Auctions with Variable Supply and Competing Auctioneers

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Chapter 1

Introduction

One of the most fascinating fields in modern microeconomics studies the process of price formation on imperfectly competitive markets. The theory of auctions is an area within this field, which developed rapidly over the last 45 years. The formal analysis of auction mechanisms as games of incomplete information, which was pioneered by Vickrey (1961), experienced an enormous growth in the past couple of decades. One of the driving forces for this development lies in the parallel advancements in game theory. The seminal works of Harsanyi (1967) and Selten (1975) facilitated this rapid development by providing important foundations and instruments for analysis. The established equilibrium concepts for games of incomplete information and dynamic games provided the tools necessary for the systematic analysis of auction games. As a result the literature on the topic burgeoned. Now it continues to grow at a rapid pace. The Econ Lit database for example contains more than thousand entries with the word "auction" or "auctions" in their titles and every major conference in economic theory has several sessions on auctions.

Auctions have been used since antiquity for the sale of a variety of goods and have a colorful history. The Babylonians auctioned wives[1] and the Romans everything from slaves to plundered booty and debtor's property[2]. In China around the seventh century

[1]Cassady (1967, p. 26), quoting "The Histories of Herodotus," translated by Henry Cary (New York: D. Appleton and Company, 1899), p. 77, reports that around 500 B.C. in Babylon once a year women of marriageable age were auctioned off on conditions that they be wed.

[2]Shubik (1983) provides an entertaining and colorful sketch on the history of auctions in the Roman and Babylonian empires. Cassady (1967, p. 28), quoting historical documents, reports that Romans, when in financial straits, employed auctions for the liquidation of property. Marcus Aurelius for example held an auction of royal heirlooms and furniture. Caligula auctioned off ornaments belonging to his family to help him meet his debts and recoup his losses. Perhaps the most preposterous auction was held in 193 A.D., when the entire Roman Empire was sold via an ascending auction by the

A.D. the belongings of deceased monks were auctioned to raise money for Buddhist monasteries and temples[3]. Auctions were extensively used in Europe toward the end of the seventeenth century. Their widespread use in Britain lead to the emergence of famous auction houses such as Sotheby's and Christie's, which were established in 1744 and 1766 respectively. Around that time these trade mechanisms were used also in America for slave trade[4]. In the beginning of the nineteenth century auctions were also widely employed at the east coast ports of the United States for a lively trade with imported goods. As early as 1887, the Netherland farmers at Broek op Langedijk organized auctions to ensure that middlemen would not be able to manipulate prices to their disadvantage.

Nowadays auctions are of special interest for economists, since in modern economies an immense volume of transactions is conducted via these allocation mechanisms. They are used by governments to sell spectrum licences for mobile phones[5], mineral rights including oil fields, foreign exchange, government debt, emission permits, import licences as well as for the allotment of procurement contracts and for the purposes of privatization of government-owned firms. The privatization of the state-owned firms in many Eastern European countries within their transition process from centrally planned to market economies, which started in the beginning of the nineties, was predominantly conducted through auctions[6]. Today a variety of goods from collectibles to items like computers, automobiles, cameras, cell-phones, video games, electronics, DvDs and movies change hands in online sites such as Amazon and Ebay, which are typically organized as auction mechanisms. Lucking-Reiley (2000) provides a survey

Praetorian Guards. The winner, and thus the next Emperor, Didius Julianus was in power for just over two months before being beheaded by Septimius Severus. This was probably the earliest and the most cruel case of the winner's curse.

[3]Cassady (1967, p. 29) reports that the auction method was one of the institutions to raise money. Besides auctions also pawn shops, mutual financing associations and lotteries were employed.

[4]This practice continued until 1808, when the importation and trade with slaves was prohibited.

[5]The US Federal Communication Commission held six radio spectrum licence auctions in the time period from July 1994 to May 1996. Cramton (1997) provides an overview of these auctions and an assessment. See also Milgrom (2004) and Cramton and Schwartz (2000). The "third generation" (3G) or (UMTS) mobile telecommunication licence auctions took place in Europe in the time period 2000-2001. Klemperer (2004, Chapter Five) provides an overview of all European auctions. For an analysis of the German auctions of the (2G) and (3G) licences see Grimm, Riedel, and Wolfstetter (2003) and Grimm, Riedel, and Wolfstetter (2002).

[6]The mass privatization programs for example in Russia, Hungary, Poland, Czechoslovakia and Bulgaria involved auctions. Prior to the auctions vouchers of a certain nominal value in the local currency were distributed to the citizens. These vouchers could then be invested in voucher funds, sold for cash or used in the bidding process to acquire shares. Wang (1991) and Atanasov (2005) provide more information on the mass privatization in these countries.

on the goods auctioned off via internet and Harden and Heyman (2002, Chapter 2) provide a brief history of internet auctions. The last couple of years witnessed the development of a new theoretical and empirical literature on online auctions[7]. Well-established works in this area are Ockenfels and Roth (2002), Bajari and Hortacsu (2003), Bajari and Hortacsu (2004), Ockenfels and Roth (2005a) and Ockenfels and Roth (2005b).

The development of auction theory has also a purpose reaching beyond the scope of its direct applications. The techniques developed by auction theorists can be employed for the study of tournaments, political contests, rent seeking, promotions in labor markets, queues, research and development races, trade wars, military and biological wars of attrition. Amann and Leininger (1996) and Baye, Kovenock, and de Vries (1996) argue that these problems and phenomena can be modelled as an all-pay auction. Amann and Leininger (1996) derive the equilibria in the all-pay auction with incomplete information and asymmetric distributions in the two-bidder case. Baye, Kovenock, and de Vries (1996) study a complete information model with an arbitrary number of bidders. Baye and Kovenock (1993) and Hillman and Riley (1989) analyze political contests. Konrad (2000) studies trade contests and Moldovanu and Sela (2005) derive the optimal design of a contest. The theory of auctions is successfully employed to study bargaining situation, because many bargaining problems can be modelled and analyzed as double auctions[8]. Auction theory bears also close connections to the theory of monopoly pricing. Bulow and Roberts (1989) uncovered an important relationship between the theories of optimal auctions and monopoly pricing. They showed how the optimal mechanism design problem can be recast only in terms of marginal revenues and marginal costs, which are terms familiar to every economist. This important contribution made the optimal auction literature more accessible to a much broader audience of economists. Another strand of literature studies the relationship between auction market games and the theory of perfect competition. Its aim is to model the functioning of a market as a strategic market game, determine the appropriate noncooperative equilibria and identify the circumstances under which they lead to the desired Walrasian outcome. In Chapter 5, which presents a model within these lines, we will review this literature.

[7] A systematical study on this topic is not available as of yet. Many of the works are at a preliminary stage of development and the literature will continue to grow rapidly.

[8] This literature has also been rapidly growing in the last two decades. Classical works on the subject are Myerson and Satterthwaite (1983) and Cramton, Gibbons, and Klemperer (1987). For the most recent treatments see Kittsteiner (2003) and Fieseler, Kittsteiner, and Moldovanu (2003).

1.1 Purpose of the thesis and approach

Despite being a very active area of research, most auction theory (at least until the beginning of the nineties) restricts attention to the analysis of auctions of a single indivisible unit[9]. Although many real-world markets function as auctions in which multiple units or divisible goods are traded, this area of auction theory remained less developed, which undoubtedly rests on the extreme complexity of these strategic market environments. In the middle of the nineties however the studies dealing with multiple unit and divisible good actions revived primarily due to the auctioning of spectrum licences[10]. This was a stimulus for the further development of the multiple unit auction theory for financial market applications, which was long overdue. This development concerns primarily the financial literature on Treasury auctions and initial public offerings (IPOs). The importance of these auctions is unmatched by any other application. The Treasury auctions take place on a regular basis (e.g., weekly, monthly or quarterly) and in these auction huge volumes of public debt are traded in a number of countries. In spite of this positive trend, the asymmetry in the development of the multiple unit auction theory compared to the auction theory of a single item remains. The current state of affairs is unsatisfactory and a lot of further research is needed.

The purpose of this thesis is to *further develop* the theory of competitive bidding on markets in which multiple units are traded. We provide models of multi-unit environments on the basis of which we *discuss and compare* the performance of some widely used and well established auction institutions: the uniform price and the discriminatory auction. Our particular interest concerns multiple unit auctions with *variable supply*. These auctions are trade institutions in which the seller does not commit to a supply quantity ex ante, but rather determines it after the bidding was completed in view of the received bids. These market institutions have not been extensively analyzed so far. They have important applications, which will be the subject of discussion in the next chapter. The performance measures that we take for the comparison of these market institutions are the standard ones in auction theory: the *expected revenue* of the seller[11]

[9] One notable exception is the paper of Wilson (1979), which deals with divisible good auction games in which the strategy spaces are continuous demand functions.

[10] See Milgrom (2004) and Klemperer (2004) for more details around the organization and the auction design of the mobile phone licence auctions in the United States and Europe.

[11] The literature on *optimal auctions* deals with the problem of determining the revenue maximizing trade mechanism of the seller from a class of mechanisms. Usually one takes the class of incentive compatible and individually rational mechanisms. This approach was adopted by Myerson (1981) in a now classical paper. We will use this approach in Chapter 6.

and *efficiency* of the equilibrium allocation. In Chapter 5 and Chapter 6 we will however go further to define a broader class of auction institutions and look for the ones within this class, which best serve a particular goal such as efficiency or revenue maximization of the auctioneer. The approach we take here is the standard one. We model the auctions as noncooperative games (with complete or incomplete information) and analyze their (noncooperative) equilibrium outcomes. These outcomes are used as a prediction of how rational players will behave in such auctions. The game-theoretical treatment appears to be very suitable for the analysis of auctions, because they specify very clear rules of trade. Although the aim of this thesis is primarily to make a theoretical contribution, we will also provide many examples of real-world markets to which the theoretical results relate.

1.2 Overview of the thesis and results

In **Chapter 2** we explain the variable supply auction format and introduce the reader to its applications on real-world markets. A special emphasis is put on the Treasury and IPO auctions, which are the most important examples.

In **Chapter 3** we present a competitive bidding model of incomplete information, in which bidders face uncertainty about the incentives of the auctioneer when bidding for one unit. The auctioneer determines supply quantity after observing the bids, which is a scenario that we model as a two-stage game. The framework is rich enough to provide a discussion of the *uniform price* and the *discriminatory* sealed bid auction formats. These auction forms are compared in terms of seller's revenue and efficiency. It is claimed that the uniform price auction outperforms the discriminatory one with respect to both criteria. These results, derived on the basis of comparison of the symmetric mixed strategy subgame perfect equilibria, concern any number of two or more bidders and a whole class of probability distributions. The two bidder case is a special case for which the results are valid even for all rationalizable strategies. The exposition of this chapter is based on Damianov and Becker (2005) and Becker and Damianov (2005).

Chapter 4 uses the same framework, but considers a more specific model of just two bidders and uniformly distributed marginal costs. This restriction makes possible the additional discussion of the standard *open* (or dynamic) auction formats, i.e. the *ascending* and the *descending* clock auctions as well as a situation in which bidders submit bids sequentially (both with uniform and discriminatory pricing). In summary, the standard uniform and discriminatory pricing rules are coupled with the following four procedures for collecting bids: sealed bid (simultaneous), sequential bid, via an ascending and descending clock auction. Contrary to the intuition from the single unit auction, with uniform pricing the sealed bid auction outperforms the ascending clock, as well as the descending clock and the sequential auction procedure. The discriminatory price auction is shown to be inferior to the uniform price auction. The exposition follows Damianov (2004b).

Chapter 5, which is based on Damianov (2004a) and Damianov (2005b), retains the same two-stage trade mechanism structure, but extends the analysis in another direction. In the new framework the bidders do not submit solely a bid price for one unit, but rather announce a price and a (maximal) quantity that they wish to purchase at that price. We provide conditions on the pricing and the rationing rules, which guarantee that the strategic equilibria of these market forms coincide with the competitive outcome. If the “pro rata on the margin” rationing rule is used, the discriminatory auction has Walrasian equilibria only, whereas the uniform price auction has additional non-Walrasian equilibria. The non-Walrasian (low-price or collusive-seeming) equilibria in the uniform price auction disappear, if the seller uses “pro rata” rationing and adheres to a simple consistency rule when selecting among several profit-maximizing quantities. These models provide a strategic foundation of the competitive equilibrium paradigm.

Chapter 6 studies a multi-unit market in which sellers of a single unit of a homogenous good design trade mechanisms in order to attract buyers. Bidders choose which trade mechanism they will participate in, learn their valuations and submit bids. We contribute to the literature on competition among auctioneers in mechanism design by providing conditions for the existence of equilibrium on this market. The equilibrium trade mechanisms have the simple structure of an auction with no reserve price, but a positive participation fee, which is derived as a function of the distribution of buyers valuations. The exposition follows Damianov (2005a).

Chapter 7 summarizes the main findings of the thesis, points to unresolved issues and provides a perspective of possible directions for future research.

Chapter 2

Multi-Unit Auctions with Variable Supply: Applications

In the traditionally analyzed (multi-unit) auctions the seller is assigned a rather passive role. The seller chooses an auction format and possibly a reserve price and leaves the price formation process entirely in the hands of the bidders. There is however a variety of market institutions for the allocation of multiple units, which are characterized by a much more active participation of the seller. The monopolist not only chooses the auction form, but also remains a player in the price-setting game. The monopoly seller might decide to reduce, postpone, reschedule or cancel the auction if announced demand is weak and bids are unsatisfactory low. Conversely, the seller might have an incentive to extend supply from an initial target quantity if announced demand is high and it is profitable to sell additional units.

One special class of auctions sharing this feature are the *multi-unit auctions with variable supply.* In these auction formats the seller does not commit to a supply quantity ex ante, but rather determines it after the bidding depending on the received bids. The next three chapters will present and discuss formal models of competitive bidding in variable supply auctions. The purpose of this chapter is to familiarize the reader with the existing real-world markets, which are organized as variable supply auctions.

The variable supply multi-unit auction is used for trade of various goods ranging from wine and art to jewelry and furniture, which are auctioned both traditionally and via the internet. However, the applications of greatest interest and largest trade volume come from the financial markets. Variable supply auctions are employed on a regular basis for the sale of financial assets like Treasury bills and initial public offerings (IPOs).

2.1 Treasury auctions

The Treasury departments in many countries employ variable supply auctions to finance the Government debt. The Governments in Mexico, Switzerland, Sweden, Finland, Germany and Italy for example reserve the right to either reject bids or change the volume of the bond issue that they intend to auction.

In the context of the Mexican Treasury procedure, Umlauf (1993, pp. 316-317) explains that the Government cancels auctions to take advantage of the interest rate declines arising from macroeconomic shocks realized in the time window between bid submission and announcement of results. In other words, bids are cancelled when the government can reissue at lower rates than those prevailing at the time of bid submission.

Heller and Lengwiler (2001, p. 420) comment that the Swiss Treasury, which uses a uniform price auction, announces the maximum number of bonds that will be issued, but usually significantly less bonds than this maximum number are sold in the auction. The Treasury department also reserves the right to cancel an auction if it does not consider the bids satisfactory[1].

Nyborg, Rydqvist, and Sundaresan (2002) report the Treasury in Sweden uses a discriminatory auction and remark (on page 422) that it will be of interest to build models of multi-unit auctions in which endogenous supply is present. They argue that many Treasury departments either reject bids or change the amount that they will sell. The data on bidding that empiricists collect reflects optimal adjustment by bidders who account for this supply uncertainty.

Keloharju, Nyborg, and Rydqvist (2005) emphasize that the Treasury department in Finland acts strategically by determining supply after observing the bids. The Finnish Treasury does not have an explicit policy regarding the choice of quantity (and stop-out price), and they do not operate with pre-announced reserve prices. The Treasury's actual choice is influenced by the long-term revenue target, market conditions, the Treasury's own opinion about the true market price and unwillingness to spoil the market by accepting too low bids. Similar behavior is documented by Rocholl (2004) for the Treasury auctions in Germany and by Scalia (1997) for the Treasury bond auctions in Italy.

[1] The Swiss Treasury also frequently chooses not to issue a newly designed bond, but instead decides to extend the volume of a previously issued series (reopenings).

The US Treasury auctions are traditionally considered to be of the fixed supply format[2]. However, Keloharju, Nyborg, and Rydqvist (2005) recently report that the US Treasury, although not giving itself the extreme flexibility with respect to determining supply as the Finnish Treasury does, reserves the right to accept or refuse to recognize any or all bids. This institutional feature has been recently established by law. In the US Federal Register, Vol. 69, No. 144, Wednesday, July 28, 2004, § 356.33 one reads the following Rules and Regulations[3]:

Does the Treasury have any discretion in the auction process?

(a) *We have the discretion to:*

1. *Accept, reject, or refuse to recognize any bids submitted in an auction;*
2. *Award more or less than the amount of securities specified in the auction announcement;*
3. *Waive any provision of this part for any bidder or submitter; and*
4. *Change the terms and conditions of an auction.*

(b) *Our decisions under this part are final. We will provide a public notice if we change any auction provision, term, or condition.*

(c) *We reserve the right to modify the terms and conditions of new securities and to depart from the customary pattern of securities offerings at any time. Participant receives a larger auction award by acquiring securities through others than it could have received had it been considered one of these types of bidders.*

Keloharju, Nyborg, and Rydqvist (2005) claim that the possible reduction in supply protects the US Treasury against very low prices. Dealers may not find it worthwhile to submit low prices, because this might cost them their primary dealer privileges.

[2] See for example Nandi (1997) and Simon (1994).

[3] This document can be downloaded at ftp.publicdebt.treas.gov/gsr31cfr356.pdf, §356.33 Reservation of rights.

2.2 Initial Public Offerings (IPOs)

The dominant trading mechanism for IPOs in the United States is the auction-like bookbuilding process. During this process the issuing firm can withdraw part or the entire issue if it considers the bids unsatisfactory and it becomes clear that the minimal acceptable price (which is usually kept secret) will not be reached. Busaba, Benveniste, and Guo (2001), Brisley and Busaba (2003) and Dunbar and Foerster (2002) provide a thorough description of the bookbuilding method, incidences of IPO withdrawals from the public press and statistics on withdrawn issues over the time period 1984-2000.

Conversely, if demand is strong, underwriters are often granted the "Greenshoe Option"[4] to increase the amount of the issue by up to 15%. For example the IT consulting firm Wincor Nixdorf went public in June 2004 and the underwriter, Goldman, Sachs & Co, issued 852,131 additional shares[5].

Google's IPO from August 2004 provides the opposite example. As a result of low bidding shareholders withdrew at about six million shares, which comprises approximately 23% of the issue[6].

Extensions and reductions of the offering are also ubiquitous in the European IPOs. The biggest IPOs in Germany for the last year exhibit these features. Postbank went public on June 21, 2004 offering 82 million shares at a target price in the range of € 31,50-36,50 per share. *Financial Times Germany* from June 23, 2004 reported that due to the lack of interest by the investors one third of the issue was taken off the market and the final price was reduced to € 28,50 per share. When Premiere (Pay-TV) went public on March 9, 2005, it offered 36,6 million shares at a target range of € 24,00-28,00 per share. The issue was heavily oversubscribed and finally a total amount of 42,1 million shares were sold at a price at the upper end of the price range[7].

[4]This option is also called over-allotment and is exercised when the IPO is oversubscribed and trading above its offer price. A typical underwriting agreement allows the underwriter to buy up to an additional 15% of shares at an offering price for a period of several weeks after the offering. This allows the underwriter to manage the aftermarket trading. The term comes for the Green Shoe Company, which was the first to have this option.

[5]The maximal volume of additional shares, which possibly could be sold was 1,2 million. This example was brought by McAdams (2005). See www.wincor-nixdorf.com for more about this IPO.

[6]See Kawamoto and Olsen (2004).

[7]The final price was € 28,00. See www.geld-und-boerse.de, March 9, 2005.

2.3 Other applications

Auctions with variable supply are also used by some auction houses like Sotheby's, Christie's and Phillips (see Ashenfelter 1989, pp. 24–25) when selling goods such as rare wine, art, jewelry and furniture. The usual practice is to keep a secret reserve price, which is solicited and made public after the bidding. Bids below the reserve price are not served and auctioneers say that the retained objects are "bought in". This means that they will be put up for sale at a later auction, sold elsewhere or taken off the market. Conditioning the supply quantity on the received bids via secret reserve prices is also practised in internet auctions (see Lucking-Reiley 2000, p. 244). Salant and Loxley (2002) report that the electricity default service procurement auctions in New Jersey also give the seller the right to decrease total quantity after the bidding[8].

[8] This example was brought also by McAdams (2005).

Chapter 3

Common Value Auctions with Variable Supply: Uniform Price versus Discriminatory

3.1 Introduction

In this chapter we study a two-stage common value model of rational bidding in a variable supply auction. In the first stage bidders submit bids to the auctioneer and in the second stage the seller decides on a supply quantity so as to maximize profit. The bidders are incompletely informed about the (constant) marginal costs of the seller, which are represented by a random variable. This feature of the model accounts for the supply uncertainty that bidders face in variable supply financial auctions such as auctions for Treasury bills and IPO auctions. Since these assets are traded on a very liquid secondary market after the auction, they are clearly of common value to the bidders. We study the mixed strategy subgame perfect equilibria (and the rationalizable strategies in the two-bidder case) of these auction forms, capturing the optimal bidders' adjustment to supply uncertainty.

Here we compare the standard pricing rules – the discriminatory and the uniform price auction. We show that, due to the supply uncertainty in a symmetric (mixed strategy) equilibrium, buyers bid higher in the uniform price auction with a probability of one. In the two-bidder case this result is even valid for all rationalizable bids. As a consequence, we demonstrate that the seller's expected profit is higher under the uniform pricing rule. We find also that (under a convexity condition) the uniform price auction generates higher average trade volume.

Besides arguing in favor of uniform pricing in the presence of supply uncertainty, our model helps to reconcile the recent theoretical studies with the empirical evidence on uniform price and discriminatory multiple-unit auctions. On the empirical side, Goldreich (2004) examines underpricing in a multi-unit common value setting. He finds that there is underpricing in both auction formats, but in the uniform price auction there is less underpricing relative to the discriminatory auction. Malvey and Archibald (1998) predict that the uniform price auction will allow the Treasury to make improvements in the efficiency of market operations and reduce the costs of financing the Treasury debt. All these stylized facts will be confirmed by our theoretical model. Umlauf (1993) also argues in favor of the uniform price auction, especially when there is *supply uncertainty*.

The theoretical literature, on the other hand, argues against the uniform price auction on different grounds. The primary concern is the existence of low-price equilibria in the uniform price auction with fixed supply. Back and Zender (1993) extend a model first introduced by Wilson (1979) and demonstrate that there are low-price (or collusive-seeming) equilibria in which bidders submit kinked demand schedules. McAdams (2000) shows that low price equilibria exist in which bidders employ linear strategies. Wang and Zender (2002) find that when the submission of non-competitive bids in a divisible good auction is allowed, there always exist equilibria of a uniform-price auction with lower expected revenue than the equilibrium revenues in a discriminatory auction. Nyborg and Strebulaev (2004) show in a model allowing for short squeezes that the discriminatory auction leads to more short squeezing and higher revenue than the uniform price auction.

In contrast to the above literature, in the auction games presented here the aspect of supply uncertainty is explicitly modelled. This aspect, which has not been extensively analyzed as of yet, pertains to financial markets such as for example the Treasury bill auctions in many countries. We show that in this setting bidders submit higher bids in the uniform price auction and explain the intuition behind this result.

3.1.1 Related theoretical literature

Despite the existing bulk of descriptive and empirical literature, theoretical models of competitive bidding in variable supply auctions appeared only recently. Most closely related to our paper is the work of Lengwiler (1999), who also analyzes a model in which the bidders are incompletely informed about the (constant) marginal costs of the seller. Whereas in Lengwiler's setting bidders choose quantities as strategic variables, in our model bidders choose prices. Our bidders submit a price for a single unit, whereas Lengwiler's bidders announce quantities on a discrete price grid consisting only of two exogenously given prices – high and low. Lengwiler showed that the uniform price and the discriminatory auctions have perfect equilibria. Since the computation of equilibria in such a setting is rather difficult, both standard auction forms could not be compared in terms of revenue for the seller or efficiency. He claimed that both formats allocate inefficiently and the inefficiency does not necessarily decrease with an increased number of bidders.

A model with no uncertainty was studied independently by Back and Zender (2001) and McAdams (2000), who analyze the uniform price Treasury auction. In their setting, the seller fixes a supply quantity, which can potentially be reduced after the bidding, and sets a reserve price of zero. The bidders have common knowledge of the asset value and compete for shares by submitting left-continuous bid functions. Demand is rationed according to the "pro rata on the margin" rule. The option to reduce supply is found to provide a lower bound on the collusive-seeming equilibria[1] known to exist in the divisible good uniform price auction with fixed supply. Damianov (2005b) showed that if the "pro rata" rationing rule is used, low price equilibria in the uniform price auction with endogenous supply do not exist and in a subgame perfect equilibrium the Walrasian quantities are traded at the Walrasian price.

3.1.2 Nash equilibria and rationalizable strategies

In this work we provide some new insights about *rational bidding behavior* in common value auctions with the variable supply feature. We use the term "rational bidding" intentionally, since in the two-bidder case we do not limit our study solely to equilibrium behavior. Rather, we extend our concept of rational bidding to the set of *rationalizable*[2]

[1]Low-price (or collusive seeming) equilibria of the uniform price multi unit auction with fixed supply are first discussed by Wilson (1979). Back and Zender (1993) bring that issue in the context of the ongoing discussion regarding how Treasury bills auctions should be organized. They argue that the uniform price auction has the potential of yielding very low revenues for the Treasury.

[2]The notion of rationalizability as a criterion for rational strategic choice was introduced independently by Bernheim (1984) and Pearce (1984). They argued that in a simultaneous game without pre-play communication one cannot expect that players will be able to fully predict their opponents' behavior and therefore can be in doubt whether in such a game Nash equilibria will be played at all. They proposed that a player's choice should only be rational given some conjectures (or beliefs) about other players' behavior. These beliefs need to be rational(izable) in a precisely defined sense but need

strategies.

Our main contribution consists in comparing the sets of symmetric mixed strategies Nash equilibria (and rationalizable strategies in the two-bidder case) of the uniform price and the discriminatory auction. We do that by identifying bounds on the supports of these sets. In the setting under consideration these bounds allow us to compare both auctions in terms of revenue for the seller and average trade volume. Our approach enables the comparison of the two auction formats without the need to explicitly compute their symmetric equilibria, which is possible only in special cases in this complex multi-unit auction setting[3]. Additionally, the bounds we provide on the rationalizable strategy sets in the two-bidder case apply to all Nash equilibria, as the set of Nash equilibria is contained in the set of rationalizable strategies (see Bernheim 1984, Pearce 1984).

The methodology to identify bounds on the set of rationalizable bids is relatively novel to the theory of auctions. Recently Battigalli and Siniscalchi (2003) study interim rationalizable bids in symmetric first-price single-unit auctions with interdependent values and affiliated signals. Cho (2003) analyzes rationalizable strategies in a single-unit first price auction with many bidders[4] and Deckel and Wolinsky (2003) provide rationalizable bidding results in first-price single-unit discrete auctions with many bidders.

This methodology has advantages reaching beyond the scope of these papers. In complex models, where the characterization of Nash equilibria proves to be intractable or possible only in special cases, one might still be able to identify some general properties of rationalizable strategy sets. These properties are valid not only for the correct, self-fulfilling beliefs required by the equilibrium notion of rationality, but also for all other not necessarily correct, but sophisticated or rational(izable) beliefs. This additional generality might be important, if one tests theoretical predictions like those presented here with experiments in which the subjects play simultaneously and are not allowed to communicate.

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not necessarily coincide with the actually played (pure or mixed) strategies as the Nash equilibrium concept requires. The rationalizability notion thus allows for more flexibility in the beliefs the players hold. Generally players can have many rationalizable strategies and in simultaneous games the set of Nash equilibria is only a subset of the set of rationalizable strategies (see Bernheim 1984).

[3]Compare also Lengwiler (1999).

[4]He extended Wilson's (1977) result that in single-unit auctions with a common value element the equilibrium price converges to the highest valuation among bidders as the number of bidder increases.

3.2 The model

3.2.1 Preliminaries

We consider a bidding game between $n \geq 2$ buyers and a monopolistic seller. The monopolist sells off multiple units of a common value asset via a variable supply auction. Each buyer $i \in \{1, 2, \ldots, n\}$ is risk neutral and submits a price bid for a single unit. The following assumptions further specify the setting of the model.

Assumptions

(A1) *No proprietary information*[5].

The buyers are identically informed about the common value $v > 0$ of the asset. The seller is uninformed about v, which explains the use of an auction as a trade mechanism.

(A2) *Private information about seller's marginal costs.*

The monopolist "produces" the good with constant, but privately observed marginal production costs c. c is a random variable with support $[0, \overline{c}]$, where $\overline{c} \geq v$. The distribution function of that random variable is denoted by $F(c)$ and the density function by $f(c)$. The latter is taken to be continuous, strictly positive in the interval $[0, \overline{c}]$ and differentiable in the interval $(0, \overline{c})$.

(A3) *Monotone hazard rate.*

Further it is assumed that the distribution function is log-concave, i.e.

$$(\ln F(c))' = \frac{f(c)}{F(c)}$$

is a monotonically decreasing function[6].

[5] The assumption and the term "no proprietary information" were introduced into the auction literature by Wilson (1979) and Bernheim and Whinston (1986). This simple information structure is often assumed in multi-unit auction models (see, e. g. Back and Zender 2001, Kremer and Nyborg 2004). The assumption is definitely a restriction, since one cannot discuss the effects on bidding behavior any more, which arise from the interaction of privately informed bidders. Such effects, which are well known from the single-unit common value auction literature, are the winner's curse and the linkage principle (see Wilson 1977, Milgrom 1981, Milgrom and Weber 1982). Since these effects are related to the private information of the bidders, they are now excluded by assumption. The assumption allows, however, to focus on the effects related particularly to multi-unit auction environments with endogenous supply uncertainty.

[6] This property of the distribution, called "monotone hazard rate" is a standard assumption in

(A4) *Bid constraints.*

Bidders are allowed to submit a price bid from the interval $M = [0, m]$, where $m > v$ is an arbitrarily large, but finite number[7].

3.2.2 Auction games

After receiving the bids the seller decides on supply quantity so as to maximize profit: a scenario that we model as a two-stage game. The payoffs of the players depend on the *bids*, the *supply quantity* and the *payment rule* of the auction. We will first introduce some general notation for the payoffs of the bidders in order to provide standard definitions of equilibrium and rationalizability for an arbitrary variable supply auction game Γ. Then we will specify these payoffs separately for the uniform price and the discriminatory auction and will analyze rational bidding in both auction formats.

3.2.2.1 Pure strategies

Each bidder i submits a price bid x_i to the auctioneer, indicating the (highest) price he is willing to pay for a unit. The vector of submitted bids is denoted by $\mathbf{x}$ and the bid vector of all bidders except bidder i by $\mathbf{x}_{-i}$. Let us consider an arbitrary trade mechanism Γ. Since the seller can condition the supply on the received bids, his strategy is a mapping from the set of bid vectors and possible values of the private information c into supply quantity:

$$\phi_\Gamma \colon M^n \times [0, \overline{c}] \to \{0, 1, 2, .., n\}.$$

Assume that after observing the bids $\mathbf{x}$ and the marginal costs c the seller supplies the quantity q. We denote his profit by $r_{\mathrm{S}}^{\Gamma}(\mathbf{x}; q, c)$ and the payoff (or the net consumer surplus) of bidder i by $r_i^{\Gamma}(\mathbf{x}; q)$. If the seller supplies according to the strategy ϕ_Γ, the expected payoff of bidder i is

$$R_i^{\Gamma}(\mathbf{x}; \phi_\Gamma) = \int_0^{\overline{c}} r_i^{\Gamma}\big(\mathbf{x}; \phi_\Gamma(\mathbf{x}, c)\big) \cdot f(c)\,\mathrm{d}c$$

auction theory. It guarantees in single-unit first-price auction models that bidders with higher valuations submit higher bids. It is satisfied by most common distributions: uniform, normal, logistic, chi-squared, exponential and Laplace. See Bagnoli and Bergstrom (1989) for a more complete list and for results allowing the identification of distributions with monotone hazard rates.

[7] The bidders are not able to pay infinitely large bid prices.

and the (ex ante) expected profit of the seller is

$$R_{\mathrm{S}}^{\Gamma}(\mathbf{x};\phi_{\Gamma}) = \int_{0}^{\bar{c}} r_{\mathrm{S}}^{\Gamma}\big(\mathbf{x};\phi_{\Gamma}(\mathbf{x},c),c\big)\cdot f(c)\,\mathrm{d}c.$$

3.2.2.2 Mixed strategies

A mixed strategy σ_i of bidder i is a probability distribution over the set of pure strategies M. The set Σ of mixed strategies is the set of probability distributions defined on $(M,\mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra on M. A mixed strategy profile of all bidders is denoted by $\boldsymbol{\sigma}$ and a mixed strategy profile of all fellow bidders of bidder i by $\boldsymbol{\sigma}_{-i}$. The payoff of bidder i in the reduced game is defined as

$$\mathfrak{R}_i^{\Gamma}(\boldsymbol{\sigma};\phi_{\Gamma}) = \int R_i^{\Gamma}(\mathbf{x};\phi_{\Gamma})\mathrm{d}\boldsymbol{\sigma}(\mathbf{x}).$$

The ex ante profit of the seller is defined as

$$\mathfrak{R}_{\mathrm{S}}^{\Gamma}(\boldsymbol{\sigma};\phi_{\Gamma}) = \int R_{\mathrm{S}}^{\Gamma}(\mathbf{x};\phi_{\Gamma})\mathrm{d}\boldsymbol{\sigma}(\mathbf{x}).$$

3.3 Definitions

Definition 3.1 (subgame perfect equilibrium)**.** *The mixed strategy profile $\boldsymbol{\sigma}^*$ and the supply function of the seller ϕ_{Γ}^* constitute a subgame perfect equilibrium (short: equilibrium) of the auction Γ, if the following conditions (SS) and (FS) hold. An equilibrium in which the bidders play pure strategies is called a pure strategy equilibrium.*

Second stage

For every vector of declared bids $\mathbf{x}$ and every value of the marginal costs c, the auctioneer sets the supply quantity so as to maximize profit:

$$\phi_{\Gamma}^*(\mathbf{x},c) \in \underset{q\in\{0,1,2,..,n\}}{\arg\max}\ r_{\mathrm{S}}^{\Gamma}(\mathbf{x},q,c). \tag{SS}$$

First stage

In the first stage of the game the strategy of every bidder i maximizes his expected payoff, given the strategies of the other bidders and optimal supply function of the seller:

$$\mathfrak{R}_i^{\Gamma}(\sigma_i^*,\boldsymbol{\sigma}_{-i}^*;\phi_{\Gamma}^*) \geq \mathfrak{R}_i^{\Gamma}(\sigma_i,\boldsymbol{\sigma}_{-i}^*;\phi_{\Gamma}^*) \quad \forall\sigma_i\in\Sigma. \tag{FS}$$

Reduced game

We will further on consider only optimal behavior of the seller in the second stage of the trade mechanisms under consideration. From now on we will, therefore, write

$$\mathfrak{R}_i^\Gamma(\sigma_i, \boldsymbol{\sigma}_{-i}) \quad \text{instead of} \quad \mathfrak{R}_i^\Gamma(\sigma_i, \boldsymbol{\sigma}_{-i}; \phi_\Gamma^*),$$

always assuming that the seller supplies a profit maximizing quantity. We will similarly write $R_i^\Gamma(x_i, \mathbf{x}_{-i})$ instead of $R_i^\Gamma(x_i, \mathbf{x}_{-i}; \phi_\Gamma^*)$. Condition (FS) requires that bidders' strategies constitute a Nash equilibrium in the reduced game.

Definition 3.2 (rationalizable strategies)**.** *Consider the trade mechanism* Γ*. Let* $\Sigma_i^{\Gamma,0} \equiv \Sigma$ *and for each* i *recursively define*

$$\Sigma_i^{\Gamma,k} = \Big\{ \sigma_i \in \Sigma_i^{\Gamma,(k-1)} : \exists \boldsymbol{\sigma}_{-i} \in \operatorname{conv} \Sigma_{-i}^{\Gamma,(k-1)} \quad \text{such that} \quad \mathfrak{R}_i^\Gamma(\sigma_i, \boldsymbol{\sigma}_{-i}) \geq \mathfrak{R}_i^\Gamma(\sigma_i', \boldsymbol{\sigma}_{-i}) \quad \text{for all} \quad \sigma_i' \in \Sigma_i^{\Gamma,(k-1)} \Big\}.$$ [8,9]

The set of rationalizable strategies for player i *in the trade mechanism* Γ *is defined as*

$$\Sigma_i^\Gamma = \bigcap_{k=0}^{\infty} \Sigma_i^{\Gamma,k}.$$

In words, the rationalizable (or strategically sophisticated) strategy profiles are (mixed) strategy profiles which survive the serial deletion of strategies not belonging to the best responses of the players. Obviously in a symmetric game the sets of rationalizable strategies for all players are equal. For notational brevity we will, therefore, omit the index i and write Σ^Γ instead of Σ_i^Γ.

Remark 3.1. *The rationalizable strategy set and the set surviving the iterated deletion of strictly dominated strategies coincide in two-player games.*

Easily accessible proofs of this statement can be found in Fudenberg and Tirole (1991, pp. 51-52) and Pearce (1984, pp. 1048-1049, Appendix B, Lemma 3). These proofs

[8] conv stands for *convex hull*. The convex hull of a set X is the smallest convex set that contains it.

[9] For brevity and ease of access we stick to the definition and the notation of Fudenberg and Tirole (1991, pp. 49 Definition 2.3). Although this definition does not introduce the notion of a belief system as the original definition does (see Bernheim 1984, pp.1013-1014, Definitions 3.1-3.3), it is equivalent to Bernheim's (1984) definition. The only difference is that Fudenberg and Tirole (1991) consider only games with a finite strategy space (see also Pearce 1984), whereas Bernheim (1984) like us considers a more general strategy space, which is a compact subset of an Euclidean space.

are conducted for games with finite strategy spaces, but the claim is also valid for the compact strategy sets of our model (for this argument consult Bernheim 1984, p. 1016). The claim will be useful later on when discussing the implications of theorem 3.4.

3.4 The uniform price and the discriminatory auctions

3.4.1 Payoffs

In both the uniform and the discriminatory auction the seller orders the bids in a descending order and serves them until the supply q is exhausted. Whereas in the uniform price auction all winning bidders pay a price equal to the lowest winning bid, which is called the *stopout price*, in the discriminatory auction the seller acts as a perfectly discriminating monopolist and all winners are charged their own bid prices. Let us introduce some additional notation to describe the players' payoffs. Take an arbitrary bid vector $\mathbf{x}$. Order the bids in a *descending* order. For that purpose define the function

$$\varphi_{\mathbf{x}} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

where $\varphi_{\mathbf{x}}(j) = k$, if bidder j submitted the k-th highest bid. If two or more bids are equal, then the function φ orders them arbitrarily. Further we define

$$\boldsymbol{\tau}(\mathbf{x}) = (\tau_1(\mathbf{x}), \tau_2(\mathbf{x}), \ldots, \tau_n(\mathbf{x})),$$

where $\tau_k(\mathbf{x})$ is the k-th highest bid if the bids are ordered in a descending order. The stopout price then is $\tau_q(\mathbf{x})$. The payoff of bidder i in the uniform price auction is

$$r_i^U(\mathbf{x}; q) = \begin{cases} v - \tau_q(\mathbf{x}) & \text{for } \varphi_{\mathbf{x}}(i) \leq q, \\ 0 & \text{for } \varphi_{\mathbf{x}}(i) > q. \end{cases}$$

The payoff of bidder i in the discriminatory price auction is

$$r_i^D(\mathbf{x}; q) = \begin{cases} v - x_i & \text{for } \varphi_{\mathbf{x}}(i) \leq q, \\ 0 & \text{for } \varphi_{\mathbf{x}}(i) > q. \end{cases}$$

The payoffs of the auctioneer in the uniform price and in the discriminatory auction are, respectively,

$$\begin{aligned} r_S^U(\mathbf{x};q) &= (\tau_q(\mathbf{x})-c)q, \\ r_S^D(\mathbf{x};q) &= \sum_{j=1}^{q}(\tau_j(\mathbf{x})-c). \end{aligned}$$

3.4.2 Discriminatory auction (D)

Theorem 3.1. *The set of rationalizable strategies of the discriminatory auction contains only one pure strategy for each bidder:*

$$\sigma\big(\{z_D\}\big)=1 \quad \textit{for all} \quad \sigma\in\Sigma^D,$$

where z_D *is the unique solution of the equation*

$$v-z=\frac{F\left(z\right)}{f\left(z\right)}. \tag{D}$$

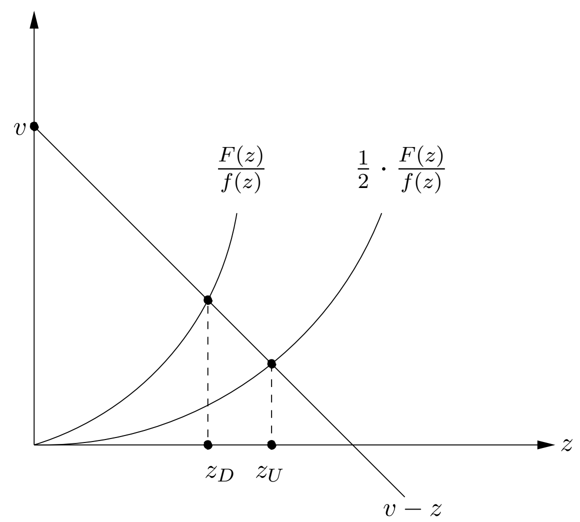

Figure 3.1: *z_D and z_U are the unique solutions of the equations (D) and (U) (see theorem 3.4).*

Proof.

Second stage:

The monopolist is serving the bids as long as he finds it profitable to sell additional units (see 3.1). Every bid which exceeds (or is at least not lower than) the marginal costs c is served. The optimal supply quantity of the auctioneer takes the form:

$$\phi^*_{\mathrm{D}}(\mathbf{x}, c) = \max\{k : \tau_k(\mathbf{x}) \geq c.\}^{10}$$

For the payoff of bidder i we obtain

$$r_i^{\mathrm{D}}(\mathbf{x}; \phi^*_D) = \begin{cases} v - x_i & \text{for } x_i \geq c, \\ 0 & \text{for } x_i < c. \end{cases}$$

Observe that the payoff of each bidder is independent of the other bids.
First stage:
The expected consumer surplus of bidder i is thus

$$R_i^{\mathrm{D}}(\mathbf{x}) = (v - x_i)F(x_i).$$

From the first order condition one obtains that the maximizer z_D is the unique solution of equation (D). Uniqueness and existence are guaranteed by assumptions (A2) and (A3)[11]. For the reduced game, the bid z_D is a strongly dominant strategy for each player, which completes the proof. □

If the strategy of the seller were to set a reservation price above which bids are served, then choosing c would have been a dominant strategy. The observation that in the discriminatory auction the optimal strategies of the bidders are independent of the strategies (or the oligopolistic structure) of the other bidders has been discussed also in Lengwiler (1999) in a setting in which bidders' strategies are quantities (announced at two different price levels) rather than prices. This observation makes the analysis of subgame perfect equilibria in the discriminatory auction simpler than that in the uniform price auction. For an analysis of a setting in which the bidders perceive the stopout price as a random variable in preparing their bids see Nautz (1995) and Nautz and Wolfstetter (1997). In their models the bidders submit entire demand functions.

[10] In fact the auctioneer is indifferent between selling or not selling units to bidders who quoted a price equal to the marginal costs. This detail is not important here, as such an event happens with probability 0, since the distribution $F(c)$ is atomless.

[11] Consider the function $G(z) = v - z - \frac{F(z)}{f(z)}$. Observe that $G(0) = v > 0$ and $G(v) = -\frac{F(v)}{f(v)} < 0$ (see A2). The continuity of $G(z)$ guarantees that the equation $G(z) = 0$ has a solution in the interval $(0, v)$ (by the Intermediate Value Theorem). (A3) requires that $\frac{F(z)}{f(z)}$ is a monotonically increasing function, therefore $G(z)$ is strictly monotonically decreasing. Thus the equation $G(z) = 0$ has a unique solution.

3.4.3 Uniform price auction (U)

3.4.3.1 The two bidder case

In this case the bid vector $\mathbf{x}$ consists only of two bids. The payoff of the monopolist is

$$R_{\mathrm{S}}^{\mathrm{U}}(\mathbf{x}, q, c) = \begin{cases} 0 & \text{for } q = 0, \\ \tau_1(\mathbf{x}) - c & \text{for } q = 1, \\ 2(\tau_2(\mathbf{x}) - c) & \text{for } q = 2. \end{cases}$$

Second stage:

The optimal supply strategy of the auctioneer is given by

$$\phi_{\mathrm{U}}^*(\mathbf{x}, c) = \begin{cases} 0 & \text{for } c > \tau_1(\mathbf{x}), \\ 1 & \text{for } \tau_1(\mathbf{x}) > c > 2 \cdot \tau_2(\mathbf{x}) - \tau_1(\mathbf{x}), \\ 2 & \text{for } 2 \cdot \tau_2(\mathbf{x}) - \tau_1(\mathbf{x}) > c. \end{cases}$$

The equalities occur with probability 0 and are therefore omitted.

First stage:

Now one can characterize the expected payoff of bidder i:

$$R_i^{\mathrm{U}}(x_i, x_{-i}) =$$

$$= \begin{cases} (v - x_i) \cdot \big(F(x_i) - F(2x_{-i} - x_i)\big) + (v - x_{-i}) \cdot F(2x_{-i} - x_i) & \text{for } x_i \geq x_{-i}, \\ (v - x_i) \cdot F(2x_i - x_{-i}) & \text{for } x_i < x_{-i}. \end{cases} \tag{3.1}$$

The next theorem establishes several important properties of the expected payoff function. See figure 3.2 for a graphical illustration.

Theorem 3.2. *The expected profit function has the following properties*[12]*:*

(i)

$$R_i^{\mathrm{U}}(x_i, x_{-i}) \quad \textit{is continuous in } (x_i, x_{-i}),$$

(ii)

$$R_i^{\mathrm{U}}(x_i, x_{-i}) = 0 \quad \textit{for} \quad 0 \leq x_i \leq \frac{x_{-i}}{2},$$

(iii)

$$\partial_i^+ R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \textit{for} \quad x_i = x_{-i} < v,$$

[12] $\partial_i^+ R_i^{\mathrm{U}}(x_i, x_{-i})$ is the partial derivative from above with respect to x_i.

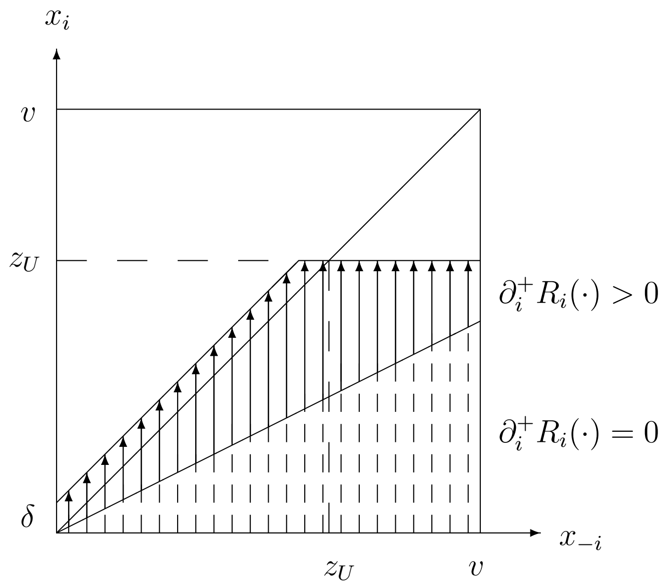

Figure 3.2: *In the dash-line area the payoff of bidder i is zero (see property (ii)). In the vector area the bidder's payoff increases in the direction of the arrows (see properties (iii), (iv) and (v)).*

(iv)

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \textit{for} \quad \frac{x_{-i}}{2} < x_i < \min\{x_{-i}, z_U\},$$

where z_U is the unique solution of the equation

$$v - z = \frac{1}{2} \cdot \frac{F(z)}{f(z)}. \qquad (U)$$

(v) There exists $\delta > 0$ such that

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \textit{for} \quad x_{-i} < x_i < \min\{x_{-i} + \delta, z_U\}.$$

Proof.
Statements *(i)* and *(ii)* are straightforward. They follow directly from expression (4.4). To prove *(i)* observe that for $x_i = x_{-i}$ both lines in (4.4) are equal to $(v - x_i) \cdot F(x_i)$. We will claim later on that the bidders' payoff function is continuous also in the general n-bidder case (see lemma 3.1). The intuition behind *(ii)* is simple. If $0 \leq x_i \leq x_i/2$, it is not profitable for the seller to serve bidder i for any realization of the marginal costs c, which means that with probability one bidder i is not served. We now move to the

proof of statement *(iii)*: For $x_i = x_{-i}$ we have

$$\begin{aligned}\partial_i{}^{+} R_i^{\mathrm{U}}(x_i, x_{-i}) = \quad & (v - x_i) \cdot (f(x_i) + f(2x_{-i} - x_i)) - F(x_i) \\ & + F(2x_{-i} - x_i) - (v - x_{-i}) \cdot f(2x_{-i} - x_i) \\ = \quad & (v - x_i) \cdot f(x_i) > 0.\end{aligned}$$

Statement *(iv)* follows from the (in)equalities:

$$\begin{aligned}\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) &= (v - x_i) \cdot f\left(2x_i - x_{-i}\right) \cdot 2 - F\left(2x_i - x_{-i}\right) \\ &= 2f\left(2x_i - x_{-i}\right)\left[(v - x_i) - \frac{F\left(2x_i - x_{-i}\right)}{2f\left(2x_i - x_{-i}\right)}\right] \\ &> 2f\left(2x_i - x_{-i}\right)\left[(v - x_i) - \frac{F\left(x_i\right)}{2f\left(x_i\right)}\right] \\ &> 2f\left(2x_i - x_{-i}\right)\left[(v - z_U) - \frac{F\left(z_U\right)}{2f\left(z_U\right)}\right] = 0.\end{aligned}$$

Notice that the last two inequalities apply because $\frac{F}{f}$ is a monotonically increasing function (see assumption (A3)). A rigorous proof of property *(v)* can be found in Appendix. Here we illustrate only the main idea. If we exploit the already proven properties *(i)* and *(iii)* and the fact that pre-images of open sets under continuous mapping are open, we will reach the conclusion that in an open neigborhood around the set

$$\{(x_i, x_{-i}) \mid x_i = x_{-i} < z_U\}$$

the partial derivative from above with respect to x_i is positive. The claim follows. □

As a consequence of Theorem 3.2 and expression (4.4), one obtains the following statement.

Corollary 3.1. *The (pure strategy) best response correspondence x_i^* of each bidder i, has the following properties:*

$$x_i^*\left(x_{-i}\right) \neq x_{-i}, \tag{3.2}$$

$$x_i^*\left(0\right) = z_D > 0, \tag{3.3}$$

$$x_i^*\left(v\right) < v. \tag{3.4}$$

Proof. (3.2) follows from *(iii)*; (3.3) and (3.4) follow from (4.4). □

(3.2) implies that the uniform price auction has no symmetric subgame perfect equilibrium in pure strategies. (3.3) and (3.4) further imply that the best response correspondence is not continuous, which points to the generic difficulty for the existence of pure strategy equilibria at all. Indeed, if the best response were continuous, it should cross the 45° line, which does not happen here because of (3.2). In subsection 3.4.5 we provide the best responses for a uniformly distributed marginal costs example. See figure 3.7 for an illustration of the best response correspondences for that numerical example. The next theorem provides an equilibrium existence result.

Theorem 3.3 (equilibrium existence)**.** *The uniform price auction has a mixed strategy equilibrium.*

Proof. The existence is guaranteed by Glicksberg's (1952) theorem, since the expected payoff functions $R_i^{\mathrm{U}}(x_i, x_{-i})$ are continuous (see property *(i)*) and the support M of the bids is a convex and compact set. □

Theorem 3.4 (rationalizable strategies)**.** *The set of rationalizable strategies of the uniform price auction contains only (mixed) strategies with support in* $[z_U, v]$*:*

$$\sigma\big([z_U, v]\big) = 1 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

The theorem applies also for the sets of mixed strategies which survive the serial deletion of strongly dominated strategies (see Remark 3.1). One can easily check[13] that $z_U > z_D$; therefore, it follows from theorems 3.1 and 3.4 that the rationalizable bids in the uniform price auction are (almost surely) higher than those in the discriminatory auction. Before providing a proof of theorem 3.4, let us intuitively explain why rational players bid higher in the uniform price than in the discriminatory auction. Consider the case in which bidder i submitted a bid at least as high as his fellow bidder ($x_i \geq x_{-i}$) and let us compare the changes in his payoff resulting from increasing of his bid under the two pricing rules. In both auction formats the winning probability $F(x_i)$ will clearly increase equally. While in the discriminatory auction the bidder has to pay his new bid with probability one, in the uniform price auction he pays on average less: he pays the bid price of his fellow bidder when both bidders are served. In this case, increasing his bid is more profitable (or at least less unprofitable) under the uniform pricing rule. Consider now the case ($x_i < x_{-i}$). In this scenario bidder i is

[13]Follows directly from the fact that z_D solves the equation (D), z_U solves the equation (U) and assumption (A3) (see also figure 3.1).

served with higher probability under the discriminatory than under the uniform price auction: $F(x_i) > F(2x_i - x_{-i})$. Therefore in the uniform price auction he will be willing to compensate for this lower probability by increasing his bid. This very characteristic of the uniform price payment rule creates incentives for higher bidding. We now move on to provide the idea of the proof of the theorem 3.4. A more technical proof can be found in Appendix 3.A.

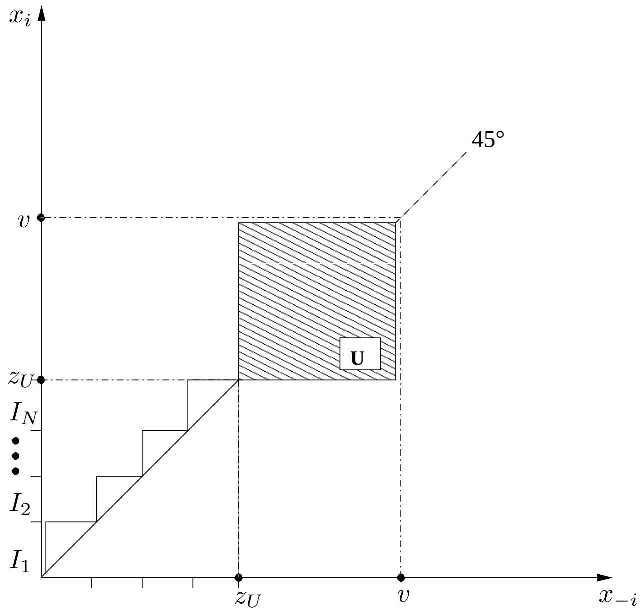

Figure 3.3: *The dark colored rectangle illustrates the boundaries of the support of the rationalizable strategies in the uniform price auction. The triangles are meant to represent the serial elimination of mixed strategies placing positive probability in the intervals $I_1, I_2, \ldots, I_N$. Such strategies are proven not to be rationalizable.*

Sketch of proof of theorem 3.4 (rationalizable strategies). The fact that rational players do not bid higher than their valuation is intuitively clear (see Part 1 of the proof in Appendix 3.A). The more interesting part is to show that bidders do not bid below z_U. Let us take a look again at statement *(v)* of theorem 3.2:
There exists $\delta > 0$ such that

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \text{for} \quad x_{-i} < x_i < \min\{x_{-i} + \delta, z_U\}.$$

This statement is captured in figure 3.2, where one can see that in a small neighborhood above the 45° line the payoff of bidder i is increasing in his bid. One can use this observation now to prove the statement:

Playing rationalizable strategies bidders do not submit bids lower than z_U :

$$\sigma\big([0, z_U]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

As the exposition of the proof is rather long and involved, we provide here only the basic idea, relegating the technical issues again to Appendix 3.A (see Part 2 of the proof). We divide the interval $[0, z_U)$ into small intervals of length δ (in the sense of statement *(v)* of theorem 3.2), where $z_U/\delta = N$ is an integer number. We denote the intervals

$$I_k = [(k-1)\cdot\delta, k\cdot\delta) \quad \text{for} \quad k = 1, 2, \ldots N; \quad I_0 = \emptyset,$$

as shown in figure 3.3. By an iterative procedure it is now shown that mixed strategies placing positive probability on $I_1, I_2, \ldots, I_N$ are not rationalizable. For that purpose we use the properties of the bidders' payoff function as stated in theorem 3.2.

3.4.3.2 The general case

In this subsection we discuss the case in which an arbitrary number of $n \geq 2$ bidders participate in the uniform price auction. We will formally derive the bidders' payoff function and will claim that it is continuous in the vector of declared bids (see lemma 3.1). This finding is used to verify that the uniform price auction has a symmetric mixed strategy equilibrium (Theorem 6.3). We further prove that in a symmetric equilibrium bids in the uniform price auction are with probability one higher than those in the discriminatory auction (Theorem 3.6).

Let $\mathbf{x}$ be an arbitrary bid vector and $q, q' \in \{0, \ldots, n\}$. Define $\tau_0(\mathbf{x}) := v$. The seller weakly prefers to sell q instead of q' units if and only if

$$\big(\tau_q(\mathbf{x}) - c\big)\cdot q \geq \big(\tau_{q'}(\mathbf{x}) - c\big)\cdot q'.$$

Thus the seller will not supply *more* than q units if and only if

$$c \geq c_q^-(\mathbf{x}) := \begin{cases} \max_{q<q'\leq n} \frac{q\cdot\tau_q(\mathbf{x}) - q'\cdot\tau_{q'}(\mathbf{x})}{q-q'} & \text{for } q < n, \\ 0 & \text{for } q = n. \end{cases}$$

He will not supply *less* than q units if and only if

$$c \leq c_q^+(\mathbf{x}) := \begin{cases} \min_{0\leq q'<q} \frac{q\cdot\tau_q(\mathbf{x}) - q'\cdot\tau_{q'}(\mathbf{x})}{q-q'} & \text{for } q \geq 1, \\ \overline{c} & \text{for } q = 0. \end{cases}$$

So, the seller optimally supplies the quantity q for $c \in [c_q^-, c_q^+]$. The set of winners then is $\{j \mid \varphi_{\mathbf{x}}(j) \leq q\}$, and all winners pay the stopout price $\tau_q(\mathbf{x})$. The expected payoff of an arbitrary bidder i is thus

$$R_i^{\mathrm{U}}(\mathbf{x}) = \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q;\mathbf{x}) \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}}(i) \leq q\}},$$

where

$$P(q;\mathbf{x}) := \text{Prob } \big(c_q^+(\mathbf{x}) > c > c_q^-(\mathbf{x})\big) = \max\big\{F\big(c_q^+(\mathbf{x})\big) - F\big(c_q^-(\mathbf{x})\big), 0\big\}$$

is the probability that exactly q units are sold.

Lemma 3.1 (continuity)**.** *$R_i^{\mathrm{U}}(\mathbf{x})$ is continuous in $\mathbf{x}$.*

See Appendix 3.A for a proof.

Theorem 3.5 (existence)**.** *The uniform price auction has a symmetric mixed strategy equilibrium.*

The theorem follows immediately from Becker and Damianov (2005). They prove that symmetric games with continuous payoffs and compact and convex strategy spaces possess symmetric mixed strategy equilibria. Their proof utilizes Glicksberg's (1952) fixed point theorem and its application to mixed strategy Nash equilibrium points similarly to the way in which Moulin (1986, pp. 115–116) shows that symmetric games satisfying the conditions of Kakutani's (1941) fixed point theorem have symmetric pure strategy equilibria.

Theorem 3.6. *In every symmetric equilibrium $\boldsymbol{\sigma}_U^*$ of the uniform price auction, bids are almost always higher than in the discriminatory auction:*

$$\sigma_U^*\big((z_D, v]\big) = 1.$$

Sketch of the proof: Although the idea of the proof is simple, the proof itself is somewhat involved and consists of a number of steps. The more technical parts can be found in Appendix 3.B. Here we provide only the basic intuition and sketch the most important arguments. The representation here will be strengthened by a graphical illustration for the case of $n = 3$ bidders.

The part that in a symmetric mixed strategy equilibrium buyers do not bid with positive probability higher than their valuation is intuitively clear. Indeed, assume that bidder i submits a bid $x_i > v$. Then for all $\mathbf{x}_{-i}$ and all realizations of the marginal costs c three cases are possible:

1. Bidder i wins and has to pay a price not higher than v,

2. Bidder i wins and has to pay a price higher than v,

3. Bidder i doesn't win.

In the first case, if the bidder submitted a bid of v instead, he would also have won, as the seller would supply the same quantity and the stop-out price would remain the same. In the second case, which would arise with positive probability in a symmetric equilibrium, the bidder has a negative payoff. In this case a bid of v would have guaranteed a payoff of at least zero. In the third case bidder i would not win with a bid of v either. All these arguments let one conclude that a deviation from the equilibrium mixed strategy, according to which bidder i shifts the probability measure he places on the interval $(v, \infty]$ to the point v would be profitable. This is a contradiction to the equilibrium assumption.

The more difficult and interesting part is to show that in a symmetric equilibrium buyers bid with probability one higher than z_D. Denote by z_* the lower bound of the support of a symmetric mixed strategy equilibrium:

$$z_* = \max\{z \mid \sigma_U^*\big([z, v]\big) = 1\}.$$

We proceed by contradiction, assuming that there exists a symmetric mixed strategy equilibrium for which $z_* \leq z_D$. We will consider a deviation strategy of an arbitrary bidder i, which shifts the probability mass of an interval $Z_*^\varepsilon := [z_*, z_* + \varepsilon)$ to the point $z_* + \varepsilon$ and show that for a sufficiently small $\varepsilon > 0$ the deviation is profitable. Thus we will reach a contradiction to the equilibrium assumption. To show this, we define the intervals

$$Z := \Big[z_*, v\Big], \qquad Z_0^\varepsilon := \Big[z_* + (n-1)^2\varepsilon, \min\{v, \frac{n}{n-1} \cdot z_*\}\Big)$$

and the sets

$$\mathcal{Z} = Z^{n-1}, \quad \mathcal{Z}^\varepsilon = \left(\{z_*\} \cup \Big[z_* + (n-1)^2\varepsilon, v\Big]\right)^{n-1},$$

$$\mathcal{Z}_0^\varepsilon = (Z_0^\varepsilon)^{n-1}, \quad \mathcal{Z}_* = \{z_*\}^{n-1}.$$

Then, we break down the set $\mathcal{Z}$ into the following four sets: $\mathcal{Z} \setminus \mathcal{Z}^\varepsilon$, $\mathcal{Z}^\varepsilon \setminus (\mathcal{Z}_0^\varepsilon \cup \mathcal{Z}_*)$, $\mathcal{Z}_0^\varepsilon$ and $\mathcal{Z}_*$. In the case of $n = 3$ bidders, taken from the perspective of bidder 3, all these sets are represented in figure 3.4.

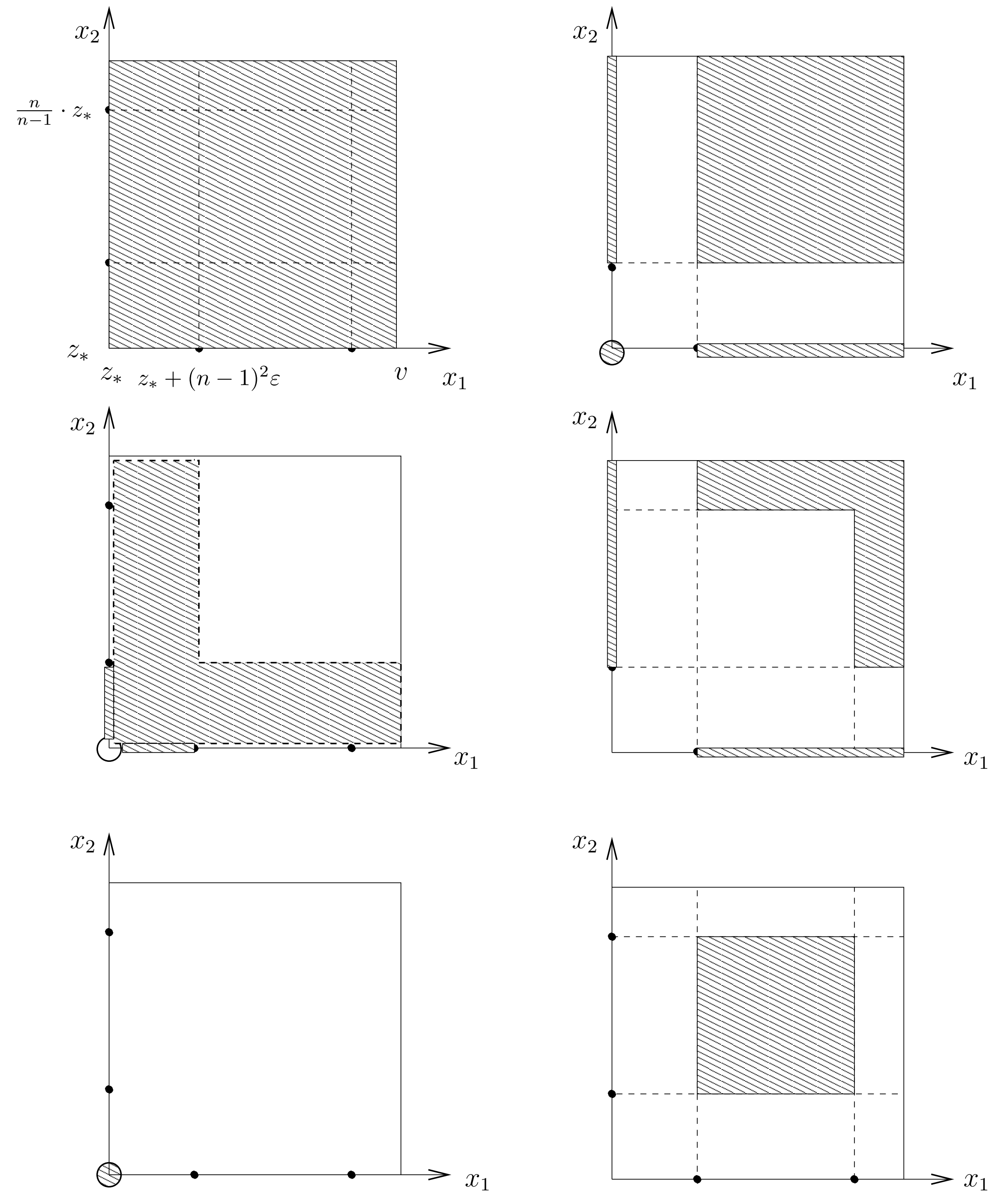

Figure 3.4: *The pattern ares represent the sets $\mathcal{Z}$ (upper-left), $\mathcal{Z}^\varepsilon$ (upper-right), $\mathcal{Z} \setminus \mathcal{Z}^\varepsilon$ (middle-left), $\mathcal{Z}^\varepsilon \setminus (\mathcal{Z}_0^\varepsilon \cup \mathcal{Z}_*)$ (middle-right), $\mathcal{Z}_*$ (lower-left) and Z_0^ε (lower-right) for $n = 3$ bidders.*

From the lemmas proved in Appendix 3.B four statements concerning the payoff of bidder i follow: For a sufficiently small ε and $x_i \in Z_*^\varepsilon$,

1. If $\mathbf{x}_{-i} \in \mathcal{Z} \setminus \mathcal{Z}^\varepsilon$, then $R_i(z_* + \varepsilon; \mathbf{x}_{-i}) - R_i(x_i; \mathbf{x}_{-i}) \geq -1 \cdot (z_* + \varepsilon - x_i)$, (see lemma 3.3),
2. If $\mathbf{x}_{-i} \in \mathcal{Z}^\varepsilon \setminus (\mathcal{Z}_0^\varepsilon \cup \mathcal{Z}_*)$, then $\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) \geq 0$, (see lemmas 3.2 [(i)&(ii)] and 3.6),
3. If $\mathbf{x}_{-i} \in \mathcal{Z}_0^\varepsilon$, then $\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) \geq \overline{\partial} > 0$, (see lemmas 3.2 [(i)&(iii)] and 3.4),
4. If $\mathbf{x}_{-i} \in \mathcal{Z}_0^\varepsilon$, then $\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) \geq \widetilde{\partial} > 0$, (see lemma 3.5).

In the last two cases $\widetilde{\partial}$ and $\overline{\partial}$ are positive constants. In Appendix 3.B we show that when bidder i plays the deviation, for a sufficiently small $\varepsilon > 0$ the possible reduction in his expected payoff arising in the set $\mathcal{Z} \setminus \mathcal{Z}^\varepsilon$ is offset by the increase in his payoff in the sets $\mathcal{Z}_0^\varepsilon$ and $\mathcal{Z}_*$[14]. Thus, the presented deviation is shown to be profitable.

3.4.4 Revenue and average trade volume

Theorem 3.4 states that in the two-bidder case the supports of the rationalizable strategy sets in the uniform price and the discriminatory auction are disjoint. In theorem 3.6 we further claim that in the general case the supports of the symmetric mixed strategy sets in the two auctions are disjoint. In the uniform price auction all bidders submit with probability one higher bids than in the discriminatory auction. In this section we will use these findings to establish a ranking of both auction formats in terms of ex ante revenue for the auctioneer and efficiency.

3.4.4.1 Revenue

Theorem 3.7 (revenue)**.** *The uniform price auction is ex ante more profitable for the seller than the discriminatory auction:*

(a) for all rationalizable strategies of the bidders in the two-bidder case

$$\mathfrak{R}_{\mathrm{S}}^U(\boldsymbol{\sigma}_U) > \mathfrak{R}_{\mathrm{S}}^D(\boldsymbol{\sigma}_D) \quad \textit{for} \quad n = 2 \quad \textit{and all} \quad \boldsymbol{\sigma}_D \in (\Sigma^D)^n, \boldsymbol{\sigma}_U \in (\Sigma^U)^n, \qquad (R_2)$$

[14] This is the case since $\lim_{\varepsilon \to 0} \boldsymbol{\sigma}_{-i}^*(\mathcal{Z} \setminus \mathcal{Z}^\varepsilon) = 0$.

(b) for all symmetric mixed strategy equilibria in the general case

$$\mathfrak{R}_S^U(\boldsymbol{\sigma}_U^*) > \mathfrak{R}_S^D(\boldsymbol{\sigma}_D^*) \quad \text{for} \quad n \geq 2. \tag{R_n}$$

Proof. We prove (R_n) using theorems 3.6 and 3.1:

$$\mathfrak{R}_S^U(\boldsymbol{\sigma}_U^*) > R_S^U(z_D, \ldots, z_D) = R_S^D(z_D, \ldots, z_D) = \mathfrak{R}_S^D(\boldsymbol{\sigma}_D^*).$$

The proof of (R_2) is analogous; apply theorems 3.4 and 3.1. □

3.4.4.2 Average trade volume

The average trading quantity[15] generated by the mechanism Γ, if the bidders employ the mixed strategy profile $\boldsymbol{\sigma}(\mathbf{x})$ is given as follows:

$$\mathfrak{Q}_\Gamma(\boldsymbol{\sigma}) = \int Q_\Gamma(\mathbf{x}) \mathrm{d}\boldsymbol{\sigma}(\mathbf{x}),$$

where

$$Q_\Gamma(\mathbf{x}) = \int_0^v \phi_\Gamma^*(\mathbf{x}, c) \cdot f(c) \mathrm{d}c.$$

Theorem 3.8 (average trade quantity)**.** *If the marginal costs' distribution function is convex ($F'' \geq 0$) the average trading quantity in the uniform price auction is higher than that in the discriminatory auction:*

(a) for all rationalizable strategies of the bidders in the two-bidder case

$$\mathfrak{Q}^U(\boldsymbol{\sigma}_U) > \mathfrak{Q}^D(\boldsymbol{\sigma}_D) \quad \text{for} \quad n = 2 \quad \text{and all} \quad \boldsymbol{\sigma}_D \in (\Sigma^D)^n, \boldsymbol{\sigma}_U \in (\Sigma^U)^n, \tag{E_2}$$

(b) for all symmetric mixed strategy equilibria in the general case

$$\mathfrak{Q}^U(\boldsymbol{\sigma}_U^*) > \mathfrak{Q}^D(\boldsymbol{\sigma}_D^*) \quad \text{for} \quad n \geq 2. \tag{E_n}$$

Proof. Take an arbitrary bid vector $\mathbf{x}$ and denote

$$P(q; \mathbf{x}) := \max\{F\big(c_q^+(\mathbf{x})\big) - F\big(c_q^-(\mathbf{x})\big), 0\}.$$

[15] The average turnover can be taken in this setting also as an efficiency measure. Note that buyers are only served when $v \geq c$ since they submit bids not higher than v and the seller does not serve bids below c. This means that in both auction forms trade takes place only when desirable ex-post. The mechanism, which induces a higher probability for sale, i.e. higher average turnover, can be therefore considered as the more efficient mechanism. One needs to point out, however, that higher average turnover does not necessarily imply higher efficiency in the Pareto sense or ex-ante higher sum of the revenues of market participants.

The average quantity sold in the uniform price auction can be written as a function of the ordered bids:

$$Q_U(\mathbf{x}) = Q_U(\boldsymbol{\tau}(\mathbf{x})) = \sum_{q=1}^{n} q \cdot P(q;\mathbf{x}) = \sum_{\{q|c_q^+ > c_q^-\}} q \cdot P(q;\mathbf{x}).$$

The last equality means that one needs to sum only over the elements $\tau_q(\mathbf{x})$ for which $c_q^+ > c_q^-$, as otherwise $P(q;\mathbf{x}) = 0$. We write these quantities in an ascending order $l_1, l_2, \ldots, l_h$ and obtain

$$\tau_{l_1}(\mathbf{x}) > \tau_{l_2}(\mathbf{x}) > \cdots > \tau_{l_h}(\mathbf{x}).$$

For the sake of brevity, we will further write τ_{l_k} instead of $\tau_{l_k}(\mathbf{x})$. We will show that

$$Q_U(\tau_{l_1}, \tau_{l_2}, ..., \tau_{l_h}) \geq Q_U(\tau_{l_2}, \tau_{l_2}, ..., \tau_{l_h}). \tag{3.5}$$

One can observe that $c_{l_1}^- = c_{l_2}^+$ and recall that $c_{l_1}^-$ is a solution of the equation

$$l_1 \cdot (\tau_{l_1} - c_{l_1}^-) = l_2 \cdot (\tau_{l_2} - c_{l_1}^-), \tag{3.6}$$

which means that the two dark rectangles in figure 3.5 cover equal areas. Equation (3.6) is equivalent to

$$\frac{\tau_{l_1} - c_{l_1}^-}{\tau_{l_2} - c_{l_1}^-} = \frac{l_2}{l_1}.$$

From the convexity of F it follows that

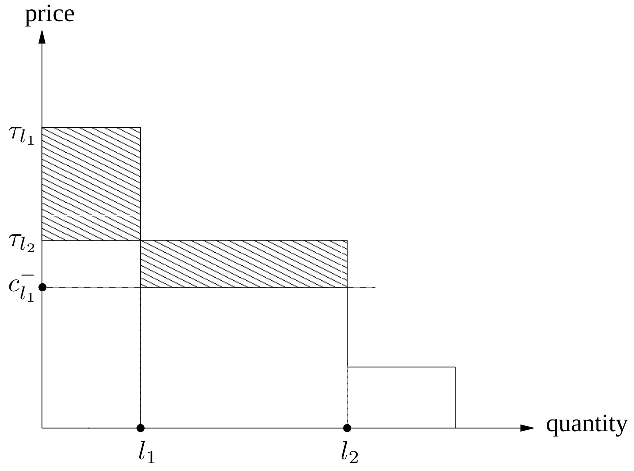

Figure 3.5: *Announced demand curve in the uniform price auction. The two pattern rectangles cover equal areas.*

$$\frac{F(\tau_{l_1}) - F(c_{l_1}^-)}{F(\tau_{l_2}) - F(c_{l_1}^-)} \geq \frac{\tau_{l_1} - c_{l_1}^-}{\tau_{l_2} - c_{l_1}^-}$$

and therefore

$$\frac{F(\tau_{l_1}) - F(c_{l_1}^-)}{F(\tau_{l_2}) - F(c_{l_1}^-)} \geq \frac{l_2}{l_1}. \tag{3.7}$$

The identities

$$\begin{aligned} Q_U(\tau_{l_1}, \tau_{l_2}, ...\tau_{l_h}) - Q_U(\tau_{l_2}, \tau_{l_2}, ...\tau_{l_h}) \\ &= l_1\big[F(\tau_{l_1}) - F(\tau_{l_2})\big] - (l_2 - l_1)\big[F(\tau_{l_2}) - F(c_{l_1}^-)\big] \\ &= l_1\big[F(\tau_{l_1}) - F(c_{l_1}^-)\big] - l_2\big[F(\tau_{l_2}) - F(c_{l_1}^-)\big] \geq 0 \end{aligned}$$

prove (3.5). The above argument can be applied iteratively $(h-1)$ times to verify the inequality

$$Q_U(\tau_{l_1}, \tau_{l_2}, ...\tau_{l_h}) \geq Q_U(\tau_{l_h}, \tau_{l_h}, ...\tau_{l_h}).$$

Now one can easily prove (E_n) applying theorem 3.6:

$$\begin{aligned} \mathfrak{Q}_U(\boldsymbol{\sigma}_U^*) &= \int Q_U(\mathbf{x})\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) \geq \int Q_U(\tau_h, \tau_h, ...\tau_h)\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) \\ &> \int Q_U(z_D, z_D, ...z_D)\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) = n \cdot F(z_D) \\ &= \int Q_D(z_D, z_D, ...z_D)\mathrm{d}\boldsymbol{\sigma}_D^*(\mathbf{x}) = \mathfrak{Q}_D(\boldsymbol{\sigma}_D^*). \end{aligned}$$

The proof of (E_2) is analogous (apply theorem 3.4). □

3.4.5 A numerical example

Consider the following two-bidder example. $v = 1$ *and the marginal costs of the auctioneer are uniformly distributed:* $f(c) = 1$ *for* $c \in [0, 1]$.

In the discriminatory auction all bidders submit a bid of $z_D = \frac{1}{2}$ with probability one, which is their only rationalizable strategy since the bid z_D solves the equation (D). All rationalizable strategies in the uniform price auction have support in the interval

$$[z_U, v] = [2/3, 1],$$

since z_U solves the equation (U). The average trade quantities of the discriminatory auction and the uniform price auctions are:

$$\begin{aligned} \mathfrak{Q}_D &= \int_0^{\frac{1}{2}} 2\mathrm{d}c = 1, \\ \mathfrak{Q}_U &> \frac{4}{3}. \end{aligned}$$

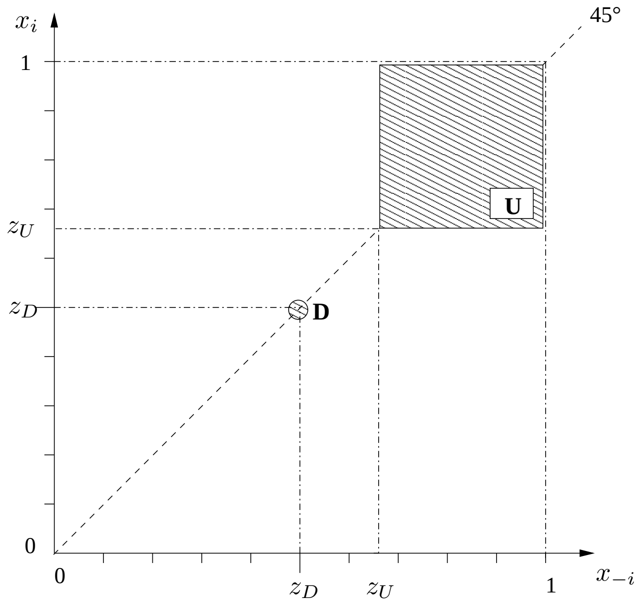

Figure 3.6: *Numerical example: $v = 1$, $n = 2$ and $f(c) = 1$ for $c \in [0, 1]$. The supports of the rationalizable strategy sets in both auction forms (the pattern areas) are disjoint. The bids in the uniform price auction are higher with probability one.*

The revenue for the seller in the two auction formats is

$$\mathfrak{R}_{\mathrm{S}}^{D} = \int_0^{\frac{1}{2}} 2(\frac{1}{2} - c)\mathrm{d}c = \frac{1}{4};$$
$$\mathfrak{R}_{\mathrm{S}}^{U} > \int_0^{\frac{2}{3}} 2(\frac{2}{3} - c)\mathrm{d}c = \frac{4}{9}.$$

The payoff of bidder i in the uniform price auction is:

$$R_i^{\mathrm{U}}(x_i, x_{-i}) = \begin{cases} (1 - x_i)x_i & \text{for } x_i > 2x_{-i}, \\ (1 - x_i)\big(2x_i - 2x_{-i}\big) + (1 - x_{-i})(2x_{-i} - x_i) & \text{for } 2x_{-i} \geq x_i \geq x_{-i}, \\ (1 - x_i)(2x_i - x_{-i}) & \text{for } \frac{x_{-i}}{2} \leq x_i < x_{-i}, \\ 0 & \text{for } x_i < \frac{x_{-i}}{2}. \end{cases}$$

For the (pure strategy) best response correspondence of bidder i in the uniform price

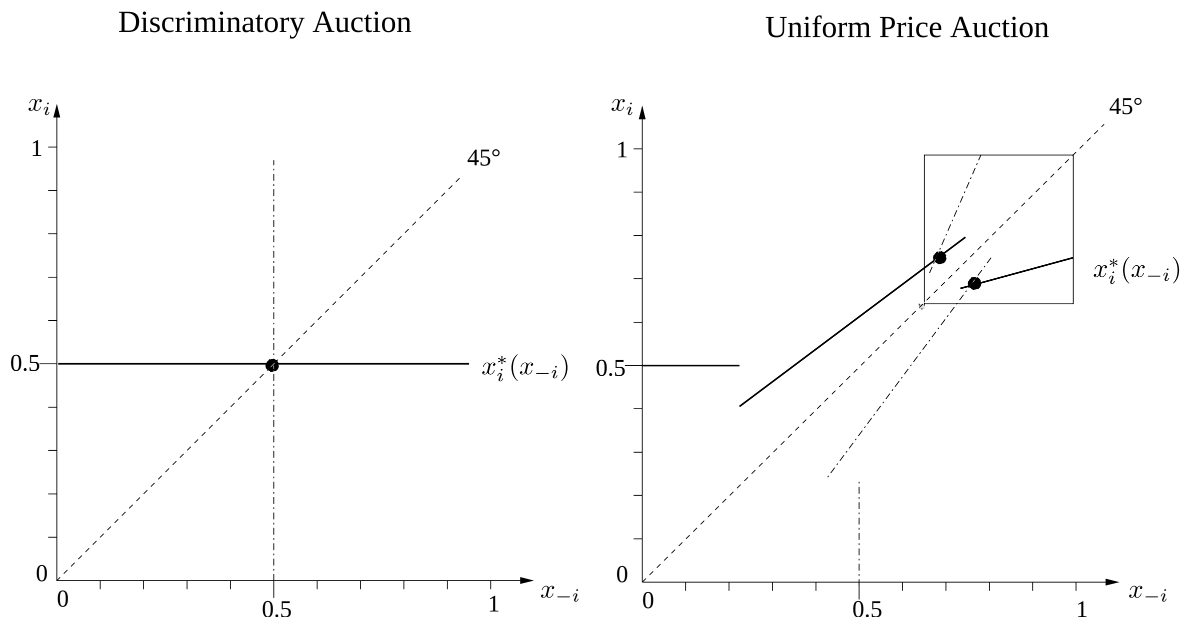

Figure 3.7: *Best responses and pure strategy equilibria (the thick dots) in the uniform price and the discriminatory auctions. The support of the rationalizable strategies of the uniform price auction lies within the square as has been proven in theorem 3.4.*

auction one obtains[16]:

$$x_i^*(x_{-i}) = \begin{cases} \frac{1}{2} & \text{for } x_{-i} < \frac{3-\sqrt{2}}{7}, \\ \{\frac{16-3\sqrt{2}}{28}, \frac{1}{2}\} & \text{for } x_{-i} = \frac{3-\sqrt{2}}{7}, \\ \frac{3x_{-i}+1}{4} & \text{for } x_{-i} = (\frac{3-\sqrt{2}}{7}, \frac{3}{4}), \\ \{\frac{11}{16}, \frac{13}{16}\} & \text{for } x_{-i} = \frac{3}{4}, \\ \frac{x_{-i}+2}{4} & \text{for } x_{-i} \in (\frac{3}{4}, 1]. \end{cases}$$

For that numerical example the uniform price auction has two asymmetric subgame perfect equilibria in pure strategies (see figure 3.7 for a graphical illustration):

$$(x_{i,U}^*, x_{-i,U}^*) \;\; = \left(\tfrac{10}{13}, \tfrac{9}{13}\right), \; i = 1, 2.$$

For the average trade quantity and the revenue of the auctioneer in equilibrium one

[16] We will further deal with this example in the next chapter. The calculations are provided there (see lemma 4.1 (C)).

obtains:

$$\mathfrak{Q}_U = \frac{18}{13} > \frac{4}{3},$$
$$\mathfrak{R}_S^U = \int_0^{\frac{8}{13}} 2(\frac{9}{13} - c)\mathrm{d}c + \int_{\frac{8}{13}}^{\frac{10}{13}} (\frac{10}{13} - c)\mathrm{d}c = \frac{82}{169} > \frac{4}{9}.$$

3.5 Concluding remarks

The standard pricing techniques, the uniform pricing rule and the price discrimination rule, are widely used by monopolists for the simultaneous sale of multiple units. When a monopolist lacks information about demand, these pricing techniques often take the form of an auction, in which the seller first collects bids from prospective customers and then decides on a supply quantity so as to maximize profit. These auction forms, called variable supply multi-unit auctions, are used on various markets ranging from Treasury bills and IPOs to rare wine and art. They differ from the fixed supply multi-unit auctions in the sense that the seller participates in the price-setting process as he controls the supply after the bidding. We model this scenario as a two-stage game and compare these variable supply pricing mechanisms in a common value model without proprietary information. In our model bidders announce bids for one unit and are uncertain about the constant marginal costs of the seller. We find that due to this uncertainty in a symmetric equilibrium the bidders bid higher in the uniform price auction than in the discriminatory auction. This finding further implies that the uniform price auction is (ex ante) more profitable for the seller and leads to higher average trade volume.

The obtained results can be given the following intuition. As has been discussed, in the discriminatory auction the winning probability of each bidder is not affected by the bids of his fellow bidders, as the seller optimally serves every bid above his marginal costs. Since the bidders share the same information, they submit equal bids. Thus, as in Lengwiler (1999), the right of the seller to discriminate among the bidders and charge them different prices has no bite. Bidders in the discriminatory auction do not compete at all; in the reduced form of the game a bidder's expected payoff is independent of the bids of the other bidders. In the uniform price auction, on the other hand, the probability of winning as well as the final price depend on all bids. Submitting higher bids in this auction format proves to be profitable as it raises the probability of winning, but not necessarily the price a bidder has to pay. This simple observation is employed to demonstrate that the uniform price auction induces a more competitive environment and leads to higher equilibrium bids with a probability of one (see theorems 3.4 and 3.6). We obtained this result without the need to compute the equilibria precisely. Rather, we exploited the properties of the bidders' payoff functions and the definitions of the equilibrium and rationalizability concepts. Finally, unlike Lengwiler (1999), we demonstrate a clear-cut revenue and average trade volume (or efficiency) ranking result. Moreover, we showed that revenue and efficiency go hand in hand[17] and are not competing goals.

Appendix 3.A

Proof of property (v) of theorem 3.2: *There exists $\delta > 0$ such that*

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \text{for} \quad x_{-i} < x_i < \min\{x_{-i} + \delta, z_U\}.$$

On the set $K := \big\{(y_i, y_{-i}) \;\big|\; 0 \le y_{-i} \le z_U, 0 \le y_i \le \frac{v - z_U}{2}\big\}$ we define a function $g\colon K \to \mathbb{R}$ by

$$g(y_i, y_{-i}) = \partial_i R_i^{\mathrm{U}}(y_i + y_{-i}, y_{-i})$$

where for $y_i = 0$ we mean the derivative from above. Notice that we simply wrote the partial derivative as a function of $y_{-i} = x_{-i}$ and the difference $y_i = x_i - x_{-i}$. g is continuous with $g(0, y_{-i}) > 0$ for every $y_{-i} \in [0; z_U]$, so the set $H := g^{-1}\big((0; \infty)\big)$ of points where the partial derivative is strictly positive is open[18] in K with $\{0\} \times [0, z_U] \subseteq H$. Therefore, as $[0, z_U]$ is compact, there exists[19] a neighborhood $[0, \delta]$, $\delta > 0$, of 0 in $[0, \frac{v - z_U}{2}]$ with $[0; \delta] \times [0, z_U] \subseteq H$.

Proof of Theorem 3.4:

Part 1: *Bidders who play rationalizable strategies do not bid higher than their valuations:*

$$\sigma\big((v, m]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

[17]Campbell, Carare, and McLean (2004) for example find that in an asymmetric setting with two bidders and two objects the discriminatory auction is inherently inefficient, but may result in higher expected revenue than the efficient Vickrey mechanism.

[18]Here we used the fact that pre-images of open sets under continuous mappings are open, see e. g. Königsberger (2002), p. 16.

[19]This follows from the so called "tube lemma", see e. g. Königsberger (2002), p. 32.

For each $\sigma \in \Sigma$, define $\hat{\sigma} \in \Sigma$ by

$$\hat{\sigma}(B) = \sigma\big(B \cap [0; v]\big) + \sigma\big((v; m]\big) \cdot \mathbb{1}_{v \in B} \quad \text{for } B \in \mathscr{B},$$

which means, a bidder with strategy $\hat{\sigma}$ bids v whenever a bidder with strategy σ would submit a bid from the interval $(v; m]$. We first remark that $\hat{\sigma}$ always weakly dominates σ, as bids above v lead to a strictly negative outcome when served. So a strategy σ_i of player i with $\sigma_i\big((v; m]\big) > 0$ will never be a best answer to a strategy σ_{-i} of player $-i$, if under the strategy combination (σ_i, σ_{-i}) player i has to pay more than v with strictly positive probability.

Using this fact we will now show by induction that, with the notation of definition 3.2, for $k = 1, 2, \ldots$

$$\sigma_i \notin \Sigma_i^{\mathrm{U},k}, \quad i = 1, 2, \quad \text{if } \sigma_i\Big(\big(\max\{v, 2^{-k}m\}; m\big]\Big) > 0. \tag{3.8}$$

We start with $k = 1$. As the other bid is never greater than m, bids from the interval $\big(\max\{v, \frac{m}{2}\}; m\big]$ are served when the costs of the seller are below v, which will happen with a strictly positive probability. So, by the introductory remark, if $\sigma_i\big(\big(\max\{v, \frac{m}{2}\}; m\big]\big) > 0$, $\hat{\sigma}_i$ will be strictly better than σ_i, regardless of what $-i$ does.

Now assume that equation (3.8) holds for $k - 1$. Bids above $\max\{v, 2^{-k}m\}$ are served when the other bidder does not submit a bid above $\max\{v, 2^{-(k-1)}m\}$ and the costs are below v, which by induction happens with strictly positive probability if the other bidder plays a strategy from $\Sigma_{-i}^{\mathrm{U},k-1}$. So for each strategy σ_i with $\sigma_i\big(\big(\max\{v, 2^{-k}m\}; m\big]\big) > 0$ the strategy $\widehat{\sigma_i}$ will be a strictly better answer to any element of $\Sigma_{-i}^{\mathrm{U},k-1}$, which proves 3.8 for k.

Part 2: *Bidders who play rationalizable strategies do not submit bids lower than z_U,*

$$\sigma\big([0, z_U]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

Denote the intervals

$$J_k = \bigcup_{l=0}^{k} I_l = [0, k\delta).$$

We will iteratively show that

$$\Sigma_i^{U,k} \subseteq \{\sigma_i \mid \sigma_i(J_k) = 0\}, \quad \text{for} \quad k = 1, 2, .., N; i = 1, 2, \tag{3.9}$$

which proves the lemma. Observe that (3.9) trivially holds for $k = 0$. Assume that it holds for $k - 1 < N$ for player $-i$. We will show that

$$\Sigma_i^{U,k} \subseteq \{\sigma_i \mid \sigma_i(J_k) = 0\}. \tag{3.10}$$

Assume on the contrary

$$\exists \sigma_i \in \Sigma_i^{U,k} \quad \text{with} \quad \sigma_i(I_k) > 0. \tag{3.11}$$

We will now demonstrate that

$$\forall \sigma_{-i} \in \operatorname{conv} \Sigma_{-i}^{U,(k-1)}, \exists \hat{\sigma}_i : \mathfrak{R}_i^U(\hat{\sigma}_i, \sigma_{-i}) \geq \mathfrak{R}_i^U(\sigma_i, \sigma_{-i}), \tag{3.12}$$

which is a contradiction to the above assumption (3.11), namely that σ_i is a best response to some mixed strategy from the set $\operatorname{conv} \Sigma_{-i}^{UP,(k-1)}$.

Case 1: $\sigma_{-i}(J_{2k}) > 0$.

Consider the strategy $\hat{\sigma_i}$:

$$\hat{\sigma_i}\,(B) = \sigma_i\,(B \cap \complement I_k) + \sigma_i\,(I_k) \cdot \mathbb{1}_{k\delta \in B} \quad \text{for } B \in \mathscr{B},$$

where $\complement I_k$ is the complement set of I_k ($\complement I_k \equiv M \backslash I_k$) and $B \in \mathscr{B}$:

$$\begin{aligned}
&\mathfrak{R}_i^U(\hat{\sigma}_i, \sigma_{-i}) - \mathfrak{R}_i^U(\sigma_i, \sigma_{-i}) \\
&\quad \geq \int \Big(\int R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\hat{\sigma}_i(x_i) - \int R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\sigma_i(x_i) \Big) \mathrm{d}\sigma_{-i}(x_{-i}) \\
&\quad = \int \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) \cdot \sigma_i(I_k) - \int_{I_k} R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\sigma_i(x_i) \Big) \mathrm{d}\sigma_{-i}(x_{-i}) \\
&\quad = \int \int_{I_k} \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) && (3.13) \\
&\quad = \int_{\complement J_{(k-1)}} \int_{I_k} \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) && (3.14) \\
&\quad > 0 && (3.15)
\end{aligned}$$

(3.14) follows from (3.13) because we assumed that (3.9) holds for $(k-1) < N$ for player $-i$. Further, from theorem 3.2 follows that

$$\begin{aligned}
R_i^{\mathrm{U}}(x_i, x_{-i}) &< R_i^{\mathrm{U}}(k\delta, x_{-i}) \quad \text{if} \quad x_i \in I_k \quad \text{and} \quad (k-1) \cdot \delta \leq x_{-i} < 2k\delta, \\
R_i^{\mathrm{U}}(x_i, x_{-i}) &\leq R_i^{\mathrm{U}}(k\delta, x_{-i}) \quad \text{if} \quad x_i \in I_k \quad \text{and} \quad (k-1) \cdot \delta \leq x_{-i}.
\end{aligned}$$

As by assumption $\sigma_{-i}(J_{2k}) > 0$ the inequality (3.15) is also valid.

Case 2: $\sigma_{-i}(J_{2k}) = 0$. For the strategy $\hat{\sigma_i}$, where

$$\hat{\sigma_i}\left(B\right) = \sigma_i\left(B \cap \complement I_k\right) + \sigma_i\left(I_k\right) \cdot \mathbb{1}_{\frac{3v}{4} \in B} \quad \text{for } B \in \mathscr{B}$$

one observes that

$$\begin{aligned} &\mathfrak{R}_i^U(\hat{\sigma_i}, \sigma_{-i}) - \mathfrak{R}_i^U(\sigma_i, \sigma_{-i}) \\ &\quad = \int_{\complement J_{2k}} \int_{I_k} \Big(R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i})\Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) \\ &\quad = \int_{\complement J_{2k}} \int_{I_k} \Big(R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i})\Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) > 0. \end{aligned} \tag{3.16}$$

The inequality (3.16) holds because $R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i}) > 0$ for $x_{-i} \in [0, v]$.

Proof of Lemma 3.1: $R_i^{\mathrm{U}}(\mathbf{x})$ *is continuous in* $\mathbf{x}$.

Let $\mathbf{x}$ be an arbitrary bid vector. We will show that for any sequence of bid vectors $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, with $\mathbf{x}^{(k)} \to \mathbf{x}$ we have $R_i^{\mathrm{U}}(\mathbf{x}^{(k)}) \to R_i^{\mathrm{U}}(\mathbf{x})$. Using the (easy to prove) inequality

$$|a'b'c' - abc| \leq |a' - a| \cdot b'c' + a \cdot |b' - b| \cdot c' + ab \cdot |c' - c|,$$

which holds for arbitrary nonnegative reals a, b, c, a', b', c', we obtain

$$\begin{aligned} &\big|R_i^{\mathrm{U}}(\mathbf{x}^{(k)}) - R_i^{\mathrm{U}}(\mathbf{x})\big| \\ &\leq \sum_{q=0}^{n} \big|\tau_q(\mathbf{x}^{(k)}) - \tau_q(\mathbf{x})\big| \cdot P(q; \mathbf{x}^{(k)}) \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} \\ &\quad + \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot |P(q; \mathbf{x}^{(k)}) - P(q; \mathbf{x})| \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} \\ &\quad + \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q; \mathbf{x}) \cdot |\mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} - \mathbb{1}_{\{\varphi_{\mathbf{x}}(i) \leq q\}}|. \end{aligned}$$

This inequality can be interpreted as a decomposition of the change in expected revenue of bidder i into a *price effect*, a *quantity effect* and an *allocation effect*. As sums, differences, products, quotients, minimums and maximums of continuous functions are continuous, so are the functions $c_q^-(\cdot)$, $c_q^+(\cdot)$, $P(q; \cdot)$, and therefore

$$\big|\tau_q(\mathbf{x}^{(k)}) - \tau_q(\mathbf{x})\big| \to 0, \quad |P(q; \mathbf{x}^{(k)}) - P(q; \mathbf{x})| \to 0$$

for $k \to \infty$, which means price and quantity effect tend to 0. To complete the proof, we will now show that the allocation effect also tends to 0. This effect can be expressed as

$$\sum_{q \in L_{\mathbf{x}}} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q; \mathbf{x}) \cdot |\mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} - \mathbb{1}_{\{\varphi_{\mathbf{x}}(i) \leq q\}}|,$$

where

$$L_{\mathbf{x}} = \{q \mid \tau_q(\mathbf{x}) > \tau_{q+1}(\mathbf{x})\}$$

because $P(q; \mathbf{x}) = 0$ for $q \notin L_{\mathbf{x}}$.[20] In words, one needs to sum only over the positions in the announced demand curve for which an increase in quantity leads to a decrease in the stopout price. This is so, because if several bids are equal, the seller serves with probability one either none or all of them. Observe now that there exists a k_0, so that for all $k \geq k_0$ we have:

$$x_j^{(k)} < x_i^{(k)} \text{ if } x_j < x_i \text{ and } x_j^{(k)} > x_i^{(k)} \text{ if } x_j > x_i \text{ for all } i, j \in \{1, \ldots, n\}. \tag{3.17}$$

Then the inequalities

$$\varphi_{\mathbf{x}}(i) \leq q \quad \text{and} \quad \varphi_{\mathbf{x}^{(k)}}(i) \leq q$$

are equivalent for $q \in L_{\mathbf{x}}$ and $k \geq k_0$, which completes the proof.

Appendix 3.B

In this appendix we prove the second part of theorem 3.6, namely that in a symmetric mixed strategy equilibrium buyers bid with probability one higher than z_{D}. In the form of five lemmas we first provide some auxiliary statements needed for the proof.

Lemma 3.2. *Let* $\mathcal{L}(\mathbf{x}_{-i}) := \big[0;\ \min(\{x_j \mid j \neq i\} \cup \{z_{\mathrm{D}}\})\big)$.

(i) For any i and any given $\mathbf{x}_{-i}$, the partial derivative $\partial_i R_i(x_i; \mathbf{x}_{-i})$ exists in all but finitely many points $x_i \in \mathcal{L}(\mathbf{x}_{-i})$.

(ii) The partial derivative of the bidder that submitted the lowest bid is nonnegative if that bidder submitted a bid not higher than z_{D}. Formally, for any $\mathbf{x}_{-i}$

$$\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) \geq 0,$$

for all $x_i \in \mathcal{L}(\mathbf{x}_{-i})$ for which $\partial_i R_i^{\mathrm{U}}$ exists.

[20] One observes that $c_q^- = \tau_q$ and $c_q^+ \leq \tau_q$. Hence $P(q; \mathbf{x}) = 0$.

(iii) *The partial derivative of the bidder that submitted the lowest bid is uniformly bounded away from* 0 *if that bidder submitted a bid not higher than* z_{D} *and is served with positive probability. Formally, there exists* $\overline{\partial} > 0$ *such that for any* $\mathbf{x}_{-i}$

$$\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) > \overline{\partial}$$

for all $x_i \in \mathcal{L}(\mathbf{x}_{-i})$ *for which* $\partial_i R_i^{\mathrm{U}}$ *exists and* $c_i^+(x_i; \mathbf{x}_{-i}) > 0$.

Proof. **(i)** The expected revenue of bidder i is given by

$$R_i^{\mathrm{U}}(\mathbf{x}) = (v - x_i) F\big(c_n^+(\mathbf{x})\big).$$

As, by assumption, F is differentiable, we only have to show the differentiability of c_n^+. Observe that

$$c_n^+(\mathbf{x}) = \min_{0 \le q < n} \frac{q\tau_q(\mathbf{x}) - nx_i}{q - n}$$

and define

$$\hat{q}(x_i; \mathbf{x}_{-i}) := \min \ \operatorname*{arg\,min}_{0 \le q < n} \frac{q\tau_q(\mathbf{x}) - nx_i}{q - n},$$

then

$$c_n^+(\mathbf{x}) = \frac{\hat{q}(\mathbf{x})\tau_{\hat{q}(\mathbf{x})}(\mathbf{x}) - nx_i}{\hat{q}(\mathbf{x}) - n}.$$

We will now show that $\hat{q}(x_i; \mathbf{x}_{-i})$ is almost everywhere differentiable in x_i and as a consequence so will be $c_n^+(x_i; \mathbf{x}_{-i})$. Since $\hat{q}(x_i; \mathbf{x}_{-i})$, as a function of x_i, takes only finitely many integer values, monotonicity will be sufficient for it to be piecewise constant and therefore differentiable in all but finitely many points. So, to complete the proof, we will show that $\hat{q}(x_i; \mathbf{x}_{-i})$ is weakly decreasing in x_i. Take x_i', x_i'' with $x_i' < x_i''$, let $q' := \hat{q}(x_i'; \mathbf{x}_{-i})$ and $q'' := \hat{q}(x_i''; \mathbf{x}_{-i})$, and assume by contradiction that $q' < q''$. Observe that according to the definition of $\hat{q}(x_i; \mathbf{x}_{-i})$ the quantity q' minimizes the quotient

$$\frac{q\tau_q(x_i', \mathbf{x}_{-i}) - nx_i'}{q - n},$$

and the quantity q'' minimizes the quotient

$$\frac{q\tau_q(x_i'', \mathbf{x}_{-i}) - nx_i''}{q - n}.$$

Considering the inequalities

$$\begin{aligned}
&\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i''}{q''-n}-\frac{q'\tau_{q'}(x_i'',\mathbf{x}_{-i})-nx_i''}{q'-n}\\
&\quad=\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i''}{q''-n}-\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i'}{q''-n}+\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i'}{q''-n}-\frac{q'\tau_{q'}(x_i'',\mathbf{x}_{-i})-nx_i''}{q'-n}\\
&\quad\geq\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i''}{q''-n}-\frac{q''\tau_{q''}(x_i'',\mathbf{x}_{-i})-nx_i'}{q''-n}+\frac{q'\tau_{q'}(x_i'',\mathbf{x}_{-i})-nx_i'}{q'-n}-\frac{q'\tau_{q'}(x_i'',\mathbf{x}_{-i})-nx_i''}{q'-n}\\
&\quad=\frac{n}{n-q''}(x_i''-x_i')-\frac{n}{n-q'}(x_i''-x_i')\\
&\quad>0,
\end{aligned}$$

we reach a contradiction to the statement that q'' minimizes the quotient

$$\frac{q\tau_q(x_i'',\mathbf{x}_{-i})-nx_i''}{q-n}.$$

(ii) For all $c_i^+(x_i;\mathbf{x}_{-i})<0$, we have $R_i^{\mathrm{U}}(x_i;\mathbf{x}_{-i})=0$ and thus $\partial_i R_i^{\mathrm{U}}(x_i;\mathbf{x}_{-i})=0$. For the case $c_i^+(x_i;\mathbf{x}_{-i})>0$ see the next part.
(iii) By assumption there are bids strictly higher than x_i, therefore $\hat{q}(\mathbf{x})\geq 1$ and $c_n^+(\mathbf{x})<x_i$. Let $\underline{f}:=\min_{c\in[0,\bar{c}]} f(c)$. As f is continuous and strictly positive in the interval $[0,\bar{c}]$ (see assumption A2) we have $\underline{f}>0$. Recall also that F/f is increasing by assumption (A3). As $c_n^+(\mathbf{x})<x_i<z_{\mathrm{D}}$, the following (in)equalities are valid for all points in which the partial derivative exists:

$$\begin{aligned}
\partial_i R_i^{\mathrm{U}}(\mathbf{x})&=(v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot\partial_i c_n^+(\mathbf{x})-F\big(c_n^+(\mathbf{x})\big)\\
&=(v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot\frac{n}{n-\hat{q}(\mathbf{x})}-F\big(c_n^+(\mathbf{x})\big)\\
&\geq(v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot\frac{n}{n-1}-F\big(c_n^+(\mathbf{x})\big)\\
&=\frac{n}{n-1}\cdot f\big(c_n^+(\mathbf{x})\big)\Big(v-x_i-\frac{n-1}{n}\cdot\frac{F\big(c_n^+(\mathbf{x})\big)}{f\big(c_n^+(\mathbf{x})\big)}\Big)\\
&>\frac{n}{n-1}\cdot\underline{f}\cdot\Big(v-z_{\mathrm{D}}-\frac{n-1}{n}\cdot\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})}\Big)=:\overline{\partial}.
\end{aligned}$$

Observe that $\overline{\partial}>0$ because

$$v-z_{\mathrm{D}}-\frac{n-1}{n}\cdot\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})}>v-z_{\mathrm{D}}-\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})}=0.$$

□

Lemma 3.3. *For any $\mathbf{x}\in[0,v]^n$ and any $\varepsilon>0$ for which $x_i+\varepsilon\leq v$ the following inequality holds:*

$$R_i(x_i+\varepsilon;\mathbf{x}_{-i})-R_i(x_i;\mathbf{x}_{-i})\geq -1\cdot\varepsilon.$$

Proof. The inequality applies because an increase in the bid of bidder i can lead to an increase in the stop-out price (with some probability), but does not lower the winning chances of that bidder. □

Lemma 3.4. *Let* $\mathbf{x}$ *be such that there exists* x *with* $x \le x_j < \frac{n}{n-1} \cdot x$ *for all* j*. Then* $c_n^+(\mathbf{x}) > 0$ *(that means, the bidder with the lowest bid is served with positive probability).*

Proof. We have

$$c_n^+(\mathbf{x}) > \frac{nx - (n-1) \cdot \frac{n}{n-1} \cdot x}{n - n + 1} = 0.$$

□

Lemma 3.5. *If all bidders except one (say, bidder* i*) submit a bid of* $x \in [0, v)$ *(that means,* $x_j = x$ *for* $j \ne i$*) then there exist* $\varepsilon > 0$ *and* $\widetilde{\partial} > 0$ *such that for* $x_i \in [x, x+\varepsilon)$ *the following holds*[21]*:* $\partial_i R_i^{\mathrm{U}}(x_i, x, \ldots, x) > \widetilde{\partial}$*.*

Proof. From

$$R_i^{\mathrm{U}}(x_i, x, \ldots, x) = (v - x) \cdot F\big(\frac{nx - x_i}{n-1}\big) + (v - x_i) \cdot \Big(F(x_i) - F\big(\frac{nx - x_i}{n-1}\big)\Big)$$

we obtain the partial derivative function

$$\begin{aligned}\partial_i R_i^{\mathrm{U}}(x_i, x, \ldots, x) = &- \frac{v - x}{n-1} \cdot f\big(\frac{nx - x_i}{n-1}\big) - \Big(F(x_i) - F\big(\frac{nx - x_i}{n-1}\big)\Big) \\ &+ (v - x_i) \cdot \Big(f(x_i) + \frac{1}{n-1} f\big(\frac{nx - x_i}{n-1}\big)\Big),\end{aligned}$$

which is continuous in x_i. As $\partial_i R_i^{\mathrm{U}}(x, x, \ldots, x) = (v - x) \cdot f(x) > 0$, there exist $\varepsilon > 0$ and $\widetilde{\partial} > 0$ such that $\partial_i R_i^{\mathrm{U}}(x_i, x, \ldots, x) > \widetilde{\partial}$ for $x_i \in [x, x+\varepsilon)$. □

Lemma 3.6. *Let* $\mathbf{x}$ *be such that there exist bidders* i*,* j*,* k *with* $x_i \ge x_k$ *and*

$$x_j > x_k + (n-1)^2 (x_i - x_k).$$

Then

$$\partial_i^+ R_i^{\mathrm{U}}(\mathbf{x}) = 0.$$

Proof. Observe that $\varphi_{\mathbf{x}}(i) \le (n-1)$.[22] We have

$$c^+_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \le \frac{x_i \cdot (n-1) - x_j}{n-2}, \quad c^-_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \ge n \cdot x_k - (n-1) \cdot x_i.$$

[21] For $x_i = x$ we mean the derivative from above.

[22] If bidder i submits also a bid of x_k, then we choose $\varphi_{\mathbf{x}}$ such that bidder i obtains a number lower than bidder k.

The identities

$$\begin{aligned}
\frac{x_i \cdot (n-1) - x_j}{n-2} &< n \cdot x_k - (n-1) \cdot x_i \Leftrightarrow \\
x_i \cdot (n-1) - x_j &< n \cdot (n-2) \cdot x_k - (n-2) \cdot (n-1) \cdot x_i \Leftrightarrow \\
x_j &> (n-1)^2 \cdot x_i - n \cdot (n-2) \cdot x_k \Leftrightarrow \\
x_j &> (n-1)^2 \cdot x_i - [(n-1)^2 - 1] \cdot x_k \Leftrightarrow \\
x_j &> x_k + (n-1)^2 (x_i - x_k),
\end{aligned}$$

verify that

$$c^+_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) < c^-_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \text{ for } x_j > x_k + (n-1)^2(x_i - x_k),$$

which completes the proof. □

Proof of theorem 3.6. We can now present the proof of the remaining part of theorem 3.6. Recall that z_* is the lower bound of the symmetric equilibrium mixed strategy:

$$z_* = \max\left\{z \mid \sigma_i^*\big([z, v]\big) = 1\right\}.$$

Assume by contradiction

$$z_* \leq z_{\mathrm{D}}.$$

Take an arbitrary bidder i and consider a deviation strategy σ_i^ε, which only shifts the probability mass of the small interval $Z_*^\varepsilon = [z_*, z_* + \varepsilon)$ to the point $z_* + \varepsilon$:

$$\sigma_i^\varepsilon(B) = \sigma_i\big(B \cap \complement Z_*^\varepsilon\big) + \sigma_i\big(Z_*^\varepsilon\big) \cdot \mathbb{1}_{\{z_* + \varepsilon \in B\}} \quad \text{for } B \in \mathscr{B}.$$

We will show that, for ε small enough, this deviation strategy will be more profitable for player i, a contradiction to the equilibrium assumption. We defined already in the exposition the following intervals and sets:

$$Z = \Big[z_*, v\Big], \qquad Z_0^\varepsilon = \Big[z_* + (n-1)^2\varepsilon, \min\{v, \frac{n}{n-1} \cdot z_*\}\Big),$$

$$\mathcal{Z} = Z^{n-1}, \quad \mathcal{Z}^\varepsilon = \bigg(\{z_*\} \cup \Big[z_* + (n-1)^2\varepsilon, v\Big]\bigg)^{n-1},$$

$$\mathcal{Z}_0^\varepsilon = (Z_0^\varepsilon)^{n-1}, \quad \mathcal{Z}_* = \{z_*\}^{n-1}.$$

Consider the difference

$$\begin{aligned}
&\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^{\varepsilon}, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \\
&\quad = \int_{\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}} \int_{Z_*^{\varepsilon}} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}^{\varepsilon}\setminus(\mathcal{Z}_0^{\varepsilon}\cup\mathcal{Z}_*)} \int_{Z_*^{\varepsilon}} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}_0^{\varepsilon}} \int_{Z_*^{\varepsilon}} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}_*} \int_{Z_*^{\varepsilon}} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}).
\end{aligned}$$

For $\varepsilon > 0$ small enough, we obtain lower bounds of the four terms by using lemma 3.3 for the first term, lemmas 3.2 [(i)&(ii)] and lemma 3.6 for the second one, lemmas 3.2 [(i)&(iii)] and lemma 3.4 for the third one[23] and lemma 3.5 for the fourth term, which leads us to the following inequality:

$$\begin{aligned}
&\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^{\varepsilon}, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \\
&\quad \geq \int_{\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}} \int_{Z_*^{\varepsilon}} (-1) \cdot (z_* + \varepsilon - x_i) \, \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}^{\varepsilon}\setminus(\mathcal{Z}_0^{\varepsilon}\cup\mathcal{Z}_*)} \int_{Z_*^{\varepsilon}} 0 \cdot (z_* + \varepsilon - x_i) \, \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}_0^{\varepsilon}} \int_{Z_*^{\varepsilon}} \overline{\partial} \cdot (z_* + \varepsilon - x_i) \, \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad + \int_{\mathcal{Z}_*} \int_{Z_*^{\varepsilon}} \tilde{\partial} \cdot \varepsilon \, \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\quad = \int_{Z_*^{\varepsilon}} (z_* + \varepsilon - x_i) \mathrm{d}\sigma_i^*(x_i) \cdot \Big((-1) \cdot \boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z} \setminus \mathcal{Z}^{\varepsilon}\big) + \overline{\partial} \cdot \boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big)\Big) + \tilde{\partial}\varepsilon\boldsymbol{\sigma}_{-i}^*(\mathcal{Z}_*).
\end{aligned}$$

We will prove that for sufficiently small $\varepsilon > 0$ the expression in the last line is positive. Indeed, observe that $\lim_{\varepsilon\to 0} \boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z} \setminus \mathcal{Z}^{\varepsilon}\big) = 0$. So, if there exists an $\varepsilon > 0$ for which $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) > 0$, then $\lim_{\varepsilon\to 0} \boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) > 0$ and consequently $\lim_{\varepsilon\to 0}\Big((-1)\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big)+\overline{\partial}\cdot \boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big)\Big) > 0$. If, on the other hand, $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) = 0$ for all $\varepsilon > 0$, then $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z} \setminus \mathcal{Z}^{\varepsilon}\big) = 0$ and $(-1)\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big)+\overline{\partial}\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) = 0$ for all $\varepsilon > 0$. In this case $\boldsymbol{\sigma}_{-i}^*(\mathcal{Z}_*) > 0$, because z_* was assumed to be the lower bound of the symmetric equilibrium mixed strategy. In either case we can state the existence of an $\varepsilon > 0$ for which the expression in the last line is positive and consequently $\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^{\varepsilon}, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) > 0$, which completes the proof.

[23]Lemma 3.4 guarantees that in the considered set $c_n^+(\mathbf{x}) > 0$, lemma 3.2 then states the existence of $\overline{\partial} > 0$.

Chapter 4

Static and Dynamic Auctions with Variable Supply

4.1 Introduction

In this chapter we will consider the same model structure as in chapter 3. However, we will focus on a narrower framework of just two bidders and uniformly distributed marginal costs of the seller. Within this setting we will additionally analyze the subgame perfect equilibrium outcomes of the standard *open* (or *dynamic*) auction formats, i.e. the *ascending* and the *descending* clock auctions. We will also discuss how the procedure of sequentially collecting bids compares to the other auction formats in terms of efficiency and ex-ante seller's revenue. More precisely, the standard uniform and discriminatory pricing rules will be studied here in conjunction with the following four procedures for collecting bids: sealed bid (simultaneous), sequential bid, via an ascending and descending clock auction. All these variants define eight auction mechanisms, the comparison of which will be the subject of this chapter.

The restriction to the two-bidder case and uniformly distributed marginal costs simplifies the analysis significantly. In this formulation pure strategy subgame perfect equilibria in all eight auction formats exist, and we will be able to explicitly derive them for all auctions. From the seller's viewpoint all these auction games have either a unique equilibrium outcome or there is a unique equilibrium, which can be selected on the basis of a reasonable criterium[1]. This circumstance facilitates the comparison of all auction formats, because the revenue and the efficiency measure in equilibrium

[1] Only the descending clock auction combined with the uniform pricing rule has multiple equilibria. One of these equilibria can be selected on the basis of dominance arguments.

can be readily computed.

4.1.1 Relation to the theoretical literature

Equilibria of ascending and simultaneous (sealed bid) uniform price auctions have been analyzed in multi-unit settings with fixed supply in a number of papers. The basic insight is that both auction formats possess low-price equilibria, which imply very low revenues for the seller. Wilson (1979) provides examples of low-price equilibria in a sealed bid uniform-price share auction. Back and Zender (1993) extend Wilson's model to allow for incomplete information. They derive a class of low-price equilibria, in which the bidders submit discontinuous demand functions. Low-price equilibria are also a matter of concern in the ascending clock multi-unit auction. Such equilibria have been identified in several contributions, which study models of complete information. Menezes (1996) and Grimm, Riedel, and Wolfstetter (2003) present models of an ascending multi-unit auction with fixed supply and a discrete price grid[2]. They solve for the subgame perfect equilibria of these auction games using backward induction arguments and obtain the result that the game will end at the initial price. Due to the simultaneous nature of the bidding these models have multiple solutions. Ausubel and Schwartz (1999) present another version of this model in which bids at each price are submitted sequentially. This structure is similar to a finite alternating offers bargaining game like the one presented in Rubinstein (1982) and Ståhl (1972). This game has a unique solution, which is at the lowest possible price and depends on who has the right to submit the first bid. These collusive equilibria have been of theoretical and practical concern, since their existence do not require the formation or existence of a bidding ring. They arise as purely noncooperative outcomes of the auction games. Therefore recent research has been revolving around the question of what the seller can actively do to prevent these purely strategic outcomes. Equilibrium collusive bidding can be alleviated if the seller retains control over the final allocation by setting supply quantity after the bidding. This argument spurred the analysis of the variable supply auction formats. In this chapter we will discuss these auction formats in an incomplete information model.

[2] At each price bidders announce quantities. If announced demand exceeds supply, then the price increases by a marginal unit.

4.1.2 Organization of the chapter

The chapter is organized as follows. In the next section we present the model: the setting and a description of the trade mechanisms we analyze. Section 4.3 contains the main results. We derive the subgame perfect equilibria of the uniform price and the discriminatory auction under four procedure for collecting bids: sealed bid (simultaneous), open (sequential), via an ascending and via a descending auction. In section 4.4 a ranking of the considered mechanisms is provided in terms of revenue for the seller, average trade volume and efficiency. The final section concludes with a discussion of the results and their relation to the existing auction literature.

4.2 The model

4.2.1 The setting

We consider auction games between a monopolistic seller and two risk neutral buyers. The seller possesses multiple (at least two) units of an asset, which can be acquired in the auction and each buyer $i \in \{1, 2\}$ is risk neutral and submits a bid for a single unit. We will assume that the asset is of common value for the bidders – an assumption which describes well a situation in which the asset acquired in the auction is traded on liquid secondary market opening after the auction[3]. Although we will not model explicitly the existence and functioning of a secondary market, auctions with resale markets appeal to the present common value setting. Further it is assumed that both bidders share the same common value estimate of $v \in [0, 1]$. This assumption, although quite restrictive, is often assumed in the multi-unit auction literature. Wilson (1979) and Bernheim and Whinston (1986) introduced it in the theoretical literature on multi-unit auctions and labelled it "no proprietary information". See also Back and Zender (2001) and Kremer and Nyborg (2004) for a more recent work using this assumption. In an auction followed by a resale market the assumption suggests that bidders have an access to the same source of information concerning the future resale value of the asset to be auctioned. In the context of an IPO or Treasury auction, in which the bidders are big institutional investors such as investment banks or mutual funds, the value v might be interpreted as the investors' expected price of alternative financial

[3]Such secondary markets usually open after auctions for financial assets like Treasury bills or IPOs.

investments such as corporate bonds[4].

We further focus attention to variable supply auction formats, i.e. we assume that after collecting the bids the seller decides on how many units to sell so as to maximize profit. Following Lengwiler (1999) and the framework of the previous chapter we introduce supply uncertainty by assuming that the marginal costs (or the reservation price) of the seller, are constant and private information as well. These costs are denoted by c and assumed to be a random variable uniformly distributed over the interval $[0, 1]$. The probability distribution of c is denoted by $F(c)$. The density function is $f(c) = 1$ for $c \in [0, 1]$ and zero otherwise.

4.2.2 Trade mechanisms

In our model a trade mechanism consists of two components: a *pricing rule* and a *procedure for collecting the bids*. Here we will focus on the standard pricing rules: uniform price (U) and discriminatory (D) and couple them with four procedures for collecting the bids, thus analyzing eight games (or trade mechanisms) in extensive form (see Table 4.1). We start with a verbal description of the four procedures for bid collection.

4.2.3 Procedures for collecting bids

[S] **Sealed bid (similtaneous) collection of bids**
Both buyers submit sealed bids to the auctioneer. Each bidder is not informed and needs to build the right expectation about the bid of the other bidder – an aspect which is captured by the notion of equilibrium.

[O] **Open (sequential) collection of bids**
Buyers bid sequentially. The bid of the first bidder is made public before the second bidder submits his bid. Thus the second bidder conditions his bid price on the observed bid.

[A] **Ascending auction clock**
The seller operates a continuously increasing auction clock. The clock starts at a price of 0 and gradually moves upwards until at least one of the bidders stops it. The price at which a bidder stops the clock (or in other words exits the auction)

[4] I would like to thank Jens Tapking for pointing this to me.

defines his bid price. Once exited, a bidder cannot reenter the bidding. If only one bidder stops the clock at a certain price, the clock is restarted from that price on and the other bidder is invited to stop the clock again. Exit prices and identities of the bidders are commonly observed. The collecting procedure is over when both bidders announced their bid prices.

[D] **Descending auction clock**
The rules of this auction clock are similar to the ascending auction clock with the following differences: the clock is started at a very high price $\overline{p}$ and moves continuously downwards. The bidding is over when both bidders stopped the clock or the clock reached a price of 0. In the latter case zero is the bid price of the bidder(s) who remained in the auction.

After the bidding the auctioneer steps in to determine the number of units sold and the identity of the winner(s) as a function of the submitted bids and her private information.

		Collecting procedures			
		S	**O**	**A**	**D**
Pricing Rules	**U**	U;S	U;O	U;A	U;D
	D	D;S	D;O	D;A	D;D

Table 4.1: *Auction forms*

4.2.4 Pricing rules: uniform price (U) and discriminatory (D)

In the uniform auction all winners pay the same price, which equals the lowest winning bid, whereas in the discriminatory auction the winning bidders are charged their own bid prices. In both auction formats the seller collects the bids and orders them in a descending order. Let us assume that the bidders submitted the bids x_1 and x_2 and define

$$\tau_1 = \max\{x_1, x_2\}, \quad \tau_2 = \min\{x_1, x_2\}.$$

We further determine the payoffs of the bidders and the auctioneer and analyze the optimal quantity decision of the auctioneer under the two pricing rules. If the seller chooses the supply quantity $q \in \{0, 1, 2\}$, her payoff in the uniform price auction will be

$$R_{\mathrm{S}}^{\mathrm{U}}(\tau_1,\tau_2,q,c)=\begin{cases}0 & \text{for } q=0,\\ \tau_1-c & \text{for } q=1,\\ 2(\tau_2-c) & \text{for } q=2,\end{cases}$$

and in the discriminatory auction

$$R_{\mathrm{S}}^{\mathrm{D}}(\tau_1,\tau_2,q,c)=\begin{cases}0 & \text{for } q=0,\\ \tau_1-c & \text{for } q=1,\\ \tau_1+\tau_2-2c & \text{for } q=2.\end{cases}$$

4.3 Analysis

4.3.1 The second stage of the game

The profit-maximizing quantity decision of the auctioneer in the uniform price auction is

$$\phi_{\mathrm{U}}^{*}(\tau_1,\tau_2,c)=\begin{cases}0 & \text{for } c>\tau_1,\\ 1 & \text{for } \tau_1>c>2\tau_2-\tau_1,^{5}\\ 2 & \text{for } 2\tau_2-\tau_1>c.\end{cases}$$

In the discriminatory auction the seller serves optimally every bidder, whose bid is not lower than the marginal costs:

$$\phi_{\mathrm{D}}^{*}(\tau_1,\tau_2,c)=\begin{cases}0 & \text{for } c>\tau_1,\\ 1 & \text{for } \tau_1>c>\tau_2,\\ 2 & \text{for } \tau_2>c.\end{cases}$$

Taking into account the optimal supply behavior of the seller after collecting the bids, one obtains for the expected consumer surplus of bidder $i=1,2$ in the uniform price auction:

$R_i^{\mathrm{U}}(x_i,x_{-i};\phi_U^*(\cdot))=$

$$=\begin{cases}(v-x_i)\cdot\big(F(x_i)-F(2x_{-i}-x_i)\big)+(v-x_{-i})\cdot F(2x_{-i}-x_i) & \text{for } x_i\geq x_{-i},\\ (v-x_i)\cdot F(2x_i-x_{-i}) & \text{for } x_i<x_{-i},\end{cases}\tag{4.1}$$

$$=\begin{cases}(v-x_i)x_i & \text{for } x_i>2x_{-i},\\ (v-x_i)\big(2x_i-2x_{-i}\big)+(v-x_{-i})(2x_{-i}-x_i) & \text{for } 2x_{-i}\geq x_i\geq x_{-i},\\ (v-x_i)(2x_i-x_{-i}) & \text{for } \frac{x_{-i}}{2}\leq x_i<x_{-i},\\ 0 & \text{for } x_i<\frac{x_{-i}}{2}.\end{cases}\tag{4.2}$$

[5] The equalities occur with probability 0 and are therefore omitted.

The consumer surplus of bidder i in the discriminatory auction is:

$$R_i^{\mathrm{D}}(x_i, x_{-i}; \phi_D^*(\cdot)) = (v - x_i)F(x_i) = (v - x_i)x_i. \tag{4.3}$$

As we will be interested in the subgame perfect equilibra of these trade mechanisms, from now on we will suppress the terms $\phi_D^*(\cdot)$ and $\phi_U^*(\cdot)$ and will write $R_i^{\mathrm{D}}(x_i, x_{-i})$ instead of $R_i^{\mathrm{D}}(x_i, x_{-i}; \phi_D^*(\cdot))$ as well as $R_i^{\mathrm{U}}(x_i, x_{-i})$ instead of $R_i^{\mathrm{U}}(x_i, x_{-i}; \phi_U^*(\cdot))$ always assuming that the seller chooses an optimal supply quantity under the two pricing rules. In the following sections we will be defining and analyzing the bidding stages of the games, which result from the different regimes of bid collection.

4.3.2 Uniform pricing (U)

In this subsection we will demonstrate that the procedure for collecting the bids has an impact on the equilibrium bids. All the four collecting procedures define different games and generate different equilibrium bids. Before we turn to the analysis of each game, we will state a lemma, which gives the solution of three problems useful for the future analysis.

Lemma 4.1. *The problems (A),(B) and (C) given below*

(A)

$$\arg\max_{x_i} \quad R_i^{\mathrm{U}}(x_i, x_{-i}) \quad s.t. \quad x_i \geq x_{-i},$$

(B)

$$\arg\max_{x_i} \quad R_i^{\mathrm{U}}(x_i, x_{-i}) \quad s.t. \quad x_i \leq x_{-i},$$

(C)

$$\arg\max_{x_i} \quad R_i^{\mathrm{U}}(x_i, x_{-i}),$$

have the following solutions[6] *for $x_{-i} \in [0, v]$:*
(A):

$$x_i^A(x_{-i}) = \begin{cases} \frac{1v}{2} & \text{for } x_{-i} \in [0, \frac{(3-\sqrt{2})v}{7}), \\ \{\frac{(16-3\sqrt{2})v}{28}, \frac{1v}{2}\} & \text{for } x_{-i} = \frac{(3-\sqrt{2})v}{7}, \\ \frac{(3x_{-i}+1)v}{4} & \text{for } x_{-i} \in (\frac{(3-\sqrt{2})v}{7}, v) \end{cases}$$

[6]We will consider the solutions in the interval [0,v] since it is clear that the bidder will not be willing to bid above their valuation v.

(B):

$$x_i^B(x_{-i}) = \begin{cases} x_{-i} & \text{for } x_{-i} \in [0, \frac{2v}{3}], \\ \frac{(x_{-i}+2)v}{4} & \text{for } x_{-i} \in (\frac{2v}{3}, v]. \end{cases}$$

(C):

$$x_i^C(x_{-i}) = \begin{cases} \frac{v}{2} & \text{for } x_{-i} \in [0, \frac{(3-\sqrt{2})v}{7}), \\ \{\frac{(16-3\sqrt{2})v}{28}, \frac{1v}{2}\} & \text{for } x_{-i} = \frac{(3-\sqrt{2})v}{7}, \\ \frac{(3x_{-i}+1)v}{4} & \text{for } x_{-i} \in (\frac{(3-\sqrt{2})v}{7}, \frac{3v}{4}), \\ \{\frac{11v}{16}, \frac{13v}{16}\} & \text{for } x_{-i} = \frac{3v}{4}, \\ \frac{(x_{-i}+2)v}{4} & \text{for } x_{-i} \in (\frac{3v}{4}, v]. \end{cases}$$

The solutions are given in figure 5.1 below. The proof of the lemma is given in the

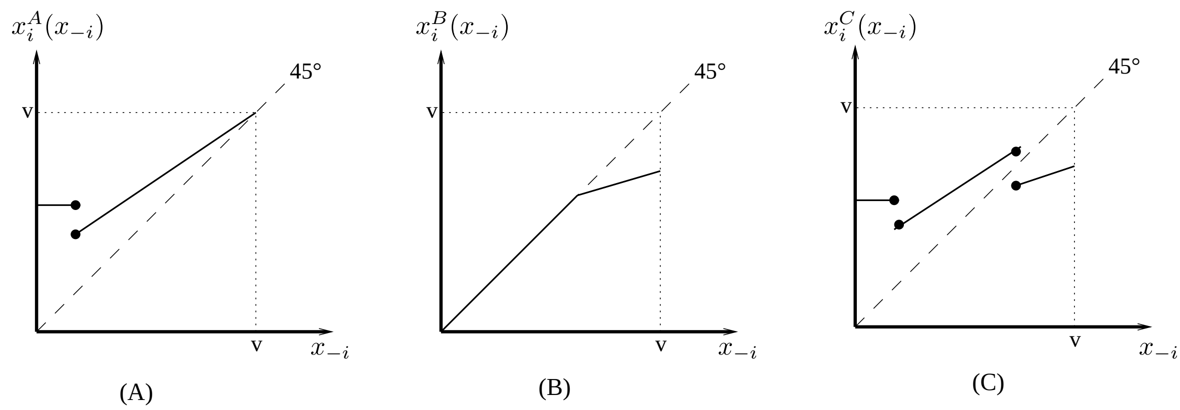

Figure 4.1: *Solutions of the problems (A),(B) and (C) of Lemma 5.1.*

Appendix 4.A and is organized as follows. We first solve problems (A) and (B), obtaining as a solution the functions $x_i^A(x_{-i})$ and $x_i^B(x_{-i})$. Then we compare the payoffs $R_i^{\mathrm{U}}(x_i^A(x_{-i}), x_{-i})$ and $R_i^{\mathrm{U}}(x_i^B(x_{-i}), x_{-i})$ to determine $x_i^C(x_{-i})$ as follows:

$$x_i^C(x_{-i}) = \begin{cases} x_i^A(x_{-i}) & \text{if } R_i^{\mathrm{U}}(x_i^A(x_{-i}), x_{-i}) > R_i^{\mathrm{U}}(x_i^B(x_{-i}), x_{-i}), \\ \{x_i^A(x_{-i}), x_i^B(x_{-i})\} & \text{if } R_i^{\mathrm{U}}(x_i^A(x_{-i}), x_{-i}) = R_i^{\mathrm{U}}(x_i^B(x_{-i}), x_{-i}), \\ x_i^B(x_{-i}) & \text{if } R_i^{\mathrm{U}}(x_i^A(x_{-i}), x_{-i}) < R_i^{\mathrm{U}}(x_i^B(x_{-i}), x_{-i}). \end{cases}$$

4.3.2.1 Simultaneous and sequential collection of bids

A Nash equilibrium of the bidding stage of the simultaneous procedure for collecting bids is a strategy profile $(x_i^{U,S}, x_{-i}^{U,S}) \in \mathbb{R}_+^2$, for which

$$x_i^{U,S} \in \arg\max_{x_i} R_i^{\mathrm{U}}(x_i, x_{-i}^{U,S}), \quad i = 1, 2.$$

Claim 4.1 (sealed bid auction)**.** *The uniform price sealed bid auction (U,S) has two (identical for the seller) asymmetric subgame perfect equilibria, in which the following bids are submitted:*

$$(x_i^{U,S}, x_{-i}^{U,S}) = \left(\tfrac{10v}{13}, \tfrac{9v}{13}\right) \textit{ for } i = 1, 2.$$

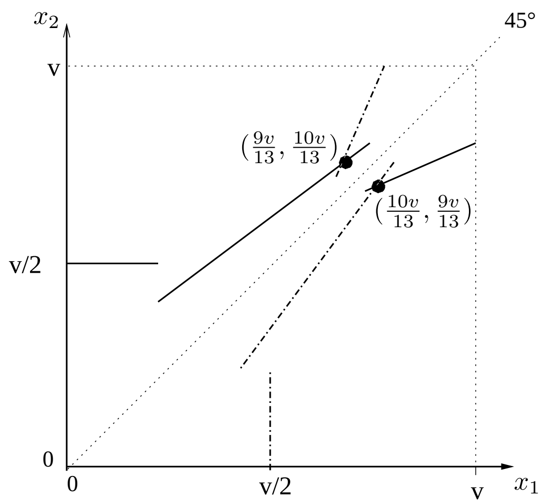

Figure 4.2: *Best responses and equilibrium bids of the uniform price sealed bid auction*

Proof. The best responses of the bidders are given by the solution of problem (C) of Lemma 1. They are plotted in figure 4.2. One observes that two asymmetric Nash equilibria emerge. They satisfy the following system of equations:

$$\left| \begin{array}{l} x_i = (3x_{-i} + v)/4, \\ x_{-i} = (x_i + 2v)/4. \end{array} \right.$$

The equilibrium bids are derived by directly solving the equation system. □

In the sequential procedure bidder 2 observes the bid submitted by bidder 1. Whereas bidder 1 has the same strategy space $x_1 \in \mathbb{R}_+$, the strategy set of the second bidder is larger, including all real-valued functions $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$. A subgame perfect equilibrium of this procedure for bid collection is a strategy profile $(x_1^{U,O}, \varphi_2^{U,O}(x_1))$ such that

$$\begin{aligned} \varphi_2^{U,O}(x_1) &\in \arg\max_{x_2} \quad R_2^{\mathrm{U}}(x_1, x_2), \\ x_1^{U,O} &\in \arg\max_{x_1} \quad R_1^{\mathrm{U}}(x_1, \varphi_2^{U,O}(x_1)). \end{aligned}$$

Claim 4.2 (sequential collection of bids)**.** *In the unique subgame perfect equilibrium of the uniform price auction with sequential collection of bids (U,O) the bidders bid as follows:*

$$(x_1^{U,O}, x_2^{U,O}) = \left(\tfrac{12v}{16}, \tfrac{11v}{16}\right).$$

Proof. The bid function of the second bidder $\varphi_2^{U,O}(x_1)$ is given by the solution of problem (C). Thus the payoff of bidder 1, if we substitute for the optimal bid(s) of bidder 2 is
$R_1^{\mathrm{U}}(x_1, \varphi_2^{U,O}(x_1)) =$

$$= \begin{cases} 0 & \text{for } x_1 \in [0, \frac{(3-\sqrt{2})v}{7}), \\ 0 & \text{for } x_1 = \frac{(3-\sqrt{2})v}{7} \text{ and } \varphi^{U,O}(\frac{(3-\sqrt{2})v}{7}) = v/2, \\ \frac{(4+\sqrt{2})(8-5\sqrt{2})}{196} & \text{for } x_1 = \frac{(3-\sqrt{2})v}{7} \text{ and } \varphi^{U,O}(\frac{(3-\sqrt{2})v}{7}) = (16-3\sqrt{2})v/4, \\ (v-x_1)(5x_1-v)/4 & \text{for } x_1 \in (\frac{(3-\sqrt{2})v}{7}, \frac{3v}{4}), \\ \frac{11v^2}{64} & \text{for } x_1 = 3v/4 \text{ and } \varphi^{U,O}(3v/4) = 13v/16, \\ \frac{29v^2}{128} & \text{for } x_1 = 3v/4 \text{ and } \varphi^{U,O}(3v/4) = 11v/16, \\ \frac{(v-x_1)(3x_1+2v)}{2} + \frac{(2v-x_1)^2}{8} & \text{for } x_1 \in (3v/4, v] \end{cases}$$

See figure 4.3 for a graphical illustration. In the appendix is shown that the global maximizer of this function is $x_1 = \frac{3v}{4}$ if bidder 2 submits $\frac{11v}{16}$ (bidder 2 is indifferent between submitting $\frac{11v}{16}$ and $\frac{13v}{16}$). □

The result implies that bids are lower under the sequential procedure: $\frac{10v}{13} > \frac{12v}{16}$ and $\frac{9v}{13} > \frac{11v}{16}$. This result can be intuitively explained as follows. As the strategy profile $(x_1^{U,S}, x_2^{U,S}) = \left(\frac{10v}{13}, \frac{9v}{13}\right)$ is an equilibrium in the sealed bid procedure, for the partial derivative with respect to bidder 1 we have

$$\left.\frac{\partial R_1^{\mathrm{U}}(x_1, x_2)}{\partial x_1}\right|_{(\frac{10v}{13}, \frac{9v}{13})} = 0.$$

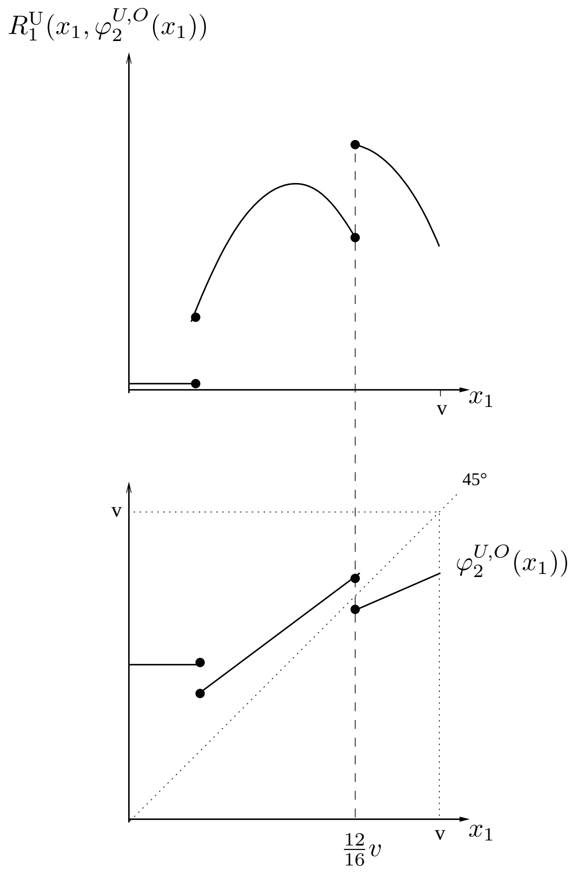

Figure 4.3: *The uniform price auction with sequential collection of bids*

We observe now that

$$\begin{aligned}
\left.\frac{\partial R_1^{\mathrm{U}}(x_1, \varphi_2^{U,O}(x_1))}{\partial x_1}\right|_{(\frac{10v}{13}, \frac{9v}{13})} &= \frac{\partial R_1^{\mathrm{U}}(x_1, x_2)}{\partial x_1} + \frac{\partial R_1^{\mathrm{U}}(x_1, x_2)}{\partial x_2} \cdot \frac{\partial \varphi_2^{U,O}(x_1)}{\partial x_1} \\
&= \frac{\partial R_1^{\mathrm{U}}(x_1, x_2)}{\partial x_2} \cdot \frac{\partial \varphi_2^{U,O}(x_1)}{\partial x_1} \\
&= (3x_1 - 4x_2) \cdot \frac{1}{4} \\
&= \frac{(3 \cdot 10 - 4 \cdot 9)v}{13} \cdot \frac{1}{4} = -\frac{3v}{26} < 0.
\end{aligned}$$

In words, the partial derivative in the sequential procedure evaluated at the point of the equilibrium bids in the sealed bid procedure is negative due to the indirect effect arising from the adjustment of the bid of the second bidder. For that reason it is profitable for the first bidder to reduce his bid, thus reducing the bid of the follower, which turns out to be profitable for the first bidder.

4.3.2.2 Ascending clock auction

The ascending clock multi-unit auction has been employed for privatization of government-owned companies. This trade mechanism has been used for example in the nineties during the privatization programs in Brazil and in former Czechoslovakia[7]. It has been also proposed by the US Treasury as a novel method to sell Treasury bills, but the idea was (temporarily) shelved. Cramton (1998) presents informal arguments in favor of the ascending auction format. He argues that this auction serves well the goals of generating high seller's revenue and efficiency.

We start by formally defining the bidding stage of the ascending auction clock as a game in extensive form. In this game the strategy space of the bidders will be more complex as it should prescribe at what price to stop the clock depending on the behavior of the other bidder as the auction clock progresses. A strategy of a bidder i in this game consists of:

- A nonnegative real number p_i, denoting the price at which to stop the clock provided that the other bidder did not stop the clock at a lower price.
- A function $r_i^A : \mathbb{R}_+ \to \mathbb{R}_+$, where $r_i^A(p_{-i}) \geq p_{-i}$. It denotes at which price to stop the clock, if the other bidder stopped the clock first at the price $p_{-i} \in \mathbb{R}_+$.

Remark 4.1. *A strategy in this game can be defined in the usual way by introducing the concept of a history of the game determined by the ascending clock and the actions of the players. Let a history for bidder i at price t be denoted by h_i^t. It can be either $(\emptyset; t)$, if the other bidder does not exit until the clock reaches the price t or $(p_{-i}; t)$, if the other bidder exits at price $p_{-i} \leq t$. Thus, the set of all histories for a bidder i at price t is*

$$H_i^t \equiv (\emptyset; t) \cup \{(p_{-i}; t) \mid p_{-i} \in [0, t]\}$$

and the set of all histories (at all prices) is $H_i \equiv \bigcup_{t=0}^{\infty} H_i^t$. Now a strategy of bidder i can be defined as a mapping $s_i : H_i \longrightarrow \mathbb{R}_+$ such that

$$s_i(h_t) = \begin{cases} p_i \in \mathbb{R}_+ & \text{for all } h_t = \{\emptyset; t\}, t \in \mathbb{R}_+, \\ r_i^A(p_{-i}), & \text{for all } h_t = \{p_{-i}; t\}, t \in \mathbb{R}_+. \end{cases} \tag{4.4}$$

The so defined strategy of a player obviously do not depend on the progressing of the clock itself (i.e. on t), but only on the actions taken by the other player. Introducing a

[7] See Menezes (1996, pp. 671-672).

richer strategy space by additionally conditioning a bidder's strategy on t is not necessary here, because every bidder can design a plan where to exit the auction prior to the bidding, which is not dependent on the clock, but only on the observed actions of his fellow bidder. Thus we can without loss of generality denote the strategy of each bidder $i = 1, 2$ *by* $(p_i, r_i^A(p_{-i}))$.

The payoffs in the auction can now be given as follows:

$$\pi_i^{U,A}(\cdot,\cdot) = \begin{cases} R_i^{\mathrm{U}}(p_i, r_{-i}^A(p_i)) & \text{for } p_i < p_{-i}, \\ R_i^{\mathrm{U}}(p_i, p_{-i}) & \text{for } p_i = p_{-i}, \\ R_i^{\mathrm{U}}(r_i^A(p_{-i}), p_{-i})) & \text{for } p_i > p_{-i}. \end{cases}$$

A subgame perfect equilibrium is a strategy profile prescribing a Nash equilibrium play in all subgames of the so defined extensive form game, the set of which for bidder i is $\{p_{-i} \mid p_{-i} \in \mathbb{R}_+\} \cup \{\emptyset\}$. In the subgames $\{p_{-i} \mid p_{-i} \in \mathbb{R}_+\}$ bidder i is the only mover and decides where to stop the clock in the cases in which the other bidder stopped the clock at p_{-i}. In the subgame $\{\emptyset\}$ both bidders (simultaneously) plan at which price to stop the clock provided that the other bidder did not stop the clock until that price.

Remark 4.2. *The ascending clock auction is strategically equivalent to the following two-stage game. In the first stage both bidders play a sealed bid auction by submitting the prices* p_i *and* p_{-i}. *If both bidders did not submit the same price a second stage takes place in which the bidder with the* ***higher*** *bid (let's call it bidder i) is allowed to revise his bid by conditioning it on* p_{-i}, *i.e. submitting* $r_i^{U,A}(p_{-i}) \geq p_{-i}$.

Formally, a subgame perfect equilibrium is a strategy profile

$$\left((p_i^{U,A}, r_i^{U,A}(\cdot)), (p_{-i}^{U,A}, r_{-i}^{U,A}(\cdot))\right)$$

which satisfies two conditions. The first one requires optimal play of bidder $i \in \{1, 2\}$ in the subgames $\{p_{-i} \mid p_{-i} \in \mathbb{R}_+\}$:

$$r_i^{U,A}(p_{-i}) \in \arg\max_{r_i} \{R_i^{\mathrm{U}}(r_i, p_{-i}) \mid r_i \geq p_{-i}\}, i = 1, 2. \tag{4.5}$$

Assuming optimal play for these subgames we define the *reduced form of the game,* and its payoffs as follows

$$\Pi_i^{U,A}(p_i, p_{-i}) = \begin{cases} R_i^{\mathrm{U}}(p_i, r_{-i}^{U,A}(p_i)) & \text{for } p_i < p_{-i}, \\ R_i^{\mathrm{U}}(p_i, p_{-i}) & \text{for } p_i = p_{-i}, \\ R_i^{\mathrm{U}}(r_i^{U,A}(p_{-i}), p_{-i})) & \text{for } p_i > p_{-i}. \end{cases}$$

The second condition requires that $(p_i^{U,A}, p_{-i}^{U,A})$ constitutes a Nash equilibrium in the *reduced form of the game* and is given as follows:

$$p_i^{U,A} \in \arg\max_{p_i} \Pi_i^{U,A}(p_i, p_{-i}^{U,A}), \quad i = 1, 2. \tag{4.6}$$

Claim 4.3 (ascending clock)**.** *The ascending auction clock has multiple (identical for the seller) asymmetric equilibria. The collected bids in all equilibria are*

$$(x_i^{U,A}, x_{-i}^{U,A}) = \left(\tfrac{7v}{10}, \tfrac{6v}{10}\right) \textit{ for } i = 1, 2.$$

Proof. Without loss of generality we will search for equilibria, for which $p_i^{U,A} \geq p_{-i}^{U,A}$. Solving the subgames starting at histories $\{p_{-i} \mid p_{-i} \in \mathbb{R}_+\}$ requires solving the problem (A). The solution takes the form:

$$r_i^{U,A}(p_{-i}) = \begin{cases} \frac{1v}{2} & \text{for } p_{-i} \in [0, \frac{(3-\sqrt{2})v}{7}), \\ \{\frac{(16-3\sqrt{2})v}{28}, \frac{1v}{2}\} & \text{for } p_{-i} = \frac{(3-\sqrt{2})v}{7}, \\ \frac{(3p_{-i}+v}{4} & \text{for } p_{-i} \in (\frac{(3-\sqrt{2})v}{7}, v]. \end{cases}$$

One observes that $r_i^{U,A}(p_{-i}) > p_{-i}$ for $p_{-i} \in [0, v)$, so a strategy profile according to which both bidders exit at the same price lower than v is not an equilibrium. The payoff of bidder $-i$ takes the form:

$$R_{-i}^{\mathrm{U}}(r_i^{U,A}(p_{-i}), p_{-i})) = \begin{cases} 0 & \text{for } p_{-i} \in [0, \frac{(3-\sqrt{2})v}{7}), \\ 0 & \text{for } p_{-i} = \frac{(3-\sqrt{2})v}{7} \text{ and } r_i^{U,A}(\frac{(3-\sqrt{2})v}{7}) = v/2, \\ \frac{(4+\sqrt{2})(8-5\sqrt{2})}{196} & \text{for } p_{-i} = \frac{(3-\sqrt{2})v}{7} \text{ and } r_i^{U,A}(\frac{(3-\sqrt{2})v}{7}) = \frac{(16-3\sqrt{2})v}{4}, \\ (v - p_{-i})(5p_{-i} - v)/4 & \text{for } p_{-i} \in (\frac{(3-\sqrt{2})v}{7}, v). \end{cases}$$

This function reaches a global maximum at $p_{-i} = \frac{6v}{10}$ (see figure 4.4 for an illustration). Bidder i submits $r_i^{U,A}(\frac{6v}{10}) = \frac{7v}{10}$. The payoff of bidder $-i$ is

$$R_{-i}^{\mathrm{U}}(\frac{7v}{10}, \frac{6v}{10}) = (v - \frac{6v}{10})(2 \cdot \frac{6v}{10} - \frac{7v}{10}) = \frac{v^2}{5}.$$

For the strategy profile $(p_i^{U,A}, p_{-i}^{U,A})$ with $p_i^{U,A} > p_{-i}^{U,A}$ to be an equilibrium profile, for the payoff of bidder i in the reduced form of the game the following inequality should hold:

$$\Pi_i^{U,A}(p_i^{U,A}, p_{-i}^{U,A}) \geq \Pi_i^{U,A}(p_i, p_{-i}^{U,A}) \quad \text{for every } p_i \in \mathbb{R}_+,$$

(which is equivalent to

$$R_i^{\mathrm{U}}(\frac{7v}{10}, \frac{6v}{10}) \geq R_i^{\mathrm{U}}(p_i, r_{-i}^{U,A}(p_i)) \quad \text{for } p_{-i} \in [0, \tfrac{6v}{10})), \tag{4.7}$$

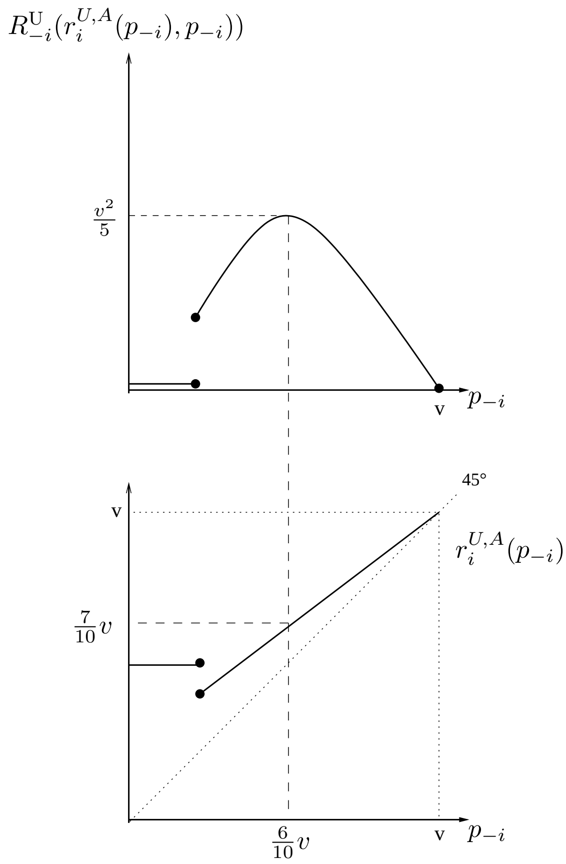

Figure 4.4: *The ascending clock auction*

and

$$\Pi^{U,A}_{-i}(p_i^{U,A}, p_{-i}^{U,A}) \geq \Pi^{U,A}_{-i}(p_i^{U,A}, p_{-i}) \quad \text{for } p_{-i} \in \mathbb{R}_+,$$

(which is equivalent to

$$p_{-i}^{U,A} = \frac{6v}{10}, \quad R^{\mathrm{U}}_{-i}(\frac{7v}{10}, \frac{6v}{10}) \geq R^{\mathrm{U}}_{-i}(p_i^{U,A}, r_{-i}^{U,A}(p_i^{U,A}))). \tag{4.8}$$

Inequality (4.7) guarantees that it is not profitable for bidder i to exit before bidder $-i$. We will show that this inequality holds. Observe that

$$R^{\mathrm{U}}_i(r_i^{U,A}(p_{-i}), p_{-i})) \geq R^{\mathrm{U}}_{-i}(r_i^{U,A}(p_{-i}), p_{-i}) \quad \text{for } p_i \in [0, v).$$

Indeed, the bidder with the higher bid is better off than the bidder with the lower bid, since if both bidders are served they pay the same price, but when the lower bidder is not served, and the higher bidder is served (which happens with positive probability), the higher bidder pays his bid price, which is less than v. Now if we recall that

$$\frac{6v}{10} = \arg\max_{p_{-i}} R^{\mathrm{U}}_{-i}(r_i^{U,A}(p_{-i}), p_{-i})),$$

the inequality follows.

Inequality (4.8) guarantees that it is unprofitable for bidder $-i$ to wait until bidder i exits first. This inequality is trivially satisfied[8] for $p_{-i}^{U,A} \geq v$ since in this case

$$R^{\mathrm{U}}_{-i}(p_i^{U,A}, r_{-i}^{U,A}(p_i^{U,A})) \leq 0 < \frac{v^2}{5} = R^{\mathrm{U}}_{-i}(\frac{7v}{10}, \frac{6v}{10}).$$

Thus we proved that the ascending auction has subgame perfect equilibria and in all the equilibria the collected bids are $(\frac{7v}{10}, \frac{6v}{10})$. Figure 4.5 provides a graphical illustration of the above described equilibria in the reduced game. □

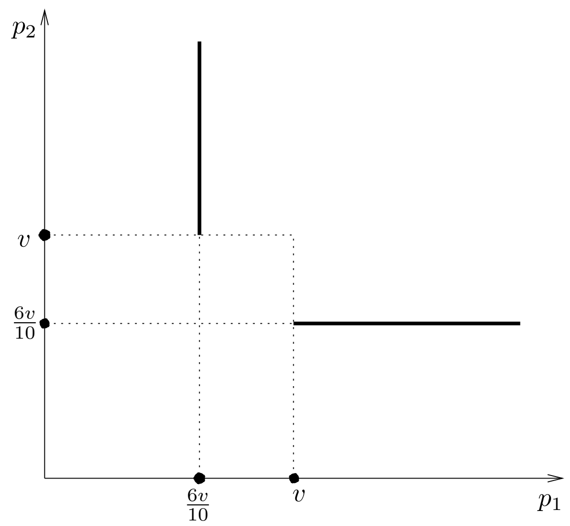

Figure 4.5: *Equilibria in the reduced ascending clock auction game (the solid lines).*

4.3.2.3 Descending clock auction

The descending clock or Dutch auction was invented in the 1870s by a Dutch cauliflower grower, a farmer who wanted to simplify the selling of his product. Today it is widely used in its multi-unit version in the Netherlands to sell flowers and pot plants.[9]

[8]The inequality has other solutions in the interval $[0, v)$, which are in fact the solutions of the inequation

$$\frac{v^2}{5} \geq (v - \frac{(3p_{-i}+1)v}{4})(2\frac{(3p_{-i}+1)v}{4} - 2p_{-i}) + (v - p_{-i})(2p_{-i} - \frac{(3p_{-i}+1)v}{4})$$

in the above interval. We will not additionally compute these equilibria, since they lead to the same final bids for the players.

[9]Famous flower auctions in the Netherlands are the Aalsmeer Flower Auction and the Tele Flower Auction (see Kambil and van Heck (2002, pp. 74-79)). In these auctions the clock starts at a high

The Dutch auction is used also on the internet. For instance a company called Intermodalex.com provides a Dutch auction for shipping firms, which matches shipping services with customers who need to send products from the North Sea to ports around the world. In Germany the Dutch auction is used on the auction site www.azubo.de to sell a variety of goods.

Here we define the strategies in the descending clock auction. Similarly to the ascending clock auction, a strategy of a bidder i consists of

- A number $p_i \in [0, \overline{p}]$ denoting the price at which to stop the clock provided that the other bidder didn't stop the clock until that price.
- A function $r_i^D(p_{-i})$, such that $r_i^D(p_{-i}) \leq p_{-i}$, which denotes at which price to stop the clock, if the other bidder stopped the clock first at the price $p_{-i} \in [0, \overline{p}]$. The difference to the ascending clock consists in the fact that the bidder who exits second defines the lower bid.

Remark 4.3. *A strategy for a bidder can be defined in a similar way as in the ascending clock auction by defining the history of the game (see remark 4.1). Here we will also use the description* $(p_i, r_i^D(p_{-i}))$ *of a strategy for each bidder* $i = 1, 2$*. Similarly as in the ascending auction, the descending auction is strategically equivalent to the following two-stage game. In the first stage both bidders play a sealed bid auction by submitting the prices* p_i *and* p_{-i}*. If both bidders did not submit the same price a second stage takes place, in which the bidder with the* ***lower*** *bid (let's call it bidder i) is allowed to revise his bid by conditioning it on* p_{-i}*, i.e. submitting* $r_i^{U,D}(p_{-i}) \leq p_{-i}$*.*

The payoffs in the descending clock auction are as follows:

$$\pi_i^{U,D}(\cdot,\cdot) = \begin{cases} R_i^{\mathrm{U}}(p_i, r_{-i}^D(p_i)) & \text{for } p_i > p_{-i}, \\ R_i^{\mathrm{U}}(p_i, p_{-i}) & \text{for } p_i = p_{-i}, \\ R_i^{\mathrm{U}}(r_i^D(p_{-i}), p_{-i})) & \text{for } p_i < p_{-i}. \end{cases}$$

price, and moves counter-clockwise to lower prices. Whenever a bidder stops the clock, by pushing a button, a sale occurs at that price. Immediately after stopping the price clock, the buyer speaks into a microphone to inform the auction staff of his desired quantity at the price on the stopped clock. Then the price clock resumes its counter-clockwise path to lower prices. The next bidder who stops the price clock buys at his chosen price, and so on until the lot of flowers or pot plants is completely sold and the auction subsequently proceeds to sell another lot. This is a fixed supply auction format, which has not been analyzed theoretically as of yet. Here we will analyze a variable supply version of it. When flowers or plants are actively being auctioned in the Netherlands, transaction prices are formed about once every four seconds on each price clock. Thus the Dutch flower auctions are very fast, which is an important feature of an auction used to sell a highly perishable commodity such as cut flowers. The Aalsmeer Flower Auction for example has 13 clocks in five auction rooms, with each price clock yielding a transaction price every few seconds.

Formally, a subgame perfect equilibrium is a strategy profile

$$\Big((p_i^{U,D}, r_i^{U,D}(\cdot)), (p_{-i}^{U,D}, r_{-i}^{U,D}(\cdot))\Big)$$

satisfying the conditions (4.9) and (4.10) given below.

$$r_i^{U,D}(p_{-i}) \in \arg\max_{r_i}\{R_i^{\mathrm{U}}(r_i, p_{-i}) \mid r_i \leq p_{-i}\}, \tag{4.9}$$

$$p_i^{U,D} \in \arg\max_{p_i} \Pi_i^{U,D}(p_i, p_{-i}^{U,D}), i = 1, 2, \tag{4.10}$$

where $\Pi_i^{U,D}(p_i, p_{-i})$ denotes the payoffs in the *reduced form of the game* and is given as follows

$$\Pi_i^{U,D}(p_i, p_{-i}) = \begin{cases} R_i^{\mathrm{U}}(p_i, r_{-i}^{U,D}(p_i)) & \text{for } p_i > p_{-i}, \\ R_i^{\mathrm{U}}(p_i, p_{-i}) & \text{for } p_i = p_{-i}, \\ R_i^{\mathrm{U}}(r_i^{U,D}(p_{-i}), p_{-i})) & \text{for } p_i < p_{-i}. \end{cases} \tag{4.11}$$

Claim 4.4 (descending clock). *The descending clock auction has multiple equilibria. The collected bids in all equilibria are given as follows:*

$$(x_i^{U,D}, x_{-i}^{U,D}) = (x, x), \textit{ where } x \in [\tfrac{v}{2}, \tfrac{1+\sqrt{11}}{2\cdot\sqrt{11}}v] \textit{ and } i = 1, 2.$$

In all the equilibria in the reduced form of the game except the equilibrium

$$(p_1^{U,D}, p_2^{U,D}) = (\frac{v}{2}, \frac{v}{2})$$

bidders employ weakly dominated strategies.

Proof. Again, without loss of generality, we will focus on equilibria for which $p_i^{U,D} \leq p_{-i}^{U,D}$. We solve the subgames starting at histories $\{p_{-i} \mid p_{-i} \in \mathbb{R}_+\}$. The optimal $r_i^{U,D}(p_{-i})$ is given by the solution of the problem (B) and takes the form:

$$r_i^{U,D}(p_{-i}) = \begin{cases} p_{-i} & \text{for } p_{-i} \in [0, \frac{2v}{3}], \\ \frac{p_{-i}+2}{4} & \text{for } p_{-i} \in (\frac{2v}{3}, v]. \end{cases} \tag{4.12}$$

Thus the payoff of bidder $-i$ for $p_i \leq p_{-i}$ in the reduced form of the game is given as follows:

$$R_{-i}^{\mathrm{U}}(r_i^{U,D}(p_{-i}), p_{-i})) = \begin{cases} (v - p_{-i})p_{-i} & \text{for } p_{-i} \in [0, \frac{2v}{3}], \\ (v - \frac{p_{-i}+2v}{4})(\frac{(p_{-i}+2v}{2} - p_{-i}) & \\ +(v - p_{-i})(2p_{-i} - \frac{p_{-i}+2v}{2}) & \text{for } p_{-i} \in (\frac{2v}{3}, v]. \end{cases}$$

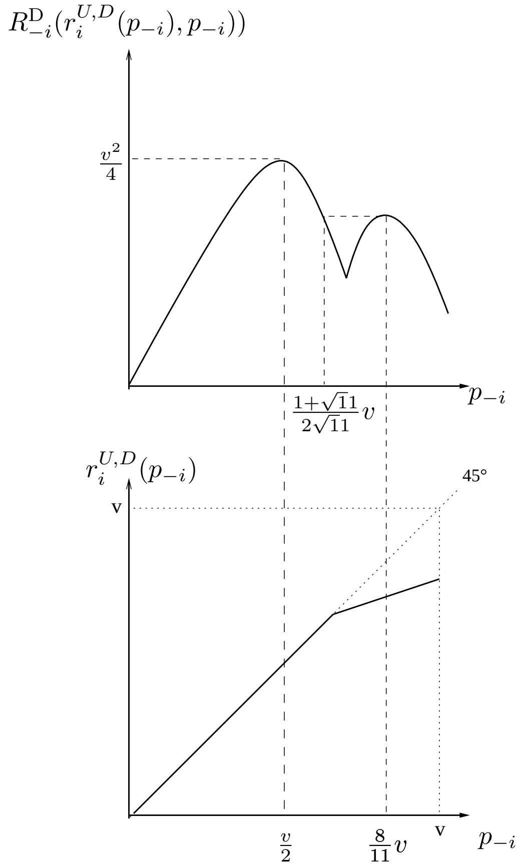

Figure 4.6: *The descending clock auction*

See figure 4.6 for an illustration. The first term takes its maximum at $p_{-i} = \frac{v}{2}$ and the second term at $p_{-i} = \frac{8v}{11}$ (see Appendix 4.A for this computation). For these values of p_{-i} one obtains:

$$r_i^{U,D}(\frac{v}{2}) = \frac{v}{2}; \quad r_i^{U,D}(\frac{8v}{11}) = \frac{15v}{22}.$$

The payoffs of bidder $-i$ are then

$$R^{\mathrm{U}}_{-i}(\frac{v}{2}, \frac{v}{2}) = \frac{v^2}{4}; \quad R^{\mathrm{U}}_{-i}(\frac{15v}{22}, \frac{8v}{11}) = \frac{5v^2}{22}.$$

Now the inequality

$$R^{\mathrm{U}}_{-i}(x, x) \geq R^{\mathrm{U}}_{-i}(\frac{15v}{22}, \frac{8v}{11}) \Leftrightarrow (v - x)x \geq \frac{5v^2}{22} \tag{4.13}$$

is easily shown to be satisfied for $x \in [\frac{v}{2}, \frac{1+\sqrt{11}}{2\sqrt{11}}v]$ and not to hold for $x > \frac{1+\sqrt{11}}{2\sqrt{11}}v$ (see again figure 4.6 for an illustration). We will prove now the first part of the claim:

a) In the reduced form of the game exiting at the same price

$$p_i^{U,D} = p_{-i}^{U,D} = x \in [\frac{v}{2}, \frac{1+\sqrt{11}}{2\sqrt{11}}v]$$

constitutes an equilibrium profile.

b) No other equilibria exist.

We begin with part a). First, if $p_{-i}^{U,D} = x$, submitting any price $p_i \leq x$ leads for bidder i to the same payoff, since in this case the information that bidder $-i$ stopped the clock at x will optimally require that bidder i also stops the clock at that very same price (see the first line of expression 4.12). Second, submitting a price $p_i > x$ leads to a strictly lower payoff for that bidder:

- If bidder i exits at a higher price: $p_i \in (x, \frac{1+\sqrt{11}}{2\sqrt{11}}v]$ bidder $-i$ will exit also at the same price (see again the first line of expression 4.12). Bidder i obtains a lower payoff since the function $\mathbb{R}_i^U(p_i, p_i)$ is monotonically decreasing in the above interval.
- The inequality (4.13) holds for $x \in (\frac{v}{2}, \frac{1+\sqrt{11}}{2\sqrt{11}}v]$, so bidder i does not increase his payoff by exiting first at any price $p_i > \frac{(1+\sqrt{11})}{2\sqrt{11}}v$.

These arguments prove that the best responses of the players in the interval $[\frac{v}{2}, \frac{1+\sqrt{11}}{2\sqrt{11}}v]$ are as shown in figure 4.7.

Let us prove now part b). First, an equilibrium for which $p_{-i}^{U,D} \in [0, \frac{v}{2})$ obviously does not exist, since bidder i is better off with the strategy $\frac{v}{2}$ than with any strategy from the interval $[0, p_{-i}^{U,D})$. See figure 4.7 for the best responses in the interval $[0, \frac{v}{2})$. Second, an equilibrium for which $p_{-i}^{U,D} \in (\frac{1+\sqrt{11}}{2\sqrt{11}}v, \frac{8v}{11})$ does not exist either. In this case the best response of bidder i is submitting $\frac{8v}{11}$ and thus defining the higher bid (see figure 4.7). Third, a strategy in which bidder $-i$ submits $p_{-i}^{U,D} \in [\frac{8v}{11}, v)$ also cannot be part of an equilibrium. In this case we show that bidder i will be better off submitting a bid of $p_i = p_{-i}^{U,D} + \epsilon$, if $\epsilon > 0$ is small enough instead of $p_i^{U,D} \leq p_{-i}^{U,D}$. Indeed for $p_{-i}^{U,D} \in (\frac{2v}{3}, v)$ the following (in)equalities apply

$$R_{-i}^{\mathrm{U}}(r_i^{U,D}(p_{-i}^{U,D}), p_{-i}^{U,D})) > R_i^{\mathrm{U}}(r_i^{U,D}(p_{-i}^{U,D}), p_{-i}^{U,D})),$$
$$\lim_{\epsilon \longrightarrow 0} R_i^{\mathrm{U}}((p_i^{U,D} + \epsilon), r_{-i}^{U,D}(p_i^{U,D} + \epsilon)) = R_{-i}^{\mathrm{U}}(r_i^{U,D}(p_{-i}^{U,D}), p_{-i}^{U,D})).$$

Therefore there exists ϵ enough small such that for $p_i = p_{-i}^{U,D} + \epsilon$ one obtains

$$R_i^{\mathrm{U}}(p_i, r_{-i}^{U,D}(p_i)) > R_i^{\mathrm{U}}(r_i^{U,D}(p_{-i}^{U,D}), p_{-i}^{U,D})),$$

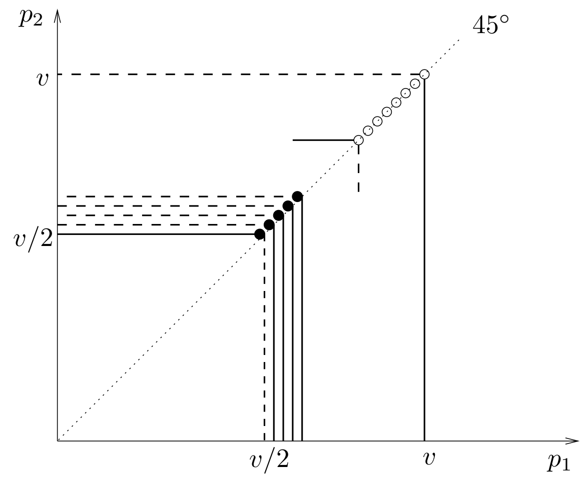

Figure 4.7: *Best responses and equilibria in the reduced descending clock auction game: The solid lines represent the best response of bidder 2 and the dash lines the best response of bidder 1. The thick dots denote the (subgame perfect) equilibrium bids. They lie in the interval* $[\frac{v}{2}, \frac{1+\sqrt{11}}{2\cdot\sqrt{11}}v]$. *In the interval* $[\frac{8v}{11}, v)$ *the best responses do not exist (see the hollow dots).*

a profitable deviation from the assumed equilibrium strategy $p_i^{U,D} \leq p_{-i}^{U,D}$. In fact in this interval the best response of bidder i is not defined since he is willing to bid higher, but still as close as possible to the bid of bidder $-i$. In figure 4.7 this is represented by the hollow dots. Finally, submitting the price $p_{-i}^{U,D} = v$ cannot be an equilibrium bid either. The best response of bidder i is to bid any $p_i < v$ (see figure 4.7).

Now it remains to show the second part of the claim, namely that the equilibria $p_i^{U,D} = p_{-i}^{U,D} = x \in (\frac{v}{2}, \frac{1+\sqrt{11}}{2\cdot\sqrt{11}}v]$ involve the use of weakly dominated strategies. The payoffs of these equilibria are given as follows:

$$R_i^{\mathrm{U}}(p_{-i}, p_{-i}) = \begin{cases} (v - p_{-i})^2 & \text{for } p_{-i} \in [0, \frac{2v}{3}], p_i \leq p_{-i}, \\ (v - p_i)^2 & \text{for } p_i \in [0, \frac{2v}{3}], p_i > p_{-i}. \end{cases}$$

As the function $(v - p_i)^2$ is monotonically decreasing in the interval $[\frac{v}{2}, \frac{2v}{3}]$ it follows that the strategy $p_i = \frac{v}{2}$ weakly dominates the strategies from the interval $(\frac{v}{2}, \frac{2v}{3}]$. The claim follows. □

4.3.3 Discriminatory pricing (D)

As has already been argued, each bidder's payoff in the bidding stage of the discriminatory auction is independent of the submitted bids of the other bidders, since each bidder with bid price above the marginal costs c will be served by the seller. Then, independent on the way how the seller collects the bids, every bidder i will choose his bid so as to maximize

$$R_i^{\mathrm{D}}(x_i, x_{-i}) = (v - x_i)x_i.$$

The function x_i reaches a maximum at $x_i = \frac{v}{2}$, which is a dominant strategy for each bidder.

4.4 Revenues, average trade volume and efficiency

Figure 4.8 illustrates graphically the equilibria in all auction formats. In this section we will define and compare the expected revenue of the seller, the average trade volume as well as the efficiency measure in the equilibria of these auctions. Not surprisingly they will be ordered in the same manner as the equilibrium bids are ordered.

Definition 4.1 (seller's ex ante revenue)**.** *The expected revenue of the seller in the uniform price auction (U) and the discriminatory auction (D), in which the bidders submit bids* (τ_1, τ_2) *such that* $\tau_1 \leq 2\tau_2$[10] *is given as follows.*
Absolute measure:

$$R_{S,a}^{\mathrm{U}}(\tau_1, \tau_2) := 2 \cdot \int_0^{2\tau_2 - \tau_1} (\tau_2 - c) dF(c) + \int_{2\tau_2 - \tau_1}^{\tau_1} (\tau_1 - c) dF(c),$$
$$R_{S,a}^{\mathrm{D}}(\tau_1, \tau_2) := 2 \cdot \int_0^{\tau_2} (\tau_2 - c) dF(c) + \int_{\tau_2}^{\tau_1} (\tau_1 - c) dF(c).$$

Relative measure:

$$R_{S,r}^{\mathrm{U}}(\tau_1, \tau_2) := \frac{R_{S,a}^{\mathrm{U}}(\tau_1, \tau_2)}{2\int_0^v (v - c) dF(c)},$$
$$R_{S,r}^{\mathrm{D}}(\tau_1, \tau_2) := \frac{R_{S,a}^{\mathrm{D}}(\tau_1, \tau_2)}{2\int_0^v (v - c) dF(c)}.$$

The relative measure is defined as the ratio between the absolute (ex ante) expected revenue and the revenue, which the seller would realize if she extracted the whole consumer surplus (i.e she sells two units at a price of v whenever $c \leq v$).

Definition 4.2 (absolute and relative average trade volume)**.** *The average trade volume in the uniform price (U) and the discriminatory (D) auction, in which the bidders*

[10]The inequality always holds for the equilibria of all auction formats.

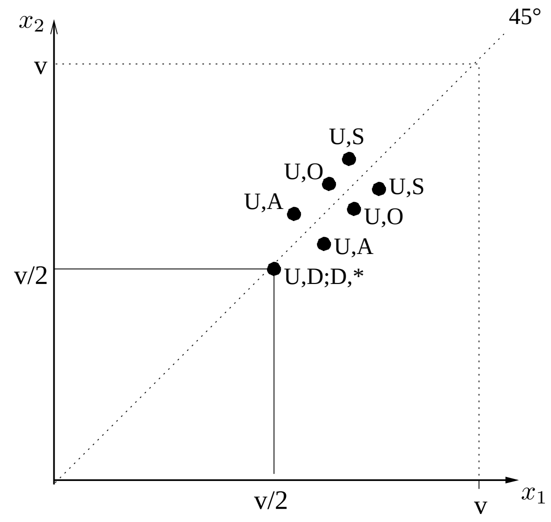

Figure 4.8: *Equilibrium bids in the analyzed trade mechanisms. D,* stands for all four bidding procedures with discriminatory pricing. In the descending sealed bid auction we selected the non-dominated strategies in the collapsed game* $(\frac{v}{2}, \frac{v}{2})$.

submitted the bids (τ_1, τ_2) *such that* $\tau_1 \leq 2\tau_2$ *is given as follows.*
Absolute measure:

$$Q_a^U(\tau_1, \tau_2) = 2 \cdot F(2\tau_2 - \tau_1) + F(\tau_1 - (2\tau_2 - \tau_1)),$$
$$Q_a^D(\tau_1, \tau_2) = 2 \cdot F(\tau_2) + F(\tau_1 - \tau_2).$$

Relative measure:

$$Q_r^U(\tau_1, \tau_2) = Q_a^U(\tau_1, \tau_2)/2 \cdot F(v),$$
$$Q_r^D(\tau_1, \tau_2) = Q_a^D(\tau_1, \tau_2)/2 \cdot F(v).$$

The relative average trade volume of an auction game relates its average trade volume to the trade volume which would result, if all the trades desirable ex post (i.e. whenever $c \leq v$) have taken place.

Definition 4.3 (absolute and relative efficiency)**.** *The absolute efficiency in the uniform price and the discriminatory auctions, in which the bidders submitted the bids* τ_1, τ_2 *such that* $\tau_1 \leq 2\tau_2$ *is defined as the sum of the ex ante consumer and producer surplus. The relative efficiency measures the relation of the absolute efficiency to the sum of the buyers' and seller's ex ante surplus, which would result if trade takes place always when desirable ex-post.*

Absolute:

$$E_a^U(\tau_1,\tau_2) = 2\cdot\int_0^{2\tau_2-\tau_1}(v-c)dF(c) + \int_{2\tau_2-\tau_1}^{\tau_1}(v-c)dF(c)$$
$$E_a^D(\tau_1,\tau_2) = 2\cdot\int_0^{\tau_2}(v-c)dF(c) + \int_{\tau_2}^{\tau_1}(v-c)dF(c)$$

Relative:

$$E_r^U(\tau_1,\tau_2) = E_a^U(\tau_1,\tau_2)/2\cdot\int_0^{v}(v-c)dF(c).$$
$$E_r^D(\tau_1,\tau_2) = E_a^D(\tau_1,\tau_2)/2\cdot\int_0^{v}(v-c)dF(c).$$

This measure of efficiency is a standard measure, which is surely weaker than the efficiency in the Pareto sense. It is however a sensible criterium according to which all the analyzed auction forms can be ordered. The auctions are not comparable in the Pareto sense, since the auction forms leading to higher bids increase the seller's average revenue but lowers the bidders' payoffs.

The relative efficiency measure is defined as the ratio between the equilibrium efficiency and the full efficiency - the efficiency measure in the case in which trade always takes place when desirable ex post (i.e. always when $c \le v$).

Claim 4.5. *The performance of each auction format in terms of the criteria defined above is summarized in table 4.2 below.*

	Trade mechanisms				
	U,S	**U,O**	**U,A**	**U,D**	**D,***
Equilibrium bids	$(\frac{10v}{13},\frac{9v}{13})$	$(\frac{12v}{16},\frac{11v}{16})$	$(\frac{7v}{10},\frac{6v}{10})$	$(\frac{v}{2},\frac{v}{2})$	$(\frac{v}{2},\frac{v}{2})$
Ex ante revenue (absolute)	$\frac{82}{169}v^2$	$\frac{61}{128}v^2$	$\frac{37}{100}v^2$	$\frac{25}{100}v^2$	$\frac{25}{100}v^2$
Ex ante revenue (relative)	$\frac{82}{169}$	$\frac{61}{128}$	$\frac{37}{100}$	$\frac{25}{100}$	$\frac{25}{100}$
Ex ante revenue (relative %)	$\approx 48,52\%$	$\approx 47,66\%$	$37,00\%$	$25,00\%$	$25,00\%$
Average volume (absolute)	$\frac{18v}{13}$	$\frac{22v}{16}$	$\frac{12v}{10}$	v	v
Average volume (relative)	$\frac{9}{13}$	$\frac{11}{16}$	$\frac{6}{10}$	$\frac{1}{2}$	$\frac{1}{2}$
Average volume (relative %)	$\approx 69,23\%$	$68,75\%$	$60,00\%$	$50,00\%$	$50,00\%$
Efficiency (absolute)	$\frac{152}{169}v^2$	$\frac{115}{128}v^2$	$\frac{83}{100}v^2$	$\frac{75}{100}v^2$	$\frac{75}{100}v^2$
Efficiency (relative)	$\frac{152}{169}$	$\frac{115}{128}$	$\frac{83}{100}$	$\frac{75}{100}$	$\frac{75}{100}$
Efficiency (relative %)	$\approx 89,94\%$	$\approx 89,84\%$	$83,00\%$	$75,00\%$	$75,00\%$

Table 4.2: *Performance of trade mechanisms*

Proof. The *absolute ex ante revenues* in the uniform and discriminatory auctions are:

$$\begin{aligned} R^{\mathrm{U}}_{S,a}(\tau_1,\tau_2) &= 2\big[\tau_2(2\tau_2-\tau_1) - \frac{(2\tau_2-\tau_1)^2}{2}\big] + \tau_1(2\tau_1-2\tau_2) - \frac{(\tau_1)^2}{2} + \frac{(2\tau_2-\tau_1)^2}{2} \\ &= (\tau_1)^2 - 2\tau_1\cdot\tau_2 + (\tau_2)^2, \end{aligned}$$

$$\begin{aligned} R^{\mathrm{D}}_{S,a}(\tau_1,\tau_2) &= 2(\tau_2\cdot\tau_2 - \frac{(\tau_2)^2}{2}) + \tau_1(\tau_1-\tau_2) - \frac{(\tau_1)^2}{2} + \frac{(\tau_2)^2}{2} \\ &= \frac{3}{2}\cdot(\tau_2)^2 - \tau_1\cdot\tau_2 + \frac{(\tau_2)^2}{2}. \end{aligned}$$

For the *relative measures* of the ex ante revenue one obtains

$$\begin{aligned} R^{\mathrm{U}}_{S,r}(\tau_1,\tau_2) &= \frac{R^{\mathrm{U}}_{S,a}(\tau_1,\tau_2)}{v^2}, \\ R^{\mathrm{D}}_{S,r}(\tau_1,\tau_2) &= \frac{R^{\mathrm{D}}_{S,a}(\tau_1,\tau_2)}{v^2}. \end{aligned}$$

For the *absolute and the relative measure of the average trade volume* in both auctions one obtains:

$$\begin{aligned} Q^U_a(\tau_1,\tau_2) &= 2\cdot(2\tau_2-\tau_1) + (\tau_1 - (2\tau_2-\tau_1)) = 2\tau_2, \\ Q^D_a(\tau_1,\tau_2) &= 2\cdot\tau_2 + \tau_1 - \tau_2 = \tau_1+\tau_2, \\ Q^U_r(\tau_1,\tau_2) &= Q^U_a(\tau_1,\tau_2)/2v, \\ Q^D_r(\tau_1,\tau_2) &= Q^D_a(\tau_1,\tau_2)/2v. \end{aligned}$$

The *efficiency measures* in both auction formats are computed as follows.
Absolute:

$$\begin{aligned} E^U_a(\tau_1,\tau_2) &= 2\tau_2(2\tau_2-\tau_1) - (2\tau_2-\tau_1)^2 + \tau_1(2\tau_1-2\tau_2) - \frac{1}{2}[\tau_1^2 - (2\tau_2-\tau_1)^2] \\ &= 2v\cdot\tau_2 + 2\tau_1\cdot\tau_2 - \tau_1^2 - 2\tau_2^2 = 2\tau_2(v+\tau_1-\tau_2) - \tau_1^2, \end{aligned}$$

$$\begin{aligned} E^D_a(\tau_1,\tau_2) &= 2(v\cdot\tau_2 - \frac{(\tau_2)^2}{2}) + v(\tau_1-\tau_2) - \frac{(\tau_1)^2}{2} + \frac{(\tau_2)^2}{2} \\ &= v(\tau_1+\tau_2) + \frac{(\tau_1)^2}{2} - \frac{(\tau_2)^2}{2}. \end{aligned}$$

Relative:

$$\begin{aligned} E^U_r(\tau_1,\tau_2) &= E^U_a(\tau_1,\tau_2)/v^2, \\ E^D_r(\tau_1,\tau_2) &= E^D_a(\tau_1,\tau_2)/v^2. \end{aligned}$$

With a direct substitution with the equilibrium bids one obtains the results in table 4.1. □

4.5 Concluding remarks

The simple model formulation of this chapter allows us to accommodate in our analysis alongside the sealed bid uniform price and discriminatory auctions, which we discussed in the last chapter, the analysis of several other standard procedures for collecting bids.

Equilibria are derived for the uniform and the discriminatory pricing rules under four procedures for collecting bids: sumutaneous, sequential, via an ascending clock and via a descending clock (see figure 4.8 and table 4.1 for summary of the results). The analysis provides several new insights. We show that the sealed bid uniform price auction is superior to the ascending and descending clock uniform price auctions, as well as to the procedure of sequential collection of the bids. Thus, contrary to the prevailing wisdom from single unit auctions, using an ascending auction and thus revealing more information about the bidding (or price formation) process is not of benefit to the seller[11]. The use of an ascending auction, which is usually associated with higher information revelation, just like the policy of making public the already submitted bids, turns out not only to be detrimental for the proceeds of the seller, but also to lead to lower average trade volume and be more inefficient. These findings further imply that the basic results from the single unit auction literature deserve a closer scrutiny when applying to multi-unit auction contexts. Failure of the linkage principle in a multi-unit environment has been discussed by Perry and Reny (1999) within a simple two-bidder, two-unit Vickrey auction. However, in the model presented here the weaker performance of the ascending auction cannot really be accrued to a "failure" of the linkage principle. The linkage principle stems from the interaction among bidders withholding private information about the common value of the auctioned good. Here we assumed that bidders are equally informed about the expected value of the items for sale, so in this model the bidders do not withhold private information and the linkage effect does not exist. We focused on the bidders' reaction to supply uncertainty under different pricing rules and procedures for collecting bids. The ascending clock uniform price auction and the uniform price auction with sequential collection of bids are shown to create strategy space and payoffs, which weaken the competition between bidders compared to the sealed bid format. The descending auction creates even less

[11] In single-unit auctions with affiliated signals and a common value element the ascending auction brings higher revenues for the seller than the second-price auction, which on its turn outperforms the first-price auction. This result is due to the revenue ranking (or linkage) principle, which was uncovered and used by Wilson (1977), Milgrom (1981) and Milgrom and Weber (1982) in nowadays influential papers.

competition. In the discriminatory price auction variants bidders do not compete with each other at all, since the payoffs of each bidder is independent of the bids submitted by the other bidder.

The conducted comparison is subject to several limitations. We focused only on the two-bidder case and assumed that the marginal costs c are uniformly distributed. In a more general model one could consider the $n-$bidder case and an arbitrary distribution function $F(c)$. The latter modification however would substantially complicate the analysis even in the $2-$bidder case. For this more general scenario the existence of equilibrium in the sealed bid uniform price auction cannot be guaranteed due to the non-convexity of the payoff function and the resulting discontinuity of the best responses in the collapsed game (see figure 4.2). Existence of equilibrium can be guaranteed in mixed strategies, however little can be said about the nature of these equilibria and how they compare to the equilibria of the other auction forms. The subgames at prices $\{p_{-i}\}$ in the ascending and the descending clock auction might have multiple solutions, which complicates additionally a possible comparison. The "no-proprietary information" assumption and the constant marginal costs assumption are also significant for the results. Relaxing the first assumption would require introducing incomplete information also among bidders. This scenario does not seem tractable in this complex setting. Such an assumption will undo the dominant strategy equilibrium in the discriminatory auctions. The same effect would induce an assumption of increasing marginal costs. In such a setting the payoff of each bidder in the collapsed games will depend on the strategies of the other bidders and exhibit discontinuities, thus causing equilibrium nonexistence problems even in mixed strategies.

The paper presents a first attempt to compare the performance of static and dynamic multi-unit auctions, in which the seller takes part in the price-setting process. In single unit auctions seller's intervention in the bidding process has been discussed in the literature on ascending auctions primarily as an instrument against collusion. Graham and Marshall (1987) explore the participation of the seller in the English thermometer (or Japanese) auction in the presence of a bidding ring[12]. In their model the auctioneer enters the auction at the moment in which the thermometer stops; then she bids together with the last active bidder, who participates possibly on behalf of the bidding ring. The intervention of the seller here is of related nature: the seller reduces

[12]See this work also for a discussion of the use of reserve prices as an instrument against collusion in second-price auctions.

the supply quantity or cancels the auction if bids are unsatisfactory, thus influencing bidder's behavior on the bidding stage.

The present analysis, although conducted within a simple framework, presents a thorough study of the traditional procedures for collecting bids for the uniform and the discriminatory pricing rules. The results cast light on the impact of the used auction mechanism on the equilibrium bids and its consequences for the expected revenue for the seller as well as the average trade volume and the efficiency of the equilibrium allocation.

Appendix 4.A

Proof of Lemma 5.1

Problem (A):

The payoff of bidder i has the form

$$R_i^{\mathrm{U}}(x_i, x_{-i}) = \begin{cases} (v - x_i)x_i & \text{for } x_i > 2x_{-i}, \\ (v - x_i)\big(2x_i - 2x_{-i}\big) + (v - x_{-i})(2x_{-i} - x_i) & \text{for } 2x_{-i} \geq x_i > x_{-i}. \end{cases}$$

The partial derivative is

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) = \begin{cases} v - 2x_i & \text{for } x_i > 2x_{-i}, \\ -4x_i + 3x_{-i} + v & \text{for } 2x_{-i} \geq x_i > x_{-i}. \end{cases}$$

The term in the first line reaches unique maximum at $x_i^{(1)}(x_{-i}) = v/2$ and the term in the second line at $x_i^{(2)}(x_{-i}) = (3x_{-i} + v)/4$. The payoffs in both cases are

$$\begin{aligned} R_i^{\mathrm{U}}(x_i^{(1)}(x_{-i}), x_{-i}) &= \frac{v^2}{4}, \\ R_i^{\mathrm{U}}(x_i^{(2)}(x_{-i}), x_{-i}) &= \big(v - \frac{3x_{-i}}{4}\big)\big(\frac{3x_{-i} + v}{2} - 2x_{-i}\big) + \big(v - x_{-i}\big)\big(2x_{-i} - \frac{3x_{-i} + v}{4}\big) \\ &= \frac{(3v - 3x_{-i})(v - x_{-i}) + (2v - 2x_{-i})(5x_{-i} - v)}{8} \\ &= \frac{-7x_{-i}^2 + 6 \cdot v \cdot x_{-i} - v^2}{8} \end{aligned}$$

Solving the quadratic inequations

$$\frac{-7x_{-i}^2 + 6 \cdot v \cdot x_{-i} - v^2}{8} \gtreqless \frac{v^2}{4}$$

one obtains

$$\begin{aligned} &R_i^{\mathrm{U}}(x_i^{(2)}(x_{-i}), x_{-i}) > R_i^{\mathrm{U}}(x_i^{(1)}(x_{-i}), x_{-i}) \quad \text{for} \quad x_{-i} \in \left(\frac{(3 - \sqrt{2})v}{7}, \frac{(3 + \sqrt{2})v}{7} \right), \\ &R_i^{\mathrm{U}}(x_i^{(2)}(x_{-i}), x_{-i}) = R_i^{\mathrm{U}}(x_i^{(1)}(x_{-i}), x_{-i}) \quad \text{for} \quad x_{-i} \in \left\{ \frac{(3 - \sqrt{2})v}{7}, \frac{(3 + \sqrt{2})v}{7} \right\}, \\ &R_i^{\mathrm{U}}(x_i^{(2)}(x_{-i}), x_{-i}) < R_i^{\mathrm{U}}(x_i^{(1)}(x_{-i}), x_{-i}) \quad \text{otherwise.} \end{aligned}$$

This leads us to the desired result[13]

$$x_i^A(x_{-i}) = \begin{cases} \frac{1v}{2} & \text{for } x_{-i} \in (0, \frac{(3-\sqrt{2})v}{7}), \\ \{\frac{(16-3\sqrt{2})v}{28}, \frac{1v}{2}\} & \text{for } x_{-i} = \frac{(3-\sqrt{2})v}{7}, \\ \frac{3x_{-i}+v}{4} & \text{for } x_{-i} \in (\frac{(3-\sqrt{2})v}{7}, v). \end{cases}$$

[13] Observe that $\frac{(3+\sqrt{2})v}{7} > \frac{v}{2}$, so that for $x_{-i} \geq \frac{(3+\sqrt{2})v}{7}$ the term in the first line applies

Problem (B):

The payoff of bidder i takes the form

$$R_i^{\mathrm{U}}(x_i, x_{-i}) = \begin{cases} (v - x_i)(2x_i - x_{-i}) & \text{for } x_i > x_{-i}/2, \\ 0 & \text{otherwise.} \end{cases}$$

For the partial derivative of the term in the first line one obtains:

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) = -4x_i + x_{-i} + 2v.$$

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) = 0 \Leftrightarrow x_i(x_{-i}) = (x_{-i} + 2v)/4.$$

Since $(x_{-i} + 2v)/4 > x_{-i}$ for $x_{-i} \in [0, \frac{2v}{3})$ one obtains the desired result

$$x_i^B\,(x_{-i}) = \begin{cases} x_{-i} & \text{for } x_{-i} \in [0, \frac{2v}{3}], \\ \frac{x_{-i}+2}{4} & \text{for } x_{-i} \in (\frac{2v}{3}, v]. \end{cases}$$

Problem (C):

The payoff of bidder i in problem (B) is given as follows:

$$R_i^{\mathrm{U}}(x_i^B\,(x_{-i})\,, x_{-i}) = \begin{cases} (v - x_{-i})x_{-i} \quad \text{for } x_{-i} \in (0, \frac{2v}{3}], \\ (v - \frac{x_{-i}+2v}{4})(\frac{x_{-i}+2v}{2} - x_{-i}) = (2v - x_{-i})^2/8 \quad \text{for } x_{-i} \in (\frac{2v}{3}, v]. \end{cases}$$

Solving the quadratic inequations

$$(2v - x_{-i})^2/8 \gtreqless (-7x_{-i}^2 + 6 \cdot v \cdot x_{-i} - v^2)/8$$

one obtains

$$\begin{aligned} R_i^{\mathrm{U}}(x_i^A\,(x_{-i})\,, x_{-i}) &> R_i^{\mathrm{U}}(x_i^B\,(x_{-i})\,, x_{-i}) \quad \text{for} \quad x_{-i} \in \big[0, \frac{3v}{4}\big), \\ R_i^{\mathrm{U}}(x_i^A\,(x_{-i})\,, x_{-i}) &= R_i^{\mathrm{U}}(x_i^B\,(x_{-i})\,, x_{-i}) \quad \text{for} \quad x_{-i} = \frac{3v}{4}, \\ R_i^{\mathrm{U}}(x_i^A(x_{-i}), x_{-i}) &< R_i^{\mathrm{U}}(x_i^B(x_{-i}), x_{-i}) \quad \text{for} \quad x_{-i} \in \big(\frac{3v}{4}, v\big], \end{aligned}$$

which is the desired result of problem (C).

Sequential collection of bids:

For $x_1 \in (\frac{(3-\sqrt{2})v}{7}, \frac{3v}{4})$ the function $R_1^{\mathrm{U}}(x_1, \varphi_2^{U,O}(x_1)) = (v - x_1)(5x_1 - v)/4$ reaches a maximum at $x_1 = \frac{6v}{10}$. For $x_1 \in [3v/4, v]$ one obtains $R_1^{\mathrm{U}}(x_1, \varphi_2^{U,O}(x_1)) = \frac{(v-x_1)(3x_1+2v)}{2} +$

$\frac{(2v-x_1)^2}{8}$ and it is easy to show that $\frac{dR_1^{\mathrm{U}}(x_1,\varphi_2^{U,O}(x_1))}{d(x_1)} < 0$. The function reaches a maximum at $x_1 = 3v/4$. The (in)equalities

$$R_1^{\mathrm{U}}(\frac{12v}{16}, \frac{11v}{16}) = \frac{58}{256}v^2 > \frac{1}{5}v^2 = R_1^{\mathrm{U}}(\frac{6v}{10}, \frac{7v}{10})$$

verify that $x_1 = 3v/4$ is a global maximum.

The descending clock auction:

For $p_{-i} \in (\frac{2v}{3}, v]$ one obtains

$$\begin{aligned}\mathbb{R}_{-i}^{D}(r_i^{U,D}(p_{-i}), p_{-i})) &= (v - \frac{p_{-i}+2v}{4})(\frac{(p_{-i}+2v}{2} - p_{-i}) + (v-p_{-i})(2p_{-i} - \frac{p_{-i}+2v}{2}) \\ &= (-11p_i^2 + 16vp_i - 4v^2)/8.\end{aligned}$$

It is now easy to see that the quadratic expression reaches a maximum at $p_{-i} = \frac{8v}{11}$. One obtains

$$\mathbb{R}_{-i}^{D}(r_i^{U,D}(\frac{8v}{11}), \frac{8v}{11})) = \frac{5v}{22}.$$

Chapter 5

Auctions with Variable Supply and the Walrasian Outcome

5.1 Introduction

The models presented in chapters 3 and 4 analyze an auction setting in which bidders submit a bid price for a single unit. The multi-unit (or divisible good) applications discussed in chapter 2 however are multi-unit auctions in a double sense. In these auctions both the seller supplies multiple units of a homogeneous good and the bidders submit bids for multiple units and can win more than one unit. The purpose of this chapter is to address this shortcoming by proposing a new model, which contains the quantity dimension missing in the previous two chapters. This extension will not be without a price. Whereas in the previous models the (constant) marginal costs were stochastic, in the current model there will be no uncertainty. The present model will have a more complex bidder strategy space, but will generally analyze games of complete information since the (weakly increasing) supply function of the auctioneer will be assumed to be common knowledge. Whereas we will again be interested in the subgame perfect equilibria of auctions with variable supply, the discussion will be framed around another point. We will ask the question of whether there are auction mechanisms, the subgame perfect equilibrium outcomes of which produce the Walrasian allocation and lead to efficient trade. A set of pricing and rationing rules will be outlined, which single out the Walrasian allocation as the only possible allocation arising in a subgame perfect equilibrium. The analysis will again focus on variable (or endogenous) supply multi-unit auctions – trade mechanisms in which the seller first collects price-quantity orders (or bids) and then decides on a supply quantity depending on the bidding. In this model the bidders will be allowed to submit a single price-quantity pair. We will argue however, that the results remain valid even if one allows bidders to submit multiple price-quantity pairs or whole (left-continuous) demand functions. The restriction to a single price quantity pair is not essential and is made for expositional clarity and in order to bring the analysis in line with the literature on competitive market games, which will be summarized here.

5.1.1 Relation to monopoly price discrimination

In his seminal work, Pigou (1920) classified the three types of monopolistic price discrimination. The perfect, or first degree, price discrimination allows the producer to capture the entire social surplus and trade is Pareto efficient. Perfect discrimination is however unlikely in practice, as it requires complete information about individual preferences, which sellers generally do not possess. Eliciting the private information of potential customers is often accomplished via auction mechanisms, in which buyers are invited to declare purchase quantities and corresponding prices. These reports, although strategically chosen, facilitate the monopolist in setting an appropriate production quantity and selling this quantity to the consumers.

Here we model these forms of trade as two-stage games and analyze their equilibrium allocations. In the first stage of the model, which we will again call the bidding stage, buyers simultaneously submit price-quantity pairs to the auctioneer. In the second stage the seller determines the profit-maximizing supply quantity. Trade is conducted in accordance with the auction rules, which specify how the supply quantity will be divided among the buyers and the payments due.

Our analysis uncovers several parallels to Pigou's (1920) discussion on monopolistic trade. We demonstrate that just like first degree price discrimination trade with the discriminatory auction is efficient[1]; the producer loses however his monopolistic power. In the only subgame perfect equilibrium outcome the Walrasian quantities are traded at the Walrasian price as in the perfect competition model (see figure 5.1 for an illustration of our results). Buyers receive a consumer rent, which is the tribute that the seller has to pay to resolve his information problem. The uniqueness of the equilibrium outcome arises out of the discriminatory rule of the auction. We show by means of an example

[1]In this formulation bids will be rationed according to the standard "pro rata on the margin" rationing rule.

that trade with the uniform price auction can lead to inefficient equilibrium allocations as in the case of simple or non-discriminatory monopoly. The inefficiency arises because one of the bidders can alter the price away from the competitive level and still be served by the seller.

This inefficiency can be resolved, if the seller employs the "pro rata" rationing rule and adheres to a simple consistency rule, when selecting among several profit-maximizing quantities. We show that under these premises the uniform price auction also leads to the Walrasian allocation. The presented models provide a strategic foundation of the competitive equilibrium paradigm.

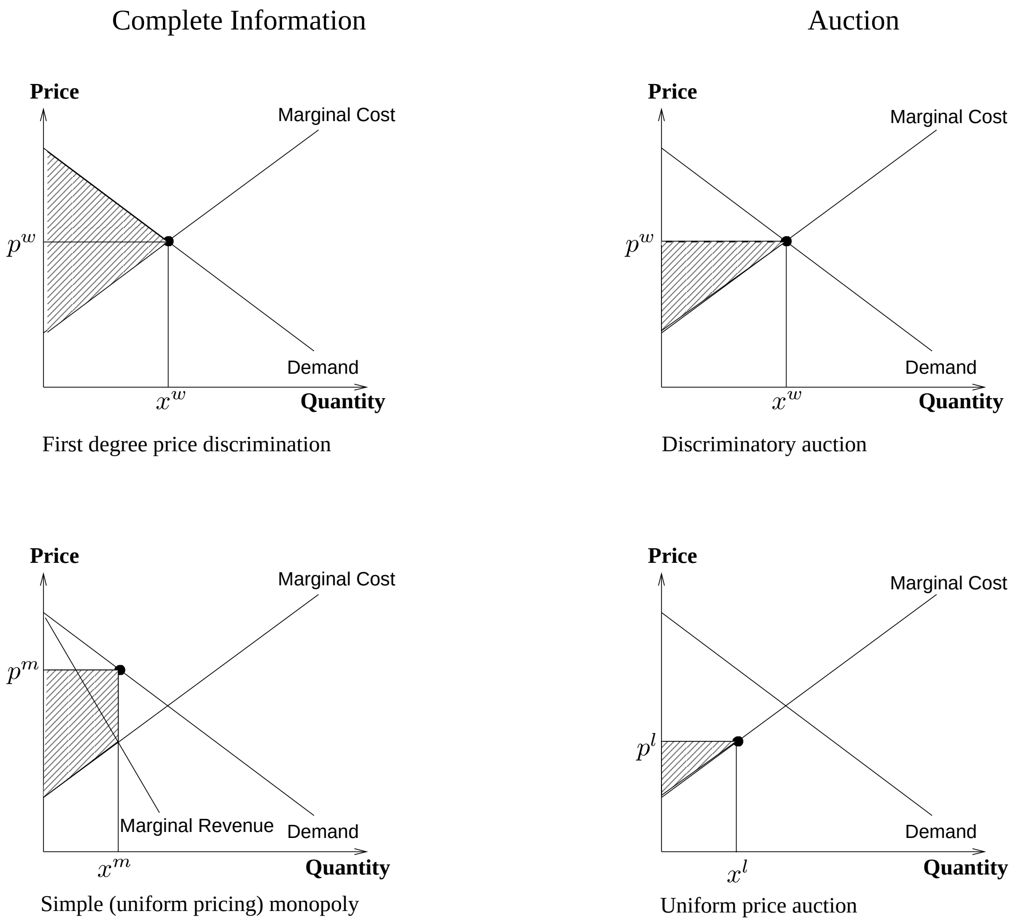

Figure 5.1: *Notation: (p^w, x^w)–Walrasian price and quantity; (p^m, x^m)–allocation in the simple monopoly case; (p^l, q^l)–non Walrasian (low price) equilibrium in the uniform price auction with the "pro rata on the margin" rationing rule. The covered areas represent the seller's profit.*

5.1.2 Relation to the literature on competitive market games

As we present here market games with Walrasian outcomes our analysis contributes to the already existing vast literature on the strategic approach to the competitive equilibrium.

The competitive market model has two very desirable properties. First, trade leads to a Pareto efficient allocation (first theorem of welfare economics) and second, agents need just information about their own preferences in order to quote optimal trading quantities. However, this traditional model is questioned from several directions. One observes, for example, that the market agents are completely excluded from the price formation process, since as price takers they choose only trading quantities. The question of how prices are determined in equlibrium when all agents quote only quantities remains unexplained, unless one imagines an auctioneer, who performs the task of finding a set of prices that clear all markets. In reality, however, there is no auctioneer, except in certain special markets. The price-taking assumption can be questioned also from another viewpoint. It readily captures situations, in which agents are negligibly small relative to the size of the market, but is inappropriate for markets, in which the demand or the supply side is represented only by one or several big traders. These traders should be modelled as participants in the price formation process and the final allocaton should arise out of the interplay between the market position and information of the players.

Extending the competitive analysis to market environments, in which the agents are not price-takers, is a recurrent and still evolving topic. One major line of research, to which this work also belongs, is the *strategic* or *noncooperative* approach. Its aim is to model the functioning of a market as a strategic market game, determine the appropriate noncooperative equilibria and identify the circumstances under which they lead to the desired Walrasian outcome.

The literature on the *strategic* approach to the competitive equilibrium[2] can generally be divided in two categories. The first one (see, e.g., Benassy (1986) , Dubey (1980), Hurwicz (1979), Schmeidler (1980), Simon (1984)), which we will call here the *axiomatic approach,* imposes conditions or axioms for competitiveness on the economic institution of trade. The trade institution is modelled by a *strategic outcome function*, which maps agents' simultaneous selection of strategies into allocations. An agent's strategy usually

[2]See Benassy (1986) for an overview of that literature.

specifies a desired trading quantity and a corresponding price for each market. Models with such strategy spaces are referred to as Bertrand–Cournot or Price-Quantity models (see, e.g. Simon (1984) or Dubey (1980)). Simon (1984), for example, presents games in which the competitive allocation emerges as Nash equilibrium, if at least two buyers and two sellers are actively trading in every market. Here we will obtain a similar result in the monopoly case.

The second category (see, e.g. Gale (1987), Rubinstein and Wolinsky (1990) and Wooders (1997)) aims at modelling in detail some particular trading process as a game in extensive form. The advantage of the first approach lies in the fact that one can identify a whole set of simultaneous games with Nash equilibria, which are Walrasian. The second approach, on the other hand, covers extensive games and considers more tangible assumptions of the trading process by spelling out the exact rules of trade.

This work refers to both categories: on the one hand, it deals with *dynamic* (two-stage) trade mechanisms and, on the other hand, it outlines a set of competitive mechanisms via conditions on the strategic outcome function. A model with similar two-stage structure has been analyzed by Wilson (1978). He studies an exchange economy in which one of the consumers is assigned the role of the auctioneer and the others are bidders. The bidders propose trade offers to the auctioneer, who then chooses which to accept. Wilson (1978, p. 583) presents a convergence to the Walrasian allocation, for the case in which bidders are infinitely replicated. We will obtain here a similar Walrasian allocation result, without needing to replicate the bidders. Our result, although applicable for two-good quasilinear economies, is true for any number of bidders with possibly asymmetric demand curves.

5.1.3 Organization of the chapter

The exposition is organized as follows. In the next section, we outline a simple two-good quasilinear production economy and define the concept of a dynamic trading mechanism quite generally through a strategic outcome function. The *variable (or endogenous) supply auctions* are then defined as a special subclass of these sale mechanisms by specifying rationing rules[3] and imposing conditions on the pricing rule. In section 5.3 we state our results: we provide conditions on the pricing rule and rationing rules, which guarantee Walrasian trade. Section 5.4 concludes.

[3] We consider the "pro rata" and the "pro rata on the margin" rationing rule.

5.2 The model

5.2.1 Preliminaries

We consider an economy with a consumption good x and a numeraire commodity y. The numeraire y is a production factor, consumption good and medium of exchange at the same time. We will call this composite commodity "money" and normalize its price to unity. Commodity x is a consumption good, which can be produced out of the resource y by a monopolist with the production function

$$x = f(y)$$

which is continuous differentiable, monotonically increasing and convex. We denote the cost function by

$$c(x) = f^{-1}(\cdot).$$

In the economy there are $n \geq 2$ consumers. The set of consumers will be denoted by $N = \{1, .., n\}$. The preferences of each customer $i \in N$ are represented by the quasi-linear utility function

$$U_i(y, x) = y + v_i(x),$$

where the *valuation function* $v_i(\cdot)$ is differentiable, monotonically increasing and concave. Each consumer i is initially endowed with money $w_i > 0$, which can be exchanged for the consumption good x. The monopolist, being uncertain about the preferences and the money endowments of the consumers, employs an *endogenous supply auction* to collect information about demand and decide on a production and sales plan, which maximizes profit[4]. The consumers participate in the auction mechanism by submitting price-quantity pairs. They are perfectly informed about each other's preferences and the production technology of the seller – an assumption which allows us to study auction mechanisms as games of complete information[5].

5.2.2 The competitive equilibrium

Although the agents in this economy clearly will not act as price takers, we introduce here the competitive equilibrium as a reference point. Because consumers' preferences

[4]If we consider the monopolist also as a consumer, then the profit maximization assumption implies that the seller is interested only in consumption of "money" and has preferences $U(x, y) = y$.

[5]See Back and Zender (2001) and McAdams (2000) for an analysis of the uniform price auction within a similar framework. The literature on competitive market games presented in section 5.1.2 also models trade as a game of complete information.

are quasi-linear, the demand of buyer i for good x is given by:

$$d_i(p) = \max\Big\{\min\{v_i'^{-1}(p), \frac{w_i}{p}\}, 0\Big\}, \forall i \in N.$$

From the above assumptions regarding the valuation function $v_i(\cdot)$, it follows that demand $d_i(p)$ is monotonically decreasing and continuous. The aggregate demand $D(p) = \sum_{i=1}^{n} d_i(p)$ has the same properties. Acting as a price taker the seller maximizes profit $p \cdot x - c(x)$. As a solution of his maximization problem, one obtains the monotonically increasing and continuous supply function

$$S(p) = \max\{c'^{-1}(p), 0\}.$$

The Walrasian price p^w clears the market for good x:

$$S(p^w) = D(p^w).$$

We assume that in equilibrium positive quantities[6] of good x are traded, $x^w = S(p^w) > 0$, and denote the equilibrium input quantity by $y^w = c(x^w)$. The individual consumption quantities are given by:

$$\begin{aligned} x_i^w &= d_i(p^w), \\ y_i^w &= w_i - p^w \cdot x_i^w, \forall i \in N. \end{aligned}$$

The seller obtains the profit $y_S^w := p^w \cdot x^w - c(x^w)$. Because of Walras' law the "money" market is also cleared:

$$\sum_{i=1}^{n} y_i^w + y_S^w = \sum_{i=1}^{n} w_i - c\left(x^w\right).$$

We will further assume that there are at least two consumers, who demand positive quantities of the consumption good at the Walrasian equilibrium, i.e.

$$\exists i, j \in N, i \neq j : d_i\left(p^w\right) > 0, d_j\left(p^w\right) > 0.$$

5.2.3 Two-stage mechanisms

We consider two-stage anonymous[7] trade mechanisms with the following structure. In the first stage the consumers (the bidders) send price-quantity messages to the seller.

[6]The equilibrium production quantity x^w is unique because the demand is continuous and monotonically decreasing function and supply is continuous and monotonically increasing. From $x^w > 0$ follows that p^w is also unique.

[7]An anonymous mechanism is a mechanism, in which the bidders' payoffs depend only on the submitted bids and not on the bidder's identity.

Bidder $i \in N$ submits the price-quantity pair $(p_i, q_i) \in \mathbb{R}^2_+$. Let us denote the vector of quantities by

$$\mathbf{q} = (q_k \mid k = 1, .., n),$$

and the vector of prices by

$$\mathbf{p} = (p_k \mid k = 1, .., n).$$

In the second stage, the auctioneer sets a supply quantity ϕ, which depends on the received bids. The *strategic outcome function* defines how the quantity assigned to each bidder $i \in N$,

$$z_i = Q_i(\mathbf{p}, \mathbf{q}; \phi)$$

and the price per unit that the bidder has to pay to the seller,

$$\pi_i = P_i(\mathbf{p}, \mathbf{q}; \phi)$$

are derived from the bids and the quantity decision of the auctioneer. The allocation prescribed by the strategic outcome function should be feasible:

$$\sum_{i=1}^{n} z_i \leq \phi.$$

Some additional notation will be needed to define a subgame perfect equilibrium. Let

$$\mathbf{q}_{-i} = (q_k \mid k \in N/\{i\}),$$
$$\mathbf{p}_{-i} = (p_k \mid k \in N/\{i\}).$$

Further, the utility of bidder i will be denoted by

$$V_i(\mathbf{p}, \mathbf{q}; \phi) = U_i(z_i, w_i - z_i \cdot \pi_i)^8$$

and the profit of the monopolist by

$$R(\mathbf{p}, \mathbf{q}; \phi) = \sum_{i=1}^{n} z_i \pi_i - c(\phi).$$

The set $\mathbf{M}(\mathbf{p}, \mathbf{q})$ denotes the set of profit maximizing quantities of the monopolist for every price-quantity vector $(\mathbf{p}, \mathbf{q})$:

$$\mathbf{M}(\mathbf{p}, \mathbf{q}) = \arg\max_{\phi \geq 0} R(\mathbf{p}, \mathbf{q}; \phi).$$

[8] Because of the quasilinear preferences of the consumers one can also write

$$U_i(z_i, \pi_i) = w_i - z_i \cdot \pi_i + \int_0^{\infty} \min\{z_i, d_i(p)\}\, dp, \forall i \in N.$$

Definition 5.1 (subgame perfect equilibrium)**.** *The strategy profile of the bidders* $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *and the supply function of the seller* $\phi^*(\cdot)$ *constitute a subgame perfect equilibrium (short equilibrium), if they satisfy the following conditions:*

(P) *Profit maximization: The seller chooses a profit maximizing quantity for each bid vector* $(\boldsymbol{p}, \boldsymbol{q})$*:*

$$\phi^*(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{M}(\boldsymbol{p}, \boldsymbol{q}).$$

(N) *Nash play on the bidding stage:*

$$V_i(p_i^*, q_i^*; \boldsymbol{p}_{-i}^*, \boldsymbol{q}_{-i}^*; \phi^*(\cdot)) \geq V_i(p_i, q_i; \boldsymbol{p}_{-i}^*, \boldsymbol{q}_{-i}^*; \phi^*(\cdot)), \quad \forall (p_i, q_i) \in \mathbb{R}_+^2, \forall i.$$

5.2.4 Variable supply auctions

Variable supply auctions are a special class of the two-stage mechanisms we introduced above. We will need some additional concepts and notation in order to describe them. Denote by S_p the subset of bidders whose bid prices are not lower than p and by S_p^+ the subset of bidders whose bid prices are higher than p :

$$S_p = \{j \in N \mid p_j \geq p\},$$
$$S_p^+ = \{j \in N \mid p_j > p\}.$$

The *announced aggregate demand function* can now be defined as

$$B_{\mathbf{p},\mathbf{q}}(p) = \sum_{j \in S_p} q_j.$$

5.2.4.1 The stopout price

The *stopout price*, which we will denote by p_S is the highest price at which announced demand equals or exceeds supply[9]:

$$p_S(\mathbf{p}, \mathbf{q}; \phi) = \max\{p \mid B_{\mathbf{p},\mathbf{q}}(p) \geq \phi\}.^{10}$$

[9]Sometimes this price is called "cutoff price". In multi-unit auctions bid prices below the stopout price are not served. The set of winning bidders is thus S_{p_S}.

[10]We use the convention that the maximum of the empty set is zero.

5.2.4.2 Rationing rules

We will consider here the two standard rationing rules: the “pro rata on the margin” (or proportional rationing on the margin) and the “pro rata” (or proportional) rationing rule.

- *Pro rata on the margin*
  The “pro rata on the margin” rationing rule is the rationing rule employed in most financial auctions. It is used for example by the US Treasury. According to this rationing rule the bidders who submitted prices above the stopout price are served fully, whereas the bidders who quoted exactly the stopout price are rationed proportionally to their demand, if there is excess demand at that price. Bids below the stopout price are not served. This assignment rule is sometimes called "price priority" rule[11] and is equivalent to an assignment procedure, in which the seller starts serving the bidders from the highest bid price and proceeds along the announced demand curve until supply is exhausted. It is formally defined as follows:

$$Q_i(\mathbf{p},\mathbf{q};\phi) = \begin{cases} q_i & \text{for } p_i > p_S(\cdot), \\ (\phi - \sum\limits_{j \in S^+_{p_S}} q_j)\frac{q_i}{\sum\limits_{j:p_j=p_S} q_j} & \text{for } p_i = p_S(\cdot), \\ 0 & \text{for } p_i < p_S. \end{cases}$$

- *Pro rata*
  The “pro rata” rationing rule does not give priority to high demand and each bidder is granted a quantity proportional tho his quantity at the clearing price. Formally it is defined as follows:

$$Q_i = \begin{cases} q_i \cdot \phi/(\sum\limits_{j \in S_{p_S}} q_j) & \text{if } p_i \geq p_S, \\ 0 & \text{if } p_i < p_S. \end{cases}$$

  This rule is used for instance to ration the bids in the IPO auctions in France (see Biais and Faugeron-Crouzet (2002)). Kremer and Nyborg (2004) provide a classification of different rationing rules. In their terminology (see p. 150, definition 3) such rationing rule is called “independent of irrelevant demand”.

[11] Benassy (1986, p. 102) for example uses that terminology.

5.2.4.3 Pricing rules

We analyze pricing rules, for which the bidders' payments depend on the stopout price and their individual bids only. We assume that the unit price a bidder i has to pay is a continuous function of his bid price and the stopout price in the market:

$$P_i(\mathbf{p}, \mathbf{q}, \phi) = P_i(p_i, p_S), \forall i \in N.$$

This assumption implies, that the unit price each bidder has to pay depends on the other bidders' reports and the supply quantity of the seller only as far as they determine the stopout price. Such pricing rules are for example the standard uniform and discriminatory rules. The restriction to use these pricing rules definitely narrows the monopolist's scope to influence consumer prices. The seller can change the consumer price of a single bidder only through the stopout price and that price is a characteristic of the whole market. This means two things. First, in doing so the seller will influence the prices of the other consumers, and second such a change can sometimes be achieved only at the cost of substantial changes in the supply quantity. The following conditions further narrow down the set of pricing rules for the winning bidders[12].

Pricing conditions:

P1: *price upper bound* – bidders do not pay more than their bid prices:

$$P_i(p_i, p_S) \leq p_i, \forall i \in S_{p_S}.$$

P2: *price lower bound*[13] – bidders do not pay prices lower than the stopout price:

$$P_i(p_i, p_S) \geq p_S, \forall i \in S_{p_S}.$$

P3: *weak stopout price monotonicity* – $P_i(p_i, p_S)$ is weakly increasing in p_S.

P4: *price discrimination* – $P_i(p_i, p_S)$ is strictly increasing in p_i. We call this condition "price discrimination" as it implies that bidders with higher bids pay more (note that we consider only anonymous mechanisms, which implies that the function $P_i(p_i, p_S)$ is the same for all bidders).

[12]Prices for the losers will not be specified, as they obtain a zero quantity.

[13]The stopout price separates the winning from the losing bids, so for that reason all winning bidders are charged at least that price.

Some of these conditions are satisfied by the most popular trade mechanisms.

Classification of auction mechanisms:

- *The discriminatory auction*: $P_i(\cdot) = p_i$. Satisfies $P1$–$P4$.

- *The uniform price auction*: $P_i(\cdot) = p_S(\cdot)$. Satisfies $P1$–$P3$.

- *Linear hybrid auctions*: $P_i(\cdot) = \alpha \cdot p_i + (1-\alpha) \cdot p_S(\cdot)$, $\alpha \in [0,1]$. Satisfy $P1$–$P3$ and $P4$ for $\alpha > 0$. Involve the uniform price and the discriminatory auction as special cases[14].

- *The Spanish Treasury auction pricing rule*[15]: Let $\overline{p}$ be the weighted average price of the winning bids:
$$\overline{p} = \frac{\sum\limits_{i \in S_{p_S}} p_i \cdot Q_i(\mathbf{p}, \mathbf{q}; \phi)}{\phi}.$$
Then
$$P_i(\mathbf{p}, \mathbf{q}; \phi) = \begin{cases} \overline{p} & \text{for } p_i > \overline{p}, \\ p_i & \text{for } \overline{p} \geq p_i \geq p_S(\cdot). \end{cases}$$
Although this pricing rule satisfies the assumptions $P1$–$P4$, it does not belong to the class $P_i(\mathbf{p}, \mathbf{q}, \phi) = P_i(p_i, p_S), \forall i \in N$ that we consider here. The payment rule $P_i(\mathbf{p}, \mathbf{q}, \phi)$ depends on the average price $\overline{p}$, and that price cannot generally be expressed as a function of p_i and $p_S(\cdot)$ only.

5.3 Analysis

We will need some additional notation which is summarized in the table below.

[14] Wang and Zender (2002) discuss these auction forms within an incomplete information Treasury auction model with fixed supply.

[15] The Spanish Treasury employs a special pricing rule, which is a combination between a price discriminatory and uniform price auction. See Alvarez, Mazon, and Cerda (2002) for a description of the institutional details of the Spanish treasury auction format.

Notation	**Terminology**
$\phi^* = \phi^*(\mathbf{p}^*, \mathbf{q}^*)$	*Equilibrium supply quantity*
$p_S^* = p_S(\mathbf{p}^*, \mathbf{q}^*; \phi^*)$	*Equilibrium stopout price*
$z_i^* = Q_i(\mathbf{p}^*, \mathbf{q}^*; \phi^*)$	*Equilibrium quantity of bidder i*
$\pi_i^* = P_i(\mathbf{p}^*, \mathbf{q}^*; \phi^*)$	*Equilibrium price of bidder i*
(p_i^D, q_i^D)	*Deviation strategy of bidder i*
$(\boldsymbol{p}^D, \boldsymbol{q}^D) = (p_i^D, \boldsymbol{p}_{-i}^*, q_i^D, \boldsymbol{q_{-i}^*})$	*Deviation price vector*
$\phi^{*D} \in \mathbf{M}(\mathbf{p}^D, \mathbf{q}^D)$	*Element of seller's best response set to a deviation vector*
$\mathbf{P}(\mathbf{p}^D, \mathbf{q}^D) = \{p_S(\mathbf{p}^D, \mathbf{q}^D; \phi^{*D}) \mid \phi^{*D} \in \mathbf{M}(\mathbf{p}^D, \mathbf{q}^D)\}$	*Set of optimal stopout prices resulting from a deviation vector*
$p_S^{*D} = p_S(\mathbf{p}^D, \mathbf{q}^D; \phi^{*D})$	*The stopout price resulting from the best response element ϕ^{*D} to a deviation vector*

Table 5.1: *Notation and terminology*

5.3.1 The main result

Theorem 5.1. *In every subgame perfect equilibrium of the variable supply auctions satisfying the pricing conditions P1–P4 and the "pro rata on the margin" rationing rule, the Walrasian quantities are traded at the Walrasian price:*

$$z_i^* = x_i^w,$$
$$\pi_i^* = p^w; \forall i \in N.$$

The proof is provided in the following four lemmas. In lemma 5.1 through lemma 5.3 we establish several properties of the subgame perfect equilibria (provided that such exist). In lemma 5.1, we show that in equilibrium all winning bidders[16] submit the same price. In lemma 5.2, we claim that this equilibrium stopout price can only be the Walrasian price. In lemma 5.3, we verify that, in equilibrium, the Walrasian quantities are traded. In the proofs of all of these lemmas we proceed by contradiction. We show that if the statement in the lemma does not hold one of the bidders has a profitable deviation strategy, a contradiction to the equilibrium assumption. The existence of an equilibrium is provided by construction in lemma 5.4.

Lemma 5.1. *In equilibrium all winning bidders quote the same price:*

$$p_i^* = p_S^*, \forall i \in S_{p_S^*}.$$

[16]In equilibrium there is at least one winning bidder. Indeed if no bidder obtains a positive quantity, then any bidder whose Walrasian quantity is positive has a profitable deviation. He can quote his Walrasian quantity and the Walrasian price and will be served by the monopolist.

Proof. We begin with some observations concerning the optimal quantity choice of the auctioneer. Take an arbitrary bid vector $(\mathbf{p}, \mathbf{q})$. Without loss of generality we can number the bidders in a descending order according to their bid prices:

$$p_1 \geq p_2 \geq ... \geq p_n.$$

For every bid price $p_k \in \{p_i \mid i \in N\}$ consider the interval of supply quantities $(l_k, h_k]$ leading to a stopout price of p_k:

$$(l_k, h_k] := \{\phi \mid p_S(\boldsymbol{p}, \boldsymbol{q}; \phi) = p_k\}.$$

Let us denote

$$\phi_k^* = \arg \max_{\phi \in [l_k, h_k]} R(\boldsymbol{p}, \boldsymbol{q}; \phi).$$

Observe that for $\phi = l_k$ the stopout price is no longer p_k, but the next highest price along the announced demand curve. With a slight abuse of notation[17] we will denote that price by p_{k-1}; $p_{k-1} > p_k$. One can verify the following statements:

Statement 1: If $h_k < S(p_k)$, then $\phi_k^* = h_k$,

Statement 2: If $l_k < S(p_k) \leq h_k$, then $\phi_k^* = S(p_k)$,

Statement 3: If $S(p_k) \leq l_k$, then $\phi_k^* = l_k$.

See figure 5.2 for an illustration.

Proof. For $\phi \in (l_k, h_k]$ the profit of the seller is

$$\begin{aligned} R(\boldsymbol{p}, \boldsymbol{q}; \phi) &= \sum_{i \in S_{p_k}^+} q_i \cdot P_i(p_i, p_k) + (\phi - \sum_{i \in S_{p_k}^+} q_i) \cdot p_k - c(\phi) \\ &= \sum_{i \in S_{p_k}^+} q_i \cdot [P_i(p_i, p_k) - p_k] + p_k \cdot \phi - c(\phi). \end{aligned} \quad \text{(P1,P2)}$$

Then

$$\frac{\partial R(\boldsymbol{p}, \boldsymbol{q}; \phi)}{\partial \phi} = p_k - c'(\phi).$$

[17] There may be more than one bidder, who quoted p_k.

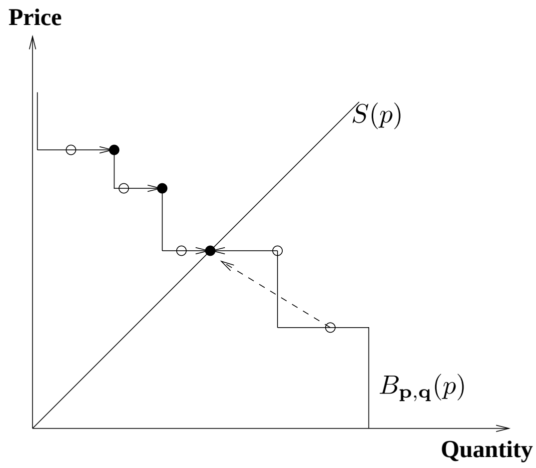

Figure 5.2: *The hollow dots denote quantities and corresponding stopout prices, which cannot be part of an equilibrium, because they do not maximize the seller's profit. The arrows point the quantity direction, in which the monopolist's profit increases. The thick dots denote possible equilibrium quantities and the corresponding stopout prices.*

As $c'(\phi)$ is monotonically increasing and $c'(\phi) = S^{-1}(\cdot)$, statements 1 and 2 follow. To verify statement 3 observe that for $\phi \in (l_k, h_k)$ and $S(p_k) \leq l_k$ ($\Leftrightarrow c'(l_k) \geq p_k$) one obtains

$$\begin{aligned} R(\boldsymbol{p}, \boldsymbol{q}; l_k) &= \sum_{i \in S^+_{p_{k-1}}} q_i \cdot P_i(p_i, p_{k-1}) + p_{k-1} \cdot \sum_{i: p_i = p_{k-1}} q_i - c(l_k) \qquad \text{(P1,P3)} \\ &\geq \sum_{i \in S^+_{p_{k-1}}} q_i \cdot P_i(p_i, p_k) + P_i(p_i, p_k) \cdot \sum_{i: p_i = p_{k-1}} q_i - c(l_k) \\ &= \sum_{i \in S^+_{p_k}} q_i \cdot P_i(p_i, p_k) - c(l_k) \\ &= \sum_{i \in S^+_{p_k}} q_i \cdot [P_i(p_i, p_k) - p_k] + p_k \cdot l_k - c(l_k) \\ &> \sum_{i \in S^+_{p_k}} q_i \cdot [P_i(p_i, p_k) - p_k] + p_k \cdot \phi - c(\phi) = R(\boldsymbol{p}, \boldsymbol{q}; \phi). \end{aligned}$$

Loosely speaking, if the monopolist sells the quantity l_k at a stopout price of p_{k-1}, he will obtain a profit not lower than the profits resulting from selling the same quantity at a stopout price of p_k due to the weak stopout price monotonicity condition $P3$. But even at the stopout price of p_k, the auctioneer would prefer to sell the quantity l_k instead of $\phi \in (l_k, h_k]$ since $c'(l_k) \geq p_k$. □

Assume now, contrary to the lemma, that in equilibrium not all winning bidders quoted the same price. Recall that the stopout price p_S^* is the highest price at which announced demand equals or exceeds supply. As by assumption not all winning bidders quoted p_S^* the highest bid exceeds the stopout price: ($p_1^* > p_S^*$). The price conditions (P2) and (P4) imply

$$\pi_1^* > p_S^*.$$

Consider the following deviation of bidder 1:

$$(p_1^D, q_1^D) = (p_S^* + \varepsilon, q_1^*),$$

where $\varepsilon > 0$ is small enough. Observe that for small enough $\varepsilon > 0$ one obtains

$$p_1^D < \min\{\pi_1^*, \{p_i^* \mid i \in S_{p_S^*}^+\}\}.$$

Claim. For a small enough $\varepsilon > 0$ every $p_S^{*D} \in \mathbf{P}(\mathbf{p}^D, \mathbf{q}^D)$ is such that $p_S^{*D} \leq p_S^*$.

In words, if the bidder who submitted the highest price changed his bid price and submitted a price slightly higher than p_S^* instead, the seller would not raise the stopout price (see figure 5.3 for an an illustration). The deviating bidder will be served. He acquires the same quantity and pays less.

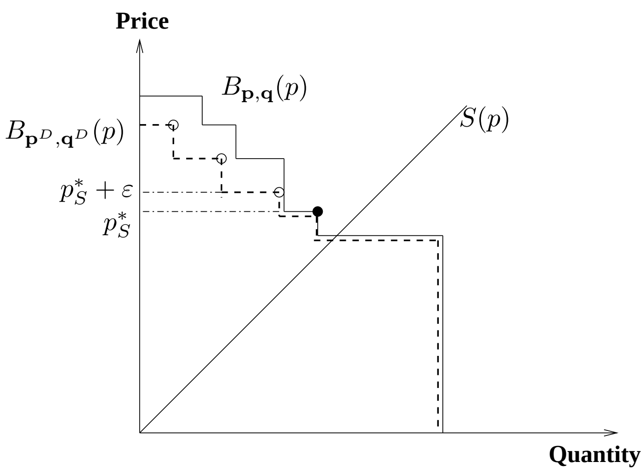

Figure 5.3: *Change in the aggregate demand function (the dash line) as a result of the deviation. Lowering the quantity to charge a higher stopout price (represented by the hollow dots) is not profitable for the seller.*

Proof. Assume on the contrary that there exists some $p_S^{*D} \in \mathbf{P}(\mathbf{p}^D, \mathbf{q}^D)$ such that $p_S^{*D} > p_S^*$. Let us analyze the two possible cases:

(L1.1): $p_S^{*D} > p_1^D$. From the inequalities

$$\begin{aligned} R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*) &= R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*) - z_1^*(\pi_1^* - \pi_1^D), \\ & \qquad \textit{(Bidder 1 pays } \pi_1^D \textit{ instead of } \pi_1^*) \\ R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^{*D}) &\leq R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^{*D}) - z_1^* \cdot \pi_1^*, \qquad \textit{(Bidder 1 is not served.)} \\ R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*) &\geq R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^{*D}), \qquad (\phi^* \in \mathbf{M}(\mathbf{p}^*, \mathbf{q}^*)) \end{aligned}$$

follows

$$R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^{*D}) < R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*),$$

a contradiction to $\phi^{*D} \in \mathbf{M}(\mathbf{p}^D, \mathbf{q}^D)$.

(L1.2): $p_S^{*D} = p_1^D$. As $\lim_{\varepsilon \to 0} p_1^D = \lim_{\varepsilon \to 0} (p_S^* + \varepsilon) = p_S^*$ and $P_i(\cdot)$ is continuous one obtains $\lim_{\varepsilon \to 0} R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^{*D})$

$$\begin{aligned} &= \sum_{i \in S^+_{p_S^*}} q_i \cdot P_i(p_i, p_S^*) - c(\phi^{*D}) \\ &< \sum_{i \in S^+_{p_S^*}} q_i \cdot P_i(p_i, p_S^*) - c(\phi^{*D}) + p_S^*(\phi^* - \phi^{*D}) - c(\phi^*) + c(\phi^{*D}) \\ &= \sum_{i \in S_{p_S^*}} q_i \cdot P_i(p_i, p_S^*) - c(\phi^*) = R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*). \end{aligned}$$

The inequality applies since for $\phi \in (\phi^{*D}, \phi^*)$ one observes that $c'(\phi) < c'(\phi^*) \leq p_S^*$ (see statements 2 and 3) and therefore

$$p_S^* \cdot (\phi^* - \phi^{*D}) - c(\phi^*) + c(\phi^{*D}) = \int_{\phi^{*D}}^{\phi^*} [p_S^* - c'(\phi)] \mathrm{d}\phi > 0.$$

For small enough $\varepsilon > 0$ follows

$$R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^{*D}) < R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*),$$

a contradiction to $\phi^{*D} \in \mathbf{M}(\mathbf{p}^D, \mathbf{q}^D)$.

□

To summarize, we presented a deviation of bidder 1, according to which he lowers his bid price, setting it slightly higher than the stopout price. Playing this deviation the bidder obtains the same quantity, as the seller does not raise the stopout price in response to the deviation. However this bidder pays less because $p_1^D < \pi_1^*$ (see P1). The deviation is thus profitable. □

Lemma 5.2. *In equilibrium trade is conducted at the Walrasian price:*

$$p_S^* = p^w.$$

Proof. Using the results already obtained in lemma 1, namely that all winning bidders submit in equilibrium the same bid price, we will show that there exists a bidder with a profitable deviation strategy whenever $p_S^* \neq p^w$. We will analyze the three cases possible:

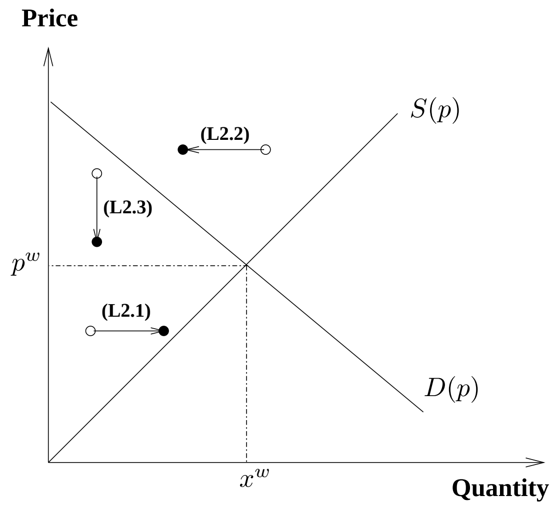

Figure 5.4: *The change in supply quantity and stopout price as a result of a profitable deviation (represented by the arrows) in the three cases of Lemma 2.*

(L2.1): $p_S^* < p^w$.

The seller supplies

$$\phi^* = \min\{\sum_{i \in S_{p_S^*}} q_i^*, S(p_S^*)\} \leq S(p_S^*) < D(p_S^*).$$

Since there is excess demand at the stopout price, there exists a bidder j for whom $z_j^* < d_j(p_S^*)$. Consider the following deviation of this bidder j:

$$(p_j^D, q_j^D) = (p_S^*, q_j^* + \varepsilon),$$

where $\varepsilon > 0$. If ε is chosen small enough, then the optimal response of the seller is:

$$\phi^{*D} = \begin{cases} \phi^* + \varepsilon & \text{for } \phi^* < S(p_S^*), \\ \phi^* & \text{for } \phi^* = S(p_S^*). \end{cases}$$

Bidder j is assigned the quantity

$$z_j^{*D} = \begin{cases} z_j^* + \varepsilon & \text{for } \phi^* < S(p_S^*), \\ \frac{q_j^* + \varepsilon}{\sum\limits_{i \in S_{p_S^*}} q_i^* + \varepsilon} \cdot \phi^* & \text{for } \phi^* = S(p_S^*). \end{cases}$$

For small enough ε one obtains[18]:

$$z_j^* < z_j^{*D} \leq d_j(p_S^*),$$

the deviation is thus profitable. An illustration of the change in the supply quantity and the corresponding stopout price as a result of the deviations in lemma 5.2 are presented in figure 5.4.

(L2.2): $p_S^* > p^w$ and $\sum\limits_{i \in S_{p_S^*}} q_i^* > D(p_S^*)$.

The seller supplies

$$\phi^* = \min\{\sum_{i \in S_{p_S^*}} q_i^*, S(p_S^*)\} > D(p_S^*).$$

Bidders are assigned in aggregate a higher quantity, than they are willing to buy. Therefore there exists j for whom $z_j^* > d_j(p_S^*)$. Consider now the following deviation of that bidder:

$$(p_j^D, q_j^D) = (p_S^*, q_j^* - \varepsilon).$$

The seller supplies

$$\phi^{*D} = \begin{cases} \sum\limits_{i \in S_{p_S^*}} q_i^* - \varepsilon & \text{for } \sum\limits_{i \in S_{p_S^*}} q_i^* - \varepsilon < S(p_S^*), \\ \phi^* & \text{for } \sum\limits_{i \in S_{p_S^*}} q_i^* - \varepsilon = S(p_S^*). \end{cases}$$

Bidder j is assigned the quantity

$$z_j^{*D} = \begin{cases} z_j^* - \varepsilon & \text{for } \phi^* < S(p_S^*), \\ \frac{q_j^* - \varepsilon}{\sum\limits_{i \in S_{p_S^*}} q_i^* - \varepsilon} \cdot \phi^* & \text{for } \phi^* = S(p_S^*). \end{cases}$$

[18]The inequality is obviously true since

$$z_j^* = \frac{q_j^*}{\sum\limits_{i \in S_{p_S^*}} q_i^*} \cdot \phi^* < \frac{q_j^* - \varepsilon}{\sum\limits_{i \in S_{p_S^*}} q_i^* - \varepsilon} \cdot \phi^* = z_j^{*D} \quad \text{for } \phi^* < S(p_S^*).$$

For small enough ε one obtains[19]:

$$z_j^* > z_j^{*D} \geq d_j(p_S^*),$$

the deviation is thus profitable.

(L2.3): $p_S^* > p^w$ and $\sum_{i \in S_{p_S^*}} q_i^* \leq D(p_S^*) < S(p_S^*)$.

The seller supplies

$$\phi^* = \sum_{i \in S_{p_S^*}} q_i^*.$$

Consider now the following deviation of an arbitrary winning bidder j:

$$(p_j^D, q_j^D) = (p_S^* - \varepsilon, q_j^*).$$

We will show that for a small enough $\varepsilon > 0$ the set $\mathbf{P}(\mathbf{p}^D, \mathbf{q}^D)$ contains only the element $p_S^{*D} = p_S^* - \varepsilon$ and the seller does not change the supply quantity ϕ^* due to the deviation. There are two possible candidates for optimal supply quantities: either ϕ^*, leading to a stopout price of $p_S^* - \varepsilon$, or $\phi^* - q_j^*$, leading to a stopout price of p_S^*. In the latter variant, the seller does not serve bidder j and his payoff is thus:

$$\begin{aligned} R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^* - q_j^*) &= p_S^* \cdot (\phi^* - q_j^*) - c(\phi^* - q_j^*) \\ &< p_S^* \cdot (\phi^*) - c(\phi^*) = R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*). \end{aligned}$$

On the other hand as $\lim_{\varepsilon \to 0} (p_S^* - \varepsilon) = p_S^*$ one obtains

$$\lim_{\varepsilon \to 0} R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*) = p_S^* \cdot (\phi^*) - c(\phi^*) = R(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*).$$

Therefore for small enough ε

$$R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^*) > R(\boldsymbol{p}^D, \boldsymbol{q}^D; \phi^* - q_j^*).$$

The seller retains the same supply quantity and bidder j obtains the same quantity at a lower stopout price. The deviation is profitable.

[19] The inequality is obviously true since:

$$z_j^* = \frac{q_j^*}{\sum_{i \in S_{p_S^*}} q_i^*} \cdot \phi^* > \frac{q_j^* - \varepsilon}{\sum_{i \in S_{p_S^*}} q_i^* - \varepsilon} \cdot \phi^* = z_j^{*D} \quad \text{for } \phi^* = S(p_S^*).$$

□

Lemma 5.3. *In every equilibrium the Walrasian quantities are traded:*

$$z_i^* = x_i^*, \forall i \in N.$$

Proof. In lemma 1 and lemma 2 we obtained the result that all bidders submit the Walrasian price. The monopolist always sells the quantity $\phi^* \leq S(p^w)$. If we assume that $\phi^* < S(p^w)$, then, because of statement 1, we know that

$$\phi^* = \sum_{i \in S_{p_S^*}} q_i^* < S(p^w) = D(p^w).$$

This means there exists a bidder j for whom $z_j^* < x_j^w$. Consider the following deviation of this bidder:

$$(p_j^D, q_j^D) = (p_S^*, q_j^* + \varepsilon).$$

For small enough ε the deviation is profitable. The seller supplies $\phi^{*D} = \phi^* + \varepsilon$ and bidder j obtains $q_j^D = z_j^* + \varepsilon$. Thus, we showed that in equilibrium the seller supplies $S(p^w)$. Let us assume now that not every bidder obtains his Walrasian quantity. Since $\sum_{i=1}^{n} x_i^w = S(p^w)$ then there exist a bidder i for whom $z_i^* < x_i^w$. Consider a deviation as the above one for that bidder. The seller supplies $\phi^* = S(p^w)$ and bidder i obtains

$$z_j^D = \frac{q_j^* + \varepsilon}{\sum_{i \in S_{p_S^*}} q_i^* + \varepsilon} \cdot \phi^* > \frac{q_j^*}{\sum_{i \in S_{p_S^*}} q_i^*} \cdot \phi^* = z_j^*.$$

The bidder is able to buy a higher quantity at the same price and it is thus profitable to deviate. □

Lemma 5.4 (existence)**.** *The strategy profiles that satisfy the following conditions (E1) and (E2) are equilibrium profiles.*

$$p_i^* = p^w, \forall i \in N. \tag{E1}$$

$$q_i^* = k \cdot x_i^w, \forall i \in N \quad \textit{where} \quad k \geq \frac{\sum_{i \in N} x_i^w}{\sum_{i \in N} x_i^w - \max_i \{x_i^w\}}. \tag{E2}$$

Proof. Observe that the seller supplies $\phi^*(\mathbf{p}^*, \mathbf{q}^*) = S(p^w)$; every bidder is granted his Walrasian quantity and has to pay the competitive price. No player has a profitable deviation, because no deviation leads to a lower stopout price. Indeed, if a bidder

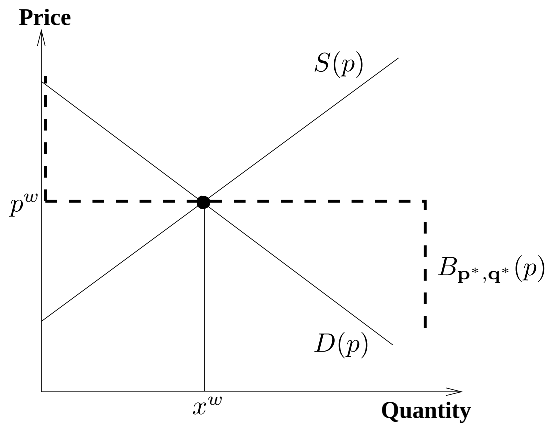

Figure 5.5: *Supply, demand and announced aggregate demand function (the dash curve) in equilibrium.*

submits a bid price lower than p^w, then the seller will retain the supply quantity of $S(p^w)$ and that bidder will not be served. With the strategy described in the lemma, every bidder is granted his Walrasian quantity at the Walrasian price, so a possible deviation can have only two effects. It can either lead to a higher stopout price or the deviating bidder will not be assigned the Walrasian quantity, both of which is not profitable for him. This strategy profile is thus an equilibrium. □

5.3.2 The uniform price auction

The uniform price auction violates the price discrimination condition $P4$. This condition is crucial for the main result as it guarantees, that in equilibrium all bidders submit equal bid prices (see lemma 5.1). Here we demonstrate by means of an example that, if the "pro rata on the margin" rationing rule is used, the uniform price auction has additional (low-price) non-Walrasian equilibria.

5.3.2.1 Example of low-price equilibrium with the "pro rata on the margin" rationing rule

Two bidders compete in a uniform price auction. Each bidder $i \in \{1, 2\}$ is endowed with money $w_i \geq 2,2$ and has the utility function

$$U_i(x, y) = \begin{cases} 2,2 \cdot x + y & \text{for } 0 \geq x \geq 1, \\ 2,2 + y & \text{for } x > 1. \end{cases}$$

The cost function of the seller is

$$c(x) = \frac{x^2}{2}.$$

For the Walrasian equilibrium one obtains

$$p^w = 2,$$
$$x_i^w = 1, i \in \{1, 2\}.$$

Except for the strategies discussed in lemma 5.4, which are (Walrasian) equilibria, the uniform price auction has also non-Walrasian equilibria.

Claim 5.1. *The bids*

$$(p_1^*, q_1^*) = (2, 1),$$
$$(p_2^*, q_2^*) = (\sqrt{3}, \sqrt{3}),$$

are equilibrium bids. The seller supplies $\phi^* = \sqrt{3}$*, the stopout price is* $p_S^* = \sqrt{3}$ *and the bidders are granted the quantities*

$$z_1^* = 1,$$
$$z_2^* = \sqrt{3} - 1.$$

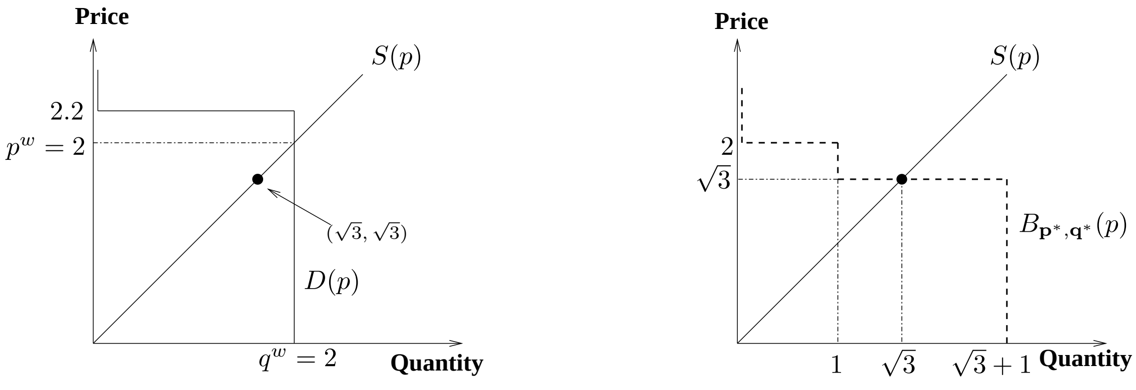

Figure 5.6: *Non-Walrasian (low-price) equilibrium allocation of the uniform price auction (the thick dot). The dash curve in the right figure represents the announced demand curve in equilibrium.*

Proof. Observe that $\mathbf{M}_{\mathbf{p}^*,\mathbf{q}^*} = \{1, \sqrt{3}\}$, so the seller can supply the quantity $\phi^* = \sqrt{3}$ and realize a maximal payoff of $R(\mathbf{p}^*, \mathbf{q}^*; \phi^*) = \frac{3}{2}$. We will show now that the bidders have no profitable deviation strategies, which asserts that the above strategy profile is an equilibrium. Bidder 1 obtains his desired quantity at the price of $\sqrt{3}$. He has no deviation, which can lower the stopout price, because if he submits a bid price lower than $\sqrt{3}$ he will not be served. In such a case the auctioneer will sell the quantity $\sqrt{3}$ to bidder 2. Bidder 2 has no profitable deviation either. If he submits a bid price lower than $\sqrt{3}$ he also will not be served. The profit of the monopolist will drop below $\frac{3}{2}$ and he will prefer to sell only one unit to bidder 1 at a price of 2. Submitting a higher price is also unprofitable for bidder 2. Indeed if he submitted a price of $p_2 > \sqrt{3}$ he could maximally obtain the quantity $p_2 - 1$ (see figure 5.6). His profit then will be

$$(2.2 - p_2)(p_2 - 1) < (2.2 - \sqrt{3}) \cdot (\sqrt{3} - 1).$$

The described strategy profile is thus an equilibrium. □

The allocation is inefficient, since the monopolist supplies less than the Walrasian quantity. For other non-Walrasian equilibrium examples of the uniform price auction with the "pro rate on the margin" rationing rule see Back and Zender (2001) and McAdams (2000). Ausubel and Cramton (2002) consider an incomplete information multi-unit auction model with fixed supply. They also find that the uniform price auction leads to demand reduction and inefficient allocation.

5.3.2.2 The uniform price auction with the "pro rata" rationing rule

The existence of low-price equilibria of the uniform-price auction is (partly) due to the applied "pro rata on the margin" rationing rule. In the exposition here, which follows Damianov (2005b), we will show that the seller can eliminate these low-price equilibria, if she employs the simple "pro rata" rule. Using a "pro rata" rationing rule and adhering to a simple consistency rule when selecting among profit-maximizing quantities, the seller can eliminate all low-price equilibria. As we will demonstrate, the interplay of these two assumptions leads to a unique equilibrium allocation, in which efficiency is restored. In equilibrium buyers pay the Walrasian price and are awarded the Walrasian quantities.

Definition 5.2. *The bid profile* $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *and the seller's supply function* $\phi^* : \mathbb{R}_+^{2n} \to \mathbb{R}_+$ *constitute a* ***consistent*** *subgame perfect equilibrium, if they constitute a subgame perfect equilibrium (i.e. satisfy the conditions* ***(P)*** *and* ***(N)****) and the quantity selection function satisfies additionally the condition:*

(C) *Consistency: For any two bid profiles* $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$ *and* $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}})$ *for which*

$$\boldsymbol{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}) \subseteq \boldsymbol{M}(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$$

we have

$$\phi^*(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}) \in \boldsymbol{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}) \Rightarrow \phi^*(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}) = \phi^*(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}).$$

The consistency condition is a refinement of the subgame perfect equilibrium concept[20]. It poses an additional restriction on the maximizer selection function $\phi^*(\mathbf{p}, \mathbf{q})$. Only a subset of the maximizer selection functions[21] defined by **(P)** satisfy **(C)**. In words, the consistency condition requires that if the seller chooses the maximizer $\phi^*(\overline{\mathbf{p}}, \overline{\mathbf{q}})$ from the set of best alternatives $\mathbf{M}(\overline{\mathbf{p}}, \overline{\mathbf{q}})$, then she should not choose another maximizer in all subgames, which offer only a subset of the set of best alternatives $\mathbf{M}(\overline{\mathbf{p}}, \overline{\mathbf{q}})$ (among which is $\phi^*(\overline{\mathbf{p}}, \overline{\mathbf{q}})$). Supplying consistently, the seller precludes situations in which a winning bidder, who deviates by extending his bidding quantity, will no longer be served. A bidder with excess demand is able to buy a larger quantity without the need to bid a higher price by simply magnifying his quantity announcement at the stopout price (see Case 1 in the proof of the next theorem). As the next theorem states, the proportional rationing rule in combination with a consistent supply function eliminates the existence of equilibria below the competitive price[22]. Simple examples of consistent supply functions are

$$\phi^*(\boldsymbol{p}, \boldsymbol{q}) = \max\{\phi : \phi \in \mathbf{M}(\mathbf{p}, \mathbf{q})\},$$
$$\phi^*(\boldsymbol{p}, \boldsymbol{q}) = \min\{\phi : \phi \in \mathbf{M}(\mathbf{p}, \mathbf{q})\}.$$

[20]This refinement is similar to other refinements of the subgame perfect equilibrium concept. Harsanyi and Selten (1988) introduce a refinement, which they term *subgame consistency* and McAfee (1993) another refinement, called *subform consistency.* Roughly speaking, both refinements require that in identical subgames the same equilibria should be played.

[21]The quantity selection conditions **(P)** and **(C)** in this game theoretical context bear some analogy to two of the axioms, which define the *Nash bargaining solution.* The term *bargaining problem* in Nash (1950) translates here into the term *subgame* and the term *bargaining solution* is here *supply function.* Eichberger (1993, pp. 249-260) provides the Nash axioms and a proof of the Nash bargaining solution. Condition **(P)** can be compared to *Axiom 2* (Pareto optimality) as the seller chooses a profit maximizing quantity. Condition **(C)** is similar to *Axiom 4* (independence of irrelevant alternatives).

[22]We will further show that if the seller supplies inconsistently low-price equilibria reemerge (see the next claim).

The consistency requirement is economically meaningful in our setting for two reasons. Firstly, the seller can commit to such a strategic choice function since she chooses in every subgame a quantity, which is among the revenue maximizers. Secondly, such behavior rewards the seller in the sense that it eliminates the so called collusive-seeming equilibria, i.e. the equilibria with stopout prices lower than the Walrasian price[23]. In the next example we provide such a collusive-seeming equilibrium of the uniform price auction for an inconsistent maximizer selection function of the seller. The consistency condition can thus also be viewed as seller's instrument against *tacit collusion*. Agents tacitly (or implicitly) collude, if they coordinate on Nash equilibrium, which is profitable for them and unprofitable for the seller. This type of collusion is sometimes called noncooperative collusion; see Tirole (1989, pp. 207,239-270,313).

Theorem 5.2. *In all* ***consistent*** *subgame perfect equilibrium of the uniform price auction with the "pro rata" rationing rule the competitive quantities are traded at the competitive price:*

$$p_S(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*(\cdot)) = p^w,$$
$$Q_i(\boldsymbol{p}^*, \boldsymbol{q}^*; \phi^*(\cdot)) = d_i(p^w), \forall i.$$

Proof. For the sake of brevity let us denote the equilibrium stopout price by p_S^* and the seller's equilibrium supply quantity by ϕ^*. We show that no equilibria exist, in which $p_S^* > p^w$ or $p_S^* < p^w$. For each of these cases, we construct a profitable deviation for one of the bidders, thereby reaching a contradiction to the equilibrium condition (**N**).

Case 1: $p_S^* < p^w$. As the seller supplies a quantity not higher than $S(p_S^*)$ there is excess demand at the stopout price and hence there exists a bidder j who received in equilibrium less than he is willing to buy at that price. Consider the following deviation of that bidder: $(p_j^D, q_j^D) = (p_S^*, q_j^* + \varepsilon)$, where $\varepsilon > 0$. If ε is chosen sufficiently small the optimal response of the seller to the deviation is $\phi^{*D} = \phi^* + \varepsilon$ for $\phi^* < S(p_S^*)$ (see condition (**P**)) and $\phi^{*D} = \phi^*$ otherwise (see condition (**C**)). Bidder j is assigned the quantity $q_j^* + \varepsilon$ for $\phi^* < S(p_S^*)$ and $\frac{q_j^*+\varepsilon}{\sum_{i:p_i^* \geq p_S^*} q_i^* + \varepsilon} \cdot \phi^*$ otherwise. For ε sufficiently small the deviation is profitable.

Case 2(a): $p_S^* > p^w$ and $\sum_{i:p_i^* \geq p_S^*} q_i^* > S(p_S^*)$. There is a bidder j who obtained more than he is willing to buy at that price. Playing the deviation $(p_j^D, q_j^D) =$

[23]This terminology was introduced by McAdams (2000).

$(p_S^*, q_j^* - \varepsilon)$, where $\varepsilon > 0$ is sufficiently small, this bidder will be able to slightly reduce the quantity assigned to him since a consistent seller (see 5.2) will not change his supply quantity $\phi^* = S(p_S^*)$ when facing such a deviation.

Case 2(b): $p_S^* > p^w$ and $\sum_{i:p_i^* \geq p_S^*} q_i^* = S(p_S^*)$. Take the bidder who submitted the highest bid price $j : p_j^* \geq p_i^*, \forall i$. Consider the following deviation of that bidder:

$$(p_j^D, q_j^D) = \Big(p_S^* - \varepsilon, \frac{q_j^*}{S(p_S^* - \varepsilon) - q_j^*} \cdot \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*\Big).$$

For a sufficiently small $\varepsilon > 0$ it will be argued that the seller supplies the quantity $\phi^{*D} = S(p_S^* - \varepsilon)$ at the stopout price of $p_S^* - \varepsilon$. Indeed observe that

$$\lim_{\varepsilon \to 0} R(p_j^D, \mathbf{p}_{-j}^*, q_j^D, \mathbf{q}_{-i}^*; \phi^{*D}) = R(\mathbf{p}^*, \mathbf{q}^*; \phi^*)$$

and consider also that if the seller decides not to serve bidder j, (which means he reduces the supply quantity to charge a stopout price higher than $p_S^* - \varepsilon$) he will obtain a payoff strictly lower than $R(\mathbf{p}^*, \mathbf{q}^*; \phi^*)$. It follows that for a sufficiently small ε it is not optimal for the seller to supply a quantity leading to a stopout price higher than $p_S^* - \varepsilon$. On the other hand supplying a quantity leading to a stopout price lower than $p_S^* - \varepsilon$ generates obviously lower payoff. As a result of the deviation bidder j obtains the quantity

$$q_j^D \cdot \frac{\phi^{*D}}{q_j^D + \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*} = \frac{q_j^* \cdot \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*}{\phi^{*D} - q_j^*} \cdot \frac{\phi^{*D}}{\frac{q_j^* \cdot \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*}{\phi^{*D} - q_j^*} + \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*}$$

$$= \frac{q_j^* \cdot \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*}{\phi^{*D} - q_j^*} \cdot \frac{\phi^{*D}}{\frac{\phi^{*D} \cdot \sum_{i:p_i^* \geq p_S^* - \varepsilon, i \neq j} q_i^*}{\phi^{*D} - q_j^*}} = q_j^*.$$

Thus playing the deviation the bidder j obtains the same quantity at a lower price. The deviation is profitable.

Case 2(c): $p_S^* > p^w$ and $\sum_{i:p_i^* \geq p_S^*} q_i^* < S(p_S^*)$. Let us denote the highest profit maximizing quantity by

$$\phi_{max}^* = \max\{\phi : \phi \in \mathbf{M}(\mathbf{p}^*, \mathbf{q}^*)\}$$

and the corresponding stopout price by p^*_{min}. In the case $\phi^*_{max} < S(p^*_{min})$ consider the following deviation of the bidder who submitted the highest bid price:

$$(p_j^D, q_j^D) = \left(p^*_{min} - \varepsilon, q_j^*\right).$$

Analogously to the Case 2(b) can be argued, that for a sufficiently small ε the seller supplies the quantity ϕ^*_{max} at the stopout price $p^*_{min} - \varepsilon$. Bidder j obtains the same quantity and pays $p^*_{min} - \varepsilon < p^*_S$. The deviation is profitable. In the case $\phi^*_{max} = S(p^*_{min})$ one considers the deviation

$$(p_j^D, q_j^D) = \left(p^*_{min}, \frac{q_j^*}{S(p^*_{min}) - q_j^*} \cdot \sum_{i:p_i^* \geq p^*_{min}, i \neq j} q_i^*\right).$$

Again analogously to the Case 2(b) can be argued, that the seller supplies ϕ^*_{max} and bidder j is granted the quantity q_j^* at the price $p_{min} < p^*_S$. The deviation is profitable, which completes the proof.

One concludes than in equilibrium (provided that one exists) trade is conducted at the competitive price p^w. The seller supplies

$$\phi^* = \min\{\sum_{i:p_i^* \geq p^w} q_i^*, S(p^w)\}.$$

If $\phi^* < S(p^w)$ there is excess demand and consequently one of the bidders obtains less than his Walrasian quantity. With the deviation presented in Case 1 this bidder will be able to obtain a larger quantity at the competitive price. It follows $\phi^* = S(p^w)$. If it is assumed that in equilibrium not all bidders obtain their Walrasian quantities, then there is a bidder, who obtains less than his Walrasian quantity and the deviation presented in Case 1 will again be profitable. The existence of an equilibrium with Walrasian trades is proven with arguments analogous to these in lemma 5.4. □

5.3.2.3 Low-price equilibrium with the "pro rata" rationing rule and inconsistent quantity selection function

We will show that the uniform price auction has non-Walrasian (low-price) equilibria, if bids are prorated according to the "pro rata" (or proportional) rationing rule and the seller behaves inconsistently. Consider the following example. Five bidders participate in a uniform price auction. Their utility function is given by

$$U_1(x,y) = \begin{cases} 5 \cdot x + y & \text{for } 0 \geq x \geq 0.8, \\ 5 + y & \text{for } x > 0.8. \end{cases}$$

$$U_i(x,y) = \begin{cases} 5 \cdot x + y & \text{for } 0 \geq x \geq 1, \\ 5 + y & \text{for } x > 1, \end{cases}$$

for $i \in \{2,3,4,5\}$. The cost function of the seller is

$$c(x) = \frac{x^2}{2}.$$

For the Walrasian equilibrium one obtains (see figure 5.6)

$$\begin{aligned} &p^w = 4.8, \\ &z_1^w = 0,8; \quad z_i^w = 1 \quad \text{for } i \in \{2,3,4,5\}. \end{aligned}$$

Except for the strategies discussed in lemma 5.4, which are (Walrasian) equilibria, the uniform price auction has also non-Walrasian equilibria as the claim below states.

Claim 5.2. *The bids*

$$\begin{aligned} (p_1^*, q_1^*) &= (8.5, 1), \\ (p_i^*, q_i^*) &= (4, 1), \end{aligned}$$

are equilibrium bids if the seller supplies inconsistently according to the supply function

$$\phi^*(q_1^*, q_2, q_3, q_4, q_5) = \begin{cases} 4 & \textit{for } q_i = 1, \\ 1 & \textit{for } q_i > 1, \end{cases}$$

for $i \in \{2,3,4,5\}$*. The stopout price is* $p_S^* = 4$ *and the bidders are granted the quantities*

$$z_1^* = z_2^* = \cdots = z_5^* = 0.8.$$

Proof. Observe that $\mathbf{M}(\mathbf{p}^*, \mathbf{q}^*) = \{1, 4\}$, so the seller can choose the quantity $\phi^* = 4$. We have $R(\mathbf{p}^*, \mathbf{q}^*, \phi) = 8$. Bidder 1 has obviously no profitable deviation. He cannot lower the stopout price by playing a deviation. If he submits a price lower than 4, he will not be served. The monopolist would sell 4 units to the four other bidders. It is also evident that bidder 1 does not find it profitable to change his quantity bid: he obtains with his current strategy exactly his Walrasian quantity. Bidders $i \in \{2,3,4,5\}$

have no profitable deviation as well. Again, if i submits a price lower than 4, he will not be served. If he tries to obtain a higher quantity simply by extending the bid quantity, he will not be served either. In this case the *inconsistent* seller will sell only one unit to bidder 1. The bidder can obtain a higher quantity $z_i = q_i$, by submitting a price-quantity pair (p_i, q_i) with $p_i > 0$ satisfying

$$p_i(1 + q_i) - (1 + q_i)^2/2 \geq 8. \tag{C2–1}$$

In that case the seller will supply the quantity $\phi = (1 + q_i)$ at the price of p_i. Bidder i will only be better off, if he acquires a quantity q_i and pays a price p_i, such that either

$$(5 - p_i) \cdot q_i > 0.8 \quad \text{for } q_i \leq 1, \tag{C2–2}$$

or

$$(5 - p_i) - (q_i - 1)p_i = 5 - q_i \cdot p_i > 0.8 \quad \text{for } q_i > 1. \tag{C2–3}$$

One can easily verify that no price-quantity pair (p_i, q_i) exists, which satisfies the conditions (C2–1) and [(C2–2) or (C2–3)]. The bidder has thus no profitable deviation.

□

5.4 Concluding remarks

In this chapter we analyzed the subgame perfect equilibrium outcomes of a class of multi-unit trade mechanisms–*auctions with variable supply.* We outlined a set of pricing rules, which guarantee efficient trade between the auctioneer and the bidders. Here we further discuss how our model relates to the classical model perfect competition as well as issues like collusion and extensions of the model.

5.4.1 The perfect competition model

The question of price formation is one of the basic and constantly recurrent topics in economics. In the classical representation of the perfect competition model, agents act as price-takers and the job of determining the market prices is left to an imaginary "auctioneer". This simple explanation leaves a lot to be questioned about how prices are formed.

This issue is resolved here in a standard way: by using noncooperative game theory to model the market interaction. We study dynamic (two-stage) market games, in which a monopolist first collects bids from his clients and then decides on supply quantity. We provide conditions on the pricing and the rationing rules, which guarantee competitive trade for this class of trade mechanisms.

The paper strengthens the noncooperative foundations of the Walrasian equilibrium in the sense that it presents trade mechanisms which actually generate the equilibrium prices of the imaginary *Walrasian auctioneer.* The agents in the model however are not price takers and the auctioneer is not imaginary and benevolent, but a profit-maximizing player of the market game. The obtained results rest on the assumption that the production function of the monopolist is common knowledge and buyers are completely informed about each other's preferences[24]. That assumption makes it possible to define the studied auction mechanisms as games of complete information.

The analysis shows that low-price equilibria of the endogenous supply uniform price auction do not exist, if the seller uses the proportional rationing rule and supplies consistently (Theorem 5.2). With discriminatory pricing competitive trades are obtained with the "pro rata on the margin" rationing rule without imposing additional conditions on the quantity selection function of the seller (Theorem 5.1). The latter result is surprising for several reasons. First, from the point of view of the theory of imperfect competition, one usually expects to observe on monopolistic markets a price higher than the competitive one. This does not happen since the monopolist plays a rather passive role. The seller is regulated by the trading mechanism and cannot make use of his monopolistic position[25]. Second, although the seller uses a discriminatory rule, price discrimination does not materialize due to the strategic behavior of the bidders. Still, this price discrimination rule eliminates the low-price equilibria known to exist in the uniform price multi-unit auction (see Wilson (1979), Back and Zender (1993), Back and Zender (2001) and our low price equilibrium example of the uniform price auction with the "pro rata on the margin" rationing rule).

5.4.2 Collusive bidding

It is straightforward that collusive behavior does not change the equilibrium allocation unless all the bidders, who demand positive quantities at the competitive price join forces to form a single coalition. Indeed, if there are two or more coalitions, each

[24]This is a standard assumption throughout the literature on competitive market games as it discusses Nash equilibrium outcomes. See section 5.1.2 for an overview of that literature.

[25]See Wilson (1978) for a model with similar structure and results.

coalition can be viewed as a single bidder with a demand function corresponding to the aggregate demand of the bidders in the coalition. The competition of two or more coalitions in the auction will lead again to the Walrasian outcome. This is the case because our analysis does not rely on symmetry considerations and works with arbitrary possibly asymmetric bidders' demand curves.

5.4.3 Extending the strategy space

In the exposition we assumed that bidders submit only a single price-quantity pair. This assumption is not essential for the results. It has been made to align the work with the literature on competitive market games, as well as for expositional clarity and in order not to introduce cumbersome notation. Our efficiency results will remain valid even if one allows the bidders to submit multiple price-quantity pairs or even entire demand schedules. This argument has been clarified in Damianov (2005b), who provides a proof of Theorem 5.2 in a framework in which bidders' strategy space include all left-continuous demand functions[26]. The key insight is that the deviations, which preclude the existence of non-Walrasian equilibria can indeed be single price-quantity pairs. Damianov's (2005b) arguments can analogously be applied to Theorem 5.1.

McAdams (2005, p. 6) claims that the existence of low-price equilibrium in the example provided in Damianov (2004a), which is presented here in subsection 5.3.2.1, is due to the restricted strategy space allowing only a single price-quantity pair. However, Theorem 1 in McAdams (2005), which ensures the non-existence of low-price equilibria, holds not because the bidders have richer strategy spaces, but because they have perfectly elastic demand curves. A departure from this assumption, especially assuming general (possibly asymmetric) demand functions as has been done in the present model, will cause McAdam's theorem to break down[27]. The low-price equilibrium presented here would exist also if bidders were allowed to submit entire demand schedules.

[26]The same framework has also been assumed in Back and Zender (2001).

[27]I am indebted to an anonymous referee for *Economic Theory* for pointing out this discrepancy.

Chapter 6

Competition among sellers by mechanism design

6.1 Introduction

As in the previous chapters in this chapter we will analyze a market in which multiple units are offered for sale. However, we will depart from the monopoly setting analyzed so far and will turn our attention to a setting in which several sellers, each in the possession of a single unit, choose a trade method in order to attract customers. We will address the question of what kind of trade mechanisms would arise in a *strategic* equilibrium of an oligopoly market, in which sellers compete for a common pool of customers by designing *trade mechanisms*.

Our analysis will be conducted within the independent private value model, in which buyers' valuations are privately observed and drawn from identical distributions. We will analyze a market game with the following time structure. In the first stage of the game sellers, who possess a single unit of a homogeneous good, simultaneously choose trade mechanisms from the class of anonymous and incentive-compatible mechanisms (see Myerson (1981)). In the second stage, upon observing the choice of the sellers, each buyer decides which seller to visit, if any. Randomizing over the sellers' mechanisms is allowed. In the third stage buyers learn their valuations and the number of their competitors participating at the same mechanism and submit their bids. Finally, the mechanisms are operated and the transactions take place.

The present inquiry belongs to a growing literature on mechanism design by competing sellers, which has been initiated by McAfee (1993). This literature studies the relationship between trade mechanisms offered by sellers and bidders' distribution across sellers as well as the consequences of this relationship for the sellers' choice of trade mechanisms in equilibrium.

It has basically two complex problems to deal with. The first one is to determine the equilibrium distribution of bidders across sellers for every profile of offered mechanisms. The second one is to solve for the equilibrium in sellers' trade mechanisms in the first stage of the game.

McAfee (1993) deals with the first problem by suggesting a new equilibrium concept, which he terms competitive subform consistent equilibrium (CSCE). It requires that every seller ignore his influence on the expected profits offered to buyers by other sellers. This assumption is applicable to a market with an infinite number of buyers and sellers, in which each seller's decision has no effect on the distribution of buyers across the other mechanisms, but is not appropriate for finite economies. Peters and Severinov (1997) propose a new limit equilibrium concept, competitive matching equilibrium (CME), which justifies McAfee's conjecture for a large number of market participants on both sides of the market.

The second problem relates to the complexity of the sellers' strategy space and is basically circumvented by significantly restricting the class of possible mechanisms. Burguet and Sakovics (1999), Peters and Severinov (1997) and Hernando-Veciana (2005) restrict the sellers' strategy space to second price auctions in which the sellers choose their reservation price. This assumption is convenient, because a trade mechanism can be described by a single variable. Burguet and Sakovics (1999) show that in the two-seller case reserve prices are not driven down to (zero) production costs and the mixed strategy[1] symmetric subgame perfect equilibrium (SPE) is inefficient. Hernando-Veciana (2005) demonstrates, that for any finite set of feasible reserve prices, reserve prices in a (SPE) go down to production costs if the numbers of auctioneers and bidders is sufficiently large, but finite.

In the present paper we will opt for neither of the solutions. On the one hand we will be interested in the (SPE) of the above described auction game, capturing all the repercussions that a change in a seller's mechanism has on the payoffs of the bidders with other mechanisms. On the other hand we will not be restricting the strategy space of the sellers, thus dealing directly with the general mechanisms design problem.

[1] In their model the existence of pure strategy equilibria is not guaranteed.

As in McAfee (1993) we will find the equilibrium trade mechanisms are auctions: the object should go to the highest value bidder. The equilibrium auctions have (zero cost) reservation price, but involve an entrance fee, which might depend on the number of bidders. We provide conditions for the existence of a unique symmetric equilibrium, in which the entrance fee does not depend on the number of participants. The entrance fee is derived as a function of the distribution of the buyer's valuations and the value buyers' value of the option of not entering the market. The paper contributes to the existing literature on the topic by solving the general mechanism design problem for the finite buyer and seller case. The use of second price auctions with entrance fees or any other payoff equivalent auction is here a derived result and not an assumption.

Generally two models have been suggested in the literature. The first one assumes that buyers learn their valuation after visiting a seller and inspecting the good on offer. The second one assumes that the buyers know their valuation prior to deciding which seller to visit. McAfee (1993), Burguet and Sakovics (1999) and Hernando-Veciana (2005) consider the former variant. Wolinsky (1988) considers the first one but in his model the matching technology of buyers and sellers is random and exogenous. Peters and Severinov (1997) analyze both cases for their limiting equilibrium concept. Both variants can be considered as benchmark cases. The former one is reasonable, if buyers need some time to study the good to form their valuation. If bidders search for some predefined attributes, the latter one will be more appropriate. It is however much more difficult to analyze, since the decision to visit a particular seller depends additionally on the valuation of the bidder. How bidders with different valuations will distribute over the sellers' mechanism according to their valuation in equilibrium is a difficult problem to solve. It still remains open issue in its finite version, in which sellers are free to choose any direct mechanism.

We show that the first problem is tractable even for a finite number of buyers and sellers and a very general sellers' strategy space. Peters and Severinov (1997, pp. 147-153) also consider this case, but restrict the strategy space to second price auctions with a reserve price. Surprisingly, this restriction of the strategy space leads to problems with the existence of a (SPE). Burguet and Sakovics (1999, p. 240) provide an example for the nonexistence of a (SPE) in pure strategies in a market of two sellers and two buyers, whose valuations are drawn from a uniform distribution with support $[0, 1]$.

The chapter is organized as follows. In the next section we will present the model: the general framework, the strategy space of buyers and sellers, their payoffs as well as the concept of a (SPE). Section 6.3 contains the main results and their proofs as well as a numerical example for a (SPE) in a market of two sellers and two buyers. Section 6.4 concludes with a discussion of how the derived results relate to the literature on the topic.

6.2 The model

6.2.1 Preliminaries

We consider an imperfectly competitive market with a number of $J \geq 2$ sellers (females) and $I \geq 2$ buyers (males). All agents are risk neutral. Each seller possesses a unit of a commodity, which she wishes to sell to a buyer. The use value equals across sellers and without loss of generality is normalized to zero. The sellers compete in the market by simultaneously choosing trade mechanisms. After observing the sellers' "offers" the buyers either choose a seller, whose trade mechanism to participate in or stay out of the market exercising an outside option. Buyers are allowed to participate only in one trading mechanism, but randomizing over the sellers' trade mechanism and the outside option is possible. Exercising the outside option[2] is associated with a sure payoff of $\beta \geq 0$ for a buyer. Once a buyer selects a seller, he learns his valuation, which is a draw from a random variable. After learning his valuation and the number of the bidders participating in his mechanism, bidders participate in the mechanism by reporting their type. Finally, the resulting allocation is implemented.

6.2.2 Notation

Buyers will be indexed by i and sellers by j. The valuation of buyer i, x_i is private information and a random draw from the interval $[0, 1]$ according to the continuously differentiable distribution function F. If a buyer participates in the mechanism of a certain seller, he will be asked to report his private valuation to the mechanism. Since the number of bidders visiting certain seller is not known ex-ante, the mechanism should prescribe an allocation and a payment rule for any number (and identity) of bidders and any realization of their valuations. Let us denote for that purpose the set

[2]The outside option can be broadly understood. It might for example be associated with the transportation costs to arrive at the marketplace or the costs which a bidder spends to learn his own valuation.

of the subsets (the power set) of all bidders by $\mathcal{I}$ and the power set of all rivals of bidder i by $\mathcal{I}^{-i}$. Let $s \in \mathcal{I}$ denotes a group of bidders and x^s the ordered vector[3] of the valuations of bidders from the group s. Further let X^s denote the set of all possible ordered vectors of their valuations. Let

$$X \equiv \bigcup_{s \in \mathcal{I}} X^s$$

denote the set of all ordered vectors of the valuations of all subsets of bidders and x an element of this set[4]. Similarly by X_{-i} one denotes the set of the ordered vectors of the valuations of all subsets of bidders except bidder i.

6.2.3 Sellers' strategy space

Before defining the sellers' strategy set, let us first denote the set

$$\mathfrak{A} \equiv \Big\{ \{(p_i, z_i)\}_{i \in I} \mid (p_i, z_i) : X \to [0,1] \times \mathbb{R} \Big\},$$

which contains all trade mechanisms satisfying the conditions (NP),(F) and (A) given below. Generally, a trade mechanism is defined by the functions $p_i(\cdot)$ for every bidder i, which determine the probability, with which every bidder i receives the item and by the functions $z_i(\cdot)$, determining the payment of every bidder i to the mechanism.

- *Non-participation condition:*

$$p_i(x^s) = 0, \quad z_i(x^s) = 0, \forall i \notin s. \tag{NP}$$

  The condition requires that bidders who do not participate in a certain mechanism don't win the object and don't pay.

- *Feasibility:*

$$\sum_{i=1}^{I} p_i(x) \leq 1, \quad \forall x \in X. \tag{F}$$

  The feasibility conditions requires that for any realization of the private information of the participating bidders the mechanism rules do not allow more units to

[3]By ordered vector x^s we refer to the vector of the valuations of the bidders from a subset s, in which the components are ordered in an ascending order according to the bidder's number.

[4]Note that the valuation of each bidder i, x_i, might or might not appear in the vector x depending on whether this bidder i participates in the mechanism or not.

be sold than physically available. Here we allow also for mechanisms for which for some realizations of x the inequality can be satisfied. This might for example be the case if the mechanism is a second-price auction with a positive reservation price. I such a case if the valuation of the participating bidders lie below the seller's reserve price, she will retain the item.

- (A) *Anonymity*:
Let
$$p(\cdot) \equiv \Big(p_1(\cdot), p_2(\cdot), \ldots, p_I(\cdot)\Big) \quad \text{and} \quad z(\cdot) \equiv \Big(z_1(\cdot), z_2(\cdot), \ldots, z_I(\cdot)\Big)$$
denote the vectors of probability and allocation functions (respectively). The anonymity condition requires that the functions p and z are *permutation invariant*. This means that permuting the valuations of any ordered vector $x \in X$ permutes the vectors $p(x)$ and $z(x)$ in the same fashion. Let (x_k, x_l, x^s) denote the ordered vector of the valuation of the bidders from the group s and the bidders $l, k \notin s$. Then the permutation invariance implies:
$$\begin{aligned} p_k(x_k, x_l, x^s) &= p_l(x_l, x_k, x^s), \\ z_k(x_k, x_l, x^s) &= z_l(x_l, x_k, x^s), \\ p_i(x_k, x_l, x^s) &= p_i(x_l, x_k, x^s), \\ z_i(x_k, x_l, x^s) &= z_i(x_l, x_k, x^s), \end{aligned}$$
$\forall l, k, \forall i \in s, \forall s$. The anonymity or equal treatment guarantees that sellers do not discriminate among buyers on characteristics different than their reports to the mechanism or in other words the chances of winning and the payment are not dependent on the buyers' identities but solely on their reports to the mechanism. McAfee (1993) and Peters (1994) also consider anonymous mechanisms and provide equivalent definitions.

The set $\mathfrak{A}$, satisfying the above conditions, is larger than the sellers' strategy set. We will further narrow down the set of possible mechanisms among which the sellers can choose by imposing additional conditions on the probability and payment functions. To define the sellers' strategy space, which we will denote by $\widehat{\mathfrak{A}}$, we impose the two additional requirements.

- *Incentive compatibility:*
Let us assume that bidder i chooses to participate in mechanism j. He learns

his valuation x_i (by inspecting the object for sale for example) and the fact that he will compete for the object with the buyers from the set $s \in \mathcal{I}^{-i}$. We denote the expected probability of winning and the expected payment of bidder i, who reports the valuation $\tilde{x}_i$, provided that the other participants report truthfully by

$$P^s(\tilde{x}_i) := \int p_i(\tilde{x}_i, x^s) dF(x^s),$$
$$Z^s(\tilde{x}_i) := \int z_i(\tilde{x}_i, x^s) dF(x^s).$$

The incentive compatibility requires, that bidder i finds it profitable to report truthfully if all other bidders do so, i.e. for every $s \in \mathcal{I}^{-i}$ and every $\tilde{x}_i \in [0,1]$ the following inequality holds:

$$\begin{aligned} E^s(\tilde{x}_i \mid x_i) &\equiv x_i \cdot P^s(\tilde{x}_i) - Z^s(\tilde{x}_i) \\ &\leq x_i \cdot P^s(x_i) - Z^s(x_i) \equiv E^s(x_i \mid x_i) =: E^s(x_i). \end{aligned} \tag{IC}$$

$E^s(\tilde{x}_i \mid x_i)$ is the expected payment of a bidder, who has a valuation of x_i and reports the valuation $\tilde{x}_i$ to the mechanism.

There is indeed no loss of generality to restrict the sellers to use incentive compatible mechanisms. In the present setting buyers submit bids after they learn their valuations and the number of their fellow bidders (but not their valuations), so the sellers' mechanisms described here are standard Bayesian games for which the *revelation principle* applies (see e.g. Myerson (1997, p. 260)).

- (R) *Regularity:*
  Let us assume that bidder i participates in a certain mechanism j with a probability of one and let all other bidders visit this mechanism with a probability of m. The regularity condition requires that the expected payoff of a bidder from participating in the mechanism j is (weakly) decreasing in the probability m. A formal definition will be given later on after we define the bidders' strategy space and their expected payoff. Roughly speaking, the condition requires that a bidder's expected payoff decreases with increased competition for this mechanism.

6.2.4 Bidders' strategy space

Conditional on observing the mechanisms offered by the sellers, the bidders decide which seller to visit. Although bidders are not allowed to visit more than one seller, randomizing over the sellers is allowed. Thus, the bidders play a *behavior strategy* as they decide on every node defined by a profile of trade mechanisms of the sellers, which seller to visit. Eichberger (1993, pp. 22-24) offers a definition and a discussion on the behavior strategy concept. The strategy of bidder i is a mapping from the set of possible vectors of trade mechanism into probabilities, with which that bidder plans to visit each seller. We will denote a strategy of bidder i by

$$m^i = \Big(m^i_o(\cdot), m^i_1(\cdot), m^i_2(\cdot) \ldots, m^i_J(\cdot)\Big),$$

where

$$m^i_j : \widehat{\mathfrak{A}}^J \to [0,1]; \quad m^i_o : \widehat{\mathfrak{A}}^J \to [0,1] \quad \text{and} \quad m^i_o(\cdot) + \sum_{j=1}^{J} m^i_j(\cdot) = 1.$$

Here $m^i_o(\cdot)$ denotes the probability with which bidder i stays out of the market. It will be useful to represent a strategy profile of the bidders by the $I \times (J+1)$ matrix

$$m(\cdot) := \begin{pmatrix} m^1_o(\cdot) & m^1_1(\cdot) & m^1_2(\cdot) & \ldots & m^1_J(\cdot) \\ m^2_o(\cdot) & m^2_1(\cdot) & m^2_2(\cdot) & \ldots & m^2_J(\cdot) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^I_o(\cdot) & m^I_1(\cdot) & m^I_2(\cdot) & \ldots & m^I_J(\cdot) \end{pmatrix}.$$

A strategy profile of all bidders except bidder i will be denoted by $m^{-i}(\cdot)$. We will say, that the bidders use a symmetric behavioral strategy, if the functions in every column of the matrix are identical. A symmetric strategy profile will be denoted by

$$\big(m_o(\cdot), m_1(\cdot), m_2(\cdot), \ldots, m_J(\cdot)\big).$$

6.2.5 Payoffs

Let us denote by $m^{-i}_j(p,z)$ the vector of probabilities with which all bidders except i visit mechanism j (this is the $j-$th column in the above matrix, except the probability of bidder i). If bidder i visits mechanism j with a probability of one, then his payoff is given by

$$R^i_j\Big((p^j, z^j); m^{-i}_j(p,z)\Big) = \sum_{s \in \mathcal{I}^{-i}} \prod_{l \in s} m^l_j(p,z) \cdot \prod_{k \in I/s,i} m^k_j(p,z) \cdot \int_0^1 E^s_j(x_i) dF(x_i).$$

In the payoff of bidder i one sums the products of the probabilities with which bidder i encounters any group of rivals and his expected payoff in case that this group of rivals is encountered. The payoff of seller j is

$$\Pi_j\Big((p,z);m(p,z)\Big) = \sum_{i=1}^{I} m_j^i(p,z)\cdot\Big(\sum_{s\in\mathcal{I}^{-i}}\prod_{l\in s} m_j^l(p,z)\cdot\prod_{k\in I/s,i} m_j^k(p,z)\cdot\int_0^1 Z_j^s(x_i)dF(x_i)\Big),$$

which is the sum of the expected payments of the bidders to the mechanism j. Since we consider anonymous mechanisms, the functions $E_j^s(x_i)$, $Z_j^s(x_i)$ and $P_j^s(x_i)$ depend only on the number of rivals of bidder i (in the set s) and not on their identity. Therefore for simplicity from now on we will use the notation $E_j^{(n)}(x_i)$, $Z_j^{(n)}(x_i)$ and $P_j^{(n)}(x_i)$ when describing the payoff, the payment and the probability with which bidder i is served when he faces $(n-1)$ rivals. If all rivals of bidder i visit mechanism j with a probability of m, then his payoff from participating with a probability of one is

$$R_j^i\Big((p^j,z^j);m\Big) = \sum_{n=1}^{I}\binom{I-1}{n-1}m^{n-1}(1-m)^{I-n}\cdot\int_0^1 E_j^{(n)}(x_i)dF(x_i).$$

If all bidders visit mechanism j with probability m, then the expected profit of seller j is

$$\Pi_j\Big((p^j,z^j);m\Big) = \sum_{n=1}^{I}\binom{I}{n}m^{n}(1-m)^{I-n}\cdot\int_0^1 Z_j^{(n)}(x_i)dF(x_i).$$

Now, we can formally define the regularity condition introduced in subsection (6.2.3).

Definition 6.1 (R). *A mechanism (p^j,z^j) is* ***regular*** *if the function $R_j^i\Big((p^j,z^j);m\Big)$ is (weakly) decreasing in m.*

The regularity condition is satisfied by the standard auction formats. We will show that all payoff equivalent mechanisms to a second price auction with an entrance fee, which does not depend on the number of participating bidders, are regular mechanisms[5] (see lemma 6.2). The regularity condition is not satisfied for example by mechanisms according to which the seller imposes high participation fees if a low number of bidders participate and a low participation fee (or even a bonus) if many bidders visit the mechanism. In such a situation an increased competition can lead to higher expected payoffs for the participants.

If bidder i employs the behavioral strategy $m^i(p,z)$, his payoff is:

$$\mathcal{R}_i\Big((p,z);m(p,z)\Big) = m_o^i(p,z)\cdot\beta + \sum_{j=1}^{J} m_j^i(p,z)\cdot R_j^i\Big((p^j,z^j),m^{-i}(p,z)\Big).$$

[5] One can easily show that the second price auction with a non-trivial reserve price is a regular mechanism as well (the proof of this claim emulates the proof of lemma 6.2, which is given in Appendix 6.A).

As already indicated, after arriving at the mechanism, buyers learn their valuation (for example by inspecting the item for sale) and the number of their fellow bidders participating at that mechanism. As we require that the mechanisms are incentive-compatible, bidders report their valuations truthfully, the mechanisms are operated and the transactions take place.

6.2.6 Equilibrium

In this model we will be interested in the symmetric subgame perfect equilibria of the model, which are defined as follows.

Definition 6.2. *The sellers' strategy profile (p^*, z^*) and the symmetric selection behavioral strategy functions of the bidders represented by the matrix $m^*(\cdot)$ constitute a* ***symmetric subgame perfect equilibrium*** *(short: equilibrium), if they satisfy the following conditions:*

1. *(Optimal selection by buyers):*

$$\mathcal{R}_i\big((p,z); m^{*i}(p,z), m^{*-i}(p,z)\big) \geq \mathcal{R}_i\big((p,z); m^i, m^{*-i}(p,z)\big), \quad \forall (p,z) \in \widehat{\mathfrak{A}}^J, \forall i, \forall m^i \in [0,1].$$

2. *(Nash equilibrium play in the reduced form of the game):*

$$\Pi_j\Big((p^{*j}, z^{*j}), (p^{*-j}, z^{*-j}); m^*(\cdot)\Big) \geq \Pi_j\Big((p^j, z^j), (p^{*-j}, z^{*-j}); m^*(\cdot)\Big), \forall (p^j, z^j) \in \widehat{\mathfrak{A}}. \tag{NE}$$

3. *(Symmetry): All sellers use the same trade mechanism.*

The first equilibrium condition requires that in each subgame defined by the sellers' choice of mechanisms the bidders randomize symmetrically over the mechanisms, i.e. they play symmetric behavior strategies, which constitute a Nash equilibrium in the second stage of the game. The second condition requires that the sellers choose mechanisms, which build a Nash equilibrium in the first stage of the game.

6.3 Analysis

6.3.1 Organization of the analysis and results

In this work we will show that in equilibrium sellers hold auctions (Theorem 6.1). Holding auctions in this setting amounts to using a trade mechanism, which assigns the unit to the participant with the highest valuation (see McAfee (1993, p. 1292) and the exposition of the next subsection). To summarize the results and explain the

	$\mathfrak{A}$	$\widehat{\mathfrak{A}}$	Ω	$\widehat{\Omega}$
Non-Participation	+	+	+	+
Feasibility	+	+	+	+
Anonymity	+	+	+	+
Incentive Compatibility		+	+	+
Regularity		+	+	+
Efficiency			+	+
Constant entrance fee				+

Table 6.1: *Mechanism sets and conditions.*

arguments, on which the proofs are based, Table 6.1 will be helpful. The rows in this table represent conditions imposed on trade mechanisms. The first five conditions are already defined. The "Efficiency" condition requires that the object should always be granted to the participant with the highest valuation. The "Constant entrance fee" condition requires that the seller uses an auction with an entrance fee, which does not depend on the number of bidders participating at the mechanism. The + sign denotes which conditions are satisfied by the mechanisms from the sets $\mathfrak{A}$, $\widehat{\mathfrak{A}}$, Ω and $\widehat{\Omega}$. The sets $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ are already defined. $\widehat{\mathfrak{A}}$ is the sellers' strategy set. Lemma 6.1 derives the profit maximizing mechanisms (for a seller) among all mechanisms from the set $\mathfrak{A}$ which give every participating bidder a constant expected payment (of R^*), provided that all bidders visit this mechanism with a certain probability (of m^*). It states that the profit-maximizing mechanism should be efficient, i.e. the object should go to the participant with the highest valuation. As a consequence of the lemma one narrows down substantially the set of mechanism, which can constitute an equilibrium in the game with strategy set $\widehat{\mathfrak{A}}$. Imposing additionally the (IC) and (R) conditions, one obtains that only the mechanisms from the set Ω are possible equilibria. This set consists only of auctions with zero reserve price and an entrance fee, which might depend on the number of participating bidders (this argument rests on standard results from auction theory). In the exposition later on we will discuss this argument in more detail. We further are restricting attention only to the set $\widehat{\Omega}$, which consists of auctions with an entrance fee independent on the number of bidders visiting the mechanism. In Theorem 6.2 we show that if a strategy profile of the sellers is an equilibrium in the game with strategy space $\widehat{\Omega}$, it is also an equilibrium in the game with a strategy space Ω. It follows that this equilibrium strategy profile constitutes an equilibrium also in the game with strategy space $\widehat{\mathfrak{A}}$. Theorem 6.3 provides conditions for the existence of equilibrium (within the set $\widehat{\Omega}$) and characterizes the equilibrium trade mechanism.

6.3.2 Theorems and proofs

Theorem 6.1. *(i) The sellers' equilibrium mechanisms (provided that an equilibrium exists) assigns the item (almost surely) to the highest-valuation bidder, if this valuation is higher than the seller's use value.*

(ii) The equilibrium mechanisms are payoff equivalent to a second price auction with a reserve price equal to the seller's valuation and an entrance fee, which might depend on the number of participating bidders.

This theorem establishes some equilibrium properties without resolving the question of existence of an equilibrium. We shall deal later on with this problem by providing conditions which guarantee the existence and uniqueness of a symmetric equilibrium in this market game for an arbitrary number of sellers and buyers. The present theorem is useful, as it restricts the type of mechanism profiles, which can constitute an equilibrium. This initial result will further be employed for the characterization of equilibrium and for the existence and uniqueness proof itself.

Proof. For part *(i)* we will proceed by contradiction. Take an equilibrium profile (p^*, z^*) and let in this equilibrium buyers visit a certain mechanism j with probability m^*. Let the expected profit of a buyer participating in the mechanism of seller j be denoted by E^*. We will show that if the equilibrium mechanism (p^{j*}, z^{j*}) does not satisfy the condition of part *(i)* of the theorem, one can construct a deviation mechanism $(p^{jD}, z^{jD}) \in \widehat{\mathfrak{A}}$ which assigns the object to the highest valuation bidder and does not change the equilibrium probability with which buyers visit that seller. We show that this mechanism is more profitable for the seller, reaching a contradiction to the equilibrium requirement (NE). We start with the following lemma.

Lemma 6.1. *The profit maximizing mechanisms for an arbitrary seller j, among all mechanisms from the set $\mathfrak{A}$ which give every participating bidder an expected payment of R^*, provided that all bidders visit this mechanism with probability m^*, assign the object with probability one to the participant with the highest valuation, if this valuation exceeds the seller's use value.*

A formal proof of the lemma is provided in the Appendix 6.A. The statement is closely related to an argument provided by McAfee and McMillan (1987b), which concerns a setting with one seller and an outside option. Their argument is useful to understand the idea of the current proof and therefore will be shortly sketched here. For any number of participating bidders the seller's expected revenue is the winning bidder's expected valuation minus the expected profit of the participating bidders. Thus, for any given number of participating bidders the seller should award the good so as to maximize the expected valuation of the winner. This can only be done by awarding the good to the highest valuation bidder whenever this valuation exceeds the seller's reserve value[6].

Observe that the lemma does not require that the mechanisms satisfy the incentive compatibility constraint (IC) or the regularity condition (R). If we found a deviation mechanism (p^{jD}, z^{jD}) which belongs to the set $\widehat{\mathfrak{A}}$ (i.e. satisfies additionally the conditions (IC) and (R)) and does not change the probability distribution of buyers across sellers, then from the lemma would follow that this deviation is profitable.

Observe that a bidder participating in the mechanism of seller j might face any number of 0 to $I-1$ bidders. Let $n \in \{1, 2, \dots, I\}$ denote the total number of bidders participating in the mechanism of that seller j. The winning probability of bidder i with valuation x_i who reports valuation $\tilde{x}_i$, if the other $(n-1)$ bidders report truthfully is $P_i^n(\tilde{x}_i) \equiv [F(\tilde{x}_i)]^{n-1}$. By the Envelope theorem one obtains for the derivative of the payoff of bidder i at an incentive-compatible mechanism[7] which awards the good to the highest valuation bidder:

$$\frac{d}{dx_i}(E^{(n)}(\tilde{x}_i \mid x_i)) = \frac{\partial}{\partial x_i}(E^n(\tilde{x}_i \mid x_i))\bigg|_{\tilde{x}_i = x_i} = [F(x_i)]^{n-1}.$$

The expected profit is

$$E^{(n)}(x_i \mid x_i) = C_n + \int_0^{x_i} [F(y)]^{n-1} dy,$$

[6]In our setting the seller's reserve value is 0 and the bidders' valuations are distributed on the interval $[0,1]$, so they are almost surely higher than the seller's reserve value.

[7]Here we follow the exposition in McAfee and McMillan (1987b).

where C_n is the expected profit of a bidder with the lowest valuation 0. C_n is thus the entrance fee or bonus, which each bidder has to pay or receives, when participating in the mechanism with $(n-1)$ other bidders. From the theory of optimal auctions[8] it is know that the (ex ante) expected payment of a bidder participating with $(n-1)$ other bidders in an incentive-compatible mechanism with 0 entrance fee, which assigns the object to the highest value bidder is

$$B_n = \int_0^1 \int_0^x [F(z)]^{n-1} dz dF(x).$$

The expected profit of a seller who auctions an item to n bidders is

$$S_n = n \cdot \int_0^1 [x \cdot f(x) + F(x) - 1] \cdot [F(x)]^{n-1} dx.$$

All these mechanism are payoff equivalent to a second price auction with a zero reservation price.

A mechanism from the set Ω can now be identified only by the participation fee for any number of participants and will be denoted by $(C_1, C_2, \ldots, C_I)$. For constructing the deviation mechanism consider a mechanism $(\underbrace{C, C, \ldots, C}_{I})$ requiring the same participation fee independent on the number of the bidders[9]. Consider the following lemma.

Lemma 6.2. *All incentive compatible mechanisms involving an entrance fee, which is independent on the number of participants, are* ***regular*** *mechanisms.*

The proof is somewhat technical and not of interest in itself. It is moved to Appendix 6.A. The lemma guarantees that this deviation mechanism indeed belongs to the set $\widehat{\mathfrak{A}}$. The expected profit of a bidder from participating in the deviation mechanism, if every other bidder participates with probability m^* is

$$E^* = \sum_{n=0}^{I-1} B_{n+1} \cdot \binom{I-1}{n} (m^*)^n (1-m^*)^{I-1-n} - C.$$

Choosing a participation fee of

$$\sum_{n=0}^{I-1} B_{n+1} \cdot \binom{I-1}{n} (m^*)^n (1-m^*)^{I-1-n} - E^*$$

[8]See for example Riley and Samuelson (1981) or McAfee and McMillan (1987a).
[9]There are many ways to construct a deviation mechanism. This is one of the variants.

would present the desired deviation mechanism. It remains to be verified, that this deviation will not reshuffle the probability distribution of bidders across sellers. This is guaranteed by the regularity condition imposed on the strategy set $\widehat{\mathfrak{A}}$. This condition precludes the cases in which a bidder participating in a certain mechanism obtain the same payoff in cases in which the other bidders visit this mechanism with a different probability. Indeed, if each bidder visits mechanism j with a probability higher than m^*, then the expected profit of each bidder will fall below E^*, whereas the expected profit with other mechanism will increase above E^*. On the other hand, if each bidder visits mechanism j with a probability lower than m^*, then the bidders' expected profit with j will rise and the expected profit with other mechanisms will fall. In sum, the bidders' selection stage of the game will not be in equilibrium for any probability of visiting j, which is different than m^*. □

Recall that we denoted the set of incentive compatible and regular mechanisms, in which the highest valuation bidder wins and the entrance fee is independent on the number of participating bidders by $\widehat{\Omega}$. One can state the following theorem.

Theorem 6.2. *If the sellers' strategy profile*

$$\Big(\underbrace{(\underbrace{C^*, C^*, \ldots, C^*}_{I}), (\underbrace{C^*, C^*, \ldots, C^*}_{I}), \ldots, (\underbrace{C^*, C^*, \ldots, C^*}_{I})}_{J}\Big)$$

is an equilibrium profile of the game with strategy space $\widehat{\Omega}$, then it is also an equilibrium profile in the game with strategy space Ω.

The theorem is useful for the proof of the existence of equilibrium in the original game (with a strategy set $\widehat{\mathfrak{A}}$). The next theorem will assert that the game with strategy space $\widehat{\Omega}$ has an equilibrium. From the present theorem and theorem 6.1 follows then that the original game has the same equilibrium profile.

Proof. Take an equilibrium strategy $(\underbrace{C^*, C^*, \ldots, C^*}_{I})$ and assume by a way of contradiction that there exists a profitable deviation of an arbitrary seller j, which we denote by $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I) \in \widehat{\mathfrak{A}}$. Let us assume that this strategy induce an equilibrium participation probability of $\tilde{m}$ and as the deviation is profitable we have

$$\Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) > \Pi_j\Big((\underbrace{C^*, C^*, \ldots, C^*}_{I}); m^*\Big).$$

The expected profit of bidder i is

$$R_j^i\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot B_n - \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot \tilde{C}_n$$

and of the seller

$$\Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot S_n + \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot n \cdot \tilde{C}_n.$$

Since we consider only *regular* mechanisms the strategy $(\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}) \in \widehat{\mathfrak{A}}$, where

$$\tilde{C} = \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot \tilde{C}_n$$

induces the same unique participation probability $\tilde{m}$ and leads to the same expected bidder's payoff:

$$R_j^i\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = R_j^i\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big).$$

The payoff of the seller is

$$\Pi_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big) = \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot S_n + I \cdot \tilde{m} \cdot \tilde{C}.$$

One can readily observe now that

$$\begin{aligned}\sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot n \cdot \tilde{C}_n &= I \cdot \tilde{m} \cdot \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot \tilde{C}_n \\ &= I \cdot \tilde{m} \cdot \tilde{C} \qquad \Leftrightarrow \\ \Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) &= \Pi_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big).\end{aligned}$$

Indeed, $(\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I})$ is so constructed that the expected payment of each bidder to seller j equals the expected payment under the deviation strategy $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I)$ if both mechanisms are visited with the same probability of $\tilde{m}$. Therefore the expected fees that the seller obtains from every bidder are equal in both mechanisms. The resulting (in)equalities

$$\Pi_j\Big((\underbrace{\tilde{C},\tilde{C},\ldots,\tilde{C}}_{I});\tilde{m}\Big) = \Pi_j\Big((\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_I);\tilde{m}\Big) > \Pi_j\Big((\underbrace{C^*,C^*,\ldots,C^*}_{I});m^*\Big)$$

establish the desired contradiction to the equilibrium assumption.

To summarize, we assumed by contradiction that a certain strategy profile is an equilibrium of the game with the strategy space $\widehat{\Omega}$ and not an equilibrium of the game with the strategy space Ω. As a consequence for one bidder a profitable deviation strategy from the set Ω exists. Then however we demonstrate that a profitable deviation strategy from the set $\widehat{\Omega}$ also exists, which poses a contradiction to the equilibrium assumption. □

The next theorem will characterize the equilibria for the game with strategy set $\widehat{\Omega}$. For that purpose we will introduce some addition notation. Let us denote the expected payoff of a bidder participating in a second price auction with a zero entrance fee if all other bidders visit this mechanism with a probability of m by

$$R_{(m)} := \sum_{n=1}^{I}\binom{I-1}{n-1}(m)^{n-1}\cdot(1-m)^{I-n}\cdot B_n.$$

Let all sellers except seller j play the strategy $(\underbrace{C,C,\ldots,C}_{I})$, and let seller j employ the strategy $(\underbrace{C^j,C^j,\ldots,C^j}_{I})$. Let $\overline{m}(C^j,C)$ be defined as the solution of the equation

$$R_{(m)} - C^j = R_{(\frac{1-m}{J-1})} - C$$

and $\underline{m}(C^j,\beta)$ be the solution of the equation

$$R_{(m)} - C^j = \beta.$$

The function $\overline{m}(C^j,C)$ determines the probability with which bidders visits seller j, provided that they use the outside option with a probability of 0. The function $\underline{m}(C^j,\beta)$ determines the probability with which bidders will visit seller j, provided that they randomize between the outside option and the mechanism of seller j. Let $\Pi_j(C^j;m)$ denote the payoff of seller j if she holds an auction with a participation fee of C^j

(independent on the number of participants) and all other bidders visit this mechanism with a probability of m. Define the functions

$$\overline{\varphi}(C) := \frac{\partial \Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)}{\partial C^j}\Bigg|_{C^j=C}$$

and

$$\underline{\varphi}(C^j, \beta) := \frac{\partial \Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)}{\partial C^j}.$$

Theorem 6.3 (equilibrium). *The game with strategy space $\widehat{\Omega}$ has a unique symmetric subgame perfect equilibrium in pure strategies if the functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave with respect to C^j. The equilibrium fee is*

$$C^*(\beta) = \begin{cases} \overline{C} & \text{for} \quad \beta \leq R_{(1/J)} - \overline{C}, \\ \max\{R_{(1/J)} - \beta, \underline{C}(\beta)\} & \text{for} \quad \beta > R_{(1/J)} - \overline{C}, \end{cases}$$

where $\overline{C}$ is the unique solution of the equation $\overline{\varphi}(C) = 0$ and $\underline{C}(\beta)$ is the unique solution of the equation $\underline{\varphi}(C^j, \beta) = 0$.

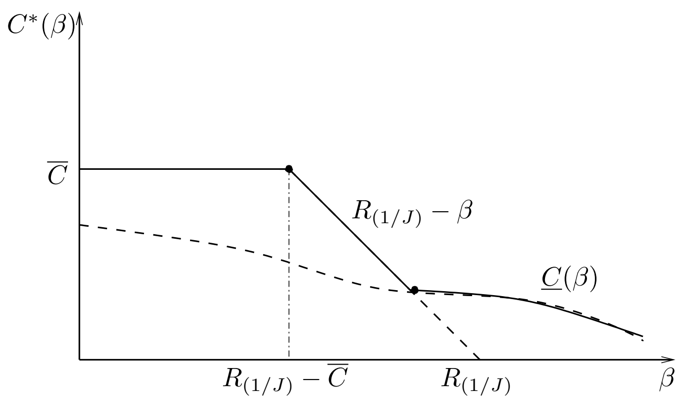

Figure 6.1: *The unique equilibrium entrance as a function of the outside option (the solid line).*

See Appendix 6.B for a proof and figure 6.1 for a graphical illustration. Next we will investigate which markets satisfy the premises of the above theorem.

6.3.3 Concavity of the payoff functions

Theorem 6.3 provides a condition for the existence and uniqueness of equilibrium, which requires that the payoff functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave in C_j. Although we could not find an example in which this property is not satisfied, we also could not show that this property is satisfied for all probability distributions F and any number of buyers and sellers. In this section we will show that in small markets (in the cases of two sellers and two or three buyers) the functions are convex for all F and there always exists a unique equilibrium. A unique equilibrium exists also for any number of at least up to 100 buyers and sellers if F is uniformly distributed. If β is sufficiently high and F is uniformly distributed an equilibrium exist for any number of buyers and sellers. The next two theorems establish these results.

Theorem 6.4. *In the cases $J = 2$ and $I \in \{2, 3\}$ the functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave in C^j for any distribution F.*

Theorem 6.5. *If F is uniformly distributed over the unit interval the function*

$$\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$$

is concave for any number of $I \geq 2$ buyers and $J \geq 2$ sellers.

The proofs are in Appendix 6.B.

6.3.4 Numerical example: two buyers and two sellers

Claim 6.1. *If two sellers compete for two buyers (i.e. I=J=2) the equilibrium entrance fee is*

$$C^*(\beta) = \begin{cases} S_2/2 & \text{for } 0 \leq \beta < B_2, \\ B_2 + S_2/2 - \beta & \text{for } B_2 \leq \beta \leq B_2 + S_2/2, \\ 0 & \text{for } \beta > B_2 + S_2/2. \end{cases}$$

See Appendix 6.B for a proof and Figure 6.2 for a graphical illustration of the equilibrium entrance fee. The equilibrium probability with which bidders visit an arbitrary seller $j \in \{1, 2, \ldots, J\}$ is given as follows[10]:

$$m_j^*(\beta) = \begin{cases} 1 & \text{for } 0 \leq \beta < B_2 + S_2/2, \\ (B_1 - \beta)/(B_1 - B_2) & \text{for } B_2 + S_2/2 \leq \beta \leq B_1, \\ 0 & \text{for } \beta > B_1. \end{cases}$$

Observe that for $\beta \in [B_2 + S_2/2, B_1]$ in equilibrium sellers lower the participation fee to allow all bidders to enter the market with a probability of one. This holds true until the

[10]See Appendix 6.B.

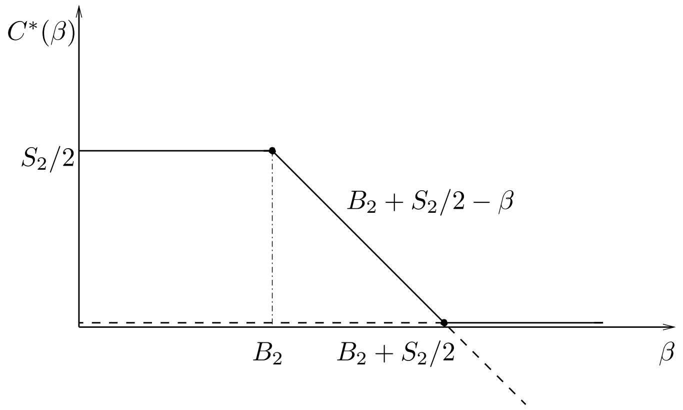

Figure 6.2: *Entrance fee in a subgame perfect equilibrium in the case $I = J = 2$ (the solid line).*

entrance fee falls down to 0. As β further increases sellers hold zero-reserve auctions with no entrance fee (or bonus). The entry probability in the market decreases linearly as β further increases. For $\beta > B_1$ bidders do not enter the market any more.

6.4 Concluding remarks

The classical auction model studies the mechanism design problem of a monopoly seller in an environment of incomplete information regarding the valuations of the bidders. The present paper departs from this framework by considering a model of two or more sellers, who compete for the same pool of customers by designing trade mechanisms.

The primary message of the paper is that in a market of finitely many buyers and sellers the equilibrium trade mechanisms will be auctions with a trivial reserve price. The model can be considered as a complement to McAfee's (1993) pioneering work, in which similar result is obtained in a model describing an infinite economy.

Such a situation is evident in many markets. In housing markets close substitutes are sold via auctions; auction houses compete by selling similar products; on internet sites such as Ebay or Amazon sellers offer identical commodities like cameras, computers and other standardized products using a variety of sale methods: posted price, English auctions, Dutch auctions, auctions with a buy-it-now option, auctions with secret reserve prices, auctions with different closing rules, etc. Generally, the trade mechanism appears to be an important instrument in the competition for customers along the characteristics of the offered product.

Auctions with a trivial reserve price, called *absolute auctions*, are used as a sale method for instance in markets for restaurant equipment and real estate. Manning (2000), a real estate and restaurant equipment auctioneer, asserts that in his experience the public response to a property sold via an absolute auction is much more enthusiastic than similar property offered at an auction with a reserve price. Another (historical) example underscoring the benefits of an absolute auction is the rapid growth of trade through the Port of New York relative to the trade through other East Coast ports of the United States following the War of 1812. Engelbrecht-Wiggans and Nonnenmacher (1999) provide evidence that in the two decades following the War, New York's trade grew significantly, while other ports stayed at roughly their 1811 level. The data they collected suggests that this growth is due primarily to the change in the law regarding auctions of imports, which discourages the setting of reservation prices. Both examples lend support to the theoretical prediction of our stylized model probably because they picture scenarios in which prospective buyers need a close scrutiny of the object to form their valuation as is assumed in our model.

Similar result concerning the optimal auction in the monopoly case, in which bidders can either exercise an outside option or enter the auction market have been derived in Engelbrecht-Wiggans (1987), McAfee and McMillan (1987b) and Engelbrecht-Wiggans (1993).

Appendix 6.A

Proof of lemma 6.2:

We have to show that the function

$$R_j^i\Big(\underbrace{(C, C, \ldots, C)}_{I}; m\Big) = \sum_{n=1}^{I} \binom{I-1}{n-1} m^{n-1}(1-m)^{I-n} \cdot B_n - C$$

is monotonically decreasing in m. Let us denote

$$G^{(l)}(m) := \binom{I-1}{l-1} m^{l-1}(1-m)^{I-l}.$$

We will first show that the functions

$$G_n(m) := \sum_{l=1}^{n} G^{(l)}(m)$$

are strictly monotonically decreasing in m for $m \in [0,1]$ and $n \in \{1\ldots,I-2\}$. We have

$$\frac{dG^{(l)}(m)}{dm} = \binom{I-1}{l-1} \cdot m^l \cdot (1-m)^{I-l-2} \cdot \big[l\cdot(1+2\cdot m) - (I-1)\cdot m\big]$$

Thus $\frac{dG^{(l)}(m)}{dm} \gtreqless 0$ for $l \gtreqless \frac{(I-1)\cdot m}{1+2\cdot m}$. Let $\bar{l}$ be the highest integer, which is not larger than $\frac{(I-1)\cdot m}{1+2\cdot m}$. Then for all $n \in \{1,\ldots,\bar{l}\}$ the the functions $G_{I-1}(m)$ are obviously monotonically decreasing. For $n \in \{\bar{l}+1,\ldots,I-2\}$ observe that since

$$G_{I-1}(m) \equiv 1$$

one obtains

$$G_n(m) = 1 - \sum_{l=n+1}^{I-1} G^{(l)}(m)$$

Since for these l we obtained $\frac{dG^{(l)}(m)}{dm} > 0$ it follows again that $G_{I-1}(m)$ are monotonically decreasing. The desired result follows from the inequalities

$$\begin{aligned}
&B_1 \cdot \frac{d}{dm}\big(G^{(0)}(m)\big) + B_2 \cdot \frac{d}{dm}\big(G^{(1)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
&B_2 \cdot \frac{d}{dm}\big(G^{(0)}(m)\big) + B_2 \cdot \frac{d}{dm}\big(G^{(1)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) = \\
&B_2 \cdot \frac{d}{dm}\big(G_1(m)\big) + B_3 \cdot \frac{d}{dm}\big(G^{(2)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
&B_3 \cdot \frac{d}{dm}\big(G_2(m)\big) + B_4 \cdot \frac{d}{dm}\big(G^{(3)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
&. \\
&. \\
&B_{l+1} \cdot \frac{d}{dm}\big(G_l(m)\big) + B_{l+1} \cdot \frac{d}{dm}\big(G^{(l)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
&. \\
&. \\
&> B_I \cdot \frac{d}{dm}\big(G_{I-1}(m)\big) = 0.
\end{aligned}$$

Proof of lemma 6.1:

The seller j solves the problem of choosing (p^j, z^j) so as to maximize

$$\Pi_j\Big((p^j,z^j);m^*\Big) = \sum_{n=1}^{I} \binom{I}{n} (m^*)^n (1-m^*)^{I-n} \cdot \int_0^1 Z_j^{(n)}(x_i)dF(x_i)$$

subject to the constraint

$$
\begin{aligned}
R^* &= R^i_j\Big((p^j, z^j); m^*\Big) \\
&= \sum_{n=1}^{I} \binom{I-1}{n-1} (m^*)^{n-1} (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P^{(n)}_j(x_i) - Z^{(n)}_j(x_i)\Big) dF(x_i).
\end{aligned}
$$

The constraint can be rewritten as

$$
I \cdot m^* \cdot R^* = \sum_{n=1}^{I} \binom{I}{n} (m^*)^{n} (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P^{(n)}_j(x_i) - Z^{(n)}_j(x_i)\Big) dF(x_i).
$$

Using this observation the maximization problem becomes equivalent to maximizing the expression

$$
\sum_{n=1}^{I} \binom{I}{n} (m^*)^{n} (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P^{(n)}_j(x_i)\Big) dF(x_i) - I \cdot m^* \cdot R^*.
$$

The expectation obviously takes a maximum if the expression

$$
\int_0^1 \Big(x_i \cdot P^{(n)}_j(x_i)\Big) dF(x_i)
$$

is maximized for every n. Since for $x_{-i} \in [0,1]^{n-1}$ we have

$$
P^{(n)}_j(x_i) = \int\limits_{[0,1]^{n-1}} p_i(x_i, x_{-i}) dF(x_{-i})
$$

and we consider anonymous mechanisms one obtains for $x \in [0,1]^n$

$$
\int_0^1 \Big(x_i \cdot P^{(n)}_i(x_i)\Big) dF(x_i) = \frac{1}{n} \cdot \int\limits_{[0,1]^n} \Big(\sum_{i=1}^{n} x_i \cdot p_i(x)\Big) dF(x).
$$

The expression takes a maximum if for every participant i the probability $p_i(x)$ is chosen so that

$$
p_i(x) = \begin{cases} 1 & \text{if } x_i \text{ is the highest valuation,} \\ 0 & \text{otherwise.} \end{cases}
$$

Appendix 6.B

Proof of theorem 6.3:

Consider the function

$$\begin{aligned}\overline{\varphi}(C) =& \frac{\partial}{\partial \overline{m}}\left(\sum_{n=1}^{I}\binom{I}{n}\overline{m}^n(1-\overline{m})^{I-n}\cdot S_n\right)\cdot \frac{\partial \overline{m}(C,C^j)}{\partial C_j}\bigg|_{C^j=C} \\ &+ I\cdot \overline{m}(C,C) \\ &+ I\cdot \frac{\partial \overline{m}(C,C^j)}{\partial C_j}\bigg|_{C^j=C}\cdot C.\end{aligned}$$

Observe that the first two terms are constants. Indeed, from the equation defining $\overline{m}(C,C^j)$ follows that $\overline{m}(C,C) = 1/J$ and that $\frac{\partial \overline{m}(\cdot)}{\partial C_j}\Big|_{C^j=C}$ is negative and constant with respect to C. The function in the last term is linear and decreasing in C. It follows that there exists a unique $\overline{C}$ for which $\overline{\varphi}(\overline{C}) = 0$. Since it is assumed that $\Pi_j\Big(C^j;\overline{m}(C,C^j)\Big)$ is concave in C^j it follows that $C^j = \overline{C}$ is the unique (global) maximizer of this function. If $\beta \leq R_{(1/J)} - \overline{C}$, then all bidders find it optimal to enter the market with a probability of one and to visit each seller with a probability of $1/J$. The equilibrium participation fee in this case is $\overline{C}$.

The function $\underline{\varphi}(C^j,\beta)$ is also decreasing in C^j because by assumption the function $\Pi_j\Big(C^j;\underline{m}(\beta,C^j)\Big)$ is concave. The unique maximizer of this function is $\underline{C}(\beta)$. If the entrance fee is $R_{(1/J)} - \beta$ and bidders enter the market with a probability of one, then their expected payoff is β. If the entrance fee is $\underline{C}(\beta) > R_{(1/J)} - \beta$, bidders exercise the outside option with positive probability. In this case $\underline{C}(\beta)$ is the equilibrium entrance fee. If $\underline{C}(\beta) \leq R_{(1/J)} - \beta$, then observe that for $C^j < R_{(1/J)} - \beta$ bidders do not exercise the outside option and one obtains

$$\begin{aligned}\frac{\partial \Pi_j\Big(C^j;\overline{m}(C^j,R_{(1/J)}-\beta)\Big)}{\partial C^j}\Bigg|_{C^j<R_{(1/J)}-\beta} &> \frac{\partial \Pi_j\Big(C^j;\overline{m}(C^j,R_{(1/J)}-\beta)\Big)}{\partial C^j}\Bigg|_{C^j=R_{(1/J)}-\beta} \\ &> \frac{\partial \Pi_j\Big(C^j;\overline{C}\Big)}{\partial C^j}\Bigg|_{C^j=\overline{C}} = 0.\end{aligned}$$

The first inequality applies due to the concavity of $\Pi_j\Big(\cdot;\cdot\Big)$ in C^j. The second inequality applies because (as we showed) $\overline{\varphi}(C) = \frac{\partial \Pi_j\left(C^j;C\right)}{\partial C^j}\Big|_{C^j=C}$ is decreasing.

For $C^j > R_{(1/J)} - \beta \geq \underline{C}(\beta)$ bidders exercise the outside option with positive probability. This is the case because every bidder will be indifferent between entering the market and exercising the outside option, if each seller uses the fee $R_{(1/J)} - \beta$, and all other bidders enter with a probability of one. One obtains

$$\frac{\partial \Pi_j\Big(C^j;\underline{m}(\beta,C^j)\Big)}{\partial C^j}\Bigg|_{C^j > R_{(1/J)}-\beta} < \frac{\partial \Pi_j\Big(C^j;\underline{m}(\beta,C^j)\Big)}{\partial C^j}\Bigg|_{C^j = \underline{C}(\beta)} = 0.$$

In this case $R_{(1/J)} - \beta$ is the equilibrium participation fee.

Proof of theorem 6.4:

Case $I = 2$.

We have

$$\Pi_j\Big(C^j;C\Big) = m^2 \cdot S_2 + 2 \cdot m \cdot (1-m)S_1 + 2 \cdot m \cdot C_j,$$

where m solves the equation

$$m \cdot B_2 + (1-m) \cdot B_1 - C_j = m \cdot B_1 + (1-m) \cdot B_2 - C.$$

Showing that this function is concave in C_j is equivalent to show that the function is concave in m. After rearranging terms we obtain

$$\Pi_j\Big(m;C\Big) = m^2 \cdot (S_2 + 2B_2 - 2B_1) + m(2S_1 + B_1 - B_2 + C).$$

Further we make use of the following lemma.

Lemma 6.3. *For any probability distribution F the equality $B_1 = B_2 + S_2$ is satisfied.*

Proof. A straightforward but somewhat cumbersome proof of this statement would be to substitute for B_1, B_2 and S_2 with the respective expressions defining these variables and check the equality. Here we offer a more intuitive argument. Let bidders 1 and 2 have the valuations x_1 and x_2. If bidder 1 participates alone in a second price auction with a zero reserve price, his payoff would be x_1. If he participates with bidder 2 then his payoff would be 0 if $x_1 < x_2$ and $x_1 - x_2$ otherwise. In the former case the seller's payoff is x_1 and in the latter case x_2. In both cases the sum of the buyers' and seller's payoff is x_1 just as in the case in which bidder 1 participates alone in a second price auction. Since this argument is valid for any x_1 and x_2 the claim follows. □

Applying this lemma it is now easy to see that $S_2 + 2B_2 - 2B_1 = B_2 - B_1 < 0$. The function is concave. The proof is analogous for the function

$$\Pi_j\Big(C^j;\beta\Big) = m^2 \cdot S_2 + 2 \cdot m \cdot (1-m)S_1 + 2 \cdot m \cdot C_j,$$

for which m solves the equation

$$m \cdot B_2 + (1-m) \cdot B_1 - C_j = \beta.$$

Case $I = 3$.
We have

$$\Pi_j\Big(C^j; C\Big) = m^3 \cdot S_3 + 3m^2(1-m) \cdot S_2 + 3 \cdot m \cdot (1-m)^2 S_1 + 3 \cdot m \cdot C_j,$$

where m solves the equation

$$m^2 \cdot B_3 + 2m(1-m) \cdot B_2 + (1-m)^2 \cdot B_1 - C_j = (1-m)^2 \cdot B_3 + 2m(1-m) \cdot B_2 + m^2 \cdot B_1 - C.$$

After solving the latter equation and substituting in the former one, we obtain

$$\Pi_j\Big(m; C\Big) = m^3 \cdot (S_3 - 3S_2) + m^2 \cdot (3S_2 - 6B_1 + 6B_2) + m \cdot (\text{term}) + (\text{another term}).$$

For the second derivative with respect to m we obtain

$$\frac{\partial^2 \Pi_j\Big(m; C\Big)}{\partial^2 m} = 6m(S_3 - 3S_2) + 2(3S_2 - 6B_1 + 6B_3).$$

It is clear that $S_3 - 3S_2 < 0$. Further

$$3S_2 - 6B_1 + 6B_2 = 3S_2 - 3S_2 - 3B_2 - 3B_1 + 6B_3 = -3B_2 - 3B_1 + 6B_3 < 0.$$

It follows that the function is concave. Analogously for the function

$$\Pi_j\Big(C^j; \beta\Big) = m^3 \cdot (S_3 + 3C) + 3m^2(1-m)(S_2 + 2C) + 3m(1-m)^2(S_1 + C),$$

where

$$m^2 B_3 + 2m(1-m)B_2 + (1-m)^2 B_1 - C = \beta$$

we obtain for the second derivative

$$\frac{\partial^2 \Pi_j\Big(m; \beta\Big)}{\partial^2 m} = 6m(S_3 + 3B_3 - 3B_2) - 6S_2 < 6S_3 - 6S_2 < 0.$$

Proof of theorem 6.5:
For the uniform distribution it is easy to show that

$$B_n = \frac{1}{n(n+1)}; S_n = \frac{n-1}{n+1}.$$

Then the function

$$\Pi_j\Big(C^j;\beta\Big) = \sum_{n=1}^{I}\binom{I}{n}\overline{m}^n(1-\overline{m})^{I-n}\cdot S_n + I\cdot m\cdot C_j,$$

where

$$R_{(m)} - C_j = \beta$$

should be shown to be concave. Again after solving the last equality and substituting in the former one, one obtains

$$\begin{aligned}
\Pi_j\Big(m;\beta\Big) &= \sum_{n=1}^{I}\binom{I}{n}m^n(1-m)^{I-n}\cdot\frac{n-1}{n+1} \\
&\quad + I\cdot\sum_{n=1}^{I}\binom{I-1}{n-1}\frac{m^n(1-m)^{I-n}}{n(n+1)} - \beta I m = \\
&= \sum_{n=1}^{I}\binom{I}{n}\frac{m^n(1-m)^{I-n}}{(n+1)}[n-1+1] - \beta I m \\
&= \sum_{n=1}^{I}\binom{I}{n}m^n(1-m)^{I-n}\cdot\frac{n}{n+1} - \beta I m \\
&= \underbrace{\sum_{n=1}^{I}\binom{I}{n}m^n(1-m)^{I-n}}_{(*)} - \underbrace{\sum_{n=1}^{I}\binom{I}{n}\frac{m^n(1-m)^{I-n}}{n+1}}_{(**)} - \beta I m.
\end{aligned}$$

Observe that

$$\begin{aligned}
(*) &= [m+(1-m)]^I - 1\cdot m^0(1-m)^I = 1-(1-m)^I, \\
(**) &= \frac{1}{(I+1)\cdot m}\sum_{n=1}^{I}\frac{I!(I+1)}{n!(I-n)!(n+1)}m^n(1-m)^{I-n} \\
&= \frac{1}{(I+1)m}\sum_{n=1}^{I}\binom{I+1}{n+1}m^{n+1}(1-m)^{I-n} \\
&= \frac{1}{m(I+1)} - \frac{(1-m)^I+1}{m(I+1)} - (1-m)^I.
\end{aligned}$$

Substituting $(*)$ and $(**)$ in the previous equation one obtains

$$\begin{aligned}\Pi_j\Big(m;\beta\Big) &= 1-(1-m)^I-\frac{1}{m(I+1)}+\frac{(1-m)^{I+1}}{m(I+1)}+(1-m)^I-mI\beta\\ &= 1+\frac{[(1-m)^{I+1}-1]}{m(I+1)}-mI\beta\\ &= 1+\frac{(-m)[(1-m)^I+(1-m)^{I-1}+\cdots+(1-m)+1]}{m(I+1)}-mI\beta\\ &= 1-\frac{1}{I+1}\sum_{n=0}^{I}(1-m)^n-mI\beta.\end{aligned}$$

For the first derivative one obtains

$$\frac{\partial\Pi_j\Big(m;\beta\Big)}{\partial m}=\frac{1}{I+1}\sum_{n=1}^{I}n(1-m)^{n-1}-I\beta$$

and for the second

$$\frac{\partial^2\Pi_j\Big(m;\beta\Big)}{\partial^2 m}=-\frac{1}{I+1}\sum_{n=2}^{I}n(n-1)(1-m)^{n-2}\leq 0.$$

The concavity of function $\Pi_j\Big(C^j;\overline{m}(C,C^j)\Big)$ is difficult to show analytically for an arbitrary number of buyers and sellers even for F uniformly distributed. It appears however that this property holds. For all $J\in\{2,3,\dots,100\}$ and $I\in\{2,3,\dots,100\}$ we computed using a C++ program the second derivative and established that it is negative at all discrete points between 0 and 1 with a step of $0,00001$. The source code is available from the author upon request.

Proof of claim 6.1:

If seller 1 charge a participation fee of C^1, seller 2 a participation fee of C and bidders enter the market with a probability of one, then $\overline{m}(C,C^1)$ solves the equation

$$\begin{aligned}m\cdot B_2+(1-m)\cdot B_1-C^1=(1-m)\cdot B_2+m\cdot B_1-C^2 &\Leftrightarrow\\ \overline{m}(C,C^1)=\frac{(C^1-C)+B_2-B_1}{2\cdot(B_2-B_1)}&.\end{aligned}$$

The expected payoff of seller 1 is

$$\Pi_1\Big(C^1;\overline{m}(C^1,C)\Big)=\overline{m}(C,C^1)^2\cdot(S_2+2C^1)+2\overline{m}(C,C^1)(1-\overline{m}(C,C^1))(S_1+C^1).$$

The equation $\overline{\varphi}(C)=0$ has the solution $\overline{C}=B_1-B_2-S_2/2$. Recall that in lemma 6.3 we showed that $B_1=B_2+S_2$. Since the *ex ante* payoff of each bidder is $B_1/2+B_2/2-$

$S_2/2 = B_2$ and bidders will enter with a probability of one if $\beta < B_2$, the participation fee is $S_2/2$. For $j = 1, 2$ the function $\underline{m}(\beta, C^j)$ satisfies the equation

$$\begin{aligned} R^i_j\Big(C^j; m\Big) &= \beta \Leftrightarrow \\ m \cdot B_2 + (1-m) \cdot B_1 - C^j &= \beta \Leftrightarrow \\ \underline{m}(\beta, C^j) &= \frac{\beta + C^j - B_1}{(B_2 - B_1)}. \end{aligned}$$

The expected payoff of seller j is

$$\Pi_j\Big((C^j, \beta; \underline{m}\Big) = \underline{m}^2 \cdot (S_2 + 2C^j) + 2\underline{m}(1-\underline{m})(S_1 + C^j).$$

The equation

$$\underline{\varphi}(C^j, \beta) = 0$$

has the solution $\underline{C} = (\beta - B_1)(B_1 - B_2 - S_2)/(S_2 - 2B_1 + 2B_2) = 0$.

Chapter 7

Conclusion

Auction theory impresses not only with its mathematical elegance and generality, but also contributes to our understanding of how real markets work and offers advice on how to design new market institutions.

On the one hand, this thesis analyzes auction models in the spirit and tradition of the received auction theory: we followed the well-established approach to model and analyze market institutions as games. On the other hand, our analysis concerns markets in which multiple units are traded. This is an area of important applications for which the theory is yet to be developed. In chapters 3, 4 and 5 a particular emphasis has been put on studying the effects of a monopolist's endogenous supply decisions on competitive bidding and their consequences for efficiency and auctioneer's profit.

Besides narrowing a gap in the theory, the virtue of the thesis lies in the *comparison* of market institutions. This comparison, which is essential for policy decision making, has been elusive in the existing multi-unit auction models with variable supply. Back and Zender (2001) and Damianov (2005b) show that the auctioneer's endogenous supply decision eliminates the low-price equilibria in the uniform price auction, but do not discuss the discriminatory auction[1]. Lengwiler (1999) derives the equilibria in both auction formats, but due to the complexity of his incomplete information model a comparison is not possible. In what environments to use the uniform price auction and in what the discriminatory, is still an open question. In chapters 3 and 4 of this thesis we presented an environment, for which a clear-cut ranking is possible. The studied setting captures some important features of the Treasury and IPO auctions in many

[1] In an earlier work, which studies a multi-unit auction model with fixed supply, Back and Zender (1993) provide a comparison of the uniform price and the discriminatory auction. This paper, which became very influential, argues against the use of a uniform price auction.

countries.

Another aspect, which differentiates the thesis from the traditional literature concerns the modelling. In the traditional framework, bidders face uncertainty about the valuations (or the signals) of the other bidders. In chapters 3 and 4 we depart from this framework by considering an incomplete information setting, in which bidders face uncertainty about the incentives of the auctioneer. As has been argued, this aspect is evident in financial auctions. Insights into bidders' rational behavior in such an environment have been provided.

A deficit of the elaboration here is that it does not capture one important aspect of the real-world variable supply markets: the bidding behavior of incompletely informed bidders, who are able to submit entire demand functions to the auctioneer. In chapters 3 and 4 we study incomplete information auction games, but restrict the bidders to submit a bid price for one unit. In chapter 5 the latter assumption is relaxed, but only at the cost of taking the uncertainty out of the model. The models lead to contradictory findings. In chapter 3 it has been demonstrated that due to the supply uncertainty bids in the uniform price auction are higher than in the discriminatory. In chapter 5, where there is no uncertainty, but the bidders' strategy space is more complex, one reaches the opposite conclusion if the "pro rata on the margin" rationing rule is used. Combining both aspects in a unified framework, albeit very difficult, would be important. It is our hope that the literature will develop in this direction in the future.

In chapter 6 an oligopoly market is analyzed, in which sellers are given the authority to design trade mechanisms to attract customers. The model contributes to the recent literature by providing a solution to the mechanism design problem for a finite number of mechanism designers.

In conclusion it should be pointed out that theoretical models alone (as the ones presented here), although helpful, are not sufficient for the understanding of human interaction within the analyzed market institutions. Doing empirical research and conducting controlled experiments is an equally important and indispensable part of the scientific process on the road to understanding human behavior and the functioning of market institutions. These research avenues will hopefully be pursued by the author in the future.

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