Quasilinear continuity equations of measures for bounded BV vector fields

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Abstract. The focus of interest here is a quasilinear form of the conservative continuity equation

$$\frac{d}{dt}\,\mu \;+\; D_x \cdot \big(f(\mu,\cdot)\;\mu\big) \;=\; 0 \qquad\qquad (\text{in } \mathbb{R}^N \times \,]0,T[)$$

together with its measure–valued distributional solutions $\mu(\cdot) : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^N)$. On the basis of Ambrosio's results about the nonautonomous linear equation $\frac{d}{dt}\,\mu \,+\, D_x \cdot (b\;\mu) \;=\; 0$ (see [1, 2]), the existence and uniqueness are investigated for

1. vector fields in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ and positive Radon measures on $\mathbb{R}^N$ with compact support,
2. bounded vector fields having bounded (spatial) variation and $\mathcal{L}^N$ absolutely continuous divergence in combination with positive measures absolutely continuous with respect to Lebesgue measure $\mathcal{L}^N$.

The step towards the nonlinear problem here relies on a further generalization of Aubin's mutational equations that is extending the notions of distribution–like solutions and "weak compactness" to a set supplied with a countable family of (possibly non–symmetric) distance functions (so–called ostensible metrics).

Contents

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1 Introduction

The homogeneous conservative continuity equation $\frac{d}{dt}\,\mu \;+\; D_x \cdot (\widetilde{b}\;\mu) \;=\; 0$ (in $\mathbb{R}^N \times [0, T[$) is the classical analytical tool for describing the conservation of some real–valued quantity $\mu = \mu(t, x)$ while "flowing" (or, rather, evolving) along a given vector field $\widetilde{b} : \mathbb{R}^N \times [0, T] \longrightarrow \mathbb{R}^N$. Thus, it is playing a key role in many applications of modelling like fluid dynamics and, it has been investigated under completely different types of assumptions about $\widetilde{b}(\cdot, \cdot)$.

Seizing the results of Ambrosio [1, 2] later, the values of all solutions considered here are positive finite Radon measures on $\mathbb{R}^N$ and, we are interested in (structurally) weak assumptions on these measures or the vector fields for proving existence of a distributional solution of the *quasilinear* continuity equation

$$\tfrac{d}{dt}\,\mu(t) \;+\; D_x \cdot \big(f(\mu(t), t)\;\mu(t)\big) \;=\; 0 \qquad\qquad \text{(in } \mathbb{R}^N \times\,]0, T[)$$

with given $f(\cdot, \cdot) : \mathcal{M}(\mathbb{R}^N) \times [0, T] \longrightarrow L^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Basically two cases are investigated more closely :

Firstly (i.e. in section 3), all vector fields are supposed to be uniformly bounded and locally Lipschitz continuous with respect to space, i.e. $f(\mu, t) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|f\|_\infty \leq C < \infty$. Due to this regularity assumption, the measure–valued solution of the continuity equation $\frac{d}{dt}\,\mu \;+\; D_x \cdot (b\;\mu) \;=\; 0$ (with such a vector field b) can be easily represented by the method of characteristics : $\mu_t := \mathbf{X}_b(t, \cdot)_\sharp\,\mu_0$ with $\mathbf{X}_b(\cdot, x_0) : [0, T] \longrightarrow \mathbb{R}^N$ denoting the absolutely continuous solution of the Cauchy problem

$$\wedge \left\{ \begin{array}{rcll} \frac{d}{dt}\,x(t) & = & b(x(t)) & \text{a.e. in } [0, T], \\ x(0) & = & x_0 & \end{array} \right.$$

and the index $\sharp$ abbreviating the push–forward of a measure (quoted in Proposition 3.5). In return for this (quite popular, but restrictive) Lipschitz continuity of vector fields, admissible Radon measures only have to be positive and to satisfy the condition of compact support. Their set is abbreviated as $\mathcal{M}^+_c(\mathbb{R}^N)$. So in particular, Hausdorff measures $\mathcal{H}^\delta$ of arbitrary dimension $\delta \in [0, N]$ – restricted to a compact $\mathcal{H}^\delta$–rectifiable subset of $\mathbb{R}^N$ – can be considered.
For specifying the continuity properties of "velocity" function $f : \mathcal{M}^+_c(\mathbb{R}^N) \times [0, T] \longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$, generalized distance functions for finite Radon measures on $\mathbb{R}^N$ are required. We use the pseudo–metrics $p_\varepsilon(\mu,\, \nu) \;:=\; \big|\big(\varphi_\varepsilon \cdot (\mu - \nu)\big)\,(\mathbb{R}^N)\big|$ $(\varepsilon \in \mathcal{J})$. Here $(\varphi_\varepsilon)_{\varepsilon \in \mathcal{J}}$ is an arbitrary countable family of smooth Schwartz functions $\mathbb{R}^N \longrightarrow\,]0, \infty[$ satisfying $|\nabla\,\varphi_\varepsilon| \leq \lambda_\varepsilon \cdot \varphi_\varepsilon$ with a constant $\lambda_\varepsilon > 0$ and two density conditions (formulated in Lemma 3.1). Then the first main result is a consequence of Proposition 3.11 :

Proposition 1.1 *Suppose for* $f : \mathcal{M}^+_c(\mathbb{R}^N) \times [0, T] \longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$:

1. $\exists\; C \in [0, \infty[\; : \|f(\mu, t)\|_\infty \leq C$ *for all* $(\mu, t) \in \mathcal{M}^+_c(\mathbb{R}^N) \times [0, T]$,
2. *("weak*" continuity of* f*)* $\|f(\mu, t) - f(\mu_n, t_n)\|_\infty \longrightarrow 0$ *if* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C_0(\mathbb{R}^N)$*),* $t_n \to t$.

Then for every initial $\mu_0 \in \mathcal{M}^+_c(\mathbb{R}^N)$*, there exists a distributional solution* $\mu(\cdot) : [0, T[\, \longrightarrow \mathcal{M}^+_c(\mathbb{R}^N)$ *of the continuity equation* $\frac{d}{dt}\,\mu(t) \;+\; D_x \cdot (f(\mu(t), t)\;\mu(t)) \;=\; 0$ *in* $\mathbb{R}^N \times [0, T[$ *with* $\mu(0) = \mu_0$.

Secondly (i.e. in section 4), the regularity condition on vector fields is weakened. We dispense with their local Lipschitz continuity and follow the track of Ambrosio [1, 2]. So now we use vector fields in

$$\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N) \;:=\; \big\{ b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^N) \;\big|\; D \cdot b \;=\; \mathrm{div}\, b\;\, \mathcal{L}^N \ll \mathcal{L}^N,\;\; \mathrm{div}\, b \in L^\infty(\mathbb{R}^N) \big\}$$

and positive measures on $\mathbb{R}^N$ that are bounded and absolutely continuous w.r.t. Lebesgue measure, i.e.

$$\mathbb{L}^{\infty \,\cap\, 1}(\mathbb{R}^N) \;:=\; \big\{ \rho\,\mathcal{L}^N \;\big|\; \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\;\; \rho \geq 0 \big\}.$$

The gap between $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ and $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ is bridged by Ambrosio's proof of his superposition principle ([1], Theorem 12). Indeed, for every function $\widetilde{b} \in L^1\big([0,T],\ \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)\big)$ with $\sup_t \big(\|\widetilde{b}(t,\cdot)\|_\infty + \|\mathrm{div}_x\, \widetilde{b}(t,\cdot)\|_\infty\big) < \infty$, each distributional solution $\mu(\cdot)$ of $\frac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\ \mu) = 0$ can be approximated narrowly by the unique solution $\mu_\delta(t) := \mu(t) * \rho_\delta$ of $\frac{d}{dt}\,\mu_\delta + D_x \cdot (\widetilde{b}_\delta\ \mu_\delta) = 0$ with $\widetilde{b}_\delta(t,\cdot) := \frac{(\widetilde{b}(t,\cdot)\ \mu(t)) * \rho_\delta}{\mu_\delta(t)}$ being in $L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ and $\rho_\delta(\cdot)$ denoting a smooth Gaussian kernel. So the approximation (with respect to narrow convergence) is the key tool for extending many estimates concerning vector fields in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty$ to vector fields in $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$.

Our main results about this type of continuity equation are consequences of Propositions 4.10, 4.11 using now the countable family of pseudo–metrics $q_\varepsilon(\mu,\ \nu) := \big|\varphi_\varepsilon \cdot (\mu - \nu)\big|(\mathbb{R}^N)\ \ (\varepsilon \in \mathcal{J})$:

Proposition 1.2 *Suppose for* $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C \in [0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$,

2. $\|f(\mu,t) - f(\mu_n,t_n)\|_{L^1(\mathbb{R}^N)} \longrightarrow 0$ *if* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C_0(\mathbb{R}^N)$*),* $t_n \searrow t$.

Then for every $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, *there exists a distributional solution* $\mu(\cdot) : [0,T[\ \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *of the continuity equation* $\frac{d}{dt}\,\mu(t) + D_x \cdot (f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N \times [0,T[$ *with* $\mu(0) = \mu_0$.

Proposition 1.3 *Suppose for* $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C \in [0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$,

2. $\forall\ \varepsilon \in \mathcal{J}\ \ \exists\ L_\varepsilon \in [0,\infty[$, *modulus of continuity* $\omega_\varepsilon(\cdot) \ge 0$:
$\big\|\varphi_\varepsilon\, |f(\mu,s) - f(\nu,t)|\,\big\|_{L^1(\mathbb{R}^N)} \le\ L_\varepsilon \cdot q_\varepsilon(\mu,\nu) + \omega_\varepsilon\,(|s-t|)$ *for all* $(\mu,s),(\nu,t) \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T]$.

Then for every initial measure $\mu_0 \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, *the distributional solution* $\mu(\cdot) : [0,T[\ \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ *of* $\frac{d}{dt}\,\mu(t) + D_x \cdot (f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N \times]0,T[$ *that is continuous with respect to each* q_ε $(\varepsilon \in \mathcal{J})$ *is unique.*

In particular, the continuity conditions on $f(\cdot,\cdot)$ are (slightly) weaker in Proposition 1.2 than in Proposition 1.1 because the L^1 norm is used instead of the L^∞ norm. Moreover, separating the time dependence from spatial measures opens the door to taking (spatially) nonlocal effects into consideration — to some extent.

Restricting our considerations to measures in $\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ has two analytical advantages in addition : The first one is the uniqueness of distributional solutions of the nonautonomous continuity equation. To be more precise, Ambrosio extends the DiPerna–Lions theory [9] to the case of (spatially) BV vector fields. Any distributional solution $w \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^N \times]0,T[\big)$ of $\frac{d}{dt}\,w + D_x \cdot (\widetilde{b}\ w) = \widetilde{c}$ is shown to be "renormalized" (in the sense of DiPerna and Lions) if $\widetilde{c} \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N \times]0,T[\big)$ and whenever the vector field $\widetilde{b} \in L^1_{\mathrm{loc}}\big([0,T],\ \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ has the distributional divergence $D_x \cdot \widetilde{b}(t,\cdot) = \mathrm{div}_x\, \widetilde{b}(t,\cdot)\ \mathcal{L}^N \ll \mathcal{L}^N$ for $\mathcal{L}^1$–almost every $t \in [0,T]$ ([1], Theorem 34). Then additional integral conditions on $\widetilde{b}$ and $\mathrm{div}_x\, \widetilde{b}$ imply the comparison principle for the continuity equation $\frac{d}{dt}\ w + D_x \cdot (\widetilde{b}\ w) = 0$ in the class $\big\{\, w \in L^\infty\big([0,T]; L^1(\mathbb{R}^N)\big) \cap L^\infty\big([0,T]; L^\infty(\mathbb{R}^N)\big) \cap C^0\big([0,T]; w^*\text{–}L^\infty(\mathbb{R}^N)\big)\big\}$ ([1], Theorem 26). This result is used here for specifying sufficient conditions on $f : \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ for the uniqueness of distributional solutions in Proposition 1.3.

The second advantage is provided by the area formula since it leads to an exponential growth condition of the total variation $\big|\mu(t)\big|(\mathbb{R}^N)$ and of $\big\|\frac{\mu(t)}{\mathcal{L}^N}\big\|_\infty$ for a solution $\mu(\cdot) : [0,T[\ \longrightarrow \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ of $\frac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\ \mu) = 0$ (i.e. for the solution induced by the Lagrangian flow specified in Proposition 4.1 and proven in [1, 2]). Such a priori estimates lay the basis for applying the compactness criterion of de la Vallée Poussin.

So all in all, the aim of this paper to extend Ambrosio's results of [1] to the *quasilinear* continuity equation $\frac{d}{dt}\,\mu(t) \;+\; D_x \cdot \big(f(\mu(t),t)\;\; \mu(t)\big) \;=\; 0 \qquad$ (in $\mathbb{R}^N \times\,]0,T[$).
For such a step from a *linear* to a *nonlinear* equation, we rely on a tool here that is hardly known in the PDE community, but we regard it as very useful indeed : *generalized mutational equations.*
Mutational equations introduced by Aubin [7, 8] are to extend ordinary differential equations to a metric space (E, d). For dispensing with any vector space structure, the key idea is to introduce "maps of elementary deformation" $\vartheta : [0,1] \times E \longrightarrow E$. Such a so–called *transition* specifies the point $\vartheta(t,x) \in E$ to which an initial point $x \in E$ has been moved after time $t \in [0,1]$. It can be interpreted as a generalized derivative of a curve $\xi : [0,T[\longrightarrow E$ at time $t \in [0,T[$ if it provides a first–order approximation in the sense of $\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot d\big(\xi(t+h),\;\; \vartheta(h, \xi(t))\big) \;=\; 0$
The theory of mutational equations has already been applied to different types of evolution problems in metric spaces (see [5, 6, 10, 12, 19], for example). So–called morphological equations are a very popular geometric example and use compact reachable sets of differential inclusions (supplied with the Pompeiu–Hausdorff distance) as transitions.
Aubin's continuity conditions on transitions, however, seem to be quite restrictive. So there have been several approaches of weakening them [13, 14, 15]. The main aspects of generalizing so far have been

1. introduce a separate real component of time, i.e. consider $\widetilde{E} := \mathbb{R} \times E$ instead of the given set E,
2. dispense with the symmetry of the metric d, i.e. we use a countable family $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$ of (*timed*) *ostensible metrics* satisfying only the reflexivity condition and the (timed) triangle inequality,
3. extend the notion of distributional solutions to so–called *timed forward solutions* in such a way that a (fixed) structural estimate is preserved while comparing shortly with the evolutions of all "test elements" $\widetilde{x} \in \widetilde{\mathcal{D}}$,
4. implement the notion of Petrov–Galerkin, i.e. "test elements" need not belong to the same set $\widetilde{E}$ as the values of generalized solutions (see [15], in particular),
5. permit many parameters of transitions to depend on the "test element" and the index ε of (timed) ostensible metric (in a word, for increasing "degrees of freedom" while extending Euler algorithm).

Up to now, first–order geometric evolutions have been our main motivation for generalizing mutational equations, i.e. compact subsets of $\mathbb{R}^N$ are to evolve according to nonlocal properties of both the set and their (limiting) normal cones at the boundary (see [14, 15], in particular). Now the continuity equation motivates two further aspects of generalization :
Firstly, we modify the continuity conditions on transitions so that some of their continuity parameters might have linear growth (with respect to the "initial element"). This feature can be particularly useful whenever the theory is applied to vector spaces – like the semilinear evolution equations in reflexive Banach spaces (mentioned in [14]) or the positive Radon measures in §§ 3, 4.
Secondly, the form of sequential compactness is weakened for proving existence of "right–hand sleek solutions". We assume each ostensible metric $\widetilde{q}_\varepsilon$ to be the supremum of (at most) countably many generalized distance functions $\widetilde{q}_{\varepsilon,\kappa}$ $(\kappa \in \mathcal{I})$. In this analytical environment, "weak sequential compactness" is to realize the notion that every "bounded" sequence has a subsequence converging with respect to each $\widetilde{q}_{\varepsilon,\kappa}$. In fact, this concept generalizes the definition of weak compactness in a (separable) real Banach space $(X, \|\cdot\|_X)$ since $\|z\|_X \;=\; \sup\; \big\{\, y^*(z) \;\big|\; y^* : X \longrightarrow \mathbb{R}$ linear, continuous, $\|y^*\|_{X^*} \le 1 \,\big\}$.
With regard to generalized mutational equations, we introduce the more general term "weakly timed transitionally compact" in Definition 2.20. A more detailed presentation of this theory is given in § 2.

Then, in § 3, it is applied to positive Radon measures on $\mathbb{R}^N$ with compact support that are "evolving" along vector fields of $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ – according to the continuity equation. Finally in § 4, we consider vector fields of $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ and positive measures of $\mathbb{L}^{\infty \,\cap\, 1}(\mathbb{R}^N)$ instead.

Notations

$C_c^0(\mathbb{R}^N)$ denotes the space of continuous functions $\mathbb{R}^N \longrightarrow \mathbb{R}$ with compact support and $C_0(\mathbb{R}^N)$ its closure with respect to the sup norm, respectively. $C_c^0(\mathbb{R}^N, \mathbb{R}_0^+)$ abbreviates the subset of functions $\varphi \in C_c^0(\mathbb{R}^N)$ with $\varphi \geq 0$ and correspondingly, $C_0(\mathbb{R}^N, \mathbb{R}_0^+)$ its closure.
Furthermore, $\mathcal{M}(\mathbb{R}^N)$ consists of all finite real–valued Radon measures on $\mathbb{R}^N$. As a consequence of Riesz theorem, it is the dual space of $C_0(\mathbb{R}^N)$ (see e.g. [4], Remark 1.57). Similarly $\mathcal{M}^+(\mathbb{R}^N)$ denotes the set of all positive Radon measures on $\mathbb{R}^N$, i.e. $\mathcal{M}^+(\mathbb{R}^N) := \{\mu \in \mathcal{M}(\mathbb{R}^N) \mid \mu(\cdot) \geq 0\}$.
Finally, $\mathcal{M}_c(\mathbb{R}^N) := \{\mu \in \mathcal{M}(\mathbb{R}^N) \,|\, \exists \text{ compact } K \subset \mathbb{R}^N\colon |\mu|(\mathbb{R}^N \setminus K) = 0\}$, $\mathcal{M}_c^+(\mathbb{R}^N) := \mathcal{M}_c(\mathbb{R}^N) \cap \mathcal{M}^+$.

2 Generalizing evolution equations to ostensible metric spaces : Mutational equations

In this section, the definitions and main results are presented although the measure–valued examples ensuing here do not require this concept in its most general form. There are basically two reasons for the complete presentation. Firstly, we want to present two additional steps of generalization motivated by both the measure–valued flow along vector fields (in §§ 3, 4) and the semilinear evolution equations in reflexive Banach spaces (mentioned in [14]): allowing a form of "linear growth" for some parameters of transitions and assuming a "weak(er)" form of sequential compactness for proving existence.
Secondly, all previous versions prove to be special cases of the subsequent modification. So in particular, the examples of [7], [14, 15] and the ensuing sections can be combined in systems arbitrarily. From our point of view, this property of generalized mutational equations is an essential advantage in comparison with viscosity solutions (and other concepts based on the maximum or inclusion principle).

General assumptions for § 2.

1. Let E and $\mathcal{D}$ denote nonempty sets (not necessarily $\mathcal{D} \subset E$),
   $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E, \quad \widetilde{\mathcal{D}} \overset{\text{Def.}}{=} \mathbb{R} \times \mathcal{D}, \quad \pi_1 : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow \mathbb{R},\ (t,x) \longmapsto t.$
2. $\mathcal{J} \subset [0,1]^K$ abbreviates a countable index set with $K \in \mathbb{N}$, $0 \in \overline{\mathcal{J}}$.
3. $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ satisfies the timed triangle inequality (for each $\varepsilon \in \mathcal{J}$),
   i.e. $\widetilde{q}_\varepsilon\big((r,x),\ (t,z)\big) \ \leq\ \widetilde{q}_\varepsilon\big((r,x),\ (s,y)\big) \ +\ \widetilde{q}_\varepsilon\big((s,y),\ (t,z)\big)$
   for all $(r,x), (s,y), (t,z) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $r \leq s \leq t$.
4. Fix $\lfloor \cdot \rfloor_\varepsilon : \widetilde{\mathcal{D}} \cup \widetilde{E} \longrightarrow [0, \infty[$ for each $\varepsilon \in \mathcal{J}$.
5. $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ fulfills $\widetilde{q}_\varepsilon(\widetilde{z},\, i_{\widetilde{\mathcal{D}}}\, \widetilde{z}) = 0$, $\pi_1\, \widetilde{z} = \pi_1\, i_{\widetilde{\mathcal{D}}}\, \widetilde{z}$, $\lfloor \widetilde{z} \rfloor_\varepsilon \geq \lfloor i_{\widetilde{\mathcal{D}}}\, \widetilde{z} \rfloor_\varepsilon$
   for every $\widetilde{z} \in \widetilde{\mathcal{D}}, \varepsilon \in \mathcal{J}$.

Assumption (4.) lays the foundation for the first new aspect in comparison with earlier definitions. For allowing a form of "linear growth" for some parameters of transitions, we need a counterpart of norms. Roughly speaking, it is to measure the absolute magnitude of any element $\widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ whereas each $\widetilde{q}_\varepsilon$ rather "compares" two elements with each other. Dispensing with any vector space structure, however, this counterpart is supposed to be just a nonnegative function that might even depend on $\varepsilon \in \mathcal{J}$, namely $\lfloor \cdot \rfloor_\varepsilon : \widetilde{\mathcal{D}} \cup \widetilde{E} \longrightarrow [0, \infty[$.
Generalizing mutational equations, the key idea is now to preserve the following structural estimate for comparing transitions $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h, \widetilde{z}),\ \ \widetilde{\tau}(t_2+h, \widetilde{y})\big) \\
\leq\ \ &\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \ \widetilde{\tau}(t_2, \widetilde{y})\big) \ \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}\ \ + \\
&h \cdot \Big(\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \cdot \big(1 + \lfloor \widetilde{y} \rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2+h)} + \zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2+h)\big)\ +\ \gamma_\varepsilon(\widetilde{\tau})\Big)\Big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}.
\end{aligned}$$

for all $\widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ \varepsilon \in \mathcal{J}$ and $0 \le t_1 \le t_2 < 1,\ h \ge 0$ with $\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}$ and $t_1 + h$ sufficiently small (depending only on $\widetilde{\vartheta}, \widetilde{z}$). In comparison with the last modification (in [15]), the new feature is the factor $\big(1 + \lfloor \widetilde{y} \rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau})\,(t_2+h)} + \zeta_\varepsilon(\widetilde{\tau})\ (t_2+h)\big) \le 1 + \lfloor \widetilde{y} \rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau})} + \zeta_\varepsilon(\widetilde{\tau})$ allowing the second summand on the right–hand side to share the "linear growth" of the compared element $\widetilde{y}$ (with respect to $\lfloor\cdot\rfloor_\varepsilon$). A corresponding dependence on $\lfloor \widetilde{z} \rfloor_\varepsilon$ can be regarded as part of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$.

For the subsequent definition of solution, this modification will be hardly relevant. Indeed, a transition $\widetilde{\vartheta}$ again induces a "first–order approximation" of a curve $\widetilde{x} : [0,T[\ \longrightarrow \widetilde{E}$ at time $t \in [0,T[$ according to the following (still vague) idea : Comparing $\widetilde{x}(t+\cdot)$ with $\widetilde{\vartheta}(\cdot, \widetilde{z})$ shortly (for any test element $\widetilde{z} \in \widetilde{\mathcal{D}},\ \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$), the same structural estimate ought to hold as if the factor $\widetilde{Q}_\varepsilon(\cdot, \cdot\,; \widetilde{z})$ was 0 (see Definitions 2.10, 2.12 for details).

2.1 Transitions as "elementary deformations"

Now we specify the new definition of "timed sleek transition" for the tuple that has an additional component in comparison with earlier versions (see [15] in particular) : $\Big(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}\Big)$.

Definition 2.1 *A map* $\widetilde{\vartheta} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ *is called* timed sleek transition *on* $(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0, \cdot) = \mathrm{Id}_{\widetilde{\mathcal{D}} \cup \widetilde{E}}$,
2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \ge 0 : \quad \limsup\limits_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0 \quad$ *and*
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \le \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x}))\big) \le \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$$
3. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} \quad \exists\ \alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in [0,\infty[, \quad \mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in\,]0,1] :$
$$\limsup_{h\downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\vartheta}(h, \widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\, \widetilde{y}) - \gamma_\varepsilon(\widetilde{\vartheta})\ h}{h\ \ \big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\, \widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\ h\big)} \right)^{+} \le \alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \qquad \forall\ 0 \le t < \mathbb{T}_\varepsilon,\ \ \widetilde{y} \in \widetilde{E}\ \ (t + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}),$$
4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) : \,]0,1] \longrightarrow [0,\infty[\quad$ *modulus of continuity*:
$$\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(s, \widetilde{y}),\ \ \widetilde{\vartheta}(t, \widetilde{y})\big) \le \beta_\varepsilon(\widetilde{\vartheta})(t-s) \cdot (1 + \lfloor \widetilde{y} \rfloor_\varepsilon) \qquad \forall\ s < t \le 1,\ \ \widetilde{y} \in \widetilde{E},$$
5. $\exists\ \zeta_\varepsilon(\widetilde{\vartheta}) \ge 0 : \quad \lfloor \widetilde{\vartheta}(t, \widetilde{x}) \rfloor_\varepsilon \le \lfloor \widetilde{x} \rfloor_\varepsilon \cdot e^{\zeta_\varepsilon(\widetilde{\vartheta})\ t} + \zeta_\varepsilon(\widetilde{\vartheta})\ t \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1],$
6. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} : \qquad \widetilde{\vartheta}(t, \widetilde{z}) \in \widetilde{\mathcal{D}} \qquad \forall\ t \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})],$
7. $\limsup\limits_{h\downarrow 0}\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}\big) \ge \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}\big) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ t \le \mathbb{T}_\varepsilon\ \ (t + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}),$
8. $$\widetilde{\vartheta}\big(h,\ (t,y)\big) \in \{t+h\} \times E \subset \widetilde{E} \qquad \forall\ (t,y) \in \widetilde{E},\quad h \in [0,1],$$
$$\pi_1\,\widetilde{\vartheta}(h,\ (t,z)) = t + \mathrm{const}(\widetilde{E}, \widetilde{\mathcal{D}}) \cdot h \le t + h \qquad \forall\ (t,z) \in \widetilde{\mathcal{D}},\quad h \in [0,1].$$
9. $$\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, i_{\widetilde{\mathcal{D}}}\,\widetilde{z})),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{z}))\big) \le \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}),$$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}),\ \ \widetilde{\vartheta}(t+h,\ \ \widetilde{z})\,\big) \le \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}).$$

Remark 2.2 Conditions (4.) and (5.) provide the only new features in comparison with earlier concepts [13, 14, 15]. Roughly speaking, property (5.) bounds the "absolute magnitude" of $\widetilde{\vartheta}(\cdot, \widetilde{x})$ to uniform exponential growth for each initial element $\widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E}$. As its main advantage, we can always restrict our considerations to bounded subsets of $\widetilde{E}$ (with respect to $\lfloor\cdot\rfloor_\varepsilon$), so for example, the Euler approximations for proving existence.

Condition (4.) now allows $\vartheta(\cdot, \widetilde{y})$ that the modulus of time continuity has linear growth with respect to the initial element $\widetilde{y} \in \widetilde{E}$.

The other conditions have already been explained in earlier concepts. For the purpose of a self–contained presentation here, we briefly motivate their key points :

Condition (2.) can be regarded as a weakened form of the semigroup property. It consists of two demands as $\widetilde{q}_\varepsilon$ need not be symmetric. Condition (3.) specifies the continuity property of $\widetilde{\vartheta}$ with respect to the initial point. In particular, the first argument of $\widetilde{q}_\varepsilon$ is restricted to elements $\widetilde{z}$ of the "test set" $\widetilde{\mathcal{D}}$ and, $\alpha_\varepsilon(\widetilde{\vartheta})$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon(\cdot) < \infty$ uniformly in some applications.

Condition (6.) guarantees that every $\widetilde{z} \in \widetilde{\mathcal{D}}$ stays in the "test set" $\widetilde{\mathcal{D}}$ for short times at least. This assumption is required because estimates using the parameter $\alpha_\varepsilon(\cdot)$ can be ensured only within this period. Further conditions on $\mathbb{T}_\varepsilon(\widetilde{\vartheta}, \cdot) > 0$ are avoidable for proving existence of solutions, but they are used for uniqueness (as in [13] and Propositions 2.14, 2.24).

Condition (7.) forms the basis for applying Gronwall's Lemma that has been extended to semicontinuous functions in [13] (see Lemma 2.6). Indeed, every function $\widetilde{y} : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{q}_\varepsilon(\widetilde{y}(t-h),\ \widetilde{y}(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}(t)\Big) \ \leq\ \limsup_{h \downarrow 0}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h, \widetilde{z}),\ \ \widetilde{y}(t-h)\Big).$$

for all elements $\widetilde{z} \in \widetilde{\mathcal{D}}$ with $\pi_1\, \widetilde{\vartheta}(\cdot, \widetilde{z}) \ \leq\ \pi_1\, \widetilde{y}(\cdot)$ and times $t \in\ ]\,0,\ \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{z})]$.

Condition (8.) describes the real "time" component of $\widetilde{\vartheta}(\cdot, \widetilde{y})$. For initial elements $\widetilde{y} \in \widetilde{E}$, the time component has to be additive whereas for "test elements" $\widetilde{z} \in \widetilde{\mathcal{D}}$, the time component of $\widetilde{\vartheta}(\cdot, \widetilde{z})$ might increase more slowly.

Finally, condition (9.) provides a counterpart of the general assumption $\widetilde{q}_\varepsilon(\widetilde{z},\, i_{\widetilde{\mathcal{D}}}\, \widetilde{z}) = 0$ for each $\widetilde{z} \in \widetilde{\mathcal{D}}$. As $\widetilde{q}_\varepsilon$ need not be symmetric, it is required for estimating the distance between a timed forward transition $\widetilde{\vartheta}(\cdot, \widetilde{z})$ and a timed right–hand forward solution (see proofs of Proposition 2.22 and Lemma 2.19).

Definition 2.3
$\widetilde{\Theta}(\widetilde{E},\, \widetilde{\mathcal{D}},\, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$ denotes a set of timed sleek transitions on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor \cdot \rfloor_\varepsilon))$ assuming

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \ :=\ \sup_{\substack{t \,<\, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}),\ \widetilde{y} \,\in \widetilde{E} \\ t + \pi_1\, \widetilde{z} \ \leq\ \pi_1\, \widetilde{y}}} \ \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y})\Big) \ - \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}\Big) \,\cdot\, e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot h}}{h \quad (1 + \lfloor \widetilde{y} \rfloor_\varepsilon)} \right)^{\!+}$$

to be finite for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E},\, \widetilde{\mathcal{D}},\, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}\ (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ \varepsilon \in \mathcal{J}$.

Remark 2.4 Due to the dependence on the initial "test element" of $\widetilde{\mathcal{D}}$, the triangle inequality for $\widetilde{Q}_\varepsilon(\cdot, \cdot; \widetilde{z})$ cannot be expected to hold in general. The factor $(1 + \lfloor \widetilde{y} \rfloor_\varepsilon)$ in the denominator is a new feature in comparison with earlier concepts and again takes the (possible) linear growth into consideration. Its consequences for the structural estimate are specified in Proposition 2.5.

Proposition 2.5 *Let $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ be timed sleek transitions on $(\widetilde{E},\, \widetilde{\mathcal{D}},\, (\widetilde{q}_\varepsilon), (\lfloor \cdot \rfloor_\varepsilon))$. Suppose $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\widetilde{y} \in \widetilde{E}$ and $0 \leq t_1 \leq t_2 \leq 1$, $h \geq 0$ with $\pi_1\, \widetilde{z} \leq \pi_1\, \widetilde{y}$, $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$, $t_2 + h \leq 1$. Then,*

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1 + h, \widetilde{z}),\ \ \widetilde{\tau}(t_2 + h, \widetilde{y})\big) \\
&\leq\ \ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \ \widetilde{\tau}(t_2, \widetilde{y})\big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}\ \ + \\
&\qquad h \cdot \Big(\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \cdot \big(1 + \lfloor \widetilde{y} \rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2 + h)} + \zeta_\varepsilon(\widetilde{\tau}) \cdot (t_2 + h)\big) + \gamma_\varepsilon(\widetilde{\tau})\Big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}.
\end{aligned}$$

Proof is based on the subsequent version of Gronwall's Lemma for semicontinuous functions. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1+h,\widetilde{z}),\ \widetilde{\tau}(t_2+h,\widetilde{y}))$ satisfies $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0} \varphi_\varepsilon(h-k)$ due to property (7.) of Definition 2.1.
Moreover it fulfills for any $h \in [0,1[$ with $t_1+h < \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})$

$$\limsup_{k\downarrow 0} \tfrac{\varphi_\varepsilon(h+k)-\varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\cdot\varphi_\varepsilon(h) + \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\cdot\big(1+\lfloor\widetilde{y}\rfloor_\varepsilon\, e^{\zeta_\varepsilon(\widetilde{\tau})(t_2+h)}+\zeta_\varepsilon(\widetilde{\tau})(t_2+h)\big) + \gamma_\varepsilon(\widetilde{\tau}).$$

Indeed, for all $k>0$ sufficiently small, the timed triangle inequality leads to

$$\begin{aligned}\varphi_\varepsilon(h+k) \;\le\;& \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h+k,\ \widetilde{z}),\quad \widetilde{\tau}(k,\ \widetilde{\tau}(t_2+h,\ \widetilde{y}))\big)\\ &+\widetilde{q}_\varepsilon\big(\widetilde{\tau}(k,\widetilde{\tau}(t_2+h,\widetilde{y})),\quad \widetilde{\tau}(t_2+h+k,\ \ \widetilde{y})\ \big)\\ \le\;& \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\ \big(1+\lfloor\widetilde{\tau}(t_2+h,\widetilde{y})\rfloor_\varepsilon\big) \cdot k + \varphi_\varepsilon(h)\ e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\,k} + \gamma_\varepsilon(\widetilde{\tau})\ k + o(k)\\ \le\;& \widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})\ \big(1+\lfloor\widetilde{y}\rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\tau})\cdot(t_2+h)}+\zeta_\varepsilon(\widetilde{\tau})\ (t_2+h)\big)\ k + \varphi_\varepsilon(h)\ e^{\alpha_\varepsilon(\widetilde{\tau},\widetilde{z})\,k} + \gamma_\varepsilon(\widetilde{\tau})\ k + o(k).\end{aligned}$$

□

Lemma 2.6 (Lemma of Gronwall for semicontinuous functions [13]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R},\ f,g \in C^0([a,b[,\mathbb{R})$ *satisfy* $f(\cdot)\ge 0$ *and*

$$\begin{aligned}\psi(t) &\le \limsup_{h\downarrow 0}\ \psi(t-h), && \forall\ t\in\,]a,\ b],\\ \psi(t) &\ge \limsup_{h\downarrow 0}\ \psi(t+h), && \forall\ t\in[a,\ b[,\\ \limsup_{h\downarrow 0}\ \tfrac{\psi(t+h)-\psi(t)}{h} &\le f(t)\cdot\limsup_{h\downarrow 0}\ \psi(t-h) + g(t) && \forall\ t\in\,]a,\ b[.\end{aligned}$$

Then, for every $t\in[a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \le \psi(a)\cdot e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds \qquad \textit{with}\ \ \mu(t) := \int_a^t f(s)\ ds.$$

□

Lemma 2.7 *For all* $\widetilde{\vartheta}\in\widetilde{\Theta}(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \widetilde{z}\in\widetilde{\mathcal{D}},\ \varepsilon\in\mathcal{J},\ \widetilde{Q}_\varepsilon$ *satisfies*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\ \widetilde{\vartheta};\ \widetilde{z}) \le 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$$

Proof follows exactly the same track as in Proposition 2.5. Fix $\varepsilon\in\mathcal{J},\ \widetilde{z}\in\widetilde{\mathcal{D}},\ \widetilde{y}\in\widetilde{E}$ and $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})[$ with $t+\pi_1\,\widetilde{z}\le\pi_1\,\widetilde{y}$ arbitrarily. Considering now

$$\varphi_\varepsilon :\ [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})-t[\ \longrightarrow\ [0,\infty[,\quad h\longmapsto\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big),$$

the semicontinuity $\varphi_\varepsilon(h)\le\limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ again results from property (7.) of Definition 2.1. Moreover we obtain for all $h\in[0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})-t[$ and small $k>0$

$$\begin{aligned}\varphi_\varepsilon(h+k) \;\le\;& \widetilde{q}_\varepsilon\big(\ \widetilde{\vartheta}(t+h+k,\ \ \widetilde{z}),\quad \widetilde{\vartheta}(k,\ \ \widetilde{\vartheta}(h,\ \widetilde{y}))\big)\\ &+\widetilde{q}_\varepsilon\big(\ \widetilde{\vartheta}(k,\ \ \widetilde{\vartheta}(h,\quad \widetilde{y})),\quad \widetilde{\vartheta}(h+k,\quad \widetilde{y})\ \big)\\ \le\;& (\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})+o(1))\cdot k\ \cdot\ \Big(\varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\ k\Big) + \varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\ k\\ &+\ (\gamma_\varepsilon(\widetilde{\vartheta})\quad +o(1))\cdot k\end{aligned}$$

and thus, $\displaystyle\limsup_{k\downarrow 0}\ \tfrac{\varphi_\varepsilon(h+k)-\varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot\varphi_\varepsilon(h) + 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$

Gronwall's Lemma 2.6 guarantees $\quad\varphi_\varepsilon(h) \le \varphi_\varepsilon(0)\cdot e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h} + h\ \cdot\ 2\,\gamma_\varepsilon(\widetilde{\vartheta})\ \cdot\ e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h}$

and finally, $\displaystyle\limsup_{h\downarrow 0}\ \tfrac{\varphi_\varepsilon(h)-\varphi_\varepsilon(0)\,\cdot\,e^{\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})\cdot h}}{h\quad(1+\lfloor\widetilde{y}\rfloor_\varepsilon)} \le 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$ □

Here we briefly mention two more transparent characterizations of $\widetilde{Q}_\varepsilon(\cdot,\cdot;\cdot)$ for the case $\alpha_\varepsilon(\cdot,\cdot)=\text{const}$. The first one clarifies the link with Aubin's original definition of the distance between two transitions ϑ,τ on a metric space (M,d), i.e. $\sup_{y\in M}\ \limsup_{h\downarrow 0}\ \frac{1}{h}\cdot d\,(\vartheta(h,y),\ \tau(h,y))$ (see [7], Definition 1.1.2). In particular, the first upper bound of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z})$ does not depend on $\widetilde{z}\in\widetilde{\mathcal{D}}$.

Such a dependence on the "test element" is preserved in the second estimate stated in Lemma 2.9. It is based on comparing only the evolutions of "test elements" $\widetilde{\vartheta}(t,\widetilde{z}) \in \widetilde{\mathcal{D}}$ for $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})[$ and will be useful for the examples of measure–valued flows in §§ 3, 4. This time however, we dispense with all additional advantages of bounded growth that $\lfloor\cdot\rfloor_\varepsilon$ might provide.

Lemma 2.8 *If* $\alpha_\varepsilon(\cdot,\cdot) = M_\varepsilon = const,$ *then*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z}) \;\le\; \sup_{\widetilde{y}\,\in\widetilde{E}} \; \limsup_{h\,\downarrow\,0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{y}),\; \widetilde{\tau}(h,\widetilde{y})\big)}{h \quad (1+\lfloor\widetilde{y}\rfloor_\varepsilon)} \right)^+ \;+\; 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})$$

for all $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E},\,\widetilde{\mathcal{D}},\,(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\,(\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}),\;\; \widetilde{z}\in\widetilde{\mathcal{D}},\;\; \varepsilon\in\mathcal{J}.$

Proof. It is not restrictive to assume additionally that the right–hand side is finite. For an upper bound of $\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\cdot)$ on the basis of Def. 2.3, we choose $\widetilde{y}\in\widetilde{E}$, $\widetilde{z}\in\widetilde{\mathcal{D}}$, $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})[$ with $t+\pi_1\widetilde{z}\le\pi_1\widetilde{y}$ arbitrarily and consider now the auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h,\widetilde{z}),\; \widetilde{\tau}(h,\widetilde{y}))$. Property (7.) of Definition 2.1 again implies $\varphi_\varepsilon(h) \;\le\; \limsup_{k\,\downarrow\,0}\; \varphi_\varepsilon(h-k)$ and thus enables us to apply Gronwall's Lemma 2.6 later. As an abbreviation, set here $\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau}) := \sup_{\widetilde{y}\,\in\widetilde{E}} \limsup_{h\,\downarrow\,0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{y}),\; \widetilde{\tau}(h,\widetilde{y})\big)}{h \quad (1+\lfloor\widetilde{y}\rfloor_\varepsilon)} \right)^+ \;<\; \infty\,.$

Choosing any $h\in[0,\;\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})-t[$ and any small $\delta>0$, the timed triangle inequality ensures for all $k>0$ sufficiently small (depending on t,h,δ)

$$\begin{array}{rcll}
\varphi_\varepsilon(h+k) &=& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t{+}h+k,\;\widetilde{z}), & \widetilde{\tau}(h{+}\;k,\;\widetilde{y})\Big)\\
&\le& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t{+}h+k,\;\widetilde{z}), & \widetilde{\vartheta}(k,\;\; \widetilde{\vartheta}(t{+}h,\widetilde{z}))\Big)\\
&& +\;\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\;\; \widetilde{\vartheta}(t{+}h,\widetilde{z})), & \widetilde{\vartheta}(k,\;\; \widetilde{\tau}(h,\;\widetilde{y}))\Big)\\
&& +\;\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\;\; \widetilde{\tau}(h,\;\widetilde{y})), & \widetilde{\tau}(k,\;\; \widetilde{\tau}(h,\;\widetilde{y}))\Big)\\
&& +\;\widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\;\; \widetilde{\tau}(h,\;\widetilde{y})), & \widetilde{\tau}(h{+}\;k,\;\widetilde{y})\Big)\\
&\le& (\gamma_\varepsilon(\widetilde{\vartheta}) \quad +\delta)\;\cdot\;k &\\
&& +\;(M_\varepsilon \quad +\delta)\;\cdot\;k\;\;(\varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\cdot k) \quad + \quad \varphi_\varepsilon(h)+\gamma_\varepsilon(\widetilde{\vartheta})\cdot k &\\
&& +\;(\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\,\widetilde{\tau})\;+\delta)\;\cdot\;k\;\;(1+\lfloor\widetilde{\tau}(h,\;\widetilde{y})\rfloor_\varepsilon) &\\
&& +\;(\gamma_\varepsilon(\widetilde{\tau}) \quad +\delta)\;\cdot\;k &
\end{array}$$

and thus,

$$\limsup_{k\,\downarrow\,0}\; \frac{\varphi_\varepsilon(h{+}k)\,-\,\varphi_\varepsilon(h)}{k} \;\;\le\;\; (M_\varepsilon+\delta)\cdot\varphi_\varepsilon(h) \;\;+\;\; (\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\,\widetilde{\tau})+\delta)\;\;(1+\lfloor\widetilde{y}\rfloor_\varepsilon\cdot e^{\zeta_\varepsilon(\widetilde{\tau})\,\cdot\,h}+\zeta_\varepsilon(\widetilde{\tau})\,h)$$
$$+\;\; 2\,\gamma_\varepsilon(\widetilde{\vartheta})+\gamma_\varepsilon(\widetilde{\tau})+2\,\delta.$$

So Gronwall's Lemma 2.6 and $\delta\downarrow 0$ imply

$$\varphi_\varepsilon(h) \;\;\le\;\; \varphi_\varepsilon(0)\cdot e^{M_\varepsilon\cdot h} \;\;+\;\; h\cdot e^{M_\varepsilon\,\cdot\,h}\,\Big(\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\,\widetilde{\tau})\;\;(1+\lfloor\widetilde{y}\rfloor_\varepsilon\cdot e^{\zeta_\varepsilon(\widetilde{\tau})\,\cdot\,h}+\zeta_\varepsilon(\widetilde{\tau})\,h)\;\;+\;\;2\,\gamma_\varepsilon(\widetilde{\vartheta})+\gamma_\varepsilon(\widetilde{\tau})\Big).$$

Finally we conclude from Definition 2.3

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z}) \;\;\le\;\; \limsup_{h\,\downarrow\,0}\; \frac{\varphi_\varepsilon(h)\,-\,\varphi_\varepsilon(0)\,\cdot\,e^{M_\varepsilon\cdot h}}{h \quad (1+\lfloor\widetilde{y}\rfloor_\varepsilon)} \;\;\le\;\; \widetilde{\Delta}_\varepsilon(\widetilde{\vartheta},\,\widetilde{\tau})+2\,\gamma_\varepsilon(\widetilde{\vartheta})+\gamma_\varepsilon(\widetilde{\tau}) \qquad \square$$

Lemma 2.9 *If* $\alpha_\varepsilon(\cdot,\cdot) = M_\varepsilon = const,$ *then*

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta},\widetilde{\tau};\widetilde{z}) \;\le\; \sup_{0\,\le\,t\,<\,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})} \; \limsup_{h\,\downarrow\,0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\,\widetilde{\vartheta}(t,\widetilde{z})),\;\widetilde{\tau}(h,\,\widetilde{\vartheta}(t,\widetilde{z}))\big)}{h} \right)^+ \;+\; \gamma_\varepsilon(\widetilde{\vartheta}) + 2\,\gamma_\varepsilon(\widetilde{\tau})$$

for all $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E},\,\widetilde{\mathcal{D}},\,(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\,(\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}),\;\; \widetilde{z}\in\widetilde{\mathcal{D}},\;\; \varepsilon\in\mathcal{J}.$

Proof follows essentially the same track as for Lemma 2.8. Again $\widetilde{y}\in\widetilde{E},\;\widetilde{z}\in\widetilde{\mathcal{D}},\;t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})[$ with $t+\pi_1\widetilde{z}\le\pi_1\widetilde{y}$ are chosen arbitrarily, but now the auxiliary function $\varphi_\varepsilon : h\longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t{+}h,\widetilde{z}),\;\widetilde{\tau}(h,\widetilde{y}))$ is estimated in a slightly different way :

From now on, let $\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$ denote the supremum on the right–hand side of the claim. Fixing any $h \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) - t[$ and any small $\delta > 0$, we obtain for all $k > 0$ sufficiently small (depending on t, h, δ)

$$\begin{array}{rcl} \varphi_\varepsilon(h+k) & \leq & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h+k,\ \widetilde{z}), \qquad \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(t+h, \widetilde{z}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\ \ \widetilde{\vartheta}(t+h, \widetilde{z})), \quad \widetilde{\tau}(k,\ \widetilde{\vartheta}(t+h, \widetilde{z}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\ \ \widetilde{\vartheta}(t+h, \widetilde{z})) \qquad \widetilde{\tau}(k,\ \widetilde{\tau}(h,\ \widetilde{y}))\Big) \\ & & +\ \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\ \ \widetilde{\tau}(h,\ \widetilde{y})), \qquad \widetilde{\tau}(h+k,\ \widetilde{y})\Big) \\ & \leq & (\gamma_\varepsilon(\widetilde{\vartheta}) \qquad + \delta) \cdot k \\ & & +\ (\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \delta) \cdot k \\ & & +\ (M_\varepsilon \qquad + \delta) \cdot k\ \ (\varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\tau}) \cdot k) \quad + \quad \varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\tau}) \cdot k \\ & & +\ (\gamma_\varepsilon(\widetilde{\tau}) \qquad + \delta) \cdot k \end{array}$$

and thus, $\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \leq (M_\varepsilon + \delta) \cdot \varphi_\varepsilon(h) + \widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \gamma_\varepsilon(\widetilde{\vartheta}) + 2\, \gamma_\varepsilon(\widetilde{\tau}) + 3\, \delta.$

Again Gronwall's Lemma 2.6 and $\delta \downarrow 0$ lead to

$$\varphi_\varepsilon(h) \ \leq\ \varphi_\varepsilon(0) \cdot e^{M_\varepsilon \cdot h} \ +\ h \cdot e^{M_\varepsilon \cdot h} \Big(\widetilde{\Delta}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \ +\ \gamma_\varepsilon(\widetilde{\vartheta}) + 2\, \gamma_\varepsilon(\widetilde{\tau})\Big)$$

and, the claim results from Definition 2.3 due to $\lfloor \cdot \rfloor_\varepsilon \geq 0$. □

2.2 Defining timed right–hand sleek solutions

Now timed sleek transitions are used for specifying "first–order approximations" of a curve $\widetilde{x}(\cdot) : [0, T[\longrightarrow \widetilde{E}$. Proposition 2.5 has just provided the structural estimate that we want to preserve. So replacing $\widetilde{\tau}(t_2 + \cdot, \widetilde{y})$ formally by $\widetilde{x}(t + \cdot)$, a sleek transition $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$ can be interpreted as "first–order approximation" of $\widetilde{x}(\cdot)$ at time $t \in [0, T[$ if for all test elements $\widetilde{z} \in \widetilde{\mathcal{D}}$ $(\pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t))$, infinitesimal $h > 0$ satisfy the corresponding estimate with the "distance term" $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z})$ equal to 0. This notion motivates the following definition :

Definition 2.10 *The curve* $\widetilde{x}(\cdot) : [0, T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand sleek primitive *of a map* $\widetilde{\vartheta}(\cdot) : [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ \ t \in [0, T[\qquad \forall\ \ \widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$:
   $\exists\ \ \widehat{\alpha}_\varepsilon(t, \widetilde{z}) \geq \alpha_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z}) \quad \exists\ \ \widehat{\gamma}_\varepsilon(t, \widetilde{z}) \geq \gamma_\varepsilon(\widetilde{\vartheta}(t))$:
   $\limsup_{h \downarrow 0} \frac{1}{h} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s+h, \widetilde{z}),\ \widetilde{x}(t+h)\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s, \widetilde{z}),\ \widetilde{x}(t)\Big) \cdot e^{\widehat{\alpha}_\varepsilon(t, \widetilde{z}) \cdot h}\Big) \leq \widehat{\gamma}_\varepsilon(t, \widetilde{z})$
   for every $s \in \big[0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z})\big[$ *with* $s + \pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$,
2. $\widetilde{x}(\cdot)$ *is uniformly continuous in time direction with respect to* $\widetilde{q}_\varepsilon$,
   i.e. there is $\omega_\varepsilon(\widetilde{x}, \cdot) :]0, T[\longrightarrow [0, \infty[$ *such that* $\limsup_{h \downarrow 0} \omega_\varepsilon(\widetilde{x}, h) = 0$,
   $\widetilde{q}_\varepsilon\big(\widetilde{x}(s), \widetilde{x}(t)\big) \leq \omega_\varepsilon(\widetilde{x}, t - s)$ *for* $0 \leq s < t < T$.
3. $\pi_1 \widetilde{x}(t) = t + \pi_1 \widetilde{x}(0)$ *for all* $t \in [0, T[$.

Remark 2.11 Considering $\widetilde{\vartheta}(\cdot, \widetilde{y}) : [0, 1[\longrightarrow \widetilde{E}$ for any $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor \cdot \rfloor_\varepsilon))$ and $\widetilde{y} \in \widetilde{E}$ fixed, timed sleek transitions induce their own sleek primitives — as a direct consequence of Definition 2.1, Proposition 2.5 and Lemma 2.7. Correspondingly, each piecewise constant function $\widetilde{\vartheta} : [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$ has a timed right–hand sleek primitive that is defined piecewise as well.

Definition 2.12 *For* $\widetilde{f} : \widetilde{E} \times [0, T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, (\lfloor \cdot \rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *given, a map* $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ *is a* timed right–hand sleek solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0, T[$.

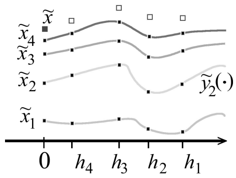
2.3 Existence of solutions due to timed transitional compactness

In particular, the new aspect of the additional term $(1 + \lfloor\cdot\rfloor_\varepsilon)$ does not have any consequence for verifying the solution property if we suppose the parameter $\zeta_\varepsilon(\widetilde{f}(\widetilde{x},t))$ to be uniformly bounded for all $\widetilde{x} \in \widetilde{E},\ t \in [0,T[$. Briefly, all Euler approximations satisfy an obvious a priori estimate with respect to $\lfloor\cdot\rfloor_\varepsilon$ (for each given initial element in $\widetilde{E}$) as stated in Lemma 2.16. So the proofs of [15] lead directly to the following Proposition 2.14 about existence :

Definition 2.13 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$.*
$\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}\Big)$ *is called* timed transitionally compact *if it fulfills :*

Let $(\widetilde{x}_n)_{n\in\mathbb{N}},\ (h_j)_{j\in\mathbb{N}}$ be any sequences in $\widetilde{E},\ ]0,1[$, respectively and $\widetilde{v} \in \widetilde{E}$ with $\sup_n\ |\pi_1\,\widetilde{x}_n| < \infty$, $\sup_n \lfloor\widetilde{x}_n\rfloor_\varepsilon < \infty$, $\sup_n \widetilde{q}_\varepsilon(\widetilde{v},\widetilde{x}_n) < \infty$ for each $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. Moreover suppose $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$ to be piecewise constant (for each index $n \in \mathbb{N}$) and satisfy the following property : for every $\varepsilon \in \mathcal{J}$, there exists $\beta_\varepsilon : [0,1] \longrightarrow [0,\infty[$ with $\limsup_{s\to 0}\ \beta_\varepsilon(s) = 0$ such that each curve $\widetilde{\vartheta}_n(t)(\cdot,\widetilde{x}) : [0,1] \longrightarrow \widetilde{E}$ (with $n \in \mathbb{N},\ t \in [0,1],\ \widetilde{x} \in \widetilde{E}$) has the modulus of continuity $(1 + \lfloor\widetilde{x}\rfloor_\varepsilon)\ \beta_\varepsilon(\cdot)$.
Each $\widetilde{\vartheta}_n$ induces a function $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{y}_n(0) = \widetilde{x}_n$ in the same (piecewise) way as timed sleek transitions induce their own primitives according to Remark 2.11 (i.e. using $\widetilde{\vartheta}_n(t_m)\,(\cdot,\ \widetilde{y}_n(t_m))$ in each interval $]t_m, t_{m+1}]$ in which $\widetilde{\vartheta}_n(\cdot)$ is constant).
Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\begin{array}{rlcl}
& \lim\limits_{k\to\infty}\ \pi_1\,\widetilde{x}_{n_k} & = & \pi_1\,\widetilde{x}, \\
& \limsup\limits_{k\longrightarrow\infty}\ \lfloor\widetilde{x}_{n_k}\rfloor_\varepsilon & \geq & \lfloor\widetilde{x}\rfloor_\varepsilon, \\
& \limsup\limits_{k\longrightarrow\infty}\ \widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\ \widetilde{x}) & = & 0, \\
\limsup\limits_{j\longrightarrow\infty} & \sup\limits_{k\geq j}\ \widetilde{q}_\varepsilon(\widetilde{x},\quad \widetilde{y}_{n_k}(h_j)) & = & 0.
\end{array}$$



A nonempty subset $\widetilde{F} \subset \widetilde{E}$ is called timed transitionally compact in $(\widetilde{E},\ (\widetilde{q}_\varepsilon),\ (\lfloor\cdot\rfloor_\varepsilon),\ \widetilde{\Theta})$ *if the same property holds for any sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x} \in \widetilde{F}$ is* not *required).*

Proposition 2.14 (Existence of timed right–hand sleek solutions)
Assume that $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon)))$ is timed transitionally compact. Let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon))$ fulfill for every $\varepsilon \in \mathcal{J},\ \widetilde{z} \in \widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z}) := \sup\limits_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2} \{\alpha_\varepsilon\big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)(h,\widetilde{z})\big) \mid 0 \leq h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2,t_2),\widetilde{z})\} < \infty,$
2. $c_\varepsilon(h) := \sup\limits_{t,\widetilde{y}}\ \beta_\varepsilon(\widetilde{f}(\widetilde{y},t))(h) < \infty, \qquad c_\varepsilon(h) \xrightarrow{h\downarrow 0} 0$
3. $\exists\ R_\varepsilon\ :\ \sup\limits_{t,\widetilde{y}}\ \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \leq R_\varepsilon < \infty,$
4. $g_\varepsilon := \sup\limits_{t,\widetilde{y}}\ \zeta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$
5. $\exists\ \widehat{\omega}_\varepsilon(\cdot)\ :\ \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2);\ \widetilde{z}\Big) \leq R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{y}_1,\widetilde{y}_2) + t_2 - t_1\Big)$
   for all $0 \leq t_1 \leq t_2 \leq T$ and $\widetilde{y}_1,\widetilde{y}_2 \in \widetilde{E}$ $(\pi_1\,\widetilde{y}_1 \leq \pi_1\,\widetilde{y}_2)$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ nondecreasing, $\limsup\limits_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) = 0.$

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand sleek solution $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 2.15 The basic notion of its proof is easy to sketch. Indeed, we again start with Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$,

$$\begin{array}{lllllll} h_n & := & \frac{T}{2^n}, & t_n^j & := & j\, h_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := & \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := & \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := & \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j) \, \big(t - t_n^j,\; \widetilde{x}_n(t_n^j)\big) & & & & \text{for } t \in \,]t_n^j,\, t_n^{j+1}],\; j \le 2^n. \end{array}$$

Assumption (4.) and subsequent Lemma 2.16 imply the uniform estimate

$$\lfloor \widetilde{x}_n(t) \rfloor_\varepsilon \;\le\; (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\; e^{g_\varepsilon\, T} \qquad \text{for } n \in \mathbb{N},\; t \in [0,T[.$$

As a key point, it lays the basis for applying earlier conclusions to the "bounded" subset $\{\lfloor\cdot\rfloor_\varepsilon \le (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T}\} \subset \widetilde{E}$ (see [15] and references there). Roughly speaking, moduli of continuity and the distances like $\widetilde{Q}_\varepsilon(\,\cdot\,,\,\cdot\,; \widetilde{z})$ occurring now just have to be multiplied by $(\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T}$ (depending on the initial element) before applying former results. This is a basic idea for verifying most of the subsequent propositions in § 2. It has already been mentioned just briefly in the example of semilinear evolution equations in a reflexive Banach space, see [14].

With respect to the existence in Proposition 2.14, we proceed with the Euler approximations : As $\mathcal{J}$ is assumed to be countable, Cantor diagonal construction in combination with timed transitional compactness leads to a function $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ satisfying : For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on ε, j) and $N_j \in \mathbb{N}$ (depending on ε, j, K_j) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \left\{ \begin{array}{rcl} \widetilde{q}_\varepsilon\big(\widetilde{x}_{N_j}(s - 2\, h_{K_j}),\;\; \widetilde{x}(s)\,\big) & \le & \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\;\; \widetilde{x}_{N_j}(t + 2\, h_{K_j})\,\big) & \le & \frac{1}{j} \\ \lfloor \widetilde{x}(s) \rfloor_\varepsilon & \le & (\lfloor \widetilde{x}_0 \rfloor_\varepsilon + 2\, g_\varepsilon\, T)\;\; e^{g_\varepsilon\, T} \end{array} \right.$$

for every $s, t \in [0,T[$. Due to [15], Convergence Theorem 3.10 for $\widetilde{x}_{N_j}(\,\cdot\, + 2\, h_{N_j} + 2\, h_{K_j})$, the limit function $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution. (For further details, see the proof of Prop. 36 in [13] or the detailed proof of subsequent Proposition 2.17 postponed to the end of this subsection 2.3.)

Lemma 2.16 *Under the assumptions of Proposition 2.14, the Euler approximations* $\widetilde{x}_n(\cdot)$, $n \in \mathbb{N}$, *defined in Remark 2.15 satisfy at each time* $t \in [0,\; T{+}h_n]$

$$\lfloor \widetilde{x}_n(t) \rfloor_\varepsilon \;\le\; \lfloor \widetilde{x}_0 \rfloor_\varepsilon \;\; e^{g_\varepsilon\, t} \;+\; g_\varepsilon \cdot (t + h_n) \cdot e^{g_\varepsilon\, t}.$$

Proof results from condition (5.) of Definition 2.1. Indeed, for every $j \in \{0 \ldots 2^n\}$ and $t \in \,]t_n^j,\, t_n^{j+1}]$,

$$\begin{array}{rcll} \lfloor \widetilde{x}_n(t) \rfloor_\varepsilon & \le & \lfloor \widetilde{x}_n(t_n^j) \rfloor_\varepsilon \cdot e^{g_\varepsilon \cdot (t - t_n^j)} & +\; g_\varepsilon \cdot (t - t_n^j) \\ & \le & \lfloor \widetilde{x}_n(t_n^j) \rfloor_\varepsilon \cdot e^{g_\varepsilon\, h_n} & +\; g_\varepsilon \cdot h_n. \end{array}$$

By means of induction with respect to j, we obtain (for each $t \in \,]t_n^j,\, t_n^{j+1}]$ again)

$$\begin{array}{rcl} \lfloor \widetilde{x}_n(t) \rfloor_\varepsilon & \le & \lfloor \widetilde{x}_0 \rfloor_\varepsilon \;\; e^{g_\varepsilon\, t} \;\; + \displaystyle\sum_{k=0}^{j} \; g_\varepsilon\; h_n \;\; e^{g_\varepsilon \cdot (j-k)\, h_n} \\ & \le & \lfloor \widetilde{x}_0 \rfloor_\varepsilon \;\; e^{g_\varepsilon\, t} \;\; + \;\; g_\varepsilon\; (t{+}h_n) \;\; e^{g_\varepsilon\, t}. \end{array} \qquad \square$$

The example of § 3, however, shows that the continuity assumption of Proposition 2.14 might be difficult to verify – particularly if $\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{y}_1, t_1),\; \widetilde{f}(\widetilde{y}_2, t_2);\; \widetilde{z}\big)$ can be estimated just independently of $\widetilde{z}, \varepsilon$. So we present an alternative hypothesis motivated by the notion to exploit information about distances with respect to *all* $\varepsilon \in \mathcal{J}$ simultaneously.

Proposition 2.17 *Assume that* $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon)))$ *is timed transitionally compact. Let* $\widetilde{f}:\widetilde{E}\times[0,T]\longrightarrow\widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon),(\lfloor\cdot\rfloor_\varepsilon))$ *fulfill for every* $\varepsilon\in\mathcal{J},\ \widetilde{z}\in\widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z}) := \sup_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2} \{\alpha_\varepsilon\big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)(h,\widetilde{z})\big)\mid 0\le h<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2,t_2),\widetilde{z})\} < \infty,$
2. $c_\varepsilon(h) := \sup_{t,\widetilde{y}} \beta_\varepsilon(\widetilde{f}(\widetilde{y},t))(h) < \infty, \qquad c_\varepsilon(h)\stackrel{h\downarrow 0}{\longrightarrow}0$
3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{y}} \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \le R_\varepsilon<\infty,$
4. $g_\varepsilon := \sup_{t,\widetilde{y}} \zeta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty,$
5. $\limsup_{j\longrightarrow\infty} \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2^j,t_2^j);\ \widetilde{z}\Big) \le R_\varepsilon$
   for any $t_1\in[0,T],\ \widetilde{y}_1\in\widetilde{E}$ *and sequences* $(t_2^j)_{j\in\mathbb{N}}$ *in* $[0,T],\ (\widetilde{y}_2^j)_{j\in\mathbb{N}}$ *in* $\widetilde{E}$ *satisfying*
   $t_1\le t_2^j,\ \ \pi_1\,\widetilde{y}_1\le\pi_1\,\widetilde{y}_2^j\quad\forall\,j\in\mathbb{N},\qquad |t_1-t_2^j|\stackrel{j\to\infty}{\longrightarrow}0,\quad \widetilde{q}_{\varepsilon'}(\widetilde{y}_1,\ \widetilde{y}_2^j)\stackrel{j\to\infty}{\longrightarrow}0\quad\forall\,\varepsilon'\in\mathcal{J}.$

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand sleek solution* $\widetilde{x}:[0,T[\longrightarrow\widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ *with* $\widetilde{x}(0)=\widetilde{x}_0$.

For the purpose of a self–contained presentation here, we postpone a detailed proof of Proposition 2.17 till the end of this subsection 2.3. Using the same Euler approximations as in Remark 2.15, the key tool now is to guarantee that the "limit function" $\widetilde{x}(\cdot):[0,T[\longrightarrow\widetilde{E}$ is a (timed right–hand sleek) solution of $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$. So we need a new convergence theorem with an adapted continuity assumption :

Proposition 2.18 (Convergence Theorem) *For each* $\varepsilon\in\mathcal{J}$ *and* $\widetilde{z}\in\widetilde{\mathcal{D}}$, *assume the following properties of*

$$\begin{array}{llcll}\widetilde{f}_m,\ \widetilde{f}: & \widetilde{E}\times[0,T[& \longrightarrow & \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},(\lfloor\cdot\rfloor_\varepsilon)) & (m\in\mathbb{N})\\ \widetilde{x}_m,\ \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E}\ : & \end{array}$$

1. $M_\varepsilon(\widetilde{z}) := \sup_{m,t,\widetilde{y}} \{\,\alpha_\varepsilon\big(\widetilde{f}_m(\widetilde{y},t),\ \widetilde{f}(\widetilde{x}(t),t)(h,\widetilde{z})\big)\mid 0\le h<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})\} < \infty,$
   $R_\varepsilon \ge \sup_{m,t,\widetilde{y},\widetilde{z},h} \{\,\widehat{\gamma}_\varepsilon(t,\ \widetilde{f}_m(\widetilde{x}_m,\cdot),\ \widetilde{f}(\widetilde{x}(t),t)(h,\widetilde{z})),\ \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y},t)),\ \ \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t))\,\},$
   $g_\varepsilon \ge \sup_{m,t,\widetilde{y}} \{\zeta_\varepsilon(\widetilde{f}_m(\widetilde{y},t))\}$
2. $\overset{\circ}{\widetilde{x}}_m(\cdot)\ \ni\ \widetilde{f}_m(\widetilde{x}_m(\cdot),\ \cdot)$ *in* $[0,T[$, (*in the sense of Definition 2.12*)
3. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m,h) < \infty$ (*moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$), $\limsup_{h\downarrow 0}\ \widehat{\omega}_\varepsilon(h) = 0,$
4. $\forall\ 0\le t_1<t_2<T\quad\exists\ (m_j)_{j\in\mathbb{N}},\ (\delta_j')_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}}$ *with* $m_j\nearrow\infty,\ \delta_j,\delta_j'\searrow 0$
   (*i*) $\limsup_{j\longrightarrow\infty}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{y}_j,s_j);\ \widetilde{z}\big)\ \le\ R_\varepsilon,$
   for any sequences $s_j\downarrow t_1$ *and* $(\widetilde{y}_j)_{j\in\mathbb{N}}$ *in* $\widetilde{E}$ *with* $\pi_1\,\widetilde{y}_j\searrow\widetilde{x}(t_1),\ \ \widetilde{q}_{\varepsilon'}(\widetilde{x}(t_1),\,\widetilde{y}_j)\stackrel{j\to\infty}{\longrightarrow}0\ \forall\,\varepsilon'$
   (*ii*) $\widetilde{q}_{\varepsilon'}\big(\widetilde{x}(t_1),\ \ \widetilde{x}_{m_j}(t_1+\delta_j')\big)\longrightarrow 0\qquad\forall\,\varepsilon'\in\mathcal{J},\qquad \pi_1\,\widetilde{x}_{m_j}(t_1+\delta_j')\ \searrow\ \pi_1\,\widetilde{x}(t_1),$
   (*iii*) $\widetilde{q}_\varepsilon\ \big(\widetilde{x}_{m_j}(t_2-\delta_j),\ \ \widetilde{x}(t_2)\big)\longrightarrow 0,\qquad \pi_1\,\widetilde{x}_{m_j}(t_2-\delta_j)\ \nearrow\ \pi_1\,\widetilde{x}(t_2).$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ *in* $[0,T[$.

Proof of Proposition 2.18. The uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (3.) :
Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\big(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2)\big) \le \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $t_1 < t_2 < T$.
Let $\varepsilon \in \mathcal{J}$, $0 \le t_1 < t_2 < T$ be arbitrary and choose $(\delta'_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, for t_1, t_2 (according to condition (5.ii), (5.iii)). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta'_j < t_2 - \delta_j$ and so,

$$\begin{aligned}\widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}(t_2)\big) &\le \widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1+\delta'_j)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_1+\delta'_j),\ \widetilde{x}_{m_j}(t_2-\delta_j)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_2-\delta_j),\ \widetilde{x}(t_2)\big)\\ &\le o(1) + \widehat{\omega}_\varepsilon(t_2 - t_1) + o(1) \qquad \text{for } j \longrightarrow \infty.\end{aligned}$$

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$ and $t \in [0, T[$, $0 \le s < s + h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t), t),\ \widetilde{z})$ be chosen arbitrarily with $s + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$. Condition (7.) of Definition 2.1 guarantees for all $k \in]0, h[$ sufficiently small

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) \le \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h-k,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) + h^2.$$

According to assumptions (4.ii) – (4.iii), there exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \ \widetilde{x}(t+h)\big) \longrightarrow 0, & \pi_1\,\widetilde{x}_{m_j}(t+h-\delta_j) \nearrow \pi_1\,\widetilde{x}(t+h),\\ \widetilde{q}_{\varepsilon'}\big(\widetilde{x}(t),\ \ \widetilde{x}_{m_j}(t+\delta'_j)\big) \longrightarrow 0 \quad \forall\, \varepsilon' \in \mathcal{J}, & \pi_1\,\widetilde{x}_{m_j}(t+\delta'_j) \searrow \pi_1\,\widetilde{x}(t).\end{cases}$$

Now subsequent Lemma 2.19 implies for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \widetilde{z}, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big)\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h-k,\ \widetilde{z}),\ \ \widetilde{x}_{m_j}(t+\delta'_j + h-k)\big)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+\delta'_j + h-k),\ \ \widetilde{x}_{m_j}(t+h - \delta_j)\big)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h - \delta_j),\ \ \widetilde{x}(t+h)\big) + h^2\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \ \widetilde{x}_{m_j}(t+\delta'_j)\big) \cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} +\\
&+ \int_0^{h-k} e^{M_\varepsilon(\widetilde{z})\cdot(h-k-\sigma)} \Big(\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot(1+\lfloor\widetilde{z}\rfloor_\varepsilon\, e^{g_\varepsilon h} + g_\varepsilon\, h) + 3\,R_\varepsilon\Big)\, d\sigma\\
&+ \widehat{\omega}_\varepsilon(k - \delta_j - \delta'_j)\\
&+ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h - \delta_j),\ \ \widetilde{x}(t+h)\big) + h^2\\
\le\ & \Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t+\delta'_j)\big)\Big) \cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} +\\
&+ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\, d\sigma \cdot (1+\lfloor\widetilde{z}\rfloor_\varepsilon\, e^{g_\varepsilon h} + g_\varepsilon\, h)\\
&+ \widehat{\omega}_\varepsilon(k) + 2\,h^2 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\, R_\varepsilon\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)\big) \cdot e^{M_\varepsilon(\widetilde{z})\,h} + \widehat{\omega}_\varepsilon(k) + 3\,h^2 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\, R_\varepsilon\\
&+ h\ e^{M_\varepsilon(\widetilde{z})\cdot h} \sup_{0\le\sigma\le h} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot (1+\lfloor\widetilde{z}\rfloor_\varepsilon\, e^{g_\varepsilon h} + g_\varepsilon\, h)
\end{aligned}$$

First $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big)\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)\big)\cdot e^{M_\varepsilon(\widetilde{z})\,h} + 0 + \mathrm{const}(\varepsilon,\widetilde{z})\cdot h\,(R_\varepsilon + h)\\
&+ h\ e^{M_\varepsilon(\widetilde{z})\,h} \cdot \limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big)\cdot (1+\lfloor\widetilde{z}\rfloor_\varepsilon\, e^{g_\varepsilon h} + g_\varepsilon\, h).
\end{aligned}$$

Finally, due to an indirect conclusion, assumptions (4.i),(4.ii) and the equi–continuity of $(\widetilde{x}_m)$ ensure

$$\limsup_{h\downarrow 0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h} \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+\sigma}\,;\ \widetilde{z}\big) \le R_\varepsilon$$

and thus,

$$\limsup_{h\downarrow 0} \tfrac{1}{h}\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\widetilde{z}),\ \widetilde{x}(t+h)\big) - \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)\big)\cdot e^{M_\varepsilon(\widetilde{z})\,h}\Big) \le \mathrm{const}(\varepsilon,\widetilde{z})\cdot R_\varepsilon.$$

□

This lemma extends the structural estimate of Proposition 2.5 to the comparison between a test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ (evolving along a fixed transition $\widetilde{\psi}$) and a timed sleek primitive $\widetilde{x}(\cdot)$:

Lemma 2.19 *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $t_1 \in [0,1[$, $t_2 \in [0,T[$, $\widetilde{z} \in \widetilde{\mathcal{D}}$.
Let $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *be a timed sleek primitive of* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ *such that for each* $\varepsilon \in \mathcal{J}$, $t \in [0,T[$, *their parameters fulfill*

$$\wedge \left\{ \begin{array}{rcl} \sup\limits_{0 \le s \le \min\{t, \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})\}} \widehat{\alpha}_\varepsilon(t, \widetilde{\psi}(s,\widetilde{z})) & \le & M_\varepsilon(t), \\ \sup\limits_{0 \le s \le \min\{t, \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})\}} \widehat{\gamma}_\varepsilon(t, \widetilde{\psi}(s,\widetilde{z})) & \le & R_\varepsilon(t), \\ \widetilde{Q}_\varepsilon(\widetilde{\psi}, \widetilde{\vartheta}(t);\ \widetilde{z}) & \le & c_\varepsilon(t) \end{array} \right.$$

with upper semicontinuous $M_\varepsilon,\ R_\varepsilon,\ c_\varepsilon : [0,T[\longrightarrow [0,\infty[$. *Set* $\mu_\varepsilon(h) := \displaystyle\int_{t_2}^{t_2+h} M_\varepsilon(s)\, ds.$
Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0,T[$ *with* $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})$, $t_2 + h < T$, $t_1 + \pi_1\, \widetilde{z} \le \pi_1\, \widetilde{x}(t_2)$

$$\begin{array}{l} \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t_2{+}h)\big) \\ \le\ \ \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1, \widetilde{z}),\ \widetilde{x}(t_2)\big)\ \cdot\ e^{\mu_\varepsilon(h)}\ +\ \displaystyle\int_0^h e^{\mu_\varepsilon(h) - \mu_\varepsilon(s)}\ \big(c_\varepsilon(t_2{+}s) \cdot (1 + \lfloor\widetilde{z}\rfloor_\varepsilon\ e^{\zeta_\varepsilon(\widetilde{\psi})\cdot(t_1+s)} + \zeta_\varepsilon(\widetilde{\psi}) \cdot (t_1{+}s)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\ 3\, R_\varepsilon(t_2{+}s)\big)\ ds. \end{array}$$

Proof. We follow the same track as in the proof of Proposition 2.5 and consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h, \widetilde{z}),\ \widetilde{x}(t_2{+}h))$. Firstly, $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ results from condition (7.) on timed sleek transitions (Definition 2.1) and the continuity of $\widetilde{x}(\cdot)$.
Moreover we show for any $h \in [0,T[$ with $t_1 + h\ <\ \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})$,

$$\limsup_{k\downarrow 0} \tfrac{\varphi_\varepsilon(h+k)\,-\,\varphi_\varepsilon(h)}{k}\ \le\ M_\varepsilon(t_2{+}h)\cdot\varphi_\varepsilon(h)\ +\ c_\varepsilon(t_2{+}h)\cdot(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon)\ +\ 3\, R_\varepsilon(t_2{+}h).$$

In particular, it implies $\varphi_\varepsilon(h)\ \ge\ \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h+k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 2.6 – after approximating $M_\varepsilon(\cdot),\ R_\varepsilon(\cdot),\ c_\varepsilon(\cdot)$ by continuous functions from above.

For small $k > 0$, the timed triangle inequality and Proposition 2.5 guarantee

$$\begin{array}{rll} \varphi_\varepsilon(h{+}k) & \le & \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1 + h + k,\ \widetilde{z}),\qquad \widetilde{\vartheta}(t_2{+}h)\,\big(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\big)\big) \\ & & +\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_2{+}\ h)\,\big(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1{+}h, \widetilde{z})\big),\qquad \widetilde{\vartheta}(t_2{+}h)\,\big(k,\qquad \widetilde{\psi}(t_1{+}h,\ \widetilde{z})\big)\big) \\ & & +\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_2{+}\ h)\,\big(k,\qquad \widetilde{\psi}(t_1{+}h, \widetilde{z})\big),\qquad \widetilde{x}(t_2 + h + k)\big) \\ & \le & \Big\{\widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t_2{+}h);\ \widetilde{z})\,\Big(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon\cdot e^{\zeta_\varepsilon(\widetilde{\vartheta}(t_2+h))\ k} + \zeta_\varepsilon(\widetilde{\vartheta}(t_2{+}h))\ k\Big) \\ & & \qquad\qquad\qquad\qquad +\,\widehat{\gamma}_\varepsilon(t_2{+}h,\ \widetilde{z})\Big\}\ e^{M_\varepsilon(t_2+h)\cdot k}\ k \\ & & +\qquad\qquad\qquad\qquad \gamma_\varepsilon(\widetilde{\vartheta}(t_2{+}h)) \qquad\qquad\qquad\qquad \cdot\, k\ +\ o(k) \\ & & +\ \varphi_\varepsilon(h)\ e^{\widehat{\alpha}_\varepsilon(t_2+h,\ \widetilde{\psi}(t_1+h,\ \widetilde{z}))\,\cdot\, k}\ +\ \widehat{\gamma}_\varepsilon(t_2{+}h,\ \widetilde{\psi}(t_1{+}h,\ \widetilde{z})) \qquad \cdot\, k\ +\ o(k) \\ & \le & \varphi_\varepsilon(h)\ e^{M_\varepsilon(t_2+h)\,\cdot\, k}\qquad +\ \big|c_\varepsilon(t)\cdot(1{+}\lfloor\widetilde{\psi}(t_1{+}h,\ \widetilde{z})\rfloor_\varepsilon) + 3\ R_\varepsilon(t)\big|_{t\,=\,t_2+h}\ \cdot\, k\ +\ o(k) \end{array}$$

since $t_1 + h + k < \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z}) \le 1$ implies $\widetilde{\psi}(t_1{+}h, \widetilde{z}),\ \widetilde{\psi}(t_1{+}h{+}k, \widetilde{z}) \in \widetilde{\mathcal{D}}$. □

Proof of Proposition 2.17 is again based on Euler method for an approximating sequence $(\widetilde{x}_n(\cdot))$ and Cantor diagonal construction for its limit $\widetilde{x}(\cdot)$. For $n{\in}\mathbb{N}$ $(2^n{>}T)$ set

$$\begin{array}{lllll} h_n & := \frac{T}{2^n}, & t_n^j & := j\ h_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\ t_n^j)\ \big(t - t_n^j,\ \ \widetilde{x}_n(t_n^j)\big) & & & \text{for } t \in\ ]t_n^j,\ t_n^{j+1}],\ \ j \le 2^n. \end{array}$$

The modulus of continuity $c_\varepsilon(\cdot)$ can be replaced by a nondecreasing convex function $[0, T{+}1] \longrightarrow [0,\infty[$ such that all $\widetilde{x}_n(\cdot)$ satisfy $\widetilde{q}_\varepsilon\big(\widetilde{x}_n(s),\ \widetilde{x}_n(t)\big)\ \le\ c_\varepsilon(t-s)$ for any $0 \le s < t < T + h_n$ and $\varepsilon \in \mathcal{J}$.

Since $\mathcal{J}$ is countable there is a sequence $(j_k)_{k\in\mathbb{N}}$ with $\{j_1, j_2 \ldots\} = \mathcal{J} \subset [0,1]^K$. Now for every $t \in]0,T[$, choose a decreasing sequence $(\delta_k(t))_{k\in\mathbb{N}}$ in $\mathbb{Q}\cdot T$ satisfying

$$0 < \delta_k(t) < \tfrac{h_k}{2}, \qquad t+\delta_k(t) < T, \qquad c_{\varepsilon_j}(\delta_k(t)) < h_k \qquad \text{for any } j \in \{j_1 \ldots j_k\}.$$

Then, $\widetilde{q}_{\varepsilon_j}\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t+\delta_k(t))\big) \le h_k$ for any $j \in \{j_1 \ldots j_k\}$, $k,n\in\mathbb{N}$
and so $\widetilde{q}_{\varepsilon}\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t+\delta_k(t))\big) \longrightarrow 0$ $(k\longrightarrow\infty)$ for every $\varepsilon\in\mathcal{J}$, uniformly in n.
For each $t\in]0,T[$ and any fixed $\varepsilon\in\mathcal{J}$, the timed transitional compactness provides sequences $m_k \nearrow \infty$, $n_k \nearrow \infty$ $(m_k \le n_k)$ of indices and an element $\widetilde{x}(t)\in\widetilde{E}$ (independent of ε) satisfying for every $k\in\mathbb{N}$

$$\wedge\begin{cases} \sup\limits_{l\ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\quad \widetilde{x}(t)\big) \le \frac{1}{k},\\ \sup\limits_{l\ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_l}(t+\delta_{m_k}(t))\big) \le \frac{1}{k}.\end{cases}$$

(In particular, each m_k, n_k may be replaced by larger indices preserving the properties.) For arbitrary $\kappa\in\mathbb{N}$, these sequences $m_k,\, n_k \nearrow\infty$ can even be chosen in such a way that the estimates are fulfilled for the finite set of parameters $t\in Q_\kappa :=]0,T[\ \cap\ \mathbb{N}\cdot h_\kappa$ and $\varepsilon\in\mathcal{J}_\kappa := \{\varepsilon_{j_1},\, \varepsilon_{j_2}\ \ldots\ \varepsilon_{j_\kappa}\} \subset \mathcal{J}$ simultaneously.

Now the Cantor diagonal construction (with respect to the index κ) provides subsequences (again denoted by) $m_k, n_k \nearrow \infty$ such that $m_k \le n_k$,

$$\wedge\begin{cases} \sup\limits_{l\ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\quad \widetilde{x}(t)\big) \le \frac{1}{k}\\ \sup\limits_{l\ge k} & \widetilde{q}_\varepsilon\big(\widetilde{x}(s),\quad \widetilde{x}_{n_l}(s+\delta_{m_k}(s))\big) \le \frac{1}{k}\end{cases}$$

for every $\kappa\in\mathbb{N}$ and all $\varepsilon\in\mathcal{J}_\kappa$, $s,t\in Q_\kappa$, $k\ge\kappa$.
In particular, $\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \le c_\varepsilon(t-s)$ for any $s,t\in Q_{\mathbb{N}} := \bigcup_\kappa Q_\kappa$ with $s<t$ and every $\varepsilon\in\mathcal{J}$.
Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k\in\mathbb{N}}$ fulfills for every $\kappa\in\mathbb{N}$ and all $t\in Q_\kappa$, $\varepsilon\in\mathcal{J}_\kappa$, $k,l\ge\kappa$

$$\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t),\ \widetilde{x}_{n_l}(t+\delta_{m_l}(t))\big) \le \tfrac{1}{k}+\tfrac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t\in]0,T[\setminus Q_{\mathbb{N}}$, we apply the timed transitional compactness to $((\widetilde{x}_{n_k}(t))_{k\in\mathbb{N}}$ and obtain a subsequence $n_{l_j}\nearrow\infty$ of indices (depending on t) and some $\widetilde{x}(t)\in\widetilde{E}$ satisfying $\forall\,\varepsilon\in\mathcal{J}$

$$\wedge\begin{cases} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\quad \widetilde{x}(t)\big) \longrightarrow 0,\\ \sup\limits_{i\ge j} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_{l_i}}(t+\delta_{m_j}(t))\big) \longrightarrow 0\end{cases} \qquad \text{for } j\longrightarrow\infty.$$

This implies the following convergence even uniformly in t (but not necessarily in $\varepsilon\in\mathcal{J}$)

$$\wedge\begin{cases} \limsup\limits_{\kappa\longrightarrow\infty} \sup\limits_{k>\kappa} & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_\kappa),\quad \widetilde{x}(t)\,\big) = 0,\\ \limsup\limits_{\kappa\longrightarrow\infty} \sup\limits_{k>\kappa} & \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_k}(t+2\,h_\kappa)\,\big) = 0.\end{cases} \tag{$*$}$$

Indeed, for $\kappa\in\mathbb{N}$ fixed arbitrarily and any $t\in]0,T[$, there exists $s=s(t,\kappa)\in Q_\kappa$ with

$$t-2\,h_\kappa < s \le t-h_\kappa \qquad\text{and}\qquad \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s),\ \widetilde{x}_{n_l}(s+\delta_{m_l}(s))\big) \le \tfrac{1}{k}+\tfrac{1}{l} \qquad \text{for all } k,l\ge\kappa.$$

So for any $k, l_j\ge\kappa$, we conclude from $\delta_{m_{l_j}}(\cdot) < \frac{1}{2}\,h_{m_{l_j}} \le \frac{1}{2}\,h_{l_j} \le \frac{1}{2}\,h_\kappa$

$$\begin{aligned} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_\kappa),\ \widetilde{x}(t)\big) \le\ & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_\kappa),\quad \widetilde{x}_{n_k}(s)\big)\\ &+\ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s),\quad \widetilde{x}_{n_{l_j}}(s+\delta_{m_{l_j}}(s))\big)\\ &+\ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(s+\delta_{m_{l_j}}(s)),\quad \widetilde{x}_{n_{l_j}}(t)\big)\\ &+\ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\quad \widetilde{x}(t)\big)\\ \le\ & c_\varepsilon(h_\kappa) + \tfrac{1}{k}+\tfrac{1}{l_j} + c_\varepsilon(2\,h_\kappa) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t), \widetilde{x}(t)\big)\end{aligned}$$

and $j\longrightarrow\infty$ leads to the estimate $\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_\kappa),\ \widetilde{x}(t)\big) \le 2\,c_\varepsilon(2\,h_\kappa)+\frac{2}{\kappa}$.
The proof of $\limsup\limits_{\kappa\longrightarrow\infty}\sup\limits_{k>\kappa} \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,h_\kappa)\,\big) = 0$ is analogous.

We reformulate the convergence property $(*)$ in the following notation: For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exists some $K_{\varepsilon,j} \in \mathbb{N}$ satisfying $K_{\varepsilon,j} > K_{\varepsilon,\,j-1}$ and for all $s,t \in [0,T[$, $k \geq \kappa \geq K_{\varepsilon,j}$,

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s - 2\,h_\kappa),\quad \widetilde{x}(s)\,\big) & \leq \quad \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_k}(t + 2\,h_\kappa)\,\big) & \leq \quad \frac{1}{j}\,. \end{cases}$$

Now Cantor diagonal construction provides a sequence $K_i \nearrow \infty$ with the corresponding property independent of $\varepsilon \in \mathcal{J}$ (in addition), i.e. for each $i \in \mathbb{N}$, the index $K_i \in \mathbb{N}$ has to satisfy for all $s,t \in [0,T[$, $\varepsilon \in \mathcal{J}_i \stackrel{\text{Def.}}{=} \{\varepsilon_{j_1}\ \ldots\ \varepsilon_{j_i}\} \subset \mathcal{J}$, $k \geq \kappa \geq K_i$

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s - 2\,h_\kappa),\quad \widetilde{x}(s)\,\big) & \leq \quad \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_k}(t + 2\,h_\kappa)\,\big) & \leq \quad \frac{1}{j}\,. \end{cases}$$

Convergence Theorem 2.18 states that $\widetilde{x}(\cdot)$ is a timed right–hand forward solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x},\,\cdot)$.

Indeed, set $N_j := n_{K_j}$ as an abbreviation. Define $\widetilde{g}_j : (\widetilde{y}, t) \longmapsto \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j}),\ t^{a+2}_{N_j}{+}2\,h_{K_j}\big)$ for $t^a_{N_j} \leq t < t^{a+1}_{N_j}$ and consider the sequence $t \longmapsto \widetilde{x}_{N_j}(t + 2\,h_{N_j} + 2\,h_{K_j})$ of solutions.

Obviously conditions (1.), (2.), (3.) of Proposition 2.18 result from the hypotheses here. For verifying its assumption (4.), we benefit from the convergence properties of the subsequence $(\widetilde{x}_{N_j})_{j\in\mathbb{N}}$ mentioned before. It ensures that for any $\widetilde{y} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$ and $s,t \in [0,T[$ (with $t^a_{N_j} \leq t < t^{a+1}_{N_j}$),

$$\begin{aligned} &\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(s),\,s),\ \widetilde{g}_j(\widetilde{y},\,t);\ \widetilde{z}\big) \;=\; \widetilde{Q}_\varepsilon\big(\widetilde{f}\,(\widetilde{x}(s),\,s)\,,\ \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j}),\ t^{a+2}_{N_j}{+}2\,h_{K_j}\big);\ \widetilde{z}\big) \\ \text{and,}\quad &\widetilde{q}_{\varepsilon'}\big(\widetilde{x}(s),\,\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j})\big) \;\leq\; \widetilde{q}_{\varepsilon'}\big(\widetilde{x}(s),\,\widetilde{x}_{N_j}(s+2\,h_{K_j})\big) \;+\; c_{\varepsilon'}(t-s+2\,h_{N_j}) \longrightarrow 0 \quad \forall\,\varepsilon' \in \mathcal{J} \\ &\big|s \;-\; (t^{a+2}_{N_j}{+}2\,h_{K_j})\big| \;\leq\; |s-t| \;+\; 2\,h_{N_j} \;+\; 2\,h_{K_j} \longrightarrow 0 \end{aligned}$$

for $t \downarrow s$, $j \longrightarrow \infty$. So the last (missing) assumption (4.i) of Convergence Theorem 2.18 results directly from hypothesis (5.) and, the proof of Proposition 2.17 is finished. □

2.4 Introducing "weak" transitional compactness

Now the examples of § 3 and § 4 demonstrate the key role of sequential compactness. In fact, it might be very difficult (or even impossible) to take the convergence with respect to each $\widetilde{q}_\varepsilon$ into consideration simultaneously. Thus, we weaken previous definitions of "timed transitionally compact" – following an idea that has already been initiated in [15], Definition 3.16.

Usually the concepts of "weak" convergence and "weak" compactness are closely related to linear forms in a topological vector space. But linear forms do not provide an adequate starting point for extending these concepts to ostensible metric spaces. Thus, we suggest another well–known relation of linear functional analysis as motivation : In every Banach space $(X, \|\cdot\|_X)$, the norm of any element z satisfies

$$\|z\|_X \;=\; \sup\ \big\{\, y^*(z)\ \big|\ y^* : X \longrightarrow \mathbb{R}\ \text{ linear, continuous, }\ \|y^*\|_{X^*} \leq 1\,\big\}.$$

So the key notion we seize here is to represent the each ostensible metric as supremum of (countably many) generalized distance functions, i.e. $\widetilde{q}_\varepsilon \;=\; \sup_{\kappa\in\mathcal{I}}\ \widetilde{q}_{\varepsilon,\kappa}$. Considering the convergence with respect to each $\widetilde{q}_{\varepsilon,\kappa}$ (instead of $\widetilde{q}_\varepsilon$) lays the basis of "weak compactness".

In return for weaker conditions on convergence, more "structural" assumptions about each $\widetilde{q}_\varepsilon$ and a "retraction" $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{\mathcal{D}}$ are used for proving existence of solutions.

Definition 2.20 *Let* $\widetilde{\Theta}$ *denote a nonempty set of maps* $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$. *Suppose* $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$.
$\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon \in \mathcal{J}\\ \kappa \in \mathcal{I}}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}},\ \widetilde{\Theta}\Big)$ *is called* weakly timed transitionally compact *if it fulfills :*

Let $(\widetilde{x}_n)_{n \in \mathbb{N}}$, $(h_j)_{j \in \mathbb{N}}$ *and* $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$, $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ *(for each* $n \in \mathbb{N}$*) satisfy the assumptions of Definition 2.13. Then there exist a sequence* $n_k \nearrow \infty$ *and* $\widetilde{x} \in \widetilde{E}$ *such that for each* $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$,

$$\begin{array}{rlcl} & \lim\limits_{k \to \infty} \ \pi_1 \widetilde{x}_{n_k} & = & \pi_1 \widetilde{x}, \\ & \limsup\limits_{k \longrightarrow \infty} \ \lfloor \widetilde{x}_{n_k} \rfloor_\varepsilon & \geq & \lfloor \widetilde{x} \rfloor_\varepsilon, \\ & \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k},\ \widetilde{x}) & = & 0, \\ \limsup\limits_{j \longrightarrow \infty} & \sup\limits_{k \geq j} \ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) & = & 0. \end{array}$$

A nonempty subset $\widetilde{F} \subset \widetilde{E}$ *is called* weakly timed transitionally compact in $(\widetilde{E}, (\widetilde{q}_\varepsilon), (\widetilde{q}_{\varepsilon,\kappa}), (\lfloor\cdot\rfloor_\varepsilon), \widetilde{\Theta})$ *if the same property holds for any sequence* $(\widetilde{x}_n)_{n \in \mathbb{N}}$ *in* $\widetilde{F}$ *(but* $\widetilde{x} \in \widetilde{F}$ *is* not *required).*

Proposition 2.21 (Existence due to weak transitional compactness)

Assume $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ *such that each* $\kappa \in \mathcal{I}$ *has counterparts* $\kappa_1, \kappa_2 \in \mathcal{I}$ *fulfilling for all* $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ $(\pi_1 \widetilde{y}_j \leq \pi_1 \widetilde{y}_{j+1})$

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \ \leq \ \widetilde{q}_{\varepsilon,\kappa_1}(\widetilde{y}_1, \widetilde{y}_2) \ + \ \widetilde{q}_{\varepsilon,\kappa_2}(\widetilde{y}_2, \widetilde{y}_3).$$

Moreover let $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ *be a "retraction" in the sense that for all* $\widetilde{y}, \widetilde{y}' \in \widetilde{E}$, $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$
$i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\, \widetilde{y} = \widetilde{y}$, $\widetilde{q}_{\varepsilon,\kappa}\big(i_{\widetilde{E}}\, \widetilde{y},\ i_{\widetilde{E}}\, \widetilde{y}'\big) \leq const(\varepsilon,\kappa) \cdot \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{y}, \widetilde{y}'\big)$, $\lfloor i_{\widetilde{E}}\, \widetilde{y} \rfloor_\varepsilon \leq \lfloor \widetilde{y} \rfloor_\varepsilon$, $\mathbb{T}_\varepsilon(\cdot,\ i_{\widetilde{E}}\, \widetilde{y}) \geq \widehat{\mathbb{T}_\varepsilon} \in]0,1[$.

Suppose $\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon \in \mathcal{J}\\ \kappa \in \mathcal{I}}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon \in \mathcal{J}},\ \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))\Big)$ *to be weakly timed transitionally compact. Let* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ *fulfill for every* $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$

1. $M_\varepsilon := \sup\limits_{t_1, t_2, \widetilde{y}_1, \widetilde{y}_2, \widetilde{z}} \alpha_\varepsilon\big(\widetilde{f}(\widetilde{y}_1, t_1),\ \widetilde{z}\big) < \infty$,
   $\exists\ M_{\varepsilon,\kappa} : \sup\limits_{t, \widetilde{y}} \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{y}, t)\,(h, \widetilde{z}_1),\ \widetilde{f}(\widetilde{y}, t)\,(h, \widetilde{z}_2)\big) \leq \widetilde{q}_{\varepsilon,\kappa}(\widetilde{z}_1, \widetilde{z}_2) \cdot e^{M_{\varepsilon,\kappa}\, h}\ \ \forall\, \widetilde{z}_1, \widetilde{z}_2 \in \widetilde{\mathcal{D}}, h \leq \widehat{\mathbb{T}_\varepsilon}$
2. $c_\varepsilon(h) := \sup\limits_{t, \widetilde{y}} \beta_\varepsilon(\widetilde{f}(\widetilde{y}, t))(h) < \infty$, $\quad c_\varepsilon(h) \stackrel{h \downarrow 0}{\longrightarrow} 0$
3. $\exists\ R_\varepsilon : \sup\limits_{t, \widetilde{y}} \gamma_\varepsilon(\widetilde{f}(\widetilde{y}, t)) \leq R_\varepsilon < \infty$,
4. $g_\varepsilon := \sup\limits_{t, \widetilde{y}} \zeta_\varepsilon(\widetilde{f}(\widetilde{y}, t)) < \infty$,
5. $\limsup\limits_{j \longrightarrow \infty} \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}, t),\ \widetilde{f}(\widetilde{y}_j, t_j);\ i_{\widetilde{E}}\, \widetilde{v}_j\Big) \leq R_\varepsilon$ *for any* $t_j \searrow t$ *and* $\widetilde{y}, \widetilde{y}_j, \widetilde{v}_j \in \widetilde{E}$ $(j \in \mathbb{N})$
   with $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{y}, \widetilde{y}_j) \longrightarrow 0$, $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{y},\ \widetilde{v}_j) \longrightarrow 0$ $\ \forall\, \varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}$, $\quad \pi_1 \widetilde{y}_j,\ \pi_1 \widetilde{v}_j \searrow \pi_1 \widetilde{y}$,

Then for every $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand sleek solution* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

The *proof* is based on the same Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$ as in Remark 2.15 – again in combination with Cantor diagonal construction (see the proof of Proposition 2.17). Due to the "weak" form of compactness (i.e. with respect to every $\widetilde{q}_{\varepsilon,\kappa}$ instead of $\widetilde{q}_\varepsilon$), we only have to modify the conclusion that the limiting function $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ is a timed right–hand sleek solution.
So an adapted convergence theorem is required. Its proof would like to follow the same track as for Proposition 2.18 – just implementing the index $\kappa \in \mathcal{I}$ (and its dependence on other indices) in addition.

The obstacle, however, is that the structural estimate (as in Proposition 2.5 or Lemma 2.19) is only available with respect to $\widetilde{q}_\varepsilon, \widetilde{Q}_\varepsilon$ and, roughly speaking, there is no obvious way "back" to some $\widetilde{q}_{\varepsilon,\kappa}$ for which the convergence of sequences is assumed. Thus, the "retraction" $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ is introduced.

Proposition 2.22 ("Weak" Convergence Theorem)
In addition to the general assumptions at the beginning of § 2, suppose $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[\ (\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ *such that each* $\kappa \in \mathcal{I}$ *has counterparts* $\kappa_1, \kappa_2 \in \mathcal{I}$ *fulfilling for all* $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ $(\pi_1\, \widetilde{y}_j \le \pi_1\, \widetilde{y}_{j+1})$

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa_1}(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa_2}(\widetilde{y}_2, \widetilde{y}_3).$$

Moreover let $i_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{D}$ *be a "retraction" in the sense that for all* $\widetilde{y}, \widetilde{y}' \in \widetilde{E}$, $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\, \widetilde{y} \;=\; \widetilde{y}, \quad \widetilde{q}_{\varepsilon,\kappa}\big(i_{\widetilde{E}}\, \widetilde{y},\ i_{\widetilde{E}}\, \widetilde{y}'\big) \;\le\; const(\varepsilon,\kappa) \cdot \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{y}, \widetilde{y}'\big), \quad \lfloor i_{\widetilde{E}}\, \widetilde{y} \rfloor_\varepsilon \le \lfloor \widetilde{y} \rfloor_\varepsilon, \quad \mathbb{T}_\varepsilon(\cdot,\, i_{\widetilde{E}}\, \widetilde{y}) \;\ge\; \widehat{\mathbb{T}_\varepsilon} \in\,]0,1[.$$

For each $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$, *assume the following properties of*

$$\begin{array}{lrcl} \widetilde{f}_m,\ \ \widetilde{f}: & \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}\big(\widetilde{E}, \widetilde{\mathcal{D}},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}\big) \qquad (m \in \mathbb{N}) \\ \widetilde{x}_m,\ \ \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E}\ : \end{array}$$

1. $$\begin{array}{lcll} M_\varepsilon & := & \sup\limits_{m,t,\widetilde{y},\widetilde{z}} & \big\{\, \alpha_\varepsilon\big(\widetilde{f}_m(\widetilde{y},t),\ \widetilde{z}\big) \big\} \;<\; \infty, \\ R_\varepsilon & \ge & \sup\limits_{m,t,\widetilde{y},\widetilde{z},h} & \big\{\, \widetilde{\gamma}_\varepsilon(t,\ \widetilde{f}_m(\widetilde{x}_m,\cdot),\ \widetilde{z}),\ \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y},t)),\ \ \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t))\, \big\}, \\ g_\varepsilon & \ge & \sup\limits_{m,t,\widetilde{y}} & \big\{\zeta_\varepsilon(\widetilde{f}_m(\widetilde{y},t))\big\} \\ \exists\ M_{\varepsilon,\kappa}: & & \sup\limits_{t,\widetilde{y}} & \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}_1),\ \ \widetilde{f}(\widetilde{y},t)\,(h,\widetilde{z}_2)\big) \;\le\; \widetilde{q}_{\varepsilon,\kappa}(\widetilde{z}_1,\widetilde{z}_2) \cdot e^{M_{\varepsilon,\kappa}\, h} \ \ \forall\, \widetilde{z}_1,\widetilde{z}_2 \in \widetilde{\mathcal{D}}, h \le \widehat{\mathbb{T}_\varepsilon} \end{array}$$
2. $\overset{\circ}{\widetilde{x}}_m(\cdot) \;\ni\; \widetilde{f}_m(\widetilde{x}_m(\cdot),\, \cdot)$ *in* $[0,T[$, *(in the sense of Definition 2.12)*
3. $\widehat{\omega}_\varepsilon(h) \;:=\; \sup\limits_m\ \omega_\varepsilon(\widetilde{x}_m, h) \;<\; \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*),* $\quad \limsup\limits_{h\,\downarrow\,0}\ \widehat{\omega}_\varepsilon(h) \;=\; 0,$
   $n_\varepsilon \;:=\; \sup\limits_{m,t}\ \lfloor \widetilde{x}_m(t) \rfloor_\varepsilon \;<\; \infty,$
4. $\forall\ 0 \le t_1 < t_2 < T \quad \exists\ (m_j)_{j\in\mathbb{N}},\ (\delta_j')_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty,\ \delta_j, \delta_j' \searrow 0$
   (i) $\limsup\limits_{j\longrightarrow\infty}\ \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{y}_j, s_j);\ \ i_{\widetilde{E}}\ \widetilde{x}_{m_j}(t_1+\delta_j')\big) \;\le\; R_\varepsilon,$
   for any sequences $s_j \downarrow t_1$, $(\widetilde{y}_j)$ *in* $\widetilde{E}$ *s.t.* $\pi_1\, \widetilde{y}_j \searrow \pi_1\, \widetilde{x}(t_1)$, $\quad \widetilde{q}_{\varepsilon',\kappa'}(\widetilde{x}(t_1),\, \widetilde{y}_j) \overset{j\to\infty}{\longrightarrow} 0 \quad \forall\, \varepsilon', \kappa'$
   (ii) $\widetilde{q}_{\varepsilon',\kappa'}\big(\widetilde{x}(t_1),\ \ \widetilde{x}_{m_j}(t_1+\delta_j')\big) \longrightarrow 0 \qquad \forall\ \varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}, \qquad \pi_1\, \widetilde{x}_{m_j}(t_1+\delta_j') \searrow\ \pi_1\, \widetilde{x}(t_1),$
   (iii) $\widetilde{q}_{\varepsilon,\kappa}\ \big(\widetilde{x}_{m_j}(t_2-\delta_j),\ \ \widetilde{x}(t_2)\big) \longrightarrow 0, \qquad \pi_1\, \widetilde{x}_{m_j}(t_2-\delta_j) \nearrow\ \pi_1\, \widetilde{x}(t_2).$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \;\ni\; \widetilde{f}(\widetilde{x}(\cdot),\, \cdot)$ *in* $[0,T[$.

Remark 2.23 "Continuity" assumption (5.) of existence theorem 2.21 can be slightly weakened – as a consequence of both condition (4.i) of this adapted convergence theorem and the Euler approximations (presented in the proof of Proposition 2.17). Indeed, for any initial element $\widetilde{x}_0 \in \widetilde{E}$ given, it suffices to consider only sequences $(v_j)_{j\in\mathbb{N}}$ in $\widetilde{E}$ satisfying $\lfloor \widetilde{v}_j \rfloor_\varepsilon \le \lfloor \widetilde{x}_0 \rfloor_\varepsilon\, e^{g_\varepsilon\, 2\,T} + g_\varepsilon\, 2\,T$ for each j, ε (in addition to $\widetilde{q}_{\varepsilon',\kappa'}(\widetilde{y},\ \widetilde{v}_j) \longrightarrow 0$ for all $\varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}$, $\pi_1\, \widetilde{v}_j \searrow \pi_1\, \widetilde{y}$).
This modification will be useful in § 4 because in the particular example of Proposition 4.10, continuity assumptions are formulated for an upper bound of $\widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{y},t),\ \widetilde{f}(\widetilde{y}_j,t_j);\ i_{\widetilde{E}}\, \widetilde{v}_j\big)$ independent of $\widetilde{v}_j, \varepsilon$.

Proof of Proposition 2.22 The uniform continuity of $\widetilde{x}(\cdot)$ w.r.t. each $\widetilde{q}_\varepsilon$ results from assumption (3.): Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\big(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2)\big) \le \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $0 \le t_1 < t_2 < T$. Let $\varepsilon \in \mathcal{J},\ \kappa \in \mathcal{I},\ 0 \le t_1 < t_2 < T$ be arbitrary. Choose $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{I}$ as counterparts of $\kappa \in \mathcal{I}$ due to applying the timed triangle inequality twice. Furthermore $t_1,\ t_2$ induce sequences $(\delta'_j)_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}}$, according to condition (4.ii), (4.iii). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta'_j < t_2 - \delta_j$ and so,

$$\begin{aligned}
&\widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{x}(t_1),\ \widetilde{x}(t_2)\big)\\
&\le\ \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1{+}\delta'_j)\big)\ +\ \widetilde{q}_{\varepsilon,\kappa_2}\big(\widetilde{x}_{m_j}(t_1{+}\delta'_j),\ \widetilde{x}_{m_j}(t_2{-}\delta_j)\big)\ +\ \widetilde{q}_{\varepsilon,\kappa_3}\big(\widetilde{x}_{m_j}(t_2{-}\delta_j),\ \widetilde{x}(t_2)\big)\\
&\le\ o(1) \qquad\qquad\qquad\qquad +\ \widehat{\omega}_\varepsilon(t_2 - t_1) \qquad\qquad\qquad +\ o(1) \qquad\qquad \text{for } j \longrightarrow \infty.
\end{aligned}$$

First we focus on the “test element” $i_{\widetilde{E}}\,\widetilde{x}(t) \in \widetilde{\mathcal{D}}$ for any $t \in [0,T[$ and choose $\varepsilon \in \mathcal{J},\ 0 \le h < \widehat{\mathbb{T}_\varepsilon}$ arbitrarily. Condition (7.) of Definition 2.1 ensures for all $k \in]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ \le\ \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ +\ \tfrac{1}{2}\,h^2$$

Now fix $\kappa \in \mathcal{I}$ (depending on ε, t, h, k) such that

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ \le\ \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ +\ h^2$$

and, $\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4 \in \mathcal{I}$ denote its counterparts due to applying the timed triangle inequality three times. According to assumptions (4.i) – (4.iii), there exist sequences $(m_j)_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}},\ (\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty,\ \ \delta_j \downarrow 0,\ \ \delta'_j \downarrow 0,\ \ \delta_j{+}\delta'_j < k$ and

$$\begin{cases}
\widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t{+}h{-}\delta_j),\ \widetilde{x}(t{+}h)\big) \longrightarrow 0, & \pi_1\,\widetilde{x}_{m_j}(t{+}h{-}\delta_j)\ \nearrow\ \pi_1\,\widetilde{x}(t{+}h),\\
\widetilde{q}_{\varepsilon',\kappa'}\big(\widetilde{x}(t),\ \ \widetilde{x}_{m_j}(t{+}\delta'_j)\big) \longrightarrow 0 \quad \forall\,\varepsilon' \in \mathcal{J}, \kappa' \in \mathcal{I}, & \pi_1\,\widetilde{x}_{m_j}(t{+}\delta'_j)\ \searrow\ \pi_1\,\widetilde{x}(t).
\end{cases}$$

Thus, Proposition 2.5 and Lemma 2.19 imply for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \kappa, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ \le\ \widetilde{q}_{\varepsilon,\kappa}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ +\ h^2\\
\le\ &\ \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j))\big)\\
&+\ \widetilde{q}_{\varepsilon,\kappa_2}\big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)),\ \ \widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k)\big)\\
&+\ \widetilde{q}_{\varepsilon,\kappa_3}\big(\widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k),\qquad \widetilde{x}_{m_j}(t{+}h - \delta_j)\big)\\
&+\ \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t{+}h - \delta_j),\qquad \widetilde{x}(t{+}h)\big) \qquad +\ h^2\\
\le\ &\ \widetilde{q}_{\varepsilon,\kappa_1}\big(i_{\widetilde{E}}\,\widetilde{x}(t),\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t+\delta'_j)\big)\ \cdot\ e^{M_{\varepsilon,\kappa_1}\cdot(h-k)}\\
&+\ \widetilde{q}_\varepsilon\ \ \big(\widetilde{f}(\widetilde{x}(t),t)\,(h{-}k,\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)),\ \ \widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k)\big)\\
&+\ \widehat{\omega}_\varepsilon(k - \delta_j - \delta'_j)\ \ +\ \ \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t{+}h - \delta_j),\ \widetilde{x}(t{+}h)\big)\ \ +\ h^2\\
\le\ &\ \ \mathrm{const}(\varepsilon,\kappa_1)\ \cdot\ \widetilde{q}_{\varepsilon,\kappa_1}\big(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t+\delta'_j)\big)\ \cdot\ e^{M_{\varepsilon,\kappa_1} h}\\
&+\ \int_0^{h-k} e^{M_\varepsilon\cdot(h-k-s)}\ \Big\{\widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s};\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)\Big)\ \cdot\\
&\qquad\qquad \big(1 + \lfloor i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)\rfloor_\varepsilon\ e^{g_\varepsilon\cdot(h-k)} + g_\varepsilon\cdot(h{-}k)\big)\ \ +\ 3\,R_\varepsilon\Big\}\ ds\\
&+\ \ \widehat{\omega}_\varepsilon(k - \delta_j - \delta'_j)\ \ +\ \ \widetilde{q}_{\varepsilon,\kappa_4}\big(\widetilde{x}_{m_j}(t{+}h - \delta_j),\ \widetilde{x}(t{+}h)\big)\ \ +\ h^2
\end{aligned}$$

Now $j \longrightarrow \infty$ (with ε, t, h, k still fixed) leads to an estimate not depending on κ any longer

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\\
\le\ &\ \ \widehat{\omega}_\varepsilon(k)\ +\ h^2\ +\\
&+\ h\ \ e^{M_\varepsilon h}\cdot \limsup_{j\longrightarrow\infty}\ \sup_{s\le h}\ \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s};\ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)\Big)\ \big(1 + n_\varepsilon\ e^{g_\varepsilon h} + g_\varepsilon\,h\big)
\end{aligned}$$

and, convergence assumption (4.i) implies (indirectly)

$$\limsup_{h\downarrow 0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\le s\le h}\ \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s};\ \ i_{\widetilde{E}}\,\widetilde{x}_{m_j}(t{+}\delta'_j)\Big)\ \le\ R_\varepsilon.$$

So after $k \longrightarrow 0$, we obtain $\displaystyle\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ i_{\widetilde{E}}\,\widetilde{x}(t)),\ \ \widetilde{x}(t+h)\big)\ \le\ R_\varepsilon\cdot(1+n_\varepsilon).$

For verifying the solution property of $\widetilde{x}(\cdot)$ at time t, let $\widetilde{z} \in \widetilde{\mathcal{D}}$ and $s \in \big[0, \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t), \widetilde{z})\big[$ be arbitrary with $s + \pi_1\,\widetilde{z} \;\le\; \pi_1\,\widetilde{x}(t)$. Then, Proposition 2.5, Lemma 2.7 and conditions (2.), (9.) of Definition 2.1 (applied to $i_{\widetilde{\mathcal{D}}}\, i_{\widetilde{E}}\, \widetilde{x}(t) \;=\; \widetilde{x}(t))$ imply

$$\begin{aligned}
&\limsup_{h\downarrow 0}\; \tfrac{1}{h}\cdot\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\,\widetilde{z}),\quad \widetilde{x}(t+h)\big) \;-\; \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\,\widetilde{z}),\;\widetilde{x}(t)\big)\cdot e^{M_\varepsilon\,h}\Big)\\
\le\;&\limsup_{h\downarrow 0}\; \tfrac{1}{h}\cdot\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\,\widetilde{z}),\quad \widetilde{f}(\widetilde{x}(t),t)\,(h,\,\widetilde{x}(t))\big) \;-\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(s,\,\widetilde{z}),\;\widetilde{x}(t)\big)\cdot e^{M_\varepsilon\,h}\\
&\qquad +\,\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\,\widetilde{x}(t)),\quad \widetilde{f}(\widetilde{x}(t),t)\,(h,\,i_{\widetilde{E}}\,\widetilde{x}(t))\big)\;+\\
&\qquad +\,\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\,i_{\widetilde{E}}\,\widetilde{x}(t)),\quad \widetilde{x}(t+h)\big)\Big)\\
\le\;& \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\;\widetilde{f}(\widetilde{x}(t),t);\;\widetilde{z}\big)\cdot(1+\lfloor\widetilde{x}(t)\rfloor_\varepsilon) \;+\; 3\,\gamma_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\big) \;+\; R_\varepsilon\cdot(1+n_\varepsilon)\\
\le\;& R_\varepsilon\;\cdot\;6\,(1+n_\varepsilon+\lfloor\widetilde{x}(t)\rfloor_\varepsilon)
\end{aligned}$$
□

2.5 Estimates comparing solutions

Finally, we are interested in bounds of the distance between solutions. However, estimating the distance between points of timed sleek transitions is available only for "test elements" of $\widetilde{\mathcal{D}}$ in the first argument of $\widetilde{q}_\varepsilon$ (as in Proposition 2.5 and Lemma 2.19). So we are using an auxiliary function instead of the distance. In the examples of the next sections, the following estimate implies uniqueness of solutions. Here assumptions about the time parameter $\mathbb{T}_\varepsilon(\cdot,\cdot) > 0$ come into play for the first time.

Proposition 2.24 (Estimate between timed right–hand sleek solutions)
Assume for $\widetilde{f} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon)),\;\; \widetilde{x}, \widetilde{y} : [0,T[\, \longrightarrow \widetilde{E}$ *and some* $\varepsilon \in \mathcal{J},\; \rho \ge 0$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\,\cdot),\;\; \overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot),\,\cdot)$ *in* $[0,T[$ *(in the sense of Def. 2.12)*, $\pi_1\,\widetilde{x}(0) = \pi_1\,\widetilde{y}(0) = 0$,
2. $M_\varepsilon \;\ge\; \sup\limits_{\widetilde{v}\in\widetilde{\mathcal{D}}\cup\widetilde{E},\; t<T,\; \widetilde{z}\in\widetilde{\mathcal{D}}} \big\{\alpha_\varepsilon(\widetilde{f}(\widetilde{v},t),\,\widetilde{z}),\;\; \widehat{\alpha}_\varepsilon(t,\widetilde{x}(\cdot),\widetilde{z}),\;\; \widehat{\alpha}_\varepsilon(t,\widetilde{y}(\cdot),\widetilde{z})\big\}$,
3. $R_\varepsilon \;\ge\; \sup\limits_{\widetilde{v}\in\widetilde{\mathcal{D}}\cup\widetilde{E},\; t<T,\; \widetilde{z}\in\widetilde{\mathcal{D}}} \big\{\gamma_\varepsilon(\widetilde{f}(\widetilde{v},t)),\;\; \widehat{\gamma}_\varepsilon(t,\widetilde{x}(\cdot),\widetilde{z}),\;\; \widehat{\gamma}_\varepsilon(t,\widetilde{y}(\cdot),\widetilde{z})\big\}$
4. $g_\varepsilon \;\ge\; \sup\limits_{\widetilde{v}\in\widetilde{\mathcal{D}}\cup\widetilde{E},\; t<T} \big\{\zeta_\varepsilon(\widetilde{f}(\widetilde{v},t))\big\}$
5. $\exists\;\widehat{\omega}_\varepsilon(\cdot) = o(1),\; L_\varepsilon :\;\; \widetilde{Q}_\varepsilon\big(\widetilde{f}(\widetilde{z},s),\;\widetilde{f}(\widetilde{v},t);\;\widetilde{z}\big) \;\le\; R_\varepsilon + L_\varepsilon\cdot\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{v}) + \widehat{\omega}_\varepsilon(t-s)$
for all $0 \le s \le t \le T$ *and* $\widetilde{v} \in \widetilde{E},\; \widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1\,\widetilde{z}_1 \le \pi_1\,\widetilde{v}$,
6. $\forall\; t \in [0,T[\,:$ *the infimum* $\varphi_\varepsilon(t) \;:=\; \inf\limits_{\substack{\widetilde{z}\,\in\,\widetilde{\mathcal{D}}:\;\; \pi_1\,\widetilde{z}\,\le\, t,\\ \lfloor\widetilde{z}\rfloor_\varepsilon\,\le\,\rho\,\cdot\,\exp(g_\varepsilon\,t)+g_\varepsilon\,t}} \big(\widetilde{q}_\varepsilon(\widetilde{z},\,\widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\,\widetilde{y}(t))\big) < \infty$
can be approximated by a minimizing sequence $(\widetilde{z}_j)_{j\,\in\,\mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *with*
$\pi_1\widetilde{z}_j \;\le\; \pi_1\widetilde{z}_{j+1} \;\le\; t,\quad \lfloor\widetilde{z}_j\rfloor_\varepsilon \le \rho\,e^{g_\varepsilon\cdot t} + g_\varepsilon\,t,\quad \frac{\sup_{k\,>\,j}\;\widetilde{q}_\varepsilon(\widetilde{z}_j,\widetilde{z}_k)}{\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\,\widetilde{z}_j)} \;\overset{j\to\infty}{\longrightarrow}\; 0.$

Then, $\qquad \varphi_\varepsilon(t) \;\le\; \big(\varphi_\varepsilon(0) \;+\; 8\,R_\varepsilon\;\;(1{+}\rho)\cdot t\big)\;\cdot\; e^{(L_\varepsilon\,(1+\rho)\,+\,M_\varepsilon)\,\cdot\,t}.$

Proof is based on a further subdifferential version of Gronwall's Lemma quoted in Lemma 2.25. $\varphi_\varepsilon(\cdot)$ satisfies $\varphi_\varepsilon(t) \;\le\; \liminf\limits_{h\downarrow 0}\; \varphi_\varepsilon(t-h)$ for every $t \in\,]0,T[$ due to the timed triangle inequality and the continuity of $\widetilde{x}(\cdot),\, \widetilde{y}(\cdot)$ (in time direction).
For showing $\qquad \liminf\limits_{h\downarrow 0}\; \frac{\varphi_\varepsilon(t+h)\,-\,\varphi_\varepsilon(t)}{h} \;\le\; (L_\varepsilon\;(1{+}\rho) \;+\; M_\varepsilon)\cdot\varphi_\varepsilon(t) \;+\; 8\,R_\varepsilon\;(1{+}\rho)\,.$
let $(\widetilde{z}_j)_{j\,\in\,\mathbb{N}}$ denote a minimizing sequence in $\widetilde{\mathcal{D}}$ such that

$$\wedge\begin{cases} \pi_1\,\widetilde{z}_j \;\le\; \pi_1\,\widetilde{z}_k \;\le\; t, \quad \lfloor\widetilde{z}_j\rfloor_\varepsilon \;\le\; \rho\;e^{g_\varepsilon\,t} + g_\varepsilon\,t & \text{for all } j < k,\\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\widetilde{z}_k) \qquad\;\; \le\; \frac{1}{2\,j}\cdot\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j,t),\;\widetilde{z}_j) & \\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\;\widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}_j,\;\widetilde{y}(t)) \quad\longrightarrow\quad \varphi_\varepsilon(t) & (j \longrightarrow \infty).\end{cases}$$

For all $h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\ \widetilde{z}_j),\ \ j < k,$ Lemma 2.19 and assumption (5.) imply

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\ (h, \widetilde{z}_j),\ \ \widetilde{x}(t+h)\Big) \quad - \quad \widetilde{q}_\varepsilon\big(\widetilde{z}_j,\ \widetilde{x}(t)\big) \ \cdot\ e^{M_\varepsilon\, h} \\
\le\ & \int_0^h e^{M_\varepsilon\cdot(h-s)} \Big\{\Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\,(\widetilde{z}_j,\ \widetilde{x}(t+s)) + \widehat{\omega}_\varepsilon(s)\Big) \ \cdot\ \big(1 + \lfloor \widetilde{z}_j \rfloor_\varepsilon\ e^{g_\varepsilon\, s} + g_\varepsilon\, s\big) \ +\ 3\,R_\varepsilon\Big\}\ ds \\
\le\ & \int_0^h e^{M_\varepsilon\cdot(h-s)} \Big(4\,R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\,(\widetilde{z}_j,\ \widetilde{x}(t+s)) + \widehat{\omega}_\varepsilon(s)\Big) \ \cdot\ \big(1 + \ \rho \quad e^{g_\varepsilon\, h} + g_\varepsilon\, h\big) \qquad ds.
\end{aligned}$$

Setting the abbreviations $h_j := \min\{\frac{1}{2}\ \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\ \widetilde{z}_j),\ \frac{1}{j}\} > 0$ and $\delta_j := 1 + \rho e^{g_\varepsilon\, h_j} + g_\varepsilon\, h_j \overset{j\to\infty}{\longrightarrow} 1+\rho,$ the approximating properties of $(\widetilde{z}_j)_{j\,\in\,\mathbb{N}}$ and the timed triangle inequality guarantee for any index $k > j$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\ (h_j, \widetilde{z}_j),\ \ \widetilde{x}(t+h_j)\Big) \\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{z}_k,\ \widetilde{x}(t)\big)\cdot e^{M_\varepsilon\, h_j} \quad + \quad \tfrac{e^{M_\varepsilon\, h_j} - 1}{M_\varepsilon} \quad \big(L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}_k,\ \widetilde{x}(t)) + L_\varepsilon \cdot \tfrac{1}{j}\ h_j \ +\ 4\,R_\varepsilon\big)\ \delta_j \\
&\quad + \ \ \tfrac{1}{j}\ h_j \quad \cdot e^{M_\varepsilon\, h_j} \quad + \quad \int_0^{h_j} e^{M_\varepsilon\cdot(h_j - s)}\ \big(L_\varepsilon \cdot \omega_\varepsilon(\widetilde{x}, s) \qquad + \widehat{\omega}_\varepsilon(s)\big)\ ds \qquad \delta_j\,.
\end{aligned}$$

The same estimate for $\widetilde{q}_\varepsilon\ \Big(\widetilde{f}(\widetilde{z}_j, t)\ (h_j, \widetilde{z}_j),\ \ \widetilde{y}(t+h_j)\Big)$ and $k \longrightarrow \infty,\ j \longrightarrow \infty$ lead to

$$\liminf_{h\,\downarrow\,0}\ \tfrac{\varphi_\varepsilon(t+h)\ -\ \varphi_\varepsilon(t)}{h} \ \le\ (L_\varepsilon\ \ (1+\rho)\ +\ M_\varepsilon)\cdot\varphi_\varepsilon(t) \ +\ 8\,R_\varepsilon\ \ (1+\rho)\,. \qquad \square$$

Lemma 2.25 (Lemma of Gronwall for semicontinuous functions II [13]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R},\ \ f, g \in C^0([a,b[, \mathbb{R})$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned}
\psi(t) \quad &\le \quad \liminf_{h\,\downarrow\,0}\ \psi(t-h), && \forall\ t \in\,]a,\ b], \\
\psi(t) \quad &\ge \quad \liminf_{h\,\downarrow\,0}\ \psi(t+h), && \forall\ t \in [a,\ b[, \\
\liminf_{h\,\downarrow\,0}\ \tfrac{\psi(t+h)-\psi(t)}{h} \quad &\le \quad f(t)\ \cdot\ \liminf_{h\,\downarrow\,0}\ \psi(t-h)\ +\ g(t) && \forall\ t \in\,]a,\ b[.
\end{aligned}$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \ \le\ \ \psi(a)\ \cdot\ e^{\mu(t)} \ \ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds \qquad \textit{with } \mu(t) := \int_a^t f(s)\ ds. \qquad \square$$

Remark 2.26 All these results are easy to apply to sets *without* separate "time" component, i.e. consider just nonempty E, $\mathcal{D}$ (instead of $\widetilde{E}$, $\widetilde{\mathcal{D}}$). Indeed, every ostensible metric $q_\varepsilon : E \times E \longrightarrow [0,\infty[$ induces a *timed* ostensible metric $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ according to

$$\widetilde{q}_\varepsilon\ ((s,x),\ (t,y))\ \ :=\ \ |s-t| + q_\varepsilon(x,y) \qquad \text{for all } (s,x),\ (t,y) \in \widetilde{E}.$$

Then every map $\vartheta : [0,1] \times (\mathcal{D} \cup E) \longrightarrow (\mathcal{D} \cup E)$ satisfying the conditions (1.)–(7.), (9.) for the tuple $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ induces a *timed* sleek transition $\widetilde{\vartheta} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ by $\widetilde{\vartheta}\,(h,\ (t,x))\ :=\ (t+h,\ \vartheta(h,x))$ for all $(t,x) \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ \ h \in [0,1].$ So skipping the separate "time" component consistently, all conclusions of this section can also be drawn for *sleek transitions* on a given tuple $(E, D, (q_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ (as used in the next section). These counterparts are usually denoted without tilde.

3 Bounded vector fields in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$

Now the results of § 2 are applied to measure–valued solutions $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^N)$ of the continuity equation
$$\tfrac{d}{dt}\, \mu_t \;+\; D_x \cdot (b\, \mu_t) \;=\; 0 \qquad \text{(in the distributional sense)}$$
with a given vector field b. We suggest a family of pseudo–metrics $(p_\varepsilon)_{\varepsilon \in \mathcal{J}}$ on $\mathcal{M}(\mathbb{R}^N)$ motivated by vague convergence and a corresponding family $(\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}}$. Assuming $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ for the moment, measure–valued solutions of the continuity equation induce sleek transitions on the tuple $(\mathcal{M}(\mathbb{R}^N),\, \mathcal{M}(\mathbb{R}^N),\, (p_\varepsilon)_{\varepsilon\in\mathcal{J}},\, (\lfloor\cdot\rfloor_\varepsilon)_{\varepsilon\in\mathcal{J}})$ with $\gamma_\varepsilon(\cdot) \equiv 0$ and all their parameters of continuity are specified in Proposition 3.7.

Restricting ourselves to the set $\mathcal{M}^+_c(\mathbb{R}^N)$ of *positive* Radon measures *with compact support*, the existence result of Proposition 2.17 provides right–hand sleek solutions of the quasilinear continuity equation
$$\tfrac{d}{dt}\, \mu_t \;+\; D_x \cdot (f(\mu_t, t)\;\, \mu_t) \;=\; 0 \qquad \text{(in the mutational sense)}$$
for every initial datum $\mu_0 \in \mathcal{M}^+_c(\mathbb{R}^N)$ if the vector field $f : \mathcal{M}^+_c(\mathbb{R}^N) \times [0,T[\longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ holds assumptions specified later in Proposition 3.11. Such a measure–valued function μ proves to be solution in the distributional sense. Finally, Corollary 3.13 gives sufficient (Lipschitz) conditions on f for the uniqueness of right–hand sleek solutions.

Lemma 3.1 *There is a countable family $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ of smooth Schwartz functions $\mathbb{R}^N \longrightarrow [0,\infty[$ such that $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ is dense in $\big(C_0(\mathbb{R}^N,\, [0,\infty[),\, \|\cdot\|_\infty\big)$,*
$C^\infty_c(\mathbb{R}^N,\, [0,\infty[)$ is contained in the closure of $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ with respect to the C^1 norm and it satisfies $|\nabla\, \varphi_\varepsilon(\cdot)| \le \lambda_\varepsilon \cdot \varphi_\varepsilon(\cdot)$ in $\mathbb{R}^N$ with a constant $\lambda_\varepsilon > 0$ for each $\varepsilon \in \mathcal{J} \subset [0,1]$.

Proof. Such a $\varphi_\varepsilon \in C^\infty(\mathbb{R}^N, [0,\infty[)$ can be generated by means of convolution.
Indeed, $C^\infty_0(\mathbb{R}^{\mathbb{N}}, [0,\infty[)$ is known to be separable with respect to $\|\cdot\|_\infty$. Now consider a countable dense subset $(f_k)_{k\in\mathbb{N}}$ of $C^\infty_0(\mathbb{R}^{\mathbb{N}}, [0,\infty[)$ together with $e_\delta : x \longmapsto c_{\delta,N}\, \cdot\, \exp(-\,\delta\, \frac{|x|^2}{1+|x|})$ (for arbitrarily large $\delta > 0$ and the constant $c_{\delta,N} > 0$ such that $\|e_\delta\|_{L^1(\mathbb{R}^N)} = 1$).
Then, each $f_k * e_\delta$ is smooth, nonnegative and satisfies $|\nabla(f_k * e_\delta)| \;=\; |f_k * (\nabla e_\delta)| \;\le\; \delta\;\, f_k * e_\delta$ since the auxiliary function $\widehat{e}_\delta : [0,\infty[\;\longrightarrow\;]0,1],\quad r \;\longmapsto\; c_{\delta,N}\, \cdot\, \exp(-\,\delta\, \frac{r^2}{1+r})$ is smooth with $\frac{d}{dr}\, \widehat{e}_\delta(r) = -\,\delta\, \frac{r\,(r+2)}{(r+1)^2}\;\, \widehat{e}_\delta(r) \;\in\; [-\delta, 0]\cdot \widehat{e}_\delta(r)$ and thus $\frac{d}{dr}\, \widehat{e}_\delta(r) = O(r)$ for $r \longrightarrow 0^+$.
Furthermore, $f_k * e_\delta$ is a Schwartz function because so is e_δ and f_k is assumed to have compact support. $(f_k * e_\delta)_{k,\delta\,\in\,\mathbb{N}}$ is dense in $\big(C_0(\mathbb{R}^N,\, [0,\infty[),\, \|\cdot\|_\infty\big)$ since so is $(f_k)_{k\,\in\,\mathbb{N}}$ and $(e_\delta)_{\delta\,\in\,\mathbb{N}}$ is a Dirac sequence.
Finally it satisfies the second required property because for any $g \in C^\infty_c(\mathbb{R}^N,\, [0,\infty[)$ and subsequence $(f_{k_j})_{j\in\mathbb{N}}$ with $\|g - f_{k_j}\|_\infty \overset{j\to\infty}{\longrightarrow} 0$, we obtain $\nabla(f_{k_j} * e_\delta) \;=\; f_{k_j} * (\nabla e_\delta) \overset{j\to\infty}{\longrightarrow} g * (\nabla e_\delta) = (\nabla g) * e_\delta$ (uniformly) and the last convolution converges uniformly to ∇g for $\delta \longrightarrow \infty$. □

Definition 3.2 *Let $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ be a countable family of Schwartz functions as described in Lemma 3.1. For each $\varepsilon \in \mathcal{J}$, define the pseudo–metric $p_\varepsilon :\; \mathcal{M}(\mathbb{R}^N) \times \mathcal{M}(\mathbb{R}^N) \;\longrightarrow\; [0,\infty[$,*
$$p_\varepsilon(\mu,\, \nu) \;:=\; \big|\varphi_\varepsilon \cdot (\mu - \nu)\, (\mathbb{R}^N)\big| \;=\; \Big|\int_{\mathbb{R}^N} \varphi_\varepsilon\;\, d(\mu - \nu)\Big|$$

Remark 3.3 Obviously, Gronwall's Lemma implies $\varphi_\varepsilon > 0$ in $\mathbb{R}^N$ unless $\varphi_\varepsilon \equiv 0$.
So assuming $\varphi_\varepsilon \not\equiv 0$ for all $\varepsilon \in \mathcal{J}$ from now on, each pseudo–metric p_ε takes all points of $\mathbb{R}^N$ into consideration – in a weighted form.

Considering measure–valued solutions $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^N)$ of the continuity equation

$$\tfrac{d}{dt}\,\mu_t \;+\; D_x \cdot (\widetilde{b}\,\mu_t) \;=\; 0 \qquad \text{(in the distributional sense)},$$

vector fields $\widetilde{b} \in L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ are known to be advantageous because the solutions are closely related to the corresponding ordinary differential equation.

Definition 3.4 *For any $\widetilde{b} \in L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$, the function $\mathbf{X}_{\widetilde{b}} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is induced by the flow along $\widetilde{b}$, i.e. $\mathbf{X}_{\widetilde{b}}(\cdot,x_0) : [0,T] \longrightarrow \mathbb{R}^N$ is the absolutely continuous solution of the Cauchy problem*

$$\wedge \begin{cases} \frac{d}{dt}\,x(t) &=& \widetilde{b}(x(t),\,t) \quad \text{a.e. in } [0,T], \\ x(0) &=& x_0. \end{cases}$$

Proposition 3.5 ([1], Proposition 4 & Remark 7)
Assume for $\widetilde{b} \in L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)\big)$ that $\frac{|\widetilde{b}|}{1+|x|} \in L^1\left([0,T], L^\infty(\mathbb{R}^N)\right)$.
For any initial datum $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$, the unique solution $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^N),\ t \longmapsto \mu_t$ of the continuity equation $\frac{d}{dt}\,\mu_t + D_x \cdot (\widetilde{b}\,\mu_t) = 0$ (in the distributional sense) is given by the push–forward $\mu_t := \mathbf{X}_{\widetilde{b}}(t,\cdot)_\sharp\,\mu_0$ at each time $t \in\,]0,T]$, i.e.

$$\int_{\mathbb{R}^N} \varphi(x)\ d\mu_t(x) \;=\; \int_{\mathbb{R}^N} \varphi(\mathbf{X}_{\widetilde{b}}(t,x))\ d\mu_0(x) \qquad \text{for all } \varphi \in C^0_c(\mathbb{R}^N). \qquad \square$$

This simple representation formula lays the foundations for investigating whether the continuity equation with *autonomous* vector fields can induce sleek transitions.

Lemma 3.6 *For each vector field $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty$, the measure–valued solutions of the continuity equation $\frac{d}{dt}\,\mu_t + D_x \cdot (b\,\mu_t) = 0$ (in the distributional sense) induce a map $\vartheta_b : [0,1] \times \mathcal{M}(\mathbb{R}^N) \longrightarrow \mathcal{M}(\mathbb{R}^N)$ satisfying the following conditions*

1. $\vartheta_b(0,\cdot) \;=\; \mathrm{Id}_{\mathcal{M}(\mathbb{R}^N)}$,
2. $\vartheta_b(h,\ \vartheta_b(t,\mu_0)) = \vartheta_b(t+h,\mu_0)$,
   $p_\varepsilon\left(\vartheta_b(h,\ \vartheta_b(t,\mu_0)),\ \ \vartheta_b(t+h,\mu_0)\right) \;=\; 0 \;=\; p_\varepsilon\left(\vartheta_b(t+h,\mu_0),\ \ \vartheta_b(h,\ \vartheta_b(t,\mu_0))\right)$
   for any initial datum $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$ and $t,h \in [0,1]$ with $t+h \le 1$.
3. $\displaystyle\sup_{\mu_0,\nu_0 \in \mathcal{M}(\mathbb{R}^N)}\ \limsup_{h\downarrow 0}\ \frac{p_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) \,-\, p_\varepsilon(\mu_0,\nu_0)}{h \quad p_\varepsilon(\mu_0,\nu_0)} \;\le\; \lambda_\varepsilon\ \|b\|_\infty$,
4. $\left|\left(\varphi_\varepsilon\ \vartheta_b(t,\mu_0)\right)(\mathbb{R}^N)\right| \;\le\; \left|\left(\varphi_\varepsilon\ \mu_0\right)(\mathbb{R}^N)\right| \cdot e^{\lambda_\varepsilon\,\|b\|_\infty\,\cdot\,t}$ *for all $\mu_0 \in \mathcal{M}(\mathbb{R}^N),\ t \in [0,1]$,*
5. $p_\varepsilon\left(\vartheta_b(s,\mu_0),\ \vartheta_b(t,\mu_0)\right) \;\le\; |t-s| \cdot \lambda_\varepsilon\ \|b\|_\infty\ e^{\lambda_\varepsilon\,\|b\|_\infty}\left|\left(\varphi_\varepsilon\ \mu_0\right)(\mathbb{R}^N)\right|$ *for all $s,t \in [0,1],\ \mu_0 \in \mathcal{M}(\mathbb{R}^N)$,*
6. $\displaystyle\limsup_{h\downarrow 0}\ \frac{p_\varepsilon\big(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\big)}{h} \;\le\; \lambda_\varepsilon\ \left|\left(\varphi_\varepsilon\ |b_1-b_2|\ \mu_0\right)(\mathbb{R}^N)\right|$
   for all bounded vector fields $b_1,b_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ and $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$.

Proof. As a direct consequence of Proposition 3.5, the measure–valued flow $\vartheta_b : [0,1] \times \mathcal{M}(\mathbb{R}^N) \longrightarrow \mathcal{M}(\mathbb{R}^N)$ satisfies the semigroup property and thus statements (1.), (2.).
Furthermore, for any $\mu_0,\nu_0 \in \mathcal{M}(\mathbb{R}^N)$, we conclude from the definition of push–forward

$$\begin{aligned} p_\varepsilon\big(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)\big) \;&=\; \left|\,\varphi_\varepsilon \cdot \left(\mathbf{X}_b(h,\cdot)_\sharp\,\mu_0 \;-\; \mathbf{X}_b(h,\cdot)_\sharp\,\nu_0\right)(\mathbb{R}^N)\right| \\ &=\; \left|\int_{\mathbb{R}^N} \varphi_\varepsilon(\mathbf{X}_b(h,\cdot))\ \ d(\mu_0-\nu_0)\right| \\ &\le\; \left|\int_{\mathbb{R}^N} \left(\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\right)\ d(\mu_0-\nu_0)\right| \;+\; \left|\left(\varphi_\varepsilon \cdot (\mu_0-\nu_0)\right)(\mathbb{R}^N)\right|. \end{aligned}$$

So the choice of φ_ε (in Lemma 3.1) implies

$$\begin{aligned}
\limsup_{h\downarrow 0} \tfrac{p_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) - p_\varepsilon(\mu_0,\nu_0)}{h} &\le \limsup_{h\downarrow 0} \tfrac{1}{h}\cdot \Big|\int_{\mathbb{R}^N} \big(\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\big)\ d(\mu_0-\nu_0)\Big| \\
&\le \Big|\int_{\mathbb{R}^N} \nabla\varphi_\varepsilon(x)\cdot b(x) \quad d(\mu_0-\nu_0)\Big| \\
&\le \|b\|_\infty \ \Big|\int_{\mathbb{R}^N} \lambda_\varepsilon\ \varphi_\varepsilon \quad d(\mu_0-\nu_0)\Big| \\
&\le \|b\|_\infty \quad \lambda_\varepsilon \cdot p_\varepsilon(\mu_0,\nu_0).
\end{aligned}$$

Applying this estimate to $\nu_0 \equiv 0$ and $\vartheta_b(t,\mu_0)$ (instead of μ_0), we conclude property (4.) from Gronwall's Lemma 2.6 because the continuous auxiliary function

$$\delta_\varepsilon : [0,1] \longrightarrow \mathbb{R},\quad t \longmapsto \big|\big(\varphi_\varepsilon\ \vartheta_b(t,\mu_0)\big)(\mathbb{R}^N)\big| \;=\; \big|\big(\varphi_\varepsilon(\mathbf{X}_b(t,\cdot))\ \mu_0\big)(\mathbb{R}^N)\big|$$

is one–sided differentiable and satisfies $\frac{d^+}{dt^+}\ \delta_\varepsilon(\cdot) \ \le\ \lambda_\varepsilon\ \|b\|_\infty \cdot \delta_\varepsilon(\cdot)$.

In basically the same way, we obtain statement (5.) considering the auxiliary function

$$\widehat{\delta}_\varepsilon : [s,1] \longrightarrow \mathbb{R},\quad t \longmapsto \big|\varphi_\varepsilon\ \big(\vartheta_b(t,\mu_0) - \vartheta_b(s,\mu_0)\big)(\mathbb{R}^N)\big| \;=\; \Big|\Big(\big(\varphi_\varepsilon(\mathbf{X}_b(t-s,\cdot)) - \varphi_\varepsilon\big)\ \vartheta_b(s,\mu_0)\Big)(\mathbb{R}^N)\Big|$$

with $s \in [0,1[$ fixed and $\frac{d^+}{dt^+}\ \widehat{\delta}_\varepsilon(t) \ \le\ \lambda_\varepsilon\ \|b\|_\infty\ \big|\big(\varphi_\varepsilon \vartheta_b(t,\mu_0)\big)(\mathbb{R}^N)\big| \ \le\ \lambda_\varepsilon\ \|b\|_\infty\ e^{\lambda_\varepsilon\ \|b\|_\infty}\ \big|\big(\varphi_\varepsilon \mu_0\big)(\mathbb{R}^N)\big|$.

Finally choose any $b_1, b_2 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty$ and initial datum $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$. Then, for every $h \in [0,1]$,

$$\begin{aligned}
\tfrac{1}{h}\cdot p_\varepsilon\big(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\big) &= \tfrac{1}{h}\cdot \Big|\int_{\mathbb{R}^N} \big(\varphi_\varepsilon(\mathbf{X}_{b_1}(h,\cdot)) \ -\ \varphi_\varepsilon(\mathbf{X}_{b_2}(h,\cdot))\big)\ d\,\mu_0\Big| \\
\limsup_{h\downarrow 0} \tfrac{p_\varepsilon\big(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\big)}{h} &\le \Big|\int_{\mathbb{R}^N} \lambda_\varepsilon\ \varphi_\varepsilon\ |b_1 - b_2|\ d\,\mu_0\Big|.
\end{aligned}$$ □

So Lemma 3.6 reveals how $\vartheta_b : [0,1]\times\mathcal{M}(\mathbb{R}^N) \longmapsto \mathcal{M}(\mathbb{R}^N)$ can be regarded as a sleek transition : Set $E := \mathcal{D} := \mathcal{M}(\mathbb{R}^N)$, $i_{\mathcal{D}} := \mathrm{Id}_{\mathcal{M}(\mathbb{R}^N)}$, $\mathbb{T}_\varepsilon(\cdot,\cdot) := 1$.
Statement (2.) implies condition (2.) of Definition 2.1 with $\gamma_\varepsilon(\vartheta_b) := 0$. Statement (3.) motivates us to choose $\alpha_\varepsilon(\vartheta_b,\mu) := \lambda_\varepsilon\ \|b\|_\infty$ uniformly for all $\mu \in \mathcal{M}(\mathbb{R}^N)$. Now statement (5.) suggests a candidate for $\lfloor\cdot\rfloor_\varepsilon$, namely

$$\lfloor\mu\rfloor_\varepsilon \ :=\ \big|\big(\varphi_\varepsilon\ \mu\big)(\mathbb{R}^N)\big| \ <\ \infty \qquad \text{for all } \mu \in \mathcal{M}(\mathbb{R}^N),\ \varepsilon \in \mathcal{J}.$$

Statement (4.) ensures growth condition (5.) on $\lfloor\vartheta_b(\cdot,\mu_0)\rfloor_\varepsilon$ of Definition 2.1 with $\zeta_\varepsilon(\vartheta_b) := \lambda_\varepsilon\ \|b\|_\infty$.
Finally condition (7.) on sleek transitions, i.e.

$$\limsup_{h\downarrow 0}\ p_\varepsilon\big(\vartheta_b(t-h,\ \mu),\ \nu\big) \ \ge\ p_\varepsilon\big(\vartheta_b(t,\ \mu),\ \nu\big) \qquad \forall\ \mu,\nu \in \mathcal{M}(\mathbb{R}^N),\ t \in [0,1]$$

is an obvious consequence of the symmetry of $p_\varepsilon(\cdot,\cdot)$ together with statement (5.). Thus, $\vartheta_b(\cdot,\cdot)$ induces a sleek transition and, Lemma 2.9 provides an upper bound of the "distance" between these transitions (in the sense of Definition 2.3)

$$P_\varepsilon(\vartheta_b,\vartheta_c;\mu) \ \stackrel{\text{Def.}}{=}\ \sup_{\substack{0\le t<1 \\ \nu\in\mathcal{M}(\mathbb{R}^N)}}\ \limsup_{h\downarrow 0} \Big(\tfrac{p_\varepsilon\big(\vartheta_b(t+h,\mu),\ \vartheta_c(h,\nu)\big) \ -\ p_\varepsilon\big(\vartheta_b(t,\mu),\ \nu\big)\ \cdot\ e^{\alpha_\varepsilon(\vartheta_c,\mu)\cdot h}}{h\quad (1+\lfloor\nu\rfloor_\varepsilon)}\Big)^+.$$

Proposition 3.7 *Choose the parameter* $C \in [0,\infty[$ *arbitrarily.*
For each $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ *with* $\|b\|_\infty \le C$, *the measure–valued solutions of the continuity equation*

$$\tfrac{d}{dt}\ \mu_t \ +\ D_x\cdot(b\ \mu_t) \ =\ 0 \qquad \textit{(in the distributional sense)}$$

induce a sleek transition ϑ_b *on* $(\mathcal{M}(\mathbb{R}^N),\ \mathcal{M}(\mathbb{R}^N),\ (p_\varepsilon),\ (\lfloor\cdot\rfloor_\varepsilon))$ *with*

$$\begin{aligned}
\alpha_\varepsilon(\vartheta_b,\mu) &\stackrel{\text{Def.}}{=} \lambda_\varepsilon\ C, & \beta_\varepsilon(\vartheta_b)(t) &\stackrel{\text{Def.}}{=} t\ \cdot \lambda_\varepsilon\ C\ e^{\lambda_\varepsilon\ C}, \\
\gamma_\varepsilon(\vartheta_b) &\stackrel{\text{Def.}}{=} 0, & \zeta_\varepsilon(\vartheta_b) &\stackrel{\text{Def.}}{=} \lambda_\varepsilon\ C, \\
\mathbb{T}_\varepsilon(\vartheta_b,\mu) &\stackrel{\text{Def.}}{=} 1, & \lfloor\mu\rfloor_\varepsilon &\stackrel{\text{Def.}}{=} \big|\big(\varphi_\varepsilon\ \mu\big)(\mathbb{R}^N)\big|, \\
P_\varepsilon(\vartheta_b,\vartheta_c;\mu) &\le \sup_{0\le s<1}\ \lambda_\varepsilon\ \big|\big(\varphi_\varepsilon\ |b-c|\ \vartheta_b(s,\mu)\big)(\mathbb{R}^N)\big|
\end{aligned}$$

for all $b, c \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ *with* $\|b\|_\infty, \|c\|_\infty \le C$ *and* $\mu \in \mathcal{M}(\mathbb{R}^N)$, $\varepsilon \in \mathcal{J}$, $t \in [0,1]$. □

For applying the existence theorem of generalized mutational equations, we now focus on the aspect of sequential compactness.
The classical compactness criterion of de la Vallée Poussin (for finite Radon measures) implies that for any bound $C > 0$, the set $\{\mu \in \mathcal{M}(\mathbb{R}^N) : |\mu|(\mathbb{R}^N) \le C\}$ is weakly* sequentially compact (see e.g. [4], Theorem 1.59). Although this condition on measures seems to be quite weak, it is not clear how to verify it in connection with the preceding sleek transitions on $(\mathcal{M}(\mathbb{R}^N), \mathcal{M}(\mathbb{R}^N), (p_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$. Indeed, an upper bound of $|\vartheta_b(t,\mu)|(\mathbb{R}^N)$ is difficult to be found without further assumptions about μ.

So we have decided to consider only *positive* Radon measures *with compact support* from now on, i.e. $\mu \in \mathcal{M}(\mathbb{R}^N)$ satisfying $\mu(\cdot) \ge 0$, $|\mu|(\mathbb{R}^N \setminus K) = 0$ for some compact set $K \subset \mathbb{R}^N$. Indeed, we are now able to draw "global" conclusions from bounds on $\lfloor\cdot\rfloor_\varepsilon$ (for each $\varepsilon \in \mathcal{J}$), i.e. for every $\mu \in \mathcal{M}^+_c(\mathbb{R}^N)$, there is some $\varepsilon \in \mathcal{J}$ satisfying $\varphi_\varepsilon|_{\text{supp}\,\mu} \ge 1$ and thus, $|\mu|(\mathbb{R}^N) = \mu(\mathbb{R}^N) \le (\varphi_\varepsilon\,\mu)(\mathbb{R}^N) \le \lfloor\mu\rfloor_\varepsilon$. Moreover, we regard $\mathcal{M}^+_c(\mathbb{R}^N)$ as a rather weak restriction (in comparison with some absolute continuity, for example). Furthermore, the separate time component is to shorten the description of transitionally compact subsets in Lemma 3.10.

Definition 3.8 *Set* $\widetilde{\mathcal{M}}(\mathbb{R}^N) := \mathbb{R}\times\mathcal{M}(\mathbb{R}^N)$, $\widetilde{p}_\varepsilon\big((s,\mu),\,(t,\nu)\big) := |t-s| + p_\varepsilon(\mu,\nu)$ *for each* $\varepsilon \in \mathcal{J}$, $\mathcal{M}^+_c(\mathbb{R}^N) := \big\{\mu \in \mathcal{M}(\mathbb{R}^N)\big|\mu(\cdot) \ge 0,\ \exists$ *compact* $K \subset \mathbb{R}^N : |\mu|(\mathbb{R}^N\setminus K) = 0\big\}$, $\widetilde{\mathcal{M}}^+_c(\mathbb{R}^N) := \mathbb{R}\times\mathcal{M}^+_c(\mathbb{R}^N)$, $\widetilde{\vartheta}_b : [0,1]\times\widetilde{\mathcal{M}}(\mathbb{R}^N) \longrightarrow \widetilde{\mathcal{M}}(\mathbb{R}^N)$, $(h,\,(t,\mu)) \longmapsto (t+h,\ \vartheta_b(h,\mu))$ *with* $\vartheta_b(\cdot,\cdot)$ *mentioned in Prop. 3.7.*

Corollary 3.9 *Fixing the parameter* $C \in [0,\infty[$ *arbitrarily, the map* $\widetilde{\vartheta}_b(\cdot,\cdot)$ *specified in Definition 3.8 is a timed sleek transition on* $(\widetilde{\mathcal{M}}(\mathbb{R}^N), \widetilde{\mathcal{M}}(\mathbb{R}^N), (\widetilde{p}_\varepsilon), (\lfloor\cdot\rfloor_\varepsilon))$ *for each vector field* $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ *with* $\|b\|_\infty \le C$ *and, it satisfies for all* $\varepsilon \in \mathcal{J}$, $h \in [0,1]$, $\widetilde{\mu},\widetilde{\nu} \in \widetilde{\mathcal{M}}(\mathbb{R}^N)$

$$\begin{array}{rclrcl}
\alpha_\varepsilon(\widetilde{\vartheta}_b,\widetilde{\mu}) & \stackrel{\text{Def.}}{=} & \lambda_\varepsilon\,C, & \beta_\varepsilon(\widetilde{\vartheta}_b)(t) & \stackrel{\text{Def.}}{=} & t\,(1+\lambda_\varepsilon\,C\ e^{\lambda_\varepsilon\,C}),\\
\gamma_\varepsilon(\widetilde{\vartheta}_b) & \stackrel{\text{Def.}}{=} & 0, & \zeta_\varepsilon(\widetilde{\vartheta}_b) & \stackrel{\text{Def.}}{=} & \lambda_\varepsilon\,C,\\
\mathbb{T}_\varepsilon(\widetilde{\vartheta}_b,\widetilde{\mu}) & \stackrel{\text{Def.}}{=} & 1, & \lfloor(t,\mu)\rfloor_\varepsilon & \stackrel{\text{Def.}}{=} & \big|(\varphi_\varepsilon\,\mu)\,(\mathbb{R}^N)\big|.
\end{array}$$
□

From now on, the set of all these maps $\widetilde{\vartheta}_b(\cdot,\cdot)$ for any vector field $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ with $\|b\|_\infty \le C$ is abbreviated as $\widetilde{\mathcal{T}}_{C,\text{Lip}}(\mathbb{R}^N)$.

Lemma 3.10 *For any constant* $C \ge 0$ *and nonempty compact set* $K \subset \mathbb{R}^N$, *define*

$$\widetilde{\mathcal{M}}^+_K(\mathbb{R}^N) := \Big\{(t,\mu) \in \mathbb{R}\times\mathcal{M}^+_c(\mathbb{R}^N)\ \Big|\ t \ge 0,\ \ |\mu|\big(\mathbb{R}^N\setminus\mathbb{B}_{C\,t}(K)\big) = 0\Big\}.$$

Then, (a) $\widetilde{\mathcal{M}}^+_K(\mathbb{R}^N)$ *is invariant for all sleek transitions* $\widetilde{\vartheta}_b \in \widetilde{\mathcal{T}}_{C,\text{Lip}}(\mathbb{R}^N)$.
(b) $\big(\widetilde{\mathcal{M}}^+_K(\mathbb{R}^N),\,(\widetilde{p}_\varepsilon),\,(\lfloor\cdot\rfloor_\varepsilon),\,\widetilde{\mathcal{T}}_{C,\text{Lip}}(\mathbb{R}^N)\big)$ *is timed transitionally compact.*

Proof. (a) results from Proposition 3.5 because its representation formula (by means of push–forward) implies $\text{supp}\,\vartheta_b(t,\mu_0) \subset \mathbb{B}_{t\,\|b\|_\infty}(\text{supp}\,\mu_0) \subset \mathbb{B}_{C\,t}(\text{supp}\,\mu_0)$ for all $\mu_0 \in \mathcal{M}_c(\mathbb{R}^N)$.

(b) is a consequence of the compactness criterion of de la Vallée Poussin. Indeed, for every sequence $((t_n,\mu_n))_{n\in\mathbb{N}}$ in $\widetilde{\mathcal{M}}_K(\mathbb{R}^N)$ with $\sup_n |t_n| < \infty$, $\sup_n \lfloor(t_n,\mu_n)\rfloor_\varepsilon < \infty$, $\sup_n \widetilde{q}_\varepsilon((0,0),(t_n,\mu_n)) < \infty$ for each $\varepsilon \in \mathcal{J}$, we conclude that the support of each μ_n is contained in the interior of the compact set $L := K + C\,(1+\sup_n |t_n|)\cdot\mathbb{B} \subset \mathbb{R}^N$. According to the choice of $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ in Lemma 3.1, there is an index $\varepsilon_o \in \mathcal{J}$ with $\varphi_{\varepsilon_o}|_L \ge 1$ and thus, $\sup_n |\mu_n|(\mathbb{R}^N) = \sup_n \mu_n(L) \le \sup_n \lfloor(t_n,\mu_n)\rfloor_{\varepsilon_o} < \infty$.
Due to this uniform bound and the compact superset L of all supports, there exist a subsequence (again denoted by) $((t_n,\mu_n))_{n\in\mathbb{N}}$ and $(t,\mu) \in \widetilde{\mathcal{M}}(\mathbb{R}^N)$ satisfying

$$\begin{cases} t_n \longrightarrow t \geq 0, \displaystyle\int_{\mathbb{R}^N} \varphi \; d\mu_n \longrightarrow \int_{\mathbb{R}^N} \varphi \; d\mu \qquad (n \longrightarrow \infty) \qquad \text{for all } \varphi \in C_0(\mathbb{R}^N), \\ \operatorname{supp} \mu \quad \subset \ \operatorname{Lim}_{n\to\infty} \ \operatorname{supp} \mu_n \ \subset \ K + C \cdot \lim\limits_{n\to\infty} t_n \ = \ \mathbb{B}_{C\,t}(K). \end{cases}$$

So $(t, \mu) \in \widetilde{\mathcal{M}}_K(\mathbb{R}^N)$ and $\widetilde{p}_\varepsilon\,((t_n, \mu_n),\ (t, \mu)) \longrightarrow 0$, $\liminf\limits_{n\to\infty} \lfloor(t_n, \mu_n)\rfloor_\varepsilon \geq \lfloor(t, \mu)\rfloor_\varepsilon$ for all $\varepsilon \in \mathcal{J}$. In regard to timed transitional compactness, the rest results now from the symmetry of each $\widetilde{p}_{\varepsilon,\kappa}$ and the estimates of Lemma 3.6. □

Proposition 3.11 *Suppose for* $f : \mathcal{M}_c^+(\mathbb{R}^N) \times [0, T] \longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$:

1. $\exists\ C \in [0, \infty[\ : \|f(\mu, t)\|_\infty \leq C$ *for all* $(\mu, t) \in \mathcal{M}_c^+(\mathbb{R}^N) \times [0, T]$,
2. *("weak*" continuity of f)* $\|f(\mu, t) - f(\mu_n, t_n)\|_\infty \longrightarrow 0$ *if* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C_0(\mathbb{R}^N)$*)*, $t_n \to t$.

Then for every initial $\mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$*, there exists a right–hand sleek solution* $\mu(\cdot) : [0, T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0) = \mu_0$. *Moreover,* $\mu(\cdot)$ *is distributional solution of the continuity equation* $\frac{d}{dt}\, \mu(t) + D_x \cdot (f(\mu(t), t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N \times [0, T[$.

Proof. For applying the compactness results of Lemma 3.10, we take a separate real component of time into consideration. Define $\widetilde{f} : \widetilde{\mathcal{M}}_c^+(\mathbb{R}^N) \times [0, T] \longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$, $((s, \mu),\ t) \longmapsto f(\mu, t)$ and let $\widetilde{\vartheta}_{\widetilde{f}(\widetilde{\mu}, t)} : [0, 1] \times \widetilde{\mathcal{M}}(\mathbb{R}^N) \longrightarrow \widetilde{\mathcal{M}}(\mathbb{R}^N)$ denote the corresponding timed sleek transition on $(\widetilde{\mathcal{M}}(\mathbb{R}^N),\ \widetilde{\mathcal{M}}(\mathbb{R}^N),\ (\widetilde{p}_\varepsilon),\ (\lfloor\cdot\rfloor_\varepsilon))$ according to Corollary 3.9. Due to Lemma 3.10 with $K := \operatorname{supp} \mu_0$, all Euler approximations (used for proving existence in § 2.3) have their values in the timed transitionally compact subset $\widetilde{\mathcal{M}}_K^+(\mathbb{R}^N)$.
Moreover, Proposition 3.7 implies for all $\widetilde{\mu}_1, \widetilde{\mu}_2 \in \widetilde{\mathcal{M}}_c^+(\mathbb{R}^N)$, $\widetilde{\nu} \in \widetilde{\mathcal{M}}(\mathbb{R}^N)$, $t_1, t_2 \in [0, T[$

$$\begin{aligned} \widetilde{P}_\varepsilon(\widetilde{\vartheta}_{\widetilde{f}(\widetilde{\mu}_1,\,t_1)}, \widetilde{\vartheta}_{\widetilde{f}(\widetilde{\mu}_2,\,t_2)}; \widetilde{\nu}) \ &\leq\ \big\|\widetilde{f}(\widetilde{\mu}_1, t_1) - \widetilde{f}(\widetilde{\mu}_2, t_2)\big\|_\infty \cdot \sup_{0 \leq s < 1} \lambda_\varepsilon \ \Big|\int_{\mathbb{R}^N} \varphi_\varepsilon \ d\vartheta_{f(\mu_1, t_1)}(s, \nu)\Big| \\ &\leq\ \big\|\widetilde{f}(\widetilde{\mu}_1, t_1) - \widetilde{f}(\widetilde{\mu}_2, t_2)\big\|_\infty \cdot \lambda_\varepsilon \ \Big|\int_{\mathbb{R}^N} \varphi_\varepsilon \ d\nu\Big| \cdot e^{\lambda_\varepsilon\, C}. \end{aligned}$$

So Proposition 2.17 ensures the existence of a timed right–hand sleek solution $\widetilde{\mu}(\cdot) : [0, T[\longrightarrow \widetilde{\mathcal{M}}_c^+(\mathbb{R}^N)$ of the generalized mutational equation $\overset{\circ}{\widetilde{\mu}}(\cdot) \ni \widetilde{\vartheta}_{\widetilde{f}(\widetilde{\mu}(\cdot),\,\cdot)}$ with $\widetilde{\mu}(0) = (0, \mu_0)$.
As the (real) time component has no explicit influence, the second (measure–valued) component of $\widetilde{\mu}(\cdot)$, abbreviated as $\mu(\cdot) : [0, T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, is right–hand sleek solution of the generalized mutational equation $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\,\cdot)}$ with $\mu(0) = \mu_0$, i.e. $\mu(\cdot)$ is uniformly continuous with respect to each p_ε $(\varepsilon \in \mathcal{J})$ and satisfies for all $\nu \in \mathcal{M}(\mathbb{R}^N)$, $t \in [0, T[$, $\varepsilon \in \mathcal{J}$

$$\limsup_{h\,\downarrow\,0} \tfrac{1}{h} \Big(p_\varepsilon\big(\vartheta_{f(\mu(t),\,t)}\,(h, \nu),\ \mu(t + h)\big) - p_\varepsilon(\nu,\ \mu(t)) \cdot e^{\lambda_\varepsilon\, C \cdot h}\Big) \ \leq\ 0.$$

Finally $\mu(\cdot)$ has to be verified as a distributional solution of the continuity equation.
Using Euler method for constructing $\widetilde{\mu}(\cdot)$ (according to § 2.3) implies that $\mu(\cdot)$ shares the common modulus of continuity of all $\widetilde{\vartheta}_b(\cdot, \widetilde{\nu}) : [0, 1] \longrightarrow \widetilde{\mathcal{M}}(\mathbb{R}^N)$ (with $\widetilde{\vartheta}_b \in \widetilde{\mathcal{T}}_{C,\mathrm{Lip}}(\mathbb{R}^N)$, $\widetilde{\nu} \in \widetilde{\mathcal{M}}(\mathbb{R}^N)$),
i.e. $p_\varepsilon(\mu(s),\ \mu(t)) \ \leq\ \widetilde{p}_\varepsilon(\widetilde{\mu}(s),\ \widetilde{\mu}(t)) \ \leq\ |t - s| \cdot \big(e^{\lambda_\varepsilon\, C\, T} \cdot \lambda_\varepsilon\, C + 1\big)$ for all $s, t \in [0, T[$, $\varepsilon \in \mathcal{J}$.
So for each $\varepsilon \in \mathcal{J}$, the function $[0, T[\longrightarrow [0, \infty[$, $t \longmapsto \displaystyle\int_{\mathbb{R}^N} \varphi_\varepsilon \ d\mu(t)$ is Lipschitz continuous and satisfies at every time t of differentiability

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \varphi_\varepsilon \ d\,\mu(t) \ &=\ \limsup_{h\,\downarrow\,0} \ \tfrac{1}{h} \cdot \int_{\mathbb{R}^N} \big(\varphi_\varepsilon(\mathbf{X}_{f(\mu(t),\,t)}(h, x)) \ -\ \varphi_\varepsilon(x)\big) \ \ d\,\mu(t)\,(x) \\ &=\ \int_{\mathbb{R}^N} \nabla\varphi_\varepsilon(x) \ \cdot\ f(\mu(t),\, t)(x) \qquad d\,\mu(t)\,(x). \end{aligned}$$

Now every $\varphi \in C^\infty_c(\mathbb{R}^N, [0,\infty[)$ can be approximated by $(\varphi_\varepsilon)_{\varepsilon \in \mathcal{J}}$ with respect to the C^1 norm (due to Lemma 3.1). Thus, $[0,T[\longrightarrow [0,\infty[, \; t \longmapsto \int_{\mathbb{R}^N} \varphi \; d\mu(t)$ is also absolutely continuous and satisfies

$$\tfrac{d}{dt} \int_{\mathbb{R}^N} \varphi \; d\,\mu(t) \;=\; \int_{\mathbb{R}^N} \nabla\varphi(x) \cdot f(\mu(t), t)(x) \;\; d\,\mu(t)\,(x) \qquad \text{for almost every } t \in [0,T[.$$

Moreover the condition $\varphi \geq 0$ is not required, i.e. the same features are guaranteed for any $\varphi \in C^\infty_c(\mathbb{R}^N)$. Indeed, choosing any auxiliary function $\xi \in C^\infty_0(\mathbb{R}^N, [0,\infty[)$ with $\xi \equiv \|\varphi\|_\infty + 1$ in $\mathbb{B}_1(\operatorname{supp}\varphi)$, we apply the previous results (about absolute continuity and its derivative) to $\varphi(\cdot) + \xi(\cdot) \geq 0, \;\; \xi(\cdot) \geq 0$.

In regard to a solution in the distributional sense, let $\psi \in C^\infty_0(\mathbb{R}^N \times [0,T[)$ be any test function. Then, the subsequent Lemma 3.12 implies

$$\begin{aligned} -\int_{\mathbb{R}^N} \psi(\cdot,0) \;\; d\,\mu_0 \;&=\; \int_0^T \tfrac{d}{dt}\Big(\int_{\mathbb{R}^N} \psi(\cdot,t) \;\; d\,\mu(t)\Big) \; dt \\ &=\; \int_0^T \Big(\int_{\mathbb{R}^N} \partial_t\,\psi(\cdot,t) \;\; d\,\mu(t) \;+\; \tfrac{\partial}{\partial s} \int_{\mathbb{R}^N} \psi(\cdot,t) \;\; d\,\mu(s)\Big|_{s=t}\Big) \; dt \\ &=\; \int_0^T \Big(\int_{\mathbb{R}^N} \partial_t\,\psi(\cdot,t) \;\; d\,\mu(t) \;+\; \int_{\mathbb{R}^N} \nabla_x \psi(x,t) \;\cdot f(\mu(t), t)(x) \;\; d\,\mu(t)\,(x)\Big) \; dt. \end{aligned}$$

□

Lemma 3.12 *Suppose* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ *to be locally absolutely continuous in each component and*

$$\wedge \left\{ \begin{array}{lcl} \limsup\limits_{k \longrightarrow 0} \; \big\| \partial_1\, h(\,\cdot\,,\, k + \cdot\,) \;-\; \partial_1\, h(\,\cdot\,,\,\cdot\,) \big\|_{L^1([0,T])} &=& 0, \\ \limsup\limits_{k \longrightarrow 0} \; \big\| \partial_2\, h(\,\cdot\,,\, k + \cdot\,) \;-\; \partial_2\, h(\,\cdot\,,\,\cdot\,) \big\|_{L^1([0,T])} &=& 0. \end{array}\right.$$

Then $[0,T] \longrightarrow \mathbb{R}^n, \;\; t \longmapsto h(t,t)$ *is absolutely continuous and*

$$\tfrac{d}{d\,t}\, h(t,t) \;=\; \Big(\tfrac{\partial}{\partial\, t_1}\, h(t_1,t_2) + \tfrac{\partial}{\partial\, t_2}\, h(t_1,t_2)\Big)\Big|_{t=t_1=t_2}.$$

Proof is given in [16], Lemma 2.5, for example.

Corollary 3.13 *Suppose for* $f : \mathcal{M}^+_c(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$:

1. $\exists\; C \in [0,\infty[\;:\quad \|f(\mu,t)\|_\infty \leq C \qquad$ *for all* $(\mu,t) \in \mathcal{M}^+_c(\mathbb{R}^N) \times [0,T]$,
2. $\forall\; \varepsilon \in \mathcal{J} \;\;\; \exists\, L_\varepsilon \in [0,\infty[, \;\; \omega_\varepsilon(\cdot) = o(1) \;:\;\; \|f(\mu,s) - f(\nu,t)\|_\infty \;\leq\; L_\varepsilon \cdot p_\varepsilon(\mu,\nu) + \omega_\varepsilon\,(|s-t|)$ *for all* $(\mu,s), (\nu,t) \in \mathcal{M}^+_c(\mathbb{R}^N) \times [0,T]$.

Then for every $\mu_0 \in \mathcal{M}^+_c(\mathbb{R}^N)$, *the right–hand sleek solution* $\mu(\cdot) : [0,T[\longrightarrow \mathcal{M}^+_c(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\mu}(\cdot) \;\ni\; \vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0) = \mu_0$ *is unique.*

Proof results from Proposition 2.24 : Indeed for right–hand sleek solutions $\mu_1, \mu_2 : [0,T[\longrightarrow \mathcal{M}^+_c(\mathbb{R}^N)$ with the same initial datum $\mu_1(0) = \mu_2(0) = \mu_0 \in \mathcal{M}^+_c(\mathbb{R}^N)$ and for sufficiently large $\rho > 0$, define the auxiliary function

$$\delta_\varepsilon(t) \;:=\; \inf \Big\{ p_\varepsilon(\nu,\, \mu_1(t)) \;+\; p_\varepsilon(\nu,\, \mu_2(t)) \;\Big|\; \nu \in \mathcal{M}(\mathbb{R}^N),\; \lfloor \nu \rfloor_\varepsilon \;\leq\; (\lfloor \mu_0 \rfloor_\varepsilon + \rho) \cdot e^{\lambda_\varepsilon\, C\, t} + \lambda_\varepsilon\, C\, t \Big\}.$$

The symmetry and triangle inequality of $p_\varepsilon(\cdot,\cdot)$ imply $\delta_\varepsilon(t) \;=\; p_\varepsilon(\mu_1(t),\, \mu_2(t))$ and, Proposition 2.24 (with $R_\varepsilon = 0,\; \delta_\varepsilon(0) = 0$) ensures $\delta_\varepsilon(\cdot) \equiv 0$ for every $\varepsilon \in \mathcal{J}$. Thus, $\mu_1 \equiv \mu_2$. □

4 Bounded vector fields of bounded variation

The main goal now is to weaken the regularity conditions on the vector fields inducing sleek transitions. Indeed, all globally supposed bounds mentioned in the preceding section are referring to the L^∞ norm of the vector fields whereas their (local) Lipschitz continuity is just to guarantee uniqueness of the solution of the corresponding ODE.

Recent results of Ambrosio [1, 2] make a suggestion how to specify a flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ along certain vector fields of bounded (spatial) variation in a unique way. This uniqueness is based on an additional condition of regularity, i.e. the absolute continuity with respect to Lebesgue measure $\mathcal{L}^N$ is preserved uniformly : For any nonnegative function $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the measure $\mu_0 := \rho\,\mathcal{L}^N$ satisfies $\mathbf{X}(t,\cdot)_\sharp\,\mu_0 \le C\,\mathcal{L}^N$ for all $t \in [0,T]$ with a constant C independent of t.

After summarizing some features of this so–called Lagrangian flow in Proposition 4.1, we exploit the corresponding vector fields of bounded variation for inducing sleek transitions on measures. In contrast to § 3, however, the tools of Ambrosio require the restriction to finite positive measures being absolutely continuous with respect to $\mathcal{L}^N$. Although limiting the class of admitted Radon measures, this assumption has the advantage in Proposition 4.10 that applying existence results to the continuity equation does not require the continuity of the right–hand side with respect to L^∞ as in Proposition 3.11 (but L^1).

Proposition 4.1 *Assume* $\widetilde{b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *to be in* $L^1\big([0,T],\ \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)\big)$ *satisfying*

1. $\frac{|\widetilde{b}|}{1+|x|} \in L^1\big([0,T],\ L^1(\mathbb{R}^N)\big) + L^1\big([0,T],\ L^\infty(\mathbb{R}^N)\big)$,
2. $D_x \cdot \widetilde{b}(t,\cdot) = \mathrm{div}_x\,\widetilde{b}(t,\cdot)\ \mathcal{L}^N \ll \mathcal{L}^N$ *for* $\mathcal{L}^1$*–almost every* $t \in [0,T]$,
3. $[\mathrm{div}_x\,\widetilde{b}]^- \in L^1\big([0,T],\ L^\infty(\mathbb{R}^N)\big)$.

Then there exists a so–called Lagrangian flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *such that*

(a) $\mathbf{X}(\cdot,x) : [0,T] \longrightarrow \mathbb{R}^N$ *is absolutely continuous for* $\mathcal{L}^N$*–almost every* $x \in \mathbb{R}^N$ *with*

$$\mathbf{X}(t,x) = x + \int_0^t \widetilde{b}\big(s,\ \mathbf{X}(s,x)\big)\ ds \qquad \text{for all } t \in [0,T],$$

(b) *there is a constant* $C > 0$ *with* $\mathbf{X}(t,\cdot)_\sharp\,(\rho\,\mathcal{L}^N) \le C\,\|\rho\|_\infty\,\mathcal{L}^N \quad \forall\ \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ t.$

Furthermore, $\mathbf{X}(t,\cdot) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is unique up to* $\mathcal{L}^N$*–negligible sets for every time* $t \in [0,T]$ *and,* $\mu(t) := \mathbf{X}(t,\cdot)_\sharp\,\mu_0$ *is the unique distributional solution of the continuity equation*

$$\tfrac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\,\mu) = 0 \qquad \text{in }]0,T[\times \mathbb{R}^N$$

for every initial datum $\mu_0 := \rho\,\mathcal{L}^N$ *with* $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \rho \ge 0.$

Mollifying each $\mu(t)$ *with a common Gaussian kernel* $\rho \in C^1(\mathbb{R}^N,]0,\infty[)$, *the measures* $\mu_\delta(t) := \mu(t) * \rho_\delta$ *solve the continuity equation* $\frac{d}{dt}\,\mu_\delta + D_x \cdot (\widetilde{b}_\delta\,\mu_\delta) = 0$ *(in the distributional sense) with* $\widetilde{b}_\delta(t,\cdot) := \frac{(\widetilde{b}(t,\cdot)\,\mu(t)) * \rho_\delta}{\mu_\delta(t)}$ *being in* $L^1\big([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)\big)$ *and,* $\mu_\delta(t) \xrightarrow{\delta\downarrow 0} \mu(t)$ *narrowly (i.e. with respect to the duality of bounded continuous functions) for every* $t \in [0,T]$.

Proof is presented in [1] (and in [2]). Indeed, extending [1], Theorem 30 to vector fields of bounded spatial variation (as stated in the end of [1], § 5), there exists a Lagrangian flow $\mathbf{X} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with the claimed properties (a),(b) and, it is unique (up to $\mathcal{L}^N$–negligible sets).

The proof of [1], Theorem 19 bridges the gap between the Lagrangian flow and the measure–valued solution of the continuity equation (by means of push–forward). The uniqueness of $\mu(\cdot)$ results from the comparison principle of the continuity equation (due to the assumptions about $\widetilde{b}$) according to [1], Theorem 26. Finally proving [1], Theorem 12 implies the narrow sequential compactness of $\eta_\delta := \big(x, \mathbf{X}_{\widetilde{b}_\delta}(\cdot,x)\big)_\sharp\ \mu_\delta(0)$ (using Prokhorov compactness theorem). So equation (9) there implies the narrow convergence of $\mu_\delta(t)$ to its unique limit point $\mu(t)$. $\square$

Lemma 4.2 *For any* $\widetilde{b} \in L^1([0,T],\ W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N))$ *with* $\operatorname{div}_x \widetilde{b} \in L^1([0,T],\ L^\infty(\mathbb{R}^N,\mathbb{R}^N))$, *the flow* $\mathbf{X}_b : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *specified in Definition 3.4 satisfies for all* $t \in [0,T]$ *and* $\mathcal{L}^N$*–a.e.* $x \in \mathbb{R}^N$

$$\exp\Big(-\int_0^t \big\| [\operatorname{div}_x \widetilde{b}(t,\cdot)]^- \big\|_\infty \ dt\Big) \ \le\ \det\ D_x\,\mathbf{X}_b(t,x) \ \le\ \exp\Big(\int_0^t \big\| [\operatorname{div}_x \widetilde{b}(t,\cdot)]^+ \big\|_\infty \ dt\Big).$$

Moreover, for all $\mu = \rho\,\mathcal{L}^N$ *with* $\rho \in L^1(\mathbb{R}^N) \cap L^\infty$, *the total variation of* $\mathbf{X}_b(t,\cdot)_\sharp\,\mu$ *fulfills*

$$\big|\mathbf{X}_b(t,\cdot)_\sharp\,\mu\big|(\mathbb{R}^N) \ \le\ |\mu|(\mathbb{R}^N)\ \cdot\ \exp\Big(2\int_0^t \|\operatorname{div}_x \widetilde{b}(t,\cdot)\|_\infty\ dt\Big)$$

Proof of the first part is mentioned in [2], Remark 6.3, for example. The second part results from the area formula and the transformation of Lebesgue integrals. Indeed, for $\mu = \rho\mathcal{L}^N$ with $\rho \in L^1(\mathbb{R}^N) \cap L^\infty$,

$$\begin{aligned}
\big|\mathbf{X}_b(t,\cdot)_\sharp\,\mu\big|(\mathbb{R}^N) \ &=\ \int_{\mathbb{R}^N} \Big|\frac{\rho}{|\det\ D_x\,\mathbf{X}_b(t,\cdot)|} \circ \mathbf{X}(t,\cdot)^{-1}\Big|\ d\mathcal{L}^N \\
&\le\ \int_{\mathbb{R}^N} \big|\rho \circ \big(\mathbf{X}(t,\cdot)^{-1}\big)\big|\ d\mathcal{L}^N \qquad\qquad\qquad \cdot\ \exp\Big(\int_0^t \big\| [\operatorname{div}_x \widetilde{b}(t,\cdot)]^- \big\|_\infty\ dt\Big) \\
&\le\ \int_{\mathbb{R}^N} |\rho|\ d\mathcal{L}^N\ \cdot\ \big\|\det\ D_x\,\mathbf{X}_b(t,\cdot)\big\|_\infty\ \cdot\ \exp\Big(\int_0^t \big\| [\operatorname{div}_x \widetilde{b}(t,\cdot)]^- \big\|_\infty\ dt\Big). \qquad \Box
\end{aligned}$$

These (mostly quoted) results motivate the following choice of vector fields and finite Radon measures. Using the notation 4.3, [1] guarantees $\mathbf{X}(t,\cdot)_\sharp\,\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N) \subset \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ for each Lagrangian flow $\mathbf{X}$.

Definition 4.3 $\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ *denotes the set of all functions* $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ *satisfying* $D \cdot b = \operatorname{div} b\ \mathcal{L}^N \ll \mathcal{L}^N$ *and* $\operatorname{div} b \in L^\infty(\mathbb{R}^N)$.
Furthermore, set $\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N) := \big\{\rho\,\mathcal{L}^N \ \big|\ \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \rho \ge 0\big\} \subset \mathcal{M}^+(\mathbb{R}^N)$ *and,*
for each $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$, *define* $\vartheta_b : [0,1] \times \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$, $(h,\mu_0) \longmapsto \mathbf{X}(h,\cdot)_\sharp\,\mu_0$ *with* $\mathbf{X}(\cdot,\cdot)$ *denoting its Lagrangian flow according to Proposition 4.1.*

Moreover, the proof of Lemma 4.2 indicates an adequate modification of the “distance” between two measures of $\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$. In § 3, $p_\varepsilon(\mu,\nu)$ was defined as absolute value of the weighted difference, i.e.

$$p_\varepsilon(\mu,\ \nu)\ :=\ \big|\varphi_\varepsilon \cdot (\mu-\nu)\,(\mathbb{R}^N)\big|\ =\ \Big|\int_{\mathbb{R}^N} \varphi_\varepsilon\ d(\mu-\nu)\Big|$$

Restricting now to measures $\mu,\nu \ll \mathcal{L}^N$, we can use the total variation of the weighted difference instead – and the key estimates of Lemma 3.6 hold essentially.

Definition 4.4 *Let* $(\varphi_\varepsilon)_{\varepsilon\in\mathcal{J}}$ *be a countable family of Schwartz functions as described in Lemma 3.1. For each* $\varepsilon \in \mathcal{J}$, *define* $q_\varepsilon : \mathcal{M}(\mathbb{R}^N) \times \mathcal{M}(\mathbb{R}^N) \longrightarrow [0,\infty[$,

$$\begin{aligned}
q_\varepsilon(\mu,\ \nu)\ &:=\ \big|\varphi_\varepsilon \cdot (\mu-\nu)\big|(\mathbb{R}^N) \\
&\overset{\text{Def.}}{=}\ \sup\Big\{\sum_{k=0}^{\infty} \Big|\int_{E_k} \varphi_\varepsilon\ d(\mu-\nu)\Big|\ \Big|\ (E_k)_{k\in\mathbb{N}} \text{ pairwise disjoint Borel sets, } \mathbb{R}^N = \bigcup_k E_k\Big\}.
\end{aligned}$$

Lemma 4.5 *For each* $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty$, $\vartheta_b : [0,1] \times \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ *satisfies*

1. $\vartheta_b(0,\cdot) = \mathrm{Id}_{\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)}$,
2. $q_\varepsilon\,(\vartheta_b(h,\ \vartheta_b(t,\mu_0)),\ \ \vartheta_b(t+h,\mu_0)) \ =\ 0\ =\ q_\varepsilon\,(\vartheta_b(t+h,\mu_0),\ \ \vartheta_b(h,\ \vartheta_b(t,\mu_0)))$
   for any initial datum $\mu_0 \in \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ *and* $t,h \in [0,1]$ *with* $t+h \le 1$.

3. $\limsup_{h \downarrow 0} \frac{q_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) \ - \ q_\varepsilon(\mu_0,\nu_0)}{h \quad q_\varepsilon(\mu_0,\nu_0)} \ \leq \ \lambda_\varepsilon \ \|b\|_\infty$ *for* $\mu_0, \nu_0 \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$,

4. $\big|\varphi_\varepsilon \ \vartheta_b(t,\mu_0))\big|(\mathbb{R}^N) \ \leq \ \big|\varphi_\varepsilon \ \mu_0\big|(\mathbb{R}^N) \cdot e^{\lambda_\varepsilon \ \|b\|_\infty \cdot t}$ *for* $\mu_0 \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N),\ t \in [0,1]$,

5. $q_\varepsilon\left(\vartheta_b(s,\mu_0),\ \vartheta_b(t,\mu_0)\right) \ \leq \ |t-s| \cdot \lambda_\varepsilon \ \|b\|_\infty \ e^{\lambda_\varepsilon \ \|b\|_\infty} \ \big|\varphi_\varepsilon \ \mu_0\big|(\mathbb{R}^N)$ *for* $s,t \in [0,1],\ \mu_0 \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$,

6. $\limsup_{h \downarrow 0} \frac{q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right)}{h} \ \leq \ \lambda_\varepsilon \ \big|\varphi_\varepsilon \ |b_1 - b_2| \ \mu_0\big|(\mathbb{R}^N) \ \leq \ \lambda_\varepsilon \ \|\varphi_\varepsilon\|_\infty \ \|\rho\|_\infty \ \cdot \ \|b_1 - b_2\|_{L^1(\mathbb{R}^N)}$
for all bounded vector fields $b_1, b_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $\mu_0 = \rho\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$.

Proof. The measure–valued flow $\vartheta_b : [0,1] \times \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N) \longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ still satisfies the semigroup property and thus statements (1.), (2.).
For any $\mu_0 = \rho\,\mathcal{L}^N,\ \nu_0 = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$, the definitions of total variation and push–forward imply

$$\begin{aligned} q_\varepsilon\big(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)\big) \ &= \ \big|\varphi_\varepsilon \cdot \big(\mathbf{X}_b(h,\cdot)_\sharp\,\mu_0 \ - \ \mathbf{X}_b(h,\cdot)_\sharp\,\nu_0\big)\big|(\mathbb{R}^N) \\ &\leq \int_{\mathbb{R}^N} \varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) \ |\rho - \sigma| \ d\,\mathcal{L}^N \\ &\leq \int_{\mathbb{R}^N} \big|\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\big| \ |\rho - \sigma| \ d\,\mathcal{L}^N \ + \ \big|\varphi_\varepsilon \cdot (\mu_0 - \nu_0)\big|(\mathbb{R}^N). \end{aligned}$$

So the choice of φ_ε (in Lemma 3.1) has the consequence

$$\begin{aligned} \limsup_{h \downarrow 0} \frac{q_\varepsilon(\vartheta_b(h,\mu_0),\ \vartheta_b(h,\nu_0)) \ - \ q_\varepsilon(\mu_0,\nu_0)}{h} \ &\leq \ \limsup_{h \downarrow 0} \frac{1}{h} \cdot \int_{\mathbb{R}^N} \big|\varphi_\varepsilon(\mathbf{X}_b(h,\cdot)) - \varphi_\varepsilon\big| \ |\rho - \sigma| \ d\,\mathcal{L}^N \\ &\leq \int_{\mathbb{R}^N} |\nabla\varphi_\varepsilon(x) \ \cdot \ b(x)| \quad |\rho - \sigma| \ d\,\mathcal{L}^N \\ &\leq \ \|b\|_\infty \int_{\mathbb{R}^N} \lambda_\varepsilon \ \varphi_\varepsilon \quad |\rho - \sigma| \ d\,\mathcal{L}^N \\ &\leq \ \|b\|_\infty \quad \lambda_\varepsilon \ \cdot \ q_\varepsilon(\mu_0,\nu_0). \end{aligned}$$

Applying this estimate to $\nu_0 \equiv 0$ and $\vartheta_b(t,\mu_0)$ (instead of μ_0), we conclude property (4.) from Gronwall's Lemma 2.6 because the lower semicontinuous auxiliary function

$$\delta_\varepsilon : [0,1] \longrightarrow \mathbb{R}, \quad t \longmapsto \big|\varphi_\varepsilon \ \vartheta_b(t,\mu_0)\big|(\mathbb{R}^N) \ = \ \big|\varphi_\varepsilon(\mathbf{X}_b(t,\cdot)) \ \mu_0\big|(\mathbb{R}^N)$$

is one–sided differentiable and satisfies $\frac{d^+}{dt^+}\,\delta_\varepsilon(\cdot) \ \leq \ \lambda_\varepsilon \ \|b\|_\infty \cdot \delta_\varepsilon(\cdot)$.
In basically the same way, we obtain statement (5.) considering the auxiliary function

$$\widehat{\delta}_\varepsilon : [s,1] \longrightarrow \mathbb{R}, \quad t \longmapsto \big|\varphi_\varepsilon \ \big(\vartheta_b(t,\mu_0) - \vartheta_b(s,\mu_0)\big)\big|(\mathbb{R}^N) \ = \ \big|(\varphi_\varepsilon(\mathbf{X}_b(t-s,\cdot)) - \varphi_\varepsilon) \ \vartheta_b(s,\mu_0)\big|(\mathbb{R}^N)$$

with $s \in [0,1[$ fixed and $\frac{d^+}{dt^+}\,\widehat{\delta}_\varepsilon(t) \ \leq \ \lambda_\varepsilon \ \|b\|_\infty \ \big|\varphi_\varepsilon \ \vartheta_b(t,\mu_0)\big|(\mathbb{R}^N) \ \leq \ \lambda_\varepsilon \ \|b\|_\infty \ e^{\lambda_\varepsilon \ \|b\|_\infty} \ \big|\varphi_\varepsilon \ \mu_0\big|(\mathbb{R}^N)$.
Last, but not least, choose any $b_1, b_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty$ and initial datum $\mu_0 = \rho\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$. Then, for every $h \in [0,1]$,

$$\begin{aligned} \tfrac{1}{h} \cdot q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right) \ &\leq \ \tfrac{1}{h} \cdot \int_{\mathbb{R}^N} \big|\varphi_\varepsilon(\mathbf{X}_{b_1}(h,\cdot)) \ - \ \varphi_\varepsilon(\mathbf{X}_{b_2}(h,\cdot))\big| \ |\rho| \ d\,\mathcal{L}^N \\ \limsup_{h \downarrow 0} \frac{q_\varepsilon\left(\vartheta_{b_1}(h,\mu_0),\ \vartheta_{b_2}(h,\mu_0)\right)}{h} \ &\leq \int_{\mathbb{R}^N} \lambda_\varepsilon \ \varphi_\varepsilon \ |b_1 - b_2| \ |\rho| \ d\,\mathcal{L}^N \\ &\leq \ \lambda_\varepsilon \ \|\varphi_\varepsilon\|_\infty \ \|\rho\|_\infty \ \cdot \ \|b_1 - b_2\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

□

In regard to the choice of $\lfloor\cdot\rfloor_\varepsilon$, there are even two candidates now. The first is the weighted total variation (as mentioned in Lemma 4.5 (4.)). Lemma 4.2, however, provides an alternative whose growth is also bounded in the required way : the total variation – not weighted by φ_ε and thus, independent of $\varepsilon \in \mathcal{J}$. For applying the compactness criterion of de la Vallée Poussin later, we prefer the total variation $|\cdot|(\mathbb{R}^N)$ and then rely on the results using "*weakly* transitionally compact" (presented in § 2.4).
In particular, subsequent Lemma 4.7 lays the basis for taking also the L^∞ norm into consideration and thus, we define (independently of $\varepsilon \in \mathcal{J}$)

$$\lfloor\mu\rfloor \ := \ |\mu|(\mathbb{R}^N) \ + \ \big\|\tfrac{\mu}{\mathcal{L}^N}\big\|_\infty \ = \ \|\sigma\|_{L^1(\mathbb{R}^N)} \ + \ \|\sigma\|_\infty \qquad \text{for every} \ \ \mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N).$$

Lemma 4.6 *For every $\mu \in \mathcal{M}(\mathbb{R}^N)$ and open set $A \subset \mathbb{R}^N$, the total variation satisfies*

$$|\mu|(A) \;=\; \sup\left\{\int_{\mathbb{R}^N} \psi \, d\,\mu \;\middle|\; \psi \in C_c^0(A),\ \|\psi\|_\infty \le 1\right\}$$

and thus, $\quad q_\varepsilon(\mu,\nu) \;=\; \sup_{\kappa\in\mathcal{I}} \; p_{\varepsilon,\kappa}(\mu,\nu) \quad$ *for all $\mu,\nu \in \mathcal{M}(\mathbb{R}^N)$*
with $\mathcal{I} \subset \mathcal{J}$ denoting the set of all indices $\kappa \in \mathcal{J}$ satisfying $\|\varphi_\kappa\|_\infty \le 1$ and

$$p_{\varepsilon,\kappa}(\mu,\nu) \;:=\; \Big|\int_{\mathbb{R}^N} \varphi_\varepsilon\,\varphi_\kappa \;\, d(\mu-\nu)\Big| \qquad \text{for each } \varepsilon \in \mathcal{J},\ \kappa \in \mathcal{I},\ \mu,\nu \in \mathcal{M}(\mathbb{R}^N).$$

Proof of the first equality is given in [4], Proposition 1.47, for example. □

Lemma 4.7 *For every vector field $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ and initial measure $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, the Radon–Nikodym derivative $\sigma_t \stackrel{\text{Def.}}{=} \frac{\vartheta_b(t,\mu)}{\mathcal{L}^N}$ of $\vartheta_b(t,\mu)$ with respect to Lebesgue measure $\mathcal{L}^N$ satisfies*

$$\begin{aligned} \|\sigma_t\|_\infty &\le \|\sigma\|_\infty \; e^{\|\mathrm{div}\, b\|_\infty\, t}, \\ \big|\vartheta_b(t,\mu)\big|(\mathbb{R}^N) \;=\; \|\sigma_t\|_{L^1} &\le \|\sigma\|_{L^1} \; e^{2\,\|\mathrm{div}\, b\|_\infty\, t}. \end{aligned}$$

Proof. The second statement results directly from Lemma 4.2 (applied to mollified vector fields b_δ) and the narrow convergence for $\delta \downarrow 0$ because the total variation is lower semicontinuous with respect to weak* convergence (see [4], Theorem 1.59, for example).
For proving the first statement, we exploit first the duality relation between L^1 and L^∞ and then use the area formula again

$$\begin{aligned} \|\sigma_t\|_\infty &= \sup \left\{ \int \psi \;\; \sigma_t \; d\mathcal{L}^N \;\middle|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &= \sup \left\{ \limsup_{\delta\downarrow 0} \int \psi \;\; d\,\vartheta_{b_\delta}(t,\mu) \;\middle|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &= \sup \left\{ \limsup_{\delta\downarrow 0} \int \psi \;\; \Big(\tfrac{\sigma}{\det\; D_x \mathbf{X}_{b_\delta}(t,\cdot)}\Big)\Big|_{\mathbf{X}_{b_\delta}(t,\cdot)^{-1}} \; d\mathcal{L}^N \;\middle|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &\le \sup \left\{ \limsup_{\delta\downarrow 0} \int \psi \;\; \|\sigma\|_\infty \;\; e^{\|\mathrm{div}\, b_\delta\|_\infty\, t} \; d\mathcal{L}^N \;\middle|\; \psi \in C_0^\infty(\mathbb{R}^N),\ \|\psi\|_{L^1} \le 1\right\} \\ &\le \|\sigma\|_\infty \;\; e^{\|\mathrm{div}\, b\|_\infty\, t}. \end{aligned}$$
□

Proposition 4.8 *For any $C \in [0,\infty[$ fixed, each $b \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\|b\|_\infty + \big\|\mathrm{div}\, b\big\|_\infty \le C$ induces the sleek transition ϑ_b on $\big(\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\lfloor\cdot\rfloor)_{\varepsilon\in\mathcal{J}}\big)$ with*

$$\begin{aligned} \alpha_\varepsilon(\vartheta_b,\mu) &\stackrel{\text{Def.}}{=} \lambda_\varepsilon\, C, & \beta_\varepsilon(\vartheta_b)(t) &\stackrel{\text{Def.}}{=} \lambda_\varepsilon\, \|\varphi_\varepsilon\|_\infty\, C\cdot t, \\ \gamma_\varepsilon(\vartheta_b) &\stackrel{\text{Def.}}{=} 0, & \zeta_\varepsilon(\vartheta_b) &\stackrel{\text{Def.}}{=} 2\,C, \\ \mathbb{T}_\varepsilon(\vartheta_b,\mu) &\stackrel{\text{Def.}}{=} 1, & i_{\mathcal{D}} &\stackrel{\text{Def.}}{=} \mathrm{Id}_{\mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)}, \end{aligned}$$

$$\begin{aligned} Q_\varepsilon(\vartheta_b,\vartheta_c;\mu) &\le \lambda_\varepsilon\; e^{2\,C}\; \big(1 + |\mu|(\mathbb{R}^N)\cdot e^{2\,C}\big)\; \big\|\tfrac{\mu}{\mathcal{L}^N}\big\|_\infty\; \big\|\varphi_\varepsilon\,|b-c|\,\big\|_{L^1(\mathbb{R}^N)} \\ p_{\varepsilon,\kappa}\big(\vartheta_b(h,\mu),\ \vartheta_b(h,\nu)\big) &\le p_{\varepsilon,\kappa}(\mu,\nu)\;\cdot\; e^{(\lambda_\varepsilon+\lambda_\kappa)\,C\,h} \end{aligned}$$

for all $b, c \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\|b\|_\infty + \big\|\mathrm{div}\, b\big\|_\infty \le C,\ \|c\|_\infty + \big\|\mathrm{div}\, c\big\|_\infty \le C$ and $\mu \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, $\varepsilon \in \mathcal{J},\ t \in [0,1]$. The set of all these sleek transitions ϑ_b is abbreviated as $\mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)$.

Proof is based on the tools of approximation provided by Proposition 4.1 :
Indeed, choose a Gaussian kernel $\rho \in C^1(\mathbb{R}^N,\ ]0,\infty[)$ and set $\rho_\delta(x) := \delta^{-N}\,\rho(\frac{x}{\delta})$ for $\delta > 0$. Fixing $\mu \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$ arbitrarily, each vector field $b_\delta := \frac{(b\,\mu)\,*\,\rho_\delta}{\mu * \rho_\delta}$ belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$ and satisfies $\|b_\delta\|_\infty \le \|b\|_\infty \le C$. So Lemma 4.5 motivates the choice of $\alpha_\varepsilon(\vartheta_{b_\delta},\mu) \stackrel{\text{Def.}}{=} \lambda_\varepsilon\, C,\ \gamma_\varepsilon(\vartheta_{b_\delta}) \stackrel{\text{Def.}}{=} 0$ and, as in the general framework of § 2.1, we conclude (from Proposition 2.5 and Lemmas 2.7)

$$q_\varepsilon\big(\vartheta_{b_\delta}(h,\ \mu),\ \ \vartheta_{b_\delta}(h,\ \nu)\big) \;\le\; q_\varepsilon(\mu,\ \nu)\;\cdot\; e^{C\,\lambda_\varepsilon\, h}$$

for all $\mu, \nu \in \mathbb{L}^{\infty\,\cap\,1}(\mathbb{R}^N)$, $h \in [0,1]$, $\delta > 0$, $\varepsilon \in \mathcal{J}$. Similarly, for any $\kappa \in \mathcal{I}$ (additionally),

$$p_{\varepsilon,\kappa}\big(\vartheta_{b_\delta}(h,\ \mu),\ \ \vartheta_{b_\delta}(h,\ \nu)\big) \;\le\; p_{\varepsilon,\kappa}(\mu,\ \nu)\;\cdot\; e^{C\,(\lambda_\varepsilon+\lambda_\kappa)\,h}$$

is proved as in Lemma 3.6 (for p_ε) – just using now $\;|\nabla\,(\varphi_\varepsilon\,\varphi_\kappa)| \;\le\; (\lambda_\varepsilon+\lambda_\kappa)\;\varphi_\varepsilon\;\varphi_\kappa$.

Considering now $\delta \downarrow 0$, the narrow convergence (mentioned in Proposition 4.1) and the lower semicontinuity of total variation (with respect to weak* convergence) provide the same estimates with b instead of b_δ for all $\nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ and, we obtain $C\,\lambda_\varepsilon$ as an admissible choice of the parameter $\alpha_\varepsilon(\vartheta_b, \mu)$. Thus, the first three conditions on sleek transitions (stated in Definition 2.1) are obviously fulfilled.

Moreover, Lemma 4.2 states $\big|\vartheta_{b_\delta}(t,\mu)\big|(\mathbb{R}^N) \;\leq\; |\mu|(\mathbb{R}^N)\,\cdot\,e^{2\,C\,t}$ for all $\mu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$, $t \in [0,1]$, $\delta > 0$ and so, the lower semicontinuity of total variation again implies

$$\big|\vartheta_b(t,\mu)\big|(\mathbb{R}^N) \;\leq\; |\mu|(\mathbb{R}^N)\,\cdot\,e^{2\,C\,t},$$

i.e. condition (5) of Definition 2.1 is satisfied with $\zeta_\varepsilon(\vartheta_b) \overset{\text{Def.}}{=} 2\,C$ (independent of $\varepsilon \in \mathcal{J}$).
Considering the continuity w.r.t. time, we obtain for every $s,t \in [0,1]$, $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$, $\varepsilon \in \mathcal{J}$

$$\begin{aligned}
q_\varepsilon\big(\vartheta_b(s,\mu),\ \vartheta_b(t,\mu)\big) &\leq \limsup_{\delta \downarrow 0}\ q_\varepsilon\big(\vartheta_{b_\delta}(s,\mu),\ \vartheta_{b_\delta}(t,\mu)\big)\\
&\leq \limsup_{\delta \downarrow 0} \int_{\mathbb{R}^N} \big|\varphi_\varepsilon\big(\mathbf{X}_{b_\delta}(s,x)\big) - \varphi_\varepsilon\big(\mathbf{X}_{b_\delta}(t,x)\big)\big|\ |\sigma(x)|\ d\,\mathcal{L}^N x\\
&\leq \limsup_{\delta \downarrow 0}\ \|\nabla\,\varphi_\varepsilon\|_\infty\ \ \|b_\delta\|_\infty\ |t-s|\ \ \|\sigma\|_{L^1(\mathbb{R}^N)}\\
&\leq \lambda_\varepsilon\ \|\varphi_\varepsilon\|_\infty\ \ C\ \ |t-s|\ \ |\mu|(\mathbb{R}^N),
\end{aligned}$$

i.e. $\beta_\varepsilon(\vartheta_b)(t) \overset{\text{Def.}}{=} \lambda_\varepsilon\,\|\varphi_\varepsilon\|_\infty\ C \cdot t$ is a modulus of continuity as required in condition (4.) (of Def. 2.1). The rest of these conditions is trivial.

Finally we specify an upper bound of

$$Q_\varepsilon(\vartheta_b, \vartheta_c; \mu) \overset{\text{Def.}}{=} \sup_{\substack{0 \leq t < 1\\ \nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)}}\ \limsup_{h \downarrow 0} \left(\frac{q_\varepsilon\big(\vartheta_b(t+h,\mu),\ \vartheta_c(h,\nu)\big) - q_\varepsilon\big(\vartheta_b(t,\mu),\ \nu\big)\cdot e^{\lambda_\varepsilon\,C\,h}}{h\quad(1+|\nu|(\mathbb{R}^N))}\right)^+$$

with $b, c \in \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying $\|b\|_\infty + \big\|\mathrm{div}\,b\big\|_\infty$, $\|c\|_\infty + \big\|\mathrm{div}\,c\big\|_\infty \leq C$ and $\mu = \sigma\,\mathcal{L}^N \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$.
Mollifying b, c in the way described above, we again obtain vector fields $b_\delta, c_\delta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|b_\delta\|_\infty$, $\|c_\delta\|_\infty \leq C$. and, Lemma 4.5 (6) states for all $\nu \in \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$

$$\limsup_{h \downarrow 0}\ \frac{q_\varepsilon\big(\vartheta_{b_\delta}(h,\nu),\ \vartheta_{c_\delta}(h,\nu)\big)}{h} \;\leq\; \lambda_\varepsilon\,\big|\varphi_\varepsilon\,|b_\delta - c_\delta|\ \nu\big|\,(\mathbb{R}^N).$$

So Lemma 2.9, the area formula and Lemma 4.2 imply

$$\begin{aligned}
Q_\varepsilon(\vartheta_{b_\delta}, \vartheta_{c_\delta}; \mu) &\leq \sup_{0 \leq t < 1}\ \lambda_\varepsilon\ \ \big|\varphi_\varepsilon\,|b_\delta - c_\delta|\ \ \vartheta_{b_\delta}(t,\mu)\big|\,(\mathbb{R}^N)\\
&= \sup_{0 \leq t < 1}\ \lambda_\varepsilon \int_{\mathbb{R}^N} \varphi_\varepsilon\,|b_\delta - c_\delta|\ \left|\frac{\sigma}{\det\,D_x\,\mathbf{X}_{b_\delta}(t,\cdot)} \circ \mathbf{X}_{b_\delta}(t,\cdot)^{-1}\right|\ d\mathcal{L}^N x\\
&\leq \sup_{0 \leq t < 1}\ \lambda_\varepsilon\ \ \big\|\varphi_\varepsilon\,|b_\delta - c_\delta|\,\big\|_{L^1(\mathbb{R}^N)}\ \ \frac{\|\sigma\|_\infty}{\|\det\,D_x\,\mathbf{X}_{b_\delta}(t,\cdot)\|_\infty}\\
&\leq \lambda_\varepsilon\ \ \big\|\varphi_\varepsilon\,|b_\delta - c_\delta|\,\big\|_{L^1(\mathbb{R}^N)}\ \ \|\sigma\|_\infty\ e^C\,.
\end{aligned}$$

Now Proposition 2.5 and the narrow convergence of $\vartheta_{b_\delta}(t,\mu)$, $\vartheta_{c_\delta}(t,\mu)$ for $\delta \downarrow 0$ (as stated in Proposition 4.1) bridge the gap to $Q_\varepsilon(\vartheta_b, \vartheta_c; \mu)$. Indeed, for all $h \in\,]0,1]$,

$$\begin{aligned}
q_\varepsilon\big(\vartheta_b(h,\mu),\ \vartheta_c(h,\mu)\big) &\leq \limsup_{\delta \downarrow 0}\ q_\varepsilon\big(\vartheta_{b_\delta}(h,\mu),\ \vartheta_{c_\delta}(h,\mu)\big)\\
&\leq \limsup_{\delta \downarrow 0}\ h \cdot Q_\varepsilon(\vartheta_{b_\delta}, \vartheta_{c_\delta}; \mu)\ \ \big(1 + |\mu|(\mathbb{R}^N)\,e^{2\,C\,h} + 2\,C\,h\big)\ \ e^{\lambda_\varepsilon\,C\,h}\\
&\leq \limsup_{\delta \downarrow 0}\ h \cdot \lambda_\varepsilon\,\|\sigma\|_\infty\,e^C\ \big\|\varphi_\varepsilon\,|b_\delta - c_\delta|\,\big\|_{L^1(\mathbb{R}^N)}\ \big(1 + |\mu|(\mathbb{R}^N)\big)\ \big(1 + O(h)\big)\\
&\leq h \cdot \lambda_\varepsilon\,\|\sigma\|_\infty\,e^C\ \big\|\varphi_\varepsilon\,|b - c|\,\big\|_{L^1(\mathbb{R}^N)}\ \ \big(1 + |\mu|(\mathbb{R}^N)\big)\ \big(1 + O(h)\big).
\end{aligned}$$

due to the construction of b_δ, c_δ. Thus, we conclude from Lemmas 2.9 and 4.7

$$\begin{aligned}
Q_\varepsilon(\vartheta_b, \vartheta_c; \mu) &\leq \sup_{0 \leq t < 1}\ \limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot q_\varepsilon\big(\vartheta_b(h,\ \vartheta_b(t,\mu)),\ \vartheta_c(h,\ \vartheta_b(t,\mu))\big)\\
&\leq \sup_{0 \leq t < 1}\ \lambda_\varepsilon\ \left\|\frac{\vartheta_b(t,\mu)}{\mathcal{L}^N}\right\|_\infty e^C\ \ \big\|\varphi_\varepsilon\,|b - c|\,\big\|_{L^1(\mathbb{R}^N)}\ \big(1 + |\vartheta_b(t,\mu)|(\mathbb{R}^N)\big)\\
&\leq \lambda_\varepsilon\ \ \|\sigma\|_\infty\,e^C\ \ e^C\ \ \big\|\varphi_\varepsilon\,|b - c|\,\big\|_{L^1(\mathbb{R}^N)}\ \big(1 + |\mu|(\mathbb{R}^N) \cdot e^{2C}\big) \qquad \square
\end{aligned}$$

Using the total variation $|\cdot|(\mathbb{R}^N)$ for the measures in $\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ proves to be particularly helpful in regard to "weak" compactness. Indeed, the classical criterion of de la Vallée Poussin has the following immediate consequence (and thus, we need not restrict ourselves to measures of compact support) :

Lemma 4.9 *The tuple* $\big(\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (p_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ (\lfloor\cdot\rfloor)_{\varepsilon\in\mathcal{J}},\ \mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)\big)$ *is weakly transitionally compact.* □

Proposition 4.10 *Suppose for* $f : \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)\times[0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C\in[0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t)\in\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)\times[0,T]$,
2. $\|f(\mu,t) - f(\mu_n,t_n)\|_{L^1(\mathbb{R}^N)} \longrightarrow 0$ *if* $\mu_n \overset{*}{\rightharpoonup} \mu$ *(w.r.t.* $C_0(\mathbb{R}^N)$*)*, $t_n \searrow t$.

Then for every initial measure $\mu_0\in\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$, *there exists a sleek solution* $\mu(\cdot) : [0,T[\longrightarrow \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0)=\mu_0$. *Moreover,* $\mu(\cdot)$ *is distributional solution of the continuity equation* $\frac{d}{dt}\,\mu(t) + D_x\cdot(f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N\times[0,T[$.

Proof. The existence of a right–hand sleek solution $\mu(\cdot)$ results from Proposition 2.21 and Remark 2.23 applied to $\big(\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N),\ \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N),\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (p_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ (|\cdot|(\mathbb{R}^N))_{\varepsilon\in\mathcal{J}},\ \mathcal{T}_{C,\mathrm{BV}}(\mathbb{R}^N)\big)$ due to the characterization of sleek transitions in Proposition 4.8.
Verifying the claimed distributional property of $t\mapsto\mu(t)=\sigma(t,\cdot)\,\mathcal{L}^N$ follows essentially the same track as in the proof of Proposition 3.11. Indeed, the Lagrangian flow $\mathbf{X}_{f(\mu(t),t)} : [0,1]\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ of each $f(\mu(t),t)\in\mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies for all $h\in[0,1]$ and $\mathcal{L}^N$– almost every $x\in\mathbb{R}^N$

$$\mathbf{X}(h,x) \;=\; x + \int_0^h f(\mu(t),t)\big(\mathbf{X}(s,x)\big)\ ds.$$

according to Proposition 4.1. As $\mu(\cdot)$ is constructed by Euler approximations being equi–Lipschitz continuous w.r.t. each q_ε $(\varepsilon\in\mathcal{J})$, it is also Lipschitz continuous w.r.t. to each q_ε and, we obtain at every time t of differentiability

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N}\varphi_\varepsilon\ d\,\mu(t) \;&=\; \limsup_{h\downarrow 0}\ \frac{1}{h}\cdot\int_{\mathbb{R}^N}\big(\varphi_\varepsilon(\mathbf{X}_{f(\mu(t),\,t)}(h,x)) \;-\; \varphi_\varepsilon(x)\big)\quad \sigma(t,x)\quad d\mathcal{L}^N x\\
&=\; \int_{\mathbb{R}^N}\nabla\varphi_\varepsilon(x)\ \cdot\ f(\mu(t),\,t)(x)\qquad\qquad \sigma(t,x)\quad d\mathcal{L}^N x.
\end{aligned}$$

Now the same steps as in the proof of Proposition 3.11 guarantee for every $\psi\in C_0^\infty(\mathbb{R}^N\times[0,T[)$

$$-\int_{\mathbb{R}^N}\psi(\cdot,0)\ d\,\mu_0 \;=\; \int_0^T\Big(\int_{\mathbb{R}^N}\partial_t\,\psi(\cdot,t)\ d\,\mu(t) \;+\; \int_{\mathbb{R}^N}\nabla_x\psi(\cdot,t)\cdot f(\mu(t),\,t)(\cdot)\ d\,\mu(t)\Big)\,dt. \qquad \square$$

Restricting our considerations to measures in $\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ has now the additional advantage of a closer relationship between distributional solutions (of the continuity equation) and right–hand sleek solutions (of the corresponding mutational equation). The key tool here is the maximum principle for distributional solutions quoted in Lemma 4.12.

Proposition 4.11 *Suppose for* $f : \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)\times[0,T] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$:

1. $\exists\ C\in[0,\infty[\ :\ \|f(\mu,t)\|_\infty + \big\|\mathrm{div}\, f(\mu,t)\big\|_\infty \le C$ *for all* $(\mu,t)\in\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)\times[0,T]$,
2. $\forall\ \varepsilon\in\mathcal{J}\quad \exists\, L_\varepsilon\in[0,\infty[,$ *modulus of continuity* $\omega_\varepsilon(\cdot)\ge 0$:
$\big\|\varphi_\varepsilon\,|f(\mu,s)-f(\nu,t)|\,\big\|_{L^1(\mathbb{R}^N)} \le\ L_\varepsilon\cdot q_\varepsilon(\mu,\nu) + \omega_\varepsilon\,(|s-t|)$ *for all* $(\mu,s),(\nu,t)\in\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)\times[0,T]$.

Then for every initial measure $\mu_0\in\mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$, *the right–hand sleek solution* $\mu(\cdot) : [0,T[\longrightarrow \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\,\cdot)}$ *with* $\mu(0)=\mu_0$ *is unique.*
So every distributional solution $\mu(\cdot) : [0,T[\longrightarrow \mathbb{L}^{\infty\cap 1}(\mathbb{R}^N)$ *of* $\frac{d}{dt}\,\mu(t) + D_x\cdot(f(\mu(t),t)\ \mu(t)) = 0$ *in* $\mathbb{R}^N\times\,]0,T[$ *that is continuous with respect to each* q_ε $(\varepsilon\in\mathcal{J})$ *is unique.*

Proof. The uniqueness of right–hand sleek solutions results from Proposition 2.24 for the same reasons as Corollary 3.13. Now suppose $\mu(\cdot) : [0,T[\longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ to be distributional solution of $\frac{d}{dt}\,\mu(t) + D_x \cdot (f(\mu(t),t)\ \mu(t)) = 0$ in $\mathbb{R}^N \times]0,T[$ that is continuous with respect to each q_ε $(\varepsilon \in \mathcal{J})$. Then for each $\widehat{t} \in]0,T[$, the restriction $\mu(\cdot)|_{[0,\widehat{t}]}$ is uniformly continuous with respect to each q_ε and thus, $f(\mu(\cdot),\cdot) : [0,\widehat{t}] \longrightarrow \mathrm{BV}^{\infty,\mathrm{div}}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies the assumption of Proposition 4.10. So there exists the unique right–hand sleek primitive $\nu(\cdot) : [0,\widehat{t}] \longrightarrow \mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ of $f(\mu(\cdot),\cdot)$ with $\nu(0) = \mu_0$ and, $\nu(\cdot)$ is also distributional solution of $\frac{d}{dt}\,\nu(t) + D_x \cdot (f(\mu(t),t)\ \nu(t)) = 0$ in $\mathbb{R}^N \times]0,\widehat{t}[$. The comparison principle of Lemma 4.12 implies $\nu(\cdot) \equiv \mu(\cdot)$. So $\mu(\cdot)$ is right–hand sleek solution of $\overset{\circ}{\mu}(\cdot) \ni \vartheta_{f(\mu(\cdot),\cdot)}$. □

Lemma 4.12 *Let $\widetilde{b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfy the assumptions of Proposition 4.1. Then the comparison principle for distributional solutions of the continuity equation*

$$\frac{d}{dt}\,\mu + D_x \cdot (\widetilde{b}\,\mu) = 0 \qquad \text{in }]0,T[\times \mathbb{R}^N$$

holds in the class $\left\{ \widetilde{w}\,\mathcal{L}^N \,\middle|\, \widetilde{w} \in L^\infty\big([0,T]; L^1(\mathbb{R}^N)\big) \cap L^\infty\big([0,T]; L^\infty(\mathbb{R}^N)\big) \cap C^0\big([0,T]; w^*\text{–}L^\infty(\mathbb{R}^N)\big) \right\}$.
In particular, distributional solutions are unique in this class.

Proof results from [1], Theorem 26 and 34. □

Acknowledgments. The author would like to thank Prof. Willi Jäger for arousing the interest in set-valued maps and geometric evolution problems and furthermore, Prof. Jean–Pierre Aubin and Hélène Frankowska for their continuous support while generalizing mutational equations. Prof. Luigi Ambrosio provided an excellent insight into the Cauchy problem for BV vector fields during the CIME course “Calculus of variations and nonlinear PDE” in June 2005. His results have proven to be the basis of the conclusions in this paper. The author is also grateful to Irina Surovtsova and Daniel Andrej for fruitful complementary discussions.
Finally he acknowledges the financial support provided through the European Community’s Human Potential Programme under contract HPRN-CT-2002-00281 [Evolution Equations] and the German Research Foundation (for participating in the CIME course) .

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