$$

it holds

$$\lambda_p(\alpha_e) = (2\theta)^{1/2} \lambda(\bar{\alpha}_e) = \tau\lambda(\bar{\alpha}_e). \tag{3.35}$$

Due to $v_1 = \tau\bar{v}_1$ and relations (3.34) and (3.35) and it suffices to show that zero is an eigenvalue of

$$-\lambda_p(\bar{\alpha}_e) \int T^t(\tau) c(\bar{v}) \otimes c(\bar{v}) T(\tau) exp(-|\bar{v}|^2) d\bar{v} + \int \bar{v}_1 T^t(\tau) c(\bar{v}) \otimes c(\bar{v}) T(\tau) exp(-|\bar{v}|^2) d\bar{v} =$$

$$= T^t(\tau) \left(-\lambda_p(\bar{\alpha}_e) \int c(\bar{v}) \otimes c(\bar{v}) exp(-|\bar{v}|^2) d\bar{v} + \int \bar{v}_1 c(\bar{v}) \otimes c(\bar{v}) exp(-|\bar{v}|^2) d\bar{v}\right) T(\tau)$$

with multiplicity less than or equal to one.

After multiplication by $T^{-t}(\tau)$ from the left and by $T^{-1}(\tau)$ from the right it remains to prove that zero is an eigenvalue with multiplicity less than or equal to one for

$$M = \int \bar{v}_1 c(\bar{v}) \otimes c(\bar{v}) exp(-|\bar{v}|^2) dv - \lambda_p(\bar{\alpha}_e) \int c(\bar{v}) \otimes c(\bar{v}) exp(-|\bar{v}|^2) dv.$$

Taking (3.22) into account and noting that $\theta(\bar{\alpha}_e) = \frac{1}{2}$ we derive that the strict entropy condition is fulfilled for the eigenvalues $\lambda_p(\cdot)$ with $\lambda_p(\bar{\alpha}_e) = (\frac{d+2}{2d})^{1/2}$ and $\lambda_p(\bar{\alpha}_e) = -(\frac{d+2}{2d})^{1/2}$.

Hence, it remains to prove that zero is an eigenvalue with multiplicity less than or equal to one for

$$M = \int v_1 c(v) \otimes c(v) exp(-|v|^2) dv - \left(\frac{d+2}{2d}\right)^{1/2} \int c(v) \otimes c(v) exp(-|v|^2) dv$$

$$(resp.\ M = \int v_1 c(v) \otimes c(v) exp(-|v|^2) dv + \left(\frac{d+2}{2d}\right)^{1/2} \int c(v) \otimes c(v) exp(-|v|^2) dv).$$

For $d = 3$ and $c(v) = (1, v_1, v_2, v_3, |v|^2, v_2^2, v_3^2, v_1 v_2, v_1 v_3, v_2 v_3, v_1^3, v_2^3, v_3^3, v_1^2 v_2, v_1^2 v_3, v_1 v_2^2, v_2^2 v_3, v_1 v_3^2, v_2 v_3^2, v_1 v_2 v_3, |v|^4)$ we get by elementary calculation:

$$\tilde{M} = \pi^{-3/2} M =$$

$$
= \begin{bmatrix}
a & \frac{1}{2} & 0 & 0 & \frac{3a}{2} & \frac{a}{2} & \frac{a}{2} & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{15a}{4} \\
\frac{1}{2} & \frac{a}{2} & 0 & 0 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{3a}{4} & 0 & 0 & 0 & 0 & \frac{a}{4} & 0 & \frac{a}{4} & 0 & 0 & \frac{35}{8} \\
0 & 0 & \frac{a}{2} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3a}{4} & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & \frac{a}{4} & 0 & 0 \\
0 & 0 & 0 & \frac{a}{2} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3a}{4} & 0 & \frac{a}{4} & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 \\
\frac{3a}{2} & \frac{5}{4} & 0 & 0 & \frac{15a}{4} & \frac{5a}{4} & \frac{5a}{4} & 0 & 0 & 0 & \frac{21}{8} & 0 & 0 & 0 & 0 & \frac{7}{8} & 0 & \frac{7}{8} & 0 & 0 & \frac{105a}{8} \\
\frac{a}{2} & \frac{1}{4} & 0 & 0 & \frac{5a}{4} & \frac{3a}{4} & \frac{a}{4} & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{35a}{8} \\
\frac{a}{2} & \frac{1}{4} & 0 & 0 & \frac{5a}{4} & \frac{a}{4} & \frac{3a}{4} & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & 0 & \frac{35a}{8} \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{a}{4} & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{a}{4} & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\
\frac{3}{4} & \frac{3a}{4} & 0 & 0 & \frac{21}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & \frac{15a}{8} & 0 & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{3a}{8} & 0 & 0 & \frac{189}{16} \\
0 & 0 & \frac{3a}{4} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{15a}{8} & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 & \frac{3a}{8} & 0 & 0 \\
0 & 0 & 0 & \frac{3a}{4} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{15a}{8} & 0 & \frac{3a}{8} & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 & \frac{a}{8} & 0 & 0 \\
0 & 0 & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{3a}{8} & 0 & \frac{a}{8} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{a}{4} & 0 & 0 & \frac{7}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{a}{8} & 0 & 0 & \frac{63}{16} \\
0 & 0 & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{a}{8} & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{a}{4} & 0 & 0 & \frac{7}{8} & \frac{1}{8} & \frac{3}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & 0 & 0 & 0 & \frac{a}{8} & 0 & \frac{3a}{8} & 0 & 0 & \frac{63}{16} \\
0 & 0 & \frac{a}{4} & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{3a}{8} & 0 & \frac{a}{8} & 0 & 0 & 0 & 0 & \frac{3a}{8} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a}{8} & 0 \\
\frac{15a}{4} & \frac{35}{8} & 0 & 0 & \frac{105a}{8} & \frac{35a}{8} & \frac{35a}{8} & 0 & 0 & 0 & \frac{189}{16} & 0 & 0 & 0 & 0 & \frac{63}{16} & 0 & \frac{63}{16} & 0 & 0 & \frac{945a}{16}
\end{bmatrix},
$$

where $a = -(\frac{5}{6})^{1/2}$ (resp. $a = (\frac{5}{6})^{1/2}$).

For $\tilde{M}$ we get

$$det(\tilde{M}) = \frac{30^{1/2} \cdot 625}{300578991243264}$$

$$(resp.\ det(\tilde{M}) = -\frac{30^{1/2} \cdot 625}{300578991243264}).$$

Hence, the matrix M is invertible.

We have calculated the determinant by Maple (version 9.01).

3.7 Appendix

3.7.1 Strict entropy condition

For the phase space density f define the density ρ, the macroscopic velocity field $\bar{u}$ and the temperature θ by the relations

$$\rho = \int_{\mathbb{R}^d} f(v)dv, \quad \bar{u} = \int_{\mathbb{R}^d} vf(v)dv, \tag{3.36}$$

$$\frac{1}{2}\rho|\bar{u}|^2 + \frac{d}{2}\rho\theta = \int_{\mathbb{R}^d} \frac{1}{2}|v|^2 f(v)dv.$$

Here and in the following, we suppress the notation of the (x,t)-dependence for reasons of convenience, i.e. instead of $\rho(x,t) = \int_{\mathbb{R}^d} f(x,t,v)dv$ we write $\rho = \int_{\mathbb{R}^d} f(v)dv$.

For a Maxwellian density

$$f(v) = \frac{\rho}{(2\pi\theta^{d/2})} exp\left(-\frac{|v-\bar{u}|^2}{2\theta}\right), \tag{3.37}$$

the local fluxes can be written as

$$\int_{\mathbb{R}^d} vv^T f(v)dv = \rho\bar{u}\bar{u}^T + \rho\theta I_d \tag{3.38}$$

and

$$\int_{\mathbb{R}^d} \frac{1}{2}|v|^2 vf(v)dv = \frac{1}{2}\rho(\bar{u}^T\bar{u})\bar{u} + \frac{d+2}{2}\rho\theta\bar{u}. \tag{3.39}$$

Plugging the relations

$$\int_{\mathbb{R}^d} \bar{c}(v)\left(f_* f'_* - ff'\right) Bd\omega dv' dv = 0$$

for

$$\bar{c}_1(v) \equiv 1, \ \bar{c}_{i+1}(v) = v_i \ (i = 1, \cdots, d), \ \bar{c}_{d+2}(v) = |v|^2$$

into (3.6) yields the Euler equations

$$\partial_t\rho + \nabla_x \cdot (\rho\bar{u}) = 0, \tag{3.40}$$

$$\partial_t(\rho\bar{u}) + \nabla_x \cdot (\rho\bar{u}\bar{u}^T) + \nabla_x(\rho\theta) = 0, \tag{3.41}$$

$$\partial_t(\frac{1}{2}\rho|\bar{u}|^2 + \frac{d}{2}\rho\theta) + \nabla_x \cdot (\frac{1}{2}\rho|\bar{u}|^2\bar{u} + \frac{d+2}{2}\rho\theta\bar{u}) = 0. \tag{3.42}$$

We can write these equations in the form (for the case $d = 1$ see [KRR], p. 278ff.)

$$\partial_t \rho + \nabla_x \cdot (\rho \bar{u}) = 0, \tag{3.43}$$

$$\partial_t (\rho \bar{u}) + \nabla_x \cdot (\rho \bar{u} \bar{u}^T) + \nabla_x p = 0, \tag{3.44}$$

$$\partial_t e + \nabla_x \cdot (\bar{u}(e + p)) = 0, \tag{3.45}$$

with $p = \rho\theta$ and e together with the equation of state

$$p = \frac{2}{d}(e - \frac{\rho}{2}|\bar{u}|^2). \tag{3.46}$$

We claim that in the variable $u = (\rho, \bar{u}_1, \cdots, \bar{u}_d, p)^T$, with $a(\rho, p) = (\frac{(2+d)}{d}\frac{p}{\rho})^{1/2}$ and with

$$\hat{A}_i(u) = \bar{u}_i I_{d+2} + \rho e_1 e_{1+i}^T + \frac{1}{\rho} e_{1+i} e_{d+2}^T + \rho a^2 e_{d+2} e_{1+i}^T \tag{3.47}$$

(where e_i denotes the i^{th} canonical base vector) for $i = 1, \cdots, d$, the equations (3.43), (3.44) and (3.45) can for smooth solutions $u = u(x, t)$ be brought into the form

$$u_t + \sum_{i=1}^{d} \hat{A}_i(u) u_{x_i} = 0. \tag{3.48}$$

These are $d + 2$ equations for $u = (\rho, \bar{u}_1, \cdots, \bar{u}_d, p)^T$.

The equations for $(u_1, \cdots, u_{d+1}) = (\rho, \bar{u}_1, \cdots, \bar{u}_d)$ in (3.48) trivially follow from (3.43) and (3.44), the equation for $u_{d+2} = p$ follows by differentiation of

$$e = \frac{d}{2}p + \frac{\rho}{2}|\bar{u}|^2$$

with respect to t and applying $\nabla_x \cdot$ to

$$(p + e)\bar{u} = \left(\frac{2 + d}{2} p + \frac{\rho |\bar{u}|^2}{2} \right) \bar{u}.$$

We obtain using (3.45) and the equation for $(u_2, \cdots, u_{d+1}) = (\bar{u}_1, \cdots, \bar{u}_d)$ in (3.48):

$$\partial_t e = \frac{d}{2}\partial_t p - \frac{1}{2}\left(|\bar{u}|^2 \bar{u} \cdot \nabla_x \rho + 3\rho|\bar{u}|^2 \nabla_x \cdot \bar{u} + 2\bar{u} \cdot \nabla_x p\right),$$

$$\nabla_x \cdot ((p+e)\bar{u}) = \frac{2+d}{2}\left(p\nabla_x \cdot \bar{u} + \bar{u} \cdot \nabla_x p\right) + \frac{1}{2}\nabla_x \cdot (\rho|\bar{u}|^2 \bar{u}).$$

Now we add these two equations and obtain the equation

$$\partial_t p + \rho a^2 \nabla_x \cdot \bar{u} + \bar{u} \cdot \nabla p = 0$$

for $u_{d+2} = p$ in (3.48).

The eigenvalues of $\hat{A}_i(u)$ are

$$\lambda_1^i(u) = \bar{u}_i - a(\rho, p), \quad \lambda_2^i(u) = \cdots = \lambda_{d+1}^i(u) = \bar{u}_i, \tag{3.49}$$

$$\lambda_{d+2}^i(u) = \bar{u}_i + a(\rho, p).$$

For the corresponding right eigenvectors

$$r_1^i(u) = \rho e_1 - a(\rho, p) e_{1+i} + \rho a(\rho, p)^2 e_{d+2}, \tag{3.50}$$

$$r_{d+2}^i(u) = \rho e_1 + a(\rho, p) e_{1+i} + \rho a(\rho, p)^2 e_{d+2},$$

$$r_j^i(\cdot) \equiv e_{j-1} \;\; for \;\; 2 \le j \le i+1,$$

$$r_j^i(\cdot) \equiv e_j \;\; for \;\; i+2 \le j \le d+1$$

there holds

$$\nabla_u \lambda_j^i(u) \cdot r_j^i(u) = 0 \;\; for \;\; j \in \{2, \cdots, d+1\}, \tag{3.51}$$

$$\nabla_u \lambda_1^i(u) \cdot r_1^i(u) = -\frac{d+1}{d} a(\rho, p) \neq 0,$$

$$\nabla_u \lambda^i_{d+2}(u) \cdot r^i_{d+2}(u) = \frac{d+1}{d} a(\rho, p) \neq 0.$$

Hence, for the first and the $(d+2)^{th}$ eigenvalue the strict entropy condition is fulfilled.

These properties are independent of the choice of the coordinates:

If $u = u(v)$ is an invertible smooth transformation, we can write system (3.48) in the form

$$v_t = \sum_{j=1}^{d} u_v^{-1}(v) \hat{A}_j(u(v)) u_v(v) v_{x_j} = 0.$$

For the eigenvalues $\bar{\lambda}^i_j(v) = \lambda^i_j(u(v))$ and the right eigenvectors $\bar{r}^i_j(v)$ of $u_v^{-1}(v)\hat{A}_j(u(v))u_v(v)$ it holds (noting that $\nabla_v \bar{\lambda}^i_j(v)$ and $\nabla_u \lambda^i_j(u(v))$ are row vectors):

$$\nabla_v \bar{\lambda}^i_j(v) = \nabla_u \lambda^i_j(u(v)) u_v(v), \quad \bar{r}^i_j(v) = u_v^{-1}(v) r^i_j(u(v)).$$

Hence,

$$\nabla_v \bar{\lambda}^i_j(v) \cdot \bar{r}^i_j(v) = \nabla_u \lambda^i_j(u(v)) \cdot r^i_j(u(v)).$$

3.7.2 Conclusions from Galilean Invariance

Now, we will prove that for n linearly independent locally integrable functions $c_1, \cdots, c_n$ the translational invariance plus the integrability of density $exp(\alpha \cdot c(\cdot))$ for each α contained in non-void open set $\mathcal{O} \subset \mathbb{R}^n$ already imply that they are polynomials.

Indeed, the c_i's have to be distributions only.

Hence, we can leave aside the assumption that the derivative has to exist at one point (cp. [J-U]).

First we fix some notation (for further details see [Y]):

The set $\mathbf{C}_0^\infty(\mathbb{R}^d)$ becomes a locally convex space through the family of seminorms

$$p_{K,r}(u) = \sum_{|\alpha|\leq r} sup_{x\in K}|D^\alpha u|, \quad r \in \mathbf{N}_0, \quad K \subset \mathbb{R}^d \ \ compact.$$

Let $\mathcal{D}$ denote this space and $\mathcal{D}'$ the linear space of continuous linear functionals on $\mathcal{D}$ and $(\cdot,\cdot) : \mathcal{D} \times \mathcal{D}' \to \mathbb{R}$ the dual pairing.

The partial derivatives $\partial_{x_k},\ k = 1, \cdots, d$ in direction e_k and the translation

$$\mathcal{D}' \to \mathcal{D}', \quad \psi \mapsto \psi(\cdot + \tau)$$

for $\tau = (\tau_1, \cdots, \tau_d) \in \mathbb{R}^d$ are defined in the distributional sense, i.e.

$$(v, \partial_{x_k} w) = -(\partial_{x_k} v, w), \quad (v, w(\cdot + \tau)) = (v(\cdot - \tau), w).$$

The embedding $\mathbf{L}^1_{loc}(\mathbb{R}^d) \hookrightarrow \mathcal{D}'$ is defined via $(\phi, g) = \int_{\mathbb{R}^d} g(x)\phi(x)dx$.

With this notation we prove the following theorem:

Theorem 3.7.1 *Assume that* $c_1, \cdots, c_n \in \mathcal{D}'$ *are linearly independent and that for each* $\tau \in \mathbb{R}^d$ *there exists a matrix* $\Lambda(\tau) \in \mathbb{R}^{n\times n}$ *such that*

$$c_i(\cdot + \tau) = \sum_{i=1}^{n} \Lambda^i_j(\tau) c_i. \tag{3.52}$$

Then there exist matrices $\Omega_1, \cdots, \Omega_d \in \mathbb{R}^{n\times n}$ *commuting with each other such that* c *can be represented as a smooth vector-valued function of the form*

$$c(x) = c(x_1, \cdots, x_d) = \left(\Pi_{k=1}^d exp(x_k \Omega_k)\right) c(0).$$

If, furthermore, the density $exp(\alpha \cdot c(\cdot))$ *is integrable for each* α *on a nonvoid open set in* $\mathbb{R}^n$*, then the functions* $c_1, \cdots, c_n$ *are polynomials.*

Proof:

First, let's remark that $c_1, \cdots, c_n$ being linearly independent implies the existence of $\phi_1, \cdots, \phi_n \in \mathcal{D}$ fulfilling the relations

$$(\phi_i, c_j) = \gamma^i_j, \quad i = 1, \cdots, n, \tag{3.53}$$

such that $\Gamma = (\gamma^i_j)_{1\leq i,j\leq n}$ is invertible.

For $A^i_j(\tau) = (\phi_j, c_i(\cdot + \tau))$ relation (3.53) yields

$$A^i_j(\tau) = \sum_{k=1}^{n} \Lambda^k_i(\tau)\,(\phi_j, c_k) = \sum_{k=1}^{n} \Lambda^k_i(\tau)\gamma^j_k, \quad i.e.\ \ \Lambda(\tau) = A(\tau)\Gamma^{-1}. \tag{3.54}$$

The derivatives in $\tau = (\tau_1, \cdots, \tau_d)$ of Λ^i_j at 0 have the form

$$\partial_{\tau_k}\Lambda^i_j(0) \overset{(3.54)}{=} \sum_{m=1}^{n} (-\partial_{x_k}\phi_j, c_m)\,(\Gamma^{-1})^i_m, \tag{3.55}$$

so that we conclude (using $A(0) \overset{(3.54)}{=} \Lambda(0)\Gamma^{-1}$ for the initial value) that the IVP

$$\Lambda(0) = I_n, \quad \partial_{\tau_k}\Lambda(\tau_1, \cdots, \tau_d) = \Omega_k\Lambda(\tau_1, \cdots, \tau_d), \quad k = 1, \cdots, d$$

have a unique solution where $\Omega_k = \lim_{\tau_k\to 0} \frac{(\Lambda(\tau_k e_k)-\Lambda(0))}{\tau_k}$ is the generator of the semigroup to Λ in the parameter τ_k which is given by relation (3.55).

From definition (3.52) of Λ it follows using the commutativity of the group of translations

$$\Lambda(\tau_k e_k)\Lambda(\tau_l e_l) = \Lambda(\tau_l e_l)\Lambda(\tau_k e_k)$$

that the matrices $\Omega_k,\ k = 1, \cdots, d$ commute with each other.

For each $l \in \{1, \cdots, d\}$ and $i \in \{1, \cdots, n\}$, there exist real numbers $\lambda_1, \lambda_2, \lambda_3$ such that $g(x_l) = c_i(x_1, \cdots, x_l, \cdots, x_d)$ has the form

$$g(x_l) = exp(\lambda_1 x_l)sin(\lambda_2 + \lambda_3 x_l)p(x_l) = exp(\lambda_1 x_l)sin(\lambda_2 + \lambda_3 x_l)\left(\sum_{i=0}^{k} \alpha_i x_l^i\right).$$

If $\lambda_1 \neq 0$ it holds $\lim_{x_l \to \infty} g(x_l) = 0$ or $\lim_{x_l \to -\infty} g(x_l) = 0$, so that $\lim_{x_l \to \infty} exp(g(x_l)) = 1$ or $\lim_{x_l \to -\infty} exp(g(x_l)) = 1$.

In this case it holds: $exp(g(\cdot)) \notin \mathbf{L}^1(\mathbb{R})$.

If $\lambda_1 = 0$, $\lambda_3 \neq 0$ denote by J_k the set

$$J_k := \{x_k \in \mathbb{R} \ | \ sin(\lambda_2 + \lambda_3 x_l)\alpha_k > |\alpha_k|/2.\}.$$

Then it holds $|p(x_l)| \geq |\alpha_k| \ \ for \ \ |x_l| > R$, $R > 0$ large enough

$$|sin(\lambda_2 + \lambda_3 x_l) p(x_x)| > |\alpha_k|/2 \ \ for \ \ |x_k| > R, \ \ x_k \in J_k.$$

As $meas(J_k \setminus B_R(0)) = \infty$ it holds $exp(g(\cdot)) \notin \mathbf{L}^1(\mathbb{R})$.

In other words:

$$\lambda_1 \neq 0 \ \ or \ \ \lambda_3 \neq 0 \ \ implies \ \ exp(g(\cdot)) \notin \mathbf{L}^1(\mathbb{R}).$$

Hence, the function c_i is polynomial. $\square$

Notation

The void box $\square$ denotes the end of proof.

If M_1 and M_2 are two sets then we denote by $M_1 \cup M_2$ their union, by $M_1 \cap M_2$ their intersection and by $M_1 \setminus M_2$ their set - theoretic difference.

The product $M_1 \times M_2$ is defined as the set of (x_1, x_2) with $x_1 \in M_1$ and $x_2 \in M_2$.

If M is a subset of a metric space we denote by ∂M (resp. $int(M)$ (resp. $\bar{M}$)) its boundary (resp. its interior (resp. its closure)).

By $\mathbb{N}$ we denote the set of strictly positive integers, by $\mathbb{N}_0$ the set of nonnegative integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{C}$ the set of complex numbers.

By $\mathbb{R}^{n\times m}$ (resp. $\mathbb{C}^{n\times m}$) we denote the set of real (resp. complex) $n \times m$ matrices where the first index is the row index and the second one is column index.

$I_n \in \mathbb{R}^{n\times n}$ denotes the identity $(n \times n)$ - matrix, i.e. the matrix with ones on the diagonal and zeroes elsewhere.

By $GL(\mathbb{R}, n)$ we denote the set of invertible matrices $M \in \mathbb{R}^{n\times n}$.

For $M \in \mathbb{C}^{m\times n}$ let $M^* \in \mathbb{C}^{n\times m}$ be the transpose of M, i.e.

$$(M^*)_{ij} = \overline{M}_{ji},$$

where $\overline{\zeta}$ denotes the complex conjugate of ζ.

For $M_i \in \mathbb{R}^{n_i\times n_i}$, $\ i = 1, \cdots, m$ the matrix $diag(M_1, \cdots, M_m) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\bar{n} = \sum_{i=1}^{m} n_i$ denotes the block diagonal square matrix with blocks $M_1, \cdots, M_m$ along the main diagonal.

Sometimes we denote the transpose by M^t or M^T instead of M^* if $M \in \mathbb{R}^{m\times n}$.

For $M \in \mathbb{C}^{n\times n}$ let $spec(M) \subset \mathbb{C}$ denote the set of eigenvalues of M, i.e. the set of complex numbers λ such that $M - \lambda I_n$ is not invertible.

As generalized eigenspace to $\lambda \in spec(M)$ we denote the space corresponding to the Jordan blocks of $M - \lambda I_n$ whose diagonal entries are zero.

We set $\mathbb{R}^n := \mathbb{R}^{n\times 1}$ equipped with the Euclidean norm

$$|x| = \sqrt{x^t x}.$$

For a strictly positive number ρ define

$$B_\rho(\bar{x}) = \{x \in \mathbb{R}^n \mid |x - \bar{x}| < \rho\}, \quad K_\rho(\bar{x}) = \partial B_\rho(\bar{x}).$$

Set $\mathbb{S}^{n-1} = K_1(0)$.

If $\mathcal{U} \subset \mathbb{R}^m$ is non-void and open then $\mathbf{C}^k(\mathcal{U}, \mathbb{R}^n)$ denotes the linear space of functions $f : \mathcal{U} \to \mathbb{R}^n$ whose k^{th} derivative exists and is continuous.

By f_x, $\partial_x f$ or sometimes $D_x f$ we denote the derivative of f in the variable x.

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