Nonsmooth shape evolutions under state constraints: A viability theorem

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Abstract

In shape analysis, the so-called velocity method (or speed method) has been a useful tool for avoiding regularity assumptions about the admitted shapes. The key idea is to "deform" the current set according to the flow of a Lipschitz vector field. Making this vector field dependent on the current shape leads to the so-called morphological equations. They can be regarded as a counterpart of evolution equations beyond the traditional border of vector spaces, namely for compact subsets of \mathbb{R}^N (supplied with the Pompeiu-Hausdorff metric).

Here we focus on the two new aspects: Firstly, Lipschitz set-valued maps (with nonempty convex values) replace the Lipschitz vector fields and thus, we consider the flow along differential inclusions for "deforming" compact subsets of \mathbb{R}^N . Secondly, more than one set-valued map is admitted for each compact subset, i.e. the morphological equation is replaced by a morphological "inclusion". Our aim now is to give necessary and sufficient conditions for the existence of (at least) one solution whose values always satisfy a given constraint. Drawing parallels with differential inclusions in \mathbb{R}^N , the main result is a viability theorem for morphological inclusions (using reachable sets of differential inclusions with bounded Lipschitz right-hand side as transitions).

Key words Shape evolutions with constraints, velocity method (speed method), morphological equations, Nagumo's theorem, viability condition.

1 Introduction

State constraints provide challenging questions in any form of dynamic system. Asking for sufficient and necessary conditions on the set of constraints, the first complete answer for ordinary differential equations was given by Nagumo in 1942 ([22]) and, this characterization (using the Bouligand tangent cone) has been rediscovered many times during the last decades.

If solutions of any given initial value problem are not unique, then two versions of this question are to be distinguished from each other: Either we demand *all* solutions to have their values in the fixed set of constraints or (just) *at least one* solution with this property has to exist. In the first case, the corresponding set of constraints is called *invariant* and, in the latter case, it is *viable* or *weakly invariant*. For autonomous differential inclusions in \mathbb{R}^N , the results are presented in Aubin's monography *Viability theory* [7], for example.

The main goal of this paper is a necessary and sufficient characterization of viability for shapes.

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To be more precise, we leave the familiar Euclidean space \mathbb{R}^N and consider evolutions of nonempty compact subsets of \mathbb{R}^N instead. Correspondingly, the trajectory $x:[0,T] \longrightarrow \mathbb{R}^N$ (of a differential inclusion) is now replaced by a curve $K:[0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ with $\mathcal{K}(\mathbb{R}^N)$ denoting the set of nonempty compact subsets of \mathbb{R}^N (usually supplied with the Pompeiu–Hausdorff distance d). The state constraints are again formulated as a subset, i.e. now $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ (instead of $V \subset \mathbb{R}^N$ for differential inclusions).

Differential inclusions with Lipschitz right-hand side for specifying time derivatives of curves in $(\mathcal{K}(\mathbb{R}^N), d)$

For formulating the viability problem in the metric space $(\mathcal{K}(\mathbb{R}^N), d)$, we have to specify how compact subsets of \mathbb{R}^N are "deformed". The so-called *velocity method* or *speed method* has led Céa, Delfour, Zolésio and others to remarkable results about shape optimization (see e.g. [10, 12, 26, 30] and references there). It is based on prescribing a vector field $v : \mathbb{R}^N \times [0, T] \longrightarrow \mathbb{R}^N$ such that the corresponding ordinary differential equation $\frac{d}{dt} x(\cdot) = v(x(\cdot), \cdot)$ induces a unique flow on \mathbb{R}^N . Indeed, supposing v to be sufficiently smooth, the Cauchy problem $\frac{d}{dt} x(\cdot) = v(x(\cdot), \cdot)$ in $[0, T], x(0) = x_0$ is always well-posed and, any compact initial set $K \subset \mathbb{R}^N$ is deformed to

$$\vartheta_v(t,K) \ := \ \left\{ \, x(t) \ \Big| \ \exists \, x(\cdot) \in C^1([0,t],\mathbb{R}^N) : \ \frac{d}{dt} \, x(\cdot) = v(x(\cdot), \cdot) \ \text{in} \ [0,t], \ x(0) \in K \right\}$$

after an arbitrary time $t \ge 0$. As a key advantage, this concept of set evolution does not require any regularity conditions on the compact set K or its topological boundary (but only on the vector field v). Roughly speaking, v can be interpreted as a "direction of deformation" for $(\mathcal{K}(\mathbb{R}^N), d)$.

Aubin seized this notion for extending ODEs to this metric space of compact subsets. The so-called *morphological equations* are sketched in [6] and then presented in [4, 5] in more detail.

The first aspect of generalization focuses on the "elementary deformation" which are to describe the directions in $(\mathcal{K}(\mathbb{R}^N), d)$. Aubin suggested reachable sets of differential inclusions as a more general alternative to the velocity method. For any set-valued map $G : \mathbb{R}^N \to \mathbb{R}^N$ and compact initial set $K \subset \mathbb{R}^N$ given, the so-called *reachable set* at time $t \ge 0$ is defined as

$$\vartheta_G(t,K) := \left\{ \begin{array}{ll} x(t) \in \mathbb{R}^N & \exists x(\cdot) \in AC([0,t], \mathbb{R}^N) : x(0) \in K, \\ & \frac{d}{d\tau} x(\tau) \in G(x(\tau)) \text{ for almost every } \tau \in [0,t] \end{array} \right\}$$

In contrast to the velocity method, this kind of "deformation" need not be reversible in time. (Geometrically speaking, "holes" can disappear.) The well-known Theorem of Filippov ensures suitable properties of $[0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(t,K) \longmapsto \vartheta_G(t,K)$ if $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty compact values and is Lipschitz continuous.

The second key contribution of Aubin is a suggestion how to interpret such a set-valued map (and its reachable sets) as a time derivative of a curve in the metric space $(\mathcal{K}(\mathbb{R}^N), d)$.

Indeed, let $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ be a curve. A bounded Lipschitz set-valued map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ (with nonempty compact values) represents a first-order approximation of $K(\cdot)$ at time $t \in [0,T[$ if



(*)

$$\limsup_{h \downarrow 0} \quad \frac{1}{h} \cdot dt \big(K(t+h), \ \vartheta_G(h, K(t)) \big) = 0.$$

Of course, such a map $G(\cdot)$ need not be unique and thus, *all* such bounded Lipschitz maps with this property (*) form the so-called *mutation* $\overset{\circ}{K}(t)$ of $K(\cdot)$ at time $t \in [0, T[$. It is a subset of $\text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, i.e. the set of all bounded Lipschitz maps $\mathbb{R}^N \to \mathbb{R}^N$ with nonempty compact values, and extends the time derivative to curves in the metric space $(\mathcal{K}(\mathbb{R}^N), d)$.

§ 1 INTRODUCTION

Solving a morphological equation with state constraints: Aubin's adaptation of Nagumo's theorem

The step from specifying a time derivative (of a curve) to formulating a (generalized) differential equation is rather small. It is based just on prescribing the time derivative as a function of the current state. In connection with nonempty compact subsets of \mathbb{R}^N , a function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. For any initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$, we are looking for $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ satisfying

1. $K(\cdot)$ is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance d,

2. $f(K(t)) \in \overset{\circ}{K}(t)$ for a.e. $t \in [0, T[$, i.e. $\lim_{h \perp 0} \frac{1}{h} \cdot d(K(t+h), \vartheta_{f(K(t))}(h, K(t))) = 0,$

3. $K(0) = K_0$.

Then, $K(\cdot)$ is called *solution* of the (autonomous) *morphological equation* $K(\cdot) \ni f(K(\cdot))$ in [0,T] with initial value K_0 .

Considering now additional state constraints, the question about existence of a solution has been answered completely by Aubin in [4], Theorem 4.1.7. In particular, the assumptions about constraints and $f(\cdot)$ justify its interpretation as a counterpart of Nagumo's theorem.

Proposition 1.1 (Nagumo's theorem for morphological equations [4, 5])

Suppose $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ to be nonempty and closed with respect to d. Let $f : (\mathcal{K}(\mathbb{R}^N), d) \longrightarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ be a continuous function satisfying

1. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \operatorname{Lip} f(M) < \infty$,

2. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} ||f(M)||_{\infty} < \infty$.

Furthermore suppose for every $M \in \mathcal{V}$: $f(M) \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is contingent to \mathcal{V} at M in the sense that $0 = \liminf_{h \downarrow 0} \frac{1}{h} \cdot \operatorname{dist}(\vartheta_{f(M)}(h, M), \mathcal{V}) \stackrel{\text{Def.}}{=} \liminf_{h \downarrow 0} \frac{1}{h} \cdot \inf_{C \in \mathcal{V}} d(\vartheta_{f(M)}(h, M), C).$

Then, from any $K_0 \in \mathcal{V}$ starts a solution $K(\cdot) : [0, \infty[\longrightarrow \mathcal{K}(\mathbb{R}^N)]$ of the morphological equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ which is viable in \mathcal{V} , i.e. $K(t) \in \mathcal{V}$ for all t.

The new step to morphological inclusions

This paper focuses on the corresponding conditions (of viability) if more than one Lipschitz map is admitted for each compact set, i.e. the single-valued function $f: \mathcal{K}(\mathbb{R}^N) \longrightarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is replaced by a set-valued map $\mathcal{F}: \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. This modification of given data leads directly to the following definition: A curve $K(\cdot): [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is called *solution* of the *morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ in [0, T]

with initial value K_0 if

- 1. $K(\cdot)$ is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance d,
- 2. $\mathcal{F}(K(t)) \cap \overset{\circ}{K}(t) \neq \emptyset$ for almost every t, i.e. there exists some $G \in \mathcal{F}(K(t)) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with $\limsup_{k \to 0} \frac{1}{h} \cdot d(K(t+h), \vartheta_G(h, K(t))) = 0,$
- 3. $K(0) = K_0$.

Considering now additional state constraints on $K(\cdot)$, the problems of invariance and viability have already been investigated for the velocity method (i.e. Lipschitz vector fields instead of Lipschitz set–valued maps).

Doyen [18] has given sufficient conditions on $F(\cdot)$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ for the *invariance* of \mathcal{V} (i.e. all solutions starting in \mathcal{V} stay in \mathcal{V}). His key notion is first to extend Filippov's existence theorem from differential inclusions (in \mathbb{R}^N) to morphological inclusions (in $\mathcal{K}(\mathbb{R}^N)$) and then to verify $\operatorname{dist}(K(\cdot), \mathcal{V}) \leq 0$ (by means of Gronwall's inequality).

The corresponding question about viability of \mathcal{V} (i.e. *at least one* solution has to stay in \mathcal{V}) has been answered by the author in [20] recently.

The main result here considers morphological inclusions in their full generality, i.e. in contrast to the velocity method, we choose the "directions of deformation" in $\text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ (and their reachable sets). It concerns sufficient conditions on $\mathcal{F}(\cdot) : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ for the *viability* of \mathcal{V} . This question (in a more general environment) was pointed out as open in § 2.3.3 of [4] and, to the best of my knowledge, it has not been answered for the morphological inclusions so far.

In fact, the following statement is very similar to the viability theorem for differential inclusions in \mathbb{R}^N (as it is discussed in Aubin's monography *Viability theory* [7] and quoted here in Theorem 3.3). Roughly speaking, \mathcal{F} is supposed to be upper semicontinuous with closed convex values — after specifying a suitable topology on $\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ in a moment — and, we require (at least) one "contingent direction" in the value $\mathcal{F}(K) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ for each $K \in \mathcal{V}$.

Theorem 1.2 (Viability theorem for morphological inclusions)

Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ be a set-valued map and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ a nonempty closed subset satisfying:

- 1.) all values of \mathcal{F} are nonempty and convex (i.e. for any $G_1, G_2 \in \mathcal{F}(K) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda \in [0, 1]$, the set-valued map $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1 \lambda) \cdot G_2(x)$ also belongs to $\mathcal{F}(K)$)
- $2.) \quad A := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \operatorname{Lip} G < \infty, \qquad B := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \|G\|_{\infty} < \infty,$
- 3.) the graph of \mathcal{F} is closed (w.r.t. locally uniform convergence in $\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$),
- 4.) for each $K \in \mathcal{V}$, some $G \in \mathcal{F}(K) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is contingent to \mathcal{V} at K

in the sense that
$$0 = \liminf_{h \downarrow 0} \frac{1}{h} \cdot \operatorname{dist}(\vartheta_G(h, K), \mathcal{V}) \stackrel{\text{\tiny Def.}}{=} \liminf_{h \downarrow 0} \frac{1}{h} \cdot \inf_{C \in \mathcal{V}} d(\vartheta_G(h, K), C).$$

Then for every initial element $K_0 \in \mathcal{V}$, there exists at least one solution $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ of the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \in \mathcal{V}$ for all $t \in [0,1]$.

The convexity mentioned in assumption (1.) implies directly that for every $G \in \mathcal{F}(K) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, the pointwise convexification $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N, x \mapsto co G(x)$ is also contained in $\mathcal{F}(K)$. Thus we can restrict our considerations to set-valued maps $G \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with convex values (in addition).

This introduction (§ 1) is reflecting the structure of the paper: Aubin's theory of morphological equations is summarized in § 2. In particular, we mention the counterparts of Filippov's and Nagumo's theorems for evolutions in the metric space $(\mathcal{K}(\mathbb{R}^N), d)$. Then, § 3 provides the step to morphological inclusions. It starts with the viability theorem about differential inclusions (in § 3.1) and extends this result to morphological inclusions (in § 3.2).

2 A brief introduction to morphological equations

Morphological equations provide a typical geometric example of so-called mutational equations. First presented in [6] and elaborated in [5, 4], mutational equations are to extend ordinary differential equations to a metric space (E, d). In a word, the key idea is to describe derivatives by means of continuous maps (called *transitions*) $\vartheta : [0,1] \times E \longrightarrow E$, $(h,x) \longmapsto \vartheta(h,x)$ instead of affine-linear maps $(h,x) \longmapsto x + h v$ (that are always used in *vector spaces*). Strictly speaking, such a transition specifies the point $\vartheta(t,x) \in E$ to which any initial point $x \in E$ has been moved after time $t \in [0,T]$. It can be interpreted as a generalized derivative of a curve $\xi : [0,T[\longrightarrow E]$ at time $t \in [0,T[$ if it provides a first-order approximation in the sense of

$$\limsup_{h \downarrow 0} \frac{1}{h} \cdot d(\xi(t+h), \ \vartheta(h, \xi(t))) = 0.$$

The so-called *morphological equations* apply this concept to the set $\mathcal{K}(\mathbb{R}^N)$ of nonempty compact subsets of \mathbb{R}^N supplied with the Pompeiu-Hausdorff distance d,

$$d(K_1, K_2) := \inf \{ \rho > 0 \mid K_1 \subset K_2 + \rho \, \mathbb{B}_1, \, K_2 \subset K_1 + \rho \, \mathbb{B}_1 \}$$

Here \mathbb{B}_1 always denotes the closed unit ball in \mathbb{R}^N , i.e. $\mathbb{B}_1 := \{x \in \mathbb{R}^N \mid |x| \leq 1\}$. This is a very general starting point for geometric evolution problems as there are no a priori restriction in regard to regularity. Motivated by the velocity method (often used in shape optimization), ordinary differential equations can lay the basis for transitions – as investigated in [20] already. Here, however, we follow a suggestion of Aubin (in [4, 5]) and consider a more general approach of evolutions instead: differential inclusions and their reachable sets.

Definition 2.1 LIP $(\mathbb{R}^N, \mathbb{R}^N)$ consists of all set-valued maps $F : \mathbb{R}^N \to \mathbb{R}^N$ satisfying

- 1. F has nonempty compact values that are uniformly bounded in \mathbb{R}^N
- 2. F is Lipschitz continuous with respect to the Pompeiu-Hausdorff distance.

 $\operatorname{Lip}(M, \mathbb{R}^N)$ consists of all bounded and Lipschitz continuous functions $M \longrightarrow \mathbb{R}^N$.

Definition 2.2 Choosing any set-valued map $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$, the so-called reachable set $\vartheta_F(t,K)$ of the initial set $K \in \mathcal{K}(\mathbb{R}^N)$ at time $t \in [0,T]$ is defined as

$$\vartheta_F(t,K) := \left\{ \begin{array}{ll} x(t) \in \mathbb{R}^N \ \middle| \ \exists \ x(\cdot) \in AC([0,t], \ \mathbb{R}^N) : \ x(0) \in K, \\ \frac{d}{d\tau} \ x(\tau) \in F(\tau, x(\tau)) \ for \ almost \ every \ \tau \in [0,t] \right\}$$

(and correspondingly for $F : \mathbb{R}^N \to \mathbb{R}^N$ and its autonomous differential inclusion).

Lemma 2.3 For every $F \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, the map $\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(h, K) \mapsto \vartheta_F(h, K)$ is well-defined and satisfies the four conditions on a transition on the metric space $(\mathcal{K}(\mathbb{R}^N), d)$ (in the sense of Aubin), i.e.

 $1. \ \vartheta_{F}(0, K) = K \qquad \qquad \text{for all } K \in \mathcal{K}(\mathbb{R}^{N}),$ $2. \ \limsup_{h \downarrow 0} \ \frac{1}{h} \cdot d \left(\vartheta_{F}(t+h, K), \ \vartheta_{F}(h, \vartheta_{F}(t, K)) \right) = 0 \quad \text{for all } K \in \mathcal{K}(\mathbb{R}^{N}), \ t \in [0, 1[,$ $3. \ \alpha(\vartheta_{F}) := \ \max\left(0, \ \sup_{K_{1} \neq K_{2}} \ \limsup_{h \downarrow 0} \ \frac{d(\vartheta_{F}(h, K_{1}), \vartheta_{F}(h, K_{2})) - d(K_{1}, K_{2})}{h \cdot d(K_{1}, K_{2})} \right) < \infty$ $4. \ \beta(\vartheta_{F}) := \ \sup_{K \in \mathcal{K}(\mathbb{R}^{N})} \ \limsup_{h \downarrow 0} \ \frac{1}{h} \cdot d(K, \ \vartheta_{F}(h, K)) \ < \ \infty.$

In fact, $\alpha(\vartheta_F) \leq \operatorname{Lip} F$ and $\beta(\vartheta_F) \leq ||F||_{\infty} \stackrel{\text{Def.}}{=} \sup_{x \in \mathbb{R}^N} \sup_{y \in F(x)} |y|.$

Furthermore, the "transitional" distance between ϑ_F , ϑ_G for any $F, G \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$,

i.e.
$$d_{\Lambda}(\vartheta_F, \, \vartheta_G) := \sup_{K \in \mathcal{K}(\mathbb{R}^N)} \limsup_{h \downarrow 0} \frac{1}{h} \cdot d\!\!\! \left(\vartheta_F(h, K), \, \vartheta_G(h, K) \right)$$

is bounded from above by $dl_{\infty}(F,G) \stackrel{\text{Def.}}{=} \sup_{x \in \mathbb{R}^N} dl(F(x),G(x)) < \infty.$ In particular, $dl(\vartheta_F(h,K_1), \vartheta_G(h,K_2)) \leq e^{\operatorname{Lip} F \cdot h} (dl(K_1,K_2) + h \cdot dl_{\infty}(F,G)).$

The proof is presented in [4], Proposition 3.7.3 – as a direct consequence of Filippov's Theorem (about trajectories of differential inclusions in \mathbb{R}^N). In particular, this lemma justifies calling ϑ_F a morphological transition on $(\mathcal{K}(\mathbb{R}^N), d)$ – in accordance with [4], Definition 3.7.2. For the sake of simplicity, $F \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is sometimes identified with its morphological transition ϑ_F .

These reachable sets provide the tools for specifying (generalized) shape derivatives of a compact-valued tube $K(\cdot) : [0, T[\rightarrow \mathbb{R}^N, \text{ i.e. a curve } K(\cdot) : [0, T[\rightarrow \mathcal{K}(\mathbb{R}^N)]$. So the next step will be to solve equations prescribing such shape derivatives.

Definition 2.4 For any compact-valued tube $K(\cdot) : [0, T[\rightarrow \mathbb{R}^N, \text{ the so-called shape mutation } \mathring{K}(t)$ at time $t \in [0, T[$ consists of all set-valued maps $F \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ satisfying $\limsup_{h \downarrow 0} \quad \frac{1}{h} \cdot d(\vartheta_F(h, K(t)), K(t+h)) = 0.$

Definition 2.5 For any given function $f : \mathcal{K}(\mathbb{R}^N) \times [0, T[\longrightarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N), a \text{ compact-valued tube} K(\cdot) : [0, T[\longrightarrow \mathbb{R}^N \text{ is called solution of the morphological equation} \overset{\circ}{K}(\cdot) \ni f(K(\cdot), \cdot)$

if 1. $K(\cdot): [0,T[\rightarrow \mathbb{R}^N \text{ is Lipschitz continuous with respect to } dl \text{ and}$

2. for almost every $t \in [0, T[, f(K(t), t) \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N) \text{ belongs to } \overset{\circ}{K}(t)$ or, equivalently, $\limsup_{h \downarrow 0} \quad \frac{1}{h} \cdot dt \left(\vartheta_{f(K(t), t)}(h, K(t)), K(t+h) \right) = 0.$

As an essential result of [4, 5], the Euler algorithm can be applied in the environment of morphological equations and so, the Cauchy–Lipschitz Theorem (about autonomous ordinary differential equations) has the following counterpart (proven in [4], Theorem 4.1.2):

Theorem 2.6 Suppose $f : (\mathcal{K}(\mathbb{R}^N), d) \longrightarrow (\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N), d_{\infty})$ to be λ -Lipschitz continuous and to satisfy $M := \sup_{K \in \mathcal{K}(\mathbb{R}^N)} \operatorname{Lip} f(K) < \infty$. For every initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$, there exists a unique solution $K(\cdot) : [0, \infty[\rightarrow \mathbb{R}^N \text{ of the morpho$ $logical equation } \overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ with $K(0) = K_0$. Furthermore every Lipschitz compact-valued tube $Q : [0, \infty[\rightarrow \mathbb{R}^N \text{ with } \overset{\circ}{Q}(t) \neq \emptyset \text{ for every } t \ge 0$ satisfies the following estimate at each time $t \ge 0$

$$d(K(t),Q(t)) \leq d(K_0, Q(0)) \cdot e^{(M+\lambda) t} + \int_0^t e^{(M+\lambda) (t-s)} \cdot \inf_{G \in \overset{\circ}{Q}(s)} d_{\infty}(f(Q(s)), G) ds.$$

In particular, the solution $K(\cdot)$ depends on the initial set K_0 and the right-hand side f in a Lipschitz continuous way.

§ 3 MORPHOLOGICAL INCLUSIONS

Existence under (additional) constraints proves to be a very interesting question for many applications. In the particular case of ordinary differential equations, Nagumo's Theorem gives a necessary and sufficient condition on the set \mathcal{V} of constraints for existence of local solutions. It uses the contingent cone (in the sense of Bouligand) and has served as a key motivation for viability theory (see e.g. [7]). In fact, Nagumo's Theorem also holds for morphological equations as shown in [4], Theorem 4.1.7:

Definition 2.7 For any nonempty subset
$$\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$$
 and $K \in \mathcal{V}$,
 $T_{\mathcal{V}}(K) := \left\{ F \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N) \mid 0 = \liminf_{h \downarrow 0} \frac{1}{h} \cdot \operatorname{dist}(\vartheta_F(h, K), \mathcal{V}) \right\}$

$$\stackrel{\text{Def.}}{=} \liminf_{h \downarrow 0} \frac{1}{h} \cdot \inf_{C \in \mathcal{V}} d(\vartheta_F(h, K), C)$$

is called contingent transition set of \mathcal{V} at K.

Remark. The "geometric" background of reachable sets implies an additional property of morphological transitions in $T_{\mathcal{V}}(K) \subset \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Indeed, for any $F \in T_{\mathcal{V}}(K)$, every function $G \in \operatorname{Lip}(\mathbb{R}^N, \mathbb{R}^N)$ with $F(\cdot) = G(\cdot)$ in a neighborhood of ∂K is also contained in $T_{\mathcal{V}}(K)$. So in other words, the criterion of $T_{\mathcal{V}}(K)$ depends only on an arbitrarily small neighborhood of the boundary ∂K .

Theorem 2.8 (Nagumo's theorem for morphological equations [4])

Suppose $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ to be nonempty and closed with respect to d.

Let $f: (\mathcal{K}(\mathbb{R}^N), d) \longrightarrow (\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N), d_{\infty})$ be a continuous function satisfying

- 1. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \operatorname{Lip} f(M) < \infty$,
- 2. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} ||f(M)||_{\infty} < \infty$.

Then from any initial state $K_0 \in \mathcal{V}$ starts at least one Lipschitz solution $K(\cdot) : [0, T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ of $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ viable in \mathcal{V} (i.e. $K(t) \in \mathcal{V}$ for all t) if and only if \mathcal{V} is a viability domain of fin the sense of $f(M) \in T_{\mathcal{V}}(M)$ for all $M \in \mathcal{V}$.

3 The step to morphological inclusions

The main aim now is to prove such a viability theorem for morphological *inclusions*, i.e. the single– valued function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ of the right–hand side is to be replaced by a set–valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

3.1 The (well-known) Viability Theorem for differential inclusions in \mathbb{R}^N

The situation has already been investigated intensively for differential inclusions in \mathbb{R}^N (see e.g. [7]). For clarifying the new aspects of morphological inclusions, we now quote the corresponding result from [7], Theorems 3.3.2, 3.3.5 after specifying the required terms (according to [7], Definitions 1.1.3, 2.2.4). **Definition 3.1** Let X and Y be normed vector spaces. A set-valued map $F: X \rightsquigarrow Y$ is called Marchaud map if it has the following properties:

- 1. F is nontrivial, i.e. Graph $F \neq \emptyset$,
- 2. F is upper semicontinuous, i.e. for any $x \in X$, neighborhood $V \supset F(x)$, there is a neighborhood $U \subset X$ of x s.t. $F(U) \subset V$,
- 3. F has compact convex values,

F has linear growth,

i.e. $\sup_{x \in X} \left(\frac{1}{|x|+1} \cdot \sup_{y \in F(x)} |y| \right) < \infty.$

Definition 3.2 Let X be a normed vector space, $V \subset X$ nonempty and $x \in V$. The contingent cone to V at x (in the sense of Bouligand) is the set $T_V(x) := \{ u \in X | \liminf_{h \to 0} \frac{1}{h} \cdot \operatorname{dist}(x+hu, V) = 0 \}.$

Theorem 3.3 (Viability theorem for differential inclusions [7]) Consider a Marchaud map $F : \mathbb{R}^N \to \mathbb{R}^N$ and a nonempty closed subset $V \subset \mathbb{R}^N$ with $F(x) \neq \emptyset$ for all $x \in \mathbb{R}^N$. Then the following two statements are equivalent:

- 1. For every point $x_0 \in V$, there is at least one solution $x(\cdot) \in AC([0, \infty[, \mathbb{R}^N) of x'(\cdot) \in F(x(\cdot)) (almost everywhere) with <math>x(0) = x_0$ and $x(t) \in V$ for all t.
- 2. $F(x) \cap T_V(x) \neq \emptyset$ for all $x \in V$.

The implication $(1.) \Longrightarrow (2.)$ is rather obvious. For proving $(2.) \Longrightarrow (1.)$, a standard approach uses an "approximating" sequence $(x_n(\cdot))_{n\in\mathbb{N}}$ in $W^{1,\infty}([0,1],\mathbb{R}^N)$ such that $\sup_t \operatorname{dist}(x_n(t), V) \xrightarrow{n \to \infty} 0$ and $(x_n(t), \frac{d}{dt}x_n(t))$ is close to Graph $F \subset \mathbb{R}^N \times \mathbb{R}^N$ for almost every t. Then the theorems of Arzela–Ascoli and Alaoglu provide a subsequence $(x_{n_j}(\cdot))_{j\in\mathbb{N}}$ and limit functions $x(\cdot) \in C^0([0,1],\mathbb{R}^N)$, $w(\cdot) \in L^{\infty}([0,1],\mathbb{R}^N)$ with $x_{n_j}(\cdot) \longrightarrow x(\cdot)$ uniformly, $\frac{d}{dt}x_{n_j}(\cdot) \longrightarrow w(\cdot)$ weakly* in $L^{\infty}([0,1],\mathbb{R}^N)$. Due to the continuous embedding $L^{\infty}([0,1],\mathbb{R}^N) \subset L^1([0,1],\mathbb{R}^N)$, we even obtain the convergence $\frac{d}{dt}x_{n_j}(\cdot) \longrightarrow w(\cdot)$ weakly in $L^1([0,1],\mathbb{R}^N)$. Thus, $w(\cdot)$ is the weak derivative of $x(\cdot)$ and, $x(\cdot)$ is Lipschitz. Furthermore, Mazur's Lemma implies $w(t) \in \bigcap_{\varepsilon > 0} \overline{co} \left(\bigcup_{z \in \mathbb{B}_{\varepsilon}(x(t))} F(z)\right) = F(x(t))$ for almost every t.

Considering now morphological inclusions on $(\mathcal{K}(\mathbb{R}^N), d)$ (instead of differential inclusions), an essential aspect changes: The derivative of a curve is not represented as a function in $L^1([0, 1], \mathbb{R}^N)$ any longer, but rather as a function $[0, 1] \longrightarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. So the classical theorems of Arzela–Ascoli, Alaoglu and Mazur might have to be replaced by their counterparts concerning functions with their values in a Banach space (instead of \mathbb{R}^N).

3.2 Adapting this concept to morphological inclusions

Definition 3.4 $\operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ denotes the set of all set-valued maps $G \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ whose (nonempty compact) values are convex in addition.

4.

Remark. The well-known Relaxation Theorem of Filippov–Ważiewski implies $\vartheta_G(t, K) = \vartheta_{\overline{co} G}(t, K)$ for every map $G \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, initial set $K \in \mathcal{K}(\mathbb{R}^N)$ and time $t \ge 0$. So in this regard, it is no geometric restriction to consider only reachable sets of set–valued maps in $\operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$. The main (technical) advantage of this additional assumption is the opportunity of measurable/Lipschitz parameterizations according to Lemma 3.7.

Theorem 3.5 (Viability theorem for morphological inclusions)

Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ be a set-valued map and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ a nonempty closed subset satisfying:

- 1.) all values of \mathcal{F} are nonempty and convex (i.e. for any $G_1, G_2 \in \mathcal{F}(K) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda \in [0, 1]$, the set-valued map $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1 \lambda) \cdot G_2(x)$ also belongs to $\mathcal{F}(K)$)
- $2.) \quad A := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \operatorname{Lip} G < \infty, \qquad B := \sup_{M \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(M)} \|G\|_{\infty} < \infty,$
- 3.) the graph of \mathcal{F} is closed (w.r.t. locally uniform convergence in LIP($\mathbb{R}^N, \mathbb{R}^N$)),
- 4.) $T_{\mathcal{V}}(K) \cap \mathcal{F}(K) \neq \emptyset$ for all $K \in \mathcal{V}$.

Then for every initial element $K_0 \in \mathcal{V}$, there exists a compact-valued Lipschitz continuous solution $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ of the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \in \mathcal{V}$ for all $t \in [0,1]$.

Remark. In assumption (3.), the topology on $\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is specified. A sequence $(G_n)_{n \in \mathbb{N}}$ in $\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is said to converge "locally uniformly" to $G \in \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ if for every nonempty compact set $M \subset \mathbb{R}^N$, $\sup_{x \in M} d(G_n(x), G(x)) \longrightarrow 0$ for $n \longrightarrow \infty$ using here the Pompeiu–Hausdorff distance d on $\mathcal{K}(\mathbb{R}^N)$.

Due to the uniform Lipschitz bounds in assumption (2.), it leads to sequential compactness of the image of \mathcal{F} (as shown in Lemma 3.12 later).

Proposition 3.6 (Constructing approximative solutions) Choose any $\varepsilon > 0$.

Under the assumptions of Viability Theorem 3.5, there exist a *B*-Lipschitz continuous function $K_{\varepsilon}(\cdot)$: [0,1] $\longrightarrow \mathcal{K}(\mathbb{R}^N)$ and a function $f_{\varepsilon}(\cdot): [0,1[\longrightarrow \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ satisfying

a) $K_{\varepsilon}(0) = K_0,$

b) dist $(K_{\varepsilon}(t), \mathcal{V}) \leq r_{\varepsilon}(t)$ with $r_{\varepsilon}(t) := \varepsilon e^{At} t$ for all $t \in [0, 1]$,

c) $f_{\varepsilon}(t) \in \overset{\circ}{K}_{\varepsilon}(t) \cap \mathcal{F}(\mathbb{B}_{R_{\varepsilon}}(K_{\varepsilon}(t))) \neq \emptyset$ with $R_{\varepsilon} := \varepsilon \ e^{A}$ for all $t \in [0, 1[$.

d) $f_{\varepsilon}(\cdot)$ is piecewise constant in the following sense: for each $t \in [0,1[$, there exists some $\delta > 0$ such that $f_{\varepsilon}(\cdot)|_{[t, t+\delta]}$ is constant.

Proof. follows the same track as [4, Aubin 99], Lemma 1.6.5 and uses Zorn's Lemma: For $\varepsilon > 0$ fixed, let $\mathcal{A}_{\varepsilon}(K_0)$ denote the set of all tuples $(T_K, K(\cdot), f(\cdot))$ consisting of some $T_K \in [0, 1]$, a *B*-Lipschitz continuous function $K(\cdot) : [0, T_K] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d)$ and some function $f(\cdot) : [0, 1[\longrightarrow \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ such that conditions (a) – (c) are satisfied for all $t \in [0, T_K[$ and condition (d) is fulfilled for all $t \in [0, 1[$. Obviously, $\mathcal{A}_{\varepsilon}(K_0)$ is not empty as it contains $(0, K(\cdot) \equiv K_0, G)$ with arbitrary $G \in \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$. Moreover, an order relation \preceq on $\mathcal{A}_{\varepsilon}(K_0)$ is specified by

 $T_K \leq T_M, \ M \Big|_{[0,T_K]} = K, \ g \Big|_{[0,T_K]} = f.$ $(T_K, K(\cdot), f(\cdot)) \preceq (T_M, M(\cdot), g(\cdot)) :\iff$ So Zorn's Lemma guarantees the existence of a maximal element $(T, K_{\varepsilon}(\cdot), f_{\varepsilon}(\cdot)) \in \mathcal{A}_{\varepsilon}(K_0)$.

Assuming T < 1 for a moment, we obtain a contradiction if $K_{\varepsilon}(\cdot), f_{\varepsilon}(\cdot)$ can be extended to a larger interval $[0, T + \delta] \subset [0, 1]$ ($\delta > 0$) preserving conditions (b), (c), (d).

Since closed bounded balls of $(\mathcal{K}(\mathbb{R}^N), d)$ are compact, the closed set \mathcal{V} contains an element $Z \in \mathcal{K}(\mathbb{R}^N)$ with $d(K_{\varepsilon}(T), Z) = \text{dist}(K_{\varepsilon}(T), \mathcal{V}) \leq r_{\varepsilon}(T)$ and, assumption (4.) of Viability Theorem 3.5 provides some $G \in T_{\mathcal{V}}(Z) \cap \mathcal{F}(Z) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$. According to Definition 2.7 of $T_{\mathcal{V}}(Z)$, there is a positive $\widehat{\delta} < 1 - T$ such that $\operatorname{dist}(\vartheta_G(h, Z), \mathcal{V}) \leq \varepsilon h$ for every $h \in [0, \widehat{\delta}]$. Now set

$$K_{\varepsilon}(t) := \vartheta_G(t - T, K_{\varepsilon}(T)), \quad f_{\varepsilon}(t) := G \quad \text{for each } t \in [T, T + \widehat{\delta}].$$

Obviously, Lemma 2.3 (2) implies $G \in \overset{\circ}{K}_{\varepsilon}(t)$ for all $t \in [T, T + \hat{\delta}]$. Furthermore, Lemma 2.3 leads to $\leq d \left(\vartheta_G(t-T, \, K_\varepsilon(T)), \ \vartheta_G(t-T, \, Z) \right) + d \left(\vartheta_G(t-T, \, Z), \ Z \right)$ $dl(K_{\varepsilon}(t), Z)$ $\leq dl (K_{\varepsilon}(T), Z) \cdot e^{A \cdot (t-T)}$ $\leq \varepsilon e^{AT} T \cdot e^{A \cdot (t-T)}$ $\begin{array}{rrr} + & B \cdot (t - T) \\ + & B \cdot (t - T) & \leq R_{\varepsilon} \end{array}$

for every $t \in [T, T + \delta[$ with $\delta := \min\{\widehat{\delta}, \varepsilon e^A \frac{1-T}{1+B}\}$ (i.e. condition (c) holds) and, dist $(K_c(t), \mathcal{V}) \leq d(\vartheta_C(t-T, K_c(T)), \vartheta_C(t-T, Z)) + dist(\vartheta_C(t-T, Z), \mathcal{V})$

$$\begin{aligned} \operatorname{Hst}(K_{\varepsilon}(t), \, \nu) &\leq & \operatorname{dl}(v_G(t-1, \, K_{\varepsilon}(T)), \, v_G(t-1, \, Z)) &+ \, \operatorname{dlst}(v_G(t-1, \, Z), \, \nu) \\ &\leq & \operatorname{dl}(K_{\varepsilon}(T), \, Z) \, \cdot \, e^{A \cdot (t-T)} &+ \, \varepsilon \cdot (t-T) \\ &\leq & \varepsilon \, \, e^{A \cdot T} \, T \, \cdot \, e^{A \cdot (t-T)} &+ \, \varepsilon \cdot (t-T) \, \leq \, r_{\varepsilon}(t) \end{aligned}$$

i.e. condition (b) is also satisfied. So $K_{\varepsilon}(\cdot)|_{[0, T+\delta]}$ and $f_{\varepsilon}(\cdot)|_{[0, T+\delta[}$ provide the wanted contradiction and thus, T = 1.

As an immediate consequence of property (d), the function $f_{\varepsilon}: [0,1] \longrightarrow \text{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ Remark. can have at most countably many points of discontinuity. This enables us later to apply preceding results about autonomous morphological equations to the approximations $K_{\varepsilon}(\cdot), f_{\varepsilon}(\cdot)$ in a "piecewise" way.

An essential step is now to "parameterize" the set-valued maps $[0,1] \times \mathbb{R}^N \to \mathbb{R}^N, (t,x) \mapsto f_{\varepsilon}(t)(x).$ This tool (quoted from [8]) provides a bridge to a vector space. In Proposition 3.8 and Lemma 3.12, for example, it is exploited for arguments of sequential compactness (with respect to suitable topologies).

Lemma 3.7 (Parameterization of bounded maps, [8], Theorem 9.7.2)

Consider a metric space X and a set-valued map $G: [a,b] \times X \to \mathbb{R}^N$ satisfying

- 1. G has nonempty compact convex values,
- 2. $G(\cdot, x) : [a, b] \rightsquigarrow \mathbb{R}^N$ is measurable for every $x \in X$,
- 3. there exists $k(\cdot) \in L^1([a,b])$ such that for every $t \in [a,b]$, the set-valued map $G(t,\cdot): X \to \mathbb{R}^N$ is k(t)-Lipschitz continuous.

Then there exists a function $g: [a,b] \times X \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ (with $\mathbb{B}_1 \stackrel{\text{Def.}}{=} \{u \in \mathbb{R}^N : |u| \le 1\}$) fulfilling

- $\begin{array}{lll} 1. & \forall \ (t, \, x) \in [a, b] \times X: \\ 2. & \forall \ (x, u) \in X \times \mathbb{B}_1: \\ 3. & \forall \ (t, \, u) \in [a, b] \times \mathbb{B}_1: \end{array} \qquad \begin{array}{lll} G(t, x) = \bigcup_{u \in \mathbb{B}_1} g(t, x, u), \\ g(\cdot, x, u) : [a, b] \longrightarrow \mathbb{R}^N \quad is \ measurable, \\ g(t, \cdot, u) : X \longrightarrow \mathbb{R}^N \quad is \ c \cdot k(t) Lipschitz \ continuous \end{array}$
- 4. $\forall t \in [a,b], x \in X, u, v \in \mathbb{B}_1 : |g(t,x,u) g(t,x,v)| \le c ||G(t,x)||_{\infty} |u-v|$

with a constant c > 0 independent of G.

Proposition 3.8 (Selecting an approximative subsequence)

Under the assumptions of Viability Theorem 3.5, there exist a constant c = c(N, A, B) > 0, sequences $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N), \ \hat{f}_n(\cdot) : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N) \ (n \in \mathbb{N}) \ and \ K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N),$ $\hat{f}(\cdot) : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N) \ such that$

with the abbreviation $\widetilde{K}_j := \mathbb{B}_{j+B}(K_0) \stackrel{\text{\tiny Def.}}{=} \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, K_0) \le j+B\} \in \mathcal{K}(\mathbb{R}^N).$

Proof. is based on the approximative solutions of Proposition 3.6, of course.

Indeed, for each $n \in \mathbb{N}$, Prop. 3.6 provides $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $f_n(\cdot) : [0,1[\longrightarrow \operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)]$ corresponding to $\varepsilon := \frac{1}{n} e^{-A}$. Now according to Lemma 3.7, the set-valued map $[0,1[\times \mathbb{R}^N \to \mathbb{R}^N, (t,x) \mapsto f_n(t,x)]$ has a parameterization $[0,1[\times \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N]$ that we interpret as function $\widehat{f_n} : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)]$. Obviously, they satisfy the claimed properties (a) – (e).

In particular, these features stay correct whenever we consider subsequences instead and again abbreviate them as $(K_n(\cdot))_{n\in\mathbb{N}}$, $(\widehat{f}_n(\cdot))_{n\in\mathbb{N}}$ respectively.

The *B*-Lipschitz continuity of each $K_n(\cdot)$ has two important consequences, i.e.

1. all $K_n(\cdot): [0,1] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d)$ $(n \in \mathbb{N})$ are equi-continuous and

2. $\bigcup_{\substack{n \in \mathbb{N} \\ t \in [0,1]}} \{K_n(t)\}$ is contained in the compact subset $\mathbb{B}_B(K_0)$ of $(\mathcal{K}(\mathbb{R}^N), d)$.

So, the Theorem of Arzela–Ascoli provides a subsequence (again denoted by) $(K_n(\cdot))_n$ converging uniformly to a function $K(\cdot): [0,1] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d)$. In particular, $K(\cdot)$ is also *B*–Lipschitz continuous with $K(0) = K_0$, i.e. properties (a) – (f) are fulfilled completely.

In regard to feature (g), we cannot follow the same track as for differential inclusions any longer. Indeed, the functions $\hat{f}_n(\cdot)$ of morphological transitions have their values in $\operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ which cannot be regarded as a dual space in an obvious way. So Alaoglu's Theorem (stating that closed balls of dual Banach spaces are weakly* compact) cannot be applied immediately.

Alternatively, we restrict our considerations to a compact neighborhood \widetilde{K} of $\bigcup_{\substack{n \in \mathbb{N} \\ t \in [0,1]}} K_n(t) \subset \mathbb{R}^N$ and use a sufficient condition on relatively weakly compact sets in $L^1([0,1], C^0(\widetilde{K} \times \mathbb{B}_1, \mathbb{R}^N))$. Here $C^0(\widetilde{K} \times \mathbb{B}_1, \mathbb{R}^N)$ (supplied with the supremum norm $\|\cdot\|_{\infty}$) denotes the Banach space of all continuous functions $\widetilde{K} \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$. According to subsequent Lemma 3.9, if $W \subset C^0(\widetilde{K} \times \mathbb{B}_1, \mathbb{R}^N)$ is weakly compact then the subset

 $\left\{ h \in L^1([0,1], \ C^0(\widetilde{K} \times \mathbb{B}_1, \ \mathbb{R}^N)) \ \Big| \ h(t) \in W \text{ for (Lebesgue) almost every } t \in [0,1] \right\}$ is relatively weakly compact in $L^1([0,1], \ C^0(\widetilde{K} \times \mathbb{B}_1, \ \mathbb{R}^N)).$

In fact, the set $\{\widehat{f}_n(t) \mid n \in \mathbb{N}, t \in [0,1]\} \subset C^0(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ is uniformly bounded and equi–continuous

(due to property (c)). So according to the Theorem of Arzela–Ascoli, the set of their restrictions to the compact set $\widetilde{K} \times \mathbb{B}_1 \subset \mathbb{R}^{2N}$

$$W := \left\{ \left. \widehat{f}_n(t) \right|_{\widetilde{K} \times \mathbb{B}_1} \left| n \in \mathbb{N}, \ t \in [0,1] \right\} \subset C^0(\widetilde{K} \times \mathbb{B}_1, \ \mathbb{R}^N) \right\}$$

is even relatively compact with respect to $\|\cdot\|_{\infty}$. Thus, $\{\widehat{f}_n(\cdot)|_{\widetilde{K}\times\mathbb{B}_1} \mid n \in \mathbb{N}\}$ is relatively weakly compact in $L^1([0,1], C^0(\widetilde{K}\times\mathbb{B}_1, \mathbb{R}^N))$ and, we obtain a subsequence (again denoted by) $(\widehat{f}_n(\cdot))_{n\in\mathbb{N}}$ and some $g(\cdot) \in L^1([0,1], C^0(\widetilde{K}\times\mathbb{B}_1, \mathbb{R}^N))$ with

 $\widehat{f}_{n}(\cdot)|_{\widetilde{K}\times\mathbb{B}_{1}} \longrightarrow g(\cdot) \quad \text{weakly in } L^{1}([0,1], C^{0}(\widetilde{K}\times\mathbb{B}_{1}, \mathbb{R}^{N})).$ Now this construction of subsequences is applied to $\widetilde{K}_{j} \stackrel{\text{Def.}}{=} \mathbb{B}_{j+B}(K_{0}) \subset \mathbb{C} \mathbb{R}^{N}$ for $j = 1, 2, 3 \dots$ successively. In combination with Cantor's diagonal construction, we obtain a subsequence (again denoted by) $(\widehat{f}_{n}(\cdot))_{n\in\mathbb{N}}$ and some $g_{j}(\cdot) \in L^{1}([0,1], C^{0}(\widetilde{K}_{j}\times\mathbb{B}_{1}, \mathbb{R}^{N}))$ (for each $j \in \mathbb{N}$) such that $\forall j$,

$$\widehat{f}_n(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \longrightarrow g_j(\cdot)$$
 weakly in $L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)).$

As this subsequence is constructed independently of $j \in \mathbb{N}$, its weak convergence implies for any j < k

$$g_j(t)(\cdot) = g_k(t)(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \in C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)$$
 for almost every $t \in [0, 1]$

and so $(g_j(\cdot))_{j\in\mathbb{N}}$ induces a single function $\widehat{f}: [0,1[\longrightarrow C^0(\mathbb{R}^N\times\mathbb{B}_1,\mathbb{R}^N)]$ defined as

 $\widehat{f}(t)(x,u) := g_j(t)(x,u) \quad \text{for } x \in \widetilde{K}_j, \ u \in \mathbb{B}_1 \text{ and almost every } t \in [0,1[.$ Finally, we verify $\widehat{f}(t) \in \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N), \ \|\widehat{f}(t,\cdot,\cdot)\|_{\infty} + \operatorname{Lip}\widehat{f}(t,\cdot,\cdot) \leq c \text{ for almost every } t.$ Indeed, as in the case of differential inclusions (§ 3.1), Mazur's Lemma ensures for each $j \in \mathbb{N}$ (fixed)

$$\widehat{f}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \in \bigcap_{n \in \mathbb{N}} \overline{co} \left\{ \widehat{f}_n(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1}, f_{n+1}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \dots \right\} \quad \text{in } L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)).$$

Thus, $\widehat{f}(\cdot)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}$ can be approximated by convex combinations of $\{\widehat{f}_{1}(\cdot)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}, \widehat{f}_{2}(\cdot)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}, \dots \}$ with respect to the L^{1} norm. A further subsequence (of these convex combinations) converges to $\widehat{f}(\cdot)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}$ almost everywhere in [0,1]. So, for almost every $t \in [0,1], \widehat{f}(t)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}$ belongs to the same compact convex subset of $(C^{0}(\widetilde{K}_{j}\times\mathbb{B}_{1}, \mathbb{R}^{N}), \|\cdot\|_{\infty})$ as $\widehat{f}_{1}(t)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}}, \widehat{f}_{2}(t)|_{\widetilde{K}_{j}\times\mathbb{B}_{1}} \dots$, namely $\{w \in \operatorname{Lip}(\widetilde{K}_{j}\times\mathbb{B}_{1}, \mathbb{R}^{N}) \mid \|w\|_{\infty} + \operatorname{Lip} w \leq c\}$.

Lemma 3.9 ([28], Proposition 7) Let (Ω, Σ, μ) be a probabilistic space, X an arbitrary Banach space and $L^1(\mu, X)$ the Banach space of Bochner integrable functions $\Omega \longrightarrow X$ equipped with its usual L^1 norm (as in [15]).

For any weakly compact subset $W \subset X$, the set $\{h \in L^1(\mu, X) \mid h(\omega) \in W \text{ for } \mu\text{-almost every } \omega \in \Omega\}$ is relatively weakly compact.

Remark. An earlier version of this result is presented in [13] and, [14] considers weak compactness of Bochner integrable functions with values in an arbitrary Banach space under weaker assumptions.

Proposition 3.10 (The limit function is a solution) Under the assumptions of Theorem 3.5, consider $K_n(\cdot), K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ and $\hat{f}_n(\cdot), \hat{f}(\cdot) : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ specified in Proposition 3.8.

Then $K(\cdot)$ is a solution of the morphological inclusion $K(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \in \mathcal{V}$ for all $t \in [0, 1]$. Proof consists of several steps. $K(t) \in \mathcal{V}$ for all $t \in [0,1]$ results directly from properties (d), (f) of Proposition 3.8 because \mathcal{V} is assumed to be a closed subset of $(\mathcal{K}(\mathbb{R}^N), d)$. So $\widehat{f}(t, \cdot, \mathbb{B}_1) \in \overset{\circ}{\mathcal{K}}(\cdot)$ is still to prove for a.e. t. This is the common consequence of the following Lemmas 3.11, 3.13 and 3.15.

First we interpret the weak convergence of the parameterized maps $\widehat{f}_n(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \longrightarrow \widehat{f}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1}$ (in L^1) with respect to the corresponding set-valued maps $[0, 1[\times \widetilde{K}_j \rightsquigarrow \mathbb{R}^N]$ and meet the topology of locally uniform convergence in $\operatorname{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

As a rather technical tool, subsequent Lemma 3.12 clarifies how the uniform Lipschitz bounds of $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ (according to assumption (2.)) imply useful compactness features which ensure that the limit map $\widehat{f}(\cdot, \cdot, \mathbb{B}_1) : [0, 1] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is related to $\mathcal{F}(K(\cdot))$ at almost every time.

Lemma 3.11 Let the sequences $K_n(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\hat{f}_n(\cdot) : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N) \ (n \in \mathbb{N})$ and the functions $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $\hat{f}(\cdot) : [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)]$ be as in Proposition 3.8. Then, for \mathcal{L}^1 almost every $t \in [0,1[$, $\operatorname{dist}\left(\hat{f}(t,x,\mathbb{B}_1), \ co\left\{\hat{f}_n(t,x,\mathbb{B}_1), \ \hat{f}_{n+1}(t,x,\mathbb{B}_1) \dots\right\}\right) \longrightarrow 0$ locally uniformly in $x \in \mathbb{R}^N$ for $n \to \infty$ with the coefficients of the approximating convex combinations being chosen independently from t, x. So in particular, $\hat{f}(t,\cdot,\mathbb{B}_1) \in \mathcal{F}(K(t)) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$.

 $\begin{array}{ll} Proof. & \operatorname{Proposition 3.8} \ (\mathrm{g}) \ \mathrm{specifies the \ convergence \ resulting \ directly \ from \ construction} \\ & \widehat{f}_n(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \longrightarrow \widehat{f}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} & \operatorname{weakly \ in} \ L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)) & \text{for each} \ j \in \mathbb{N} \\ \text{with the abbreviation} \ \widetilde{K}_j \ := \ \mathbb{B}_{j+B}(K_0) \stackrel{\text{def.}}{=} \left\{ x \in \mathbb{R}^N \ \big| \ \mathrm{dist}(x, K_0) \leq j+B \right\} \in \mathcal{K}(\mathbb{R}^N). \\ \text{Fixing the index} \ j \in \mathbb{N} \ \text{of compact sets arbitrarily, Mazur's Lemma provides a sequence} \ \left(h_{j,n}(\cdot)\right)_{n \in \mathbb{N}} \\ \text{with} \ h_{j,n}(\cdot) \ \in \ co \left\{ \widehat{f}_n(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1}, \ \widehat{f}_{n+1}(\cdot)|_{\widetilde{K}_j \times \mathbb{B}_1} \dots \right\} \ \subset \ L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)) \\ & \text{ and} \\ h_{j,n}(\cdot) \ \longrightarrow \ \widehat{f}(\cdot) \qquad (n \longrightarrow \infty) \qquad \text{strongly in} \ L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)). \\ \text{Considering subsequences (again denoted by)} \ \left(h_{j,n}(\cdot)\right)_{n \in \mathbb{N}} \\ & \text{instead, we even obtain convergence for} \\ \mathcal{L}^1 \ \text{almost every} \ t \in [0,1], \qquad h_{j,n}(t) \ \longrightarrow \ \widehat{f}(t)|_{\widetilde{K}_j \times \mathbb{B}_1} \ (n \longrightarrow \infty) \qquad \text{in} \left(C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N), \| \cdot \|_{\infty}\right), \\ & \text{i.e. uniformly in} \ \widetilde{K}_j \times \mathbb{B}_1 \subset \mathbb{R}^N \times \mathbb{R}^N. \ \text{So the first claim is proved.} \end{array}$

In particular, all values of $\widehat{f}(t,\cdot,\mathbb{B}_1):\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ are convex since so has each $\widehat{f}_n(t,\cdot,\mathbb{B}_1) \in \operatorname{im} \mathcal{F}$.

Furthermore, we obtain the following inclusions for
$$\mathcal{L}^{1}$$
 almost every $t \in [0,1]$ (and each index $j \in \mathbb{N}$)
 $\widehat{f}(t, \cdot, \mathbb{B}_{1})|_{\widetilde{K}_{j}} \in \bigcap_{n \in \mathbb{N}} \overline{h_{j,n}(t)(\cdot, \mathbb{B}_{1})|_{\widetilde{K}_{j}} \cup h_{j,n+1}(t)(\cdot, \mathbb{B}_{1})|_{\widetilde{K}_{j}} \cup \ldots}$ in a pointwise way in \widetilde{K}_{j}
 $\subset \bigcap_{n \in \mathbb{N}} \overline{co} \bigcup_{\substack{m \ge n \\ m \ge n}} \widehat{f}_{m}(t, \cdot, \mathbb{B}_{1})|_{\widetilde{K}_{j}}$ due to Proposition 3.8 (e)
 $\subset \bigcap_{\varepsilon > 0} \overline{co} \mathcal{F}(\mathbb{B}_{\varepsilon}(K(t)))|_{\widetilde{K}_{j}}$ since $d(K_{m}(t), K(t)) \to 0$.

Here, to be more precise, the closed convex hull (in the last line) denotes the following set-valued map

$$\widetilde{K}_j \rightsquigarrow \mathbb{R}^N, \qquad x \mapsto \overline{co} \bigcup_{\substack{M \in \mathcal{K}(\mathbb{R}^N) \\ d(K(t),M) \leq \epsilon}} \bigcup_{G \in \mathcal{F}(M)} G(x).$$

Fixing now $j \in \mathbb{N}$ and $\delta > 0$ arbitrarily, we introduce the abbreviation

$$\mathcal{B}_{\delta}\left(\mathcal{F}(K(t)); \ \widetilde{K}_{j}\right) := \left\{ G \in \operatorname{LIP}_{\overline{co}}(\mathbb{R}^{N}, \mathbb{R}^{N}) \mid \delta \geq \operatorname{dist}(G(\cdot)|_{\widetilde{K}_{j}}, \ \mathcal{F}(K(t))|_{\widetilde{K}_{j}}) \\ \stackrel{\text{Def.}}{=} \inf_{\substack{Z \in \mathcal{F}(K(t))}} \sup_{x \in \widetilde{K}_{j}} d(G(x), \ Z(x)) \right\}$$

for the "ball" around the set $\mathcal{F}(K(t)) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ containing all maps $G \in \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ whose restriction to \widetilde{K}_j has the "uniform distance" $\leq \delta$ from $\mathcal{F}(K(t))$.

For any $\delta > 0$, there exists a radius $\rho > 0$ with $\mathcal{F}(\mathbb{B}_{\rho}(K(t))) \subset \mathcal{B}_{\delta}(\mathcal{F}(K(t)); K_j)$ because otherwise there would exist sequences $(M_k)_{k \in \mathbb{N}}$, $(G_k)_{k \in \mathbb{N}}$ in $\mathcal{K}(\mathbb{R}^N)$, $\operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$, respectively, with

 $d\!l (M_k, K(t)) \leq \frac{1}{k}, \qquad G_k \in \mathcal{F}(M_k), \qquad \inf_{\substack{Z \in \mathcal{F}(K(t)) \\ x \in \widetilde{K}_j}} \sup_{x \in \widetilde{K}_j} d\!l (G_k(x), Z(x)) > \delta \qquad \text{for each } k \in \mathbb{N}$

and, Lemma 3.12 would lead to a contradiction.

Obviously, $\mathcal{B}_{\delta}(\mathcal{F}(K(t)); \widetilde{K}_j) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed with respect to locally uniform convergence. Moreover, it is convex (with regard to pointwise convex combinations) because $\mathcal{F}(K(t))$ is supposed to be convex. Thus, we even obtain the inclusion $\overline{co} \mathcal{F}(\mathbb{B}_{\rho}(K(t))) \subset \mathcal{B}_{\delta}(\mathcal{F}(K(t)); \widetilde{K}_j)$, i.e.

$$\widehat{f}(t, \cdot, \mathbb{B}_1) \in \bigcap_{\delta > 0} \mathcal{B}_{\delta}(\mathcal{F}(K(t)); \widetilde{K}_j)$$
 for each $j \in \mathbb{N}$.

In particular, there exists some $Z_j \in \mathcal{F}(K(t))$ satisfying $\sup_{x \in \widetilde{K}_j} d(f(t, x, \mathbb{B}_1), Z(x)) \leq \frac{1}{j}$ and, the compactness property of Lemma 3.12 implies $\widehat{f}(t, \cdot, \mathbb{B}_1) \in \mathcal{F}(K(t))$ for almost every time t.

Lemma 3.12 (Sequential compactness in the image and graph of $\mathcal{F}(\cdot)$)

In addition to the hypotheses of Viability Theorem 3.5, let $(G_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) = \bigcup_{M\in\mathcal{K}(\mathbb{R}^N)} \mathcal{F}(M) \subset \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N).$ Then, there exist a subsequence $(G_{k_j})_{j\in\mathbb{N}}$ and a set-valued map $G \in \operatorname{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ such that for any compact set $K \subset \mathbb{R}^N$, $\sup_{x \in K} d(G_{k_j}(x), G(x)) \longrightarrow 0$ $(j \longrightarrow \infty)$ and $\operatorname{Lip} G \leq A$, $||G||_{\infty} \leq B$.

Let now $(K_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{K}(\mathbb{R}^N)$ such that $\bigcup_{k\in\mathbb{N}} K_k \subset \mathbb{R}^N$ is bounded and $G_k \in \mathcal{F}(K_k)$ for each $k \in \mathbb{N}$. Then there exist subsequences $(K_{k_j})_{j\in\mathbb{N}}$, $(G_{k_j})_{j\in\mathbb{N}}$, a set $K \in \mathcal{K}(\mathbb{R}^N)$ and a set-valued map $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ with

Proof. Applying the parameterization theorem 3.7 to the autonomous maps $G_k : \mathbb{R}^N \to \mathbb{R}^N$ provides a sequence $(g_k)_{k \in \mathbb{N}}$ of Lipschitz functions $\mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ with $g_k(\cdot, \mathbb{B}_1) = G_k$ for each $k \in \mathbb{N}$ and $\sup_k (||g_k||_{\infty} + \operatorname{Lip} g_k) \leq \operatorname{const}(A, B) < \infty$.

So for any nonempty compact set $K \subset \mathbb{R}^N$, the Theorem of Arzela–Ascoli guarantees a subsequence $(g_{k_j})_{j \in \mathbb{N}}$ converging uniformly in $K \times \mathbb{B}_1$. In combination with Cantor's diagonal construction, we obtain even a subsequence (again denoted by) $(g_{k_j})_{j \in \mathbb{N}}$ converging uniformly in each of the countably many compact sets $\mathbb{B}_m(0) \times \mathbb{B}_1 \subset \mathbb{R}^N \times \mathbb{R}^N$ $(m \in \mathbb{N})$.

Let $h_m : \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ denote an arbitrary Lipschitz function with $\sup_{\mathbb{B}_m(0) \times \mathbb{B}_1} |g_{k_j}(\cdot) - h_m(\cdot)| \xrightarrow{j \to \infty} 0$. Then we obtain the unique limit function $h : \mathbb{R}^N \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ by setting $h(x, \cdot) := h_m(x, \cdot)$ for all $x \in \mathbb{B}_m(0), m \in \mathbb{N}$ and, $g_{k_j} \longrightarrow h$ $(j \to \infty)$ locally uniformly in $\mathbb{R}^N \times \mathbb{B}_1$. In particular, $h(\cdot)$ is also Lipschitz continuous and has the same global Lipschitz bounds as $(g_k)_{k\in\mathbb{N}}$. So, $G := h(\cdot, \mathbb{B}_1) : \mathbb{R}^N \to \mathbb{R}^N$ provides a set-valued map being Lipschitz continuous and satisfying $\sup_{x \in K} d(G_{k_j}(x), G(x)) \leq \sup_{x \in K} \sup_{u \in \mathbb{B}_1} |g_{k_j}(x, u) - h(x, u)| \longrightarrow 0 \quad (j \longrightarrow \infty) \quad \text{for any } K \in \mathcal{K}(\mathbb{R}^N).$ This convergence of $(G_{k_j})_{j \in \mathbb{N}}$ implies directly Lip $G \leq A$, $\|G\|_{\infty} \leq B$ and the convexity of all values of G. So the first claim is proved.

For verifying the second claim, we extract a convergent subsequence $(K_{k_l})_{l \in \mathbb{N}}$ as all sets $K_k, k \in \mathbb{N}$, are contained in one and the same compact subset of \mathbb{R}^N . So, there is $K \in \mathcal{K}(\mathbb{R}^N)$ with $d(K_{k_l}, K) \xrightarrow{l \to \infty} 0$. Following the same track as in the first part, we obtain subsequences (again denoted by) $(K_{k_i})_{i \in \mathbb{N}}$, $(G_{k_i})_{i\in\mathbb{N}}$ such that in addition, the latter converges to some $G\in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ locally uniformly. According to assumption (3.) of Viability Theorem 3.5, Graph $\mathcal{F} \subset \mathcal{K}(\mathbb{R}^N) \times \text{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed with respect to these topologies and thus, it contains (K, G).

Lemma 3.13 (K(t) as a reachable set of $\hat{f}(\cdot, \cdot, \mathbb{B}_1)$)

Let the sequences $K_n(\cdot): [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N), \ \widehat{f}_n(\cdot): [0,1[\longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N) \ (n \in \mathbb{N})$ and the functions $K(\cdot): [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N), \ \widehat{f}(\cdot): [0,1] \longrightarrow \operatorname{Lip}(\mathbb{R}^N \times \mathbb{B}_1, \mathbb{R}^N)$ be as in Proposition 3.8.

Then, for any $x(\cdot) \in C^0([0,1], \mathbb{R}^N)$ and Lebesgue measurable set $J \subset [0,1]$, $I\left(\int \widehat{f}\left(a,m(e) \mathbb{D}\right) de \int \widehat{f}\left(a,m(e) \mathbb{D}_{e}\right) de \right) \xrightarrow{n \to \infty} 0$

$$d\left(\int_{J} f_{n}(s, x(s), \mathbb{B}_{1}) \ ds, \int_{J} f(s, x(s), \mathbb{B}_{1}) \ ds\right) \longrightarrow 0$$

So in particular, $\vartheta_{\widehat{f}(\cdot, \mathbb{B}_{1})}(t, K_{0}) = K(t)$ for every $t \in [0, 1]$.

According to the definition of Aumann integral, Proof.

$$\int_{J} \widehat{f}(s, x(s), \mathbb{B}_{1}) \ ds \stackrel{\text{Def.}}{=} \Big\{ \int_{J} \widehat{f}(s, x(s), u(s)) \ ds \ \Big| \ u(\cdot) \in L^{1}(J, \mathbb{B}_{1}) \Big\}.$$

Fixing $u(\cdot) \in L^{1}(J, \mathbb{B}_{1})$ and $x(\cdot) \in C^{0}([0, 1], \mathbb{R}^{N})$ arbitrarily, we conclude from Proposition 3.8 (g)

 $\int_{I} \widehat{f}_n(s, x(s), u(s)) \ ds \ \longrightarrow \ \int_{I} \widehat{f}(s, x(s), u(s)) \ ds$

 $L^1([0,1], C^0(\widetilde{K}_j \times \mathbb{B}_1, \mathbb{R}^N)) \longrightarrow \mathbb{R}, \quad h \longmapsto \int_I h(s)(x(s), u(s)) \, ds$ is continuous and linear since whenever $x([0,1]) \subset \widetilde{K}_j$. This implies both $\operatorname{dist}\left(\int_J \widehat{f}_n(s,x(s),\mathbb{B}_1) \ ds, \int_J \widehat{f}(s,x(s),\mathbb{B}_1) \ ds\right) \longrightarrow 0$ and $\operatorname{dist}\left(\int_J \widehat{f}(s,x(s),\mathbb{B}_1) \ ds, \int_J \widehat{f}_n(s,x(s),\mathbb{B}_1) \ ds\right) \longrightarrow 0$ for $n \to \infty$.

So the first claim holds

Due to Proposition 3.8 (d), each $\widehat{f}_n(\cdot, x, \mathbb{B}_1) : [0, 1] \rightsquigarrow \mathbb{R}^N$ $(n \in \mathbb{N}, x \in \mathbb{R}^N)$ is piecewise constant and thus, it has at most countably many points of discontinuity. So applying the Cauchy–Lipschitz– type Theorem 2.6 (of Aubin) in a piecewise way with respect to time, we conclude from its uniqueness $\vartheta_{\widehat{f}(\mathbf{K}_{0})}(t,K_{0}) = K_{n}(t)$ for every $t \in [0, 1]$ and $n \in \mathbb{N}$.

$$d(K_n(t), K(t)) \longrightarrow 0 \quad \text{has already been mentioned in Proposition 3.8 (f). So we now still have to verify} d\left(\vartheta_{\widehat{f}_n(\cdot,\mathbb{B}_1)}(t,K_0), \quad \vartheta_{\widehat{f}(\cdot,\mathbb{B}_1)}(t,K_0)\right) \longrightarrow 0 \quad \text{for every } t \in [0,1] \text{ and } n \to \infty.$$

If $K_0 \subset \mathbb{R}^N$ consists of only one point, then this convergence results directly from an earlier theorem of Stassinopoulos and Vinter [27] quoted subsequently in Lemma 3.14.

For extending it to arbitrary initial sets $K_0 \in \mathcal{K}(\mathbb{R}^N)$, we exploit the Lipschitz dependence (of reachable sets) on the initial sets according to Lemma 2.3. It implies here for any $M_1, M_2 \in \mathcal{K}(\mathbb{R}^N)$ and $t \in [0, 1]$

$$\begin{cases} d \left(\vartheta_{\widehat{f}_{n}(\cdot,\mathbb{B}_{1})}(t,M_{1}), \quad \vartheta_{\widehat{f}_{n}(\cdot,\mathbb{B}_{1})}(t,M_{2}) \right) &\leq e^{A} d \left(M_{1},M_{2}\right) \\ d \left(\vartheta_{\widehat{f}(\cdot,\mathbb{B}_{1})}(t,M_{1}), \quad \vartheta_{\widehat{f}(\cdot,\mathbb{B}_{1})}(t,M_{2}) \right) &\leq e^{A} d \left(M_{1},M_{2}\right) \end{cases}$$

Lemma 3.14 ([27], Theorem 7.1) Let $D : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ and each $D_n : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ $(n \in \mathbb{N})$ satisfy the following assumptions:

- 1. D and D_n have nonempty convex compact values,
- 2. $D(\cdot, x), D_n(\cdot, x) : [0, 1] \rightsquigarrow \mathbb{R}^N$ are measurable for every $x \in \mathbb{R}^N$,
- 3. there exists $k(\cdot) \in L^1([0,1])$ such that $D(t,\cdot), D_n(t,\cdot) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ are k(t)-Lipschitz for a.e. t,
- 4. there exists $h(\cdot) \in L^1([0,1])$ such that $\sup_{y \in D(t,x) \cup D_n(t,x)} |y| \le h(t)$ for every $x \in \mathbb{R}^N$ and a.e. t.

Fixing the initial point $a \in \mathbb{R}^N$ arbitrarily, the absolutely continuous solutions of

$$\wedge \begin{cases} y'(\cdot) \in D_n(\cdot, y(\cdot)) & a.e. \ in \ [0,1] \\ y(0) = a \end{cases} \quad and \quad \wedge \begin{cases} y'(\cdot) \in D(\cdot, y(\cdot)) & a.e. \ in \ [0,1] \\ y(0) = a \end{cases}$$

respectively form compact subsets of $(C^0([0,1],\mathbb{R}^N), \|\cdot\|_{\infty})$ denoted by $\mathcal{D}_n \ (n \in \mathbb{N}), \mathcal{D}$.

Then, \mathcal{D}_n converges to \mathcal{D} (w.r.t. the Pompeiu-Hausdorff metric on compact subsets of $C^0([0,1],\mathbb{R}^N)$) if and only if for every solution $d(\cdot) \in \mathcal{D}$, $D_n(\cdot, d(\cdot)) : [0,1] \rightsquigarrow \mathbb{R}^N$ converges to $D(\cdot, d(\cdot)) : [0,1] \rightsquigarrow \mathbb{R}^N$ weakly in the following sense

$$d\left(\int_{J} D_{n}(s, d(s)) \ ds, \quad \int_{J} D(s, d(s)) \ ds\right) \xrightarrow{n \to \infty} 0 \qquad \text{for every measurable subset } J \subset [0, 1].$$

For completing the proof of Proposition 3.10, the final step focuses on the solution property of the reachable map $t \mapsto \vartheta_{\widehat{f}(\cdot,\mathbb{B}_1)}(t,K_0)$ at almost every time $t \in [0,1]$. Here we benefit from an earlier result of Frankowska, Plaskacz and Rzeżuchowski [19] about the infinitesimal behavior of reachable maps:

Lemma 3.15 ([19], Theorem 2.5) Let V be a separable metric space and $G : [0, T] \times \mathbb{R}^N \times V \to \mathbb{R}^N$ a set-valued map satisfying

- 1. G has nonempty closed convex values,
- 2. $\mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$, $(x, v) \mapsto G(t, x, v)$ is continuous for almost all $t \in [0, T]$,
- 3. $[0,T] \rightsquigarrow \mathbb{R}^N, t \mapsto G(t,x,v)$ is measurable for all $(x,v) \in \mathbb{R}^N \times V$,

4. there exists $h(\cdot) \in L^1([0,T])$ with $\sup_{y \in G(t,x,v)} |y| \le h(t)$ for all $(x,v) \in \mathbb{R}^N \times V$ and a.e. $t \in [0,T]$. Then, there exists a set $J \subset [0,T]$ of full Lebesgue measure (i.e. $\mathcal{L}^1([0,T] \setminus J) = 0$) such that for every $(t,x,v) \in J \times \mathbb{R}^N \times V$, $dl \left(\frac{1}{h} \cdot \left(\vartheta_{G(t+\cdot,\cdot,v)}(h,x) - x\right), G(t,x,v)\right) \longrightarrow 0$ for $h \downarrow 0$.

$$\frac{1}{h} \cdot d\!\!\left(\vartheta_{G(t+\cdot,\cdot,v)}(h,\,K_t), \bigcup_{x \in K_t} (x+h \cdot G(t,x,v))\right) \longrightarrow 0 \qquad \text{for } h \downarrow 0.$$

Applying this result to its autonomous counterpart $G(t, \cdot, \cdot) : \mathbb{R}^N \times V \rightsquigarrow \mathbb{R}^N$ (with arbitrary $t \in J$), the corresponding limit exists on the whole time interval [0, T - t] and thus,

$$\frac{1}{h} \cdot d\!l \Big(\vartheta_{G(t, \cdot, v)} (h, K_t), \qquad \bigcup_{x \in K_t} (x + h \cdot G(t, x, v)) \Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0 \text{ and all } K_t \in \mathcal{K}(\mathbb{R}^N), v \in V.$$

Combing these two asymptotic features, we conclude for any $t \in J, K_t \in \mathcal{K}(\mathbb{R}^N), v \in V$

$$\frac{1}{h} \cdot dl \Big(\vartheta_{G(t, \cdot, v)} (h, K_t), \quad \vartheta_{G(t+ \cdot, \cdot, v)}(h, K_t) \Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0,$$

i.e. fixing the initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ arbitrarily, there exists a set $J \subset [0,T]$ of full Lebesgue measure such that $G(t, \cdot, v)$ belongs to the shape mutation of the reachable map

$$[0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N), \qquad s \longmapsto \vartheta_{G(\cdot,\cdot,v)}(s,K_0)$$

at every time $t \in J$.

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