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The Lie algebra of the

$N=2$-string

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Abstract

The theory of generalized Kac-Moody algebras is a generalization of the theory of finite dimensional simple Lie algebras. The physical states of some compactified strings give realizations of generalized Kac-Moody algebras. For example the physical states of a bosonic string moving on a 26 dimensional torus form a generalized Kac-Moody algebra and the physical states of a $N=1$-string moving on a 10 dimensional torus form a generalized Kac-Moody superalgebra. A natural question is whether the physical states of the compactified $N=2$-string also realize such an algebra.
In this thesis we construct the Lie algebra of the compactified $N=2$-string, study its properties and show that it is not a generalized Kac-Moody algebra.
The Fock space of a $N=2$-string moving on a 4 dimensional torus can be described by a vertex algebra constructed from a rational lattice of signature $(8,4)$. Here 6 coordinates with signature $(4,2)$ come from the matter part and 6 coordinates with signature $(4,2)$ come from the ghost part. The physical states are represented by the cohomology of the BRST-operator. The vertex algebra induces a product on the vector space of physical states that defines the structure of a Lie algebra on this space. This Lie algebra shares many properties with generalized Kac-Moody algebras but we will show that it is not a generalized Kac-Moody algebra.

Zusammenfassung

Die Theorie der verallgemeinerten Kac-Moody Algebren ist eine Verallgemeinerung der Theorie der endlichdimensionalen einfachen Lie Algebren. Die physikalischen Zustände einiger kompaktifizierter Strings liefern Realisierungen von verallgemeinerten Kac-Moody Algebren. Beispielsweise werden die physikalischen Zustände eines bosonischen Strings, der sich auf einem 26-dimensionalen Torus bewegt, durch eine verallgemeinerte Kac-Moody Algebra beschrieben, die physikalischen Zustände eines N=1- Strings auf einem 10-dimensionalen Torus durch eine verallgemeinerte Kac-Moody Superalgebra. Eine natürliche Frage ist, ob die physikalischen Zustände eines $N=2$-Strings, der auf einem 4-dimensionalen Torus kompaktifiziert ist, auch eine verallgemeinerte Kac-Moody Algebra bilden.
In dieser Arbeit konstruieren wir die Lie Algebra des $N=2$-Strings, untersuchen ihre Eigenschaften und zeigen insbesondere, daß diese keine verallgemeinerte Kac-Moody Algebra darstellt.
Der Fock Raum eines $N=2$-Strings, der sich auf einem 4-dimensionalen Torus bewegt, kann durch eine Vertex Algebra zu einem rationalen Gitter der Signatur $(8,4)$ beschrieben werden. Hierbei kommen 6 Koordinaten mit Signatur $(4,2)$ vom Materieanteil und ebensoviele vom Geistanteil. Die physikalischen Zustände sind durch die Kohomologie des BRST-Operators gegeben. Die Vertex Algebra induziert ein Produkt auf dem Vektorraum der physikalischen Zustände, das die Struktur einer Lie Algebra definiert. Die Lie Algebra weist viele Eigenschaften einer verallgemeinerten Kac-Moody Algebra auf, bildet aber selbst, wie wir zeigen werden, keine verallgmeinerte Kac-Moody Algebra.

Contents

Chapter 1

Introduction

Today there are some different attempts in physics to build up a theory that unifies the theory of relativity and quantum mechanics. One of them is string theory. In string theory particles are not described as points but correspond to different vibration modes of a one-dimensional string.
Sometimes strings can be described by vertex algebras. That are infinite dimensional vector spaces with a state field correspondence which associates to each element in the vector space (the state) a formal Laurent series (the field). The fields act on the states. The vertex algebra of the string is acted on by some extension of the Virasoro algebra of central charge zero.
The different string theories can be distinguished by the representation of these extensions they carry. In this way we obtain three string theories, the $N=0$-string (bosonic string), the $N=1$-string (superstring) and the $N=2$-string. For example the $N=2$-string is acted on by the $N=2$-extension of the Virasoro algebra. The representation is used to construct the so-called BRST-operator. The physical states of the string are represented by the cohomology of this operator.
In [B3] it is shown that the physical states of the $N=0$-string moving on a torus form a Lie algebra called Fake monster algebra. This algebra is the simplest example of a generalized Kac-Moody algebra. The physical states of a compactified $N=1$-string give a realization of a generalized Kac-Moody superalgebra, the fake monster superalgebra (cf. [S1]). Hence a natural question is whether the physical states of the $N=2$-string moving on a torus also form such an algebra.

In this paper we construct the Lie algebra of the $N=2$-string moving on a torus, describe some properties of it and show that it is not a generalized Kac-Moody algebra.

Now we explain the construction of the Lie algebra of the $N=2$-string in more detail.

The $N=2$-string moving on a torus is described by a vertex algebra constructed from the rational lattice $L = II_{2,2} \oplus D'_{2,2} \oplus \mathbb{Z}^4$. This vertex algebra carries a representation of the N=2-extension of the Virasoro algebra, which is used to construct the BRST-operator Q satisfying $Q^2 = 0$. The physical states are represented by the cohomology of this operator. More precisely for nonzero momentum there are infinitely many isomorphic copies of vector spaces called pictures representing the physical states. We restrict to the $(0,0)$ picture. The vertex algebra induces a product on the vector space of physical states that defines the structure of a Lie algebra on this space. This Lie algebra is graded by the 4 dimensional momentum lattice $II_{2,2}$. The Lie algebra decomposes into root spaces which have dimension one. All roots have zero norm. We show that this Lie algebra is not a generalized Kac-Moody algebra.

Most of the ideas necessary for the above construction come from physics and are described in the physics literature (cf. [LT], [P]). The construction is analogous to the construction of the Lie algebra of the superstring (cf. [S1]). References for the $N=2$-string ([BL], [BKL], [KL], [JL1], [JL2]) are also important for this work.

We describe the chapter of this thesis in more detail.

As mentioned above strings sometimes can be described by vertex algebras. In chapter two we define the term vertex algebra and state some calculation rules that will be needed in the following computations. We define the Virasoro algebra. The Fock space of the N=2-string will be realized by a vertex algebra constructed from a rational lattice. Therefore we describe the construction of a lattice vertex algebra. To get later an invariant bilinear form on the Lie algebra we define first an invariant bilinear form on the vertex algebra. We state the terms and results about invariant bilinear forms on vertex algebras that we will need.

In the third chapter we describe some results about generalized Kac-Moody algebras (cf. [B1], [B2]). The theory of generalized Kac-Moody algebras is a generalization of the theory of finite dimensional simple Lie algebras.
Let G be a finite dimensional simple Lie algebra over $\mathbb{C}$. Then G has a Cartan subalgebra H and an invariant bilinear form $(x,y) = tr(ad(x)ad(y))$ where $x, y \in G$. G decomposes into a sum over H and one-dimensional root spaces G_α, i.e.

$$G = H \oplus \bigoplus_{\alpha \in \Delta} G_\alpha.$$

We choose a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}, n = \dim H$. Then the symmetrized Cartan matrix $a_{ij} = ((\alpha_i, \alpha_j))_{i,j}$ has the following properties

- $a_{ii} > 0$,
- $a_{ij} = a_{ji}$,
- $a_{ij} \leq$ for $i \neq j$,
- $2a_{ij}/a_{ii} \in \mathbb{Z}$.

In this way we can associate a matrix to G. G is completely determined by the symmetrized Cartan matrix because of the Theorem of Serre which states that a Lie algebra with generators e_i, f_i, h_i and relations

- $[e_i, f_j] = \delta_{ij} h_i$,
- $[h_i, e_j] = a_{ij} e_j \ , \ [h_i, f_j] = -a_{ij} f_j$,
- $(ad\, e_i)^{1-2a_{ij}/a_{ii}} e_j = (ad\, f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \ , \ i \neq j$,

is isomorphic to G. Generalized Kac-Moody algebras are now obtained by applying Serre's construction to matrices which might have diagonal entries smaller or equal to zero. That means generalized Kac-Moody algebras might have imaginary simple roots. We recall some basic definitions and results about generalized Kac-Moody algebras and describe the results that will be needed to show that the Lie algebra of the N=2-string is not a generalized Kac-Moody algebra.

In chapter four we construct a vertex algebra that represents the Fock space of the chiral $N=2$-string moving on a torus. The $N=2$-string has critical dimension 4, i.e. it can be defined consistently only in 4 dimensions and has 4 bosonic and 4 fermionic degrees of freedom. We represent the bosonic degrees by a 4 dimensional even lattice with signature $(2,2)$ and the fermionic degrees in bosonized form by the weight lattice of the Lie algebra D_2. Because of the bosonized form the fermions give two degrees of freedom, together with the four bosonic degrees of freedom we get six matter degrees of freedom. In order to obtain a representation of the N=2-superconformal algebra of central charge zero we have to introduce six ghost coordinates. Two of them carry a negative metric and must be described together with the fermionic lattice. Hence we need a rational lattice L of signature $(8,4)$ to describe the N=2-string moving on a torus. The Fock space of the string is then given by the vertex algebra of L.

In chapter five we show that the vertex algebra of the $N=2$-string carries a representation of the $N=2$-extension of the Virasoro algebra of central charge zero and use it to construct the BRST-operator. Thereto we define a matter part representation of central charge 6 and a ghost part representation of central charge -6 so that they sum to a representation of central charge 0. The vertex algebra of the N=2-string contains physical and non-physical states. By means of the representation of the N=2-extension of the Virasoro algebra we construct the BRST-operator Q. The physical states are described then by the cohomology of this operator.

In chapter six we construct the Lie algebra of physical states of the N=2-string moving on a torus. The physical states are realized by the cohomology of the BRST-operator Q. There are many isomorphic copies of the space of physical states and the picture changing operators produce isomorphisms between them. We define a subalgebra V_S of the vertex algebra of the compactified N=2-string called small algebra. This restriction will allow us to construct the picture changing operator. In order to construct a bilinear form we restrict further to the vertex superalgebra V_S^{GSO}. We determine the cohomology groups of Q in V_S^{GSO}. As we will see the only nonvanishing cohomology groups for nonzero $\alpha \in II_{2,2}$ are the cohomology groups with $\alpha^2 = 0$ and ghostnumber one. We define the Lie algebra of the N=2-string. Thereto we define a product $\{,\}$ on V_S^{GSO} and describe some properties of it. We show that $\{,\}$ closes on the cohomology. To define the Lie algebra G we restrict to the $(0,0)$-picture and define the Lie bracket as $[u,v] = \{u,v\}, u, v \in G$. To obtain a bilinear form on G we define a bilinear form on the cohomology. As we will see later the bilinear form vanishes on the root spaces.

In chapter seven we calculate explicitly the cohomology group in the $(0,0)$ picture. This will be needed to calculate the bilinear form and the bracket. Thereto we have to construct the vector spaces $C^0_{0,0}(\alpha)$, $C^1_{0,0}(\alpha)$ of picture $(0,0)$ and ghost number 0 and 1 respectively. We determine the Cartan subalgebra

$$H = \text{Ker } Q|_{C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}, \qquad \alpha \in II_{2,2}, \alpha = 0,$$

and the the root spaces

$$G_\alpha = \text{Ker } Q|_{C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}, \quad \alpha \in II_{2,2}, \alpha \neq 0, \alpha^2 = 0.$$

In chapter eight we describe some properties of the Lie algebra G of the N=2-string. In particular we show that G is not a generalized Kac-Moody algebra. G decomposes as

$$G = H \oplus \bigoplus_{\alpha \in II_{2,2} \setminus \{0\}} G_\alpha.$$

We show that the invariant bilinear form on G is symmetric and pairs G_α with $G_{-\alpha}$ trivially. We also prove that there is no nondegenerate invariant bilinear form on G. We show that the subalgebra H of G is abelian. The bilinear form on G is nondegenerate on H and gives an isometry from H to $II_{2,2} \bigotimes_{\mathbb{Z}} \mathbb{C}$. Therewith we show that the subalgebra H measures the momentum, i.e. $[h, x] = (h, \alpha)x$ for $h \in H$ and $x \in G_\alpha$. Because of these properties we call H Cartan subalgebra of G. Then we calculate the commutator $[x, y]$ for $x \in G_\alpha, y \in G_\beta$. We show that the Lie algebra G of the $N{=}2$-string is not a generalized Kac-Moody algebra with Cartan subalgebra H.

Chapter 2

Vertex algebras

Sometimes strings can be described by vertex algebras. These are vector spaces with a state-field correspondence which associates to each element in the vector space (the state) a formal Laurent series (the field). The modes of the field act on the states.
In this chapter we define the term vertex algebra and state some calculation rules that will be needed in the following computations. We recall the definition of the Virasoro algebra. The Fock space of the $N=2$-string will be realized by a vertex algebra constructed from a rational lattice. In the second section of this chapter we will describe the construction of a lattice vertex algebra. To get later an invariant bilinear form on the Lie algebra we define first an invariant bilinear form on the vertex algebra. In the third section we state the terms and results about invariant bilinear forms on vertex algebras that we will need.

2.1 Definition and some properties

Let Γ be a finite abelian group, g an exponent of Γ, $\Delta : \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$ a symmetric map bilinear mod $\mathbb{Z}$ and $\eta : \Gamma \times \Gamma \to \mathbb{C}^*$ bimultiplicative.

We define

Definition 2.1
Let

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma$$

be a Γ-graded complex vector space and

$$\begin{array}{rcl} V & \to & (\operatorname{End} V)[[z^{\frac{1}{g}}, z^{-\frac{1}{g}}]] \\ a & \mapsto & a(z) = \sum\limits_{n \in \frac{1}{g}\mathbb{Z}} a_n z^{-n-1} \end{array}$$

a parity preserving **state-field correspondence**, *i.e. for* $a \in V_{\gamma_1}, b \in V_{\gamma_2}$,

$$a(z)b = \sum_{n \in \mathbb{Z} + \Delta(\gamma_1,\gamma_2)} a_n b z^{-n-1}$$

and

$$a_n b \in V_{\gamma_1+\gamma_2}$$

is zero for n *sufficiently large.*
V *is a* Γ*-graded* **vertex algebra** *if it satisfies the following conditions.*

- *There is an element* $1 \in V_0$ *called vacuum with* $1(z)a = a$ *and* $a(z)1|_{z=0} = a$.
- *The operator* D *on* V *defined by* $Da = a_{-2}1$ *satisfies*

$$[D, a(z)] = \partial a(z).$$

- *The locality condition*

$$i_{z,w}(z-w)^n a(z)b(w) - \eta(\gamma_1,\gamma_2) i_{w,z}(z-w)^n b(w)a(z) = 0$$

holds for $a \in V_{\gamma_1}, b \in V_{\gamma_2}$ *and* $n \in \mathbb{Z} + \Delta(\gamma_1,\gamma_2)$ *sufficiently large. Here* $i_{z,w}(z-w)^n$ *is the binomial expansion of* $(z-w)^n$ *in the domain* $|z| > |w|$, *i.e.* $i_{z,w}(z-w)^n = \sum_{k \geq 0} (-1)^k \binom{n}{k} z^{n-k} w^k$.

Proposition 2.2
Let $a \in V_{\gamma_1}, b \in V_{\gamma_2}, c \in V_{\gamma_3}$ *and* $n \in \mathbb{Z} + \Delta(\gamma_1,\gamma_2), k \in \mathbb{Z} + \Delta(\gamma_1,\gamma_3)$. *Then we have* **Bocherds' identity**

$$\sum_{j \geq 0} \binom{n}{j} (-1)^j \{a_{n+k-j} b_{m+j} c - \eta(\gamma_1,\gamma_2) e^{i\pi n} b_{m+n-j} a_{k+j} c\}$$

$$= \sum_{j \geq 0} \binom{k}{j} (a_{n+j} b)_{k+m-j} c.$$

Proof:
cf. [R]. $\square$

If $\Delta(\gamma_1,\gamma_3) = 0$ we can put $k = 0$ to get the **associativity formula**

$$(a_n b)_m c = \sum_{j \geq 0} \binom{n}{j} (-1)^j \{a_{n-j} b_{m+j} c - \eta(\gamma_1,\gamma_2) e^{i\pi n} b_{m+n-j} a_j c\}$$

and if $\Delta(\gamma_1,\gamma_2) = 0$ we get the **commutator formula**

$$a_k b_m c - \eta(\gamma_1,\gamma_2) b_m a_k c = \sum_{j \geq 0} \binom{k}{j} (a_j b)_{k+m-j} c.$$

We call the term

$$a(z)b = \sum a_n b z^{-n-1}$$

operator product expansion (OPE). Since the commutator is determined by the terms $a_j b$ with $j \geq 0$ (Proposition 2.2) we will write

$$a(z)b = \sum_{n \geq 0} a_n b z^{-n-1} + \underbrace{\dots}_{nonsingular\ terms}.$$

We will need the following notations.

Definition 2.3
Let $a_k, b_m \in \operatorname{End} V$.
We define the **commutator-symbol** $[,]$ *by* $[a_k, b_m] = a_k b_m - b_m a_k$ *and the* **anticommutator-symbol** $\{,\}$ *by* $\{a_k, b_m\} = a_k b_m + b_m a_k$.

Hence we have

$$\begin{aligned}
[a_m, b_n] &= -[b_n, a_m]. \\
\{c_m, d_n\} &= \{d_n, c_m\}. \\
a_m b_n &= [a_m, b_n] + b_n a_m. \\
c_m d_n &= \{c_m, d_n\} - d_n c_m.
\end{aligned}$$

Proposition 2.4
For the vacuum $1 \in V$ holds $1_n = \delta_{n+1}$ and $D1 = 0$.

Proof:
Definition 2.1 implies

$$1(z)a = \sum_{n \in \frac{1}{g}\mathbb{Z}} 1_n a z^{-n-1} = a.$$

Hence

$$1_n a = \begin{cases} a & \text{for} \quad n = -1, \\ 0 & \quad\quad \text{else}, \end{cases}$$

hence $1_n = \delta_{n+1}$. We get $D1 = 1_{-2}1 = \delta_{-2+1}1 = 0$. □

The following result is also useful for calculations.

Proposition 2.5
Let $v \in V$. Then we have

$$(Dv)_n = -n v_{n-1}.$$

Proof:

$$
\begin{aligned}
(Dv)_n x &= (v_{-2}1)_n x \\
&= \sum_{j\geq 0} \binom{-2}{j} (-1)^j \{ v_{-2-j} \underbrace{1_{n+j}x}_{\delta_{n+j+1}x} + \underbrace{1_{n-2-j}v_j x}_{\delta_{n-1-j}v_j x} \} \\
&= \binom{-2}{-n-1} (-1)^{-n-1} v_{n-1}x + \binom{-2}{n-1} (-1)^{n-1} v_{n-1}x \\
&= -nv_{n-1}x.
\end{aligned}
$$

□

For later use we need the definition of a vertex superalgebra.

In a $\mathbb{Z}_2$-graded vectorspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ we put $|v| = 0$ if $v \in V_{\bar{0}}$ and $|v| = 1$ if $v \in V_{\bar{1}}$. Elements in $V_{\bar{0}}$ or $V_{\bar{1}}$ are called homogeneous. Whenever $|v|$ is written, it is to be understood that v is homogeneous.

A vertex superalgebra is a special case of a Γ-graded vertex algebra where Γ is $\mathbb{Z}_2$ and the fields have expansions in integral powers of z and z^{-1}. A more explicit and equivalent definition is the following.

Definition 2.6
A **vertex superalgebra** *is a $\mathbb{Z}_2$-graded vectorspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with an infinite number of products, written as $u_n v$ for $u, v \in V, n \in \mathbb{Z}$, satisfying the following axioms.*

- *The products respect the grading, i.e. $|u_n v| = |u| + |v|$.*
- *$u_n v = 0$ for n sufficiently large.*
- *There is an element $1 \in V_{\bar{0}}$, called vacuum, such that $v_n 1 = 0$ for $n \geq 0$ and $v_{-1}1 = v$.*
- *The Borcherds identity*

$$
\sum_{k\geq 0} \binom{m}{k} (u_{l+k}v)_{m+n-k}w =
$$
$$
\sum_{k\geq 0} \binom{l}{k} (-1)^k \{ u_{m+l-k}(v_{n+k}w) - (-1)^l (-1)^{|u||v|} v_{n+l-k}(u_{m+k}w) \}
$$

holds.

Sometimes vertex algebras carry symmetries, i.e. representations of Lie algebras.

Definition 2.7
Let V be a vertex algebra. An element $\omega \in V$ is called **Virasoro element of central charge** c *if it satisfies the following conditions.*

- *The operators $L_m = \omega_{m+1}$ give a representation of the* **Virasoro algebra** *of central charge c, i.e.*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n}c.$$

- *L_0 is diagonalizable on V.*
- *$D = L_{-1}$.*

2.2 Construction from rational lattices

The Fock space of the $N{=}2$-string moving on a torus is realized by a lattice vertex algebra, i.e. a vertex algebra constructed from a rational lattice. In this section we describe this construction.

First we recall some basic terms about rational lattices.

Definition 2.8
A **rational lattice** *is a free $\mathbb{Z}$-module L of finite rank with a nondegenerate symmetric rational bilinear form $(,) : L \times L \to \mathbb{Q}$.*
For a lattice L the **dual lattice** *L' is defined by*

$$L' = Hom(L, \mathbb{Z}) \cong \{x \in \mathbb{R}^n \,|(x,y) \in \mathbb{Z}\ \forall y \in L\}.$$

An **even lattice** *is a lattice L with*

$$x^2 = (x,x) \equiv 0 \bmod 2 \qquad \forall x \in L.$$

Definition 2.9
Let L be a rational lattice of rank n. Then we can diagonalize the bilinear form on $V = \mathbb{Q}\bigotimes_{\mathbb{Z}} L$. Let n^+ be the number of the basic vectors with positive norm, n^- the number of the basic vectors with negative norm. Then $n = n^+ + n^-$ and (n^+, n^-) is called the **signature** *of L.*

Definition 2.10
Let L be a rational lattice with basis $\{e_1, \ldots, e_n\}$. The matrix $A = ((e_i, e_j))_{i,j}$ is called **Gram-matrix**. *The determinant of the Gram-matrix is independent of the choice of the basis and is denoted by $det(L)$.*

Definition 2.11
An **integral lattice** *L is a lattice with integral bilinear form $(,) : L \times L \to \mathbb{Z}$.*
A rational lattice L is called **unimodular** *if $|det(L)| = 1$.*

Now we describe the construction of the lattice vertex algebra.
Let L be a rational lattice of finite rank and $L_0 \neq 0$ an even sublattice of L with $L \subset L_0'$.
Let

$$L = L_0 \cup L_1 \cup \ldots \cup L_n \text{ with } L_i = \delta_i + L_0$$

be the coset decomposition of L with respect to L_0.
Put

$$\Gamma = L/L_0 = \{\gamma_0, \gamma_1, \ldots, \gamma_n\},$$

where γ_i corresponds to L_i, and let g be an exponent of Γ.
Define

$$\Delta : \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$$

by

$$\Delta(\gamma_i, \gamma_j) = -(\delta_i, \delta_j) \bmod \mathbb{Z}.$$

This map is well-defined and bilinear mod$\mathbb{Z}$.
Let $\eta : \Gamma \times \Gamma \to \mathbb{C}^*$ be a bimultiplicative map with

$$\eta(\gamma_j, \gamma_j) = e^{i\pi(\delta_j, \delta_j)}. \tag{2.1}$$

Let $\{\alpha_1, \ldots, \alpha_m\}$ be a $\mathbb{Z}$-basis of L.
Define $\epsilon : L \times L \to \mathbb{C}^*$ by

$$\begin{aligned} \epsilon(\alpha_i, \alpha_j) &= a_{ij} & i \leq j, \\ \epsilon(\alpha_i, \alpha_j) &= B(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) & i > j, \end{aligned}$$

where $B : L \times L \to \mathbb{C}^*$ with

$$B(\alpha, \beta) = e^{-i\pi(\alpha,\beta)}\eta(\gamma_i, \gamma_j), \tag{2.2}$$

and with some $a_{ij} \in \mathbb{C}^*$ and extend to L by bimultiplicativity.
Then ϵ is a 2-cocycle, i.e.

$$\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$$

and

$$\epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma)$$

holds.

Let $h_L = L \bigotimes_{\mathbb{Z}} \mathbb{C}$.

Definition 2.12
Define the infinite dimensional **Heisenberg algebra**

$$\hat{h} = h_L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with products $[h_1(m), h_2(n)] = m\delta_{m+n}(h_1, h_2)c$ *and* $[h_1(m), c] = 0$*,*
where $h_i(m)$ *is a short form for* $h \otimes t^m$ *,* $h \in h_L$*.*

Then $\hat{h}^- = h \otimes t^{-1}\mathbb{C}[t^{-1}]$ is an abelian subalgebra of $\hat{h}$ and $S(\hat{h}^-)$ is the symmetric algebra of polynomials in $\hat{h}^-$.
Let $\mathbb{C}[L]$ be the group algebra of L with basis $\{e^\alpha | \alpha \in L\}$ and products $e^\alpha e^\beta = e^{\alpha+\beta}$.

Definition 2.13
We define the vector space V by

$$V = S(\hat{h}^-) \otimes \mathbb{C}[L].$$

V decomposes as

$$V = \bigoplus_{\gamma_i \in \Gamma} V_{\gamma_i}$$

where

$$V_{\gamma_i} = S(\hat{h}^-) \otimes \mathbb{C}[L_i].$$

Now we define the vertex operator for the element $e^\alpha \in \mathbb{C}[L]$ and the element $h(-n-1) \in S(\hat{h}^-)$.

Definition 2.14
We define the **vertex operator of** $e^\alpha \in \mathbb{C}[L]$ *as*

$$e^\alpha(z) = e^\alpha(z)^+ e^\alpha(z)^-$$

where

$$e^\alpha(z)^+ = e^\alpha c_\alpha exp\left(\sum_{m>0} \alpha(-m)\frac{z^m}{m}\right) = e^\alpha c_\alpha \sum_{m\geq 0} S_m(\alpha) z^m$$

and

$$e^\alpha(z)^- = z^{\alpha(0)} exp\left(-\sum_{m>0} \alpha(m)\frac{z^{-m}}{m}\right).$$

The linear operator c_α acts on $\mathbb{C}[L]$ as $c_\alpha e^\beta = \epsilon(\alpha,\beta)e^\beta$.

The $S_m(\alpha)$ are called **Schur polynomials**.

Definition 2.15
For $h(-n-1) \in S(\hat{h}^-), n \geq 0$, put

$$h(-n-1)(z) = \partial_z^{(n)} h(z)$$

with $h(z) = \sum_{n\in\mathbb{Z}} h(n) z^{-n-1}$ and $\partial_n^{(n)} = \frac{\partial_z}{n!}$.

Now we can define the vertex operator for an element $v \in V$.

Definition 2.16
The **bosonic normal ordering** $: h_1(n_1)\dots h_k(n_k) :$ *of Heisenberg generators is defined by putting all $h(n)$ with $n<0$ (creation operators) to the left of those with $n\geq 0$ (annihilation operators).*
We define

$$\begin{gathered}(h_1(-n_1-1)\dots h_k(-n_k-1)e^\alpha)(z) = \\ e^\alpha(z)^+ : h_1(-n_1-1)(z)\dots h_k(-n_k-1)(z) : e^\alpha(z)^- \end{gathered} \tag{2.3}$$

and extend this definition linearly to V to get a parity preserving state-field correspondence $V \to (\operatorname{End} V)[[z^{\frac{1}{g}}, z^{-\frac{1}{g}}]]$.

Theorem 2.17
With this structure V is a vertex algebra graded by Γ. The vacuum is given by $1\otimes e^0$.

Proof:
cf. [S1]. □

Proposition 2.18
The first 4 Schur polynomials are

$$\begin{aligned} S_0(\alpha) &= 1, \\ S_1(\alpha) &= \alpha(-1), \\ S_2(\alpha) &= \frac{1}{2}(\alpha(-1)^2+\alpha(-2)), \\ S_3(\alpha) &= \frac{1}{6}\alpha(-1)^3+\frac{1}{2}\alpha(-1)\alpha(-2)+\frac{1}{3}\alpha(-3). \end{aligned}$$

Proof:
We have

$$exp\left(\sum_{m>0}\alpha(-m)\frac{z^m}{m}\right) = S_m(\alpha)z^m$$

(Definition 2.14). Comparison of coefficients gives the assertion. □

Proposition 2.19
For $e^\alpha, e^\beta \in \mathbb{C}[L]$ we have

$$e^\alpha_n e^\beta = \epsilon(\alpha,\beta)S_{-n-1-(\alpha,\beta)}(\alpha)e^{\alpha+\beta}.$$

Proof:
For $m\geq 0$ we have $\alpha(m)e^\beta = \delta_m(\alpha,\beta)e^\beta$. Hence

$$\begin{aligned} & exp\left(-\sum_{m>0}\alpha(m)\frac{z^{-m}}{m}\right)e^\beta = \sum_{l=0}^{\infty}\frac{1}{l!}\left(-\sum\alpha(m)\frac{z^{-m}}{m}\right)^l e^\beta \\ = & \left(1-\sum_{m>0}\alpha(m)\frac{z^{-m}}{m}+\frac{1}{2}\left(-\sum_{m>0}\alpha(m)\frac{z^{-m}}{m}\right)^2+\dots\right)e^\beta = e^\beta. \end{aligned}$$

$$\alpha(0)e^{\beta} = (\alpha,\beta)e^{\beta}$$

implies

$$z^{\alpha(0)}e^{\beta} = z^{(\alpha,\beta)}e^{\beta}.$$

Hence

$$e^{\alpha}c_{\alpha}z^{\alpha(0)}e^{\beta} = \epsilon(\alpha,\beta)z^{(\alpha,\beta)}e^{\alpha+\beta}.$$

With this we have

$$\begin{aligned}
e^{\alpha}(z)e^{\beta} &= \sum_{n\in\mathbb{Z}} e^{\alpha}_{n}e^{\beta}z^{-n-1} \\
&= e^{\alpha}c_{\alpha}exp\left(\sum_{m>0}\alpha(-m)\frac{z^{m}}{m}\right)z^{\alpha(0)}\underbrace{exp\left(-\sum_{m>0}\alpha(m)\frac{z^{-m}}{m}\right)e^{\beta}}_{=e^{\beta}} \\
&= e^{\alpha}c_{\alpha}\left(\sum_{m\geq 0}S_{m}(\alpha)z^{m}\right)z^{\alpha(0)}e^{\beta} \\
&= \epsilon(\alpha,\beta)\sum_{m\geq 0}S_{m}(\alpha)z^{m+(\alpha,\beta)}e^{\alpha+\beta} \\
&= \epsilon(\alpha,\beta)\sum_{n=-\infty}^{-1-(\alpha,\beta)}S_{-n-1-(\alpha,\beta)}(\alpha)e^{\alpha+\beta}z^{-n-1} \\
&= \epsilon(\alpha,\beta)\sum_{n\in\mathbb{Z}}S_{-n-1-(\alpha,\beta)}(\alpha)e^{\alpha+\beta}z^{-n-1}.
\end{aligned}$$

This proves the proposition. □

The following result is helpful for calculations.

Proposition 2.20

For $k(-t)\in S(\hat{h}^{-})$ we have

$$k(-t)_{m} = \binom{-m+t-2}{t-1}k(m+1-t).$$

Proof:

$$\begin{aligned}
k(-t)(z) &= \sum_{m\in\mathbb{Z}} k(-t)_{m}z^{-m-1} = \partial_{z}^{(t-1)}k(z) \\
&= \sum_{n\in\mathbb{Z}} \partial_{z}^{(t-1)}k(n)z^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(t-1)!}(-n-1)\cdot(-n-2)\cdot\ldots\cdot(-n+1-t)k(n)z^{-n-t} \\
&= \sum_{m\in\mathbb{Z}} \frac{1}{(t-1)!}(-m-2+t)(-m-3+t) \\
&\qquad \cdot\ldots\cdot(-m)k(m+1-t)z^{-m-1}.
\end{aligned}$$

$$\Rightarrow\ k(-t)_m = \frac{1}{(t-1)!}(-m+t-2)(-m+t-3)\cdot\ldots\cdot(-m)k(m+1-t).$$

With

$$\begin{aligned}(-m+t-2)(-m+t-3)\cdot\ldots\cdot(-m) &= \frac{(-m+t-2)!}{(-m-1)!}\\ &= (t-1)!\binom{-m+t-2}{t-1}\end{aligned}$$

follows

$$k(-t)_m = \binom{-m+t-2}{t-1}k(m+1-t).$$

□

In particular we get for $t=1$

$$k(-1)_m = k(m). \tag{2.4}$$

2.3 Invariant bilinear forms

Here we describe some results on invariant bilinear forms on vertex superalgebras. They will be used to construct an invariant bilinear form on the vertex algebra of the N=2-string and on the corresponding Lie algebra. For more details confer [S1] and [S2].

Definition 2.21
Let V be a vertex superalgebra with Virasoro element. Suppose that L_1 acts locally nilpotent on V and that the eigenvalues of L_0 are all integral.
We define the **adjoint vertex operator** *of $a \in V$ with $L_0 a = ha$ by*

$$a(z)^* = \sum_{n\in\mathbb{Z}} a_n^* z^{-n-1}$$

where

$$a_n^* = (-1)^h \sum_{m\geq 0}\left(\frac{L_1^m}{m!}a\right)_{2h-n-m-2}.$$

Definition 2.22
A bilinear form $(,)$ on a vertex superalgebra V is called **invariant** *if*

$$(a_n b, c) = (-1)^{|a||b|}(b, a_n^* c).$$

Define $V_h := \{a \in V | L_0 a = ha\}$.

Theorem 2.23
The space of invariant bilinear forms on V is naturally isomorphic to the dual of V_0/L_1V_1.

Proof:
cf. [S1], [S2]. □

Chapter 3

Generalized Kac-Moody algebras

In this chapter we describe some results about generalized Kac-Moody algebras (cf. [B1], [B2]). The theory of generalized Kac-Moody algebras is a generalization of the theory of finite dimensional simple Lie algebras.
Let G be a finite dimensional simple Lie algebra over $\mathbb{C}$. Then G has a Cartan subalgebra H and an invariant bilinear form $(x, y) = tr(ad(x)ad(y))$ where $x, y \in G$. G decomposes into a sum over H and one-dimensional root spaces G_α, i.e.

$$G = H \oplus \bigoplus_{\alpha \in \Delta} G_\alpha.$$

We choose a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}, n = \dim H$. Then the symmetrized Cartan matrix $a_{ij} = ((\alpha_i, \alpha_j))_{i,j}$ has the following properties.

- $a_{ii} > 0$,
- $a_{ij} = a_{ji}$,
- $a_{ij} \leq$ for $i \neq j$,
- $2a_{ij}/a_{ii} \in \mathbb{Z}$.

In this way we can associate a matrix to G. G is completely determined by the symmetrized Cartan matrix because of the Theorem of Serre which states that a Lie algebra with generators e_i, f_i, h_i and relations

- $[e_i, f_j] = \delta_{ij} h_i$,
- $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$,
- $(ad\, e_i)^{1-2a_{ij}/a_{ii}} e_j = (ad\, f_i)^{1-2a_{ij}/a_{ii}} f_j = 0$, $i \neq j$,

is isomorphic to G. Generalized Kac-Moody algebras are now obtained by applying Serre's construction to matrices which might have diagonal entries smaller or equal to zero. That means generalized Kac-Moody algebras might have imaginary simple roots. In the first section we recall some basic definitions and results about generalized Kac-Moody algebras. In the second section we describe the results that will be needed to show that the Lie algebra of the N=2-string is not a generalized Kac-Moody algebra.

3.1 Basic definitions

We state some basic definitions and results about generalized Kac-Moody algebras from [B1].

Definition 3.1
A **generalized Kac-Moody algebra** *G will be constructed from the following objects.*

- *A real vector space H with a symmetric bilinear inner product $(,)$.*
- *A set of elements h_i of H indexed by a countable set I, such that $(h_i, h_j) \leq 0$ if $i \neq j$ and $2\frac{(h_i,h_j)}{(h_i,h_i)}$ is an integer if (h_i, h_i) is positive.*

We write a_{ij} for (h_i, h_j) and call the matrix a_{ij} the **symmetrized Cartan matrix** *of G.*
The **generalized Kac-Moody algebra** *G associated to these object is defined to be the Lie algebra generated by H and elements e_i and f_i for i in I with the following relations.*

- *The image of H in G is commutative.*
- *If h is in H then $[h, e_i] = (h, h_i)e_i$ and $[h, f_i] = -(h, h_i)f_i$.*
- *$[e_i, f_j] = h_i$ if $i = j$, $[e_i, f_j] = 0$ if $i \neq j$.*
- *$(ad\ e_i)^{1-2a_{ij}/a_{ii}}e_j = 0$ and $(ad\ f_i)^{1-2a_{ij}/a_{ii}}f_j = 0$ if $a_{ii} > 0$.*
- *$[e_i, e_j] = [f_i, f_j] = 0$ if $a_{ij} = 0$.*

H is called the **Cartan subalgebra** *of G.*

Note that the h_i need not be linearly independent.

Definition 3.2
The **root lattice Q** *is defined to be the free abelian group generated by elements α_i for i in I, i.e. $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$, and Q has a real valued bilinear form defined by $(\alpha_i, \alpha_j) = a_{ij}$. The Lie algebra G is graded by Q by letting H have degree 0, e_i have degree α_i and f_i have degree $-\alpha_i$.*

Hence we can write

$$G = \bigoplus_{\alpha \in Q} G_\alpha = H \oplus \bigoplus_{\alpha \in Q \setminus \{0\}} G_\alpha.$$

We call the nonvanishing G_α with $\alpha \neq 0$ **root spaces**.
Of course we have

$$[G_\alpha, G_\beta] \subset G_{\alpha+\beta} \text{ for } \alpha, \beta \in Q.$$

Definition 3.3
$\alpha \in Q \setminus \{0\}$ is called **root** $:\Leftrightarrow$ *$\exists x \in G, x \neq 0, : deg\, x = \alpha$.*
Let Δ be the set of the roots.
$\alpha \in \Delta$ is called **simple** $:\Leftrightarrow$ *α is one of the α_i that generates Q.*
$\alpha \in \Delta$ is called **real** $:\Leftrightarrow$ *$(\alpha, \alpha) > 0$.*
$\alpha \in \Delta$ is called **imaginary** $:\Leftrightarrow$ *$(\alpha, \alpha) \leq 0$.*
Let Δ^{re} (resp. Δ^{im}) be the set of real roots (resp. imaginary roots).
Let Π denote the set of simple roots and Π^{re} the set of real simple roots.

We define N_+ as the subalgebra generated by the e_i, N_- as the subalgebra generated by the f_i.
For $\alpha_i \in \Pi$ we define

$$\alpha_i(h) := (h, h_i)$$

for $h \in H$. We also define for $\alpha \in Q$

$$G_\alpha = \{x \in G | [h, x] = \alpha(h)x \ \forall h \in H\}.$$

Let $Q_+ = \bigoplus \mathbb{Z}_+ \alpha_i$. Then

Proposition 3.4
G decomposes as follows.

$$G = N_+ \oplus H \oplus N_-$$

and

$$N_\pm = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} G_{\pm\alpha}.$$

Proof:
Analogous to the proof in [K1]. □

Definition 3.5
$\alpha \in \Delta$ is called **positive** $:\Leftrightarrow$ *$\alpha = \sum k_i \alpha_i,\ k_i \geq 0\ \forall i$.*
$\alpha \in \Delta$ is called **negative** $:\Leftrightarrow$ *$-\alpha$ is positive.*
Let $\Delta^+ (\Delta^-)$ be the set of positive (resp. negative) roots.

The Definition implies

$$-\Delta^+ = \Delta^-. \tag{3.1}$$

For the set of the roots we have

Proposition 3.6

$$\Delta = \Delta^+ \sqcup \Delta^-.$$

Proof:
This follows from Proposition 3.4. $\square$

Definition 3.7
For $\alpha = \sum k_i \alpha_i \in Q$ *we define the* **support of** $\alpha : supp(\alpha) = \{\alpha_i | k_i \neq 0\}$. *If* $\alpha = \alpha_i$ *we say that* $supp(\alpha)$ *is connected. If* α *is not a simple root we say that* $supp(\alpha)$ *is connected if for each* $\alpha_i \in supp(\alpha)$ *there exists an* $\alpha_j \in supp(\alpha)$ *with* $i \neq j$ *and* $(\alpha_i, \alpha_j) \neq 0$.

Proposition 3.8
Let $I_1, I_2 \subset \{1, \ldots, n\}$ *be disjoint subsets such that* $a_{ij} = a_{ji} = 0$ *if* $i \in I_1, j \in I_2$. *Let* $\alpha_s = \sum_{i \in I_s} k_i^{(s)} \alpha_i \in \Delta$ $(k_i^{(s)} \in \mathbb{Z}, s \in \{1, 2\})$. *Suppose that* $\alpha = \alpha_1 + \alpha_2 \in \Delta$. *Then* $\alpha_1 = 0$ *or* $\alpha_2 = 0$.

Proof:
Let $i \in I_1, j \in I_2$. Then $a_{ij} = a_{ji} = 0$ and $[h_i, e_j] = [h_j, e_i] = [e_i, f_j] = [e_j, f_i] = [e_i, e_j] = [f_i, f_j] = 0$. Hence $G^{(1)}$ and $G^{(2)}$, where $G^{(s)}$ denotes the subalgebra generated by e_i and f_i with $i \in I_s$, commute. Since G_α lies in the subalgebra generated by $G^{(1)}$ and $G^{(2)}$, we deduce that G_α lies either in $G^{(1)}$ or in $G^{(2)}$. $\square$

Proposition 3.9
For all $\alpha \in \Delta : supp(\alpha)$ *is connected.*

Proof:
This follows from Proposition 3.8. $\square$

3.2 The root system

In this section we state the result that we need to show that the Lie algebra of the N=2-string is not a generalized Kac-Moody algebra (cf. [B1]).

Lemma 3.10
For $\alpha \in \Delta^+$ *we have* $(\alpha, \alpha_i) \leq 0$ *for all* $\alpha_i \in \Pi^{im}$.

Proof:
Let $\alpha_i \in \Pi^{im}$.
We have $\alpha = \sum k_j \alpha_j$, $k_j \geq 0 \; \forall j$. Hence

$$(\alpha, \alpha_i) = k_i \underbrace{(\alpha_i, \alpha_i)}_{\leq 0} + \sum_{j \neq i} k_j \underbrace{(\alpha_j, \alpha_i)}_{\leq 0} \leq 0.$$

This proves the lemma. □

Proposition 3.11
Let $\Delta = \Delta^{im}$. For $\alpha \in \Delta^+$ we have: If $\alpha^2 = 0$ then $\alpha \in \Pi$.

Proof:
$\alpha \in \Delta^+$, hence $\alpha = \sum k_i \alpha_i$, $k_i \geq 0$, $\alpha_i \in \Pi = \Pi^{im}$. We have $k_i > 0 \; \forall \alpha_i \in supp(\alpha)$. Then

$$0 = (\alpha, \alpha) = \sum k_i \underbrace{(\alpha, \alpha_i)}_{\leq 0 \text{ , } Lemma \text{ 3.10}} \leq 0.$$

$$\Rightarrow \; (\alpha, \alpha_i) = 0 \; \forall \alpha_i \in supp(\alpha).$$

$$0 = (\alpha, \alpha_i) = k_i \underbrace{(\alpha_i, \alpha_i)}_{\leq 0} + \sum_{j \neq i} k_j \underbrace{(\alpha_j, \alpha_i)}_{\leq 0}.$$

$$\Rightarrow \; (\alpha_j, \alpha_i) = 0 \quad \forall \alpha_i, \alpha_j \in supp(\alpha).$$

As $supp(\alpha)$ is connected, there exists i with $\alpha = \alpha_i$, hence $\alpha \in \Pi$. □

Chapter 4

The vertex algebra of the $N = 2$-string

In this chapter we construct a vertex algebra that represents the Fock space of the chiral $N = 2$ string moving on a torus. Thereto we use the lattice construction described in section 2.2.

The N=2-string has critical dimension 4 , i.e. it can be defined consistently only in 4 dimensions and has 4 bosonic and 4 fermionic degrees of freedom. We represent the bosonic degrees by a 4 dimensional even lattice with signature $(2, 2)$ and the fermionic degrees in bosonized form by the weight lattice of the Lie algebra D_2. Because of the bosonized form the fermions give two degrees of freedom, together with the four bosonic degrees of freedom we get six matter degrees of freedom. In order to obtain a representation of the $N = 2$-superconformal algebra of central charge zero we have to introduce six ghost coordinates. Two of them carry a negative metric and must be described together with the fermionic lattice.

Hence we need a rational lattice L of signature $(8, 4)$ to describe the N=2-string moving on a torus. The Fock space of the string is then given by the vertex algebra of L.

4.1 The lattice L

We define

$$L = L^X \bigoplus L^{\psi,\phi,\tilde{\phi}} \bigoplus L^{\chi,\sigma,\tilde{\chi},\tilde{\sigma}}$$

where

$$\begin{aligned} L^X \quad = \quad & \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^{2,2} \,|(\text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \frac{1}{2}) \text{ and} \\ & x_1 + x_2 - x_3 - x_4 \equiv 0 \bmod 2\} \end{aligned}$$

is the unique 4 dimensional even unimodular lattice of signature $(2,2)$ and describes the 4 bosonic degrees of freedom,

$$L^{\psi,\phi,\tilde{\phi}} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^{2,2} |\text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \frac{1}{2}\}$$

describes the two fermionic degrees of freedom and the two ghost degrees of freedom that carry a negative metric and

$$L^{\chi,\sigma,\tilde{\chi},\tilde{\sigma}} = \mathbb{Z}^4$$

describes four ghost degrees of freedom.

Of course we have

$$L^X \cong I\!I_{2,2} \cong I\!I_{1,1} \oplus I\!I_{1,1}.$$

Now we want to decompose the lattice L defined above with respect to an even sublattice of L. First we decompose the lattice $L^{\psi,\phi,\tilde{\phi}}$. Thereto we need an even sublattice of $L^{\psi,\phi,\tilde{\phi}}$.

Proposition 4.1
$L_0^{\psi,\phi,\tilde{\phi}} := \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^{2,2} | x_i \in \mathbb{Z} \wedge x_1 + x_2 - x_3 - x_4 \equiv 0 \bmod 2\}$ *is an even sublattice of* $L^{\psi,\phi,\tilde{\phi}}$.

Proof:
Squaring $x_1 + x_2 - x_3 - x_4 = 2k$ we get
$x_4^2 = x_1^2 + 2x_1x_2 - 2x_1x_3 - 4kx_1 + x_2^2 - 2x_2x_3 - 4kx_2 + x_3^2 + 4kx_3 + 4k^2$
so that

$$\begin{aligned}(x,x) &= x_1^2 + x_2^2 - x_3^2 - x_4^2 \\ &= 2(-x_3^2 - x_1x_2 + x_1x_3 + x_2x_3 + 2kx_1 + 2kx_2 - 2kx_3 - 2k^2) \in 2\mathbb{Z}.\end{aligned}$$

□

Proposition 4.2

$$L^{\psi,\phi,\tilde{\phi}}/L_0^{\psi,\phi,\tilde{\phi}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof:
Let $y = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) \in L^{\psi,\phi,\tilde{\phi}}$. Because $\det(L^{\psi,\phi,\tilde{\phi}}) = \frac{1}{4}$ and $\det(L_0^{\psi,\phi,\tilde{\phi}}) = 1$ the order of $L^{\psi,\phi,\tilde{\phi}}/L_0^{\psi,\phi,\tilde{\phi}}$ is four. Assume $L^{\psi,\phi,\tilde{\phi}}/L_0^{\psi,\phi,\tilde{\phi}} = \mathbb{Z}/4\mathbb{Z}$. Then there exists $z \in L^{\psi,\phi,\tilde{\phi}}$ with

$$y = 2z + x \text{ for } \text{ some } x \in L_0^{\psi,\phi,\tilde{\phi}} \quad (*),$$

but $z = (z_1,z_2,z_3,z_4)$ with $2z_i \in \mathbb{Z}$, $i \in \{1,\ldots,4\}$, i.e. the coefficients on the right side of $(*)$ are integers in contradiction to the assumption. □

We write $\Gamma(L^{\psi,\phi,\tilde{\phi}}) := \{0, V, S, C\} := L^{\psi,\phi,\tilde{\phi}}/L_0^{\psi,\phi,\tilde{\phi}}$.

It is easy to see that

Proposition 4.3
We have the coset decomposition

$$L^{\psi,\phi,\tilde{\phi}} = L_0^{\psi,\phi,\tilde{\phi}} \cup L_V^{\psi,\phi,\tilde{\phi}} \cup L_S^{\psi,\phi,\tilde{\phi}} \cup L_C^{\psi,\phi,\tilde{\phi}}$$

with

$$\begin{aligned}
L_0^{\psi,\phi,\tilde{\phi}} &= \{x \in L^{\psi,\phi,\tilde{\phi}} | x_i \in \mathbb{Z} \ \wedge \ x_1 + x_2 - x_3 - x_4 \equiv 0 \bmod 2\}, \\
L_V^{\psi,\phi,\tilde{\phi}} &= (0,0,0,1) + L_0^{\psi,\phi,\tilde{\phi}}, \\
L_S^{\psi,\phi,\tilde{\phi}} &= (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) + L_0^{\psi,\phi,\tilde{\phi}}, \\
L_C^{\psi,\phi,\tilde{\phi}} &= (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}) + L_0^{\psi,\phi,\tilde{\phi}}.
\end{aligned}$$

We can write the conjugacy classes of $L^{\psi,\phi,\tilde{\phi}}$ also in terms of the classes of D_2'. The lattice D_n' has the decomposition $0(D_n') \cup V(D_n') \cup S(D_n') \cup C(D_n')$ where $0(D_n')$ is the even lattice with elements $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $\sum x_i$ is even and

$$\begin{aligned}
V(D_n') &= (0,\ldots,0,1) + 0(D_n'), \\
S(D_n') &= (\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}) + 0(D_n'), \\
C(D_n') &= (\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2}) + 0(D_n').
\end{aligned}$$

The quotient of D_n' by $0(D_n')$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ for n even and $\mathbb{Z}_4$ for n odd. With this notation we can write

$L^{\psi,\phi,\tilde{\phi}}$	(D_2', D_2')
0	$(0,0) \cup (V,V)$
V	$(0,V) \cup (V,0)$
S	$(S,S) \cup (C,C)$
C	$(C,S) \cup (C,S)$

Because in the vertex algebra constructed on the lattice L there are states that represents particles moving faster than light (tachyons) we construct a vertex algebra on a sublattice of L. For this we need the following sublattice of $L^{\psi,\phi,\tilde{\phi}}$.

Proposition 4.4
The sublattice $E_{2,2} = L_0^{\psi,\phi,\tilde{\phi}} \cup L_S^{\psi,\phi,\tilde{\phi}}$ *is an even unimodular lattice.*

Proof:

$$v_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad v_2 = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$
$$v_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \quad v_4 = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$$

is a $\mathbb{Z}$-basis of $E_{2,2}$ with $v_1 v_2 = v_2 v_1 = v_3 v_4 = v_4 v_3 = -1$ and all other scalar products vanish. Hence the gram-matrix of $E_{2,2}$ is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $E_{2,2}$ is even and unimodular. □

The lattice $L^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}} = \mathbb{Z}^2$ can be decomposed as $L^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}} = L_0^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}} \cup L_1^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}}$ into the elements of even and odd norm. The quotient $\Gamma(L^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}}) = \{0,1\}$ is isomorphic to $\mathbb{Z}_2$. The sum

$$L^X \bigoplus L_0^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}} \bigoplus L_0^{\psi,\phi,\tilde{\phi}}$$

is an even sublattice of L. The quotient $\Gamma(L)$ of L by this sublattice is $\Gamma(L^{\psi,\phi,\tilde{\phi}}) \times \Gamma(L^{\chi,\tilde{\chi},\sigma,\tilde{\sigma}})$.

Now we define a basis of L.

Definition 4.5
We choose

$$\phi^1 = (1,0,0,0), \phi^2 = (0,1,0,0), s^3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), s^4 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$$

as a $\mathbb{Z}$*-basis of* $L^{\psi,\phi,\tilde{\phi}}$*,*

$$\chi = (1,0,0,0), \sigma = (0,1,0,0), \tilde{\chi} = (0,0,1,0), \tilde{\sigma} = (0,0,0,1)$$

as a $\mathbb{Z}$*-basis of* $L^{\chi,\sigma,\tilde{\chi},\tilde{\sigma}}$ *and let* $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ *be any basis of* L^X*. Then* $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \phi^1, \phi^2, s^3, s^4, \chi, \sigma, \tilde{\chi}, \tilde{\sigma}\}$ *is an ordered basis of* L*.*

4.2 Construction of the vertex algebra

Now we define the vertex algebra of the $N\!=\!2$ string. Therefore we need a 2-cocycle ϵ as described in section 2.2. In the following we denote the above basis of L by $\{\alpha^1, \dots, \alpha^{12}\}$.

Definition 4.6
We define

$$\epsilon : \begin{array}{ccc} L \times L & \to & \mathbb{C}^* \\ (\alpha^i, \alpha^j) & \mapsto & \epsilon(\alpha^i, \alpha^j) := a_{ij} \end{array}$$

with

$$a_{ij} = \begin{cases} (-1)^{\frac{1}{2}(\alpha^i, \alpha^j)} & if \quad i,j \in \{1, \dots, 4\} \wedge i = j, \\ 1 & if \quad i,j \in \{5, \dots, 12\} \wedge i \leq j, \\ B(\alpha^i, \alpha^j)\epsilon(\alpha^j, \alpha^i) & if \quad i,j \in \{5, \dots, 12\} \wedge i > j, \end{cases}$$

on the basis of L.

The following lemma holds for the 2-cocycle ϵ. It will be needed later to calculate $e^\alpha_n e^\beta = \epsilon(\alpha, \beta) S_{-n-1-(\alpha,\beta)}(\alpha) e^{\alpha+\beta}$, $e^\alpha, e^\beta \in \mathbb{C}[L]$.
It is easy to prove that

Lemma 4.7

$$\begin{array}{rcl} 1 & = & \epsilon(\alpha, \beta)\epsilon(-\alpha, \beta), \\ 1 & = & \epsilon(\alpha, -\beta)\epsilon(\alpha, \beta), \\ 1 & = & \epsilon(-\alpha, -\beta)\epsilon(\alpha, -\beta). \end{array}$$

Hence

$$\begin{array}{lll} \text{If } \epsilon(\alpha, \alpha) = 1 & \text{then} & \epsilon(-\alpha, \alpha) = \epsilon(\alpha, -\alpha) = \epsilon(-\alpha, -\alpha) = 1. \\ \text{If } \epsilon(\alpha, \beta) = 1 & \text{then} & \epsilon(\alpha, -\beta) = \epsilon(-\alpha, \beta) = \epsilon(-\alpha, -\beta) = 1. \end{array}$$

To construct a vertex algebra from L we need a bilinear map

$$\eta : \Gamma(L) \times \Gamma(L) \to \mathbb{C}^* .$$

There is a number of such maps but we need a special one to achieve that the modes of the physical bosons and fermions that we will define in the next chapter have the correct commutator and anticommutator properties. The following definition of $\eta : \Gamma(L) \times \Gamma(L) \to \mathbb{C}^*$ turns out to be appropriate:

	$(0,0)$	$(V,0)$	$(S,0)$	$(C,0)$	$(0,1)$	$(V,1)$	$(S,1)$	$(C,1)$
$(0,0)$	1	1	1	1	1	1	1	1
$(V,0)$	1	-1	y	$-y$	-1	1	$-y$	y
$(S,0)$	1	$-y$	1	$-y$	y	-1	y	-1
$(C,0)$	1	y	y	1	$-y$	-1	-1	$-y$
$(0,1)$	1	-1	y	$-y$	-1	1	$-y$	y
$(V,1)$	1	1	1	1	1	1	1	1
$(S,1)$	1	y	y	1	$-y$	-1	-1	$-y$
$(C,1)$	1	$-y$	1	$-y$	y	-1	y	-1

$y \in \{\pm 1\}$.

The column denotes the first and the row the second argument of η. Note that η is not symmetric.

We define the vertex algebra V of the compactified N=2-string as the vertex algebra constructed from the lattice L with η and ϵ as above.

$$V = V_{(0,0)} \oplus V_{(V,0)} \oplus V_{(S,0)} \oplus V_{(C,0)} \oplus V_{(0,1)} \oplus V_{(V,1)} \oplus V_{(S,1)} \oplus V_{(C,1)}.$$

V can be decomposed into the following two sectors.
Neveu-Schwarz sector

$$V^{NS} = V_{(0,0)} \oplus V_{(V,0)} \oplus V_{(0,1)} \oplus V_{(V,1)}.$$

Ramond-sector

$$V^{R} = V_{(S,0)} \oplus V_{(C,0)} \oplus V_{(S,1)} \oplus V_{(C,1)}.$$

As mentioned above there are states in V that represent tachyons. To avoid these particles we define

$$V^{GSO} = V_{(0,0)} \oplus V_{(S,1)} \oplus V_{(0,1)} \oplus V_{(S,0)}.$$

This vector space is a vertex superalgebra with even part

$$V_{(0,0)} \oplus V_{(S,1)}$$

and odd part

$$V_{(0,1)} \oplus V_{(S,0)}$$

and describes the Fock space of a so-called GSO-projected N=2 string.

Definition 4.8
*Let $v \in V$. We call v_n the n-**mode** of v.*

Lemma 4.9
Let $v \in V_{\gamma_v}$. The possible nonvanishing n-modes v_n acting on $w \in V_{\gamma_w}$ depend on γ_v and γ_w. The corresponding n are given in the following table.

	$(0,0)$	$(V,0)$	$(0,1)$	$(V,1)$	$(S,0)$	$(C,0)$	$(S,1)$	$(C,1)$
$(0,0)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$(V,0)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$
$(0,1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$(V,1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$
$(S,0)$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$
$(C,0)$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$
$(S,1)$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$
$(C,1)$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$

Proof:
By Definition 2.1 the n of the nonvanishing n-modes v_n acting on w lie in $\mathbb{Z}+\Delta(\gamma_v,\gamma_w) = \mathbb{Z}-(\delta_v,\delta_w) \bmod \mathbb{Z}$. $\square$

Chapter 5

The symmetries of the compactified $N=2$-string

In this chapter we show that V carries a representation of the $N=2$-extension of the Virasoro algebra and use it to construct the BRST-operator. Thereto we define in the first section of this chapter a matter part representation of central charge 6 and in the second section a ghost part representation of central charge -6 so that they sum to a representation of central charge 0. In the third section we define the BRST-operator Q and state some properties of Q.

5.1 Matter part

In this section we define elements $\omega^M, \tau^{M\pm}, j^M \in V$ whose modes generate a representation of the $N=2$-superconformal constraint algebra with central charge 6. Thereto we define bosonic coordinates $z^{\pm\mu}$ and real fermions $\psi^{\pm\mu}, \mu \in \{\pm\}$.

Definition 5.1
Let $\{z^{++}, z^{+-}, z^{-+}, z^{--}\}$ be a basis of $\mathbb{C}\bigotimes_{\mathbb{Z}} L^X$ with

$$(z^{\pm\mu}, z^{\mp\nu}) = -2\eta^{\mu\nu} \quad (\mu, \nu \in \{\pm\}),$$

i.e.

$$(z^{+\mu}, z^{+\nu}) = (z^{-\mu}, z^{-\nu}) = 0,$$
$$(z^{+\mu}, z^{-\nu}) = -2\eta^{\mu\nu} = (z^{-\mu}, z^{+\nu}) = -2\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the diagonal matrix with entries $\eta^{++} = -1, \eta^{--} = 1$. We call the $z^{\pm\mu}$ the **bosonic coordinates**.

Note we use Einsteins summation convention, that means,

$$z^{+\mu}(-1)z^{-}{}_{\mu}(-1)$$

is a short form for

$$\begin{aligned}\sum_{\mu\in\{\pm\}} z^{+\mu}(-1)z^{-}{}_{\mu}(-1) &= \sum_{\mu,\nu\in\{\pm\}} \eta_{\mu\nu} z^{+\mu}(-1)z^{-\nu}(-1) \\ &= -z^{++}(-1)z^{-+}(-1) + z^{+-}(-1)z^{--}(-1).\end{aligned}$$

Lemma 5.2
We write $\alpha, \beta \in L^X$ with respect to the basis $\{z^{++}, z^{+-}, z^{-+}, z^{--}\}$.

$$\begin{aligned}\alpha = k^{-}{}_{\mu} z^{+\mu} + k^{+}{}_{\mu} z^{-\mu} &= -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--}, \\ \beta = t^{-}{}_{\mu} z^{+\mu} + t^{+}{}_{\mu} z^{-\mu} &= -t^{-+}z^{++} + t^{--}z^{+-} - t^{++}z^{-+} + t^{+-}z^{--}.\end{aligned}$$

Then we have

$$(\alpha, \beta) = 2(k^{++}t^{-+} + k^{-+}t^{++} - k^{+-}t^{--} - k^{--}t^{+-}).$$

Proof:
This follows from Definition 5.1. □

For the bosonic coordinates we have the following operator product expansion.

Proposition 5.3
For the bosonic coordinates we have the following operator product expansion

$$z^{\pm\mu}(-1)(z)z^{\mp\nu}(-1) = -2\eta^{\mu\nu}z^{-2} + \dots, \qquad \mu, \nu \in \{\pm\}.$$

This is a short form for

$$\begin{aligned} z^{+\mu}(-1)(z)z^{-\nu}(-1) &= -2\eta^{\mu\nu}z^{-2} + \dots, \qquad \mu, \nu \in \{\pm\}, \\ z^{-\mu}(-1)(z)z^{+\nu}(-1) &= -2\eta^{\mu\nu}z^{-2} + \dots, \qquad \mu, \nu \in \{\pm\}, \\ z^{+\mu}(-1)(z)z^{+\nu}(-1) &= 0, \qquad \mu, \nu \in \{\pm\}, \\ z^{-\mu}(-1)(z)z^{-\nu}(-1) &= 0, \qquad \mu, \nu \in \{\pm\}.\end{aligned}$$

Proof:
For $k \geq 0$ we have

$$\begin{aligned} z^{\pm\mu}(-1)_k z^{\mp\nu}(-1) &= z^{\pm\mu}(k)z^{\mp\nu}(-1) = k\delta_{k-1}(z^{\pm\mu}, z^{\mp\nu}) \\ &= -2\eta^{\mu\nu}\delta_{k-1}.\end{aligned}$$

□

To define the real fermions we first have to define complex fermions.

Definition 5.4
Define **complex fermions** $\Psi^{\pm i} \in V$ *by*

$$\Psi^{\pm i} = e^{\pm \phi^i}, \qquad i \in \{1, 2\}.$$

The complex fermions are elements of $V_{(V,0)}$, so that their vertex operators have integral expansions when acting on the Neveu-Schwarz sector and half-integral expansions when acting on the Ramond sector (Lemma 4.9).

Lemma 5.5

$$\epsilon(\phi^2, \phi^1) = -1.$$

Proof:
With (2.2) follows:

$$\epsilon(\phi^2, \phi^1) = e^{-i\pi(\phi^2,\phi^1)}\eta((V,0),(V,0)) = -1.$$

□

The following proposition is useful for calculations.

Proposition 5.6
We have for $k \geq 0$*.*

$$\Psi^i_k \Psi^i = 0, \quad i \in \{-2, -1, 1, 2\},$$

$$\Psi^{-i}_k \Psi^i = \Psi^i_k \Psi^{-i} = \delta_k, \quad i \in \{1, 2\},$$

$$\Psi^i_k \Psi^j = 0, \quad i \neq j, \quad i, j \in \{-2, -1, 1, 2\}.$$

We have

$$\Psi^{-i}_{-1} \Psi^i = -\phi^i(-1), \quad \Psi^i_{-1} \Psi^{-i} = \phi^i(-1), \quad i \in \{1, 2\},$$

$$\Psi^i_{-1} \Psi^j = 1 \ , \ \Psi^j_{-1} \Psi^i = -1 \ , \ i \in \{-1, 1\}, j \in \{-2, 2\}$$

and

$$\Psi^i_{-2} \Psi^i = 1, \quad i \in \{-2, -1, 1, 2\},$$

$$\Psi^i_{-2} \Psi^{-i} = \frac{1}{2}\phi^i(-1)\phi^i(-1) + \frac{1}{2}\phi^i(-2) \ , \ i \in \{1, 2\},$$

$$\Psi^{-i}_{-2} \Psi^i = \frac{1}{2}\phi^i(-1)\phi^i(-1) - \frac{1}{2}\phi^i(-2) \ , \ i \in \{1, 2\},$$

$$\Psi^1_{-2} \Psi^{\pm 2} = \phi^1(-1)e^{\phi^1 \pm \phi^2}, \quad \Psi^{-1}_{-2} \Psi^{\pm 2} = -\phi^1(-1)e^{-\phi^1 \pm \phi^2}$$

$$\Psi^2_{-2} \Psi^{\pm 1} = -\phi^2(-1)e^{\phi^2 \pm \phi^1}, \quad \Psi^{-2}_{-2} \Psi^{\pm 1} = \phi^2(-1)e^{-\phi^2 \pm \phi^1}.$$

Proof:
This follows from Proposition 2.19. □

Definition 5.7
We define the **real fermions** $\psi^{\pm\mu} \in V, \mu \in \{\pm\}$, *by*

$$\begin{aligned}
\psi^{++} &= -\Psi^{-1} - i\Psi^{-2}, \\
\psi^{-+} &= -\Psi^{1} + i\Psi^{2}, \\
\psi^{+-} &= -\Psi^{-1} + i\Psi^{-2}, \\
\psi^{--} &= \Psi^{1} + i\Psi^{2}.
\end{aligned}$$

The real fermions have the following commutation relation.

Proposition 5.8

$$\{\psi^{\pm\mu}_m, \psi^{\mp\nu}_n\} = -2\eta^{\mu\nu}\delta_{m+n+1}$$

where m and n are either both integral or both half-integral depending on whether the ψ's act on the Neveu-Schwarz or Ramond sector.

Proof:
By Proposition 5.6 we have the following operator product expansion for the real fermions

$$\psi^{\pm\mu}(z)\psi^{\mp\nu} = -2\eta^{\mu\nu}z^{-1} + \ldots, \qquad \mu, \nu \in \{\pm\}.$$

Using the commutator formula (Proposition 2.2) this implies the assertion.□

Now we can define the element $\omega^M \in V$.

Definition 5.9
The **energy momentum tensor** *for the matter sector $\omega^M \in V$ is*

$$\begin{aligned}
\omega^M &= -\frac{1}{4}z^{+\mu}(-1)z^{-}{}_{\mu}(-1) - \frac{1}{4}z^{-\mu}(-1)z^{+}{}_{\mu}(-1) \\
&\quad -\frac{1}{4}(D\psi^{+\mu})_{-1}\psi^{-}{}_{\mu} - \frac{1}{4}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}.
\end{aligned}$$

Lemma 5.10
We can also write the energy momentum tensor as

$$\begin{aligned}
\omega^M &= -\frac{1}{4}z^{+\mu}(-1)z^{-}{}_{\mu}(-1) - \frac{1}{4}z^{-\mu}(-1)z^{+}{}_{\mu}(-1) \\
&\quad +\frac{1}{2}\phi^1(-1)_{-1}\phi^1(-1) + \frac{1}{2}\phi^2(-1)_{-1}\phi^2(-1).
\end{aligned}$$

Proof:
By Proposition 5.6 we have

$$
\begin{aligned}
&-\frac{1}{4}(D\psi^{+\mu})_{-1}\psi^{-}{}_{\mu} - \frac{1}{4}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu} = \\
&\quad -\frac{1}{4}\Big(-2\phi^1(-1)\phi^1(-1) - 2\phi^2(-1)\phi^2(-1) + i\phi^1(-1)e^{\phi^1-\phi^2} \\
&\quad -i\phi^1(-1)e^{\phi^1-\phi^2} - i\phi^2(-1)e^{\phi^2-\phi^1} + i\phi^2(-1)e^{\phi^2-\phi^1} + i\phi^1(-1)e^{-\phi^1+\phi^2} \\
&\quad -i\phi^1(-1)e^{-\phi^1+\phi^2} - i\phi^2(-1)e^{\phi^1-\phi^2} + i\phi^2(-1)e^{\phi^1-\phi^2}\Big) \\
&= \quad \frac{1}{2}\phi^1(-1)\phi^1(-1) + \frac{1}{2}\phi^2(-1)\phi^2(-1).
\end{aligned}
$$

□

We can easily calculate the operator $L^M_m = \omega^M_{m+1}$.

Proposition 5.11
We have

$$
\begin{aligned}
L^M_m \quad = \quad & -\frac{1}{4}\epsilon\{z^{+\mu}\left(\frac{m}{2}\right)z^{-}{}_{\mu}\left(\frac{m}{2}\right) + z^{-\mu}\left(\frac{m}{2}\right)z^{+}{}_{\mu}\left(\frac{m}{2}\right) - 2\phi^1\left(\frac{m}{2}\right)\phi^1\left(\frac{m}{2}\right) \\
& -2\phi^2\left(\frac{m}{2}\right)\phi^2\left(\frac{m}{2}\right)\} - \frac{1}{2}\sum_{k>\frac{m}{2}}\{z^{-}{}_{\mu}(m-k)z^{+\mu}(k) \\
& +z^{+}{}_{\mu}(m-k)z^{-\mu}(k) \\
& -2\phi^1(m-k)\phi^1(k) - 2\phi^2(m-k)\phi^2(k)\}
\end{aligned}
$$

with

$$
\epsilon = \begin{cases} 1 & for \quad m \quad even, \\ 0 & for \quad m \quad odd. \end{cases}
$$

Proof:

By Definition 2.16 we have for $a = z^{+\mu}(-1)z^{-}_{\mu}(-1)$

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} w_m z^{-m-1} \quad &= \quad w(z) \overset{(2.3)}{=} : \underbrace{z^{+\mu}(-1)(z)}_{z^{+\mu}(z)}\,\underbrace{z^{-}{}_{\mu}(-1)(z)}_{z^{-}{}_{\mu}(z)} : \\
&= \quad : \sum_{n_1\in\mathbb{Z}} z^{+\mu}(n_1)z^{-n_1-1}\sum_{n_2\in\mathbb{Z}} z^{-}{}_{\mu}(n_2)z^{-n_2-1} : \\
&= \quad \sum_{n_1,n_2\in\mathbb{Z}} : z^{+\mu}(n_1)z^{-}{}_{\mu}(n_2) : z^{-n_1-n_2-2} \\
&= \quad \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} : z^{+\mu}(n)z^{-}_{\mu}(k-n) : z^{-k-2} \\
&= \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} : z^{+\mu}(n)z^{-}_{\mu}(m-1-n) : z^{-m-1}.
\end{aligned}
$$

Comparing coefficients gives

$$w_m = \sum_{n\in\mathbb{Z}} : z^{+\mu}(n) z^-_{\ \mu}(m-1-n) : .$$

Hence

$$\begin{aligned}
L_m^M &= -\frac{1}{2}\sum_{n\in\mathbb{Z}} : z^{+\mu}(n) z^-_{\ \mu}(m-n) + z^{-\mu}(n) z^+_{\ \mu}(m-n) + \phi^1(n)\phi^1(m-n) \\
&\quad +\phi^2(n)\phi^2(m-n) : \\
&= -\frac{1}{4}\epsilon\{z^{+\mu}\left(\frac{m}{2}\right) z^-_{\ \mu}\left(\frac{m}{2}\right) + z^{-\mu}\left(\frac{m}{2}\right) z^+_{\ \mu}\left(\frac{m}{2}\right) + 2\phi^1\left(\frac{m}{2}\right)\phi^1\left(\frac{m}{2}\right) \\
&\quad +2\phi^2\left(\frac{m}{2}\right)\phi^2\left(\frac{m}{2}\right)\} - \frac{1}{2}\sum_{k>\frac{m}{2}}\{z^-_{\ \mu}(m-k) z^{+\mu}(k) + z^+_{\ \mu}(m-k) z^{-\mu}(k) \\
&\quad +2\phi^1(m-k)\phi^1(k) + 2\phi^2(m-k)\phi^2(k)\}
\end{aligned}$$

with

$$\epsilon = \begin{cases} 1 & for \quad m \quad even, \\ 0 & for \quad m \quad odd. \end{cases}$$

□

Now we define the elements $\tau^{M\pm}, j^M \in V$.

Definition 5.12
Define the **supercurrents** *of the matter sector* $\tau^{M+}, \tau^{M-}, \tau^M \in V$ *by*

$$\begin{aligned}
\tau^{M+} &= z^{-\mu}(-1)\psi^+_{\ \mu}, \\
\tau^{M-} &= z^{+\mu}(-1)\psi^-_{\ \mu}, \\
\tau^M &= \tau^{M+} + \tau^{M-}.
\end{aligned}$$

Definition 5.13
We define $j^M \in V$ *by*

$$j^M = -\frac{1}{2}(\psi^{-\mu})_{-1}\psi^+_{\ \mu}.$$

Proposition 5.14

$$j^M = \phi^1(-1) + \phi^2(-1).$$

Proof:

$$\begin{aligned}
j^M &= -\frac{1}{2}(-\psi^{-+}_{-1}\psi^{++} + \psi^{--}_{-1}\psi^{+-}) \\
&\overset{Prop.\ 5.6}{=} -\frac{1}{2}(-\phi^1(-1) - \phi^2(-1) - 2i - \phi^1(-1) - \phi^2(-1) + 2i) \\
&= \phi^1(-1) + \phi^2(-1).
\end{aligned}$$

□

The following proposition will be used in Proposition 5.44.

Proposition 5.15

$$
\begin{aligned}
\omega^M(z)z^{\pm\mu}(-1) &= z^{\pm\mu}(-1)z^{-2} + Dz^{\pm\mu}(-1)z^{-1} + \dots, \\
\omega^M(z)\psi^{\pm\mu} &= \frac{1}{2}\psi^{\pm\mu}z^{-2} + D\psi^{\pm\mu}z^{-1} + \dots, \\
\omega^M(z)e^{\alpha} &= -\frac{1}{2}(z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)e^{\alpha}, \\
&\quad +(z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)e^{\alpha})z^{-1} + \dots, \\
\tau^M(z)z^{\pm\mu}(-1) &= -2\psi^{\pm\mu}z^{-2} - 2D\psi^{\pm\mu}z^{-1} + \dots, \\
\tau^M(z)\psi^{\pm\mu} &= -2z^{\pm\mu}(-1)z^{-1} + \dots, \\
\tau^{M\pm}(z)e^{\alpha} &= (z^{\mp\mu},\alpha)(\psi^{\pm}{}_{\mu})_{-1}e^{\alpha}z^{-1} + \dots, \\
j^M(z)\psi^{\pm\mu} &= \mp\psi^{\pm\mu}z^{-1} + \dots \ .
\end{aligned}
$$

Proof:
Let $m \geq -1$. Then

$$
\begin{aligned}
&L^M_m z^{+\mu}(-1) = \\
&\quad -\frac{1}{4}\epsilon\{z^{+\nu}\left(\frac{m}{2}\right)z^{-}{}_{\nu}\left(\frac{m}{2}\right)z^{+\mu}(-1) + z^{-\nu}\left(\frac{m}{2}\right)z^{+}{}_{\nu}\left(\frac{m}{2}\right)z^{+\mu}(-1)\} \\
&\quad -\frac{1}{2}\sum_{k>0}\{z^{-}{}_{\nu}(m-k)z^{+\nu}(k)z^{+\mu}(-1) + z^{+}{}_{\nu}(m-k)z^{-\nu}(k)z^{+\mu}(-1)\} \\
&= -\frac{1}{4}\epsilon\{z^{+\nu}\left(\frac{m}{2}\right)\eta_{\nu i}[z^{-i}\left(\frac{m}{2}\right), z^{+\mu}(-1)] \\
&\quad +z^{-\nu}\left(\frac{m}{2}\right)\eta_{\nu i}[z^{+i}\left(\frac{m}{2}\right), z^{+\mu}(-1)]\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{z^{-}{}_{\nu}(m-k)[z^{+\nu}(k), z^{+\mu}(-1)] \\
&\quad +z^{+}{}_{\nu}(m-k)[z^{-\nu}(k), z^{+\mu}(-1)]\} \\
&= 2(\frac{1}{4})\eta_{\nu i}\eta^{i\mu}z^{+\nu}(1)1 - 2(\frac{1}{2})\eta^{\nu\mu}z^{+}{}_{\nu}(m-1)1 \\
&= \delta_m z^{+\mu}(-1) + \delta_{m+1}z^{+\mu}(-2)
\end{aligned}
$$

and for $L^M_m z^{-\mu}(-1)$ analogously.
For $k \geq 0$ we have

$$(\phi^1(k)\phi^1(k) + \phi^2(k)\phi^2(k))\psi^{\pm\mu} = \delta_k\psi^{\pm\mu}$$

and

$$De^{\pm\phi^i} = \pm\phi^i(-1)e^{\pm\phi^i}.$$

Hence we have for $m \geq -1$:

$$
\begin{aligned}
L_m^M \psi^{\pm\mu} &= \frac{1}{2}\epsilon\{\phi^1\left(\frac{m}{2}\right)\phi^1\left(\frac{m}{2}\right) + \phi^2\left(\frac{m}{2}\right)\phi^2\left(\frac{m}{2}\right)\}\psi^{\pm\mu} \\
&\quad + \sum_{k>\frac{m}{2}}\{\phi^1(m-k)\phi^1(k) + \phi^2(m-k)\phi^2(k)\}\psi^{\pm\mu} \\
&= \frac{1}{2}\delta_m\psi^{\pm\mu} + \delta_{m+1}(D\psi^{\pm\mu}). \\
L_m^M e^\alpha &= -\frac{1}{4}\epsilon\{z^{+\mu}\left(\frac{m}{2}\right)z^{-}{}_{\mu}\left(\frac{m}{2}\right)e^\alpha + z^{-\mu}\left(\frac{m}{2}\right)z^{+}{}_{\mu}\left(\frac{m}{2}\right)e^\alpha\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}} z^{-}{}_{\mu}(m-k)z^{+\mu}(k)e^\alpha + z^{+}{}_{\mu}(m-k)z^{-\mu}(k)e^\alpha\} \\
&= -\frac{1}{4}\epsilon\{z^{+\mu}\left(\frac{m}{2}\right)[z^{-}{}_{\mu}\left(\frac{m}{2}\right), e^\alpha_{-1}]1 + z^{+\mu}\left(\frac{m}{2}\right)e^\alpha_{-1}z^{-}{}_{\mu}\left(\frac{m}{2}\right)1 \\
&\quad + z^{-\mu}\left(\frac{m}{2}\right)[z^{+}{}_{\mu}\left(\frac{m}{2}\right), e^\alpha_{-1}]1 + z^{-\mu}\left(\frac{m}{2}\right)e^\alpha_{-1}z^{+}{}_{\mu}\left(\frac{m}{2}\right)1\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{z^{-}{}_{\mu}(m-k)[z^{+\mu}(k), e^\alpha_{-1}]1 + z^{-}{}_{\mu}(m-k)e^\alpha_{-1}z^{+\mu}(k)1 \\
&\quad + z^{+}{}_{\mu}(m-k)[z^{-\mu}(k), e^\alpha_{-1}]1 + z^{+}{}_{\mu}(m-k)e^\alpha_{-1}z^{-\mu}(k)1\} \\
&= -\frac{1}{4}\epsilon\{(z^{-}{}_{\mu},\alpha)z^{+\mu})\left(\frac{m}{2}\right)e^\alpha_{\frac{m}{2}-1}1 + (z^{+}{}_{\mu},\alpha)z^{-\mu}\left(\frac{m}{2}\right)e^\alpha_{\frac{m}{2}-1}1\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{z^{+\mu},\alpha)z^{-}{}_{\mu}(m-k)e^\alpha_{k-1}1 + (z^{-\mu},\alpha)z^{+}{}_{\mu}(m-k)e^\alpha_{k-1}1\} \\
&= -\frac{1}{4}\delta_m(z^{-}{}_{\mu},\alpha)z^{+\mu}(0)e^\alpha - \frac{1}{4}\delta_m(z^{+}{}_{\mu},\alpha)z^{-\mu}(0)e^\alpha \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{\delta_k(z^{+\mu},\alpha)z^{-}{}_{\mu}(m)e^\alpha + \delta_k(z^{-\mu},\alpha)z^{+}{}_{\mu}(m)e^\alpha\} \\
&= -\frac{1}{4}\delta_m(z^{-}{}_{\mu},\alpha)[z^{+\mu}(0), e^\alpha_{-1}]1 - \frac{1}{4}\delta_m(z^{+}{}_{\mu},\alpha)[z^{-\mu}(0), e^\alpha_{-1}]1 \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{\delta_k(z^{+\mu},\alpha)[z^{-}{}_{\mu}(m), e^\alpha_{-1}]1 + \delta_k(z^{+\mu},\alpha)e^\alpha_{-1}z^{-}{}_{\mu}(m)1\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{\delta_k(z^{-\mu},\alpha)[z^{+}{}_{\mu}(m), e^\alpha_{-1}]1 + \delta_k(z^{-\mu},\alpha)e^\alpha_{-1}z^{+}{}_{\mu}(m)1\} \\
&= -\frac{1}{4}\delta_m\underbrace{(z^{-}{}_{\mu},\alpha)(z^{+\mu},\alpha)}_{=0,\ Lem.\ 5.2}e^\alpha - \frac{1}{4}\delta_m(z^{+}{}_{\mu},\alpha)(z^{-\mu},\alpha)e^\alpha \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\delta_k(z^{+\mu},\alpha)(z^{-}{}_{\mu},\alpha)e^\alpha_{m-1}1 - \frac{1}{2}\delta_{m+1}(z^{+\mu},\alpha)e^\alpha_{-1}z^{-}{}_{\mu}(-1) \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\delta_k(z^{-\mu},\alpha)(z^{+}{}_{\mu},\alpha)e^\alpha_{m-1}1 - \frac{1}{2}\delta_{m+1}(z^{-\mu},\alpha)e^\alpha_{-1}z^{+}{}_{\mu}(-1)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\delta_{m+1}(z^{+\mu},\alpha)[e^{\alpha}_{-1}, z^{-}{}_{\mu}(-1)]1 - \frac{1}{2}\delta_{m+1}(z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)e^{\alpha} \\
&\quad -\frac{1}{2}\delta_{m+1}(z^{-\mu},\alpha)[e^{\alpha}_{-1}, z^{+}{}_{\mu}]1 - \frac{1}{2}\delta_{m+1}(z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)e^{\alpha} \\
&= -\frac{1}{2}\delta_{m+1}((z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)e^{\alpha} + (z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)e^{\alpha}).
\end{aligned}
$$

$$
\begin{aligned}
\tau^{M+}_{k} z^{+\mu}(-1) &= \sum_{j\geq 0}\{z^{-\nu}(-1)_{-1-j}(\psi^{+}{}_{\nu})_{k+j}z^{+\mu}(-1) \\
&\quad +(\psi^{+}{}_{\nu})_{k-j-1}z^{-\nu}(-1)_{j}z^{+\mu}(-1)\} \\
&= -2\psi^{+\mu}_{k-2}1 = -2\delta_{k}(D\psi^{+\mu}) - 2\delta_{k-1}\psi^{+\mu}
\end{aligned}
$$

and for $\tau^{M-}_{k} z^{-\mu}(-1)$ analogously. For $k \geq 0$ we have

$$
\begin{aligned}
\tau^{M+}_{k}\psi^{-\mu} &= \sum_{j\geq 0}\{z^{-\nu}(-1)_{-1-j}(\psi^{+}{}_{\nu})_{k+j}\psi^{-\mu} + (\psi^{+}{}_{\nu})_{k-j-1}z^{-\nu}(-1)_{j}\psi^{-\mu}\} \\
&= \sum_{j\geq 0}\eta_{\nu i}z^{-\nu}(-1)_{-1-j}\{\psi^{+i}_{k+j}, \psi^{-\mu}_{-1}\}1 \\
&= -2\eta_{\nu i}\eta^{i\mu}\delta_{k}z^{-\nu}(-1) = -2\delta_{k}z^{-\mu}(-1)
\end{aligned}
$$

and for $\tau^{M-}_{k}\psi^{+\mu}$ analogously.

$$
\begin{aligned}
\tau^{M\pm}_{k}e^{\alpha} &= (z^{\mp\mu}(-1)_{-1}\psi^{\pm}{}_{\mu})_{k}e^{\alpha} \\
&= \sum_{j\geq 0}\{z^{\mp\mu}(-1)_{-1-j}(\psi^{\pm}{}_{\mu})_{k+j}e^{\alpha} + (\psi^{\pm}{}_{\mu})_{k-1-j}z^{\mp\mu}(-1)_{j}e^{\alpha}\} \\
&= \sum_{j\geq 0}(\psi^{\pm}{}_{\mu})_{k-1-j}[z^{\mp\mu}(-1)_{j}, e^{\alpha}_{-1}]1 \\
&= (\psi^{\pm}{}_{\mu})_{k-1}(z^{\mp\mu},\alpha)e^{\alpha} = \delta_{k}(z^{\mp\mu},\alpha)(\psi^{\pm}{}_{\mu})_{-1}e^{\alpha}.
\end{aligned}
$$

$$
\begin{aligned}
j^{M}_{k}\psi^{+\mu} &= -\frac{1}{2}\sum_{j\geq 0}\{\psi^{-\nu}_{-1-j}(\psi^{+}{}_{\nu})_{k+j}\psi^{+\mu} - (\psi^{+}{}_{\nu})_{k-1-j}\psi^{-\nu}_{j}\psi^{+\mu}\} \\
&= -\frac{1}{2}\sum_{j\geq 0}\eta_{\nu i}\psi^{-\nu}_{-1-j}\{\psi^{+i}_{k+j}, \psi^{+\mu}_{-1}\}1 - (\psi^{+}{}_{\nu})_{k-1-j}\{\psi^{-\nu}_{j}, \psi^{+\mu}_{-1}\}1\} \\
&= -\psi^{+\mu}_{k-1}1 = -\delta_{k}\psi^{+\mu}. \\
j^{M}_{k}\psi^{-\mu} &= -\frac{1}{2}\sum_{j\geq 0}\{\eta_{\nu i}\psi^{-\nu}_{-1-j}\{\psi^{+i}_{k+j}, \psi^{-\mu}_{-1}\}1 - (\psi^{+}{}_{\nu})_{k-1-j}\{\psi^{-\nu}_{j}, \psi^{-\mu}_{-1}\}1\} \\
&= \eta_{\nu i}\eta^{i\mu}\delta_{k}\psi^{-\nu} = \delta_{k}\psi^{-\mu}.
\end{aligned}
$$

□

The next proposition supplies all information necessary to conclude that the matter fields give a representation of the N=2-extension of the Virasoro algebra of central charge 6.

Proposition 5.16
The above fields satisfy the operator product expansions

$$
\begin{aligned}
\omega^M(z)\omega^M &= 3z^{-4} + 2\omega^M z^{-2} + D\omega^M z^{-1} + \dots, \\
\omega^M(z)\tau^{M\pm} &= \frac{3}{2}\tau^{M\pm} z^{-2} + D\tau^{M\pm} z^{-1} + \dots, \\
\omega^M(z) j^M &= j^M z^{-2} + Dj^M z^{-1} + \dots, \\
j^M(z)\tau^{M\pm} &= \mp\tau^{M\pm} z^{-1} + \dots \\
j^M(z) j^M &= 2z^{-2} + \dots, \\
\tau^{M+}(z)\tau^{M-} &= 8z^{-3} - 4j^M z^{-2} + 4(\omega^M - \frac{1}{2}Dj^M)z^{-1} + \dots, \\
\tau^{M-}(z)\tau^{M+} &= 8z^{-3} + 4j^M z^{-2} + 4(\omega^M + \frac{1}{2}Dj^M)z^{-1} + \dots \quad .
\end{aligned}
$$

$\tau^{M+}(z)\tau^{M+}$ and $\tau^{M-}(z)\tau^{M-}$ contain only nonsingular terms.

Proof:
Let $m \geq -1$. Then

$$
\begin{aligned}
L^M_m \tau^{M+} &= -\frac{1}{4}\epsilon\{z^{+\mu}\left(\frac{m}{2}\right) z^{-}{}_{\mu}\left(\frac{m}{2}\right) z^{-\nu}(-1)\psi^{+}{}_{\nu} \\
&\quad + z^{-\mu}\left(\frac{m}{2}\right) z^{+}{}_{\mu}\left(\frac{m}{2}\right) z^{-\nu}(-1)\psi^{+}{}_{\nu} \\
&\quad -2(\phi^1\left(\frac{m}{2}\right)\phi^1\left(\frac{m}{2}\right) + \phi^2\left(\frac{m}{2}\right)\phi^2\left(\frac{m}{2}\right))z^{-\nu}(-1)\psi^{+}{}_{\nu}\} \\
&\quad -\frac{1}{2}\sum_{k>\frac{m}{2}}\{z^{-}{}_{\mu}(m-k)[z^{+\mu}(k), z^{-\nu}(-1)]\psi^{+}{}_{\nu} \\
&\quad + z^{+}{}_{\mu}(m-k)z^{-\mu}(k)z^{-\nu}(-1)\psi^{+}{}_{\nu} \\
&\quad -2(\phi^1(m-k)\phi^1(k) + \phi^2(m-k)\phi^2(k))z^{-\nu}(-1)\psi^{+}{}_{\nu}\} \\
&= \frac{1}{2}\delta_m\tau^{M+} + \delta_m\tau^{M+} + \delta_{m+1}D\tau^{M+}. \\
\omega^M_k j^M &= \frac{1}{8}\sum_{j\geq 0}(D\psi^{+\mu})_{-1-j}(\psi^{-}{}_{\mu})_{k+j}\psi^{-\nu}_{-1}\psi^{+}{}_{\nu} \\
&\quad -\frac{1}{8}\sum_{j\geq 0}(\psi^{-}{}_{\mu})_{k-1-j}(D\psi^{+\mu})_j\psi^{-\nu}_{-1}\psi^{+}{}_{\nu} \\
&\quad +\frac{1}{8}\sum_{j\geq 0}(D\psi^{-\mu})_{-1-j}(\psi^{+}{}_{\mu})_{k+j}\psi^{-\nu}_{-1}\psi^{+}{}_{\nu} \\
&\quad -\frac{1}{8}\sum_{j\geq 0}(\psi^{+}{}_{\mu})_{k-1-j}(D\psi^{-\mu})_j\psi^{-\nu}_{-1}\psi^{+}{}_{\nu} \\
&= -\frac{1}{8}\sum_{j\geq 0}(D\psi^{+\mu})_{-1-j}\psi^{-\nu}_{-1}\{(\psi^{-}{}_{\mu})_{k+j}, (\psi^{+}{}_{\nu})_{-1}\}1
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{8}\sum_{j\geq 0}(-j)(\psi^{-}{}_{\mu})_{k-1-j}\{\psi^{+\mu}_{j-1},\psi^{-\nu}_{-1}\}\psi^{+}{}_{\nu} \\
& +\frac{1}{8}\sum_{j\geq 0}\eta_{\mu i}(D\psi^{-\mu})_{-1-j}\{\psi^{+i}_{k+j},\psi^{-\nu}_{-1}\}\psi^{+}{}_{\nu} \\
& +\frac{1}{8}\sum_{j\geq 0}(-j)\eta_{\nu i}(\psi^{+}{}_{\mu})_{k-1-j}\psi^{-\nu}_{-1}\{\psi^{-\mu}_{j-1},\psi^{+i}_{-1}\}1 \\
= \;& \delta_k(-2)\eta_{\mu\nu}(D\psi^{+\mu})_{-1}\psi^{-\nu}-\frac{1}{8}(-1)(-2)\eta^{\mu\nu}(\psi^{-}{}_{\mu})_{k-2}\psi^{+}{}_{\nu} \\
& +\frac{1}{8}\delta_k(-2)\eta^{i\nu}\eta_{\mu i}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\nu}+\frac{1}{8}(-2)(-1)\eta^{\mu i}\eta_{\nu i}(\psi^{+}{}_{\mu})_{k-2}\psi^{-\nu} \\
= \;& \frac{1}{4}\delta_k(D\psi^{+\mu})_{-1}\psi^{-}{}_{\mu}-\frac{1}{4}\eta_{\nu i}\{\psi^{-\nu}_{k-2},\psi^{+i}_{-1}\}1+\frac{1}{4}(\psi^{+}{}_{\nu})_{-1}\psi^{-\nu}_{k-2}1 \\
& -\frac{1}{4}\delta_k(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}+\frac{1}{4}\eta_{\nu i}\{\psi^{+i}_{k-2},\psi^{-\nu}_{-1}\}1-\frac{1}{4}\psi^{-\nu}_{-1}(\psi^{+}{}_{\nu})_{k-2}1 \\
= \;& \frac{1}{4}\delta_k(D\psi^{+\mu})_{-1}\psi^{-}{}_{\mu}-\frac{1}{4}(-2)\eta_{\nu i}\eta^{\nu i}\delta_{k-2}+\frac{1}{4}\delta_k(\psi^{+}{}_{\nu})_{-1}(D\psi^{+\nu}) \\
& +\frac{1}{4}\delta_{k-1}(\psi^{+}{}_{\nu})_{-1}\psi^{-\nu}-\frac{1}{4}\delta_k(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}+\frac{1}{4}(-2)\eta_{\nu i}\eta^{i\nu}\delta_{k-2} \\
& -\frac{1}{4}\delta_k\psi^{-\nu}_{-1}(D\psi^{+}{}_{\nu})-\frac{1}{4}\delta_{k-1}\psi^{-\nu}_{-1}\psi^{+}{}_{\nu} \\
= \;& \frac{1}{4}\delta_k(D\psi^{+\mu})_{-1}\psi^{-}_{\mu}-\frac{1}{4}\delta_k(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}+\frac{1}{4}\delta_k(\psi^{+}{}_{\nu})_{-1}(D\psi^{-\nu}) \\
& -\frac{1}{4}\delta_k\psi^{-\nu}_{-1}(D\psi^{+}{}_{\nu})+\frac{1}{4}\delta_{k-1}(\psi^{+}{}_{\nu})_{-1}\psi^{-\nu}-\frac{1}{4}\delta_{k-1}\psi^{\nu}_{-1}\psi^{+}{}_{\nu} \\
& +\delta_{k-2}-\delta_{k-2} \\
= \;& \delta_k(-\frac{1}{4}\psi^{-\mu}_{-1}(D\psi^{+}{}_{\mu})-\frac{1}{4}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}-\frac{1}{4}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu} \\
& -\frac{1}{4}\psi^{-\mu}_{-1}(D\psi^{+}{}_{\mu}))+\delta_{k-1}(-\frac{1}{4}\psi^{-\mu}_{-1}\psi^{+}{}_{\mu}-\frac{1}{4}\psi^{-\mu}_{-1}\psi^{+}{}_{\mu}) \\
= \;& \delta_k Dj^M+\delta_{k-1}j^M. \\
j^M_k\tau^{M+} = \;& -\frac{1}{2}\sum_{j\geq 0}\{\psi^{-\nu}_{-1-j}(\psi^{+}{}_{\nu})_{k+j}z^{-\mu}(-1)_{-1}\psi^{+}{}_{\mu} \\
& -(\psi^{+}{}_{\nu})_{k-1-j}\psi^{-\nu}_{j}z^{-\mu}(-1)_{-1}\psi^{+}{}_{\mu}\} \\
= \;& \frac{1}{2}\sum_{j\geq 0}\eta_{\mu i}(\psi^{+}{}_{\nu})_{k-1-j}z^{-\mu}(-1)_{-1}\{\psi^{-\nu}_{j},\psi^{+i}_{-1}\}1 \\
= \;& -2(\frac{1}{2})\eta_{\mu i}\eta^{\nu i}z^{-\mu}(-1)_{-1}(\psi^{+}{}_{\nu})_{k-1}1=-\delta_k z^{-}{}_{i}(-1)_{-1}\psi^{+i} \\
= \;& -\delta_k z^{-\mu}(-1)_{-1}\psi^{+}{}_{\mu}=-\delta_k\tau^{M+}.
\end{aligned}
$$

For $j^M_k\tau^{M-}$ analogously.

$$
\begin{aligned}
j^M_k j^M &= \frac{1}{4}\sum_{j\geq 0}\{\psi^{-\mu}_{-1-j}(\psi^+{}_\mu)_{k+j}\psi^{-\nu}_{-1}\psi^+{}_\nu - (\psi^+{}_\mu)_{k-1-j}\psi^{-\mu}_j\psi^{-\nu}_{-1}\psi^+{}_\nu\} \\
&= \frac{1}{4}\sum_{j\geq 0}\eta_{\mu i}\psi^{-\mu}_{-1-j}\{\psi^{+i}_{k+j},\psi^{-\nu}_{-1}\}\psi^+{}_\nu \\
&\quad +\frac{1}{4}\sum_{j\geq 0}\eta_{\nu i}(\psi^+{}_\mu)_{k-1-j}\psi^{-\nu}_{-1}\{\psi^{-\mu}_j,\psi^{+i}_{-1}\}1 \\
&= \frac{1}{4}\delta_k\eta_{\nu i}\eta^{\nu i}(-2)\psi^{-\mu}_{-1}\psi^+{}_\nu + \frac{1}{4}(-2)\eta_{\nu i}\eta^{\mu i}(\psi^+{}_\mu)_{k-1}\psi^{-\nu} \\
&= -\frac{1}{2}\delta_k\psi^{-\mu}_{-1}\psi^+{}_\mu - \frac{1}{2}\eta_{\nu i}\{\psi^{+i}_{k-1},\psi^{-\nu}_{-1}\}1 + \frac{1}{2}\psi^{-\nu}_{-1}(\psi^+{}_\nu)_{k-1}1 \\
&= -\frac{1}{2}\delta_k\psi^{-\mu}_{-1}\psi^+{}_\mu - \frac{1}{2}(-2)\eta_{\nu i}\eta^{i\nu}\delta_{k-1} + \frac{1}{2}\delta_k\psi^{-\nu}_{-1}\psi^+{}_\nu \\
&= \delta_k j^M - \delta_k j^M + 2\delta_{k-1}.
\end{aligned}
$$

We have

$$
\begin{aligned}
Dj^M &= -\frac{1}{2}(\psi^{-\mu}_{-1}\psi^+{}_\mu)_{-2}1 \\
&= -\frac{1}{2}\sum_{j\geq 0}\{\psi^{-\mu}_{-1-j}(\psi^+{}_\mu)_{-2+j}1 - (\psi^+{}_\mu)_{-3-j}\psi^{-\mu}_j 1\} \\
&= -\frac{1}{2}\psi^{-\mu}_{-1}(D\psi^+{}_\mu) - \frac{1}{2}(D\psi^{-\mu})_{-1}(\psi^+{}_\mu), \\
4\omega^M - 2Dj^M &= -z^{+\mu}(-1)z^-{}_\mu(-1) - z^{-\mu}(-1)z^+{}_\mu(-1), \\
&\quad -(D\psi^{+\mu})_{-1}\psi^-{}_\mu - (D\psi^{-\mu})_{-1}\psi^+{}_\mu + \psi^{-\mu}_{-1}(D\psi^+{}_\mu) \\
&\quad +(D\psi^{-\mu})_{-1}\psi^+{}_\mu - 2z^{-\mu}(-1)z^+{}_\mu(-1) + 2\psi^{-\mu}_{-1}(D\psi^+{}_\mu) \\
4\omega^M + 2Dj^M &= -2z^{+\mu}(-1)z^-{}_\mu(-1) - 2(D\psi^-{}_\mu)_{-1}(\psi^+{}_\mu).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\tau^{M+}_k\tau^{M-} &= \sum_{j\geq 0}z^{-\mu}(-1)_{-1-j}(\psi^+{}_\mu)_{k+j}z^{+\nu}(-1)_{-1}\psi^-{}_\nu \\
&\quad +\sum_{j\geq 0}(\psi^+{}_\mu)_{k-1-j}z^{-\mu}(-1)_j z^{+\nu}(-1)\psi^-{}_\nu \\
&= \sum_{j\geq 0}z^{-\mu}(-1)_{-1-j}z^{+\nu}(-1)\{(\psi^+{}_\mu)_{k+j},(\psi^-{}_\nu)_{-1}\}1 \\
&\quad +\sum_{j\geq 0}(\psi^+{}_\mu)_{k-1-j}[z^{-\mu}(-1)_j,z^{+\nu}(-1)]\psi^-{}_\nu
\end{aligned}
$$

$$
\begin{aligned}
&= -2\delta_k\eta_{\mu\nu}z^{-\mu}(-1)z^{+\nu}(-1) - 2\eta^{\mu\nu}\{(\psi^+{}_\mu)_{k-2},(\psi^-{}_\nu)_{-1}\}1 \\
&\quad +2\eta^{\mu\nu}(\psi^-{}_\nu)_{-1}(\psi^+{}_\mu)_{k-2}1 \\
&= -2\delta_k z^{-\mu}(-1)z^+{}_\mu(-1) - 2\delta_k\psi^{-\mu}_{-1}(D\psi^+{}_\mu) + 4\eta^{\mu\nu}\eta_{\mu\nu}\delta_{k-2} \\
&\quad +2\delta_{k-1}\psi^{-\mu}_{-1}(\psi^+{}_\mu) \\
&= -2\delta_k(z^{-\mu}(-1)_{-1}z^+{}_\mu(-1) - \psi^{-\mu}_{-1}(D\psi^+{}_\mu)) + 2\delta_{k-1}\psi^{-\mu}_{-1}(\psi^+{}_\mu) \\
&\quad +8\delta_{k-2}
\end{aligned}
$$

and

$$
\begin{aligned}
\tau^{M-}_k\tau^{M+} &= (z^{+\mu}(-1)\psi^-{}_\mu)_k z^{-\nu}(-1)\psi^+{}_\nu \\
&= \sum_{j\geq 0}\{z^{+\mu}(-1)_{-1-j}(\psi^-{}_\mu)_{k+j}z^{-\nu}(-1)\psi^+{}_\nu \\
&\quad +(\psi^-{}_\mu)_{k-1-j}z^{+\mu}(-1)_j z^{-\nu}(-1)\psi^+{}_\nu\} \\
&= \sum_{j\geq 0} z^{+\mu}(-1)_{-1-j}z^{-\nu}(-1)\{(\psi^-{}_\mu)_{k+j},(\psi^+{}_\nu)_{-1}\}1 \\
&\quad +(\psi^-{}_\mu)_{k-1-j}[z^{+\mu}(-1)_j, z^{-\nu}(-1)]\psi^+{}_\nu\} \\
&= -2\delta_k z^{+\mu}(-1)_{-1}z^-{}_\mu(-1) - 2\eta^{\mu\nu}\{(\psi^-{}_\mu)_{k-2},(\psi^+{}_\nu)_{-1}\}1 \\
&\quad +2\eta^{\mu\nu}(\psi^+{}_\nu)_{-1}(\psi^-{}_\mu)_{k-2}1 \\
&= -2\delta_k z^{+\mu}(-1)z^-{}_\mu(-1) + 4\eta^{\mu\nu}\eta_{\mu\nu}\delta_{k-2} + 2\delta_k(\psi^{+\mu})_{-1}(D\psi^-{}_\mu) \\
&\quad +2\delta_{k-1}\psi^{+\mu}_{-1}\psi^-{}_\mu \\
&= -2\delta_k(z^{+\mu}(-1)z^-{}_\mu(-1) + (D\psi^-{}_\mu)_{-1}(\psi^+{}_\mu)) + 8\delta_{k-2} \\
&\quad -2\delta_{k-1}(\psi^{-\mu})_{-1}(\psi^+{}_\mu).
\end{aligned}
$$

□

Proposition 5.17
The operators $L^M_m = \omega^M_{m+1}, G^{M\pm}_n = \tau^{M\pm}_{n+\frac{1}{2}}$ *and* j^M_m *give a representation of the N=2-extension of the Virasoro algebra of central charge 6, i.e.*

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{2}\delta_{m+n}, \\
[L_m, G^\pm_n] &= (\frac{1}{2}m-n)G^\pm_{m+n}, \\
[L_m, j_n] &= -nj_{m+n}, \\
[j_m, G^\pm_n] &= \mp G^\pm_{m+n}, \\
[j_m, j_n] &= 2m\delta_{m+n}, \\
\{G^+_m, G^-_n\} &= 4L_{m+n} - 2(m-n)j_{m+n} + (4m^2-1)\delta_{m+n}, \\
\{G^+_m, G^+_n\} &= \{G^-_m, G^-_n\} = 0
\end{aligned}
$$

holds.

Proof:
This follows from Proposition 5.16 and the commutator formula. □

Lemma 5.18
We have

$$\tau_0^M \tau^M = 8\omega^M.$$

Proof:
This follows from Proposition 5.16. □

5.2 Ghost part

In this section we construct a representation of central charge -6 coming from the ghost part of V. In order to define the elements $\omega^{Gh}, \tau^{Gh\pm}, j^{Gh} \in V$ generating this representation we have to define ghost fields.

Definition 5.19
Let $\phi = (0,0,1,0) \in L^{\psi,\phi,\tilde{\phi}}$ *and* $\tilde{\phi} = (0,0,0,1) \in L^{\psi,\phi,\tilde{\phi}}$.
We define the **ghost fields** $b, c, \tilde{b}, \tilde{c}, \beta^{\pm}, \gamma^{\pm} \in V$ *by*

$$\begin{gathered} b = e^{-\sigma}, \quad c = e^{\sigma}, \quad \tilde{b} = e^{-\tilde{\sigma}}, \quad \tilde{c} = e^{\tilde{\sigma}}, \\ \beta^+ = \chi(-1)_{-1} e^{-\phi+\chi}, \quad \gamma^- = e^{\phi-\chi}, \\ \beta^- = \tilde{\chi}(-1)_{-1} e^{-\tilde{\phi}+\tilde{\chi}}, \quad \gamma^+ = e^{\tilde{\phi}-\tilde{\chi}}. \end{gathered}$$

Lemma 5.20

$$\begin{gathered} \epsilon(\phi,\phi) = \epsilon(\tilde{\phi},\tilde{\phi}) = -1. \\ \epsilon(\chi,\phi) = -1. \\ \epsilon(\phi,\chi) = 1. \end{gathered}$$

Hence

$$\epsilon(\phi-\chi, -\phi+\chi) = \epsilon(-\phi+\chi, \phi-\chi) = 1.$$

Proof:
$\phi = -\phi^1 + s^3 + s^4,\ \tilde{\phi} = -\phi^2 + s^3 - s^4.$
Then

$$\begin{aligned} \epsilon(\phi,\phi) &= \epsilon(-\phi^1,-\phi^1)\epsilon(-\phi^1,s^3)\epsilon(-\phi^1,s^4)\epsilon(s^3,-\phi^1)\epsilon(s^3,s^3)\epsilon(s^3,s^4) \\ &\quad \epsilon(s^4,-\phi^1)\epsilon(s^4,s^3)\epsilon(s^4,s^4). \\ \epsilon(s^3,-\phi^1) &= e^{-i\pi(s^3,-\phi^1)}\eta((V,0),(S,0)) \\ &= -iy, \qquad y \in \{\pm 1\}. \end{aligned}$$

$$\begin{array}{rcl}
\epsilon(s^4,-\phi^1) & = & -iy. \\
\epsilon(s^4,s^3) & = & 1. \\
\Rightarrow \quad \epsilon(\phi,\phi) & = & -1. \\
\epsilon(\tilde{\phi},\tilde{\phi}) & = & \epsilon(-\phi^2,-\phi^2)\epsilon(-\phi^2,s^3)\epsilon(-\phi^2,-s^4)\epsilon(s^3,-\phi^2)\epsilon(s^3,s^3) \\
 & & \epsilon(s^3,-s^4)\epsilon(-s^4,-\phi^2)\epsilon(-s^4,s^3)\epsilon(-s^4,-s^4). \\
\epsilon(s^3,-\phi^2) & = & -iy. \\
\epsilon(-s^4,-\phi^2) & = & -iy. \\
\epsilon(-s^4,s^3) & = & 1. \\
\Rightarrow \quad \epsilon(\tilde{\phi},\tilde{\phi}) & = & -1. \\
\epsilon(\chi,\phi) & = & \epsilon(\chi,-\phi^1)\epsilon(\chi,s^3)\epsilon(\chi,s^4). \\
\epsilon(\chi,\phi^1) & = & e^{-i\pi(\chi,\phi^1)}\eta((0,1),(V,0)) = -1. \\
\epsilon(\chi,s^3) & = & e^{-i\pi(\chi,s^3)}\eta((0,1),(V,0)) = y. \\
\epsilon(\chi,s^4) & = & y. \\
\Rightarrow \epsilon(\chi,\phi) & = & -1. \\
\epsilon(\phi,\chi) & = & \epsilon(-\phi^1,\chi)\epsilon(s^3,\chi)\epsilon(s^4,\chi) = 1.
\end{array}$$

□

Lemma 5.21
We can write $\beta^+ = -(De^\chi)_{-1}e^{-\phi}$.

Proof:

$$\begin{array}{rcl}
-(De^\chi)_{-1}e^{-\phi} & = & -e^\chi_{-2}e^{-\phi} = -\epsilon(\chi,-\phi)S_{2-1}(\chi)e^{\chi-\phi} \\
 & = & S_1(\chi)e^{\chi-\phi} = \chi(-1)e^{\chi-\phi} = \beta^+.
\end{array}$$

□

Note that

$$c,b,\tilde{c},\tilde{b} \in V_{(0,1)},$$
$$z^{\pm\mu}(-1) \in V_{(0,0)},$$
$$\gamma^\pm,\beta^\pm \in V_{(V,1)},$$
$$\psi^{\pm\mu} \in V_{(V,0)}.$$

Proposition 5.22
The only nontrivial commutation relations are

$$[\gamma^\mp_m,\beta^\pm_n] = \delta_{m+n+1},$$

$$\{c_m,b_n\} = \delta_{m+n+1}.$$

Proof:
For $k \geq 0$ we have

$$\gamma_k^- e^\chi = \delta_k e^\phi$$

and

$$\gamma_k^- e^{-\phi} = 0.$$

Hence

$$\begin{aligned}
\gamma_k^- \beta^+ &= -\gamma_k^- (De^\chi)_{-1} e^{-\phi} \\
&= -[\gamma_k^-, e_{-2}^\chi] e^{-\phi} - (De^\chi)_{-1} [\gamma_k^-, e_{-1}^{-\phi}] 1 \\
&= -e_{k-2}^\phi e^{-\phi} = -\epsilon(\phi, -\phi) \delta_k 1 = \delta_k 1. \\
\Rightarrow \quad [\gamma_m^-, \beta_n^+] &= \sum_{k \geq 0} \binom{m}{k} (\gamma_k^- \beta^+)_{m+n-k} \\
&= 1_{m+n} = \delta_{m+n+1}.
\end{aligned}$$

For $[\gamma_m^+, \beta_n^-]$ analogously. For $k \geq 0$ we have $e_k^\sigma e^{-\sigma} = S_{-k-1+1}(\sigma) e^0 = \delta_k$.

$$\Rightarrow \quad \{e_m^\sigma, e_n^{-\sigma}\} = \sum_{k \geq 0} \binom{m}{k} (e_k^\sigma e^{-\sigma})_{m+n-k} = 1_{m+n} = \delta_{m+n+1}.$$

□

Now we define the element $\omega^{Gh} \in V$.

Definition 5.23
We define the **energy momentum tensor** $\omega^{Gh} \in V$ *of the ghost part as*

$$\begin{aligned}
\omega^{Gh} &= 2(Dc)_{-1} b - (Db)_{-1} c + (D\tilde{c})_{-1} \tilde{b} \\
&\quad -\frac{3}{2}(D\gamma^+)_{-1} \beta^- - \frac{1}{2}(D\beta^-)_{-1} \gamma^+ \\
&\quad -\frac{3}{2}(D\gamma^-)_{-1} \beta^+ - \frac{1}{2}(D\beta^+)_{-1} \gamma^-.
\end{aligned}$$

Proposition 5.24
The energy momentum tensor ω^{Gh} *can be written as*

$$\omega^{Gh} = \omega^\sigma + \omega^{\tilde{\sigma}} + \omega^\phi + \omega^{\tilde{\phi}} + \omega^\chi + \omega^{\tilde{\chi}}$$

with

$$\begin{aligned}
\omega^\sigma &= \frac{1}{2}\sigma(-1)_{-1}\sigma(-1) + \frac{3}{2}\sigma(-2), \\
\omega^{\tilde{\sigma}} &= \frac{1}{2}\tilde{\sigma}(-1)_{-1}\tilde{\sigma}(-1) + \frac{1}{2}\tilde{\sigma}(-2),
\end{aligned}$$

$$
\begin{aligned}
\omega^{\phi} &= -\frac{1}{2}\phi(-1)_{-1}\phi(-1) - \phi(-2), \\
\omega^{\tilde{\phi}} &= -\frac{1}{2}\tilde{\phi}(-1)_{-1}\tilde{\phi}(-1) - \tilde{\phi}(-2), \\
\omega^{\chi} &= \frac{1}{2}\chi(-1)_{-1}\chi(-1) + \frac{1}{2}\chi(-2), \\
\omega^{\tilde{\chi}} &= \frac{1}{2}\tilde{\chi}(-1)_{-1}\tilde{\chi}(-1) + \frac{1}{2}\tilde{\chi}(-2).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
2(Dc)_{-1}b - (Db)_{-1}c &= 2S_2(\sigma) - S_2(-\sigma) \\
&= 2(\frac{1}{2}\sigma(-1)_{-1}\sigma(-1) + \frac{1}{2}\sigma(-2)) \\
&\quad -(\frac{1}{2}\sigma(-1)_{-1}\sigma(-1) - \frac{1}{2}\sigma(-2)) \\
&= \frac{1}{2}\sigma(-1)_{-1}\sigma(-1) + \frac{3}{2}\sigma(-2) = \omega^{\sigma}. \\
(D\tilde{c})_{-1}\tilde{b} &= S_2(\tilde{\sigma}) \\
&= \frac{1}{2}\tilde{\sigma}(-1)_{-1}\tilde{\sigma}(-1) + \frac{1}{2}\tilde{\sigma}(-2). \\
-\frac{1}{2}\beta^{+}_{-2}\gamma^{-} &= \frac{1}{2}((De^{\chi})_{-1}e^{-\phi})_{-2}e^{\phi-\chi} \\
&= \frac{1}{2}\sum_{j\geq 0}\{(De^{\chi})_{-1-j}e^{-\phi}_{-2+j}e^{\phi-\chi} - e^{-\phi}_{-3-j}(De^{\chi})_j e^{\phi-\chi}\} \\
&= -\frac{1}{2}e^{\chi}_{-2}e^{-\chi} - \frac{1}{2}e^{-\phi}_{-4}e^{\phi} \\
&= -\frac{1}{2}S_{2-1+1}(\chi)e^{0} + \frac{1}{2}S_{4-1-1}(-\phi) \\
&= -\frac{1}{2}S_2(\chi) + \frac{1}{2}S_2(-\phi). \\
-\frac{3}{2}\gamma^{-}_{-2}\beta^{+} &= \frac{3}{2}e^{\phi-\chi}_{-2}e^{\chi}_{-2}e^{-\phi} \\
&= \frac{3}{2}[e^{\phi-\chi}_{-2}, e^{\chi}_{-2}]e^{-\phi} + \frac{3}{2}e^{\chi}_{-2}e^{\phi-\chi}_{-2}e^{-\phi} \\
&= \frac{3}{2}e^{\phi}_{-4}e^{-\phi} + \frac{3}{2}e^{\chi}_{-2}e^{-\chi} \\
&= -\frac{3}{2}S_{4-1-1}(\phi) + \frac{3}{2}S_{2-1+1}(\chi) \\
&= -\frac{3}{2}S_2(\phi) + \frac{3}{2}S_2(\chi).
\end{aligned}
$$

Hence

$$\begin{aligned}
-\frac{1}{2}\beta^+_{-2}\gamma^- - \frac{3}{2}\gamma^-_{-2}\beta^+ &= -\frac{1}{4}\chi(-1)\chi(-1) - \frac{1}{2}\chi(-2) \\
&+\frac{1}{4}\phi(-1)\phi(-1) - \frac{1}{4}\phi(-2) - \frac{3}{4}\phi(-1)\phi(-1) \\
&-\frac{3}{4}\phi(-2) + \frac{3}{4}\chi(-1)\chi(-1) + \frac{3}{4}\chi(-2).
\end{aligned}$$

□

The elements $\omega^\sigma, \omega^{\tilde{\sigma}}, \omega^\phi, \omega^{\tilde{\phi}}, \omega^\chi, \omega^{\tilde{\chi}}$ generate representations of the Virasoro algebra on V of central charge $-26, -2, 13, 13, -2, -2$.

Proposition 5.25
We have

$$\begin{aligned}
L_m^{Gh} &= \frac{1}{2}\epsilon\sigma\left(\frac{m}{2}\right)\sigma\left(\frac{m}{2}\right) + \frac{1}{2}\epsilon\tilde{\sigma}\left(\frac{m}{2}\right)\tilde{\sigma}\left(\frac{m}{2}\right) \\
&+\frac{1}{2}\epsilon\chi\left(\frac{m}{2}\right)\chi\left(\frac{m}{2}\right) + \frac{1}{2}\epsilon\tilde{\chi}\left(\frac{m}{2}\right)\tilde{\chi}\left(\frac{m}{2}\right) \\
&-\frac{1}{2}\epsilon\phi\left(\frac{m}{2}\right)\phi\left(\frac{m}{2}\right) - \frac{1}{2}\epsilon\tilde{\phi}\left(\frac{m}{2}\right)\tilde{\phi}\left(\frac{m}{2}\right) \\
&+\sum_{k>\frac{m}{2}}\{\sigma(m-k)\sigma(k) + \tilde{\sigma}(m-k)\tilde{\sigma}(k) \\
&+\chi(m-k)\chi(k) + \tilde{\chi}(m-k)\tilde{\chi}(k) \\
&-\phi(m-k)\phi(k) - \tilde{\phi}(m-k)\tilde{\phi}(k)\} \\
&+\frac{3}{2}(-m-1)\sigma(m) + \frac{1}{2}(-m-1)\tilde{\sigma}(m) \\
&+\frac{1}{2}(-m-1)\chi(m) + \frac{1}{2}(-m-1)\tilde{\chi}(m) \\
&-(-m-1)\phi(m) - (-m-1)\tilde{\phi}(m)
\end{aligned}$$

with

$$\epsilon = \begin{cases} 1 & for \quad m \quad even, \\ 0 & for \quad m \quad odd. \end{cases}$$

Proof:
Analogously to the proof of Proposition 5.11. □

Now we define the elements $\tau^{Gh\pm}, j^{Gh} \in V$.

Definition 5.26
We define the **supercurrents** $\tau^{Gh+}, \tau^{Gh-} \in V$ *of the ghost part by*

$$\begin{aligned}
\tau^{Gh+} &= c_{-1}(D\beta^{+}) + \frac{3}{2}(Dc)_{-1}\beta^{+} - 4b_{-1}\gamma^{+}, \\
&\quad -4\tilde{b}_{-1}(D\gamma^{+}) - 2(D\tilde{b})_{-1}\gamma^{+} - \tilde{c}_{-1}\beta^{+}, \\
\tau^{Gh-} &= c_{-1}(D\beta^{-}) + \frac{3}{2}(Dc)_{-1}\beta^{-} - 4b_{-1}\gamma^{-} \\
&\quad +4\tilde{b}_{-1}(D\gamma^{-}) + 2(D\tilde{b})_{-1}\gamma^{-} + \tilde{c}_{-1}\beta^{-}.
\end{aligned}$$

Definition 5.27
We define $j^{Gh} \in V$ *by*

$$j^{Gh} = \gamma^{+}_{-1}\beta^{-} - \beta^{+}_{-1}\gamma^{-} - (D\tilde{b})_{-1}c + (Dc)_{-1}\tilde{b}.$$

Proposition 5.28
We can write

$$j^{Gh} = -\phi(-1) + \tilde{\phi}(-1) - (D\tilde{b})_{-1}c + (Dc)_{-1}\tilde{b}.$$

Proof:

$$\begin{aligned}
\beta^{+}_{-1}\gamma^{-} &= -((De^{\chi})_{-1}e^{-\phi})_{-1}e^{\phi-\chi} \\
&= -\sum_{j\geq 0}\{(De^{\chi})_{-1-j}e^{-\phi}_{-1+j}e^{\phi-\chi} - e^{-\phi}_{-2-j}(De^{\chi})_{j}e^{\phi-\chi}\} \\
&= e^{-\phi}_{-3}e^{\phi} = -S_1(-\phi) = \phi(-1).
\end{aligned}$$

□

The next proposition supplies all information necessary to conclude that the ghost fields give a representation of the $N{=}2$ extension of the Virasoro algebra of central charge -6.

Proposition 5.29
The above fields satisfy the operator product expansions

$$\begin{aligned}
\omega^{Gh}(z)\omega^{Gh} &= -3z^{-4} + 2\omega^{Gh}z^{-2} + D\omega^{Gh}z^{-1} + \ldots, \\
\omega^{Gh}(z)\tau^{Gh\pm} &= \frac{3}{2}\tau^{Gh\pm}z^{-2} + D\tau^{Gh\pm}z^{-1} + \ldots, \\
\omega^{Gh}(z)j^{Gh} &= j^{Gh}z^{-2} + Dj^{Gh}z^{-1} + \ldots, \\
j^{Gh}(z)\tau^{Gh\pm} &= \mp\tau^{Gh\pm}z^{-1} + \ldots, \\
j^{Gh}(z)j^{Gh} &= -2z^{-2} + \ldots, \\
\tau^{Gh+}(z)\tau^{Gh-} &= 8z^{-3} - 4j^{Gh}z^{-2} + 4(\omega^{Gh} - \frac{1}{2}Dj^{Gh})z^{-1} + \ldots, \\
\tau^{Gh-}(z)\tau^{Gh+} &= 8z^{-3} + 4j^{Gh}z^{-2} + 4(\omega^{Gh} + \frac{1}{2}Dj^{Gh})z^{-1} + \ldots \;.
\end{aligned}$$

$\tau^{Gh+}(z)\tau^{Gh+}$ *and* $\tau^{Gh}(z)\tau^{Gh-}$ *contain only nonsingular terms.*

Proof:
Let $k \geq 0$.

$$
\begin{aligned}
&\omega^{Gh}_k \omega^{Gh} = \\
&\quad 4\sum_{j\geq 0}(Dc)_{-1-j}\{b_{k+j}, c_{-2}\}b + 2\sum_{j\geq 0}(Dc)_{-1-j}(Db)_{-1}\{b_{k+j}, c_{-1}\}1 \\
&\quad +4\sum_{j\geq 0}(-j)b_{k-1-j}(Dc)_{-1}\{c_{j-1}, b_{-1}\}1 + 2\sum_{j\geq 0}(-j)b_{k-1-j}\{c_{j-1}, b_{-2}\}c \\
&\quad +2\sum_{j\geq 0}(Db)_{-1-j}(Dc)_{-1}\{c_{k+j}, b_{-1}\}1 + \sum_{j\geq 0}(Db)_{-1-j}\{c_{k+j}, b_{-2}\}c \\
&\quad +\sum_{j\geq 0}(D\tilde{c})_{-1-j}\{\tilde{b}_{k+j}, \tilde{c}_{-2}\}\tilde{b} + 2\sum_{j\geq 0}(-j)c_{k-1-j}\{b_{j-1}, c_{-2}\}b \\
&\quad +\sum_{j\geq 0}(-j)c_{k-1-j}(Db)_{-1}\{b_{j-1}, c_{-1}\}1 + \sum_{j\geq 0}(-j)\tilde{b}_{k-1-j}(D\tilde{c})_{-1}\{\tilde{c}_{j-1}, \tilde{b}_{-1}\}1 \\
&\quad +\frac{9}{4}\sum_{j\geq 0}(D\gamma^+)_{-1-j}[\beta^-_{k+j}, \gamma^+_{-2}]\beta^- + \frac{3}{4}\sum_{j\geq 0}(D\gamma^+)_{-1-j}(D\beta^-)_{-1}[\beta^-_{k+j}, \gamma^+_{-1}]1 \\
&\quad +\frac{3}{4}\sum_{j\geq 0}(D\beta^-)_{-1-j}(D\gamma^+)_{-1}[\gamma^+_{k+j}, \beta^-_{-1}]1 \\
&\quad +\frac{9}{4}\sum_{j\geq 0}(-j)\beta^-_{k-1-j}(D\gamma^+)_{-1}[\gamma^+_{j-1}, \beta^-_{-1}]1 \\
&\quad +\frac{3}{4}\sum_{j\geq 0}(-j)\beta^-_{k-1-j}[\gamma^+_{j-1}, \beta^-_{-2}]\gamma^+ + \frac{1}{4}\sum_{j\geq 0}(D\beta^-)_{-1-j}[\gamma^+_{k+j}, \beta^-_{-2}]\gamma^+ \\
&\quad +\frac{3}{4}\sum_{j\geq 0}(-j)\gamma^+_{k-1-j}[\beta^-_{j-1}, \gamma^+_{-2}]\beta^- + \frac{1}{4}\sum_{j\geq 0}(-j)\gamma^+_{k-1-j}(D\beta^-)_{-1}[\beta^-_{j-1}, \gamma^+_{-1}]1 \\
&\quad +\frac{9}{4}\sum_{j\geq 0}(-j)\beta^+_{k-1-j}(D\gamma^-)_{-1}[\gamma^-_{j-1}, \beta^+_{-1}]1 + \frac{9}{4}\sum_{j\geq 0}(D\gamma^-)_{-1-j}[\beta^+_{k+j}, \gamma^-_{-2}]\beta^+ \\
&\quad +\frac{3}{4}\sum_{j\geq 0}(D\gamma^-)_{-1-j}(D\beta^+)_{-1}[\beta^+_{k+j}, \gamma^-_{-1}]1 + \frac{3}{4}\sum_{j\geq 0}(-j)\beta^+_{k-1-j}[\gamma^-_{j-1}, \beta^+_{-2}]\gamma^- \\
&\quad +\frac{3}{4}\sum_{j\geq 0}(D\beta^+)_{-1-j}(D\gamma^-)_{-1}[\gamma^-_{k+j}, \beta^+_{-1}]1 + \frac{3}{4}\sum_{j\geq 0}(-j)\gamma^-_{k-1-j}[\beta^+_{j-1}, \gamma^-_{-2}]\beta^+ \\
&\quad +\frac{1}{4}\sum_{j\geq 0}(D\beta^+)_{-1-j}[\gamma^-_{k+j}, \beta^+_{-2}]\gamma^- + \frac{1}{4}\sum_{j\geq 0}(-j)\gamma^-_{k-1-j}[(D\beta^+)_{j-1}, \gamma^-_{-1}]1 \\
&= \quad 8\delta_k c_{-3}b + 4\delta_{k-1}(Dc)_{-1}b + 2\delta_k(Dc)_{-1}(Db) - 4\{b_{k-2}, c_{-2}\}1 \\
&\quad +4(Dc)_{-1}b_{k-2}1 - 4\{b_{k-3}, c_{-1}\}1 + 4c_{-1}b_{k-3}1 \\
&\quad +2\delta_k(Db)_{-1}(Dc) + 2\delta_k b_{-3}c + \delta_{k-1}(Db)_{-1}c + 2\delta_k c_{-3}\tilde{b} + \delta_{k-1}(D\tilde{c})_{-1}\tilde{b} \\
&\quad -4\{c_{k-3}, b_{-1}\}1 + 4b_{-1}c_{k-3}1 - \{c_{k-2}, b_{-2}\}1 + (Db)_{-1}c_{k-2}1
\end{aligned}
$$

$$
\begin{aligned}
& -\{\tilde{b}_{k-2},\tilde{c}_{-2}\}1+(D\tilde{c})_{-1}\tilde{b}_{k-2}1-\frac{9}{2}\delta_k\gamma^+_{-3}\beta^- -\frac{9}{4}\delta_{k-1}(D\gamma^+)_{-1}\beta^- \\
& +\frac{3}{4}\delta_k(D\beta^-)_{-1}(D\gamma^+)-\frac{9}{4}[\beta^-_{k-2},\gamma^+_{-2}]1-\frac{9}{4}(D\gamma^+)_{-1}\beta^-_{k-2}1 \\
& -\frac{3}{2}[\beta^-_{k-3},\gamma^+_{-1}]1-\frac{3}{2}\gamma^+_{-1}\beta^-_{k-3}1+\frac{1}{2}\delta_k\beta^-_{-3}\gamma^+ +\frac{1}{4}\delta_{k-1}(D\beta^-)_{-1}\gamma^+ \\
& +\frac{3}{2}[\gamma^+_{k-3},\beta^-_{-1}]1+\frac{3}{2}\beta^-_{-1}\gamma^+_{k-3}1+\frac{1}{4}[\gamma^+_{k-2},\beta^-_{-2}]1 \\
& +\frac{1}{4}(D\beta^-)_{-1}\gamma^+_{k-2}1-\frac{9}{4}[\beta^+_{k-2},\gamma^-_{-2}]1-\frac{9}{4}(D\gamma^-)_{-1}\beta^+_{k-2}1 \\
& -\frac{9}{2}\delta_k\gamma^-_3\beta^+ -\frac{9}{4}\delta_{k-1}(D\gamma^-)_{-1}\beta^+ -\frac{3}{4}\delta_k(D\gamma^-)_{-1}(D\beta^+) \\
& -\frac{3}{2}[\beta^+_{k-3},\gamma^-_{-1}]1-\frac{3}{2}\gamma^-_{-1}\beta^+_{k-3}1+\frac{3}{4}\delta_k(D\beta^+)_{-1}(D\gamma^-) \\
& +\frac{3}{2}[\gamma^-_{k-3},\beta^+_{-1}]1+\frac{3}{2}\beta^+_{-1}\gamma^-_{k-3}1+\frac{1}{2}\delta_k\beta^+_{-3}\gamma^- \\
& +\frac{1}{4}\delta_{k-1}(D\beta^+)_{-1}\gamma^- +\frac{1}{4}[\gamma^-_{k-2},\beta^+_{-2}]1+\frac{1}{4}(D\beta^+)_{-1}\gamma^-_{k-2}1 \\
= \; & \delta_k(8c_{-3}b+2(Dc)_{-1}(Db)+4(Dc)_{-1}(Db)+4c_{-1}b_{-3}1 \\
& +2(Db)_{-1}(Dc)+2b_{-3}c+2\tilde{c}_{-3}\tilde{b}+4b_{-1}c_{-3}1 \\
& +(Db)_{-1}(Dc)+(D\tilde{c})_{-1}(D\tilde{b})-\frac{9}{2}\gamma^+_{-3}\beta^- -\frac{3}{4}(D\gamma^+)_{-1}(D\beta^-) \\
& +\frac{3}{4}(D\beta^-)_{-1}(D\gamma^+)-\frac{9}{4}(D\gamma^+)_{-1}(D\beta^-)-\frac{3}{2}\gamma^+_{-1}\beta^-_{-3}1 \\
& +\frac{1}{2}\beta^-_{-3}\gamma^+ +\frac{3}{2}\beta^-_{-1}\gamma^+_{-3}1+\frac{1}{4}(D\beta^-)_{-1}(D\gamma^+)-\frac{9}{4}(D\gamma^-)_{-1}(D\beta^+) \\
& -\frac{9}{2}\gamma^-_{-3}\beta^+ -\frac{3}{4}(D\gamma^-)_{-1}(D\beta^+)-\frac{3}{2}\gamma^-_{-1}\beta^+_{-3}1+\frac{3}{4}(D\beta^+)_{-1}(D\gamma^-) \\
& +\frac{3}{2}\beta^+_{-1}\gamma^-_{-3}1+\frac{1}{2}\beta^+_{-3}\gamma^- +\frac{1}{4}(D\beta^+)_{-1}(D\gamma^-)) \\
& +\delta_{k-1}(4(Dc)_{-1}b+4(Dc)_{-1}b+4c_{-1}(Db)+(Db)_{-1}c \\
& +(D\tilde{c})_{-1}\tilde{b}+4b_{-1}(Dc)+(Db)_{-1}c+(D\tilde{c})_{-1}\tilde{b} \\
& -\frac{9}{4}(D\gamma^+)_{-1}\beta^- -\frac{9}{4}(D\gamma^+)_{-1}\beta^- -\frac{3}{2}\gamma^+_{-1}(D\beta^-) \\
& +\frac{1}{4}(D\beta^-)_{-1}\gamma^+ +\frac{3}{2}\beta^-_{-1}(D\gamma^+)+\frac{1}{4}(D\beta^-)_{-1}\gamma^+ \\
& -\frac{9}{4}(D\gamma^-)_{-1}\beta^+ -\frac{9}{4}(D\gamma^-)_{-1}\beta^+ -\frac{3}{2}\gamma^-_{-1}(D\beta^+) \\
& +\frac{3}{2}\beta^+_{-1}(D\gamma^-)+\frac{1}{4}(D\beta^+)_{-1}\gamma^- +\frac{1}{4}(D\beta^+)_{-1}\gamma^-) \\
& +\delta_{k-2}(4c_{-1}b+4b_{-1}c-\frac{3}{2}\gamma^+_{-1}\beta^- +\frac{3}{2}\beta^-_{-1}\gamma^+ -\frac{3}{2}\gamma^-_{-1}\beta^+ +\frac{3}{2}\beta^+_{-1}\gamma^+) \\
& +\delta_{k-3}(-4-4-4-1-1+\frac{9}{4}+\frac{3}{2}+\frac{3}{2}+\frac{1}{4}+\frac{9}{4}+\frac{3}{2}+\frac{3}{2}+\frac{1}{4}) \\
& +\delta_k(4c_{-3}b+3(Dc)_{-1}(Db)-2b_{-3}c+2\tilde{c}_{-3}\tilde{b}+(D\tilde{c})_{-1}(D\tilde{b})-3\gamma^+_{-3}\beta^-
\end{aligned}
$$

$$
\begin{aligned}
& -2(D\gamma^+)_{-1}(D\beta^-) - \gamma^+_{-1}\beta^-_{-3}1 - 2(D\gamma^-)_{-1}(D\beta^+) - 3\gamma^-_{-3}\beta^+ - \beta^+_{-3}\gamma^-) \\
& +\delta_{k-1}(4(Dc)_{-1}b + 2c_{-1}(Db) + 2(D\tilde{c})_{-1}\tilde{b} - 3(D\gamma^+)_{-1}\beta^- - (D\beta^-)_{-1}\gamma^+ \\
& -3(D\gamma^-)_{-1}\beta^+ - (D\beta^+)_{-1}\gamma^-) - \delta_{k-3} \\
= \quad & D\omega^{Gh}\delta_k + 2\omega^{Gh}\delta_{k-1} - 3\delta_{k-3}.
\end{aligned}
$$

The operator product expansion $\omega^{Gh}(z)\omega^{Gh}$ follows also from $\omega^{Gh} = \omega^\sigma + \omega^{\tilde{\sigma}} + \omega^\phi + \omega^{\tilde{\phi}} + \omega^\chi + \omega^{\tilde{\chi}}$ (Proposition 5.24) and the fact that the elements $\omega^\sigma, \omega^{\tilde{\sigma}}, \omega^\phi, \omega^{\tilde{\phi}}, \omega^\chi, \omega^{\tilde{\chi}}$ generate a representation of the Virasoro algebra on V of central charge $-26, -2, 13, 13, -2, -2$.

$$
\begin{aligned}
& \omega^{Gh}_k \tau^{Gh+} = \\
& \quad 2\sum_{j\geq 0}(Dc)_{-1-j}(D\beta^+)_{-1}\{b_{k+j}, c_{-1}\}1 + 3\sum_{j\geq 0}(Dc)_{-1-j}\beta^+_{-1}\{b_{k+j}, c_{-2}\}1 \\
& \quad -8\sum_{j\geq 0} j b_{k-1-j}\gamma^+_{-1}\{c_{j-1}, b_{-1}\}1 + 4\sum_{j\geq 0}(Db)_{-1-j}\gamma^+_{-1}\{c_{j+k}, b_{-1}\}1 \\
& \quad -\sum_{j\geq 0} j c_{k-1-j}(D\beta^+)_{-1}\{b_{j-1}, c_{-1}\}1 - \frac{3}{2}\sum_{j\geq 0} c_{k-1-j}\beta^+_{-1}\{b_{j-1}, c_{-2}\}1 \\
& \quad -\sum_{j\geq 0}(D\tilde{c})_{-1-j}\beta^+_{-1}\{\tilde{b}_{k+j}, \tilde{c}_{-1}\}1 - 4\sum_{j\geq 0}\tilde{b}_{k-1-j}(D\gamma^+)_{-1}\{\tilde{c}_{j-1}, \tilde{b}_{-1}\}1 \\
& \quad -2\sum_{j\geq 0}\tilde{b}_{k-1-j}\gamma^+_{-1}\{\tilde{c}_{j-1}, \tilde{b}_{-2}\}1 + 6\sum_{j\geq 0}(D\gamma^+)_{-1-j}b_{-1}[\beta^-_{k+j}, \gamma^+_{-1}]1 \\
& \quad +6\sum_{j\geq 0}(D\gamma^+)_{-1-j}\tilde{b}_{-1}[\beta^-_{k+j}, \gamma^+_{-2}]1 + 3\sum_{j\geq 0}(D\gamma^+)_{-1-j}(D\tilde{b})_{-1}[\beta^+_{k+j}, \gamma^+_{-1}]1 \\
& \quad -2\sum_{j\geq 0} j\gamma^+_{k-1-j}b_{-1}[\beta^-_{j-1}, \gamma^+_{-1}]1 - 2\sum_{j\geq 0} j\gamma^+_{k-1-j}\tilde{b}_{-1}[\beta^-_{j-1}, \gamma^+_{-2}]1 \\
& \quad -\sum_{j\geq 0} j\gamma^+_{k-1-j}(D\tilde{b})_{-1}[\beta^-_{j-1}, \gamma^+_{-1}]1 + \frac{3}{2}\sum_{j\geq 0}\beta^+_{k-1-j}c_{-1}[\gamma^-_{j-1}, \beta^+_{-2}]1 \\
& \quad +\frac{9}{4}\sum_{j\geq 0} j\beta^+_{k-1-j}(Dc)_{-1}[\gamma^-_{j-1}, \beta^+_{-1}]1 - \frac{3}{2}\sum_{j\geq 0} j\beta^+_{k-1-j}\tilde{c}_{-1}[\gamma^-_{j-1}, \beta^+_{-1}]1 \\
& \quad -\frac{1}{2}\sum_{j\geq 0}(D\beta^+)_{-1-j}c_{-1}[\gamma^-_{k+j}, \beta^+_{-2}]1 \\
& \quad -\frac{3}{4}\sum_{j\geq 0}(D\beta^+)_{-1-j}(Dc)_{-1}[\gamma^-_{k+j}, \beta^+_{-1}]1 \\
& \quad +\frac{1}{2}\sum_{j\geq 0}(D\beta^+)_{-1-j}\tilde{c}_{-1}[\gamma^-_{k+j}, \beta^+_{-1}]1
\end{aligned}
$$

$$
\begin{aligned}
= \;& \delta_k \left(\frac{5}{2}(Dc)_{-1}(D\beta^+) + 2c_{-1}\beta^+_{-3}1 + 3c_{-3}\beta^+ - 4(Db)_{-1}\gamma^+ \right. \\
& -4b_{-1}(D\gamma^+) - 8(D\tilde{b})_{-1}(D\gamma^+) - 8\tilde{b}_{-1}\gamma^+_{-3}1 - 4\tilde{b}_{-3}\gamma^+ \\
& \left. -(D\tilde{c})_{-1}\beta^+ - \tilde{c}_{-1}(D\beta^+) \right) \\
& +\delta_{k-1} \left(\frac{3}{2}c_{-1}(D\beta^+) + \frac{9}{4}(Dc)_{-1}\beta^+ - 6b_{-1}\gamma^+ - 6\tilde{b}_{-1}(D\gamma^+) \right. \\
& \left. -3(D\tilde{b})_{-1}\gamma^+ - \frac{3}{2}\tilde{c}_{-1}\beta^+ \right) \\
& +\delta_{k-2}(-3c_{-1}\beta^+ - 4\tilde{b}_{-1}\gamma^+ + 4\gamma^+_{-1}\tilde{b} + 3\beta^+_{-1}c) \\
= \;& \delta_k D\tau^{Gh+} + \delta_{k-2}\frac{3}{2}\tau^{Gh+}
\end{aligned}
$$

and for $\omega_k^{Gh}\tau^{Gh-}$ analogously.

$$
\begin{aligned}
\omega_k^{Gh} j^{Gh} = \;& \\
& -2\sum_{j\geq 0}(d)_{-1-j}b_{k+j}(D\tilde{b})_{-1}c + 2\sum_{j\geq 0}(Dc)_{-1-j}b_{k+j}(Dc)_{-1}\tilde{b} \\
& -\sum_{j\geq 0}c_{k-1-j}(Db)_j(D\tilde{b})_{-1}c + \sum_{j\geq 0}c_{k-1-j}(Db)_j(Dc)_{-1}\tilde{b} \\
& +\sum_{j\geq 0}\tilde{b}_{k-1-j}(D\tilde{c})_j(D\tilde{b})_{-1}c - \sum_{j\geq 0}\tilde{b}_{k-1-j}(D\tilde{c})_j(Dc)_{-1}\tilde{b} \\
& -\frac{3}{2}\sum_{j\geq 0}(D\gamma^+)_{-1-j}[\beta^-_{k+j},\gamma^+_{-2}]\beta^- - \frac{3}{2}\sum_{j\geq 0}(-j)\beta^-_{k-1-j}[\gamma^+_{j-1},\beta^-_{-1}]\gamma^+ \\
& -\frac{1}{2}\sum_{j\geq 0}(D\beta^-)_{-1-j}[\gamma^+_{k+j},\beta^-_{-1}]\gamma^+ - \frac{1}{2}\sum_{j\geq 0}(-j)\gamma^+_{k-1-j}[\beta^-_{j-1},\gamma^+_{-1}]\beta^- \\
& +\frac{3}{2}\sum_{j\geq 0}(D\gamma^-)_{-1-j}[\beta^+_{k+j},\gamma^-_{-1}]\beta^+ + \frac{3}{2}\sum_{j\geq 0}(-j)\beta^+_{k-1-j}[\gamma^-_{j-1},\beta^+_{-1}]\gamma^- \\
& +\frac{1}{2}\sum_{j\geq 0}(D\beta^+)_{-1-j}[\gamma^-_{k+j},\beta^+_{-1}]\gamma^- + \frac{1}{2}\sum_{j\geq 0}(-j)\gamma^-_{k-1-j}[\beta^+_{j-1},\gamma^-_{-1}]\beta^+ \\
= \;& 2\sum_{j\geq 0}(Dc)_{-1-j}(D\tilde{b})_{-1}\{b_{k+j},c_{-1}\}1 + 2\sum_{j\geq 0}(Dc)_{-1-j}\{b_{k+j},c_{-2}\}\tilde{b} \\
& +\sum_{j\geq 0}(-j)c_{k-1-j}(D\tilde{b})_{-1}\{b_{j-1},c_{-1}\}1 + \sum_{j\geq 0}(-j)c_{k-1-j}\{b_{j-1},c_{-2}\}\tilde{b} \\
& +\sum_{j\geq 0}(-j)\tilde{b}_{k-1-j}\{\tilde{c}_{j-1},\tilde{b}_{-2}\}c + \sum_{j\geq 0}(-j)\tilde{b}_{k-1-j}(Dc)_{-1}\{\tilde{c}_{j-1},\tilde{b}_{-1}\}1 \\
& +\frac{3}{2}\delta_k(D\gamma^+)_{-1}\beta^- + \frac{3}{2}[\beta^-_{k-2},\gamma^+_{-1}]1 + \frac{3}{2}\gamma^+_{-1}\beta^-_{k-2}1 \\
& -\frac{1}{2}\delta_k(D\beta^-)_{-1}\gamma^+ - \frac{1}{2}[\gamma^+_{k-2},\beta^-_{-1}]1 - \frac{1}{2}\beta^-_{-1}\gamma^+_{k-2}1
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3}{2}\delta_k(D\gamma^-)_{-1}\beta^+ - \frac{3}{2}[\beta^+_{k-2},\gamma^-_{-1}]1 - \frac{3}{2}\gamma^-_{-1}\beta^+_{k-2}1 \\
& +\frac{1}{2}\delta_k(D\beta^+)_{-1}\gamma^- + \frac{1}{2}[\gamma^-_{k-2},\beta^+_{-1}]1 + \frac{1}{2}\beta^+_{-1}\gamma^-_{k-2}1 \\
= \quad & 2\delta_k(Dc)_{-1}(D\tilde{b}) + 4\delta_k c_{-3}\tilde{b} + 2\delta_{k-1}(Dc)_{-1}\tilde{b} - c_{k-2}(D\tilde{b}) \\
& -2c_{k-3}\tilde{b} - 2\tilde{b}_{k-3}c - \tilde{b}_{k-2}(Dc) + \frac{3}{2}\delta_k(D\gamma^+)_{-1}\beta^- \\
& -\frac{3}{2}\delta_{k-2} + \frac{3}{2}\delta_k\gamma^+_{-1}(D\beta^-) + \frac{3}{2}\delta_{k-1}\gamma^+_{-1}\beta^- - \frac{1}{2}\delta_k(D\beta^-)_{-1}\gamma^+ \\
& -\frac{1}{2}\delta_{k-2} - \frac{1}{2}\delta_k\beta^-_{-1}(D\gamma^+) - \frac{1}{2}\delta_{k-1}\beta^-_{-1}\gamma^+ - \frac{3}{2}\delta_k(D\gamma^-)_{-1}\beta^+ \\
& +\frac{3}{2}\delta_{k-2} - \frac{3}{2}\delta_k\gamma^-_{-1}(D\beta^+) - \frac{3}{2}\delta_{k-1}\gamma^-_{-1}\beta^+ + \frac{1}{2}\delta_k(D\beta^+)_{-1}\gamma^- \\
& +\frac{1}{2}\delta_{k-2} + \frac{1}{2}\delta_k\beta^+_{-1}(D\gamma^-) + \frac{1}{2}\delta_{k-1}\beta^+_{-1}\gamma^- \\
= \quad & 2\delta_k(Dc)_{-1}(D\tilde{b}) + 4\delta_k c_{-3}\tilde{b} + 2\delta_{k-1}(Dc)_{-1}\tilde{b} - \delta_k(Dc)_{-1}(D\tilde{b}) \\
& -\delta_{k-1}c_{-1}(D\tilde{b}) - 2\delta_k c_{-3}\tilde{b} - 2\delta_{k-1}(Dc)_{-1}\tilde{b} - 2\delta_{k-2}c_{-1}\tilde{b} \\
& -\delta_k(D\tilde{b})_{-1}(Dc) - \delta_{k-1}\tilde{b}_{-1}(Dc) + \frac{3}{2}\delta_k(D\gamma^+)_{-1}\beta^- - \frac{3}{2}\delta_{k-2} \\
& +\frac{3}{2}\delta_k\gamma^+_{-1}(D\beta^-) + \frac{3}{2}\delta_{k-1}\gamma^+_{-1}\beta^- - \frac{1}{2}\delta_k(D\beta^-)_{-1}\gamma^+ + \frac{1}{2}\delta_{k-2} \\
& -\frac{1}{2}\delta_k\beta^-_{-1}(D\gamma^-) - \frac{1}{2}\delta_{k-1}\beta^-_{-1}\gamma^+ - \frac{3}{2}\delta_k(D\gamma^-)_{-1}\beta^+ + \frac{3}{2}\delta_{k-2} \\
& -\frac{3}{2}\delta_k\gamma^-_{-1}(D\beta^+) - \frac{3}{2}\delta_{k-1}\gamma^-_{-1}\beta^+ + \frac{1}{2}\delta_k(D\beta^+)_{-1}\gamma^- - \frac{1}{2}\delta_{k-2}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\delta_k\beta^+_{-1}(D\gamma^-) + \frac{1}{2}\delta_{k-1}\beta^+_{-1}\gamma^- - 2\delta_k\tilde{b}_{-3}c - 2\delta_{k-1}(D\tilde{b})_{-1}c \\
& -2\delta_{k-2}\tilde{b}_{-1}c \\
= \quad & \delta_k(2(Dc)_{-1}(D\tilde{b}) - (Dc)_{-1}(D\tilde{b}) - (D\tilde{b})_{-1}(Dc) + 4c_{-3}\tilde{b} \\
& -2c_{-3}\tilde{b} - 2\tilde{b}_{-3}c - \frac{3}{2}(D\gamma^-)_{-1}\beta^+ + \frac{1}{2}\beta^+_{-1}(D\gamma^-) \\
& -\frac{3}{2}\gamma^-_{-1}(D\beta^+) + \frac{1}{2}(D\beta^+)_{-1}\gamma^- + \frac{3}{2}(D\gamma^+)_{-1}\beta^- - \frac{1}{2}\beta^-_{-1}(D\gamma^+) \\
& +\frac{3}{2}\gamma^+_{-1}(D\beta^-) - \frac{1}{2}(D\beta^-)_{-1}\gamma^+) \\
& +\delta_{k-1}(2(Dc)_{-1}\tilde{b} - 2(Dc)_{-1}\tilde{b} - \tilde{b}_{-1}(Dc) - c_{-1}(D\tilde{b}) \\
& -2(D\tilde{b})_{-1}c + \frac{3}{2}\gamma^+_{-1}\beta^- - \frac{1}{2}\beta^-_{-1}\gamma^+ - \frac{3}{2}\gamma^-_{-1}\beta^+ \\
& +\frac{1}{2}\beta^+_{-1}\gamma^-) \\
& +\delta_{k-2}(-2c_{-1}\tilde{b} - \frac{3}{2} + \frac{1}{2} + \frac{3}{2} - \frac{1}{2} - 2\tilde{b}_{-1}c)
\end{aligned}
$$

$$
\begin{aligned}
&= \delta_k(2(Dc)_{-1}(D\tilde{b}) + 2c_{-3}\tilde{b} - 2\tilde{b}_{-3}c - (D\gamma^-)_{-1}\beta^+ \\
&\quad -\gamma^-_{-1}(D\beta^+) + (D\gamma^+)_{-1}\beta^- + \gamma^+_{-1}(D\beta^-)) \\
&\quad +\delta_{k-1}((Dc)_{-1}\tilde{b} - (D\tilde{b})_{-1}c + \gamma^+_{-1}\beta^- - \beta^+_{-1}\gamma^-) \\
&= \delta_k Dj^{Gh} + \delta_{k-1}j^{Gh}. \\
j^{Gh}_k\tau^{Gh+} &= -4\sum_{j\geq 0}\gamma^+_{-1-j}b_{-1}[\beta^-_{k+j},\gamma^+_{-1}]1 - 4\sum_{j\geq 0}\gamma^+_{-1-j}\tilde{b}_{-1}[\beta^-_{k+j},\gamma^+_{-2}]1 \\
&\quad -2\sum_{j\geq 0}\gamma^+_{-1-j}(D\tilde{b})_{-1}[\beta^-_{k+j},\gamma^+_{-1}]1 - \sum_{j\geq 0}\beta^+_{-1-j}c_{-1}[\gamma^-_{k+j},\beta^+_{-2}]1 \\
&\quad -\frac{3}{2}\sum_{j\geq 0}\beta^+_{-1-j}(Dc)_{-1}[\gamma^-_{k+j},\beta^+_{-1}]1 + \sum_{j\geq 0}\beta^+_{-1-j}\tilde{c}_{-1}[\gamma^-_{k+j},\beta^+_{-1}]1 \\
&\quad +4\sum_{j\geq 0}(D\tilde{b})_{-1-j}\gamma^+_{-1}\{c_{k+j},b_{-1}\}1 + \sum_{j\geq 0}jc_{k-j-1}\beta^+_{-1}\{\tilde{b}_{j-1},\tilde{c}_{-1}\}1 \\
&\quad -\sum_{j\geq 0}(Dc)_{-1-j}\beta^+_{-1}\{\tilde{b}_{k+j},\tilde{c}_{-1}\}1 - 4\sum_{j\geq 0}j\tilde{b}_{k-j-1}\gamma^+_{-1}\{c_{j-1},b_{-1}\}1 \\
&= 4\delta_k\gamma^+_{-1}b + 4\delta_{k-1}\gamma^+_{-1}\tilde{b} + 4\delta_k(D\gamma^+)_{-1}\tilde{b} + 2\delta_k\gamma^+_{-1}(D\tilde{b}) \\
&\quad -\delta_{k-1}\beta^+_{-1}c - \delta_k(D\beta^+)_{-1}c - \frac{3}{2}\delta_k\beta^+_{-1}(Dc) + \delta_k\beta^+_{-1}\tilde{c} \\
&\quad +4\delta_k(D\tilde{b})_{-1}\gamma^+ + \delta_k(Dc)_{-1}\beta^+ + \delta_{k-1}c_{-1}\beta^+ \\
&\quad -\delta_k(Dc)_{-1}\beta^+ - 4\delta_k(D\tilde{b})_{-1}\gamma^+ - 4\delta_{k-1}\tilde{b}_{-1}\gamma^+ \\
&= \delta_k(4\gamma^+_{-1}b + 4(D\gamma^+)_{-1}\tilde{b} + 2\gamma^+_{-1}(D\tilde{b}) - (D\beta^+)_{-1}c \\
&\quad -\frac{3}{2}\beta^+_{-1}(Dc) + \beta^+_{-1}\tilde{c}) = -\delta_k\tau^{Gh+}. \\
j^{Gh}_k\tau^{Gh-} &= \sum_{j\geq 0}\beta^-_{k-j-1}c_{-1}[\gamma^+_j,\beta^-_{-2}]1 + \frac{3}{2}\sum_{j\geq 0}\beta^-_{k-j-1}(Dc)_{-1}[\gamma^+_j,\beta^-_{-1}]1 \\
&\quad +\sum_{j\geq 0}\beta^-_{k-j-1}\tilde{c}_{-1}[\gamma^+_j,\beta^-_{-1}]1 + 4\sum_{j\geq 0}\gamma^-_{k-j-1}b_{-1}[\beta^+_j,\gamma^-_{-1}]1 \\
&\quad -4\sum_{j\geq 0}\gamma^-_{k-1-j}\tilde{b}_{-1}[\beta^+_j,\gamma^-_{-2}]1 - 2\sum_{j\geq 0}\gamma^-_{k-1-j}(D\tilde{b})_{-1}[\beta^+_j,\gamma^-_{-1}]1 \\
&\quad +4\sum_{j\geq 0}(D\tilde{b})_{-1-j}\gamma^-_{-1}\{c_{k+j},b_{-1}\}1 - \sum_{j\geq 0}jc_{k-1-j}\beta^-_{-1}\{\tilde{b}_{j-1},\tilde{c}_{-1}\}1 \\
&\quad +\sum_{j\geq 0}(Dc)_{-1-j}\beta^-_{-1}\{\tilde{b}_{k+j},\tilde{c}_{-1}\}1 - 4\sum_{j\geq 0}j\tilde{b}_{k-1-j}\gamma^-_{-1}\{c_{j-1},b_{-1}\}1 \\
&= \delta_k(D\beta^-)_{-1}c + \frac{3}{2}\delta_k\beta^-_{-1}(Dc) + \delta_k\beta^-_{-1}\tilde{c} - 4\delta_k\gamma^-_{-1}b \\
&\quad +4\delta_k(D\gamma^-)_{-1}\tilde{b} + 2\delta_k\gamma^-_{-1}(D\tilde{b}) + 4\delta_k(D\tilde{b})_{-1}\gamma^- - \delta_k(Dc)_{-1}\beta^- \\
&\quad +\delta_k(Dc)_{-1}\beta^- - 4\delta_k(D\tilde{b})_{-1}\gamma^- + \delta_{k-1}\beta^-_{-1}c + 4\delta_{k-1}\gamma^-_{-1}\tilde{b} \\
&\quad -\delta_{k-1}c_{-1}\beta^- - 4\delta_{k-1}\tilde{b}_{-1}\gamma^- = \delta_k\tau^{Gh-}.
\end{aligned}
$$

$$
\begin{aligned}
j_k^{Gh} j^{Gh} &= \sum_{j\geq 0} \gamma^+_{-1-j}[\beta^-_{k+j}, \gamma^+_{-1}]\beta^- + \sum_{j\geq 0} \beta^-_{k-1-j}[\gamma^+_j, \beta^-_{-1}]\gamma^+ \\
&\quad + \sum_{j\geq 0} \beta^+_{-1-j}[\gamma^-_{k+j}, \beta^+_{-1}]\gamma^- + \sum_{j\geq 0} \gamma^-_{k-1-j}[\beta^+_j, \gamma^-_{-1}]\beta^+ \\
&= -\delta_k \gamma^+_{-1}\beta^- + \beta^-_{k-1}\gamma^+ + \delta_k \beta^+_{-1}\gamma^- - \gamma^-_{k-1}\beta^+ \\
&= -\delta_k \gamma^+_{-1}\beta^- + [\beta^-_{k-1}, \gamma^+_{-1}] + \gamma^+_{-1}\beta^-_{k-1}1 \\
&\quad +\delta_k \beta^+_{-1}\gamma^- - [\gamma^-_{k-1}, \beta^+_{-1}]1 - \beta^+_{-1}\gamma^-_{k-1}1 \\
&= -\delta_k \gamma^+_{-1}\beta^- - \delta_{k-1} + \delta_k \gamma^+_{-1}\beta^- + \delta_k \beta^+_{-1}\gamma^- - \delta_{k-1} - \delta_k \beta^+_{-1}\gamma^- \\
&= -2\delta_{k-1}.
\end{aligned}
$$

$$
\begin{aligned}
\tau_k^{Gh+}\tau^{Gh-} &= 4\sum_{j\geq 0}(k+j)c_{-1-j}b_{-1}[\beta^+_{k+j-1}, \gamma^-_{-1}]1 \\
&-4\sum_{j\geq 0}(D\beta^+)_{k-1-j}\{c_j, b_{-1}\}\gamma^- \\
&-4\sum_{j\geq 0}(k+j)c_{-1-j}\tilde{b}_{-1}[\beta^+_{k+j-1}, \gamma^-_{-2}]1 \\
&-2\sum_{j\geq 0}(k+j)c_{-1-j}(D\tilde{b})_{-1}[\beta^+_{k-1-j}, \gamma^-_{-1}]1 \\
&-6\sum_{j\geq 0}(Dc)_{-1-j}b_{-1}[\beta^+_{k+j}, \gamma^-_{-1}]1 - 6\sum_{j\geq 0}(-j)\beta^+_{k-1-j}\{c_{j-1}, b_{-1}\}\gamma^- \\
&+6\sum_{j\geq 0}(Dc)_{-1-j}b_{-1}[\beta^+_{k+j}, \gamma^-_{-2}]1 + 3\sum_{j\geq 0}(Dc)_{-1-j}(D\tilde{b})_{-1}[\beta^+_{k+j}, \gamma^-_{-1}]1 \\
&-4\sum_{j\geq 0}b_{-1-j}c_{-1}[\gamma^+_{k+j}, \beta^-_{-2}]1 - 4\sum_{j\geq 0}\gamma^+_{k-1-j}\{b_j, c_{-1}\}(D\beta^-) \\
&-6\sum_{j\geq 0}b_{-1-j}(Dc)_{-1}[\gamma^+_{k+j}, \beta^-_{-1}]1 - 6\sum_{j\geq 0}\gamma^+_{k-1-j}\{b_j, c_{-2}\}\beta^- \\
&-4\sum_{j\geq 0}b_{-1-j}\tilde{c}_{-1}[\gamma^+_{k+j}, \beta^-_{-1}]1 + 4\sum_{j\geq 0}(k+j)b_{-1-j}c_{-1}[\gamma^+_{k+j-1}, \beta^-_{-2}]1 \\
&-4\sum_{j\geq 0}(D\gamma^+)_{k-1-j}\{\tilde{b}_j, \tilde{c}_{-1}\}\beta^- \\
&+6\sum_{j\geq 0}(k+j)\tilde{b}_{-1-j}(Dc)_{-1}[\gamma^+_{k+j-1}, \beta^-_{-1}]1 \\
&-2\sum_{j\geq 0}(D\tilde{b})_{-1-j}c_{-1}[\gamma^+_{k+j}, \beta^-_{-2}]1 - 2\sum_{j\geq 0}(-j)\gamma^+_{k-1-j}\{\tilde{b}_{j-1}, \tilde{c}_{-1}\}\beta^- \\
&-3\sum_{j\geq 0}(D\tilde{b})_{-1-j}(Dc)_{-1}[\gamma^+_{k+j}, \beta^-_{-1}]1 - 2\sum_{j\geq 0}(D\tilde{b})_{-1-j}\tilde{c}_{-1}[\gamma^+_{k+j}, \beta^-_{-1}]1
\end{aligned}
$$

$$
\begin{aligned}
& +4\sum_{j\geq 0}\tilde{c}_{-1-j}b_{-1}[\beta^+_{k+j},\gamma^-_{-1}]1-4\sum_{j\geq 0}\beta^+_{k-1-j}\{\tilde{c}_j,\tilde{b}_{-1}\}(D\gamma^-)\\
& -4\sum_{j\geq 0}\tilde{c}_{-1-j}\tilde{b}_{-1}[\beta^+_{k+j},\gamma^-_{-2}]1-2\sum_{j\geq 0}\beta^+_{k-1-j}\{\tilde{c}_j,\tilde{b}_{-2}\}\gamma^-\\
& -2\sum_{j\geq 0}\tilde{c}_{-1-j}(D\tilde{b})_{-1}[\beta^+_{k+j},\gamma^-_{-1}]1\\
= & -4\delta_k(Dc)_{-1}b+8\delta_k c_{-3}\tilde{b}+2\delta_k(Dc)_{-1}(D\tilde{b})+6\delta_k(Dc)_{-1}b\\
& -12\delta_k c_{-3}b-4\delta_k(Db)_{-1}c-4\delta_{k-1}c_{-1}b+8\delta_{k-1}(Dc)_{-1}\tilde{b}\\
& +2\delta_{k-1}c_{-1}(D\tilde{b})-4\delta_k\tilde{b}_{-3}c-6\delta_{k-1}(Dc)_{-1}b-4\delta_{k-1}b_{-1}c\\
& +8\delta_{k-2}c_{-1}\tilde{b}-2\delta_{k-1}(D\tilde{b})_{-1}c+6\delta_k(D\tilde{b})_{-1}(Dc)+4\delta_k(D\tilde{b})_{-1}\tilde{c}\\
& +6\delta_{k-1}\tilde{b}_{-1}(Dc)+4\delta_{k-1}\tilde{b}_{-1}\tilde{c}-3\delta_k(Dc)_{-1}(D\tilde{b})-6\delta_k b_{-1}(Dc)\\
& -4\delta_k b_{-1}\tilde{c}+8\delta_k\tilde{b}_{-3}c-3\delta_k(D\tilde{b})_{-1}(Dc)-2\delta_k(D\tilde{b})_{-1}\tilde{c}\\
& +4\delta_k(D\tilde{c})_{-1}\tilde{b}-4\delta_k\tilde{c}_{-1}b+8\delta_{k-1}(D\tilde{b})_{-1}c+4\delta_{k-1}\tilde{c}_{-1}\tilde{b}\\
& +8\delta_{k-2}\tilde{b}_{-1}c+2\delta_k\tilde{c}_{-1}(D\tilde{b})\\
& -4(D\beta^+)_{k-1}\gamma^-(=4\delta_{k-2}-4\delta_k\gamma^-_{-1}(D\beta^+))\\
& +6\beta^+_{k-2}\gamma^-(=-6\delta_{k-2}+6\delta_k\gamma^-_{-1}(D\beta^+)+6\delta_{k-1}\gamma^-_{-1}\beta^+)\\
& -4\gamma^+_{k-1}(D\beta^-)(=-4\delta_{k-2}-4\delta_k(D\beta^-)_{-1}\gamma^+)\\
& -6\gamma^+_{k-2}\beta^-(=-6\delta_{k-2}-6\delta_k\beta^-_{-1}(D\gamma^+)-6\delta_{k-1}\beta^-_{-1}\gamma^+)\\
& -4(D\gamma^+)_{k-1}\beta^-(=-4\delta_{k-2}-4\delta_k\beta^-_{-1}(D\gamma^+))\\
& +2\gamma^+_{k-2}\beta^-(=2\delta_{k-2}+2\delta_k\beta^-_{-1}(D\gamma^+)+2\delta_{k-1}\beta^-_{-1}\gamma^+)\\
& -4\beta^+_{k-1}(D\gamma^-)(=+4\delta_{k-2}-4\delta_k(D\gamma^-)_{-1}\beta^+)\\
& -2\beta^+_{k-2}\gamma^-(=2\delta_{k-2}-2\delta_k\gamma^-_{-1}(D\beta^+)-2\delta_{k-1}\gamma^-_{-1}\beta^+)\\
= & \ \delta_k(8(Dc)_{-1}b-4(Db)_{-1}c+4(D\tilde{c})_{-1}\tilde{b}+4\tilde{b}_{-3}c\\
& -4c_{-3}\tilde{b}+4(D\tilde{b})_{-1}(Dc)-8(D\gamma^+)_{-1}\beta^--4(D\beta^-)_{-1}\gamma^+\\
& -4(D\gamma^-)_{-1}\beta^+)\\
& +\delta_{k-1}(-4\gamma^+_{-1}\beta^-+4\beta^+_{-1}\gamma^-+4(D\tilde{b})_{-1}c-4(Dc)_{-1}\tilde{b})\\
& -8\delta_{k-2}\\
= & \ (4\omega^{Gh}-2Dj^{Gh})\delta_k-4j^{Gh}\delta_{k-1}-8\delta_{k-2}.
\end{aligned}
$$

□

Proposition 5.30
The operators $L_m^{Gh} = \omega_{m+1}^{Gh}, G_n^{Gh\pm} = \tau_{n+\frac{1}{2}}^{Gh\pm}$ *and* j_m^{Gh} *give a representation of the* N*=2-extension of the Virasoro algebra of central charge* 6*, i.e.*

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} - \frac{m^3-m}{2}\delta_{m+n}c, \\
[L_m, G_n^{\pm}] &= (\frac{1}{2}m - n)G_{m+n}^{\pm}, \\
[L_m, j_n] &= -nj_{m+n}, \\
[j_m, G_n^{\pm}] &= \mp G_{m+n}^{\pm}, \\
[j_m, j_n] &= -2m\delta_{m+n}, \\
\{G_m^+, G_n^-\} &= 4L_{m+n} - 2(m-n)j_{m+n} - (4m^2-1)\delta_{m+n}, \\
\{G_m^+, G_n^+\} &= \{G_m^-, G_n^-\} = 0
\end{aligned}
$$

holds.

Proof:
This follows from Proposition 5.29. □

Theorem 5.31
Let $\omega = \omega^M + \omega^{Gh}, \tau^{\pm} = \tau^{M\pm} + \tau^{Gh\pm}$ *and* $j = j^M + j^{Gh}$.
Then the operators $L_n = \omega_{n+1}, G_m^{\pm} = \tau_{m+\frac{1}{2}}^{\pm}$ *and* j_m *give a representation of the* N*=2-extension of the Virasoro algebra of central charge zero, i.e.*

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \\
[L_m, G_n^{\pm}] &= (\frac{1}{2}m - n)G_{m+n}^{\pm}, \\
[L_m, j_n] &= -nj_{m+n}, \\
[j_m, G_n^{\pm}] &= \mp G_{m+n}^{\pm}, \\
[j_m, j_n] &= 0, \\
\{G_m^+, G_n^-\} &= 4L_{m+n} - 2(m-n)j_{m+n}, \\
\{G_m^+, G_n^+\} &= \{G_m^-, G_n^-\} = 0
\end{aligned}
$$

holds.

Proof:
This follows from Proposition 5.17 and Proposition 5.30. □

Definition 5.32
Let $a \in V$ *be an eigenvector of* L_0*. We call the* L_0*-eigenvalue of* a *the* **conformal weight** *of* a *and denote it by* $h(a)$*.*

For calculations it is useful to have a formula for the eigenvalues of the operator L_0.

Proposition 5.33
Let $k_i \in L \otimes_{\mathbb{Z}} \mathbb{C}$, $i \in \{1,\ldots,t\}, \mu \in \{\pm\}, m_i, r_1, r_2, n_j \in \mathbb{Z}, m_i > 0$, $i \in \{1,\ldots,t\}, j \in \{1,\ldots,6\}, \alpha \in L^X$. *For*
$v = k_1(-m_1)\cdot \ldots \cdot k_t(-m_t)e^{\alpha}_{-1}e^{r_1\phi^1}_{-1}e^{r_2\phi^2}_{-1}e^{n_1\sigma+n_2\tilde{\sigma}+n_3\chi+n_4\tilde{\chi}+n_5\phi+n_6\tilde{\phi}}$
in V *we have*

$$\begin{aligned} L_0 v &= \left(\sum_{i=1}^{t} m_i + \frac{1}{2}\alpha^2 + \frac{1}{2}n_1(n_1-3) + \frac{1}{2}n_2(n_2-1) + \frac{1}{2}n_3(n_3-1)\right. \\ &\left. +\frac{1}{2}n_4(n_4-1) - \frac{1}{2}n_5(n_5+2) - \frac{1}{2}n_6(n_6+2) + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2\right) v. \end{aligned}$$

Proof:

$$\begin{aligned} \frac{1}{2}\sigma(0)\sigma(0)e^{n_1\sigma} - \frac{3}{2}\sigma(0)e^{n_1\sigma} &= (\frac{1}{2}n_1^2 - \frac{3}{2}n_1)e^{n_1\sigma}, \\ \frac{1}{2}\tilde{\sigma}(0)\tilde{\sigma}(0)e^{n_2\tilde{\sigma}} - \frac{1}{2}\tilde{\sigma}(0)e^{n_2\tilde{\sigma}} &= (\frac{1}{2}n_2^2 - \frac{1}{2}n_2)e^{n_2\tilde{\sigma}}, \\ \frac{1}{2}\chi(0)\chi(0)e^{n_3\chi} - \frac{1}{2}\chi(0)e^{n_3\chi} &= (\frac{1}{2}n_3^2 - \frac{1}{2}n_3)e^{n_3\chi}, \\ -\frac{1}{2}\phi(0)\phi(0)e^{n_5\phi} + \phi(0)e^{n_5\phi} &= (-\frac{1}{2}n_5^2 - n_5)e^{n_5\phi}, \\ \frac{1}{2}\phi^1(0)\phi^1(0)e^{r_1\phi^1} &= \frac{1}{2}(r_1)^2 e^{r_1\phi^1}. \end{aligned}$$

For $n > 0$ we have

$$\begin{aligned} \sum_{k>0} \sigma(-k)\underbrace{[\sigma(k), \sigma(-n)]}_{k\delta_{k-n}} &= n\sigma(-n), \\ \sum_{k>0} -\phi(-k)\underbrace{[\phi(k), \phi(-n)]}_{-k\delta_{k-n}} &= n\phi(-n), \\ \sum_{k>0} \phi^1(-k)[\phi^1(k), \phi^1(-n)] &= n\phi^1(-n), \\ -\frac{1}{2}\sum_{k>0} z^{-}{}_{\mu}(-k)\underbrace{[z^{+\mu}(k), z^{-\nu}(-n)]}_{-2k\delta_{k-n}\eta^{\mu\nu}} &= nz^{-\nu}(-n). \end{aligned}$$

This implies the assertion. $\square$

Proposition 5.34
The conformal weights h of the matter fields are.

matter field	h
$z^{\pm\mu}(-1)$	1
$\psi^{\pm\mu}$	$\frac{1}{2}$

with $\mu \in \{\pm\}$.
The conformal weights h of the ghostfields are.

ghostfield	h
b	2
c	-1
$\tilde{b}$	1
$\tilde{c}$	0
$\gamma^{\pm}$	$-\frac{1}{2}$
$\beta^{\pm}$	$\frac{3}{2}$

Proof:
This follows from Proposition 5.33. $\square$

For the next proposition we need the following lemma.

Lemma 5.35
Let $V_h = \{v \in V | L_0 v = hv\}$. Then

$$(V_p)_n(V_q) \subset V_{p+q-n-1}.$$

Proof: (cf. [S2]).
We have $L_{m-1} = \omega_m$. Hence for $v \in V$

$$\begin{aligned} [L_{m-1}, v_n] &= [\omega_m, v_n] \\ &= \sum_{k \geq 0} \binom{m}{k} (\omega_k v)_{m+n-k} = \sum_{k \geq 0} \binom{m}{k} (L_{k-1} v)_{m+n-k} \end{aligned}$$

so that for $v \in V_p$

$$\begin{aligned} [L_0, v_n] &= \sum_{k \geq 0} \binom{1}{k} (L_{k-1} v)_{1+n-k} \\ &= (L_{-1} v)_{n+1} + (L_0 v)_n \\ &= (Dv)_{n+1} + (L_0 v)_n \\ &= -(n+1) v_n + p v_n = (p-n-1) v_n. \end{aligned}$$

Finally if $v \in V_p, w \in V_q$ then

$$\begin{aligned} L_0 v_n w &= [L_0, v_n]w + v_n L_0 w \\ &= (p-n-1)v_n w + q v_n w. \end{aligned}$$

$\square$

Hence for homogeneous $u, v \in V$ we have

$$h(u_{-1}v) = h(u) + h(v).$$

Recall the definition of the adjoint operators from section 2.3.

Proposition 5.36
We have

$$\begin{gathered} c_n^* = -c_{-n-4}, \quad b_n^* = b_{2-n}, \\ \tilde{c}_n = \tilde{c}^*_{-n-2}, \quad \tilde{b}_n^* = -\tilde{b}_{-n}, \\ (z^{\pm\mu}(-1)_n)^* = -z^{\pm\mu}(-1)_{-n}, \quad (e^\alpha)_n^* = e^\alpha_{-n-2}, \\ \gamma^{\mp}_{-1}\psi^{\pm\mu})^*_n = (\gamma^{\mp}_{-1}\psi^{\pm\mu})_{-n-2}, \quad (\psi^{-\mu}_{-1}\psi^{+\nu})^*_n = -(\psi^{-\mu}_{-1}\psi^{+\nu})_{-n}. \end{gathered}$$

Proof:
We have $L_1 x = 0$ for $x \in \{c, \tilde{c}, b, \tilde{b}, \gamma^{\mp}_{-1}\psi^{\pm\mu}, z^{\pm\mu}, e^\alpha, \gamma^{\mp}_{-1}\psi^{\pm\mu}\}$.
Then Proposition 5.34 implies.

$$\begin{aligned} c_n^* &= -c_{-2-n-2} = -c_{-4-n}, \\ \tilde{b}_n^* &= -\tilde{b}_{2-n-2} = -\tilde{b}_{-n}. \end{aligned}$$

By Lemma 5.35 we have

$$h(\gamma^{\mp}_{-1}\psi^{\pm\mu}) = -\frac{1}{2} + \frac{1}{2} = 0$$

so that

$$(\gamma^{\mp}_{-1}\psi^{\pm\mu})^*_n = (\gamma^{\mp}_{-1}\psi^{\pm\mu})_{-n-2}.$$

The other cases follows analogously. $\square$

5.3 The BRST-operator

The vertex algebra of the $N=2$-string contains physical and non-physical states. By means of the representation of the $N=2$-extension of the Virasoro algebra we construct in this section the BRST-operator Q. The physical states are then described by the cohomology of this operator.
To define the BRST-operator we need the element $j^{BRST} \in V$.

Definition 5.37
We define the **BRST-current** $j^{BRST} \in V$ *by*

$$\begin{aligned} j^{BRST} &= c_{-1}(\omega^M + \frac{1}{2}\omega^{Gh} + \frac{1}{2}a) \\ &+\tilde{c}_{-1}(j^M + \frac{1}{2}j^{Gh} + \frac{1}{2}d) \\ &+\gamma^+_{-1}(\tau^{M-} + \frac{1}{2}\tau^{Gh-}) \\ &+\gamma^-_{-1}(\tau^{M+} + \frac{1}{2}\tau^{Gh+}) \end{aligned}$$

with

$$\begin{aligned} a &= (Db)_{-1}c + (D\tilde{c})_{-1}\tilde{b} \quad \in V, \\ d &= (D\tilde{b})_{-1}c - (Dc)_{-1}\tilde{b} \quad \in V. \end{aligned}$$

We want to construct operators that induce gradings on V. For this we have to know the following.

Proposition 5.38
Let

$$j_0^X = a_1\phi(0) + a_2\tilde{\phi}(0) + a_3\chi(0) + a_4\tilde{\chi}(0) + a_5\sigma(0) + a_6\tilde{\sigma}(0),$$

$a_i \in \mathbb{C}, i \in \{1, \ldots, 6\}$. *Then*

$$j^{BRST} \text{ is eigenvector of } j_0^X \quad \Leftrightarrow \quad \begin{array}{l} a_5 + a_2 + a_4 = 0 \text{ and } a_5 = a_6 \text{ and} \\ a_1 + a_3 = a_2 + a_4. \end{array}$$

Proof:

$$\begin{aligned} j_0^X j^{BRST} &= a_5 c_{-1}(\omega^M + \frac{1}{2}\omega^{Gh} + \frac{1}{2}a) \\ &+a_6\tilde{c}_{-1}(j^M + \frac{1}{2}j^{Gh} + \frac{1}{2}d) \\ &+(-a_2 - a_4)\gamma^+_{-1}(\tau^{M-} + \frac{1}{2}\tau^{Gh-}) \\ &+(-a_1 - a_3)\gamma^-_{-1}(\tau^{M+} + \frac{1}{2}\tau^{Gh+}) \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}a_5\gamma^+_{-1}c_{-1}(D\beta^-) + \frac{1}{2}(a_2+a_4)\gamma^+_{-1}c_{-1}(D\beta^-) \\
& +\frac{1}{2}a_5\gamma^-_{-1}c_{-1}(D\beta^+) + \frac{1}{2}(a_1+a_3)\gamma^-_{-1}c_{-1}(D\beta^+) \\
& +\frac{3}{4}a_5\gamma^+_{-1}(Dc)_{-1}\beta^- + \frac{3}{4}(a_2+a_4)\gamma^+_{-1}(Dc)_{-1}\beta^- \\
& +\frac{3}{4}a_5\gamma^-_{-1}(Dc)_{-1}\beta^+ + \frac{3}{4}(a_1+a_3)\gamma^-_{-1}(Dc)_{-1}\beta^+ \\
& +2a_5\gamma^+_{-1}b_{-1}\gamma^- + 2(a_1+a_3)\gamma^+_{-1}b_{-1}\gamma^- \\
& +2a_5\gamma^-_{-1}b_{-1}\gamma^+ + 2(a_2+a_4)\gamma^-_{-1}b_{-1}\gamma^+ \\
& -2a_6\gamma^+_{-1}\tilde{b}_{-1}(D\gamma^-) - 2(a_1+a_3)\gamma^+_{-1}\tilde{b}_{-1}(D\gamma^-) \\
& -a_6\gamma^+_{-1}(D\tilde{b})_{-1}\gamma^- - (a_1+a_3)\gamma^+_{-1}(D\tilde{b})_{-1}\gamma^- \\
& +2a_6\gamma^-_{-1}\tilde{b}_{-1}(D\gamma^+) + 2(a_2+a_4)\gamma^-_{-1}\tilde{b}_{-1}(D\gamma^+) \\
& +a_6\gamma^-_{-1}(D\tilde{b})_{-1}\gamma^+ + (a_2+a_4)\gamma^-_{-1}(D\tilde{b})_{-1}\gamma^+ \\
& +\frac{1}{2}a_6\gamma^+_{-1}\tilde{c}_{-1}\beta^- + \frac{1}{2}(a_2+a_4)\gamma^+_{-1}\tilde{c}_{-1}\beta^- \\
& -\frac{1}{2}a_6\gamma^-_{-1}\tilde{c}_{-1}\beta^+ - \frac{1}{2}(a_1+a_3)\gamma^-_{-1}\tilde{c}_{-1}\beta^+ \\
= \quad & a_5c_{-1}(\omega^M + \frac{1}{2}\omega^{Gh} + \frac{1}{2}a) \\
& +a_6\tilde{c}_{-1}(j^M + \frac{1}{2}j^{Gh} + \frac{1}{2}d) \\
& +(-a_2-a_4)\gamma^+_{-1}(\tau^{M-} + \frac{1}{2}\tau^{Gh-}) \\
& +(-a_1-a_3)\gamma^-_{-1}(\tau^{M+} + \frac{1}{2}\tau^{Gh+}) \\
& +\gamma^+_{-1}\left(\frac{1}{2}(a_5+a_2+a_4)c_{-1}(D\beta^-) + \frac{3}{4}(a_5+a_2+a_4)(Dc)_{-1}\beta^-\right. \\
& +2(a_5+a_1+a_3)b_{-1}\gamma^- - 2(a_6+a_1+a_3)\tilde{b}_{-1}(D\gamma^-) \\
& \left. -(a_6+a_1+a_3)(D\tilde{b})_{-1}\gamma^- + \frac{1}{2}(a_6+a_2+a_4)\tilde{c}_{-1}\beta^-\right) \\
& +\gamma^-_{-1}\left(\frac{1}{2}(a_5+a_1+a_3)c_{-1}(D\beta^+) + \frac{3}{4}(a_5+a_1+a_3)(Dc)_{-1}\beta^+\right. \\
& +2(a_5+a_2+a_4)b_{-1}\gamma^+ + 2(a_6+a_2+a_4)\tilde{b}_{-1}(D\gamma^+) \\
& \left. +(a_6+a_2+a_4)(D\tilde{b})_{-1}\gamma^+ - \frac{1}{2}(a_6+a_1+a_3)\tilde{c}_{-1}\beta^+\right).
\end{aligned}
$$

Hence j^{BRST} is a eigenvector of j_0 iff $a_5 + a_2 + a_4 = 0$ and $a_5 = a_6$ and $a_1 + a_3 = a_2 + a_4$. $\square$

Proposition 5.38 implies

Proposition 5.39
For $j_0^X = a_1\phi(0) + a_2\tilde{\phi}(0) + a_3\chi(0) + a_4\tilde{\chi}(0) + a_5\sigma(0) + a_6\tilde{\sigma}(0)$ *with* $a_5 + a_1 + a_4 = 0, a_5 = a_6, a_1 + a_3 = a_2 + a_4$, $a_i \in \mathbb{C}$ *for* $i \in \{1, \ldots, 6\}$, *we have*

$$\begin{aligned} j_0^X j^{BRST} &= a_5 c_{-1}(\omega^M + \frac{1}{2}\omega^{Gh} + \frac{1}{2}a) \\ &\quad + a_6 \tilde{c}_{-1}(j^M + \frac{1}{2}j^{Gh} + \frac{1}{2}d) \\ &\quad + (-a_2 - a_4)\gamma^+_{-1}(\tau^{M-} + \frac{1}{2}\tau^{Gh-}) \\ &\quad + (-a_1 - a_3)\gamma^-_{-1}(\tau^{M+} + \frac{1}{2}\tau^{Gh+}) \\ &= a_5 j^{BRST}. \end{aligned}$$

To get gradings on V we define the following elements.

Definition 5.40
We define elements $j^N, j^{P+}, j^{P-} \in V$ *by*

$$j^N = -\chi(-1) - \tilde{\chi}(-1) + \sigma(-1) + \tilde{\sigma}(-1),$$

$$j^{P+} = -\phi(-1) + \chi(-1),$$

$$j^{P-} = -\tilde{\phi}(-1) + \tilde{\chi}(-1)$$

and call the zero mode j_0^N *the* **ghost number operator** *and the zero modes* $j_0^{P\pm}$ *the* **ghost picture operators**.

Proposition 5.41
We have

	j_0^N	j_0^{P+}	j_0^{P-}
$e^{n\phi}$	0	n	0
$e^{n\tilde{\phi}}$	0	0	n
$e^{n\chi}$	$-n$	n	0
$e^{n\tilde{\chi}}$	$-n$	0	n
$e^{n\sigma}$	n	0	0
$e^{n\tilde{\sigma}}$	n	0	0
j^{BRST}	1	0	0

Proof:

$$\begin{aligned}
j_0^N e^{n\phi} &= -\chi(0)e^{n\phi} - \tilde{\chi}(0)e^{n\phi} + \sigma(0)e^{n\phi} + \tilde{\sigma}(0)e^{n\phi} = 0, \\
j_0^{P+} e^{n\phi} &= -\phi(0)e^{n\phi} + \chi(0)e^{n\phi} \\
&= -(n\phi, \phi)e^{n\phi} = -n\underbrace{(\phi, \phi)}_{=-1} e^{n\phi} = ne^{n\phi}, \\
j_0^{P^-} e^{n\phi} &= -\tilde{\phi}(0)e^{n\phi} + \tilde{\chi}(0)e^{n\phi} = 0.
\end{aligned}$$

The other cases are analogous. The last line follows from Prop. 5.39: $j_0^N : a_3 = -1, a_4 = -1, a_5 = a_6 = 1, a_1 = a_2 = 0$. Hence $j_0^N j^{BRST} = j^{BRST}$. The proof of the other two rows in the last line of the table are analogous. □

Definition 5.42
We define the **BRST-Operator** Q *as the zero mode of the BRST-current*

$$Q = j_0^{BRST}.$$

Proposition 5.43

$$[j_0^N, Q] = Q, \qquad [J_0^{P+}, Q] = 0, \qquad [j_0^{P-}, Q] = 0.$$

Proof:
Because of

$$\begin{aligned}
[j_0^x, Q] &= \sum_{k \geq 0} \binom{0}{k} (j_k^x j^{BRST})_k \\
&= (j_0^x j^{BRST})_0, \qquad x \in \{N, P^+, P^-\},
\end{aligned}$$

this follows from Proposition 5.41. □

Hence the BRST-operator increases the ghost number by one and leaves the ghost pictures unchanged.

The following commutation relations are useful for calculations.

Theorem 5.44
Let $\alpha \in L^X$ and $\mu \in \{\pm\}$. Then

$$\begin{aligned}
[Q, z^{\pm\mu}(-1)_n] &= \Big((Dz^{\pm\mu}(-1))_{-1}c + z^{\pm\mu}(-1)_{-1}(Dc) - 2\psi^{\pm\mu}_{-1}(D\gamma^{\pm}) \\
&\quad -2(D\psi^{\pm\mu})_{-1}\gamma^{\pm}\Big)_n, \\
\{Q, \psi^{\pm\mu}_n\} &= \left(-(D\psi^{\pm\mu})_{-1}c - \frac{1}{2}\psi^{\pm\mu}_{-1}(Dc) - 2z^{\pm\mu}(-1)_{-1}\gamma^{\pm} \mp \tilde{c}_{-1}\psi^{\pm\mu}\right)_n, \\
[Q, e^{\alpha}_n] &= \Big(-\frac{1}{2}(z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)c_{-1}e^{\alpha} - \frac{1}{2}(z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)c_{-1}e^{\alpha} \\
&\quad +(z^{-\mu},\alpha)\gamma^{-}_{-1}(\psi^{+}{}_{\mu})_{-1}e^{\alpha} + (z^{+\mu},\alpha)\gamma^{+}_{-1}(\psi^{-}{}_{\mu})_{-1}e^{\alpha}\Big)_n, \\
\{Q, b_n\} &= \omega_n, \\
\{Q, \tilde{b}_n\} &= j_n, \\
\{Q, c_n\} &= \left(c_{-1}(Dc) - 4\gamma^{+}_{-1}\gamma^{-}\right)_n, \\
\{Q, \tilde{c}_n\} &= \left(c_{-1}(D\tilde{c}) + 2\gamma^{+}_{-1}(D\gamma^{-}) - 2\gamma^{-}_{-1}(D\gamma^{+})\right)_n, \\
[Q, \gamma^{\pm}_n] &= (c_{-1}(D\gamma^{\pm}) - \frac{1}{2}\gamma^{\pm}_{-1}(Dc) \mp \tilde{c}_{-1}\gamma^{\pm})_n, \\
[Q, \beta^{\pm}_n] &= \tau^{\pm}_n, \\
[Q, \omega_n] &= 0, \\
[Q, \tau^{\pm}_n] &= 0.
\end{aligned}$$

Proof: We have

$$[Q, v_n] = [j_0^{BRST}, v_n] = \sum_{k\geq 0} \binom{0}{k} \left(j_k^{BRST} v\right)_{n-k} = \left(j_0^{BRST} v\right)_n \quad , \ v \in V,$$

and

$$\begin{aligned}
Qz^{\pm\mu}(-1) &= \sum_{k\geq 0} c_{-1-k}\omega^M_k z^{\pm\mu}(-1) \\
&\quad + \sum_{k\geq 0} \gamma^{+}_{-1-k}\tau^{M-}_k z^{\pm\mu}(-1) + \sum_{k\geq 0} \gamma^{-}_{-1-k}\tau^{M+}_k z^{\pm\mu}(-1) \\
&\overset{Prop.\ 5.15}{=} c_{-1}(Dz^{\pm\mu}(-1)) + (Dc)_{-1}z^{\pm\mu}(-1) - 2(D\gamma^{\pm})_{-1}\psi^{\pm\mu} \\
&\quad -2\gamma^{\pm}_{-1}(D\psi^{\pm\mu}), \\
Q\psi^{\pm\mu} &= \sum_{k\geq 0} c_{-1-k}\omega^M_k \psi^{\pm\mu} + \sum_{k\geq 0} \tilde{c}_{-1-k} j^M_k \psi^{\pm\mu} \\
&\quad + \sum_{k\geq 0} \gamma^{+}_{-1-k}\tau^{M-}_k \psi^{\pm\mu} + \sum_{k\geq 0} \gamma^{-}_{-1-k}\tau^{M+}_k \psi^{\pm\mu} \\
&\overset{Prop.\ 5.15}{=} c_{-1}(D\psi^{\pm\mu}) + \frac{1}{2}(Dc)_{-1}\psi^{\pm\mu} \mp \tilde{c}_{-1}\psi^{\pm\mu} - 2\gamma^{\pm}_{-1}z^{\pm\mu}(-1),
\end{aligned}$$

$$
\begin{aligned}
Qe^{\alpha} &= \sum_{k\geq 0} c_{-1-k}\omega^{M}_{k}e^{\alpha} + \sum_{k\geq 0}\gamma^{+}_{-1-k}\tau^{M-}_{k}e^{\alpha} \\
&\quad + \sum_{k\geq 0}\gamma^{-}_{-1-k}\tau^{M+}_{k}e^{\alpha} \\
&= -\frac{1}{2}(z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)_{-1}c_{-1}e^{\alpha} - \frac{1}{2}(z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)c_{-1}e^{\alpha} \\
&\quad +(z^{-\mu},\alpha)\gamma^{-}_{-1}(\psi^{+}{}_{\mu})_{-1}e^{\alpha} + (z^{+\mu},\alpha)\gamma^{+}_{-1}(\psi^{-}{}_{\mu})_{-1}e^{\alpha}, \\
Qc &= \sum_{k\geq 0} c_{-1-k}((Dc)_{-1}b)_{k}c - (\frac{1}{2}\cdot 4)\sum_{k\geq 0}\gamma^{+}_{-1-k}(b_{-1}\gamma^{-})_{k}c \\
&\quad -(\frac{1}{2}\cdot 4)\sum_{k\geq 0}\gamma^{-}_{-1-k}(b_{-1}\gamma^{+})_{k}c \\
&= \delta_{k}\sum_{k\geq 0} c_{-1-k}(Dc) - 2\sum_{k\geq 0}\gamma^{+}_{-1-k}\gamma^{-}_{k-1}1 - 2\sum_{k\geq 0}\gamma^{-}_{-1-k}\gamma^{+}_{k-1}1 \\
&= c_{-1}(Dc) - 2\gamma^{+}_{-1}\gamma^{-} - 2\gamma^{-}_{-1}\gamma^{+} \\
&= c_{-1}(Dc) - 4\gamma^{+}_{-1}\gamma^{-}, \\
Q\tilde{c} &= \sum_{k\geq 0} c_{-1-k}((D\tilde{c})_{-1}\tilde{b})_{k}\tilde{c} \\
&\quad +(\frac{1}{2}\cdot 4)\sum_{k\geq 0}\gamma^{+}_{-1-k}(\tilde{b}_{-1}(D\gamma^{-}))_{k}\tilde{c} + (\frac{1}{2}\cdot 2)\sum_{k\geq 0}\gamma^{+}_{-1-k}((D\tilde{b})_{-1}\gamma^{-})_{k}\tilde{c} \\
&\quad -(\frac{1}{2}\cdot 4)\sum_{k\geq 0}\gamma^{-}_{-1-k}(\tilde{b}_{-1}(D\gamma^{+}))_{k}\tilde{c} - (\frac{1}{2}\cdot 2)\sum_{k\geq 0}\gamma^{-}_{-1-k}((D\tilde{b})_{-1}\gamma^{+})_{k}\tilde{c} \\
&= \sum_{k\geq 0}\delta_{k}c_{-1-k}(D\tilde{c}) + 2\sum_{k\geq 0}\delta_{k}\gamma^{+}_{-1-k}(D\gamma^{-}) \\
&\quad -\sum_{k\geq 0}\gamma^{+}_{-1-k}\gamma^{-}_{k-2}1 - 2\sum_{k\geq 0}\delta_{k}\gamma^{-}_{-1-k}(D\gamma^{+}) + \sum_{k\geq 0}\gamma^{-}_{-1-k}\gamma^{+}_{k-2}1 \\
&= c_{-1}(D\tilde{c}) + 2\gamma^{+}_{-1}(D\gamma^{-}) - \gamma^{+}_{-1}(D\gamma^{-}) - (D\gamma^{+})_{-1}\gamma^{-} \\
&\quad -2\gamma^{-}_{-1}(D\gamma^{+}) + \gamma^{-}_{-1}(D\gamma^{+}) + (D\gamma^{-})_{-1}\gamma^{+} \\
&= c_{-1}(D\tilde{c}) + 2\gamma^{+}_{-1}(D\gamma^{-}) - 2\gamma^{-}_{-1}(D\gamma^{+}), \\
Qb &= \sum_{k\geq 0}\omega^{M}_{-1-k}c_{k}b + \frac{1}{2}\sum_{k\geq 0}\omega^{Gh}_{-1-k}c_{k}b + \frac{1}{2}\sum_{k\geq 0}a_{-1-k}c_{k}b \\
&\quad +(\frac{1}{2}\cdot 2)\sum_{k\geq 0}c_{-1-k}((Dc)_{-1}b)_{k}b \\
&\quad +\frac{1}{2}\sum_{k\geq 0}\gamma^{+}_{-1-k}(c_{-1}(D\beta^{-}))_{k}b + (\frac{1}{2}\cdot\frac{3}{2})\sum_{k\geq 0}\gamma^{+}_{-1-k}((Dc)_{-1}\beta^{-})_{k}b \\
&\quad +\frac{1}{2}\sum_{k\geq 0}\gamma^{-}_{-1-k}(c_{-1}(D\beta^{+}))_{k}b + (\frac{1}{2}\cdot\frac{3}{2})\sum_{k\geq 0}\gamma^{-}_{-1-k}((Dc)_{-1}\beta^{+})_{k}b
\end{aligned}
$$

$$
\begin{aligned}
&= \omega^M + \frac{1}{2}\omega^{Gh} + \frac{1}{2}a + \sum_{k\geq 0} c_{-1-k}b_{k-2}1 \\
&\quad -\frac{1}{2}(k-1)\sum_{k\geq 0}\gamma^+_{-1-k}\beta^-_{k-2}1 - (\frac{1}{2}\cdot\frac{3}{2})\sum_{k\geq 0}\gamma^+_{-1-k}\beta^-_{k-2}1 \\
&\quad -\frac{1}{2}(k-1)\sum_{k\geq 0}\gamma^-_{-1-k}\beta^+_{k-2}1 - (\frac{1}{2}\cdot\frac{3}{2})\sum_{k\geq 0}\gamma^-_{-1-k}\beta^+_{k-2}1 \\
&= \omega^M - \frac{1}{2}(Db)_{-1}c + (Dc)_{-1}b + \frac{1}{2}(D\tilde{c})_{-1}\tilde{b} - \frac{1}{4}(D\beta^-)_{-1}\gamma^+ \\
&\quad -\frac{3}{4}(D\gamma^+)_{-1}\beta^- - \frac{1}{4}(D\beta^+)_{-1}\gamma^- - \frac{3}{4}(D\gamma^-)_{-1}\beta^+ + \frac{1}{2}(Db)_{-1}c \\
&\quad +\frac{1}{2}(D\tilde{c})_{-1}\tilde{b} + c_{-1}(Db) + (Dc)_{-1}b + \frac{1}{2}\gamma^+_{-1}(D\beta^-) \\
&\quad -\frac{3}{4}\gamma^+_{-1}(D\beta^-) - \frac{3}{4}(D\gamma^+)_{-1}\beta^- - \frac{1}{4}\gamma^-_{-1}(D\beta^+) - \frac{3}{4}(D\gamma^-)_{-1}\beta^+ \\
&= \omega, \\
Q\tilde{b} &= \sum_{k\geq 0} j^M_{-1-k}\tilde{c}_k\tilde{b} + \frac{1}{2}\sum_{k\geq 0} j^{Gh}_{-1-k}\tilde{c}_k\tilde{b} + \frac{1}{2}\sum_{k\geq 0} d_{-1-k}\tilde{c}_k\tilde{b} \\
&\quad +\sum_{k\geq 0} c_{-1-k}((D\tilde{c})_{-1}\tilde{b})_k\tilde{b} + \frac{1}{2}\sum_{k\geq 0}\gamma^+_{-1-k}(\tilde{c}_{-1}\beta^-)_k\tilde{b} \\
&\quad -\frac{1}{2}\sum_{k\geq 0}\gamma^-_{-1-k}(\tilde{c}_{-1}\beta^+)_k\tilde{b} \\
&= j^M + \frac{1}{2}j^{Gh} + \frac{1}{2}d + c_{-1}(D\tilde{b}) + (Dc)_{-1}\tilde{b} + \frac{1}{2}\gamma^+_{-1}\beta^- - \frac{1}{2}\gamma^-_{-1}\beta^+ \\
&= j^M + \frac{1}{2}\gamma^+_{-1}\beta^- - \frac{1}{2}\beta^+_{-1}\gamma^- \\
&\quad +c_{-1}(D\tilde{b}) + (Dc)_{-1}\tilde{b} + \frac{1}{2}\gamma^+_{-1}\beta^- - \frac{1}{2}\gamma^-_{-1}\beta^+ \\
&= j, \\
Q\gamma^\pm &= -\frac{3}{4}\sum_{k\geq 0} c_{-1-k}((D\gamma^\pm)_{-1}\beta^\mp)_k\gamma^\pm \\
&\quad -\frac{1}{4}\sum_{k\geq 0} c_{-1-k}((D\beta^\mp)_{-1}\gamma^\pm)_k\gamma^\pm - \frac{1}{2}\sum_{k\geq 0}\tilde{c}_{-1-k}(\gamma^\pm_{-1}\beta^\mp)_k\gamma^\pm \\
&\quad +\frac{1}{2}\sum_{k\geq 0}\gamma^\pm_{-1-k}(c_{-1}(D\beta^\mp))_k\gamma^\pm + \frac{3}{4}\sum_{k\geq 0}\gamma^\pm_{-1-k}((Dc)_{-1}\beta^\mp)_k\gamma^\pm \\
&\quad \pm\frac{1}{2}\sum_{k\geq 0}\gamma^\pm_{-1-k}(\tilde{c}_{-1}\beta^\mp)_k\gamma^\pm \\
&= \frac{3}{4}c_{-1}(D\gamma^\pm) - \frac{1}{4}c_{-1}(D\gamma^\pm) - \frac{1}{4}(Dc)_{-1}\gamma^\pm
\end{aligned}
$$

$$
\begin{aligned}
& \mp\frac{1}{2}\tilde{c}_{-1}\gamma^{\pm} + \frac{1}{2}(D\gamma^{\pm})_{-1}c + \frac{1}{2}\gamma^{\pm}_{-1}(Dc) - \frac{3}{4}\gamma^{\pm}_{-1}(Dc) \mp \frac{1}{2}\gamma^{\pm}_{-1}\tilde{c} \\
= \;& c_{-1}(D\gamma^{\pm}) - \frac{1}{2}\gamma^{\pm}_{-1}(Dc) \mp \tilde{c}_{-1}\gamma^{\pm}, \\
Q\beta^{\pm} = \;& \sum_{k\geq 0}\tau^{M\pm}_{-1-k}\gamma^{\mp}_{k}\beta^{\pm} + \frac{1}{2}\sum_{k\geq 0}\tau^{Gh\pm}_{-1-k}\gamma^{\mp}_{k}\beta^{\pm} \\
& -\frac{3}{4}\sum_{k\geq 0}c_{-1-k}((D\gamma^{\mp})_{-1}\beta^{\pm})_{k}\beta^{\pm} - \frac{1}{4}\sum_{k\geq 0}c_{-1-k}((D\beta^{\pm})_{-1}\gamma^{\mp})_{k}\beta^{\pm} \\
& \mp\frac{1}{2}\sum_{k\geq 0}\tilde{c}_{-1-k}(\beta^{\pm}_{-1}\gamma^{\mp})_{k}\beta^{\pm} - 2\sum_{k\geq 0}\gamma^{\pm}_{-1-k}(b_{-1}\gamma^{\mp})_{k}\beta^{\pm} \\
& \pm 2\sum_{k\geq 0}\gamma^{\pm}_{-1-k}(\tilde{b}_{-1}(D\gamma^{\mp}))_{k}\beta^{\pm} \pm \sum_{k\geq 0}\gamma^{\pm}_{-1-k}((D\tilde{b})_{-1}\gamma^{\mp})_{k}\beta^{\pm} \\
= \;& \tau^{M\pm} + \frac{1}{2}\tau^{Gh\pm} + \frac{3}{4}c_{-1}(D\beta^{\pm}) + \frac{3}{4}(Dc)_{-1}\beta^{\pm} - \frac{1}{4}c_{-1}(D\beta^{\pm}) \\
& \mp\frac{1}{2}\tilde{c}_{-1}\beta^{\pm} - 2\gamma^{\pm}_{-1}b \mp 2(D\gamma^{\pm})_{-1}\tilde{b} \mp 2\gamma^{\pm}_{-1}(D\tilde{b}) \pm \gamma^{\pm}_{-1}(D\tilde{b}) \\
= \;& \tau^{M\pm} + \frac{1}{2}c_{-1}(D\beta^{\pm}) + \frac{3}{4}(Dc)_{-1}\beta^{\pm} - 2b_{-1}\gamma^{\pm} \mp 2\tilde{b}_{-1}(D\gamma^{\pm}) \\
& \mp(D\tilde{b})_{-1}\gamma^{\pm} \mp \frac{1}{2}\tilde{c}_{-1}\beta^{\pm} + \frac{1}{2}c_{-1}(D\beta^{\pm}) + \frac{3}{4}(Dc)_{-1}\beta^{\pm} \\
& -2\gamma^{\pm}_{-1}b \mp 2(D\gamma^{\pm})_{-1}\tilde{b} \mp \gamma^{\pm}(D\tilde{b}) \mp \frac{1}{2}\tilde{c}_{-1}\beta^{\pm} = \tau^{\pm}.
\end{aligned}
$$

□

The following result is fundamental because it will allowing us to construct cohomology sequences.

Theorem 5.45
$Q^2 = 0$.

Proof:

$$
\begin{aligned}
Qj^{BRST} = \;& Qc_{-1}\omega^{M} + \frac{1}{2}Qc_{-1}(\omega^{Gh} + a) + Q\tilde{c}_{-1}j^{M} + \frac{1}{2}Q\tilde{c}_{-1}(j^{Gh} + d) \\
& +Q\gamma^{+}_{-1}\tau^{M} + \frac{1}{2}Q\gamma^{+}_{-1}\tau^{Gh-} + Q\gamma^{-}_{-1}\tau^{M+} + \frac{1}{2}Q\gamma^{-}_{-1}\tau^{Gh+} \\
= \;& -\frac{1}{4}\{Q, c_{-1}\}z^{+\mu}(-1)_{-1}z^{-}{}_{\mu}(-1) - \frac{1}{4}\{Q, c_{-1}\}z^{-\mu}(-1)_{-1}z^{+}{}_{\mu}(-1) \\
& -\frac{1}{4}\{Q, c_{-1}\}(D\psi^{+\mu})_{-1}\psi^{-}{}_{\mu} - \frac{1}{4}\{Q, c_{-1}\}(D\psi^{-\mu})_{-1}\psi^{+}{}_{\mu} \\
& +\frac{1}{4}c_{-1}[Q, z^{+\mu}(-1)_{-1}]z^{-}{}_{\mu}(-1) + \frac{1}{4}c_{-1}z^{+\mu}(-1)_{-1}[Q, z^{-}{}_{\mu}(-1)]1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}c_{-1}[Q,z^{-\mu}(-1)]z^{+}{}_{\mu}(-1)+\frac{1}{4}c_{-1}z^{+}{}_{\mu}(-1)[Q,z^{-\mu}(-1)]1\\
&+\frac{1}{4}c_{-1}\{Q,\psi^{+\mu}_{-2}\}\psi^{-}{}_{\mu}-\frac{1}{4}c_{-1}(D\psi^{+\mu})_{-1}\{Q,(\psi^{-}{}_{\mu})_{-1}\}1\\
&+\frac{1}{4}c_{-1}\{Q,\psi^{-\mu}_{-2}\}\psi^{+}{}_{\mu}-\frac{1}{4}c_{-1}(D\psi^{-\mu})_{-1}\{Q,(\psi^{+}{}_{\mu})_{-1}\}1\\
&+\{Q,c_{-1}\}(Dc)_{-1}b+\{Q,c_{-1}\}(D\tilde{c})_{-1}\tilde{b}\\
&-\frac{3}{4}\{Q,c_{-1}\}(D\gamma^{+})_{-1}\beta^{-}-\frac{1}{4}\{Q,c_{-1}\}(D\beta^{-})_{-1}\gamma^{+}\\
&-\frac{3}{4}\{Q,c_{-1}\}(D\gamma^{-})_{-1}\beta^{+}-\frac{1}{4}\{Q,c_{-1}\}(D\beta^{+})_{-1}\gamma^{-}\\
&-c_{-1}\{Q,c_{-2}\}b+c_{-1}(Dc)_{-1}\{Q,b_{-1}\}1\\
&-c_{-1}\{Q,\tilde{c}_{-2}\}\tilde{b}+c_{-1}(D\tilde{c})_{-1}\{Q,\tilde{b}_{-1}\}1\\
&+\frac{3}{4}c_{-1}[Q,\gamma^{+}_{-2}]\beta^{-}+\frac{3}{4}c_{-1}(D\gamma^{+})_{-1}[Q,\beta^{-}_{-1}]1\\
&+\frac{1}{4}c_{-1}[Q,\beta^{-}_{-2}]\gamma^{+}+\frac{1}{4}c_{-1}(D\beta^{-})_{-1}[Q,\gamma^{+}_{-1}]1\\
&+\frac{3}{4}c_{-1}[Q,\gamma^{-}_{-2}]\beta^{+}+\frac{3}{4}c_{-1}(D\gamma^{-})_{-1}[Q,\beta^{+}_{-1}]1\\
&+\frac{1}{4}c_{-1}[Q,\beta^{+}_{-2}]\gamma^{-}+\frac{1}{4}c_{-1}(D\beta^{+})_{-1}[Q,\gamma^{-}_{-1}]1\\
&-\frac{1}{2}\{Q,\tilde{c}_{-1}\}(\psi^{-\mu})_{-1}\psi^{+}{}_{\mu}+\frac{1}{2}\tilde{c}_{-1}\{Q,(\psi^{-\mu})_{-1}\}\psi^{+}{}_{\mu}\\
&-\frac{1}{2}\tilde{c}_{-1}\psi^{-\mu}_{-1}\{Q,(\psi^{+}{}_{\mu})_{-1}\}1+\frac{1}{2}\{Q,\tilde{c}_{-1}\}\gamma^{+}_{-1}\beta^{-}\\
&-\frac{1}{2}\{Q,\tilde{c}_{-1}\}\beta^{+}_{-1}\gamma^{-}-\frac{1}{2}\tilde{c}_{-1}[Q,\gamma^{+}_{-1}]\beta^{-}\\
&-\frac{1}{2}\tilde{c}_{-1}\gamma^{+}_{-1}[Q,\beta^{-}_{-1}]1+\frac{1}{2}\tilde{c}_{-1}[Q,\beta^{+}_{-1}]\gamma^{-}\\
&+\frac{1}{2}\tilde{c}_{-1}\beta^{+}_{-1}[Q,\gamma^{-}_{-1}]1+[Q,\gamma^{+}_{-1}]z^{+\mu}(-1)\psi^{-}{}_{\mu}\\
&+\gamma^{+}_{-1}[Q,z^{+\mu}(-1)]\psi^{-}{}_{\mu}+\gamma^{+}_{-1}z^{+\mu}(-1)\{Q,(\psi^{-}{}_{\mu})_{-1}\}1\\
&+[Q,\gamma^{-}_{-1}]z^{-\mu}(-1)\psi^{+}{}_{\mu}+\gamma^{-}_{-1}[Q,z^{-\mu}(-1)]\psi^{+}{}_{\mu}\\
&+\gamma^{-}_{-1}z^{-\mu}(-1)\{Q,(\psi^{+}{}_{\mu})_{-1}\}1+\frac{1}{2}[Q,\gamma^{+}_{-1}]c_{-1}(D\beta^{-})\\
&+\frac{3}{4}[Q,\gamma^{+}_{-1}](Dc)_{-1}\beta^{-}-2[Q,\gamma^{+}_{-1}]b_{-1}\gamma^{-}\\
&+2[Q,\gamma^{+}_{-1}]\tilde{b}_{-1}(D\gamma^{-})+[Q,\gamma^{+}_{-1}](D\tilde{b})_{-1}\gamma^{-}\\
&+\frac{1}{2}[Q,\gamma^{+}_{-1}]\tilde{c}_{-1}\beta^{-}+\frac{1}{2}\gamma^{+}_{-1}\{Q,c_{-1}\}(D\beta^{-})\\
&-\frac{1}{2}\gamma^{+}_{-1}c_{-1}[Q,\beta^{-}_{-2}]1+\frac{3}{4}\gamma^{+}_{-1}\{Q,c_{-2}\}\beta^{-}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}\gamma^+_{-1}(Dc)_{-1}[Q,\beta^-_{-1}]1 - 2\gamma^+_{-1}\{Q,b_{-1}\}\gamma^- \\
&+2\gamma^+_{-1}b_{-1}[Q,\gamma^-_{-1}]1 + 2\gamma^+_{-1}\{Q,\tilde{b}_{-1}\}(D\gamma^-) \\
&-2\gamma^+_{-1}\tilde{b}_{-1}[Q,\gamma^-_{-2}]1 + \gamma^+_{-1}\{Q,\tilde{b}_{-2}\}\gamma^- \\
&-\gamma^+_{-1}(D\tilde{b})_{-1}[Q,\gamma^-_{-1}]1 + \frac{1}{2}\gamma^+_{-1}\{Q,\tilde{c}_{-1}\}\beta^- \\
&-\frac{1}{2}\gamma^+_{-1}\tilde{c}_{-1}[Q,\beta^-_{-1}]1 + \frac{1}{2}[Q,\gamma^-_{-1}]c_{-1}(D\beta^+) \\
&+\frac{3}{4}[Q,\gamma^-_{-1}](Dc)_{-1}\beta^+ - 2[Q,\gamma^-_{-1}]b_{-1}\gamma^+ \\
&-2[Q,\gamma^-_{-1}]\tilde{b}_{-1}(D\gamma^+) - [Q,\gamma^-_{-1}](D\tilde{b})_{-1}\gamma^+ \\
&-\frac{1}{2}[Q,\gamma^-_{-1}]\tilde{c}_{-1}\beta^+ + \frac{1}{2}\gamma^-_{-1}\{Q,c_{-1}\}(D\beta^+) \\
&-\frac{1}{2}\gamma^-_{-1}c_{-1}[Q,\beta^+_{-2}]1 + \frac{3}{4}\gamma^-_{-1}\{Q,c_{-2}\}\beta^+ \\
&-\frac{3}{4}\gamma^-_{-1}(Dc)_{-1}[Q,\beta^+_{-1}]1 - 2\gamma^-_{-1}\{Q,b_{-1}\}\gamma^+ \\
&+2\gamma^-_{-1}b_{-1}[Q,\gamma^+_{-1}]1 - 2\gamma^-_{-1}\{Q,\tilde{b}_{-1}\}(D\gamma^+) \\
&+2\gamma^-_{-1}\tilde{b}_{-1}[Q,\gamma^+_{-2}]1 - \gamma^-_{-1}\{Q,\tilde{b}_{-2}\}\gamma^+ \\
&+\gamma^-_{-1}(D\tilde{b})_{-1}[Q,\gamma^+_{-1}]1 - \frac{1}{2}\gamma^-_{-1}\{Q,\tilde{c}_{-1}\}\beta^+ \\
&+\frac{1}{2}\gamma^-_{-1}\tilde{c}_{-1}[Q,\beta^+_{-1}]1.
\end{aligned}
$$

A tedious calculation using Theorem 5.44 shows

$$Qj^{BRST} = Dv$$

with

$$
\begin{aligned}
v \;=\;& -2\gamma^+_{-1}\gamma^-_{-1}c_{-1}b - 4\gamma^+_{-1}\gamma^-_{-1}\tilde{c}_{-1}\tilde{b} + \gamma^+_{-1}\gamma^-_{-1}\gamma^+_{-1}\beta^- \\
&-\gamma^+_{-1}\gamma^-_{-1}\beta^+_{-1}\gamma^- + c_{-1}\tilde{c}_{-1}\gamma^+_{-1}\beta^- - c_{-1}\tilde{c}_{-1}\beta^+_{-1}\gamma^- \\
&+4\gamma^+_{-1}\gamma^-_{-1}j^M + \frac{1}{2}c_{-1}\tilde{c}_{-1}j^M - 6\gamma^-_{-1}(D\gamma^+) \\
&-6(D\gamma^-)_{-1}\gamma^+ - c_{-1}\gamma^+_{-1}(D\gamma^-)_{-1}\tilde{b} \\
&-c_{-1}(D\gamma^+)_{-1}\tilde{b}_{-1}\gamma^- - \frac{3}{2}c_{-3}c - c_{-1}\gamma^+_{-1}\tau^{M-} - c_{-1}\gamma^-_{-1}\tau^{M+}.
\end{aligned}
$$

Hence we have

$$Q^2 = (j_0^{BRST}j^{BRST})_0 = (Dv)_0 = 0.$$

This proves the theorem. □

Chapter 6

The Lie algebra of physical states

In this chapter we construct the Lie algebra of physical states of the N=2-string moving on a torus. As we mentioned above the physical states are realized by the cohomology of the BRST-operator Q. There are many isomorphic copies of the space of physical states and the picture changing operators produce isomorphisms between them. In the first section we define a subalgebra V_S of the vertex algebra of the compactified N=2-string called small algebra. This restriction will allow us to construct the picture changing operators. In order to construct a bilinear form we restrict further to the vertex superalgebra V_S^{GSO}. In the second section we determine the cohomology groups of Q in V_S^{GSO}. As we will see the only nonvanishing cohomology groups for nonzero $\alpha \in L^X$ are the cohomology groups with $\alpha^2 = 0$ and ghostnumber 1. In the third section we define the Lie algebra of the N=2-string. Thereto we define a product $\{,\}$ on V_S^{GSO} and describe some properties of it. We show that $\{,\}$ closes on the cohomology. To define the Lie algebra G we restrict to the $(0,0)$-picture and define the Lie bracket as $[u,v] = \{u,v\}\ ,\ u,v \in G$. To obtain a bilinear form on G we define a bilinear form on the cohomology. As we will see later the bilinear form vanishes on the root spaces.

6.1 The small algebra

First we define a subalgebra of the vertex algebra of the N=2-string which is called small algebra. This is necessary to define the picture changing operator which we need for the isomorphisms between the cohomology groups in different pictures. We also construct a bilinear form on the small algebra.

Definition 6.1
We define the elements $\xi^+, \xi^-, \eta^+, \eta^- \in V$ *by*

$$\xi^+ = e^{\chi}, \quad \xi^- = e^{\tilde{\chi}}, \quad \eta^+ = e^{-\tilde{\chi}}, \quad \eta^- = e^{-\chi}.$$

Proposition 6.2
The only nontrivial commutation relations are

$$\begin{aligned} \{\xi^{\pm}_m, \eta^{\mp}_n\} &= \delta_{m+n+1}, \\ \{(D\xi^{\pm})_m, \eta^{\mp}_n\} &= -m\delta_{m+n}. \end{aligned}$$

Proof:
For $k \geq 0$ we have

$$\begin{aligned} \xi^+_k \eta^- &= e^{\chi}_k e^{-\chi} = \underbrace{\epsilon(\chi, -\chi)}_{=1} S_{-k-1-(\chi,-\chi)}(\chi)e^0 \\ &= S_{-k}(\chi) = \delta_k, \\ (D\xi^+)_k \eta^- &= -k\xi^+_{k-1}\eta^- = -ke^{\chi}_{k-1}e^{-\chi} \\ &= -kS_{-k+1-1+1}(\chi)e^0 = -k\delta_{k-1} \end{aligned}$$

and for the other cases analogously.
Hence

$$\begin{aligned} \xi^{\pm}(z)\eta^{\mp} &= z^{-1} + \dots, \\ (D\xi^{\pm})(z)\eta^{\mp} &= -z^{-2} + \dots, \end{aligned}$$

so that

$$\begin{aligned} \{\xi^{\pm}_m, \eta^{\mp}_n\} &= \sum_{k \geq 0} \binom{m}{k} (\underbrace{\xi^{\pm}_k \eta^{\mp}}_{=\delta_k})_{m+n-k} = 1_{m+n} = \delta_{m+n+1}, \\ \{(D\xi^{\pm})_m, \eta^{\mp}_n\} &= \sum_{k \geq 0} \binom{m}{k} (\underbrace{(D\xi^{\pm})_k \eta^{\mp}}_{=-k\delta_{k-1}})_{m+n-k} = -m1_{m+n-1} \\ &= -m\delta_{m+n} \end{aligned}$$

and we get the assertion. □

Definition 6.3
We define the **small algebra** V_S *as the subalgebra of* V *generated by the states* $e^{\gamma}, D\xi^{\pm}, \eta^{\pm}, b, c$ *with* $\gamma \in L^X \bigoplus L^{\psi,\phi,\tilde{\phi}}$.

We have

Theorem 6.4

$$\begin{aligned}
Q\xi^+ &= c_{-1}(D\xi^+) + \tau^{M+}_{-1}e^{\phi} + 4\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}b + 4D(\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}\tilde b) \\
&\quad -2\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}(D\tilde b) - 4\eta^+_{-1}(De^{\phi})_{-1}e^{\tilde\phi}_{-1}\tilde b, \\
Q\xi^- &= c_{-1}(D\xi^-) + \tau^{M-}_{-1}e^{\tilde\phi} + 4\eta^-_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}b + 4D(\eta^-_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}\tilde b) \\
&\quad -2\eta^-_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}(D\tilde b) - 4\eta^-_{-1}e^{\phi}_{-1}(De^{\tilde\phi})_{-1}\tilde b.
\end{aligned}$$

Proof:

$$\begin{aligned}
Q\xi^- &= \frac{1}{2}(c_{-1}w^{Gh})_0\xi^- + \frac{1}{2}(\tilde c_{-1}j^{Gh})_0\xi^- + (\gamma^+_{-1}\tau^{M-})_0\xi^- \\
&\quad +\frac{1}{2}(\gamma^+_{-1}\tau^{Gh-})_0\xi^- + \frac{1}{2}(\gamma^-_{-1}\tau^{Gh+})_0\xi^- \\
&= c_{-1}(D\xi^-) + \tau^{M-}_{-1}e^{\tilde\phi} + \frac{1}{2}(4b_{-1}\eta^-_{-1}e^{\phi}_{-1}e^{\tilde\phi} - 2\tilde b_{-2}\eta^-_{-1}e^{\phi}_{-1}e^{\tilde\phi} \\
&\quad -4\tilde b_{-1}(De^{\phi})_{-1}e^{\tilde\phi}_{-1}\eta^- - 4(D\eta^-)_{-1}e^{\phi}_{-1}e^{\tilde\phi}) \\
&\quad +\frac{1}{2}(-4\gamma^-_{-1}b_{-1}e^{\tilde\phi} + 2\gamma^-_{-1}\tilde b_{-2}e^{\tilde\phi} + 4\gamma^-_{-2}\tilde b_{-1}e^{\tilde\phi}) \\
&= c_{-1}(D\xi^+) + \tau^{M+}_{-1}e^{\phi} + 4\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}b + 4D(\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}\tilde b) \\
&\quad -2\eta^+_{-1}e^{\phi}_{-1}e^{\tilde\phi}_{-1}(D\tilde b) - 4\eta^+_{-1}(De^{\phi})_{-1}e^{\tilde\phi}_{-1}\tilde b.
\end{aligned}$$

For $Q\xi^+$ analogously. □

Definition 6.5
Define $X^+, X^- \in V$ as

$$X^+ := Q\xi^+\ ,\ X^- := Q\xi^-.$$

We define the **picture changing operators** *as the modes*

$$X^{\pm}_{-1} = (Q\xi^{\pm})_{-1} = \{Q, \xi^{\pm}_{-1}\}.$$

Recall that V^{GSO} is a vertex superalgebra.

Definition 6.6
We define the vertex superalgebra

$$V_S^{GSO} = V_S \cap V^{GSO}.$$

Now we want to define a bilinear form on V_S^{GSO}. For this we need the following results.

Proposition 6.7
The eigenvalues of L_0 on V_S^{GSO} are all integral.

Proof:
For

$$v = k_1(-m_1) \cdot \ldots \cdot k_t(-m_t) e^{\alpha}_{-1} e^{r_1\phi^1}_{-1} e^{r_2\phi^2}_{-1} e^{n_1\sigma + N_2\tilde{\sigma} + n_3\chi + n_4\tilde{\chi} + n_5\phi + n_6\tilde{\phi}}$$

the eigenvalues of L_0 are given by

$$\begin{aligned} L_0 v &= \left(\sum_{i=1}^{t} m_i + \frac{1}{2}\alpha^2 + \frac{1}{2}n_1(n_1-3) + \frac{1}{2}n_2(n_2-1) + \frac{1}{2}n_3(n_3-1) \right. \\ & \left. + \frac{1}{2}n_4(n_4-1) - \frac{1}{2}n_5(n_5+2) - \frac{1}{2}n_6(n_6+2) + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2 \right) v \end{aligned}$$

(Proposition 5.33). It holds

$$n_1(n_1-3) \equiv 0(2), \; n_i(n_i-1) \equiv 0 \bmod 2, \quad i \in \{2,3,4\}.$$

Recall that

$$\phi = (0,0,1,0), \tilde{\phi} = (0,0,0,1), \phi^1 = (1,0,0,0), \phi^2 = (0,1,0,0) \in L^{\psi,\phi,\tilde{\phi}}.$$

In order that

$$v \in V^{GSO} = V_{(0,0)} \bigoplus V_{(S,1)} \bigoplus V_{(0,1)} \bigoplus V_{(S,0)}$$

we need

$$r_1 + r_2 - n_5 - n_6 \equiv 0(2).$$

Hence either none, two or all of the coefficients r_1, r_2, n_5, n_6 are odd numbers.
Hence

$$(-n_5(n_5+2) - n_6(n_6+2) + r_1^2 + r_2^2) \equiv 0 \bmod 2.$$

Hence

$$\begin{aligned} \sum_{i=1}^{t} m_i + \frac{1}{2}\{ n_1(n_1-3) + n_2(n_2-1) + n_3(n_3-1) + n_4(n_4-1) - n_5(n_5+2) \\ -n_6(n_6+2) + r_1^2 + r_2^2 \} \in \mathbb{Z}. \end{aligned}$$

□

Proposition 6.8
L_1 acts locally nilpotent on V_S^{GSO}.

Proof:
We have

$$\begin{aligned} L_1 &= -\frac{1}{2} \sum_{k>\frac{1}{2}} \{ z^-{}_{\mu}(1-k) z^{+\mu}(k) + z^+{}_{\mu}(1-k) z^{-\mu}(k) \\ & -2\phi^1(1-k)\phi^1(k) - 2\phi^2(1-k)\phi^2(k) \\ & +\sigma(1-k)\sigma(k) + \tilde{\sigma}(1-k)\tilde{\sigma}(k) + \chi(1-k)\chi(k) \\ & +\tilde{\chi}(1-k)\tilde{\chi}(k) - \phi(1-k)\phi(k) - \tilde{\phi}(1-k)\tilde{\phi}(k) \} \\ & -3\sigma(1) - \tilde{\sigma}(1) - \chi(1) - \tilde{\chi}(1) + 2\phi(1) + 2\tilde{\phi}(1). \end{aligned}$$

From this it is easy to see that L_1 actually is locally nilpotent on V. $\square$

With Proposition 6.7, 6.8 we can apply the results of section 2.3 to construct a bilinear form on V_S^{GSO}. Note that $e^{3\sigma}$ is not in $L_1 V_1$.

Definition 6.9
We define an invariant bilinear form $(,)$ *on* V_S^{GSO} *by*

$$(e^{3\sigma}, 1) = 1.$$

6.2 The cohomology spaces

Now we construct the cohomology spaces of Q and and describe some properties of them.

Definition 6.10
$B = V_S^{GSO} \cap \operatorname{Ker} b_1 \cap \operatorname{Ker} \tilde{b}_0$.

The space B is graded by the lattice L^X. We denote the homogeneous subspaces by $B(\alpha)$. The operators $L_0, j_0, j_0^N, j_0^{P+}, j_0^{P-}$ are simultaneously diagonalizable on $B(\alpha)$ so that we can write

$$B(\alpha) = \bigoplus B(\alpha)^{k,\tilde{k},n}_{p+,p-}$$

where $k, \tilde{k}, n, p+, p-$ denote the eigenvalues of $L_0, j_0, j_0^N, j_0^{P+}, j_0^{P-}$.

Definition 6.11
We define vector spaces

$$C(\alpha)^n_{p+,p-} = B(\alpha)^{0,0,n}_{p+,p-},$$

$$C(\alpha) = \bigoplus C(\alpha)^n_{p+,p-} \quad , \; C = \bigoplus C(\alpha).$$

Since $b_1^* = b_1$ and $\{b_1, c_{-2}\} = 1$ the bilinear form on C induced from $(,)$ is zero.

Definition 6.12
We define a bilinear form on C.

$$(u, v)_C = (c_{-2} u, v).$$

Because of $Q^2 = 0$ the sequence

$$\dots \xrightarrow{Q} C(\alpha)^{n-1}_{p^+,p^-} \xrightarrow{Q} C(\alpha)^n_{p^+,p^-} \xrightarrow{Q} C(\alpha)^{n+1}_{p^+,p^-} \xrightarrow{Q} \dots$$

defines a complex. We denote the cohomology groups by $H(\alpha)^n_{p^+,p^-}$.

Definition 6.13
We define a bilinear form on $H := \bigoplus_{\alpha \in L^X} H(\alpha)^n_{p^+,p^-}$.

$$(u,v)_H = (u,v)_C.$$

The maps X^+_{-1}, X^-_{-1} are defined on V_S^{GSO}. Because of $Q^2 = 0$ it is easy to see that X^+_{-1}, X^-_{-1} are well defined on the cohomology.

Theorem 6.14
For $\alpha \in L^X, \alpha \neq 0$*, the maps*

$$X^+_{-1} : H(\alpha)^n_{p^+,p^-} \to H(\alpha)^n_{p^++1,p^-},$$

$$X^-_{-1} : H(\alpha)^n_{p^+,p^-} \to H(\alpha)^n_{p^+,p^-+1}$$

are isomorphisms.

Proof:
cf. [JL1]. □

Theorem 6.15
Let $\alpha \in L^X, \alpha \neq 0$.
If $\alpha^2 \neq 0$ *then*

$$H(\alpha)^n_{p^+,p^-} = 0 \qquad \forall n.$$

If $\alpha^2 = 0$ *then*

$$dim\ H(\alpha)^n_{p^+,p^-} = \begin{cases} 1 & for \quad n = 1, \\ 0 & else. \end{cases}$$

Proof:
Let $\alpha \in L^X$ with $\alpha^2 = 0$. In chapter seven we will calculate explicitly the cohomology group $H(\alpha \neq 0)^1_{0,0}$. As we will see there the dimension of $H(\alpha \neq 0)^1_{0,0}$ is one. Then Theorem 6.14 implies $\dim H(\alpha)^1_{p,q} = 1$. The rest of the proof can be found in [JL1]. □

6.3 The Lie algebra and the Lie bracket

Now we want to construct the Lie bracket. Therefore we define a product $\{,\}$ on V_S^{GSO}.

Definition 6.16
For $u, v \in V_S^{GSO}$ *we define (cf. [LZ1])*

$$\{u,v\} = (-1)^{|u|}(b_0 u)_0 v.$$

Now we want to show that $\{,\}$ closes in the cohomology. For this we need the following results.

Proposition 6.17
For $u, v, w \in V_S^{GSO}$ we have

$$u_0(v_0 w) = (u_0 v)_0 w + (-1)^{|u||v|} v_0(u_0 w).$$

Proof:
V_S^{GSO} is a vertex superalgebra, hence we have

$$(u_n v)_m w = \sum_{j\geq 0} \binom{n}{j} (-1)^j \{u_{n-j} v_{m+j} w - (-1)^n (-1)^{|u||v|} v_{m+n-j} u_j w\}.$$

Then

$$\begin{aligned}
(u_0 v)_0 w &= \sum_{k\geq 0} \binom{0}{k} (-1)^k \{u_{-k} v_k w - (-1)^0 (-1)^{|u||v|} v_{-j} u_j w\} \\
&= u_0(v_0 w) - (-1)^{|u||v|} v_0(u_0 w). \\
\Rightarrow \quad u_0(v_0 w) &= (u_0 v)_0 w + (-1)^{|u||v|} v_0(u_0 w).
\end{aligned}$$

□

Proposition 6.18
For $u, v \in V_S^{GSO}$ we have

$$(-1)^{|u|}\{u, v\} = b_1(u_{-1} v) - (b_1 u)_{-1} v + (-1)^{|u|} u_{-1}(b_1 v).$$

$$Q\{u, v\} = \{Qu, v\} - (-1)^{|u|+1}\{u, Qv\}.$$

Proof:
With Proposition 6.17 we have

$$b_0(u_0 v) = (b_0 u)_0 v + (-1)^{|u|} u_0(b_0 v). \tag{6.1}$$

Then

$$\begin{aligned}
(b_1 u)_{-1} v &= \sum_{j\geq 0} \binom{1}{j} (-1)^j \{b_{1-j} u_{-1+j} v - (-1)^1 (-1)^{|u|} u_{-j} b_j v\} \\
&= b_1(u_{-1} v) + (-1)^{|u|} u_0(b_0 v) - b_0(u_0 v) + (-1)^{|u|} u_{-1}(b_1 v). \\
\overset{(6.1)}{\Rightarrow} (b_1 u)_{-1} v &= b_1(u_{-1} v) + (-1)^{|u|} u_0(b_0 v) - (b_0 u)_0 v - (-1)^{|u|} u_0(b_0 v) \\
&\quad +(-1)^{|u|} u_{-1}(b_1 v). \\
(b_1 u)_{-1} v &= b_1(u_{-1} v) - (b_0 u)_0 v - (-1)^{|u|} u_{-1}(b_1 v). \\
\Rightarrow (-1)^{|u|}\{u, v\} &= (-1)^{|u|}(-1)^{|u|}(b_0 u)_0 v \\
&= b_1(u_{-1} v) - (b_1 u)_{-1} v + (-1)^{|u|} u_{-1}(b_1 v).
\end{aligned}$$

We have $|s_m t| = |s| + |t|$ for $s, t \in V^{GSO}$ and $|Q| = 1$, $|b| = 1$ with respect to the grading in V^{GSO}.

$$\Rightarrow \quad [Q, (b_0 u)_0]v = Q(b_0 u)_0 v - (-1)^{|Q||b_0 u|}(b_0 u)_0 (Qv).$$

Then

$$\begin{aligned}
Q\{u, v\} &= (-1)^{|u|} Q(b_0 u)_0 v \\
&= (-1)^{|u|}[Q, (b_0 u)_0]v + (-1)^{|u|}(-1)^{|u|+1}(b_0 u)_0 (Qv) \\
&= (-1)^{|u|} \sum_{k \geq 0} \binom{0}{k} (j_k^{BRST}(b_0 u))_{-k} v + (-1)^{|u|}(-1)^{|u|+1}(b_0 u)_0 (Qv) \\
&= (-1)^{|u|}(Q(b_0 u))_0 v + (-1)^{|u|}(-1)^{|u|+1}(b_0 u)_0 (Qv).
\end{aligned}$$

We have $D = \omega_0 = \{Q, b_0\}$.

$$\Rightarrow \quad Du = \{Q, b_0\}u = Q(b_0 u) - (-1)b_0(Qu).$$

We have

$$-(-1)^{|u|}(b_0(Qu))_0 v = (-1)^{|Q|+|u|}(b_0(Qu))_0 v = (-1)^{|Qu|}(b_0(Qu))_0 v. \quad (6.2)$$

Then

$$\begin{aligned}
Q\{u, v\} &= (-1)^{|u|}(Du)_0 v - (-1)^{|u|}(b_0(Qu))_0 v \\
&\quad +(-1)^{|u|}(-1)^{|u|+1}(b_0 u)_0 (Qv) \\
&\overset{(6.2),\, Prop.\ 2.5}{=} \{Qu, v\} - (-1)^{|u|+1}\{u, Qv\}.
\end{aligned}$$

□

Proposition 6.19
We have

$$j_0 b = 0,$$
$$b_1^2 = 0.$$

Proof:

$$\begin{aligned}
j_0 b &= \phi^1(-1)_0 b + \phi^2(-1)_0 b + (\gamma^+_{-1}\beta^-)_0 b - (\beta^+_{-1}\gamma^-)_0 b - ((D\tilde{b})_{-1}c)_0 b \\
&\quad +((Dc)_{-1}\tilde{b})_0 b \\
&= -\sum_{j \geq 0}\{(D\tilde{b})_{-1-j}c_j b - c_{-1-j}(D\tilde{b})_j b\} \\
&\quad +\sum_{j \geq 0}\{(Dc)_{-1-j}\tilde{b}_j b - \tilde{b}_{-1-j}(Dc)_j b\} \\
&= -\tilde{b}_{-2}e^0 + \tilde{b}_{-2}e^0 = 0.
\end{aligned}$$

Let $v \in V$. Then

$$\begin{aligned}
(b_1 b)_1 v &= \sum_{j \geq 0} \binom{1}{j} (-1)^j \{b_{1-j} b_{1+j} v - b_{2-j} b_j v\} \\
&= b_1 b_1 v - b_2 b_0 v - b_0 b_2 v + b_1 b_1 v. \\
\Rightarrow \; b_1^2 v &= \frac{1}{2} (b_1 b)_1 v.
\end{aligned}$$

With

$$b_1 b = e_1^{-\sigma} e^{-\sigma} = S_{-1-1-1}(-\sigma) e^{-2\sigma} = 0$$

follows that $b_1^2 v = 0$. □

Proposition 6.20
For all $u, v \in H$ we have

$$\begin{aligned}
L_0\{u,v\} &= 0, \\
j_0\{u,v\} &= 0, \\
\tilde{b}_0\{u,v\} &= 0.
\end{aligned}$$

Proof:
We have $\{Q, b_1\} = L_0$. Hence by Proposition 6.18 and Proposition 6.19.

$$L_0\{u,v\} = Q b_1 \{u,v\} - b_1 Q \{u,v\} = 0.$$

We have

$$j_0((\underbrace{b_0 u}_{=w})_0 v) \overset{Prop.\ 6.17}{=} (j_0 w)_0 v + (-1)^{|J||w|} w_0 \underbrace{(j_0 v)}_{=0,\ v \in H}.$$

$$j_0 w = j_0(b_0 u) \overset{Prop.\ 6.17}{=} (\underbrace{j_0 b}_{=0, Prop.\ 6.19})_0 u + (-1)^{|J||w|} b_0 (j_0 u).$$

Because $\tilde{b}_0 b = 0$ the proof that $\tilde{b}_0\{u,v\} = 0$ is given analogous to the proof that $j_0\{u,v\} = 0$. □

Hence we have

Proposition 6.21

$$\{u,v\} \in H \quad \textit{for } u, v \in H.$$

Proof:
This follows from Propositions 6.18, 6.19, 6.20. □

Corollary 6.22

$$\{,\} : H(\alpha)^m_{p^+,p^-} \times H(\beta)^n_{q^+,q^-} \to H(\alpha+\beta)^{m+n-1}_{p^+ + q^+, p^- + q^-}.$$

Proposition 6.23
Let u, v, w be representatives of elements in H. Then

$$\{u,v\} + (-1)^{(|u|+1)(|v|+1)}\{v,u\} = 0.$$

$$(-1)^{(|u|+1)(|w|+1)}\{u,\{v,w\}\} + (-1)^{(|v|+1)(|u|+1)}\{v,\{w,u\}\}$$
$$+(-1)^{(|w|+1)(|v|+1)}\{w,\{u,v\}\} = 0.$$

Proof:
cf. [LZ1]. □

Now we define the Lie algebra of the $N{=}2$- string. For the Cartan subalgebra we use the symbol H. This symbol is also used for the cohomology group of Q. It will be clear from the context what is meant.

Definition 6.24
For $\alpha \in L^X, \alpha = 0$, we define

$$H := H(\alpha = 0)^1_{0,0}$$

and for $\alpha \in L^X, \alpha \neq 0$, we define

$$G_\alpha := H(\alpha)^1_{0,0}.$$

Now we have

Theorem 6.25
$$G = H \oplus \bigoplus_{\alpha \in L^X, \alpha \neq 0} G_\alpha$$

is a Lie algebra under the bracket

$$\begin{aligned} [u,v] &= \{u,v\} \\ &= (-1)^{|u|}(b_0 u)_0 v, \quad u, v \in G. \end{aligned}$$

Proof:
This follows from Theorem 6.23 □

Chapter 7

Calculation of the cohomology groups in the $(0,0)$-picture

Now we calculate explicitly the cohomology groups in the $(0,0)$-picture. This will be needed for the calculation of the bilinear form and the bracket. Thereto we construct the vector spaces $C^0_{0,0}(\alpha)$ and $C^1_{0,0}(\alpha)$ for $\alpha \in L^X$ with $\alpha^2 = 0$. Recall that $C^n_{0,0}(\alpha) = \{v \in V_S^{GSO} | v \in \operatorname{Ker} L_0 \cap \operatorname{Ker} j_0 \cap \operatorname{Ker} b_1 \cap \operatorname{Ker} \tilde{b}_0 \wedge j_0^N v = nv, j_0^{P+} v = 0,\ j_0^{P-} v = 0, n \in \{0,1\}\}$. In the second section we calculate the Cartan subalgebra

$$H = \text{Ker } Q|_{C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}, \alpha = 0$$

and in the third section we calculate the root spaces

$$G_\alpha = \text{Ker } Q|_{C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}, \alpha \neq 0.$$

7.1 The spaces $C(\alpha)^0_{0,0}, C(\alpha)^1_{0,0}$

For the calculation we need the following lemma.

Lemma 7.1
For

$$v = \sigma(-1)e^{n_1\sigma + n_2\tilde{\sigma}}$$

or

$$v = \tilde{\sigma}(-1)e^{n_1\sigma + n_2\tilde{\sigma}}$$

or

$$v = e^{n_1\sigma + n_2\tilde{\sigma}}$$

where $n_1 \in \{0,1,2\}, n_2 \in \{-1,0,1\}$, *we have*

$$\{((Dc)_{-1}\tilde{b})_0 - ((D\tilde{b})_{-1}c)_0\}v = 0.$$

Proof:
By explicit calculation. $\square$

Proposition 7.2
Let $\alpha \in L^X, \alpha^2 = 0$ (we allow $\alpha = 0$).
$C^0_{0,0}(\alpha)$ has dimension 2 and the vectors

$$\begin{aligned} v_1 &= e^\alpha, \\ v_2 &= \tilde{b}_{-1} c_{-1} e^\alpha \end{aligned}$$

form a basis of it.
$C^1_{0,0}(\alpha)$ has dimension 19 and the vectors

$$\begin{aligned} v_1 &= \beta^+_{-1}\gamma^-_{-1}c_{-1}e^\alpha, \\ v_2 &= \gamma^+_{-1}\beta^-_{-1}c_{-1}e^\alpha, \\ v_3 &= (\psi^{-\mu})_{-1}(\psi^+_\mu)_{-1}c_{-1}e^\alpha, \\ v_4 &= (Dc)_{-1}e^\alpha, \\ v_5 &= \tilde{c}_{-1}\tilde{b}_{-1}c_{-1}e^\alpha, \\ v_6^{\pm\mu} &= z^{\pm\mu}(-1)_{-1}c_{-1}e^\alpha, \\ v_7^{\pm\mu} &= \gamma^\mp_{-1}\psi^{\pm\mu}_{-1}e^\alpha, \\ v_8^{\pm\mu} &= \gamma^\mp_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c_{-1}e^\alpha, \\ v_9 &= \gamma^-_{-1}\gamma^+_{-1}\tilde{b}_{-1}e^\alpha, \\ v_{10} &= (\psi^{--}_{-1}\psi^{++} - \psi^{-+}_{-1}\psi^{+-})_{-1}c_{-1}e^\alpha, \end{aligned}$$

where $\mu \in \{\pm\}$, form a basis of it.

Proof:
An element of V can be written as a linear combination of elements of the form

$$v = k_1(-m_1)\cdot \ldots \cdot k_t(-m_t)e^\alpha_{-1}e^{r_1\phi^1}_{-1}e^{r_2\phi^2}_{-1}e^{n_1\sigma+n_2\tilde{\sigma}+n_3\chi+n_4\tilde{\chi}+n_5\phi+n_6\tilde{\phi}},$$

where $k_i \in L\otimes_{\mathbb{Z}}\mathbb{C},\ i\in\{1,\ldots,t\}, \mu\in\{\pm\}, m_i, r_1, r_2, n_j \in \mathbb{Z}, m_i > 0$, $i \in \{1,\ldots,t\}, j\in\{1,\ldots,6\}$. If v has to be in $C^0_{0,0}(\alpha)$ or $C^1_{0,0}(\alpha)$ respectively we get the following conditions

$$v \in \operatorname{Ker} L_0 \cap \operatorname{Ker} j_0 \cap \operatorname{Ker} b_1 \cap \operatorname{Ker} \tilde{b}_0$$

and

$$\begin{aligned} j_0^N v &= nv, \quad n\in\{0,1\}, \\ j_0^{P+} v &= j_0^{P-} v = 0. \end{aligned}$$

With this we determine in the following some conditions for the n_i, r_j in v. We denote the eigenvalue of j_0^N, j_0^{P+}, j^{P-} by n, p^+ and p^-. Then

$$\begin{aligned} p^+ &= n_3 + n_5 \quad , \ p^- = n_4 + n_6, && (7.1) \\ n &= n_1 + n_2 - (n_3 + n_4). && (7.2) \end{aligned}$$

Because we have the ghostpicture $(0,0)$ it follows from (7.1)

$$n_5 = -n_3 \ , \ n_6 = -n_4. \tag{7.3}$$

With (7.2) it follows

$$(n_3 + n_4) = n_1 + n_2 - n, \quad n \in \{0,1\}. \tag{7.4}$$

From Proposition 5.33 follows

$$\begin{aligned} L_0 v &= \left(\sum_{i=1}^{t} m_i + \frac{1}{2}\alpha^2 + \underbrace{\frac{1}{2}n_1(n_1-3)}_{=:T_1} + \underbrace{\frac{1}{2}n_2(n_2-1)}_{=:T_2} + \underbrace{\frac{1}{2}n_3(n_3-1)}_{=:T_3} \right. \\ & \left. + \underbrace{\frac{1}{2}n_4(n_4-1)}_{=:T_4} - \underbrace{\frac{1}{2}n_5(n_5+2)}_{=:T_5} - \underbrace{\frac{1}{2}n_6(n_6+2)}_{=:T_6} + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2 \right) v. \end{aligned}$$

From (7.3) we get

$$\sum_{j=3}^{6} T_j = \frac{1}{2}(n_3 + n_4),$$

so that by (7.4)

$$L_0 v = \underbrace{\left(\sum m_i + \underbrace{\frac{1}{2}n_1(n_1-2)}_{=:T(n_1)} + \frac{1}{2}n_2^2 + \frac{1}{2}(r_1)^2 + \frac{1}{2}(r_2)^2 - \frac{1}{2}n \right)}_{:=T} v.$$

$$\begin{aligned} \text{If } n_1 \in \{0,2\} &\quad \text{we} \quad \text{get} \quad T(n_1) = 0, \\ \text{if } n_1 = 1 &\quad \text{we} \quad \text{get} \quad T(n_1) = -\frac{1}{2}, \\ \text{if } n_1 \leq -1 \vee n_1 \geq 3 &\quad \text{we} \quad \text{get} \quad T(n_1) \geq \frac{1}{2} \ \Rightarrow \ T > 0. \end{aligned}$$

Hence we get from the condition $v \in \operatorname{Ker} L_0$ that

$$n_1 \in \{0,1,2\}. \tag{7.5}$$

If $|n_2| \geq 2$ then $T > 0$, hence

$$n_2 \in \{-1, 0, 1\}. \tag{7.6}$$

If $|r_i| \geq 2$, $i \in \{1, 2\}$, then $T > 0$, hence

$$r_i \in \{-1, 0, 1\}. \tag{7.7}$$

If $n \neq 1 \wedge r_1 \neq 0 \wedge r_2 \neq 0$ then $T > 0$, hence

$$r_1 \neq 0 \wedge r_2 \neq 0 \text{ only} \quad \text{for } n = 1. \tag{7.8}$$

If $\sum m_i \geq 2$ then $T > 0$, hence

$$\sum m_i \leq 1. \tag{7.9}$$

From the **condition** $v \in \text{Ker } j_0$ we get the following:

$$j_0 = \underbrace{-\phi(0) + \tilde{\phi}(0) + \phi^1(0) + \phi^2(0)}_{=:l_1} + \underbrace{\Big((Dc)_{-1}\tilde{b})_0 - ((D\tilde{b})_{-1}c\Big)_0}_{=:l_2}$$

Because of (7.5), (7.6), (7.9) it follows with Lemma 7.1 that

$$l_2 v = 0.$$

We have

$$l_1 e^{n_5\phi + n_6\tilde{\phi} + r_1\phi^1 + r_2\phi^2} = (n_5 - n_6 + r_1 + r_2)e^{n_5\phi + n_6\tilde{\phi} + r_1\phi^1 + r_2\phi^2},$$

where $n_5 - n_6 + r_1 + r_2 \stackrel{!}{=} 0$ so that

$$n_6 = n_5 + r_1 + r_2. \tag{7.10}$$

Hence with (7.5), (7.6), (7.7), (7.8), (7.9), (7.10) we obtain the following tables.

For ghostnumber $n = 0$

n_1	n_2	n_3	n_4	n_5	n_6	r_i	$(+)$	$\sum m_i$	
0	0	0	0	0	0	0	0	0	1
0	-1	0	-1	0	1	1	1	$\star$	
0	-1	-1	0	1	0	-1	1	$\star$	
0	1	0	1	0	-1	-1	1	$\star$	
0	1	1	0	-1	0	1	1	$\star$	
1	0	1	0	-1	0	1	0	0	$\notin V_S^{GSO}$
1	0	0	1	0	-1	-1	0	0	$\notin V_S^{GSO}$
1	-1	0	0	0	0	0	0	0	$e^{\sigma}_{-1}e^{-\sigma}$
1	1	1	1	-1	-1	0	0	0	$\notin V_S^{GSO}$
2	0	1	1	-1	-1	0	0	0	$\notin V_S^{GSO}$
2	-1	0	1	0	-1	-1	1	$\star$	
2	-1	1	0	-1	0	1	1	$\star$	
2	1	2	1	-2	-1	1	1	$\star$	
2	1	1	2	-1	-2	-1	1	$\star$	

r_i: By (7.8) one of the r_i has to be zero. We give the value of the nonzero r_i.
$(+) : \sum T_j + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2$.
$\star : -1$ in contradiction to $\sum m_i \geq 0$.

For ghostnumber $n = 1$

n	r_i	r_1	r_2	$(+)$	$\sum m_i$	
$(0,0,0,-1,0,1)$	1			0	0	$e^{-\tilde{\chi}}_{-1}e^{\tilde{\phi}}_{-1}e^{\phi^\nu}$
$(0,0,-1,0,1,0)$	-1			0	0	$e^{-\chi}_{-1}e^{\phi}_{-1}e^{-\phi^\nu}$
$(0,-1,-1,-1,1,1)$	0			0	0	$e^{-\tilde{\sigma}}_{-1}e^{-\chi}_{-1}e^{-\tilde{\chi}}_{-1}e^{\phi}_{-1}e^{\tilde{\phi}}$
$(0,1,0,0,0,0)$	0			0	0	$e^{\tilde{\sigma}} \notin \mathrm{Ker}\,\tilde{b}_0$
$(1,0,0,0,0,0)$	0			-1	1	$k(-1)e^{\sigma}$
$(1,0,-1,1,1,-1)$		-1	-1	0	0	$\notin V_S^{GSO}$
$(1,0,1,-1,-1,1)$		1	1	0	0	$\notin V_S^{GSO}$
$(1,-1,-1,0,1,0)$	-1			0	0	$e^{\sigma}_{-1}e^{-\tilde{\sigma}}_{-1}e^{-\chi}_{-1}e^{\phi}_{-1}e^{-\phi^\nu}$
$(1,-1,0,-1,0,1)$	1			0	0	$e^{\sigma}_{-1}e^{-\tilde{\sigma}}_{-1}e^{-\tilde{\chi}}_{-1}e^{\tilde{\phi}}_{-1}e^{\phi^\nu}$
$(1,1,0,1,0,-1)$	-1			0	0	$\notin V_S^{GSO}$
$(1,1,1,0,-1,0)$	1			0	0	$\notin V_S^{GSO}$
$(2,0,1,0,-1,0)$	1			0	0	$\notin V_S^{GSO}$
$(2,0,0,1,0,-1)$	-1			0	0	$\notin V_S^{GSO}$
$(2,-1,0,0,0,0)$	0			0	0	$e^{2\sigma}_{-1}e^{-\tilde{\sigma}} \notin \mathrm{Ker}\,b_1$
$(2,1,1,1,-1,-1)$	0			0	0	$\notin V_S^{GSO}$

$n = (n_1, n_2, n_3, n_4, n_5, n_6)$.
r_i: By (7.8) one of the r_i has to be zero. We give the value of the nonzero r_i.
$(+)$: $\sum T_j + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2$.
$\nu \in \{1, 2\}$. □

7.2 The subalgebra H

Proposition 7.3
Let $\alpha \in L^X, \alpha = 0$. Then
Ker $Q|_{C^1_{0,0}(\alpha)} = \{x \in C^1_{0,0}(\alpha) | Qx = 0\}$ *has dimension 5 and the vectors*

$$\begin{aligned} v^{\pm\mu} &= z^{\pm\mu}(-1)_{-1}c - 2\gamma^{\mp}_{-1}\psi^{\pm\mu}, \\ v &= \gamma^{+}_{-1}\beta^{-}_{-1}c - \beta^{+}_{-1}\gamma^{-}_{-1}c - \frac{1}{2}(\psi^{-\mu})_{-1}(\psi^{+}_{\mu})_{-1}c + 4\tilde{b}_{-1}\gamma^{+}_{-1}\gamma^{-}, \end{aligned}$$

where $\mu \in \{\pm\}$, form a basis.

Proof:
Let $a_1, a_2, a_3, a_4, a_5, a_6^{\pm\mu}, a_7^{\pm\mu}, a_8^{\pm\mu}, a_9, a_{10} \in \mathbb{C},\ \mu \in \{\pm\}$. We have

$$\begin{aligned}
(c_{-1}(D\beta^+))_{-1}\gamma^-_{-1}c &= \sum_{j\geq 0}\{c_{-1-j}(D\beta^+)_{-1+j}\gamma^-_{-1}c \\
&\quad +(D\beta^+)_{-2-j}c_j\gamma^-_{-1}c\} \\
&= \sum_{j\geq 0} -(-1+j)c_{-1-j}\beta^+_{-2+j}\gamma^-_{-1}c \\
&= \sum_{j\geq 0} -(-1+j)c_{-1-j}\underbrace{[\beta^+_{-2+j},\gamma^-_{-1}]}_{=-\delta_{j-2}}c \\
&\quad +\sum_{j\geq 0} -(-1+j)c_{-1-j}\gamma^-_{-1}c_{-1}\beta^+_{-2+j}1 \\
&= c_{-3}c + c_{-1}\gamma^-_{-1}c_{-1}(D\beta^+) \\
&= c_{-3}c. \\
\frac{3}{2}((Dc)_{-1}\beta^+)_{-1}\gamma^-_{-1}c &= \frac{3}{2}\sum_{j\geq 0}\{(Dc)_{-1-j}\beta^+_{-1+j}\gamma^-_{-1}c \\
&\quad +\beta^+_{-2-j}(Dc)_j\gamma^-_{-1}c\} \\
&= \frac{3}{2}\sum_{j\geq 0}(Dc)_{-1-j}[\beta^+_{-1+j},\gamma^-_{-1}]c \\
&\quad +\frac{3}{2}\sum_{j\geq 0}(Dc)_{-1-j}\gamma^-_{-1}c_{-1}\beta^+_{-1+j}1 \\
&= -\frac{3}{2}(Dc)_{-2}c + \frac{3}{2}(Dc)_{-1}\gamma^-_{-1}c_{-1}\beta^+. \\
&= -3c_{-3}c + \frac{3}{2}(Dc)_{-1}\gamma^-_{-1}c_{-1}\beta^+ \\
-4(b_{-1}\gamma^+)_{-1}\gamma^-_{-1}c &= -4\sum_{j\geq 0}\{b_{-1-j}\gamma^+_{-1+j}\gamma^-_{-1}c + \gamma^+_{-2-j}b_j\gamma^-_{-1}c\} \\
&= -4b_{-1}\gamma^+_{-1}\gamma^-_{-1}c - 4(D\gamma^+)_{-1}\gamma^-. \\
-(\tilde{c}_{-1}\beta^+)_{-1}\gamma^-_{-1}c &= -\sum_{j\geq 0}\{\tilde{c}_{-1-j}\beta^+_{-1+j}\gamma^-_{-1}c + \beta^+_{-2-j}\tilde{c}_j\gamma^-_{-1}c\} \\
&= (D\tilde{c})_{-1}c - \tilde{c}_{-1}\gamma^-_{-1}c_{-1}\beta^+. \\
(c_{-1}(D\gamma^+))_{-1}\beta^-_{-1}c &= \sum_{j\geq 0} c_{-1-j}(D\gamma^+)_{-1+j}\beta^-_{-1}c \\
&= -\sum_{j\geq 0}(-1+j)c_{-1-j}[\gamma^+_{-2+j},\beta^-_{-1}]c \\
&\quad -\sum_{j\geq 0}(-1+j)c_{-1-j}\beta^-_{-1}c_{-1}\gamma^+_{-2+j}1 \\
&= -c_{-3}c + c_{-1}\beta^-_{-1}c_{-1}(D\gamma^+).
\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{2}(\gamma^+_{-1}(Dc))_{-1}\beta^-_{-1}c &= -\frac{1}{2}\sum_{j\geq 0}\{\gamma^+_{-1-j}(Dc)_{-1+j}\beta^-_{-1}c \\
&\quad +(Dc)_{-2-j}[\gamma^+_j,\beta^-_{-1}]c\} \\
&= \frac{1}{2}\sum_{j\geq 0}(-1+j)\gamma^+_{-1-j}\beta^-_{-1}c_{-1}c_{-2+j}1 \\
&\quad -\frac{1}{2}(Dc)_{-2}c \\
&= -\frac{1}{2}\gamma^+_{-1}(Dc)_{-1}\beta^-_{-1}c - c_{-3}c. \\
-(\tilde{c}_{-1}\gamma^+)_{-1}\beta^-_{-1}c &= -\sum_{j\geq 0}\tilde{c}_{-1-j}[\gamma^+_{-1+j},\beta^-_{-1}]c \\
&\quad -\sum_{j\geq 0}\tilde{c}_{-1-j}\beta^-_{-1}c_{-1}\gamma^+_{-1+j}1 \\
&= -(D\tilde{c})_{-1}c - \tilde{c}_{-1}\beta^-_{-1}c_{-1}\gamma^+. \\
-4\gamma^+_{-1}(b_{-1}\gamma^-)_{-1}c &= -4\sum_{j\geq 0}\{\gamma^+_{-1}b_{-1-j}\gamma^-_{-1+j}c + \gamma^+_{-1}\gamma^-_{-2-j}b_jc\} \\
&= -4\gamma^+_{-1}b_{-1}\gamma^-_{-1}c - 4\gamma^+_{-1}(D\gamma^-). \\
-((D\psi^{-\mu})_{-1}c)_{-1}(\psi^+{}_\mu)_{-1}c &= -\sum_{j\geq 0}\{(D\psi^{-\mu})_{-1-j}c_{-1+j}(\psi^+{}_\mu)_{-1}c \\
&\quad -c_{-2-j}(D\psi^{-\mu})_j(\psi^+{}_\mu)_{-1}c\} \\
&= -(D\psi^{-\mu})_{-1}c_{-1}(\psi^+{}_\mu)_{-1}c \\
&\quad +\sum_{j\geq 0}(-j)\eta_{\mu\nu}c_{-2-j}\underbrace{\{\psi^{-\mu}_{j-1},\psi^{+\nu}_{-1}\}}_{=-2\eta^{\mu\nu}\delta_{j-1}}1 \\
&= 2\eta_{\mu\nu}\eta^{\mu\nu}c_{-3}c \\
&= 4c_{-3}c. \\
-\frac{1}{2}(\psi^{-\mu}_{-1}(Dc))_{-1}(\psi^+{}_\mu)_{-1}c &= -\frac{1}{2}\sum_{j\geq 0}\{\psi^{-\mu}_{-1-j}(Dc)_{-1+j}(\psi^+{}_\mu)_{-1}c \\
&\quad -(Dc)_{-2-j}\psi^{-\mu}_j(\psi^+{}_\mu)_{-1}c\} \\
&= -\frac{1}{2}\psi^{-\mu}_{-1}(Dc)_{-1}(\psi^+{}_\mu)_{-1}c \\
&\quad +\frac{1}{2}\sum_{j\geq 0}\eta_{\mu\nu}(Dc)_{-2-j}\underbrace{\{\psi^{-\mu}_j,\psi^{+\nu}_{-1}\}}_{=-2\eta^{\mu\nu}\delta_j}1 \\
&= -\frac{1}{2}\psi^{-\mu}_{-1}(Dc)_{-1}(\psi^+{}_\mu)_{-1}c - 2(Dc)_{-2}c \\
&= -\frac{1}{2}\psi^{-\mu}_{-1}(Dc)_{-1}(\psi^+{}_\mu)_{-1}c - 4c_{-3}c.
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{c}_{-1}\psi^{-\mu})_{-1}(\psi^{+}{}_{\mu})_{-1}c &= \sum_{j\geq 0}\tilde{c}_{-1-j}\psi^{-\mu}_{-1+j}(\psi^{+}{}_{\mu})_{-1}c \\
&= \sum_{j\geq 0}\eta_{\mu\nu}\tilde{c}_{-1-j}\{\psi^{-\mu}_{-1+j},\psi^{+\nu}_{-1}\}c \\
&\quad +\sum_{j\geq 0}\tilde{c}_{-1-j}(\psi^{+}{}_{\mu})_{-1}c_{-1}\psi^{-\mu}_{-1+j}1 \\
&= -4(D\tilde{c})_{-1}c+\tilde{c}(\psi^{+}{}_{\mu})_{-1}c_{-1}\psi^{-\mu}. \\
(c_{-1}(Dc))_{-2}1 &= \sum_{j\geq 0}\{c_{-1-j}(Dc)_{-2+j}1-(Dc)_{-1-j}c_j1\} \\
&= -\sum_{j\geq 0}(-2-j)c_{-1-j}c_{-3+j}1 \\
&= 2c_{-1}c_{-3}1+c_{-2}c_{-2}1. \\
-4(\gamma^{+}_{-1}\gamma^{-})_{-2}1 &= -4\sum_{j\geq 0}\{\gamma^{+}_{-1-j}\gamma^{-}_{-2+j}1+\gamma^{-}_{-3-j}\gamma^{+}_{j}1\} \\
&= -4\gamma^{+}_{-1}(D\gamma^{-})-4(D\gamma^{+})_{-1}\gamma^{-}. \\
(c_{-1}(D\tilde{c}))_{-1}\tilde{b}_{-1}c &= \sum_{j\geq 0}c_{-1-j}(D\tilde{c})_{-1+j}\tilde{b}_{-1}c \\
&= -\sum_{j\geq 0}(-1+j)c_{-1-j}\{\tilde{c}_{-2+j},\tilde{b}_{-1}\}c \\
&\quad +\sum_{j\geq 0}(-1+j)c_{-1-j}\tilde{b}_{-1}c_{-1}\tilde{c}_{-2+j}1 \\
&= -c_{-3}c-c_{-1}\tilde{b}_{-1}c_{-1}(D\tilde{c}). \\
\pm(\tilde{c}_{-1}\gamma^{\mp})_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c &= \pm\sum_{j\geq 0}\{\tilde{c}_{-1-j}\gamma^{\mp}_{-1+j}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c \\
&\quad +\gamma^{\mp}_{-2-j}\psi^{\pm\mu}_{-1}\{\tilde{c}_j,\tilde{b}_{-1}\}c\} \\
&= \pm\tilde{c}_{-1}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c \\
&\quad \pm((D\gamma^{\mp})_{-1}\psi^{\pm\mu})_{-1}\psi^{\pm\mu}_{-1}c. \\
\mp\gamma^{\mp}_{-1}(\tilde{c}_{-1}\psi^{\pm\mu})_{-1}\tilde{b}_{-1}c &= \mp\sum_{j\geq 0}\{\gamma^{\mp}_{-1}\tilde{c}_{-1-j}\psi^{\pm\mu}_{-1+j}\tilde{b}_{-1}c \\
&\quad -\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-2-j}\{\tilde{c}_j,\tilde{b}_{-1}\}c\} \\
&= \mp\gamma^{\mp}_{-1}\tilde{c}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c \\
&\quad \pm\gamma^{\mp}_{-1}(D\psi^{\pm\mu})_{-1}c. \\
(\tilde{c}_{-1}\gamma^{-})_{-1}\gamma^{+}_{-1}\tilde{b} &= \sum_{j\geq 0}\{\tilde{c}_{-1-j}\gamma^{-}_{-1+j}\gamma^{+}_{-1}\tilde{b} \\
&\quad +\gamma^{-}_{-2-j}\gamma^{+}_{-1}\{\tilde{c}_j,\tilde{b}_{-1}\}1\} \\
&= \tilde{c}_{-1}\gamma^{-}_{-1}\gamma^{+}_{-1}\tilde{b}+(D\gamma^{-})_{-1}\gamma^{+}.
\end{aligned}
$$

$$
\begin{aligned}
-\gamma^-_{-1}(\tilde{c}_{-1}\gamma^+)_{-1}\tilde{b} &= -\sum_{j\geq 0}\{\gamma^-_{-1}\tilde{c}_{-1-j}\gamma^+_{-1+j}\tilde{b} \\
&\quad +\gamma^-_{-1}\gamma^+_{-2-j}\{\tilde{c}_j,\tilde{b}_{-1}\}1\} \\
&= -\gamma^-_1\tilde{c}_{-1}\gamma^+_{-1}\tilde{b}-\gamma^-_{-1}(D\gamma^+).
\end{aligned}
$$

Using Theorem 5.44 we get

$$
\begin{gathered}
Qa_1\beta^+_{-1}\gamma^-_{-1}c+Qa_2\gamma^+_{-1}\beta^-_{-1}c+Qa_3\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}c+Qa_4(Dc)+Qa_5\tilde{c}_{-1}\tilde{b}_{-1}c \\
+Qa_6^{\pm\mu}z^{\pm\mu}(-1)_{-1}c+Qa_7^{\pm\mu}\gamma^\mp_{-1}\psi^{\pm\mu}+Qa_8^{\pm\mu}\gamma^\mp_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c+Qa_9\gamma^-_{-1}\gamma^+_{-1}\tilde{b} \\
+Qa_{10}\psi^{--}_{-1}\psi^{++}_{-1}c-Qa_{10}\psi^{-+}_{-1}\psi^{+-}_{-1}c
\end{gathered}
$$

$$
\begin{aligned}
= \;& a_1[Q,\beta^+_{-1}]\gamma^-_{-1}c+a_1\beta^+_{-1}[Q,\gamma^-_{-1}]c+a_1\beta^+_{-1}\gamma^-_{-1}\{Q,c_{-1}\}1 \\
&+a_2[Q,\gamma^+_{-1}]\beta^-_{-1}c+a_2\gamma^+_{-1}[Q,\beta^-_{-1}]c+a_2\gamma^+_{-1}\beta^-_{-1}\{Q,c_{-1}\}1 \\
&+a_3\{Q,\psi^{-\mu}_{-1}\}(\psi^+{}_\mu)_{-1}c-a_3\psi^{-\mu}_{-1}\{Q,(\psi^+{}_\mu)_{-1}\}c \\
&+a_3\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}\{Q,c_{-1}\}1+a_4\{Q,c_{-2}\}1+a_5\{Q,\tilde{c}_{-1}\}\tilde{b}_{-1}c \\
&-a_5\tilde{c}_{-1}\{Q,\tilde{b}_{-1}\}c+a_5\tilde{c}_{-1}\tilde{b}_{-1}\{Q,c_{-1}\}1+a_6^{\pm\mu}[Q,z^{\pm\mu}(-1)]c \\
&+a_6^{\pm\mu}z^{\pm\mu}(-1)\{Q,c_{-1}\}1+a_7^{\pm\mu}[Q,\gamma^\mp_{-1}]\psi^{\pm\mu}+a_7^{\pm\mu}\gamma^\mp_{-1}\{Q,\psi^{\pm\mu}_{-1}\}1 \\
&+a_8^{\pm\mu}[Q,\gamma^\mp_{-1}]\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c+a_8^{\pm\mu}\gamma^\mp_{-1}\{Q,\psi^{\pm\mu}_{-1}\}\tilde{b}_{-1}c-a_8^{\pm\mu}\gamma^\mp_{-1}\psi^{\pm\mu}_{-1}\{Q,\tilde{b}_{-1}\}c \\
&+a_8^{\pm\mu}\gamma^\mp_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}\{Q,c_{-1}\}1+a_9[Q,\gamma^-_{-1}]\gamma^+_{-1}\tilde{b}+a_9\gamma^-_{-1}[Q,\gamma^+_{-1}]\tilde{b} \\
&+a_9\gamma^-_{-1}\gamma^+_{-1}\{Q,\tilde{b}_{-1}\}1+a_{10}\{Q,\psi^{--}_{-1}\}\psi^{++}_{-1}c-a_{10}\psi^{--}_{-1}\{Q,\psi^{++}_{-1}\}c \\
&+a_{10}\psi^{--}_{-1}\psi^{++}_{-1}\{Q,c_{-1}\}1-a_{10}\{Q,\psi^{-+}_{-1}\}\psi^{+-}_{-1}c+a_{10}\psi^{-+}_{-1}\{Q,\psi^{+-}_{-1}\}c \\
&-a_{10}\psi^{-+}_{-1}\psi^{+-}_{-1}\{Q,c_{-1}\}1 \\
= \;& a_1z^{-\mu}(-1)_{-1}(\psi^+{}_\mu)_{-1}\gamma^-_{-1}c+a_1(c_{-1}(D\beta^+))_{-1}\gamma^-_{-1}c \\
&+\frac{3}{2}a_1((Dc)_{-1}\beta^+)_{-1}\gamma^-_{-1}c-4a_1(b_{-1}\gamma^+)_{-1}\gamma^-_{-1}c-4a_1\tilde{b}_{-1}(D\gamma^+)_{-1}\gamma^-_{-1}c \\
&-2a_1(D\tilde{b})_{-1}\gamma^+_{-1}\gamma^-_{-1}c-a_1(\tilde{c}_{-1}\beta^+)_{-1}\gamma^-_{-1}c+a_1\beta^+_{-1}c_{-1}(D\gamma^-)_{-1}c \\
&-\frac{1}{2}a_1\beta^+_{-1}\gamma^-_{-1}(Dc)_{-1}c+a_1\beta^+_{-1}\tilde{c}_{-1}\gamma^-_{-1}c+a_1\beta^+_{-1}\gamma^-_{-1}c_{-1}(Dc) \\
&-4a_1\beta^+_{-1}\gamma^-_{-1}\gamma^+_{-1}\gamma^-+a_2(c_{-1}(D\gamma^+))_{-1}\beta^-_{-1}c \\
&-\frac{1}{2}a_2(\gamma^+_{-1}(Dc))_{-1}\beta^-_{-1}c-a_2(\tilde{c}_{-1}\gamma^+)_{-1}\beta^-_{-1}c+a_2\gamma^+_{-1}z^{+\mu}(-1)_{-1}(\psi^-{}_\mu)_{-1}c \\
&+a_2\gamma^+_{-1}c_{-1}(D\beta^-)_{-1}c+\frac{3}{2}a_2\gamma^+_{-1}(Dc)_{-1}\beta^-_{-1}c \\
&-4a_2\gamma^+_{-1}(b_{-1}\gamma^-)_{-1}c+4a_2\gamma^+_{-1}\tilde{b}_{-1}(D\gamma^-)_{-1}c+2a_2\gamma^+_{-1}(D\tilde{b})_{-1}\gamma^-_{-1}c \\
&+a_2\gamma^+_{-1}\tilde{c}_{-1}\beta^-_{-1}c+a_2\gamma^+_{-1}\beta^-_{-1}c_{-1}(Dc)-4a_2\gamma^+_{-1}\beta^-_{-1}\gamma^+_{-1}\gamma^-
\end{aligned}
$$

$$
\begin{aligned}
&-a_3((D\psi^{-\mu})_{-1}c)_{-1}(\psi^+{}_\mu)_{-1}c-\frac{1}{2}a_3(\psi^{-\mu}_{-1}(Dc))_{-1}(\psi^+{}_\mu)_{-1}c\\
&-2a_3z^{-\mu}(-1)_{-1}\gamma^-_{-1}(\psi_\mu)_{-1}c+a_3(\tilde{c}_{-1}\psi^{-\mu})_{-1}(\psi^+{}_\mu)_{-1}c\\
&+a_3\psi^{-\mu}_{-1}(D\psi^+{}_\mu)_{-1}c_{-1}c+\frac{1}{2}a_3\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}(Dc)_{-1}c\\
&+2a_3\psi^{-\mu}_{-1}z^+{}_\mu(-1)_{-1}\gamma^+_{-1}c\\
&+a_3\psi^{-\mu}_{-1}\tilde{c}_{-1}(\psi^+{}_\mu)_{-1}c+a_3\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}c_{-1}(Dc)-4a_3\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}\gamma^+_{-1}\gamma^-\\
&+a_4(c_{-1}(Dc))_{-2}1-4a_4(\gamma^+_{-1}\gamma^-)_{-2}1+a_5(c_{-1}(D\tilde{c}))_{-1}\tilde{b}_{-1}c\\
&+2a_5\gamma^+_{-1}(D\gamma^-)_{-1}\tilde{b}_{-1}c-2a_5\gamma^-_{-1}(D\gamma^+)_{-1}\tilde{b}_{-1}c+\frac{1}{2}a_5\tilde{c}_{-1}\psi^{-\mu}_{-1}(\psi^+{}_\mu)_{-1}c\\
&-a_5\tilde{c}_{-1}\gamma^+_{-1}\beta^-_{-1}c+a_5\tilde{c}_{-1}\beta^+_{-1}\gamma^-_{-1}c+a_5\tilde{c}_{-1}(D\tilde{b})_{-1}c_{-1}c\\
&-a_5\tilde{c}_{-1}(Dc)_{-1}\tilde{b}_{-1}c+a_5\tilde{c}_{-1}\tilde{b}_{-1}c_{-1}(Dc)-4a_5\tilde{c}_{-1}\tilde{b}_{-1}\gamma^+_{-1}\gamma^-\\
&+a_6^{\pm\mu}(Dz^{\pm\mu})_{-1}c_{-1}c+a_6^{\pm\mu}z^{\pm\mu}(-1)_{-1}(Dc)_{-1}c-2a_6^{\pm\mu}(D\gamma^{\mp})_{-1}\psi^{\pm\mu}_{-1}c\\
&-2a_6^{\pm\mu}\gamma^{\mp}_{-1}(D\psi^{\pm\mu})_{-1}c+a_6^{\pm\mu}z^{\pm\mu}(-1)_{-1}c_{-1}(Dc)\\
&-4a_6^{\pm\mu}z^{\pm\mu}(-1)_{-1}\gamma^+_{-1}\gamma^-+a_7^{\pm\mu}c_{-1}(D\gamma^{\mp})_{-1}\psi^{\pm\mu}-\frac{1}{2}a_7^{\pm\mu}\gamma^{\mp}_{-1}(Dc)_{-1}\psi^{\pm\mu}\\
&\pm a_7^{\pm\mu}\tilde{c}_{-1}\gamma^{\mp}_{-1}\psi^{\pm\mu}-a_7^{\pm\mu}\gamma^{\mp}_{-1}(D\psi^{\pm\mu})_{-1}c-\frac{1}{2}a_7^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(Dc)\\
&-2a_7^{\pm\mu}\gamma^{\mp}_{-1}z^{\pm\mu}(-1)_{-1}\gamma^{\pm}\mp a_7^{\pm\mu}\gamma^{\mp}_{-1}\tilde{c}_{-1}\psi^{\pm\mu}+a_8^{\pm\mu}c_{-1}(D\gamma^{\mp})_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c\\
&-\frac{1}{2}a_8^{\pm\mu}\gamma^{\mp}_{-1}(Dc)_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c\pm a_8^{\pm\mu}(\tilde{c}_{-1}\gamma^{\mp})_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c\\
&-a_8^{\pm\mu}\gamma^{\mp}_{-1}(D\psi^{\pm\mu})_{-1}c_{-1}\tilde{b}_{-1}c-\frac{1}{2}a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(Dc)_{-1}\tilde{b}_{-1}c\\
&-2a_8^{\pm\mu}\gamma^{\mp}_{-1}z^{\pm\mu}(-1)_{-1}\gamma^{\pm}_{-1}\tilde{b}_{-1}c\mp a_8^{\pm\mu}\gamma^{\mp}_{-1}(\tilde{c}_{-1}\psi^{\pm\mu})_{-1}\tilde{b}_{-1}c\\
&+\frac{1}{2}a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(\psi^{\nu})_{-1}(\psi^+{}_\nu)_{-1}c\\
&-a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\gamma^+_{-1}\beta^-_{-1}c+a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\beta^+_{-1}\gamma^-_{-1}c+a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(D\tilde{b})_{-1}c_{-1}c\\
&-a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(Dc)_{-1}\tilde{b}_{-1}c+a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c_{-1}(Dc)\\
&-4a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}\gamma^+_{-1}\gamma^-\\
&+a_9c_{-1}(D\gamma^-)_{-1}\gamma^+_{-1}\tilde{b}-\frac{1}{2}a_9\gamma^-_{-1}(Dc)_{-1}\gamma^+_{-1}\tilde{b}+a_9(\tilde{c}_{-1}\gamma^-)_{-1}\gamma^+_{-1}\tilde{b}\\
&+a_9\gamma^-_{-1}c_{-1}(D\gamma^+)_{-1}\tilde{b}-\frac{1}{2}a_9\gamma^-_{-1}\gamma^+_{-1}(Dc)_{-1}\tilde{b}-a_9\gamma^-_{-1}(\tilde{c}_{-1}\gamma^+)_{-1}\tilde{b}\\
&-\frac{1}{2}a_9\gamma^-_{-1}\gamma^+_{-1}(\psi^{-\mu})_{-1}\psi^+{}_\mu+a_9\gamma^-_{-1}\gamma^+_{-1}\gamma^+_{-1}\beta^--a_9\gamma^-_{-1}\gamma^+_{-1}\beta^+_{-1}\gamma^-\\
&-a_9\gamma^-_{-1}\gamma^+_{-1}(D\tilde{b})_{-1}c+a_9\gamma^-_{-1}\gamma^+_{-1}(Dc)_{-1}\tilde{b}-2a_{10}z^{--}(-1)\gamma^-_{-1}\psi^{++}_{-1}c\\
&+2a_{10}z^{++}(-1)\psi^{--}_{-1}\gamma^+_{-1}c+2a_{10}z^{-+}(-1)\gamma^-_{-1}\psi^{+-}_{-1}c-2z^{+-}(-1)\psi^{-+}_{-1}\gamma^+_{-1}c\\
&-4a_{10}\psi^{--}_{-1}\psi^{++}_{-1}\gamma^+_{-1}\gamma^-+4_{10}\psi^{-+}_{-1}\psi^{+-}_{-1}\gamma^+_{-1}\gamma^-
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (a_1-2a_3)z^{-\mu}(-1)_{-1}(\psi^{+}{}_{\mu})_{-1}\gamma^{-}_{-1}c+(a_1-3a_1-a_2-a_2+4a_3-4a_3 \\
& -2a_4-a_5)c_{-3}c \\
& +(\frac{3}{2}a_1-\frac{1}{2}a_1-a_1)(Dc)_{-1}\gamma^{-}_{-1}c_{-1}\beta^{+}+(-4a_1-4a_2)b_{-1}\gamma^{+}_{-1}\gamma^{-}_{-1}c \\
& +(-4a_1-4a_4-a_9)(D\gamma^{+})_{-1}\gamma^{-}+(-4a_1-2a_5-a_9)\tilde{b}_{-1}(D\gamma^{+})_{-1}\gamma^{-}_{-1}c \\
& +(-2a_1+2a_2-a_9)(D\tilde{b})_{-1}\gamma^{+}_{-1}\gamma^{-}_{-1}c+(a_1-a_2-4a_3)(D\tilde{c})_{-1}c \\
& +(-a_1+a_1+a_5)\tilde{c}_{-1}\gamma^{-}_{-1}c_{-1}\beta^{+}+(-4a_1-a_9)\beta^{+}_{-1}\gamma^{-}_{-1}\gamma^{+}_{-1}\gamma^{-} \\
& +(\frac{1}{2}a_2+a_2-\frac{3}{2}a_2)\gamma^{+}_{-1}\beta^{-}_{-1}c_{-1}(Dc)+(-a_2+a_2-a_5)\tilde{c}_{-1}\beta^{-}_{-1}c_{-1}\gamma^{+} \\
& +(a_2+2_3)z^{+\mu}(-1)_{-1}\gamma^{+}_{-1}(\psi^{-}{}_{\mu})_{-1}c+(-4a_2-4a_4+a_9)\gamma^{+}_{-1}(D\gamma^{-}) \\
& +(4a_2+2a_5-a_9)\gamma^{+}_{-1}\tilde{b}_{-1}(D\gamma^{-})_{-1}c+(-4a_2+a_9)\gamma^{+}_{-1}\beta^{-}_{-1}\gamma^{+}_{-1}\gamma^{-} \\
& +(-\frac{1}{2}a_3-\frac{1}{2}a_3+a_3)\psi^{-\mu}_{-1}(Dc)_{-1}(\psi^{+}{}_{\mu})_{-1}c \\
& +(a_3-a_3+\frac{1}{2}a_5)\tilde{c}_{-1}(\psi^{+}{}_{\mu})_{-1}c_{-1}\psi^{-\mu}+(-4a_3-\frac{1}{2}a_9)\psi^{-\mu}_{-1}(\psi^{+}{}_{\mu})_{-1}\gamma^{+}_{-1}\gamma^{-} \\
& +a_4c_{-2}c_{-2}1+(-a_5+a_5)\tilde{c}_{-1}(Dc)_{-1}\tilde{b}_{-1}c \\
& +(-4a_5+a_9-a_9)\tilde{c}_{-1}\tilde{b}_{-1}\gamma^{+}_{-1}\gamma^{-}+(a_6^{\pm\mu}-a_6^{\pm\mu})z^{\pm\mu}(-1)_{-1}(Dc)_{-1}c \\
& +(-2a_6^{\pm\mu}-a_7^{\pm\mu}\pm a_8^{\pm\mu})(D\gamma^{\mp})_{-1}\psi^{\pm\mu}_{-1}c \\
& +(-2a_6^{\pm\mu}-a_7^{\pm\mu}\pm a_8^{\pm\mu})\gamma^{\mp}_{-1}(D\psi^{\pm\mu})_{-1}c \\
& +(-4a_6^{\pm\mu}-2a_7^{\pm\mu})z^{\pm\mu}(-1)_{-1}\gamma^{+}_{-1}\gamma^{-} \\
& +(-\frac{1}{2}a_7^{\pm\mu}+\frac{1}{2}a_7^{\pm\mu})\gamma^{\mp}_{-1}(Dc)_{-1}\psi^{\pm\mu}+(\pm a_7^{\pm\mu}\mp a_7^{\pm\mu})\tilde{c}_{-1}\gamma^{\mp}_{-1}\psi^{\pm\mu} \\
& +(-\frac{1}{2}a_8^{\pm\mu}+\frac{1}{2}a_8^{\pm\mu}+a_8^{\pm\mu}-a_8^{\pm\mu})\gamma^{\mp}_{-1}(Dc)_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c \\
& +(\pm a_8^{\pm\mu}\mp a_8^{\pm\mu})\tilde{c}_{-1}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}c-2a_8^{\pm\mu}z^{\pm\mu}(-1)_{-1}\gamma^{\mp}_{-1}\gamma^{\pm}_{-1}\tilde{b}_{-1}c \\
& +\frac{1}{2}a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}(\psi^{-\nu})_{-1}(\psi^{+}{}_{\nu})_{-1}c-a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\gamma^{+}_{-1}\beta^{-}_{-1}c \\
& +a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\beta^{+}_{-1}\gamma^{-}_{-1}c-4a_8^{\pm\mu}\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}\tilde{b}_{-1}\gamma^{+}_{-1}\gamma^{-} \\
& +(-\frac{1}{2}a_9-\frac{1}{2}a_9+a_9)\gamma^{-}_{-1}(Dc)_{-1}\gamma^{+}_{-1}\tilde{b}-2a_{10}z^{--}(-1)\gamma^{-}_{-1}\psi^{++}_{-1}c \\
& +2a_{10}z^{++}(-1)\psi^{--}_{-1}\gamma^{+}_{-1}c+2a_{10}z^{-+}(-1)\gamma^{-}_{-1}\psi^{+-}_{-1}c \\
& -2z^{+-}(-1)\psi^{-+}_{-1}\gamma^{+}_{-1}c-4a_{10}\psi^{--}_{-1}\psi^{++}_{-1}\gamma^{+}_{-1}\gamma^{-}+4_{10}\psi^{-+}_{-1}\psi^{+-}_{-1}\gamma^{+}_{-1}\gamma^{-} \\
\stackrel{!}{=} \quad & 0.
\end{aligned}
$$

Since the elements are linearly independent it follows

$$\begin{aligned} a_4 = a_5 &= a_8^{\pm\mu} = a_{10} = 0, \\ a_1 &= -a_2, \\ a_3 &= -\frac{1}{2}a_2, \\ a_9 &= 4a_2, \\ a_7^{\pm\mu} &= -2a_6^{\pm\mu}. \end{aligned}$$

□

Proposition 7.4
Let $\alpha \in L^X, \alpha = 0$. Then
Im $Q|_{C^0_{0,0}(\alpha)} = \{Qx | x \in C^0_{0,0}(\alpha)\}$ *has dimension one with basis*

$$\gamma^+_{-1}\beta^-_{-1}c - \beta^+_{-1}\gamma^-_{-1}c - \frac{1}{2}(\psi^{-\mu})_{-1}(\psi^+_\mu)_{-1}c + 4\tilde{b}_{-1}\gamma^+_{-1}\gamma^-.$$

Proof:
Theorem 5.44 implies

$$\begin{aligned} Q\tilde{b}_{-1}c &= \{Q, \tilde{b}_{-1}\}c - \tilde{b}_{-1}\{Q, c_{-1}\}1 \\ &= -\frac{1}{2}(\psi^{-\mu})_{-1}(\psi^+{}_\mu)_{-1}c + \gamma^+_{-1}\beta^-_{-1}c - \beta^+_{-1}\gamma^-_{-1}c - (D\tilde{b})_{-1}\underbrace{c_{-1}c}_{=0} \\ &\quad +(Dc)_{-1}\tilde{b}_{-1}c - \tilde{b}_{-1}c_{-1}(Dc) + 4\tilde{b}_{-1}\gamma^+_{-1}\gamma^- \\ &= -\frac{1}{2}(\psi^{-\mu})_{-1}(\psi^+{}_\mu)_{-1}c + \gamma^+_{-1}\beta^-_{-1}c - \beta^+_{-1}\gamma^-_{-1}c + 4\tilde{b}_{-1}\gamma^+_{-1}\gamma^-. \end{aligned}$$

We also have $Q1 = 0$. This proves the proposition. □

Theorem 7.5
Let $\alpha \in L^X, \alpha = 0$. Then $H = \mathrm{Ker}\, Q|_{C^1_{0,0}(\alpha)} / \mathrm{Im}\, Q|_{C^0_{0,0}(\alpha)}$ has dimension 4 and the vectors

$$v^{\pm\mu} = z^{\pm\mu}(-1)_{-1}c - 2\gamma^{\mp}_{-1}\psi^{\pm\mu},$$

where $\mu \in \{\pm\}$,form a basis.

Proof:
This follows from Proposition 7.3 and Proposition 7.4. □

7.3 The root spaces G_α

Now we calculate the cohomology group

$$G_\alpha = H(\alpha)^1_{0,0} = \text{Ker } Q|_{C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}$$

for $\alpha \in L^X$ with $\alpha^2 = 0$ and $\alpha \neq 0$.

Proposition 7.6
Let
$$\alpha = -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--} \in L^X$$
with $\alpha \neq 0$ *and* $\alpha^2 = 0$*. Then*
$\mathrm{Ker}\ Q|_{C^1_{0,0}(\alpha)} = \{v \in C(\alpha)^1_{0,0} | Qv = 0\}$ *has dimension* 3 *and the vectors*

$$\begin{aligned}&k^+{}_\mu z^{-\mu}(-1)_{-1}c_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)_{-1}c_{-1}e^\alpha \\ &-(k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu})_{-1}c_{-1}e^\alpha - k^+{}_\mu(\psi^{-\mu})_{-1}\gamma^+_{-1}e^\alpha + k^-{}_\mu(\psi^{+\mu})_{-1}\gamma^-_{-1}e^\alpha\end{aligned}$$

and

$$\begin{aligned}&\gamma^+_{-1}\beta^-_{-1}c_{-1}e^\alpha - \beta^+_{-1}\gamma^-_{-1}c_{-1}e^\alpha - \frac{1}{2}(\psi^{-\mu})_{-1}(\psi^+{}_\mu)_{-1}c_{-1}e^\alpha \\ &+4\tilde{b}_{-1}\gamma^+_{-1}\gamma^-_{-1}e^\alpha + (z^{-\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^-_{-1}(\psi^+{}_\mu)_{-1}e^\alpha \\ &+(z^{+\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^+_{-1}(\psi^-{}_\mu)_{-1}e^\alpha\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{2}(z^{+\mu},\alpha)z^-{}_\mu(-1)c_{-1}e^\alpha - \frac{1}{2}(z^{-\mu},\alpha)z^+{}_\mu(-1)c_{-1}e^\alpha \\ &+(z^{-\mu},\alpha)\gamma^-_{-1}(\psi^+{}_\mu)_{-1}e^\alpha + (z^{+\mu},\alpha)\gamma^+_{-1}(\psi^-{}_\mu)_{-1}e^\alpha\end{aligned}$$

form a basis.

Proof:
Analogous to the proof of Proposition 7.3. □

Now we calculate $\mathrm{Im}\ Q|_{C^0_{0,0}(\alpha)} = \{Qv | v \in C^0_{0,0}(\alpha)\}$ where $\alpha^2 = 0$ and $\alpha \neq 0$.

Proposition 7.7
Let $\alpha \in L^X, \alpha^2 = 0, \alpha \neq 0$*. Then*
$\mathrm{Im}\ Q|_{C^0_{0,0}(\alpha)}$ *has dimension* 2 *and the vectors*

$$\begin{aligned}&\gamma^+_{-1}\beta^-_{-1}c_{-1}e^\alpha - \beta^+_{-1}\gamma^-_{-1}c_{-1}e^\alpha - \frac{1}{2}(\psi^{-\mu})_{-1}(\psi^+{}_\mu)_{-1}c_{-1}e^\alpha \\ &+4\tilde{b}_{-1}\gamma^+_{-1}\gamma^-_{-1}e^\alpha + (z^{-\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^-_{-1}(\psi^+{}_\mu)_{-1}e^\alpha \\ &+(z^{+\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^+_{-1}(\psi^-{}_\mu)_{-1}e^\alpha\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{2}(z^{+\mu},\alpha)z^-{}_\mu(-1)c_{-1}e^\alpha - \frac{1}{2}(z^{-\mu},\alpha)z^+{}_\mu(-1)c_{-1}e^\alpha \\ &+(z^{-\mu},\alpha)\gamma^-_{-1}(\psi^+{}_\mu)_{-1}e^\alpha + (z^{+\mu},\alpha)\gamma^+_{-1}(\psi^-{}_\mu)_{-1}e^\alpha\end{aligned}$$

form a basis.

Proof:
Let $a_1, a_2 \in \mathbb{C}$. Then

$$\begin{aligned}
&Qa_1\tilde{b}_{-1}c_{-1}e^{\alpha} + Qa_2e^{\alpha} = \\
&\quad a_1\gamma^{+}_{-1}\beta^{-}_{-1}c_{-1}e^{\alpha} - a_1\beta^{+}_{-1}\gamma^{-}_{-1}c_{-1}e^{\alpha} - \frac{1}{2}a_1(\psi^{-\mu})_{-1}(\psi^{+}{}_{\mu})_{-1}c_{-1}e^{\alpha} \\
&\quad +4a_1\tilde{b}_{-1}\gamma^{+}_{-1}\gamma^{-}_{-1}e^{\alpha} + a_1(z^{-\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^{-}_{-1}(\psi^{+}{}_{\mu})_{-1}e^{\alpha} \\
&\quad +a_1(z^{+\mu},\alpha)\tilde{b}_{-1}c_{-1}\gamma^{+}_{-1}(\psi^{-}{}_{\mu})_{-1}e^{\alpha} \\
&\quad -\frac{1}{2}a_2(z^{+\mu},\alpha)z^{-}{}_{\mu}(-1)c_{-1}e^{\alpha} - \frac{1}{2}a_2(z^{-\mu},\alpha)z^{+}{}_{\mu}(-1)c_{-1}e^{\alpha} \\
&\quad +(z^{-\mu},\alpha)a_2\gamma^{-}_{-1}(\psi^{+}{}_{\mu})_{-1}e^{\alpha} + (z^{+\mu},\alpha)a_2\gamma^{+}_{-1}(\psi^{-}{}_{\mu})_{-1}e^{\alpha}.
\end{aligned}$$

□

Proposition 7.8
Let $\alpha = -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--} \in L^X$ with $\alpha^2 = 0$ and $\alpha \neq 0$. Then $G_\alpha = \text{Ker } Q_{|C^1_{0,0}(\alpha)} / \text{Im } Q|_{C^0_{0,0}(\alpha)}$ has dimension 1 and is generated by

$$\begin{aligned}
&k^{+}{}_{\mu}z^{-\mu}(-1)_{-1}c_{-1}e^{\alpha} - k^{-}{}_{\mu}z^{+\mu}(-1)_{-1}c_{-1}e^{\alpha} \\
&-(k^{+}{}_{\mu}\psi^{-\mu})_{-1}(k^{-}{}_{\nu}\psi^{+\nu})_{-1}c_{-1}e^{\alpha} - k^{+}{}_{\mu}(\psi^{-\mu})_{-1}\gamma^{+}_{-1}e^{\alpha} \\
&+k^{-}{}_{\mu}(\psi^{+\mu})_{-1}\gamma^{-}_{-1}e^{\alpha}.
\end{aligned}$$

Proof:
This follows from Proposition 7.6 and Proposition 7.7. □

Chapter 8

Properties of the Lie algebra

In this chapter we describe some properties of the Lie algebra G of the N=2-string. In particular we show that G is not a generalized Kac-Moody algebra. In the first section we show that the invariant bilinear form on G is symmetric and pairs G_α with $G_{-\alpha}$ trivially and that there is no nondegenerate invariant bilinear form on G. We show that the subalgebra H of G is abelian. The bilinear form on G is nondegenerate on H and gives an isometry from H to $L^X \bigotimes_{\mathbb{Z}} \mathbb{C}$. We show that H measures the momentum, i.e. $[h,x] = (h,\alpha)x$ for $h \in H$ and $x \in G_\alpha$. Because of the properties we call H Cartan subalgebra of G. Then we calculate the commutator $[x,y]$ for $x \in G_\alpha, y \in G_\beta$. In the second section we show that the Lie algebra G of the N=2-string is not a generalized Kac-Moody algebra with Cartan subalgebra H. In the third section we summarize the properties of the Lie algebra G.

8.1 The bilinear form and the commutator

We define a bilinear form on G.

Definition 8.1
For $u, v \in G$ we define

$$< u, v >= (u,v)_H.$$

Since $C^1_{0,0}(\alpha) \subset V_{(0,1)}$ we have

$$\forall u \in G : |u| = 1 \quad (\text{parity with respect to } V^{GSO}). \tag{8.1}$$

The bilinear form $<,>$ on G has the following properties.

Proposition 8.2
$<,>$ is symmetric, invariant and pairs G_α with $G_{-\alpha}$.

Proof:

Since
$$a_n^* = (-1)^h \sum_{m \geq 0} \left(\frac{L_1^m}{m!} a \right)_{2h-n-m-2}$$
and
$$c_n^* = -c_{-n-4}, \quad b_n^* = b_{2-n}$$
and
$$(-1)^{|u|}\{u,v\} = b_1(u_{-1}v) \quad \text{(Theorem 6.18)}$$
we have for $u, v \in G$
$$< u, v >= (c_{-2}u, v) = (u, c_{-2}v) = (c_{-2}v, u) =< v, u > .$$
Also for $u, v \in G$
$$\begin{aligned} < [u,v], w > &= - < [v,u], w >=< b_1 v_{-1} u, w > \\ &= (c_{-2} b_1 v_{-1} u, w) = -(b_1 v_{-1} u, c_{-2} w) \\ &= -(v_{-1}u, b_1 c_{-2} w) = -(v_{-1}u, w) = -(u, v_{-1}w) \end{aligned}$$
and
$$\begin{aligned} < u, [v,w] > &= -(c_{-2}u, b_1 v_{-1} w) = -(b_1 v_{-1} w, c_{-2} u) \\ &= -(v_{-1}w, b_1 c_{-2} u) = -(v_{-1}w, u) = -(u, v_{-1}w). \end{aligned}$$
Hence
$$< [u,v], w >=< u, [v,w] > .$$

□

It is clear from the definition of $<,>$ that it pairs G_α with $G_{-\alpha}$. We show now that the pairing of G_α and $G_{-\alpha}$ is trivial.

Recall that $\{z^{++}, z^{+-}, z^{-+}, z^{--}\}$ is a basis of $L^X \bigotimes \mathbb{C}$ with
$$\begin{aligned} (z^{+\mu}, z^{+\nu}) &= (z^{-\mu}, z^{-\nu}) = 0, \\ (z^{+\mu}, z^{-\nu}) &= (z^{-\mu}, z^{+\nu}) = 2\eta^{\mu\nu}, \quad \mu, \nu \in \{\pm\}. \end{aligned}$$
For
$$\alpha = -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--} \in L^X$$
and
$$\beta = -t^{-+}z^{++} + t^{--}z^{+-} - t^{++}z^{-+} + t^{+-}z^{--} \in L^X$$
we define
$$(\alpha, \beta \tilde{)} = k^{++}t^{-+} - k^{+-}t^{--} + k^{--}t^{+-} - k^{-+}t^{++}.$$

Then $(,\tilde{)}$ is an antisymmetric bilinear form on $L^X \bigotimes \mathbb{C}$.
For $\alpha \in L^X$ with $\alpha^2 = 0$ and $\alpha \neq 0$ we define

$$
\begin{aligned}
|\alpha\rangle \quad = \quad & k^+{}_\mu z^{-\mu}(-1)c_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)c_{-1}e^\alpha \\
& -(k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu})_{-1}c_{-1}e^\alpha - k^+{}_\mu(\psi^{-\mu})_{-1}\gamma^+_{-1}e^\alpha \\
& +k^-{}_\mu\psi^{+\mu}_{-1}\gamma^-_{-1}e^\alpha.
\end{aligned}
$$

Proposition 8.3
Let $\alpha, \beta \in L^X \setminus \{0\}$ with $\alpha^2 = \beta^2 = 0$. Then

$$\langle|\alpha\rangle, |\beta\rangle\rangle = 0.$$

Proof:
We have (Proposition 5.36)

$$
\begin{gathered}
(e^\alpha)^*_n = e^\alpha_{-n-2},\ c^*_{-2} = -c_{-2},\ c^*_{-1} = -c_{-3},\ z^{\pm\mu}(-1)^*_{-1} = -z^{\pm\mu}(-1)_1, \\
((k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu}))^*_{-1} = -((k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu})_1, \\
(\psi^{\pm\mu}_{-1}\gamma^\mp)^*_{-1} = (\psi^{\pm\mu}_{-1}\gamma^\mp)_{-1}
\end{gathered}
$$

and

$$
\begin{aligned}
& ((k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu}))_1(t^+{}_\xi\psi^{-\xi})_{-1}(t^-{}_\theta\psi^{+\theta}) \\
= \quad & \sum_{j\geq 0}\{(k^+{}_\mu\psi^{-\mu})_{-1-j}(k^-{}_\nu\psi^{+\nu})_{1+j}(t^+{}_\xi\psi^{-\xi})_{-1}(t^-{}_\theta\psi^{+\theta}) \\
& +(k^-{}_\nu\psi^{+\nu})_{-j}(t^+{}_\xi\psi^{-\xi})_{-1}(k^+{}_\mu\psi^{-\mu})_j(t^-{}_\theta\psi^{+\theta})\} \\
= \quad & -2\eta^{\mu\theta}k^+{}_\mu t^-{}_\theta k^-{}_\nu\psi^{+\nu}_0 t^+{}_\xi\psi^{-\xi} \\
= \quad & 4(k^{+\theta}t^-{}_\theta)(k^{-\xi}t^+{}_\xi).
\end{aligned}
$$

Let

$$
\begin{aligned}
|\alpha\rangle \quad = \quad & k^+{}_\mu z^{-\mu}(-1)c_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)c_{-1}e^\alpha \\
& -(k^+{}_\mu\psi^{-\mu})_{-1}(k^-{}_\nu\psi^{+\nu})_{-1}c_{-1}e^\alpha - k^+{}_\mu(\psi^{-\mu})_{-1}\gamma^+_{-1}e^\alpha \\
& +k^-{}_\mu\psi^{+\mu}_{-1}\gamma^-_{-1}e^\alpha, \\
|\beta\rangle \quad = \quad & t^+{}_\xi z^{-\xi}(-1)c_{-1}e^\beta - t^-{}_\xi z^{+\xi}(-1)c_{-1}e^\beta \\
& -(t^+{}_\xi\psi^{-\xi})_{-1}(t^-{}_\theta\psi^{+\theta})_{-1}c_{-1}e^\beta - t^+{}_\xi(\psi^{-\xi})_{-1}\gamma^+_{-1}e^\beta + t^-{}_\xi\psi^{+\xi}_{-1}\gamma^-_{-1}e^\beta.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\langle|\alpha\rangle,|\beta\rangle\rangle = (c_{-2}|\alpha\rangle,|\beta\rangle) \\
&= -(1, k^{+}{}_{\mu}t^{-}{}_{\xi}e^{\alpha}_{-1}c_{-3}c_{-2}[z^{-\mu}(-1)_1, z^{+\xi}(-1)]c_{-1}e^{\beta}) \\
&\quad -(1, k^{-}{}_{\mu}t^{+}{}_{\xi}e^{\alpha}_{-1}c_{-3}c_{-2}[z^{+\mu}(-1)_1, z^{-\xi}(-1)]c_{-1}e^{\beta}) \\
&\quad -(1, e^{\alpha}_{-1}c_{-3}c_{-2}((k^{+}{}_{\mu}\psi^{-\mu})_{-1}(k^{-}{}_{\nu}\psi^{+\nu}))_1(t^{+}{}_{\xi}\psi^{-\xi})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}e^{\beta}) \\
&= (2\underbrace{(k^{+\xi}t^{-}{}_{\xi} + k^{-\xi}t^{+}{}_{\xi})}_{=-(\alpha,\beta)} + 4(\underbrace{k^{+\theta}t^{-}{}_{\theta})}_{=-\frac{1}{2}(\alpha,\tilde{\beta})}\underbrace{(k^{-\xi}t^{+}{}_{\xi})}_{=\frac{1}{2}(\alpha,\tilde{\beta})})\underbrace{(1, e^{3\sigma}_{-1}e^{\alpha+\beta})}_{=0\ for\ \beta\neq-\alpha}.
\end{aligned}
$$

Now $(1, e^{3\sigma}_{-1}e^{\alpha+\beta}) = 0$ for $\beta \neq -\alpha$ and $(\alpha,\beta) = 0, (\alpha,\tilde{\beta}) = 0$ for $\beta = -\alpha$ implies the assertion. $\square$

Next we show that the subalgebra H is abelian.

Proposition 8.4

For $h_1, h_2 \in H$ we have

$$[h_1, h_2] = 0.$$

Proof:

Let $a_1^{\pm\mu}, a_2^{\pm\nu} \in \mathbb{C}$ and

$$
\begin{aligned}
h_1 &= \sum_{\mu\in\{\pm\}} a_1^{+\mu}\left(z^{+\mu}(-1)c - 2\gamma^{-}_{-1}\psi^{+\mu}\right) \\
&\quad + \sum_{\mu\in\{\pm\}} a_1^{-\mu}\left(z^{-\mu}(-1)c - 2\gamma^{+}_{-1}\psi^{-\mu}\right), \\
h_2 &= \sum_{\nu\in\{\pm\}} a_2^{+\nu}\left(z^{+\nu}(-1)c - 2\gamma^{-}_{-1}\psi^{+\nu}\right) \\
&\quad + \sum_{\nu\in\{\pm\}} a_2^{-\nu}\left(z^{-\nu}(-1)c - 2\gamma^{+}_{-1}\psi^{-\nu}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
[h_1, h_2] &= -(b_0h_1)_0h_2 \\
&= -\sum_{\mu\in\{\pm\}}\sum_{j\geq 0}\left\{a_1^{+\mu}z^{+\mu}(-1)_{-1-j}1_j + a_1^{+\mu}1_{-2-j}z^{+\mu}(-1)_j\right\}h_2 \\
&\quad -\sum_{\mu\in\{\pm\}}\sum_{j\geq 0}\left\{a_1^{-\mu}z^{-\mu}(-1)_{-1-j}1_j + a_1^{-\mu}1_{-2-j}z^{-\mu}(-1)_j\right\}h_2 \\
&= -\sum_{\mu\nu\in\{\pm\}}\sum_{j\geq 0} a_1^{+\mu}a_2^{-\nu}1_{-2-j}[z^{+\mu}(-1)_j, z^{-\nu}(-1)]c \\
&\quad -\sum_{\mu\nu\in\{\pm\}}\sum_{j\geq 0} a_1^{-\mu}a_2^{+\nu}1_{-2-j}[z^{-\mu}(-1)_j, z^{+\nu}(-1)]c \\
&= \sum_{\mu,\nu\in\{\pm\}} 2\eta^{\mu\nu}a_1^{+\mu}a_2^{-\nu}\underbrace{1_{-3}c}_{=0} + \sum_{\mu,\nu\in\{\pm\}} 2\eta^{\mu\nu}a_1^{-\mu}a_2^{+\nu}\underbrace{1_{-3}c}_{=0} = 0.
\end{aligned}
$$

$\square$

Proposition 8.5
Let $\{v^{++}, v^{+-}, v^{-+}, v^{--}\}$ be the basis of H given in Theorem 7.5. Then

$$< v^{\pm\mu}, v^{\mp\nu} >= -2\eta^{\mu\nu}$$

where $\mu, \nu \in \{\pm\}$.
Hence the map from H to $\mathbb{C}\bigoplus_{\mathbb{Z}} L^X$ defined by

$$v^{\pm\mu} \mapsto z^{\pm\mu}$$

is an isometry.

Proof:
Proposition 5.36 implies

$$\begin{aligned}
< v^{\pm\mu}, v^{\mp\nu} >= & \\
& (z^{\pm\mu}(-1)_{-1}c_{-2}c, z^{\mp\nu}(-1)_{-1}c) - 2(z^{\pm\mu}(-1)_{-1}c_{-2}c, \gamma^{\pm}_{-1}\psi^{\mp\nu}) \\
& -2(c_{-2}\gamma^{\mp}_{-1}\psi^{\pm\mu}, z^{\mp\nu}(-1)_{-1}c) + 4(c_{-2}\gamma^{\mp}_{-1}\psi^{\pm\mu}, \gamma^{\pm}_{-1}\psi^{\mp\nu}) \\
= & \ (1, c_{-3}c_{-2}z^{\pm\mu}(-1)_1 z^{\mp\nu}(-1)c) - 2(1, \underbrace{c_{-3}c_{-2}z^{\pm\mu}(-1)_1\gamma^{\pm}_{-1}\psi^{\mp\nu})}_{=0} \\
& -2(1, \underbrace{\gamma^{\mp}_{-1}\psi^{\pm\mu}_{-1}c_{-2}z^{\mp\nu}(-1)c})_{\neq e^{3\sigma}} + 4(1, \underbrace{(\gamma^{\mp}_{-1}\psi^{\pm\mu})_{-1}c_{-2}\gamma^{\pm}_{-1}\psi^{\mp\nu})}_{\neq e^{3\sigma}} \\
= & \ -2\eta^{\mu\nu}(1, c_{-3}c_{-2}c) = -2\eta^{\mu\nu}(1, e^{3\sigma}) \\
= & \ -2\eta^{\mu\nu}.
\end{aligned}$$

□

Now we show that the subalgebra H measures the momentum.

Proposition 8.6
Let $h \in H$, $x \in G_\alpha$. Then

$$[h, x] = (h, \alpha)x.$$

Proof:
Let $a, a^{\pm\mu} \in \mathbb{C}$ and

$$\begin{aligned}
h &= \sum_{\mu\in\{\pm\}} \left(a^{+\mu}z^{+\mu}(-1)_{-1}c - 2a^{+\mu}\gamma^{-}_{-1}\psi^{+\mu}\right) \\
& \quad \sum_{\mu\in\{\pm\}} \left(a^{-\mu}z^{-\mu}(-1)_{-1}c - 2a^{-\mu}\gamma^{+}_{-1}\psi^{-\mu}\right), \\
x &= a\left(k^{+}{}_{\nu}z^{-\nu}(-1)_{-1}ce^{\alpha} - k^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}e^{\alpha}\right. \\
& \quad -(k^{+}{}_{\nu}\psi^{-\nu})_{-1}(k^{-}{}_{\xi}\psi^{+\xi})_{-1}c_{-1}e^{\alpha} - k^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}e^{\alpha} \\
& \quad \left.+k^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}e^{\alpha}\right).
\end{aligned}$$

Then

$$
\begin{aligned}
(b_0 h)_0 x \quad = \quad & a \sum_{\mu \in \{\pm\}} \left\{ a^{+\mu} \sum_{j \geq 0} k^{+}{}_{\nu} 1_{-1-j} [z^{+\mu}(-1)_j, z^{-\nu}(-1)_{-1}] c_{-1} e^{\alpha} \right. \\
& + a^{+\mu} \sum_{j \geq 0} 1_{-1-j} k^{+}{}_{\nu} z^{-\nu}(-1)_{-1} c_{-1} z^{+\mu}(-1)_j e^{\alpha} \\
& + a^{-\mu} \sum_{j \geq 0} 1_{-1-j} k^{+}{}_{\nu} z^{-\nu}(-1)_{-1} c_{-1} z^{-\mu}(-1)_j e^{\alpha} \\
& - a^{-\mu} \sum_{j \geq 0} 1_{-1-j} k^{-}{}_{\nu} [z^{-\mu}(-1)_j, z^{+\nu}(-1)_{-1}] c_{-1} e^{\alpha} \\
& - a^{+\mu} \sum_{j \geq 0} 1_{-1-j} k^{-}{}_{\nu} z^{+\nu}(-1)_{-1} c_{-1} z^{+\mu}(-1)_j e^{\alpha} \\
& - a^{-\mu} \sum_{j \geq 0} 1_{-1-j} k^{-}{}_{\nu} z^{+\nu}(-1)_{-1} c_{-1} z^{-\mu}(-1)_j e^{\alpha} \\
& - a^{+\mu} \sum_{j \geq 0} 1_{-1-j} (k^{+}{}_{\nu} \psi^{-\nu})_{-1} (k^{-}{}_{\xi} \psi^{+\xi})_{-1} c_{-1} z^{+\mu}(-1)_j e^{\alpha} \\
& - a^{-\mu} \sum_{j \geq 0} 1_{-1-j} (k^{+}{}_{\nu} \psi^{-\nu})_{-1} (k^{-}{}_{\xi} \psi^{+\xi})_{-1} c_{-1} z^{-\mu}(-1)_j e^{\alpha} \\
& - a^{+\mu} \sum_{j \geq 0} 1_{-1-j} k^{+}{}_{\nu} \psi^{-\nu}_{-1} \gamma^{+}_{-1} z^{+\mu}(-1)_j e^{\alpha} \\
& - a^{-\mu} \sum_{j \geq 0} 1_{-1-j} k^{+}{}_{\nu} \psi^{-\nu}_{-1} \gamma^{+}_{-1} z^{-\mu}(-1)_j e^{\alpha} \\
& + a^{+\mu} \sum_{j \geq 0} 1_{-1-j} k^{-}{}_{\nu} \psi^{+\nu}_{-1} \gamma^{-}_{-1} z^{+\mu}(-1)_j e^{\alpha} \\
& \left. + a^{-\mu} \sum_{j \geq 0} 1_{-1-j} k^{-}{}_{\nu} \psi^{+\nu}_{-1} \gamma^{-}_{-1} z^{-\mu}(-1)_j e^{\alpha} \right\} \\
= \quad & a \sum_{\mu \in \{\pm\}} (z^{+\mu}, \alpha) a^{+\mu} x + a \sum_{\mu \in \{\pm\}} (z^{-\mu}, \alpha) a^{-\mu} x \overset{Prop.\ 8.5}{=} (h, \alpha) x.
\end{aligned}
$$

□

Hence H is an abelian subalgebra with $[h, x] = (h, \alpha)x$ for $h \in H$ and $x \in G_\alpha$. Therefore we call H Cartan subalgebra.

Now we calculate the Lie bracket on G. Note that

$$[G_\alpha, G_\beta] \subset G_{\alpha+\beta}$$

so that $[G_\alpha, G_\beta]$ can only be nontrivial if $\alpha^2 = \beta^2 = (\alpha, \beta) = 0$.

Theorem 8.7
Let

$$\alpha = k^-{}_\mu z^{+\mu} + k^+{}_\mu z^{-\mu} = -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--} \in L^X,$$

$$\beta = t^-{}_\mu z^{+\mu} + t^+{}_\mu z^{-\mu} = -t^{-+}z^{++} + t^{--}z^{+-} - t^{++}z^{-+} + t^{+-}z^{--} \in L^X,$$

with $\alpha^2 = \beta^2 = (\alpha, \beta) = 0$. *Then we have*

$$[|\alpha\rangle, |\beta\rangle] = 2\widetilde{(\alpha,\beta)}|\alpha+\beta\rangle.$$

Proof:
Let

$$\begin{aligned} u &= k^+{}_\mu z^{-\mu}(-1)_{-1}c_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)_{-1}c_{-1}e^\alpha \\ &\quad -(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}c_{-1}e^\alpha - k^+{}_\mu \psi^{-\mu}_{-1}\gamma^+_{-1}e^\alpha \\ &\quad +k^-{}_\mu \psi^{+\mu}_{-1}\gamma^-_{-1}e^\alpha \end{aligned}$$

and

$$\begin{aligned} v &= t^+{}_\nu z^{-\nu}(-1)_{-1}c_{-1}e^\beta - t^-{}_\nu z^{+\nu}(-1)_{-1}c_{-1}e^\beta \\ &\quad -(t^+{}_\nu \psi^{-\nu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^\beta - t^+{}_\nu \psi^{-\nu}_{-1}\gamma^+_{-1}e^\beta \\ &\quad +t^-{}_\nu \psi^{+\nu}_{-1}\gamma^-_{-1}e^\beta. \end{aligned}$$

We have

$$b_0 u = k^+{}_\mu z^{-\mu}(-1)_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)_{-1}e^\alpha - (k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}e^\alpha$$

and

$$\begin{aligned} (b_0 u)_0 &= \sum_{j\geq 0}\{k^+{}_\mu z^{-\mu}(-1)_{-1-j}e^\alpha_j + e^\alpha_{-1-j}k^+{}_\mu z^{-\mu}(-1)_j\} \\ &\quad -\sum_{j\geq 0}\{k^-{}_\mu z^{+\mu}(-1)_{-1-j}e^\alpha_j + e^\alpha_{-1-j}k^-{}_\mu z^{+\mu}(-1)_j\} \\ &\quad -\sum_{j\geq 0}\{(k^+{}_\mu \psi^{-\mu})_{-1-j}((k^-{}_\xi \psi^{+\xi})_{-1}e^\alpha)_j \\ &\quad -((k^-{}_\xi \psi^{+\xi})_{-1}e^\alpha)_{-1-j}k^+{}_\mu \psi^{-\mu}_j\} \end{aligned}$$

with

$$((k^-{}_\xi \psi^{+\xi})_{-1}e^\alpha)_j = \sum_{k\geq 0}\{k^-{}_\xi \psi^{+\xi}_{-1-k}e^\alpha_{j+k} + e^\alpha_{j-1-k}k^-{}_\xi \psi^{+\xi}_k\}.$$

Furthermore

$$\begin{aligned} (z^{-\mu},\beta)k^+{}_\mu &= -(z^{-+},\beta)k^{++} + (z^{--},\beta)k^{+-} \\ &= 2(t^{-+}k^{++} - t^{--}k^{+-}), \\ (z^{+\mu},\beta)k^-{}_\mu &= -(z^{++},\beta)k^{-+} + (z^{+-},\beta)k^{--} \end{aligned}$$

$$
\begin{aligned}
&= 2(k^{-+}t^{++} - k^{--}t^{+-}),\\
(z^{+\nu},\alpha)t^{-}{}_{\nu} &= -(z^{++},\alpha)t^{-+} + (z^{+-},\alpha)t^{--}\\
&= 2(k^{++}t^{-+} - k^{+-}t^{--}),\\
(z^{-\nu},\alpha)t^{+}{}_{\nu} &= -(z^{-+},\alpha)t^{++} + (z^{--},\alpha)t^{+-}\\
&= 2(k^{-+}t^{++} - k^{--}t^{+-}),\\
k^{+\nu}t^{-}{}_{\nu} &= -k^{++}t^{-+} + k^{+-}t^{--},\\
k^{-\nu}t^{+}{}_{\nu} &= -k^{-+}t^{++} + k^{--}t^{+-}
\end{aligned}
$$

so that

$$
\begin{aligned}
(b_0u)_0v = &\sum_{j\geq 0} k^{+}{}_{\mu}z^{-\mu}(-1)_{-1-j}e^{\alpha}_{j}t^{+}{}_{\nu}z^{-\nu}(-1)_{-1}c_{-1}e^{\beta}\\
&-\sum_{j\geq 0} k^{+}{}_{\mu}z^{-\mu}(-1)_{-1-j}e^{\alpha}_{j}t^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{+}{}_{\nu}\underbrace{[z^{-\mu}(-1)_{j},z^{-\nu}(-1)_{-1}]}_{=0}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{+}{}_{\nu}z^{-\nu}(-1)_{-1}c_{-1}z^{-\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{-}{}_{\nu}\underbrace{[z^{-\mu}(-1)_{j},z^{+\nu}(-1)]}_{=-2\delta_{j-1}\eta^{\mu\nu}}c_{-1}e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}z^{-\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}(t^{+}{}_{\nu}\psi^{-\nu})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}z^{-\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}z^{-\mu}(j)e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{+}{}_{\mu}t^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}z^{-\mu}(-1)z^{-\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0} k^{-}{}_{\mu}z^{+\mu}(-1)_{-1-j}t^{+}{}_{\nu}[e^{\alpha}_{j},z^{-\nu}(-1)]c_{-1}e^{\beta}\\
&+\sum_{j\geq 0} k^{-}{}_{\mu}z^{+\mu}(-1)_{-1-j}t^{-}{}_{\nu}[e^{\alpha}_{j},z^{+\nu}(-1)]c_{-1}e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-}{}_{\mu}t^{+}{}_{\nu}[z^{+\mu}(j),z^{-\nu}(-1)]c_{-1}e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-}{}_{\mu}t^{+}{}_{\nu}z^{-\nu}(-1)_{-1}c_{-1}z^{+\mu}(j)e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{1-j}k^{-}{}_{\mu}t^{-}{}_{\nu}[z^{+\mu}(j),z^{+\nu}(-1)]c_{-1}e^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-}{}_{\mu}t^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}z^{+\mu}(j)e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-}{}_{\mu}(t^{+}{}_{\nu}\psi^{-\nu})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}z^{+\mu}(j)e^{\beta}\\
&+\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-}{}_{\mu}t^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}z^{+\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0} e^{\alpha}_{-1-j}k^{-\mu}t^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}z^{+\mu}(j)e^{\beta}\\
&-\sum_{j\geq 0}(k^{+}{}_{\mu}\psi^{-\mu})_{-1-j}\sum_{k\geq 0}k^{-}{}_{\xi}\psi^{+\xi}_{-1-k}e^{\alpha}_{j+k}t^{+}{}_{\nu}z^{-\nu}(-1)_{-1}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0}(k^{+}{}_{\mu}\psi^{-\mu})_{-1-j}\sum_{k\geq 0}k^{-}{}_{\xi}\psi^{+\xi}_{-1-k}e^{\alpha}_{j+k}t^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0}(k^{+}{}_{\mu}\psi^{-\mu})_{-1-j}\sum_{k\geq 0}e^{\alpha}_{-1-k+j}k^{-}{}_{\xi}\psi^{+\xi}_{k}(t^{+}{}_{\nu}\psi^{-\nu})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0}(k^{+}{}_{\mu}\psi^{-\mu})_{-1-j}\sum_{k\geq 0}e^{\alpha}_{-1-k+j}k^{-}{}_{\xi}\psi^{+\xi}_{k}t^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}e^{\beta}\\
&-\sum_{j\geq 0}(k^{+}{}_{\mu}\psi^{-\mu})_{-1-j}\sum_{k\geq 0}e^{\alpha}_{-1-k+j}k^{-}{}_{\xi}\psi^{+\xi}_{k}t^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}e^{\beta}\\
&+\sum_{j\geq 0}((k^{-}{}_{\xi}\psi^{+\xi})_{-1}e^{\alpha})_{-1-j}k^{+}{}_{\mu}\psi^{-\mu}_{j}(t^{+}{}_{\nu}\psi^{-\nu})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}e^{\beta}\\
&+\sum_{j\geq 0}((k^{-}{}_{\xi}\psi^{+\xi})_{-1}e^{\alpha})_{-1-j}k^{+}{}_{\mu}\psi^{-\mu}_{j}t^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}e^{\beta}\\
&-\sum_{j\geq 0}((k^{-}{}_{\xi}\psi^{+\xi})_{-1}e^{\alpha})_{-1-j}k^{+}{}_{\mu}\psi^{-\mu}_{j}t^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}e^{\beta}\\
= \quad & -(z^{-\nu},\alpha)k^{+}{}_{\mu}t^{+}{}_{\nu}z^{-\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&+(z^{+\nu},\alpha)k^{+}{}_{\mu}t^{-}{}_{\nu}z^{-\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&+(z^{-\mu},\beta)k^{+}{}_{\mu}t^{+}{}_{\nu}[e^{\alpha}_{-1},z^{-\nu}(-1)]c_{-1}e^{\beta}\\
&+(z^{-\mu},\beta)k^{+}{}_{\mu}t^{+}{}_{\nu}z^{-\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&+2\eta^{\mu\nu}k^{+}{}_{\mu}t^{-}{}_{\nu}\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&-(z^{-\mu},\beta)k^{+}{}_{\mu}t^{-}{}_{\nu}[e^{\alpha}_{-1},z^{+\nu}(-1)]c_{-1}e^{\beta}\\
&-(z^{-\mu},\beta)k^{+}{}_{\mu}t^{-}{}_{\nu}z^{+\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&-(z^{-\mu},\beta)k^{+}{}_{\mu}(t^{+}{}_{\nu}\psi^{-\nu})_{-1}(t^{-}{}_{\theta}\psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta}\\
&-(z^{-\mu},\beta)k^{+}{}_{\mu}t^{+}{}_{\nu}\psi^{-\nu}_{-1}\gamma^{+}_{-1}e^{\alpha+\beta}\\
&+(z^{-\mu},\beta)k^{+}{}_{\mu}t^{-}{}_{\nu}\psi^{+\nu}_{-1}\gamma^{-}_{-1}e^{\alpha+\beta}\\
&+(z^{-\nu},\alpha)k^{-}{}_{\mu}t^{+}{}_{\nu}z^{+\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta}\\
&-(z^{+\nu},\alpha)t^{-}{}_{\nu}k^{-}{}_{\mu}z^{+\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta}
\end{aligned}
$$

$$
\begin{aligned}
& +2\eta^{\mu\nu}k^-{}_\mu t^+{}_\nu \alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& -(z^{+\mu},\beta)k^-{}_\mu t^+{}_\nu [e^\alpha_{-1}, z^{-\nu}(-1)]c_{-1}e^\beta \\
& -(z^{+\mu},\beta)k^-{}_\mu t^+{}_\nu z^{-\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu t^-{}_\nu [e^\alpha_{-1}, z^{+\nu}(-1)]c_{-1}e^\beta \\
& +(z^{+\mu},\beta)k^-{}_\mu t^-{}_\nu z^{+\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu (t^+{}_\nu \psi^{-\nu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu t^+{}_\nu \psi^{-\nu}_{-1}\gamma^+_{-1}e^{\alpha+\beta} \\
& -(z^{+\mu},\beta)k^-{}_\mu t^-{}_\mu \psi^{+\nu}_{-1}\gamma^-_{-1}e^{\alpha+\beta} \\
& +t^+{}_\nu (z^{-\nu},\alpha)(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}c_{-1}e^{\alpha+\beta} \\
& -t^-{}_\nu (z^{+\nu},\alpha)(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}c_{-1}e^{\alpha+\beta} \\
& -2\eta^{\xi\nu}t^+{}_\nu k^-{}_\xi (k^+{}_\mu \psi^{-\mu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta} \\
& -2\eta^{\xi\nu}k^-{}_\xi t^+{}_\nu k^+{}_\mu \psi^{-\mu}_{-1}\gamma^+_{-1}e^{\alpha+\beta} \\
& -2\eta^{\mu\theta}k^+{}_\mu t^-{}_\theta (k^-{}_\xi \psi^{+\xi})_{-1}(t^+{}_\nu \psi^{-\nu})_{-1}c_{-1}e^{\alpha+\beta} \\
& +2\eta^{\mu\nu}k^+{}_\mu t^-{}_\nu k^-{}_\xi \psi^{+\xi}_{-1}\gamma^-_{-1}e^{\alpha+\beta} \\
= \; & ((z^{+\nu},\alpha)t^-{}_\nu - (z^{-\nu},\alpha)t^+{}_\nu)k^+{}_\mu z^{-\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +((z^{-\mu},\beta)k^+{}_\mu - (z^{+\mu},\beta)k^-{}_\mu)t^+{}_\nu z^{-\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{-\nu},\alpha)t^+{}_\nu - (z^{+\nu},\alpha)t^-{}_\nu)k^-{}_\mu z^{+\mu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +((z^{+\mu},\beta)k^-{}_\mu - (z^{-\mu},\beta)k^+{}_\mu)t^-{}_\nu z^{+\nu}(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{-\mu},\beta)k^+{}_\mu t^+{}_\nu (z^{-\nu},\alpha)\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{-\mu},\beta)k^+{}_\mu t^-{}_\nu (z^{+\nu},\alpha)\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu t^+{}_\nu (z^{-\nu},\alpha)\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu t^-{}_\nu (z^{+\nu},\alpha)\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& +(2k^{+\nu}t^-{}_\nu + 2k^{-\nu}t^+{}_\nu)\alpha(-1)_{-1}c_{-1}e^{\alpha+\beta} \\
& -(z^{-\mu},\beta)k^+{}_\mu (t^+{}_\nu \psi^{-\nu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu (t^+{}_\nu \psi^{-\nu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta} \\
& +t^+{}_\nu (z^{-\nu},\alpha)(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}c_{-1}e^{\alpha+\beta} \\
& -t^-{}_\nu (z^{+\nu},\alpha)(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\xi \psi^{+\xi})_{-1}c_{-1}e^{\alpha+\beta} \\
& -4k^{-\nu}t^+{}_\nu (k^+{}_\mu \psi^{-\mu})_{-1}(t^-{}_\theta \psi^{+\theta})_{-1}c_{-1}e^{\alpha+\beta} \\
& -4k^{+\theta}t^-{}_\theta (k^-{}_\xi \psi^{+\xi})_{-1}(t^+{}_\nu \psi^{-\nu})_{-1}c_{-1}e^{\alpha+\beta} \\
& -(z^{-\mu},\beta)k^+{}_\mu t^+{}_\nu \psi^{-\nu}_{-1}\gamma^+_{-1}e^{\alpha+\beta} \\
& +(z^{+\mu},\beta)k^-{}_\mu t^+{}_\nu \psi^{-\nu}_{-1}\gamma^+_{-1}e^{\alpha+\beta} \\
& -4k^{-\nu}t^+{}_\nu k^+{}_\mu \psi^{-\mu}_{-1}\gamma^+_{-1}e^{\alpha+\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+(z^{-\mu},\beta)k^{+}{}_{\mu}t^{-}{}_{\nu}\psi_{-1}^{+\nu}\gamma_{-1}^{-}e^{\alpha+\beta}\\
&-(z^{+\mu},\beta)k^{-}{}_{\mu}t^{-}{}_{\nu}\psi_{-1}^{+\nu}\gamma_{-1}^{-}e^{\alpha+\beta}\\
&-4k^{+\nu}t^{-}{}_{\nu}k^{-}{}_{\xi}\psi_{-1}^{+\xi}\gamma_{-1}^{-}e^{\alpha+\beta}.
\end{aligned}
$$

The last expression is a representative of $2\widetilde{(\alpha,\beta)}|\alpha+\beta\rangle$.
This proves the theorem. □
Note that for $\alpha+\beta=0$ we have $[|\alpha\rangle,|\beta\rangle]=0$.
The last proposition implies that there is no nondegenerate invariant bilinear form on G.

Proposition 8.8
Let $\{,\}_i$ be an invariant symmetric bilinear form on G. Then $\{H,G_\alpha\}_i=0$ and $\{G_\alpha,G_\beta\}_i=0$ for all $\alpha,\beta\in L^X$ with $\alpha^2=\beta^2=0$ and $\alpha\neq0,\beta\neq0$.

Proof:
By Proposition 8.4 and Proposition 8.6

$$0=\{[h_1,h_2],|\alpha\rangle\}_i=\{h_1,[h_2,|\alpha\rangle]\}_i=(h_2,\alpha)\,\{h_1,|\alpha\rangle\}_i\quad\forall h_1,h_2\in H.$$

Hence $\{h_1,|\alpha\rangle\}_i=0$. This proves the first statement.
First we treat the case $\alpha+\beta\neq0$. Then by Proposition 8.6

$$
\begin{aligned}
(h,\alpha)\,\{|\alpha\rangle,|\beta\rangle\}_i &= \{[h,|\alpha\rangle]\,,|\beta\rangle\}_i\\
&= -\{|\alpha\rangle,[h,|\beta\rangle]\}_i=-(h,\beta)\,\{|\alpha\rangle,|\beta\rangle\}_i\quad\forall h\in H\\
\Leftrightarrow\quad & (h,\alpha+\beta)\{|\alpha\rangle,|\beta\rangle\}_i=0\quad\forall h\in H.
\end{aligned}
$$

This implies $\{|\alpha\rangle,|\beta\rangle\}_i=0$. Now we treat the case $\alpha+\beta=0$. Then by Proposition 8.7 we have $[|\alpha\rangle,|\beta\rangle]=0$ so that

$$0=\{h,[|\alpha\rangle,|\beta\rangle]\}_i=\{[h,|\alpha\rangle]\,,|\beta\rangle\}_i=(h,\alpha)\,\{|\alpha\rangle,|\beta\rangle\}_i\ \forall h\in H$$

and $\{|\alpha\rangle,|\beta\rangle\}_i=0$. This proves the second statement. □

8.2 Is G a generalized Kac-Moody algebra ?

Now we want to show that the Lie algebra of the $N=2$-string is not a generalized Kac-Moody algebra with Cartan subalgebra H. Therefor we need the following proposition. Let

$$\Delta:=\{\alpha\in L^X:\alpha^2=0\wedge\alpha\neq0\}.$$

Proposition 8.9

$$\exists\alpha,\beta,\gamma\in\Delta,\alpha\neq\beta,\beta\neq\gamma,\alpha\neq\gamma:(\alpha,\beta)>0,(\beta,\gamma)<0,(\alpha,\gamma)<0.$$

Proof:
Let $\{v_1, v_2, v_3, v_4\}$ be a $\mathbb{Z}$-basis of L^X with $v_1v_4 = v_4v_1 = v_2v_3 = v_3v_2 = 1$ and all other scalar products vanish. Hence for $x, y \in L^X, x = \sum x_i v_i, y = \sum y_j v_j$, we have $(x,y) = \sum_{i,j} x_i y_j v_i v_j = x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1$.
Then there are α, β, γ with $(\alpha,\beta) > 0, (\beta,\gamma) < 0, (\alpha,\gamma) < 0$. For example if

$$\begin{aligned}
\alpha &= (0,2,0,1), \\
\gamma &= (-2,1,0,0), \\
\beta &= (-1,1,1,1),
\end{aligned}$$

then

$$(\alpha,\gamma) = -2, \quad (\alpha,\beta) = 1, \quad (\beta,\gamma) = -1.$$

$\square$

With this proposition we can prove the following theorem. This will be needed to show that the Lie algebra G is not a generalized Kac-Moody algebra with Cartan subalgebra H.

Theorem 8.10
For any decomposition of Δ into a disjoint union

$$\Delta = M^+ \sqcup M^-$$

with

$$-M^+ = M^-$$

we have

$$\exists \alpha, \beta \in M^+, \alpha \neq \beta : (\alpha,\beta) > 0.$$

Proof:
Let

$$\Delta = M^+ \sqcup M^-$$

be an arbitrary decomposition of Δ such that

$$-M^+ = M^-.$$

Note that we have for $\alpha \in \Delta : \alpha \in M^+$ or $-\alpha \in M^+$.
Assume that

$$(*) \quad \forall \alpha, \beta \in \Delta \quad \text{with} \quad (\alpha,\beta) > 0 \quad \text{we have} \quad \alpha, -\beta \in M^+ \vee -\alpha, \beta \in M^+.$$

By Proposition 8.9 we have

$$\exists\ \alpha, \beta, \gamma \in \Delta, \alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma : (\alpha,\beta) > 0, (\alpha,\gamma) < 0, (\beta,\gamma) < 0.$$

We can assume that $\alpha, -\beta \in M^+$.
If $\gamma \in M^+$ then $(-\beta,\gamma) = -(\beta,\gamma) > 0$.
If $-\gamma \in M^+$ then $(\alpha,-\gamma) = -(\alpha,\gamma) > 0$.
This contradicts the assumption. $\square$

Theorem 8.11
G is not a generalized Kac-Moody algebra with Cartan subalgebra H.

Proof:
Assume
$$G = H \oplus \bigoplus_{\alpha\in\Delta} G_\alpha$$
is a generalized Kac-Moody algebra with Cartan subalgebra H. Then Δ is the set of roots. By (3.1) we have
$$-\Delta^+ = \Delta^-$$
and Proposition 3.6 implies that
$$\Delta = \Delta^+ \sqcup \Delta^-.$$
The roots of G have norm zero so that by Proposition 3.11 all positive roots are simple roots. Now Theorem 8.10 shows that independent of the decomposition of Δ into positive and negative roots there are always positive roots, i.e. simple roots, α and β with $\alpha \neq \beta$ and $(\alpha,\beta) > 0$. That is impossible if G is a generalized Kac-Moody algebra (cf. Definition 3.1). $\square$

8.3 Summary of the properties of G

Now we summarize the properties of the Lie algebra of the N=2-string.

Theorem 8.12
The Lie algebra G of the N=2-string moving on the torus $\mathbb{R}^{2,2}/I\!I_{2,2}$ decomposes as
$$G = H \oplus \bigoplus_{\alpha\in\Delta} G_\alpha$$
where H is an abelian subalgebra of G and $\Delta = \{\alpha \in I\!I_{2,2} | \alpha^2 = 0 \wedge \alpha \neq 0\}$. There is an invariant symmetric bilinear form on G which is nondegenerate on H and vanishes on the root spaces G_α.
H has dimension 4 and the vectors
$$v^{\pm\mu} = z^{\pm\mu}(-1)_{-1}c - 2\gamma^{\mp}_{-1}\psi^{\pm\mu}$$
form a basis of H. The map $H \to I\!I_{2,2} \otimes \mathbb{C}, v^{\pm\mu} \mapsto z^{\pm\mu}$ is an isometry.
Let $h \in H$ and $x \in G_\alpha$. Then
$$[h,x] = (h,\alpha)x.$$
Let $\alpha \in I\!I_{2,2}$ with $\alpha^2 = 0$ and $\alpha \neq 0$. Write
$$\alpha = -k^{-+}z^{++} + k^{--}z^{+-} - k^{++}z^{-+} + k^{+-}z^{--}.$$

Then G_α has dimension one and is generated by

$$|\alpha\rangle = k^+{}_\mu z^{-\mu}(-1)c_{-1}e^\alpha - k^-{}_\mu z^{+\mu}(-1)c_{-1}e^\alpha$$

$$-(k^+{}_\mu \psi^{-\mu})_{-1}(k^-{}_\nu \psi^{+\nu})_{-1}c_{-1}e^\alpha - k^+{}_\mu(\psi^{-\mu})_{-1}\gamma^+_{-1}e^\alpha + k^-{}_\mu \psi^{+\mu}_{-1}\gamma^-_{-1}e^\alpha.$$

Let $\alpha, \beta \in I\!I_{2,2}$ with $\alpha^2 = \beta^2 = (\alpha,\beta) = 0$ and $\alpha \neq 0, \beta \neq 0$. Then

$$[|\alpha\rangle, |\beta\rangle] = 2\widetilde{(\alpha,\beta)}|\alpha+\beta\rangle$$

where

$$\widetilde{(\alpha,\beta)} = k^{++}t^{-+} - k^{+-}t^{--} + k^{--}t^{+-} - k^{-+}t^{++}.$$

There is no nondegenerate invariant bilinear form on G.
G is not a generalized Kac-Moody algebra with Cartan subalgebra H.

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