Quasilinear continuity equations of measures for bounded BV vector fields

Thomas Lorenz [1]

Abstract. The focus of interest here is a quasilinear form of the conservative continuity equation

$$\tfrac{d}{dt}\,\mu \;+\; D_x \cdot \big(f(\mu,\cdot)\;\mu\big) \;=\; 0 \qquad\qquad (\text{in } \mathbb{R}^N \times\,]0,T[)$$

together with its measure–valued distributional solutions $\mu(\cdot) : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^N)$. On the basis of Ambrosio's results about the nonautonomous linear equation $\frac{d}{dt}\,\mu \;+\; D_x \cdot (b\;\;\mu) \;=\; 0$ (see [1, 2]), the existence and uniqueness of solutions are investigated for coefficients being bounded vector fields with bounded (spatial) variation and $\mathcal{L}^N$ absolutely continuous divergence in combination with positive measures absolutely continuous with respect to Lebesgue measure $\mathcal{L}^N$.

The step towards the nonlinear problem here relies on a further generalization of Aubin's mutational equations that is extending the notions of distribution–like solutions and "weak compactness" to a set supplied with a countable family of (possibly non–symmetric) distance functions (so–called ostensible metrics).

Contents

[1] Interdisciplinary Center for Scientific Computing (IWR)
Ruprecht–Karls–University of Heidelberg
Im Neuenheimer Feld 294, 69120 Heidelberg (Germany)
thomas.lorenz@iwr.uni-heidelberg.de

First preprint: December 22, 2005
This revised form: January 14, 2008

1 Introduction

The continuity equation $\frac{d}{dt}\,\mu \;+\; D_x \cdot (\widetilde{b}\;\mu) \;=\; 0$ (in $\mathbb{R}^N \times [0,T[$) is the classical analytical tool for describing the conservation of some real–valued quantity $\mu = \mu(t,x)$ while "flowing" (or, rather, evolving) along a given vector field $\widetilde{b} : \mathbb{R}^N \times [0,T] \longrightarrow \mathbb{R}^N$. Thus, it is playing a key role in many applications of modelling like fluid dynamics and, it has been investigated under completely different types of assumptions about $\widetilde{b}(\cdot,\cdot)$.

Using the results of Ambrosio [1, 2] later, the values of all solutions considered here are positive finite Radon measures on $\mathbb{R}^N$ and, we are interested in structurally weak assumptions about these measures or the vector fields for proving existence of a distributional solution to the *quasilinear* continuity equation

$$\tfrac{d}{dt}\,\mu(t) \;+\; D_x \cdot \big(f(\mu(t),t)\;\;\mu(t)\big) \;=\; 0 \qquad\qquad (\text{in } \mathbb{R}^N \times\,]0,T[)$$

with given $f(\cdot,\cdot) : \mathcal{M}(\mathbb{R}^N) \times [0,T] \longrightarrow L^\infty(\mathbb{R}^N,\mathbb{R}^N)$. Basically two cases are investigated more closely:

In [14], all vector fields were supposed to be uniformly bounded and Lipschitz continuous with respect to space, i.e. $f(\mu,t) \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$ with $\|f\|_{W^{1,\infty}} \le C < \infty$. Due to this regularity assumption, the measure–valued solution to the continuity equation $\frac{d}{dt}\,\mu \;+\; D_x \cdot (b\;\mu) \;=\; 0$ (with such a vector field b) can be easily represented by the method of characteristics: $\mu_t \;:=\; \mathbf{X}_b(t,\cdot)_\sharp\,\mu_0$ with $\mathbf{X}_b(\cdot,x_0) : [0,T] \longrightarrow \mathbb{R}^N$ denoting the absolutely continuous solution to the Cauchy problem

$$\wedge \left\{ \begin{array}{rcll} \frac{d}{dt}\,x(t) &=& b(x(t)) & \text{a.e. in } [0,T], \\ x(0) &=& x_0 & \end{array} \right.$$

and the index $\sharp$ abbreviating the push–forward of a measure. In return for this (quite popular, but restrictive) Lipschitz continuity of vector fields, admissible measures did not have to be absolutely continuous with respect to Lebesgue, but all positive Radon measures with compact support were taken into consideration. Their set is abbreviated as $\mathcal{M}_c^+(\mathbb{R}^N)$. So in particular, Hausdorff measures $\mathcal{H}^\delta$ of arbitrary dimension $\delta \in [0,N]$ – restricted to a compact $\mathcal{H}^\delta$–rectifiable subset of $\mathbb{R}^N$ – can be considered. For specifying the continuity properties of "velocity" function $f : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$, generalized distance functions for finite Radon measures on $\mathbb{R}^N$ are required. In [14], the weak* topology on Radon measures with compact support was metrized and, one of the main results there is :

Proposition 1.1 *Suppose for* $f : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$:

1. $\exists\; C \in [0,\infty[\; : \|f(\mu,t)\|_{W^{1,\infty}} \le C$ *for all* $(\mu,t) \in \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T]$,
2. *("weak*" continuity of* f*)* $\|f(\mu,t) - f(\mu_n,t_n)\|_\infty \longrightarrow 0$ *if* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C_0^0(\mathbb{R}^N)$*)*, $t_n \to t$.

Then for every initial $\mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, *there exists a distributional solution* $\mu(\cdot) : [0,T[\, \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$ *to the continuity equation* $\frac{d}{dt}\,\mu(t) \;+\; D_x \cdot (f(\mu(t),t)\;\;\mu(t)) \;=\; 0$ *in* $\mathbb{R}^N \times\; [0,T[$ *with* $\mu(0) = \mu_0$.

In this paper, the regularity condition on vector fields is weakened. We dispense with their local Lipschitz continuity and follow the track of Ambrosio [1, 2]. So now we use vector fields in

$$\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N) \;:=\; \big\{ b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^N) \;\big|\; D \cdot b \;=\; \mathrm{div}\, b\;\; \mathcal{L}^N \ll \mathcal{L}^N,\;\; \mathrm{div}\, b \in L^\infty(\mathbb{R}^N) \big\}$$

and positive measures on $\mathbb{R}^N$ that are bounded and absolutely continuous w.r.t. Lebesgue measure, i.e.

$$\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \;:=\; \big\{ \rho\,\mathcal{L}^N \;\big|\; \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\;\; \rho \ge 0 \big\}.$$

The gap between $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ and $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ is bridged by Ambrosio's proof of his superposition principle [1, Theorem 12]. Indeed, for every function $\widetilde{b} \in L^1\big([0,T],\ \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)\big)$ with $\sup_t \big(\|\widetilde{b}(t,\cdot)\|_\infty + \|\mathrm{div}_x\, \widetilde{b}(t,\cdot)\|_\infty\big) < \infty$, each distributional solution $\mu(\cdot)$ of $\frac{d}{dt}\,\mu\; +\; D_x \cdot (\widetilde{b}\;\mu) = 0$ can be approximated narrowly by the unique solution $\mu_\delta(t) := \mu(t) * \rho_\delta$ of $\frac{d}{dt}\,\mu_\delta\; +\; D_x \cdot (\widetilde{b}_\delta\;\mu_\delta) = 0$ with $\widetilde{b}_\delta(t,\cdot) := \frac{(\widetilde{b}(t,\cdot)\,\mu(t)) * \rho_\delta}{\mu_\delta(t)}$ being in $L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)\big)$ and $\rho_\delta(\cdot)$ denoting a smooth Gaussian kernel. So the approximation (with respect to narrow convergence) is the key tool for extending many estimates concerning vector fields in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ to vector fields in $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$.

Our main results about this type of continuity equation are consequences of Propositions 3.14, 3.16 using now the pseudo–metrics $q_\varepsilon(\mu,\,\nu) := \big|\varphi_\varepsilon \cdot (\mu - \nu)\big|(\mathbb{R}^N)$ $(\varepsilon \in \mathcal{J})$ with a countable dense family of suitable Schwartz functions $\varphi_\varepsilon(\cdot)$ (specified in Lemma 3.6 below):

Proposition 1.2 *Suppose for* $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C \in [0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$,

2. $\forall\ \varepsilon \in \mathcal{J}:\ \ \big\|\varphi_\varepsilon\, |f(\mu,t) - f(\mu_n,t_n)|\big\|_{L^1(\mathbb{R}^N)} \longrightarrow 0$ *whenever* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C^0_0(\mathbb{R}^N)$*)*, $t_n \searrow t$.

Then for every $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, *there exists a distributional solution* $\mu(\cdot) : [0,T[\, \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *to the continuity equation* $\frac{d}{dt}\,\mu(t)\ +\ D_x \cdot (f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N \times [0,T[$ *with* $\mu(0) = \mu_0$.

Proposition 1.3 *Suppose for* $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C \in [0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$,

2. $\forall\ \varepsilon \in \mathcal{J}\quad \exists\, L_\varepsilon \in [0,\infty[$, *modulus of continuity* $\omega_\varepsilon(\cdot) \ge 0$:
$\big\|\varphi_\varepsilon\, |f(\mu,s) - f(\nu,t)|\,\big\|_{L^1(\mathbb{R}^N)} \le\ L_\varepsilon \cdot q_\varepsilon(\mu,\nu) + \omega_\varepsilon\,(|s-t|)$ *for all* $(\mu,s),(\nu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$.

Then for every initial measure $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, *the distributional solution* $\mu(\cdot) : [0,T[\, \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *of* $\frac{d}{dt}\,\mu(t)\ +\ D_x \cdot (f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N \times]0,T[$ *that is continuous with respect to each* q_ε $(\varepsilon \in \mathcal{J})$ *is unique.*

In particular, the continuity conditions on $f(\cdot,\cdot)$ are (slightly) weaker in Proposition 1.2 than in Proposition 1.1 because the weighted L^1 norms are used instead of the L^∞ norm. Moreover, separating the time dependence from spatial measures opens the door to taking spatially nonlocal effects into consideration — to some extent.

Restricting our considerations to measures in $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ has two analytical advantages in addition: The first one is the uniqueness of distributional solutions to the nonautonomous continuity equation. To be more precise, Ambrosio extends the DiPerna–Lions theory [9] to the case of spatially BV vector fields. Any distributional solution $w \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^N \times]0,T[\big)$ of $\frac{d}{dt}\,w\ +\ D_x \cdot (\widetilde{b}\;w) = \widetilde{c}$ is shown to be "renormalized" (in the sense of DiPerna and Lions) if $\widetilde{c} \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N \times]0,T[\big)$ and whenever the vector field $\widetilde{b} \in L^1_{\mathrm{loc}}\big([0,T],\ \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ has the distributional divergence $D_x \cdot \widetilde{b}(t,\cdot) = \mathrm{div}_x\, \widetilde{b}(t,\cdot)\ \mathcal{L}^N \ll \mathcal{L}^N$ for $\mathcal{L}^1$–almost every $t \in [0,T]$ [1, Theorem 34]. Then additional integral conditions on $\widetilde{b}$ and $\mathrm{div}_x\, \widetilde{b}$ imply the comparison principle for the continuity equation $\frac{d}{dt}\ w\ +\ D_x \cdot (\widetilde{b}\ w) = 0$ in the class $\big\{\, w \in L^\infty\big([0,T]; L^1(\mathbb{R}^N)\big) \cap L^\infty\big([0,T]; L^\infty(\mathbb{R}^N)\big) \cap C^0\big([0,T]; w^*\text{–}L^\infty(\mathbb{R}^N)\big) \big\}$ [1, Theorem 26]. This result is used here for specifying sufficient conditions on $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ for the uniqueness of distributional solutions in Proposition 1.3.

The second advantage is provided by the area formula since it leads to an exponential growth condition of the total variation $\big|\mu(t)\big|(\mathbb{R}^N)$ and of $\big\|\frac{\mu(t)}{\mathcal{L}^N}\big\|_\infty$ for a solution $\mu(\cdot) : [0,T[\, \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ of $\frac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\;\mu) = 0$ (i.e. for the solution induced by the Lagrangian flow specified in Proposition 3.3 and proven in [1, 2]). Such a priori estimates lay the basis for applying the compactness criterion of de la Vallée Poussin.

So all in all, the aim of this paper to extend Ambrosio's results of [1] to the *quasilinear* continuity equation $\frac{d}{dt}\,\mu(t) \;+\; D_x \cdot \big(f(\mu(t),t)\;\mu(t)\big) \;=\; 0$ (in $\mathbb{R}^N \times\,]0,T[$). For such a step from a *linear* to a *nonlinear* equation, we rely on a technique here that is hardly known in the PDE community, but we regard it as very useful indeed: *generalized mutational equations.*

Mutational equations introduced by Aubin [7, 8] are to extend ordinary differential equations to a metric space (E,d). For dispensing with any linear structure, the key idea is to introduce "maps of elementary deformation" $\vartheta : [0,1] \times E \longrightarrow E$. Such a so–called *transition* specifies the point $\vartheta(t,x) \in E$ to which an initial point $x \in E$ has been moved after time $t \in [0,1]$. It can be interpreted as a generalized derivative of a curve $\xi : [0,T[\longrightarrow E$ at time $t \in [0,T[$ if it provides a first–order approximation in the sense of

$$\limsup_{h \downarrow 0}\; \tfrac{1}{h} \cdot d\big(\xi(t+h),\;\; \vartheta(h,\xi(t))\big) \;=\; 0.$$

The theory of mutational equations has already been applied to different types of evolution problems in metric spaces (see [5, 6, 10, 11, 21], for example). So–called morphological equations are a very popular geometric example and use compact reachable sets of differential inclusions (supplied with the Pompeiu–Hausdorff distance) as transitions.

Aubin's continuity conditions on transitions, however, seem to be quite restrictive. So there have been several approaches of weakening them [12, 13, 16]. The main aspects of generalizing so far have been

1. introduce a separate real component of time, i.e. consider $\widetilde{E} := \mathbb{R} \times E$ instead of the given set E,
2. dispense with the symmetry of the metric d, i.e. we use a countable family $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$ of (*timed*) *ostensible metrics* satisfying only the reflexivity condition and the (timed) triangle inequality,
3. extend the notion of distributional solutions to so–called *timed forward solutions* in such a way that a (fixed) structural estimate is preserved while comparing shortly with the evolutions of all "test elements" $\widetilde{x} \in \widetilde{\mathcal{D}}$,
4. implement the notion of Petrov–Galerkin, i.e. "test elements" do not have to belong to the same set $\widetilde{E}$ as the values of generalized solutions (see [12], in particular),
5. permit many parameters of transitions to depend on the "test element" and the index ε of (timed) ostensible metric (in a word, for increasing "degrees of freedom" while extending Euler algorithm).

Up to now, first–order geometric evolutions have been our main motivation for generalizing mutational equations, i.e. compact subsets of $\mathbb{R}^N$ are to evolve according to nonlocal properties of both the set and their (limiting) normal cones at the boundary (see [12, 13], in particular). Now the continuity equation motivates two further aspects of generalization:

Firstly, we modify the continuity conditions on transitions so that some of their continuity parameters might have linear growth (with respect to the "initial element"). This feature can be particularly useful whenever the theory is applied to vector spaces – like the semilinear evolution equations in reflexive Banach spaces (mentioned in [17]), the positive Radon measures in [14] or the $\mathcal{L}^N$ absolutely continuous measures here in § 3.

Secondly, the form of sequential compactness is weakened for proving existence of "right–hand sleek solutions". We assume each ostensible metric $\widetilde{q}_\varepsilon$ to be the supremum of (at most) countably many generalized distance functions $\widetilde{q}_{\varepsilon,\kappa}$ ($\kappa \in \mathcal{I}$). In this analytical environment, "weak sequential compactness" is to realize the notion that every "bounded" sequence has a subsequence converging with respect to each $\widetilde{q}_{\varepsilon,\kappa}$. In fact, this concept generalizes the definition of weak compactness in a (separable) real Banach space $(X, \|\cdot\|_X)$ since $\|z\|_X \;=\; \sup\,\big\{\, y^*(z) \;\big|\; y^* : X \longrightarrow \mathbb{R}$ linear, continuous, $\|y^*\|_{X^*} \le 1 \,\big\}$.

With regard to generalized mutational equations, we introduce the more general term "weakly timed transitionally compact" in Definition 2.20. A more detailed presentation of this theory is given in § 2. Then, in § 3, we consider positive measures of $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ that are "evolving" along vector fields of $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$.

Notations

$C^0_c(\mathbb{R}^N)$ denotes the space of continuous functions $\mathbb{R}^N \longrightarrow \mathbb{R}$ with compact support and $C^0_0(\mathbb{R}^N)$ its closure with respect to the sup norm, respectively. $C^0_c(\mathbb{R}^N, \mathbb{R}^+_0)$ abbreviates the subset of functions $\varphi \in C^0_c(\mathbb{R}^N)$ with $\varphi \geq 0$ and correspondingly, $C^0_0(\mathbb{R}^N, \mathbb{R}^+_0)$ its closure.
Furthermore, $\mathcal{M}(\mathbb{R}^N)$ consists of all finite real–valued Radon measures on $\mathbb{R}^N$. As a consequence of Riesz theorem, it is the dual space of $C^0_0(\mathbb{R}^N)$ (see e.g. [4, Remark 1.57]). Similarly $\mathcal{M}^+(\mathbb{R}^N)$ denotes the set of all positive Radon measures on $\mathbb{R}^N$, i.e. $\mathcal{M}^+(\mathbb{R}^N) := \{\mu \in \mathcal{M}(\mathbb{R}^N) \mid \mu(\cdot) \geq 0\}$.

2 Generalizing evolution equations to ostensible metric spaces: Mutational equations

In this section, the definitions and main results are presented although the measure–valued examples ensuing here do not require this concept in its most general form. There are basically two reasons for the complete presentation. Firstly, we want to present two additional steps of generalization motivated by both the measure–valued flow along vector fields (in § 3) and the semilinear evolution equations in reflexive Banach spaces (mentioned in [17]): allowing a form of "linear growth" for some parameters of transitions and assuming a "weak(er)" form of sequential compactness for proving existence.
Secondly, all previous versions prove to be special cases of the subsequent modification. So in particular, the examples of [7], [12, 13] and the ensuing section can be combined in systems arbitrarily. From our point of view, this property of generalized mutational equations is an essential advantage in comparison with viscosity solutions (and other concepts based on the maximum or inclusion principle).

General assumptions for § 2.

1. Let E and $\mathcal{D}$ denote nonempty sets (not necessarily $\mathcal{D} \subset E$),
   $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R} \times E, \quad \widetilde{\mathcal{D}} \stackrel{\text{Def.}}{=} \mathbb{R} \times \mathcal{D}, \quad \pi_1 : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow \mathbb{R},\ (t,x) \longmapsto t.$
2. $\mathcal{J} \neq \emptyset$ abbreviates a countable index set.
3. $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ satisfies the timed triangle inequality (for each $\varepsilon \in \mathcal{J}$),
   i.e. $\widetilde{q}_\varepsilon\big((r,x),\ (t,z)\big) \ \leq\ \widetilde{q}_\varepsilon\big((r,x),\ (s,y)\big) \ +\ \widetilde{q}_\varepsilon\big((s,y),\ (t,z)\big)$
   for all $(r,x), (s,y), (t,z) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $r \leq s \leq t$.
4. Fix $\lfloor \cdot \rfloor_\varepsilon : \widetilde{\mathcal{D}} \cup \widetilde{E} \longrightarrow [0, \infty[$ for each $\varepsilon \in \mathcal{J}$.
5. $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ fulfills $\widetilde{q}_\varepsilon(\widetilde{z},\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}) = 0,\ \ \pi_1\,\widetilde{z} = \pi_1\, i_{\widetilde{\mathcal{D}}}\,\widetilde{z},\ \ \lfloor \widetilde{z} \rfloor_\varepsilon \geq \lfloor i_{\widetilde{\mathcal{D}}}\,\widetilde{z} \rfloor_\varepsilon$
   for every $\widetilde{z} \in \widetilde{\mathcal{D}}, \varepsilon \in \mathcal{J}$.

Assumption (4.) lays the foundation for the first new aspect in comparison with earlier definitions. For allowing a form of "linear growth" for some parameters of transitions, we need a counterpart of norms. Roughly speaking, it is to measure the absolute magnitude of any element $\widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ whereas each $\widetilde{q}_\varepsilon$ rather "compares" two elements with each other. Dispensing with any vector space structure, however, this counterpart is supposed to be just a nonnegative function that might even depend on $\varepsilon \in \mathcal{J}$, namely $\lfloor \cdot \rfloor_\varepsilon : \widetilde{\mathcal{D}} \cup \widetilde{E} \longrightarrow [0, \infty[$.
Generalizing mutational equations, the key idea is now to preserve the following structural estimate for comparing transitions $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$

$$\begin{aligned} &\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h, \widetilde{z}),\ \ \widetilde{\tau}(t_2+h, \widetilde{y})\big) \\ \leq\ \ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \ \widetilde{\tau}(t_2, \widetilde{y})\big) \ \cdot\ e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h} \ + \\ & h \cdot \Big(\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \cdot \big(1 + \lfloor \widetilde{y} \rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2+h)} + \zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2 + h)\big) \ +\ \gamma_\varepsilon(\widetilde{\tau})\Big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}. \end{aligned}$$

for all $\widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ \varepsilon \in \mathcal{J}$ and $0 \leq t_1 \leq t_2 < 1,\ h \geq 0$ with $\pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y}$ and $t_1 + h$ sufficiently small (depending only on $\widetilde{\vartheta}, \widetilde{z}$). In comparison with the last modification (in [12]), the new feature is the factor $\big(1 + \lfloor \widetilde{y} \rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau})\,(t_2+h)} + \zeta_\varepsilon(\widetilde{\tau})\,(t_2+h)\big) \leq 1 + \lfloor \widetilde{y} \rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau})} + \zeta_\varepsilon(\widetilde{\tau})$ allowing the second summand on the right–hand side to share the "linear growth" of the compared element $\widetilde{y}$ (with respect to $\lfloor\cdot\rfloor_\varepsilon$). A corresponding dependence on $\lfloor \widetilde{z} \rfloor_\varepsilon$ can be regarded as part of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$.

For the subsequent definition of solution, this modification will be hardly relevant. Indeed, a transition $\widetilde{\vartheta}$ again induces a "first–order approximation" of a curve $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ at time $t \in [0,T[$ according to the following (still vague) idea: Comparing $\widetilde{x}(t + \cdot)$ with $\widetilde{\vartheta}(\cdot, \widetilde{z})$ shortly (for any test element $\widetilde{z} \in \widetilde{\mathcal{D}},\ \pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{x}(t)$), the same structural estimate ought to hold as if the factor $\widetilde{Q}_\varepsilon(\cdot,\cdot\,;\widetilde{z})$ was 0 (see Definitions 2.10, 2.12 for details).

2.1 Transitions as "elementary deformations"

Now we specify the new definition of "timed sleek transition" for the tuple that has an additional component in comparison with earlier versions (see [12] in particular): $\Big(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}\Big)$.

Definition 2.1 *A map* $\widetilde{\vartheta} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ *is called* timed sleek transition *on* $(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0,\cdot) = \mathrm{Id}_{\widetilde{\mathcal{D}} \cup \widetilde{E}}$,
2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 :\ \limsup_{h\downarrow 0}\ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[$,
$\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[$,
3. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} \quad \exists\ \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \in [0,\infty[,\quad \mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \in\,]0,1] :$
$$\limsup_{h\downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\widetilde{y}) - \gamma_\varepsilon(\widetilde{\vartheta})\,h}{h\quad \big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\big)}\right)^{+} \leq \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \qquad \forall\ 0 \leq t < \mathbb{T}_\varepsilon,\ \ \widetilde{y} \in \widetilde{E}\ \ (t + \pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y}),$$
4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) \geq 0 :\quad \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(s,\widetilde{y}),\ \widetilde{\vartheta}(t,\widetilde{y})\big) \leq \beta_\varepsilon(\widetilde{\vartheta})\cdot(t-s)\cdot(1 + \lfloor\widetilde{y}\rfloor_\varepsilon) \qquad \forall\ s < t \leq 1,\ \ \widetilde{y} \in \widetilde{E}$,
5. $\exists\ \zeta_\varepsilon(\widetilde{\vartheta}) \geq 0 :\quad \lfloor\widetilde{\vartheta}(t,\widetilde{x})\rfloor_\varepsilon \leq \lfloor\widetilde{x}\rfloor_\varepsilon \cdot e^{\zeta_\varepsilon(\widetilde{\vartheta})\,t} + \zeta_\varepsilon(\widetilde{\vartheta})\,t \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1]$,
6. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} :\quad \widetilde{\vartheta}(t,\widetilde{z}) \in \widetilde{\mathcal{D}} \qquad \forall\ t \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})]$,
7. $\limsup_{h\downarrow 0}\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}\big) \geq \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}\big) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ t \leq \mathbb{T}_\varepsilon\ \ (t + \pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y})$,
8. $\widetilde{\vartheta}\big(h,\ (t,y)\big) \in \{t+h\} \times E \subset \widetilde{E} \qquad \forall\ (t,y) \in \widetilde{E},\ \ h \in [0,1]$,
$\pi_1\,\widetilde{\vartheta}(h,\ (t,z)) \leq t + h$ *nondecreasing w.r.t.* $h \qquad \forall\ (t,z) \in \widetilde{\mathcal{D}},\ \ h \in [0,1]$.
9. $\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, i_{\widetilde{\mathcal{D}}}\,\widetilde{z})),\ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{z}))\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})$,
$\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{z})\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})$.

Remark 2.2 Conditions (4.) and (5.) provide the only new features in comparison with earlier concepts [16, 17, 12]. Roughly speaking, property (5.) bounds the "absolute magnitude" of $\widetilde{\vartheta}(\cdot,\widetilde{x})$ to uniform exponential growth for each initial element $\widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E}$. As its main advantage, we can always restrict our considerations to bounded subsets of $\widetilde{E}$ (with respect to $\lfloor\cdot\rfloor_\varepsilon$), so for example, the Euler approximations for proving existence.
Condition (4.) now allows $\vartheta(\cdot,\widetilde{y})$ that the Lipschitz constant (with respect to time) has linear growth with respect to the initial element $\widetilde{y} \in \widetilde{E}$.

The other conditions have already been explained in earlier concepts. For the purpose of a self–contained presentation here, we briefly motivate their key points:

Condition (2.) can be regarded as a weakened form of the semigroup property. It consists of two demands as $\widetilde{q}_\varepsilon$ does not have to be symmetric. Condition (3.) specifies the continuity property of $\widetilde{\vartheta}$ with respect to the initial point. In particular, the first argument of $\widetilde{q}_\varepsilon$ is restricted to elements $\widetilde{z}$ of the "test set" $\widetilde{\mathcal{D}}$ and, $\alpha_\varepsilon(\widetilde{\vartheta})$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon(\cdot) < \infty$ uniformly in some applications.

Condition (6.) guarantees that every $\widetilde{z} \in \widetilde{\mathcal{D}}$ stays in the "test set" $\widetilde{\mathcal{D}}$ for short times at least. This assumption is required because estimates using the parameter $\alpha_\varepsilon(\cdot)$ can be ensured only within this period. Further conditions on $\mathbb{T}_\varepsilon(\widetilde{\vartheta}, \cdot) > 0$ are avoidable for proving existence of solutions, but they are used for uniqueness (as in [16] and subsequent Propositions 2.14, 2.23).

Condition (7.) forms the basis for applying Gronwall's Lemma that has been extended to semicontinuous functions in [16] (see Lemma 2.6 below). Indeed, every function $\widetilde{y} : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{q}_\varepsilon(\widetilde{y}(t-h),\ \widetilde{y}(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}(t)\Big) \ \le\ \limsup_{h\downarrow 0}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{z}),\ \ \widetilde{y}(t-h)\Big).$$

for all elements $\widetilde{z} \in \widetilde{\mathcal{D}}$ with $\pi_1\, \widetilde{\vartheta}(\cdot, \widetilde{z}) \ \le\ \pi_1\, \widetilde{y}(\cdot)$ and times $t \in\ ]\,0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})]$.

Condition (8.) describes the real "time" component of $\widetilde{\vartheta}(\cdot, \widetilde{y})$. For initial elements $\widetilde{y} \in \widetilde{E}$, the time component has to be additive whereas for "test elements" $\widetilde{z} \in \widetilde{\mathcal{D}}$, the time component of $\widetilde{\vartheta}(\cdot, \widetilde{z})$ might increase more slowly.

Finally, condition (9.) provides a counterpart of the general assumption $\widetilde{q}_\varepsilon(\widetilde{z},\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}) = 0$ for each $\widetilde{z} \in \widetilde{\mathcal{D}}$. As $\widetilde{q}_\varepsilon$ does not have to be symmetric, it is required for estimating the distance between a timed sleek transition $\widetilde{\vartheta}(\cdot, \widetilde{z})$ and a timed right–hand sleek solution (see proofs of Proposition 2.22 and Lemma 2.19).

Definition 2.3
$\widetilde{\Theta}(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ denotes a set of timed sleek transitions on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ assuming

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \ :=\ \sup_{\substack{t\,<\,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z}),\ \ \widetilde{y}\,\in\widetilde{E} \\ t+\pi_1\,\widetilde{z}\ \le\ \pi_1\,\widetilde{y}}}\ \ \limsup_{h\downarrow 0}\ \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\widetilde{z}),\ \widetilde{\tau}(h,\widetilde{y})\big)\ -\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}\big)\cdot e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\cdot h}}{h\quad (1+\lfloor\widetilde{y}\rfloor_\varepsilon)}\right)^{+}$$

to be finite for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ \varepsilon \in \mathcal{J}$.

Remark 2.4 Due to the dependence on the initial "test element" of $\widetilde{\mathcal{D}}$, the triangle inequality for $\widetilde{Q}_\varepsilon(\cdot,\cdot;\widetilde{z})$ cannot be expected to hold in general. The factor $(1\ +\ \lfloor\widetilde{y}\rfloor_\varepsilon)$ in the denominator is a new feature in comparison with earlier concepts and again takes the (possible) linear growth into consideration. Its consequences for the structural estimate are specified in Proposition 2.5.

Proposition 2.5 *Let $\widetilde{\vartheta}, \widetilde{\tau} : [0,1]\times(\widetilde{\mathcal{D}}\cup\widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}}\cup\widetilde{E})$ be timed sleek transitions on $(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$. Suppose $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\widetilde{y} \in \widetilde{E}$ and $0 \le t_1 \le t_2 \le 1$, $h \ge 0$ with $\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}$, $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$, $t_2 + h \le 1$. Then,*

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1{+}h, \widetilde{z}),\ \ \widetilde{\tau}(t_2{+}h, \widetilde{y})\big)\\
\le\ \ &\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \ \widetilde{\tau}(t_2, \widetilde{y})\big)\cdot e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\ h}\ \ +\\
&h\cdot\ \Big(\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})\cdot\big(1+\lfloor\widetilde{y}\rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau})\cdot(t_2+h)} + \zeta_\varepsilon(\widetilde{\tau})\cdot(t_2+h)\big) + \gamma_\varepsilon(\widetilde{\tau})\Big)\ \Big)\cdot e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\ h}.
\end{aligned}$$

Proof is based on the subsequent version of Gronwall's Lemma for semicontinuous functions. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1+h,\widetilde{z}),\ \widetilde{\tau}(t_2+h,\widetilde{y}))$ satisfies $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ due to property (7.) of Definition 2.1.
Moreover it fulfills for any $h \in [0,1[$ with $t_1+h < \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})$

$$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k)-\varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\cdot\varphi_\varepsilon(h) + \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\cdot\big(1+\lfloor\widetilde{y}\rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau})(t_2+h)}+\zeta_\varepsilon(\widetilde{\tau})(t_2+h)\big) + \gamma_\varepsilon(\widetilde{\tau}).$$

Indeed, for all $k>0$ sufficiently small, the timed triangle inequality leads to

$$\begin{aligned}\varphi_\varepsilon(h+k) \ &\le\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h+k,\ \widetilde{z}),\quad \widetilde{\tau}(k,\ \widetilde{\tau}(t_2+h,\ \widetilde{y}))\big)\\ &\quad + \widetilde{q}_\varepsilon\big(\widetilde{\tau}(k,\widetilde{\tau}(t_2+h,\widetilde{y})),\quad \widetilde{\tau}(t_2+h+k,\quad \widetilde{y})\ \big)\\ &\le\ \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\ \big(1+\lfloor\widetilde{\tau}(t_2+h,\widetilde{y})\rfloor_\varepsilon\big) \cdot k+\varphi_\varepsilon(h)\ e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\ k} + \gamma_\varepsilon(\widetilde{\tau})\ k+o(k)\\ &\le\ \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\ \big(1+\lfloor\widetilde{y}\rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau})\cdot(t_2+h)}+\zeta_\varepsilon(\widetilde{\tau})\ (t_2+h)\big)\ k+\varphi_\varepsilon(h)\ e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\ k} + \gamma_\varepsilon(\widetilde{\tau})\ k+o(k).\end{aligned}$$

□

Lemma 2.6 (Lemma of Gronwall for semicontinuous functions [16]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R},\ f,g \in C^0([a,b[,\mathbb{R})$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned}\psi(t) &\le \limsup_{h\downarrow 0}\ \psi(t-h), && \forall\ t\in\ ]a,\ b],\\ \psi(t) &\ge \limsup_{h\downarrow 0}\ \psi(t+h), && \forall\ t\in [a,\ b[,\\ \limsup_{h\downarrow 0}\ \frac{\psi(t+h)-\psi(t)}{h} &\le f(t)\ \cdot\ \limsup_{h\downarrow 0}\ \psi(t-h)\ +\ g(t) && \forall\ t\in\ ]a,\ b[.\end{aligned}$$

Then, for every $t\in[a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \le\ \psi(a)\ \cdot\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds \qquad \textit{with}\ \ \mu(t) := \int_a^t f(s)\ ds.$$

□

Lemma 2.7 *For all* $\widetilde{\vartheta}\in\widetilde{\Theta}(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \widetilde{z}\in\widetilde{\mathcal{D}},\ \varepsilon\in\mathcal{J},\ \widetilde{Q}_\varepsilon$ *satisfies*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\vartheta};\ \widetilde{z})\ \le\ 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$$

Proof follows exactly the same track as in Proposition 2.5. Fix $\varepsilon\in\mathcal{J},\ \widetilde{z}\in\widetilde{\mathcal{D}},\ \widetilde{y}\in\widetilde{E}$ and $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})[$ with $t+\pi_1\,\widetilde{z}\ \le\ \pi_1\,\widetilde{y}$ arbitrarily. Considering now

$$\varphi_\varepsilon :\ [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})-t[\ \longrightarrow\ [0,\infty[,\quad h\longmapsto \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big),$$

the semicontinuity $\varphi_\varepsilon(h)\ \le\ \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ again results from property (7.) of Definition 2.1. Moreover we obtain for all $h\in[0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})-t[$ and small $k>0$

$$\begin{aligned}\varphi_\varepsilon(h+k)\ &\le\ \widetilde{q}_\varepsilon\big(\ \widetilde{\vartheta}(t+h+k,\ \ \widetilde{z}),\quad \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\ \widetilde{y}))\big)\\ &\quad +\ \widetilde{q}_\varepsilon\big(\ \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\quad \widetilde{y})),\quad \widetilde{\vartheta}(h+k,\quad \widetilde{y})\ \big)\\ &\le\ (\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})+o(1))\cdot k\ \cdot\ \Big(\varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\ k\Big)\ +\ \varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\ k\\ &\quad +\ (\gamma_\varepsilon(\widetilde{\vartheta})\quad +o(1))\cdot k\end{aligned}$$

and thus, $\displaystyle\limsup_{k\downarrow 0}\ \frac{\varphi_\varepsilon(h+k)-\varphi_\varepsilon(h)}{k}\ \le\ \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\ \cdot\ \varphi_\varepsilon(h)\ +\ 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$

Gronwall's Lemma 2.6 guarantees $\varphi_\varepsilon(h)\ \le\ \varphi_\varepsilon(0)\cdot e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h}\ +\ h\ \cdot\ 2\,\gamma_\varepsilon(\widetilde{\vartheta})\ \cdot\ e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h}$

and finally, $\displaystyle\limsup_{h\downarrow 0}\ \frac{\varphi_\varepsilon(h)-\varphi_\varepsilon(0)\cdot e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h}}{h\quad(1+\lfloor\widetilde{y}\rfloor_\varepsilon)}\ \le\ 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$ □

Here we briefly mention two more transparent characterizations of $\widetilde{Q}_\varepsilon(\cdot,\cdot;\cdot)$ for the case $\alpha_\varepsilon(\cdot,\cdot)=\text{const}$. The first one clarifies the link with Aubin's original definition of the distance between two transitions ϑ,τ on a metric space (M,d), i.e. $\sup_{y\in M}\ \limsup_{h\downarrow 0}\ \frac{1}{h}\cdot d\,(\vartheta(h,y),\ \tau(h,y))$ (see [7, Definition 1.1.2]). In particular, the first upper bound of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})$ does not depend on $\widetilde{z}\in\widetilde{\mathcal{D}}$.

Such a dependence on the "test element" is preserved in the second estimate stated in Lemma 2.9. It is based on comparing only the evolutions of "test elements" $\widetilde{\vartheta}(t, \widetilde{z}) \in \widetilde{\mathcal{D}}$ for $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})[$. This time however, we dispense with all additional advantages of bounded growth that $\lfloor \cdot \rfloor_\varepsilon$ might provide.

Lemma 2.8 *If* $\alpha_\varepsilon(\cdot,\cdot) = M_\varepsilon = const,$ *then*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \;\le\; \sup_{\widetilde{y} \in \widetilde{E}} \; \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{y}),\ \widetilde{\tau}(h,\widetilde{y})\big)}{h \quad (1 + \lfloor \widetilde{y} \rfloor_\varepsilon)} \right)^+ \;+\; 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})$$

for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \varepsilon \in \mathcal{J}.$

Proof. It is not restrictive to assume additionally that the right–hand side is finite. For an upper bound of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \cdot)$ on the basis of Def. 2.3, we choose $\widetilde{y} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})[$ with $t + \pi_1 \widetilde{z} \le \pi_1 \widetilde{y}$ arbitrarily and consider now the auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y}))$. Property (7.) of Definition 2.1 again implies $\varphi_\varepsilon(h) \le \limsup_{k \downarrow 0} \varphi_\varepsilon(h-k)$ and thus enables us to apply Gronwall's Lemma 2.6 later. As an abbreviation, set here $\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}) := \sup_{\widetilde{y} \in \widetilde{E}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{y}),\ \widetilde{\tau}(h,\widetilde{y})\big)}{h \quad (1 + \lfloor \widetilde{y} \rfloor_\varepsilon)} \right)^+ < \infty$.
Choosing any $h \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) - t[$ and any small $\delta > 0$, the timed triangle inequality ensures for all $k > 0$ sufficiently small (depending on t, h, δ)

$$\begin{array}{rcl}
\varphi_\varepsilon(h+k) &=& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h+k,\ \widetilde{z}),\qquad \widetilde{\tau}(h+k,\ \widetilde{y})\Big)\\
&\le& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h+k,\ \widetilde{z}),\qquad \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(t+h, \widetilde{z}))\Big)\\
&& +\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\ \ \widetilde{\vartheta}(t+h, \widetilde{z})),\qquad \widetilde{\vartheta}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y}))\Big)\\
&& +\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y})),\qquad \widetilde{\tau}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y}))\Big)\\
&& +\ \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y})),\qquad \widetilde{\tau}(h+k,\ \widetilde{y})\Big)\\
&\le& (\gamma_\varepsilon(\widetilde{\vartheta}) \quad + \delta) \cdot k\\
&& +\ (M_\varepsilon \quad + \delta) \cdot k\ \ (\varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\vartheta}) \cdot k) \quad + \quad \varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\vartheta}) \cdot k\\
&& +\ (\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\tau}) \ + \delta) \cdot k\ \ (1 + \lfloor \widetilde{\tau}(h,\ \widetilde{y}) \rfloor_\varepsilon)\\
&& +\ (\gamma_\varepsilon(\widetilde{\tau}) \quad + \delta) \cdot k
\end{array}$$

and thus,

$$\begin{array}{rl}
\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \;\le\; (M_\varepsilon + \delta) \cdot \varphi_\varepsilon(h) &+\ (\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\tau}) + \delta)\ \ (1 + \lfloor \widetilde{y} \rfloor_\varepsilon \cdot e^{\zeta_\varepsilon(\widetilde{\tau}) \cdot h} + \zeta_\varepsilon(\widetilde{\tau})\ h)\\
&+\ 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}) + 2\,\delta.
\end{array}$$

So Gronwall's Lemma 2.6 and $\delta \downarrow 0$ imply

$$\varphi_\varepsilon(h) \;\le\; \varphi_\varepsilon(0) \cdot e^{M_\varepsilon \cdot h} \;+\; h \cdot e^{M_\varepsilon \cdot h} \Big(\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\tau})\ \ (1 + \lfloor \widetilde{y} \rfloor_\varepsilon \cdot e^{\zeta_\varepsilon(\widetilde{\tau}) \cdot h} + \zeta_\varepsilon(\widetilde{\tau})\ h) \;+\; 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}) \Big).$$

Finally we conclude from Definition 2.3

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \;\le\; \limsup_{h \downarrow 0} \frac{\varphi_\varepsilon(h) - \varphi_\varepsilon(0) \cdot e^{M_\varepsilon \cdot h}}{h \quad (1 + \lfloor \widetilde{y} \rfloor_\varepsilon)} \;\le\; \widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\tau}) + 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}) \qquad \square$$

Lemma 2.9 *If* $\alpha_\varepsilon(\cdot,\cdot) = M_\varepsilon = const,$ *then*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \;\le\; \sup_{0 \le t < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})} \; \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{z})),\ \widetilde{\tau}(h,\ \widetilde{\vartheta}(t,\widetilde{z}))\big)}{h} \right)^+ \;+\; \gamma_\varepsilon(\widetilde{\vartheta}) + 2\,\gamma_\varepsilon(\widetilde{\tau})$$

for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \varepsilon \in \mathcal{J}.$

Proof follows essentially the same track as for Lemma 2.8. Again $\widetilde{y} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})[$ with $t + \pi_1 \widetilde{z} \le \pi_1 \widetilde{y}$ are chosen arbitrarily, but now the auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y}))$ is estimated in a slightly different way:

From now on, let $\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$ denote the supremum on the right–hand side of the claim. Fixing any $h \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) - t[$ and any small $\delta > 0$, we obtain for all $k > 0$ sufficiently small (depending on t, h, δ)

$$\begin{array}{rcl} \varphi_\varepsilon(h+k) & \leq & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h+k,\ \widetilde{z}), \qquad \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(t+h, \widetilde{z}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\ \ \widetilde{\vartheta}(t+h, \widetilde{z})), \quad \widetilde{\tau}(k,\ \widetilde{\vartheta}(t+h, \widetilde{z}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\ \ \widetilde{\vartheta}(t+h, \widetilde{z})) \quad \widetilde{\tau}(k,\ \widetilde{\tau}(h,\ \widetilde{y}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y})), \qquad \widetilde{\tau}(h+k,\ \widetilde{y})\Big) \\ & \leq & (\gamma_\varepsilon(\widetilde{\vartheta}) \qquad +\delta) \cdot k \\ & & +\ (\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \delta) \cdot k \\ & & +\ (M_\varepsilon \qquad +\delta) \cdot k\ \ (\varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\tau}) \cdot k) \quad + \quad \varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\tau}) \cdot k \\ & & +\ (\gamma_\varepsilon(\widetilde{\tau}) \qquad +\delta) \cdot k \end{array}$$

and thus, $\displaystyle\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \leq (M_\varepsilon + \delta) \cdot \varphi_\varepsilon(h) + \widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \gamma_\varepsilon(\widetilde{\vartheta}) + 2\, \gamma_\varepsilon(\widetilde{\tau}) + 3\, \delta.$

Again Gronwall's Lemma 2.6 and $\delta \downarrow 0$ lead to

$$\varphi_\varepsilon(h) \ \leq\ \varphi_\varepsilon(0) \cdot e^{M_\varepsilon \cdot h} \ +\ h \cdot e^{M_\varepsilon \cdot h} \Big(\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \ +\ \gamma_\varepsilon(\widetilde{\vartheta}) + 2\, \gamma_\varepsilon(\widetilde{\tau})\Big)$$

and, the claim results from Definition 2.3 due to $\lfloor \cdot \rfloor_\varepsilon \geq 0$. □

2.2 Defining timed right–hand sleek solutions

Now timed sleek transitions are used for specifying "first–order approximations" of a curve $\widetilde{x}(\cdot) : [0, T[\longrightarrow \widetilde{E}$. Proposition 2.5 has just provided the structural estimate that we want to preserve. So replacing $\widetilde{\tau}(t_2 + \cdot, \widetilde{y})$ formally by $\widetilde{x}(t + \cdot)$, a sleek transition $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$ can be interpreted as "first–order approximation" of $\widetilde{x}(\cdot)$ at time $t \in [0, T[$ if for all test elements $\widetilde{z} \in \widetilde{\mathcal{D}}$ ($\pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$), infinitesimal $h > 0$ satisfy the corresponding estimate with the "distance term" $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$ equal to 0. This notion motivates the following definition:

Definition 2.10 *The curve* $\widetilde{x}(\cdot) : [0, T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand sleek primitive *of a map* $\widetilde{\vartheta}(\cdot) : [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ \ t \in [0, T[\qquad \forall\ \ \widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$:
   $\exists\ \ \widehat{\alpha}_\varepsilon(t, \widetilde{z}) \geq \alpha_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z}) \quad \exists\ \ \widehat{\gamma}_\varepsilon(t, \widetilde{z}) \geq \gamma_\varepsilon(\widetilde{\vartheta}(t))$:
   $\displaystyle\limsup_{h \downarrow 0} \frac{1}{h} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s+h, \widetilde{z}),\ \widetilde{x}(t+h)\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s, \widetilde{z}),\ \widetilde{x}(t)\Big) \cdot e^{\widehat{\alpha}_\varepsilon(t, \widetilde{z}) \cdot h}\Big) \leq \widehat{\gamma}_\varepsilon(t, \widetilde{z})$
   for every $s \in \big[0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z})\big[$ *with* $s + \pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$,
2. $\widetilde{x}(\cdot)$ *is uniformly continuous in time direction with respect to* $\widetilde{q}_\varepsilon$,
   i.e. there is $\omega_\varepsilon(\widetilde{x}, \cdot) :]0, T[\longrightarrow [0, \infty[$ *such that* $\displaystyle\limsup_{h \downarrow 0}\ \omega_\varepsilon(\widetilde{x}, h) = 0$,
   $\widetilde{q}_\varepsilon\big(\widetilde{x}(s), \widetilde{x}(t)\big) \leq \omega_\varepsilon(\widetilde{x}, t - s) \qquad$ *for* $0 \leq s < t < T$.
3. $\pi_1 \widetilde{x}(t) \ =\ t\ +\ \pi_1 \widetilde{x}(0) \qquad$ *for all* $t \in [0, T[$.

Remark 2.11 Considering $\widetilde{\vartheta}(\cdot, \widetilde{y}) : [0, 1[\longrightarrow \widetilde{E}$ for any $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor \cdot \rfloor_\varepsilon))$ and $\widetilde{y} \in \widetilde{E}$ fixed, timed sleek transitions induce their own sleek primitives — as a direct consequence of Definition 2.1, Proposition 2.5 and Lemma 2.7. Correspondingly, each piecewise constant function $\widetilde{\vartheta} : [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$ has a timed right–hand sleek primitive that is defined piecewise as well.

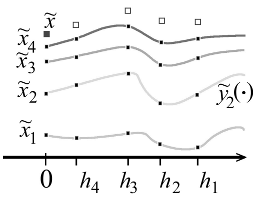
Definition 2.12 *For* $\widetilde{f} : \widetilde{E} \times [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *given, a map* $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ *is a* timed right–hand sleek solution *to the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand sleek primitive of* $\widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0, T[$.

2.3 Existence of solutions due to timed transitional compactness

In particular, the new aspect of the additional term $(1 + \lfloor\cdot\rfloor_\varepsilon)$ does not have any consequence for verifying the solution property if we suppose the parameter $\zeta_\varepsilon(\widetilde{f}(\widetilde{x}, t))$ to be uniformly bounded for all $\widetilde{x} \in \widetilde{E}$, $t \in [0, T[$. Briefly, all Euler approximations satisfy an obvious a priori estimate with respect to $\lfloor\cdot\rfloor_\varepsilon$ (for each given initial element in $\widetilde{E}$) as stated in Lemma 2.16. So the proofs of [12] lead directly to the following Proposition 2.14 about existence:

Definition 2.13 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$.*
$\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}\Big)$ *is called* timed transitionally compact *if it fulfills the following condition:*
Let $(\widetilde{x}_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$, $(\widetilde{\vartheta}_n(\cdot))_{n\in\mathbb{N}}$ be any sequences in $\widetilde{E}$, $]0,1[$ and $[0,1] \longrightarrow \widetilde{\Theta}$ respectively satisfying
1.) $\sup_n |\pi_1\, \widetilde{x}_n| < \infty$, $\sup_n \lfloor\widetilde{x}_n\rfloor_\varepsilon < \infty$, $\sup_n \widetilde{q}_\varepsilon(\widetilde{x}_0, \widetilde{x}_n) < \infty$ *for each $\varepsilon \in \mathcal{J}$,*
2.) $h_j \longrightarrow 0$ *for $j \to \infty$,*
3.) $\sup_{n,t} \beta(\widetilde{\vartheta}_n(t)) < \infty$,
4.) each $\widetilde{\vartheta}_n(\cdot) : [0,1] \longrightarrow \widetilde{\Theta}$ is piecewise constant, i.e. for each $n \in \mathbb{N}$, there exists a finite partition $0 = s_{n,0} < s_{n,1} < \ldots < s_{n,k_n} = 1$ such that $\vartheta_n(\cdot)$ is constant in each $[s_{n,i}, s_{n,i+1}[$.
For each $n \in \mathbb{N}$, define the function $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{y}_n(0) := \widetilde{x}_n$ in the piecewise way as $\widetilde{y}_n(t) := \widetilde{\vartheta}_n(s_{n,i})\,(t - s_{n,i},\ \widetilde{y}_n(s_{n,i}))$ for all $t \in]s_{n,i},\ s_{n,i+1}]$.
Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\begin{array}{llcl}
 & \lim_{k\to\infty} \pi_1\, \widetilde{x}_{n_k} & = & \pi_1\, \widetilde{x}, \\
 & \limsup_{k\longrightarrow\infty} \lfloor\widetilde{x}_{n_k}\rfloor_\varepsilon & \geq & \lfloor\widetilde{x}\rfloor_\varepsilon, \\
 & \limsup_{k\longrightarrow\infty} \widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\ \widetilde{x}) & = & 0, \\
\limsup_{j\longrightarrow\infty} & \sup_{k\geq j} \widetilde{q}_\varepsilon(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) & = & 0.
\end{array}$$



A nonempty subset $\widetilde{F} \subset \widetilde{E}$ is called timed transitionally compact in $(\widetilde{E},\ (\widetilde{q}_\varepsilon),\ (\lfloor\cdot\rfloor_\varepsilon),\ \widetilde{\Theta})$ *if the same property holds for any sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x} \in \widetilde{F}$ is* not *required).*

Proposition 2.14 (Existence of timed right–hand sleek solutions)
Assume that $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon)))$ is timed transitionally compact. Let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ fulfill for every $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z}) := \sup_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2} \{\alpha_\varepsilon\big(\widetilde{f}(\widetilde{y}_1, t_1),\ \widetilde{f}(\widetilde{y}_2, t_2)(h, \widetilde{z})\big) \mid 0 \leq h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2, t_2), \widetilde{z})\} < \infty,$

2. $c_\varepsilon := \sup_{t,\widetilde{y}} \beta_\varepsilon(\widetilde{f}(\widetilde{y}, t)) < \infty,$

3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{y}} \gamma_\varepsilon(\widetilde{f}(\widetilde{y}, t)) \leq R_\varepsilon < \infty,$

4. $g_\varepsilon := \sup_{t,\widetilde{y}} \zeta_\varepsilon(\widetilde{f}(\widetilde{y}, t)) < \infty,$

5. $\exists\ \widehat{\omega}_\varepsilon(\cdot) : \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}_1, t_1),\ \widetilde{f}(\widetilde{y}_2, t_2);\ \widetilde{z}\Big) \leq R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{y}_1, \widetilde{y}_2) + t_2 - t_1\Big)$
for all $0 \leq t_1 \leq t_2 \leq T$ and $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{E}$ $(\pi_1\, \widetilde{y}_1 \leq \pi_1\, \widetilde{y}_2)$,
$\widehat{\omega}_\varepsilon(\cdot) \geq 0$ nondecreasing, $\limsup_{s\downarrow 0} \widehat{\omega}_\varepsilon(s) = 0.$

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand sleek solution $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ to the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 2.15 The basic notion of its proof is easy to sketch. Indeed, we again start with Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$,

$$\begin{array}{llllll} h_n & := \frac{T}{2^n}, & t_n^j & := j\, h_n & & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j)\, \big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\big) & & & & \text{for } t \in\,]t_n^j,\, t_n^{j+1}],\ j \le 2^n. \end{array}$$

Assumption (4.) and subsequent Lemma 2.16 imply the uniform estimate

$$\lfloor \widetilde{x}_n(t) \rfloor_\varepsilon \;\le\; (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\; e^{g_\varepsilon\, T} \qquad \text{for } n \in \mathbb{N},\ t \in [0,T[.$$

As a key point, it lays the basis for applying earlier conclusions to the "bounded" subset $\{\lfloor \cdot \rfloor_\varepsilon \le (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T}\} \subset \widetilde{E}$ (see [12] and references there). Roughly speaking, moduli of continuity and the distances like $\widetilde{Q}_\varepsilon(\,\cdot\,,\,\cdot\,;\widetilde{z})$ occurring now just have to be multiplied by $(\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T}$ (depending on the initial element) before applying former results. This is a basic idea for verifying most of the subsequent propositions in § 2. It has already been mentioned just briefly in the example of semilinear evolution equations in a reflexive Banach space, see [17].
With respect to the existence in Proposition 2.14, we proceed with the Euler approximations: As $\mathcal{J}$ is assumed to be countable, Cantor diagonal construction in combination with timed transitional compactness leads to a function $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ satisfying: For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on ε, j) and $N_j \in \mathbb{N}$ (depending on ε, j, K_j) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{N_j}(s - 2\, h_{K_j}),\;\; \widetilde{x}(s)\,\big) & \le \;\; \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\;\; \widetilde{x}_{N_j}(t + 2\, h_{K_j})\,\big) & \le \;\; \frac{1}{j} \\ \qquad\qquad\qquad \lfloor \widetilde{x}(s) \rfloor_\varepsilon & \le \;\; (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T} \end{cases}$$

for every $s,t \in\, [0,T[$. Due to subsequent Proposition 2.18 applied to for $\widetilde{x}_{N_j}(\,\cdot\, + 2\, h_{N_j} + 2\, h_{K_j})$, the limit function $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution. (For further details, see the proof of Prop. 36 in [16] or the detailed proof of Proposition 2.17 postponed to the end of this subsection 2.3.)

Lemma 2.16 *Under the assumptions of Proposition 2.14, the Euler approximations* $\widetilde{x}_n(\cdot)$, $n \in \mathbb{N}$, *defined in Remark 2.15 satisfy at each time* $t \in [0,\, T{+}h_n]$

$$\lfloor \widetilde{x}_n(t) \rfloor_\varepsilon \;\le\; \lfloor \widetilde{x}_0 \rfloor_\varepsilon\;\; e^{g_\varepsilon\, t} \;+\; g_\varepsilon \cdot (t + h_n) \cdot e^{g_\varepsilon\, t}.$$

Proof results from condition (5.) of Definition 2.1. Indeed, for every $j \in \{0 \ldots 2^n\}$ and $t \in\,]t_n^j,\, t_n^{j+1}]$,

$$\begin{array}{lll} \lfloor \widetilde{x}_n(t) \rfloor_\varepsilon & \le & \lfloor \widetilde{x}_n(t_n^j) \rfloor_\varepsilon \cdot e^{g_\varepsilon \cdot (t - t_n^j)} \;\;+\;\; g_\varepsilon \cdot (t - t_n^j) \\ & \le & \lfloor \widetilde{x}_n(t_n^j) \rfloor_\varepsilon \cdot e^{g_\varepsilon\, h_n} \qquad\;\; +\;\; g_\varepsilon \cdot h_n. \end{array}$$

By means of induction with respect to j, we obtain (for each $t \in\,]t_n^j,\, t_n^{j+1}]$ again)

$$\begin{array}{lll} \lfloor \widetilde{x}_n(t) \rfloor_\varepsilon & \le & \lfloor \widetilde{x}_0 \rfloor_\varepsilon\;\; e^{g_\varepsilon\, t} \;\;+\;\; \displaystyle\sum_{k=0}^{j} \; g_\varepsilon\; h_n\;\; e^{g_\varepsilon \cdot (j-k)\, h_n} \\ & \le & \lfloor \widetilde{x}_0 \rfloor_\varepsilon\;\; e^{g_\varepsilon\, t} \;\;+\;\; g_\varepsilon\; (t{+}h_n)\;\; e^{g_\varepsilon\, t}. \end{array} \qquad \square$$

Some examples show, however, that the continuity assumption of Proposition 2.14 might be difficult to verify – particularly if $\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{y}_1, t_1),\; \widetilde{f}(\widetilde{y}_2, t_2);\;\; \widetilde{z}\big)$ can be estimated just independently of $\widetilde{z}, \varepsilon$. So we present an alternative hypothesis motivated by the notion to exploit information about distances with respect to *all* $\varepsilon \in \mathcal{J}$ simultaneously.

Proposition 2.17 *Assume that* $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon)))$ *is timed transitionally compact. Let* $\widetilde{f} : \widetilde{E}\times[0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon))$ *fulfill for every* $\varepsilon\in\mathcal{J},\ \widetilde{z}\in\widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z}) := \sup_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2} \{\alpha_\varepsilon\big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)(h,\widetilde{z})\big) \mid 0\le h< \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2,t_2),\widetilde{z})\} < \infty,$
2. $c_\varepsilon := \sup_{t,\widetilde{y}} \beta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$
3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{y}} \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \le R_\varepsilon < \infty,$
4. $g_\varepsilon := \sup_{t,\widetilde{y}} \zeta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$
5. $\limsup_{j\longrightarrow\infty} \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2^j,t_2^j);\ \widetilde{z}\Big) \le R_\varepsilon$
   for any $t_1\in[0,T],\ \widetilde{y}_1\in\widetilde{E}$ *and sequences* $(t_2^j)_{j\in\mathbb{N}}$ *in* $[0,T],\ (\widetilde{y}_2^j)_{j\in\mathbb{N}}$ *in* $\widetilde{E}$ *satisfying*
   $t_1\le t_2^j,\ \pi_1\,\widetilde{y}_1\le\pi_1\,\widetilde{y}_2^j \quad \forall\, j\in\mathbb{N}, \qquad |t_1-t_2^j| \stackrel{j\to\infty}{\longrightarrow} 0, \quad \widetilde{q}_{\varepsilon'}(\widetilde{y}_1,\ \widetilde{y}_2^j) \stackrel{j\to\infty}{\longrightarrow} 0 \quad \forall\,\varepsilon'\in\mathcal{J}.$

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand sleek solution* $\widetilde{x} : [0,T[\longrightarrow\widetilde{E}$ *to the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *with* $\widetilde{x}(0)=\widetilde{x}_0$.

For the purpose of a self–contained presentation here, we postpone a detailed proof of Proposition 2.17 till the end of this subsection 2.3. Using the same Euler approximations as in Remark 2.15, the key tool now is to guarantee that the "limit function" $\widetilde{x}(\cdot) : [0,T[\longrightarrow\widetilde{E}$ is a timed right–hand sleek solution of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$. So we need a new convergence theorem with an adapted continuity assumption:

Proposition 2.18 (Convergence Theorem) *For each* $\varepsilon\in\mathcal{J}$ *and* $\widetilde{z}\in\widetilde{\mathcal{D}}$, *assume the following properties of*

$$\begin{array}{lll} \widetilde{f}_m,\ \widetilde{f}: & \widetilde{E}\times[0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},(\lfloor\cdot\rfloor_\varepsilon)) & (m\in\mathbb{N}) \\ \widetilde{x}_m,\ \widetilde{x}: & [0,T[\longrightarrow \widetilde{E}\ : & \end{array}$$

1. $M_\varepsilon(\widetilde{z}) := \sup_{m,t,\widetilde{y}} \{\,\alpha_\varepsilon\big(\widetilde{f}_m(\widetilde{y},t),\ \widetilde{f}(\widetilde{x}(t),t)(h,\widetilde{z})\big) \mid 0\le h<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})\} < \infty,$
   $R_\varepsilon \ge \sup_{m,t,\widetilde{y},\widetilde{z},h} \{\,\widehat{\gamma}_\varepsilon(t,\ \widetilde{f}_m(\widetilde{x}_m,\cdot),\ \widetilde{f}(\widetilde{x}(t),t)(h,\widetilde{z})),\ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y},t)),\ \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t))\,\},$
   $g_\varepsilon \ge \sup_{m,t,\widetilde{y}} \{\zeta_\varepsilon(\widetilde{f}_m(\widetilde{y},t))\}$
2. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ni \widetilde{f}_m(\widetilde{x}_m(\cdot),\cdot)$ *in* $[0,T[$, *(in the sense of Definition 2.12)*
3. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m,h) < \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*),* $\limsup_{h\downarrow 0} \widehat{\omega}_\varepsilon(h) = 0,$
   $n_\varepsilon := \sup_{m,t} \lfloor\widetilde{x}_m(t)\rfloor_\varepsilon < \infty,$
4. $\forall\ 0\le t_1<t_2<T \quad \exists\ (m_j)_{j\in\mathbb{N}},\ (\delta'_j)_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}}$ *with* $m_j\nearrow\infty,\ \delta_j,\delta'_j\searrow 0$
   (i) $\limsup_{j\longrightarrow\infty} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{y}_j,s_j);\ \widetilde{z}\big) \le R_\varepsilon,$
   for any sequences $s_j\downarrow t_1$ *and* $(\widetilde{y}_j)_{j\in\mathbb{N}}$ *in* $\widetilde{E}$ *with* $\pi_1\,\widetilde{y}_j\searrow\widetilde{x}(t_1),\ \widetilde{q}_{\varepsilon'}(\widetilde{x}(t_1),\widetilde{y}_j) \stackrel{j\to\infty}{\longrightarrow} 0\ \forall\,\varepsilon'$
   (ii) $\widetilde{q}_{\varepsilon'}\big(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1+\delta'_j)\big)\longrightarrow 0 \quad \forall\,\varepsilon'\in\mathcal{J}, \qquad \pi_1\,\widetilde{x}_{m_j}(t_1+\delta'_j)\searrow\pi_1\,\widetilde{x}(t_1),$
   (iii) $\widetilde{q}_\varepsilon\ \big(\widetilde{x}_{m_j}(t_2-\delta_j),\ \widetilde{x}(t_2)\big)\longrightarrow 0, \qquad \pi_1\,\widetilde{x}_{m_j}(t_2-\delta_j)\nearrow\pi_1\,\widetilde{x}(t_2).$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot)\ni\widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *in* $[0,T[$.

Proof of Proposition 2.18. The uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (3.):
Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\big(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2)\big) \le \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $t_1 < t_2 < T$.
Let $\varepsilon \in \mathcal{J}$, $0 \le t_1 < t_2 < T$ be arbitrary and choose $(\delta'_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, for $t_1,\ t_2 \in [0,T]$ according to condition (5.ii), (5.iii). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta'_j < t_2 - \delta_j$ and so,

$$\begin{aligned}\widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}(t_2)\big) &\le \widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1{+}\delta'_j)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_1{+}\delta'_j),\ \widetilde{x}_{m_j}(t_2{-}\delta_j)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_2{-}\delta_j),\ \widetilde{x}(t_2)\big)\\ &\le o(1) + \widehat{\omega}_\varepsilon(t_2 - t_1) + o(1) \qquad \text{for } j \longrightarrow \infty.\end{aligned}$$

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$ and $t \in [0,T[$, $0 \le s < s+h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{z})$ be chosen arbitrarily with $s + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$. Condition (7.) of Definition 2.1 guarantees for all $k \in\,]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) \le \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h{-}k,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) + h^2.$$

According to assumptions (4.ii) – (4.iii), there exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j{+}\delta'_j < k$ and

$$\begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t{+}h{-}\delta_j),\ \ \widetilde{x}(t{+}h)\big) \longrightarrow 0, & \pi_1\,\widetilde{x}_{m_j}(t{+}h{-}\delta_j) \nearrow \pi_1\,\widetilde{x}(t{+}h),\\ \widetilde{q}_{\varepsilon'}\big(\widetilde{x}(t),\ \ \widetilde{x}_{m_j}(t{+}\delta'_j)\big) \longrightarrow 0 \quad \forall\,\varepsilon' \in \mathcal{J}, & \pi_1\,\widetilde{x}_{m_j}(t{+}\delta'_j) \searrow \pi_1\,\widetilde{x}(t).\end{cases}$$

Now subsequent Lemma 2.19 implies for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \widetilde{z}, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big)\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h{-}k,\ \widetilde{z}),\ \ \widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k)\big)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k),\ \ \widetilde{x}_{m_j}(t{+}h - \delta_j)\big)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t{+}h - \delta_j),\ \ \widetilde{x}(t{+}h)\big) + h^2\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}_{m_j}(t{+}\delta'_j)\big)\cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} +\\
&+ \int_0^{h-k} e^{M_\varepsilon(\widetilde{z})\cdot(h-k-\sigma)}\,\Big(\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot(1{+}\lfloor\widetilde{z}\rfloor_\varepsilon\,e^{g_\varepsilon h}{+}g_\varepsilon\,h) + 3\,R_\varepsilon\Big)\,d\sigma\\
&+ \widehat{\omega}_\varepsilon(k - \delta_j - \delta'_j)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t{+}h - \delta_j),\ \ \widetilde{x}(t{+}h)\big) + h^2\\
\le\ & \Big(\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)) + \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t{+}\delta'_j)\big)\Big)\cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} +\\
&+ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\,d\sigma\cdot(1{+}\lfloor\widetilde{z}\rfloor_\varepsilon\,e^{g_\varepsilon h}{+}g_\varepsilon\,h)\\
&+ \widehat{\omega}_\varepsilon(k) + 2\,h^2 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\,R_\varepsilon\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)\big)\cdot e^{M_\varepsilon(\widetilde{z})\,h} + \widehat{\omega}_\varepsilon(k) + 3\,h^2 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\,R_\varepsilon\\
&+ h\ \ e^{M_\varepsilon(\widetilde{z})\cdot h}\sup_{0\le\sigma\le h}\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot(1{+}\lfloor\widetilde{z}\rfloor_\varepsilon\,e^{g_\varepsilon h}{+}g_\varepsilon\,h)
\end{aligned}$$

First $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big)\\
\le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\cdot e^{M_\varepsilon(\widetilde{z})\,h} + 0 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\,(R_\varepsilon + h)\\
&+ h\ e^{M_\varepsilon(\widetilde{z})\,h}\cdot\limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot(1{+}\lfloor\widetilde{z}\rfloor_\varepsilon\,e^{g_\varepsilon h}{+}g_\varepsilon\,h).
\end{aligned}$$

Finally, assumptions (4.i),(4.ii) and the equi–continuity of $(\widetilde{x}_m)$ ensure indirectly

$$\limsup_{h\downarrow 0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big) \le R_\varepsilon$$

and thus,

$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\widetilde{z}),\ \widetilde{x}(t{+}h)\big) - \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\cdot e^{M_\varepsilon(\widetilde{z})\,h}\Big) \le \mathrm{const}(\varepsilon,\widetilde{z})\cdot R_\varepsilon.$$

□

This lemma extends the structural estimate of Proposition 2.5 to the comparison between a test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ (evolving along a fixed transition $\widetilde{\psi}$) and a timed sleek primitive $\widetilde{x}(\cdot)$:

Lemma 2.19 *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $t_1 \in [0,1[$, $t_2 \in [0,T[$, $\widetilde{z} \in \widetilde{\mathcal{D}}$.
Let $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *be a timed sleek primitive of* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ *such that for each* $\varepsilon \in \mathcal{J}$, $t \in [0,T[$, *their parameters fulfill*

$$\wedge \left\{ \begin{array}{rcl} \sup\limits_{0 \le s \le \min\{t, \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})\}} \widehat{\alpha}_\varepsilon(t, \widetilde{\psi}(s,\widetilde{z})) & \le & M_\varepsilon(t), \\ \sup\limits_{0 \le s \le \min\{t, \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})\}} \widehat{\gamma}_\varepsilon(t,\ \widetilde{\psi}(s,\widetilde{z})) & \le & R_\varepsilon(t), \\ \widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t);\ \widetilde{z}) & \le & c_\varepsilon(t) \end{array} \right.$$

with upper semicontinuous M_ε, R_ε, $c_\varepsilon : [0,T[\longrightarrow [0,\infty[$. *Set* $\mu_\varepsilon(h) := \displaystyle\int_{t_2}^{t_2+h} M_\varepsilon(s)\, ds$.
Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0,T[$ *with* $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$, $t_2 + h < T$, $t_1 + \pi_1\, \widetilde{z} \le \pi_1\, \widetilde{x}(t_2)$

$$\begin{array}{l} \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t_2{+}h)\big) \\ \le\ \ \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1,\widetilde{z}),\ \widetilde{x}(t_2)\big) \cdot e^{\mu_\varepsilon(h)} \ +\ \displaystyle\int_0^h e^{\mu_\varepsilon(h)-\mu_\varepsilon(s)}\ \big(c_\varepsilon(t_2{+}s)\cdot(1+\lfloor\widetilde{z}\rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\psi})\cdot(t_1+s)} + \zeta_\varepsilon(\widetilde{\psi})\cdot(t_1{+}s)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad +\ 3\, R_\varepsilon(t_2{+}s)\big)\ ds. \end{array}$$

Proof. We follow the same track as in the proof of Proposition 2.5 and consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h,\widetilde{z}),\ \widetilde{x}(t_2{+}h))$. Firstly, $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ results from condition (7.) on timed sleek transitions (Definition 2.1) and the continuity of $\widetilde{x}(\cdot)$.
Moreover we show for any $h \in [0,T[$ with $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$,

$$\limsup_{k\downarrow 0} \tfrac{\varphi_\varepsilon(h+k)\,-\,\varphi_\varepsilon(h)}{k}\ \le\ M_\varepsilon(t_2{+}h)\cdot\varphi_\varepsilon(h)\ +\ c_\varepsilon(t_2{+}h)\cdot(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon)\ +\ 3\, R_\varepsilon(t_2{+}h).$$

In particular, it implies $\varphi_\varepsilon(h) \ge \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h+k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 2.6 – after approximating $M_\varepsilon(\cdot)$, $R_\varepsilon(\cdot)$, $c_\varepsilon(\cdot)$ by continuous functions from above.

For small $k > 0$, the timed triangle inequality and Proposition 2.5 guarantee

$$\begin{array}{rll} \varphi_\varepsilon(h{+}k) & \le & \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1{+}h+k,\ \widetilde{z}), \qquad \widetilde{\vartheta}(t_2{+}h)\,\big(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\big)\big) \\ & & +\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_2{+}\,h)\,\big(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1{+}h,\widetilde{z})\big), \qquad \widetilde{\vartheta}(t_2{+}h)\,\big(k,\ \ \widetilde{\psi}(t_1{+}h,\ \widetilde{z})\big)\big) \\ & & +\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_2{+}\,h)\,\big(k,\ \ \widetilde{\psi}(t_1{+}h,\widetilde{z})\big), \qquad \widetilde{x}(t_2+h+k)\big) \\ & \le & \Big\{\widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t_2{+}h);\ \widetilde{z})\ \Big(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon\cdot e^{\zeta_\varepsilon(\widetilde{\vartheta}(t_2+h))\ k} + \zeta_\varepsilon(\widetilde{\vartheta}(t_2{+}h))\ k\Big) \\ & & \qquad\qquad +\ \widehat{\gamma}_\varepsilon(t_2{+}h,\ \widetilde{z})\Big\}\ e^{M_\varepsilon(t_2+h)\cdot k}\ k \\ & & +\qquad \gamma_\varepsilon(\widetilde{\vartheta}(t_2{+}h)) \qquad \cdot k\ +\ o(k) \\ & & +\ \varphi_\varepsilon(h)\ e^{\widehat{\alpha}_\varepsilon(t_2+h,\ \widetilde{\psi}(t_1+h,\ \widetilde{z}))\cdot k}\ +\ \widehat{\gamma}_\varepsilon(t_2{+}h,\ \widetilde{\psi}(t_1{+}h,\ \widetilde{z})) \qquad \cdot k\ +\ o(k) \\ & \le & \varphi_\varepsilon(h)\ e^{M_\varepsilon(t_2+h)\cdot k}\ +\ \big|c_\varepsilon(t)\cdot(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon) + 3\ R_\varepsilon(t)\big|_{t\,=\,t_2+h} \cdot k\ +\ o(k) \end{array}$$

since $t_1 + h + k < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z}) \le 1$ implies $\widetilde{\psi}(t_1{+}h,\widetilde{z}),\ \widetilde{\psi}(t_1{+}h{+}k,\widetilde{z}) \in \widetilde{\mathcal{D}}$. □

Proof of Proposition 2.17 is again based on Euler method for an approximating sequence $(\widetilde{x}_n(\cdot))$ and Cantor diagonal construction for its limit $\widetilde{x}(\cdot)$. For $n \in \mathbb{N}$ $(2^n > T)$ set

$$\begin{array}{lll} h_n := \frac{T}{2^n}, & t_n^j := j\ h_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) := \widetilde{x}_0, & \widetilde{x}_0(\cdot) := \widetilde{x}_0, & \\ \widetilde{x}_n(t) := \widetilde{f}(\widetilde{x}_n(t_n^j),\ t_n^j)\ \big(t - t_n^j,\ \ \widetilde{x}_n(t_n^j)\big) & & \text{for } t \in\ ]t_n^j,\ t_n^{j+1}],\ \ j \le 2^n. \end{array}$$

Since $\mathcal{J}$ is countable there is a sequence $(j_k)_{k \in \mathbb{N}}$ with $\{j_1, j_2 \ldots\} = \mathcal{J}$. Now for every $t \in]0, T[$, choose a decreasing sequence $(\delta_k(t))_{k \in \mathbb{N}}$ in $\mathbb{Q} \cdot T$ satisfying

$$0 < \delta_k(t) < \tfrac{h_k}{2}, \qquad t + \delta_k(t) < T, \qquad c_{\varepsilon_j} \cdot \delta_k(t) < h_k \qquad \text{for any } j \in \{j_1 \ldots j_k\}.$$

Then, $\widetilde{q}_{\varepsilon_j}\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t + \delta_k(t))\big) \le h_k$ for any $j \in \{j_1 \ldots j_k\}$, $k, n \in \mathbb{N}$
and so $\widetilde{q}_{\varepsilon}\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t + \delta_k(t))\big) \longrightarrow 0$ $(k \longrightarrow \infty)$ for every $\varepsilon \in \mathcal{J}$, uniformly in n.
For each $t \in]0, T[$ and any fixed $\varepsilon \in \mathcal{J}$, the timed transitional compactness provides sequences $m_k \nearrow \infty$, $n_k \nearrow \infty$ $(m_k \le n_k)$ of indices and an element $\widetilde{x}(t) \in \widetilde{E}$ (independent of ε) satisfying for every $k \in \mathbb{N}$

$$\wedge \begin{cases} \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\ \ \widetilde{x}(t)\big) \le \frac{1}{k}, \\ \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \ \widetilde{x}_{n_l}(t + \delta_{m_k}(t))\big) \le \frac{1}{k}. \end{cases}$$

(In particular, each m_k, n_k may be replaced by larger indices preserving the properties.) For arbitrary $\kappa \in \mathbb{N}$, these sequences $m_k,\ n_k \nearrow \infty$ can even be chosen in such a way that the estimates are fulfilled for the finite set of parameters $t \in Q_\kappa := \ ]0, T[\ \cap \mathbb{N} \cdot h_\kappa$ and $\varepsilon \in \mathcal{J}_\kappa := \{\varepsilon_{j_1}, \varepsilon_{j_2} \ldots \varepsilon_{j_\kappa}\} \subset \mathcal{J}$ simultaneously.

Now the Cantor diagonal construction (with respect to the index κ) provides subsequences (again denoted by) $m_k, n_k \nearrow \infty$ such that $m_k \le n_k$,

$$\wedge \begin{cases} \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\ \ \widetilde{x}(t)\big) \le \frac{1}{k} \\ \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}(s),\ \ \widetilde{x}_{n_l}(s + \delta_{m_k}(s))\big) \le \frac{1}{k} \end{cases}$$

for every $\kappa \in \mathbb{N}$ and all $\varepsilon \in \mathcal{J}_\kappa$, $s, t \in Q_\kappa$, $k \ge \kappa$.
In particular, $\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \le c_\varepsilon(t - s)$ for any $s, t \in Q_{\mathbb{N}} := \bigcup_\kappa Q_\kappa$ with $s < t$ and every $\varepsilon \in \mathcal{J}$.
Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k \in \mathbb{N}}$ fulfills for every $\kappa \in \mathbb{N}$ and all $t \in Q_\kappa$, $\varepsilon \in \mathcal{J}_\kappa$, $k, l \ge \kappa$

$$\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t),\ \widetilde{x}_{n_l}(t + \delta_{m_l}(t))\big) \le \tfrac{1}{k} + \tfrac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t \in]0, T[\setminus Q_{\mathbb{N}}$, we apply the timed transitional compactness to $((\widetilde{x}_{n_k}(t))_{k \in \mathbb{N}}$ and obtain a subsequence $n_{l_j} \nearrow \infty$ of indices (depending on t) and some $\widetilde{x}(t) \in \widetilde{E}$ satisfying $\forall\, \varepsilon \in \mathcal{J}$

$$\wedge \begin{cases} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\ \ \widetilde{x}(t)\big) \longrightarrow 0, \\ \sup\limits_{i \ge j} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \ \widetilde{x}_{n_{l_i}}(t + \delta_{m_j}(t))\big) \longrightarrow 0 \end{cases} \qquad \text{for } j \longrightarrow \infty.$$

This implies the following convergence even uniformly in t (but not necessarily in $\varepsilon \in \mathcal{J}$)

$$\wedge \begin{cases} \limsup\limits_{\kappa \longrightarrow \infty} \ \sup\limits_{k > \kappa} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t - 2\,h_\kappa),\ \ \widetilde{x}(t)\,\big) = 0, \\ \limsup\limits_{\kappa \longrightarrow \infty} \ \sup\limits_{k > \kappa} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \ \widetilde{x}_{n_k}(t + 2\,h_\kappa)\,\big) = 0. \end{cases} \qquad (*)$$

Indeed, for $\kappa \in \mathbb{N}$ fixed arbitrarily and any $t \in]0, T[$, there exists $s = s(t, \kappa) \in Q_\kappa$ with

$$t - 2\,h_\kappa < s \le t - h_\kappa \qquad \text{and} \qquad \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s),\ \widetilde{x}_{n_l}(s + \delta_{m_l}(s))\big) \le \tfrac{1}{k} + \tfrac{1}{l} \qquad \text{for all } k, l \ge \kappa.$$

So for any $k, l_j \ge \kappa$, we conclude from $\delta_{m_{l_j}}(\cdot) < \frac{1}{2}\,h_{m_{l_j}} \le \frac{1}{2}\,h_{l_j} \le \frac{1}{2}\,h_\kappa$

$$\begin{aligned} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t - 2\,h_\kappa),\ \widetilde{x}(t)\big) \le \ & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t - 2\,h_\kappa),\ \ \widetilde{x}_{n_k}(s)\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s),\ \ \widetilde{x}_{n_{l_j}}(s + \delta_{m_{l_j}}(s))\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(s + \delta_{m_{l_j}}(s)),\ \ \widetilde{x}_{n_{l_j}}(t)\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\ \ \widetilde{x}(t)\big) \\ \le \ & c_\varepsilon(h_\kappa) + \tfrac{1}{k} + \tfrac{1}{l_j} + c_\varepsilon(2\,h_\kappa) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t), \widetilde{x}(t)\big) \end{aligned}$$

and $j \longrightarrow \infty$ leads to the estimate $\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t - 2\,h_\kappa),\ \widetilde{x}(t)\big) \le 2\,c_\varepsilon(2\,h_\kappa) + \frac{2}{\kappa}$.
The proof of $\limsup\limits_{\kappa \longrightarrow \infty} \sup\limits_{k > \kappa} \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t{+}2\,h_\kappa)\,\big) = 0$ is analogous.

We reformulate the convergence property (∗) in the following notation: For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exists some $K_{\varepsilon,j} \in \mathbb{N}$ satisfying $K_{\varepsilon,j} > K_{\varepsilon,\,j-1}$ and for all $s, t \in [0,T[,\ \ k \geq \kappa \geq K_{\varepsilon,j}$,

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s - 2\,h_\kappa), \quad \widetilde{x}(s)\,\big) & \leq \ \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t), \quad \widetilde{x}_{n_k}(t + 2\,h_\kappa)\,\big) & \leq \ \frac{1}{j}\,. \end{cases}$$

Now Cantor diagonal construction provides a sequence $K_i \nearrow \infty$ with the corresponding property independent of $\varepsilon \in \mathcal{J}$ (in addition), i.e. for each $i \in \mathbb{N}$, the index $K_i \in \mathbb{N}$ has to satisfy for all $s, t \in [0,T[,\ \ \varepsilon \in \mathcal{J}_i \stackrel{\text{Def.}}{=} \{\varepsilon_{j_1} \dots \varepsilon_{j_i}\} \subset \mathcal{J},\ \ k \geq \kappa \geq K_i$

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s - 2\,h_\kappa), \quad \widetilde{x}(s)\,\big) & \leq \ \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t), \quad \widetilde{x}_{n_k}(t + 2\,h_\kappa)\,\big) & \leq \ \frac{1}{j}\,. \end{cases}$$

Convergence Theorem 2.18 states that $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution to the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}, \cdot)$.

Indeed, set $N_j := n_{K_j}$ as an abbreviation. Define $\widetilde{g}_j : (\widetilde{y}, t) \longmapsto \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j} + 2\,h_{K_j}),\ t^{a+2}_{N_j} + 2\,h_{K_j}\big)$ for $t^{a}_{N_j} \leq t < t^{a+1}_{N_j}$ and consider the sequence $t \longmapsto \widetilde{x}_{N_j}(t + 2\,h_{N_j} + 2\,h_{K_j})$ of solutions.

Obviously conditions (1.), (2.), (3.) of Proposition 2.18 result from the hypotheses here. For verifying its assumption (4.), we benefit from the convergence properties of the subsequence $(\widetilde{x}_{N_j})_{j \in \mathbb{N}}$ mentioned before. It ensures that for any $\widetilde{y} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$ and $s, t \in [0,T[$ (with $t^{a}_{N_j} \leq t < t^{a+1}_{N_j}$),

$$\begin{array}{lll} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(s),\, s),\ \widetilde{g}_j(\widetilde{y},\, t);\ \widetilde{z}\big) & = & \widetilde{Q}_\varepsilon\big(\widetilde{f}\,(\widetilde{x}(s),\, s)\,,\ \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j} + 2\,h_{K_j}),\ t^{a+2}_{N_j} + 2\,h_{K_j}\big);\ \widetilde{z}\big) \\ \text{and,}\ \ \widetilde{q}_{\varepsilon'}\big(\widetilde{x}(s),\ \widetilde{x}_{N_j}(t^{a+2}_{N_j} + 2\,h_{K_j})\big) & \leq & \widetilde{q}_{\varepsilon'}\big(\widetilde{x}(s),\ \widetilde{x}_{N_j}(s + 2\,h_{K_j})\big) \ + \ c_{\varepsilon'}(t - s + 2\,h_{N_j}) \longrightarrow 0 \quad \forall\, \varepsilon' \in \mathcal{J} \\ \big|s \ - \ (t^{a+2}_{N_j} + 2\,h_{K_j})\big| & \leq & |s - t| \ + \ 2\,h_{N_j} \ + \ 2\,h_{K_j} \qquad\qquad \longrightarrow 0 \end{array}$$

for $t \downarrow s,\ \ j \longrightarrow \infty$. So the last (missing) assumption (4.i) of Convergence Theorem 2.18 results directly from hypothesis (5.) and, the proof of Proposition 2.17 is finished. □

2.4 Introducing "weak" transitional compactness

Now the example of § 3 demonstrates the key role of sequential compactness. In fact, it might be very difficult (or even impossible) to take the convergence with respect to each $\widetilde{q}_\varepsilon$ into consideration simultaneously. Thus, we weaken previous definitions of "timed transitionally compact" – following an idea that has already been initiated in [12, Definition 5.3].

Usually the concepts of "weak" convergence and "weak" compactness are closely related to linear forms in a topological vector space. But linear forms do not provide an adequate starting point for extending these concepts to ostensible metric spaces. Thus, we suggest another well–known relation of linear functional analysis as motivation: In every Banach space $(X, \|\cdot\|_X)$, the norm of any element z satisfies

$$\|z\|_X \ = \ \sup\ \big\{\, y^*(z) \ \big|\ y^* : X \longrightarrow \mathbb{R} \ \text{linear, continuous,} \ \ \|y^*\|_{X^*} \leq 1 \,\big\}.$$

So the key notion we seize here is to represent each ostensible metric as supremum of (at most countably many) generalized distance functions, i.e. $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$. Considering the convergence with respect to each $\widetilde{q}_{\varepsilon,\kappa}$ (instead of $\widetilde{q}_\varepsilon$) lays the basis of "weak compactness".

In return for weaker conditions on convergence, more "structural" assumptions about each $\widetilde{q}_\varepsilon$ and a "retraction" $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{\mathcal{D}}$ are used for proving existence of solutions.

Definition 2.20 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$. Suppose $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ with (at most) countably many $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$.*
$\Big(\widetilde{E},\, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\, (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\, \widetilde{\Theta}\Big)$ is called weakly timed transitionally compact *if it fulfills:*

Let $(\widetilde{x}_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ and $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$, $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ (for each $n \in \mathbb{N}$) satisfy the assumptions of Definition 2.13. Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ such that for each $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$,

$$\begin{array}{llll} & \lim\limits_{k\to\infty} & \pi_1\, \widetilde{x}_{n_k} & = \pi_1\, \widetilde{x}, \\ & \limsup\limits_{k\longrightarrow\infty} & \lfloor \widetilde{x}_{n_k} \rfloor_\varepsilon & \geq \lfloor \widetilde{x} \rfloor_\varepsilon, \\ & \limsup\limits_{k\longrightarrow\infty} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k},\, \widetilde{x}) & = 0, \\ \limsup\limits_{j\longrightarrow\infty} & \sup\limits_{k\geq j} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x},\quad \widetilde{y}_{n_k}(h_j)) & = 0. \end{array}$$

A nonempty subset $\widetilde{F} \subset \widetilde{E}$ is called weakly timed transitionally compact in *$(\widetilde{E},\, (\widetilde{q}_\varepsilon),\, (\widetilde{q}_{\varepsilon,\kappa}),\, (\lfloor\cdot\rfloor_\varepsilon),\, \widetilde{\Theta})$ if the same property holds for any sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x} \in \widetilde{F}$ is* not *required).*

Proposition 2.21 (Existence due to weak transitional compactness)
Assume $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ with (at most) countably many $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon\in\mathcal{J}, \kappa\in\mathcal{I})$ such that each $\kappa \in \mathcal{I}$ has counterparts $\kappa_1, \kappa_2 \in \mathcal{I}$ fulfilling for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ $(\pi_1\, \widetilde{y}_j \leq \pi_1\, \widetilde{y}_{j+1})$

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\leq\; \widetilde{q}_{\varepsilon,\kappa_1}(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa_2}(\widetilde{y}_2, \widetilde{y}_3).$$

Moreover let $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ be a "retraction" in the sense that for all $\widetilde{y}, \widetilde{y}' \in \widetilde{E}$, $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\, \widetilde{y} = \widetilde{y}, \quad \widetilde{q}_{\varepsilon,\kappa}\big(i_{\widetilde{E}}\, \widetilde{y},\, i_{\widetilde{E}}\, \widetilde{y}'\big) \leq \mathit{const}(\varepsilon,\kappa) \cdot \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{y}, \widetilde{y}'\big), \quad \lfloor i_{\widetilde{E}}\, \widetilde{y} \rfloor_\varepsilon \leq \lfloor \widetilde{y} \rfloor_\varepsilon, \quad \mathbb{T}_\varepsilon(\cdot,\, i_{\widetilde{E}}\, \widetilde{y}) \geq \widehat{\mathbb{T}_\varepsilon} \in\,]0,1[.$$

Suppose $\Big(\widetilde{E},\, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\, (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\, \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))\Big)$ to be weakly timed transitionally compact. Let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ fulfill for every $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $h \in [0, \widehat{\mathbb{T}_\varepsilon}]$

1. a) $\widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}),\; \widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}_n)\big) \stackrel{n\to\infty}{\longrightarrow} 0$ *for every $\widetilde{y} \in \widetilde{E}, t \in [0,T[$ and any $(\widetilde{z}_n)_{n\in\mathbb{N}}$ in $\widetilde{\mathcal{D}}$ with* $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{z}, \widetilde{z}_n) \stackrel{n\to\infty}{\longrightarrow} 0$ *and* $\sup\limits_{n\in\mathbb{N}} \lfloor \widetilde{z}_n \rfloor_{\varepsilon'} < \infty \quad \forall\, \varepsilon', \kappa'$

b) $M_\varepsilon := \sup\limits_{t,\widetilde{y},\widetilde{z}} \; \alpha_\varepsilon\big(\widetilde{f}(\widetilde{y},t),\; \widetilde{z}\big) < \infty,$

2. $c_\varepsilon := \sup\limits_{t,\widetilde{y}} \; \beta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$

3. $\exists\; R_\varepsilon : \sup\limits_{t,\widetilde{y}} \; \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \leq R_\varepsilon < \infty,$

4. $g_\varepsilon := \sup\limits_{t,\widetilde{y}} \; \zeta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$

5. $\limsup\limits_{j\longrightarrow\infty} \; \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y},t),\; \widetilde{f}(\widetilde{y}_j,t_j);\; i_{\widetilde{E}}\, \widetilde{v}_j\Big) \leq R_\varepsilon$ *for any $t_j \searrow t$ and $\widetilde{y}, \widetilde{y}_j, \widetilde{v}_j \in \widetilde{E}$ $(j\in\mathbb{N})$*
with $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{y}, \widetilde{y}_j) \longrightarrow 0, \;\; \widetilde{q}_{\varepsilon',\kappa'}(\widetilde{y},\, \widetilde{v}_j) \longrightarrow 0, \;\; \sup\limits_j \{\lfloor y_j \rfloor_{\varepsilon'}, \lfloor v_j \rfloor_{\varepsilon'}\} < \infty \quad \forall\, \varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I},$
$\pi_1\, \widetilde{y}_j \searrow \pi_1\, \widetilde{y}, \quad \pi_1\, \widetilde{v}_j \searrow \pi_1\, \widetilde{y}.$

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand sleek solution $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ to the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\, \cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$.

The *proof* is based on the same Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$ as in Remark 2.15 – again in combination with Cantor diagonal construction (see the proof of Proposition 2.17). Due to the "weak" form of compactness (i.e. with respect to every $\widetilde{q}_{\varepsilon,\kappa}$ instead of $\widetilde{q}_\varepsilon$), we only have to modify the conclusion that the limiting function $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ is a timed right–hand sleek solution.
So an adapted convergence theorem is required. Its proof would like to follow the same track as for Proposition 2.18 – just implementing the index $\kappa \in \mathcal{I}$ (and its dependence on other indices) in addition.

The obstacle, however, is that the structural estimate (as in Proposition 2.5 or Lemma 2.19) is only available with respect to $\widetilde{q}_\varepsilon, \widetilde{Q}_\varepsilon$ and, roughly speaking, there is no obvious way "back" to some $\widetilde{q}_{\varepsilon,\kappa}$ for which the convergence of sequences is assumed. Thus, the "retraction" $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ is introduced.

Proposition 2.22 ("Weak" Convergence Theorem)
In addition to the general assumptions at the beginning of § 2, suppose $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ *such that each* $\kappa \in \mathcal{I}$ *has counterparts* $\kappa_1, \kappa_2 \in \mathcal{I}$ *fulfilling for all* $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ $(\pi_1 \widetilde{y}_j \le \pi_1 \widetilde{y}_{j+1})$

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa_1}(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa_2}(\widetilde{y}_2, \widetilde{y}_3).$$

Moreover let $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ *be a "retraction" in the sense that for all* $\widetilde{y}, \widetilde{y}' \in \widetilde{E}$, $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\, \widetilde{y} = \widetilde{y}, \quad \widetilde{q}_{\varepsilon,\kappa}\big(i_{\widetilde{E}}\, \widetilde{y},\; i_{\widetilde{E}}\, \widetilde{y}'\big) \le \mathit{const}(\varepsilon,\kappa) \cdot \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{y}, \widetilde{y}'\big), \quad \lfloor i_{\widetilde{E}}\, \widetilde{y} \rfloor_\varepsilon \le \lfloor \widetilde{y} \rfloor_\varepsilon, \quad \mathbb{T}_\varepsilon(\cdot,\, i_{\widetilde{E}}\, \widetilde{y}) \ge \widehat{\mathbb{T}_\varepsilon} \in]0,1[.$$

For each $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$, *assume the following properties of*

$$\begin{array}{rcll} \widetilde{f}_m,\;\; \widetilde{f}: & \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}\big(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}}\big) \qquad (m \in \mathbb{N}) \\ \widetilde{x}_m,\;\; \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E}\;: \end{array}$$

1. $M_\varepsilon := \sup_{m,t,\widetilde{y},\widetilde{z}} \big\{ \alpha_\varepsilon\big(\widetilde{f}_m(\widetilde{y},t),\, \widetilde{z}\big) \big\} < \infty,$

   $R_\varepsilon \ge \sup_{m,t,\widetilde{y},\widetilde{z},h} \big\{ \widehat{\gamma}_\varepsilon(t,\, \widetilde{f}_m(\widetilde{x}_m,\cdot),\, \widetilde{z}),\;\; \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y},t)),\;\; \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \big\},$

   $g_\varepsilon \ge \sup_{m,t,\widetilde{y}} \big\{ \zeta_\varepsilon(\widetilde{f}_m(\widetilde{y},t)) \big\}$

   $\widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}),\;\; \widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}_n)\big) \longrightarrow 0$ *for any* $h \le \widehat{\mathbb{T}_\varepsilon}$ *and* $(\widetilde{z}_n)_{n \in \mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *such that* $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{z}, \widetilde{z}_n) \longrightarrow 0$, $\sup_j \lfloor \widetilde{z}_j \rfloor_{\varepsilon'} < \infty \quad \forall\, \varepsilon', \kappa'$
2. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ni \widetilde{f}_m(\widetilde{x}_m(\cdot),\, \cdot)$ *in* $[0,T[$, *(in the sense of Definition 2.12)*
3. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m, h) < \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*),* $\limsup_{h \downarrow 0} \widehat{\omega}_\varepsilon(h) = 0,$

   $n_\varepsilon := \sup_{m,t} \lfloor \widetilde{x}_m(t) \rfloor_\varepsilon < \infty,$
4. $\forall\; 0 \le t_1 < t_2 < T \quad \exists\; (m_j)_{j \in \mathbb{N}},\; (\delta_j')_{j \in \mathbb{N}},\; (\delta_j)_{j \in \mathbb{N}}$ *with* $m_j \nearrow \infty$, $\delta_j, \delta_j' \searrow 0$

   (i) $\limsup_{j \longrightarrow \infty} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t_1), t_1),\;\; \widetilde{f}_{m_j}(\widetilde{y}_j, s_j);\;\; i_{\widetilde{E}}\, \widetilde{x}_{m_j}(t_1 + \delta_j')\big) \le R_\varepsilon,$
   for any sequences $s_j \downarrow t_1$, $(\widetilde{y}_j)_{j \in \mathbb{N}}$ *in* $\widetilde{E}$ *such that* $\pi_1 \widetilde{y}_j \searrow \pi_1 \widetilde{x}(t_1)$ *and* $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{x}(t_1), \widetilde{y}_j) \overset{j \to \infty}{\longrightarrow} 0,\;\; \sup_j \lfloor \widetilde{y}_j \rfloor_{\varepsilon'} < \infty \quad \forall\, \varepsilon' \in \mathcal{J},\, \kappa' \in \mathcal{I}$

   (ii) $\widetilde{q}_{\varepsilon',\kappa'}\big(\widetilde{x}(t_1),\;\; \widetilde{x}_{m_j}(t_1 + \delta_j')\big) \longrightarrow 0 \quad \forall\; \varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}, \qquad \pi_1\, \widetilde{x}_{m_j}(t_1 + \delta_j') \searrow \pi_1\, \widetilde{x}(t_1),$

   (iii) $\widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{x}_{m_j}(t_2 - \delta_j),\;\; \widetilde{x}(t_2)\big) \longrightarrow 0, \qquad \pi_1\, \widetilde{x}_{m_j}(t_2 - \delta_j) \nearrow \pi_1\, \widetilde{x}(t_2).$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\, \cdot)$ *in* $[0,T[$.

Proof of Proposition 2.22 The uniform continuity of $\widetilde{x}(\cdot)$ w.r.t. each $\widetilde{q}_\varepsilon$ results from assumption (3.): Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\big(\widetilde{x}_m(t_1),\, \widetilde{x}_m(t_2)\big) \le \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $0 \le t_1 < t_2 < T$.
Let $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$, $0 \le t_1 < t_2 < T$ be arbitrary. Choose $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{I}$ as counterparts of $\kappa \in \mathcal{I}$ due to applying the timed triangle inequality twice. Furthermore t_1, t_2 induce sequences $(\delta_j')_{j \in \mathbb{N}}$, $(\delta_j)_{j \in \mathbb{N}}$, according to condition (4.ii), (4.iii). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta_j' < t_2 - \delta_j$ and so,

$$\begin{array}{ll} & \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{x}(t_1),\, \widetilde{x}(t_2)\big) \\ \le & \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{x}(t_1),\, \widetilde{x}_{m_j}(t_1 + \delta_j')\big) + \widetilde{q}_{\varepsilon,\kappa_2}\big(\widetilde{x}_{m_j}(t_1 + \delta_j'),\, \widetilde{x}_{m_j}(t_2 - \delta_j)\big) + \widetilde{q}_{\varepsilon,\kappa_3}\big(\widetilde{x}_{m_j}(t_2 - \delta_j),\, \widetilde{x}(t_2)\big) \\ \le & o(1) \qquad\qquad\qquad\qquad + \widehat{\omega}_\varepsilon(t_2 - t_1) \qquad\qquad\qquad\qquad + o(1) \qquad \text{for } j \longrightarrow \infty. \end{array}$$

First we focus on the “test element” $i_{\widetilde{E}}\,\widetilde{x}(t) \in \widetilde{\mathcal{D}}$ for any $t \in [0,T[$ and choose $\varepsilon \in \mathcal{J}$, $0 \le h < \widehat{\mathbb{T}_\varepsilon}$ arbitrarily. Condition (7.) of Definition 2.1 ensures for all $k \in]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\;\le\;\; \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\;+\;\; \tfrac{1}{2}\,h^2$$

Now fix $\kappa \in \mathcal{I}$ (depending on ε, t, h, k) such that

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\;\le\;\; \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\;+\;\; h^2$$

and, $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathcal{I}$ denote its counterparts due to applying the timed triangle inequality three times. According to assumptions (4.i) – (4.iii), there exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\begin{cases} \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t+h-\delta_j),\;\; \widetilde{x}(t+h)\big) \;\longrightarrow\; 0, & \pi_1\,\widetilde{x}_{m_j}(t+h-\delta_j) \;\nearrow\; \pi_1\,\widetilde{x}(t+h), \\ \widetilde{q}_{\varepsilon',\kappa'}\big(\widetilde{x}(t),\;\; \widetilde{x}_{m_j}(t+\delta'_j)\big) \;\longrightarrow\; 0 \quad \forall\, \varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}, & \pi_1\,\widetilde{x}_{m_j}(t+\delta'_j) \;\searrow\; \pi_1\,\widetilde{x}(t). \end{cases}$$

Thus, Proposition 2.5 and Lemma 2.19 imply for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \kappa, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\le\; \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;+\; h^2 \\
\le\;\; & \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j))\big) \\
& + \widetilde{q}_{\varepsilon,\kappa_2}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)),\;\; \widetilde{x}_{m_j}(t+\delta'_j+h-k)\big) \\
& + \widetilde{q}_{\varepsilon,\kappa_3}\big(\widetilde{x}_{m_j}(t+\delta'_j+h-k),\qquad \widetilde{x}_{m_j}(t+h-\delta_j)\big) \\
& + \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t+h-\delta_j),\qquad \widetilde{x}(t+h)\big) \qquad +\;\; h^2 \\
\le\;\; & \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h-k,\, i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j))\big) \\
& + \int_0^{h-k} e^{M_\varepsilon\cdot(h-k-s)}\;\Big\{\widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\;\; \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\,;\;\; i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)\Big)\,\cdot \\
&\qquad\qquad \big(1+\lfloor i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)\rfloor_\varepsilon\;\; e^{g_\varepsilon\cdot(h-k)} + g_\varepsilon\cdot(h-k)\big) \;\;+\;\; 3\,R_\varepsilon\Big\}\;\; ds \\
& + \;\; \widehat{\omega}_\varepsilon(k-\delta_j-\delta'_j) \;\;+\;\; \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t+h-\delta_j),\;\; \widetilde{x}(t+h)\big) \;\;+\;\; h^2
\end{aligned}$$

Now $j \longrightarrow \infty$ (with ε, t, h, k still fixed) leads to an estimate not depending on κ any longer

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \\
\le\;\; & 0 \;+\; \widehat{\omega}_\varepsilon(k) \;+\; h^2 \;+ \\
& +\; h\;\; e^{M_\varepsilon\, h}\cdot \limsup_{j\longrightarrow\infty}\; \sup_{s\,\le\, h}\; \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\;\widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\,;\, i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)\Big)\;\big(1+n_\varepsilon\;e^{g_\varepsilon\,h} + g_\varepsilon\,h\big)
\end{aligned}$$

and, convergence assumption (4.i) implies (indirectly)

$$\limsup_{h\,\downarrow\,0}\;\; \limsup_{j\longrightarrow\infty}\;\; \sup_{0\,\le\, s\,\le\, h}\; \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\; \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\,;\;\; i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)\Big) \;\;\le\;\; R_\varepsilon.$$

So after $k \longrightarrow 0$, we obtain $\displaystyle\limsup_{h\,\downarrow\,0}\;\; \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\; \widetilde{x}(t+h)\big) \;\le\; R_\varepsilon\cdot(1+n_\varepsilon).$

For verifying the solution property of $\widetilde{x}(\cdot)$ at time t, let $\widetilde{z} \in \widetilde{\mathcal{D}}$ and $s \in \big[0, \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})\big[$ be arbitrary with $s + \pi_1\,\widetilde{z} \;\le\; \pi_1\,\widetilde{x}(t)$. Then, Proposition 2.5, Lemma 2.7 and conditions (2.), (9.) of Definition 2.1 (applied to $i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\,\widetilde{x}(t) \;=\; \widetilde{x}(t)$) imply

$$\begin{aligned}
&\limsup_{h\,\downarrow\,0}\;\; \tfrac{1}{h}\cdot\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\,\widetilde{z}),\qquad \widetilde{x}(t+h)\big) \;\;-\;\; \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\,\widetilde{z}),\;\; \widetilde{x}(t)\big)\cdot e^{M_\varepsilon\,h}\Big) \\
\le\;\; & \limsup_{h\,\downarrow\,0}\;\; \tfrac{1}{h}\cdot\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\,\widetilde{z}),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h,\,\widetilde{x}(t))\big) \quad -\, \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\,\widetilde{z}),\,\widetilde{x}(t)\big)\cdot e^{M_\varepsilon\,h} \\
&\qquad + \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\,\widetilde{x}(t)),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t))\big) \;+ \\
&\qquad + \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\, i_{\widetilde{E}}\,\widetilde{x}(t)),\;\;\; \widetilde{x}(t+h)\big)\Big) \\
\le\;\; & \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\; \widetilde{f}(\widetilde{x}(t),t);\; \widetilde{z}\big)\cdot\big(1+\lfloor\widetilde{x}(t)\rfloor_\varepsilon\big) \;\;+\;\; 3\;\gamma_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\big) \;\;+\;\; R_\varepsilon\cdot(1+n_\varepsilon) \\
\le\;\; & R_\varepsilon\;\cdot\; 6\,\big(1+n_\varepsilon+\lfloor\widetilde{x}(t)\rfloor_\varepsilon\big) \qquad\qquad \square
\end{aligned}$$

2.5 Estimates comparing solutions

Finally, we are interested in bounds of the distance between solutions. However, estimating the distance between points of timed sleek transitions is available only for "test elements" of $\widetilde{\mathcal{D}}$ in the first argument of $\widetilde{q}_\varepsilon$ (as in Proposition 2.5 and Lemma 2.19). So we are using an auxiliary function instead of the distance. In the example of the next section, the following estimate implies uniqueness of solutions. Here assumptions about the time parameter $\mathbb{T}_\varepsilon(\cdot,\cdot) > 0$ play a decisive role for the first time.

Proposition 2.23 (Estimate between timed right–hand sleek solutions)
Assume for $\widetilde{f} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon)),\quad \widetilde{x}, \widetilde{y} : [0,T[\, \longrightarrow \widetilde{E}$ *and some* $\varepsilon \in \mathcal{J},\ \rho \geq 0$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot),\quad \overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *(in the sense of Def. 2.12)*, $\quad \pi_1\, \widetilde{x}(0) = \pi_1\, \widetilde{y}(0) = 0,$
2. $M_\varepsilon \ \geq \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t<T,\ \widetilde{z} \in \widetilde{\mathcal{D}}} \big\{\alpha_\varepsilon(\widetilde{f}(\widetilde{v},t),\ \widetilde{z}),\ \ \widehat{\alpha}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{z}),\ \ \widehat{\alpha}_\varepsilon(t, \widetilde{y}(\cdot), \widetilde{z})\big\},$
3. $R_\varepsilon \ \geq \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t<T,\ \widetilde{z} \in \widetilde{\mathcal{D}}} \big\{\gamma_\varepsilon(\widetilde{f}(\widetilde{v},t)),\ \ \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{z}),\ \ \widehat{\gamma}_\varepsilon(t, \widetilde{y}(\cdot), \widetilde{z})\big\}$
4. $g_\varepsilon \ \geq \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t<T} \big\{\zeta_\varepsilon(\widetilde{f}(\widetilde{v},t))\big\}$
5. $\exists\ \widehat{\omega}_\varepsilon(\cdot) = o(1),\ L_\varepsilon :\ \ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{z},s),\ \widetilde{f}(\widetilde{v},t);\ \widetilde{z}\big) \ \leq\ R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z},\widetilde{v}) + \widehat{\omega}_\varepsilon(t-s)$
   for all $0 \leq s \leq t \leq T$ *and* $\widetilde{v} \in \widetilde{E},\ \widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1\, \widetilde{z}_1 \leq \pi_1\, \widetilde{v}$,
6. $\forall\ t \in [0,T[\,:$ *the infimum* $\varphi_\varepsilon(t) := \inf\limits_{\substack{\widetilde{z} \in \widetilde{\mathcal{D}}:\ \pi_1\, \widetilde{z} \leq t,\\ \lfloor\widetilde{z}\rfloor_\varepsilon \leq \rho\, \cdot\, \exp(g_\varepsilon\, t) + g_\varepsilon\, t}} \big(\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) \ +\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{y}(t))\big) < \infty$
   can be approximated by a minimizing sequence $(\widetilde{z}_j)_{j \in \mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *with*
   $\pi_1 \widetilde{z}_j \ \leq\ \pi_1 \widetilde{z}_{j+1} \ \leq\ t,\quad \lfloor\widetilde{z}_j\rfloor_\varepsilon \leq \rho\, e^{g_\varepsilon \cdot t} + g_\varepsilon\, t,\quad \dfrac{\sup_{k>j}\ \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k)}{\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\, \widetilde{z}_j)} \ \stackrel{j \to \infty}{\longrightarrow}\ 0.$

Then, $$\varphi_\varepsilon(t) \ \leq\ \big(\varphi_\varepsilon(0) \ +\ 8\, R_\varepsilon\ \ (1{+}\rho) \cdot t\big) \ \cdot\ e^{(L_\varepsilon\ (1+\rho)\ +\ M_\varepsilon) \cdot t}.$$

Proof is based on a further subdifferential version of Gronwall's Lemma quoted in Lemma 2.24. $\varphi_\varepsilon(\cdot)$ satisfies $\varphi_\varepsilon(t) \ \leq\ \liminf\limits_{h \downarrow 0}\ \varphi_\varepsilon(t-h)$ for every $t \in\,]0,T[$ due to the timed triangle inequality and the continuity of $\widetilde{x}(\cdot),\, \widetilde{y}(\cdot)$ (in time direction).

For showing $$\liminf_{h \downarrow 0}\ \frac{\varphi_\varepsilon(t{+}h) - \varphi_\varepsilon(t)}{h} \ \leq\ (L_\varepsilon\ (1{+}\rho) \ +\ M_\varepsilon) \cdot \varphi_\varepsilon(t) \ \ +\ \ 8\, R_\varepsilon\ (1{+}\rho)\,,$$
let $(\widetilde{z}_j)_{j \in \mathbb{N}}$ denote a minimizing sequence in $\widetilde{\mathcal{D}}$ such that
$$\wedge \begin{cases} \pi_1\, \widetilde{z}_j \ \leq\ \pi_1\, \widetilde{z}_k \ \leq\ t, \quad \lfloor\widetilde{z}_j\rfloor_\varepsilon \ \leq\ \rho\, e^{g_\varepsilon\, t} + g_\varepsilon\, t & \\ \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k) \qquad \leq\ \frac{1}{2\,j} \cdot \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j) & \text{for all } j < k, \\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\ \widetilde{x}(t)) \ +\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\ \widetilde{y}(t)) \quad \longrightarrow \quad \varphi_\varepsilon(t) & (j \longrightarrow \infty). \end{cases}$$

For all $h \ <\ \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j),\ \ j < k,$ Lemma 2.19 and assumption (5.) imply
$$\begin{aligned} &\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h,\widetilde{z}_j),\ \ \widetilde{x}(t{+}h)\Big) \quad - \quad \widetilde{q}_\varepsilon\big(\widetilde{z}_j,\ \widetilde{x}(t)\big) \ \cdot\ e^{M_\varepsilon\, h} \\ \leq\ &\int_0^h e^{M_\varepsilon\, \cdot\, (h-s)}\ \Big\{\Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\, (\widetilde{z}_j,\ \widetilde{x}(t{+}s)) + \widehat{\omega}_\varepsilon(s)\Big) \ \cdot\ \big(1 + \lfloor\widetilde{z}_j\rfloor_\varepsilon\ e^{g_\varepsilon\, s} + g_\varepsilon\, s\big) \ +\ 3\, R_\varepsilon\Big\}\ \ ds \\ \leq\ &\int_0^h e^{M_\varepsilon\, \cdot\, (h-s)}\ \Big(4\, R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\, (\widetilde{z}_j,\ \widetilde{x}(t{+}s)) + \widehat{\omega}_\varepsilon(s)\Big) \ \cdot\ \big(1 +\ \rho \qquad e^{g_\varepsilon\, h} + g_\varepsilon\, h\big) \qquad ds. \end{aligned}$$

Setting the abbreviations $h_j := \min\{\frac{1}{2}\ \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\, \widetilde{z}_j),\ \frac{1}{j}\} > 0$ and $\delta_j := 1{+}\rho e^{g_\varepsilon\, h_j} {+} g_\varepsilon\, h_j \stackrel{j \to \infty}{\longrightarrow} 1{+}\rho,$ the approximating properties of $(\widetilde{z}_j)_{j \in \mathbb{N}}$ and the timed triangle inequality guarantee for any index $k > j$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h_j,\widetilde{z}_j),\ \ \widetilde{x}(t{+}h_j)\Big)\\
&\le\ \widetilde{q}_\varepsilon\big(\widetilde{z}_k,\ \widetilde{x}(t)\big)\cdot e^{M_\varepsilon\,h_j}\ +\ \tfrac{e^{M_\varepsilon\,h_j}-1}{M_\varepsilon}\ \big(L_\varepsilon\cdot\widetilde{q}_\varepsilon(\widetilde{z}_k,\ \widetilde{x}(t))+L_\varepsilon\cdot\tfrac{1}{j}\,h_j\ +\ 4\,R_\varepsilon\big)\,\delta_j\\
&\quad+\ \tfrac{1}{j}\,h_j\ \cdot e^{M_\varepsilon\,h_j}\ +\ \int_0^{h_j} e^{M_\varepsilon\cdot(h_j-s)}\,\big(L_\varepsilon\cdot\omega_\varepsilon(\widetilde{x},s)\ +\widehat{\omega}_\varepsilon(s)\big)\ ds\qquad \delta_j\,.
\end{aligned}$$

The same estimate for $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h_j,\widetilde{z}_j),\ \ \widetilde{y}(t{+}h_j)\Big)$ and $k\longrightarrow\infty,\ j\longrightarrow\infty$ lead to

$$\liminf_{h\downarrow 0}\ \tfrac{\varphi_\varepsilon(t+h)\,-\,\varphi_\varepsilon(t)}{h}\ \le\ (L_\varepsilon\ (1{+}\rho)\ +\ M_\varepsilon)\cdot\varphi_\varepsilon(t)\ +\ 8\,R_\varepsilon\ (1{+}\rho)\,. \qquad\Box$$

Lemma 2.24 (Lemma of Gronwall for semicontinuous functions II [16]**)**
Let $\psi:[a,b]\longrightarrow\mathbb{R},\ f,g\in C^0([a,b[,\mathbb{R})$ *satisfy* $f(\cdot)\ge 0$ *and*

$$\begin{aligned}
\psi(t)\ &\le\ \liminf_{h\downarrow 0}\ \psi(t-h), && \forall\ t\in\,]a,\ b],\\
\psi(t)\ &\ge\ \liminf_{h\downarrow 0}\ \psi(t+h), && \forall\ t\in[a,\ b[,\\
\liminf_{h\downarrow 0}\ \tfrac{\psi(t+h)-\psi(t)}{h}\ &\le\ f(t)\ \cdot\ \liminf_{h\downarrow 0}\ \psi(t-h)\ +\ g(t) && \forall\ t\in\,]a,\ b[.
\end{aligned}$$

Then, for every $t\in[a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \le\ \psi(a)\ \cdot\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds \qquad \textit{with } \mu(t):=\int_a^t f(s)\ ds. \qquad\Box$$

Remark 2.25 All these results are easy to apply to sets *without* separate "time" component, i.e. consider just nonempty $E,\ \mathcal{D}$ (instead of $\widetilde{E},\ \widetilde{\mathcal{D}}$). Indeed, every ostensible metric $q_\varepsilon: E\times E\longrightarrow[0,\infty[$ induces a *timed* ostensible metric $\widetilde{q}_\varepsilon:\widetilde{E}\times\widetilde{E}\longrightarrow[0,\infty[$ according to

$$\widetilde{q}_\varepsilon\,((s,x),\ (t,y))\ :=\ |s-t|+q_\varepsilon(x,y) \qquad \text{for all } (s,x),\ (t,y)\in\widetilde{E}.$$

Then every map $\vartheta:[0,1]\times(\mathcal{D}\cup E)\longrightarrow(\mathcal{D}\cup E)$ satisfying the conditions (1.)–(7.), (9.) for the tuple $(E,D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ induces a *timed* sleek transition $\widetilde{\vartheta}:[0,1]\times(\widetilde{\mathcal{D}}\cup\widetilde{E})\longrightarrow(\widetilde{\mathcal{D}}\cup\widetilde{E})$ on $(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ by $\widetilde{\vartheta}\,(h,\ (t,x))\ :=\ (t+h,\ \vartheta(h,x))$ for all $(t,x)\in\widetilde{\mathcal{D}}\cup\widetilde{E},\ h\in[0,1]$. So skipping the separate "time" component consistently, all conclusions of this section can also be drawn for *sleek transitions* on a given tuple $(E,D,\ (q_\varepsilon),\ (\lfloor\cdot\rfloor_\varepsilon))$ (as used in the next section). These counterparts are usually denoted without tilde.

3 The continuity equation with bounded BV vector fields

Now the results of § 2 are applied to measure–valued solutions $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^N)$ to the continuity equation $\frac{d}{dt}\,\mu_t \;+\; D_x \cdot (b\,\mu_t) \;=\; 0$ (in the distributional sense) with a given vector field b. We suggest a family of pseudo–metrics $(q_\varepsilon)_{\varepsilon\in\mathcal{J}}$ on $\mathcal{M}(\mathbb{R}^N)$ motivated by vague convergence and a corresponding family $(\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}$.
Assuming $b \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)\cap L^\infty$ for the moment, measure–valued solutions to the continuity equation can be characterized explicitly via the flow along b :

Definition 3.1 *For any* $\widetilde{b} \in L^1\big([0,T],\; W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$, *the function* $\mathbf{X}_{\widetilde{b}} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is induced by the flow along* $\widetilde{b}$, *i.e.* $\mathbf{X}_{\widetilde{b}}(\cdot,x_0) : [0,T] \longrightarrow \mathbb{R}^N$ *is the absolutely continuous solution to the Cauchy problem*

$$\wedge\begin{cases} \frac{d}{dt}\,x(t) &=\; \widetilde{b}(x(t),\,t) \quad a.e.\ in\ [0,T], \\ \quad x(0) &=\; x_0. \end{cases}$$

Proposition 3.2 ([1, Proposition 4 & Remark 7])
Assume for $\widetilde{b} \in L^1\big([0,T],\; W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ *that* $\frac{|\widetilde{b}|}{1+|x|} \in L^1\left([0,T], L^\infty(\mathbb{R}^N)\right)$.
For any initial datum $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$, *the unique solution* $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^N)$, $t \longmapsto \mu_t$ *to the continuity equation* $\frac{d}{dt}\,\mu_t \;+\; D_x \cdot (\widetilde{b}\,\mu_t) \;=\; 0$ *(in the distributional sense) is given by the push–forward* $\mu_t \;:=\; \mathbf{X}_{\widetilde{b}}(t,\cdot)_\sharp\,\mu_0$ *at each time* $t \in\,]0,T]$, *i.e.*

$$\int_{\mathbb{R}^N} \varphi(x)\;\,d\mu_t(x) \;\;=\;\; \int_{\mathbb{R}^N} \varphi(\mathbf{X}_{\widetilde{b}}(t,x))\;\,d\mu_0(x) \qquad \textit{for all}\;\; \varphi \in C^0_c(\mathbb{R}^N). \qquad \square$$

In [14], a metric on the space $\mathcal{M}_1^+(\mathbb{R}^N)$ of positive Radon measures with finite first moment has been proposed so that these measure-valued solutions to the linear problem induce sleek transitions on the set $\mathcal{M}_c^+(\mathbb{R}^N)$ of positive Radon measures with compact support. Applying the theory of mutational equations, sufficient conditions on the bounded Lipschitz vector fields are specified in [14] for ensuring existence and uniqueness of measure-valued solutions to the nonlinear continuity equations. (Meanwhile the restriction to compact supports of positive Radon measures has proved to be dispensable. All finite Radon measures can be considered instead.)

The main goal here is to weaken the regularity conditions on the vector fields considered as coefficients in the continuity equation. In particular, spatial vector fields $b(\cdot)$ of bounded variation have aroused interest for weakening the assumption of (local) Lipschitz continuity.
Recent results of Ambrosio [1, 2] make a suggestion how to specify a flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ along certain vector fields of bounded (spatial) variation in a unique way. This uniqueness is based on an additional condition of regularity, i.e. the absolute continuity with respect to Lebesgue measure $\mathcal{L}^N$ is preserved uniformly: For any nonnegative function $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the measure $\mu_0 := \rho\,\mathcal{L}^N$ satisfies $\mathbf{X}(t,\cdot)_\sharp\,\mu_0 \;\le\; C\,\mathcal{L}^N$ for all $t \in [0,T]$ with a constant C independent of t.
After summarizing some features of this so–called Lagrangian flow in subsequent Proposition 3.3, we exploit the corresponding vector fields of bounded variation for inducing sleek transitions on measures. In contrast to [14], however, the tools of Ambrosio require the restriction to finite positive measures being absolutely continuous and bounded with respect to $\mathcal{L}^N$.
Although limiting the class of admitted Radon measures, this assumption has the advantage in subsequent Proposition 3.14 that applying existence results to the continuity equation here requires the continuity of the vector fields (representing the right–hand side) with respect to L^1 whereas in [14, Theorem 4.6], continuity with respect to L^∞ was assumed.

Proposition 3.3 *Assume* $\widetilde{b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *to be in* $L^1\big([0,T],\ \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)\big)$ *satisfying*

1. $\frac{|\widetilde{b}|}{1+|x|} \in L^1\big([0,T],\ L^1(\mathbb{R}^N)\big) + L^1\big([0,T],\ L^\infty(\mathbb{R}^N)\big)$,
2. $D_x \cdot \widetilde{b}(t,\cdot) = \mathrm{div}_x\, \widetilde{b}(t,\cdot)\ \mathcal{L}^N \ll \mathcal{L}^N$ *for* $\mathcal{L}^1$*–almost every* $t \in [0,T]$,
3. $[\mathrm{div}_x\, \widetilde{b}]^- \in L^1\big([0,T],\ L^\infty(\mathbb{R}^N)\big)$.

Then there exists a so–called Lagrangian flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *such that*

(a) $\mathbf{X}(\cdot,x) : [0,T] \longrightarrow \mathbb{R}^N$ *is absolutely continuous for* $\mathcal{L}^N$*–almost every* $x \in \mathbb{R}^N$ *with*
$$\mathbf{X}(t,x) = x + \int_0^t \widetilde{b}\big(s,\ \mathbf{X}(s,x)\big)\ ds \qquad \text{for all } t \in [0,T],$$

(b) *there is a constant* $C > 0$ *with* $\mathbf{X}(t,\cdot)_\sharp\,(\rho\,\mathcal{L}^N) \le C\ \|\rho\|_\infty\ \mathcal{L}^N \quad \forall\ \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ t.$

Furthermore, $\mathbf{X}(t,\cdot) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is unique up to* $\mathcal{L}^N$*–negligible sets for every time* $t \in [0,T]$ *and,* $\mu(t) := \mathbf{X}(t,\cdot)_\sharp\,\mu_0$ *is the unique distributional solution to the continuity equation*
$$\tfrac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\,\mu) = 0 \qquad \text{in }]0,T[\, \times \mathbb{R}^N$$
for every initial datum $\mu_0 := \rho\,\mathcal{L}^N$ *with* $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \rho \ge 0.$

Mollifying each $\mu(t)$ *with a common Gaussian kernel* $\rho \in C^1(\mathbb{R}^N,]0,\infty[)$*, the measures* $\mu_\delta(t) := \mu(t) * \rho_\delta$ *solve the continuity equation* $\tfrac{d}{dt}\,\mu_\delta + D_x \cdot (\widetilde{b}_\delta\,\mu_\delta) = 0$ *(in the distributional sense) with* $\widetilde{b}_\delta(t,\cdot) := \frac{(\widetilde{b}(t,\cdot)\,\mu(t)) * \rho_\delta}{\mu_\delta(t)}$ *being in* $L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ *and,* $\mu_\delta(t) \xrightarrow{\delta \downarrow 0} \mu(t)$ *narrowly (i.e. with respect to the duality of bounded continuous functions) for every* $t \in [0,T]$.

Proof is presented in [1] (and in [2]). Indeed, extending [1, Theorem 30], to vector fields of bounded spatial variation (as stated in the end of [1, § 5]), there exists a Lagrangian flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with the claimed properties (a),(b) and, it is unique (up to $\mathcal{L}^N$–negligible sets).
The proof of [1, Theorem 19] bridges the gap between the Lagrangian flow and the measure–valued solution to the continuity equation (by means of push–forward). The uniqueness of $\mu(\cdot)$ results from the comparison principle of the continuity equation (due to the assumptions about $\widetilde{b}$) according to [1, Theorem 26]. Finally proving [1, Theorem 12] implies the narrow sequential compactness of $\eta_\delta := \big(x, \mathbf{X}_{\widetilde{b}_\delta}(\cdot,x)\big)_\sharp\ \mu_\delta(0)$ (using Prokhorov compactness theorem). So equation (9) there implies the narrow convergence of $\mu_\delta(t)$ to its unique limit point $\mu(t)$. □

Lemma 3.4 *For any* $\widetilde{b} \in L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ *with* $\mathrm{div}_x\,\widetilde{b} \in L^1\big([0,T],\ L^\infty(\mathbb{R}^N,\mathbb{R}^N)\big)$*, the flow* $\mathbf{X}_b : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *specified in Definition 3.1 satisfies for all* $t \in [0,T]$ *and* $\mathcal{L}^N$*–a.e.* $x \in \mathbb{R}^N$
$$\exp\Big(-\int_0^t \big\|\,[\mathrm{div}_x\,\widetilde{b}(t,\cdot)]^-\,\big\|_\infty\ dt\Big) \ \le\ \det\ D_x\,\mathbf{X}_b(t,x) \ \le\ \exp\Big(\int_0^t \big\|\,[\mathrm{div}_x\,\widetilde{b}(t,\cdot)]^+\,\big\|_\infty\ dt\Big).$$
Moreover, for all $\mu = \rho\,\mathcal{L}^N$ *with* $\rho \in L^1(\mathbb{R}^N) \cap L^\infty$*, the total variation of* $\mathbf{X}_b(t,\cdot)_\sharp\,\mu$ *fulfills*
$$\big|\mathbf{X}_b(t,\cdot)_\sharp\,\mu\big|(\mathbb{R}^N) \ \le\ |\mu|(\mathbb{R}^N)\ \cdot\ \exp\Big(2\int_0^t \|\mathrm{div}_x\,\widetilde{b}(t,\cdot)\|_\infty\ dt\Big)$$

Proof of the first part is mentioned in [2, Remark 6.3]. The second part results from the area formula and the transformation of Lebesgue integrals. Indeed, for $\mu = \rho\,\mathcal{L}^N$ with $\rho \in L^1(\mathbb{R}^N) \cap L^\infty$,
$$\begin{aligned}
\big|\mathbf{X}_b(t,\cdot)_\sharp\,\mu\big|(\mathbb{R}^N) &= \int_{\mathbb{R}^N} \Big|\frac{\rho}{|\det\ D_x\,\mathbf{X}_b(t,\cdot)|} \circ \mathbf{X}_b(t,\cdot)^{-1}\Big|\ d\mathcal{L}^N \\
&\le \int_{\mathbb{R}^N} \big|\rho \circ \big(\mathbf{X}_b(t,\cdot)^{-1}\big)\big|\ d\mathcal{L}^N \quad \cdot\ \exp\Big(\int_0^t \big\|\,[\mathrm{div}_x\,\widetilde{b}(t,\cdot)]^-\,\big\|_\infty\ dt\Big) \\
&\le \int_{\mathbb{R}^N} |\rho|\ d\mathcal{L}^N\ \cdot\ \big\|\det\ D_x\,\mathbf{X}_b(t,\cdot)\big\|_\infty \quad \cdot\ \exp\Big(\int_0^t \big\|\,[\mathrm{div}_x\,\widetilde{b}(t,\cdot)]^-\,\big\|_\infty\ dt\Big). \quad \square
\end{aligned}$$

These (mostly quoted) results motivate the following choice of vector fields and finite Radon measures. Using the notation of following Definition 3.5, the results of [1] guarantee for each Lagrangian flow $\mathbf{X}$

$$\mathbf{X}(t,\cdot)_\sharp\, \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \;\subset\; \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N).$$

Definition 3.5 $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ *denotes the set of all functions* $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ *satisfying* $D\cdot b = \operatorname{div} b\ \mathcal{L}^N \ll \mathcal{L}^N$ *and* $\operatorname{div} b \in L^\infty(\mathbb{R}^N)$.
Furthermore, set $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) := \big\{\rho\,\mathcal{L}^N \;\big|\; \rho \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\ \rho \geq 0\big\} \subset \mathcal{M}^+(\mathbb{R}^N)$ *and, for each* $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$, *define* $\vartheta_b : [0,1]\times \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, $(h,\mu_0) \longmapsto \mathbf{X}(h,\cdot)_\sharp\,\mu_0$ *with* $\mathbf{X}(\cdot,\cdot)$ *denoting its Lagrangian flow according to Proposition 3.3.*

Moreover, the proof of Lemma 3.4 indicates an adequate choice of the "distance" between two measures of $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$. We use the weighted total variation – with a countable family $(\varphi_\varepsilon)_\varepsilon$ of smooth positive weight functions whose gradient can be estimated by the function itself.

Lemma 3.6 *There is a countable family* $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ *of smooth Schwartz functions* $\mathbb{R}^N \longrightarrow [0,\infty[$ *such that* $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ *is dense in* $\big(C^0_0(\mathbb{R}^N,\,[0,\infty[),\ \|\cdot\|_\infty\big)$,
$C^\infty_c(\mathbb{R}^N,\,[0,\infty[)$ *is contained in the closure of* $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ *with respect to the* C^1 *norm and it satisfies* $|\nabla\,\varphi_\varepsilon(\cdot)| \leq \lambda_\varepsilon\cdot\varphi_\varepsilon(\cdot)$ *in* $\mathbb{R}^N$ *with a constant* $\lambda_\varepsilon > 0$ *for each* $\varepsilon\in\mathcal{J}$.

Proof. Such a $\varphi_\varepsilon \in C^\infty(\mathbb{R}^N,[0,\infty[)$ can be generated by means of convolution.
Indeed, $C^\infty_0(\mathbb{R}^\mathbb{N},[0,\infty[)$ is known to be separable with respect to $\|\cdot\|_\infty$. Now consider a countable dense subset $(f_k)_{k\in\mathbb{N}}$ of $C^\infty_c(\mathbb{R}^\mathbb{N},[0,\infty[)$ together with $e_\delta : \mathbb{R}^\mathbb{N} \longrightarrow\,]0,\infty[,\ \ x \longmapsto c_{\delta,N}\cdot\exp(-\,\delta\,\frac{|x|^2}{1+|x|})$ (for arbitrarily large $\delta>0$ and the constant $c_{\delta,N}>0$ such that $\|e_\delta\|_{L^1(\mathbb{R}^N)}=1$).
Then, each $f_k * e_\delta$ is smooth, nonnegative and satisfies $|\nabla(f_k*e_\delta)| = |f_k*(\nabla e_\delta)| \leq \delta\ f_k*e_\delta$ since the auxiliary function $\widehat{e}_\delta : [0,\infty[\ \longrightarrow\ ]0,1]$, $r \longmapsto c_{\delta,N}\cdot\exp(-\,\delta\,\frac{r^2}{1+r})$ is smooth with $\frac{d}{dr}\,\widehat{e}_\delta(r) = -\,\delta\,\frac{r\,(r+2)}{(r+1)^2}\ \widehat{e}_\delta(r) \in [-\delta,0]\cdot\widehat{e}_\delta(r)$ and thus $\frac{d}{dr}\,\widehat{e}_\delta(r) = O(r)$ for $r\longrightarrow 0^+$.
Furthermore, f_k*e_δ is a Schwartz function because so is e_δ and f_k is assumed to have compact support. $(f_k*e_\delta)_{k,\delta\,\in\,\mathbb{N}}$ is dense in $\big(C^0_0(\mathbb{R}^N,\,[0,\infty[),\ \|\cdot\|_\infty\big)$ since so is $(f_k)_{k\,\in\,\mathbb{N}}$ and $(e_\delta)_{\delta\,\in\,\mathbb{N}}$ is a Dirac sequence.
Finally it satisfies the second required property because for any $g\in C^\infty_c(\mathbb{R}^N,\,[0,\infty[)$ and subsequence $(f_{k_j})_{j\in\mathbb{N}}$ with $\|g-f_{k_j}\|_\infty \stackrel{j\to\infty}{\longrightarrow} 0$, we obtain $\nabla(f_{k_j}*e_\delta) = f_{k_j}*(\nabla e_\delta) \stackrel{j\to\infty}{\longrightarrow} g*(\nabla e_\delta) = (\nabla g)*e_\delta$ uniformly and the last convolution converges uniformly to ∇g for $\delta\longrightarrow\infty$. □

Definition 3.7 *Let* $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ *be a countable family of Schwartz functions as described in Lemma 3.6. For each* $\varepsilon\in\mathcal{J}$, *define* $q_\varepsilon:\ \mathcal{M}(\mathbb{R}^N)\times\mathcal{M}(\mathbb{R}^N) \longrightarrow [0,\infty[$,

$$\begin{aligned} q_\varepsilon(\mu,\,\nu) \;&:=\; \big|\varphi_\varepsilon\cdot(\mu-\nu)\big|(\mathbb{R}^N) \\ &\stackrel{\text{Def.}}{=}\; \sup\Big\{\sum_{k=0}^{\infty}\Big|\int_{E_k}\varphi_\varepsilon\ d(\mu-\nu)\Big|\ \Big|\ (E_k)_{k\in\mathbb{N}}\ \text{pairwise disjoint Borel sets, } \mathbb{R}^N=\bigcup_k E_k\Big\}. \end{aligned}$$

Remark 3.8 Obviously, Gronwall's Lemma implies $\varphi_\varepsilon>0$ in $\mathbb{R}^N$ unless $\varphi_\varepsilon\equiv 0$.
So assuming $\varphi_\varepsilon\not\equiv 0$ for all $\varepsilon\in\mathcal{J}$ from now on, each pseudo–metric q_ε takes all points of $\mathbb{R}^N$ into consideration – in a weighted form.

Now we first investigate the regularity features of $\vartheta_b:[0,1]\times\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)\longrightarrow\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ for more regular vector field $b\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\cap L^\infty$ with respect to each pseudo–metric $q_\varepsilon(\varepsilon\in\mathcal{J})$. Afterwards the approximation via convolution (mentioned in Proposition 3.3) lays the basis for extending the estimates to $b\in\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ in subsequent Proposition 3.12.

Lemma 3.9 *For each* $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty$, $\vartheta_b : [0,1] \times \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *satisfies*

1. $\vartheta_b(0,\cdot) \;=\; \mathrm{Id}_{\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)}$,
2. $q_\varepsilon\left(\vartheta_b(h,\ \vartheta_b(t,\mu_0)),\ \ \vartheta_b(t+h,\mu_0)\right) \;=\; 0 \;=\; q_\varepsilon\left(\vartheta_b(t+h,\mu_0),\ \ \vartheta_b(h,\ \vartheta_b(t,\mu_0))\right)$ *for any initial datum* $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *and* $t,h \in [0,1]$ *with* $t+h \le 1$.
3. $\limsup_{h\downarrow 0} \frac{q_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) \,-\, q_\varepsilon(\mu_0,\nu_0)}{h} \;\le\; \lambda_\varepsilon\,\|b\|_\infty \cdot q_\varepsilon(\mu_0,\nu_0)$ *for* $\mu_0,\nu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$,
4. $\left|\varphi_\varepsilon\ \vartheta_b(t,\mu_0))\right|(\mathbb{R}^N) \;\le\; \left|\varphi_\varepsilon\ \mu_0\right|(\mathbb{R}^N)\cdot e^{\lambda_\varepsilon\,\|b\|_\infty\,\cdot\, t}$ *for* $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ t\in[0,1]$,
5. $q_\varepsilon\left(\vartheta_b(s,\mu_0),\ \vartheta_b(t,\mu_0)\right) \;\le\; |t-s|\cdot\lambda_\varepsilon\,\|b\|_\infty\ e^{\lambda_\varepsilon\,\|b\|_\infty}\ \left|\varphi_\varepsilon\ \mu_0\right|(\mathbb{R}^N)$ *for* $s,t\in[0,1],\ \mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$,
6. $\limsup_{h\downarrow 0} \frac{q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right)}{h} \;\le\; \lambda_\varepsilon\,\left|\varphi_\varepsilon\ |b_1-b_2|\ \mu_0\right|(\mathbb{R}^N) \;\le\; \lambda_\varepsilon\,\|\rho\|_\infty \ \cdot\ \left\|\varphi_\varepsilon\ |b_1-b_2|\right\|_{L^1(\mathbb{R}^N)}$ *for all bounded vector fields* $b_1,b_2 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)\cap L^\infty$ *and* $\mu_0 = \rho\,\mathcal{L}^N \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$.

Proof. The measure–valued flow $\vartheta_b : [0,1]\times\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ still satisfies the semigroup property and thus statements (1.), (2.).

For any $\mu_0 = \rho\,\mathcal{L}^N,\ \nu_0 = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, the definitions of total variation and push–forward imply

$$\begin{aligned} q_\varepsilon\big(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)\big) \;&=\; \left|\,\varphi_\varepsilon\cdot\big(\mathbf{X}_b(h,\cdot)_\sharp\,\mu_0 \;-\; \mathbf{X}_b(h,\cdot)_\sharp\,\nu_0\big)\right|(\mathbb{R}^N)\\ &\le\; \int_{\mathbb{R}^N} \varphi_\varepsilon(\mathbf{X}_b(h,\cdot))\ \ |\rho-\sigma|\ \ d\,\mathcal{L}^N\\ &\le\; \int_{\mathbb{R}^N} \left|\,\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\right|\ \ |\rho-\sigma|\ \ d\,\mathcal{L}^N \;+\; \left|\varphi_\varepsilon\cdot(\mu_0-\nu_0)\right|(\mathbb{R}^N). \end{aligned}$$

So the choice of φ_ε (in Lemma 3.6) has the consequence

$$\begin{aligned} \limsup_{h\downarrow 0} \frac{q_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) \,-\, q_\varepsilon(\mu_0,\nu_0)}{h} \;&\le\; \limsup_{h\downarrow 0}\ \frac{1}{h}\cdot \int_{\mathbb{R}^N} \left|\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\right|\ |\rho-\sigma|\ \ d\,\mathcal{L}^N\\ &\le\; \int_{\mathbb{R}^N} |\nabla\varphi_\varepsilon(x)\ \cdot\ b(x)|\qquad |\rho-\sigma|\ \ d\,\mathcal{L}^N\\ &\le\; \|b\|_\infty \int_{\mathbb{R}^N} \lambda_\varepsilon\ \ \varphi_\varepsilon \qquad |\rho-\sigma|\ \ d\,\mathcal{L}^N\\ &\le\; \|b\|_\infty\quad \lambda_\varepsilon\ \cdot\ q_\varepsilon(\mu_0,\nu_0). \end{aligned}$$

Applying this estimate to $\nu_0 \equiv 0$ and $\vartheta_b(t,\mu_0)$ (instead of μ_0), we conclude property (4.) from Gronwall's Lemma 2.6 because the lower semicontinuous auxiliary function

$$\delta_\varepsilon : [0,1] \longrightarrow \mathbb{R},\quad t \longmapsto \left|\varphi_\varepsilon\ \ \vartheta_b(t,\mu_0)\right|(\mathbb{R}^N) \;=\; \left|\varphi_\varepsilon(\mathbf{X}_b(t,\cdot))\ \mu_0\right|(\mathbb{R}^N)$$

is one–sided differentiable and satisfies $\frac{d^+}{dt^+}\,\delta_\varepsilon(\cdot) \;\le\; \lambda_\varepsilon\,\|b\|_\infty\cdot\delta_\varepsilon(\cdot)$.
In basically the same way, we obtain statement (5.) considering the auxiliary function

$$\widehat{\delta}_\varepsilon : [s,1] \longrightarrow \mathbb{R},\quad t \longmapsto \left|\varphi_\varepsilon\ \ \big(\vartheta_b(t,\mu_0) - \vartheta_b(s,\mu_0)\big)\right|(\mathbb{R}^N) \;=\; \left|(\varphi_\varepsilon(\mathbf{X}_b(t-s,\cdot)) - \varphi_\varepsilon)\ \ \vartheta_b(s,\mu_0)\right|(\mathbb{R}^N)$$

with $s\in[0,1[$ fixed and $\frac{d^+}{dt^+}\,\widehat{\delta}_\varepsilon(t) \;\le\; \lambda_\varepsilon\,\|b\|_\infty\,\left|\varphi_\varepsilon\ \ \vartheta_b(t,\mu_0)\right|(\mathbb{R}^N) \;\le\; \lambda_\varepsilon\,\|b\|_\infty\,e^{\lambda_\varepsilon\,\|b\|_\infty}\,\left|\varphi_\varepsilon\ \mu_0\right|(\mathbb{R}^N)$.
Last, but not least, choose any $b_1,b_2 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)\cap L^\infty$ and initial datum $\mu_0 = \rho\,\mathcal{L}^N \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$. Then, for every $h\in[0,1]$,

$$\begin{aligned} \frac{1}{h}\cdot q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right) \;&\le\; \frac{1}{h}\cdot \int_{\mathbb{R}^N} \left|\varphi_\varepsilon(\mathbf{X}_{b_1}(h,\cdot)) \;-\; \varphi_\varepsilon(\mathbf{X}_{b_2}(h,\cdot))\right|\ |\rho|\ \ d\,\mathcal{L}^N\\ \limsup_{h\downarrow 0} \frac{q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right)}{h} \;&\le\; \int_{\mathbb{R}^N} \lambda_\varepsilon\ \varphi_\varepsilon\ \ |b_1-b_2|\ \ |\rho|\ \ d\,\mathcal{L}^N\\ &\le\; \lambda_\varepsilon\ \|\rho\|_\infty\ \cdot\ \left\|\varphi_\varepsilon\ |b_1-b_2|\right\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

□

In regard to the choice of $\lfloor\cdot\rfloor_\varepsilon$, there are even two candidates now. The first is the weighted total variation (as mentioned in Lemma 3.9 (4.)). Lemma 3.4, however, provides an alternative whose growth is also bounded in the required way: the total variation – not weighted by φ_ε and thus, independent of $\varepsilon \in \mathcal{J}$. For applying the compactness criterion of de la Vallée Poussin later, we prefer the total variation $|\cdot|(\mathbb{R}^N)$ and then rely on the results using "*weakly* transitionally compact" (presented in § 2.4).
In particular, subsequent Lemma 3.11 lays the basis for taking also the L^∞ norm into consideration and thus, we define $\lfloor\mu\rfloor := |\mu|(\mathbb{R}^N) + \left\|\frac{\mu}{\mathcal{L}^N}\right\|_\infty = \|\sigma\|_{L^1(\mathbb{R}^N)} + \|\sigma\|_\infty$ for $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$.

Lemma 3.10 *For every $\mu \in \mathcal{M}(\mathbb{R}^N)$ and open set $A \subset \mathbb{R}^N$, the total variation satisfies*
$$|\mu|(A) = \sup\left\{\int_{\mathbb{R}^N} \psi\, d\mu \;\Big|\; \psi \in C_c^0(A),\ \|\psi\|_\infty \le 1\right\}$$
and thus, $q_\varepsilon(\mu,\nu) = \sup_{\kappa\in\mathcal{I}}\ p_{\varepsilon,\kappa}(\mu,\nu)$ *for all $\mu,\nu \in \mathcal{M}(\mathbb{R}^N)$*
with $\mathcal{I} \subset \mathcal{J}^2$ denoting the set of all indices $\kappa = (\kappa_1,\kappa_2) \in \mathcal{J}^2$ satisfying $0 < \varphi_{\kappa_1}(\cdot),\ \varphi_{\kappa_2}(\cdot) \le 1$,
$$p_{\varepsilon,\kappa}(\mu,\nu) := \left|\int_{\mathbb{R}^N} \varphi_\varepsilon\ (\varphi_{\kappa_1} - \varphi_{\kappa_2})\ d(\mu-\nu)\right| \qquad \text{for all } \varepsilon \in \mathcal{J},\ \kappa \in \mathcal{I}.$$

Proof of the first equality is given in [4, Proposition 1.47].
As a consequence of Lemma 3.6, the set $\{\varphi_{\kappa_1} \mid \kappa \in \mathcal{I}\}$ is dense in $\big(C_0^0(\mathbb{R}^N,\ [0,1]),\ \|\cdot\|_\infty\big)$ and thus, $\{\varphi_{\kappa_1} - \varphi_{\kappa_2} \mid \kappa \in \mathcal{I}\}$ is dense in $\big(C_0^0(\mathbb{R}^N,\ [-1,1]),\ \|\cdot\|_\infty\big)$. So the first equality implies for $\mu \in \mathcal{M}(\mathbb{R}^N)$
$$\int_{\mathbb{R}^N} \varphi_\varepsilon\ d|\mu| = \sup_{(\kappa_1,\kappa_2)\in\mathcal{I}} \int_{\mathbb{R}^N} \varphi_\varepsilon\ (\varphi_{\kappa_1} - \varphi_{\kappa_2})\ d\mu. \qquad \square$$

Lemma 3.11 *For every vector field $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ and initial measure $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$, the Radon–Nikodym derivative $\sigma_t \overset{\text{Def.}}{=} \frac{\vartheta_b(t,\mu)}{\mathcal{L}^N}$ of $\vartheta_b(t,\mu)$ with respect to Lebesgue measure $\mathcal{L}^N$ satisfies*
$$\begin{aligned} \|\sigma_t\|_\infty &\le \|\sigma\|_\infty\ e^{\|\mathrm{div}\, b\|_\infty\, t}, \\ \big|\vartheta_b(t,\mu)\big|(\mathbb{R}^N) = \|\sigma_t\|_{L^1} &\le \|\sigma\|_{L^1}\ e^{2\,\|\mathrm{div}\, b\|_\infty\, t}. \end{aligned}$$

Proof. The second statement results directly from Lemma 3.4 (applied to mollified vector fields b_δ) and the narrow convergence for $\delta \downarrow 0$ because the total variation is lower semicontinuous with respect to weak* convergence (see [4, Theorem 1.59]). For proving the first statement, we exploit first the duality relation between L^1 and L^∞ and then use the area formula
$$\begin{aligned} \|\sigma_t\|_\infty &= \sup\ \left\{\int \psi\ \sigma_t\ d\mathcal{L}^N \;\Big|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &= \sup\ \left\{\limsup_{\delta\downarrow 0} \int \psi\ d\,\vartheta_{b_\delta}(t,\mu) \;\Big|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &= \sup\ \left\{\limsup_{\delta\downarrow 0} \int \psi\ \left(\frac{\sigma}{\det\ D_x\mathbf{X}_{b_\delta}(t,\cdot)}\right)\Big|_{\mathbf{X}_{b_\delta}(t,\cdot)^{-1}} d\mathcal{L}^N \;\Big|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &\le \sup\ \left\{\limsup_{\delta\downarrow 0} \int \psi\ \|\sigma\|_\infty\ e^{\|\mathrm{div}\, b_\delta\|_\infty\, t}\ d\mathcal{L}^N \;\Big|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &\le \|\sigma\|_\infty\ e^{\|\mathrm{div}\, b\|_\infty\, t}. \end{aligned} \qquad \square$$

Proposition 3.12 *For any $C \in [0,\infty[$ fixed, each $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\|b\|_\infty + \big\|\mathrm{div}\, b\big\|_\infty \le C$ induces the sleek transition ϑ_b on $\big(\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N),\ \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor)_{\varepsilon\in\mathcal{J}}\big)$ with*
$$\begin{aligned} \alpha_\varepsilon(\vartheta_b,\mu) &\overset{\text{Def.}}{=} \lambda_\varepsilon\ C, & \beta_\varepsilon(\vartheta_b) &\overset{\text{Def.}}{=} \lambda_\varepsilon\ \|\varphi_\varepsilon\|_\infty\ C, \\ \gamma_\varepsilon(\vartheta_b) &\overset{\text{Def.}}{=} 0, & \zeta_\varepsilon(\vartheta_b) &\overset{\text{Def.}}{=} 2\,C, \\ \mathbb{T}_\varepsilon(\vartheta_b,\mu) &\overset{\text{Def.}}{=} 1, & i_{\mathcal{D}} &\overset{\text{Def.}}{=} \mathrm{Id}_{\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)}, \\ Q_\varepsilon(\vartheta_b,\vartheta_c;\mu) &\le \lambda_\varepsilon\ e^{2\,C}\ \big(1 + |\mu|(\mathbb{R}^N)\cdot e^{2\,C}\big)\ \left\|\tfrac{\mu}{\mathcal{L}^N}\right\|_\infty\ \big\|\varphi_\varepsilon\,|b-c|\,\big\|_{L^1(\mathbb{R}^N)} \end{aligned}$$
for all $b, c \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\|b\|_\infty + \big\|\mathrm{div}\, b\big\|_\infty \le C$, $\|c\|_\infty + \big\|\mathrm{div}\, c\big\|_\infty \le C$ and $\mu \in \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$, $\varepsilon \in \mathcal{J}$, $t \in [0,1]$. The set of all these sleek transitions ϑ_b is abbreviated as $\mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)$.
$\vartheta_b(h,\cdot) : (\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N), \text{weak*}) \longrightarrow (\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N), p_{\varepsilon,\kappa})$ is continuous for every $h \in [0,1], \varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$.

Proof is based on the tools of approximation provided by Proposition 3.3 and Lemma 3.9: Indeed, choose a Gaussian kernel $\rho \in C^1(\mathbb{R}^N, \,]0,\infty[)$ and set $\rho_\delta(x) := \delta^{-N} \rho(\frac{x}{\delta})$ for $\delta > 0$. Fixing $\mu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ arbitrarily, each vector field $b_\delta := \frac{(b\,\mu) * \rho_\delta}{\mu * \rho_\delta}$ belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ and satisfies $\|b_\delta\|_\infty \le \|b\|_\infty \le C$. So Lemma 3.9 motivates the choice of $\alpha_\varepsilon(\vartheta_{b_\delta}, \mu) \stackrel{\text{Def.}}{=} \lambda_\varepsilon\, C$, $\gamma_\varepsilon(\vartheta_{b_\delta}) \stackrel{\text{Def.}}{=} 0$ and, as in the general framework of § 2.1, we conclude from Proposition 2.5 and Lemma 2.7

$$q_\varepsilon\big(\vartheta_{b_\delta}(h,\ \mu),\ \ \vartheta_{b_\delta}(h,\ \nu)\big) \ \le\ q_\varepsilon(\mu,\ \nu)\ \cdot\ e^{C\,\lambda_\varepsilon\, h}$$

Considering now $\delta \downarrow 0$, the narrow convergence (mentioned in Proposition 3.3) and the lower semicontinuity of total variation (with respect to weak* convergence) provide the same estimates with b instead of b_δ for all $\mu, \nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ and, we obtain $C\,\lambda_\varepsilon$ as an admissible choice of the parameter $\alpha_\varepsilon(\vartheta_b, \mu)$. Thus, the first three conditions on sleek transitions (stated in Definition 2.1) are obviously fulfilled.

Moreover, Lemma 3.4 states $\big|\vartheta_{b_\delta}(t,\mu)\big|(\mathbb{R}^N) \ \le\ |\mu|(\mathbb{R}^N)\ \cdot\ e^{2\,C\,t}$ for all $\mu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$, $t \in [0,1]$, $\delta > 0$ and so, the lower semicontinuity of total variation again implies

$$\big|\vartheta_b(t,\mu)\big|(\mathbb{R}^N) \ \le\ |\mu|(\mathbb{R}^N)\ \cdot\ e^{2\,C\,t},$$

i.e. condition (5.) of Definition 2.1 is satisfied with $\zeta_\varepsilon(\vartheta_b) \stackrel{\text{Def.}}{=} 2\,C$ (independent of $\varepsilon \in \mathcal{J}$).

Considering the continuity w.r.t. time, we obtain for every $s, t \in [0,1]$, $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$, $\varepsilon \in \mathcal{J}$

$$\begin{aligned} q_\varepsilon\big(\vartheta_b(s,\mu),\ \vartheta_b(t,\mu)\big) \ &\le\ \limsup_{\delta \downarrow 0}\ \ q_\varepsilon\big(\vartheta_{b_\delta}(s,\mu),\ \vartheta_{b_\delta}(t,\mu)\big) \\ &\le\ \limsup_{\delta \downarrow 0}\ \int_{\mathbb{R}^N} \big|\varphi_\varepsilon\big(\mathbf{X}_{b_\delta}(s,x)\big) \ -\ \varphi_\varepsilon\big(\mathbf{X}_{b_\delta}(t,x)\big)\big| \quad |\sigma(x)|\ d\,\mathcal{L}^N x \\ &\le\ \limsup_{\delta \downarrow 0}\quad \|\nabla\,\varphi_\varepsilon\|_\infty \quad \|b_\delta\|_\infty\ \ |t-s| \qquad \|\sigma\|_{L^1(\mathbb{R}^N)} \\ &\le\ \qquad \lambda_\varepsilon\ \|\varphi_\varepsilon\|_\infty \quad C \quad |t-s| \qquad |\mu|(\mathbb{R}^N), \end{aligned}$$

i.e. $\beta_\varepsilon(\vartheta_b) \stackrel{\text{Def.}}{=} \lambda_\varepsilon\ \|\varphi_\varepsilon\|_\infty\ C$ is the relevant part of a Lipschitz constant as required in condition (4.) of Definition 2.1. The rest of these conditions on sleek transitions is trivial.

Now we specify an upper bound of

$$Q_\varepsilon(\vartheta_b, \vartheta_c; \mu) \ \stackrel{\text{Def.}}{=}\ \sup_{\substack{0 \le t < 1 \\ \nu \,\in\, \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)}}\ \ \limsup_{h \downarrow 0}\ \Big(\frac{q_\varepsilon\big(\vartheta_b(t+h,\mu),\ \vartheta_c(h,\nu)\big) \ -\ q_\varepsilon\big(\vartheta_b(t,\mu),\ \nu\big)\ \cdot\ e^{\lambda_\varepsilon\, C\, h}}{h\quad (1 + |\nu|(\mathbb{R}^N))}\Big)^+$$

with $b, c \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying $\|b\|_\infty + \big\|\mathrm{div}\, b\big\|_\infty$, $\|c\|_\infty + \big\|\mathrm{div}\, c\big\|_\infty \le C$ and $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$. Mollifying b, c in the way described above, we again obtain vector fields $b_\delta, c_\delta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|b_\delta\|_\infty$, $\|c_\delta\|_\infty \le C$. and, Lemma 3.9 (6) states for all $\nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$

$$\limsup_{h \downarrow 0}\ \frac{q_\varepsilon\big(\vartheta_{b_\delta}(h,\nu),\ \vartheta_{c_\delta}(h,\nu)\big)}{h} \ \le\ \lambda_\varepsilon\ \big|\varphi_\varepsilon\ |b_\delta - c_\delta|\ \nu\big|\,(\mathbb{R}^N).$$

So Lemma 2.9, the area formula and Lemma 3.4 imply

$$\begin{aligned} Q_\varepsilon(\vartheta_{b_\delta}, \vartheta_{c_\delta}; \mu) \ &\le\ \sup_{0 \le t < 1}\ \ \lambda_\varepsilon \quad \big|\varphi_\varepsilon\ |b_\delta - c_\delta| \quad \vartheta_{b_\delta}(t,\mu)\big|\,(\mathbb{R}^N) \\ &=\ \sup_{0 \le t < 1}\ \ \lambda_\varepsilon \int_{\mathbb{R}^N} \varphi_\varepsilon\ |b_\delta - c_\delta| \quad \Big|\frac{\sigma}{\det\ D_x\,\mathbf{X}_{b_\delta}(t,\cdot)} \circ \mathbf{X}_{b_\delta}(t,\cdot)^{-1}\Big| \quad d\mathcal{L}^N x \\ &\le\ \sup_{0 \le t < 1}\ \ \lambda_\varepsilon \quad \big\|\varphi_\varepsilon\ |b_\delta - c_\delta|\,\big\|_{L^1(\mathbb{R}^N)} \quad \frac{\|\sigma\|_\infty}{\|\det\ D_x\,\mathbf{X}_{b_\delta}(t,\cdot)\|_\infty} \\ &\le\ \qquad \lambda_\varepsilon \quad \big\|\varphi_\varepsilon\ |b_\delta - c_\delta|\,\big\|_{L^1(\mathbb{R}^N)} \quad \|\sigma\|_\infty\ \ e^C\,. \end{aligned}$$

Now Proposition 2.5 and the narrow convergence of $\vartheta_{b_\delta}(t,\mu)$, $\vartheta_{c_\delta}(t,\mu)$ for $\delta \downarrow 0$ (as stated in Proposition 3.3) bridge the gap to $Q_\varepsilon(\vartheta_b, \vartheta_c; \mu)$. Indeed, for all $h \in \,]0,1]$,

$$\begin{aligned} q_\varepsilon\big(\vartheta_b(h,\mu),\ \vartheta_c(h,\mu)\big) \ &\le\ \limsup_{\delta \downarrow 0}\ \ q_\varepsilon\big(\vartheta_{b_\delta}(h,\mu),\ \vartheta_{c_\delta}(h,\mu)\big) \\ &\le\ \limsup_{\delta \downarrow 0}\ \ h \cdot Q_\varepsilon(\vartheta_{b_\delta}, \vartheta_{c_\delta}; \mu) \quad \big(1 + |\mu|(\mathbb{R}^N)\ e^{2\,C\,h} + 2\,C\,h\big)\ \ e^{\lambda_\varepsilon\, C\, h} \end{aligned}$$

$$\begin{array}{rcll} q_\varepsilon\big(\vartheta_b(h,\mu),\ \vartheta_c(h,\mu)\big) & \leq & \limsup\limits_{\delta\downarrow 0} & h\cdot\lambda_\varepsilon\ \|\sigma\|_\infty\ e^C\ \big\|\varphi_\varepsilon\ |b_\delta-c_\delta|\ \big\|_{L^1(\mathbb{R}^N)}\ \ \big(1+|\mu|(\mathbb{R}^N)\big)\ \ (1+O(h)) \\ & \leq & & h\cdot\lambda_\varepsilon\ \|\sigma\|_\infty\ e^C\ \big\|\varphi_\varepsilon\ |b-c|\ \big\|_{L^1(\mathbb{R}^N)}\ \ \big(1+|\mu|(\mathbb{R}^N)\big)\ \ (1+O(h)). \end{array}$$

due to the construction of b_δ, c_δ. Thus, we conclude from Lemmas 2.9 and 3.11

$$\begin{array}{rclll} Q_\varepsilon(\vartheta_b,\vartheta_c;\mu) & \leq & \sup\limits_{0\,\leq\,t\,<\,1} & \limsup\limits_{h\,\downarrow\,0} & \frac{1}{h}\cdot q_\varepsilon\big(\vartheta_b(h,\ \vartheta_b(t,\mu)),\ \ \vartheta_c(h,\ \vartheta_b(t,\mu))\big) \\ & \leq & \sup\limits_{0\,\leq\,t\,<\,1} & \lambda_\varepsilon & \Big\|\frac{\vartheta_b(t,\mu)}{\mathcal{L}^N}\Big\|_\infty\ e^C\ \ \ \big\|\varphi_\varepsilon\ |b-c|\ \big\|_{L^1(\mathbb{R}^N)}\ \ \big(1+|\vartheta_b(t,\mu)|(\mathbb{R}^N)\big) \\ & \leq & & \lambda_\varepsilon & \|\sigma\|_\infty\ e^C\ \ \ e^C\ \ \ \big\|\varphi_\varepsilon\ |b-c|\ \big\|_{L^1(\mathbb{R}^N)}\ \ \big(1+|\mu|(\mathbb{R}^N)\cdot e^{2\,C}\big). \end{array}$$

Finally, we have to verify that $\vartheta_b(h,\cdot):(\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\text{weak*})\longrightarrow(\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),p_{\varepsilon,\kappa})$ is continuous for every $h\in[0,1],\varepsilon\in\mathcal{J},\kappa\in\mathcal{I}$.
Let $\big(\mu_n=\sigma_n\,\mathcal{L}^N\big)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ converging weakly* to $\mu=\sigma\,\mathcal{L}^N\in\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$. Choose $h\in\,]0,1]$, $\delta>0$ and $\varphi\in C_0^0(\mathbb{R}^N)$ arbitrarily. Using the same Gaussian kernel $\rho\in C^1(\mathbb{R}^N,\,]0,\infty[)$ as before, we conclude from the well-known features of convolution and Proposition 3.3

$$\begin{array}{rcl} \displaystyle\int_{\mathbb{R}^N}\varphi*\rho_\delta\ \ d\,\vartheta_b(h,\mu) & = & \displaystyle\int_{\mathbb{R}^N}\varphi\ \ d\,\big(\vartheta_b(h,\mu)*\rho_\delta\big)\ \ =\ \ \int_{\mathbb{R}^N}\varphi\ \ d\,\vartheta_{b_\delta}(h,\mu) \\ & = & \displaystyle\int_{\mathbb{R}^N}\varphi\big(\mathbf{X}_{b_\delta}(h,\cdot)\big)\ \ \sigma\ \ \ d\mathcal{L}^N \\ & = & \displaystyle\lim_{n\,\to\,\infty}\int_{\mathbb{R}^N}\varphi\big(\mathbf{X}_{b_\delta}(h,\cdot)\big)\ \ \sigma_n\ \ \ d\mathcal{L}^N\ \ =\ \ \ldots \\ & = & \displaystyle\lim_{n\,\to\,\infty}\int_{\mathbb{R}^N}(\varphi*\rho_\delta)\ \ d\,\vartheta_b(h,\mu_n). \end{array}$$

Due to the uniform bound of total variation, i.e. $\sup\limits_{n\in\mathbb{N}}|\vartheta_b(h,\mu_n)|(\mathbb{R}^N)\ \leq\ \sup\limits_{n\in\mathbb{N}}|\mu_n|(\mathbb{R}^N)\cdot e^{2\,C}\ <\infty$, we obtain $\vartheta_b(h,\mu_n)\longrightarrow\vartheta_b(h,\mu)$ weakly* (with respect to $C_0^0(\mathbb{R}^N)$) for $n\longrightarrow\infty$ and, thus the claimed continuity of $\vartheta_b(h,\cdot)$ with respect to every $p_{\varepsilon,\kappa}$. □

Using the total variation $|\cdot|(\mathbb{R}^N)$ for the measures in $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ proves to be particularly helpful in regard to “weak” compactness. Indeed, the classical criterion of de la Vallée Poussin has the following immediate consequence:

Lemma 3.13 *The tuple* $\big(\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (p_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ (\lfloor\cdot\rfloor)_{\varepsilon\in\mathcal{J}},\ \mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)\big)$ *is weakly transitionally compact.* □

Proposition 3.14 *Suppose for* $f:\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)\times[0,T]\longrightarrow\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C\in[0,\infty[\ :\ \|f(\mu,t)\|_\infty+\big\|\mathrm{div}\,f(\mu,t)\big\|_\infty\leq C$ *for all* $(\mu,t)\in\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)\times[0,T]$,

2. $\forall\,\varepsilon\in\mathcal{J}:\quad\big\|\varphi_\varepsilon\ |f(\mu,t)-f(\mu_n,t_n)|\ \big\|_{L^1(\mathbb{R}^N)}\longrightarrow 0$ *whenever* $\mu_n\overset{*}{\rightharpoonup}\mu$ *(w.r.t.* $C_0^0(\mathbb{R}^N)$*)*, $t_n\searrow t$.

Then for every initial measure $\mu_0\in\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, *there exists a sleek solution* $\mu(\cdot):[0,T[\longrightarrow\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *to the generalized mutational equation* $\overset{\circ}{\mu}(\cdot)\ni\vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0)=\mu_0$. *Moreover,* $\mu(\cdot)$ *is distributional solution to the continuity equation* $\frac{d}{dt}\,\mu(t)\ +\ D_x\cdot\big(f(\mu(t),t)\ \mu(t)\big)\ =\ 0$ *in* $\mathbb{R}^N\times[0,T[$.

Proof. The existence of a right–hand sleek solution $\mu(\cdot)$ results from Proposition 2.21 applied to $\big(\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (p_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ (|\cdot|(\mathbb{R}^N))_{\varepsilon\in\mathcal{J}},\ \mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)\big)$ due to the characterization of sleek transitions in Proposition 3.12.
Now we verify the claimed distributional property of $t\mapsto\mu(t)=\sigma(t,\cdot)\,\mathcal{L}^N$. Indeed, the Lagrangian flow $\mathbf{X}_{f(\mu(t),t)}:[0,1]\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ of each $f(\mu(t),t)\in\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies for all $h\in[0,1]$ and $\mathcal{L}^N$–almost every $x\in\mathbb{R}^N$

$$\mathbf{X}_{f(\mu(t),t)}(h,x) \;=\; x + \int_0^h f(\mu(t),t)\big(\mathbf{X}_{f(\mu(t),t)}(s,x)\big)\; ds.$$

according to Proposition 3.3. As $\mu(\cdot)$ is constructed by Euler approximations being equi–Lipschitz continuous w.r.t. each q_ε ($\varepsilon \in \mathcal{J}$), it is also Lipschitz continuous w.r.t. to each q_ε and, we obtain at every time t of differentiability

$$\begin{aligned}
\tfrac{d}{dt} \int_{\mathbb{R}^N} \varphi_\varepsilon \; d\,\mu(t) \;&=\; \limsup_{h\downarrow 0}\; \tfrac{1}{h}\cdot \int_{\mathbb{R}^N} \big(\varphi_\varepsilon(\mathbf{X}_{f(\mu(t),\,t)}(h,x)) \;-\; \varphi_\varepsilon(x)\big) \;\; \sigma(t,x) \;\; d\mathcal{L}^N x\\
&=\; \int_{\mathbb{R}^N} \nabla\varphi_\varepsilon(x) \;\cdot\; f(\mu(t),\,t)(x) \qquad \sigma(t,x) \;\; d\mathcal{L}^N x.
\end{aligned}$$

Now every $\varphi \in C_c^\infty(\mathbb{R}^N,\, [0,\infty[)$ can be approximated by $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ with respect to the C^1 norm due to Lemma 3.6. Thus, $[0,T[\; \longrightarrow [0,\infty[,\;\; t \longmapsto \int_{\mathbb{R}^N} \varphi \;\; d\mu(t)$ is also absolutely continuous and satisfies

$$\tfrac{d}{dt} \int_{\mathbb{R}^N} \varphi \;\, d\,\mu(t) \;\;=\;\; \int_{\mathbb{R}^N} \nabla\varphi(x) \;\cdot\; f(\mu(t),\,t)(x) \;\;\; d\,\mu(t)\,(x) \qquad\qquad \text{for almost every } t \in [0,T[.$$

Moreover the condition $\varphi \geq 0$ is not required, i.e. the same features are guaranteed for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Indeed, choosing any auxiliary function $\xi \in C_c^\infty(\mathbb{R}^N,\, [0,\infty[)$ with $\xi \equiv \|\varphi\|_\infty + 1$ in $\mathbb{B}_1(\operatorname{supp}\varphi)$, we apply the previous results (about absolute continuity and its derivative) to $\varphi(\cdot) + \xi(\cdot) \geq 0,\;\; \xi(\cdot) \geq 0$. In regard to a solution in the distributional sense, let $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T[)$ be any test function. Then, the subsequent Lemma 3.15 implies

$$\begin{aligned}
-\int_{\mathbb{R}^N} \psi(\cdot,0)\;\; d\,\mu_0 \;\;&=\;\; \int_0^T \tfrac{d}{dt} \Big(\int_{\mathbb{R}^N} \psi(\cdot,t)\;\; d\,\mu(t)\Big)\;\, dt\\
&=\;\; \int_0^T \Big(\int_{\mathbb{R}^N} \partial_t\,\psi(\cdot,t)\;\; d\,\mu(t) \;+\;\; \tfrac{\partial}{\partial s} \int_{\mathbb{R}^N} \psi(\cdot,t)\;\; d\,\mu(s)\Big|_{s=t}\Big)\;\, dt\\
&=\;\; \int_0^T \Big(\int_{\mathbb{R}^N} \partial_t\,\psi(\cdot,t)\;\; d\,\mu(t) \;+\; \int_{\mathbb{R}^N} \nabla_x\psi(x,t)\;\,\cdot f(\mu(t),\,t)(x) \;\;\; d\,\mu(t)\,(x)\Big)\; dt.
\end{aligned}$$

□

Lemma 3.15 ([15, Lemma 2.5]) *Suppose* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ *to be locally absolutely continuous in each component and*

$$\wedge \left\{ \begin{array}{lcl}
\limsup\limits_{k\longrightarrow 0} \;\; \big\|\,\partial_1\, h(\,\cdot\,,\, k+\cdot\,) \;\;-\;\; \partial_1\, h(\,\cdot\,,\,\cdot\,)\,\big\|_{L^1([0,T])} &=& 0,\\
\limsup\limits_{k\longrightarrow 0} \;\; \big\|\,\partial_2\, h(\,\cdot\,,\, k+\cdot\,) \;\;-\;\; \partial_2\, h(\,\cdot\,,\,\cdot\,)\,\big\|_{L^1([0,T])} &=& 0.
\end{array}\right.$$

Then $[0,T] \longrightarrow \mathbb{R}^n,\;\; t \longmapsto h(t,t)$ *is absolutely continuous and*

$$\tfrac{d}{d\,t}\, h(t,t) \;\;=\;\; \Big(\tfrac{\partial}{\partial\, t_1}\, h(t_1,t_2) + \tfrac{\partial}{\partial\, t_2}\, h(t_1,t_2)\Big)\Big|_{t=t_1=t_2}.$$

Restricting our considerations to measures in $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ has now the additional advantage of a closer relationship between distributional solutions (to the continuity equation) and right–hand sleek solutions (to the corresponding mutational equation). The key tool here is the maximum principle for distributional solutions quoted in Lemma 3.17.

Proposition 3.16 *Suppose for* $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\; C \in [0,\infty[\; : \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \leq C$ *for all* $(\mu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$,

2. $\forall\; \varepsilon \in \mathcal{J}\;\;\; \exists\, L_\varepsilon \in [0,\infty[$, *modulus of continuity* $\omega_\varepsilon(\cdot) \geq 0$:
$\big\|\varphi_\varepsilon\, |f(\mu,s) - f(\nu,t)|\big\|_{L^1(\mathbb{R}^N)} \leq\; L_\varepsilon \cdot q_\varepsilon(\mu,\nu) + \omega_\varepsilon\,(|s-t|)$ *for all* $(\mu,s),(\nu,t)\in\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)\times[0,T]$.

Then for every initial measure $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$*, the right–hand sleek solution* $\mu(\cdot) : [0,T[\; \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *to the generalized mutational equation* $\overset{\circ}{\mu}(\cdot) \;\ni\; \vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0) = \mu_0$ *is unique.*
So every distributional solution $\mu(\cdot) : [0,T[\; \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *of* $\frac{d}{dt}\,\mu(t) \;+\; D_x \cdot (f(\mu(t),t)\;\; \mu(t)) \;=\; 0$ *in* $\mathbb{R}^N \times\,]0,T[$ *that is continuous with respect to each* q_ε ($\varepsilon \in \mathcal{J}$) *is unique.*

Proof. The uniqueness of right–hand sleek solutions results from Proposition 2.23. Indeed for right–hand sleek solutions $\mu_1, \mu_2 : [0,T[\longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ with the same initial datum $\mu_1(0) = \mu_2(0) = \mu_0 \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ and for sufficiently large $\rho > 0$, define the auxiliary function

$$\delta_\varepsilon(t) := \inf \Big\{ q_\varepsilon(\nu,\, \mu_1(t)) + q_\varepsilon(\nu,\, \mu_2(t)) \;\Big|\; \nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N),\ \lfloor \nu \rfloor_\varepsilon \le (\lfloor \mu_0 \rfloor_\varepsilon + \rho) \cdot e^{\lambda_\varepsilon C\, t} + \lambda_\varepsilon\, C\, t \Big\}.$$

The symmetry and triangle inequality of $q_\varepsilon(\cdot,\cdot)$ imply $\delta_\varepsilon(t) = q_\varepsilon(\mu_1(t),\, \mu_2(t))$ and, Proposition 2.23 (with $R_\varepsilon = 0,\ \delta_\varepsilon(0) = 0$) ensures $\delta_\varepsilon(\cdot) \equiv 0$ for every $\varepsilon \in \mathcal{J}$. Thus, $\mu_1 \equiv \mu_2$.

Now suppose $\mu(\cdot) : [0,T[\longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ to be distributional solution of

$$\tfrac{d}{dt}\, \mu(t) + D_x \cdot \big(f(\mu(t),t)\ \mu(t)\big) = 0 \qquad \text{in } \mathbb{R}^N \times\,]0,T[$$

that is continuous with respect to each q_ε $(\varepsilon \in \mathcal{J})$. Then for each $\widehat{t} \in\,]0,T[$, the restriction $\mu(\cdot)|_{[0,\widehat{t}]}$ is uniformly continuous with respect to each q_ε and thus, $f(\mu(\cdot),\cdot) : [0,\widehat{t}] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies the assumption of Proposition 3.14. So there exists the unique right–hand sleek primitive $\nu(\cdot) : [0,\widehat{t}] \longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ of $f(\mu(\cdot),\cdot)$ with $\nu(0) = \mu_0$ and, $\nu(\cdot)$ is also distributional solution of $\frac{d}{dt}\, \nu(t) + D_x \cdot \big(f(\mu(t),t)\ \nu(t)\big) = 0$ in $\mathbb{R}^N \times\,]0,\widehat{t}\,[$. The comparison principle of Lemma 3.17 implies $\nu(\cdot) \equiv \mu(\cdot)$. So $\mu(\cdot)$ is right–hand sleek solution of $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\,\cdot)}$ and thus unique. □

Lemma 3.17 *Let $\widetilde{b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfy the assumptions of Proposition 3.3.*
Then the comparison principle for distributional solutions to the continuity equation

$$\tfrac{d}{dt}\, \mu + D_x \cdot (\widetilde{b}\, \mu) = 0 \qquad \textit{in }]0,T[\, \times \mathbb{R}^N$$

holds in the class $\big\{ \widetilde{w}\, \mathcal{L}^N \,\big|\, \widetilde{w} \in L^\infty\big([0,T]; L^1(\mathbb{R}^N)\big) \cap L^\infty\big([0,T]; L^\infty(\mathbb{R}^N)\big) \cap C^0\big([0,T]; w^*\text{–}L^\infty(\mathbb{R}^N)\big) \big\}$.
In particular, distributional solutions are unique in this class.

Proof results from [1, Theorems 26, 34] (see also [18]). □

Acknowledgments. The author would like to thank Prof. Willi Jäger for arousing the interest in set-valued maps and geometric evolution problems and furthermore, Prof. Jean–Pierre Aubin and Hélène Frankowska for their continuous support while generalizing mutational equations. Prof. Luigi Ambrosio provided an excellent insight into the Cauchy problem for BV vector fields during the CIME course “Calculus of variations and nonlinear PDE” in June 2005. His results have proven to be the basis of the conclusions in this paper. The author is also grateful to Irina Surovtsova and Daniel Andrej for fruitful complementary discussions.
Finally he acknowledges the financial support provided through the European Community’s Human Potential Programme under contract HPRN-CT-2002-00281 [Evolution Equations] and the German Research Foundation (for participating in the CIME course) .

References

[1] Ambrosio, L. (2005): Transport equation and Cauchy problem for non–smooth vector fields, Lecture notes of C.I.M.E. course “Calculus of variations and non linear PDE” 2005 (to appear)

[2] Ambrosio, L. (2004): Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* **158**, No.2, pp. 227-260

[3] Ambrosio, L. (2000): Geometric evolution problems, distance function and viscosity solutions, in: Buttazzo, G. (ed.) et al., *Calculus of variations and partial differential equations. Topics on geometrical evolution problems and degree theory*, Springer-Verlag

[4] Ambrosio, L., Fusco, N. & Pallara, D. (2000): *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs

[5] Aubin, J.-P. & Lesne, A. (2005): Constructing and exploring wells of energy landscapes. *J. Math. Phys.* **46**, No.4, 043508, 16 pp.

[6] Aubin, J.-P. & Murillo-Hernandez, A. (2004): Morphological equations and sweeping processes, in: *Nonsmooth Mechanics and Analysis: Theoretical and Numerical Advances*, chapter 21, pp. 249–259, Kluwer

[7] Aubin, J.-P. (1999) : *Mutational and Morphological Analysis : Tools for Shape Evolution and Morphogenesis*, Birkhäuser-Verlag, Systems and Control: Foundations and Applications

[8] Aubin, J.-P. (1993): Mutational equations in metric spaces, *Set-Valued Analysis* **1**, pp. 3-46

[9] Di Perna, R.J. & Lions, P.-L.(1989): Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98**, pp. 511-547

[10] Doyen, L. (1994): Shape Lyapunov functions and stabilization of reachable tubes of control problems. *J. Math. Anal. Appl.* **184**, No.2, pp. 222–228.

[11] Gorre, A. (1997): Evolutions of tubes under operability constraints. *J. Math. Anal. Appl.* **216**, No.1, pp. 1–22

[12] Lorenz, Th. (2008): Generalizing mutational equations for uniqueness of some nonlocal first–order geometric evolutions, to appear in *Set–Valued Analysis*. (IWR Preprint available at `http://www.ub.uni-heidelberg.de/archiv/5580`)

[13] Lorenz, Th. (2007): Evolution equations in ostensible metric spaces: First-order evolutions of nonsmooth sets with nonlocal terms, to appear in *Discussiones Mathematicae. Diff. Inclusions, Control and Optim.*, Preprint available at `http://www.ub.uni-heidelberg.de/archiv/7392`

[14] Lorenz, Th. (2007): Radon measures solving the Cauchy problem of the nonlinear transport equation, Preprint available at `http://www.ub.uni-heidelberg.de/archiv/7252`

[15] Lorenz, Th. (2006): Reynold's transport theorem for differential inclusions, *Set–Valued Analysis* **14**, No.3, pp. 209-247

[16] Lorenz, Th. (2005) : Evolution equations in ostensible metric spaces: Definitions and existence. IWR Preprint available at `http://www.ub.uni-heidelberg.de/archiv/5519`

[17] Lorenz, Th. (2005) : First–order evolutions of nonsmooth boundaries with nonlocal terms. IWR Preprint available at `http://www.ub.uni-heidelberg.de/archiv/5520`

[18] Maniglia, St. (2007): Probabilistic representation and uniqueness results for measure-valued solutions of transport equations, *J. Math. Pures Appl.* (9) **87**, no. 6, pp. 601-626

[19] Panasyuk, A.I. (1995): Quasidifferential equations in a complete metric space under conditions of the Carathéodory type. I, *Differ. Equations* **31**, No.6, pp. 901-910

[20] Panasyuk, A.I. (1995): Quasidifferential equations in a complete metric space under Carathéodory-type conditions. II, *Differ. Equations* **31**, No.8, pp. 1308-1317

[21] Pichard, K. & Gautier, S. (2000): Equations with delay in metric spaces: the mutational approach. *Numer. Funct. Anal. Optim.* **21**, No.7–8, pp.917–932.