Radon measures solving the Cauchy problem of the nonlinear transport equation

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Abstract. The focus of interest is the Cauchy problem of the nonlinear transport equation

$$\frac{d}{dt}\,\mu \;+\; \mathrm{div}_x(f(\mu,\cdot)\;\mu) \;=\; g(\mu,\cdot)\,\mu \qquad\qquad (\text{in } \mathbb{R}^N \times\,]0,T[)$$

together with its distributional solutions $\mu(\cdot) : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$ whose values are positive Radon measures on $\mathbb{R}^N$ with compact support. The coefficients $f(\mu,t)$, $g(\mu,t)$ are assumed to be uniformly bounded and Lipschitz continuous vector fields on $\mathbb{R}^N$.

Sufficient conditions on the coefficients $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ for existence, uniqueness and even for stability of these distributional solutions are presented. Starting from the well-known results about the corresponding linear problem, the step towards the nonlinear problem here relies on Aubin's mutational equations, i.e. dynamical systems in a metric space (with a new slight modification).

Contents

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1 Introduction

The scalar transport equation $\frac{d}{dt}\, u \;+\; \mathrm{div}_x(\widetilde{\mathbf{b}}\; u) \;=\; \widetilde{c}\; u$ (in $\mathbb{R}^N \times [0,T[$) is the classical analytical tool for describing a real–valued quantity $u = u(x,t)$ while "flowing" (or, rather, evolving) along a given vector field $\widetilde{\mathbf{b}} : \mathbb{R}^N \times [0,T] \longrightarrow \mathbb{R}^N$ and exploiting a form of source (described by the scalar field $\widetilde{c} : \mathbb{R}^N \times [0,T] \longrightarrow \mathbb{R}$). Thus, it is playing a key role in many applications of modelling like fluid dynamics and, it has been investigated under completely different types of assumptions about $\widetilde{\mathbf{b}}(\cdot,\cdot)$, $\widetilde{c}(\cdot,\cdot)$.

Due to well-known difficulties in regard to smooth solutions, the values of all solutions considered here are positive finite Radon measures on $\mathbb{R}^N$ with compact support (whose set is abbreviated as $\mathcal{M}_c^+(\mathbb{R}^N)$) and, we are interested in (structurally) simple sufficient conditions on the coefficients for proving existence, uniqueness and stability of a distributional solution of the *nonlinear* transport equation

$$\begin{cases} \frac{d}{dt}\, \mu(t) \;+\; \mathrm{div}_x(F_1(\mu(t),t)\;\, \mu(t)) \;=\; F_2(\mu(t),t)\;\, \mu(t) \qquad \text{in } \mathbb{R}^N \times]0,T[\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mu(0) \;=\; \mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N) \end{cases}$$

with given $F = (F_1, F_2) : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N,\mathbb{R})$.
As a main result in this paper, suitable continuity of the coefficients $F(\cdot,\cdot)$ implies existence:

Proposition 1.1 (Existence)
Let $F : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N,\mathbb{R})$ *satisfy the following conditions:*

1.) $\sup_{\mu_0,t} \big(\|F_1(\mu_0,t)\|_{W^{1,\infty}} + \|F_2(\mu_0,t)\|_{W^{1,\infty}}\big) \;<\; \infty$

2.) F_1, F_2 *are continuous with respect to narrow convergence in* $\mathcal{M}_c^+(\mathbb{R}^N)$ *and* L^∞ *norm of spatial fields.*

Then, for any initial datum $\nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, *there exists a narrowly continuous weak solution* $\mu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, $t \longmapsto \mu_t$ *of the nonlinear transport equation*

$$\begin{cases} \partial_t\, \mu_t \;+\; \mathit{div}_x\, (F_1(\mu_t,t)\;\, \mu_t) \;=\; F_2(\mu_t,t)\;\, \mu_t \quad \text{in } [0,T] \\ \qquad\qquad\qquad\qquad\quad\;\; \mu_0 \;=\; \nu_0 \end{cases} \tag{1}$$

This global existence result has three essential advantages in common with the subsequent statements about uniqueness and stability: Firstly, the structural conditions on the considered measures are rather weak. In particular, these positive Radon measures with compact support need not be absolutely continuous with respect to Lebesgue measure. We can apply all results of this paper to the evolution of lower dimensional Hausdorff measures (with compact support), for example. Secondly, there is no restriction imposed on the initial datum – such as "small norm" (in any sense). Thirdly, the coefficient function $F = (F_1, F_2)$ is assumed to be defined on $\mathcal{M}_c^+(\mathbb{R}^N) \times [0,T]$ in a very general way obeying merely continuity hypotheses. So in particular, nonlocal information about the Radon measures can be taken into consideration explicitly (such as nonlinear functions of weighted integral means).

Solving this nonlinear Cauchy problem (1) is based on a decomposition: The first step focuses on the corresponding autonomous linear problem

$$\left\{ \begin{array}{rcll} \partial_t \, \mu_t \; + \; \mathrm{div}_x \, (\mathbf{b} \, \mu_t) & = & c \, \mu_t & \text{in } [0,T] \\ \mu_0 & = & \nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N) & \end{array} \right. \tag{2}$$

for given $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and $c \in L^\infty(\mathbb{R}^N, \mathbb{R})$. Here we investigate the regularity of the weak solution with respect to time, initial datum ν_0 and coefficients $\mathbf{b}(\cdot), c(\cdot)$. The bridge to the nonlinear Cauchy problem is then gapped by a form of "feedback", i.e. the coefficients are prescribed as a function of current measure and time. Let us discuss this decomposition in more detail:

The solution of the corresponding autonomous linear problem can be characterized explicitly in a rather easy way. Indeed, the Lipschitz continuity of the spatial vector fields implies that the characteristics are well-defined. So considering the Cauchy problem (2), the weak solution $\mu : [0,T[\longrightarrow \mathcal{M}^+(\mathbb{R}^N)$, $t \longmapsto \mu_t$ is uniquely described by

$$\int_{\mathbb{R}^N} \varphi \, d\mu_t \; = \; \int_{\mathbb{R}^N} \varphi(\mathbf{X_b}(t,x)) \cdot \exp\left(\int_0^t c(\mathbf{X_b}(s,x)) \, ds \right) \, d\mu_0(x) \qquad \text{for all } \varphi \in C_c^0(\mathbb{R}^N)$$

with $\mathbf{X_b}(\cdot,\cdot) : [0,\infty[\times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ denoting the flow along the bounded Lipschitz continuous vector field $\mathbf{b}$. Thus, the autonomous linear Cauchy problem induces a C^0 semigroup on $\mathcal{M}^+(\mathbb{R}^N)$ (with respect to narrow convergence).

The crucial step to the *nonlinear* transport equation is now based on a tool that is hardly known in the PDE community, but we regard it as very useful indeed : Aubin's concept of *mutational equations* [5, 6, 7]. Its goal is to extend ordinary differential equations to any metric space (E,d) so that the "derivative" of the wanted curve can be prescribed as a function of the current state. Thus, it has many similarities with the so-called *quasidifferential equations* or *funnel equations* introduced independently by Panasyuk and others (see e.g. [21, 22, 23]). For dispensing with any vector space structure, Aubin's starting point is to introduce "maps of elementary deformation" $\vartheta : [0,1] \times E \longrightarrow E$. Such a so–called *transition* specifies the point $\vartheta(t,x) \in E$ to which an initial point $x \in E$ has been moved after time $t \in [0,1]$. It can be interpreted as a generalized derivative of a curve $\xi : [0,T[\longrightarrow E$ at time $t \in [0,T[$ if it provides a first–order approximation in the sense of $\limsup_{h \downarrow 0} \frac{1}{h} \cdot d(\xi(t+h), \; \vartheta(h, \xi(t))) \; = \; 0.$
So correspondingly to ordinary differential equations in a vector space, such a transition is prescribed as a function of state and time, i.e. $f : (x,t) \longmapsto \vartheta$, and, we are then interested in a continuous curve $\xi(\cdot)$ satisfying the condition

$$\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot d(\xi(t+h), \;\; f(\xi(t),t)\,(h, \xi(t))) \; = \; 0 \qquad \text{at each time } t.$$

For succeeding in such a construction, each transition $\vartheta : [0,1] \times E \longrightarrow E$ is to satisfy some continuity conditions (in both arguments), of course. They will be specified in § 2. As the new point here in comparison with Aubin's original version, the continuity parameters of a transition need not be uniform in the whole set E, but they are the same for all initial elements in "closed balls" $B_r := \{x \in E \mid [\![x]\!] \le r\}$ (for each $r \ge 0$) where some given function $[\![\cdot]\!] : E \longrightarrow [0, \infty[$ is to play the role of a norm (but it does not have to satisfy any structural conditions like homogeneity or triangular inequality).

For applying this abstract tool to nonlinear transport equations, we need both a metric and an appropriate "absolute magnitude" $[\![\cdot]\!]$ for the positive Radon measures that we want to consider. The linear Wasserstein metric has proven to be a very powerful tool for probability measures μ, ν on $\mathbb{R}^N$ in connection with the optimal mass transportation problem and gradient flows (see e.g. [4]). Its dual representation for measures with bounded support, however,

$$d_1(\mu, \nu) \;=\; \sup \Big\{ \int_{\mathbb{R}^N} \psi \; d(\mu - \nu) \;\Big|\; \psi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\; \mathrm{Lip}\;\psi \le 1 \Big\}$$

reveals that this supremum might be ∞ whenever $\mu(\mathbb{R}^N) \ne \nu(\mathbb{R}^N)$ (independently from the additional assumption of compact support). As an extension of $d_1(\cdot, \cdot)$, we propose here

$$\begin{aligned}
\rho(\mu,\, \nu) := \; & \sup_{r>0} \Big\{ \tfrac{1}{e^r} \int_{\mathbb{R}^N} \psi \; d(\mu - \nu) \;\Big|\; \psi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\; \mathrm{Lip}\;\psi \le 1,\, \inf\, |\psi^{-1}(0)| \le r \Big\} \\
= \; & \sup_{r>0} \Big\{ \tfrac{1}{e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0)) \; d(\mu - \nu) \;\Big|\; \varphi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\; \mathrm{Lip}\;\varphi \le 1,\, |x_0| \le r \Big\}.
\end{aligned}$$

It is finite if μ, ν have finite first moments and, it coincides with $d_1(\mu, \nu)$ if $\mu(\mathbb{R}^N) = \nu(\mathbb{R}^N)$. Moreover, for all positive Radon measures with compact support in an arbitrarily fixed ball $\mathbb{B}_r(0) := \{z \in \mathbb{R}^N \,|\, |z| \le r\}$, $r \ge 0$, the convergence with respect to ρ proves to be equivalent to narrow convergence. This property – together with an easy compactness criterion – has been the key motivation for restricting our considerations to positive Radon measures *with compact support* and for defining $[\![\cdot]\!] : \mathcal{M}_c^+(\mathbb{R}^N) \longrightarrow [0, \infty[$ as

$$[\![\mu]\!] \;:=\; |\mu(\mathbb{R}^N)| \;+\; \inf\{r > 0 \mid \mathrm{supp}\;\mu \subset \mathbb{B}_r(0)\} \qquad \text{for } \mu \in \mathcal{M}_c^+(\mathbb{R}^N).$$

Indeed, each "ball" $\{\mu \in \mathcal{M}_c^+(\mathbb{R}^N) \mid [\![\mu]\!] \le R\}$ (with $R \ge 0$) is sequentially compact with respect to narrow convergence (due to Prokhorov criterion) and thus with respect to ρ.

After supplying $\mathcal{M}_c^+(\mathbb{R}^N)$ with the metric ρ and the "absolute magnitude" $[\![\cdot]\!]$, the Cauchy problem (2) of the autonomous linear transport equation lays the basis for transitions $\vartheta_{\mathbf{b},c}(\cdot,\cdot)$ on $\mathcal{M}_c^+(\mathbb{R}^N)$ depending on the coefficients $\mathbf{b}(\cdot), c(\cdot)$. For ensuring appropriate continuity properties, we assume $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and (slightly stronger than before) $c \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$. Prescribing these coefficients as a function of the current measure and time

$$F = (F_1, F_2) : \;\; \mathcal{M}_c^+(\mathbb{R}^N) \times [0, T] \;\longrightarrow\; W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R}),$$

leads to mutational equations whose solutions $\mu : [0, T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, $t \longmapsto \mu_t$ prove to be weak solutions of the nonlinear transport equation (1).

So in this setting, the existence of a weak solution mentioned in Proposition 1.1 results directly from the counterpart of Peano's theorem about mutational equations. Furthermore, Gronwall's inequality ensures that "local" Lipschitz continuity of the coefficient function F implies uniqueness of the (mutational) solution. Exploiting now the well-known result that the corresponding *nonautonomous linear* Cauchy problem has unique *weak* solutions [1, 4, 20], we can even draw conclusions about the uniqueness of *weak* measure-valued solutions:

Proposition 1.2 (Uniqueness)
Let $F : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N,\mathbb{R})$ *satisfy the following conditions:*

1.) $\sup\limits_{\mu_0,t} \; \big(\|F_1(\mu_0,t)\|_{W^{1,\infty}} + \|F_2(\mu_0,t)\|_{W^{1,\infty}}\big) \;<\; \infty$

2.) *for any* $R > 0$, *there is a constant* $L_R > 0$ *and a modulus* $\omega_R(\cdot)$ *of continuity with*

$$\|F_1(\mu,s) - F_1(\nu,t)\|_\infty + \|F_2(\mu,s) - F_2(\nu,t)\|_\infty \;\le\; L_R \cdot \rho(\mu,\nu) + \omega_R(|t-s|)$$

for all $\mu,\nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ *with* $\|\mu\|, \|\nu\| \le R$.

Then the narrowly continuous weak solution $\mu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),\; t \longmapsto \mu_t$ *of the Cauchy problem (1) with* $\sup_t \|\mu_t\| < \infty$ *is unique.*

Last, but not least, solutions of mutational equations are stable with respect to the coefficients. This rather simple consequence of Gronwall's inequality implies an interesting statement about the stability of weak solutions of the nonlinear transport equation:

Proposition 1.3 (Stability)
Assume for $F, G : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N,\mathbb{R})$:

1.) $M_F := \sup\limits_{\mu_0,t} \; \big(\|F_1(\mu_0,t)\|_{W^{1,\infty}} + \|F_2(\mu_0,t)\|_{W^{1,\infty}}\big) \;<\; \infty$

$M_G := \sup\limits_{\mu_0,t} \; \big(\|G_1(\mu_0,t)\|_{W^{1,\infty}} + \|G_2(\mu_0,t)\|_{W^{1,\infty}}\big) \;<\; \infty$

2.) *for any* $R > 0$, *there is a constant* $L_R > 0$ *and a modulus* $\omega_R(\cdot)$ *of continuity with*

$$\|F_1(\mu,s) - F_1(\nu,t)\|_\infty + \|F_2(\mu,s) - F_2(\nu,t)\|_\infty \;\le\; L_R \cdot \rho(\mu,\nu) + \omega_R(|t-s|)$$

for all $\mu,\nu \in \mathcal{M}_c^+(\mathbb{R}^N)$, $\|\mu\|, \|\nu\| \le R$.

3.) G_1, G_2 *are continuous with respect to narrow convergence in* $\mathcal{M}_c^+(\mathbb{R}^N)$ *and* L^∞ *norm of spatial fields.*

Let $\nu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),\; t \longmapsto \nu_t$ *be a narrowly continuous weak solution of*

$$\partial_t\, \nu_t \;+\; div_x\,(F_1(\nu_t,t)\;\nu_t) \;=\; F_2(\nu_t,t)\;\nu_t$$

with $\sup_t \|\nu_t\| < \infty$.

Then, for every initial measure $\mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, *there exists a narrowly continuous weak solution* $\mu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),\; t \longmapsto \mu_t$ *of*

$$\partial_t\, \mu_t \;+\; div_x\,(G_1(\mu_t,t)\;\mu_t) \;=\; G_2(\mu_t,t)\;\mu_t \quad in\ [0,T]$$

satisfying $\sup_t \|\mu_t\| < \infty$ *and for all* $t \in [0,T[$

$$\rho(\mu_t,\nu_t) \;\le\; \Big(\rho(\mu_0,\nu_0) + t\cdot const(M_F, M_G, \|\mu_0\|, \|\nu_0\|) \cdot (\|F_1 - G_1\|_\infty + \|F_2 - G_2\|_\infty)\Big)\; e^{\,const(F)\cdot t}.$$

If the function $G(\cdot,\cdot)$ satisfies even a Lipschitz condition corresponding to hypothesis (2.), then the weak solution μ is unique, of course, due to preceding Proposition 1.2.

This stability with respect to coefficients belongs to the main results of this paper and demonstrates an essential advantage of the presented approach. As mentioned before, it is based on a decomposition of the nonlinear problem into the autonomous linear problem (being easy to solve properly) and a form of "feedback".
The characteristics lay the foundations for solving the linear Cauchy problem and thus, former results about the Glimm scheme, wave-front tracking algorithm etc. are not used. We are free to consider arbitrary space dimension instead. Although it is not presented explicitly in this paper, all the following results can be extended to systems easily because the systems of mutational equations can be solved in exactly the same way.
Furthermore, the form of decomposition reflects the key difference from the so-called *Standard Riemann Semigroup* (SRS) that considers a hyperbolic $n \times n$ system of conservation laws $\partial_t u + \partial_x F(u) = 0$ in one space dimension. In a word, the values of SRS are usually Lebesgue-integrable functions $R \longrightarrow \mathbb{R}^N$ of (sufficiently) small total variation. The existence of such an appropriate semigroup (with additional continuity conditions) was first proved for 2×2 systems in [13] and then extended to general $n \times n$ systems in [12]. The Standard Riemann Semigroup takes the full nonlinearity into consideration immediately and thus, its stability with respect to coefficients is not easy at all to investigate (e.g. [8, 16]). In [10], Bressan suggests how to combine the Standard Riemann Semigroup with the quasidifferential equations of Panasyuk. His goal there is to draw conclusions about uniqueness – rather than proving existence of solutions (see also [15]). Due to SRS as starting point, however, Bressan needs a rather complicated metric on the domain of SRS for obtaining a "locally nonexpansive" semigroup (roughly speaking) and, he makes several suggestions seizing notions of Riemannian metrics on a manifold (see also [9, 11, 14], for example). In this paper, we prefer a rather simple metric on $\mathcal{M}_c^+(\mathbb{R}^N)$ (independent of the transport equation) and exploit then the regularity of the linear Cauchy problem as preparatory step for solving the full nonlinear problem. (To the best of the author's knowledge, mutational equations for solving PDEs were first proposed in [18, 19].)

Coming to the end of this introduction, we briefly sketch the next paragraphs: § 2 provides a short, but self-contained survey of mutational equations. In § 3, we introduce the metric ρ on $\mathcal{M}_c^+(\mathbb{R}^N)$ and investigate the relation between $(\mathcal{M}_c^+(\mathbb{R}^N), \rho)$ and the more popular topology of narrow convergence. Finally, in § 4, more detailed results about the autonomous linear Cauchy problem (2) lay the basis for solving the nonlinear Cauchy problem (1) by the means of § 2.

Notation $C_c^0(\mathbb{R}^N)$ denotes the space of continuous functions $\mathbb{R}^N \longrightarrow \mathbb{R}$ with compact support and $C_0(\mathbb{R}^N)$ its closure with respect to the sup norm, respectively.
Furthermore, $\mathcal{M}(\mathbb{R}^N)$ consists of all finite real–valued Radon measures on $\mathbb{R}^N$. As a consequence of Riesz theorem, it is the dual space of $C_0(\mathbb{R}^N)$ (see e.g.[3], Remark 1.57). $\mathcal{M}^+(\mathbb{R}^N)$ denotes the set of all positive finite Radon measures on $\mathbb{R}^N$: $\mathcal{M}^+(\mathbb{R}^N) := \{\mu \in \mathcal{M}(\mathbb{R}^N) \mid \mu(\cdot) \geq 0\}$.
Finally, $\mathcal{M}_c^+(\mathbb{R}^N)$ consists of all positive Radon measures on $\mathbb{R}^N$ with compact support and $\mathcal{M}_1^+(\mathbb{R}^N) := \Big\{\mu \in \mathcal{M}^+(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} (1+|x|)\; d\mu(x) < \infty\Big\}$.

2 Mutational equations on a metric space

Assumptions of § 2 E is a nonempty set and $d : E \times E \longrightarrow [0,\infty[$ a metric on E. Furthermore let $\lfloor\!\lfloor \cdot \rfloor\!\rfloor : E \longrightarrow [0,\infty[$ be an arbitrary function (that is to play the role of a norm on E, but need not satisfy structural conditions like homogeneity or triangular inequality).

Now we specify the primary tools for describing deformations in the tuple $(E, d, \lfloor\!\lfloor \cdot \rfloor\!\rfloor)$. A map $\vartheta : [0,1] \times E \longrightarrow E$ is to define which point $\vartheta(t,x) \in E$ is reached from the initial point $x \in E$ after time t. Of course, ϑ has to fulfill some regularity conditions so that it may form the basis for a calculus of differentiation.

Definition 2.1 *A function* $\vartheta : [0,1] \times E \longrightarrow E$ *is called* transition *on* $(E, d, \lfloor\!\lfloor \cdot \rfloor\!\rfloor)$ *if it satisfies the following conditions:*

1.) $\vartheta(0,\cdot) \;=\; \mathrm{Id}_E,$

2.) $\lim_{h \downarrow 0} \frac{1}{h} \cdot d\left(\vartheta(t+h,x),\ \vartheta(h, \vartheta(t,x))\right) \;=\; 0$ *for all* $x \in E,\ t \in [0,1[,$

3.) *there exists a parameter function* $\alpha(\vartheta;\,\cdot\,) : [0,\infty[\longrightarrow [0,\infty[$ *such that*
$\limsup_{h \downarrow 0} \frac{d(\vartheta(h,x),\,\vartheta(h,y)) \,-\, d(x,y)}{h} \;\leq\; \alpha(\vartheta; r) \cdot d(x,y)$ *for all* $x, y \in E,\ r$ *with* $\lfloor\!\lfloor x \rfloor\!\rfloor, \lfloor\!\lfloor y \rfloor\!\rfloor \leq r,$

4.) *there exists a parameter function* $\beta(\vartheta;\,\cdot\,) : [0,\infty[\longrightarrow [0,\infty[$ *such that*
$d(\vartheta(s,x),\ \vartheta(t,x)) \;\leq\; \beta(\vartheta;\, r) \,\cdot\, |t-s|$ *for all* $x \in E,\ r, s, t$ *with* $\lfloor\!\lfloor x \rfloor\!\rfloor \leq r,$

5.) *there exists a constant* $\zeta(\vartheta) \in [0,\infty[$ *such that*
$\lfloor\!\lfloor \vartheta(h,x) \rfloor\!\rfloor \;\leq\; \lfloor\!\lfloor x \rfloor\!\rfloor \cdot e^{\zeta(\vartheta)\,h} \;+\; \zeta(\vartheta)\,h$ *for all* $x \in E,\ h \in [0,1].$

Remark. The first two conditions are very similar to the definition of a semigroup. The only difference here is that a condition on the "first–order change" (with respect to time) suffices completely.
Property (3.) is to ensure an appropriate form of continuity with respect to the initial element. It implies that the initial distance of two points $x,\ y \in E$ may grow at most exponentially in time while evolving along the same transition ϑ and, the corresponding exponent can be chosen uniformly on each "ball" $\{x \in E \mid \lfloor\!\lfloor x \rfloor\!\rfloor \leq r\},\ r \geq 0.$
Property (4.) ensures Lipschitz continuity of $\vartheta(\cdot, x)$ for each initial point $x \in E$. Similarly to property (3.), the Lipschitz constant may depend on $\lfloor\!\lfloor x \rfloor\!\rfloor$ and, these dependencies are the new aspects of this paper in comparison with Aubin's original definition of transitions [5, 6].
Last, but not least, we need a bound of the "absolute magnitude" $\lfloor\!\lfloor \vartheta(h,x) \rfloor\!\rfloor$ depending on both arguments. The combination of exponential and linear growth mentioned here has the key advantage that for any continuous curve $x : [0,T[\longrightarrow E$ defined piecewise by finitely many transitions $\vartheta_1 \ldots \vartheta_n$ with $\widehat{\zeta} := \sup_j \zeta(\vartheta_j) < \infty$ (as in the proof of Theorem 2.7 later), we conclude from Gronwall's lemma: $\lfloor\!\lfloor x(t) \rfloor\!\rfloor \;\leq\; \lfloor\!\lfloor x(0) \rfloor\!\rfloor\, e^{\widehat{\zeta}\,t} \;+\; \widehat{\zeta}\,t$ for all $t \in [0,T[$.

Definition 2.2 *$\Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor) \neq \emptyset$ denotes a set of transitions on $(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor)$ assuming*

$$D(\vartheta, \tau;\ r) \ := \ \sup \ \left\{ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d(\vartheta(h, x),\ \tau(h, x)) \ \middle| \ x \in E,\ \ \lfloor\!\lfloor x \rfloor\!\rfloor \le r \right\} \ < \ \infty$$

for any $\vartheta, \tau \in \Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor)$ and $r \ge 0$. (If $\{x \in E \mid \lfloor\!\lfloor x \rfloor\!\rfloor \le r\} = \emptyset$, set $D(\,\cdot\,, \cdot\,;\ r) := 0$.)

Obviously, $D(\cdot, \cdot; r) : \Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor) \times \Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor) \longrightarrow [0, \infty[$ is symmetric and satisfies the triangular inequality for each $r \ge 0$. Moreover, it lays the basis for estimating the distance between two initial points $x,\ y \in E$ after evolving along two transitions $\vartheta,\ \tau \in \Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor)$, respectively, for some time $h \in [0, 1]$. The way how to derive this estimate from (a not very popular form of) Gronwall's Lemma is very characteristic for mutational equations and will be reused for similar inequalities later (see Lemma 2.8).

Lemma 2.3 *Let $\vartheta, \tau \in \Theta(E, d, \lfloor\!\lfloor\cdot\rfloor\!\rfloor)$ be any transitions. Then for every time $h \in [0, 1]$ and initial points $x, y \in E$ with $\lfloor\!\lfloor x \rfloor\!\rfloor, \lfloor\!\lfloor y \rfloor\!\rfloor \le r$, the distance between $\vartheta(h, x)$ and $\tau(h, y)$ satisfies (with the abbreviation $R := r\ e^{\max\{\zeta(\vartheta),\, \zeta(\tau)\}} + \max\{\zeta(\vartheta),\ \zeta(\tau)\}$)*

$$d\,(\vartheta(h, x),\ \tau(h, y)) \ \le \ \Big(d(x, y) \ + \ h \cdot D(\vartheta, \tau;\ R) \Big)\ e^{\alpha(\tau; R)\, h}\,.$$

Proof. According to property (5.) of transitions, $\lfloor\!\lfloor x \rfloor\!\rfloor,\ \lfloor\!\lfloor y \rfloor\!\rfloor \le r$ implies $\lfloor\!\lfloor \vartheta(h, x) \rfloor\!\rfloor \le R$ and $\lfloor\!\lfloor \tau(h, y) \rfloor\!\rfloor \le R$ for all $h \in [0, 1]$.
The auxiliary function $\varphi : [0, 1] \longrightarrow [0, \infty[,\ \ h \longmapsto d(\vartheta(h, x),\ \tau(h, y))$ is continuous due to property (4.) and the triangular inequality of d. Furthermore, we obtain for every $h \in [0, 1[$ and $k \in [0, 1 - t[$

$$\begin{aligned}
\varphi(h + k) \ \le & \quad d(\vartheta(h + k, x),\ \ \vartheta(k,\ \vartheta(h, x))) \ && + \ d(\vartheta(k,\ \vartheta(h, x)),\ \ \tau(k,\ \vartheta(h, x))) \\
& + \ d(\tau(k,\ \vartheta(h, x)),\ \ \tau(k,\ \tau(h, y))) \ && + \ d(\tau(k,\ \tau(h, y)),\ \ \tau(h + k, y)) \\
\le & \quad o(k) && + \ D(\vartheta, \tau;\ R) \ \cdot \ k \ + \ o(k) \\
& + \ \varphi(h) + k \cdot \alpha(\tau; R)\ \varphi(h) + o(k) && + \ o(k)
\end{aligned}$$

and thus, $\displaystyle \limsup_{k \downarrow 0} \ \frac{\varphi(h+k) - \varphi(h)}{k} \ \le \ \alpha(\tau; R) \cdot \varphi(h) \ + \ D(\vartheta, \tau;\ R).$

The claimed estimate results now directly from subsequent version of Gronwall's Lemma. □

Lemma 2.4 (Lemma of Gronwall for upper Dini derivatives)
Let $\psi, f, g \in C^0([a, b[, \mathbb{R})$ satisfy $f(\cdot) \ge 0$ and

$$\limsup_{h \downarrow 0} \ \frac{\psi(t+h) - \psi(t)}{h} \ \le \ f(t) \ \cdot \ \psi(t) \ + \ g(t) \qquad \textit{for all } t \in\,]a,\ b[.$$

Then, for every $t \in [a, b[$, the function $\psi(\cdot)$ fulfills the upper estimate

$$\psi(t) \ \le \ \psi(a) \ \cdot \ e^{\mu(t)} \ + \ \int_a^t e^{\mu(t) - \mu(s)}\ g(s)\ ds$$

with $\displaystyle \mu(t) := \int_a^t f(s)\ ds.$

Proof. Let $\delta > 0$ be arbitrarily small. The proof is based on comparing ψ with the auxiliary function $\varphi_\delta : [a,b] \longrightarrow \mathbb{R}$ that uses $\psi(a) + \delta,\ g(\cdot) + \delta$ instead of $\psi(a),\ g(\cdot)$:

$$\varphi_\delta(t) \ := \ (\psi(a) + \delta)\, e^{\mu(t)} \ + \ \int_a^t e^{\mu(t)-\mu(s)}\, (g(s) + \delta)\, ds.$$

Then, $\varphi'_\delta(t) \ = \ f(t)\, \varphi_\delta(t) + g(t) + \delta$ in $[a,b[$ and, $\varphi_\delta(t) > \psi(t)$ for all $t \in [a,b[$ close to a. Assume now that there is some $t_0 \in\,]a,b]$ with $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1 \ := \ \inf\ \{t \in [a, t_0] \mid \varphi_\delta(t) < \psi(t)\ \},$$

we obtain $\varphi_\delta(t_1) \ = \ \psi(t_1)$ and $a < t_1 < t_0$ because

$$\begin{aligned} \varphi_\delta(t_1) \ &= \ \lim_{h\downarrow 0} \ \varphi_\delta(t_1 - h) \ \geq \ \limsup_{h\downarrow 0} \ \psi(t_1 - h) \ \geq \ \psi(t_1), \\ \varphi_\delta(t_1) \ &= \ \lim_{\substack{h\to 0 \\ h\geq 0}} \ \varphi_\delta(t_1 + h) \ \leq \ \limsup_{\substack{h\to 0 \\ h\geq 0}} \ \psi(t_1 + h) \ \leq \ \psi(t_1). \end{aligned}$$

Thus, we conclude from the definition of t_1

$$\begin{aligned} \liminf_{h\downarrow 0} \ \tfrac{\varphi_\delta(t_1+h)-\varphi_\delta(t_1)}{h} \ &\leq \ \limsup_{h\downarrow 0} \ \tfrac{\psi(t_1+h)-\psi(t_1)}{h} \\ \varphi'_\delta(t_1) \ &\leq \ f(t_1)\ \cdot\ \psi(t_1) \ + \ g(t_1) \\ f(t_1)\, \varphi_\delta(t_1) + g(t_1) + \delta \ &\leq \ f(t_1)\ \cdot\ \varphi_\delta(t_1) \ + \ g(t_1) \end{aligned}$$

— a contradiction. So $\varphi_\delta(\cdot) \geq \psi(\cdot)$ for any $\delta > 0$. □

A transition $\vartheta \in \Theta(E, d, \llbracket\cdot\rrbracket)$ can be interpreted as "time derivative" of curve $x(\cdot) : [0,T[\longrightarrow E$ at time $t \in [0,T[$ if it induces a first-order approximation, i.e. the evolution of $x(t)$ along ϑ differs from the curve $x(t + \cdot)$ "up to order" $\frac{1}{h}$: $\lim_{h\downarrow 0} \frac{1}{h} \cdot d\,(\vartheta(h,\ x(t)),\ x(t{+}h)) \ = \ 0.$ This condition may be fulfilled by more than one transition, of course. So we collect all these transitions in the so-called mutation of $x(\cdot)$ at time t. The main notion of a mutational equation is then to prescribe a transition in the mutation (of the wanted continuous curve) as function of the current state and time.

Definition 2.5 *Let $x(\cdot) : [0,T[\longrightarrow E$ be a curve in E. The so-called* mutation $\overset{\circ}{x}(t)$ *of $x(\cdot)$ at time $t \in [0,T[$ consists of all transitions $\vartheta \in \Theta(E, d, \llbracket\cdot\rrbracket)$ satisfying*

$$\limsup_{h\downarrow 0} \ \tfrac{1}{h} \cdot d\,(\vartheta(h,\ x(t)),\ \ x(t + h)) \ = \ 0.$$

Definition 2.6 *For a given function $f : E \times [0,T[\longrightarrow \Theta(E, d, \llbracket\cdot\rrbracket)$, a curve $x(\cdot) : [0,T[\longrightarrow E$ is called* solution *of the* mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ *in $[0,T[$ if it fulfills the following conditions:*

1.) for every $t \in [0,T[$, $f(x(t),t) \in \overset{\circ}{x}(t)$, i.e. $\lim_{h\downarrow 0} \frac{1}{h} \cdot d\,(f(x(t),t)\,(h,\ x(t)),\ \ x(t+h)) = 0,$

2.) $x(\cdot)$ is continuous with respect to d,

3.) $\sup_{0\leq t<T} \ \llbracket x(t) \rrbracket \ < \ \infty.$

Remark. For any transition $\vartheta \in \Theta(E, d, \llbracket\cdot\rrbracket)$ and initial element $x_0 \in E$, the curve $[0,1[\longrightarrow E,\ h \longmapsto \vartheta(h,x)$ is a solution of the mutational equation $\overset{\circ}{x}(\cdot) \ni \vartheta$ in $[0,1[$ (with constant right-hand side). This results directly from property (2.) of transitions in Def. 2.1.

Proposition 2.7 (Existence) *Let (E,d) be a metric space and $\llbracket\cdot\rrbracket : E \longrightarrow [0,\infty[$ such that each "ball" $\{x \in E \mid \llbracket x\rrbracket \le r\}$, $r \ge 0$, is compact in (E,d).*
Furthermore suppose $f : (E,d) \times [0,T[\longrightarrow (\Theta(E,d),\ D(\cdot,\cdot\,;r))$ to be continuous with

$$\begin{aligned} \widehat{\alpha}(r) &:= \sup_{x,t} \alpha(f(x,t);\, r) < \infty, \\ \widehat{\beta}(r) &:= \sup_{x,t} \beta(f(x,t);\, r) < \infty, \\ \widehat{\zeta} &:= \sup_{x,t} \zeta(f(x,t)) < \infty \end{aligned}$$

for each $r \ge 0$.

Then for every initial element $x_0 \in E$, there exists a solution $x(\cdot) : [0,T[\longrightarrow E$ of the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$ with $x(0) = x_0$.

Proof is based on Euler approximations $x_n(\cdot) : [0,T] \longrightarrow E$ $(n \in \mathbb{N})$ in combination with the Arzela–Ascoli theorem (see [17], for example). Indeed, for each $n \in \mathbb{N}$, set

$$\begin{aligned} h_n &:= \tfrac{T}{2^n}, & t_n^j &:= j\, h_n & &\text{for } j = 0 \ldots 2^n, \\ x_n(0) &:= x_0, & x_0(\cdot) &:= x_0, \\ x_n(t) &:= f(x_n(t_n^j),\, t_n^j)\Big(t - t_n^j,\ x_n(t_n^j)\Big) & & & &\text{for } t \in\,]t_n^j,\, t_n^{j+1}],\ j < 2^n. \end{aligned}$$

This piecewise construction of each $x_n(\cdot)$ implies firstly $\llbracket x_n(t)\rrbracket \le \llbracket x_0\rrbracket \cdot e^{\widehat{\zeta}\, T} + \widehat{\zeta}\, T =: R$ for all $t \in [0,T]$, $n \in \mathbb{N}$. So all values of the Euler approximations $x_n(\cdot), n \in \mathbb{N}$, are contained in the compact set $K_R := \{y \in E \mid \llbracket y\rrbracket \le R\}$. Secondly, the triangle inequality ensures

$$d\,(x_n(s),\ x_n(t)) \le \widehat{\beta}(R)\ |t-s| \qquad \text{for all } s,t \in [0,T],\ n \in \mathbb{N}$$

and thus, the sequence $(x_n(\cdot))_{n\in\mathbb{N}}$ is equicontinuous.
The classical Theorem of Arzela–Ascoli states that the set $\{x_n(\cdot) \mid n \in \mathbb{N}\} \subset C^0([0,T], K_R)$ is precompact (with respect to uniform convergence) and so, there is a subsequence $\big(x_{n_j}(\cdot)\big)_{j\in\mathbb{N}}$ converging uniformly to a function $x(\cdot) \in C^0([0,T], K_R)$.

Finally, we verify that $x(\cdot)$ solves the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$.
Indeed, $x(\cdot)$ is continuous with respect to d and, it satisfies $\sup_t \llbracket x(t)\rrbracket \le R$ due to its construction. Furthermore, using the notation $\delta_n := \sup_{[0,T]} d(x_n(\cdot), x(\cdot))$, we conclude from subsequent Lemma 2.8 for any times $t \in [0,T[$, $h \in [0, T-t[$ and arbitrary $n \in \mathbb{N}$

$$\begin{aligned} & d\,(f(x(t),t)\,(h,\ x(t)),\ \ x(t+h)) \\ \le\ & d\Big(f(x(t),t)\,(h,\ x(t)),\ \ x_n(t+h)\Big) + d\,(x_n(t+h),\ x(t+h)) \\ \le\ & \Big(\delta_n + h \cdot \sup_{\substack{-h_n \le s \le h\ (t+s\ge 0) \\ y:\ d(y,x(t+s)) \le \delta_n}} D\,(f(x(t),t),\ f(y,t+s),\ R)\Big)\, e^{\widehat{\alpha}(R)\, h} + \delta_n\,. \end{aligned}$$

Due to the continuity of f with respect to $D(\cdot,\cdot;R)$, the limit for $n \longrightarrow \infty$ implies

$$d\Big(f(x(t),t)\,(h,\ x(t)),\ \ x(t+h)\Big) \ \le\ h \cdot \sup_{0\le s\le h} D\,(f(x(t),t),\ \ f(x(t+s),t+s),\ R)\ \ e^{\widehat{\alpha}(R)\,h}$$

and thus, $$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot d\Big(f(x(t),t)\,(h,\ x(t)),\ \ x(t+h)\Big) \ \le\ 0.$$ □

Lemma 2.8 *Assume for* $f,g : E\times[0,T[\longrightarrow \Theta(E,d,\lfloor\!\lfloor\cdot\rfloor\!\rfloor)$ *and the curves* $x,y : [0,T[\longrightarrow E$ *that* $x(\cdot)$ *is a solution of the mutational equation* $\overset{\circ}{x}(\cdot)\ni f(x(\cdot),\cdot)$ *in* $[0,T[$ *and* $y(\cdot)$ $\overset{\circ}{y}(\cdot)\ni g(y(\cdot),\cdot)$ *in* $[0,T[$.
Furthermore, let $R>0,\ M>0$ *and* $\psi\in C^0([0,T[)$ *satisfy*

$$\begin{cases} \lfloor\!\lfloor x(t)\rfloor\!\rfloor,\ \lfloor\!\lfloor y(t)\rfloor\!\rfloor & \le\ R \\ \alpha(g(y(t),t);\ R) & \le\ M \\ D\,(f(x(t),t),\ g(y(t),t);\ R) & \le\ \psi(t) \end{cases} \qquad \textit{for all } t\in[0,T[.$$

Then, $$d(x(t),\ y(t)) \ \le\ \Big(d(x(0),\ y(0))\ +\ \int_0^t \psi(s)\,e^{-Ms}\,ds\Big)\ e^{Mt} \qquad \textit{for any } t\in[0,T[.$$

Proof follows exactly the same track as for Lemma 2.3 (comparing the evolutions along fixed transitions): The auxiliary function $\varphi : [0,1]\longrightarrow[0,\infty[,\ t\longmapsto d(x(t),\ y(t))$ is continuous due to the triangular inequality of d. Furthermore, we obtain for every $t\in[0,T[$ and $h\in[0,T-t[$

$$\begin{aligned}
&\varphi(t+h)\\
\le\ & d(x(t+h),\ \ f(x(t),t)\,(h,\ x(t))) \quad + d(f(x(t),t)\,(h,\ x(t)),\ \ g(y(t),t)(h,\ x(t)))\\
&+ d(g(y(t),t)(h,\ x(t)),\ \ g(y(t),t)(h,\ y(t))) \ + d(g(y(t),t)(h,\ y(t)),\ \ y(t+h))\\
\le\ & o(h) \quad + D(f(x(t),t),\ g(y(t),t);\ R)\cdot h + o(h)\\
&+ \varphi(t)+h\cdot M\,\varphi(t)+o(h) \quad + o(h)
\end{aligned}$$

and thus, $\limsup_{h\downarrow 0}\ \frac{\varphi(t+h)-\varphi(t)}{h}\ \le\ M\cdot\varphi(h)+\psi(t).$ So the claim results from Gronwall's Lemma 2.4. □

Proposition 2.9 (Uniqueness) *Suppose* $f : (E,d)\times[0,T[\longrightarrow(\Theta(E,d),\ D(\cdot,\cdot;r))$ *to be* λ_r*–Lipschitz continuous in the first argument with* $\widehat{\alpha}(r) := \sup_{x,t}\alpha(f(x,t);\ r) < \infty$ *for any* $r\ge 0$.

Then for every initial element $x_0\in E$, *the solution* $x(\cdot) : [0,T[\longrightarrow E$ *of the mutational equation* $\overset{\circ}{x}(\cdot)\ni f(x(\cdot),\cdot)$ *in* $[0,T[$ *with* $x(0)=x_0$ *is unique and, it depends on* x_0 *in a locally Lipschitz way.*

Proof is based on the estimate in Lemma 2.8. Let $x(\cdot),y(\cdot) : [0,T[\longrightarrow E$ be two solutions of the same mutational equation $\overset{\circ}{x}(\cdot)\ni f(x(\cdot),\cdot)$ in $[0,T[$ (but not necessarily with the same initial element). Then, $R := \sup_t\{\lfloor\!\lfloor x(t)\rfloor\!\rfloor,\ \lfloor\!\lfloor y(t)\rfloor\!\rfloor\} < \infty$ (due to the definition of solutions) and, $t\longmapsto d(x(t),y(t))$ is continuous. As a consequence of

$$D(f(x(t),t),\ \ f(y(t),t))\ \le\ \lambda_R\cdot d(x(t),\ y(t)),$$

Lemma 2.8 implies for any $t \in [0,T[$

$$d(x(t),\ y(t)) \ \le\ d(x(0),\ y(0)) \cdot e^{\widehat{\alpha}(R)\cdot t} \ +\ \int_0^t \lambda_R \cdot d(x(s),\ y(s))\ e^{\widehat{\alpha}(R)\cdot(t-s)}\ ds$$

and, Gronwall's Lemma (in its well-known integral form) guarantees

$$d(x(t),\ y(t)) \ \le\ d(x(0),\ y(0)) \ \cdot\ e^{(\widehat{\alpha}(R)+\lambda_R)\cdot t} \qquad \text{for all } \ t \in [0,T[. \qquad \square$$

Proposition 2.10 (Stability) *Suppose* $f : (E,d) \times [0,T[\longrightarrow (\Theta(E,d),\ D(\cdot,\cdot;r))$ *to be* λ_r*–Lipschitz continuous in the first argument with* $\widehat{\alpha}(r) := \sup_{x,t} \alpha(f(x,t);\ r) < \infty$ *for any* $r \ge 0$. *Assume for* $g : E \times [0,T[\longrightarrow \Theta(E,d,\llbracket\cdot\rrbracket)$ *that* $\sup_{z,s} D(f(z,s),\ g(z,s);\ r) < \infty$ *for each* $r \ge 0$.

(*a*) *Let* $y(\cdot) : [0,T[\longrightarrow E$ *be a solution of the mutational equation* $\overset{\circ}{y}(\cdot) \ni g(y(\cdot),\cdot)$ *in* $[0,T[$. *Then, every solution* $x(\cdot) : [0,T[\longrightarrow E$ *of the mutational equation* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ *in* $[0,T[$ *satisfies the following estimate for all* $t \in [0,T[$

$$d(x(t),\ y(t)) \ \le\ \Big(d(x(0),\ y(0)) \ +\ t \cdot \sup_{\substack{z:\ \llbracket z\rrbracket \le R\\ 0 \le s < T}} D(f(z,s),\ g(z,s);\ R)\Big) \ \cdot\ e^{(\widehat{\alpha}(R)+\lambda_R)\cdot t}$$

with the abbreviation $R \ :=\ \sup_t \ \{\llbracket x(t)\rrbracket, \llbracket y(t)\rrbracket\} < \infty$.

(*b*) *Let* $x(\cdot) : [0,T[\longrightarrow E$ *be a solution of the mutational equation* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ *in* $[0,T[$. *Then, every solution* $y(\cdot) : [0,T[\longrightarrow E$ *of the mutational equation* $\overset{\circ}{y}(\cdot) \ni g(y(\cdot),\cdot)$ *in* $[0,T[$ *fulfills for all* $t \in [0,T[$

$$d(x(t),\ y(t)) \ \le\ \Big(d(x(0),\ y(0)) \ +\ t \cdot \sup_{\substack{z:\ \llbracket z\rrbracket \le R\\ 0 \le s < T}} D(f(z,s),\ g(z,s);\ R)\Big) \ \cdot\ e^{(\widehat{\alpha}(R)+\lambda_R)\cdot t}$$

using again the abbreviation $R \ :=\ \sup_t \ \{\llbracket x(t)\rrbracket, \llbracket y(t)\rrbracket\} < \infty$.

Proof results from Lemma 2.8 similarly to the preceding Proposition 2.9 because $D(\cdot,\cdot;R)$ satisfies the triangular inequality (as an obvious consequence of its Definition 2.2) and thus,

$$\begin{aligned} D\Big(f(x(t),t),\ g(y(t),t);\ R\Big) \ &\le\ D\Big(f(x(t),t),\ f(y(t),t);\ R\Big) \ +\ D\Big(f(y(t),t),\ g(y(t),t);\ R\Big) \\ &\le\ \lambda_R \ \cdot\ d(x(t),\ y(t)) \ +\ \sup_{\substack{z:\ \llbracket z\rrbracket \le R\\ 0 \le s < T}} D\Big(f(z,s),\ g(z,s);\ R\Big) \end{aligned}$$

$\square$

3 A metric for positive Radon measures on $\mathbb{R}^N$ with compact support

Motivated by the dual representation of the linear Wasserstein metric on probability measures with compact support, we introduce

Definition 3.1 *The distance function* $\rho : \mathcal{M}(\mathbb{R}^N) \times \mathcal{M}(\mathbb{R}^N) \longrightarrow [0, \infty]$ *is defined by*

$$\begin{aligned} \rho(\mu,\,\nu) \;&:=\; \sup_{r>0} \;\Big\{ \tfrac{1}{e^r} \int_{\mathbb{R}^N} (\psi - \psi(x_0))\; d(\mu-\nu) \;\Big|\; \psi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\ \mathrm{Lip}\;\psi \le 1,\, |x_0| \le r \Big\} \\ &=\; \sup_{\lambda, r>0} \;\Big\{ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\; d(\mu-\nu) \;\Big|\; \varphi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\ \mathrm{Lip}\;\varphi \le \lambda,\, |x_0| \le r \Big\}. \end{aligned}$$

Furthermore, set

$$\begin{aligned} \lfloor \mu \rfloor \;&:=\; \int_{\mathbb{R}^N} (1+|x|)\;\; d\mu(x) && \text{for } \mu \in \mathcal{M}^+(\mathbb{R}^N), \\ \lfloor\!\lfloor \mu \rfloor\!\rfloor \;&:=\; |\mu(\mathbb{R}^N)| \;+\; \inf\{r>0 \mid \mathrm{supp}\,\mu \subset \mathbb{B}_r(0)\} && \text{for } \mu \in \mathcal{M}_c^+(\mathbb{R}^N). \end{aligned}$$

Lemma 3.2

1.) $\rho(\mu,\nu) < \infty$ *for any* $\mu, \nu \in \mathcal{M}_1^+(\mathbb{R}^N)$, *i.e. Radon measures* $\mu, \nu \ge 0$ *with* $\lfloor\mu\rfloor + \lfloor\nu\rfloor < \infty$.

2.) For probability measures μ, ν *on* $\mathbb{R}^N$ *with compact support, the distance* $\rho(\mu,\nu)$ *coincides with the linear Wasserstein distance (in its dual representation)*

$$\rho(\mu,\nu) \;=\; \sup \Big\{ \int_{\mathbb{R}^N} \psi \;\, d(\mu-\nu) \;\;\Big|\;\; \psi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\;\; \mathrm{Lip}\;\;\psi \le 1 \Big\}.$$

3.) For any sequence $(\mu_n)_{n\in\mathbb{N}}$ *in* $\mathcal{M}_1^+(\mathbb{R}^N)$ *and* $\mu \in \mathcal{M}_1^+(\mathbb{R}^N)$ *with* $\mu_n(\mathbb{R}^N) \longrightarrow \mu(\mathbb{R}^N) > 0$, *the convergence* $\rho(\mu_n, \mu) \longrightarrow 0$ *is equivalent to* $\rho\Big(\frac{1}{\mu_n(\mathbb{R}^N)}\,\mu_n,\; \frac{1}{\mu(\mathbb{R}^N)}\,\mu\Big) \longrightarrow 0$

4.) For any measures $\mu, \nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ *and radius* $R > 0$ *with* $\mathrm{supp}\,\mu \,\cup\, \mathrm{supp}\,\nu \;\subset\; \mathbb{B}_R(0)$,

$$\begin{aligned} \rho(\mu,\nu) \;&=\; \sup_{\lambda>0} \Big\{ \tfrac{1}{\lambda\, e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\;\, d(\mu-\nu) \;\Big|\; \varphi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R}),\ \mathrm{Lip}\;\;\varphi \le \lambda,\; |x_0| \le 3R{+}2 \Big\} \\ &\ge\; \tfrac{1}{e^{R+1}}\;\; \big|\mu(\mathbb{R}^N) - \nu(\mathbb{R}^N)\big|\,. \end{aligned}$$

5.) Let $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}_c^+(\mathbb{R}^N)$ *with* $\sup_n \lfloor\!\lfloor \mu_n \rfloor\!\rfloor < \infty$. *Then* $(\mu_n)_{n\in\mathbb{N}}$ *converges to* $\mu \in \mathcal{M}_c^+(\mathbb{R}^N)$ *with respect to* ρ *if and only if* $\mu_n \longrightarrow \mu$ *narrowly, i.e.*

$$\int_{\mathbb{R}^N} \varphi \; d\mu_n \;\;\longrightarrow\;\; \int_{\mathbb{R}^N} \varphi \; d\mu \qquad\quad \text{for all bounded } \varphi \in C^0(\mathbb{R}^N).$$

6.) For each $\delta > 0$, *the set* $\{\mu \in \mathcal{M}_c^+(\mathbb{R}^N) \mid \lfloor\!\lfloor\mu\rfloor\!\rfloor \le \delta\}$ *is sequentially compact w.r.t.* ρ.

Proof. 1.) Let μ be any positive Radon measure with $\lfloor\mu\rfloor < \infty$. Then we obtain for any $\varphi \in \mathrm{Lip}\,(\mathbb{R}^N, \mathbb{R})$ and $x \in \mathbb{R}^N$ with $\mathrm{Lip}\,\varphi \le \lambda$, $|x| \le r$

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\;\, d\mu \;\;&\le\;\; \int_{\mathbb{R}^N} \mathrm{Lip}\,\varphi \cdot \;\; |x - x_0| \;\;\, d\mu(x) \\ &\le\;\; \lambda \;\cdot\; \int_{\mathbb{R}^N} (|x| + r) \;\;\, d\mu(x) \end{aligned}$$

and thus,

$$\rho(\mu, 0) \;\;\le\;\; \tfrac{1}{e^r} \cdot \int_{\mathbb{R}^N} (|x| + r) \;\;\, d\mu(x) \;\;\le\;\; \lfloor\mu\rfloor \;\;<\;\infty.$$

Finally, statement (1.) results from the obvious triangle inequality of $\rho(\cdot,\cdot)$.

2.) For any $\mu,\ \nu \in \mathcal{M}^+(\mathbb{R}^N)$ with $\mu(\mathbb{R}^N) = \nu(\mathbb{R}^N) < \infty$, we obtain

$$\begin{aligned}
\rho(\mu,\nu) &= \sup_{\lambda, r>0} \Big\{ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu-\nu) \ \Big|\ \varphi \in \mathrm{Lip}\,(\mathbb{R}^N,\mathbb{R}),\ \mathrm{Lip}\ \varphi \le \lambda,\ |x_0| \le r \Big\} \\
&= \sup_{\lambda>0} \Big\{ \tfrac{1}{\lambda} \int_{\mathbb{R}^N} \varphi \ d(\mu-\nu) \ \Big|\ \varphi \in \mathrm{Lip}\,(\mathbb{R}^N,\mathbb{R}),\ \mathrm{Lip}\ \varphi \le \lambda \Big\} \\
&= \sup \Big\{ \int_{\mathbb{R}^N} \psi \ d(\mu-\nu) \ \Big|\ \psi \in \mathrm{Lip}\,(\mathbb{R}^N,\mathbb{R}),\ \mathrm{Lip}\ \psi \le 1 \Big\}.
\end{aligned}$$

3.) results easily from Definition 3.1 of $\rho(\mu_n,\mu)$ and the following relation for any measures $\mu_n, \mu \in \mathcal{M}_1^+(\mathbb{R}^N)$ with $\mu_n(\mathbb{R}^N), \mu(\mathbb{R}^N) > 0$

$$\begin{aligned}
& \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d\Big(\tfrac{\mu_n}{\mu_n(\mathbb{R}^N)} - \tfrac{\mu}{\mu(\mathbb{R}^N)}\Big) \\
=\ & \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d\Big(\tfrac{\mu_n}{\mu_n(\mathbb{R}^N)} - \tfrac{\mu}{\mu_n(\mathbb{R}^N)} + \tfrac{\mu}{\mu_n(\mathbb{R}^N)} - \tfrac{\mu}{\mu(\mathbb{R}^N)}\Big) \\
=\ & \tfrac{1}{\mu_n(\mathbb{R}^N)}\ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu_n - \mu) \\
& + \Big(\tfrac{1}{\mu_n(\mathbb{R}^N)} - \tfrac{1}{\mu(\mathbb{R}^N)}\Big)\ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d\mu \\
\in\ & \tfrac{1}{\mu_n(\mathbb{R}^N)}\ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu_n - \mu) \\
& + \Big(\tfrac{1}{\mu_n(\mathbb{R}^N)} - \tfrac{1}{\mu(\mathbb{R}^N)}\Big)\ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} \lambda\,(|x| + r)\ d\mu(x) \ \cdot [-1,1] \\
=\ & \tfrac{1}{\mu_n(\mathbb{R}^N)}\ \tfrac{1}{\lambda\, e^r} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu_n - \mu) + \Big(\tfrac{1}{\mu_n(\mathbb{R}^N)} - \tfrac{1}{\mu(\mathbb{R}^N)}\Big) \lfloor\mu\rfloor \cdot [-1,1]
\end{aligned}$$

with arbitrary $\varphi \in \mathrm{Lip}\,(\mathbb{R}^N,\mathbb{R})$ and $x_0 \in \mathbb{R}^N$ satisfying $\mathrm{Lip}\,\varphi \le \lambda,\ |x_0| \le r$.

4.) For $\mu,\nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ and $R > 0$, assume $\mathrm{supp}\,\mu \cup \mathrm{supp}\,\nu \subset \mathbb{B}_R(0)$.
The estimate $\frac{1}{e^{R+1}}\ |\mu(\mathbb{R}^N) - \nu(\mathbb{R}^N)| \le \rho(\mu,\nu)$ is an obvious consequence of Definition 3.1 due to

$$\varphi: \mathbb{R}^N \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 1 & \text{if} \quad |x| \le R \\ 1 + R - |x| & \text{if} \quad R < |x| \le R+1 \\ 0 & \text{if} \quad |x| > R+1 \end{cases}.$$

Now choose $\varphi \in \mathrm{Lip}\,(\mathbb{R}^N,\mathbb{R})$ and $x_0 \in \mathbb{R}^N$ arbitrarily with $\mathrm{Lip}\,\varphi \le \lambda,\ |x_0| > 3R+2$, $\int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu-\nu) \ge 0$. The key idea is to construct a Lipschitz continuous function $\psi: \mathbb{R}^N \longrightarrow \mathbb{R}$ with compact support in $\mathbb{B}_{2R}(0)$ and

$$\tfrac{1}{\lambda\, e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d(\mu-\nu) \ \le\ \tfrac{1}{\mathrm{Lip}\,\psi\,\cdot\, e^{2R}} \int_{\mathbb{R}^N} (\psi - \psi(2R,0\ldots0))\ d(\mu-\nu).$$

Indeed, set
$$\psi(x) := \begin{cases} \varphi(x) - \varphi(x_0) & \text{if} \quad |x| \le R \\ (\varphi(x) - \varphi(x_0)) \left(\frac{2R-|x|}{R}\right)^2 & \text{if} \quad R < |x| \le 2R \\ 0 & \text{if} \quad |x| > 2R \end{cases}.$$

Obviously, $\psi \in C^0(\mathbb{R}^N)$ has compact support in $\mathbb{B}_{2R}(0)$ and is differentiable almost everywhere with

$$\nabla\psi(x) = \begin{cases} \nabla\varphi(x) & \text{if} \quad |x| < R \\ \nabla\varphi(x) \left(\frac{2R-|x|}{R}\right)^2 + (\varphi(x) - \varphi(x_0)) \cdot 2\ \frac{2R-|x|}{R}\ \frac{-x}{|x|} & \text{if} \quad R < |x| \le 2R \end{cases}.$$

Thus, ψ is Lipschitz continuous with $\mathrm{Lip}\,\psi \le \lambda + \lambda(2R+|x_0|)\cdot 2$. Due to $\mathrm{supp}\,(\mu-\nu) \subset \mathbb{B}_R(0)$,

$$\begin{aligned}
\frac{1}{\lambda\, e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\; d(\mu-\nu) &= \frac{1}{\lambda\, e^{|x_0|}} \int_{\mathbb{R}^N} \psi\; d(\mu-\nu) \\
&= \frac{\lambda\,(1{+}4R{+}2|x_0|)\cdot e^{2R}}{\lambda\, e^{|x_0|}} \;\; \frac{1}{\lambda\,(1{+}4R{+}2|x_0|)\cdot e^{2R}} \int_{\mathbb{R}^N} \psi\; d(\mu-\nu) \\
&\le \frac{1+4R+2|x_0|}{e^{|x_0|-2R}} \;\; \frac{1}{\lambda\,(1{+}4R{+}2|x_0|)\cdot e^{2R}} \int_{\mathbb{R}^N} \psi\; d(\mu-\nu)
\end{aligned}$$

The auxiliary function $[3R+2,\infty[\longrightarrow \mathbb{R},\ \ \xi \longmapsto \frac{1+4R+2\xi}{e^{\xi-2R}}$ has negative derivative and thus, it is strictly decreasing. So its upper bound is $\frac{1+4R+2\xi}{e^{\xi-2R}}\Big|_{\xi=3R+2} = \frac{10\,R+5}{e^{R+2}} < 1$ for any $R>0$.

5.) The assumptions $\sup_n \llbracket \mu_n \rrbracket < \infty$ and $\mu \in \mathcal{M}_c^+(\mathbb{R}^N)$ imply

(∗) $\operatorname{supp}\mu \;\cup\; \bigcup_n \operatorname{supp}\mu_n \;\subset\; \mathbb{B}_R(0) \;\subset\; \mathbb{R}^N$ for some $R \in\,]0,\infty[$.

So the property is obvious if $\mu \equiv 0$. In case of $\mu \neq 0$, the equivalence is based on the link with the linear Wasserstein metric (in statement 2) and subsequent Proposition 3.3. Indeed,

$$\begin{aligned}
\rho(\mu_n,\mu) \longrightarrow 0 \;&\overset{(3.,4.)}{\Longleftrightarrow}\; \rho\Big(\tfrac{1}{\mu_n(\mathbb{R}^N)}\,\mu_n,\ \tfrac{1}{\mu(\mathbb{R}^N)}\,\mu\Big) \longrightarrow 0 \quad\text{and}\quad \mu_n(\mathbb{R}^N) \longrightarrow \mu(\mathbb{R}^N) \\
&\overset{(2.)}{\Longleftrightarrow}\; \tfrac{1}{\mu_n(\mathbb{R}^N)}\,\mu_n \longrightarrow \tfrac{1}{\mu(\mathbb{R}^N)}\,\mu \ \text{narrowly} \quad\text{and}\quad \mu_n(\mathbb{R}^N) \longrightarrow \mu(\mathbb{R}^N) \quad\text{and} \\
&\qquad \Big(\tfrac{1}{\mu_n(\mathbb{R}^N)}\,\mu_n\Big)_n \ \text{has uniformly integrable first moments} \\
&\overset{(*)}{\Longleftrightarrow}\; \tfrac{1}{\mu_n(\mathbb{R}^N)}\,\mu_n \longrightarrow \tfrac{1}{\mu(\mathbb{R}^N)}\,\mu \ \text{narrowly} \quad\text{and}\quad \mu_n(\mathbb{R}^N) \longrightarrow \mu(\mathbb{R}^N) \\
&\overset{\mu(\mathbb{R}^N)>0}{\Longleftrightarrow}\; \mu_n \longrightarrow \mu \quad \text{narrowly}
\end{aligned}$$

with the last step resulting from estimates similar to the preceding ones for statement (3.).

6.) results from statement (5.) and subsequent Proposition 3.3. Indeed, let $(\mu_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{M}_c^+$ with $\llbracket \mu_n \rrbracket \le \delta$. In particular, $\operatorname{supp}\mu_n \subset \mathbb{B}_\delta(0)$ for all $n \in \mathbb{N}$.

Then, either $\liminf_{n\longrightarrow\infty} \mu_n(\mathbb{R}^N) = 0$ (and thus, $\mu_{n_j} \longrightarrow 0$ narrowly for some subsequence)

or $\liminf_{n\longrightarrow\infty} \mu_n(\mathbb{R}^N) > 0$. In the latter case, there exists a subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$ such that $\mu_{n_j}(\mathbb{R}^N) \longrightarrow p > 0$ and $\inf\{r>0 \,|\, \operatorname{supp}\mu_{n_j} \subset \mathbb{B}_r(0)\} \longrightarrow q \le \delta - p$ for $j \longrightarrow \infty$. According to Proposition 3.3, $\Big\{\frac{1}{\mu_{n_j}(\mathbb{R}^N)}\,\mu_{n_j} \;\Big|\; j \in \mathbb{N}\Big\} \subset \mathcal{P}_1(\mathbb{R}^N)$ is relatively compact with respect to the linear Wasserstein metric. So, there exists a probability measure $\nu \in \mathcal{M}^+(\mathbb{R}^N)$ and an infinite subset $J \subset \mathbb{N}$ with $\frac{1}{\mu_{n_j}(\mathbb{R}^N)}\,\mu_{n_j} \longrightarrow \nu$ narrowly for $j \longrightarrow \infty$ $(j \in J)$.
In particular, this characterization implies $\operatorname{supp}\nu \subset \mathbb{B}_q(0)$ and, $\rho\Big(\frac{1}{\mu_{n_j}(\mathbb{R}^N)}\,\mu_{n_j},\ \nu\Big) \longrightarrow 0$ due to statement (5.)
So finally, $\mu := p\,\nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ satisfies $\mu(\mathbb{R}^N) = p = \lim_{j\to\infty} \mu_{n_j}(\mathbb{R}^N) > 0$ and, statement (3.) ensures $\rho(\mu_{n_j},\mu) \longrightarrow 0$ for $j \longrightarrow \infty$ $(j \in J)$. □

Proposition 3.3 *The subset of probability measures on* $\mathbb{R}^N$

$$\mathcal{P}_1(\mathbb{R}^N) := \Big\{\mu \in \mathcal{M}^+(\mathbb{R}^N) \;\Big|\; \mu(\mathbb{R}^N) = 1,\ \int_{\mathbb{R}^N} |x|\; d\mu(x) < \infty\Big\} \subset \mathcal{M}_1^+(\mathbb{R}^N)$$

endowed with the linear Wasserstein metric is a complete separable metric space.

A set $\mathcal{K} \subset \mathcal{P}_1(\mathbb{R}^N)$ *is relatively compact if and only if it is tight and its first moments are uniformly integrable. In particular, for a given sequence* $(\mu_n)_{n\in\mathbb{N}}$ *in* $\mathcal{P}_1(\mathbb{R}^N)$*, we have*

$$\mu_n \longrightarrow \mu \text{ in } \mathcal{P}_1(\mathbb{R}^N) \quad \Longleftrightarrow \quad \wedge \begin{cases} \mu_n \longrightarrow \mu & \text{narrowly} \quad \text{for } n \longrightarrow \infty \\ (\mu_n)_{n\in\mathbb{N}} & \text{has uniformly integrable first moments.} \end{cases}$$

Proof results directly from [4], Proposition 7.1.5. □

4 The Cauchy problem for positive Radon measures on $\mathbb{R}^N$ with compact support

Assumptions of § 4 Let $\mathbf{b} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $c : \mathbb{R}^N \longrightarrow \mathbb{R}$ be bounded and Lipschitz continuous. For given $\nu_0 \in \mathcal{M}(\mathbb{R}^N)$, the linear problem here focuses on a measure–valued distributional solution $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^N)$, $t \longmapsto \mu_t$ of

$$\begin{cases} \partial_t \mu_t + \operatorname{div}_x (\mathbf{b}\, \mu_t) = c\, \mu_t & \text{in } [0,T] \\ \mu_0 = \nu_0 \end{cases} \tag{3}$$

i.e. $$\int_{\mathbb{R}^N} \varphi(x)\; d\mu_t(x) - \int_{\mathbb{R}^N} \varphi(x)\; d\nu_0(x) = \int_0^t \int_{\mathbb{R}^N} \Big(\nabla\varphi(x) \cdot \mathbf{b}(x) + c(x)\Big)\, d\mu_s(x)\; ds$$
for every $t \in [0,T]$ and any test function $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$.

Definition 4.1 $\mathbf{X_b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is induced by the flow along* $\mathbf{b}$*, i.e.* $\mathbf{X_b}(\cdot, x_0) : [0,T] \longrightarrow \mathbb{R}^N$ *is the continuously differentiable solution of the Cauchy problem*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) = \mathbf{b}(x(t)) & \text{in } [0,T], \\ x(0) = x_0. \end{cases}$$

As a well-known result about ODEs, $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and Gronwall's Lemma imply

Lemma 4.2 $\mathbf{X_b} : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is continuously differentiable with*

$$\begin{aligned} \operatorname{Lip} \mathbf{X_b}(t, \cdot) &\leq e^{\operatorname{Lip} \mathbf{b} \cdot t}, \\ \|\mathbf{X_b}(t, \cdot) - \mathbf{X}_{\widetilde{\mathbf{b}}}(t, \cdot)\|_\infty &\leq \|\mathbf{b} - \widetilde{\mathbf{b}}\|_\infty \cdot t\, e^{t \cdot \operatorname{Lip} \widetilde{\mathbf{b}}} \quad \text{for any } \widetilde{\mathbf{b}} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N). \end{aligned}$$
□

Proposition 4.3 *For any initial datum* $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$*, a solution* $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^N)$, $t \longmapsto \mu_t$ *of the linear problem* (3) *(in the distributional sense) is given by*

$$\int_{\mathbb{R}^N} \varphi\, d\mu_t = \int_{\mathbb{R}^N} \varphi(\mathbf{X_b}(t,x)) \cdot \exp\left(\int_0^t c(\mathbf{X_b}(s,x))\, ds\right) d\mu_0(x) \qquad \text{for all } \varphi \in C_c^0(\mathbb{R}^N).$$

If the initial Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$ *is positive with compact support then so are all these measures* μ_t *and* $\operatorname{supp} \mu_t \subset \mathbb{B}_{\|b\|_\infty \cdot t}(\operatorname{supp} \mu_0) \subset \mathbb{R}^N$.

Proof. First, we verify that the right–hand side provides a distributional solution of the linear problem with the initial datum μ_0. In fact, it is absolutely continuous with respect to t because for any subinterval $[s,t] \subset [0,T]$,

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \varphi \; d\mu_t \; - \; \int_{\mathbb{R}^N} \varphi \; d\mu_s \Big| \\
= \; & \Big| \int_{\mathbb{R}^N} \Big(\varphi(\mathbf{X_b}(t,x)) \cdot e^{\int_0^t c(\mathbf{X_b}(r,x))\, dr} - \varphi(\mathbf{X_b}(s,x)) \cdot e^{\int_0^s c(\mathbf{X_b}(r,x))\, dr} \Big) \; d\mu_0(x) \Big| \\
\leq \; & \int_{\mathbb{R}^N} \Big(|\varphi(\mathbf{X_b}(t,x)) - \varphi(\mathbf{X_b}(s,x))| \; e^{t\,\|c\|_\infty} + |\varphi(\mathbf{X_b}(s,x))| \; \Big[e^{\int_0^\sigma c(\mathbf{X_b}(r,x))\, dr} \Big]_{\sigma=s}^{\sigma=t} \Big) d|\mu_0(x)| \\
\leq \; & \Big(\|\nabla\varphi\|_\infty \; \|\mathbf{b}\|_\infty \; (t-s) \quad e^{t\,\|c\|_\infty} + \|\varphi\|_\infty \quad e^{t\,\|c\|_\infty} \; \|c\|_\infty \, (t-s) \Big) \; |\mu_0|(\mathbb{R}^N)
\end{aligned}$$

At Lebesgue–almost every time $t \in [0,T]$, the weak derivative of the right–hand side is

$$\begin{aligned}
& \int_{\mathbb{R}^N} \Big(\nabla\varphi(\mathbf{X_b}(t,x)) \cdot \mathbf{b}(\mathbf{X_b}(t,x)) \; + \; \varphi(\mathbf{X_b}(t,x)) \; c(\mathbf{X_b}(t,x)) \Big) \; e^{\int_0^t c(\mathbf{X_b}(r,x))\, dr} \; d\mu_0(x) \\
= \; & \int_{\mathbb{R}^N} \Big(\nabla\varphi(y) \cdot \mathbf{b}(y) \; + \; \varphi(y) \; c(y) \Big) \; d\mu_t(y).
\end{aligned}$$

and thus, it solves the linear problem (3).

In case of $\mu_0 \in \mathcal{M}^+(\mathbb{R}^N)$, the positivity of measures is obviously preserved. Finally, the preceding representation implies for all $t \in [0,T]$

$$\text{supp}\; \mu_t \;\subset\; \mathbf{X_b}(t,\cdot)^{-1}\,(\text{supp}\; \mu_0) \;\subset\; \mathbf{X_{-b}}(t,\cdot)\,(\text{supp}\; \mu_0) \;\subset\; \mathbb{B}_{\|\mathbf{b}\|_\infty\, t}(\text{supp}\; \mu_0)$$

So in particular, the compactness of $\text{supp}\; \mu_0 \subset \mathbb{R}^N$ is also preserved. □

Remark. The uniqueness of this distributional solution and more details about its representation are given in [20], § 3 (quoted here in subsequent Lemma 4.8, see also [24]).

Lemma 4.4 *For each* $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $c \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$, *the measure–valued solutions of the linear problem* $\frac{d}{dt}\,\mu_t + div_x \cdot (\mathbf{b}\,\mu_t) = c\,\mu_t$ *mentioned in Proposition 4.3 induce a map* $\vartheta_{\mathbf{b},c} : [0,1] \times \mathcal{M}_c^+(\mathbb{R}^N) \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N), (t,\mu_0) \longmapsto \mu_t$ *satisfying the following conditions for any* $\mu_0, \nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, $t,h \in [0,1]$, $\widetilde{\mathbf{b}} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $\widetilde{c} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$ *with* $t+h \leq 1$, $\text{supp}\;\mu_0 \cup \text{supp}\;\nu_0 \subset \mathbb{B}_R(0)$

1. $\vartheta_{\mathbf{b},c}(0,\cdot) \;=\; \text{Id}_{\mathcal{M}_c^+(\mathbb{R}^N)}$,
2. $\vartheta_{\mathbf{b},c}(h,\; \vartheta_{\mathbf{b},c}(t,\mu_0)) = \vartheta_{\mathbf{b},c}(t+h,\mu_0)$
3. $\vartheta_{\mathbf{b},c}(h,\mu_0)\,(\mathbb{R}^N) \;\leq\; \mu_0(\mathbb{R}^N) \cdot e^{\|c\|_\infty \cdot h}$
   $\llbracket \vartheta_{\mathbf{b},c}(h,\mu_0) \rrbracket \;\leq\; \llbracket \mu_0 \rrbracket \cdot e^{\|c\|_\infty \cdot h} + \|\mathbf{b}\|_\infty\, h$
4. $\rho\,(\mu_0,\; \vartheta_{\mathbf{b},c}(h,\mu_0)) \;\leq\; h\,(\|\mathbf{b}\|_\infty + \|c\|_\infty)\; e^{\|c\|_\infty}\; (\llbracket \mu_0 \rrbracket + \|\mathbf{b}\|_\infty + 1)^2$
5. $\rho\,(\vartheta_{\mathbf{b},c}(h,\mu_0),\; \vartheta_{\mathbf{b},c}(h,\nu_0)) \;\leq\; \rho(\mu_0,\nu_0) \cdot e^{4\,(R+\|b\|_\infty+1)\,(\|\mathbf{b}\|_{W^{1,\infty}}+\|c\|_{W^{1,\infty}})\cdot h}$
6. $\rho\,\Big(\vartheta_{\mathbf{b},c}(h,\mu_0),\; \vartheta_{\widetilde{\mathbf{b}},\widetilde{c}}(h,\mu_0)\Big) \;\leq\; h \cdot (\|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty + \|c-\widetilde{c}\|_\infty) \cdot e^{\text{Lip}\,\widetilde{\mathbf{b}}+\max\{\|c\|_\infty,\|\widetilde{c}\|_\infty\}+\text{Lip}\,\widetilde{c}}$
   $(\llbracket \mu_0 \rrbracket + 1 + \max\{\|\mathbf{b}\|_\infty, \|\widetilde{\mathbf{b}}\|_\infty\})^2$

Proof. As a direct consequence of Proposition 4.3, $\vartheta_{\mathbf{b},c} : [0,1] \times \mathcal{M}_c^+(\mathbb{R}^N) \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$ satisfies the semigroup property and thus statements (1.), (2.).
Furthermore, the constant function $\varphi \equiv 1$ in a sufficiently large ball leads to

$$\vartheta_{\mathbf{b},c}(h,\mu_0)\,(\mathbb{R}^N) \;=\; \int_{\mathbb{R}^N} e^{\int_0^h c(\mathbf{X_b}(s,x))\,ds}\; d\mu_0 \;\le\; e^{\|c\|_\infty\, h}\;\mu_0(\mathbb{R}^N)$$

and thus, statement (3.) results from $\operatorname{supp}\,\vartheta_{\mathbf{b},c}(h,\mu_0) \subset \mathbb{B}_{\|b\|_\infty\, h}(\operatorname{supp}\mu_0)$.

For proving estimate (5.), choose any $\mu_0,\nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, $\lambda, R > 0$, $h \in [0,1]$, $x_0 \in \mathbb{R}^N$, $\varphi \in \operatorname{Lip}(\mathbb{R}^N,\mathbb{R})$ with $\operatorname{Lip}\varphi \le \lambda$, $\operatorname{supp}\mu \cup \operatorname{supp}\nu \subset \mathbb{B}_R(0)$, $|x_0| \le r_h := 3(R + \|b\|_\infty\, h) + 2$.
Then Proposition 4.3 implies $\operatorname{supp}\vartheta_{\mathbf{b},c}(h,\mu_0) \cup \operatorname{supp}\vartheta_{\mathbf{b},c}(h,\nu_0) \subset \mathbb{B}_{R+\|b\|_\infty\, h}(0)$ and

$$\begin{aligned}
&\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi(x) - \varphi(x_0))\; d\left(\vartheta_{\mathbf{b},c}(h,\mu_0) - \vartheta_{\mathbf{b},c}(h,\nu_0)\right)(x)\\
=\;&\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi(\mathbf{X_b}(h,x)) - \varphi(x_0)) \cdot e^{\int_0^h c(\mathbf{X_b}(s,x))\,ds}\; d\,(\mu_0 - \nu_0)\,(x).
\end{aligned}$$

The latter integrand, $\psi_h : \mathbb{R}^N \longrightarrow \mathbb{R}$, $x \longmapsto (\varphi(\mathbf{X_b}(h,x)) - \varphi(x_0)) \cdot e^{\int_0^h c(\mathbf{X_b}(s,x))\,ds}$ is locally Lipschitz continuous for each fixed $h \in [0,1]$, $x_0 \in \mathbb{R}^N$ and, restricted to the closed ball $\mathbb{B}_R(0) \subset \mathbb{R}^N$, its Lipschitz constant satisfies

$$\begin{aligned}
&\operatorname{Lip}\psi_h|_{\mathbb{B}_R(0)}\\
\le\;& \operatorname{Lip}\varphi\cdot\operatorname{Lip}\mathbf{X_b}(h,\cdot)\cdot e^{\|c\|_\infty\, h} + \sup_{\mathbb{B}_R(0)} |\varphi(\mathbf{X_b}(h,\cdot)) - \varphi(x_0)|\cdot e^{\|c\|_\infty\, h}\int_0^h \operatorname{Lip} c\cdot\operatorname{Lip}\mathbf{X_b}(s,\cdot)\,ds\\
\le\;& \operatorname{Lip}\varphi\cdot e^{\operatorname{Lip}\mathbf{b}\cdot h}\cdot e^{\|c\|_\infty\, h} + \operatorname{Lip}\varphi\cdot(R + \|\mathbf{b}\|_\infty\, h + r_h)\cdot e^{\|c\|_\infty\, h}\;\operatorname{Lip} c\cdot e^{\operatorname{Lip}\mathbf{b}\cdot h}\; h\\
\le\;& \lambda\cdot e^{\operatorname{Lip}\mathbf{b}\cdot h}\cdot e^{\|c\|_\infty\, h} + \lambda\cdot(R + \|\mathbf{b}\|_\infty\, h + r_h)\cdot e^{\|c\|_\infty\, h}\;\operatorname{Lip} c\cdot e^{\operatorname{Lip}\mathbf{b}\cdot h}\; h\\
\le\;& \lambda\cdot e^{(\operatorname{Lip}\mathbf{b}+\|c\|_\infty)\cdot h}\left(1 + h\cdot\operatorname{Lip} c\cdot 4\,(R + \|b\|_\infty\, h + 1)\right)
\end{aligned}$$

as a consequence of the product rule (applied to the partial derivatives with respect to x) and Lemma 4.2. Furthermore, ψ_h has a root at $\mathbf{X_b}(h,\cdot)^{-1}(x_0) = \mathbf{X}_{-\mathbf{b}}(h,x_0) \subset \mathbb{B}_{|x_0|+\|\mathbf{b}\|_\infty\cdot h}(0)$.
So Definition 3.1 of $\rho(\mu_0,\nu_0)$ implies

$$\int_{\mathbb{R}^N} \psi_h(x)\; d(\mu_0 - \nu_0)(x) \;\le\; \operatorname{Lip}\psi_h|_{\mathbb{B}_R(0)}\;\; e^{|x_0|+\|\mathbf{b}\|_\infty\cdot h}\cdot\rho(\mu_0,\nu_0)$$

and thus,

$$\begin{aligned}
&\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi(x) - \varphi(x_0))\; d\left(\vartheta_{\mathbf{b},c}(h,\mu_0) - \vartheta_{\mathbf{b},c}(h,\nu_0)\right)(x)\\
\le\;& e^{(\|\mathbf{b}\|_{W^{1,\infty}}+\|c\|_\infty)\cdot h}\left(1 + h\cdot\operatorname{Lip} c\cdot 4\,(R + \|b\|_\infty\, h + 1)\right)\rho(\mu_0,\nu_0)\\
\le\;& e^{(\|\mathbf{b}\|_{W^{1,\infty}}+\|c\|_\infty)\cdot h}\; e^{h\cdot\operatorname{Lip} c\cdot 4\,(R+\|b\|_\infty+1)}\;\rho(\mu_0,\nu_0)\\
\le\;& e^{4\,(R+\|b\|_\infty+1)\,(\|\mathbf{b}\|_{W^{1,\infty}}+\|c\|_{W^{1,\infty}})\cdot h}\;\rho(\mu_0,\nu_0).
\end{aligned}$$

As a consequence of Lemma 3.2 (4.), $\rho\left(\vartheta_{\mathbf{b},c}(h,\mu_0),\ \vartheta_{\mathbf{b},c}(h,\nu_0)\right)$ has the same upper bound, i.e. estimate (5.) holds.

In regard to statement (6.), choose any $\mathbf{b}, \widetilde{\mathbf{b}} \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$ and $c, \widetilde{c} \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R})$.
Furthermore, let $\mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, $\lambda, R > 0$, $h \in [0,1]$, $x_0 \in \mathbb{R}^N$ and $\varphi \in \operatorname{Lip}(\mathbb{R}^N,\mathbb{R})$ satisfy $\operatorname{Lip}\varphi \le \lambda$, $\operatorname{supp}\mu \subset \mathbb{B}_R(0)$.

Then,

$$\begin{aligned}
&\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi(x) - \varphi(x_0)) \;\; d\Big(\vartheta_{\mathbf{b},c}(h,\mu_0) - \vartheta_{\widetilde{\mathbf{b}},\widetilde{c}}(h,\mu_0)\Big)(x) \\
= \; &\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} \Big((\varphi(\mathbf{X_b}(h,x)) - \varphi(x_0)) \cdot e^{\int_0^h c(\mathbf{X_b}(s,x))\,ds} - \\
&\qquad (\varphi(\mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)) - \varphi(x_0)) \cdot e^{\int_0^h \widetilde{c}(\mathbf{X}_{\widetilde{\mathbf{b}}}(s,x))\,ds} \Big) \qquad d\mu_0(x)
\end{aligned}$$

For $h \in [0,1]$, $x \in \mathbb{B}_R(0)$ fixed, the auxiliary function $\psi : [0,1] \longrightarrow \mathbb{R}$ (in the last integrand)

$$\psi(t) \;:=\; \Big(\varphi\big(t\cdot \mathbf{X_b}(h,x) + (1-t)\cdot \mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)\big) - \varphi(x_0)\Big)\cdot e^{t\cdot\int_0^h c(\mathbf{X_b}(s,x))\,ds \,+\, (1-t)\cdot\int_0^h \widetilde{c}(\mathbf{X}_{\widetilde{\mathbf{b}}}(s,x))\,ds}$$

is continuously differentiable and using $\widehat{b} := \max\{\|\mathbf{b}\|_\infty, \|\widetilde{\mathbf{b}}\|_\infty\}$, $\widehat{c} := \max\{\|c\|_\infty, \|\widetilde{c}\|_\infty\}$,

$$\begin{aligned}
\psi'(t) \;=\; & e^{t\cdot\int_0^h c(\mathbf{X_b}(s,x))\,ds \,+\, (1-t)\cdot\int_0^h \widetilde{c}(\mathbf{X}_{\widetilde{\mathbf{b}}}(s,x))\,ds} \cdot \\
&\Big(\nabla\varphi|_{t\cdot\mathbf{X_b}(h,x)+(1-t)\cdot\mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)} \cdot \Big(\mathbf{X_b}(h,x) - \mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)\Big) \; + \\
&\Big(\varphi|_{t\cdot\mathbf{X_b}(h,x)+(1-t)\cdot\mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)} - \varphi(x_0)\Big) \cdot \int_0^h \big(c(\mathbf{X_b}(s,x)) - \widetilde{c}(\mathbf{X}_{\widetilde{\mathbf{b}}}(s,x))\;ds\Big) \\
\le\; & e^{\widehat{c}\,h} \;\Big(\mathrm{Lip}\,\varphi \cdot \Big|\mathbf{X_b}(h,x) - \mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)\Big| \; + \\
&\mathrm{Lip}\,\varphi\,(R + \widehat{b}\,h + |x_0|)\cdot h\;\Big(\|c-\widetilde{c}\|_\infty \;+\; \mathrm{Lip}\,\widetilde{c}\cdot \max_{0\le s\le h} |\mathbf{X_b}(h,x) - \mathbf{X}_{\widetilde{\mathbf{b}}}(h,x)|\Big)\Big) \\
\le\; & e^{\widehat{c}\,h} \;\Big(\mathrm{Lip}\,\varphi \cdot h \;\;\|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty\, e^{h\,\cdot\,\mathrm{Lip}\,\widetilde{\mathbf{b}}} \; + \\
&\mathrm{Lip}\,\varphi\,(R + \widehat{b}\,h + |x_0|)\cdot h\;\Big(\|c-\widetilde{c}\|_\infty \;+\; \mathrm{Lip}\,\widetilde{c}\,\cdot\, \|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty\;\; e^{h\,\cdot\,\mathrm{Lip}\,\widetilde{\mathbf{b}}}\,h\Big)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} (\varphi(x) - \varphi(x_0)) \;\; d\Big(\vartheta_{\mathbf{b},c}(h,\mu_0) - \vartheta_{\widetilde{\mathbf{b}},\widetilde{c}}(h,\mu_0)\Big)(x) \\
= \; &\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} \psi(1) - \psi(0) \quad d\mu_0(x) \\
= \; &\frac{1}{\lambda\cdot e^{|x_0|}} \int_{\mathbb{R}^N} \int_0^1 \psi'(t)\;dt\;\,d\mu_0(x) \\
\le \; &\frac{1}{\lambda\cdot e^{|x_0|}} \;\; \mu_0(\mathbb{R}^N) \quad e^{\widehat{c}\,h}\;\Big(\lambda \cdot h \;\;\|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty\, e^{h\,\cdot\,\mathrm{Lip}\,\widetilde{\mathbf{b}}} \; + \\
&\qquad \lambda\,(R + \widehat{b}\,h + |x_0|)\cdot h\;\Big(\|c-\widetilde{c}\|_\infty + \mathrm{Lip}\,\widetilde{c}\cdot \|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty\, e^{h\,\cdot\,\mathrm{Lip}\,\widetilde{\mathbf{b}}}\,h\Big)\Big) \\
\le \; & h \quad \mu_0(\mathbb{R}^N)\;\, e^{\widehat{c}+\mathrm{Lip}\,\widetilde{\mathbf{b}}}\;\,(\|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty + \|c-\widetilde{c}\|_\infty) \quad (R + \widehat{b} + 1) \quad (1 + \mathrm{Lip}\,\widetilde{c})
\end{aligned}$$

ensuring estimate (6.):

$$\rho\Big(\vartheta_{\mathbf{b},c}(h,\mu_0),\;\vartheta_{\widetilde{\mathbf{b}},\widetilde{c}}(h,\mu_0)\Big) \;\le\; h\,\cdot\,(\|\mathbf{b}-\widetilde{\mathbf{b}}\|_\infty + \|c-\widetilde{c}\|_\infty)\,\cdot\, e^{\widehat{c}+\mathrm{Lip}\,\widetilde{c}+\mathrm{Lip}\,\widetilde{\mathbf{b}}}\,(\llbracket\mu_0\rrbracket + \widehat{b} + 1)^2.$$

Statement (4.) is an immediate consequence (using $\widetilde{\mathbf{b}} \equiv 0$, $\widetilde{c} \equiv 0$) due to $\vartheta_{0,0}(h,\cdot) = \mathrm{Id}_{\mathbb{R}^N}$.

□

This Lemma 4.4 lays the basis for identifying the parameters of $\vartheta_{\mathbf{b},c}$ as transition on $\big(\mathcal{M}_c^+(\mathbb{R}^N),\;\rho,\;\llbracket\cdot\rrbracket\big)$. So seizing the notation of Definition 2.1 (and taking the bounds of $\llbracket\vartheta_{\mathbf{b},c}(\cdot,\mu_0)\rrbracket$ on $[0,1]$ into consideration properly), we obtain:

Proposition 4.5 *For any functions* $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $c \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$, *the map* $\vartheta_{\mathbf{b},c} : [0,1] \times \mathcal{M}_c^+(\mathbb{R}^N) \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N), (t, \mu_0) \longmapsto \mu_t$ *defined by the solutions of the linear problem* $\frac{d}{dt}\,\mu_t + div_x \cdot (\mathbf{b}\;\mu_t) = c\;\mu_t$ *is a transition on* $(\mathcal{M}_c^+(\mathbb{R}^N),\ \rho,\ \llbracket\cdot\rrbracket)$ *in the sense of Def. 2.1 with*

$$
\begin{array}{llcl}
1.) & \alpha(\vartheta_{\mathbf{b},c}\,;\ r) & \le & 4\,(r + \|b\|_\infty + 1)\ (\|\mathbf{b}\|_{W^{1,\infty}} + \|c\|_{W^{1,\infty}}) \\
2.) & \beta(\vartheta_{\mathbf{b},c}\,;\ r) & \le & (\|\mathbf{b}\|_\infty + \|c\|_\infty)\ e^{\|c\|_\infty}\ (r \cdot e^{\|c\|_\infty} + 2\,\|\mathbf{b}\|_\infty + 1)^2 \\
3.) & \zeta(\vartheta_{\mathbf{b},c}) & \le & \max\{\|\mathbf{b}\|_\infty,\ \|c\|_\infty\} \\
4.) & D(\vartheta_{\mathbf{b},c},\ \vartheta_{\widetilde{\mathbf{b}},\widetilde{c}}\,;\ r) & \le & (\|\mathbf{b} - \widetilde{\mathbf{b}}\|_\infty + \|c - \widetilde{c}\|_\infty)\ \cdot\ e^{\mathrm{Lip}\,\widetilde{\mathbf{b}} + \max\{\|c\|_\infty, \|\widetilde{c}\|_\infty\} + \mathrm{Lip}\,\widetilde{c}} \\
 & & & \qquad (r + 1 + \max\{\|\mathbf{b}\|_\infty, \|\widetilde{\mathbf{b}}\|_\infty\})^2
\end{array}
$$
□

Theorem 4.6 (Existence) *Let* $F : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$ *satisfy the following conditions:*

1.) $M := \sup_{\mu_0, t}\ (\|F_1(\mu_0, t)\|_{W^{1,\infty}} + \|F_2(\mu_0, t)\|_{W^{1,\infty}})\ <\ \infty$

2.) $\|F_1(\mu_n, s_n) - F_1(\mu_0, t)\|_\infty + \|F_2(\mu_n, s_n) - F_2(\mu_0, t)\|_\infty \longrightarrow 0$ *whenever* $\rho(\mu_n, \mu_0) \longrightarrow 0$, $s_n \longrightarrow t$ *(for* $n \longrightarrow \infty$*) and* $\sup_n \llbracket\mu_n\rrbracket < \infty$.

Then, for any initial datum $\nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, *there exists a narrowly continuous weak solution* $\mu : [0, T[\ \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, $t \longmapsto \mu_t$ *of the nonlinear transport equation*

$$
\left\{ \begin{array}{rcl} \partial_t\,\mu_t\ +\ div_x\,(F_1(\mu_t, t)\ \mu_t) & = & F_2(\mu_t, t)\ \mu_t \quad in\ [0, T] \\ \mu_0 & = & \nu_0 \end{array} \right. \tag{4}
$$

Remark. Due to Lemma 3.2 (5.), hypothesis (2.) of this existence theorem can be reformulated in the following way: $F : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow \big(W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R}),\ \|\cdot\|_\infty\big)$ is assumed to be continuous on all "cylinders" $\{\mu \in \mathcal{M}_c^+(\mathbb{R}^N) \mid \llbracket\mu\rrbracket \le \delta\} \times [0,T]$, $\delta > 0$, with respect to narrow convergence on $\mathcal{M}_c^+(\mathbb{R}^N)$.

Proof. Choose the initial datum $\nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$ arbitrarily. After identifying each value $F(\mu, t) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$ with the corresponding transition on $\mathcal{M}_c^+(\mathbb{R}^N)$

$$\vartheta_{F_1(\mu,t), F_2(\mu,t)} : [0,1] \times \mathcal{M}_c^+(\mathbb{R}^N) \longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),$$

Proposition 2.7 (about mutational equations) ensures the existence of a Lipschitz–continuous solution $\mu : [0, T[\ \longrightarrow (\mathcal{M}_c^+(\mathbb{R}^N), \rho)$, $t \longmapsto \mu_t$ of the mutational equation $\overset{\circ}{\mu}_t \ni F(\mu_t, t)$ with $\mu_0 = \nu_0$, i.e. according to Definition 2.6

1.) $\limsup_{h \downarrow 0}\ \frac{1}{h} \cdot \rho\left(\vartheta_{F_1(\mu_t,t), F_2(\mu_t,t)}(h, \mu_t),\ \ \mu_{t+h}\right)\ =\ 0$ for every $t \in [0, T[$,

2.) there is a constant $L > 0$ with $\rho(\mu_s, \mu_t)\ \le\ L\ |s - t|$ for any $s, t \in [0, T[$,

3.) $\sup_{0 \le t < T}\ \llbracket\mu_t\rrbracket\ <\ \infty$.

So $\mu : t \longmapsto \mu_t$ is narrowly continuous due to Lemma 3.2 (5.) and, we still have to verify that μ is a weak solution of the nonlinear transport equation (4). Choose any $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$. Then,

$\psi : [0,T[\longrightarrow \mathbb{R}.\ t \longmapsto \int_{\mathbb{R}^N} \varphi(x)\, d\mu_t(x)$ is Lipschitz continuous since for any $x_0 \in \mathbb{R}^N \setminus \operatorname{supp} \varphi$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} \varphi\, d\mu_t - \int_{\mathbb{R}^N} \varphi\, d\mu_s\Big| &= \Big|\int_{\mathbb{R}^N} (\varphi - \varphi(x_0))\ d\,(\mu_t - \mu_s)\Big| \\ &\leq \|\nabla\varphi\|_\infty\ e^{|x_0|} \cdot \rho(\mu_t, \mu_s) \\ &\leq \|\nabla\varphi\|_\infty\ e^{|x_0|} \cdot L\,|t-s|.\end{aligned}$$

So choosing now $t \in [0,T[$ as a point of differentiability of ψ, we obtain for $h \in]0,1]$

$$\begin{aligned}&\int_{\mathbb{R}^N} \varphi\ d\mu_{t+h} - \int_{\mathbb{R}^N} \varphi\ d\mu_t \\ &= \int_{\mathbb{R}^N} \varphi\ \ d\left(\mu_{t+h} - \vartheta_{F(\mu_t,t)}(h,\mu_t)\right) + \int_{\mathbb{R}^N} \varphi\ \ d\left(\vartheta_{F(\mu_t,t)}(h,\mu_t) - \mu_t\right) \\ &= \|\nabla\varphi\|_\infty\, e^{|x_0|} \cdot \rho\left(\mu_{t+h},\ \vartheta_{F(\mu_t,t)}(h,\mu_t)\right) + \int_0^h \int_{\mathbb{R}^N} \Big(\nabla\varphi(x)\cdot F_1(\mu_t,t)(x) + F_2(\mu_t,t)(x)\Big) \\ &\qquad\qquad d\vartheta_{F(\mu_t,t)}(s,\mu_t)\,(x)\ \ ds\end{aligned}$$

After dividing by $h > 0$, the first summand is tending to 0 for $h \downarrow 0$ due to property (1.) of μ.

Thus, $\psi'(t) = \lim_{h\downarrow 0} \frac{1}{h} \cdot \int_0^h \int_{\mathbb{R}^N} \Big(\nabla\varphi(x)\cdot F_1(\mu_t,t)(x) + F_2(\mu_t,t)(x)\Big)\ d\vartheta_{F(\mu_t,t)}(s,\mu_t)\,(x)\ ds.$

Finally, $\psi'(t) = \int_{\mathbb{R}^N} \Big(\nabla\varphi(x)\cdot F_1(\mu_t,t)(x) + F_2(\mu_t,t)(x)\Big)\, d\mu_t(x).$ Indeed, choose $R > 0$ such that $\bigcup_{\substack{s\in[0,T[\\ \tau\in[0,1]}} \operatorname{supp}\ \vartheta_{F(\mu_s,s)}(\tau,\mu_s)\ \cup\ \operatorname{supp}\varphi\ \subset\ \mathbb{B}_R(0)$ (depending only on $\llbracket \nu_0 \rrbracket, M, \varphi$).

Then, for any $s \in]0,1]$ and $x_0 \in \mathbb{R}^N$ with $|x_0| > 3R + 2$, Lemmas 3.2 (4.) and 4.4 (4.) imply

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(\nabla\varphi(x)\cdot F_1(\mu_t,t)(x) + F_2(\mu_t,t)(x)\Big)\ \ d\left(\vartheta_{F(\mu_t,t)}(s,\mu_t) - \mu_t\right)(x) \\ \leq\ &\operatorname{const}(M, \|\varphi\|_{W^{2,\infty}})\, e^{|x_0|} \cdot \rho\left(\vartheta_{F(\mu_t,t)}(s,\mu_t),\ \mu_t\right) \\ \leq\ &\operatorname{const}(M, \|\varphi\|_{W^{2,\infty}})\, e^{|x_0|} \cdot \operatorname{const}(M, \sup_\tau \llbracket \mu_\tau \rrbracket)\ \ s.\end{aligned}$$

So the last representation of $\psi'(t)$ at every point t of differentiability leads to

$$\int_{\mathbb{R}^N} \varphi\ d\mu_t - \int_{\mathbb{R}^N} \varphi\ d\nu_0 = \int_0^t \int_{\mathbb{R}^N} \Big(\nabla\varphi(x)\cdot F_1(\mu_t,t)(x) + F_2(\mu_t,t)(x)\Big)\, d\mu_s(x)\ \ ds$$

for every time $t \in [0,T]$ and any test function $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$, i.e. μ is a weak solution of Cauchy problem (4). □

Theorem 4.7 (Uniqueness) *Let* $F : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$ *satisfy the following conditions:*

1.) $\sup_{\mu_0,t} \ (\|F_1(\mu_0,t)\|_{W^{1,\infty}} + \|F_2(\mu_0,t)\|_{W^{1,\infty}}) < \infty$

2.) for any $R > 0$*, there is a constant* $L_R > 0$ *and a modulus* $\omega_R(\cdot)$ *of continuity with*

$$\|F_1(\mu,s) - F_1(\nu,t)\|_\infty + \|F_2(\mu,s) - F_2(\nu,t)\|_\infty \ \leq\ L_R \cdot \rho(\mu,\nu) + \omega_R(|t-s|)$$

for all $\mu,\nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ *with* $\llbracket \mu \rrbracket, \llbracket \nu \rrbracket \leq R$.

Then the narrowly continuous weak solution $\mu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),\ \ t \longmapsto \mu_t$ *of Cauchy problem (4) with* $\sup_t \llbracket \mu_t \rrbracket < \infty$ *is unique.*

Remark. Correspondingly to the remark about Theorem 4.6, assumption (2.) can be reinterpreted in the following way: Local Lipschitz continuity of the vector fields $F(\cdot,\cdot)$ (with respect to ρ and $\|\cdot\|_\infty$) implies uniqueness of the "bounded" weak solution with values in $\mathcal{M}_c^+(\mathbb{R}^N)$.

Proof. Let $\nu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N),\ t \longmapsto \nu_t$ be any weak solution of

$$\partial_t \nu_t + \operatorname{div}_x (F_1(\nu_t,t)\,\nu_t) = F_2(\nu_t,t)\,\nu_t$$

that is narrowly continuous with $R := 1 + \sup_{0 \le t < T} \llbracket \nu_t \rrbracket < \infty$. Exploiting the uniqueness statement about nonautonomous linear transport equations in subsequent Lemma 4.8, we now prove that ν is a solution of the corresponding mutational equation $\overset{\circ}{\nu}_t \ni F(\nu_t,t)$ in $[0,T[$ and thus, Proposition 2.9 ensures its uniqueness for the initial datum $\nu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$.

ν is continuous with respect to ρ according to Lemma 3.2 (5.). So the composition

$$[0,T[\longrightarrow \big(W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N,\mathbb{R}),\ \|\cdot\|_\infty\big), \qquad t \longmapsto F(\nu_t,t)$$

is continuous and, Theorem 4.6 provides a continuous solution $\mu : [0,T[\longrightarrow (\mathcal{M}_c^+(\mathbb{R}^N),\rho)$ of the mutational equation $\overset{\circ}{\mu}_t \ni F(\nu_t,t)$ in $[0,T[$ with $\mu_0 = \nu_0$, $\sup_t \llbracket \mu_t \rrbracket < \infty$ that is also weak solution of the nonautonomous linear transport equation

$$\partial_t \mu_t + \operatorname{div}_x (F_1(\nu_t,t)\,\mu_t) = F_2(\nu_t,t)\,\mu_t.$$

Subsequent Lemma 4.8 (2.) guarantees the uniqueness of weak solutions of the linear Cauchy problem and thus, $\mu \equiv \nu$, i.e. ν is solution of the mutational equation $\overset{\circ}{\nu}_t \ni F(\nu_t,t)$ in $[0,T[$. Due to Proposition 2.9, Cauchy problems of this mutational equation have unique solutions. □

Lemma 4.8 *Let $v : t \longmapsto v_t$ be a Borel vector field in $L^1\big([0,T];\ W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)\big)$ and c a Borel bounded and locally Lipschitz continuous (w.r.t. the space variable) scalar function.*

(1.) For each $\nu_0 \in \mathcal{M}^+(\mathbb{R}^N)$ with $\nu_0(\mathbb{R}^N) = 1$, there exists a unique narrowly continuous $\mu : [0,T] \longrightarrow \mathcal{M}^+(\mathbb{R}^N),\ t \longmapsto \mu_t$ solving the initial value problem (in the distributional sense)

$$\partial_t \mu_t + \mathit{div}_x (v_t\,\mu_t) = c_t\,\mu_t \quad \text{in }]0,T[\times\mathbb{R}^N, \qquad \mu_0 = \nu_0.$$

(2.) The comparison principle holds in the following sense: Let $\sigma : t \longmapsto \sigma_t$ be a narrowly continuous family of signed measures solving $\partial_t \sigma_t + \mathit{div}_x (v_t\,\sigma_t) = c_t\,\sigma_t$ in $]0,T[\times\mathbb{R}^N$ with $\sigma_0 \le 0$ and

$$\int_0^T \int_{\mathbb{R}^N} \Big(|v_t(x)| + |c_t(x)|\Big)\ d|\sigma_t|(x)\ dt < \infty$$

$$\int_0^T \Big(|\sigma_t|(B) + \sup_B |v_t| + \operatorname{Lip} v_t|_B\Big)\ dt < \infty$$

$$\int_0^T \Big(|\sigma_t|(B) + \sup_B |c_t| + \operatorname{Lip} c_t|_B\Big)\ dt < \infty$$

for any bounded closed set $B \subset \mathbb{R}^N$. Then, $\sigma_t \le 0$ for any $t \in [0,T[$.

Proof is given in [20], Lemma 3.5 and Proposition 3.6, for example (see also [1, 4]).

Theorem 4.9 (Stability)
Assume for $F, G : \mathcal{M}_c^+(\mathbb{R}^N) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times W^{1,\infty}(\mathbb{R}^N, \mathbb{R})$:

1.) $M_F := \sup_{\mu_0, t} \left(\|F_1(\mu_0,t)\|_{W^{1,\infty}} + \|F_2(\mu_0,t)\|_{W^{1,\infty}} \right) < \infty$

$M_G := \sup_{\mu_0, t} \left(\|G_1(\mu_0,t)\|_{W^{1,\infty}} + \|G_2(\mu_0,t)\|_{W^{1,\infty}} \right) < \infty$

2.) *for any* $R > 0$, *there is a constant* $L_R > 0$ *and a modulus* $\omega_R(\cdot)$ *of continuity with*

$$\|F_1(\mu,s) - F_1(\nu,t)\|_\infty + \|F_2(\mu,s) - F_2(\nu,t)\|_\infty \le L_R \cdot \rho(\mu,\nu) + \omega_R(|t-s|)$$

for all $\mu, \nu \in \mathcal{M}_c^+(\mathbb{R}^N)$ *with* $\llbracket\mu\rrbracket, \llbracket\nu\rrbracket \le R$.

3.) $\|G_1(\mu_n, s_n) - G_1(\mu_0, t)\|_\infty + \|G_2(\mu_n, s_n) - G_2(\mu_0,t)\|_\infty \longrightarrow 0$ *whenever* $\rho(\mu_n, \mu_0) \longrightarrow 0$, $s_n \longrightarrow t$ *(for* $n \longrightarrow \infty$*) and* $\sup_n \llbracket\mu_n\rrbracket < \infty$.

Let $\nu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, $t \longmapsto \nu_t$ *be a narrowly continuous weak solution of*

$$\partial_t \, \nu_t + \mathrm{div}_x \, (F_1(\nu_t, t) \;\, \nu_t) = F_2(\nu_t, t) \;\, \nu_t$$

with $\sup_t \llbracket\nu_t\rrbracket < \infty$.

Then, for every initial measure $\mu_0 \in \mathcal{M}_c^+(\mathbb{R}^N)$, *there exists a narrowly continuous weak solution* $\mu : [0,T[\longrightarrow \mathcal{M}_c^+(\mathbb{R}^N)$, $t \longmapsto \mu_t$ *of*

$$\partial_t \, \mu_t + \mathrm{div}_x \, (G_1(\mu_t, t) \;\, \mu_t) = G_2(\mu_t, t) \;\, \mu_t \quad \text{in } [0,T]$$

satisfying $\sup_t \llbracket\mu_t\rrbracket < \infty$ *and for all* $t \in [0,T[$

$$\rho(\mu_t, \nu_t) \le \Big(\rho(\mu_0, \nu_0) + t \cdot \mathit{const}(M_F, M_G, \llbracket\mu_0\rrbracket, \llbracket\nu_0\rrbracket) \cdot (\|F_1 - G_1\|_\infty + \|F_2 - G_2\|_\infty) \Big) \, e^{\mathit{const}(F) \cdot t}.$$

Proof results directly from Proposition 2.10 (b) about the stability of solutions of mutational equations – in the combination with Proposition 4.5 (4.) estimating the "distance" between two transitions here. Indeed, due to assumption (2.), ν solves the corresponding mutational equation with the coefficient function $F = (F_1, F_2)$ as shown in the proof of Theorem 4.7. So there exists a solution μ of the mutational equation with the coefficients G satisfying the claimed estimate, and finally, μ is a weak solution of the corresponding nonlinear transport equation according to the proof of Theorem 4.6. □

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