Shape evolutions under state constraints:
A viability theorem

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Abstract

The aim of this paper is to adapt the Viability Theorem from differential inclusions (governing the evolution of vectors in a finite dimensional space) to so-called morphological inclusions (governing the evolution of nonempty compact subsets of the Euclidean space).
In this morphological framework, the evolution of compact subsets of $\mathbb{R}^N$ is described by means of flows along differential inclusions with bounded and Lipschitz continuous right-hand side. This approach is a generalization of using flows along bounded Lipschitz vector fields introduced in the so-called velocity method alias speed method in shape analysis.
Now for each compact subset, more than just one differential inclusion is admitted for prescribing the future evolution (up to first order) – correspondingly to the step from ordinary differential equations to differential inclusions for vectors in the Euclidean space.

We specify sufficient conditions on the given data such that for every initial compact set, at least one of these compact-valued evolutions satisfies fixed state constraints in addition. The proofs follow an approximative track similar to the standard approach for differential inclusions in $\mathbb{R}^N$, but they use tools about weak compactness and weak convergence of Banach-valued functions. Finally the viability condition is applied to constraints of nonempty intersection and inclusion, respectively, in regard to a fixed closed set $M \subset \mathbb{R}^N$.

Key words Nagumo's theorem, viability condition, Velocity method (speed method), morphological equations, reachable sets of differential inclusions

1 Introduction

Viability is a very important feature of dynamic systems under state constraints whose initial value problems do not ensure uniqueness of solutions. Indeed, lacking uniqueness leads to two different questions how to satisfy state constraints at each time: Either we demand *all* solutions to have their values in the fixed constrained set or (just) *at least one* solution with this property has to exist. In the first case, the corresponding constrained set is called *invariant* and, in the latter case, it is *viable* (or *weakly invariant*). For autonomous differential inclusions in $\mathbb{R}^N$ and other Banach spaces, sufficient and necessary conditions of viability have been investigated in great detail (see e.g. [7]).

The main goal of this paper is a sufficient characterization of viability for shapes.
To be more precise, we leave the familiar Euclidean space $\mathbb{R}^N$ and consider evolutions of nonempty compact subsets of $\mathbb{R}^N$ instead. Correspondingly, the trajectory $x : [0,T] \longrightarrow \mathbb{R}^N$ (of a differential inclusion) is now replaced by a curve $K : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ with $\mathcal{K}(\mathbb{R}^N)$ denoting the set of nonempty compact subsets of $\mathbb{R}^N$ (usually supplied with the Pompeiu–Hausdorff distance $d\!l$). The state constraints are again formulated as a subset, i.e. now $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ (instead of $V \subset \mathbb{R}^N$ for differential inclusions).

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Sketching motivation

Such a step beyond the traditional border of vector spaces is always required whenever shapes come into play and start evolving – without any regularity restriction of their boundaries. “Shapes [...] are basically sets, not even smooth” [4]. Thus, we consider nonempty compact subsets of $\mathbb{R}^N$ and want to construct a continuous function $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ whose evolution is determined by a “feedback loop” at each time $t \in [0,T]$, i.e. it depends on its current state $K(t) \in \mathcal{K}(\mathbb{R}^N)$ according to a given rule. In the Euclidean space $\mathbb{R}^N$, ordinary differential equations provide the classical tool serving as conceptional model here. Such a generalization of dynamic systems to $\mathcal{K}(\mathbb{R}^N)$ has already been applied to image segmentation [25] and vision–based control of robots [17], for example. Furthermore it was suggested for describing equilibrium conditions on moving bodies [23] and provides enormous potential for modelling the spatial evolution of epidemics (as mentioned in [30]) and other biological populations (such as how to manage a fishery without exhausting [20]).

Differential inclusions with Lipschitz right–hand side
for specifying time derivatives of curves in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$

For formulating the viability problem in the metric space $\big(\mathcal{K}(\mathbb{R}^N),\, d\!\!l\big)$, we have to specify how compact subsets of $\mathbb{R}^N$ are “deformed”. The so–called *velocity method* or *speed method* has led Céa, Delfour, Zolésio and others to remarkable results about shape optimization (see e.g. [10, 12, 13, 31, 36] and references there). It is based on prescribing a vector field $v : \mathbb{R}^N \times [0,T] \longrightarrow \mathbb{R}^N$ such that the corresponding ordinary differential equation $\frac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot)$ induces a unique flow on $\mathbb{R}^N$. Indeed, supposing v to be sufficiently smooth, the Cauchy problem

$$\tfrac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot) \text{ in } [0,T], \qquad x(0) = x_0 \in \mathbb{R}^N$$

is always well–posed and, any compact initial set $K \subset \mathbb{R}^N$ is deformed to

$$\vartheta_v(t,K) \;:=\; \big\{\, x(t) \;\big|\; \exists\, x(\cdot) \in C^1([0,t], \mathbb{R}^N) :\; \tfrac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot) \text{ in } [0,t], \;\; x(0) \in K \big\}$$

after an arbitrary time $t \geq 0$. As a key advantage, this concept of set evolution does not require any regularity conditions on the compact set K or its topological boundary (but only on the vector field v). In a word, v can be interpreted as a “direction of deformation” for $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$. So it is “possible to define directional derivatives and speak of shape gradient and shape Hessian with respect to the associated vector space of velocities. This [...] approach has been known in the literature as the *velocity method*” [12, Chapter 1, § 6].

Aubin seized this notion for extending ODEs to this metric space of compact subsets. The so–called *morphological equations* are sketched in [6] and then presented in [4, 5] in more detail. (They seem to be closer to ODEs in $\mathbb{R}^N$ than Panasyuk’s concept of “quasidifferential equations” [27, 28, 29].)

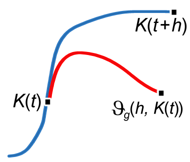
The first aspect of generalization focuses on the “elementary deformation” which are to describe the directions in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$. Aubin suggested reachable sets of differential inclusions as a more general alternative to the velocity method. For any set–valued map $G : \mathbb{R}^N \leadsto \mathbb{R}^N$ and initial set $K \subset \mathbb{R}^N$ given, the so–called *reachable set* at time $t \geq 0$ is defined as

$$\begin{aligned} \vartheta_G(t,K) \;\; := \;\; \Big\{ x(t) \in \mathbb{R}^N \;\Big|\; & \exists\;\, x(\cdot) \in W^{1,1}([0,t],\, \mathbb{R}^N) :\; x(0) \in K, \\ & \tfrac{d}{d\tau}\, x(\tau) \in G(x(\tau)) \;\text{ for almost every }\; \tau \in [0,t] \Big\} \end{aligned}$$

In contrast to the velocity method, this kind of "deformation" need not be reversible in time. (Geometrically speaking, "holes" can disappear.) The well–known Theorem of Filippov ensures suitable properties of $[0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(t, K) \longmapsto \vartheta_G(t, K)$ if $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty compact values and is bounded Lipschitz continuous. In fact, the Relaxation Theorem of Filippov–Ważiewski (e.g. [3, § 2.4, Theorem 2]) implies no changes of reachable sets if each value of G is replaced by its convex hull. So we are always free to consider bounded Lipschitz continuous maps $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact *and convex* values instead.

The second key contribution of Aubin is a suggestion how to interprete such a set–valued map (and its reachable sets) as time derivative of a curve in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.

Indeed, let $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ be a curve. A bounded Lipschitz set–valued map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ (with nonempty compact values) represents a first–order approximation of $K(\cdot)$ at time $t \in [0, T[$ if



$$\lim_{h \downarrow 0} \quad \tfrac{1}{h} \cdot d\!\!l\big(K(t+h),\ \vartheta_G(h,\ K(t))\big) \ = \ 0. \qquad (*)$$

Of course, such a map $G(\cdot)$ need not be unique and thus, *all* such bounded Lipschitz maps with this property $(*)$ form the so–called *morphological mutation* $\overset{\circ}{K}(t)$ of $K(\cdot)$ at time $t \in [0, T[$. It is a subset of $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ denoting the set of all bounded Lipschitz maps $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact values. Correspondingly, $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ consists of all bounded Lipschitz maps $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact *and convex* values. $\overset{\circ}{K}(t)$ extends the time derivative to curves in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.

Solving a morphological equation with state constraints: Aubin's adaptation of Nagumo's theorem

The step from specifying a time derivative (of a curve) to formulating a (generalized) differential equation is rather small. It is based just on prescribing the time derivative as a function of the current state. In connection with nonempty compact subsets of $\mathbb{R}^N$, a function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is given. For any initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$, we are looking for $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ satisfying

1. $K(\cdot)$ is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance $d\!\!l$,
2. $f(K(t)) \in \overset{\circ}{K}(t)$ for a.e. $t \in [0, T[$, i.e. $\lim\limits_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l\ \big(K(t+h),\ \vartheta_{f(K(t))}(h,\ K(t))\big) = 0$,
3. $K(0) = K_0$.

Then, $K(\cdot)$ is called *solution* of the (autonomous) *morphological equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ in $[0, T]$ with initial value K_0.

Considering now additional state constraints, the question about existence of a solution has been answered completely by Aubin in [4, Theorem 4.1.7]. In particular, the assumptions about constraints and $f(\cdot)$ justify its interpretation as a counterpart of Nagumo's theorem [26]. Here we use the notation $\|G\|_\infty := \ \sup_{x \in \mathbb{R}^N} \ \sup_{y \in G(x)} \ |y|$ for any set–valued map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$.

Proposition 1.1 (Nagumo's theorem for morphological equations [4, 5]**)**
Suppose $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *to be nonempty and closed with respect to* $d\!\!l$.
Let $f : (\mathcal{K}(\mathbb{R}^N), d\!\!l) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *be a continuous function satisfying*

1. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \ \mathrm{Lip}\, f(M) \ < \ \infty$ (*uniform bound of the Lipschitz constants*)
2. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \ \|f(M)\|_\infty \ < \ \infty$ (*uniform bound of the set values*).

Furthermore suppose for every $M \in \mathcal{V}$: $f(M) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is contingent *to $\mathcal{V}$ at M in the sense that* $\quad 0 \;=\; \liminf_{h \downarrow 0} \; \frac{1}{h} \cdot \mathrm{dist}\big(\vartheta_{f(M)}(h, M),\; \mathcal{V}\big) \;\stackrel{\text{Def.}}{=}\; \liminf_{h \downarrow 0} \; \frac{1}{h} \cdot \inf_{C \in \mathcal{V}} d\!\!l\big(\vartheta_{f(M)}(h, M),\; C\big).$

Then, from any $K_0 \in \mathcal{V}$ starts a solution $K(\cdot) : [0, \infty[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ of the morphological equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ which is viable in $\mathcal{V}$, i.e. $K(t) \in \mathcal{V}$ for all t.

The new step to morphological inclusions

This paper focuses on the corresponding conditions (of viability) if more than one Lipschitz map is admitted for each compact set, i.e. the single–valued function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is replaced by a set–valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.
This modification of given data leads directly to the following definition: A curve $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is called *solution* of the *morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ in $[0, T[$ with initial value K_0 if

1. $K(\cdot)$ is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance $d\!\!l$,
2. $\mathcal{F}(K(t)) \cap \overset{\circ}{K}(t) \neq \emptyset$ for almost every t, i.e. there exists some $G \in \mathcal{F}(K(t)) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with $\lim_{h \downarrow 0} \frac{1}{h} \cdot d\!\!l\,(K(t{+}h),\; \vartheta_G(h,\; K(t))) = 0$,
3. $K(0) = K_0$.

Considering now additional state constraints on $K(\cdot)$, the problems of invariance and viability have already been investigated for the velocity method (i.e. bounded Lipschitz vector fields instead of Lipschitz set–valued maps). Indeed, Doyen [19] has given sufficient and some necessary conditions on $\mathcal{F}(\cdot)$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ for the *invariance* of $\mathcal{V}$ (i.e. *all continuous* solutions starting in $\mathcal{V}$ stay in $\mathcal{V}$). His key notion is first to extend Filippov's existence theorem from differential inclusions (in $\mathbb{R}^N$) to morphological inclusions in $\mathcal{K}(\mathbb{R}^N)$ [19, Theorem 7.1] and then to verify $\mathrm{dist}(K(\cdot), \mathcal{V}) \leq 0$ (under the assumption that the values of $\mathcal{F}(\cdot)$ are always contained in the corresponding *contingent cone* to $\mathcal{V}$) [19, Theorem 8.2].

The corresponding question about viability of $\mathcal{V}$ (i.e. *at least one* Lipschitz solution has to stay in $\mathcal{V}$) was pointed out as open in [4, § 2.3.3]. Recently, the author has specified sufficient conditions for the special case of velocity method, i.e. using flows along the bounded Lipschitz vector fields in $\mathrm{Lip}(\mathbb{R}^N, \mathbb{R}^N)$ (instead of set-valued maps in $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$) [24]. For this first answer, some tools of Banach-valued functions (quoted here in § 3.3) opened the door to applying Haddad's classical concept of approximation developed for finite-dimensional vector spaces. Indeed, the counterparts of time derivatives there were functions $[0, 1] \longrightarrow \mathrm{Lip}(\mathbb{R}^N, \mathbb{R}^N)$, but the linear structure of these vector fields is now lost.

The main result of this paper considers morphological inclusions in their full generality, i.e. in contrast to velocity method, we choose the "directions of deformation" in $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ (and their reachable sets). It concerns sufficient conditions on $\mathcal{F}(\cdot) : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\, \mathbb{R}^N)$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ for the *viability* of $\mathcal{V}$.
In fact, the following statement is very similar to the viability theorem for differential inclusions in $\mathbb{R}^N$ (as it is discussed in [7] and quoted here in Theorem 3.3). Roughly speaking, $\mathcal{F}$ is supposed to be upper semicontinuous with closed convex values — after specifying a suitable topology on $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ in a moment — and, we require (at least) one "contingent direction" in the value $\mathcal{F}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ for each $K \in \mathcal{V}$.

Theorem 1.2 (Viability theorem for morphological inclusions)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *a nonempty closed subset satisfying:*

1.) all values of $\mathcal{F}$ *are nonempty and convex (i.e. for any* $G_1, G_2 \in \mathcal{F}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $\lambda \in [0,1]$, *the set–valued map* $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1-\lambda) \cdot G_2(x)$ *also belongs to* $\mathcal{F}(K)$*)*

2.) $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \ \sup_{G \in \mathcal{F}(M)} \big(\|G\|_\infty + \mathrm{Lip}\, G\big) < \infty,$

3.) the graph of $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*),*

4.) for each $K \in \mathcal{V}$, *some* $G \in \mathcal{F}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *is* contingent *to* $\mathcal{V}$ *at* K *in the sense that* $0 = \liminf_{h \downarrow 0} \frac{1}{h} \cdot \mathrm{dist}\big(\vartheta_G(h,K), \mathcal{V}\big) \overset{\mathrm{Def.}}{=} \liminf_{h \downarrow 0} \frac{1}{h} \cdot \inf_{C \in \mathcal{V}} d\!\!l\big(\vartheta_G(h,K), C\big).$

Then for every initial set $K_0 \in \mathcal{V}$, *there exists at least one solution* $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ *of the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \in \mathcal{V}$ *for all* $t \in [0,1]$.

An example: Shape evolutions under operability constraints

Seizing the examples of [23], we consider two types of constraints with a given closed set $M \subset \mathbb{R}^N$:

1. $K(t) \cap M \neq \emptyset$ for every t,
2. $K(t) \subset M$ for every t.

These constraints occur, for example, while a person (or a robot) is standing without tumbling. Indeed, the balance of the human body is closely related to the following condition: While standing, the feet exert pressure on the ground in a spatially nonhomogenous way. The convex hull of the corresponding support area has to cover the projection of the center of gravity since otherwise the body is tumbling. The center of gravity, though, is always fluctuating in unpredictable directions as the human body is very large and complex. So in the interest of its permanent stability, we prefer even a small neighborhood of this center to satisfy the condition. Mathematically speaking, this constraint on the projection $K(t)$ of the neighborhood has the form $K(t) \subset M$ with M denoting the convex hull of the pressure regions. Following the notation of [23], each curve $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ satisfying this inclusion at every time $t \in [0,T]$ is called *strongly operable* on $M \subset \mathbb{R}^N$. So every successful strategy of balancing the human body has to be strongly operable in this regard.

Constraint (1.), i.e. $K(t) \cap M \neq \emptyset$, makes a weaker impression at first glance, but it is structurally different. Such a curve $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is called *operable.* In [23], Gorre investigated the contingent transition sets of these two constraints (needed for assumption (4.) of Viability theorem 1.2) and characterized them by means of the tangential features of $M \subset \mathbb{R}^N$. Combining her results with ours, reveals

Theorem 1.3 (Compact–valued solutions "operable" in M)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $M \subset \mathbb{R}^N$ *a closed subset satisfying:*

1.) all values of $\mathcal{F}$ *are nonempty, convex (as in Theorem 1.2) and have the global bounds*
$$\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \ \sup_{G \in \mathcal{F}(K)} \big(\|G\|_\infty + \mathrm{Lip}\, G\big) < \infty,$$

2.) the graph of $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*),*

3.) for any $K \in \mathcal{K}(\mathbb{R}^N)$ *with* $K \cap M \neq \emptyset$, *there exist* $G \in \mathcal{F}(K)$, $x \in K \cap M$ *such that some* $v \in G(x)$ *is paratingent to* M *relative to* K *at* x *in the sense of Bouligand (see Definition 4.1).*

Then for every compact set $K_0 \subset \mathbb{R}^N$ *with* $K_0 \cap M \neq \emptyset$, *there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ *of the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \cap M \neq \emptyset$ *for all* $t \in [0,1]$.

Theorem 1.4 (Compact–valued solutions "strongly operable" in M)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $M \subset \mathbb{R}^N$ *a closed subset satisfying:*

1.) *all values of* $\mathcal{F}$ *are nonempty, convex (as in Theorem 1.2) and have the global bounds*
$$\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(K)} \big(\|G\|_\infty + \mathrm{Lip}\, G\big) < \infty,$$
2.) *the graph of* $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*),*

3.) *for any compact set* $K \subset M$, *there exist a set-valued map* $G \in \mathcal{F}(K)$ *such that for each* $x \in K$, *every vector* $v \in G(x)$ *is contingent to* M *at* x *in the sense of Bouligand (see Definition 2.8).*

Then for every nonempty compact set $K_0 \subset M$, *there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ *of the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \subset M$ *for all* $t \in [0,1]$.

This introduction (§ 1) is reflecting the structure of the paper: Aubin's theory of morphological equations is summarized in § 2. In particular, we mention the counterparts of Filippov's and Nagumo's theorems for evolutions in the metric space $\big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$. Then, § 3 provides the step to morphological inclusions. It starts with the viability theorem about differential inclusions (in § 3.1) and extends this result to morphological inclusions (in § 3.2). §§ 3.3, 3.4 collect some useful results about Banach-valued functions and set-valued maps, respectively, for proving the main theorem in detail in § 3.5. Finally, in § 4, we specify the results of Gorre for solving viability problems under operability constraints.

2 A brief introduction to morphological equations

Morphological equations provide typical geometric examples of so–called mutational equations. First presented in [6] and elaborated in [5, 4], mutational equations are to extend ordinary differential equations to a metric space (E, d). In a word, the key idea is to describe derivatives by means of continuous maps (called *transitions*) $\vartheta : [0,1] \times E \longrightarrow E$, $(h,x) \longmapsto \vartheta(h,x)$ instead of affine–linear maps $(h,x) \longmapsto x + h\, v$ (that are usually used in *vector spaces*). Strictly speaking, such a transition specifies the point $\vartheta(t,x) \in E$ to which any initial point $x \in E$ has been moved after time $t \in [0,1]$. It can be interpreted as a first–order approximation of a curve $\xi : [0,T[\longrightarrow E$ at time $t \in [0,T[$ if

$$\lim_{h \downarrow 0} \tfrac{1}{h} \cdot d\big(\xi(t+h),\ \vartheta(h,\xi(t))\big) \ = \ 0.$$

The so–called *morphological equations* apply this concept to the set $\mathcal{K}(\mathbb{R}^N)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu–Hausdorff distance $d\!\!\!l$,

$$\begin{aligned} d\!\!\!l(K_1, K_2) \ &:= \ \sup_{\substack{x \in K_1,\\ y \in K_2}} \ \Big\{ \mathrm{dist}(x, K_1),\ \ \mathrm{dist}(y, K_2) \Big\} \\ &= \ \inf \ \big\{ \rho > 0 \,\big|\, K_1 \subset K_2 + \rho\, \mathbb{B}_1,\ K_2 \subset K_1 + \rho\, \mathbb{B}_1 \big\}. \end{aligned}$$

Here $\mathbb{B}_1$ always denotes the closed unit ball in $\mathbb{R}^N$, i.e. $\mathbb{B}_1 := \{x \in \mathbb{R}^N \mid |x| \le 1\}$. This is a very general starting point for geometric evolution problems as there are no a priori restriction in regard to the regularity of sets and their boundaries. Motivated by the velocity method (often used in shape optimization, e.g. [10, 12, 13, 31, 36]), ordinary differential equations can lay the basis for transitions – as investigated in [24] already. Here, however, we follow a suggestion of Aubin (in [4, 5]) and consider a more general approach of evolutions instead: autonomous differential inclusions and their reachable sets.

Definition 2.1 ([4, Definition 3.7.1]) $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *consists of all set–valued maps* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *satisfying*

1. *F has nonempty compact values that are uniformly bounded in $\mathbb{R}^N$,*
2. *F is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance.*

$\mathrm{Lip}(M, \mathbb{R}^N)$ *consists of all bounded and Lipschitz continuous functions* $M \longrightarrow \mathbb{R}^N$.

Definition 2.2 *Choosing any set–valued map* $F : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, *the so–called* reachable set $\vartheta_F(t,K)$ *of the initial set* $K \in \mathcal{K}(\mathbb{R}^N)$ *at time* $t \in [0,T]$ *is defined as*

$$\vartheta_F(t,K) \ := \ \Big\{ x(t) \in \mathbb{R}^N \ \Big|\ \exists\ x(\cdot) \in W^{1,1}([0,t],\ \mathbb{R}^N):\ x(0) \in K,\ \tfrac{d}{d\tau}\, x(\tau) \in F(\tau, x(\tau)) \ \textit{for almost every}\ \tau \in [0,t] \Big\}$$

(*and correspondingly for* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and its autonomous differential inclusion*).

The special case of constant functions $F(\cdot) \equiv \{v\}$ (with an arbitrary vector $v \in \mathbb{R}^N$) leads to the Minkowski sum $\vartheta_F(t,K) \ = \ K + h \cdot v \subset \mathbb{R}^N$ and, for an initial set $K = \{x\}$ with just one element, in particular, we return to the familiar affine–linear map $(h,x) \longmapsto x + h \cdot v$ that has already been mentioned as motivation.

An essential contribution of Aubin was to specify appropriate continuity conditions on the maps $\vartheta : [0,1] \times E \longrightarrow E,\ (h,x) \longmapsto \vartheta(h,x)$ so that the familiar track of ordinary differential equations can be followed in a metric space (E,d). Here we quote his definition introduced in the monograph [4] (emphasizing the local features slightly more than his original version in [5]). Reachable sets of every set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ satisfy these conditions in the metric space $(\mathcal{K}(\mathbb{R}^N),\ d\!\!\!l)$:

Definition 2.3 ([4, Definition 1.1.2]) *Let (E,d) be a metric space. A map $\vartheta : [0,1] \times E \longrightarrow E$ is called* transition *on (E,d) if it satisfies the following conditions:*

1. $\vartheta(0,x) = x$ *for all $x \in E$,*
2. $\lim\limits_{h \downarrow 0} \frac{1}{h} \cdot d\left(\vartheta(t+h,x),\ \vartheta(h,\ \vartheta(t,x))\right) = 0$ *for all $x \in E,\ t \in [0,1[$,*
3. $\alpha(\vartheta) := \max\Big(0,\ \sup\limits_{x \neq y}\ \limsup\limits_{h \downarrow 0} \frac{d(\vartheta(h,x),\ \vartheta(h,y))\ -\ d(x,y)}{h\ \cdot\ d(x,y)}\Big) < \infty$
4. $\beta(\vartheta) := \sup\limits_{x \in E}\ \limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot d(x,\ \vartheta(h,x)) < \infty.$

For any two transitions $\vartheta_1, \vartheta_2 : [0,1] \times E \longrightarrow E$ on the same metric space (E,d), the transitional distance between ϑ_1 and ϑ_2 is defined by

$$d_\Lambda(\vartheta_1,\ \vartheta_2) := \sup_{x \in E}\ \limsup_{h \downarrow 0}\ \frac{1}{h} \cdot d\left(\vartheta_1(h,x),\ \vartheta_2(h,x)\right).$$

Lemma 2.4 *For every set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, the map $\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(h,K) \longmapsto \vartheta_F(h,K)$ of reachable sets (as introduced in Definition 2.2) is a well–defined transition on the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$ according to Definition 2.3.*

To be more precise, the reachable sets satisfy for all initial sets $K, K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$, set-valued maps $F, G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and times $t, h \geq 0$

$$\begin{aligned}
\vartheta_F(0,K) &= K, \\
\vartheta_F(t+h,K) &= \vartheta_F(h,\ \vartheta_F(t,K)), \\
d\!\!\!l(\vartheta_F(h,K_1),\ \ \vartheta_F(h,K_2)) &\leq d\!\!\!l(K_1,K_2) \cdot \quad e^{\mathrm{Lip}\,F \cdot h} \\
d\!\!\!l(\vartheta_F(h,K),\ \ \vartheta_G(h,K)\,) &\leq d\!\!\!l_\infty(F,G)\ \cdot h\ \ e^{\mathrm{Lip}\,F \cdot h} \\
d\!\!\!l(\vartheta_F(t,K),\ \vartheta_F(t+h,K)) &\leq \|F\|_\infty\ h
\end{aligned}$$

with $\quad \|F\|_\infty \stackrel{\text{Def.}}{=} \sup\limits_{x \in \mathbb{R}^N}\ \sup\limits_{y \in F(x)} |y| < \infty$

$\qquad d\!\!\!l_\infty(F,G) \stackrel{\text{Def.}}{=} \sup\limits_{x \in \mathbb{R}^N} d\!\!\!l(F(x), G(x)) < \infty$

and thus, $\alpha(\vartheta_F) \leq \mathrm{Lip}\,F, \quad \beta(\vartheta_F) \leq \|F\|_\infty, \quad d_\Lambda(\vartheta_F, \vartheta_G) \leq d\!\!\!l_\infty(F,G).$
In particular, $\quad d\!\!\!l\left(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\right) \leq e^{\mathrm{Lip}\,F \cdot h}\ \left(d\!\!\!l(K_1,K_2) + h \cdot d\!\!\!l_\infty(F,G)\right).$

The proof is presented in [4, Proposition 3.7.3] – as a direct consequence of Filippov's Theorem (about solutions of differential inclusions in $\mathbb{R}^N$). In particular, this lemma justifies calling ϑ_F a *morphological transition* on $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$ [4, Definition 3.7.2]. For the sake of simplicity, $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is sometimes identified with its morphological transition ϑ_F.
These reachable sets provide the tools for specifying (generalized) shape derivatives of a compact–valued tube $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$, i.e. a curve $K(\cdot) : [0,T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$. So the next step will be to solve equations prescribing an element of the morphological mutation.

Definition 2.5 ([4, Defintion 3.7.9 (2)]) *For any compact–valued tube* $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$, *the* morphological mutation $\overset{\circ}{K}(t)$ *at time* $t \in [0,T[$ *consists of all set–valued maps* $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *satisfying*

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l\,(\vartheta_F(h,\ K(t)),\ K(t+h)) \ = \ 0.$$

Definition 2.6 *For any given function* $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, *a compact–valued tube* $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ *is called solution of the morphological equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$

if 1. $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ *is Lipschitz continuous with respect to* $d\!\!l$ *and*

2. *for almost every* $t \in [0,T[$, $f(K(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *belongs to* $\overset{\circ}{K}(t)$ *or, equivalently,* $\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l\,\big(\vartheta_{f(K(t))}(h,\ K(t)),\ K(t+h)\big) \ = \ 0.$

At first glance, the symbol $\ni$ here seems to be contradictory to the term "equation". The mutation $\overset{\circ}{K}(t)$, however, is defined as *subset* of all morphological transitions providing a first-order approximation of $K(t+\cdot)$ and so, the "right-hand side" $f(K(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ should be one of its elements. (In the classical framework of differentiable functions and vector spaces, the mutation consists of just one vector.)

As an essential result of [4, 5], the Euler algorithm can be applied in the environment of morphological equations and so, the Cauchy–Lipschitz Theorem (about autonomous ordinary differential equations) has the following counterpart:

Theorem 2.7 ([4, Theorem 4.1.2]) *Suppose* $f : (\mathcal{K}(\mathbb{R}^N), d\!\!l) \longrightarrow \big(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N),\ d\!\!l_\infty\big)$ *to be* λ*–Lipschitz continuous and to satisfy* $M := \sup_{K \in \mathcal{K}(\mathbb{R}^N)} \mathrm{Lip}\, f(K) < \infty$.
For every initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ *and time* $T \in]0,\infty[$, *there exists a unique solution* $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ *of the morphological equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ *with* $K(0) = K_0$.
Furthermore every Lipschitz compact–valued tube $Q : [0,T[\rightsquigarrow \mathbb{R}^N$ *with* $\overset{\circ}{Q}(t) \neq \emptyset$ *for every* $t \in [0,T[$ *satisfies the following estimate at each time* $t \in [0,T[$

$$d\!\!l(K(t), Q(t)) \ \le \ d\!\!l(K_0, Q(0)) \cdot e^{(M+\lambda)\,t} + \int_0^t e^{(M+\lambda)\,(t-s)} \cdot \inf_{G \in \overset{\circ}{Q}(s)} d\!\!l_\infty(f(Q(s)),\ G)\ ds.$$

In particular, the solution $K(\cdot)$ *depends on the initial set* K_0 *and the right–hand side* f *in a Lipschitz continuous way.*

Existence under (additional) state constraints proves to be a very interesting question for many applications. In the particular case of ordinary differential equations, Nagumo's Theorem gives a necessary and sufficient condition on the constrained set $\mathcal{V}$ for existence of local solutions. It uses the contingent cone (in the sense of Bouligand) and has served as a key motivation for viability theory (see e.g. [7]).

Definition 2.8 ([7, Definition 1.1.3]) *Let* X *be a normed vector space,* $V \subset X$ *nonempty and* $x \in V$. *The* contingent cone *to* V *at* x *(in the sense of Bouligand) is*

$$T_V(x) \ := \ \big\{\, u \in X \,\big|\, \liminf_{h \downarrow 0} \ \tfrac{1}{h} \cdot \mathrm{dist}(x + h\,u,\ V) \ = \ 0 \big\}.$$

This classical definition of contingent cone in a vector space is now extended to the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ by using the morphological transitions of $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$:

Definition 2.9 ([4, Definition 1.5.2]) *For a nonempty subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ and any element $K \in \mathcal{V}$,*

$$T_{\mathcal{V}}(K) := \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; 0 = \liminf_{h \downarrow 0} \tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_F(h, K),\ \mathcal{V}\big) \stackrel{\text{Def.}}{=} \liminf_{h \downarrow 0} \tfrac{1}{h} \cdot \inf_{C \in \mathcal{V}} d\!\!l\big(\vartheta_F(h, K),\ C\big) \Big\}$$

is called contingent transition set *of $\mathcal{V}$ at K (in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$).*

The "geometric" background of reachable sets implies an additional property of morphological transitions in $T_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Indeed, for any $F \in T_{\mathcal{V}}(K)$, every map $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with $F(\cdot) = G(\cdot)$ in an open neighborhood of the compact set K is also contained in $T_{\mathcal{V}}(K)$ because $\vartheta_F(t, K) = \vartheta_G(t, K)$ for sufficiently small $t > 0$. So in other words, the criterion of $T_{\mathcal{V}}(K)$ depends only on an arbitrarily small neighborhood of the current set K. (The corresponding statement even holds for an open neighborhood of the boundary ∂K as a closer investigation of the boundaries $\partial \vartheta_F(t, K) \subset \vartheta_F(t, \partial K)$ reveals. These details, however, will not be used in the following.)

In fact, Nagumo's Theorem also holds for morphological equations:

Theorem 2.10 (Nagumo's theorem for morphological equations [4, Theorem 4.1.7]**)**
Suppose $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ to be nonempty and closed with respect to $d\!\!l$.
Let $f : (\mathcal{K}(\mathbb{R}^N), d\!\!l) \longrightarrow \big(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N),\ d\!\!l_\infty\big)$ be a continuous function satisfying

1. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \ \mathrm{Lip}\, f(M) < \infty$ (*uniform bound of Lipschitz constants*),
2. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \ \|f(M)\|_\infty < \infty$ (*uniform bound of compact values*).

Then from any initial state $K_0 \in \mathcal{V}$ starts at least one Lipschitz solution $K(\cdot) : [0, T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ of $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ viable in $\mathcal{V}$ (i.e. $K(t) \in \mathcal{V}$ for all t) if and only if $\mathcal{V}$ is a viability domain *of f in the sense of $f(M) \in T_{\mathcal{V}}(M)$ for each $M \in \mathcal{V}$.*

3 The step to morphological inclusions

The main aim now is to prove the corresponding existence of viable solutions for morphological *inclusions*, i.e. the single–valued function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ of the right–hand side is to be replaced by a set–valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Correspondingly to Definition 2.6, we introduce the solution of a morphological inclusion in the following way:

Definition 3.1 *For any given function $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, a compact–valued tube $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ is called* solution *of the morphological inclusion*

$$\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$$

if 1. $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ is Lipschitz continuous with respect to $d\!\!l$ and

2. $\mathcal{F}(K(t)) \cap \overset{\circ}{K}(t) \neq \emptyset$ for almost every t, i.e. a set-valued map $G \in \mathcal{F}(K(t)) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to $\overset{\circ}{K}(t)$ or, equivalently, $\lim_{h \downarrow 0} \frac{1}{h} \cdot d\!\!l\,(K(t+h),\ \vartheta_G(h,\ K(t))) = 0$,

3.1 The (well–known) Viability Theorem for differential inclusions

The situation has already been investigated intensively for differential inclusions in $\mathbb{R}^N$ (see e.g. [7, 3]). For clarifying the new aspects of morphological inclusions, we now quote the corresponding result from [7, Theorems 3.3.2, 3.3.5] after specifying the required terms.

Definition 3.2 ([7, Definition 2.2.4]) *Let X and Y be normed vector spaces. A set–valued map $F : X \rightsquigarrow Y$ is called* Marchaud map *if it has the following properties:*

1. *F is nontrivial,* *i.e.* Graph $F \neq \emptyset$,
2. *F is upper semicontinuous,* *i.e. for any $x \in X$ and neighborhood $V \supset F(x)$, there is a neighborhood $U \subset X$ of x such that $F(U) \subset V$,*
3. *F has compact convex values,*
4. *F has linear growth,* *i.e.* $\sup_{y \in F(x)} |y| \leq C\,(1 + |x|)$ *for all $x \in X$.*

Theorem 3.3 (Viability theorem for differential inclusions [7, Theorems 3.3.2, 3.3.5]**)**
Consider a Marchaud map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ and a nonempty closed subset $V \subset \mathbb{R}^N$ with $F(x) \neq \emptyset$ for all $x \in V$. Then for any finite time $T \in]0, \infty[$, the following two statements are equivalent:

1. *For every point $x_0 \in V$, there is at least one solution $x(\cdot) \in W^{1,1}([0,T],\ \mathbb{R}^N)$ of $x'(\cdot) \in F(x(\cdot))$ (almost everywhere) with $x(0) = x_0$ and $x(t) \in V$ for all t.*
2. *$F(x) \cap T_V(x) \neq \emptyset$ for all $x \in V$.*

The implication (1.) $\Longrightarrow$ (2.) is rather obvious. For proving (2.) $\Longrightarrow$ (1.), a standard approach uses an "approximating" sequence $\big(x_n(\cdot)\big)_{n \in \mathbb{N}}$ in $W^{1,\infty}([0,1], \mathbb{R}^N)$ such that $\sup_t\ \mathrm{dist}(x_n(t),\ V) \longrightarrow 0$ $(n \to \infty)$ and $\big(x_n(t),\ \frac{d}{dt}\, x_n(t)\big)$ is close to Graph $F \subset \mathbb{R}^N \times \mathbb{R}^N$ for almost every t. Then the theorems of Arzela–Ascoli and Alaoglu provide a subsequence $\big(x_{n_j}(\cdot)\big)_{j \in \mathbb{N}}$ and limits $x(\cdot) \in C^0([0,1], \mathbb{R}^N)$, $w(\cdot) \in L^\infty([0,1], \mathbb{R}^N)$ with

$$x_{n_j}(\cdot) \longrightarrow x(\cdot) \ \text{ uniformly}, \qquad \qquad \tfrac{d}{dt}\, x_{n_j}(\cdot) \longrightarrow w(\cdot) \ \text{ weakly* in } L^\infty([0,1], \mathbb{R}^N).$$

Due to the continuous embedding $L^\infty([0,1], \mathbb{R}^N) \subset L^1([0,1], \mathbb{R}^N)$, we even obtain the convergence $\frac{d}{dt}\, x_{n_j}(\cdot) \longrightarrow w(\cdot)$ weakly in $L^1([0,1], \mathbb{R}^N)$. Thus, $w(\cdot)$ is the weak derivative of $x(\cdot)$ in $[0,1]$ and, $x(\cdot)$ is Lipschitz continuous. Finally Mazur's Lemma 3.7 implies

$$w(t) \in \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\Big(\bigcup_{z \in \mathbb{B}_\varepsilon(x(t))} F(z) \Big) \ = \ F(x(t)) \qquad \text{for almost every } t.$$

Considering now morphological inclusions on $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ (instead of differential inclusions), an essential aspect changes: The derivative of a curve is not represented as a function in $L^1([0,1], \mathbb{R}^N)$ any longer, but rather as a function $[0,1] \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. So the classical theorems of Arzela–Ascoli, Alaoglu and Mazur might have to be replaced by their counterparts concerning functions with their values in a Banach space (instead of $\mathbb{R}^N$).

3.2 Adapting this concept to morphological inclusions: The main theorem.

Now $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and a constrained set $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ are given.
Correspondingly to Theorem 3.3 about differential inclusions, we focus on the so-called *viability condition* demanding from each compact set $K \in \mathcal{V}$ that the value $\mathcal{F}(K)$ and the contingent transition set $T_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ have at least one morphological transition in common. Lacking a concrete counterpart of Aumann integral in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, the question of its necessity (for the existence of "in $\mathcal{V}$ viable" solutions) is more complicated than for differential inclusions in $\mathbb{R}^N$ and thus, we skip it here deliberately. The main contribution of this paper is that in combination with appropriate assumptions about $\mathcal{F}(\cdot)$ and $\mathcal{V}$, the viability condition is *sufficient*.

Convexity again comes into play, but we have to distinguish between (at least) two aspects: First, assuming $\mathcal{F}$ to have convex values in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and second, supposing each set-valued map $G \in \mathcal{F}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ (with $K \in \mathcal{K}(\mathbb{R}^N)$) to have convex values in $\mathbb{R}^N$. The latter, however, does not really provide a geometric restriction on morphological transitions. Indeed, the well–known Relaxation Theorem of Filippov–Ważiewski (e.g. [3, § 2.4, Theorem 2]) implies $\vartheta_G(t, K) = \vartheta_{\overline{co}\, G}(t, K)$ for every map $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, initial set $K \in \mathcal{K}(\mathbb{R}^N)$ and time $t \geq 0$. So we suppose the values of $\mathcal{F}$ to be in $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$:

Definition 3.4 $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *denotes the set of all set–valued maps* $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *whose* (*nonempty compact*) *values are convex in addition.*

Theorem 3.5 (Viability theorem for morphological inclusions)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *a nonempty closed subset satisfying*:

1.) *all values of* $\mathcal{F}$ *are nonempty and convex* (*i.e. for any* $G_1, G_2 \in \mathcal{F}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $\lambda \in [0, 1]$, *the set–valued map* $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1 - \lambda) \cdot G_2(x)$ *also belongs to* $\mathcal{F}(K)$)
2.) $A := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \mathrm{Lip}\, G < \infty$,
$B := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \|G\|_\infty < \infty$,
3.) *the graph of* $\mathcal{F}$ *is closed* (*w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$),
4.) $T_{\mathcal{V}}(K) \cap \mathcal{F}(K) \neq \emptyset$ *for all* $K \in \mathcal{V}$.

Then for every initial set $K_0 \in \mathcal{V}$, *there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0, 1] \rightsquigarrow \mathbb{R}^N$ *of the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \in \mathcal{V}$ *for all* $t \in [0, 1]$.

Remark. In assumption (3.), the topology on $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is specified. A sequence $(G_n)_{n \in \mathbb{N}}$ in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is said to converge "locally uniformly" to $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ if for every nonempty compact set $M \subset \mathbb{R}^N$, $d\!\!l_\infty(G_n(\cdot)|_M, G(\cdot)|_M) \overset{\text{Def.}}{=} \sup_{x \in M} d\!\!l(G_n(x), G(x)) \longrightarrow 0$ for $n \longrightarrow \infty$ using here the Pompeiu–Hausdorff distance $d\!\!l$ on $\mathcal{K}(\mathbb{R}^N)$.
Due to the uniform bounds in assumption (2.), the image $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is sequentially compact with respect to this topology (as we prove in subsequent Lemma 3.11). So $\mathcal{F}$ is upper semi-continuous (in the sense of Bouligand and Kuratowski) according to [8, Proposition 1.4.8].

3.3 Tools for functions with values in metric or Banach spaces

Before adapting the concept for finite-dimensional differential inclusions (sketched in § 3.1) to morphological inclusions, we collect the main tools briefly. In this subsection, they consist mainly of (particularly weakly sequential) compactness criteria for Bochner-integrable functions on a probabilistic space. In following § 3.4, we summarize several results about parameterizing set-valued maps and differential inclusions.

First of all, the theorems of Arzela–Ascoli and Mazur do not change significantly. Indeed, we always use the following general versions in this paper:

Proposition 3.6 (Arzela–Ascoli in metric spaces [22]**)**
Let (E_1, d_1), (E_2, d_2) be precompact metric spaces, i.e. for any $\varepsilon > 0$, each set E_i $(i = 1, 2)$ can be covered by finitely many ε-balls with respect to metric d_i. Moreover, suppose the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $E_1 \longrightarrow E_2$ to be uniformly equicontinuous (i.e. with a common modulus of continuity in E_1). Then there exists a sequence $n_j \nearrow \infty$ such that $(f_{n_j})_{j\in\mathbb{N}}$ is Cauchy sequence with respect to uniform convergence. If (E_2, d_2) is complete in addition, then $(f_{n_j})_{j\in\mathbb{N}}$ converges uniformly to a continuous function $E_1 \longrightarrow E_2$.

Proposition 3.7 (Mazur's Lemma, e.g. [35, § V.1, Theorem 2]**)**
For any weakly converging sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space, its weak limit is contained in the closed convex hull of $\{x_n \mid n \in \mathbb{N}\}$.

The so-called Bochner integral extends the familiar concept of integration from real-valued functions to Banach-valued functions on the basis of "simple" functions.

Definition 3.8 ([16]**)** *Let (Ω, Σ, μ) be a finite measure space and X a Banach space. A function $f : \Omega \longrightarrow X$ is called* simple *if there exist $x_1, x_2 \ldots x_n \in X$ and $E_1, E_2 \ldots E_n \in \Sigma$ such that $f = \sum_{j=1}^{n} x_j\, \chi_{E_j}$ with $\chi_{E_j} : \Omega \longrightarrow \{0, 1\}$ denoting the characteristic function of $E_j \subset \Omega$.*
*A function $f : \Omega \longrightarrow X$ is called μ–*measurable *if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $\Omega \longrightarrow X$ with $\|f - f_n\|_X \longrightarrow 0$ μ–almost everywhere for $n \to \infty$.*
A μ–measurable function $f : \Omega \longrightarrow X$ is called Bochner integrable *if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $\Omega \longrightarrow X$ such that*
$$\lim_{n\to\infty} \int_\Omega \|f - f_n\|_X \, d\mu \;=\; 0.$$
Then, the Bochner integral of f over $E \in \Sigma$ is defined by $\displaystyle\int_E f\, d\mu := \lim_{n\to\infty} \int_E f_n\, d\mu$.
Let $L^1(\mu, X)$ denote the Banach space of Bochner integrable functions $\Omega \longrightarrow X$ equipped with its usual L^1 norm.

In the nineties, Ülger proved that restricting the values of Bochner integrable functions to a weakly compact subset of X implies the relative weak compactness of these functions in $L^1(\mu, X)$. For real-valued Lebesgue integrable functions, this is closely related with Alaoglu's Theorem and a compact embedding. An earlier version of this result is presented in [14] and, [15] considers weak compactness of Bochner integrable functions with values in an arbitrary Banach space under weaker assumptions (see also [9]).

Proposition 3.9 ([33, Proposition 7]**)** *Let (Ω, Σ, μ) be a probabilistic space, X an arbitrary Banach space. For any weakly compact subset $W \subset X$, the set*

$$\big\{\, h \in L^1(\mu, X) \;\big|\; h(\omega) \in W \text{ for } \mu\text{–almost every } \omega \in \Omega \big\}$$

is relatively weakly compact.

3.4 Tools for set-valued maps and differential inclusions.

Drawing parallels with differential inclusions in $\mathbb{R}^N$, the derivative of the wanted curve $K : [0,1] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ has now to be represented as a function $[0,1] \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$. This form, however, does not provide any obvious criteria in regard to sequential compactness. So as an essential tool, we prefer parameterizing set-valued maps (of time and space) for obtaining links with Banach-valued functions. This "detour" lays the basis for concluding (weak sequential) compactness properties of morphological transitions from Proposition 3.9.

Proposition 3.10 (Parameterization of bounded maps, [8, Theorem 9.7.2]**)**
Consider a metric space X and a set–valued map $G : [a,b] \times X \leadsto \mathbb{R}^N$ satisfying
1. *G has nonempty compact convex values,*
2. *$G(\cdot, x) : [a,b] \leadsto \mathbb{R}^N$ is measurable for every $x \in X$,*
3. *there exists $k(\cdot) \in L^1([a,b])$ such that for every $t \in [a,b]$, the set–valued map $G(t,\cdot) : X \leadsto \mathbb{R}^N$ is $k(t)$–Lipschitz continuous.*

Then there exists a function $g : [a,b] \times X \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ (with $\mathbb{B}_1 \stackrel{\text{Def.}}{=} \{u \in \mathbb{R}^N : |u| \le 1\}$) fulfilling
1. $\forall\ (t, x) \in [a,b] \times X :$ $\quad G(t,x) = \bigcup_{u \in \mathbb{B}_1} g(t,x,u)$,
2. $\forall\ (x,u) \in X \times \mathbb{B}_1 :$ $\quad g(\cdot, x, u) : [a,b] \longrightarrow \mathbb{R}^N$ *is measurable,*
3. $\forall\ (t, u) \in [a,b] \times \mathbb{B}_1 :$ $\quad g(t, \cdot, u) : X \longrightarrow \mathbb{R}^N$ *is $c \cdot k(t)$–Lipschitz continuous*
4. $\forall\ t \in [a,b],\ x \in X,\ u, v \in \mathbb{B}_1 :$ $\quad |g(t,x,u) - g(t,x,v)| \le c\, \|G(t,x)\|_\infty\, |u - v|$

with a constant $c > 0$ independent of G.

As a first result of the parameterization technique, we draw now useful conclusions about (sequential) compactness of $\mathrm{Graph}\, \mathcal{F} \subset \mathcal{K}(\mathbb{R}^N) \times \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ and the values of $\mathcal{F}$. They are based on supposing uniform Lipschitz bounds of all set-valued maps in the image of $\mathcal{F}$ (hypothesis (2.) of Theorem 3.5).

Lemma 3.11 (Sequential compactness in the image and graph of $\mathcal{F}(\cdot)$)
In addition to the hypotheses of Viability Theorem 3.5, let $(G_k)_{k \in \mathbb{N}}$ be an arbitrary sequence in the image $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) = \bigcup_{M \in \mathcal{K}(\mathbb{R}^N)} \mathcal{F}(M) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$.

Then, there exist a subsequence $(G_{k_j})_{j \in \mathbb{N}}$ and a set–valued map $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ such that for any compact set $M \subset \mathbb{R}^N$, $\sup_{x \in M} d\!\!l(G_{k_j}(x), G(x)) \longrightarrow 0$ $(j \longrightarrow \infty)$ and $\mathrm{Lip}\, G \le A$, $\|G\|_\infty \le B$.

Let now $(K_k)_{k\in\mathbb{N}}$ *be an arbitrary sequence in* $\mathcal{K}(\mathbb{R}^N)$ *such that* $\bigcup_{k\in\mathbb{N}} K_k \subset \mathbb{R}^N$ *is bounded and* $G_k \in \mathcal{F}(K_k)$ *for each* $k \in \mathbb{N}$. *Then there exist subsequences* $(K_{k_j})_{j\in\mathbb{N}}$, $(G_{k_j})_{j\in\mathbb{N}}$, *a set* $K \in \mathcal{K}(\mathbb{R}^N)$ *and a set–valued map* $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *with*
$d\!\!l(K_{k_j},\ K) \overset{j\to\infty}{\longrightarrow} 0, \qquad \sup_{x\in M} d\!\!l(G_{k_j}(x),\ G(x)) \overset{j\to\infty}{\longrightarrow} 0$ *for any* $M \in \mathcal{K}(\mathbb{R}^N)$ *and* $G \in \mathcal{F}(K)$.

Proof. Applying the parameterization theorem 3.10 to the autonomous maps $G_k : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ provides a sequence $(g_k)_{k\in\mathbb{N}}$ of Lipschitz functions $\mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ with $g_k(\cdot, \mathbb{B}_1) = G_k$ for each $k \in \mathbb{N}$ and $\sup_k (\|g_k\|_\infty + \mathrm{Lip}\, g_k) \le \mathrm{const}(A, B) < \infty$.
So for any nonempty compact set $K \subset \mathbb{R}^N$, the Theorem of Arzela–Ascoli (Proposition 3.6) guarantees a subsequence $(g_{k_j})_{j\in\mathbb{N}}$ converging uniformly in $K \times \mathbb{B}_1$. In combination with Cantor's diagonal construction, we obtain even a subsequence (again denoted by) $(g_{k_j})_{j\in\mathbb{N}}$ converging uniformly in each of the countably many compact sets $\mathbb{B}_m(0) \times \mathbb{B}_1 \subset \mathbb{R}^N \times \mathbb{R}^N$ $(m \in \mathbb{N})$.
Let $h_m : \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ denote an arbitrary Lipschitz function with $\sup_{\mathbb{B}_m(0)\times\mathbb{B}_1} |g_{k_j}(\cdot) - h_m(\cdot)| \overset{j\to\infty}{\longrightarrow} 0$.
Then we obtain the unique limit function $h : \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ by setting $h(x,\cdot) := h_m(x,\cdot)$ for all $x \in \mathbb{B}_m(0)$, $m \in \mathbb{N}$ and, $g_{k_j} \longrightarrow h$ $(j \to \infty)$ locally uniformly in $\mathbb{R}^N \times \mathbb{B}_1$.
In particular, $h(\cdot)$ is also Lipschitz continuous and has the same global Lipschitz bounds as $(g_k)_{k\in\mathbb{N}}$.
So, $G := h(\cdot, \mathbb{B}_1) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ provides a set–valued map being Lipschitz continuous and satisfying
$\sup_{x\in M} d\!\!l(G_{k_j}(x),\ G(x)) \le \sup_{x\in M} \sup_{u\in\mathbb{B}_1} |g_{k_j}(x,u) - h(x,u)| \longrightarrow 0 \quad (j \longrightarrow \infty)$ for any $M \in \mathcal{K}(\mathbb{R}^N)$.
This convergence of $(G_{k_j})_{j\in\mathbb{N}}$ implies directly $\mathrm{Lip}\, G \le A$, $\|G\|_\infty \le B$ and the convexity of all values of G. So the first claim is proved.

For verifying the second claim, we extract a convergent subsequence $(K_{k_l})_{l\in\mathbb{N}}$ as all sets $K_k, k \in \mathbb{N}$, are contained in one and the same compact subset of $\mathbb{R}^N$. So, there is $K \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(K_{k_l},\ K) \overset{l\to\infty}{\longrightarrow} 0$.
Following the same track as in the first part, we obtain subsequences (again denoted by) $(K_{k_j})_{j\in\mathbb{N}}$, $(G_{k_j})_{j\in\mathbb{N}}$ such that in addition, the latter converges to some $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ locally uniformly.
According to assumption (3.) of Viability Theorem 3.5, Graph $\mathcal{F} \subset \mathcal{K}(\mathbb{R}^N) \times \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed with respect to these topologies and thus, it contains (K, G). □

The next proposition focuses on solutions of nonautonomous differential inclusions in $\mathbb{R}^N$. In a word, this earlier theorem of Stassinopoulos and Vinter [32] characterizes perturbations (of the set-valued right-hand side) that have vanishing effect on the sets of continuous solutions. We will use it in subsequent § 3.5 for verifying that the limit of an approximative subsequence has led to a solution of the morphological inclusion (see Lemma 3.18).

Proposition 3.12 ([32, Theorem 7.1]) *Let* $D : [0,1] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and each* $D_n : [0,1] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ $(n \in \mathbb{N})$ *satisfy the following assumptions:*

1. D *and* D_n *have nonempty convex compact values,*
2. $D(\cdot, x),\ D_n(\cdot, x) : [0,1] \rightsquigarrow \mathbb{R}^N$ *are measurable for every* $x \in \mathbb{R}^N$,
3. *there exists* $k(\cdot) \in L^1([0,1])$ *such that* $D(t,\cdot),\ D_n(t,\cdot) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *are* $k(t)$*–Lipschitz for a.e.* t,
4. *there exists* $h(\cdot) \in L^1([0,1])$ *such that* $\sup_{y\in D(t,x)\,\cup\, D_n(t,x)} |y| \le h(t)$ *for every* $x \in \mathbb{R}^N$ *and a.e.* t.

Fixing the initial point $a \in \mathbb{R}^N$ *arbitrarily, the absolutely continuous solutions of*

$$\wedge \begin{cases} y'(\cdot) \in D_n(\cdot, y(\cdot)) & a.e. \text{ in } [0,1] \\ y(0) = a \end{cases} \quad \text{and} \quad \wedge \begin{cases} y'(\cdot) \in D(\cdot, y(\cdot)) & a.e. \text{ in } [0,1] \\ y(0) = a \end{cases}$$

respectively form compact subsets of $\big(C^0([0,1], \mathbb{R}^N), \|\cdot\|_\infty\big)$ *denoted by* $\mathcal{D}_n$ $(n \in \mathbb{N})$, $\mathcal{D}$.

Then, $\mathcal{D}_n$ *converges to* $\mathcal{D}$ *(w.r.t. the Pompeiu–Hausdorff metric on compact subsets of* $C^0([0,1], \mathbb{R}^N)$*) if and only if for every solution* $d(\cdot) \in \mathcal{D}$, $D_n(\cdot, d(\cdot)) : [0,1] \rightsquigarrow \mathbb{R}^N$ *converges to* $D(\cdot, d(\cdot)) : [0,1] \rightsquigarrow \mathbb{R}^N$ *weakly in the following sense*

$$d\!\!l\Big(\int_J D_n(s, d(s))\ ds, \quad \int_J D(s, d(s))\ ds\Big) \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{for every measurable subset } J \subset [0,1].$$

Reachable sets of differential inclusions provide candidates for solutions of morphological equations (and morphological inclusions, respectively).
This statement is rather obvious for every *autonomous* set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Indeed, the semigroup property implies $\vartheta_F(t+h, K_0) = \vartheta_F(h, \vartheta_F(t, K_0))$ for every $K_0 \in \mathcal{K}(\mathbb{R}^N)$, $t, h \geq 0$ (as stated in Lemma 2.4) and thus, $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the morphological mutation of $[0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $s \longmapsto \vartheta_F(s, K_0)$ at every time $t \in [0, T[$ and initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ according to Definition 2.5. A similar statement holds also for reachable sets of *nonautonomous* differential inclusions and *almost every* time. For this purpose, we benefit from an earlier result of Frankowska, Plaskacz and Rzeżuchowski [21] about the infinitesimal behavior of reachable maps:

Proposition 3.13 ([21, Theorem 2.5]) *Let* V *be a separable metric space and* $G : [0,T] \times \mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$ *a set–valued map satisfying*

1. G *has nonempty closed convex values,*
2. $\mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$, $(x, v) \mapsto G(t, x, v)$ *is continuous for almost all* $t \in [0,T]$,
3. $[0,T] \rightsquigarrow \mathbb{R}^N$, $t \mapsto G(t, x, v)$ *is measurable for all* $(x, v) \in \mathbb{R}^N \times V$,
4. *there exists* $h(\cdot) \in L^1([0,T])$ *with* $\sup_{y \in G(t,x,v)} |y| \leq h(t)$ *for all* $(x, v) \in \mathbb{R}^N \times V$ *and a.e.* $t \in [0,T]$.

Then, there exists a set $J \subset [0,T]$ *of full Lebesgue measure (i.e.* $\mathcal{L}^1([0,T] \setminus J) = 0$*) such that for every* $(t, x, v) \in J \times \mathbb{R}^N \times V$,

$$d\!\!l\Big(\tfrac{1}{h} \cdot \big(\vartheta_{G(t+\cdot\,,\,\cdot\,,v)}(h,\ x) - x\big),\ \ G(t,x,v)\Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0.$$

Corollary 3.14 *Let* V *be a separable metric space and the set-valued map* $G : [0,T] \times \mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$ *satisfy the assumptions of Proposition 3.13.*
Then for each $K_0 \in \mathcal{K}(\mathbb{R}^N)$, *there exists a set* $J \subset [0,T]$ *of full Lebesgue measure such that at every time* $t \in J$ *and for any* $v \in V$, $G(t, \cdot, v)$ *belongs to the morphological mutation of the reachable map*

$$[0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N), \qquad s \longmapsto \vartheta_{G(\cdot,\cdot,v)}(s, K_0).$$

at time t.

Proof. The detailed proof of [21, Theorem 2.5] even implies that the limit (of Pompeiu–Hausdorff distances) is locally uniform in $x \in \mathbb{R}^N$. So we obtain for any $K_t \in \mathcal{K}(\mathbb{R}^N)$ and all $t \in J, v \in V$

$$\tfrac{1}{h} \cdot d\!\!l\Big(\vartheta_{G(t+\cdot\,,\,\cdot\,,v)}(h,\ K_t),\ \bigcup_{x \in K_t} (x + h \cdot G(t,x,v))\Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0.$$

Applying this result to its autonomous counterpart $G(t,\cdot,\cdot) : \mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$ (with arbitrary $t \in J$), the corresponding limit exists again for each $t \in J$ and satisfies

$$\tfrac{1}{h} \cdot d\!\!l\Big(\vartheta_{G(t,\,\cdot\,,v)}\,(h,\ K_t), \qquad \bigcup_{x\in K_t}\ (x + h\cdot G(t,x,v))\Big) \ \longrightarrow\ 0 \qquad \text{for } h\downarrow 0 \text{ and all } K_t\in\mathcal{K}(\mathbb{R}^N), v\in V.$$

Combining these asymptotic features via triangle inequality, we conclude for any $t \in J, K_t\in\mathcal{K}(\mathbb{R}^N), v\in V$

$$\tfrac{1}{h} \cdot d\!\!l\Big(\vartheta_{G(t,\,\cdot\,,v)}\,(h,\ K_t), \quad \vartheta_{G(t+\,\cdot\,,\,\cdot\,,v)}(h,\ K_t)\Big) \ \longrightarrow\ 0 \qquad \text{for } h\downarrow 0,$$

i.e. fixing the initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ arbitrarily, there exists a set $J \subset [0,T]$ of full Lebesgue measure such that at every time $t \in J$, $G(t,\,\cdot\,,v)$ belongs to the morphological mutation of the reachable map

$$[0,T] \ \longrightarrow\ \mathcal{K}(\mathbb{R}^N), \qquad s \ \longmapsto\ \vartheta_{G(\cdot,\cdot,v)}(s, K_0) \qquad \square$$

3.5 Proof of main theorem 3.5

The proof of Viability Theorem 3.5 seizes the notion of approximation developed by Haddad and others for differential inclusions in $\mathbb{R}^N$ (and sketched in § 3.1).

For any given "threshold" $\varepsilon > 0$, we verify the existence of an approximative solution $K_\varepsilon(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ such that its values have distance $\leq \varepsilon$ from the constrained set $\mathcal{V}$. In addition, each $K_\varepsilon(\cdot)$ is induced by a piecewise constant function $f_\varepsilon(\cdot) : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ of morphological transitions such that $(K_\varepsilon(t),\ f_\varepsilon(t))$ is close to $\mathrm{Graph}\,\mathcal{F}$ at every time $t \in [0,T[$ (Lemma 3.15). Proposition 3.10 about parameterization bridges the gap between $f_\varepsilon(\cdot) : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ and the auxiliary function $\widehat{f}_\varepsilon(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\,\mathbb{R}^N)$ whose single values are additionally in the Banach space $\big(C^0(\mathbb{R}^N\times\mathbb{B}_1,\,\mathbb{R}^N),\ \|\cdot\|_\infty\big)$.

Then, letting $\varepsilon > 0$ tend to 0, we obtain subsequences denoted by $(K_n(\cdot))_{n\in\mathbb{N}}$, $\big(\widehat{f}_n(\cdot)\big)_{n\in\mathbb{N}}$ that are converging to some $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ and $\widehat{f} : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\,\mathbb{R}^N)$, respectively, in an appropriate sense – due to compactness arguments specified in § 3.3 (see subsequent Lemma 3.16).

Last, but not least, we prove that these limits satisfy for Lebesgue almost every $t \in [0,T[$

$$\widehat{f}(t)(\cdot,\mathbb{B}_1) \ \in \overset{\circ}{K}(t)\ \cap\ \mathcal{F}(K(t))\ \neq\ \emptyset.$$

Indeed, Lemma 3.17 concludes $\widehat{f}(t)(\cdot,\mathbb{B}_1) \in \mathcal{F}(K(t))$ for almost every $t \in [0,T[$ from Lemma 3.11 stating that the graph of $\mathcal{F}$ is sequentially compact. Furthermore, $K(\cdot)$ can be characterized as reachable set due to Proposition 3.12, i.e. $\vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0) = K(t)$ at every time $t \in]0,1]$ (Lemma 3.18). So finally, preceding Corollary 3.14 implies $\widehat{f}(t)(\cdot,\mathbb{B}_1) \in \overset{\circ}{K}(t)$ for *almost* every $t \in]0,1[$.

Let us now follow this track in detail:

Lemma 3.15 (Constructing approximative solutions) *Choose any* $\varepsilon > 0$.

Under the assumptions of Viability Theorem 3.5, there exist a B–Lipschitz continuous function $K_\varepsilon(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ *and a function* $f_\varepsilon(\cdot) : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\,\mathbb{R}^N)$ *satisfying with* $R_\varepsilon := \varepsilon\ e^A$

a) $K_\varepsilon(0) = K_0$,

b) $\mathrm{dist}\big(K_\varepsilon(t),\ \mathcal{V}\big)\ \leq\ R_\varepsilon$ *for all* $t \in [0,1]$,

c) $f_\varepsilon(t)\ \in\ \overset{\circ}{K}_\varepsilon(t)\ \cap\ \mathcal{F}\big(\mathbb{B}_{R_\varepsilon}(K_\varepsilon(t))\big)\ \neq\ \emptyset$ *for all* $t \in [0,1[$,

d) $f_\varepsilon(\cdot)$ *is piecewise constant in the following sense: for each* $t \in [0,1[$, *there exists some* $\delta > 0$ *such that* $f_\varepsilon(\cdot)|_{[t,\,t+\delta[}$ *is constant.*

Proof follows the same track as [4, Lemma 1.6.5] and uses Zorn's Lemma: For $\varepsilon > 0$ fixed, let $\mathcal{A}_\varepsilon(K_0)$ denote the set of all tuples $(\tau_K,\ K(\cdot),\ f(\cdot))$ consisting of some $\tau_K \in [0,1]$, a B–Lipschitz continuous function $K(\cdot) : [0, \tau_K] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and some piecewise constant function $f(\cdot) : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\ \mathbb{R}^N)$ such that

a) $K(0) = K_0$,

b') 1.) $\mathrm{dist}\big(K(\tau_K),\ \mathcal{V}\big) \ \le\ r_\varepsilon(\tau_K)$ with $r_\varepsilon(t) := \varepsilon\, e^{A\,t}\, t$,

2.) $\mathrm{dist}\big(K(t),\ \mathcal{V}\big) \ \le\ R_\varepsilon$ for all $t \in [0, \tau_K]$,

c) $f(t) \ \in\ \overset{\circ}{K}(t) \ \cap\ \mathcal{F}\big(\mathbb{B}_{R_\varepsilon}(K(t))\big) \ \ne\ \emptyset$ for all $t \in [0, \tau_K[$.

Obviously, $\mathcal{A}_\varepsilon(K_0)$ is not empty since it contains $(0,\ K(\cdot) \equiv K_0,\ f(\cdot) \equiv f_0)$ with arbitrary $f_0 \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\ \mathbb{R}^N)$. Moreover, an order relation $\preceq$ on $\mathcal{A}_\varepsilon(K_0)$ is specified by

$$(\tau_K,\ K(\cdot),\ f(\cdot)) \ \preceq\ (\tau_M,\ M(\cdot),\ g(\cdot)) \quad :\Longleftrightarrow \quad \tau_K \le \tau_M,\ M\big|_{[0,\tau_K]} = K,\ g\big|_{[0,\tau_K[} = f.$$

So Zorn's Lemma provides a maximal element $\big(\tau,\ K_\varepsilon(\cdot),\ f_\varepsilon(\cdot)\big) \in \mathcal{A}_\varepsilon(K_0)$.
As all considered functions with values in $\mathcal{K}(\mathbb{R}^N)$ have been supposed to be B–Lipschitz continuous, $K_\varepsilon(\cdot)$ is well-defined on the closed interval $[0, \tau] \subset [0, 1]$.

Assuming $\tau < 1$ for a moment, we obtain a contradiction if $K_\varepsilon(\cdot),\ f_\varepsilon(\cdot)$ can be extended to a larger interval $[0, \tau + \delta] \ \subset\ [0, 1]$ $(\delta > 0)$ preserving conditions (b'), (c).
Since closed bounded balls of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ are compact, the closed set $\mathcal{V}$ contains an element $Z \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(K_\varepsilon(\tau), Z) \ =\ \mathrm{dist}(K_\varepsilon(\tau),\ \mathcal{V}) \ \le\ r_\varepsilon(\tau)$ and, assumption (4.) of Viability Theorem 3.5 provides a set-valued map

$$G \ \in\ T_{\mathcal{V}}(Z) \ \cap\ \mathcal{F}(Z) \ \subset\ \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\ \mathbb{R}^N).$$

Due to Definition 2.9 of the contingent transition set $T_{\mathcal{V}}(Z)$, there is a sequence $h_m \downarrow 0$ in $]0, 1{-}\tau[$ such that $\mathrm{dist}(\vartheta_G(h_m, Z),\ \mathcal{V}) \le \varepsilon\, h_m$ for all $m \in \mathbb{N}$. Now set

$$K_\varepsilon(t) \ :=\ \vartheta_G\big(t - \tau,\ K_\varepsilon(\tau)\big), \qquad f_\varepsilon(t) \ :=\ G \qquad\qquad \text{for each } t \in [\tau,\ \tau + h_1[.$$

Obviously, Lemma 2.4 implies $G \ \in\ \overset{\circ}{K}_\varepsilon(t)$ for all $t \in [\tau,\ \tau + h_1[$. Moreover, it leads to

$$\begin{aligned} d\!\!l\big(K_\varepsilon(t),\ Z\big) \quad &\le\ d\!\!l\big(\vartheta_G(t - \tau,\ K_\varepsilon(\tau)),\ \ K_\varepsilon(\tau)\big) &&+ d\!\!l\big(K_\varepsilon(\tau),\ \ Z\big) \\ &\le\ B \cdot (t - \tau) &&+ \varepsilon\ \ e^{A\,\tau}\ \ \tau \qquad \le\ R_\varepsilon \end{aligned}$$

for every $t \in [\tau,\ \tau + \delta[$ with $\delta \ :=\ \min\big\{h_1,\ \ \varepsilon\, e^{A}\ \frac{1-\tau}{1+B}\big\}$, i.e. conditions (b')(2.) and (c) hold in the interval $[\tau, \tau + \delta]$. For any index $m \in \mathbb{N}$ with $h_m < \delta$,

$$\begin{aligned} \mathrm{dist}\big(K_\varepsilon(\tau{+}h_m),\ \mathcal{V}\big) \ &\le\ d\!\!l\big(\vartheta_G(h_m,\ K_\varepsilon(\tau)),\ \ \vartheta_G(h_m,\ Z)\big) + \mathrm{dist}\big(\vartheta_G(h_m,\ Z),\ \ \mathcal{V}\big) \\ &\le\ d\!\!l\big(K_\varepsilon(\tau),\ Z\big)\ \cdot\ e^{A\,h_m} \qquad\quad + \varepsilon \cdot h_m \\ &\le\ \varepsilon\ \ e^{A\,\tau}\ \ \tau \qquad \cdot\ e^{A\,h_m} \qquad\quad + \varepsilon \cdot h_m \ \ \le\ r_\varepsilon(\tau + h_m), \end{aligned}$$

i.e. condition (b')(1.) is also satisfied at time $t = \tau + h_m$ with any large $m \in \mathbb{N}$.
So $K_\varepsilon(\cdot)\big|_{[0,\ \tau+h_m]}$ and $f_\varepsilon(\cdot)\big|_{[0,\ \tau+h_m[}$ provide the wanted contradiction and thus, $\tau = 1$. □

Remark. As a direct consequence of property (d), the function $f_\varepsilon : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ can have at most countably many points of discontinuity. This enables us to apply preceding results about autonomous morphological equations (§ 2) to the approximations $K_\varepsilon(\cdot), f_\varepsilon(\cdot)$ in a "piecewise" way.

Now the "threshold of accuracy" $\varepsilon > 0$ is tending to 0. The "detour" of parameterization (Proposition 3.10) and the statements about sequential compactness in § 3.3 lay the basis for extracting subsequences with additional features of convergence:

Lemma 3.16 (Selecting an approximative subsequence)
Under the assumptions of Viability Theorem 3.5, there exist a constant $c = c(N, A, B) > 0$, *sequences* $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}_n(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ $(n \in \mathbb{N})$ *and* $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ *such that for every* $j, n \in \mathbb{N}$, $t \in [0,1[$, $x \in \mathbb{R}^N$, $u \in \mathbb{B}_1 \subset \mathbb{R}^N$

a) $K_0 \;=\; K_n(0) \;=\; K(0)$,

b) $K(\cdot)$ *and* $K_n(\cdot)$ *are* B*–Lipschitz continuous w.r.t.* $d\!l$,

c) $\widehat{f}_n(\cdot)(x,u)$ *is piecewise constant (in the sense of Lemma 3.15 (d)),*
$\|\widehat{f}_n(t)(\cdot,\cdot)\|_\infty + \mathrm{Lip}\,\widehat{f}_n(t)(\cdot,\cdot) \;\le\; c \;<\; \infty$,

d) $\mathrm{dist}\big(K_n(t),\ \mathcal{V}\big) \;\le\; \frac{1}{n}$

e) $\widehat{f}_n(t)(\cdot, \mathbb{B}_1) \;\in\; \overset{\circ}{K}_n(t) \;\cap\; \mathcal{F}\big(\mathbb{B}_{1/n}(K_n(t))\big) \;\neq\; \emptyset$

f) $d\!l\big(K_m(\cdot),\ K(\cdot)\big) \longrightarrow 0$ *uniformly in* $[0,1]$ *for* $m \longrightarrow \infty$,

g) $\widehat{f}_m(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \;\longrightarrow\; \widehat{f}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1}$ *weakly in* $L^1\big([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)\big)$ *for* $m \longrightarrow \infty$,

h) $\|\widehat{f}(t)(\cdot,\cdot)\|_\infty + \mathrm{Lip}\,\widehat{f}(t)(\cdot,\cdot) \;\le\; c \;<\; \infty$,

i) $K(t) \in \mathcal{V}$

with the abbreviation $\widetilde{K}_j \;:=\; \mathbb{B}_{j+B}(K_0) \;\overset{\text{Def.}}{=}\; \big\{x \in \mathbb{R}^N \;\big|\; \mathrm{dist}(x, K_0) \le j + B\big\} \;\in\; \mathcal{K}(\mathbb{R}^N)$.

Proof is based on the approximative solutions of Lemma 3.15, of course.
Indeed, for each $n \in \mathbb{N}$, Lemma 3.15 provides $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $f_n(\cdot) : [0,1[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ corresponding to $\varepsilon := \frac{1}{n}\, e^{-A}$. Now according to Proposition 3.10, the set–valued map $[0,1[\times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $(t,x) \mapsto f_n(t)(x)$ has a parameterization $[0,1[\times \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ that we interprete as function $\widehat{f}_n : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$. Obviously, they satisfy the claimed properties (a) – (e).
In particular, these features stay correct whenever we consider subsequences instead and again abbreviate them as $(K_n(\cdot))_{n \in \mathbb{N}}$, $(\widehat{f}_n(\cdot))_{n \in \mathbb{N}}$ respectively.

For property (f) about uniform convergence of $(K_n(\cdot))$ *with respect to* $d\!l$:

The B–Lipschitz continuity of each $K_n(\cdot)$ has two important consequences, i.e.

1. all $K_n(\cdot) : [0,1] \longrightarrow \big(\mathcal{K}(\mathbb{R}^N), d\!l\big)$ $(n \in \mathbb{N})$ are equi–continuous and
2. $\bigcup_{\substack{n \in \mathbb{N} \\ t \in [0,1]}} \big\{K_n(t)\big\}$ is contained in the compact subset $\mathbb{B}_B(K_0)$ of $\big(\mathcal{K}(\mathbb{R}^N),\ d\!l\big)$.

So, the Theorem of Arzela–Ascoli (Proposition 3.6) provides a subsequence (again denoted by) $(K_n(\cdot))_n$ converging uniformly to a function $K(\cdot) : [0,1] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!l)$. In particular, $K(\cdot)$ is also B–Lipschitz continuous with $K(0) = K_0$, i.e. properties (a) – (f) are fulfilled completely.

For property (g) about weak convergence of $f_n(\cdot)|_{\widetilde{K}}$ *with a fixed compact subset* $\widetilde{K} \subset \mathbb{R}^N$:

We cannot follow the same track as for differential inclusions in $\mathbb{R}^N$ any longer. Indeed, the functions $\widehat{f}_n(\cdot)$ of morphological transitions have their values in $\mathrm{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ which cannot be regarded as a dual space in an obvious way. So Alaoglu's Theorem (stating that closed balls of dual Banach spaces are weakly* compact) cannot be applied similarly to differential inclusions (§ 3.1).

Alternatively, we restrict our considerations to a compact neighborhood $\widetilde{K}$ of $\bigcup_{\substack{n\in\mathbb{N}\\ t\in[0,1]}} K_n(t) \subset \mathbb{R}^N$ and use a sufficient condition on relatively weakly compact sets in $L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big)$. Here $C^0(\widetilde{K}\times\mathbb{B}_1, \mathbb{R}^N)$ (supplied with the supremum norm $\|\cdot\|_\infty$) denotes the Banach space of all continuous functions $\widetilde{K}\times\mathbb{B}_1 \longrightarrow \mathbb{R}^N$. According to Proposition 3.9, if $W \subset C^0(\widetilde{K}\times\mathbb{B}_1, \mathbb{R}^N)$ is weakly compact then the subset

$$\Big\{ h \in L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big) \ \Big|\ h(t) \in W \ \text{ for (Lebesgue) almost every } \ t\in[0,1]\Big\}$$

is relatively weakly compact in $L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big)$.
In fact, the set $\big\{\widehat{f}_n(t) \,\big|\, n\in\mathbb{N}, t\in[0,1]\big\} \subset C^0(\mathbb{R}^N\times\mathbb{B}_1,\ \mathbb{R}^N)$ is uniformly bounded and equi–continuous (due to property (c)). So according to the Theorem of Arzela–Ascoli (Proposition 3.6), the set of their restrictions to the compact set $\widetilde{K}\times\mathbb{B}_1 \subset \mathbb{R}^N\times\mathbb{R}^N$

$$W \ := \ \Big\{ \widehat{f}_n(t)|_{\widetilde{K}\times\mathbb{B}_1} \ \Big|\ n\in\mathbb{N},\ t\in[0,1]\Big\} \ \subset \ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)$$

is relatively compact with respect to $\|\cdot\|_\infty$. Thus, $\big\{\widehat{f}_n(\cdot)|_{\widetilde{K}\times\mathbb{B}_1} \,\big|\, n\in\mathbb{N}\big\}$ is relatively weakly compact in $L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big)$ and, we obtain a subsequence (again denoted by) $(\widehat{f}_n(\cdot))_{n\in\mathbb{N}}$ and some $g(\cdot)\in L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big)$ with

$$\widehat{f}_n(\cdot)|_{\widetilde{K}\times\mathbb{B}_1} \stackrel{n\to\infty}{\longrightarrow} g(\cdot) \qquad\qquad \text{weakly in } \ L^1\big([0,1],\ C^0(\widetilde{K}\times\mathbb{B}_1,\ \mathbb{R}^N)\big).$$

For property (g) about $f_n(\cdot)|_{\widetilde{K}_j}$ with every compact $\widetilde{K}_j \stackrel{\text{Def.}}{=} \mathbb{B}_{j+B}(K_0)\subset\mathbb{R}^N$ $(j\in\mathbb{N})$:
Now this construction of subsequences is applied to $\widetilde{K}_j \stackrel{\text{Def.}}{=} \mathbb{B}_{j+B}(K_0) = \big\{x\in\mathbb{R}^N \,\big|\, \operatorname{dist}(x,K_0)\le j+B\big\}$ for $j=1,2,3\ldots$ successively. By means of Cantor's diagonal construction, we obtain a subsequence (again denoted by) $(\widehat{f}_n(\cdot))_{n\in\mathbb{N}}$ and some $g_j(\cdot)\in L^1\big([0,1],\ C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N)\big)$ (for each $j\in\mathbb{N}$) such that for every index $j\in\mathbb{N}$,

$$\widehat{f}_n(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \stackrel{n\to\infty}{\longrightarrow} g_j(\cdot) \qquad\qquad \text{weakly in } \ L^1\big([0,1],\ C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N)\big).$$

As restrictions to $\widetilde{K}_j\times\mathbb{B}_1$ of one and the same subsequence $(\widehat{f}_n(\cdot))_{n\in\mathbb{N}}$ converge weakly for each $j\in\mathbb{N}$, the inclusion $\widetilde{K}_j\subset\widetilde{K}_{j+1}$ implies for any indices $j<k$

$$g_j(t)(\cdot) \ = \ g_k(t)(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \in C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N) \qquad\qquad \text{for almost every } t\in[0,1]$$

and, so $(g_j(\cdot))_{j\in\mathbb{N}}$ induces a single function $\widehat{f}:[0,1[\ \longrightarrow C^0(\mathbb{R}^N\times\mathbb{B}_1,\mathbb{R}^N)$ defined as

$$\widehat{f}(t)(x,u) \ := \ g_j(t)(x,u) \qquad\qquad \text{for } \ x\in\widetilde{K}_j,\ u\in\mathbb{B}_1 \ \text{ and almost every } \ t\in[0,1[.$$

For property (h) about Lipschitz continuity and bounds of limit function $f(\cdot)$:
Finally, we verify $\widehat{f}(t)\in\operatorname{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\ \mathbb{R}^N)$, $\|\widehat{f}(t,\cdot,\cdot)\|_\infty + \operatorname{Lip}\widehat{f}(t,\cdot,\cdot)\le c$ for almost every $t\in[0,1[$.
Indeed, as in the case of differential inclusions (§ 3.1), Mazur's Lemma 3.7 ensures for each $j\in\mathbb{N}$ (fixed)

$$\widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \in \bigcap_{n\in\mathbb{N}} \overline{co}\,\big\{\widehat{f}_n(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1},\ f_{n+1}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1}\ \ldots\big\} \qquad \text{in } \ L^1\big([0,1],\ C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N)\big).$$

Thus, $\widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1}$ can be approximated by convex combinations of $\big\{\widehat{f}_1(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1},\ \widehat{f}_2(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1}\ \ldots\big\}$ with respect to the L^1 norm. A further subsequence (of these convex combinations) converges to $\widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1}$ almost everywhere in $[0,1]$. So, for almost every $t\in[0,1]$, $\widehat{f}(t)|_{\widetilde{K}_j\times\mathbb{B}_1}$ belongs to the same compact convex subset of $\big(C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N),\ \|\cdot\|_\infty\big)$ as $\widehat{f}_1(t)|_{\widetilde{K}_j\times\mathbb{B}_1}$, $\widehat{f}_2(t)|_{\widetilde{K}_j\times\mathbb{B}_1}\ \ldots$, namely $\big\{w\in\operatorname{Lip}(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N) \,\big|\, \|w\|_\infty + \operatorname{Lip}\, w \le c\big\}$. As the index $j\in\mathbb{N}$ is fixed arbitrarily, we obtain property (h).

Property (i), i.e. $K(t)\in\mathcal{V}$ for every $t\in[0,1]$, results directly from statements (d), (f) and the assumption that $\mathcal{V}$ is closed in $\big(\mathcal{K}(\mathbb{R}^N), d\!\!\!l\big)$. This completes the proof of Lemma 3.16. □

The last step is to verify at Lebesgue almost every time $t \in [0,1[$ that $\widehat{f}(t)(\cdot,\mathbb{B}_1) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ belongs to both $\mathcal{F}(K(t))$ and the morphological mutation $\overset{\circ}{K}(t)$.

First we interprete the weak convergence of the parameterized maps $\widehat{f}_n(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \longrightarrow \widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1}$ (in L^1) with respect to the corresponding set–valued maps $[0,1[\times\widetilde{K}_j \rightsquigarrow \mathbb{R}^N$ and meet the topology of locally uniform convergence in $\mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$.
As a rather technical tool, preceding Lemma 3.11 (in § 3.3) clarifies how the uniform Lipschitz bounds of $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ (according to assumption (2.)) imply useful compactness features which ensure that the limit map $\widehat{f}(t)(\cdot,\mathbb{B}_1) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is related to $\mathcal{F}(K(t))$ at almost every time t.

Lemma 3.17 *Let the sequences* $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}_n(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\,\mathbb{R}^N)$ $(n\in\mathbb{N})$ *and the functions* $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\,\mathbb{R}^N)$ *be as in Lemma 3.16.*
Then, for $\mathcal{L}^1$ *almost every* $t\in[0,1[$,

$$\mathrm{dist}\Big(\widehat{f}(t)(x,\mathbb{B}_1),\;\; co\big\{\widehat{f}_n(t)(x,\mathbb{B}_1),\; \widehat{f}_{n+1}(t)(x,\mathbb{B}_1)\;\ldots\big\}\Big) \overset{n\to\infty}{\longrightarrow} 0 \qquad \textit{locally uniformly in } x\in\mathbb{R}^N$$

with the coefficients of the approximating convex combinations being chosen independently from t, x.
So in particular, $\widehat{f}(t)(\cdot,\mathbb{B}_1) \in \mathcal{F}(K(t)) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$.

Proof. Lemma 3.16 (g) specifies the convergence resulting directly from construction

$$\widehat{f}_n(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \overset{n\to\infty}{\longrightarrow} \widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \quad \text{weakly in } L^1\big([0,1], C^0(\widetilde{K}_j\times\mathbb{B}_1,\,\mathbb{R}^N)\big) \quad \text{for each } j\in\mathbb{N}$$

with the abbreviation $\widetilde{K}_j := \mathbb{B}_{j+B}(K_0) \overset{\text{Def.}}{=} \big\{x\in\mathbb{R}^N \;\big|\; \mathrm{dist}(x,K_0)\le j+B\big\} \in \mathcal{K}(\mathbb{R}^N)$.
Fixing the index $j\in\mathbb{N}$ of compact sets arbitrarily, Mazur's Lemma 3.7 provides a sequence $\big(h_{j,n}(\cdot)\big)_{n\in\mathbb{N}}$ with $h_{j,n}(\cdot) \in co\big\{\widehat{f}_n(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1},\, \widehat{f}_{n+1}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \cdots\big\} \subset L^1\big([0,1], C^0(\widetilde{K}_j\times\mathbb{B}_1,\,\mathbb{R}^N)\big)$ and

$$h_{j,n}(\cdot) \longrightarrow \widehat{f}(\cdot)|_{\widetilde{K}_j\times\mathbb{B}_1} \qquad (n\longrightarrow\infty) \qquad \text{strongly in } L^1\big([0,1], C^0(\widetilde{K}_j\times\mathbb{B}_1,\,\mathbb{R}^N)\big).$$

For a subsequence $\big(h_{j,n_k}(\cdot)\big)_{k\in\mathbb{N}}$, we even obtain convergence for $\mathcal{L}^1$ almost every $t\in[0,1]$,

$$h_{j,n_k}(t) \longrightarrow \widehat{f}(t)|_{\widetilde{K}_j\times\mathbb{B}_1} \qquad (k\longrightarrow\infty) \qquad \text{in } \big(C^0(\widetilde{K}_j\times\mathbb{B}_1,\mathbb{R}^N),\; \|\cdot\|_\infty\big),$$

i.e. uniformly in $\widetilde{K}_j\times\mathbb{B}_1 \subset \mathbb{R}^N\times\mathbb{R}^N$. So the first claim is proved. In particular, all values of $\widehat{f}(t)(\cdot,\mathbb{B}_1) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ are convex since each $\widehat{f}_n(t)(\cdot,\mathbb{B}_1) \in \mathrm{im}\,\mathcal{F} \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ has convex values.

Furthermore, we obtain the following inclusions for $\mathcal{L}^1$ almost every $t\in[0,1]$ (and each index $j\in\mathbb{N}$)

$$\begin{aligned}
\widehat{f}(t)(\,\cdot\,,\mathbb{B}_1)\big|_{\widetilde{K}_j} &\in \bigcap_{n\in\mathbb{N}} \overline{h_{j,n}(t)(\,\cdot\,,\mathbb{B}_1)\big|_{\widetilde{K}_j} \cup h_{j,\,n+1}(t)(\,\cdot\,,\mathbb{B}_1)\big|_{\widetilde{K}_j} \cup \ldots} && \text{in a pointwise way in } \widetilde{K}_j\\
&\subset \bigcap_{n\in\mathbb{N}} \overline{co} \bigcup_{m\ge n} \widehat{f}_m(t)(\,\cdot\,,\mathbb{B}_1)\big|_{\widetilde{K}_j}\\
&\subset \bigcap_{n\in\mathbb{N}} \overline{co} \bigcup_{m\ge n} \mathcal{F}\big(\mathbb{B}_{1/m}(K_m(t))\big)\big|_{\widetilde{K}_j} && \text{due to Lemma 3.16 (e)}\\
&\subset \bigcap_{\varepsilon>0} \overline{co}\;\; \mathcal{F}\big(\mathbb{B}_\varepsilon(K(t))\big)\big|_{\widetilde{K}_j} && \text{since } d\!\!\!l(K_m(t),K(t)) \to 0.
\end{aligned}$$

Here, to be more precise, the closed convex hull (in the last line) denotes the following set–valued map

$$\widetilde{K}_j \rightsquigarrow \mathbb{R}^N, \qquad x \mapsto \overline{co} \bigcup_{\substack{M\in\mathcal{K}(\mathbb{R}^N)\\ d\!\!\!l(K(t),M)\le\varepsilon}} \;\bigcup_{G\in\mathcal{F}(M)} G(x).$$

Fixing now $j \in \mathbb{N}$ and $\delta > 0$ arbitrarily, we introduce the abbreviation

$$\mathcal{B}_\delta\left(\mathcal{F}(K(t));\ \widetilde{K}_j\right) \;:=\; \Big\{G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N) \;\Big|\; \delta \;\geq\; \operatorname{dist}\left(G(\cdot)|_{\widetilde{K}_j},\ \mathcal{F}(K(t))|_{\widetilde{K}_j}\right) \stackrel{\text{Def.}}{=} \inf_{Z \in \mathcal{F}(K(t))} \sup_{x \in \widetilde{K}_j} d\!\!l(G(x),\ Z(x))\Big\}$$

for the "ball" around the set $\mathcal{F}(K(t)) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ containing all maps $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ whose restriction to $\widetilde{K}_j$ has the "uniform distance" $\leq \delta$ from $\mathcal{F}(K(t))$.
For any $\delta > 0$ and $j \in \mathbb{N}$, there exists a radius $\rho > 0$ with $\mathcal{F}(\mathbb{B}_\rho(K(t))) \subset \mathcal{B}_\delta\big(\mathcal{F}(K(t));\ \widetilde{K}_j\big)$ because otherwise there would exist sequences $(M_k)_{k\in\mathbb{N}}$, $(G_k)_{k\in\mathbb{N}}$ in $\mathcal{K}(\mathbb{R}^N)$ and $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$, respectively, with $d\!\!l\,(M_k,\ K(t)) \leq \frac{1}{k}$, $\quad G_k \in \mathcal{F}(M_k) \setminus \mathcal{B}_\delta\big(\mathcal{F}(K(t));\ \widetilde{K}_j\big)$ for each $k \in \mathbb{N}$ and, Lemma 3.11 would lead to a contradiction (similarly to [8, Proposition 1.4.8] about closed graph and upper semicontinuity of set-valued maps between metric spaces).
Obviously, $\mathcal{B}_\delta\big(\mathcal{F}(K(t));\ \widetilde{K}_j\big) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ is closed with respect to locally uniform convergence. Moreover, it is convex (with regard to pointwise convex combinations) because $\mathcal{F}(K(t))$ is supposed to be convex. Thus, we even obtain the inclusion $\overline{co}\,\mathcal{F}(\mathbb{B}_\rho(K(t))) \subset \mathcal{B}_\delta\big(\mathcal{F}(K(t));\ \widetilde{K}_j\big)$, i.e.

$$\widehat{f}(t)(\,\cdot\,,\mathbb{B}_1)\big|_{\widetilde{K}_j} \;\in\; \bigcap_{\delta > 0} \mathcal{B}_\delta\big(\mathcal{F}(K(t));\ \widetilde{K}_j\big) \qquad \text{for almost every } t \text{ and each } j \in \mathbb{N}.$$

In particular, there exists some $Z_j \in \mathcal{F}(K(t))$ satisfying $\sup_{x \in \widetilde{K}_j} d\!\!l\,(f(t)(x,\mathbb{B}_1),\ Z_j(x)) \leq \frac{1}{j}$ and, the compactness property of Lemma 3.11 implies $\widehat{f}(t)(\,\cdot\,,\mathbb{B}_1) \in \mathcal{F}(K(t))$ for almost every time t. □

So last, but not least, we have to prove $\widehat{f}(t)(\cdot,\mathbb{B}_1) \in \overset{\circ}{K}(t)$ at $\mathcal{L}^1$ almost every time $t \in [0,1]$. Due to Corollary 3.14 in § 3.4, we can restrict our considerations to describing $K(t)$ as reachable set of a nonautonomous differential inclusion, i.e. $\vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0) \;=\; K(t)$ for every $t \in]0,1]$.
Proposition 3.12 of Stassinopoulos and Vinter lays the basis for initial sets with a single element.

Lemma 3.18 **($K(t)$ as a reachable set of $\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)$)**
Let the sequences $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}_n(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\ \mathbb{R}^N)$ $(n \in \mathbb{N})$ *and the functions* $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\widehat{f}(\cdot) : [0,1[\longrightarrow \mathrm{Lip}(\mathbb{R}^N\times\mathbb{B}_1,\ \mathbb{R}^N)$ *be as in Lemma 3.16.*

Then, for any $x(\cdot) \in C^0([0,1],\mathbb{R}^N)$ *and Lebesgue measurable set* $J \subset [0,1]$,

$$d\!\!l\Big(\int_J \widehat{f}_n(s)(x(s),\mathbb{B}_1)\ ds,\ \int_J \widehat{f}(s)(x(s),\mathbb{B}_1)\ ds\Big) \;\stackrel{n\to\infty}{\longrightarrow}\; 0.$$

So in particular, $\vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0) \;=\; K(t)$ *for every* $t \in]0,1]$.

Proof. According to the definition of Aumann integral (e.g. [8, § 8.6]),

$$\int_J \widehat{f}(s)(x(s),\mathbb{B}_1)\ ds \;\stackrel{\text{Def.}}{=}\; \Big\{\int_J \widehat{f}(s)(x(s),u(s))\ ds \;\Big|\; u(\cdot) \in L^1(J,\mathbb{B}_1)\Big\}.$$

Fixing $u(\cdot) \in L^1(J,\mathbb{B}_1)$ and $x(\cdot) \in C^0([0,1],\mathbb{R}^N)$ arbitrarily, we conclude from Lemma 3.16 (g)

$$\int_J \widehat{f}_n(s)(x(s),u(s))\ ds \;\longrightarrow\; \int_J \widehat{f}(s)(x(s),u(s))\ ds \qquad \text{for } n \to \infty$$

since $L^1\big([0,1],C^0(\widetilde{K}_j\times\mathbb{B}_1,\ \mathbb{R}^N)\big) \longrightarrow \mathbb{R}$, $h \longmapsto \int_J h(s)(x(s),u(s))\ ds$ is continuous and linear whenever $x([0,1]) \subset \widetilde{K}_j$. This implies

$$\text{both} \qquad \operatorname{dist}\Big(\int_J \widehat{f}_n(s)(x(s),\mathbb{B}_1)\ ds,\ \int_J \widehat{f}(s)(x(s),\mathbb{B}_1)\ ds\Big) \;\longrightarrow\; 0$$

$$\text{and} \qquad \operatorname{dist}\Big(\int_J \widehat{f}(s)(x(s),\mathbb{B}_1)\ ds,\ \int_J \widehat{f}_n(s)(x(s),\mathbb{B}_1)\ ds\Big) \;\longrightarrow\; 0 \qquad \text{for } n \to \infty.$$

So the first claim holds.

Due to Lemma 3.16 (c), each $\widehat{f_n}(\cdot)(x,\mathbb{B}_1) : [0,1[\rightsquigarrow \mathbb{R}^N$ $(n \in \mathbb{N},\ x \in \mathbb{R}^N)$ is piecewise constant and thus, it has at most countably many points of discontinuity. So applying the Cauchy–Lipschitz–type Theorem 2.7 (of Aubin) in a piecewise way with respect to time, we conclude from its uniqueness and from the B–Lipschitz continuity of $K_n(\cdot)$

$$\vartheta_{\widehat{f_n}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0) \;=\; K_n(t) \qquad \text{for every } \; t \in \,]0,1] \;\text{ and }\; n \in \mathbb{N}.$$

$d\!\!l(K_n(t),\ K(t)) \longrightarrow 0$ has already been mentioned in Lemma 3.16 (f). So we now still have to verify

$$d\!\!l\left(\vartheta_{\widehat{f_n}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0),\quad \vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}(t,K_0)\right) \longrightarrow 0 \qquad \text{for every } \; t \in \,]0,1] \;\text{ and }\; n \to \infty.$$

If $K_0 \subset \mathbb{R}^N$ consists of only one point, then this convergence results directly from Proposition 3.12. For extending it to arbitrary initial sets $K_0 \in \mathcal{K}(\mathbb{R}^N)$, we exploit two features: first, the reachable set of a union is always the union of the corresponding reachable sets and second, the Lipschitz dependence (of reachable sets) on the initial sets according to Lemma 2.4, i.e. for any $M_1, M_2 \in \mathcal{K}(\mathbb{R}^N)$ and $t \in [0,1]$

$$\begin{cases} d\!\!l\big(\vartheta_{\widehat{f_n}(\cdot)(\cdot,\mathbb{B}_1)}(t,M_1), \quad \vartheta_{\widehat{f_n}(\cdot)(\cdot,\mathbb{B}_1)}(t,M_2)\big) & \le \quad e^A \;\; d\!\!l(M_1,M_2) \\ d\!\!l\big(\vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}\;(t,M_1), \quad \vartheta_{\widehat{f}(\cdot)(\cdot,\mathbb{B}_1)}\;(t,M_2)\big) & \le \quad e^A \;\; d\!\!l(M_1,M_2) \end{cases}$$

The latter case of nonautonomous differential inclusions is covered by generalized Filippov's Theorem (e.g. [34, Theorem 2.4.3]) correspondingly to Lemma 2.4. □

4 Evolution of shapes under operability constraints

Now Viability Theorem 3.5 is applied to two very special forms of constraints successively:

$$\begin{aligned} \mathcal{V}_1 \;&:=\; \big\{K \in \mathcal{K}(\mathbb{R}^N) \;\big|\; K \cap M \neq \emptyset\big\} \\ \mathcal{V}_2 \;&:=\; \big\{K \in \mathcal{K}(\mathbb{R}^N) \;\big|\; K \subset M\big\} \end{aligned}$$

with some (arbitrarily fixed) nonempty closed subset $M \subset \mathbb{R}^N$. Consequently, we obtain sufficient conditions on $M \subset \mathbb{R}^N$ and $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\,\mathbb{R}^N)$ for the existence of a Lipschitz continuous solution $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ satisfying (respectively)

$$\begin{cases} \overset{\circ}{K}(\cdot)\,\cap\,\mathcal{F}(K(\cdot)) \neq \emptyset & \text{a.e. in } [0,1] \\ \qquad K(t)\,\cap\,M \neq \emptyset & \text{for each } t \in [0,1] \end{cases} \quad \text{and} \quad \begin{cases} \overset{\circ}{K}(\cdot)\,\cap\,\mathcal{F}(K(\cdot)) \neq \emptyset & \text{a.e. in } [0,1] \\ \qquad K(t)\,\subset\,M & \text{for each } t \in [0,1] \end{cases}$$

Here we benefit from earlier results of Anne Gorre [23] considering the corresponding problems with morphological *equations* (instead of inclusions). In a word, she proved $\mathcal{V}_1$, $\mathcal{V}_2$ to be closed subsets of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and characterized their contingent transition sets completely by means of the tangential properties of the closed set $M \subset \mathbb{R}^N$. Then she applied Nagumo's theorem for morphological equations (quoted here in Theorem 2.10). Now we seize her characterizations in Lemmas 4.2, 4.4 for combining them directly with Viability Theorem 3.5.

Let us first introduce a modification of Bouligand's contingent cone (mentioned in Definition 2.8).

Definition 4.1 ([8, Definition 4.5.4]) *Let K and L be subsets of a normed vector space X. The so–called* Bouligand paratingent cone $P_K^L(x)$ *to K relative to L at a point $x \in \overline{K} \cap \overline{L}$ is defined by*

$$\begin{aligned} P_K^L(x) \;:=&\; \Big\{ v \in X \;\Big|\; \liminf_{\substack{y \to x,\; y \in L \\ h \downarrow 0}} \; \tfrac{1}{h}\cdot \mathrm{dist}(y + h\cdot v,\ K) \;=\; 0 \Big\} \\ =&\; \Big\{ v \in X \;\Big|\; \exists\, h_n \downarrow 0,\ (y_n)_{n\in\mathbb{N}} \text{ in } L,\ (v_n)_{n\in\mathbb{N}} \text{ in } X :\;\; y_n \to x,\ v_n \to v,\;\; y_n + h_n\cdot v_n \in K\ \forall\, n \Big\}. \end{aligned}$$

Lemma 4.2 ([23, Theorem 3.3]) *Let $M \subset \mathbb{R}^N$ be closed. For every nonempty compact set $K \in \mathcal{V}_1$ (i.e. $K \cap M \neq \emptyset$) and each set–valued map $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$, the following two conditions are equivalent:*

1. *$G \in T_{\mathcal{V}_1}(K)$, i.e. G belongs to the contingent transition set of $\mathcal{V}_1$ at K (Definition 2.9).*
2. *there exists $x \in K \cap M$ with $G(x) \cap P_M^K(x) \neq \emptyset$.* □

Theorem 4.3 (Compact–valued solutions "operable" in M)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ be a set–valued map and $M \subset \mathbb{R}^N$ a closed subset satisfying:

1.) all values of $\mathcal{F}$ are nonempty, convex (as in Theorem 3.5) and have the global bounds
$$\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \ \sup_{G \in \mathcal{F}(K)} \big(\|G\|_\infty + \mathrm{Lip}\, G \big) < \infty,$$
2.) the graph of $\mathcal{F}$ is closed (w.r.t. locally uniform convergence in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$),

3.) for any $K \in \mathcal{K}(\mathbb{R}^N)$ with $K \cap M \neq \emptyset$, there exist $G \in \mathcal{F}(K)$, $x \in K \cap M$ with $G(x) \cap P_M^K(x) \neq \emptyset$.

Then for every compact set $K_0 \subset \mathbb{R}^N$ with $K_0 \cap M \neq \emptyset$, there exists a compact–valued Lipschitz continuous solution $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ of the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \cap M \neq \emptyset$ for all $t \in [0,1]$.

In [23], a compact–valued map $K(\cdot) : [0,T] \rightsquigarrow \mathbb{R}^N$ satisfying this condition (of nonempty intersection with M at each time) is called *operable* on $M \subset \mathbb{R}^N$. Obviously, it is a stronger condition of analytically different nature if we require $K(t)$ to be contained in the closed set M at every time t. This property is called *strongly operable* on M [23].

Lemma 4.4 ([23, Theorem 4.3]) *Let $M \subset \mathbb{R}^N$ be closed and nonempty. For every nonempty compact set $K \in \mathcal{V}_2$ (i.e. $K \subset M$) and each set–valued map $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$, the following two conditions are equivalent:*

1. *$G \in T_{\mathcal{V}_2}(K)$, i.e. G belongs to the contingent transition set of $\mathcal{V}_2$ at K (Definition 2.9).*
2. *$G(x) \subset T_M(x)$ for every $x \in K$, i.e. $G(x)$ is contained in Bouligand's contingent cone of M at each point $x \in K \subset M$ (Definition 2.8).* □

Theorem 4.5 (Compact–valued solutions "strongly operable" in M)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ be a set–valued map and $M \subset \mathbb{R}^N$ a closed subset satisfying:

1.) all values of $\mathcal{F}$ are nonempty, convex (as in Theorem 3.5) and have the global bounds
$$\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \ \sup_{G \in \mathcal{F}(K)} \big(\|G\|_\infty + \mathrm{Lip}\, G \big) < \infty,$$
2.) the graph of $\mathcal{F}$ is closed (w.r.t. locally uniform convergence in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$),

3.) for any compact set $K \subset M$, there exist $G \in \mathcal{F}(K)$ with $G(x) \subset T_M(x)$ for every $x \in K$.

Then for every nonempty compact set $K_0 \subset M$, there exists a compact–valued Lipschitz continuous solution $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ of the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \subset M$ for all $t \in [0,1]$.

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