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Viscous Fluid Flow

in Bifurcating Channels and Pipes

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Abstract

In the present thesis the flow of a viscous Newtonian fluid in a bifurcation of thin three-dimensional pipes with a diameter-to-length ratio of order $O(\epsilon)$ is studied. The model is based on the steady-state Navier-Stokes equations with pressure conditions on the in- and outflow boundaries. Existence and local uniqueness is proven under the assumption of small data represented by a Reynolds number Re_ϵ of order $O(\epsilon)$.
Our aim is to construct an asymptotic expansion in powers of ϵ and Re_ϵ for the solution of this Navier-Stokes problem. In the first part of the thesis we therefore present a formal method of computing the pressure drop and the flux based on Poiseuille flow. In contrast to the existing literature, we also analyze the influence of the bifurcation geometry on the fluid flow by introducing local Stokes problems in the junction. We show that the solutions of these Stokes problems in the junction of diameter $O(M)$ approximate the solutions of the corresponding Leray problems in the infinite bifurcation up to an error decaying exponentially in M.
In the second part of the thesis, the construction of the approximation for the Navier-Stokes solution is presented and its properties are discussed. The approximation is based on the idea of a continuous matching of the Poiseuille velocity to the solution of the junction problem on each pipe-junction interface.
The main result of our analysis is the derivation of error estimates for the approximation in powers of ϵ and Re_ϵ according to the designated approximation accuracy. The obtained results generalize and improve the existing ones in literature. In addition, our results show that Kirchhoff's law of the balancing fluxes has to be corrected in $O(\epsilon)$ in order to obtain an adequate error estimate for the gradient of velocity.

Zusammenfassung

Die vorliegende Arbeit behandelt die Strömung einer viskosen Newtonschen Flüssigkeit in einer Verzweigung dreidimensionaler Kapillaren, deren Verhältnis von Durchmesser zu Länge von Ordnung $O(\epsilon)$ *ist. Ausgangspunkt des Modells sind die stationären Navier-Stokes-Gleichungen mit Druckrandbedingungen an den Zu- bzw. Abflußrändern. Unter der Annahme kleiner Daten, d.h. einer Reynolds-Zahl* Re_ϵ *von Ordnung* $O(\epsilon)$*, wird ein Existenz- und lokales Eindeutigkeitsresultat bewiesen.*
Ziel ist die Konstruktion einer asymptotischen Entwicklung in Potenzen von ϵ *und* Re_ϵ*, um die Lösung dieses Navier-Stokes-Problems zu approximieren. Im ersten Teil der Arbeit stellen wir dazu eine formale Methode zur Berechnung von Druckabfall und Durchfluß basierend auf Poiseuille-Strömungen vor. Im Gegensatz zu bisherigen Ergebnissen in der Literatur untersuchen wir dabei auch den Einfluß der Geometrie der Verzweigung auf die Strömung durch die Einführung lokaler Stokes-Probleme im Verzweigungsbereich. Wir zeigen, daß die Lösungen dieser Stokes-Probleme in der Verzweigung von Durchmesser* $O(M)$ *die Lösungen der entsprechenden Leray-Probleme in der unendlichen Verzweigung bis auf einen in* M *exponentiell abfallenden Fehler approximieren.*
Im zweiten Teil der Arbeit werden der Aufbau der Approximation für die Navier-Stokes-Lösung dargestellt und ihre Eigenschaften diskutiert. Die Approximation basiert dabei auf der Idee, die Poiseuille-Geschwindigkeiten jeder Röhre auf den Grenzflächen mit der Verzweigung stetig an die Lösung des Stokes-Problems anzufügen.
Als Hauptresultat unserer Analyse werden Fehlerabschätzungen in Potenzen von ϵ *und* Re_ϵ *gemäß der verwendeten Approximationsgenauigkeit abgeleitet. Die erzielten Ergebnisse verallgemeinern und verbessern die bisher in der Literatur existierenden Resultate. Weiterhin wird gezeigt, daß das Kirchhoffsche Gesetz des Gleichgewichts der Flüsse in Ordnung* $O(\epsilon)$ *korrigiert werden muß, um eine hinreichend genaue Approximation für den Geschwindigkeitsgradienten zu erhalten.*

Contents

List of notations and abbreviations

Domains and boundaries	(cf. section 2.1)
Ω^ϵ	Bifurcation
Ω_0 $(\Omega_0^\epsilon = \epsilon\Omega_0)$	(Scaled) junction
Ω_j^ϵ	j^{th} pipe (of cross-section ϵS_j and length L_j)
Ω^M $(\Omega^{M,\epsilon} = \epsilon\Omega^M)$	(Scaled) extended junction domain
Z_j^M $(Z_j^{M,\epsilon} = \epsilon Z_j^M)$	(Scaled) cylinder of length M and cross-section S_j
Ω^∞, Ω_j^∞	Bifurcation with infinitely long pipes Ω_j^∞
Γ^ϵ (Γ_0)	Lateral boundary of Ω^ϵ (Ω_0)
Σ_j^ϵ	In-/outflow boundary of the j^{th} pipe (at $x_1^j = L_j$)
γ_j^M $(\gamma_j^{M,\epsilon} = \epsilon\gamma_j^M)$	Cross-section at position $y_1^j = M$ $(x_1^j = \epsilon M)$ in the j^{th} pipe
$n_j = e_1^j$	outer normal vector on Σ_j^ϵ (direction of pipe axis)
$\chi^\epsilon := \chi_{\Omega^{M,\epsilon}}$	Characteristic function of $\Omega^{M,\epsilon}$
$\chi_j^\epsilon := \chi_{\Omega_j^\epsilon \setminus Z_j^{M,\epsilon}}$	Characteristic function of $\Omega_j^\epsilon \setminus Z_j^{M,\epsilon}$
Navier-Stokes equations	(cf. section 2.2)
(v^ϵ, p^ϵ)	Solution of the Navier-Stokes problem
p_j	Given constant pressure values on the boundary Σ_j^ϵ
Approximations	(cf. chapter 5)
$(u_{k,l}^\epsilon, q_{k,l}^\epsilon)$	Approximation for velocity and pressure (v^ϵ, p^ϵ) including terms up to the order $(k,l) \leftrightarrow \epsilon^k Re_\epsilon^l$
$(u_0^\epsilon, q_0^\epsilon) \equiv (u_{0,0}^\epsilon, q_{0,0}^\epsilon)$	Zero-order approximation
Poiseuille flow	(cf. section 2.3, chapter 5)
w_j	Poiseuille velocity profile in the j^{th} pipe
$c_j := \dfrac{1}{L_j}\displaystyle\int_{S_j} w_j$	Conductance of the j^{th} pipe
$V_j^{k,l} := -w_j(\tilde{y}^j)\, C_j^{k,l}\, e_1^j$	Poiseuille velocity of order $\epsilon^k Re_\epsilon^l$ in the j^{th} pipe
$P_j^{k,l} := C_j^{k,l}\,(x_1^j - L_j)$	Poiseuille pressure of order (k,l) in the j^{th} pipe
$(V_j^0, P_j^0) \equiv (V_j^{0,0}, P_j^{0,0})$	Zero-order Poiseuille flow in the j^{th} pipe

List of notations and abbreviations (contd)

Junction flow	(cf. chapters 4, 5)
$(\omega^{k,l}, \pi^{k,l})$	Stokes flow in the junction of order (k,l)
$(\omega^0, \pi^0) \equiv (\omega^{0,0}, \pi^{0,0})$	Zero-order junction flow
$(\tilde{\omega}^{k,l}, \tilde{\pi}^{k,l})$	Inertial (nonlinear) correction of order (k,l)

Effective pressure quantities	(cf. chapter 5)
Leading order	
$q^0 \equiv q^{0,0} := \dfrac{\sum_j c_j\, p_j}{\sum_j c_j}$	Weighted mean value of the pressures p_j
$C_j^0 \equiv C_j^{0,0} := \dfrac{p_j - q^0}{L_j}$	Zero-order Poiseuille pressure profile in the j^{th} pipe
$\tau_j^0 \equiv \tau_j^{0,0} := \pi^0 - C_j^0 y_1^j$	Difference of zero-order junction pressure and Poiseuille pressure profile in j^{th} pipe
Higher order (k,l) (recursively)	
$\tau_j^{k,l} := \pi^{k,l} - C_j^{k,l}\, y_1^j$	Difference of junction pressure and Poiseuille pressure profile in j^{th} pipe
$\langle \tau_j^{k,l} \rangle := \lvert S_j \rvert^{-1} \displaystyle\int_{\gamma_j^M} \tau_j^{k,l}$	Mean value of $\tau_j^{k,l}$ on γ_j^M
$\langle \tilde{\pi}^{k,l} \rangle_j := \lvert S_j \rvert^{-1} \displaystyle\int_{\gamma_j^M} \tilde{\pi}^{k,l}$	Mean value of $\tilde{\pi}^{k,l}$ on γ_j^M
$q^{k,l} := -\dfrac{\sum_j c_j \left(\langle \tau_j^{k-1,l} \rangle + \langle \tilde{\pi}^{k-1,l} \rangle_j\right)}{\sum_j c_j}$	Weighted mean value of order (k,l), $k \geq 1$, $l \geq 0$ (for $l = 0$ no inertial term)
$C_j^{k,l} := -\dfrac{q^{k,l} + \langle \tau_j^{k-1,l} \rangle + \langle \tilde{\pi}^{k-1,l} \rangle_j}{L_j}$	Poiseuille pressure profile of order (k,l), $k \geq 1$, $l \geq 0$ (for $l = 0$ no inertial term)

Chapter 1

Introduction

1.1 Fluid flow in branching structures

The study of fluid flow through branching structures is of special interest in many applications from different sciences, like e.g. biology, physiology, and engineering. Water and nutrients in plants are transported through complex networks of vessels. The arterial-venous system or the structure of the lungs in the human body are typical physiological examples. The water supply in big cities occurs through complex networks of pipes.

Very often different scales are inherent in these systems as shown in Fig. 1.1 (cf. e.g. [J], [MLA]). The extension in one space direction can be much larger than in the other ones. In such a case the ratio of these different lengths defines a small parameter ϵ. This is e.g. the case for large arteries, like the carotid artery, and partially also for small ones, which in general have to be treated separately (cf. [CPS] and [O]). Here we encounter one of the main problems in modeling complex physiological systems: There are many different scales in different parts of the system, making a *global* description very difficult or even impossible. The circulation system shown in Fig. 1.2 is an example in this respect.

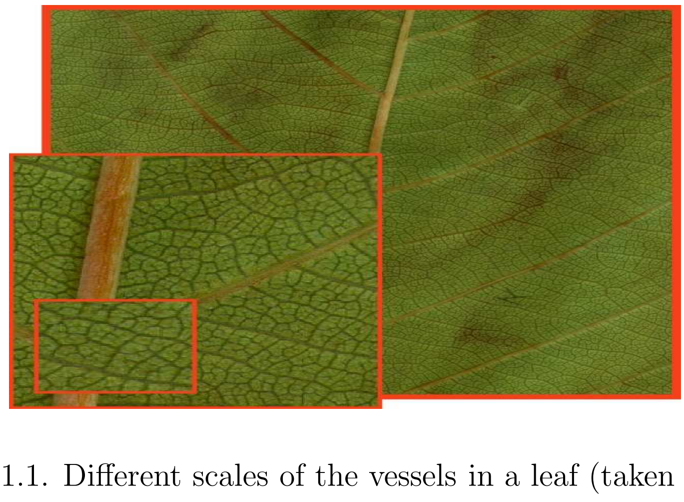

Fig. 1.1. Different scales of the vessels in a leaf (taken from [J])

The subject of the present work is the analysis of the flow of a viscous Newtonian fluid in a three-dimensional network of capillaries. The network consists of long and thin *pipes* with a diameter-to-length ratio of order $O(\epsilon)$ which are connected to each other by *junction domains* of diameter $O(\epsilon)$. The fluid flow is modeled by the *Navier-Stokes equations* which physically describe the conservation of mass and momentum. On the in- and outflow boundaries of the network the pressure is supposed to be given.
Describing such a network as a one-dimensional graph, the junction domains and the pipes correspond to the node points and the lines, respectively. The transition from thin channels and junctions of diameter $O(\epsilon)$ to lines and nodes obviously means a reduction of the three-dimensional structure to a one-dimensional graph.

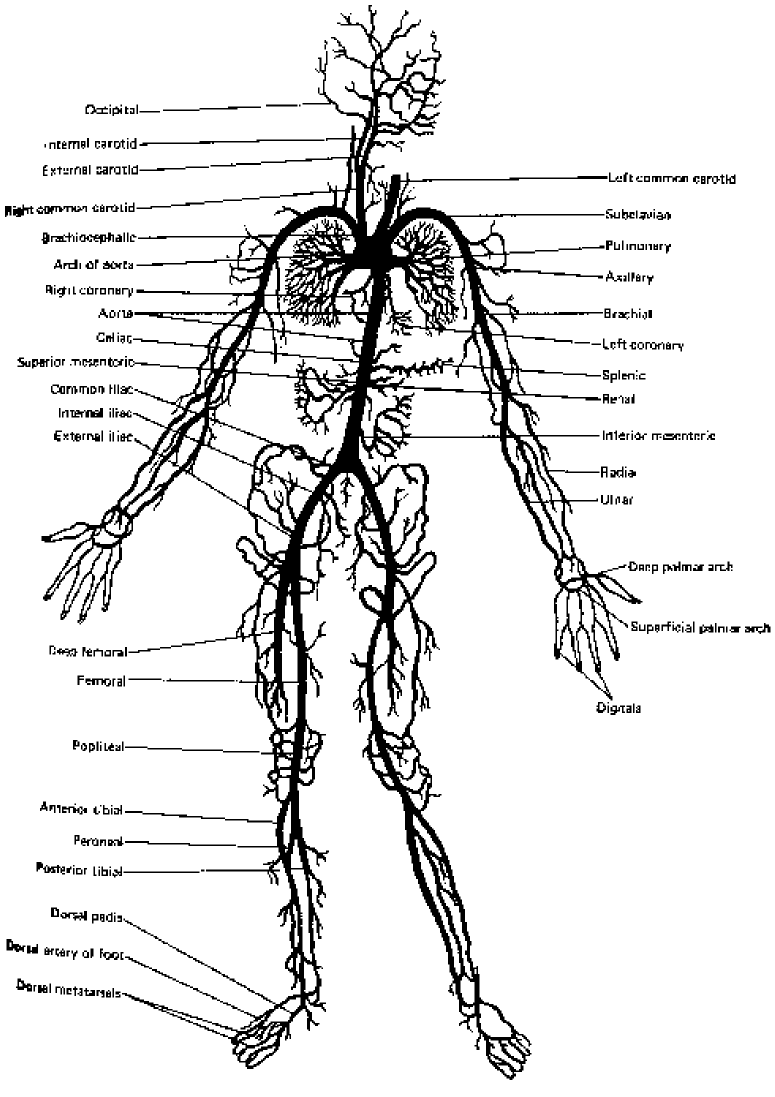

Fig. 1.2. The arterial tree of human body (taken from [SSA])

The corresponding 1d-model for the fluid flow is based on a linear relation between flux and pressure drop in-between each pair of node points of the network. The *Kirchhoff law* states the balance of the fluxes at each node point, thus providing a linear system of equations (cf. chapter 2). In order to compute the node pressures and the fluxes, the *conductance* of each channel has to be known. The conductance is the *effective quantity* which contains the information of the three-dimensional geometric structure. If the channels are cylindrical pipes of constant cross-sections, the conductance can be computed *assuming* a *Poiseuille flow* in each pipe for which velocity and pressure can be given explicitly (cf. section 2.3).
Our aim is to relate the three-dimensional *exact* description based on the *Navier-Stokes equations* to the *effective* one based on *Kirchhoff's law*. We show that the Navier-Stokes solution can be approximated by Poiseuille flows which are driven by a linear pressure drop in each pipe according to Kirchhoff's law. In particular, the fluxes computed from the 1d-model are adequate approximations if the diameter-to-length ratio ϵ of the channels is sufficiently small (cf. chapters 2 and 6). In order to simplify the analysis, we consider the case of one bifurcation, consisting of at least two pipes connected by a junction domain. Away from the junction we expect Poiseuille flow. We particularly aim at analyzing how this Poiseuille flow is influenced by the flow through the junction and at which distance from the junction it represents an appropriate approximation. Therefore, we solve a Stokes problem in the junction domain (called *junction problem* in the following) and construct an approximation matching its solution to the Poiseuille flows inside the pipes (cf. chapters 4 and 5). We thereby assume the nonlinear terms to be of higher order in ϵ and analyze them separately, using additional correction problems in the junction.

The development of a Poiseuille velocity profile in the pipes can also be confirmed numerically. Fig. 1.3 and 1.4 show the velocity components of a two-dimensional flow in a symmetric and a non-symmetric junction, respectively, obtained by solving numerically the Navier-Stokes equations (cf. [C]).

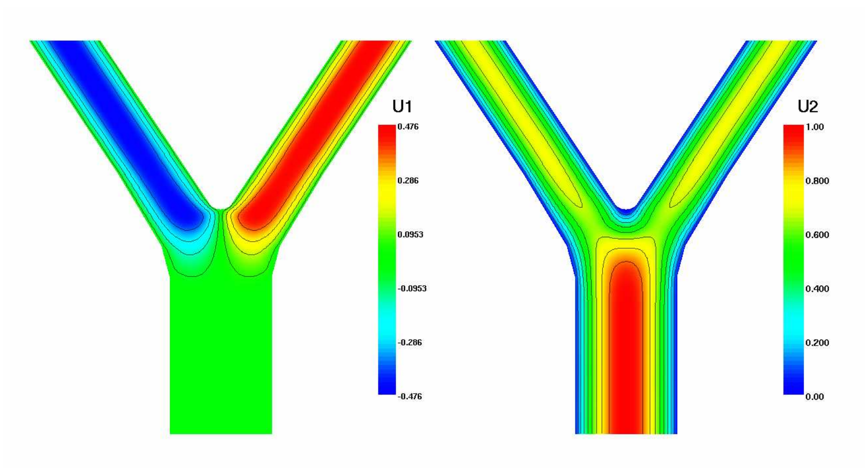

Fig. 1.3. Flow in a symmetric junction
(taken from [C])

At the upstream exit (bottom of Fig. 1.3 and Fig. 1.4) a Poiseuille velocity profile is prescribed, i.e. the transversal velocity component U_1 (left figure) equals zero and the axial velocity component U_2 is parabolic. As usual, the *no-slip* condition is posed on the lateral boundary of the junction. The downstream exits are somehow *artificial* assuming the channels to continue further and therefore the *do nothing-condition* is chosen. It relates the normal derivative of the velocity U and the pressure P on the outflow boundary in the following way: $\mu \nabla U\, n - P\, n = 0$, where μ denotes the viscosity of the fluid (cf. [HRT]). This condition is also called *natural boundary condition* as long as no further information is known on the continuation of the flow behind the exit.
The Poiseuille profile in the downstream branchings can be recognized clearly, both in the symmetric and in the asymmetric bifurcation. As expected, the flux through the small side channel of the asymmetric bifurcation is considerably lower as in the main branch.

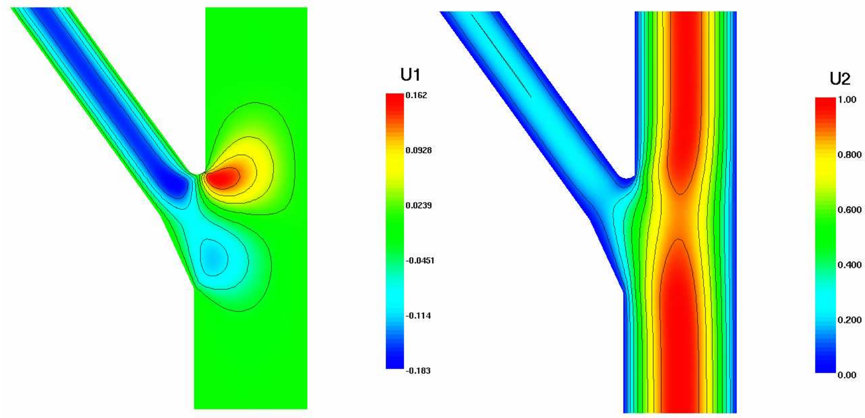

Fig. 1.4. Flow in a non-symmetric junction
(taken from [C])

In homogenization theory for fluid flow in periodic porous media local cell problems are solved in order to prove the effective filtration law, i.e. *Darcy's law*, which has been established empirically according to physical considerations (cf. [H] and the references therein). The situation we consider in the present work is similar: The linear flux-pressure relation combined with Kirchhoff's law is the one-dimensional analogon to Darcy's law and the incompressibility condition. Our aim is to trace back this effective description to the Navier-Stokes equations. In particular, we do not know how the fluid flow is affected by the junction part of the bifurcation. Furthermore, corrections due to the nonlinearity occur in powers of a local Reynolds number similar to those established for Darcy's law in porous media, cf. [BMM].

We conclude this section by giving an overview of the literature on viscous fluid flow in thin channels as far as it is related to our analysis. The starting point of the present work is [MP3], where the Navier-Stokes equations with pressure boundary condition in the junction of thin pipes are considered. An approximation based on *infinite* junction problems (called Leray's problem, cf. chapter 3) is constructed therein. In contrast, we show that the solution of Leray's problem can be approximated on *finite* junction domains (cf. chapter 4) thus avoiding an additional matching of the Poiseuille flow in the pipes to the Leray flow in the junction.

The Stokes and Navier-Stokes equations in tubelike structures are also discussed in [BGP] (construction of a Poiseuille flow approximation for Dirichlet boundary conditions), [MP2] (flow in curved pipes), [MM1] (flow in a periodic network of thin channels), [MM2] (Poiseuille flow approximation in thin pipes via two-scale convergence) and [A] (analysis of Leray's problem). In particular, our analysis of Leray's problem is based on [G] where Stokes flow in infinite channels is discussed in detail summarizing the previous results in literature (cf. the references therein).
The question of existence and uniqueness of the solution of the steady-state Navier-Stokes equations with pressure boundary condition has been studied in [CMP]. There, the *dynamic pressure* $(p+1/2\,v^2)$ is prescribed and the existence of a solution is proved without constraint on the data. But in order to get *uniqueness*, a sufficiently large viscosity or, equivalently, sufficiently small pressure values on the boundary are required. Prescribing boundary values for the *static* pressure p, we are not able to prove existence in general, but only for sufficiently small data (cf. the discussion in section 2.2). In case of the unsteady problem with static pressure boundary conditions, existence and uniqueness can be assured in a (possibly very small) time interval from the initial time t_0 to $t_0 + T$, for some $T > 0$ (cf. [JM2]). Regularity of solutions of such non-standard Navier-Stokes problems has been considered in [B], extending the results from [CMP].

1.2 Outline of the thesis

In this section we give an overview of the main ideas and results of our analysis, which are then presented in detail in the next chapters. The structure of this section is the following: We start with a short description of the geometry and specify the governing equations. Then we present the main result of our work. Next, the major steps in the construction of the approximation of the Navier-Stokes solution based on Poiseuille flow are enumerated. In the last part of the section we give the motivation for our construction and specify the properties of the approximation.

1.2.1 Mathematical model

We consider a three-dimensional *branching structure* consisting of several *pipes* and a *junction* domain (cf. section 2.1 for a detailed definition of the geometry). The *pipes* have (possibly different) constant cross-sections of diameter $O(\epsilon)$ and lengths $O(1)$, and are connected to each other by the *junction*. In the junction domain, we do not distinguish between different scales, meaning that its diameter is of order $O(\epsilon)$.

Summarizing, the *branching* Ω^ϵ, assumed to be smooth except of the edges at the in- and outflow boundaries Σ_j^ϵ, can be divided into the *pipes* Ω_j^ϵ $(j = 1, ..., N)$ of length L_j and cross-section $S_j^\epsilon = \epsilon S_j$, and the *junction domain* $\Omega_0^\epsilon = \epsilon\Omega_0$. The lateral boundary of Ω^ϵ is denoted by Γ^ϵ.
We also define the *infinite bifurcation* Ω^∞ consisting of pipes Ω_j^∞ of infinite length and cross-section S_j connected by the junction Ω_0.
Furthermore, we introduce the *extended junction domain* $\Omega^{M,\epsilon}$ consisting of the junction Ω_0^ϵ prolongated by the cylinders $Z_j^{M,\epsilon}$ of length ϵM. The interface between the pipes Ω_j^ϵ and the extended junction domain $\Omega^{M,\epsilon}$ is denoted by $\gamma_j^{M,\epsilon}$.

In Ω^ϵ we consider the following Navier-Stokes system for velocity v^ϵ and pressure p^ϵ (cf. section 2.2):

$$(1.1) \qquad \begin{cases} -\mu_0\epsilon^2\Delta v^\epsilon + \epsilon Re_\epsilon(v^\epsilon\cdot\nabla)v^\epsilon + \nabla p^\epsilon &= 0 \quad \text{in } \Omega^\epsilon, \\ div\, v^\epsilon &= 0 \quad \text{in } \Omega^\epsilon, \\ v^\epsilon &= 0 \quad \text{on } \Gamma^\epsilon, \\ v^\epsilon \times n_j = 0,\ p^\epsilon &= p_j \quad \text{on } \Sigma_j^\epsilon, \end{cases}$$

where $p_j \in \mathbb{R}$ are given constants and n_j denotes the outer normal vector on Σ_j^ϵ.

Since the diameter of the pipes is of order $O(\epsilon)$, we scale the viscosity by a factor ϵ^2 to obtain velocity and pressure of order $O(1)$. In order to analyze the influence of the nonlinearity, we define the *Reynolds number* Re_ϵ, which indicates the order of magnitude of the convective nonlinear term compared to the viscous one. We then formally have $\epsilon Re_\epsilon(v^\epsilon\cdot\nabla)v^\epsilon = O(Re_\epsilon)$. Pressure boundary conditions are considered on the in- and outflow cross-sections of the pipes. We prove existence and local uniqueness of the solution $(v^\epsilon, p^\epsilon) \in (H^1 \times L^2)(\Omega^\epsilon)$ under the condition $Re_\epsilon \leq O(\epsilon)$, using Banach's fixed point theorem.

For the solution (v^ϵ, p^ϵ) of (1.1) we aim at constructing an approximation $(u_{k,l}^\epsilon, q_{k,l}^\epsilon)$ in powers of ϵ and Re_ϵ, which is based on Poiseuille flow in the pipes away from the junction. The Poiseuille flow is characterized by a parabolic velocity profile, which does not change along the axial direction of the cylindrical pipe, and a linear pressure drop between the ends of the pipe, the pressure being constant in the cross-sectional direction (cf. section 2.3). The zero-order Poiseuille flow in the j^{th} pipe is driven by the pressure drop $(q^0 - p_j)$, where q^0 is the weighted mean value of the pressures p_k, $k = 1, \ldots, N$, the weights being the conductances of the pipes (cf. equation (2.7)). In the *extended* junction domain $\Omega^{M,\epsilon}$, we establish a Stokes flow which is matched continuously to the Poiseuille flow on the interfaces $\gamma_j^{M,\epsilon}$.

These ideas are generalized in order to include higher order corrections. The approximation $(u_{k,l}^\epsilon, q_{k,l}^\epsilon)$ then is an expansion in powers of ϵ and Re_ϵ, including correction terms up to the order $O(\epsilon^k Re_\epsilon^l)$ (cf. the discussion below).

1.2.2 Main result

The *main result* of our analysis is summarized in the following *theorem*. It compares the approximation $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$ with the solution (v^ϵ, p^ϵ) of the Navier-Stokes system (1.1). The proof of the error estimates is given in section 6.2 (cf. Theorem 6.1 and Corollary 6.3).

Theorem. *If* $Re_\epsilon \leq O(\epsilon)$*, then the following estimates hold for the approximation* $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$*: There exist constants* $C, \tilde{\sigma} > 0$*, independent of* ϵ *and* M*,* $\rho^\epsilon := |\Omega^\epsilon|^{1/2}$*, such that*

$$\frac{1}{\rho^\epsilon} \left\| \nabla \left(v^\epsilon - u^\epsilon_{k,l} \right) \right\|_{L^2(\Omega^\epsilon)} \leq C\, \epsilon^{-\frac{1}{2}}\, max \left\{ e^{-\tilde{\sigma} M}, \epsilon^k, Re_\epsilon^{l+1} \right\}, \tag{1.2}$$

$$\frac{1}{\rho^\epsilon} \left\| v^\epsilon - u^\epsilon_{k,l} \right\|_{L^2(\Omega^\epsilon)} \leq C\, \epsilon^{\frac{1}{2}}\, max \left\{ e^{-\tilde{\sigma} M}, \epsilon^k, Re_\epsilon^{l+1} \right\}, \tag{1.3}$$

$$\frac{1}{\rho^\epsilon} \left\| p^\epsilon - q^\epsilon_{k,l} \right\|_{L^2(\Omega^\epsilon)/\mathbb{R}} \leq C\, \epsilon^{\frac{1}{2}}\, max \left\{ e^{-\tilde{\sigma} M}, \epsilon^k, Re_\epsilon^{l+1} \right\}, \tag{1.4}$$

for all $M \geq 1$ *and every* $k, l \in \mathbb{N}_0$.

Remark. In section 6.2 we show that the inequalities (1.2)-(1.4) actually hold for $Re_\epsilon \leq O(\epsilon^{1/2})$ if the solution v^ϵ of (1.1) satisfies

$$\|\nabla v^\epsilon\|_{L^2(\Omega^\epsilon)} \leq O(\epsilon^{\frac{1}{2}}\, Re_\epsilon^{-1}). \tag{1.5}$$

In section 2.2 we prove the existence of a unique solution such that (1.5) holds if $Re_\epsilon \leq O(\epsilon)$.

1.2.3 Construction of the approximation

The construction of the approximation $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$ is based on the analysis of two different types of *junction problems* in the domain Ω^M (cf. section 4.1), namely:

(*i*) The *Poiseuille junction problem*, which is a homogeneous Stokes equation with prescribed Poiseuille velocities as in- and outflow boundary conditions on the cross-sections γ_j^M. The solution of this problem is denoted in zero-order by (ω^0, π^0) and analogously by $(\omega^{k,l}, \pi^{k,l})$ according to the order $\epsilon^k Re^l_\epsilon$, $k, l \in \mathbb{N}_0$.

(*ii*) The *inertial correction problem*, which is a Stokes problem with the right-hand side $f := -(\omega^0 \cdot \nabla)\omega^0$ and zero velocity on the whole boundary of Ω^M. The solution is denoted by $(\tilde{\omega}^{0,1}, \tilde{\pi}^{0,1})$. It is generalized to higher orders in chapter 4.

We then introduce for the problems (*i*) and (*ii*) the scaled functions

$$\omega^{0,\epsilon}(x) := \omega^0(\frac{x}{\epsilon}), \qquad \pi^{0,\epsilon}(x) := q^0 + \epsilon\pi^0(\frac{x}{\epsilon}),$$

and

$$\tilde{\omega}^{0,1,\epsilon}(x) := Re_\epsilon\, \tilde{\omega}^{0,1}(\frac{x}{\epsilon}), \qquad \tilde{\pi}^{0,1,\epsilon}(x) := \epsilon Re_\epsilon\, \tilde{\pi}^{0,1}(\frac{x}{\epsilon}),$$

respectively, defined on $\Omega^{M,\epsilon} = \epsilon\,\Omega^M$. The scaled functions solve the corresponding Stokes systems (with a scaling of viscosity by ϵ^2) in $\Omega^{M,\epsilon}$ and represent the approximation of the flow in the junction domain.

Having established Poiseuille flow in the pipes and Stokes flow in the junction, the zero-order approximation then reads (cf. chapter 5):

$$(1.6) \qquad \begin{cases} u_0^\epsilon(x) &:= \displaystyle\sum_j V_j^0(\frac{\tilde{x}^j}{\epsilon})\,\chi_j^\epsilon + \omega^0(\frac{x}{\epsilon})\chi^\epsilon, \\ q_0^\epsilon(x) &:= \displaystyle\sum_j P_j^0(x_1^j)\,\chi_j^\epsilon + \Big(q^0 + \epsilon\pi^0(\frac{x}{\epsilon})\Big)\chi^\epsilon. \end{cases}$$

Here $x^j = (x_1^j, \tilde{x}^j)$ denotes the coordinates of the j^{th} pipe (cf. section 2.1).

This zero-order approximation is not appropriate for the Navier-Stokes solution (v^ϵ, p^ϵ) and some additional corrections are necessary, taking into account the pressure stabilization constants and the inertial terms. We give a short description of these corrections:

Pressure decay correction

We define the difference between the zero-order junction pressure and the Poiseuille pressure profile by

$$\tau_j^0(y) := \pi^0(y) - C_j^0\, y_1^j.$$

Its mean value over the cross-section γ_j^M is defined by

$$(1.7) \qquad \langle \tau_j^0 \rangle := \frac{1}{|\gamma_j^M|} \int_{\gamma_j^M} \tau_j^0.$$

Then, the first-order Poiseuille flow correction is:

$$(1.8) \qquad V_j^{1,0}(\tilde{y}^j) := -w_j(\tilde{y}^j)\, C_j^{1,0}\, e_1^j, \;\; \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon},$$

$$(1.9) \qquad P_j^{1,0}(x_1^j) := q^{1,0} + \langle \tau_j^0 \rangle + C_j^{1,0}\, x_1^j,$$

where $C_j^{1,0} := -\dfrac{q^{1,0} + \langle \tau_j^0 \rangle}{L_j}$ and $q^{1,0} = -\dfrac{\sum_k c_k \langle \tau_k^0 \rangle}{\sum_k c_k}$. This Poiseuille flow is balanced by the junction flow $(\omega^{1,0}, \pi^{1,0})$, i.e. by the solution of the corresponding junction problem of type (i).

Inertial Correction

We take into account the nonlinear term of the Navier-Stokes equation $(1.1)_1$ by solving the junction problem of type (ii). We then proceed analogously to the pressure decay correction: The pressure $\tilde{\pi}^{0,1}$ has mean value $\langle\tilde{\pi}^{0,1}\rangle_j$ on γ_j^M. We define $q^{1,1}$ in such a way that the weighted mean value of $\left(q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j\right)$, $j = 1, ..., N$, equals zero. The Poiseuille flow in the j^{th} pipe, $(\tilde{V}_j^{1,1}, \tilde{P}_j^{1,1})$, is driven by the pressure drop $(q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j)$. The Stokes flow in the junction domain then is corrected by solving a problem of type (i), imposing the velocities $\tilde{V}_j^{1,1}$ on γ_j^M. Its solution is denoted by $(\omega^{1,1}, \pi^{1,1})$.

The approximation $(u_{1,1}^\epsilon, q_{1,1}^\epsilon)$ is thus composed of the following parts:

Approximation term	Poiseuille flow in j^{th} pipe	Stokes flow in junction
Zero order	(V_j^0, P_j^0)	$(\omega^0, q^0 + \epsilon\pi^0)$
Pressure correction I	$\epsilon(V_j^{1,0}, P_j^{1,0})$	$\epsilon(\omega^{1,0}, q^{1,0} + \epsilon\pi^{1,0})$
Nonlinear correction		$Re_\epsilon(\tilde{\omega}^{0,1}, \epsilon\tilde{\pi}^{0,1})$
Pressure correction II	$\epsilon Re_\epsilon(\tilde{V}_j^{1,1}, \tilde{P}_j^{1,1})$	$\epsilon Re_\epsilon(\omega^{1,1}, q^{1,1} + \epsilon\pi^{1,1})$

Higher order terms ($k \geq 1$, $l \geq 0$) can be established *recursively*, repeating the procedures of pressure decay and inertial correction as above (cf. section 5).

1.2.4 Motivation and approximation properties

Our approach is based on the physical assumption of a fast (exponential) decay of velocity and pressure to the Poiseuille flow inside the pipes with increasing distance from the junction. The corresponding mathematical confirmation is the analysis of *Leray's problem*, which is a Stokes problem in Ω^∞ with prescribed Poiseuille velocities at infinity (cf. section 3.1).

The Stokes flow in the junction decays to different Poiseuille flows in the pipes. But the pressure drop which drives these Poiseuille flows and the distance from the junction at which Poiseuille flow represents an adequate approximation are not a priori known. Therefore, we consider an *extended junction domain* $\Omega^{M,\epsilon}$ in which we solve the junction problems (i) and (ii) specified in subsection 1.2.3. Our aim is to establish error estimates for the approximation $(u_{k,l}^\epsilon, q_{k,l}^\epsilon)$ which depend *explicitly* on the parameter M. Having set up these estimates, we are then able to choose M for numerical computations such that the difference between the exact solution of the Navier-Stokes system and its approximation is below a given tolerance.
From a one-dimensional point of view it seems reasonable to use the weighted mean value q^0 to define the Poiseuille flows, and we impose the corresponding velocities at the outflow boundaries of Ω^M in order to obtain a continuous velocity across the interfaces γ_j^M. But we want to point out that q^0 in general does *not* provide an appropriate

approximation for the pressure drop if the unscaled junction Ω_0 has strictures (unless ϵ is sufficiently small, cf. the discussion in section 2.4).

We now discuss the properties of the approximation. Being continuous on the interfaces $\gamma_j^{M,\epsilon} = \epsilon\gamma_j^M$, the zero-order approximation velocity u_0^ϵ is a solenoidal function in $H^1(\Omega^\epsilon)$. But a *jump* in the normal derivative of u_0^ϵ and in the pressure occurs on $\gamma_j^{M,\epsilon}$. Therefore, the zero-order approximation solves the Navier-Stokes system (1.1) only up to an error term, consisting of the jumps

$$[\mu_0\epsilon^2\nabla u_0^\epsilon n_j - q_0^\epsilon n_j] \quad \text{on } \gamma_j^{M,\epsilon}, \text{ with } n_j \text{ the normal vector,} \tag{1.10}$$

and the nonlinearity of order Re_ϵ. This fact is analyzed in detail in chapter 6.

The *key point* of our approximation is the following: All jump terms decay exponentially with growing M. In order to show this exponential decay, we compare the solutions of the junction problems to those of the corresponding Leray problems in the domain Ω^∞ (cf. chapter 4). We obtain the following result: *The solution of the junction problem on the finite domain Ω^M approximates the solution of the corresponding Leray problem on the infinite domain Ω^∞ up to an error decaying exponentially in M.*

Now we return to the approximation $(u_{1,1}^\epsilon, q_{1,1}^\epsilon)$ and discuss the correction terms which appear therein.

The additional pressure correction of order $O(\epsilon)$ (*pressure correction I*) is connected with the decay properties of the solution of Leray's problem (cf. chapter 3). There we encounter additional pressure stabilization constants, i.e. the pressure in each pipe tends to the linear Poiseuille profile, which has to be prescribed in order to solve Leray's problem, plus some constants, which in general are different for each pipe. These constants overrule the exponential decay since they show up in the same order of ϵ. Therefore, we consider the difference τ_j^0 between the junction pressure and the linear Poiseuille pressure profile. In order to approximate these stabilization constants we define the mean value of τ_j^0 on γ_j^M. The Poiseuille flow correction $(V_j^{1,0}, P_j^{1,0})$ is constructed such that its pressure is zero on Σ_j^ϵ since the pressure boundary condition in (1.1) is already fulfilled in zero-order. On $\gamma_j^{M,\epsilon}$ we have $P_j^{1,0} = q^{1,0} + \langle\tau_j^0\rangle + O(\epsilon M)$. Due to the scaling, the Poiseuille flow correction is of order $O(\epsilon)$. Therefore, in order $O(\epsilon)$ the pressure jump $(\tau_j^0|_{\gamma_j^M} - \langle\tau_j^0\rangle)$ occurs on the pipe-junction interface γ_j^M. In section 6.1 we establish the exponential decay for this type of pressure jumps which we do not have without the correction. Therefore it turns out that the zero-order approximation $(u_0^\epsilon, q_0^\epsilon)$ does not provide an adequate estimate for the velocity gradient ∇v^ϵ in $L^2(\Omega^\epsilon)$ (cf. section 6.2).

By solving the inertial correction problem (junction problem of type (*ii*)), we aim at reducing the approximation error by the factor Re_ϵ (cf. chapters 5 and 6). An additional jump of type (1.10) then occurs on γ_j^M and a further pressure correction (*pressure correction II*) is necessary. As in the case of the Poiseuille junction problem, the pressure $\tilde\pi^{0,1}$ approximates the corresponding Leray pressure $\tilde\pi_L^{0,1}$ up to an exponentially decreasing error (cf. chapter 4). Therefore, the decay properties of $\tilde\pi_L^{0,1}$ apply to the junction pressure $\tilde\pi^{0,1}$. In each pipe, the function $\tilde\pi_L^{0,1}$ tends to some stabilization constant at infinity. In order to reduce the approximation error, these stabilization constants have to be removed (cf. chapter 6). This is done in the same way as in the first order pressure correction, defining the Poiseuille flow $(\tilde{V}_j^{1,1}, \tilde{P}_j^{1,1})$ and the junction flow $(\omega^{1,1}, \pi^{1,1})$.

1.2.5 Corrections to Kirchhoff's law

We conclude this section with some remarks concerning *Kirchhoff's law* (cf. subsection 6.3.3).

According to the first-order approximation $(u^\epsilon_{1,1}, q^\epsilon_{1,1})$, the Poiseuille flow in the reduced pipes $\Omega^\epsilon_j \setminus Z^{M,\epsilon}_j$ is given by

$$\begin{aligned} V^\epsilon_j(\tfrac{\tilde{x}^j}{\epsilon}) &:= w_j(\tfrac{\tilde{x}^j}{\epsilon})\,\frac{\langle q^\epsilon_j\rangle - p_j}{L_j}\, e^j_1, \\ P^\epsilon_j(x^j_1) &:= \frac{p_j - \langle q^\epsilon_j\rangle}{L_j}\, x^j_1 + \langle q^\epsilon_j\rangle, \end{aligned}$$

where

$$\langle q^\epsilon_j\rangle := q^0 + \epsilon\left(q^{1,0} + \langle\tau^0_j\rangle\right) + \epsilon Re_\epsilon\left(q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j\right).$$

The *effective junction pressure* $\langle q^\epsilon_j\rangle$ consists of the weighted mean value q^0 (cf. (2.7)) expected from Kirchhoff's law and a *higher order correction*, which is determined by the solution of the Stokes problems (i) and (ii) and reflects the geometry of the junction.

Chapter 2

Fluid flow in pipes and junctions

2.1 Geometry of the bifurcating channels

In this section we characterize the geometry of the bifurcation domains. We start with

Definition 2.1. (Junction, pipes, and bifurcation) A *junction* is a domain $\Omega_0 \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) of diameter $O(1)$ which has $N \geq 2$ cylindrical outlets of (smooth) cross-sections S_j, $j = 1, ..., N$. The junction is assumed to be smooth, except of the outflow boundary edges. The *scaled junction* Ω_0^ϵ of diameter $O(\epsilon)$ is defined by $\Omega_0^\epsilon = \epsilon\, \Omega_0$.

The junction domain Ω_0^ϵ connects the *pipes* (called *channels* in two dimensions)

$$\Omega_j^\epsilon := \{0 < x_1^j < L_j,\, \tilde{x}^j = (x_2^j, x_3^j) \in S_j^\epsilon\}$$

of constant cross-sections $S_j^\epsilon = \epsilon S_j$ and length $L_j = O(1)$. For every pipe we fix a local coordinate system $\{O_j, (e_k^j)_{k=1,2,3}\}$, obtained from the global system by rotation of the basis and translation of the origin.

Fig. 2.1. The junction domain Ω_0 and the pipes Ω_j

The *bifurcation (or branching)* Ω^ϵ is defined as the union of the pipes Ω_j^ϵ and the junction Ω_0^ϵ (including the interfaces $\gamma_j^{0,\epsilon}$ at $x_1^j = 0$).

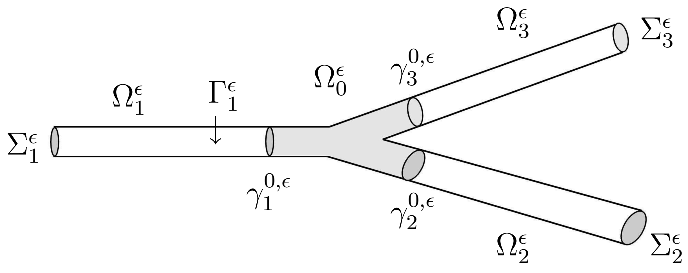

Fig. 2.2. The bifurcation Ω^ϵ

The lateral boundary of Ω^ϵ is denoted by $\Gamma^\epsilon := \bigcup_k \Gamma_k^\epsilon$, where Γ_k^ϵ, $k = 0, 1, ..., N$, is the lateral boundary of the pipe Ω_k^ϵ and the junction Ω_0^ϵ, respectively. The in- and outflow boundaries, respectively, are defined by

$$\Sigma_j^\epsilon := \{x^j = (L_j, \tilde{x}^j),\ \tilde{x}^j \in S_j^\epsilon\}.$$

Furthermore, we define the interfaces

$$\gamma_j^M := \{y^j = (M, \tilde{y}^j),\ \tilde{y}^j \in S_j\}, \quad \gamma_j^{M,\epsilon} = \epsilon\gamma_j^M$$

and the cylinders

$$Z_j^M := \{0 < y_1^j < M,\ \tilde{y}^j = (y_2^j, y_3^j) \in S_j\}, \quad Z_j^{M,\epsilon} = \epsilon Z_j^M.$$

The domain consisting of the junction part Ω_0 (Ω_0^ϵ) and the cylinders Z_j^M ($Z_j^{M,\epsilon}$), including the interfaces γ_j^0 ($\gamma_j^{0,\epsilon}$), is denoted by Ω^M ($\Omega^{M,\epsilon}$).

In order to analyze Leray's problem, we consider the following *infinite* bifurcation domains:

Definition 2.2. (Infinite branching) An infinite bifurcation

$$\Omega^\infty := \Omega_0 \cup \bigcup_j \Omega_j^\infty$$

consists of a junction Ω_0 and infinitely long pipes

$$\Omega_j^\infty := \{0 \le y_1^j < \infty,\ \tilde{y}^j = (y_2^j, y_3^j) \in S_j\}.$$

2.2 Navier-Stokes equations with pressure boundary conditions

We establish a model for viscous fluid flow in bifurcating pipes which is based on the Navier-Stokes equations with pressure boundary conditions. The fluid flow is assumed to be stationary with constant pressure on the in- and outflow boundaries in order to relate the three-dimensional model to the situation of stationary flux and pressure drop in the corresponding one-dimensional network. We introduce scalings of viscosity and pressure, generalizing the model presented in [MP3], in order to analyze the effects of the nonlinear convective term on the effective flow. Furthermore, we prove an existence and uniqueness result.

We consider the following Navier-Stokes problem with a scaling of viscosity and pressure:

$$
\left\{
\begin{array}{rcll}
-\mu_0 \epsilon^\beta \Delta v^\epsilon + (v^\epsilon \cdot \nabla) v^\epsilon + \nabla p^\epsilon & = & 0 & \text{in } \Omega^\epsilon, \\
div\, v^\epsilon & = & 0 & \text{in } \Omega^\epsilon, \\
v^\epsilon & = & 0 & \text{on } \Gamma^\epsilon, \\
v^\epsilon \times n_j \,=\, 0,\ p^\epsilon & = & \epsilon^\gamma p_j & \text{on } \Sigma_j^\epsilon,
\end{array}
\right. \tag{2.1}
$$

where $n_j := e_1^j$ is the outer normal vector on Σ_j^ϵ, $p_j \in \mathbb{R}$, $j = 1, ..., N$, are given constants, and $\beta, \gamma \in \mathbb{R}$.

In pipes of diameter $O(\epsilon)$ the velocity v^ϵ of a fluid of viscosity $O(\epsilon^\beta)$, which is driven by a pressure gradient of order $O(\epsilon^\gamma)$, (formally) is of order $O(\epsilon^{2-\beta+\gamma})$. By rescaling, i.e. $v^\epsilon = \epsilon^{2-\beta+\gamma}\, \tilde{v}^\epsilon$, $p^\epsilon = \epsilon^\gamma\, \tilde{p}^\epsilon$, we get velocity and pressure of order $O(1)$. We define the *Reynolds number* $Re_\epsilon := \epsilon^{3-2\beta+\gamma}$ which reflects the order of magnitude of the nonlinear term. Rewriting v^ϵ instead of $\tilde{v}^\epsilon$, system (2.1) then reads

$$
\left\{
\begin{array}{rcll}
-\mu_0 \epsilon^2 \Delta v^\epsilon + \epsilon Re_\epsilon (v^\epsilon \cdot \nabla) v^\epsilon + \nabla p^\epsilon & = & 0 & \text{in } \Omega^\epsilon, \\
div\, v^\epsilon & = & 0 & \text{in } \Omega^\epsilon, \\
v^\epsilon & = & 0 & \text{on } \Gamma^\epsilon, \\
v^\epsilon \times n_j \,=\, 0,\ p^\epsilon & = & p_j & \text{on } \Sigma_j^\epsilon.
\end{array}
\right. \tag{2.2}
$$

The *weak (variational)* formulation of problem (2.2) is

$$
\mu_0\, \epsilon^2 \int_{\Omega^\epsilon} \nabla v^\epsilon \nabla \phi + \epsilon\, Re_\epsilon \int_{\Omega^\epsilon} (v^\epsilon \cdot \nabla) v^\epsilon\, \phi + \sum_{k=1}^N p_k \int_{\Sigma_k^\epsilon} \phi \cdot n_k = 0 \tag{2.3}
$$

for all $\phi \in V^\epsilon$, where

$$
V^\epsilon := \left\{ u \in H^1(\Omega^\epsilon)^3 : div\, u = 0,\ u|_{\Gamma^\epsilon} = 0,\ u \times n_j|_{\Sigma_j^\epsilon} = 0,\ j = 1, ..., N \right\}.
$$

We first show an existence and uniqueness result:

Theorem 2.1. (Existence and Uniqueness) *There exists a constant $C > 0$, depending on μ_0 and $p_j, j = 1, ..., N$, such that for all $Re_\epsilon \leq C\epsilon$ the Navier-Stokes system (2.2) has a weak solution $v^\epsilon \in V^\epsilon$.*

The solution is unique in the ball

$$B^\epsilon := \left\{ \varphi \in V^\epsilon : \|\nabla\varphi\|_{L^2(\Omega^\epsilon)} \leq K\epsilon^{\frac{1}{2}} Re_\epsilon^{-1} \right\}, \tag{2.4}$$

where $K := \dfrac{\mu_0}{3\, C^2_{L^4,H^1}}$ (cf. (B.3)).

Remark: The condition $Re_\epsilon \leq C\epsilon$ can be reformulated in terms of the scaling powers β and γ of viscosity and pressure as $\gamma \geq 2\beta - 2 + \frac{ln\, C}{ln\, \epsilon}$.

Proof. We proceed as in [MP3], using a *fixed point* argument.

The proof consists of several steps:

(1) We define for $v \in B^\epsilon$ the bilinear form

$$a_v(u,\phi) := \mu_0\, \epsilon^2 \int_{\Omega^\epsilon} \nabla u \nabla \phi \, + \epsilon Re_\epsilon \int_{\Omega^\epsilon} (v\cdot\nabla)u\,\phi$$

for $u, \phi \in V^\epsilon$ and show:

(i) *a_v is V^ϵ-elliptic*, i.e.

$$a_v(u,u) = \mu_0\, \epsilon^2 \int_{\Omega^\epsilon} |\nabla u|^2 \, + \epsilon Re_\epsilon \int_{\Omega^\epsilon} (v\cdot\nabla)u\,u \geq \frac{2}{3}\,\mu_0\,\epsilon^2\,\|\nabla u\|^2_{L^2(\Omega^\epsilon)}.$$

(ii) *Estimate for the boundary values:*
We use the definition $d := \left(\sum_j |p_j - q|^2\right)^{\frac{1}{2}}$ (also possible $d := max_j |p_j - q|$). The value of d is minimal if q is taken as the arithmetic mean $q = \frac{1}{N}\sum_j p_j$.

Then the following estimate holds:

$$\left| \sum_j p_j \int_{\Sigma_j^\epsilon} \phi \cdot n_j \right| \leq C_0\, \epsilon^{\frac{3}{2}} \left(max_j |\Sigma_j|^{\frac{1}{2}} \right) d\, \|\nabla\phi\|_{L^2(\Omega^\epsilon)} \ \text{ for all } \ \phi \in V^\epsilon.$$

Proof. (i) Applying the embedding $H^1 \hookrightarrow L^4$ (cf. Lemma B.2(i)) we obtain, since $v \in B^\epsilon$,

$$\begin{aligned}\epsilon Re_\epsilon \left| \int_{\Omega^\epsilon} (v \cdot \nabla) u\, u \right| &\leq \epsilon Re_\epsilon \, \|v\|_{L^4(\Omega^\epsilon)} \, \|\nabla u\|_{L^2(\Omega^\epsilon)} \, \|u\|_{L^4(\Omega^\epsilon)} \\ &\leq C^2_{L^4,H^1} \, \epsilon^{\frac{3}{2}} Re_\epsilon \, \|\nabla u\|^2_{L^2(\Omega^\epsilon)} \, \|\nabla v\|_{L^2(\Omega^\epsilon)} \\ &\leq \frac{\mu_0}{3} \, \epsilon^2 \, \|\nabla u\|^2_{L^2(\Omega^\epsilon)}.\end{aligned}$$

(ii) For $\phi \in V^\epsilon$ we have $div\, \phi = 0$, therefore

$$\sum_j p_j \int_{\Sigma_j^\epsilon} \phi \cdot n_j = \sum_j (p_j - q) \int_{\Sigma_j^\epsilon} \phi \cdot n_j$$

for all constants $q \in \mathbb{R}$. We estimate

$$\begin{aligned}\left| \sum_j (p_j - q) \int_{\Sigma_j^\epsilon} \phi \cdot n_j \right| &\leq d \left(\sum_j |\Sigma_j^\epsilon| \, \|\phi\|^2_{L^2(\Sigma_j^\epsilon)} \right)^{1/2} \\ &\leq d\, \epsilon \left(max_j |\Sigma_j|^{\frac{1}{2}} \right) \sum_j \|\phi\|_{L^2(\Sigma_j^\epsilon)},\end{aligned}$$

since $\Sigma_j^\epsilon = \epsilon \Sigma_j$. Using the trace estimate (cf. Lemma B.3)

$$\|\phi\|_{L^2(\Sigma_j^\epsilon)} \leq C_0 \sqrt{\epsilon} \, \|\nabla \phi\|_{L^2(\Omega_j^\epsilon)},$$

we get the result.

(2) We define the mapping $T : B^\epsilon \to V^\epsilon$ by $T(v) := u$, where u is solution of the equation

$$a_v(u, \phi) + \sum_j p_j \int_{\Sigma_j^\epsilon} \phi \cdot n_j = 0 \quad \text{for all} \quad \phi \in V^\epsilon. \tag{2.5}$$

There exists a unique solution u due to (1) and Lax-Milgram's theorem.

(3) We now prove the existence of a fixed point for the mapping T:

(i) *T maps B^ϵ into itself*, i.e. $T(B^\epsilon) \subset B^\epsilon$.

(ii) *T is a contractive mapping on B^ϵ.*

Banach's fixed point theorem (also known as contraction mapping theorem) then yields the existence and uniqueness of the solution *in the ball B^ϵ*.

Proof. (i) We estimate the L^2-norm of $\nabla T(v)$:

$$\begin{aligned}\|\nabla T(v)\|^2_{L^2(\Omega^\epsilon)} = \|\nabla u\|^2_{L^2(\Omega^\epsilon)} &\le \frac{3}{2\,\mu_0\,\epsilon^2}\, a_v(u,u)\\ &\le \frac{3}{2\,\mu_0}\,\epsilon^{-\frac{1}{2}}\, d\,\left(max_j|\Sigma_j|^{\frac{1}{2}}\right)\, C_0\,\|\nabla T(v)\|_{L^2(\Omega^\epsilon)}\\ \Rightarrow \|\nabla T(v)\|_{L^2(\Omega^\epsilon)} &\le \frac{3}{2\,\mu_0}\,\epsilon^{-\frac{1}{2}}\, d\,\left(max_j|\Sigma_j|^{\frac{1}{2}}\right)\, C_0.\end{aligned}$$

Therefore, T maps B^ϵ into itself if

$$\frac{3}{2\,\mu_0}\,\epsilon^{-\frac{1}{2}}\, d\,\left(max_j|\Sigma_j|^{\frac{1}{2}}\right)\, C_0 \le \frac{\mu_0}{3\,C^2_{L^4,H^1}}\,\epsilon^{\frac{1}{2}} Re_\epsilon^{-1},$$

or, equivalently, $Re_\epsilon \le C\epsilon$ where

$$C = 2\mu_0^2\left[9\,C^2_{L^4,H^1}\, d\,\left(max_j|\Sigma_j|^{\frac{1}{2}}\right)\, C_0\right]^{-1},$$

depending in particular on the viscosity μ_0 and the given pressures p_j, $j = 1, ..., N$.

(ii) Let $v, w \in B^\epsilon$ and $T(v), T(w)$ the corresponding solution of (2.5). We then estimate as follows:

$$\begin{aligned}&\mu_0\,\epsilon^2\|\nabla(T(v)-T(w))\|^2_{L^2(\Omega^\epsilon)}\\ &= -\epsilon Re_\epsilon\int_{\Omega^\epsilon}((v-w)\cdot\nabla)T(v)\,(T(v)-T(w))\\ &\qquad\qquad - \epsilon Re_\epsilon\int_{\Omega^\epsilon}(w\cdot\nabla)(T(v)-T(w))\,(T(v)-T(w))\\ &\le C^2_{L^4,H^1}\,\epsilon^{\frac{3}{2}} Re_\epsilon\Big(\|\nabla(v-w)\|_{L^2(\Omega^\epsilon)}\,\|\nabla T(v)\|_{L^2(\Omega^\epsilon)}\,\|\nabla(T(v)-T(w))\|_{L^2(\Omega^\epsilon)}\\ &\qquad\qquad + \|\nabla w\|_{L^2(\Omega^\epsilon)}\,\|\nabla(T(v)-T(w))\|^2_{L^2(\Omega^\epsilon)}\Big)\\ &\le \frac{\mu_0}{3}\epsilon^2\left(\|\nabla(v-w)\|_{L^2(\cdot)}\|\nabla(T(v)-T(w))\|_{L^2(\cdot)} + \|\nabla(T(v)-T(w))\|^2_{L^2(\cdot)}\right),\end{aligned}$$

using $w, T(v) \in B^\epsilon$ in the last estimate.

Therefore

$$\|\nabla(T(v)-T(w))\|_{L^2(\Omega^\epsilon)} \le \frac{1}{2}\,\|\nabla(v-w)\|_{L^2(\Omega^\epsilon)},$$

which concludes the proof.

∎

Remark 2.1. Even for *small data* (i.e. small Reynolds number Re_ϵ) we are only able to prove that the solution is unique in the ball B^ϵ. The radius of this ball increases for $\epsilon \to 0$, but there possibly could exist solutions with larger norms outside B^ϵ. We do not have the appropriate *a priori*-estimates to remove this deficiency.
In the *two-dimensional* situation the result can be improved: Theorem 2.1 holds for all $Re_\epsilon \le O(\epsilon^{1/2})$ and the factor $\epsilon^{1/2}$ in definition (2.4) of the ball B^ϵ cancels. This is due to the improvement of the power of ϵ in the embedding $H^1 \hookrightarrow L^4$ on Ω^ϵ, cf. Remark B.1.

We now turn to

Theorem 2.2. (Existence of the pressure) *There exists a pressure $p^\epsilon \in L^2(\Omega^\epsilon)$ such that equation $(2.2)_1$ holds in the sense of distributions. It is unique up to an additive constant.*
Furthermore, the boundary condition $p^\epsilon = p_j$ on Σ_j^ϵ holds in the dual space

$$H_n^{-1/2}(\Sigma_j^\epsilon) = (H_n^{1/2}(\Sigma_j^\epsilon))',$$

where

$$H_n^{1/2}(\Sigma_j^\epsilon) := \Big\{ \phi \in H^{1/2}(\Sigma_j^\epsilon)^3 : \phi \times n_j = 0 \Big\}.$$

Proof. The construction of the pressure is the same as in [MP3] (cf. Theorem 2 therein). We briefly sketch the main steps.

As in the case of Dirichlet boundary conditions there exists $p^\epsilon \in L^2(\Omega^\epsilon)$ such that

$$-\mu_0 \epsilon^2 \Delta v^\epsilon + \epsilon Re_\epsilon (v^\epsilon \cdot \nabla) v^\epsilon = -\nabla p^\epsilon \text{ in } (V^\epsilon)'.$$

We define

$$Z^\epsilon := \Big\{ (u,q) \in V^\epsilon \times L^2(\Omega^\epsilon) : \sigma := -\mu_0 \epsilon^2 \nabla u + qI \in L^2(\Omega^\epsilon)^{3\times 3}, \;\; div\, \sigma \in L^{6/5}(\Omega^\epsilon)^3 \Big\}.$$

For $\phi \in H_n^{1/2}(\Sigma_j^\epsilon)$ $(j = 1, ..., N)$, there exists an extension $\tilde{\phi} \in H^1(\Omega^\epsilon)$, such that $\tilde{\phi}|_{\Sigma_j^\epsilon} = \phi$ and $\tilde{\phi}|_{\partial\Omega^\epsilon \setminus \Sigma_j^\epsilon} = 0$. Then we can define a normal trace operator $tr : Z^\epsilon \to H_n^{-1/2}(\Sigma_j^\epsilon)$, $tr(\sigma) := n_j \cdot \sigma n_j$, characterized by

$$\langle tr(\sigma), \phi \cdot n_j \rangle_{H_n^{-1/2}, H_n^{1/2}} = \int_{\Omega^\epsilon} div\, \sigma\, \tilde{\phi} + \int_{\Omega^\epsilon} \sigma \nabla \tilde{\phi}.$$

In addition, the following estimate holds:

$$\|tr(\sigma)\|_{H_n^{-1/2}(\Sigma_j^\epsilon)} \le C \left(\|\sigma\|_{L^2(\Omega^\epsilon)} + \|div\, \sigma\|_{L^{6/5}(\Omega^\epsilon)} \right).$$

Since in three dimensions the Sobolev embedding $H^1 \hookrightarrow L^r$ holds for $r \leq 6$, we require at least $div\,\sigma \in L^{6/5}(\Omega^\epsilon)$ in order to define the normal trace for $\sigma \in L^2(\Omega^\epsilon)$.

For $(u, q) \in Z^\epsilon$ we have $div\,u = 0$ and $u \times n_j = 0$, therefore we get $tr(\sigma) = n_j \cdot \sigma n_j = q$ since the boundary Σ_j^ϵ is flat. Taking $(u = v^\epsilon, q = p^\epsilon) \in Z^\epsilon$, we obtain

$$\sigma^\epsilon = -\mu_0 \epsilon^2 \nabla v^\epsilon + p^\epsilon I \in L^2(\Omega^\epsilon)^{3\times 3}$$

and $div\,\sigma^\epsilon = -\epsilon Re_\epsilon (v^\epsilon \cdot \nabla) v^\epsilon \in L^{3/2}(\Omega^\epsilon)^3$ and thus there exists the trace $tr(\sigma^\epsilon) = p^\epsilon$ in $H_n^{-1/2}(\Sigma_j^\epsilon)$. As usual, the variational formulation (2.3) then implies $p^\epsilon = p_j$ on Σ_j^ϵ.

□

2.3 Poiseuille flow and Kirchhoff's law

Our *aim* is to approximate the solution (v^ϵ, p^ϵ) of the Navier-Stokes system (2.2) by a Poiseuille flow far away from the junction. In particular, the meaning of the word *far* has to be specified. Roughly speaking, the distance from the junction has to be sufficiently large, otherwise Poiseuille flow is not appropriate. For quantitative results we refer to the *error estimates* proved in section 6.2. In this section we give the definition of the Poiseuille flow and discuss the Kirchhoff law for flow in a one-dimensional network.

In the pipes Ω_k of constant cross-section S_k and length L_k, the *Poiseuille profile* $w_k = w_k(y_2, y_3)$ is given by

$$\left\{ \begin{aligned} -\mu_0 \Delta w_k &= 1 \quad \text{in } S_k, \\ w_k &= 0 \quad \text{on } \partial S_k. \end{aligned} \right. \tag{2.6}$$

The corresponding flux through the k^{th} pipe is described by the *conductance*

$$c_k := \frac{\langle w_k \rangle}{L_k}, \text{ where } \langle w_k \rangle := \int_{S_k} w_k,$$

and the *pressure drop* in the pipe, cf. equation (2.8) below. We define q^0 as mean value of the outflow boundary values p_k, weighted with the conductances c_k, i.e.

$$q^0 := \frac{\sum_k c_k p_k}{\sum_k c_k}. \tag{2.7}$$

Poiseuille velocity and pressure then read

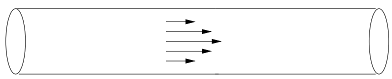

Fig. 2.3. Sketch of the parabolic Poiseuille velocity profile

$$V_k(\tilde{y}^k) := w_k(\tilde{y}^k)\frac{q^0 - p_k}{L_k}\, e_1^k,$$

$$P_k(y_1^k) := \frac{p_k - q^0}{L_k}\, y_1^k + q^0.$$

2.3.1 The Kirchhoff Law

We consider a network of *one-dimensional* pipes, as e.g. Fig. 2.4. It represents a *diverging-converging* network (or *arterial-venous* network, cf. [M]). Between the flux F_i and the pressure drop $(p_i - q_i)$ in the i^{th} pipe exists a linear relation, which can be seen as the one-dimensional analogon to Darcy's law:

$$F_i = c_i(p_i - q_i). \tag{2.8}$$

Here, p_i and q_i denote the pressure values at the end and node points of the network, respectively. *Kirchhoff's law* then states that $\sum_i F_i = 0$ in each node point, corresponding to the incompressibility of the fluid. The sum is thereby taken over all fluxes F_i which meet at the node i. Given the values p_i at the end points of the network, one can compute the unknown pressures q_i at the nodes by means of a linear system of equations. In the simplest case of only one branching node we obtain the weighted mean value q^0 as defined in (2.7).

In [M] a computational algorithm for the pressure and flow-rate distributions in tree-like networks is presented.

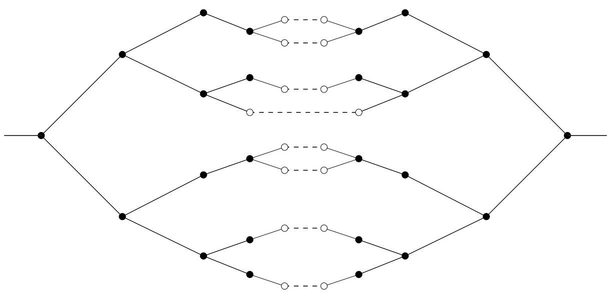

Fig. 2.4. Example of a diverging-converging network

The approximation of the Navier-Stokes flow in a domain of branching pipes by such an algebraic system of equations means a reduction of the three-dimensional geometry to a network of one-dimensional pipes by means of the Kirchhoff law. In the following, we analyze to what extend the geometry of the junction and the nonlinear character of the Navier-Stokes equations effect the Poiseuille flows in the pipes and thus Kirchhoff's law concerning the fluxes.

2.4 The pressure drop in the pipes

In this section we *formally* analyze the pressure drop which drives the Poiseuille flow in the pipes, in dependence of the diameter of the junction domain. It turns out that not only the diameter but also the flux inside the junction has an important influence. The detailed computation is given in appendix A.

We consider a junction Ω_0^δ connecting the pipes Ω_1^δ and Ω_2^δ (cf. Fig. 2.5). The *idea* now is to solve two Stokes problems for velocity ω_k and pressure π_k in the rescaled junction Ω_0 with the pressure boundary condition $\pi_k = \delta_{jk}$ on the pipe-junction interfaces γ_j, $j,k = 1,2$ (cf. 2.9). Due to the linearity of the Stokes problem we obtain the solution for prescribed constant pressure values q_j on γ_j as a linear combination of the functions ω_k and π_k. The scaled functions $(\omega^\delta, \pi^\delta)$ then solve the corresponding Stokes problem on the domain Ω_0^δ (cf. (A.2)).

$$
\left\{
\begin{array}{rcll}
-\Delta_y \omega_k + \nabla_y \pi_k &=& 0 & \text{in } \Omega_0, \\
div_y\, \omega_k &=& 0 & \text{in } \Omega_0, \\
\omega_k &=& 0 & \text{on } \Gamma_0, \\
\omega_k \times n_j &=& 0 & \text{on } \gamma_j, \\
\pi_k &=& \delta_{jk} & \text{on } \gamma_j,
\end{array}
\right.
\tag{2.9}
$$

In the pipes Ω_j^δ we assume a Poiseuille flow (V_j^δ, P_j^δ) such that $P_j^\delta = p_j$ on the in- and outflow boundaries of the pipes. In order to compute the unknown pressure values q_j on γ_j^δ we have to establish a relation between the Poiseuille flow in the pipes and the Stokes flow in the junction. This is done by two *physical assumptions*: The pressure has to be continuous on the interfaces γ_j^δ and the fluxes have to be balanced.

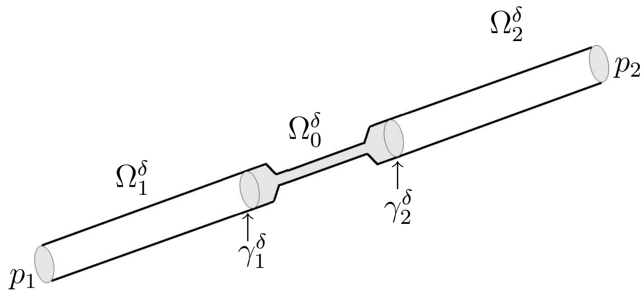

Fig. 2.5. A constricted junction

We then get the following system of linear equations:

$$
\sum_j F_{ij}^\delta\, q_j = -c_i p_i,
\tag{2.10}
$$

where

$$
F_{ij}^\delta := \int_{S_i} \omega_j \cdot n_j - \delta\, c_i \delta_{ij} \quad \text{for} \quad i,j = 1,2.
$$

The unknown pressure values q_j can be computed from (2.10) if the dimensionless parameter $\alpha := \frac{c_1 c_2}{c_1 + c_2} \frac{\delta}{F} \neq 1$:

$$q_j = \delta^{-1} \left(\frac{c_1 p_1 + c_2 p_2}{c_1 + c_2} - \alpha p_j \right) (1 - \alpha)^{-1}, \qquad j = 1, 2. \tag{2.11}$$

Here c_1, c_2 are the conductivities of the pipes (cf. section 2.3) and F is the flux through the junction Ω_0 if the pressure drop between the interfaces γ_1 and γ_2 is equal to 1, i.e. $F := \int_{\gamma_1} \omega_1 \cdot n_1$.

Expanding with respect to α, the *results of the formal computation* can be summarized as follows:

Lemma 2.1. *If $\alpha := \delta \frac{c_1 c_2}{c_1 + c_2} F^{-1} \ll 1$, then the pressure drop in each pipe is determined by the mean value*

$$q^0 := \frac{c_1 p_1 + c_2 p_2}{c_1 + c_2}.$$

If $\alpha \gg 1$, then the pressure drop is of order $O(\alpha^{-1})$:

$$\delta q_j - p_j = \alpha^{-1}(p_j - q^0) + O(\alpha^{-2}). \tag{2.12}$$

We conclude the discussion with *some remarks*:

(*i*) For $\alpha = 1$, the linear system of equations (2.10) has no solution since

$$det\,(F^{\delta}_{ij})_{ij} = \delta\ [c_1\, c_2\, \delta - (c_1 + c_2)F]$$

vanishes if $c_1 c_2\, \delta = (c_1 + c_2)F$.

(*ii*) A junction domain can be characterized by the *length*

$$\lambda := \frac{c_1 + c_2}{c_1 c_2} F.$$

The computation shows that the ratio $\alpha = \delta/\lambda$ is the important quantity in order to decide whether the pressure drop in the pipes is given by the mean value q^0 or not. For a given junction domain, the value of F is fixed by its geometric structure. The conductances c_j are determined by the diameter and length of the pipes. Therefore, for all $\alpha_0 > 0$, $\delta_0 = \alpha_0\, \lambda$, we have $\alpha \leq \alpha_0$ if $\delta \leq \delta_0$. In other words, *for sufficiently small diameter δ, the mean value q^0 is an appropriate approximation.*

(*iii*) There are different *limits* to be distinguished here: The first one is $\delta \to 0$ for fixed λ, corresponding to remark (*ii*). The second one is the limit $\lambda \to 0$ (or $F \to 0$, resp.) for a fixed diameter δ, describing, roughly speaking, the pinching of some parts of the junction domain: For a pressure drop from 1 to 0 in-between the outflow boundaries of the junction, the flux is reduced by deforming the *unscaled* junction Ω_0, such that it exerts increasing resistance on the fluid flow. Clearly, in this case the mean value approximation of the pressure drop is not adequate and the pressure q_j at the pinched subdomain tends to the given boundary pressure p_j at the end of the pipe (cf. (2.12)).
If $\delta \to 0$ and $\lambda \to 0$ simultaneously, the ratio α of these parameters determines the pressure drop in the pipes. In particular,

- if $\delta/\lambda = o(1)$, then the mean value-approximation is appropriate. The junction does not exert essential influence on the fluid flow in the pipes if its diameter δ is small enough.
- if $\lambda/\delta = o(1)$, the flow through the junction is highly reduced and the pressure drop in the pipes decreases to zero.

2.5 How to construct an approximation ?

We discuss two different approaches of building an approximation for the solution (v^ϵ, p^ϵ) of the Navier-Stokes problem (2.2). The first one is motivated by the formal computation performed in the previous section, describing the fluid flow using normalized pressure values on the pipe-junction interfaces. We briefly sketch the main ideas and the problems which occur therein in subsection 2.5.1.
In contrast, the second approach is based on the mathematical theory of Leray's problem, prescribing a Poiseuille velocity on the pipe-junction interfaces. Our subsequent analysis is based on this *Leray-Problem approach* (cf. subsection 2.5.2).

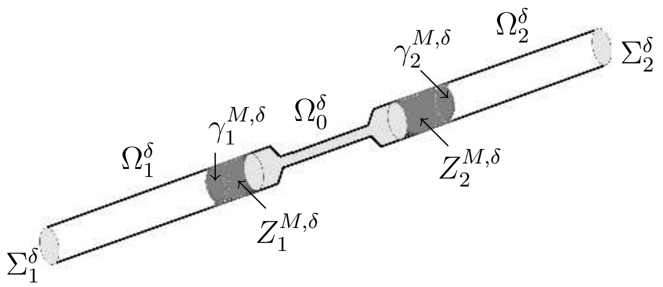

Fig. 2.6. The extended junction $\Omega^{M,\delta}$ (*shaded*)

To fix the main ideas, we refer as in the previous section to the following simplified situation: Two pipes Ω_1^δ and Ω_2^δ are connected by a junction domain Ω_0^δ. Fig. 2.6 shows the *extended junction* $\Omega^{M,\delta}$ consisting of the junction Ω_0^δ and the cylinders $Z_j^{M,\delta} \subset \Omega_j^\delta$ of length δM, $j = 1, 2$.

2.5.1 The normalized pressure approach

In our problem the pressure values on the outflow boundaries of the pipes are given, being of order $O(1)$. Assuming Poiseuille flow in the pipes, i.e. in particular a linear pressure drop in each pipe, we have pressure values $q_j = O(1)$ on the interfaces $\gamma_j^{0,\delta}$. The flow through the junction is influenced by its geometric structure. A pressure drop from 1 to 0 between the outflow boundaries $\gamma_j^{0,\delta}$ of the junction causes different fluxes for different junction domains. If e.g. some parts are pinched, then the flux is essentially reduced compared to the case of a larger diameter.

The main idea of this approach is to solve Stokes problems in the junction domain *with normalized pressures* (or *normal forces*, resp.) on the interfaces γ_j^M, $M \geq 0$, $k = 1, 2$ (cf. (A.1)):

$$
(2.13) \qquad \left\{ \begin{array}{rcll} -\Delta_y \omega_k + \nabla_y \pi_k & = & 0 & \text{in } \Omega^M, \\ div_y \, \omega_k & = & 0 & \text{in } \Omega^M, \\ \omega_k & = & 0 & \text{on } \Gamma^M, \\ -\nabla_y \omega_k n_j + \pi_k n_j & = & \delta_{jk} \, n_j & \text{on } \gamma_j^M, \quad j = 1, 2. \end{array} \right.
$$

This approach involves some *difficulties* concerning the transition from Poiseuille flow in the pipes to Stokes flow in the junction: By construction, the *normal force* is continuous on the interfaces γ_j^M, but a jump of velocity occurs there, which is a major obstacle in the construction of an appropriate approximation. We require the velocity to be a function in the space H^1, therefore a correction of these jumps is needed. Following the ideas of [JM2], *boundary layer problems* on infinite pipes including jumps of velocity have to be considered (cf. also [JM1] and [JMN]). The correction velocity tends exponentially to zero and the corresponding pressure tends to some stabilization constants at infinity. The problem coming up then is to construct a correction for these constants in such a way that no further jumps of velocity occur.

Besides these velocity and pressure corrections, we have to be aware of the fact that the linear system of equations (2.10) does not always have a solution. There are combinations of pipe conductivities c_j and junction geometries (represented by the flux value F, cf. section 2.4) for which the pressure values q_j do not exist (or become arbitrarily large which is physically impossible). This e.g. can happen if the diameter-to-length ratio δ of the pipes is *not small* but of order $O(1)$. It can be interpreted physically in such a way that the length of the pipes is not sufficiently large to fully develop a Poiseuille flow away from the junction. Regarding the computation of the previous section the assumption of Poiseuille flow is only adequate in case of sufficiently small or large values of the parameter α, cf. Lemma 2.1. This means that either the diameter-to-length-ratio of the pipes or the flux through the junction (e.g. due to constriction) is small; in both cases the velocity is small as well (with respect to viscosity of order $O(1)$).

Summarizing, the *normalized pressure approach* poses the following severe problems:

- ▷ Velocity jumps on the pipe-junction interfaces have to be corrected in order to get H^1-estimates.

- ▷ The pressure values q_j *cannot* be computed for any combination of pipes and junctions.

We now discuss the second approach which circumvents these difficulties. It is based on the theory of Leray's problem and carried out in detail in the following chapters.

2.5.2 The Leray-Problem approach

Our aim is to construct an approximation which is continuous in velocity in order to avoid the correction problems mentioned above. Therefore, we take the Poiseuille flow in the pipes as in-/outflow boundary condition for the Stokes problem in the junction. The main difficulty then is the following: We do not know the pressure drop (or flux, resp.) in each pipe, only the pressures at the end of the pipes are given. In this respect, dealing with velocity or flux boundary conditions is simpler. Regarding the computation in section 2.4, we assume the pressure drop in the pipes to be determined by the weighted mean value q^0 of the boundary pressures p_k, cf. equation (2.7).
From (2.11) we can expect this value to be an appropriate approximation if the diameter-to-length ratio δ is sufficiently small and the junction domain is not changed (i.e. the flux F is fixed) as δ tends to zero ($\alpha \ll 1$). We then use the results concerning *Leray's problem* on infinite junction domains, i.e. Stokes equations with a given Poiseuille velocity profile at infinity (cf. chapter 3). Its solution tends exponentially to Poiseuille flow, therefore an approximation can be build with exponentially decaying error terms. This is discussed in detail in the chapters 4 - 6, including the derivation of error estimates.
In this approach, a Poiseuille velocity is prescribed on the pipe-junction interfaces instead of the normal force as it is the case in the normalized pressure approach. This leads then to a jump of the *normal force* (instead of a jump of velocity) which can be made exponentially small due to the properties of the Leray solution. Since the velocity is continuous on the pipe-junction interfaces, we do not need further corrections in order to get H^1-estimates.

We proceed as follows:

- ▷ We analyze Leray's problem in chapter 3, in particular the exponential decay of its solution to Poiseuille flow.

- ▷ We then introduce the corresponding *junction problems*, i.e. Stokes problems with Poiseuille velocity on the pipe-junction interfaces (section 4.1).

- ▷ In the next part of our analysis, we show that the solution of Leray's problem can be approximated by the solution of the corresponding junction problem up to an exponentially decaying error (section 4.2).

▷ Finally, these results allow to build an approximation for the solution of the Navier-Stokes system (2.2) and to prove adequate error estimates (chapters 5 and 6).

Chapter 3

Leray's problem and related equations on infinite domains

In this chapter we summarize the theory of Leray's problem on the domain Ω^∞, consisting of the junction Ω_0 and infinitely long pipes Ω_k^∞, $k = 1, ..., N$, of constant cross-sections S_k (cf. Definition 2.2). We thereby follow [G], chapter VI.1 and VI.2. Furthermore, we analyze a related Stokes problem extending the results of [G]. For simplicity we assume the domain Ω^∞ to be of class C^∞.

3.1 Leray's problem

In the domain Ω^∞ we consider a Stokes problem with asymptotic Poiseuille velocities V_k carrying the fluxes F_k, such that the total flux $\sum_{k=1}^N F_k$ is zero. This type of problem is called *Leray's problem* in the literature:

$$(3.1) \qquad \begin{cases} -\mu_0 \Delta\omega + \nabla\pi &= 0 \qquad \text{in } \Omega^\infty, \\ div\, \omega &= 0 \qquad \text{in } \Omega^\infty, \\ \omega &= 0 \qquad \text{on } \partial\Omega^\infty, \\ \lim\limits_{x_1^k \to \infty} \omega(x) &= V_k(x) \quad \text{in } \Omega_k^\infty. \end{cases}$$

We first discuss existence and uniqueness of the solution.

3.1.1 Existence and uniqueness of the solution

The solution of (3.1) can be written in the form $\omega = u + a$, where $a \in H^2_{loc}(\overline{\Omega^\infty})$ is a solenoidal extension of the Poiseuille velocity fields V_k (cf. subsection 3.1.3) and the function $u \in H_0^1(\Omega^\infty)$, $div\, u = 0$, is the solution of the equation

$$(3.2) \qquad \int_{\Omega^\infty} \nabla u \nabla \phi = \int_{\Omega^\infty} \Delta a\, \phi \quad \text{for all } \phi \in C_0^\infty(\Omega^\infty),\ div\, \phi = 0.$$

The extension a is constructed in such a way that $a = V_k$ in the pipes $\Omega^\infty_{k,R}$ defined as follows:

$$\Omega^\infty_{k,R} := \{x \in \Omega^\infty_k : x^k_1 > R\}, \quad \text{for some } R > 0.$$

Therefore we have

$$(3.3) \qquad \begin{cases} -\mu_0 \Delta u + \nabla \tau_k &= 0 \quad \text{in } \Omega^\infty_{k,R}, \\ div\, u &= 0 \quad \text{in } \Omega^\infty_{k,R}, \\ u &= 0 \quad \text{on } \partial\Omega^\infty_{k,R} \setminus \Sigma_{k,R}, \\ \displaystyle\int_{\Sigma_{k,R}} u \cdot n &= 0, \end{cases}$$

where $\tau_k = \pi - C_k\, x^k_1$ and $\Sigma_{k,R} := \{x \in \Omega^\infty_k : x^k_1 = R\}$. The constant C_k is given by the flux of the Poiseuille flow, namely $F_k = -\, C_k \displaystyle\int_{S_k} w_k$ (cf. section 2.3).

We now state the existence and uniqueness result established in [G] (Theorem VI.1.2):

Theorem 3.1. *For any Poiseuille velocities V_k, satisfying the compatibility condition $\sum_{k=1}^N F_k = 0$, problem (3.1) admits a unique solution $\omega \in C^\infty(\overline{\Omega'})$, $\pi \in C^\infty(\overline{\Omega'})$ for every bounded subset Ω' of Ω^∞. Furthermore, for each multi-index α with $|\alpha| \geq 0$,*

$$|D^\alpha u(x)| \to 0 \quad as \quad |x| \to \infty \quad in\ \Omega^\infty_k$$

and

$$|D^\alpha \nabla \tau_k(x)| \to 0 \quad as \quad |x| \to \infty \quad in\ \Omega^\infty_k.$$

In other words, the velocity ω and the pressure gradient $\nabla\pi$, together with all their derivatives of arbitrary order, tend to the corresponding Poiseuille flow $(V_k,\, C_k\, e^k_1)$ in Ω^∞_k as $|x| \to \infty$.

This theorem provides an estimate for the pressure gradient, but for later purpose we also need the decay property of the pressure function itself. Since $\nabla\tau_k$ tends to zero in Ω^∞_k for $|x| \to \infty$, we can deduce that τ_k itself stabilizes to some constant τ^∞_k using the *mean value theorem*, cf. [G], Remark VI.2.1. These constants τ^∞_k, $k = 1, ..., N$, are uniquely determined up to one additional constant which can be chosen such that e.g. $\tau^\infty_1 = 0$. But in general the remaining constants are non-zero, i.e. $\tau^\infty_j \neq 0$, $j = 2, ..., N$. Therefore, they have to be taken into account in the construction of an approximation for the junction flow (cf. the discussion in section 5.2). For the proof of the pressure decay we also refer to [MP1] (Theorem 5.1 therein) where Leray's problem is generalized to non-newtonian fluids.

We summarize:

Corollary 3.1. (Decay of the pressure) *There exist constants $\tau^\infty_k \in \mathbb{R}$, $k = 1, ..., N$, such that $|\tau_k(x) - \tau^\infty_k| \to 0$ as $|x| \to \infty$.*

We now establish the exponential decay of the solution to Poiseuille flow.

3.1.2 Exponential decay to Poiseuille flow

The next theorem states the *exponential* decay of Leray's solution to Poiseuille flow (cf. [G], Theorem VI.2.2). This result is the main key to our further analysis.

We first define the notion of a *regular solution* of problem (3.3).

Definition 3.1. A solution (u, τ_k) of (3.3) is called a *regular solution*, if it is infinitely differentiable in the closure of any bounded subset of $\Omega^\infty_{k,R}$.

Since u vanishes on the boundary of any cross-section Σ_k of Ω^∞_k, there exists a constant $c_P = c_P(\Sigma_k) > 0$ such that the *Poincaré inequality* holds on Σ_k:

$$\|u\|^2_{L^2(\Sigma_k)} \leq c_P \,\|\nabla u\|^2_{L^2(\Sigma_k)}. \tag{3.4}$$

We state the main result on the decay of Leray's solution:

Theorem 3.2. *Let (u, τ_k) be a regular solution of (3.3) satisfying*

$$\liminf_{x_1^k \to \infty} \left(\int_0^{x_1^k} \int_{\Sigma_k} \nabla u \cdot \nabla u \right) e^{-\alpha_k x_1^k} = 0,$$

where

$$\alpha_k := \left[\left(c_0 + \frac{1}{2} \right) \sqrt{c_P} \right]^{-1}.$$

Then $\|\nabla u\|_{L^2(\Omega^\infty_{k,R})} < \infty$ and for all $r > 0$ and $m \geq 0$ the following inequality holds:

$$\|u\|_{H^{m+2}(\Omega^\infty_{k,R+r+1})} + \|\nabla \tau\|_{H^m(\Omega^\infty_{k,R+r+1})} \leq C_1 \,\|u\|_{H^1(\Omega^\infty_{k,R})}\, e^{-\sigma_k r}, \tag{3.5}$$

with

$$(C_1)^2 = c(m, \Sigma_k)\, \frac{(c_0^2+2)^{1/2}}{(c_0^2+2)^{1/2} - c_0}, \quad \sigma_k = \frac{1}{2c_P} \left(\sqrt{c_0^2 + 2} - c_0 \right). \tag{3.6}$$

The constant $c_0 = c_0(k)$ is specified by the following problem on the domain

$$\Omega^k_{s,s+1} := \Omega^\infty_{k,R} \cap \left\{ x : s < x_1^k < s+1 \right\}, \; s \geq R,$$

(cf. [G], Proof of Theorem VI.2.1):

$$\begin{cases} \nabla \cdot w = u \cdot e_1^k \;\; \text{in} \;\; \Omega^k_{s,s+1}, \\ w \in H^1_0(\Omega^k_{s,s+1}), \\ \|\nabla w\|_{L^2(\Omega^k_{s,s+1})} \leq c_0 \,\|u \cdot e_1^k\|_{L^2(\Omega^k_{s,s+1})}. \end{cases} \tag{3.7}$$

In particular, c_0 is independent of s.

Clearly, if (ω, π) is the solution of Leray's problem and a is an extension of the Poiseuille flows V_k, then $(u := \omega - V_k,\ \tau_k := \pi - C_k\, x_1^k)$ is a regular solution of (3.3) and satisfies the assumption of the theorem.

From inequality (3.5) we obtain the pointwise exponential decay of (u, τ_k) from the Sobolev embeddings on the semi-infinite cylinder Ω_k^∞. Since Ω_k^∞ can be divided into cylinders $Z_{k,s} := \{x \in \Omega_k^\infty : s < x_1^k < s+1\}$, $s \geq 0$, the Sobolev embedding can be applied for $H^m(Z_{k,s})$, $m \geq 0$, with constants *independent* of s (due to the constant cross-section, the estimates are invariant under translation of the x_1^k-coordinate.)

Corollary 3.2. *For every $x \in \Omega_{k,R}^\infty$ with $x_1^k \geq R+1$ and every $|\alpha| = m \geq 0$ the following inequality holds:*

$$|D^\alpha u(x)| + |D^\alpha \nabla \tau_k(x)| \leq C_2\, \|u\|_{H^1(\Omega_{k,R}^\infty)}\, e^{-\sigma_k (x_1^k - R - 1)}. \tag{3.8}$$

The constant C_2 only depends on m and the cross-section Σ_k of Ω_k^∞.

Regarding inequality (3.8), it remains to establish an estimate for the H^1-norm of u on the domain $\Omega_{k,R}^\infty$. We cannot obtain this estimate directly from problem (3.3), since we do not know the trace of u on $\Sigma_{k,R}$. Therefore, we have to derive from (3.2) an estimate on the whole domain Ω^∞, which clearly dominates the norm on $\Omega_{k,R}^\infty$. In order to do this, we first take a closer look on the extension a of the Poiseuille velocity fields V_k.

We briefly sketch the main steps of the construction, following [G].

3.1.3 Construction of the extended Poiseuille velocity

(i) The Poiseuille velocity V_k in the k^{th} pipe is cut off at some distance $R > 0$ in the coordinate system of the corresponding pipe. We set

$$V := \sum_k \eta_k V_k$$

with smooth cut-off functions η_k.
Then $V \in C^\infty(A_R)$, where $A_R := \Omega^\infty \setminus (\bigcup_k \overline{\Omega}_{k,R}^\infty)$ consists of the junction domain and the shortened pipes of length R. Without loss of generality we can set $R = 1$.

(ii) We consider the problem

$$\begin{cases} \nabla \cdot w = -\nabla \cdot V \ \text{ in } A_R, \\ w \in H_0^2(A_R), \\ \|w\|_{H^2(A_R)} \leq c\, \|\nabla \cdot V\|_{H^1(A_R)}. \end{cases} \tag{3.9}$$

We have $\nabla \cdot V \in H_0^1(A_R)$ and $\int_{A_R} \nabla \cdot V = 0$, thus the theory of the divergence problem yields existence of a solution w (cf. [G], ch. III.3). Extending w by zero outside A_R, we get $w \in H^2(\Omega^\infty)$.
Then $a := V + w$ is a solenoidal extension of the Poiseuille velocities in $H^2_{loc}(\overline{\Omega}^\infty)$.

(*iii*) *Estimate for the constant c in (3.9):* The domain A_R is bounded and locally lipschitzian, therefore it admits the following decomposition:

$$A_R = \bigcup_{i=1}^{I} A_R^i, \; I \geq 1,$$

where each A_R^i is *star-shaped* with respect to some open ball B_i of radius r_i with $\overline{B}_i \subset A_R^i$. This property holds in general even for domains only satisfying a cone condition (cf. [G], Remark III.3.3).
The smallest radius of these balls is denoted by $r_{min} := min_i \, r_i$. Then the following estimate holds (cf. [G], Lemma III.3.2 and III.3.4):

$$c \leq C_0 \left(\frac{diam(A_R)}{r_{min}} \right)^n \left(1 + \frac{diam(A_R)}{r_{min}} \right), \tag{3.10}$$

the constant $C_0 > 1$ depending on the space dimension $n \geq 2$ and the decomposition of A_R in star-shaped subdomains.

We finally obtain the following estimate:

Corollary 3.3. *There exists a constant $C > 0$, depending only on the Poincaré constant of Ω^∞, such that*

$$\|u\|_{H^1(\Omega^\infty)} \leq C \left[1 + C_0 \left(\frac{diam(A_R)}{r_{min}} \right)^n \left(1 + \frac{diam(A_R)}{r_{min}} \right) \right] \|V\|_{H^2(A_R)}.$$

Proof. We analyze the right-hand side of (3.2):

$$\int_{\Omega^\infty} \Delta a \, \phi = \int_{A_R} \Delta a \, \phi + \sum_k \int_{\Omega^\infty_{k,R}} \Delta a \, \phi \tag{3.11}$$

for all $\phi \in C_0^\infty(\Omega^\infty)$, $div \, \phi = 0$ (by definition of A_R).

In the pipes $\Omega^\infty_{k,R}$ the extension a coincides with the Poiseuille velocity V_k. Since $V_k = -w_k \, C_k \, e_1^k$ and $-\mu_0 \Delta w_k = 1$ (cf. section 2.3) we get

$$\int_{\Omega^\infty_{k,R}} \Delta a \, \phi = \frac{C_k}{\mu_0} \int_{\Omega^\infty_{k,R}} \phi \cdot e_1^k = \frac{C_k}{\mu_0} \int_R^\infty \int_{\Sigma_k} \phi \cdot e_1^k = 0,$$

due to the fact that ϕ carries no flux.
The solution u of (3.2) in particular is an element of the completion of the set

$$\{\phi \in C_0^\infty, div\, \phi = 0\}$$

in the seminorm of $H^1(\Omega^\infty)$. Therefore, from (3.11) and the Poincaré inequality on Ω^∞ (with the constant $C_P = C_P(\Omega^\infty) > 0$) we obtain the estimate

$$\|\nabla u\|_{L^2(\Omega^\infty)} \leq C_P \, \|\Delta a\|_{L^2(A_R)}.$$

From (3.9), (3.10) and the Poincaré inequality we finally get the result.

□

We conclude this section summarizing the main results on Leray's problem from the subsections 3.1.1 - 3.1.3.

Theorem 3.3. *Problem (3.1) has a unique weak solution (ω, π), which is infinitely differentiable on any bounded subset of Ω^∞. It decays pointwise exponentially to Poiseuille flow:*

$$|D^\alpha (\omega(x) - V_k(x))| + \left|D^\alpha \nabla \left(\pi(x) - C_k\, x_1^k\right)\right| \leq C_L\, e^{-\sigma_k x_1^k} \tag{3.12}$$

for every $x \in \Omega^\infty_{k,R}$ with $x_1^k \geq R+1$ $(k = 1, ..., N)$ and every $|\alpha| = m \geq 0$.

The constants σ_k are specified in Theorem 3.2, cf. (3.6), and there exists a constant $C = C(m, R, \sigma_j|_{j=1}^N, \Sigma_j|_{j=1}^N, C_P)$ such that

$$C_L \leq C\, max_j |F_j| \left[1 + C_0 \left(\frac{diam(A_R)}{r_{min}}\right)^3 \left(1 + \frac{diam(A_R)}{r_{min}}\right)\right]. \tag{3.13}$$

In particular, there exists $\tilde{C}_L > 0$ such that

$$|\omega(x) - V_k(x)| \,+\, |\nabla\omega(x) - \,\nabla V_k(x)| \,+\, |\pi(x) - C_k\, x_1^k - \tau_k^\infty| \leq \tilde{C}_L\, e^{-\sigma_k x_1^k} \tag{3.14}$$

for every $x \in \Omega^\infty_{k,R}$ with $x_1^k \geq R+1$, where the constant $\tilde{C}_L$ admits an estimate of type (3.13). The asymptotic pressure profile is linear, shifted by the stabilization constant τ_k^∞ (cf. Corollary 3.1).

Remark 3.1. In the following, all constants which allow an estimate of type (3.13) are denoted by C_L. For simplicity, we define

$$\sigma_L := min_k\, \sigma_k \tag{3.15}$$

and replace σ_k (cf. (3.6)) by σ_L in the corresponding decay estimates.

3.2 A generalization of Leray's problem

We extend the results of the previous section to Stokes flow in infinite bifurcation domains Ω^∞ driven by a force $f \in L^2(\Omega^\infty)$ having some decay properties (cf. the *assumptions* below):

$$(3.16) \qquad \begin{cases} -\mu_0 \Delta \tilde{\omega} + \nabla \tilde{\pi} &= f \quad \text{in } \Omega^\infty, \\ div\, \tilde{\omega} &= 0 \quad \text{in } \Omega^\infty, \\ \tilde{\omega} &= 0 \quad \text{on } \partial\Omega^\infty, \\ \int_\Sigma \tilde{\omega} \cdot n &= 0 \quad \text{for any cross-section } \Sigma \text{ of } \Omega^\infty, \\ \lim_{|x|\to\infty} \tilde{\omega}(x) &= 0 \quad \text{in } \Omega_k^\infty,\ k = 1, ..., N. \end{cases}$$

Extending the Poiseuille velocities V_k as shown in the previous section, we can rewrite Leray's problem (3.1) in the form (3.16) for an appropriate function f. In this respect, problem (3.16) is a generalization of Leray's problem (3.1).

3.2.1 Existence and regularity of the solution

For the present section we fix the following **assumptions**:

(*i*) The domain Ω^∞ has a smooth boundary.

(*ii*) *Regularity* of the force f:

$$(3.17) \qquad f \in C^\infty(\overline{\Omega'}) \text{ for any bounded subset } \Omega' \subset \Omega^\infty.$$

(*iii*) *Decay property* of the force f:

There exist constants $\sigma_f, C_f, R > 0$ such that

$$(3.18) \qquad |D^\alpha f(x)| \le C_f\, e^{-\sigma_f x_1^k}$$

for all $x \in \Omega^\infty$, $x_1^k \ge R$, $k = 1, ..., N$, and every $m = |\alpha| \ge 0$, the constant C_f possibly depending on m and R.

Proposition. From assumption (*iii*) we particularly obtain $f \in H^m(\Omega^\infty)$ and $\tilde{C}_f > 0$ such that

$$(3.19) \qquad \|f\|_{H^m(\Omega_{k,s}^\infty)} \le \tilde{C}_f\, e^{-\sigma_f s}$$

for all $m \ge 0$ and $s \ge R$, where $\Omega_{k,s}^\infty := \{x \in \Omega_k^\infty : x_1^k > s\}$.

Proof. Taking into account the definition of the H^m-norm and the decay property (3.18), we have

$$\begin{aligned}
\|f\|^2_{H^m(\Omega^\infty_{k,s})} &= \sum_{0\le|\alpha|\le m} \|D^\alpha f\|^2_{L^2(\Omega^\infty_{k,s})} \\
&\le |\Sigma_k| \left(\sum_{0\le|\alpha|\le m} (C_f(|\alpha|))^2 \right) \int_s^\infty e^{-2\sigma_f x_1^k}\, dx_1^k \\
&= \frac{|\Sigma_k|}{2\sigma_f} \left(\sum (C_f(|\alpha|))^2 \right) e^{-2\sigma_f s} \quad \text{for all } s \ge R.
\end{aligned} \tag{3.20}$$

□

In the following we discuss

- Existence, uniqueness, and smoothness of the solution, and
- Exponential decay of the solution.

We give detailed proofs in case of remarkable differences to those of the analogous results for Leray's problem. In particular, we show exponential decay of the solution generalizing the ideas from [G].

As in the case of Leray's problem analyzed in section 3.1 we have

Theorem 3.4. (Existence and uniqueness of weak solution) *There exists a unique function $\tilde\omega \in H^1_0(\Omega^\infty)$, (weakly) divergence-free in Ω^∞, such that*

$$\mu_0 \int_{\Omega^\infty} \nabla\tilde\omega\, \nabla\phi = \int_{\Omega^\infty} f\,\phi \;\; \textit{for all} \;\; \phi \in C_0^\infty(\Omega^\infty),\ \textit{div}\,\phi = 0. \tag{3.21}$$

Additionally, there exists a pressure function $\tilde\pi \in L^2_{loc}(\Omega^\infty)$ (unique up to an additive constant) such that

$$\mu_0 \int_{\Omega^\infty} \nabla\tilde\omega\, \nabla\psi = \int_{\Omega^\infty} \tilde\pi\, \nabla\cdot\psi + \int_{\Omega^\infty} f\,\psi \;\; \textit{for all} \;\; \psi \in C_0^\infty(\Omega^\infty). \tag{3.22}$$

In fact, the weak solution $\tilde\omega$ and the corresponding pressure $\tilde\pi$ are smooth, since the domain and the data are assumed to be smooth.

Theorem 3.5. (Regularity) *Let $(\tilde\omega, \tilde\pi)$ be the weak solution of (3.16) as specified in Theorem 3.4. Then $\tilde\omega, \tilde\pi \in C^\infty(\overline{\Omega}')$ for any bounded subset $\Omega' \subset \Omega^\infty$.*

3.2.2 Exponential decay of the solution

In order to establish exponential decay, we first note that the solution actually decays pointwise to zero. From [G], Lemma VI.1.2, we conclude

Theorem 3.6. (Decay property) *The velocity $\tilde{\omega}$, together with all its derivatives of arbitrary order, asymptotically tends to zero:*

$$|D^\alpha \tilde{\omega}(x)| \to 0 \quad as \quad |x| \to \infty \quad in \quad \Omega_k^\infty \tag{3.23}$$

for each multi-index α with $|\alpha| \geq 0$.

The same is true for the pressure gradient $\nabla\tilde{\pi}$, i.e.

$$|D^\alpha \nabla\tilde{\pi}(x)| \to 0 \quad as \quad |x| \to \infty \quad in \quad \Omega_k^\infty. \tag{3.24}$$

In analogy to Corollary 3.1 we have the pointwise decay of $\tilde{\pi}$ to possibly different stabilization constants in each pipe Ω_k^∞.

The following result concerning differential inequalities generalizes Lemma VI.2.2 from [G]. It is the essential tool in order to show that the decay is exponential.

Lemma 3.1. *Let $y \in C^0[0,\infty) \cap C^1(0,\infty)$, $y(t) \geq 0$ for all $t \in [0,\infty)$ and $\lim_{t\to\infty} y(t) = 0$. Furthermore, y satisfies the integro-differential inequality*

$$y'(t) + a\int_t^\infty y(s)ds \leq b\,y(t) + c\,e^{-dt} \quad for\ all \quad t \in (0,\infty), \tag{3.25}$$

with $a > 0$ and $b, c, d \geq 0$.

Let $\delta := \frac{1}{2}\left(b + \sqrt{b^2 + 4a}\right)$ and $\sigma := \delta - b$.

(i) If $d \neq \sigma$, then

$$y(t) \leq \left(\frac{\sigma+\delta}{\sigma}y(0) \,+\, c\left(\frac{\delta}{\sigma(d+\delta)} - \frac{1}{\sigma - d}\right)\right)e^{-\sigma t} + \frac{c}{\sigma - d}\,e^{-dt} \tag{3.26}$$

for all $t \in (0,\infty)$.

(ii) If $d = \sigma$, then

$$y(t) \leq \left(\frac{\sigma+\delta}{\sigma}y(0) + \frac{c\,\delta}{\sigma(\sigma+\delta)} + ct\right)e^{-\sigma t} \tag{3.27}$$

for all $t \in (0,\infty)$.

Proof. (*i*) For $d \neq \sigma$ and $\beta \leq \infty$, we set

$$F(t) := \psi(t) + \delta \int_t^\beta e^{-b(t-s)}\, \psi(s) ds - \frac{c}{\sigma - d}\, e^{-(b+d)t}, \tag{3.28}$$

where

$$\psi(t) := y(t)\, e^{-bt}. \tag{3.29}$$

With this change of variable, inequality (3.25) reads

$$\psi'(t) + a \int_t^\beta \psi(s)\, e^{-b(t-s)} ds \leq c\, e^{-(b+d)t}.$$

From (3.28) we obtain by differentiation with respect to t

$$\begin{aligned} F'(t) + \delta F(t) = & \psi'(t) + a \int_t^\beta \psi(s)\, e^{-b(t-s)} ds \\ & + (\delta^2 - b\delta - a) \int_t^\beta \psi(s)\, e^{-b(t-s)} ds - c\, e^{-(b+d)t}. \end{aligned}$$

Since δ is chosen such that

$$\delta^2 - b\delta - a = 0,$$

we have $F'(t) + \delta F(t) \leq 0$. Integrating this differential inequality, we get

$$F(t) \leq F(0)\, e^{-\delta t}. \tag{3.30}$$

Replacing F in (3.30) by its definition (3.28) and taking into account the change of variable (3.29), we obtain

$$y(t) + \delta \int_t^\beta y(s) ds \leq F(0) e^{-\sigma t} + \frac{c}{\sigma - d}\, e^{-dt}. \tag{3.31}$$

Next we establish an estimate for $F(0)$ in terms of $y(0)$: From (3.31) we get

$$\begin{aligned} -\frac{d}{dt}\left[e^{-\delta t} \int_t^\beta y(s) ds\right] &= \left(y(t) + \delta \int_t^\beta y(s)\right) e^{-\delta t} \\ &\leq F(0)\, e^{-(\sigma+\delta)t} + \frac{c}{\sigma - d}\, e^{-(d+\delta)t}, \end{aligned}$$

and integration from 0 to β yields

$$\begin{aligned} \int_0^\beta y(s) ds \leq & \int_0^\beta \left(F(0)\, e^{-(\sigma+\delta)t} + \frac{c}{\sigma - d}\, e^{-(d+\delta)t}\right) dt \\ \leq & F(0) \frac{1 - e^{-(\sigma+\delta)\beta}}{\sigma + \delta} + \frac{c}{(\sigma - d)\,(d + \delta)} \left(1 - e^{-(d+\delta)\beta}\right). \end{aligned}$$

For the special case $\beta = \infty$ we thus obtain

$$\int_0^\infty y(s)ds \leq \frac{F(0)}{\sigma+\delta} + \frac{c}{(\sigma-d)\,(d+\delta)}. \tag{3.32}$$

Returning to (3.28) we have for $t = 0$:

$$F(0) = y(0) + \delta \int_0^\infty y(s)ds - \frac{c}{\sigma - d}.$$

Using inequality (3.32), $F(0)$ can be estimated as follows (note that $\sigma > 0$ since $a > 0$ by assumption):

$$F(0) \leq \frac{\sigma+\delta}{\sigma} y(0) + \, c\left(\frac{\delta}{\sigma(d+\delta)} - \frac{1}{\sigma - d}\right).$$

Introducing this estimate for $F(0)$ in (3.31) then yields inequality (3.26), since y is a non-negative function on $(0,\infty)$.

(ii) In case of $d = \sigma$, we set

$$\tilde{F}(t) := \psi(t) + \delta \int_t^\beta e^{-b(t-s)}\, \psi(s)ds - c\,t\,e^{-\delta t}.$$

Making the same computations as above, we obtain

$$\tilde{F}'(t) + \delta\tilde{F}(t) \leq 0,$$

and by integration

$$y(t) + \delta \int_t^\beta y(s)ds \leq \left(\tilde{F}(0) + ct\right) e^{-\sigma t}. \tag{3.33}$$

We estimate $\tilde{F}(0)$ (for $\beta = \infty$) in the same way as above, first establishing the inequality

$$\begin{aligned}\int_0^\infty y(s)ds &\leq \frac{\tilde{F}(0)}{\sigma+\delta} + c\int_0^\infty t\,e^{-(\sigma+\delta)t}dt \\ &= \frac{\tilde{F}(0)}{\sigma+\delta} + \frac{c}{(\sigma+\delta)^2}.\end{aligned}$$

Thus we get

$$\tilde{F}(0) \leq \frac{\sigma+\delta}{\sigma} y(0) + \frac{c\,\delta}{\sigma(\sigma+\delta)},$$

and finally inequality (3.27) follows by inserting this estimate into (3.33).

□

Remark 3.2. If inequality (3.25) holds for all $t \geq R$ with some $R > 0$ (instead of $t > 0$) we introduce the change of variable $\tilde{t} := t - R$, $\tilde{t} \geq 0$, and define $\tilde{y}(\tilde{t}) := y(\tilde{t} + R)$ for which

$$\tilde{y}'(\tilde{t}) + a \int_{\tilde{t}}^{\infty} \tilde{y}(\tilde{s})\, d\tilde{s} \leq b\, \tilde{y}(\tilde{t}) + \tilde{c}\, e^{-d\tilde{t}} \quad \text{for all} \quad \tilde{t} > 0, \tag{3.34}$$

where $\tilde{c} := c\, e^{-dR}$. Applying Lemma 3.1 we then obtain (3.26) and (3.27), respectively, for the function $\tilde{y}$, where the constant c is replaced by $\tilde{c}$. Reversing the change of variable leads to the following estimates for the function y:

(i) If $d \neq \sigma$, then

$$y(t) \leq \left[\frac{\sigma + \delta}{\sigma} y(0) \,+\, \tilde{c} \left(\frac{\delta}{\sigma(d+\delta)} - \frac{1}{\sigma - d}\right)\right] e^{-\sigma(t-R)} + \frac{c}{\sigma - d}\, e^{-dt} \tag{3.35}$$

for all $t \geq R$.

If $d = \sigma$, then

$$y(t) \leq \left(\frac{\sigma + \delta}{\sigma} y(0)\, e^{\sigma R} + \frac{c\,\delta}{\sigma(\sigma + \delta)} + c\,(t - R)\right) e^{-\sigma t} \tag{3.36}$$

for all $t \geq R$.

By means of these results we are now able to prove

Theorem 3.7. (Exponential decay) *For every pipe Ω_k^∞, $k = 1, ..., N$, we define*

$$c_1 := 2c_0 c_P^{1/2} \left(1 + \frac{1}{2} c_0 c_P^{3/2}\right), \tag{3.37}$$

$$c_2 := \left(\frac{c_P}{2\mu_0 \sigma_f} + 1\right) \frac{C_f^2}{2\mu_0 \sigma_f} |\Sigma_k|, \tag{3.38}$$

$$\tilde{\sigma}_k := \frac{1}{2} \left(\sqrt{\left(\frac{c_1}{c_P}\right)^2 + \frac{4}{c_P}} - \frac{c_1}{c_P}\right) \tag{3.39}$$

$$\tilde{\delta}_k := \tilde{\sigma}_k + \frac{c_1}{c_P}. \tag{3.40}$$

Then the velocity $\tilde{\omega}$, specified in the Theorems 3.4-3.6, has the following additional decay properties:

(i) If $2\sigma_f \neq \tilde{\sigma}_k$, then

$$\|\tilde{\omega}\|^2_{H^1\left(\Omega^\infty_{k,R+r}\right)} \leq \left(C_1 \|\tilde{\omega}\|^2_{H^1(\Omega_k^\infty)} + C_2\right) e^{-\tilde{\sigma}_k r} + C_3\, e^{-2\sigma_f r} \quad \textit{for all } r \geq 0, \tag{3.41}$$

where $\Omega_{k,s}^{\infty} := \{x \in \Omega_k^{\infty} : x_1^k > s\}$, $s \geq 0$, *and the constants are given by*

$$C_1 := \frac{\tilde{\sigma}_k + \tilde{\delta}_k}{\tilde{\sigma}_k}\, e^{\tilde{\sigma}_k R}, \tag{3.42}$$

$$C_2 := \frac{c_2}{c_P}\left(\frac{\tilde{\delta}_k}{\tilde{\sigma}_k\,(2\sigma_f + \tilde{\delta}_k)} - \frac{1}{\tilde{\sigma}_k - 2\sigma_f}\right) e^{(\tilde{\sigma}_k - 2\sigma_f)R}, \tag{3.43}$$

$$C_3 := \frac{c_2}{c_P\,(\tilde{\sigma}_k - 2\sigma_f)}. \tag{3.44}$$

(ii) If $2\sigma_f = \tilde{\sigma}_k$, *then*

$$\|\tilde{\omega}\|^2_{H^1\left(\Omega^{\infty}_{k,R+r}\right)} \leq \left(C_1\,\|\tilde{\omega}\|^2_{H^1(\Omega_k^{\infty})} + \tilde{C}_2 + \tilde{C}_3\, r\right) e^{-\tilde{\sigma}_k r} \quad \textit{for all } r \geq 0, \tag{3.45}$$

where

$$\tilde{C}_2 := \frac{c_2\,\delta}{c_P\,\tilde{\sigma}_k(\tilde{\sigma}_k + \delta)} \quad \textit{and} \quad \tilde{C}_3 := \frac{c_2}{c_P}.$$

Proof. The *idea* of the proof is to establish an inequality of type (3.25) for

$$H(t) := \int_t^{\infty} \left(\int_{\Sigma(\tau)} |\nabla\tilde{\omega}|^2 d\Sigma\right) d\tau \tag{3.46}$$

and to apply Lemma 3.1.

We multiply equation $(3.16)_1$ with $\tilde{\omega}$ and integrate from $x_1^k = x_0$ to $x_1^k = x_1$ on Ω_k^{∞}, denoting by $\Sigma(\xi) \equiv \Sigma_k$ the constant cross-section of the pipe at position ξ. Applying partial integration leads to

$$\begin{aligned} \mu_0 \int_{x_0}^{x_1}\int_{\Sigma(\xi)} |\nabla\tilde{\omega}|^2 \;&=\; \int_{\Sigma(x_1)} (\mu_0 \nabla\tilde{\omega}\, n\cdot\tilde{\omega} - \tilde{\pi}\,\tilde{\omega}\cdot n) \\ &\quad + \int_{\Sigma(x_0)} (\tilde{\pi}\,\tilde{\omega}\cdot n - \mu_0\nabla\tilde{\omega}\, n\cdot\tilde{\omega}) + \int_{x_0}^{x_1}\int_{\Sigma(\xi)} f\,\tilde{\omega} \end{aligned} \tag{3.47}$$

where $n = e_1^k$ is the normal vector on Σ_k.

Proposition. The first integral on the right-hand side of (3.47) tends to zero as x_1 tends to ∞, i.e.

$$\int_{\Sigma(x_1)} (\mu_0\nabla\tilde{\omega}\, n\cdot\tilde{\omega} - \tilde{\pi}\,\tilde{\omega}\cdot n) \to 0 \;\text{ as }\; x_1 \to \infty.$$

Proof of proposition. Due to Theorem 3.6 we have $\tilde\omega$ and all its derivatives of arbitrary order tending to zero as $|x| \to \infty$. Therefore

$$\int_{\Sigma(x_1)} \nabla\tilde\omega\, n \cdot \tilde\omega \to 0 \quad \text{as } x_1 \to \infty.$$

In order to estimate the pressure term we define the mean value of $\tilde\pi$ on $\Sigma(x_1)$ as

$$\langle\tilde\pi\rangle := \frac{1}{|\Sigma_k|}\int_{\Sigma_k} \tilde\pi(x_1,\tilde x)\, d\tilde x.$$

From $(3.16)_4$ and the Poincaré inequality

$$\|\tilde\pi - \langle\tilde\pi\rangle\|_{L^2(\Sigma)} \le c_P\, \|\nabla\tilde\pi\|_{L^2(\Sigma)}$$

we obtain

$$\int_{\Sigma(x_1)} (\tilde\pi - \langle\tilde\pi\rangle)\, \tilde\omega(x_1,\tilde x)\cdot n\, d\tilde x \le c_P\, \|\nabla\tilde\pi\|_{L^2(\Sigma(x_1))} \|\tilde\omega\|_{L^2(\Sigma(x_1))},$$

which tends to 0 due to (3.23) and (3.24).

□

Using the result of the proposition and the definition (3.46) of the function H, equation (3.47) can be rewritten as follows:

$$\mu_0 H(x_0) = \int_{\Sigma(x_0)} (\tilde\pi\tilde\omega\cdot n - \mu_0\nabla\tilde\omega\, n\cdot\tilde\omega) + \int_{x_0}^{\infty}\int_{\Sigma(\xi)} f\,\tilde\omega.$$

Integrating this equation from $x_0 = t+l$ to $x_0 = t+l+1$, $l \in \mathbb{N}_0$, yields

$$\mu_0 \int_{t+l}^{t+l+1} H(x_0) = \int_{t+l}^{t+l+1}\int_{\Sigma(x_0)} \tilde\pi\,\tilde\omega\cdot n - \int_{t+l}^{t+l+1}\int_{\Sigma(x_0)} \mu_0\nabla\tilde\omega\, n\cdot\tilde\omega + \int_{t+l}^{t+l+1}\int_{x_0}^{\infty}\int_{\Sigma(\xi)} f\,\tilde\omega. \tag{3.48}$$

We denote the integrals on the right-hand side of (3.48) by I_j $(j = 1,2,3)$, i.e.

$$\mu_0\int_{t+l}^{t+l+1} H(x_0) = I_1 + I_2 + I_3,$$

and estimate each term separately. We first establish the

(1) *Estimate for I_3:* We define

$$G(x_0) := \int_{\Omega_{x_0}} f\,\tilde\omega = \int_{x_0}^{\infty}\int_{\Sigma(\xi)} f\,\tilde\omega,$$

where $\Omega_{x_0} := \{x \in \Omega_k^\infty : x_1^k > x_0\}$. Then we have

$$\begin{aligned} |G(x_0)| &\le \|f\|_{L^2(\Omega_{x_0})} \|\tilde{\omega}\|_{L^2(\Omega_{x_0})} \\ &\le \frac{\epsilon}{2} \|\tilde{\omega}\|^2_{L^2(\Omega_{x_0})} + \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega_{x_0})}, \end{aligned}$$

using the *Cauchy inequality* $2ab \le \epsilon a^2 + \frac{b^2}{\epsilon}$, for all $a, b \ge 0$, $\epsilon > 0$.

The Poincaré inequality (3.4) can be extended to Ω_{x_0}:

$$\|\tilde{\omega}\|^2_{L^2(\Omega_{x_0})} \le c_P(\Sigma_k) \|\nabla\tilde{\omega}\|^2_{L^2(\Omega_{x_0})}. \tag{3.49}$$

Since $H(x_0) = \|\nabla\tilde{\omega}\|^2_{L^2(\Omega_{x_0})}$, we obtain

$$|G(x_0)| \le \frac{\epsilon}{2} c_P\, H(x_0) + \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega_{x_0})}.$$

Finally,

$$I_3 \equiv \int_{t+l}^{t+l+1} G(x_0) \le \frac{\epsilon}{2} c_P \int_{t+l}^{t+l+1} H(x_0) + \frac{1}{2\epsilon} \int_{t+l}^{t+l+1} \|f\|^2_{L^2(\Omega_{x_0})}. \tag{3.50}$$

The last integral in (3.50) can be estimated using the decay property of f, cf. *assumption (iii)* above. It yields

$$\int_{t+l}^{t+l+1} \|f\|^2_{L^2(\Omega_{x_0})} \;\le\; \frac{C_f^2}{4\sigma_f^2} |\Sigma_k| \left(e^{-2\sigma_f(t+l)} - e^{-2\sigma_f(t+l+1)}\right) \quad \text{for all } t \ge R. \tag{3.51}$$

In order to absorb the first term on the right-hand side of (3.50) into the left-hand side of (3.48), we choose $\epsilon = \dfrac{\mu_0}{c_P}$. Therefore, we have for all $t \ge R$

$$I_3 \;\le\; \frac{\mu_0}{2} \int_{t+l}^{t+l+1} H(x_0) + c_P \frac{C_f^2}{8\mu_0\, \sigma_f^2} |\Sigma_k| \left(e^{-2\sigma_f(t+l)} - e^{-2\sigma_f(t+l+1)}\right). \tag{3.52}$$

(2) *Estimate for I_1:* In order to estimate

$$I_1 \equiv \int_{t+l}^{t+l+1} \int_{\Sigma(x_0)} \tilde{\pi}\, \tilde{\omega} \cdot n,$$

we consider the following problem:

$$\begin{cases} \nabla \cdot u = \tilde{\omega} \cdot n \;\text{ in } \Omega_{t+l}, \\ u \in H_0^1(\Omega_{t+l}), \\ \|\nabla u\|_{L^2(\Omega_{t+l})} \le c_0 \|\tilde{\omega} \cdot n\|_{L^2(\Omega_{t+l})}, \end{cases} \tag{3.53}$$

where

$$\Omega_{t+l} := \Omega_k^\infty \cap \Big\{ x : t+l < x_1^k < t+l+1 \Big\}.$$

Due to the constant cross-section of the pipe Ω_k^∞, the constant c_0 is independent of t and l. We rewrite I_1 using (3.53) and get

$$I_1 = -\int_{\Omega_{t+l}} \nabla\tilde{\pi}\cdot u.$$

Testing equation $(3.16)_1$ with the solution $u \in H_0^1(\Omega_{t+l})$ of (3.53), we obtain for all $t \geq R$:

$$\begin{aligned}
I_1 &\leq \mu_0 \int_{\Omega_{t+l}} \nabla\tilde{\omega}\,\nabla u - \int_{\Omega_{t+l}} f\,u \\
&\leq \mu_0\, c_0\, \|\nabla\tilde{\omega}\|_{L^2(\Omega_{t+l})}\, \|\tilde{\omega}\cdot n\|_{L^2(\Omega_{t+l})} + \|f\|_{L^2(\Omega_{t+l})}\, \|u\|_{L^2(\Omega_{t+l})} \\
&\leq \mu_0\, c_0\, \|\nabla\tilde{\omega}\|_{L^2(\Omega_{t+l})}\, \|\tilde{\omega}\cdot n\|_{L^2(\Omega_{t+l})} + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega_{t+l})} + \|u\|^2_{L^2(\Omega_{t+l})} \right) \\
&\leq \mu_0\, c_0\, c_P^{1/2} \left(1 + \frac{1}{2} c_0\, c_P^{3/2} \right) \|\nabla\tilde{\omega}\|^2_{L^2(\Omega_{t+l})} + \frac{C_f^2}{4\sigma_f}\, |\Sigma_k| \left(e^{-2\sigma_f(t+l)} - e^{-2\sigma_f(t+l+1)} \right).
\end{aligned}$$

Here we use the Poincaré inequality (cf. (3.49)) on Ω_{t+l} for $\tilde{\omega}$ and u, respectively, the Cauchy inequality with $\epsilon = 1$, and the decay property of f.

(3) We rewrite the second integral:

$$\begin{aligned}
I_2 \equiv -\mu_0 \int_{t+l}^{t+l+1} \int_{\Sigma(x_0)} \nabla\tilde{\omega}\, n\cdot\tilde{\omega} &= -\frac{\mu_0}{2} \int_{t+l}^{t+l+1} \int_{\Sigma(x_0)} \frac{\partial \tilde{\omega}^2}{\partial x_1^k} \\
&= -\frac{\mu_0}{2} \int_{\Sigma(t+l+1)} \tilde{\omega}^2 + \frac{\mu_0}{2} \int_{\Sigma(t+l)} \tilde{\omega}^2.
\end{aligned}$$

From (1), (2) and (3) we thus get

$$\begin{aligned}
\int_{t+l}^{t+l+1} H(x_0) \leq c_1 \|\nabla\tilde{\omega}\|^2_{L^2(\Omega_{t+l})} &+ \int_{\Sigma(t+l)} \tilde{\omega}^2 - \int_{\Sigma(t+l+1)} \tilde{\omega}^2 \\
&+ c_2 \left(e^{-2\sigma_f(t+l)} - e^{-2\sigma_f(t+l+1)} \right),
\end{aligned} \tag{3.54}$$

for all $t \geq R$, where

$$c_1 := 2 c_0 c_P^{1/2} \left(1 + \frac{1}{2} c_0 c_P^{3/2} \right), \tag{3.55}$$

$$c_2 := \left(\frac{c_P}{2\mu_0\sigma_f} + 1 \right) \frac{C_f^2}{2\mu_0\sigma_f} |\Sigma_k|. \tag{3.56}$$

We sum up both sides of inequality (3.54) from $l = 0$ to $l = \infty$. Since (cf. Theorem 3.6)

$$\lim_{|x|\to\infty} \int_{\Sigma(x)} \tilde{\omega}^2 = 0,$$

we have

$$\sum_{l=0}^{\infty} \left(\int_{\Sigma(t+l)} \tilde{\omega}^2 - \int_{\Sigma(t+l+1)} \tilde{\omega}^2 \right) = \int_{\Sigma(t)} \tilde{\omega}^2,$$

In the same way we have

$$\sum_{l=0}^{\infty} \left(e^{-2\sigma_f (t+l)} - e^{-2\sigma_f (t+l+1)} \right) = e^{-2\sigma_f t}.$$

We thus obtain

$$\int_t^{\infty} H(x_0) \leq c_1 H(t) + \int_{\Sigma(t)} \tilde{\omega}^2 + c_2 \, e^{-2\sigma_f t}.$$

The Poincaré inequality (3.4) yields

$$\int_{\Sigma(t)} \tilde{\omega}^2 \leq c_P \int_{\Sigma(t)} |\nabla \tilde{\omega}|^2 = -c_P \, H'(t),$$

and we finally get the following inequality of type (3.25):

$$H'(t) + \frac{1}{c_P} \int_t^{\infty} H \leq \frac{c_1}{c_P} H(t) + \frac{c_2}{c_P} e^{-2\sigma_f t}, \quad \text{for all } t \geq R. \tag{3.57}$$

The decay result now follows from Lemma 3.1 and the related Remark 3.2.

□

From inequalities (3.41) and (3.45) we get, as in the case of Leray's problem in the previous section (cf. Theorem 3.2 and Corollary 3.2), the exponential decay of higher order H^m-norms and the pointwise exponential decay:

Corollary 3.4. (*i*) *Exponential decay of H^m-norms:*

If $2\sigma_f \neq \tilde{\sigma}_k$, then there exist constants $K_1, K_2 > 0$, such that

$$\|\tilde{\omega}\|_{H^{m+2}\left(\Omega^{\infty}_{k,R+r+1}\right)} + \|\nabla \tilde{\pi}\|_{H^m\left(\Omega^{\infty}_{k,R+r+1}\right)} \leq K_1 \, e^{-\frac{\tilde{\sigma}_k}{2} r} + K_2 \, e^{-\sigma_f r}, \tag{3.58}$$

for all $r \geq 0$ and every $m \geq 0$. In particular, the constants K_1, K_2 depend on m.

If $2\sigma_f = \tilde{\sigma}_k$, then the following inequality holds:

$$\|\tilde{\omega}\|_{H^{m+2}\left(\Omega^\infty_{k,R+r+1}\right)} + \|\nabla\tilde{\pi}\|_{H^m\left(\Omega^\infty_{k,R+r+1}\right)} \leq \left(K_3 + K_4\sqrt{r}\right) e^{-\frac{\tilde{\sigma}_k}{2}r}, \tag{3.59}$$

for all $r \geq 0$ and every $m \geq 0$.

The explicit forms of the constants K_1, K_2, K_3, and K_4 are given below.

(ii) Pointwise exponential decay:

If $2\sigma_f \neq \tilde{\sigma}_k$, we have

$$|D^\alpha\tilde{\omega}(x)| + |D^\alpha\nabla\tilde{\pi}(x)| \leq C\left(K_1\, e^{-\frac{\tilde{\sigma}_k}{2}\,(x_1^k - R-1)} + K_2\, e^{-\sigma_f\,(x_1^k-R-1)}\right) \tag{3.60}$$

for every $x \in \Omega_k^\infty$, $x_1^k \geq R+1$ and every $m = |\alpha| \geq 0$. The constant C (from the corresponding Sobolev embedding) only depends on m and the cross-section Σ_k of Ω_k^∞.

If $2\sigma_f = \tilde{\sigma}_k$ such a pointwise estimate holds analogously.

Proof. We apply Lemma VI.1.2 and inequality (VI.1.19) from [G], estimating higher derivatives:

$$\|\tilde{\omega}\|_{H^{m+2}\left(\Omega^\infty_{k,R+r+1}\right)} + \|\nabla\tilde{\pi}\|_{H^m\left(\Omega^\infty_{k,R+r+1}\right)} \leq C_m\left(\|\tilde{\omega}\|_{H^1\left(\Omega^\infty_{k,R+r}\right)} + \|f\|_{H^m\left(\Omega^\infty_{k,R+r}\right)}\right).$$

Regarding inequality (3.19) and Theorem 3.7, it remains to establish an estimate for $\|\tilde{\omega}\|_{H^1(\Omega_k^\infty)}$, cf. (3.41), (3.45). The Poincaré inequality (3.49) and equation (3.21) yield

$$\|\tilde{\omega}\|^2_{H^1(\Omega_k^\infty)} \leq (1+c_P)\,\|\nabla\tilde{\omega}\|^2_{L^2(\Omega_k^\infty)} \leq (1+c_P)\,\frac{c_P}{\mu_0^2}\,\|f\|^2_{L^2(\Omega^\infty)}.$$

From Theorem 3.7 and the decay property of f we thus get, if $2\sigma_f \neq \tilde{\sigma}_k$,

$$\|\tilde{\omega}\|_{H^{m+2}\left(\Omega^\infty_{k,R+r+1}\right)} + \|\nabla\tilde{\pi}\|_{H^m\left(\Omega^\infty_{k,R+r+1}\right)} \leq K_1\, e^{-\frac{\tilde{\sigma}_k}{2}r} + K_2\, e^{-\sigma_f r}$$

where

$$K_1 := C_m\left(C_1\,(1+c_P)\,\frac{c_P}{\mu_0^2}\,\|f\|^2_{L^2(\Omega^\infty)} + C_2\right)^{1/2},$$
$$K_2 := C_m\left(\sqrt{C_3} + \tilde{C}_f\right).$$

Analogously, if $2\sigma_f = \tilde{\sigma}_k$, then

$$\|\tilde{\omega}\|_{H^{m+2}\left(\Omega^\infty_{k,R+r+1}\right)} + \|\nabla\tilde{\pi}\|_{H^m(\Omega^\infty_{k,R+r+1})} \leq (K_3 + K_4\sqrt{r})\, e^{-\frac{\sigma}{2}r}$$

where

$$K_3 := C_m \left\{ \left(C_1 \left(1 + c_P\right) \frac{c_P}{\mu_0^2} \|f\|^2_{L^2(\Omega^\infty)} + \tilde{C}_2 \right)^{1/2} + \tilde{C}_f \right\},$$
$$K_4 := C_m \sqrt{\tilde{C}_3}.$$

The *pointwise exponential decay* of (*ii*) now follows immediately from (*i*) using the Sobolev embedding on the cylinders Ω_k^∞ (cf. section 3.1).

□

Chapter 4

Approximation of Leray-type problems on finite domains

The exponential decay of Leray's solution to Poiseuille flow allows to cut off the *infinite* bifurcation and to consider analogous Stokes problems on *finite* junction domains Ω^M prescribing Poiseuille velocities on the in- and outflow boundaries. With the help of general regularity estimates established in section 4.1 we prove the corresponding approximation property in section 4.2.
The approximation of Leray's solution on finite junction domains of length $O(M)$ can also be interpreted as a method for *numerical computations*.

4.1 Stokes equations in the junction

On the extended junction domain Ω^M, consisting of the junction Ω_0 and pipes Z_j^M of length M (cf. definition of the geometry in section 2.1), we consider the Stokes equations with force f and given velocities g_j on the in-/outflow boundaries γ_j^M:

$$(4.1) \qquad \left\{ \begin{array}{rcll} -\mu_0 \Delta v + \nabla p &=& f & \text{in } \Omega^M, \\ div\, v &=& 0 & \text{in } \Omega^M, \\ v &=& 0 & \text{on } \Gamma^M, \\ v &=& g_j & \text{on } \gamma_j^M, \\ \displaystyle\sum_k \int_{\gamma_k^M} g_k \cdot e_1^k &=& 0, & \end{array} \right.$$

under the following *regularity assumptions on the data:*

(*i*) $f \in L^2(\Omega^M)$.

(*ii*) There exists an extension $g \in H^2(\Omega^M)$ of the boundary values of v, i.e. $g|_{\Gamma^M} = 0$ and $g|_{\gamma_j^M} = g_j$. For this it is necessary to have $g_j \in H^{3/2}(\gamma_j^M)$, $g_j = 0$ on $\partial\gamma_j^M$. Additionally, in order to provide regularity of the solution (v, p), we assume $div\, g \in H_0^1(\Omega^M)$, i.e. in particular $div\, g = 0$ on the *cylinder edges* $\partial\gamma_j^M$ (cf. [D2]).

Existence and uniqueness of the solution $v \in H^1(\Omega^M)$ and $p \in L^2(\Omega^M)$ is evident from the theory of Stokes equations, cf. e.g. [G] and [T]. Actually, the solution (v, p) is of higher regularity since the boundary of the domain Ω^M is assumed to be smooth apart from the edges $\partial\gamma_j^M$ of the cylinders Z_j^M at the in-/outflow boundaries and the data is of higher regularity as well. Therefore, on any subdomain of Ω^M having positive distance from these edges the solution is smooth (cf. Lemma 4.1 below).
For regularity near the boundary we refer to the general results of [D2] for zero boundary values, stated in Theorem B.1 and applied to the case of non-zero boundary conditions in Corollary B.1.

Theorem 4.1. (Existence and uniqueness) *Under the assumptions* (i) *and* (ii) *the Stokes system (4.1) has a unique solution* $v \in H^2(\Omega^M)$ *and* $p \in H^1(\Omega^M)$ *with* $\displaystyle\int_{\Omega^M} p = 0.$

4.1.1 Definition of the junction problems

We consider the following types of *junction problems*:

(i) **Poiseuille junction problem**

For given in-/outflow Poiseuille velocities $g_j = V_j$ and $f \equiv 0$, there exists a unique solution $\omega^M \in H^2(\Omega^M)$, $\pi^M \in H^1(\Omega^M)$ (with smooth cut-off functions η_j we can define $g := \sum_j \eta_j V_j$ as a suitable extension satisfying assumption (ii)). We show that it approximates the solution of the corresponding Leray problem with Poiseuille velocities V_j at infinity up to an exponentially decaying error.

(ii) **Inertial correction problem**

The Stokes problem (4.1) with $g_j \equiv 0$ and $f := (u \cdot \nabla)w$, where u, w are such that $f \in L^2(\Omega^M)$, is called *inertial correction problem.* It admits a unique solution $(\tilde{\omega}^M, \tilde{\pi}^M) \in (H^2 \times H^1)(\Omega^M)$.
Since $\omega^M \in H^2(\Omega^M)$ and due to the embedding $H^1 \hookrightarrow L^4(\Omega^M)$ (cf. Lemma B.2), the convective term $(\omega^M \cdot \nabla)\omega^M$ of the solution of the Poiseuille junction problem (i) is a function in $L^2(\Omega^M)$. Taking $f = -(\omega^M \cdot \nabla)\omega^M$ as the right-hand side, this type of junction problem is used in order to correct the leading order nonlinear term when building an approximation for the solution of the Navier-Stokes problem (2.2) in chapter 5.
In higher order approximations there occur *inertial* terms of three different types: $(\omega_i^M \cdot \nabla)\omega_j^M$, $(\tilde{\omega}_i^M \cdot \nabla)\omega_j^M + (\omega_i^M \cdot \nabla)\tilde{\omega}_j^M$ and $(\tilde{\omega}_i^M \cdot \nabla)\tilde{\omega}_j^M$, where the functions ω_k^M are the solutions of (possibly different) Poiseuille junction problems and $\tilde{\omega}_k^M$ are the solutions of inertial correction problems of lower order (cf. definitions (5.23)-(5.25) of section 5.3).
Then, the right-hand side of (4.1) is $f^M := -\displaystyle\sum_{i,j}(\omega_i^M \cdot \nabla)\omega_j^M$, where each ω_k^M is

either the solution of a Poiseuille junction or an inertial correction problem. The corresponding Leray-type problem on the infinite domain is (3.16) with right-hand side $f_L := -\sum_{i,j}(\omega_i \cdot \nabla)\omega_j$, where for each function ω_k^M we denote by ω_k the solution of the corresponding Leray problem (cf. (4.28) for the decay estimate of f_L).

4.1.2 Regularity estimates

In addition to the existence and uniqueness result, we later need regularity estimates of the solution by the given data. The constants involved therein depend in general on the domain, in particular they may depend on its diameter. Our aim is the approximation of the solution of Leray's problem by the solution of the junction problem in dependence of the parameter M. Therefore we have to establish estimates without constants implicitly depending on this length.

We start with the following

Lemma 4.1. *For the solution (v, p) of problem (4.1) the following estimates hold:*

(a) Inside the junction domain: Assume the lateral boundary Γ^M and the in-/outflow boundaries γ_j^M of the branching domain Ω^M to be smooth (at least of class C^2). Let $\Omega' \subset \Omega^M$ be a subset having a positive distance from the cylinder edges $\partial\gamma_j^M$, i.e. $dist(\partial\Omega', \partial\gamma_j^M) > 0$ $(j = 1, ..., N)$, and denote $\Sigma := \partial\Omega' \cap \partial\Omega^M$, $v|_\Sigma =: v^*$.

Then, there exists a constant $C > 0$ such that

$$\|v\|_{H^2(\Omega'')} + \|p\|_{H^1(\Omega'')} \leq C \left(\|f\|_{L^2(\Omega')} + \|v^*\|_{H^{3/2}(\Sigma)} + \|v\|_{H^1(\Omega')} + \|p\|_{L^2(\Omega')}\right) \tag{4.2}$$

for all $\Omega'' \subset \Omega'$, such that $\partial\Omega''$ is a strictly interior subset of Σ, the constant C depending on Ω' and Ω''.

(b) Near the in-/outflow boundaries:
Let $0 < s < M$, $Z_j^s := \{x \in Z_j^M : M - s < x_1^j < M\}$. Then, for $0 < l < M - 1$, there exists a constant $C = C(Z_j^{l+1})$ such that

$$\|v\|_{H^2(Z_j^l)} + \|p\|_{H^1(Z_j^l)} \leq C \left(\|f\|_{L^2(Z_j^{l+1})} + \|g\|_{H^2(Z_j^{l+1})} + \|v\|_{H^1(Z_j^{l+1})} + \|p\|_{L^2(Z_j^{l+1})}\right). \tag{4.3}$$

Proof. (a) Cf. [G], Theorem IV.5.1.

(b) We define a smooth cut-off function $\eta_j = \eta_j(x_1^j)$, which is identical 1 for $x_1^j \geq M - l$ and identical 0 in $\Omega^M \setminus Z_j^{l+1}$. In particular there are constants $C_j > 0$ such that

$max_{x_1^j}|D^i\eta_j(x_1^j)| \leq C_j$, $i = 0, 1, 2$.

For $v_j := \eta_j v$, $p_j := \eta_j p$ and $\tilde{g}_j := \eta_j g$ the following equations hold:

$$\begin{cases} -\mu_0\Delta v_j + \nabla p_j &= f\,\eta_j - \mu_0(v\Delta\eta_j + 2\nabla v\nabla\eta_j) + p\,\nabla\eta_j & \text{in } Z_j^{l+1}, \\ div\, v_j &= v\,\nabla\eta_j & \text{in } Z_j^{l+1}, \\ v_j &= \tilde{g}_j|_{\partial Z_j^{l+1}} & \text{on } \partial Z_j^{l+1}. \end{cases}$$

From Corollary B.1 we get

$$\|v_j\|_{H^2(\cdot)}+\|p_j\|_{H^1(\cdot)/\mathbb{R}} \leq C\left(\|f\|_{L^2(\cdot)} + \|g\|_{H^2(\cdot)} + \|v\|_{H^1(\cdot)} + \|p\|_{L^2(\cdot)/\mathbb{R}}\right)$$

with a constant $C = C(Z_j^{l+1})$, all norms being taken on Z_j^{l+1}.

Since $\|p_j\|_{L^2(\cdot)/\mathbb{R}} \leq \|p_j\|_{L^2(\cdot)}$ and $\|p_j\|_{L^2(\cdot)} + \|p_j\|_{H^1(\cdot)/\mathbb{R}} \geq \|p_j\|_{H^1(\cdot)}$, we obtain (with a constant $\tilde{C} > 0$)

$$\|v_j\|_{H^2(Z_j^l)} + \|p_j\|_{H^1(Z_j^l)} \leq \tilde{C}\left(\|f\|_{L^2(Z_j^{l+1})} + \|g\|_{H^2(Z_j^{l+1})} + \|v\|_{H^1(Z_j^{l+1})} + \|p\|_{L^2(Z_j^{l+1})}\right)$$

By construction we have $v_j = v$ and $p_j = p$ in Z_j^l which yields the result.

□

We are now able to prove

Theorem 4.2. (Regularity estimates) *If the pressure mean value on Ω^M is fixed to zero, i.e. $\int_{\Omega^M} p = 0$, then there exists $C^{(r)} > 0$ independent of M such that*

(4.4) $$\|v\|_{H^2(\Omega^M)} + \|p\|_{H^1(\Omega^M)} \leq C^{(r)}\,M\left(\|f\|_{L^2(\Omega^M)} + \|g\|_{H^2(\Omega^M)}\right)$$

for all $M \geq 1$.

Remark 4.1. (i) Without normalizing the pressure, we have from (4.4)

(4.5) $$\|v\|_{H^2(\Omega^M)} + \|p\|_{H^1(\Omega^M)/\mathbb{R}} \leq C^{(r)}\,M\left(\|f\|_{L^2(\Omega^M)} + \|g\|_{H^2(\Omega^M)}\right)$$

for all $M \geq 1$.

(ii) If the boundary values g_j carry no flux, i.e.

(4.6) $$\int_{\gamma_j^M} g_j \cdot e_1^j = 0, \qquad j = 1, 2, ..., N,$$

then there exists a constant $C_r > 0$ independent of M such that

(4.7) $$\|v\|_{H^1(\Omega^M)} \leq C_r\left(\|g\|_{H^1(\Omega^M)} + \|f\|_{L^2(\Omega^M)}\right)$$

for all $M \geq 1$.

Proof. Without loss of generality we restrict to the case of values $M \in \mathbb{N}$; otherwise $M = N + r$, with $N \in \mathbb{N}$, $r \in [0, 1)$, and we can substitute Ω_0 by Ω^r, i.e. by the extended junction with pipes Z_j^r of length r.

Idea: We first prove the estimate

(4.8)
$$\|v\|_{H^2(\Omega^M)} + \|p\|_{H^1(\Omega^M)} \le C \left(\|f\|_{L^2(\Omega^M)} + \|g\|_{H^2(\Omega^M)} + \|v\|_{H^1(\Omega^M)} + \|p\|_{L^2(\Omega^M)}\right),$$

with a constant independent of M. In a second step we then estimate $\|v\|_{H^1(\Omega^M)}$ and $\|p\|_{L^2(\Omega^M)}$.

(1) We use (4.2) from Lemma 4.1 on subcylinders of length 1 and $1 + 2\delta$, $0 < \delta < 1$, respectively, defined as follows (cf. Fig. 4.1):

$$Z_{j,k} := \{x \in Z_j^M : k \le x_1^j \le k + 1\} \text{ for } j = 1, ..., N, \;\; k = 0, 1, ..., M - 1$$

and

$$Z_{j,k}^\delta := \{x \in Z_j^M : k - \delta \le x_1^j \le k + 1 + \delta\} \text{ for } j = 1, ..., N, \;\; k = 1, 2, ..., M - 2.$$

Taking $\Omega' = Z_{j,k}^\delta$, $\Omega'' = Z_{j,k}$ in Lemma 4.1(a), we get

$$\|v\|_{H^2(Z_{j,k})} + \|p\|_{H^1(Z_{j,k})} \le C_j \left(\|f\|_{L^2(Z_{j,k}^\delta)} + \|v\|_{H^1(Z_{j,k}^\delta)} + \|p\|_{L^2(Z_{j,k}^\delta)}\right), \quad (4.9)$$

for $k = 1, 2, ..., M - 2$, with a constant $C_j = C_j(\delta, S_j)$ *independent* of k:
The above estimate is invariant under the change of variable $x_1^j \to x_1^j - \xi$, $\xi \ge 0$, due to the constant cross-section S_j of the cylinder Z_j^M. Therefore, inequality (4.9) for $k = 1$ already yields the estimate for all $k \ge 1$ and the constant C_j is thus independent of k.

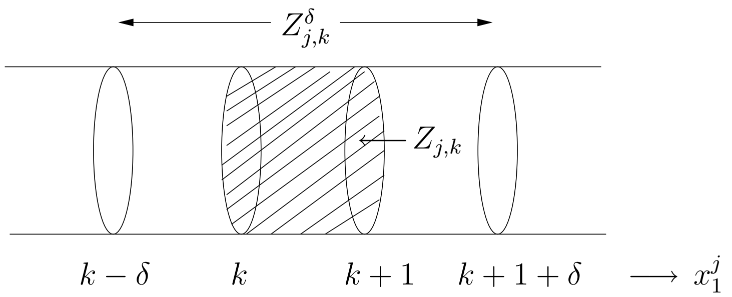

Fig. 4.1 The cylinders $Z_{j,k}$ (*hatched*) and $Z_{j,k}^\delta$

From (4.2) we also get such an estimate on the junction part $\Omega_0 \cup \left(\bigcup_j Z_{j,0}\right)$. For the remaining cylinders $Z_{j,M-1}$ at the end of the pipes we apply inequality (4.3) for $l = 1$.

By adding up all these inequalities we obtain estimate (4.8) on the whole domain Ω^M, the constant being independent of M. Note that for $k \geq 2$ only the cylinders $Z^\delta_{j,k-1}$ and $Z^\delta_{j,k+1}$ have non-empty intersection with $Z_{j,k}$.

Remark: Inequality (4.8) remains unaffected if we replace p by $p + c$, for any $c \in \mathbb{R}$. Taking $inf_{c\in\mathbb{R}}$ on both sides, we obtain

(4.10)
$$\|v\|_{H^2(\Omega^M)} + \|p\|_{H^1(\Omega^M)/\mathbb{R}} \leq C \left(\|f\|_{L^2(\Omega^M)} + \|g\|_{H^2(\Omega^M)} + \|v\|_{H^1(\Omega^M)} + \|p\|_{L^2(\Omega^M)/\mathbb{R}}\right)$$

with $\|p\|_{L^2(\Omega^M)/\mathbb{R}} \equiv inf_{c\in\mathbb{R}} \|p + c\|_{L^2(\Omega^M)}$.

(2) We now establish an *estimate for* (v, p) *in* $(H^1 \times L^2)(\Omega^M)$.

At first, we construct a divergence-free extension of the boundary values of v. By *regularity assumption* (*ii*) there exists an extension $g \in H^2(\Omega^M)$ for which in general $div\, g \neq 0$. Thus we define $W := g + \tilde{g}$, where $\tilde{g}$ is a solution of (cf. Lemma B.4)

$$\text{(4.11)} \qquad \begin{cases} div\, \tilde{g} &= -div\, g \quad \text{in } \Omega^M, \\ \tilde{g} &= 0 \qquad\quad \text{on } \partial\Omega^M, \\ \|\tilde{g}\|_{H^1(\Omega^M)} &\leq C\, M\, \|div\, g\|_{L^2(\Omega^M)}. \end{cases}$$

Then $W \in H^1(\Omega^M)$ is a solenoidal extension of the boundary values of v and

$$\text{(4.12)} \qquad \|W\|_{H^1(\Omega^M)} \leq C\, M \|g\|_{H^1(\Omega^M)}.$$

The (unique) solution of (4.1) can now be written in the form $v = w + W$, where $w \in H^1_0(\Omega^M)$, $div\, w = 0$, such that

$$\text{(4.13)} \qquad \mu_0 \int_{\Omega^M} \nabla w\, \nabla \phi = \int_{\Omega^M} f\, \phi - \mu_0 \int_{\Omega^M} \nabla W\, \nabla \phi$$

for all $\phi \in H^1_0(\Omega^M)$, $div\, \phi = 0$.

According to the theorem of *Lax-Milgram* there exists a unique function w with these properties. Using the Poincaré inequality

$$\|w\|_{L^2(\Omega^M)} \leq C_p\, \|\nabla w\|_{L^2(\Omega^M)}$$

(cf. Lemma B.1(*ii*), C_p independent of M) we then get the estimate

$$\|\nabla w\|_{L^2(\Omega^M)} \leq \mu_0^{-1} C_p\, \|f\|_{L^2(\Omega^M)} + \|\nabla W\|_{L^2(\Omega^M)}$$

and from (4.12)

$$\|w\|_{H^1(\Omega^M)} \leq \mu_0^{-1} C_p\, \|f\|_{L^2(\Omega^M)} + C\, M\, \|g\|_{H^1(\Omega^M)}.$$

The results shown up to now immediately yield

$$\|v\|_{H^1(\Omega^M)} \leq C\, M \left(\|g\|_{H^1(\Omega^M)} + \|f\|_{L^2(\Omega^M)}\right). \tag{4.14}$$

Remark: If the boundary values g_j in (4.1) carry no flux, i.e. $\displaystyle\int_{\gamma_j^M} g_j \cdot e_1^j = 0$, we can easily construct a solenoidal extension $W \in H^1(\Omega^M)$ which is zero outside the cylinders Z_j^1 (cf. definition in Lemma 4.1(b)), cutting off the extension g and solving (4.11) on Z_j^1. Then $(4.11)_3$ holds on Z_j^1 without the factor M. Extending the solution $\tilde{g}_j$ by zero outside Z_j^1 we then have $W := g + \sum_j \tilde{g}_j$ which admits an estimate of type (4.12) independent of M. Therefore, also inequality (4.14) does not depend on M (cf. Remark 4.1(ii)).

We now establish an estimate for the L^2-norm of the pressure in Ω^M. For this purpose, we consider the problem

$$\left\{\begin{array}{rcll} div\,\psi & = & p & \text{in } \Omega^M, \\ \psi & = & 0 & \text{on } \partial\Omega^M, \\ \|\psi\|_{H^1(\Omega^M)} & \leq & C_0\,\|p\|_{L^2(\Omega^M)}. & \end{array}\right. \tag{4.15}$$

If $\displaystyle\int_{\Omega^M} p = 0$, there exists at least one solution $\psi \in H_0^1(\Omega^M)$ due to Lemma B.4. Testing equation $(4.1)_1$ with such a function ψ yields

$$\|p\|_{L^2(\Omega^M)} \leq C_0 \left(\mu_0 \|\nabla v\|_{L^2(\Omega^M)} + C_P \|f\|_{L^2(\Omega^M)}\right) \tag{4.16}$$

where $C_p > 0$ is the Poincaré constant in (B.2) (independent of M). From Lemma B.4 we have $C_0 = O(M)$. Combining inequality (4.16) and (4.14) we obtain

$$\|v\|_{H^1(\Omega^M)} + \|p\|_{L^2(\Omega^M)} \leq C\, M \left(\|f\|_{L^2(\Omega^M)} + \|g\|_{H^1(\Omega^M)}\right)$$

and together with (4.8) we finally get the result of the theorem.

Remark: It is not possible to get estimate $(4.15)_3$ with a constant independent of M since the compatibility condition of zero pressure mean value cannot be satisfied on every $Z_{j,k}$, $j = 1, ..., N$, $k = 0, ..., M-1$, at the same time (cf. proof of Lemma B.4).

□

4.2 The approximation result

In this section we show that the solution of the Stokes problem in the extended junction domain Ω^M approximates the solution of the corresponding Leray-type problem with an exponentially decaying error.

4.2.1 Error estimates for the Poiseuille junction problem

The solution (ω, π) of Leray's problem, restricted to the domain Ω^M, solves the following equations:

$$\left\{ \begin{array}{rcll} -\mu_0 \Delta\omega + \nabla\pi & = & 0 & \text{in } \Omega^M, \\ div\,\omega & = & 0 & \text{in } \Omega^M, \\ \omega & = & 0 & \text{on } \Gamma^M, \\ \omega & = & \omega_j & \text{on } \gamma_j^M. \end{array} \right. \tag{4.17}$$

where $\omega_j := \omega|_{\gamma_j^M}$.

Defining $u_j := \omega_j - V_j$, with V_j the Poiseuille flow in the j^{th} pipe, we have

$$\int_{\gamma_j^M} u_j \cdot e_1^j = 0, \qquad j = 1, ..., N, \tag{4.18}$$

since ω carries the flux of the Poiseuille flow in each pipe.

Furthermore, we can easily extend u_j to the whole domain Ω^M. Let $\eta_j = \eta_j(x_1^j)$ be a smooth cut-off function, i.e. $\eta_j(x_1^j) = 1$ for $x_1^j \geq M - \delta$ for some $0 < \delta < 1$, $\eta_j(x_1^j) = 0$ for $x_1^j \leq M - 1$, such that $max_{x_1^j} |D^i \eta_j(x_1^j)| \leq C_j = O(1)$, $i = 0, 1, 2$. Then $U := \sum_j \eta_j\,(\omega - V_j)$ is a smooth extension of the boundary values, vanishing outside the cylinders $Z_{j,M-1} := \left\{ x \in Z_j^M : M - 1 < x_1^j < M \right\}$, $j = 1, ..., N$.

The difference $(v := \omega - \omega^M,\ p := \pi - \pi^M)$ between the solutions of Leray's problem (ω, π) and the corresponding Poiseuille junction problem (cf. section 4.1) then solves

$$\left\{ \begin{array}{rcll} -\mu_0 \Delta v + \nabla p & = & 0 & \text{in } \Omega^M, \\ div\,v & = & 0 & \text{in } \Omega^M, \\ v & = & 0 & \text{on } \Gamma^M, \\ v & = & u_j & \text{on } \gamma_j^M. \end{array} \right. \tag{4.19}$$

From the regularity results for Stokes equations, shown in the previous section, the functions $v \in H^2(\Omega^M)$ and $p \in H^1(\Omega^M)$ can be estimated by the extension U of the boundary values. Since the solution of Leray's problem decays exponentially to the corresponding Poiseuille flows (cf. Theorem 3.2), this yields: There exists $C_L > 0$ such that

$$\begin{aligned} \|U\|_{H^2(\Omega^M)} &= \sum_j \|\eta_j\,(\omega - V_j)\|_{H^2(Z_{j,M-1})} \\ &\leq \sum_j C_j \,\|\omega - V_j\|_{H^2(Z_{j,M-1})} \leq C_L\, e^{-\sigma_L M} \end{aligned} \tag{4.20}$$

for all $M \geq 1$, where the constant $C_L > 0$ admits an estimate of type (3.13).

Remark: More precisely, the results of section 3.1 at first yield that for some $M_0 > 1$ there is $C > 0$ (independent of M_0) such that

$$\|\omega - V_j\|_{H^2(Z_{j,M-1})} \leq C\, e^{-\sigma_L\,(M-M_0)} \tag{4.21}$$

for all $M \geq M_0$. With $R > 0$ as defined in section 3.1 (cf. (3.3)) inequality (4.21) holds for $M_0 \geq R+2$.
Since $\|\omega - V_j\|_{H^2}$ is bounded on $Z_{j,M-1}$ for $1 \leq M \leq M_0$, inequality (4.21) also holds for $1 \leq M \leq M_0$, taking the constant sufficiently large. For simplicity, we include the factor $e^{\sigma_L M_0}$ in the constant C. In the following, we always proceed in this way when applying the corresponding decay results of chapter 3.

From Theorem 4.2 and (4.20) we get the following

Theorem 4.3. (Approximation estimates for Poiseuille junction problem) *The solution (ω^M, π^M) of the Poiseuille junction problem (i.e. $g_j = V_j$ and $f \equiv 0$ in (4.1)) approximates the solution (ω, π) of Leray's problem (3.1) in the following sense:*

If $\displaystyle\int_{\Omega^M} \left(\pi^M - \pi\right) = 0$, then

$$\|\omega^M - \omega\|_{H^2(\Omega^M)} + \|\pi^M - \pi\|_{H^1(\Omega^M)} \leq C_a\, M\, e^{-\sigma_L M} \tag{4.22}$$

for all $M \geq 1$, where $C_a := C^{(r)}\, C_L$. The constants C_L, σ_L and $C^{(r)}$ are specified in (3.13), (3.15) and (4.4).

Remark: Since $(\omega - \omega^M)$ carries no flux we have due to (4.7)

$$\|\omega - \omega^M\|_{H^1(\Omega^M)} \leq C_r\, C_L\, e^{-\sigma_L M} \tag{4.23}$$

for all $M \geq 1$.

4.2.2 Error estimates for the inertial correction problem

We now consider the case of the *inertial correction problem.* Let ω_k^M, $k \in I$ (where I is any finite index set), be the solutions of different Poiseuille junction problems and ω_k the solutions of the corresponding Leray problems. Then $(\tilde{\omega}^M, \tilde{\pi}^M)$ is the solution of the inertial correction problem as defined in the previous section with the right-hand side $f^M := -\displaystyle\sum_{i,j \in I} (\omega_i^M \cdot \nabla)\omega_j^M$. We denote by $(\tilde{\omega}, \tilde{\pi})$ the solution of the corresponding generalized Leray problem (3.16) with $f_L := -\displaystyle\sum_{i,j \in I} (\omega_i \cdot \nabla)\omega_j$.

The difference ($\tilde{v} := \tilde{\omega}^M - \tilde{\omega}$, $\tilde{\pi} := \tilde{\pi}^M - \tilde{\pi}$) in $(H^2 \times H^1)(\Omega^M)$ then is the solution of the Stokes problem

$$
(4.24) \qquad \begin{cases} -\mu_0 \Delta \tilde{v} + \nabla \tilde{p} &= \tilde{f} \quad \text{in } \Omega^M, \\ div\, \tilde{v} &= 0 \quad \text{in } \Omega^M, \\ \tilde{v} &= 0 \quad \text{on } \Gamma^M, \\ \tilde{v} &= \tilde{\omega}_j \quad \text{on } \gamma_j^M, \end{cases}
$$

where $\tilde{\omega}_j := -\tilde{\omega}|_{\gamma_j^M}$ and $\tilde{f} := f^M - f_L$. We now establish an estimate for this difference between the inertial terms.

Lemma 4.2. *Let* $\tilde{f} := \sum_{i,j} \big((\omega_i \cdot \nabla)\omega_j - (\omega_i^M \cdot \nabla)\omega_j^M\big)$ *where* ω_k^M *are the solutions of Poiseuille junction problems and* ω_k *the solutions of the corresponding Leray problems. Then there exists a constant* $C_{\tilde{f}} > 0$ *independent of* M *such that*

$$
(4.25) \qquad \|\tilde{f}\|_{L^2(\Omega^M)} \le C_{\tilde{f}}\, M^{1/2}\, e^{-\sigma_L M}
$$

for all $M \ge 1$.

The constant σ_L *determines the exponential decay of Leray's solution and is specified in section 3.1, cf. (3.15).*

Proof. With the embedding $H^1(\Omega^M) \hookrightarrow L^4(\Omega^M)$ (cf. Lemma B.2 (ii)) we get

$$
\begin{aligned}
(4.26) \qquad \|(\omega_i \cdot \nabla)\omega_j - (\omega_i^M \cdot \nabla)\omega_j^M\|_{L^2} & \\
&\le \|\omega_i \cdot \nabla\,(\omega_j - \omega_j^M)\|_{L^2} + \|(\omega_i - \omega_i^M) \cdot \nabla\, \omega_j^M\|_{L^2} \\
&\le \|\omega_i\|_{L^\infty}\, \|\nabla(\omega_j - \omega_j^M)\|_{L^2} + \|\, \omega_i - \omega_i^M\|_{L^4}\, \|\nabla \omega_j^M\|_{L^4} \\
&\le \|\omega_i\|_{L^\infty} \|\nabla(\omega_j - \omega_j^M)\|_{L^2} + C^2_{L^4,H^1}\, \|\omega_j^M\|_{H^2}\, \|\omega_i - \omega_i^M\|_{H^1},
\end{aligned}
$$

where all norms are taken on Ω^M.

For our estimates we use the following properties:

- From (4.22)-(4.23) we obtain for any $k \in I$

  (i) $\|\omega_k - \omega_k^M\|_{H^1(\Omega^M)} \le C_r\, C_{L,k}\, e^{-\sigma_L M}$
  and
  (ii) $\|\omega_k - \omega_k^M\|_{H^2(\Omega^M)} \le C^{(r)}\, C_{L,k}\, M\, e^{-\sigma_L M}$

  for all $M \ge 1$.

- Since ω_k, together with all its derivatives, tends pointwise exponentially to Poiseuille flow, we have (*iii*) $\|D^\alpha \omega_k\|_{L^\infty(\Omega^M)} \le C_k^\alpha$, with C_k^α independent of M, for all $|\alpha| \ge 0$, and

(*iv*) $\|\omega_k\|_{H^2(\Omega^M)} \le C_k\, |\Omega^M|^{1/2}$, where C_k is independent of M and $|\Omega^M| = O(M)$.

Returning to the definition of $\tilde f$ and considering (4.26), we obtain

$$\|\tilde f\|_{L^2(\Omega^M)} \le \sum_{i,j} C_{i,j} \left(\|\omega_i\|_{L^\infty} + \|\omega_j^M\|_{H^2}\right) e^{-\sigma_L M} \tag{4.27}$$

for $M \ge 1$, which together with the properties (*iii*)-(*iv*) yields the result (4.25).

□

It remains to establish an estimate for the extension of the boundary values $\tilde\omega_j$ in (4.24). Therefore, we have to take a closer look on the decay of the right-hand side f_L of the generalized Leray problem (3.16) since it determines the decay of $\tilde\omega$ (cf. Corollary 3.4).

The exponential decay of Leray's solution ω to Poiseuille flow (cf. Theorem 3.3) yields the following estimate in the k^{th} pipe, $k = 1, ..., N$:

$$|f_L(x)| = \left|\sum_{i,j} \left((\omega_i - V_k^i)\cdot \nabla \omega_j(x) + (V_k^i \cdot \nabla)(\omega_j - V_k^j)(x)\right)\right| \le C\, e^{-\sigma_L x_1^k} \tag{4.28}$$

for all $x \in \Omega_k^\infty$, $x_1^k \ge 1$, the constant $C > 0$ depending on f_L. Here we use the fact that for Poiseuille flow $(V_k^i \cdot \nabla)V_k^j \equiv 0$ for all $i, j \in I$. Analogous estimates hold for any $|D^\alpha f_L|$, $|\alpha| \ge 0$.

Using cut-off functions η_k as above for u_k and applying Corollary 3.4 with $\sigma_{f_L} = \sigma_L$ we finally have

Lemma 4.3. *Let* $f_L = -\sum_{i,j\in I} (\omega_i \cdot \nabla)\omega_j$ *where the functions* ω_i, $i \in I$, *are the solutions of Leray problems (3.1) for different Poiseuille velocities at infinity. Then, there are constants* $C_k = C_k(f_L, \tilde\sigma_k)$, $k = 1, ..., N$, *such that*

$$\text{if } \tilde\sigma_k \ne 2\sigma_L:\ \|\eta_k\, \tilde\omega\|_{H^2(Z_{k,M-1})} \le C_k \left(e^{-\frac{\tilde\sigma_k}{2} M} + e^{-\sigma_L M}\right); \tag{4.29}$$

$$\text{if } \tilde\sigma_k = 2\sigma_L:\ \|\eta_k\, \tilde\omega\|_{H^2(Z_{k,M-1})} \le C_k\, M^{1/2}\, e^{-\sigma_L M}, \tag{4.30}$$

for all $M \ge 1$.

Having established these estimates, inequality (4.4) of Theorem 4.2 now implies the following approximation result:

Theorem 4.4. (Approximation estimates for inertial correction problem) *The solution* $(\tilde{\omega}^M, \tilde{\pi}^M)$ *of the inertial correction problem (i.e.* $g_j \equiv 0$ *and* $f \equiv f^M$ *in (4.1)) approximates the solution* $(\tilde{\omega}, \tilde{\pi})$ *of the generalized Leray problem (3.16) with* $f \equiv f_L$ *and* $\displaystyle\int_{\Omega^M} \left(\tilde{\pi}^M - \tilde{\pi}\right) = 0$ *in the following sense:*

There exists $\tilde{C}_a > 0$ *such that*

$$\|\tilde{\omega}^M - \tilde{\omega}\|_{H^2(\Omega^M)} + \|\tilde{\pi}^M - \tilde{\pi}\|_{H^1(\Omega^M)} \leq \tilde{C}_a \, M^{3/2} \, e^{-\tilde{\sigma} M} \tag{4.31}$$

for all $M \geq 1$*, where* $\tilde{\sigma} := min_k(\sigma_L, \tilde{\sigma}_k/2)$ *and* $\tilde{C}_a$ *depends on* $C_{\tilde{f}}, C^{(r)}$ *and on the constants* K_i *of Corollary 3.4.*

4.2.3 Generalization of the approximation results

We conclude this section with a generalization of the approximation results established so far. This is necessary in order to estimate inertial terms of higher order occurring in the construction of the approximation for the solution of the Navier-Stokes problem (2.2), cf. sections 5.3 and 6.1. To this aim we set $f^{M,0} := f^M$ and $f_L^0 := f_L$, with f^M, f_L defined above, and denote by $(\tilde{\omega}^{M,0}, \tilde{\omega}^0)$ the solutions of the corresponding junction problem (4.1) and Leray's problem (3.16), respectively. We then define $f^{M,1} := -\sum_{i,j} (\omega_i^M \cdot \nabla)\omega_j^M$, where each function ω_k^M, $k \in I$, is either the solution of a Poiseuille junction problem (as above) or *in addition* may be equal to $\tilde{\omega}^{M,0}$; the function f_L^1 is defined analogously:
$f_L^1 := -\sum_{i,j} (\omega_i \cdot \nabla)\omega_j$, where each function ω_k is either the solution of a Leray problem or equal to $\tilde{\omega}^0$. The solutions of the corresponding junction and Leray problems with right-hand side $f^{M,1}$ and f_L^1, are denoted by $\tilde{\omega}^{M,1}$ and $\tilde{\omega}^1$, respectively.

In order to show that $\tilde{\omega}^{M,1}$ is an adequate approximation for $\tilde{\omega}^1$, we proceed as above in the *zero-order* case, first generalizing Lemma 4.2:

The following estimate holds for $\tilde{f}^1 := f^{M,1} - f_L^1$:

$$\|\tilde{f}^1\|_{L^2(\Omega^M)} \leq C_{\tilde{f}^1} \, M^2 \, e^{-\tilde{\sigma} M} \tag{4.32}$$

for all $M \geq 1$, the constant $C_{\tilde{f}^1}$ being independent of M. The proof follows the same lines as those of Lemma 4.2, applying the results of Theorem 4.4 (*ii*).

We now have to distinguish three different cases concerning the *decay rates.* Let $\tilde{\sigma}_l := min_k \, \tilde{\sigma}_k$, where $l \in \{1, ..., N\}$.

(*i*) If $\tilde{\sigma}_l > 2\sigma_L$: From Lemma 4.3 we get, since $\tilde{\sigma}_k \neq 2\sigma_L$ for all $k = 1, ..., N$,

$$\|\eta_k \, \tilde{\omega}^0\|_{H^2(Z_{k,M-1})} \leq C_k^0 \, e^{-\sigma_L M} \text{ for all } k.$$

Thus, f_L^1 decays exponentially with rate σ_L and from Corollary 3.4 we obtain

$$\|\eta_k\,\tilde{\omega}^1\|_{H^2(Z_{k,M-1})} \le C_k^1\,e^{-\sigma_L M} \text{ for all } k.$$

Then, proceeding recursively, we define $\tilde{\omega}^j$, $j \ge 1$, as the solution of the generalized Leray problem with the right-hand side f_L^j which includes all the functions up to $\tilde{\omega}^{j-1}$, $j \ge 1$. Then f_L^j decays exponentially with rate σ_L for all $j \ge 1$ and we get

$$\|\eta_k\,\tilde{\omega}^j\|_{H^2(Z_{k,M-1})} \le C_k^j\,e^{-\sigma_L\,M} \text{ for all } k = 1, ..., N,\ j \ge 1. \tag{4.33}$$

(ii) If $\tilde{\sigma}_l = 2\sigma_L$: In this case inequality (4.30) applies (at least) for $k = l$ and thus the decay of f_L^1 is not *purely* exponential but an additional growth factor $M^{1/2}$ occurs. In order to apply Corollary 3.4 on the decay of Leray's solution $\tilde{\omega}^1$, which is proven under the assumption (3.18) of purely exponential decay of the right-hand side f, we reduce the exponential decay rate in order to absorb this growth factor: For any $0 < \sigma' < \sigma_L$ there is $C = C(\sigma')$ such that

$$|f_L^1(x)| \le C\,e^{-\sigma' x_1^k}. \tag{4.34}$$

The same arguments as in case (i) then yield recursively for all $j \ge 1$ the following estimate: For all $0 < \sigma' < \sigma_L$ there are constants $C_k^j = C_k^j(\sigma')$ such that

$$\|\eta_k\,\tilde{\omega}^j\|_{H^2(Z_{k,M-1})} \le C_k^j\ e^{-\sigma' M} \text{ for all } k = 1, ..., N. \tag{4.35}$$

(iii) If $\tilde{\sigma}_l < 2\sigma_L$: From Lemma 4.3 we obtain that any $\eta_k\,\tilde{\omega}^0$, $k = 1, ..., N$, decays exponentially (at least) with the rate $\tilde{\sigma}_l/2$, which implies as above the same decay rate for f_L^1. Applying Corollary 3.4 with $2\sigma_{f_L^1} = \tilde{\sigma}_l$ yields

$$\|\eta_l\,\tilde{\omega}^1\|_{H^2(Z_{l,M-1})} \le C_l^1\ M^{1/2}\,e^{-\frac{\tilde{\sigma}_l}{2}M}.$$

Since the next order f_L^2 may include a term with the function $\tilde{\omega}^1$, its decay is thus not anymore purely exponential and we reduce the decay rate as in case (ii), cf. (4.34). This leads to an estimate similar to (4.35): For all $j \ge 1$ and all $0 < \sigma' < \tilde{\sigma}_l/2$ there exist constants $C_k^j = C_k^j(\sigma')$ such that

$$\|\eta_k\,\tilde{\omega}^j\|_{H^2(Z_{k,M-1})} \le C_k^j\ e^{-\sigma' M} \text{ for all } k = 1, ..., N. \tag{4.36}$$

Summarizing, we thus have established an approximation result for $(\tilde{\omega}^{M,1}, \tilde{\pi}^{M,1})$ analogous to Theorem 4.4: The solution $(\tilde{\omega}^{M,1}, \tilde{\pi}^{M,1})$ of the inertial correction problem in the junction Ω^M approximates the corresponding Leray's solution $(\tilde{\omega}^1, \tilde{\pi}^1)$ up to an error decaying exponentially with the junction length M.

Proceeding recursively, we define $f^{M,j} := -\sum_{k,l}(\omega_k^M \cdot \nabla)\omega_l^M$, $j \in \mathbb{N}$, where each function ω_i^M, $i \in I$, is either the solution of a Poiseuille junction problem or one of the solutions $\tilde{\omega}^{M,k}$, $k = 0, 1, ..., j-1$, of the inertial correction problems with right-hand side $f^{M,k}$.

The results obtained so far allow a *generalization* of (4.32) for $\tilde{f}^j := f^{M,j} - f_L^j$, $j \geq 2$: For all $0 < \sigma' < \tilde{\sigma}_l/2$ and all $j \geq 2$, there are constants $r_j > 0$ and $\tilde{C}_j = \tilde{C}_j(\sigma') > 0$ such that the following estimate holds:

$$\text{If } \ min_k\, \tilde{\sigma}_k > 2\sigma_L: \ \|\tilde{f}^j\|_{L^2(\Omega^M)} \leq \tilde{C}_j\, M^{r_j}\, e^{-\sigma_L M}, \tag{4.37}$$

$$\text{If } \ min_k\, \tilde{\sigma}_k \leq 2\sigma_L: \ \|\tilde{f}^j\|_{L^2(\Omega^M)} \leq \tilde{C}_j\, e^{-\sigma' M}, \tag{4.38}$$

for all $M \geq 1$, the constants $\tilde{C}_j$ being independent of M.

Thus, we have shown the following generalization of Theorem 4.4:

Corollary 4.1. *The solution $(\tilde{\omega}^{M,j}, \tilde{\pi}^{M,j})$, $j \geq 1$, of the inertial correction problem with $f^M \equiv f^{M,j}$ in (4.1) approximates the solution $(\tilde{\omega}^j, \tilde{\pi}^j)$ of the generalized Leray problem (3.16) with $f_L \equiv f_L^j$ and $\int_{\Omega^M} \left(\tilde{\pi}^{M,j} - \tilde{\pi}^j\right) = 0$ in the following way:*

(i) If $min_k\, \tilde{\sigma}_k > 2\sigma_L$, then there exist constants $s_j, C_j > 0$ such that

$$\|\tilde{\omega}^{M,j} - \tilde{\omega}^j\|_{H^2(\Omega^M)} + \|\tilde{\pi}^{M,j} - \tilde{\pi}^j\|_{H^1(\Omega^M)} \leq C_j\, M^{s_j}\, e^{-\sigma_L M} \tag{4.39}$$

for all $M \geq 1$.

(ii) If $min_k\, \tilde{\sigma}_k \leq 2\sigma_L$, then for any $0 < \sigma' < min_k\, \tilde{\sigma}_k/2$ there exist constants $C_j > 0$, depending on σ', such that

$$\|\tilde{\omega}^{M,j} - \tilde{\omega}^j\|_{H^2(\Omega^M)} + \|\tilde{\pi}^{M,j} - \tilde{\pi}^j\|_{H^1(\Omega^M)} \leq C_j\, e^{-\sigma' M} \tag{4.40}$$

for all $M \geq 1$.

Remark: The exponents r_j and s_j in (4.37) and (4.39), respectively, are related by $r_j = s_{j-1} + \frac{1}{2}$, $j \geq 1$. They can be given explicitly: $s_j = \dfrac{3}{2}(1+j)$ and $r_j = \dfrac{1}{2} + \dfrac{3}{2}j$ for all $j \geq 0$.

Chapter 5

Approximation of the Navier-Stokes solution

In this chapter we construct the approximation for the Navier-Stokes system (2.2). First, we fix its general structure and define the leading order terms. Then, due to the decay properties of Leray's pressure to a linear profile *shifted by some additional stabilization constant* (cf. section 3.1), we establish a higher order correction which is necessary to approximate the gradient of velocity. We finally take into account the nonlinear term of the Navier-Stokes system (2.2) by adding further corrections in powers of the Reynolds number Re_ϵ.

5.1 General structure and leading order terms

We start this section by summarizing the strategy, referring to the discussion in section 2.5 (*Leray-Problem approach*).
In the pipes we assume a Poiseuille flow, driven by the pressure drop which is computed from Kirchhoff's law (*pressure mean value* q^0). Thus, we have satisfied the velocity and pressure conditions on the in-/outflow boundaries. On the interfaces $\gamma_j^{M,\epsilon}$, which are at distance ϵM from the in-/outflow cross-sections $\gamma_j^{0,\epsilon}$ of the junction Ω_0^ϵ, we match the Poiseuille velocity continuously to the (scaled) junction velocity, i.e. to the solution of the Poiseuille junction problem. We show that such an approximation fulfills system (2.2) up to an error consisting of two parts: the jumps of the *normal forces* on the interfaces $\gamma_j^{M,\epsilon}$ and the *inertial* terms. Since $Re_\epsilon \leq O(\epsilon)$ (cf. Theorem 2.1), the nonlinear term of (2.2) is of higher order. Therefore, the error is determined by the jump terms which we estimate by using the decay properties of the solution of Leray's problem. We can apply them directly to the solution of the junction problem due to the approximation results of section 4.2.

We first define the *general structure* of the approximation: The *zero-order* approximation is given by

$$u_{0,0}^\epsilon := \mathcal{V}_{0,0}^\epsilon + \mathcal{W}_{0,0}^\epsilon, \tag{5.1}$$

$$q_{0,0}^\epsilon := \mathcal{P}_{0,0}^\epsilon + \Pi_{0,0}^\epsilon, \tag{5.2}$$

and the approximation of order (k,l), $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, is defined by

$$u^\epsilon_{k,l} := u^\epsilon_{0,0} + \sum_{\kappa=1}^{k}\sum_{\lambda=0}^{l} \epsilon^\kappa Re^\lambda_\epsilon \left(\mathcal{V}^\epsilon_{\kappa,\lambda} + \mathcal{W}^\epsilon_{\kappa,\lambda}\right) + \sum_{\kappa=1}^{k}\sum_{\lambda=1}^{l} \epsilon^{\kappa-1} Re^\lambda_\epsilon \tilde{\mathcal{W}}^\epsilon_{\kappa-1,\lambda}, \tag{5.3}$$

$$q^\epsilon_{k,l} := q^\epsilon_{0,0} + \sum_{\kappa=1}^{k}\sum_{\lambda=0}^{l} \epsilon^\kappa Re^\lambda_\epsilon \left(\mathcal{P}^\epsilon_{\kappa,\lambda} + \Pi^\epsilon_{\kappa,\lambda}\right) + \sum_{\kappa=1}^{k}\sum_{\lambda=1}^{l} \epsilon^{\kappa-1} Re^\lambda_\epsilon \tilde{\Pi}^\epsilon_{\kappa-1,\lambda}, \tag{5.4}$$

where $(\mathcal{V}^\epsilon_{\kappa,\lambda}, \mathcal{P}^\epsilon_{\kappa,\lambda})$ is the Poiseuille flow of order $\epsilon^\kappa Re^\lambda_\epsilon$, $(\mathcal{W}^\epsilon_{\kappa,\lambda}, \Pi^\epsilon_{\kappa,\lambda})$ denotes the solution of the corresponding junction problem and $(\tilde{\mathcal{W}}^\epsilon_{\kappa,\lambda}, \tilde{\Pi}^\epsilon_{\kappa,\lambda})$ includes the velocity and pressure corrections for the inertial terms:

(*i*) *Poiseuille flow:*

$$\mathcal{V}^\epsilon_{\kappa,\lambda}(x) := \sum_{j=1}^{N} V_j^{\kappa,\lambda}\left(\frac{\tilde{x}^j}{\epsilon}\right) \chi^\epsilon_j(x), \tag{5.5}$$

$$\mathcal{P}^\epsilon_{\kappa,\lambda}(x) := \sum_{j=1}^{N} P_j^{\kappa,\lambda}(x^j_1)\, \chi^\epsilon_j(x), \tag{5.6}$$

where $(V_j^{\kappa,\lambda}, P_j^{\kappa,\lambda})$ is defined recursively by (5.39)-(5.40).

(*ii*) *Junction flow:*

$$\mathcal{W}^\epsilon_{\kappa,\lambda}(x) := \omega^{\kappa,\lambda}\left(\frac{x}{\epsilon}\right) \chi^\epsilon(x), \tag{5.7}$$

$$\Pi^\epsilon_{\kappa,\lambda}(x) := \left(q^{\kappa,\lambda} + \epsilon\, \pi^{\kappa,\lambda}\left(\frac{x}{\epsilon}\right)\right) \chi^\epsilon(x), \tag{5.8}$$

where $(\omega^{\kappa,\lambda}, \pi^{\kappa,\lambda})$ is the solution of the *Poiseuille junction problem* with in- and outflow velocities $V_j^{\kappa,\lambda}$, $j = 1, ..., N$. The constants $q^{\kappa,\lambda}$ are defined as weighted mean values (in analogy to (2.7)) in order to balance the flux through the junction (cf. sections 5.2 and 5.3 below). The additional factor ϵ in (5.8) reflects the $O(\epsilon)$-diameter of the junction domain Ω^ϵ_0.

(*iii*) *Inertial corrections:*

$$\tilde{\mathcal{W}}^\epsilon_{\kappa,\lambda}(x) := \tilde{\omega}^{\kappa,\lambda}\left(\frac{x}{\epsilon}\right) \chi^\epsilon(x), \tag{5.9}$$

$$\tilde{\Pi}^\epsilon_{\kappa,\lambda}(x) := \epsilon\, \tilde{\pi}^{\kappa,\lambda}\left(\frac{x}{\epsilon}\right) \chi^\epsilon(x), \tag{5.10}$$

where $(\tilde{\omega}^{\kappa,\lambda}, \tilde{\pi}^{\kappa,\lambda})$ is the solution of the *inertial correction problem* (5.33) corresponding to the order (κ, λ).

The *characteristic functions* χ_j^ϵ and χ^ϵ are defined as follows:
$\chi_j^\epsilon := \chi_{\Omega_j^\epsilon \setminus Z_j^{M,\epsilon}}$ on the j^{th} pipe Ω_j^ϵ without the cylinder $Z_j^{M,\epsilon}$ of length ϵM and $\chi^\epsilon := \chi_{\Omega^{M,\epsilon}}$ on the extended junction $\Omega^{M,\epsilon}$, consisting of Ω_0^ϵ and the cylinders $Z_j^{M,\epsilon}$, $j = 1, ..., N$.

Remark: Since by definition $\tilde{\omega}^{\kappa,\lambda}$ carries no flux through the junction, we do not need a balancing Poiseuille flow for it in the pipes. Therefore, in (5.10) there are no constants $\tilde{q}^{\kappa,\lambda}$ analogous to $q^{\kappa,\lambda}$ in (5.8). Actually, there are orders for which no Poiseuille and junction flow are present: $(\mathcal{V}_{0,\lambda}^\epsilon, \mathcal{P}_{0,\lambda}^\epsilon) \equiv 0$ and $(\mathcal{W}_{0,\lambda}^\epsilon, \Pi_{0,\lambda}^\epsilon) \equiv 0$ for all $\lambda \geq 1$. For $l = 0$ the inertial correction term is omitted.
Due to an additional pressure correction the approximation of order (k, l) includes inertial corrections only up to the order $k - 1$ (cf. section 5.3).

We define the leading order terms of the approximation as follows:

Definition. (Zero-order approximation) The leading order Poiseuille flow is defined as

(5.11) $$V_j^{0,0}(\tilde{y}^j) = w_j(\tilde{y}^j) \frac{q^{0,0} - p_j}{L_j} e_1^j, \quad \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon},$$

(5.12) $$P_j^{0,0}(x_1^j) = q^{0,0} + \frac{p_j - q^{0,0}}{L_j} x_1^j,$$

where $q^{0,0} \equiv q^0$ is the weighted mean value of the (constant) pressure values p_k, $k = 1, ..., N$ (cf. equation (2.7)). The corresponding solution of the junction problem (4.1) with in-/outflow velocities $V_k^{0,0}$ is denoted by $(\omega^{0,0}, \pi^{0,0})$.

We thus have as zero-order approximation

(5.13) $$u_{0,0}^\epsilon(x) := \sum_j V_j^{0,0}(\frac{\tilde{x}^j}{\epsilon}) \chi_j^\epsilon(x) + \omega^{0,0}(\frac{x}{\epsilon}) \chi^\epsilon(x),$$

(5.14) $$q_{0,0}^\epsilon(x) := \sum_j P_j^{0,0}(x_1^j)\, \chi_j^\epsilon(x) + \left(q^{0,0} + \epsilon \pi^{0,0}(\frac{x}{\epsilon})\right) \chi^\epsilon(x).$$

5.2 Pressure decay correction

In this section we construct velocity and pressure corrections to the zero-order approximation $(u_{0,0}^\epsilon, q_{0,0}^\epsilon)$, taking into account the stabilization constants which occur in the exponential pressure decay of Leray's problem, cf. equation (3.14). These corrections are necessary in order to obtain an approximation for the solution of problem (2.2) which allows appropriate error estimates for velocity and pressure *including* the velocity gradient.

We do not know *a priori* the flux through the pipes since we deal with pressure boundary conditions. Therefore, we apply the weighted mean value of the given outflow pressures to approximate the flux, simplifying the pipe-junction network as a *one-dimensional* structure. Having neglected the real *three-dimensional* geometry of the junction, which is of diameter $O(\epsilon)$ compared to the lengths of the pipes, we get an appropriate approximation for the flux only in leading order. Thus, we expect an error of order $O(\epsilon)$ in velocity which yields an error of order $O(1)$ in its gradient (since the pipes are of diameter $O(\epsilon)$). We refer to chapter 6 for the detailed discussion of the error estimates.

On the interface $\gamma_j^{M,\epsilon}$ $(j = 1, ..., N)$ there is a jump from the junction pressure $\pi^0 \equiv \pi^{0,0}$ to the linear Poiseuille pressure profile. Due to the approximation results of section 4.2, π^0 approximates the corresponding Leray pressure π_L^0 which tends to the Poiseuille pressure profile plus some stabilization constant in each pipe. In order to obtain the exponential decay of the pressure jump on $\gamma_j^{M,\epsilon}$ we have to add a correction for these pressure constants (cf. Lemma 6.2 and the corresponding Remark 6.2).

The first-order pressure decay correction is constructed as follows: The functions $\tau_{L,j}^0(y) = \pi_L^0(y) - C_j^0\, y_1^j$, denoting the difference between the Leray pressure π_L^0 and the linear Poiseuille profile in the j^{th} pipe Ω_j^∞ given by $C_j^0 := \dfrac{p_j - q^0}{L_j}$, tend to some stabilization constants τ_j^∞ as $y_1^j \to \infty$. Since π_L^0 is approximated by the junction pressure π^0 (cf. section 4.2), we define $\tau_j^0(y) := \pi^0(y) - C_j^0\, y_1^j$ and take its mean value over the cross-section γ_j^M, i.e.

$$\langle \tau_j^0 \rangle := \frac{1}{|\gamma_j^M|} \int_{\gamma_j^M} \tau_j^0. \tag{5.15}$$

Due to the approximation properties of π^0 the constants τ_j^∞ can be corrected by $\langle \tau_j^0 \rangle$ up to an exponentially decreasing error (for the corresponding estimates see section 6.1).

Since the mean values $\langle \tau_j^0 \rangle$, $j = 1, ..., N$, are non-zero in general, we have to introduce an additional Poiseuille flow of the following type:

$$V_j^{1,0}(\tilde{y}^j) := -w_j(\tilde{y}^j)\, C_j^{1,0}\, e_1^j, \quad \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon}, \tag{5.16}$$

$$P_j^{1,0}(x_1^j) := q^{1,0} + \langle \tau_j^0 \rangle + C_j^{1,0}\, x_1^j, \tag{5.17}$$

where $C_j^{1,0} := -\dfrac{q^{1,0} + \langle \tau_j^0 \rangle}{L_j}$ and $q^{1,0}$ is taken such that the fluxes in the pipes are balanced, i.e. $q^{1,0} := -\dfrac{\sum_k c_k \langle \tau_k^0 \rangle}{\sum_k c_k}$. The pressure boundary conditions of (2.2) are already fulfilled by the leading order term, therefore we have set $P_j^{1,0}(L_j) = 0$. Since the pressures π_L^0 and π^0 are unique only up to an additive constant, we can choose them such that the mean value $q^{1,0}$ equals zero, which simplifies the approximation.

The Poiseuille flow $(V_j^{1,0}, P_j^{1,0})$ has to be balanced by the junction flow $(\omega^{1,0}, \pi^{1,0})$, i.e. by the solution of the junction problem

$$\text{(5.18)} \qquad \begin{cases} -\mu_0 \Delta_y \omega^{1,0} + \nabla_y \pi^{1,0} &= 0 \quad \text{in } \Omega^M, \\ div_y\, \omega^{1,0} &= 0 \quad \text{in } \Omega^M, \\ \omega^{1,0} &= 0 \quad \text{on } \Gamma^M, \\ \omega^{1,0} &= V_j^{1,0} \quad \text{on } \gamma_j^M. \end{cases}$$

Summarizing, we have

Definition. (First order approximation)

$$\text{(5.19)} \qquad u_{1,0}^\epsilon(x) := u_{0,0}^\epsilon(x) + \epsilon \left(\sum_j V_j^{1,0}(\frac{\tilde{x}^j}{\epsilon}) \chi_j^\epsilon(x) + \omega^{1,0}(\frac{x}{\epsilon}) \chi^\epsilon(x) \right),$$

$$\text{(5.20)} \qquad q_{1,0}^\epsilon(x) := q_{0,0}^\epsilon(x) + \epsilon \left(\sum_j P_j^{1,0}(x_1^j)\, \chi_j^\epsilon(x) + \epsilon \pi^{1,0}(\frac{x}{\epsilon}) \chi^\epsilon(x) \right).$$

The same arguments as above now apply to the pressure $\pi^{1,0}$ and the corresponding Leray pressure $\pi_L^{1,0}$, which tends in the j^{th} pipe to the linear profile given by $C_j^{1,0}$ plus some stabilization constant. Recursively, we thus can define for any higher order $(\kappa, 0)$, $\kappa \geq 1$, the pressure function $\tau_j^{\kappa,0}(y) := \pi^{\kappa,0}(y) - C_j^{\kappa,0}\, y_1^j$ in Ω_j^∞, where

$$C_j^{\kappa,0} := -\frac{q^{\kappa,0} + \langle \tau_j^{\kappa-1,0} \rangle}{L_j}, \ \kappa \geq 1.$$

In the same way as above all weighted mean values $q^{\kappa,0}$, $\kappa \geq 1$, can be set to zero. The Poiseuille flow of order $(\kappa, 0)$, $\kappa \geq 1$, then reads

$$\text{(5.21)} \qquad V_j^{\kappa,0}(\tilde{y}^j) := -w_j(\tilde{y}^j)\, C_j^{\kappa,0}\, e_1^j, \ \ \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon},$$

$$\text{(5.22)} \qquad P_j^{\kappa,0}(x_1^j) := \langle \tau_j^{\kappa-1,0} \rangle + C_j^{\kappa,0}\, x_1^j,$$

and the corresponding junction flow $(\omega^{\kappa,0}, \pi^{\kappa,0})$ solves (5.18) with boundary condition $\omega^{\kappa,0} = V_j^{\kappa,0}$ on γ_j^M.

5.3 Inertial corrections

In order to define recursively the correction terms, we first analyze formally the non-linear term $\epsilon Re_\epsilon (u_{k,l}^\epsilon \cdot \nabla) u_{k,l}^\epsilon$. Since for Poiseuille flow these convective terms are identical zero, only the junction flow remains. The following three types of inertial terms $g_m^{i,j} = g_m^{i,j}(y)$, $m = 1, 2, 3$, occur in the order $O(\epsilon^i Re_\epsilon^{j+1})$, for $(k, l) \in \mathbb{N}_0 \times \mathbb{N}_0$, with $0 \le i \le 2k$, $0 \le j \le 2l$:

$$g_1^{i,j} := \sum_{0\le\kappa,\kappa'\le k} \sum_{0\le\lambda,\lambda'\le l} (\omega^{\kappa,\lambda} \cdot \nabla_y)\omega^{\kappa',\lambda'}, \quad \text{with } \kappa+\kappa' = i,\ \lambda+\lambda' = j, \quad \text{and if } i = 0 \text{ then } j = 0, \tag{5.23}$$

$$g_2^{i,j} := \sum_{\substack{0\le\kappa\le k\\ 1\le\kappa'\le k}} \sum_{\substack{0\le\lambda\le l\\ 1\le\lambda'\le l}} \Big((\omega^{\kappa,\lambda} \cdot \nabla_y)\tilde{\omega}^{\kappa'-1,\lambda'} + (\tilde{\omega}^{\kappa'-1,\lambda'} \cdot \nabla_y)\omega^{\kappa,\lambda} \Big), \quad \text{with } \kappa+\kappa'-1 = i,\ \lambda+\lambda' = j,\ j \ge 1, \tag{5.24}$$

$$g_3^{i,j} := \sum_{1\le\kappa,\kappa'\le k} \sum_{1\le\lambda,\lambda'\le l} (\tilde{\omega}^{\kappa-1,\lambda} \cdot \nabla_y)\tilde{\omega}^{\kappa'-1,\lambda'}, \quad \text{with } \kappa+\kappa'-2 = i,\ \lambda+\lambda' = j,\ j \ge 2. \tag{5.25}$$

Note that if $i = 0$ then only the term $g_1^{0,0}$ is present and no $g_1^{0,j}$-terms with $j \ge 1$ occur. The terms $g_2^{i,j}$ and $g_3^{i,j}$ are defined for $j \ge 1$ and $j \ge 2$, respectively.
Due to the regularity properties of the junction flows and the inertial correction velocities the functions $g_m^{i,j}$ are in $L^2(\Omega^M)$.

In order to remove the leading nonlinear (*inertial*) term $g_1^{0,0} = (\omega^{0,0} \cdot \nabla)\omega^{0,0}$ of order $O(Re_\epsilon)$, we add the solution $(\tilde{\omega}^{0,1}, \tilde{\pi}^{0,1})$ of the following junction problem to the approximation:

$$\left\{ \begin{array}{rcll} -\mu_0 \Delta_y \tilde{\omega}^{0,1} + \nabla_y \tilde{\pi}^{0,1} & = & -(\omega^{0,0} \cdot \nabla)\omega^{0,0} & \text{in } \Omega^M, \\ div_y\, \tilde{\omega}^{0,1} & = & 0 & \text{in } \Omega^M, \\ \tilde{\omega}^{0,1} & = & 0 & \text{on } \partial\Omega^M. \end{array} \right. \tag{5.26}$$

The situation now is similar to the case of the Poiseuille junction problem: Due to the approximation results of section 4.2, the decay properties of the corresponding Leray pressure $\tilde{\pi}_L^{0,1}$ (cf. section 3.2) apply directly to the junction pressure $\tilde{\pi}^{0,1}$. In each pipe, the function $\tilde{\pi}_L^{0,1}$ tends to some stabilization constant at infinity. Since in general these constants are non-zero and different from each other, they have to be corrected, otherwise the approximation error is *not* improved by including the solution of (5.26) (cf. chapter 6).

We proceed as in the previous section and establish an additional Poiseuille flow in the pipes and the corresponding junction flow. To this end, we define the pressure mean values

$$\langle \tilde{\pi}^{0,1} \rangle_j := \frac{1}{|\gamma_j^M|} \int_{\gamma_j^M} \tilde{\pi}^{0,1}, \qquad j = 1, ..., N, \tag{5.27}$$

and the weighted mean value $q^{1,1}$ of the $(-\langle\tilde{\pi}^{0,1}\rangle_j)$'s in analogy to q^0, cf. (2.7).

As above, we define a Poiseuille flow correction which is driven by the pressure drop $(q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j)$:

$$\tilde{V}_j^{1,1}(\tilde{y}^j) = w_j(\tilde{y}^j)\frac{q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j}{L_j}\, e_1^j, \quad \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon}, \tag{5.28}$$

$$\tilde{P}_j^{1,1}(x_1^j) = q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j - \frac{q^{1,1} + \langle\tilde{\pi}^{0,1}\rangle_j}{L_j}\, x_1^j. \tag{5.29}$$

This Poiseuille flow generates an additional flux through the junction, therefore we have to solve a junction problem with in-/outflow velocities $\tilde{V}_j^{1,1}$:

$$\left\{ \begin{array}{rcll} -\mu_0\Delta_y\omega^{1,1} + \nabla_y\pi^{1,1} & = & 0 & \text{in } \Omega^M, \\ div_y\,\omega^{1,1} & = & 0 & \text{in } \Omega^M, \\ \omega^{1,1} & = & 0 & \text{on } \Gamma^M, \\ \omega^{1,1} & = & \tilde{V}_j^{1,1} & \text{on } \gamma_j^M. \end{array} \right. \tag{5.30}$$

The improved approximation now is defined as follows:

Definition. (Approximation including first order nonlinear corrections)

$$u_{1,1}^\epsilon(x) := u_{1,0}^\epsilon(x) + Re_\epsilon \left\{ \sum_j \epsilon\tilde{V}_j^{1,1}(\frac{\tilde{x}^j}{\epsilon})\chi_j^\epsilon(x) + \left(\tilde{\omega}^{0,1}(\frac{x}{\epsilon}) + \epsilon\,\omega^{1,1}(\frac{x}{\epsilon})\right)\chi^\epsilon(x) \right\}, \tag{5.31}$$

$$q_{1,1}^\epsilon(x) := q_{1,0}^\epsilon(x) + \epsilon Re_\epsilon \left\{ \sum_j \tilde{P}_j^{1,1}(x_1^j)\,\chi_j^\epsilon(x) + \left(\tilde{\pi}^{0,1}(\frac{x}{\epsilon}) + \epsilon\pi^{1,1}(\frac{x}{\epsilon})\right)\chi^\epsilon(x) \right\}, \tag{5.32}$$

where we have set $q^{1,1} = 0$, being possible due to the fact that $\tilde{\pi}^{0,1}$ is uniquely determined only up to an additive constant.

Higher order terms are established recursively: The approximation of order (k, l), $k, l \in \mathbb{N}$, includes inertial correction terms up to the order $O(\epsilon^{k-1}Re_\epsilon^l)$, i.e. for all terms $g_m^{i,j}$, $m = 1, 2, 3$, in (5.23)-(5.25) where $i \leq k-1$, $j \leq l-1$. Generalizing problem (5.26), the solution $(\tilde{\omega}^{\kappa,\lambda}, \tilde{\pi}^{\kappa,\lambda})$, $0 \leq \kappa \leq k-1$, $1 \leq \lambda \leq l$, is defined such that it corrects all inertial terms of order $O(\epsilon^\kappa Re_\epsilon^\lambda)$, or, in other words, all $g_m^{i,j}$ for which $i = \kappa$ and $j = \lambda - 1$:

$$\left\{ \begin{array}{rcll} -\mu_0\Delta_y\tilde{\omega}^{\kappa,\lambda} + \nabla_y\tilde{\pi}^{\kappa,\lambda} & = & -g^{\kappa,\lambda-1} & \text{in } \Omega^M, \\ div_y\,\tilde{\omega}^{\kappa,\lambda} & = & 0 & \text{in } \Omega^M, \\ \tilde{\omega}^{\kappa,\lambda} & = & 0 & \text{on } \partial\Omega^M, \end{array} \right. \tag{5.33}$$

where $g^{i,j} := \sum_{m=1}^{3} g_m^{i,j}$.

In analogy to the first order correction above we define for each solution $(\tilde{\omega}^{\kappa,\lambda}, \tilde{\pi}^{\kappa,\lambda})$ of (5.33) the following Poiseuille flow in order to correct the pressure decay:

$$\tilde{V}_j^{\kappa+1,\lambda}(\tilde{y}^j) = w_j(\tilde{y}^j)\frac{\langle\tilde{\pi}^{\kappa,\lambda}\rangle_j}{L_j}\, e_1^j, \quad \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon}, \tag{5.34}$$

$$\tilde{P}_j^{\kappa+1,\lambda}(x_1^j) = \langle\tilde{\pi}^{\kappa,\lambda}\rangle_j - \frac{\langle\tilde{\pi}^{\kappa,\lambda}\rangle_j}{L_j}\, x_1^j. \tag{5.35}$$

As above we have set without loss of generality the weighted mean values $q^{\kappa+1,\lambda} = 0$.

Finally, the junction flow $(\omega^{\kappa+1,\lambda}, \pi^{\kappa+1,\lambda})$ balances this Poiseuille flow in the pipes in analogy to (5.30).

Remark: For the correction of the inertial term of order $O(\epsilon^\kappa Re_\epsilon^\lambda)$ a Poiseuille/junction flow in $O(\epsilon^{\kappa+1} Re_\epsilon^\lambda)$ is needed in order to remove the pressure decay constants. Therefore, the approximation of order (k, l) only corrects inertial terms up to the order $k-1$.

Applying the pressure decay correction as described in section 5.2 to $(\omega^{1,1}, \pi^{1,1})$, we get a Poiseuille flow analogous to (5.16)-(5.17) in order $O(\epsilon^2 Re_\epsilon)$. On the other hand there occurs an additional Poiseuille flow in the same order due to the pressure decay correction for the solution $(\tilde{\omega}^{1,1}, \tilde{\pi}^{1,1})$ of (5.33). In order to summarize these terms we define $\tau_j^{\kappa,\lambda} := \pi^{\kappa,\lambda} - C_j^{\kappa,\lambda}\, y_1^j$ for all $\kappa \in \mathbb{N}$, $\lambda \in \mathbb{N}_0$, where

$$C_j^{\kappa,0} := -\frac{q^{\kappa,0} + \langle\tau_j^{\kappa-1,0}\rangle}{L_j}, \quad \kappa \geq 1, \tag{5.36}$$

$$C_j^{1,\lambda} := -\frac{q^{1,\lambda} + \langle\tilde{\pi}^{0,\lambda}\rangle_j}{L_j}, \quad \lambda \geq 1, \tag{5.37}$$

$$C_j^{\kappa,\lambda} := -\frac{q^{\kappa,\lambda} + \langle\tau_j^{\kappa-1,\lambda}\rangle + \langle\tilde{\pi}^{\kappa-1,\lambda}\rangle_j}{L_j}, \quad \kappa \geq 2, \lambda \geq 1. \tag{5.38}$$

Without loss of generality we can fix $q^{\kappa,\lambda} = 0$ for all $\kappa \in \mathbb{N}, \lambda \in \mathbb{N}_0$.

Thus, the Poiseuille flow for any order $\kappa \in \mathbb{N}$, $\lambda \in \mathbb{N}_0$ is defined by

$$V_j^{\kappa,\lambda}(\tilde{y}^j) := -w_j(\tilde{y}^j)\, C_j^{\kappa,\lambda}\, e_1^j, \quad \tilde{y}^j = \frac{\tilde{x}^j}{\epsilon}, \tag{5.39}$$

$$P_j^{\kappa,\lambda}(x_1^j) := C_j^{\kappa,\lambda}\,(x_1^j - L_j). \tag{5.40}$$

Remark: The correction terms $(\tilde{\omega}^{0,1}, \tilde{\pi}^{0,1})$ (cf. equation (5.26)) have been computed numerically in [C].

Chapter 6

The approximation error

We prove *error estimates* for the approximations defined in the previous chapter, proceeding in two steps: At first we show that the Navier-Stokes system (2.2) is satisfied up to an error consisting of *jumps of the normal force* on the pipe-junction interfaces and *inertial terms* due to the nonlinearity. We then establish the required estimates (cf. section 6.1). In the second part we are then able to prove estimates comparing the approximation $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$ with the solution (v^ϵ, p^ϵ) of system (2.2).

6.1 Approximation properties and jump estimates

6.1.1 Jumps and inertial terms

We start with the definition of the error terms.

Definition 6.1. (Jump terms) For the approximation of order (k, l) the following *jumps of the normal force* occur on the pipe-junction interfaces γ^M_j, $j = 1, ..., N$ with normal vector $n_j = e^j_1$:

(i) Jumps of the normal force due to the transition from junction flow to Poiseuille flow:

(6.1) $$f^{\kappa,\lambda}_j(y) := -\mu_0 \nabla_y \omega^{\kappa,\lambda}(y) n_j|_{\gamma^M_j} + \tau^{\kappa,\lambda}_j(y)\, n_j|_{\gamma^M_j},$$

for $\kappa, \lambda = 0$ and $1 \leq \kappa \leq k$, $0 \leq \lambda \leq l$.

(ii) Jumps of the normal force due to the inertial correction:

(6.2) $$\tilde{f}^{\kappa,\lambda}_j(y) := -\mu_0 \nabla_y \tilde{\omega}^{\kappa,\lambda}(y) n_j|_{\gamma^M_j} + \left(\tilde{\pi}^{\kappa,\lambda}(y) - \langle \tilde{\pi}^{\kappa,\lambda}\rangle_j\right) n_j|_{\gamma^M_j},$$

for $0 \leq \kappa \leq k-1$ and $1 \leq \lambda \leq l$.

Furthermore, summing over all j, we define

$$f^{\kappa,\lambda} := \sum_{j=1}^N f^{\kappa,\lambda}_j\, \delta^\epsilon_j, \quad \tilde{f}^{\kappa,\lambda} := \sum_{j=1}^N \tilde{f}^{\kappa,\lambda}_j\, \delta^\epsilon_j.$$

Including the pressure decay correction, we define

$$f_\tau^{\kappa,\lambda} := \sum_{j=1}^{N} \left(f_j^{\kappa,\lambda} - \langle \tau_j^{\kappa,\lambda} \rangle n_j \right) \delta_j^\epsilon.$$

Here $\delta_j^\epsilon := \delta_{\gamma_j^{M,\epsilon}}$ denotes the *Dirac distribution* on the interface $\gamma_j^{M,\epsilon}$ defined by $\langle \delta_j^\epsilon, \phi \rangle := \int_{\gamma_j^{M,\epsilon}} \phi$ for all $\phi \in H^1(\Omega^\epsilon)$.

Due to the $H^2 \times H^1$-regularity of the junction flow $(\omega^{\kappa,\lambda}, \pi^{\kappa,\lambda})$ and the inertial correction $(\tilde{\omega}^{\kappa,\lambda}, \tilde{\pi}^{\kappa,\lambda})$, we have $f_j^{\kappa,\lambda}, \tilde{f}_j^{\kappa,\lambda} \in H^{1/2}(\gamma_j^M)$.

The *total error* due to these jump terms is given by (cf. Lemma 6.1 below)

$$F_{0,0}^\epsilon(x) := \epsilon f^{0,0}(\frac{x}{\epsilon}), \tag{6.3}$$

$$F_{1,0}^\epsilon(x) := \epsilon \left(f_\tau^{0,0}(\frac{x}{\epsilon}) + \epsilon f^{1,0}(\frac{x}{\epsilon}) \right), \tag{6.4}$$

$$F_{1,l}^\epsilon(x) := F_{1,0}^\epsilon + \epsilon \sum_{1 \le \lambda \le l} Re_\epsilon^\lambda \left(\tilde{f}^{0,\lambda}(\frac{x}{\epsilon}) + \epsilon f^{1,\lambda}(\frac{x}{\epsilon}) \right), \quad l \ge 1, \tag{6.5}$$

and

$$\begin{aligned} F_{k,l}^\epsilon(x) := \epsilon f_\tau^{0,0} &+ \sum_{\substack{2 \le \kappa \le k \\ 0 \le \lambda \le l}} \epsilon^\kappa Re_\epsilon^\lambda \, f_\tau^{\kappa-1,\lambda}(\frac{x}{\epsilon}) \; + \epsilon^{k+1} \sum_{0 \le \lambda \le l} Re_\epsilon^\lambda f^{k,\lambda}(\frac{x}{\epsilon}) \\ &+ \sum_{\substack{1 \le \kappa \le k \\ 1 \le \lambda \le l}} \epsilon^\kappa \, Re_\epsilon^\lambda \, \tilde{f}^{\kappa-1,\lambda}(\frac{x}{\epsilon}), \end{aligned} \tag{6.6}$$

for $k \ge 2$, $l \ge 0$, the last sum being omitted for $l = 0$.

Remark: Note that the pressure decay correction $\langle \tau_j^{k,l} \rangle$ for the Poiseuille junction flow first occurs in the term $F_{k+1,l}$ whereas the term $\langle \tilde{\pi}^{k,l} \rangle_j$ for the inertial correction is already included in the jump term $\tilde{f}_j^{k,l}$.

Definition 6.2. (Inertial terms) With the definitions (5.23)-(5.25) of the inertial terms $g_m^{i,j}$ and $g^{i,j} := \sum_{m=1}^{3} g_m^{i,j}$ the *total error* is given by (cf. Lemma 6.1 below)

$$G_{k,0}^\epsilon(x) := Re_\epsilon \sum_{0 \le i \le 2k} \epsilon^i g_1^{i,0}(\frac{x}{\epsilon}) \chi^\epsilon(x) \;\text{ for } k \ge 0, \tag{6.7}$$

and

$$G_{k,l}^\epsilon := G_{k,l}^{1,\epsilon} + G_{k,l}^{2,\epsilon} \;\text{ for } k, l \ge 1, \tag{6.8}$$

where

$$G^{1,\epsilon}_{k,l}(x) := Re_\epsilon \sum_{k\le i\le 2k} \sum_{0\le j\le 2l} \epsilon^i Re^j_\epsilon\, g^{i,j}(\frac{x}{\epsilon})\chi^\epsilon(x), \tag{6.9}$$

$$G^{2,\epsilon}_{k,l}(x) := Re_\epsilon \sum_{0\le i\le k-1} \sum_{l\le j\le 2l} \epsilon^i Re^j_\epsilon\, g^{i,j}(\frac{x}{\epsilon})\chi^\epsilon(x). \tag{6.10}$$

The total error $G^\epsilon_{k,l}$ only includes terms $g^{i,j}$ where $i \ge k$ or $j \ge l$, i.e. all terms with $0 \le i \le k-1$ and $0 \le j \le l-1$ are corrected.

In particular, we have for the zero and first-order approximations

$$G^\epsilon_{0,0}(x) := Re_\epsilon\, g^{0,0}_1(\frac{x}{\epsilon})\chi^\epsilon(x), \tag{6.11}$$

$$G^\epsilon_{1,0}(x) := Re_\epsilon \sum_{i=0}^{2} \epsilon^i g^{i,0}_1(\frac{x}{\epsilon})\chi^\epsilon(x), \tag{6.12}$$

$$G^\epsilon_{1,1}(x) := Re_\epsilon \sum_{\substack{i,j=0\\ i+j\ge 1}}^{2} \epsilon^i Re^j_\epsilon\, g^{i,j}(\frac{x}{\epsilon})\chi^\epsilon(x). \tag{6.13}$$

6.1.2 Approximation properties

The following lemma now states the main properties of the approximations defined in chapter 5.

Lemma 6.1. *The approximation $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$, $k, l \in \mathbb{N}_0$, defined in the previous chapter, is a function of the space $(H^1 \times L^2)(\Omega^\epsilon)$ and satisfies the Navier-Stokes system (2.2) with an error term $E^\epsilon_{k,l}$ on the right-hand side of equation $(2.2)_1$. This error consists of jumps in the normal force on the pipe-junction interfaces $\gamma^{M,\epsilon}_j$ and inertial terms in the domain $\Omega^{M,\epsilon}$: $E^\epsilon_{k,l} := F^\epsilon_{k,l} + G^\epsilon_{k,l}$.*

Proof. By construction, all approximations are in $(H^1 \times L^2)(\Omega^\epsilon)$ since $u^\epsilon_{k,l}$ is continuous on the pipe-junction interfaces. We consider the approximation $(u^\epsilon_{k,l}, q^\epsilon_{k,l})$ for which the following equation holds in distributional sense:

$$-\mu_0\epsilon^2\Delta u^\epsilon_{k,l} + \epsilon Re_\epsilon(u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l} + \nabla q^\epsilon_{k,l} \;=\; E^\epsilon_{k,l} \qquad \text{in } \Omega^\epsilon. \tag{6.14}$$

The *error* is given by

$$E^\epsilon_{k,l} = \sum_j \left[\mu_0\epsilon^2\nabla u^\epsilon_{k,l}\, n_j - q^\epsilon_{k,l}\, n_j\right]_{\gamma^{M,\epsilon}_j} \delta^\epsilon_j + G^\epsilon_{k,l}$$

where $[h]_\Sigma$ denotes the *jump* $(h^+ - h^-)$ of the traces of a function $h \in H^1(\Omega^\epsilon_j \setminus \Sigma)$ on the cross-section Σ of the j^{th} pipe from the positive and negative side (with respect to n_j). Due to the regularity of Poiseuille and junction flow we have $\nabla u^\epsilon_{k,l}$ and $q^\epsilon_{k,l}$ in

$H^1(\Omega^\epsilon \setminus \bigcup_j \gamma_j^{M,\epsilon})$. By a simple calculation we then get the jumps as defined in (6.1) and (6.2).

For almost every $x \in \Omega^{M,\epsilon}$ we have

$$\epsilon Re_\epsilon\, (u^\epsilon_{k,l} \cdot \nabla) u^\epsilon_{k,l}(x) = Re_\epsilon \sum_{0 \le i \le 2k} \sum_{0 \le j \le 2l} \epsilon^i\, Re^j_\epsilon\, g^{i,j}(\frac{x}{\epsilon}).$$

The approximation of order (k,l), $k,l \ge 1$, includes corrections for all terms $g^{i,j}$, $0 \le i \le k-1$, $0 \le j \le l-1$. Therefore, only the error terms $k \le i \le 2k$, $0 \le j \le 2l$ and $0 \le i \le k-1$, $l \le j \le 2l$ remain, summarized in $G^\epsilon_{k,l}$. For $l = 0$ there is no inertial correction and the error is given by (6.7).

Since $u^\epsilon_{k,l}$ consists of divergence-free functions and is continuous on $\gamma_j^{M,\epsilon}$, $j = 1, ..., N$, its divergence is zero on the whole domain Ω^ϵ. Finally, the boundary conditions of problem (2.2) hold by construction (cf. chapter 5).

□

For test functions $\phi \in H^1(\Omega^\epsilon)$ we define the error terms as follows:

$$\langle F^\epsilon_{0,0}, \phi \rangle := \epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} f_j^{0,0}(\frac{x}{\epsilon})\, \phi, \tag{6.15}$$

$$\langle F^\epsilon_{1,0}, \phi \rangle := \epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} \left(f_j^{0,0}(\frac{x}{\epsilon}) - \langle \tau_j^{0,0} \rangle n_j + \epsilon f_j^{1,0}(\frac{x}{\epsilon}) \right) \phi, \tag{6.16}$$

$$\langle F^\epsilon_{1,l}, \phi \rangle := \langle F^\epsilon_{1,0}, \phi \rangle + \epsilon \sum_{1 \le \lambda \le l} Re^\lambda_\epsilon \sum_j \left(\tilde{f}_j^{0,\lambda}(\frac{x}{\epsilon}) + \epsilon f_j^{1,\lambda}(\frac{x}{\epsilon}) \right) \phi, \quad l \ge 1, \tag{6.17}$$

$$\begin{aligned} \langle F^\epsilon_{k,l}, \phi \rangle := {} & \epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} \left(f_j^{0,0}(\frac{x}{\epsilon}) - \langle \tau_j^{0,0} \rangle n_j \right) \phi \\ & + \sum_{\substack{2 \le \kappa \le k \\ 0 \le \lambda \le l}} \epsilon^\kappa Re^\lambda_\epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} \left(f_j^{\kappa-1,\lambda}(\frac{x}{\epsilon}) - \langle \tau_j^{\kappa-1,\lambda} \rangle n_j \right) \phi \\ & + \epsilon^{k+1} \sum_{0 \le \lambda \le l} Re^\lambda_\epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} f_j^{k,\lambda}(\frac{x}{\epsilon})\, \phi \\ & + \sum_{\substack{1 \le \kappa \le k \\ 1 \le \lambda \le l}} \epsilon^\kappa\, Re^\lambda_\epsilon \sum_j \int_{\gamma_j^{M,\epsilon}} \tilde{f}_j^{\kappa-1,\lambda}(\frac{x}{\epsilon})\, \phi, \quad k \ge 2,\, l \ge 0, \end{aligned} \tag{6.18}$$

$$\langle G^\epsilon_{k,l}, \phi \rangle := \int_{\Omega^\epsilon} G^\epsilon_{k,l}\, \chi^\epsilon \phi, \quad k,l \in \mathbb{N}_0 \tag{6.19}$$

$$\langle E^\epsilon_{k,l}, \phi \rangle := \langle F^\epsilon_{k,l}, \phi \rangle + \langle G^\epsilon_{k,l}, \phi \rangle, \quad k,l \in \mathbb{N}_0. \tag{6.20}$$

The weak formulation corresponding to (6.14) with the boundary conditions of (2.2) then reads

$$\mu_0 \epsilon^2 \int_{\Omega^\epsilon} \nabla u^\epsilon_{k,l} \nabla \phi + \epsilon Re_\epsilon \int_{\Omega^\epsilon} (u^\epsilon_{k,l} \cdot \nabla) u^\epsilon_{k,l}\, \phi + \sum_k p_k \int_{\Sigma^\epsilon_k} \phi \cdot n_k = \left\langle E^\epsilon_{k,l}, \phi \right\rangle \tag{6.21}$$

for all $\phi \in V^\epsilon$, where

$$V^\epsilon = \Big\{ \psi \in H^1(\Omega^\epsilon) : div\, \psi = 0, \psi|_{\Gamma^\epsilon} = 0,\ \psi \times n_j|_{\Sigma_j^\epsilon} = 0,\ j = 1, ..., N \Big\}.$$

We now establish estimates for the jumps and inertial terms occurring on the right-hand side of (6.21). At first, we have for any $\phi \in H^1(\Omega^\epsilon)$

$$\big|\big\langle F^\epsilon_{0,0}, \phi \big\rangle\big| \le \epsilon \sum_j \left| \int_{\gamma_j^{M,\epsilon}} f_j^{0,0}(\frac{x}{\epsilon})\, \phi(x) dx \right| \le \epsilon^2 \sum_j \|f_j^{0,0}\|_{L^2(\gamma_j^M)} \|\phi\|_{L^2(\gamma_j^{M,\epsilon})},$$

and analogous estimates for higher order terms, including the L^2-norms of the jumps $f_j^{\kappa,\lambda}$, $(f_j^{\kappa,\lambda} - \langle \tau_j^{\kappa,\lambda} \rangle\, n_j)$ and $\tilde{f}_j^{\kappa,\lambda}$ on γ_j^M (cf. (6.4)-(6.6)). The additional factor ϵ occurs due to the rescaling of the cross-section $\gamma_j^{M,\epsilon} = \epsilon\, \gamma_j^M$.

For the inertial error term we obtain, abbreviating the notation of (6.7)-(6.10),

$$\begin{aligned} \|G^\epsilon_{k,l}\|_{L^2(\Omega^\epsilon)} &\le \sum_{i,j} \epsilon^i Re_\epsilon^{j+1} \left(\int_{\Omega^{M,\epsilon}} |g^{i,j}(\frac{x}{\epsilon})|^2 dx \right)^{1/2} \\ &= \sum_{i,j} \epsilon^{i+3/2} Re_\epsilon^{j+1} \left(\int_{\Omega^M} |g^{i,j}(y)|^2 dy \right)^{1/2}. \end{aligned}$$

Thus we have

$$\begin{aligned} \big|\langle G^\epsilon_{k,l}, \phi \rangle\big| &\le \|G^\epsilon_{k,l}\|_{L^2(\Omega^\epsilon)}\, \|\phi\|_{L^2(\Omega^\epsilon)} \\ &\le \sum_{i,j} \epsilon^{i+3/2} Re_\epsilon^{j+1} \|g^{i,j}\|_{L^2(\Omega^M)}\, \|\phi\|_{L^2(\Omega^\epsilon)}. \end{aligned}$$

Using the Poincaré inequality (cf. Lemma B.1)

$$\|\phi\|_{L^2(\Omega^\epsilon)} \le C\, \epsilon\, \|\nabla \phi\|_{L^2(\Omega^\epsilon)}$$

and the trace estimate (cf. Lemma B.3)

$$\|\phi\|_{L^2(\gamma_j^{M,\epsilon})} \le C \sqrt{\epsilon}\, \|\nabla \phi\|_{L^2(\Omega^\epsilon)}$$

for $\phi \in H^1(\Omega^\epsilon)$, $\phi = 0$ on Γ^ϵ, we finally get the following *estimate of the total error*:

$$\big|\big\langle E^\epsilon_{k,l}, \phi \big\rangle\big| \le C\, \epsilon^{5/2} \left(I^\epsilon_{k,l} + J^\epsilon_{k,l} \right) \|\nabla \phi\|_{L^2(\Omega^\epsilon)}, \tag{6.22}$$

for all $\phi \in H^1(\Omega^\epsilon)$, $\phi = 0$ on Γ^ϵ.

Here the *total inertial error* is denoted by

$$I^\epsilon_{k,0} := \sum_{0 \le i \le 2k} \epsilon^i \|g_1^{i,j}\|_{L^2(\Omega^M)} \ \text{ for } \ k \ge 0, \tag{6.23}$$

and

$$I^{\epsilon}_{k,l} := \sum_{k\le i\le 2k}\sum_{0\le j\le 2l} \epsilon^i Re_\epsilon^{j+1}\,\|g^{i,j}\|_{L^2(\Omega^M)} + \sum_{0\le i\le k-1}\sum_{l\le j\le 2l} \epsilon^i Re_\epsilon^{j+1}\,\|g^{i,j}\|_{L^2(\Omega^M)}, \tag{6.24}$$

for $k\ge 1$, $l\ge 0$ (cf. (6.7)-(6.10)).

The total jump error is given by

$$J^{\epsilon}_{0,0} := \sum_j \|f_j^{0,0}\|_{L^2(\gamma_j^M)} \tag{6.25}$$

$$J^{\epsilon}_{1,0} := \sum_j \|f_j^{0,0} - \langle \tau_j^{0,0}\rangle\, n_j\|_{L^2(\gamma_j^M)} + \epsilon \sum_j \|f_j^{1,0}\|_{L^2(\gamma_j^M)} \tag{6.26}$$

$$J^{\epsilon}_{1,l} := J^{\epsilon}_{1,0} + \sum_{1\le\lambda\le l} Re_\epsilon^\lambda \sum_j \left(\|\tilde f_j^{0,\lambda}\|_{L^2(\gamma_j^M)} + \epsilon \|f_j^{1,\lambda}\|_{L^2(\gamma_j^M)} \right), \quad l\ge 1, \tag{6.27}$$

and for $k\ge 2$, $l\ge 0$

$$\begin{aligned} J^{\epsilon}_{k,l} := {} & \sum_j \|f_j^{0,0} - \langle \tau_j^{0,0}\rangle\, n_j\|_{L^2(\gamma_j^M)} + \sum_{\substack{2\le\kappa\le k\\ 0\le\lambda\le l}} \epsilon^{\kappa-1}\, Re_\epsilon^\lambda \sum_j \|f_j^{\kappa-1,\lambda} - \langle \tau_j^{\kappa-1,\lambda}\rangle\, n_j\|_{L^2(\gamma_j^M)} \\ & + \epsilon^k \sum_{0\le\lambda\le l} Re_\epsilon^\lambda \sum_j \|f_j^{k,\lambda}\|_{L^2(\gamma_j^M)} + \sum_{\substack{1\le\kappa\le k\\ 1\le\lambda\le l}} \epsilon^{\kappa-1}\, Re_\epsilon^\lambda \sum_j \|\tilde f_j^{\kappa-1,\lambda}\|_{L^2(\gamma_j^M)}. \end{aligned} \tag{6.28}$$

6.1.3 Estimates for the jumps and inertial terms

The next aim is to estimate the L^2-norms of the jumps and of the inertial terms. In section 4.2 we have shown that Leray's solution can be approximated by the solution of a finite junction problem up to an error term which decays exponentially with growing distance from the junction. Applying the Theorems 4.3 and 4.4, Corollary 4.1 and Lemma B.3 (ii) we obtain

Corollary 6.1. (Trace estimates)

(i) Poiseuille junction problem:

For the solution $(\omega^{\kappa,\lambda}, \pi^{\kappa,\lambda})$, $\kappa,\lambda\in\mathbb{N}_0$, of the Poiseuille junction problem of order (κ,λ) and the solution $(\omega_L^{\kappa,\lambda}, \pi_L^{\kappa,\lambda})$ of the corresponding Leray problem the following estimate holds:

If $\displaystyle\int_{\Omega^M} \left(\pi^{\kappa,\lambda} - \pi_L^{\kappa,\lambda}\right) = 0$, then there exist constants $C_{\kappa,\lambda} > 0$ independent of M (cf. Theorem 4.3) such that

$$\mu_0 \|(\nabla\omega^{\kappa,\lambda} - \nabla\omega_L^{\kappa,\lambda})\, n_j\|_{L^2(\gamma_j^M)} + \|\pi^{\kappa,\lambda} - \pi_L^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \le C_{\kappa,\lambda}\, M\, e^{-\sigma_L M} \tag{6.29}$$

for all $M\ge 1$.

(ii) Inertial correction problem:

There exist constants $0 < \tilde{\sigma} < \sigma_L$ *and* $\tilde{C}_{\kappa,\lambda} > 0$, $\kappa \in \mathbb{N}_0, \lambda \in \mathbb{N}$, *independent of* M *(cf. Theorem 4.4 and Corollary 4.1) such that*

$$\mu_0 \|(\nabla\tilde{\omega}^{\kappa,\lambda} - \nabla\tilde{\omega}_L^{\kappa,\lambda})\, n_j\|_{L^2(\gamma_j^M)} + \|\tilde{\pi}^{\kappa,\lambda} - \tilde{\pi}_L^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \leq \tilde{C}_{\kappa,\lambda}\, e^{-\tilde{\sigma}M} \tag{6.30}$$

for all $M \geq 1$, *where* $(\tilde{\omega}^{\kappa,\lambda}, \tilde{\pi}^{\kappa,\lambda})$ *is determined by (5.33),* $(\tilde{\omega}_L^{\kappa,\lambda}, \tilde{\pi}_L^{\kappa,\lambda})$ *is the solution of the corresponding generalized Leray problem, and* $\displaystyle\int_{\Omega^M} \left(\tilde{\pi}^{\kappa,\lambda} - \tilde{\pi}_L^{\kappa,\lambda}\right) = 0.$

Remark 6.1. (i) Due to Theorem 4.3 and (3.13) the constants $C_{\kappa,\lambda}$ in (6.29) depend on the fluxes of the Poiseuille flows $V_j^{\kappa,\lambda}$, $j = 1, ..., N$, which are determined by the pressure mean values $\langle \tau_j^{\kappa-1,\lambda} \rangle$ and $\langle \tilde{\pi}^{\kappa-1,\lambda} \rangle_j$ on γ_j^M, respectively (cf. (5.39)). These quantities can be bounded *independent* of M if $\tau_j^{\kappa-1,\lambda}$ and $\tilde{\pi}^{\kappa-1,\lambda}$ approximate the corresponding Leray pressures $\tau_{j,L}^{\kappa-1,\lambda}$ and $\tilde{\pi}_L^{\kappa-1,\lambda}$ in the previous order. This means that the estimates (6.29)-(6.30) are established inductively: Starting in zero-order, the constant $C_{0,0}$ being independent of M, the approximation property carries forward to all higher orders $\kappa, \lambda \in \mathbb{N}_0$.

(ii) For further application we have simplified the approximation result of Corollary 4.1 summarizing the two different cases therein. Using the precise statement, we have, if $min_j\, \tilde{\sigma}_j > 2\sigma_L$ $(j = 1, ..., N)$,

$$\mu_0 \|(\nabla\tilde{\omega}^{\kappa,\lambda} - \nabla\tilde{\omega}_L^{\kappa,\lambda})\, n_j\|_{L^2(\gamma_j^M)} + \|\tilde{\pi}^{\kappa,\lambda} - \tilde{\pi}_L^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \leq \tilde{C}_{\kappa,\lambda} M^{s_\lambda}\, e^{-\sigma_L M}$$

or, if $min_j\, \tilde{\sigma}_j \leq 2\sigma_L$,

$$\mu_0 \|(\nabla\tilde{\omega}^{\kappa,\lambda} - \nabla\tilde{\omega}_L^{\kappa,\lambda})\, n_j\|_{L^2(\gamma_j^M)} + \|\tilde{\pi}^{\kappa,\lambda} - \tilde{\pi}_L^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \leq \tilde{C}_{\kappa,\lambda}(\sigma')\, e^{-\sigma' M}$$

for any $0 < \sigma' < min_j\, \tilde{\sigma}_j/2$.

We now apply these results in order to establish

Lemma 6.2. (Jump estimates)

There exist constants $C_{\kappa,\lambda} > 0$, $\kappa, \lambda \in \mathbb{N}_0$, *such that*

$$\|f_j^{\kappa,\lambda} - \langle \tau_j^{\kappa,\lambda} \rangle n_j\|_{L^2(\gamma_j^M)} \leq C_{\kappa,\lambda}\, M\, e^{-\sigma_L M} \tag{6.31}$$

for all $M \geq 1$ *and all* $j = 1, ..., N$.

An analogous result holds for the jumps of the nonlinear correction $\tilde{f}_j^{\kappa,\lambda}$, $\kappa \geq 0$, $\lambda \geq 1$:

$$\|\tilde{f}_j^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \leq \tilde{C}_{\kappa,\lambda}\, e^{-\tilde{\sigma}M} \tag{6.32}$$

for all $M \geq 1$ *and all* $j = 1, ..., N$.

Proof. In order to show inequality (6.31), we take into account definition (6.1) for $f_j^{\kappa,\lambda}$ and insert the solution $(\omega_L^{\kappa,\lambda}, \pi_L^{\kappa,\lambda})$ of Leray's problem. We obtain

(6.33)
$$\begin{aligned}\|f_j^{\kappa,\lambda} - \langle\tau_j^{\kappa,\lambda}\rangle n_j\|_{L^2} \le \mu_0 \, \|(\nabla\omega^{\kappa,\lambda} - \nabla\omega_L^{\kappa,\lambda})\, n_j\|_{L^2} + \|\pi^{\kappa,\lambda} - \pi_L^{\kappa,\lambda}\|_{L^2} \\ + \mu_0 \, \|\nabla\omega_L^{\kappa,\lambda}\, n_j\|_{L^2} + \left\|\pi_L^{\kappa,\lambda} - C_j^{\kappa,\lambda}\, M - \langle\tau_j^{\kappa,\lambda}\rangle\right\|_{L^2},\end{aligned}$$

where all norms are taken on γ_j^M.

From (6.29) we get the exponential decay of the first and second term. The exponential decay of the third term is evident: The velocity gradient of Leray's solution tends (pointwise) exponentially to the gradient of the Poiseuille velocity which has a vanishing normal component (cf. chapter 3).

In order to apply the decay property of Leray's pressure $\pi_L^{\kappa,\lambda}$ (cf. inequality (3.14)), we estimate the last term of (6.33) as follows:

$$\left\|\pi_L^{\kappa,\lambda} - C_j^{\kappa,\lambda}\, M - \langle\tau_j^{\kappa,\lambda}\rangle\right\|_{L^2} \le \left\|\pi_L^{\kappa,\lambda} - C_j^{\kappa,\lambda}\, M - \tau_j^{\infty,\kappa,\lambda}\right\|_{L^2} + \left\|\tau_j^{\infty,\kappa,\lambda} - \langle\tau_j^{\kappa,\lambda}\rangle\right\|_{L^2}.$$

By definition, $\tau_j^{\infty,\kappa,\lambda}$ is the limit of $\tau_{j,L}^{\kappa,\lambda} := \pi_L^{\kappa,\lambda} - C_j^{\kappa,\lambda}\, y_1^j$ for $y_1^j \to \infty$ (cf. section 5.2). In (3.14) we have the pointwise exponential decay of $\tau_{j,L}^{\kappa,\lambda}$ to $\tau_j^{\infty,\kappa,\lambda}$, in particular

$$\left\|\tau_j^{\infty,\kappa,\lambda} - \langle\tau_{j,L}^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)} = O(e^{-\sigma_L M}),$$

where $\langle\tau_{j,L}^{\kappa,\lambda}\rangle$ denotes the mean value of $\tau_{j,L}^{\kappa,\lambda}$ on γ_j^M. Thus we have

$$\begin{aligned}(6.34)\quad \left\|\tau_j^{\infty,\kappa,\lambda} - \langle\tau_j^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)} &\le \left\|\tau_j^{\infty,\kappa,\lambda} - \langle\tau_{j,L}^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)} + \left\|\langle\tau_{j,L}^{\kappa,\lambda}\rangle - \langle\tau_j^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)} \\ &\le C\, e^{-\sigma_L M} + \left\|\langle\pi_L^{\kappa,\lambda}\rangle - \langle\pi^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)}.\end{aligned}$$

The second term on the right-hand side of (6.34) can be estimated by (6.29) since

$$\left\|\langle\pi_L^{\kappa,\lambda}\rangle - \langle\pi^{\kappa,\lambda}\rangle\right\|_{L^2(\gamma_j^M)} = |\gamma_j^M|^{1/2} |\langle\pi_L^{\kappa,\lambda} - \pi^{\kappa,\lambda}\rangle| \le \left\|\pi_L^{\kappa,\lambda} - \pi^{\kappa,\lambda}\right\|_{L^2(\gamma_j^M)}.$$

Thus inequality (6.31) is proved.

The proof of inequality (6.32) follows the same lines inserting the corresponding solution of the generalized Leray problem and using the decay and approximation estimates of section 3.2 and Theorem 6.1 (*ii*).

□

Remark 6.2. In particular, we get from (6.31) the estimate

$$\|f_j^{\kappa,\lambda}\|_{L^2(\gamma_j^M)} \leq |\langle\tau_j^{\kappa,\lambda}\rangle|\,|\gamma_j^M|^{1/2} + C_{\kappa,\lambda}\, M\, e^{-\sigma_L M}. \tag{6.35}$$

Since in general $\langle\tau_j^{0,0}\rangle \neq 0$, the zero-order jump $f_j^{0,0}$ *does not get exponentially small* with increasing distance M from the junction. Therefore, the error $F_{0,0}^{\epsilon}$ is of order $O(\epsilon)$ (cf. (6.3)) which is *not sufficient* in order to get an adequate error estimate for the gradient of velocity (cf. section 6.2).

We now establish estimates for the inertial terms.

Lemma 6.3. (Inertial terms) *For all $i,j \in \mathbb{N}_0$ there exist constants $C_{i,j} > 0$, independent of M, such that*

$$\|g^{i,j}\|_{L^2(\Omega^M)} \leq C_{i,j} \tag{6.36}$$

for all $M \geq 0$.

Proof. Since all inertial terms are L^2-functions it is clear that their norms are bounded on Ω^M. We only have to ensure that these bounds, i.e. the constants $C_{i,j}$, are independent of M.
Let $(\omega^1\cdot\nabla)\omega^2$ be any inertial term occurring in (5.23)-(5.25). By comparison with the corresponding term from Leray's solution we get

$$\|(\omega^1\cdot\nabla)\omega^2\|_{L^2(\Omega^M)} \leq \|(\omega^1\cdot\nabla)\omega^2 - (\omega_L^1\cdot\nabla)\omega_L^2\|_{L^2(\Omega^M)} + \|(\omega_L^1\cdot\nabla)\omega_L^2\|_{L^2(\Omega^M)}, \tag{6.37}$$

where all norms are taken on Ω^M.

For the first term on the right-hand side of (6.37) we proceed as in section 4.2 where we have established recursively an exponential decay estimate (cf. Lemma 4.2 and its generalization in subsection 4.2.3). Therefore it can be bounded independent of M.

For the second term we use the exponential decay of Leray's solution to Poiseuille flow: Since for any two Poiseuille flows V_j^1, V_j^2 the inertial term $(V_j^1\cdot\nabla)V_j^2$ vanishes, we have

$$(\omega_L^1\cdot\nabla)\omega_L^2 = (\omega_L^1 - V_j^1)\cdot\nabla\omega_L^2 + (V_j^1\cdot\nabla)(\omega_L^2 - V_j^2)$$

which is bounded independent of M (cf. section 4.2).
If ω_L^1 or ω_L^2 is the solution of a generalized Leray problem corresponding to an inertial correction problem in the junction, then it exponentially tends to zero and the second term on the right-hand side of (6.37) is bounded independent of M as well.

□

We summarize the results of this section in the following

Corollary 6.2. *The total error $E^{\epsilon}_{k,l}$, $k,l \in \mathbb{N}_0$, can be estimated as follows: There exist constants $0 < \tilde{\sigma} < \sigma_L$ and $C(k,l) > 0$ such that*

$$\left|\left\langle E^{\epsilon}_{k,l}, \phi\right\rangle\right| \leq C(k,l)\, \epsilon^{5/2}\, max\left(e^{-\tilde{\sigma}M}, \epsilon^k, Re_{\epsilon}^{l+1}\right) \|\nabla \phi\|_{L^2(\Omega^{\epsilon})} \tag{6.38}$$

for all $M \geq 1$ and any $\phi \in H^1(\Omega^{\epsilon})$.

Remark: The constant $C(k,l)$ in (6.38) can be bounded independent of (k,l) if ϵ is sufficiently small, i.e. there exists $\epsilon_0 = \epsilon_0(k,l) > 0$ and $C > 0$ such that $C(k,l) \leq C$ for all $k,l \in \mathbb{N}_0$ if $\epsilon, Re_{\epsilon} \leq \epsilon_0(k,l)$.

6.2 Main result

In section 2.2 we have established the existence and local uniqueness of the solution $(v^{\epsilon}, p^{\epsilon})$ of Navier-Stokes system (2.2) under the assumption of a sufficiently small non-linear term (i.e. $Re_{\epsilon} \leq O(\epsilon)$). Due to the results of the previous sections we are now able to prove error estimates for the velocity and pressure approximations defined in chapter 5.

We use the following *notation*: For $a_i \in \mathbb{R}$, $i = 1, ..., n$, let

$$\{a_1, ..., a_n\} := max\{a_1, ..., a_n\}$$

and $\rho^{\epsilon} := |\Omega^{\epsilon}|^{1/2}$ denote the volume measure of the domain. Since the domain Ω^{ϵ} shrinks as ϵ tends to zero, we have to weight the L^2-norm with the factor $(\rho^{\epsilon})^{-1}$.

The following theorem states the main result of our analysis:

Theorem 6.1. (Error estimates) *Let $(v^{\epsilon}, p^{\epsilon})$ in $(H^1 \times L^2)(\Omega^{\epsilon})$ be a solution of the Navier-Stokes system (2.2) such that (cf. Theorem 2.1 and Remark 6.3 below)*

$$\|\nabla v^{\epsilon}\|_{L^2(\Omega^{\epsilon})} \leq K \epsilon^{\frac{1}{2}}\, Re_{\epsilon}^{-1}. \tag{6.39}$$

For every $k,l \in \mathbb{N}_0$ there exists a constant $C_{Re} = C_{Re}(k,l) > 0$ such that the following estimates hold if $Re_{\epsilon} \leq C_{Re}\, \epsilon^{1/2}$:

There exist constants $0 < \tilde{\sigma} < \sigma_L$ and $C = C(k,l) > 0$, independent of ϵ and M, such that

$$\frac{1}{\rho^{\epsilon}} \left\|\nabla\left(v^{\epsilon} - u^{\epsilon}_{k,l}\right)\right\|_{L^2(\Omega^{\epsilon})} \leq C\, \epsilon^{-\frac{1}{2}} \left\{e^{-\tilde{\sigma}M}, \epsilon^k, Re_{\epsilon}^{l+1}\right\}, \tag{6.40}$$

$$\frac{1}{\rho^{\epsilon}} \left\|v^{\epsilon} - u^{\epsilon}_{k,l}\right\|_{L^2(\Omega^{\epsilon})} \leq C\, \epsilon^{\frac{1}{2}} \left\{e^{-\tilde{\sigma}M}, \epsilon^k, Re_{\epsilon}^{l+1}\right\}, \tag{6.41}$$

$$\frac{1}{\rho^{\epsilon}} \left\|p^{\epsilon} - q^{\epsilon}_{k,l}\right\|_{L^2(\Omega^{\epsilon})/\mathbb{R}} \leq C\, \epsilon^{\frac{1}{2}} \left\{e^{-\tilde{\sigma}M}, \epsilon^k, Re_{\epsilon}^{l+1}\right\}, \tag{6.42}$$

for all $M \geq 1$.

Remark 6.3. Due to Theorem 2.1 we know that there exists a unique solution of (2.2) such that (6.39) holds *if* $Re_\epsilon \leq O(\epsilon)$. For higher Reynolds numbers the question of existence (and uniqueness) for the Navier-Stokes system (2.2) is unsolved (cf. [MP3] and the references therein).
In *two dimensions* the result of Theorem 6.1 holds for $Re_\epsilon \leq O(1)$ if

$$\|\nabla v^\epsilon\|_{L^2(\Omega^\epsilon)} \leq K\, Re_\epsilon^{-1}, \tag{6.43}$$

cf. Remark 2.1.

Proof. We proceed in two steps, first establishing (6.40) and (6.41). The estimate (6.42) for the pressure then follows using an appropriate test function (cf. [MP3]).

(i) We subtract equation (6.14) from equation $(2.2)_1$, test with $\phi \in V^\epsilon$, and integrate by parts:

$$\mu_0\epsilon^2 \int_{\Omega^\epsilon} \nabla(v^\epsilon - u^\epsilon_{k,l})\nabla\phi + \epsilon Re_\epsilon \int_{\Omega^\epsilon} \big((v^\epsilon\cdot\nabla)v^\epsilon - (u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l}\big)\ \phi = -\langle E^\epsilon_{k,l}, \phi\rangle. \tag{6.44}$$

We take $\phi = (v^\epsilon - u^\epsilon_{k,l}) \in V^\epsilon$ and obtain

$$\begin{aligned}\mu_0\epsilon^2 \int_{\Omega^\epsilon} |\nabla(v^\epsilon - u^\epsilon_{k,l})|^2 \leq{}& \big|\langle E^\epsilon_{k,l}, (v^\epsilon - u^\epsilon_{k,l})\rangle\big| \\ &+ \epsilon\, Re_\epsilon \left|\int_{\Omega^\epsilon} \big((v^\epsilon\cdot\nabla)v^\epsilon - (u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l}\big)\ (v^\epsilon - u^\epsilon_{k,l})\right|.\end{aligned} \tag{6.45}$$

For the first term we have already established an estimate in section 6.1, cf. Corollary 6.2. The nonlinear term of (6.44) can be separated into

$$\begin{aligned}&\int_{\Omega^\epsilon} \big((v^\epsilon\cdot\nabla)v^\epsilon - (u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l}\big)\ (v^\epsilon - u^\epsilon_{k,l}) \\ &\quad= \int_{\Omega^\epsilon} (v^\epsilon - u^\epsilon_{k,l})\cdot\nabla u^\epsilon_{k,l}\,(v^\epsilon - u^\epsilon_{k,l}) + \int_{\Omega^\epsilon} (v^\epsilon\cdot\nabla)(v^\epsilon - u^\epsilon_{k,l})\,(v^\epsilon - u^\epsilon_{k,l}).\end{aligned} \tag{6.46}$$

According to the proof of Theorem 2.1 (cf. $(1(i))$ therein), the second term on the right-hand side of (6.46) can be absorbed into the left hand side of (6.45) (due to (6.39)). For the first term, we use the Hölder inequality and the Sobolev embedding $H^1 \hookrightarrow L^4$ (cf. (B.3)) and obtain:

$$\begin{aligned}\epsilon\, Re_\epsilon \int_{\Omega^\epsilon} (v^\epsilon - u^\epsilon_{k,l})\cdot\nabla u^\epsilon_{k,l}\,(v^\epsilon - u^\epsilon_{k,l}) &\\ \leq C^2_{L^4,H^1}\,\epsilon^{3/2}\, Re_\epsilon \|\nabla(v^\epsilon - u^\epsilon_{k,l})\|^2_{L^2(\Omega^\epsilon)}\|\nabla u^\epsilon_{k,l}\|_{L^2(\Omega^\epsilon)} &\\ \leq C^2_{L^4,H^1}\, C_0\,\epsilon^{3/2}\, Re_\epsilon \|\nabla(v^\epsilon - u^\epsilon_{k,l})\|^2_{L^2(\Omega^\epsilon)}, &\end{aligned} \tag{6.47}$$

where we use the estimate

$$\|\nabla u^\epsilon_{k,l}\|_{L^2(\Omega^\epsilon)} \leq C_0, \tag{6.48}$$

which follows directly from the construction of $u^\epsilon_{k,l}$, the constant C_0 possibly depending on (k,l). For $Re_\epsilon \le C_{Re}\,\epsilon^{1/2}$ with

$$C_{Re} := \frac{\mu_0}{2\,C^2_{L^4,H^1}\,C_0}, \tag{6.49}$$

the first term on the right-hand side of (6.46) can also be absorbed into the left hand side of (6.45).

We now have established the estimate for the velocity gradient. Applying the Poincaré inequality (B.1), we immediately get the L^2-estimate for the velocity.

(*ii*) In order to estimate the pressure, we proceed as in [MP3]. First, the *a priori* estimate for the velocity gradient ∇v^ϵ of the solution of (2.2), is improved with the help of the approximation result. From (6.40) and (6.48) we get the existence of a constant $C > 0$ such that

$$\|\nabla v^\epsilon\|_{L^2(\Omega^\epsilon)} \le C. \tag{6.50}$$

We define $w^\epsilon \in H^1_0(\Omega^\epsilon)$ as solution of

$$\left\{\begin{array}{rcll} div\, w^\epsilon &=& p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle & \text{in } \Omega^\epsilon, \\ w^\epsilon &=& 0 & \text{on } \partial\Omega^\epsilon, \end{array}\right. \tag{6.51}$$

where

$$\langle p^\epsilon - q^\epsilon_{k,l}\rangle := \frac{1}{|\Omega^\epsilon|}\int_{\Omega^\epsilon} (p^\epsilon - q^\epsilon_{k,l})$$

denotes the mean value of $(p^\epsilon - q^\epsilon_{k,l})$ on Ω^ϵ. According to Lemma B.5 (cf. (B.23)) we have

$$\|w^\epsilon\|_{H^1(\Omega^\epsilon)} \le C\,\epsilon^{-1}\,\|p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle\|_{L^2(\Omega^\epsilon)}. \tag{6.52}$$

As in the first step of the proof we subtract equation (6.14) from $(2.2)_1$, test with w^ϵ and integrate by parts:

$$\begin{aligned} \|p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle\|^2_{L^2(\Omega^\epsilon)} &= \mu_0\,\epsilon^2 \int_{\Omega^\epsilon} \nabla(v^\epsilon - u^\epsilon_{k,l})\nabla w^\epsilon \\ &+ \epsilon\, Re_\epsilon \int_{\Omega^\epsilon} \big((v^\epsilon\cdot\nabla)v^\epsilon - (u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l}\,\big)\; w^\epsilon + \langle E^\epsilon_{k,l}, w^\epsilon\rangle. \end{aligned} \tag{6.53}$$

The first term on the right-hand side of (6.53) can be estimated due to (6.40) and (6.52):

$$\begin{aligned} &\mu_0\,\epsilon^2 \int_{\Omega^\epsilon} \nabla(v^\epsilon - u^\epsilon_{k,l})\nabla w^\epsilon \\ &\qquad \le C\,\epsilon^{\frac{3}{2}}\,\left\{e^{-\tilde\sigma M}, \epsilon^k, Re_\epsilon^{l+1}\right\}\,\|p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle\|_{L^2(\Omega^\epsilon)}. \end{aligned}$$

The last term of (6.53) can be estimated according to Corollary 6.2 and (6.52) by

$$\begin{aligned}\left|\langle E^\epsilon_{k,l}, w^\epsilon\rangle\right| &\le C\,\epsilon^{\frac{5}{2}}\left\{e^{-\tilde\sigma M}, \epsilon^k, Re_\epsilon^{l+1}\right\}\|\nabla w^\epsilon\|_{L^2(\Omega^\epsilon)}\\ &\le C\,\epsilon^{\frac{3}{2}}\left\{e^{-\tilde\sigma M}, \epsilon^k, Re_\epsilon^{l+1}\right\}\|p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle\|_{L^2(\Omega^\epsilon)}.\end{aligned}$$

Finally, we show an estimate for the inertial term (cf. (6.46) and (6.47)):

$$\begin{aligned}&\epsilon\, Re_\epsilon \int_{\Omega^\epsilon}\left((v^\epsilon\cdot\nabla)v^\epsilon - (u^\epsilon_{k,l}\cdot\nabla)u^\epsilon_{k,l}\right)\,w^\epsilon\\ &= \epsilon\, Re_\epsilon \int_{\Omega^\epsilon}\left((v^\epsilon - u^\epsilon_{k,l})\cdot\nabla u^\epsilon_{k,l} + (v^\epsilon\cdot\nabla)(v^\epsilon - u^\epsilon_{k,l})\right)\,w^\epsilon\\ &\le C\,\epsilon^{3/2}\,Re_\epsilon\|\nabla(v^\epsilon - u^\epsilon_{k,l})\|_{L^2(\Omega^\epsilon)}\left(\|\nabla u^\epsilon_{k,l}\|_{L^2(\Omega^\epsilon)} + \|\nabla v^\epsilon\|_{L^2(\Omega^\epsilon)}\right)\|\nabla w^\epsilon\|_{L^2(\Omega^\epsilon)}\\ &\le C\,\epsilon\, Re_\epsilon\left\{e^{-\tilde\sigma M}, \epsilon^k, Re_\epsilon^{l+1}\right\}\|p^\epsilon - q^\epsilon_{k,l} - \langle p^\epsilon - q^\epsilon_{k,l}\rangle\|_{L^2(\Omega^\epsilon)},\end{aligned}$$

using (6.40), (6.48), (6.50) and (6.52). Since, by assumption we have $Re_\epsilon \le O(\epsilon^{1/2})$, the pressure estimate (6.42) is proved.

□

Remark 6.4. The exponential decay rate $\tilde\sigma$ is bounded from above by the decay rate σ_L of the solution of Leray's problem given in (3.15).
We have simplified the exponential decay term in the estimates (6.40)-(6.42). It can be specified precisely as follows (cf. chapters 3 and 4): If $min_{j=1}^N\,\tilde\sigma_j > 2\sigma_L$, then we actually have a decay of the type $M^\alpha\, e^{-\sigma_L M}$ for some $\alpha > 0$ depending on l. If $min_j\,\tilde\sigma_j \le 2\sigma_L$, then the decay is faster than any $e^{-\sigma' M}$ with $0 < \sigma' < min_j\tilde\sigma_j/2$.

The constants $C = C(k,l)$ in (6.40)-(6.42) and C_{Re} defined by (6.49) possibly depend on the approximation order (k,l) but can be uniformly bounded if ϵ is sufficiently small: There exists $\epsilon_0(k,l), C > 0$ such that $C(k,l), C_{Re} \le C$ for all $k,l \in \mathbb{N}_0$ if $\epsilon, Re_\epsilon \le \epsilon_0(k,l)$.

In particular, we have for the zero- and first-order approximation:

Corollary 6.3. *Under the assumptions of Theorem 6.1 the following estimates hold:*
(i) Zero-order approximation:

$$\frac{1}{\rho^\epsilon}\left\|v^\epsilon - u^\epsilon_{0,0}\right\|_{L^2(\Omega^\epsilon)} + \frac{1}{\rho^\epsilon}\left\|p^\epsilon - q^\epsilon_{0,0}\right\|_{L^2(\Omega^\epsilon)/\mathbb{R}} \le C\,\epsilon^{\frac{1}{2}}. \tag{6.54}$$

The approximation of the velocity gradient ∇v^ϵ fails since equation (6.40) shows an error of $O(\epsilon^{-1/2})$.

(ii) First-order approximation including inertial corrections:

$$\frac{1}{\rho^\epsilon}\left\|\nabla\left(v^\epsilon - u^\epsilon_{1,1}\right)\right\|_{L^2(\Omega^\epsilon)} \le C\,\epsilon^{-\frac{1}{2}}\left\{e^{-\tilde\sigma M}, \epsilon\right\}, \tag{6.55}$$

$$\frac{1}{\rho^\epsilon}\left\|v^\epsilon - u^\epsilon_{1,1}\right\|_{L^2(\Omega^\epsilon)} + \frac{1}{\rho^\epsilon}\left\|p^\epsilon - q^\epsilon_{1,1}\right\|_{L^2(\Omega^\epsilon)/\mathbb{R}} \le C\,\epsilon^{\frac{1}{2}}\left\{e^{-\tilde\sigma M}, \epsilon\right\}, \tag{6.56}$$

for all $M \ge 1$.

In the following section we now discuss these results.

6.3 Some remarks concerning the estimates

6.3.1 Pressure decay correction

As already pointed out in section 5.2, an additional higher order correction is necessary in order to establish an adequate estimate for the velocity in $H^1(\Omega^\epsilon)$, cf. Corollary 6.3. This is due to the fact that the *jump error* includes the pressure decay constants of each pipe: The pressure from Leray's problem decays to a linear profile *plus some stabilization constant* in each pipe of the bifurcation, cf. [G] and [MP1]. In general these constants are non-zero and different for each pipe. Thus it is not possible to take them as zero by adding just one normalization constant for Leray's pressure. The approximation presented in [MP3] neglects this correction.

6.3.2 Approximation via the solution of Leray's problem

Instead of constructing the approximation (5.3)-(5.4) using the solution (ω^M, π^M) on the *finite* junction Ω^M, it is also possible to use the solution of Leray's problem on the *infinite* domain Ω^∞. From the theory of section 3.1 one immediately obtains the exponential decay to Poiseuille flow. The error of such an approximation is then given by the nonlinear term of order Re_ϵ, since the jumps of the normal forces decrease exponentially with growing distance from the junction if the stabilization constants are corrected.
In [MP3] this approach is carried out. The problem coming up there is the *matching* of Poiseuille flow and Leray's solution on the interfaces $\gamma_j^{M,\epsilon}$. An additional correction has to be introduced on each pipe in order to remove the jump of velocity on these interfaces. Otherwise the approximation velocity is not in H^1 and thus an estimate for the velocity gradient would be ruled out. Our approach avoids these difficulties by constructing an approximation for the velocity which is continuous on the pipe-junction interfaces $\gamma_j^{M,\epsilon}$, using Poiseuille flow as boundary condition in the finite junction problem. We thus get an approximation which consists only of Poiseuille flow away from the junction and does not need an additional correction, which is indeed small but cannot be neglected in order to get H^1-regularity.

Clearly, the solution of the Stokes equations can only be computed numerically on *finite* branching domains. The junction problem of type (4.1) provides an approximation of Leray's problem on finite domains of length $O(M)$. Fixing the parameter $\epsilon = \epsilon_0$ as the diameter-to-length ratio of the domain under consideration, the parameter $M = M(\epsilon_0)$ can be chosen such that the exponential decaying part of the approximation error is less than the two other error terms ($M = O(ln(1/\epsilon_0))$). Depending on the approximation order, the error is then determined in powers of ϵ_0 and Re_{ϵ_0} (cf. Theorem 6.1).
The estimates (6.40)-(6.42) are of *qualitative* character, showing the asymptotic behavior for large $M(\to \infty)$ and small $\epsilon(\to 0)$, since we do not have *quantitative* estimates

for the constants $C(k,l)$ and $\tilde{\sigma}$. In particular, we are not able to relate their order of magnitude to the geometry of the junction. In this respect it should be noted that in order to get convergence of the approximation to the solution of Navier-Stokes system (2.2) for $\epsilon \to 0$, we have to consider junction problems on infinite domains ("$M \to \infty$") since for finite M an error of order $O(e^{-\tilde{\sigma}M})$ remains.

6.3.3 Corrections to Kirchhoff's law

In the pipes $\Omega_j^\epsilon \setminus Z_j^{M,\epsilon}$ the first-order approximation $(u_{1,1}^\epsilon, q_{1,1}^\epsilon)$ consists of the Poiseuille flow

$$\begin{aligned} V_j^\epsilon(\tilde{x}^j) &:= w_j(\tfrac{\tilde{x}^j}{\epsilon})\,\frac{\langle q_j^\epsilon\rangle - p_j}{L_j}\,e_1^j, \\ P_j^\epsilon(x_1^j) &:= \frac{p_j - \langle q_j^\epsilon\rangle}{L_j}\,x_1^j \,+\, \langle q_j^\epsilon\rangle, \end{aligned}$$

where $\langle q_j^\epsilon\rangle := q^0 + \epsilon\left(\langle\tau_j^0\rangle + Re_\epsilon\langle\tilde{\pi}^{0,1}\rangle_j\right)$, fixing $q^{1,0} = q^{1,1} = 0$.

The Poiseuille flow is thereby determined by the weighted mean value q^0 (cf. (2.7)) and a *higher order correction* due to pressure decay (cf. section 5.2) and inertial terms (cf. section 5.3).
Regarding the Kirchhoff law for one-dimensional networks (cf. section 2.3) the estimates of Theorem 6.1 can be interpreted in the following way: If the diameter-to-length ratio ϵ of the pipes is *sufficiently* small, then the flux through any cross-section $S_j^\epsilon = \epsilon S_j$ of the j^{th} pipe $\Omega_j^\epsilon \setminus Z_j^{M,\epsilon}$ is given by

$$\begin{aligned} \epsilon^{-2}\tilde{F}_j^\epsilon : &= \epsilon^{-2}\int_{S_j^\epsilon} V_j^\epsilon \cdot e_1^j = c_j\left(\langle q_j^\epsilon\rangle - p_j\right) \\ &= c_j\,(q^0 - p_j) + \epsilon\, c_j\left(\langle\tau_j^0\rangle + Re_\epsilon\langle\tilde{\pi}^{0,1}\rangle_j\right) \end{aligned}$$

with the conductivities c_j, cf. section 2.3. Comparing $\tilde{F}_j^\epsilon$ with the flux

$$F_j^\epsilon := \int_{S_j^\epsilon} v^\epsilon \cdot e_1^j$$

of the solution of (2.2) we obtain from Corollary 6.3 and the trace inequality (cf. Lemma B.3)

$$\begin{aligned} |F_j^\epsilon - \tilde{F}_j^\epsilon| = \left|\int_{S_j^\epsilon}(v^\epsilon - V_j^\epsilon)\cdot e_1^j\right| &\le \epsilon\,|S_j|^{1/2}\|v^\epsilon - V_j^\epsilon\|_{L^2(S_j^\epsilon)} \\ &\le C\epsilon^{3/2}\|\nabla(v^\epsilon - V_j^\epsilon)\|_{L^2(\Omega_j^\epsilon\setminus Z_j^{M,\epsilon})} \le C\,\epsilon^2\,max\{e^{-\tilde{\sigma}M}, \epsilon\}. \end{aligned}$$

Summarizing, the weighted mean value q^0, computed from Kirchhoff's law, admits an adequate approximation of the flux through a junction of thin or long pipes in leading order $O(1)$. If approximations of higher accuracy are needed, then q^0 has to be corrected by taking into account *local* Stokes problems in the junction of diameter $O(\epsilon)$. In this way, the influence of the geometric structure of the junction on the fluid flow is resolved.

In analogy to V_j^ϵ the Poiseuille flow of order (k,l), $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, is given by

$$V_{j,(k,l)}^\epsilon(\tilde{x}^j) := V_j^{0,0}(\frac{\tilde{x}^j}{\epsilon}) + \sum_{\kappa=1}^{k} \sum_{\lambda=0}^{l} \epsilon^\kappa Re_\epsilon^\lambda V_j^{\kappa,\lambda}(\frac{\tilde{x}^j}{\epsilon}), \quad \tilde{x}^j \in \Omega_j^\epsilon.$$

Proceeding as above we then get for the corresponding flux $\tilde{F}_{j,(k,l)}^\epsilon := \int_{S_j^\epsilon} V_{j,(k,l)}^\epsilon \cdot e_1^j$ the following estimate:

$$|F_j^\epsilon - \tilde{F}_{j,(k,l)}^\epsilon| \le C\,\epsilon^2\, max\{e^{-\tilde{\sigma} M}, \epsilon^k, Re_\epsilon^{l+1}\}. \tag{6.57}$$

Therefore, the flux F_j^ϵ of the Navier-Stokes velocity in the j^{th} pipe can be approximated in any order (k,l), $k, l \in \mathbb{N}_0$, by the flux of the Poiseuille velocities $V_{j,(k,l)}^\epsilon$.

Chapter 7

Summary

In this chapter we resume the main results of our analysis and give some concluding remarks. An outlook on some open problems related to the present work completes the thesis.

The following enumeration summarizes the *key points* of the chapters 2-6.

In *chapter 2* a model for viscous fluid flow in bifurcating pipes based on the Navier-Stokes equations with pressure boundary conditions is presented. Existence and local uniqueness are proven under the assumption of small data (i.e. Reynolds number Re_ϵ of higher order), using a fixed point argument (cf. section 2.2). Flux and pressure drop of the Poiseuille flow in the pipes are analyzed by means of a *formal computation*, taking into account the geometry of the junction domain (cf. section 2.4).

In *chapter 3* Leray's problem is discussed by generalizing the results from [G] and the exponential decay of the solution to Poiseuille flow is shown.

In *chapter 4* the solution of Leray's problem is approximated by the solution of the corresponding Stokes problem on *finite subdomains* of diameter $O(M)$ up to an error decaying exponentially in M.

In *chapters 5 and 6* an *approximation procedure* for the solution of the Navier-Stokes model is presented, which is based on Poiseuille flow in the pipes and Stokes flow in the junction domain. Using the decay properties of the solution of Leray's problem *error estimates* in powers of ϵ and Re_ϵ are established, depending on the junction length M. Higher order corrections for pressure decay and inertial terms are included such that any order of approximation accuracy can be achieved (cf. section 6.2).

Conclusion

Our analysis of viscous fluid flow in bifurcating pipes allows the following conclusions:

- An accurate model of viscous fluid flow in branching channels and pipes requires the analysis of local Stokes problems in the junction domain (*junction problems*), coupling the different Poiseuille flows in the pipes.

- Poiseuille flow is an appropriate approximation in the pipes for the solution of the Navier-Stokes problem, the error decaying exponentially with increasing distance M from the junction.

- For given constant pressure values on the in- and outflow boundaries of the pipes, the flux in the bifurcation can be computed from Kirchhoff's law in zero-order approximation *only* if the diameter-to-length ratio ϵ of the pipes is sufficiently small. If e.g. the junction domain has *constrictions*, then the weighted mean value from Kirchhoff's law does not provide an appropriate approximation for the flux *unless* ϵ is sufficiently small.

- The pressure decay from Leray's problem to *possibly different* constants in the pipes plays an important role in the construction of the approximation. In contrast to previous results in literature, we show that higher order corrections due to this pressure stabilization are necessary in order to obtain an appropriate approximation for the solution of the Navier-Stokes problem *including the gradient of velocity.*
  The nonlinearity of the Navier-Stokes problem generates inertial terms of higher order. Their correction requires an additional type of junction problem and the generalization of Leray's problem.

- The presented *approximation scheme* for the Navier-Stokes equations based on *finite* junction problems is adequate for numerical computations.

Outlook

We finally give a short overview on some open problems concerning viscous fluid flow in bifurating channels and pipes.
As far as the *modeling* is concerned, branchings with pipes of variable diameter or bifurcations including curved pipes have to be considered in order to describe fluid flow in complex structures as e.g. the arterial-venous system of the human body. Furthermore, elastic boundaries have to be taken into account and the model for the fluid flow has to be coupled to the equations describing the displacement of the wall of the pipes. In these situations the Poiseuille flow approach is not appropriate and other types of effective laws have to be deduced from microscopic fluid-structure models.
In order to describe the fluid flow in a network consisting of many bifurcations, the local junction problems presented in this work have to be coupled. A further difficulty arises from the fact that in physiological networks, as e.g. the circulatory system, many different length scales occur. The problem of constructing a global approximation for the fluid flow in such networks still remains unsolved.
From the *mathematical* point of view the question of existence and uniqueness of the solution of the Navier-Stokes problem with pressure boundary conditions remains open unless the pressure data is assumed to be sufficiently small. If the nonlinear term is of leading order, i.e. $Re_\epsilon = O(1)$, then we do not have a Stokes problem in the junction anymore. Since in this case the nonlinear effects are dominating, it is not clear how to realize the construction of an asymptotic approximation.

Appendix A

Computation of the pressure drop in the pipes

In order to simplify the computation, we consider a domain Ω^δ consisting of a junction $\Omega_0^\delta = \delta\,\Omega_0$ with only two outlets, linking the pipes Ω_1^δ and Ω_2^δ. The diameter of the junction is chosen to be of order $O(\delta)$, the pipes are of length $O(1)$ and cross-section $O(\delta^2)$ (cf. Fig. 2.5). The computation can be carried out for the junction of $N \geq 3$ pipes in the same way. We assume given constant pressure values p_j on the outflow boundaries of the pipes Ω_j^δ.

We now consider the following Stokes systems ($k = 1, 2$) on the rescaled junction Ω_0, prescribing normalized pressure values on the in-/outflow boundaries γ_j ($j = 1, 2$):

$$\text{(A.1)} \qquad \left\{ \begin{array}{rcll} -\Delta_y \omega_k + \nabla_y \pi_k &=& 0 & \text{in } \Omega_0, \\ div_y\, \omega_k &=& 0 & \text{in } \Omega_0, \\ \omega_k &=& 0 & \text{on } \Gamma_0, \\ \omega_k \times n_j &=& 0 & \text{on } \gamma_j, \\ \pi_k &=& \delta_{jk} & \text{on } \gamma_j, \end{array} \right.$$

where Γ_0 denotes the lateral boundary of Ω_0.

For $k = 1$ we have $\pi_1 = 1$ on γ_1, $\pi_1 = 0$ on γ_2; for $k = 2$ we have $\pi_2 = 0$ on γ_1, $\pi_2 = 1$ on γ_2. Due to the linearity of the equation, $\omega := q_1\,\omega_1 + q_2\,\omega_2$ and $\pi := q_1\,\pi_1 + q_2\,\pi_2$ is the solution for prescribed constant pressure values q_1 and q_2 on the in-/outflow boundaries γ_j. The scaled functions $\omega^\delta(x) := \omega(\frac{x}{\delta})$ and $\pi^\delta(x) := \delta\,\pi(\frac{x}{\delta})$ then solve the following Stokes problem on the domain Ω_0^δ:

$$\text{(A.2)} \qquad \left\{ \begin{array}{rcll} -\delta^2 \Delta \omega^\delta + \nabla \pi^\delta &=& 0 & \text{in } \Omega_0^\delta, \\ div\, \omega^\delta &=& 0 & \text{in } \Omega_0^\delta, \\ \omega^\delta &=& 0 & \text{on } \Gamma_0^\delta, \\ \omega^\delta \times n &=& 0 & \text{on } \gamma_j^\delta, \\ \pi^\delta &=& \delta q_j & \text{on } \gamma_j^\delta. \end{array} \right.$$

Remark: Due to the scaling of the junction domain $\Omega_0^\delta = \delta\,\Omega_0$, the factor δ also occurs in the scaling of the pressure $\pi^\delta = \delta\pi(\frac{x}{\delta})$.

In the pipes Ω_j^δ we assume the Poiseuille flow

$$\begin{aligned} V_j^\delta(x) &:= w_j(\tfrac{\tilde{x}^j}{\delta})\frac{\tilde{q}_j - p_j}{L_j}\, e_1^j, \\ P_j^\delta(x) &:= \frac{p_j - \tilde{q}_j}{L_j}\, x_1^j + \tilde{q}_j, \end{aligned}$$

where w_j denotes the Poiseuille velocity profile (cf. (2.6)) and $\tilde{q}_j$ represents the pressure value on γ_j^δ. The flux in the j^{th} pipe is given by $F_j^P = c_j\,(\tilde{q}_j - p_j)$, with the conductivity $c_j := \dfrac{1}{L_j}\displaystyle\int_{S_j} w_j$.

The pressure values p_j are assumed to be given, but the pressures $\tilde{q}_j$ are unknown. In order to compute these values, we have to establish a relation between the Poiseuille flow in the pipes and the Stokes flow in the junction.

From *physical considerations*, we assume the following:

(1) Continuity of the pressure at the interfaces γ_j^δ: $\tilde{q}_j = \delta q_j$ $(j = 1, 2)$.

(2) Balance of the fluxes: $F_i^P = F_i^S$, where F_i^S is the flux through the junction, given by

$$F_i^S := \sum_j q_j F_{ij} := \sum_j q_j \int_{S_i} \omega_j \cdot n_j.$$

We define the *flux* matrix $\mathcal{F} := (F_{ij})_{i,j=1,2}$.

From these assumptions, we obtain a system of two linear equations for the unknown pressure values q_j:

$$\sum_j F_{ij}^\delta\, q_j = -c_i p_i, \quad i = 1, 2, \tag{A.3}$$

where $F_{ij}^\delta := \displaystyle\int_{S_i} \omega_j \cdot n_j - \delta\, c_i\, \delta_{ij}$. With $\mathcal{C} := (c_i \delta_{ij})_{i,j=1,2}$ we define

$$\mathcal{F}^\delta := (F_{ij}^\delta)_{i,j=1,2} := \mathcal{F} - \delta\,\mathcal{C}.$$

The matrix $\mathcal{F}$ has a special structure:

- The sum of the elements in each row equals 0, i.e. $\sum_j F_{ij} = 0$ for $i = 1, 2$: Since $(\omega_1 + \omega_2, \pi_1 + \pi_2)$ is the solution of the Stokes problem (A.1) with pressure equal to 1 on γ_1 and γ_2, we have $\omega_1 + \omega_2 = 0$ due to the uniqueness of the solution. Therefore, the sum of the fluxes vanishes as well.

- The sum of the elements in each column equals 0: $\sum_i F_{ij} = 0$ for $j = 1, 2$, due to the incompressibility of the flow ($div\, \omega_j = 0$).

Therefore, we have

$$\mathcal{F} = \begin{pmatrix} F & -F \\ -F & F \end{pmatrix} \tag{A.4}$$

where $F := F_{11} = \displaystyle\int_{S_1} \omega_1 \cdot n_1$.

The matrix $\mathcal{F}^\delta$ is the difference between the singular matrix $\mathcal{F}$ and the regular matrix $\delta\, \mathcal{C}$, i.e. there are combinations of values F, c_1, c_2 and δ, such that the determinant of $\mathcal{F}^\delta$ vanishes. Assuming $det\, \mathcal{F}^\delta \neq 0$, we can easily compute its inverse

$$(\mathcal{F}^\delta)^{-1} = \Big[-\delta(c_1 + c_2)F + \delta^2 c_1 c_2 \Big]^{-1} \begin{pmatrix} F - \delta c_2 & F \\ F & F - \delta c_1 \end{pmatrix}.$$

Finally, the solution of (A.3) is given by

$$q_j = \delta^{-1} \frac{c_1 p_1 + c_2 p_2 - \delta\, c_1\, c_2\, F^{-1}\, p_j}{c_1 + c_2 - \delta\, c_1\, c_2\, F^{-1}}, \quad j = 1, 2. \tag{A.5}$$

We rewrite this expression in the following way:

$$q_j = \delta^{-1} \left(\frac{c_1 p_1 + c_2 p_2}{c_1 + c_2} - \alpha p_j \right) (1 - \alpha)^{-1}, \tag{A.6}$$

where $\alpha := \dfrac{c_1 c_2}{c_1 + c_2} \dfrac{\delta}{F}$ is a dimensionless parameter.
Without loss of generality we can assume $F > 0$, therefore we also have $\alpha > 0$.

Expanding with respect to α, we get

$$q_j \;=\; \delta^{-1} \left(\frac{c_1 p_1 + c_2 p_2}{c_1 + c_2} - \alpha p_j \right) (1 + \alpha + O(\alpha^2)),\; j = 1, 2.$$

Appendix B

Technical results

B.1 Inequalities and trace theorem

Lemma B.1. (Poincaré inequality)
(i) There exists a constant $C > 0$ independent of ϵ, such that

(B.1) $$\|\phi\|_{L^2(\Omega^\epsilon)} \leq C\,\epsilon\,\|\nabla\phi\|_{L^2(\Omega^\epsilon)}$$

for all $\phi \in H^1(\Omega^\epsilon)$, $\phi = 0$ on Γ^ϵ.

(ii) There exists a constant $C > 0$ independent of M, such that

(B.2) $$\|\phi\|_{L^2(\Omega^M)} \leq C\,\|\nabla\phi\|_{L^2(\Omega^M)}$$

for all $\phi \in H^1(\Omega^M)$, $\phi = 0$ on Γ^M.

Proof. (i) Cf. Lemma 7 in [MP1] and Lemma A.1 in [MP2].

(ii) In order to show that the constant is independent of M, we decompose the domain Ω^M into the junction part Ω_0 and the pipes Z_j^M and apply the Poincaré inequality on each of these subdomains, where the occurring constants are independent of M. Summing up all contributions gives the result.

□

Lemma B.2. (Embedding theorem)
(i) There exists a constant $C_{L^4,H^1} > 0$ independent of ϵ, such that

(B.3) $$\|\phi\|_{L^4(\Omega^\epsilon)} \leq C_{L^4,H^1}\,\epsilon^{\frac{1}{4}}\,\|\nabla\phi\|_{L^2(\Omega^\epsilon)}$$

for all $\phi \in H^1(\Omega^\epsilon)$, $\phi = 0$ on Γ^ϵ.

(ii) There exists $C > 0$ independent of M, such that

(B.4) $$\|\phi\|_{L^4(\Omega^M)} \leq C\,\|\phi\|_{H^1(\Omega^M)}$$

for all $\phi \in H^1(\Omega^M)$, $M \geq 1$.

Proof. (i) We use the interpolation inequality for L^p-spaces (cf. Lemma 8 in [MP2]):

$$\|\phi\|_{L^4(\Omega^\epsilon)} \leq \|\phi\|^{\frac{1}{4}}_{L^2(\Omega^\epsilon)} \|\phi\|^{\frac{3}{4}}_{L^6(\Omega^\epsilon)}. \tag{B.5}$$

The second factor is to be estimated. On every pipe Ω_j^ϵ, $j = 1, ..., N$, we can extend ϕ such that the extension $\tilde{\phi} \in H^1(\Omega_j^1)$ (i.e. $\epsilon = 1$). The embedding $H^1(\Omega_j^1) \hookrightarrow L^6(\Omega_j^1)$ then is independent of ϵ, i.e.

$$\|\tilde{\phi}\|_{L^6(\Omega_j^1)} \leq C_j \, \|\nabla\tilde{\phi}\|_{L^2(\Omega_j^1)},$$

the constants $C_j > 0$ being independent of ϵ, where we use the Poincaré inequality in order to estimate the H^1-norm (as well independent of ϵ). The norms of $\tilde{\phi}$ on Ω_j^1 coincide with those of ϕ on Ω_j^ϵ. For the junction domain $\Omega_0^\epsilon = \epsilon\Omega_0$ we obtain by change of variable the estimate

$$\|\tilde{\phi}\|_{L^6(\Omega_0^\epsilon)} \leq C_0 \, \epsilon^{1/3} \, \|\nabla\tilde{\phi}\|_{L^2(\Omega_0^\epsilon)},$$

the constant $C_0 > 0$ independent of ϵ. The additional factor $\epsilon^{1/3}$ is due to the $O(\epsilon)$-diameter of the junction and cannot be obtained for the estimates in the pipes Ω_j^ϵ, $j = 1, ..., N$, cf. Remark B.1 below. Finally, we get the claim from (B.5) using the Poincaré inequality (B.1).

(ii) Without loss of generality we assume $M \in \mathbb{N}$ (cf. proof of Theorem 4.2). The domain Ω^M consists of the junction Ω_0 and the pipes Z_j^M, $j = 1, ..., N$. In Ω_0 we have the $H^1 \hookrightarrow L^4$-embedding with a constant clearly independent of M. The pipes Z_j^M can be divided into subcylinders $Z_{j,k} := \{x \in Z_j^M : k \leq x_1^j \leq k+1\}$, $k = 0, 1, ..., M-1$, for each of which the inequality

$$\|\phi\|_{L^4(Z_{j,k})} \leq C_j \, \|\phi\|_{H^1(Z_{j,k})}$$

holds. The constant C_j is *independent* of k since the inequality is invariant under translation of the x_1^j-variable (due to the constant cross-section of the pipe). Summing over all k we thus get $\|\phi\|_{L^4(Z_j^M)} \leq C_j \, \|\phi\|_{H^1(Z_j^M)}$ for all j, which together with the estimate for Ω_0 yields the result.

□

Remark B.1. The power of ϵ occurring in (B.3) is optimal. In the three-dimensional case the Sobolev embedding $H^1 \hookrightarrow L^p$ holds for all $p \in [2, 6]$. The exponent of the L^2-norm in the interpolation inequality (B.5) then is maximal. In two dimensions we have $H^1 \hookrightarrow L^p$ for all $p \in [2, \infty)$ and estimate (B.3) then can be improved to $\epsilon^{\frac{1}{2}}$.

Lemma B.3. (Trace theorem)

(i) There exists a constant $C_0 > 0$ independent of ϵ, such that

$$\|\phi\|_{L^2(\Sigma_j^\epsilon)} \leq C_0 \sqrt{\epsilon} \, \|\nabla\phi\|_{L^2(\Omega^\epsilon)} \tag{B.6}$$

for all $\phi \in H^1(\Omega^\epsilon)$, $\phi = 0$ *on* Γ^ϵ.

An analogous estimate holds for $\|\phi\|_{L^2(\gamma_j^{M,\epsilon})}$, $M \geq 1$, *with a constant independent of* ϵ *and* M.

(ii) There exists a constant $C_1 > 0$ *independent of* M*, such that*

$$\|\phi\|_{L^2(\gamma_j^M)} \leq C_1 \|\phi\|_{H^1(\Omega^M)}$$

for all $\phi \in H^1(\Omega^M)$ *and* $M \geq 1$.

Proof. We consider the cross-section $\sigma_j^\epsilon = \epsilon\sigma_j$ at $x_1^j = \epsilon l_j$, $1 \leq l_j \leq L_j/\epsilon$ and show that there is $C > 0$ independent of ϵ and l_j, such that

$$\|\phi\|_{L^2(\sigma_j^\epsilon)} \leq C\sqrt{\epsilon}\,\|\nabla\phi\|_{L^2(\Omega^\epsilon)}.$$

Let $\tilde{\phi}(y) := \phi(\epsilon y)$ with $y \in Z_j := (l_j - \delta, l_j) \times S_j$, $0 < \delta < 1$ fixed, $Z_j^\epsilon := \epsilon Z_j$. The trace theorem for σ_j and Z_j yields

$$\|\tilde{\phi}\|_{L^2(\sigma_j)} \leq C\,\|\nabla\tilde{\phi}\|_{L^2(Z_j)} \tag{B.7}$$

with $C > 0$ independent of l_j, the inequality being invariant under translation of the y_1^j-variable; in order to get the constant independent of j we simply can take the maximum for all pipes.

We compute

$$\|\nabla\tilde{\phi}\|^2_{L^2(Z_j)} = \int_{Z_j} |\nabla_y\tilde{\phi}(y)|^2 dy \;=\; \epsilon^{-1}\int_{Z_j^\epsilon} |\nabla_x\phi(x)|^2 dx = \epsilon^{-1}\,\|\nabla\phi\|^2_{L^2(Z_j^\epsilon)}, \tag{B.8}$$

$$\|\tilde{\phi}\|^2_{L^2(\sigma_j)} = \int_{\sigma_j} |\tilde{\phi}(y)|^2 dy \;=\; \epsilon^{-2}\int_{\sigma_j^\epsilon} |\phi(x)|^2 dx = \epsilon^{-2}\,\|\phi\|^2_{L^2(\sigma_j^\epsilon)}. \tag{B.9}$$

From (B.7), (B.8) and (B.9) we get the estimates in (i).

The estimate in (ii) follows analogously, using the trace estimate

$$\|\phi\|_{L^2(\sigma_j)} \leq C\,\|\phi\|_{H^1(Z_j)},$$

the constant being independent of l_j due to the translation invariance.

□

B.2 Regularity results for Stokes equations

We state the main regularity results from [D2] for Stokes equations in domains with edges and corners, right-hand side f, non-zero divergence g and vanishing velocity on the boundary.

Theorem B.1. (Regularity of Stokes equations in cylindrical domains)
Let Ω be any bounded cylinder with constant and smooth cross-section (or a bounded smooth domain with cylindrical outlets). Then, for data $f \in H^{s-1}(\Omega)$, $g \in H^s(\Omega)$, $0 \leq s < 2$, the solution of the Stokes system

$$\text{(B.10)} \qquad \begin{cases} -\mu_0 \Delta v + \nabla p &= f \quad in\, \Omega, \\ div\, v &= g \quad in\, \Omega, \\ v &= 0 \quad on\, \partial\Omega, \\ \displaystyle\int_\Omega g &= 0, \end{cases}$$

has the regularity $v \in H^{s+1}(\Omega)$, $p \in H^s(\Omega)$, provided g vanishes on the cylinder edges of the boundary (if $s \geq 1$).

Furthermore, the following inequality holds:

$$\text{(B.11)} \qquad \|v\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)/\mathbb{R}} \leq C(\Omega) \left(\|f\|_{H^{s-1}(\Omega)} + \|g\|_{H^s(\Omega)} \right).$$

In the case of non-zero boundary conditions we have the following

Corollary B.1. *Let Ω be as in Theorem B.1, the data $f \in L^2(\Omega)$, $g \in H^1(\Omega)$ vanishing on the cylinder edges, and $\tilde{V} \in H^2(\Omega)$ an extension of the boundary values V such that $div\, \tilde{V} = 0$ on the cylinder edges. Then the Stokes problem*

$$\text{(B.12)} \qquad \begin{cases} -\mu_0 \Delta v + \nabla p &= f \qquad in\, \Omega, \\ div\, v &= g \qquad in\, \Omega, \\ v &= V \qquad on\, \partial\Omega, \\ \displaystyle\int_\Omega g &= \displaystyle\int_{\partial\Omega} V \cdot n, \end{cases}$$

has a unique solution $v \in H^2(\Omega)$, $p \in H^1(\Omega)$ and the following estimate holds with a constant $C = C(\Omega)$:

$$\text{(B.13)} \qquad \|v\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|\tilde{V}\|_{H^2(\Omega)} \right).$$

Proof. We set $v_0 = v - \tilde{V}$ and obtain a Stokes problem with homogeneous boundary values:

$$\text{(B.14)} \qquad \begin{cases} -\mu_0 \Delta v_0 + \nabla p &= f + \mu_0 \Delta \tilde{V} \quad \text{in } \Omega, \\ div\, v_0 &= g - div\, \tilde{V} \quad \text{in } \Omega, \\ v_0 &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note, that

$$\int_\Omega (g - div\, \tilde{V}) = 0$$

due to the compatibility condition of the boundary values V.

Applying Theorem B.1 to system (B.14) yields the result.

□

We conclude this section with some remarks concerning the regularity results listed above. They are based on the corresponding regularity theory for elliptic boundary value problems developed in [D1]. The problem of regularity is thereby related to some general *Fredholm properties* of the elliptic operator. Characteristic conditions are given for the domain and the operator in order to have these properties. If e.g. the domain Ω is a two-dimensional polygon, then such conditions are related to the angle openings of Ω.
In [D2] these conditions are specified also for three-dimensional domains with edges and corners (such as e.g. a polyhedron or a cylinder) and are extended to the Stokes operator

$$\mathcal{S}_n : [(H_0^1 \cap H^2)(\Omega)]^n \times H^1(\Omega) \to L^2(\Omega)^n \times H^1(\Omega),$$
$$(v, p) \longmapsto (f, g),$$

given by (B.10) with Dirichlet boundary conditions (where $n = 2, 3$ denotes the space dimension).

This type of regularity results for Stokes equations on *polygonal* or *polyhedral* domains can also be found in [GR].

B.3 The divergence-problem in the junction

Lemma B.4. *For $f \in L^2(\Omega^M)$, $M \geq 1$, $\int_{\Omega^M} f = 0$, the divergence-problem*

$$\text{(B.15)} \qquad \begin{cases} div\, u &= f \quad \text{in } \Omega^M, \\ u &= 0 \quad \text{on } \partial\Omega^M, \end{cases}$$

admits (at least) one solution $u \in H_0^1(\Omega^M)$ which can be estimated as follows: There exists $C > 0$ independent of M such that

$$\text{(B.16)} \qquad \|u\|_{H^1(\Omega^M)} \leq C\, M\, \|f\|_{L^2(\Omega^M)}.$$

Remark B.2. From the theory of the divergence-problem (cf. [G], ch. III.3) we obtain the following estimate:

$$\|u\|_{H^1(\Omega^M)} \leq C\, M^{n+1}\, \|f\|_{L^2(\Omega^M)}, \tag{B.17}$$

where $n = 2, 3$ is the space dimension and $C > 0$ depends on the cross-sections Σ_j of the pipes Z_j^M and the geometry of the junction domain Ω_0. Note that the power of M is worse compared to (B.16).

Proof. Without loss of generality we assume $M \in \mathbb{N}$ (cf. proof of Theorem 4.2 and Remark B.3 below).

We decompose the cylinders Z_j^M into M subcylinders

$$Z_{j,k} := \left\{x \in Z_j^M : k \leq x_1^j \leq k+1\right\},\ k = 0, 1, ..., M-1.$$

We then consider the following divergence-problems on $Z_{j,k}$:

$$\begin{cases} div\, u &= f & \text{in } Z_{j,k}, \\ u &= 0 & \text{on } \partial Z_{j,k} \cap \partial\Omega^M, \\ u &= -\dfrac{1}{|S_j|}\left(\displaystyle\sum_{l=m}^{M-1} \int_{Z_{j,l}} f\right) e_1^j & \text{on } \Sigma_{j,m},\ m = k, k+1, \end{cases} \tag{B.18}$$

where

$$\Sigma_{j,k} := \left\{x \in Z_j^M : x_1^j = k\right\},\ k = 0, 1, ..., M-1.$$

It is known (cf. e.g. [G], ch. III.3 and [MP2], Lemma 9) that there is a solution $u \in H^1(Z_{j,k})$ with

$$\|u\|_{H^1(Z_{j,k})} \leq C_j \left(\sum_{l=k}^{M-1} \|f\|_{L^2(Z_{j,l})}\right). \tag{B.19}$$

The constant $C_j > 0$ is independent of k due to the translation invariance of the inequality (constant cross-section of the cylinder Z_j^M).

In the junction Ω_0 we choose $u \in H^1(\Omega_0)$ as solution of

$$\begin{cases} div\, u &= f & \text{in } \Omega_0, \\ u &= 0 & \text{on } \partial\Omega_0 \cap \partial\Omega^M, \\ u &= -\left(\dfrac{1}{|S_j|}\displaystyle\int_{Z_j^M} f\right) e_1^j & \text{on } \Sigma_{j,0}. \end{cases} \tag{B.20}$$

Since $\displaystyle\int_{\Omega^M} f = 0$, we have

$$\int_{\Omega_0} f = \sum_j \int_{\Sigma_{j,0}} u|_{\Sigma_{j,0}} \cdot e_1^j = -\sum_j \int_{Z_j^M} f$$

and thus problem (B.20) is well-posed. Furthermore,

$$\|u\|_{H^1(\Omega_0)} \leq C(\Omega_0)\, \|f\|_{L^2(\Omega^M)}. \tag{B.21}$$

By construction, u is well matched on the interfaces $\Sigma_{j,k}$ and thus an element of $H^1(\Omega^M)$. Summing up the estimates (B.19) and (B.21) we finally have

$$\begin{aligned}
\|u\|_{H^1(\Omega^M)} &= \|u\|_{H^1(\Omega_0)} + \sum_{j,k} \|u\|_{H^1(Z_{j,k})} \\
&\leq C(\Omega_0)\, \|f\|_{L^2(\Omega^M)} + (max_j\, C_j) \sum_{j,k} \left(\sum_{l=k}^{M-1} \|f\|_{L^2(Z_{j,l})} \right) \\
&\leq C(\Omega_0)\, \|f\|_{L^2(\Omega^M)} + (max_j\, C_j)\, M \sum_{j} \|f\|_{L^2(Z_j^M)} \\
&\leq C\, M\, \|f\|_{L^2(\Omega^M)}.
\end{aligned}$$

Remark: If the mean value of f is zero on each cylinder $Z_{j,k}$, then we can solve the divergence problem (B.18) with zero boundary conditions on the whole boundary $\partial Z_{j,k}$. Then, the sum in (B.19) can be replaced by $\|f\|_{L^2(Z_{j,k})}$ and therefore we obtain the H^1-estimate for the solution u of (B.15) independent of M.

□

Remark B.3. If $M = N + r$ with some $N \in \mathbb{N}$, $r \in [0, 1)$, the junction Ω_0 can be replaced by Ω^{1+r}, i.e. the prolongated junction with pipes of length $1 + r$. Due to the theory of the divergence problem, the constant $C(\Omega^{1+r})$ then occuring in estimate (B.21) admits an estimate of type (3.10) (cf. [G]) and in particular can be bounded independent of r.

In a similar way the following result concerning the divergence-problem in Ω^ϵ can be shown (cf. Lemma A.3 from [MP3]):

Lemma B.5. *For $f \in L^2(\Omega^\epsilon)$, $\int_{\Omega^\epsilon} f = 0$, the divergence-problem*

$$\begin{cases} div\, u &= f \quad in\ \Omega^\epsilon, \\ u &= 0 \quad on\ \partial\Omega^\epsilon, \end{cases} \tag{B.22}$$

admits at least one solution $u \in H_0^1(\Omega^\epsilon)$ which can be estimated as follows: There exists a constant $C > 0$ independent of ϵ such that

$$\|u\|_{H^1(\Omega^\epsilon)} \leq \frac{C}{\epsilon}\, \|f\|_{L^2(\Omega^\epsilon)}. \tag{B.23}$$

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