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Singular Hypersurfaces as Stratifolds

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Abstract

In dieser Dissertation wird das Klassifikationsproblem für Hyperflächen in $\mathbb{C}\mathrm{P}^4$ mit einer isolierten Singularität als Stratifolds betrachtet. Das wichtigste Hilfsmittel für die Klassifikation ist die modifizierte Surgery-Theorie. Für einfache Singularitäten des Typs A_{2k} erhält man ein vollständiges System von Invarianten; für Singularitäten des Typs A_{2k+1} erhält man ein vollständiges System von Invarianten für die Hyperflächen, deren Grad d relativ klein verglichen mit der Komplexität der Singularität ist ($d < (k+5)/2$). Beispiele von Hyperflächen, die die Voraussetzungen erfüllen, werden konstruiert.

In this dissertation the classification problem of hypersurfaces in $\mathbb{C}\mathrm{P}^4$ with an isolated singularity as stratifolds is considered. The main machinery of the classification is the modified surgery. For simple singularities of type A_{2k}, a complete system of invariants is obtained; for simple singularities of type A_{2k+1}, a complete system of invariants is obtained for the hypersurfaces whose degree d is relatively small compared with the complexity of the singularity ($d < (k+5)/2$). Examples of hypersurfaces fulfilling the assumptions are constructed.

Introduction

An algebraic hypersurface is the set of zeros of a homogeneous polynomial in some complex projective space. Algebraic hypersurfaces are important topological objects, arising naturally in algebra, geometry and topology. It was first noted by Thom, that for smooth hypersurfaces, the diffeomorphism type depends only on the degree of the defining polynomial; i.e., two n-dimensional smooth hypersurfaces in $\mathbb{C}\mathrm{P}^{n+1}$ are diffeomorphic if and only if they have equal degree. For singular hypersurfaces, the situation is more complicated. A general answer to the classification problem of singular hypersurfaces is: let $P(n, d)$ be the moduli space of hypersurfaces in $\mathbb{C}\mathrm{P}^n$ of degree d, then there is a Whitney stratification on $P(n, d)$, such that two pairs $(\mathbb{C}\mathrm{P}^n, V_f)$ and $(\mathbb{C}\mathrm{P}^n, V_g)$ are topologically equivalent when the hypersurfaces V_f and V_g belong to the same connected component of a stratum of this stratification. (see e.g. [D1].)

Instead of considering the homeomorphism type of the pair $(\mathbb{C}\mathrm{P}^n, V)$, I shall consider the classification of singular hypersurfaces viewed as some kind of smooth objects with singularites. There is a notion of stratified smooth spaces developed in recent years by Matthias Kreck ([Kr1]), which is called stratifolds. Roughly speaking, a stratifold is a stratified space whose strata are smooth manifolds of different dimensions with certain conditions of smoothness on the joining of the strata. Stratifolds are a counter part of algebraic varieties in the topological world, and each algebraic variety carries a canonical stratifold structure. Thus it is natural to ask for a classification of singular hypersurfaces as stratifolds. In general this problem is very difficult. In this dissertation we consider hypersurfaces in $\mathbb{C}\mathrm{P}^4$ with an isolated singularity. With some restrictions on the link of the singularity we classify these objects in the above sense. The main results of this dissertation are the following:

Theorem. *For $i = 1, 2$, let $V_i \subset \mathbb{C}P^4$ be a hypersurface of degree d_i with a* *unique singularity* $p_i \in V_i$*, such that the link of* p_i *is diffeomorphic to* $S^2 \times S^3$*. Suppose that the second integral homology group of the nonsingular part of* V_i *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$*, and that* d_i *is square-free. Let* μ_i *be the Milnor number of* p_i*. Then* V_1 *and* V_2 *are diffeomorphic as stratifolds if and only if* $d_1 = d_2$ *and* $\mu_1 = \mu_2$*.*

The Milnor number is a basic invariant reflecting the topology of a singularity. For the definition of the Milnor number see section 1.2.

As a consequence of this theorem, we have a classification of hypersurfaces with an A_k-singularity.

Corollary. *For* $i = 1, 2$*, let* $V_i \subset \mathbb{C}P^4$ *be a hypersurface of degree* d_i *with a unique singularity of type* A_{2k_i+1} $(k_i \geq 0)$*. If* $d_i < (k_i + 5)/2$ *and is square-free, then* V_1 *and* V_2 *are diffeomorphic as stratifolds if and only if* $d_1 = d_2$ *and* $k_1 = k_2$*.*

For $i = 1, 2$*, let* $V_i \subset \mathbb{C}P^4$ *be a hypersurface of degree* d_i *with a unique singularity of type* A_{2k_i}*, then* V_1 *and* V_2 *are diffeomorphic as stratifolds if and only if* $d_1 = d_2$ *and* $k_1 = k_2$*.*

It is interesting to compare this result with a well-known result about the topology of hypersurfaces with isolated singularities. It is known that if the degree of the hypersurface is big enough compared with the number and the complexity of the singularities, then the space of the hypersurfaces with given degree and singularities is connected and hence the topological type of these hypersurfaces is constant ([D1]). More precisely, let d be the degree of V, k the number of isolated singularities on V and s_i the $\mathscr{K}$-determinancy order of the singularity q_i, then the space of such hypersurfaces is connected if $d \geq s_1 + \cdots + s_k + k - 1$. For the definition of the $\mathscr{K}$-determinancy order, I refer to [D3]. Since the $\mathscr{K}$-determinancy order of an A_k-singularity is $k + 1$, we see that when $d \geq 2k + 2$, the pair (d, k) is a complete invariant of hypersurfaces of degree d with a unique singularity of type A_{2k+1}. On the other hand, the corollary above gives information for d relatively small $(d < (k + 5)/2)$.

Hypersurfaces with isolated singularities fall into a special class of stratifolds, which are obtained by collapsing boundaries of a smooth manifold to corresponding singularities. Therefore, the classification of such objects is equivalent to a classification problem of certain smooth manifolds with boundaries. More concretely, in our case, we need to classify simply-connected

6-manifolds with boundary diffeomorphic to $S^2 \times S^3$, fulfilling certain conditions on homology and cohomology. Here the fact that the boundaries are diffeomorphic to $S^2 \times S^3$ is a consequence of the topology of 3-dimensional A_k-singularities. The vanishing of the fundamental group and the conditions on homology and cohomology come from the topolgical properties of a hypersurface. We will apply the modified surgery theory to this classification problem and get a complete invariants system of such manifolds. It turns out that the invariants are the cohomology ring, the second Stiefel-Whitney class and the first Pontrjagin class of the manifold. That is the first step towards a proof of the main theorem above.

After obtaining a complete invariant system of such manifolds, we need to calculate it for hypersurfaces. It is in general a difficult problem to compute the homology and cohomolgy of a singular hypersurface. Partial results on the cohomology groups with rational coefficients have been obtained by several authors by using the theory of rational differential forms and linear systems. If the singularities are of type A_k, then by applying a result of Dimca [D2] we have an estimation of the second integral homology group. The assumption in the result of [D2] then turns out here to be the condition $d < (k+5)/2$ in the corollary above. Furthermore, in the case when the second integral homology group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, by a standard diagram-chasing argument, we find out that the ring structure on cohomology is also determined by d and k. That is the background of the corollary.

In the third part of this dissertation, we construct concrete examples of hypersurfaces which fulfill the assumptions of the corollary. Precisely, we construct two families of cubic hypersurfaces in $\mathbb{C}\mathrm{P}^4$ with an A_5-singularity. Then by the corollary, they are diffeomorphic to each other. On the other hand, it is not clear whether one family can be deformed to the other.

In the appendix we consider the corresponding problem for 2-dimensional complete intersections. In this dimension a classification upto homeomorphism is possible. Vogel [Vo] gives a classification of simply-connected 4-manifolds with a fixed connected boundary. We will show that under certain conditions the smooth part of a 2-dimensional complete intersection with a unique A_k-singularity falls into this class. In this case the boundary is a 3-dimensional lens space $L^3(k+1, 1)$. We then show that all the invariants appearing in Vogel's classification can be calculated from k and the multidegree $\mathbf{d}$ of the complete intersection. Furthermore, by applying a theorem of Nikulin [Ni], we obtain the topological classification of certain complex surfaces.

Theorem. *Let $V \subset \mathbb{C}\mathrm{P}^3$ be a complex surface of degree d with a unique singularity which is of type A_k, under the assumption that k is even, d is even, $T(A_k) = \{0\}$ and*

$$\begin{cases} d(d^2 - 4d + 6) - 5 & \geq & k \\ d(2d^2 - 6d + 7) - 3 & > & 3k \end{cases}$$

the homeomorphism type of V is completely determined by (d, k).

For the definition of $T(A_k)$, see section 1.2 of the appendix.

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Chapter 1

Preliminaries

In this chaper we introduce the necessary preliminaries in this dissertation, including the definition of stratifolds, the topology of hypersurface singularities and complete intersections, and the modified surgery theory as a machinery for classifying manifolds..

1.1 Stratifolds

In this section we introduce the notion of stratifolds. Since this is a relatively new notion in topology, for the reason of completeness, we give the general definition of stratifolds here, although we will only need stratifolds with isolated singularities in this dissertation. All notions in this section come from [Kr1]. The reader can find more about stratifolds there.

Definition 1.1. *Let X be a topological space, and $\mathcal{C} \subset C^0(X)$ be a subalgebra of the algebra of continuous functions on X. For a subspace $Y \subset X$ we denote by $\mathcal{C}(Y) \subset C^0(Y)$ the subalgebra consisting of all $f \in C^0(Y)$ such that for all $y \in Y$, there is an open neighbourhood U of y and an element $g \in \mathcal{C}$ such that $g|U = f|U$. We say that $\mathcal{C}$ is* ***locally detectable*** *if a function $f \in C^0(X)$ is contained in $\mathcal{C}$ if and only if for all $x \in X$ there is an open neighbourhood V such that $f|V \in \mathcal{C}(V)$.*

Definition 1.2. *A* ***differential space*** *is a pair $(X, \mathcal{C})$ where $\mathcal{C}$ is a locally detectable subalgebra of $C^0(X)$ such that if $f_1, \ldots, f_k \in \mathcal{C}$ and $g \in C^\infty(\mathbb{R}^k)$, then $g(f_1, \ldots, f_k) \in \mathcal{C}$. A continuous map $f : X \to X'$ between two differential spaces $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ is called* ***smooth*** *if for all $g \in \mathcal{C}'$ we have*

$f \circ g \in \mathcal{C}$. A bijective map $f : X \to X'$ such that f and f^{-1} are smooth is called an ***isomorphism****.*

Definition 1.3. *Let $(X, \mathcal{C})$ be a differential space. For $x \in X$, we call $f, g \in \mathcal{C}$* ***equivalent*** *if there is an open neighbourhood U of x such that $f|U = g|U$. The equivalence classes are called* ***germs****. Denote $\mathcal{C}_x$ the algebra of germs at x.*

Definition 1.4. *A* ***derivation*** *of $\mathcal{C}_x$ is a map $\alpha : \mathcal{C}_x \to \mathbb{R}$ such that*

1. *α is an additive homomorphism;*

2. *α satisfies the Leibniz rule, i.e. $\alpha(fg) = \alpha(f)g(x) + f(x)\alpha(g)$.*

The derivations of $\mathcal{C}_x$ form a vector space over $\mathbb{R}$, denoted by $T_x(X, \mathcal{C})$, called the ***tangent space*** *of $(X, \mathcal{C})$ at x.*

Definition 1.5. *Let $(X, \mathcal{C})$ be a differential space,*

$$X^i = \{x \in X |\ \dim T_x(X, \mathcal{C}) = i\}$$

is called the i-th ***stratum*** *of X and $\Sigma^i = \bigcup_{r \leq i} X^r$ is called the i-th* ***skeleton*** *of X.*

Definition 1.6. *A k-dimensional* ***stratifold*** *is a differential space $(\mathscr{S}, \mathcal{C})$, where $\mathscr{S}$ is a locally compact, Hausdorff space with countable basis, the skeleta Σ^i's are closed subspaces, and for each $j > i$ we require that $\overline{\mathscr{S}^i} \cap \mathscr{S}^j = \emptyset$. In addition we assume*

1. *$(\mathscr{S}^i, \mathcal{C}(\mathscr{S}^i))$ is a smooth manifold of dimension i and for each $x \in \mathscr{S}^i$ the restriction gives an isomorphism $i^* : \mathcal{C}_x \to C^\infty(\mathscr{S}^i)_x$.*

2. *$\dim T_x \mathscr{S}^i \leq k$ for all $x \in \mathscr{S}$, i.e. all tangent spaces have dimension not bigger than k.*

3. *for each $x \in \mathscr{S}$ and an open neighbourhood U of x, there exits a function $\eta \in \mathcal{C}$ such that $\eta(y) \geq 0$ for all $y \in \mathscr{S}$, $\eta(x) \neq 0$ and supp$(\eta) \subset U$.*

Two stratifolds are called ***diffeomorphic*** *if they are isomorphic as differential spaces.*

Example. Let W be a connected n-dimensional smooth manifold with boundary, $M_1^{(1)}, \ldots, M_{i_1}^{(1)}, \ldots, M_1^{(k)}, \ldots, M_{i_k}^{(k)}$ be the boundary components. Let $\mathscr{S}$ be the space obtained by collapsing all $M_i^{(j)}$'s, $1 \leq i \leq i_j$, to a point p_j, then $\mathscr{S}$ is a stratifold with 0-stratum $\{p_1, \ldots, p_k\}$ and top-stratum $\mathring{W}$. Here the algebra $\mathcal{C}$ is taken as the subalgebra of $C^0(\mathscr{S})$ consisting of those functions which are smooth in the interior of W and locally constant in a neighbourhood of the p_i's. Such an $\mathscr{S}$ is called a **parametrized stratifold with isolated singularities**. We will see later that a complete intersection with isolated singularities belongs to this class of stratifolds. It is clear that two parametrized stratifolds with isolated singularities $\mathscr{S}$ and $\mathscr{S}'$ are diffeomorphic if and only if there is a diffeomorphism $f : W \to W'$ preserving the corresponding boundary components.

1.2 Topology of Singularities

In this section we will describe the Milnor fibration associated to a hypersurface singularity. In the case of an isolated singularity, the topology of the Milnor fiber, and of its boundary, is relatively simple. Especially, we will identify the link of an A_k-singularity on a 3-dimensional hypersurface in $\mathbb{C}^4$.

Let $f(z_1, \cdots, z_{n+1})$ be a non-constant polynomial of $(n+1)$ complex variables, and let V be the algebraic set consisting of all $(n+1)$-tuples of complex numbers $\mathbf{z} = (z_1, \cdots, z_{n+1})$ with $f(\mathbf{z}) = 0$. Such a set V is called an *affine complex hypersurface* in $\mathbb{C}^{n+1}$. A point $\mathbf{z}^0 \in V$ is called *regular* if some partial derivative $\partial f / \partial z_j$ does not vanish at $\mathbf{z}^0$. Otherwise $\mathbf{z}^0$ is called a *singularity* of V. In order to study the topology of V in a neighborhood of $\mathbf{z}^0$, a basic idea, due to Brauner [Br], is to look at the intersection of V with a small sphere S_ε in $\mathbb{C}^{n+1}$ centered at $\mathbf{z}^0$. Then the topology of V within the disk bounded by S_ε is closely related to the topology of the set $K = V \cap S_\varepsilon$. More precisely, let D_ε denote the closed ball consisting of all $\mathbf{z} \in \mathbb{C}^{n+1}$ with $\parallel \mathbf{z} - \mathbf{z}^0 \parallel \leq \varepsilon$, and let S_ε denote the boundary of D_ε, then we have

Theorem 1.7 ([Mi])**.** *If $\mathbf{z}^0$ is a regular point or an isolated singularity, then for small ε the intersection of V and D_ε is homeomorphic to the cone over $K = V \cap S_\varepsilon$. In fact the pair $(D_\varepsilon, V \cap D_\varepsilon)$ is homeomorphic to the pair consisting of the cone over S_ε and the cone over K.*

Definition 1.8. *According to this theorem, the topological space K is well-defined for ε small. It is called the* ***link*** *of* $\mathbf{z}^0$.

Remark. This theorem also holds for non-isolated singularities, which is referred in the literature as "the conic structure of analytic sets". This is due to Burghelea-Verona [B-V].

Remark. Because of the conic structure near an isolated singularity, it is clear that a complex hypersurface with isolated singularities admits a structure of parameterized stratifold with isolated singularities in a natural way.

Theorem 1.9 (Fibration Theorem, [Mi])**.** *If* $\mathbf{z}^0$ *is any point of the complex hypersurface* $V = f^{-1}(0)$ *and if* S_ε *is a sufficiently small sphere centered at* $\mathbf{z}^0$*, then the mapping*

$$\phi(\mathbf{z}) = f(\mathbf{z}) / \parallel f(\mathbf{z}) \parallel$$

from $S_\varepsilon - K$ *to the unit circle is the projection map of a smooth fiber bundle. Each fiber*

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\varepsilon - K$$

is a smooth $2n$*-dimensional manifold.*

Definition 1.10. *The fibration in theorem 1.9 is often called the* ***Milnor fibration****, and the manifold* F_θ *is called the* ***Milnor fiber****.*

Remark. For the proof of theorem 1.9, Milnor shows at first that the map ϕ is a submersion. As the fibers are not compact, one can not apply Ehresmann's fibration theorem directly. But one can make use of the same idea as in the proof of Ehresmann's fibration theorem. Namely, one constructs a vector field on $S_\varepsilon - K$ which is transverse to the fibers and whose integral curves exist for all $t \in \mathbb{R}$. Then the flow generated by this vector field gives the local trivialization.

An alternative description of the Milnor fiber F_θ which we will use as a local information in the study of projective hypersurfaces is the following: Let D_ε be a small closed disk centered at $\mathbf{z}^0$ as before and c be a sufficiently small complex constant, then

Theorem 1.11 ([Mi])**.** *The complex hypersurface* $f^{-1}(c)$ *intersects* $\mathring{D}_\varepsilon$ *in a smooth manifold which is diffeomorphic to the fiber* F_θ*.*

By applying Morse theory to the functions $\log|f| : S_\varepsilon - K \to \mathbb{R}$ and $\log|f| : F_\theta \to \mathbb{R}$ Milnor shows the following nice properties of the Milnor fibration.

Theorem 1.12 ([Mi])**.** *In the Milnor fibration, each fiber F_θ is parallelizable, and has the homotopy type of a finite CW-complex of dimension n. The space $K = V \cap S_\varepsilon$ is $(n-2)$-connected.*

For isolated singularities, we have an even better understanding of the topology of the fiber F_θ and of the link K.

Theorem 1.13 ([Mi])**.** *If $\mathbf{z}^0$ is an isolated singularity of V, then each fiber F_θ has the homotopy type of a bouquet of n-spheres $S^n \vee \cdots \vee S^n$, the number of spheres in this bouquet being strictly positive. Each fiber can be considered as the interior of a smooth compact manifold-with-boundary, $\overline{F}_\theta = F_\theta \cup K$, where the commen boundary K is an $(n-2)$-connected, $(2n-1)$-dimensional manifold.*

Definition 1.14. *The number μ of the spheres in the bouquet is called the **Milnor number** of the singularity.*

By applying the h-cobordism theorem, we get a sharper statement of the topology of the closed Milnor fiber $\overline{F}$.

Theorem 1.15 ([Mi])**.** *For $n \neq 2$, the manifold $\overline{F}_\theta$ is diffeomorphic to a handle body, obtained from the disk D^{2n} by simultaneously attaching a number of handles of index equal to n.*

Remark. In general, let $s = \dim(V_{\text{sing}}, \mathbf{z}^0)$ be the dimension of the singular locus of V near $\mathbf{z}^0$, then the Milnor fiber is $(n-s-1)$-connected. This general case is due to Kato-Matsumoto [K-M]. On the other hand, from theorem 1.12 we see that the connectivity of the link K is independent of how bad the singularity is.

Example. Consider the nondegenerate quadratic singularity at the origin of $\mathbb{C}^{n+1}$ defined by the polynomial

$$A_1 : \ f = z_0^2 + \cdots + z_n^2, \ n \geq 1.$$

This singularity is said to be of type A_1. The link K of an A_1-singularity is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of orthonormal 2-frames in $\mathbb{R}^{n+1}$. This can be seen easily by the following computation:

Since $K = V \cap S_\varepsilon$, it is thus defined by the equations

$$z_0^2 + \cdots + z_n^2 = 0, \ |z_0|^2 + \cdots + |z_n|^2 = \varepsilon^2.$$

Let $z_j = x_j + \sqrt{-1}y_j$, then we get the equations

$$\sum_{j=0}^{n} x_j^2 + \sum_{j=0}^{n} y_j^2 = \varepsilon^2, \ \sum_{j=0}^{n} x_j^2 = \sum_{j=0}^{n} y_j^2, \ \sum_{j=0}^{n} x_j y_j = 0.$$

These equations describe exactly the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{n+1}$.

Furthermore, since the Milnor fiber F is homotopy equivalent to S^n, it must be diffeomorphic to the total space of the tangent bundle of S^n.

For an isolated singularity, since the closed Milnor fiber $\overline{F}$ is an orientable $2n$-dimensional manifold, there is a $(-1)^n$-symmetric bilinear form

$$s : H_n(\overline{F}) \times H_n(\overline{F}) \to \mathbb{Z}, \ s(\alpha, \beta) = s'(j_*\alpha, \beta),$$

where j_* is the projection map in the homology exact sequence

$$H_n(\overline{F}) \xrightarrow{j_*} H_n(\overline{F}, K) \xrightarrow{\partial} H_{n-1}(K) \longrightarrow 0$$

for the pair $(\overline{F}, K)$, and $s' : H_n(\overline{F}, K) \times H_n(\overline{F}) \longrightarrow \mathbb{Z}$ is the intersection pairing of $\overline{F}$.

Definition 1.16. *The bilinear form s is called the **Milnor lattice** of the singularity.*

Remark. Note that the rank of the Milnor lattice is just the Milnor number of the singularity.

The Milnor lattice induces a natural homomorphism

$$H_n(\overline{F}) \to H_n(\overline{F})^*, \ x \mapsto s(x, -).$$

Under the identification $H_n(\overline{F})^* \cong H^n(\overline{F}) \cong H_n(\overline{F}, K)$, it is seen that this homomorphism is just the projection j_* in the homology exact sequence. Therefore $H_{n-1}(K)$ is isomorphic to the cokernel of the homomorphism

$$H_n(\overline{F}) \to H_n(\overline{F})^*.$$

In the rest part of this section we study the topology associated to a special class of hypersurface singularities, namely, the simple singularities of type A_k.

Consider the polynomial ($k \geq 1$)

$$A_k: \; f(z_1, \cdots z_{n+1}) = z_1^2 + \cdots + z_n^2 + z_{n+1}^{k+1}.$$

It is clear that the origin is an isolated singularity of the affine hypersurface defined by f. A singularity which is equivalent to this one is called a **simple singularity of type** A_k. Here two singularities are called equivalent if there is a local holomorphic co-ordinated change, transforming the defining polynomial of one singularity to that of the other one.

Remark. From the viewpoint of singularity theory, A_1-singularity is the only non-degenerate singularity in the complex world and for $k \geq 2$, A_k-singularities are exactly singularities of corank 1, which are the simplest ones after the non-degenerate singularity A_1 (c.f. [AGV1]).

When $n = 2$, we are in the case of surface singularities. For the following description of the *resolution graph* of a surface singularity, and of the graph associated to a Milnor lattice we refer to [D1, section 2.4]. The resolution graph of a surface singularity is the dual graph associated to a very good resolution of this singularity: its vertices correspond to the exceptional curves E_i (i.e. the preimage of the singularity under the resolution), and two vertices E_i and E_j are joined by an edge if and only if $E_i \cap E_j \neq \emptyset$. (Note that for a very good resolution if $i \neq j$ then the intersection of the exceptional curves E_i and E_j consists of at most one point.) For a surface A_k-singularity, its resolution graph coinsides with the Dykin diagram of the simple Lie algebra A_k. On the other hand, one can also associate a graph to the Milnor lattice s of the singularity. Namely, one chooses a basis of $H_2(\overline{F})$ which is represented by vanishing cycles $\Delta_1, \cdots \Delta_\mu$ (this basis is called a *distinguished basis*), then a vertex in this graph corresponds to a basis element Δ_i and two vertices Δ_i

and Δ_j are joined by l edges (resp. l dotted edges) if the value $s(\Delta_i, \Delta_j)$ is l (resp. $-l$.). (Note that the self-intersection number $s(\Delta_i, \Delta_i)$ of a vanishing cycle Δ_i is always -2.) A striking fact about the A_k-singularity is that the graph associated to the Milnor lattice coinsides with its resolution graph [D1, appenix A], therefore also coinsides with the Dykin diagram of the simple Lie algbra A_k. Thus under the distinguished basis, the Milnor lattice is represented by the matrix

$$\begin{pmatrix} -2 & 1 & 0 & \dots & \dots \\ 1 & -2 & 1 & \dots & \dots \\ 0 & 1 & -2 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \vdots & 1 & -2 \end{pmatrix}_{k\times k}$$

To understand the A_k-singularities in dimensions higher than 2, we need the notion of stabilization.

Definition 1.17. *Let $f(z_1, \cdots, z_{n+1})$ be a polynomial with the origin an isolated singularity. Consider the polynomial*

$$g(z_1, \cdots, z_{n+1}, y_1, \cdots, y_m) = f(z_1, \cdots, z_{n+1}) + y_1^2 + \cdots + y_m^2.$$

*Clearly, the origin is an isolated singularity of g. It is called the **stabilization** of the singularity defined by f.*

It is seen from the definition that an A_k-singularity of dimension higher than 2 is a stabilization of the surface A_k-singularity.

The behavior of the Milnor lattice of a singularity under stabilization has been studied and we have

Theorem 1.18 ([D1])**.** *There is a distinguished basis $\overline{\Delta}_1, \cdots, \overline{\Delta}_\mu$ of the stabilization of a singularity, which corresponds to a distinguished basis $\Delta_1, \cdots,$ Δ_μ of the original singularity such that*

$$\tilde{s}(\overline{\Delta}_i, \overline{\Delta}_j) = sign(j-i)^m(-1)^{(n+1)m+m(m-1)/2}s(\Delta_i, \Delta_j), \ \text{for } i \neq j.$$

where $\tilde{s}$ and s denotes the Milnor lattice of the stabilization and the original singularity respectively.

Remark. Stabilization is a special case of a more general construction which is called the Thom-Sebastiani construction. Given polynomials f and g of n, resp. m complex variables, both having the origin as an isolated singularity. Then the polynomial $f+g$ of $(n+m)$ variables has the origin as an isolated singularity. This is the so-called *Thom-Sebastiani construction*. It is shown in [AGV2] that the Milnor lattice of the Thom-Sebastiani construction can be expressed explicitely by the Milnor lattices of the two given singularities, and the above theorem is a special case of the general result.

Now we can identify the link of an A_k-singularity on a 3-dimensional hypersurface in $\mathbb{C}^4$.

Proposition 1.19. *The link of an A_k-singularity on a 3-dimensional hypersurface is diffeomorphic to $S^2 \times S^3$ for k odd, and S^5 for k even.*

Proof. Let K be the link of such a singularity and $\overline{F}$ be the closed Milnor fiber. Then from theorem 1.13, it is seen that K is a simply-connected 5-manifold. K is spin since it is the boundary of a parallelizable manifold $\overline{F}$. The second homology group $H_2(K)$ is isomorphic to the cokernel of the homomorphism $H_3(\overline{F}) \to H_3(\overline{F})^*$ determined by the Milnor lattice of the singularity. Now the Milnor lattice is a skew-symmetric bilinear form and according to theorem 1.18, under a distinguished basis, it is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ -1 & 0 & 1 & \dots & \dots \\ 0 & -1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \vdots & -1 & 0 \end{pmatrix}_{k\times k}$$

After a simple calculation it is seen that this bilinear form is equivalent to a bilinear form represented by the matrix

$$\bigoplus_{k/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for k even and

$$\bigoplus_{(k-1)/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bigoplus (\mathbb{Z}, ()_0)$$

for k odd, where $(\mathbb{Z}, ()_0)$ denotes the trivial form on $\mathbb{Z}$. Therefore $H_2(K)$ is trivial for k even and isomorphic to $\mathbb{Z}$ for k odd. Then by the classification of simply-connected 5-manifolds of Smale [Sm], we see that the link of an A_k-singularity is diffeomorphic to $S^2 \times S^3$ for k odd and S^5 for k even. ■

1.3 Topology of Complete Intersections

In this section we discuss the topology of complete intersections and show the famous Lefschetz theorem which gives a very strong restriction on the topology of complete intersections. And then we discuss shortly about the topology of smooth complete intersections.

Definition 1.20. *Let $f_1, \cdots, f_c$ be complex homogeneous polynomials of $(n+c+1)$ variables $z_0, \cdots, z_{n+c}$. A variety $V \subset \mathbb{C}\mathrm{P}^{n+c}$ defined by the equations $f_1 = \cdots = f_c = 0$ is called a* ***complete intersection*** *if the codimension of V is equal to the number of polynomials used to define it. In other words, we have*

$$\dim_{\mathbb{C}} V = n, \ \operatorname{codim}_{\mathbb{C}} V = c.$$

When $c = 1$, we set $f_1 = f$ and $V(f = 0) \subset \mathbb{C}\mathrm{P}^{n+1}$ is called a ***(projective) hypersurface****.*

Remark. For a complete intersection V, we can define the *affine cone* CV, which is the set of zeros of the same polynomials f_1, ..., f_c in $\mathbb{C}^{n+c+1}$. If V is smooth, then it is easy to see that the origin is an isolated singularity of CV and the link of this singularity is just the total space of the sphere bundle of the restriction of the Hopf bundle over $\mathbb{C}\mathrm{P}^{n+c}$ on V. Thus even though we are interested in smooth varieties, singularities come naturally into the study of them. This viewpoint was pioneered by Lefschetz [Le].

A fundamental and deep fact about the topological property of complete intersections is the following

Lefschetz Theorem. *Let $V \subset \mathbb{C}\mathrm{P}^{n+c}$ be an n-dimensional complete intersection. Then the inclusion $j : V \to \mathbb{C}\mathrm{P}^{n+c}$ is an n-equivalence.*

This theorem can be viewed as a consequence of the following theorem.

Theorem 1.21 (Zariski Theorem of Lefschetz Type)**.** *Let $X \subset \mathbb{C}\mathrm{P}^n$ be a complete intersection of dimension m, $H \subset \mathbb{C}\mathrm{P}^n$ be a hyperplane of dimension $n-1$. Then the inclusion $j : X \cap H \to X$ is an $(m-1)$-equivalence.*

Here we only cite a simpler version of Zariski theorem of Lefschetz type. For a general (stratified) version and a proof (using essentially Morse theory on manifolds with boundary), see for example [H1].

Proof of Lefschetz Theorem assuming Zariski theorem. In order to make use of Zariski Theorem of Lefschetz Type, we have to interpret a complete intersection as a hyperplane section. To do this, we use the Veronese embedding. Let $P(n,d)$ denote the vector space of all homogenous polynomials in $\mathbb{C}[x_0,\cdots,x_n]$ of degree d and let $D = \dim P(n,d)$. Let $p_0,\cdots,p_{D-1}$ be a basis of the vector space $P(n,d)$ and consider the *Veronese embedding*

$$v_d : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^{D-1},\ v_d([x]) = [p_0(x),\cdots,p_{D-1}(x)]$$

The image $V_{n,d} = v_d(\mathbb{C}\mathrm{P}^n)$ is called the *Veronese variety* (c.f. [La]). Any $f \in P(n,d)$ can be written as $f = \sum_{i=0}^{D-1} a_i p_i$. Consider the hyperplane $H_f \subset \mathbb{C}\mathrm{P}^{D-1}$ defined by the linear equation $\sum_{i=0}^{D-1} a_i y_i = 0$. Then it is easy to see that under the Veronese embedding, the image of the hypersurface $V(f = 0) \subset \mathbb{C}\mathrm{P}^n$ is equal to the intersection of $V_{n,d}$ and H_f. In other words, we have $V_{n,d} \cap H_f = v_d(V_f)$. Therefore V_f is a hyperplane section and according to theorem 1.3 the inclusion $V_f \to \mathbb{C}\mathrm{P}^n$ is an $(n-1)$-equivalence.

Now suppose we have c polynomials $f_1,\cdots,f_c$. From the above argument, it is seen that $V_{f_1} \to \mathbb{C}\mathrm{P}^{n+c}$ is an $(n+c-1)$-equivalence. Now consider the Veronese embedding associated to the second polynomial

$$v_{d_2} : \mathbb{C}\mathrm{P}^{n+c} \to \mathbb{C}\mathrm{P}^{D_{2-1}}.$$

Let V_{f_1,f_2} be the complete intersection defined by f_1 and f_2, then we have $v_{d_2}(V_{f_1,f_2}) = v_{d_2}(V_{f_1}) \cap H_{f_2}$. Therefore the inclusion $V_{f_1,f_2} \to V_{f_1}$ is an $(n+c-2)$-equivalence, and so is the inclusion $V_{f_1,f_2} \to \mathbb{C}\mathrm{P}^{n+c}$. Inductively we can prove that the inclusion $V = V_{f_1,\cdots,f_c} \to \mathbb{C}\mathrm{P}^{n+c}$ is an n-equivalence. ■

Remark. There is an alternative proof of the Lefschetz theorem which doesn't make use of the Zariski theorem, but depends on a deeper understanding of the link of a complete intersection singularity. As mentioned before, for a complete intersection V in $\mathbb{C}\mathrm{P}^{n+c}$, we can consider the affine cone CV over V, which is a variety in $\mathbb{C}^{n+c+1}$ defined by the same equations. Clearly, the origin is a singularity of CV. Let $K_V = S^{2(n+c)+1} \cap CV$ be the link, then there is a circle bundle $S^1 \to K_V \to V$ which is the restriction of the Hopf bundle over $\mathbb{C}\mathrm{P}^{n+c}$ to V. Then the proof is based on the following facts:

1. Analogous to theorem 1.12, it is shown in [H2] that the link K_V of a complete intersection singularity is $(n-2)$-connected as well. When $n \geq 2$, this implies that V is simply-connected.

2. By looking at the Gysin sequences of the circle bundles

$$\begin{array}{ccccc} S^1 & \longrightarrow & K_V & \longrightarrow & V \\ \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^{2(n+c)+1} & \longrightarrow & \mathbb{C}\mathrm{P}^{n+c} \end{array}$$

   one shows that the induced homomorphisms on homology

$$j_* : H_k(V) \to H_k(\mathbb{C}\mathrm{P}^{n+c})$$

   is an isomorphism for $k < n$ and an epimorphism for $k = n$. Then by Whitehead theorem the inclusion is an n-equivalence.

Next we discuss the basic topological properties of smooth complete intersections. If a complete intersection V^n in $\mathbb{C}\mathrm{P}^{n+c}$ is smooth, then the diffeomorphism type of the smooth manifold V, as well as that of the embedding $V^n \subset \mathbb{C}\mathrm{P}^{n+c}$, are completely determined by the unordered tuple $(d_1, \cdots, d_c)$, where $d_i = \deg f_i$ for $i = 1, \cdots, c$. We call $\mathbf{d} = (d_1, \cdots, d_c)$ the **multidegree**, $d = d_1 \cdots d_c$ the **total degree** and denote a complex n-dimensional smooth complete intersection of a given multidegree by $X^n(\mathbf{d})$. This fact was first noted by R. Thom and can be seen as follows (see e.g. [D1]).

In the case of smooth hypersurfaces ($c = 1$), consider the space

$$Z = \{(x, f) \in \mathbb{C}\mathrm{P}^n \times P(n, d) | f(x) = 0\}.$$

Here $P(n, d)$ is the same as in the proof of Lefschetz theorem. Z is a smooth manifold since the first projection $p_1 : Z \to \mathbb{C}\mathrm{P}^n$ is a vector bundle. Consider the second projection $p_2 : Z \to P(n, d)$. After a calculation it is seen that $f \in P(n, d)$ is a regular value of p_2 if and only if the corresponding hypersurface $V(f = 0)$ is smooth. From the viewpoint of the Veronese embedding, all critical points of p_2 form the affine cone over the dual of the Veronese variety, $C(\hat{V}_{n,d})$, in $P(n, d)$. The complement $P(n, d) - C(\hat{V}_{n,d})$ is connected and by applying Ehresmann's fibration theorem, we see that *the diffeomorphism type of the pair* $(\mathbb{C}\mathrm{P}^n, V(f = 0))$ *is constant for all smooth hypersurfaces of a given degree d.* The statement for complete intersections can be proved similarly.

For smooth complete intersections, their algebraic topological invariants can be expressed explicitly by the multidegree $\mathbf{d}$. First of all, since the inclusion $i : X^n(\mathbf{d}) \to \mathbb{C}\mathrm{P}^{n+c}$ is an n-equivalence, for $k < n$, $H_k(X^n(\mathbf{d}))$ is isomorphic to $\mathbb{Z}$ for k even, and zero for k odd. Then by Poincaré duality and the universal coefficient theorem, for $n < k < 2n$, $H_k(X^n(\mathbf{d}))$ is isomorphic to $\mathbb{Z}$ for k even, zero for k odd, and $H_n(X^n(\mathbf{d}))$ is torsion free, whose rank will be determined in the next paragraph. Furthermore, the image of the fundamental class $[X^n(\mathbf{d})]$ (note that $X^n(\mathbf{d})$ is a complex manifold and possesses a natural orientation) under i_* is d times the canonical generator of $H_{2n}(\mathbb{C}\mathrm{P}^{n+c})$. This can be seen by intersecting $X^n(\mathbf{d})$ with a hyperplane of codimension n and counting the intersecting points.

The normal bundle of the embedding $X^n(\mathbf{d})$ in $\mathbb{C}\mathrm{P}^{n+c}$ is the Whitney sum of the normal bundles of the smooth hypersurfaces $V(f_j = 0)$ restricted to $X^n(\mathbf{d})$. Therefore we have

$$TX^n(\mathbf{d}) \oplus i^*H^{\otimes d_1} \oplus \cdots \oplus i^*H^{\otimes d_c} = i^*T\mathbb{C}\mathrm{P}^{n+c},$$

where $H^{\otimes d_j}$ denotes the d_j-fold tensor product of the canonical line bundle H over $\mathbb{C}\mathrm{P}^{n+c}$. Therefore for the Chern classes we have

$$\begin{aligned} c(X^n(\mathbf{d})) &= i^*c(\mathbb{C}\mathrm{P}^{n+c})/\Pi\ i^*c(H^{\otimes d_j}) \\ &= i^*((1+x)^{n+c+1}/\Pi(1+d_jx)) \end{aligned}$$

where $x \in H^2(\mathbb{C}\mathrm{P}^{n+c})$ is the first Chern class of H. Thus we see that the k-th Chern class of $X^n(\mathbf{d})$ is a multiple of a^k, where $a = i^*x$ is a generator of $H^2(X^n(\mathbf{d}))$, and the coefficient is determined by the multidegree $\mathbf{d}$ explicitly. Especially, the Euler characteristic

$$\chi(X^n(\mathbf{d})) = \langle c_n(X^n(\mathbf{d})),\ [X^n(\mathbf{d})]\rangle$$

is determined by $\mathbf{d}$ and hence the n-th Betti number b_n. The same is true for the Pontrjagin classes, hence for n even, by the Hirzebruch index theorem, the signature of $X^n(\mathbf{d})$

$$\tau(X^n(\mathbf{d})) = \langle L_{\frac{n}{2}}(p(X^n(\mathbf{d}))),\ [X^n(\mathbf{d})]\rangle$$

is expressed by $\mathbf{d}$ explicitly.

For n even, another important algebraic topological invariant is the type of the intersection form on $H_n(X^n(\mathbf{d}))$. It is also determined by the multidegree. We just state the result here, for a calculation of the type we refer to [D1].

Proposition 1.22. *For a smooth complete intersection $X^{2k}(\mathbf{d})$ with multi-degree $\mathbf{d} = (d_1, \ldots, d_c)$, the intersection form on $H_{2k}(X^{2k}(\mathbf{d}))$ is even if and only if the integer*

$$\binom{k+s}{k}$$

is even, where $s = \sharp\{d_j | d_j \text{ is even}\}$.

1.4 Modified Surgery

Surgery theory is a powerful tool for classification problems of manifolds. In the classical case, one starts from a fixed normal homotopy type (X, ξ) and try to classify all smooth manifolds which are normal homotopy equivalent to (X, ξ) ([Brw], [W2]). In our situation, the classical surgery method doesn't apply since we have seen from the above sections that the homotopy type of a complete intersection is only known upto the middle dimension. In general the homotopy classification is still an unsolved problem. (For a summary of the knowledge about the homotopy classification of smooth complete intersections, see [AGMP].) In this section we introduce the modified surgery theory developed by Matthias Kreck as a method for classifying manifolds. All notions and results in this section come from [Kr2].

Definition 1.23. *Let M be an n-dimensional smooth manifold and*

$$p : B \to BO$$

be a fibration over BO.

1. *A **normal B-structure** on M is a lift $\bar{\nu}$ of the stable normal Gauss map $\nu : M \to BO$ to B.*
2. *A normal B-structure $\bar{\nu} : M \to B$ is a **normal k-smoothing**, if it is a $(k+1)$-equivalence.*
3. *We say that B is **k-universal** if the fiber of $p : B \to BO$ is connected and its homotopy groups vanish in dimension $\geq k+1$.*

For each manifold M, there exists a k-universal fibration B^k over BO such that the stable normal Gauss map $\nu : M \to BO$ lifts to a normal k-smoothing $\bar{\nu} : M \to B^k$. Furthermore, if B' is another k-universal fibration over BO admitting a normal k-smoothing of M, then the two fibrations B' and B^k are fiber homotopy equivalent. Therefore the fiber homotopy type of this fibration B^k over BO is a well-defined invariant of M and we call it the **normal k-type** of M.

In the modified surgery theory, instead of looking at the normal homotopy type, we consider a weaker information of a manifold M, namely, its normal k-type $B^k(M)$ and will try to classify manifolds with a given normal k-type. If k is larger than the dimension of M, then $B^k(M)$ is equivalent to the normal homotopy type of M, which is the starting point of the classical surgery theory.

There is a bordism relation on closed n-dimensional manifolds with normal B-structures. Two n-dimensional manifolds with normal B-structures $(M_0, \bar{\nu}_0)$ and $(M_1, \bar{\nu}_1)$ are said to be B-bordant if there exists a $(n+1)$-dimensional manifold with normal B-structure $(W, \bar{\nu})$ such that $\partial W = M_0 \cup (-M_1)$ and $\bar{\nu}|_{M_i} = \bar{\nu}_i$ for $i = 0,\ 1$. The corresponding bordism group is denoted by $\Omega_n(B; p)$.

As in the classical situation, given a B-bordism $(W, \bar{\nu})$ between $(M_0, \bar{\nu}_0)$ and $(M_1, \bar{\nu}_1)$, we want to do surgery on $(W, \bar{\nu})$ to make it into an s-cobordism. When $k \geq [n/2] - 1$, the only obstruction for this is an algebraic object $\theta(W, \bar{\nu})$, lying in a monoid depending on the fundamental group $\pi_1(B)$ and the orientation character $w_1(B)$. When $k > [n/2] - 1$, the obstruction is in a subgroup of the monoid which is isomorphic to Wall's L-group and we return to the classical situation. In the extreme case $k = [n/2] - 1$, the obstruction may be complicated, even if B is simply-connected. But in many cases, theorem 5 in [Kr2] makes us to be able to avoid analysing this complicated object. The following theorem, which is a special case of corollary 4 of [Kr2], will be the main machinery for the classification of certain 6-manifolds in the next chapter.

Theorem 1.24. *Let M_0 and M_1 be connected $(4q+2)$-dimensional simply-connected manifolds ($q \geq 1$) with the same Euler characteristic and let $f : \partial M_0 \to \partial M_1$ be a diffeomorphism. Suppose that there are normal $2q$-smoothings of M_i in a fibration B over BO compatible with f. Then f extends to a diffeomorphism between M_0 and M_1 compatible with the normal B-structures if and only if $M_0 \cup_f M_1$ is zero bordant in $\Omega_n(B; p)$.*

Chapter 2

Classification

In this chaper we study the topology of 3-dimensional complex hypersurfaces with a unique singularity and prove the main results stated in the introduction. In section 2.1 we study the topology of the non-singular part of a singular hypersurface and compute the topological invariants of it, such as the Euler characteristic, the first Pontrjagin class, and the homology and cohomology groups. We will see that under certain assumptions all these invariants are determined by the degree of the hypersurface and the type of the singularity. Then in the next section we find out a complete invariants system of diffeomorphism type of a certain class of 6-dimensional manifolds, into which the non-singular part of a hypersurface falls. In section 2.3 the main results of this dissertation will be proved using the classification in section 2.2 and the calculation in section 2.1

2.1 Topology of the Smooth Part

Let $V \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface of degree d, with a unique singularity p. Because of the conic structure of the hypersurface near the singularity p (see theorem 1.7), there exits a small open ball $D_\varepsilon \subset \mathbb{C}\mathrm{P}^4$ centered at p, such that $\overline{D}_\varepsilon \cap V$ is homeomorphic to the cone $C(\partial \overline{D}_\varepsilon \cap V)$, where $\partial \overline{D}_\varepsilon \cap V = K$ is the link of the singularity p. Let $M = V - D_\varepsilon$, then M is a smooth manifold with boundary K, the interior of M is diffeomorphic to the nonsingular part $V - \{p\}$, and V is homeomorphic to $M/\partial M$. We call M the **smooth part** of V. It is a 6-dimensional smooth manifold with boundary and has the following properties.

Lemma 2.1. *Let M be the smooth part of V, then the Euler characteristic $\chi(M)$ and the first Pontrjagin class $p_1(M)$ are determined by the degree d and the Milnor number μ_p of the singularity p. M is spin if and only if d is odd.*

Proof. At first let's consider a smooth hypersurface $i : V_0 \subset \mathbb{C}\mathrm{P}^4$ of degree d. Let $\nu(i)$ be the normal bundle of the embedding, then

$$TV_0 \oplus \nu(i) = i^* T\mathbb{C}\mathrm{P}^4.$$

It is known that $\nu(i) = i^*(H^{\otimes d})$, where H is the canonical complex line bundle over $\mathbb{C}\mathrm{P}^4$. Thus

$$c(V_0) = i^* c(T\mathbb{C}\mathrm{P}^4)/i^* c(H^{\otimes d}) = i^*((1+x)^5/(1+dx)),$$

where $x = c_1(H)$. Therefore

$$\chi(V_0) = \langle c_3(TV_0), [V_0] \rangle \text{ and } p_1(V_0) = -c_2(TV_0 \oplus \overline{TV}_0)$$

are determined by d, where $\overline{TV}_0$ is the conjugate bundle of TV_0. Since $c_1(V_0) = (5-d)i^*(x)$ and $c_1(V_0) \equiv w_2(V_0) \pmod 2$, V_0 is spin if and only if d is odd.

Now let V_0 be a small deformation of V such that V_0 is smooth. Then according to theorem 1.11, $\overline{D}_\varepsilon \cap V_0$ can be identified with the closed Milnor fiber of the singularity p and M is diffeomorphic to $V_0 - D_\varepsilon$ (c.f. [D1, page 163]). Then $\chi(M) = \chi(V_0) - \chi(V_0 \cap \overline{D}_\varepsilon)$. Since the Milnor fiber is homotopy equivalent to a wedge of μ copies of 3-spheres, the Euler characteristic of $V_0 \cap \overline{D}_\varepsilon$ is $1 - \mu_p$. Therefore $\chi(M)$ is determined by d and μ_p.

If we identify M and $V_0 - D_\varepsilon$, then the inclusion $j : M \to V_0$ induces an isomorphism

$$j^* : H^4(V_0) \to H^4(M),$$

therefore $p_1(M) = j^* p_1(V_0) \in H^4(M) \cong \mathbb{Z}$ is determined by d.

If d is odd, then V_0 is spin and therefore M is spin. If d is even, V_0 is nonspin. We show that M is nonspin: if M is spin, since $\overline{D}_\varepsilon \cap V_0$ is also spin and there is a unique spin structure on ∂M (∂M is simply-connected), the spin structures on M and $\overline{D}_\varepsilon \cap V_0$ fit together to give rise to a spin structure on V_0, which is a contradiction. Therefore M is nonspin. ∎

Next we will study the homology and cohomology of the smooth part M. By Lefschetz theorem, V is simply-connected and $H_2(V)$ is isomorphic to $\mathbb{Z}$, therefore by Van-Kampen theorem M is simply-connected and there is an exact sequence

$$H_2(\partial M) \xrightarrow{i_*} H_2(M) \longrightarrow H_2(M, \partial M) \longrightarrow 0.$$

Since $H_2(M, \partial M) \cong H_2(V) \cong \mathbb{Z}$, it is seen that $H_2(M)$ is isomorphic to $\mathbb{Z} \oplus \mathrm{Im} i_*$. **From now on we assume that Im$i_* \cong \mathbb{Z}$.** Later on in lemma 2.3 we will show that if p is an A_{2k+1}-singularity and $d < (k+5)/2$, then this assumption is fulfilled.

Under this assumption let us consider the cohomology of M. Let $V^* = V - \{p\}$ be the nonsingular part and $U = \mathbb{C}\mathrm{P}^4 - V$ be the complement. Since the embedding $i : V^* \hookrightarrow \mathbb{C}\mathrm{P}^4 - \{p\}$ is proper, we have a Gysin sequence (c.f. [Do, page 314, 321]):

$$\cdots \to H^k(\mathbb{C}\mathrm{P}^4 - \{p\}) \xrightarrow{j^*} H^k(U) \xrightarrow{R} H^{k-1}(V^*) \xrightarrow{\delta} H^{k+1}(\mathbb{C}\mathrm{P}^4 - \{p\}) \to \cdots,$$

where $j : U \subset \mathbb{C}\mathrm{P}^4 - \{p\}$ denotes the inclusion, and the homomorphism R is the so-called Poincaré-Lerray residue ([D1]). For $k = 3$, we have a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^3(U) & \xrightarrow{R} & H^2(V^*) & \xrightarrow{\delta} & H^4(\mathbb{C}\mathrm{P}^4 - \{p\}) \cong \mathbb{Z} \\
 & & & & & \nwarrow^{i^*} & \uparrow \\
 & & & & & & H^2(\mathbb{C}\mathrm{P}^4 - \{p\}) \cong \mathbb{Z}
\end{array}$$

The composition $\delta \circ i^* : H^2(\mathbb{C}\mathrm{P}^4 - \{p\}) \to H^4(\mathbb{C}\mathrm{P}^4 - \{p\})$ is a multiplication by d, and by the assumption above, $H^2(V^*) \cong H^2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$, therefore $H^3(U)$ is isomorphic to $\mathbb{Z}$. Let $u \in H^3(U)$ be a generator. We have the following

Lemma 2.2. *For any $y \in H^2(V^*)$, the cup product $y \cup R(u)$ is 0.*

Proof. Let T be a tubular neighbourhood of V^* in $\mathbb{C}\mathrm{P}^4 - \{p\}$, $T_0 = T - V^*$ be the complement of the zero section and $j_0 : T_0 \to T$ be the inclusion. Since

$H^2(V^*)$ and $H^4(V^*)$ are torsion free, it suffices to compute $y \cup R(u)$ with $\mathbb{Q}$-coefficients. We consider the associated Gysin sequence with $\mathbb{Q}$-coefficients:

$$\cdots \longrightarrow H^k(T_0) \xrightarrow{R_0} H^{k-1}(T) \longrightarrow H^{k+1}(T) \xrightarrow{j_0^*} H^{k+1}(T_0) \longrightarrow \cdots,$$

where R_0 is the composition

$$H^k(T_0) \xrightarrow{\delta} H^{k+1}(T, T_0) \xrightarrow{\varphi} H^{k-1}(T),$$

where φ is the Thom isomorphism.

For $k = 3$, the map $H^2(V^*) \cong H^2(T) \to H^4(T) \cong H^4(V^*)$ is just the cup product with the Euler class of the normal bundle, hence is surjective.

Let $b \in H^1(T_0) \cong \mathbb{Z}$ be a generator, then for any $x \in H^3(T_0)$, we have

$$\begin{aligned} R_0((j_0^* R_0(x)) \cup b) &= \varphi \circ \delta((j_0^* R_0(x)) \cup b) \\ &= \varphi(R_0(x) \cup \delta(b)) \\ &= R_0(x) \cup R_0(b) = R_0(x), \end{aligned}$$

since δ is an $H^*(T_0)$-mod map and the Thom isomorphism φ is an $H^*(T)$-mod map.

Thus $j_0^* R_0(x) \cup b - x = j_0^*(z)$ for some $z \in H^3(T)$. For any $y \in H^2(T)$, we claim that $y \cup R_0(x) = 0$. This follows from the following calculation:

$$\begin{aligned} y \cup R_0(x) &= \varphi(y \cup \delta(x)) \\ &= \varphi\delta(j_0^*(y) \cup x) \\ &= R_0(j_0^*(y) \cup x) \\ &= R_0(j_0^*(y) \cup (j_0^* R_0(x) \cup b - j_0^*(z))) \\ &= R_0(j_0^*(y \cup R_0(x)) \cup b - j_0^*(y \cup z)). \end{aligned}$$

Note that $j_0^*(y \cup R_0(x)) = 0$ since $j_0^* : H^4(T) \to H^4(T_0)$ is a trivial map and $y \cup z = 0$ since $H^5(T) = 0$. Therefore $y \cup R_0(x) = 0$.

The statement of this lemma is then proved by the commutative diagram

$$\begin{array}{ccc} H^3(U) & \xrightarrow{R} & H^2(V^*) \\ \downarrow & & \uparrow \cong \\ H^3(T_0) & \xrightarrow{R_0} & H^2(T) \end{array}$$

where the homomorphism $H^3(U) \to H^3(T_0)$ is induced by the inclusion $T_0 \subset U$. ∎

Lemma 2.3. *Let $V \subset \mathbb{CP}^4$ be a hypersurface of degree d with a unique singularity p of type A_{2k+1}, M be the smooth part. Then if $d < (k+5)/2$, $H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$; if $d \geq (k+5)/2$, the rank of $H_2(M)$ is 1.*

Proof. The link of an A_{2k+1}-singularity is diffeomorphic to $S^2 \times S^3$. Thus in the exact sequence

$$H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to 0$$

$H_2(\partial M)$ and $H_2(M, \partial M)$ are both isomorphic to $\mathbb{Z}$. To prove $H_2(M)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, it suffices to show that $\dim H_2(M; \mathbb{C}) = 2$. For this we apply a result of A. Dimca:

Proposition 2.4 ([D2], Proposition 3.4)**.** *Let $V \subset \mathbb{P}(w_0, \dots, w_{2m})$ be a hypersurface of degree N such that the set Σ' of essential singularities satisfies:*

- *(i) Σ' is contained in the hyperplane $H_0 : x_0 = 0$.*
- *(ii) any transversal singularity (Y_i, a_i) corresponding to a point $a_i \in \Sigma'$ is of type A_{2k+1} for some k and $(Y_i \cap H_0, a_i)$ is an A_1-singularity in (H_0, a_i).*

Let $\Sigma_k = \{a_i \in \Sigma' | (Y_i, a_i)$ is of type $A_{2k+1}\}$ and for any k with $\Sigma_k \neq \emptyset$ consider the linear system

$$\mathscr{S}_k = \{h \in \bar{S}_{mN-w-kw_0} | h(\Sigma_k) = 0\}.$$

Then the only possible nonzero Hodge number of $H_0^{2m}(V)$ is given by

$$h^{m,m}(H_0^{2m}(V)) = \sum_{k, \Sigma_k \neq \emptyset} \operatorname{def}(\mathscr{S}_k).$$

Here $\mathbb{P}(w_0, \dots, w_{2m})$ is a weighted projective space, in our situation we have $m = 2$ and $w_0 = \cdots = w_4 = 1$, $w = w_0 + \cdots + w_4 = 5$. $H_0^{2m}(V)$ is the primitive cohomology group of V, which is defined as the cokernel of the map $H^{2m}(\mathbb{P}(w_0, \dots, w_{2m}); \mathbb{C}) \to H^{2m}(V; \mathbb{C})$. $\bar{S}$ denotes the polynomial ring $\mathbb{C}[x_1, \cdots, x_{2m}]$ graded by $\deg(x_i) = w_i$, and $\bar{S}_m$ is the homogeneous part of degree m. $\mathscr{S}_k$ is a subspace of the vector space $\bar{S}_{mN-w-kw_0}$. The defect $\operatorname{def}(\mathscr{S}_k)$ of the linear system $\mathscr{S}_k$ is defined as $\sharp\Sigma_k - \operatorname{codim}\mathscr{S}_k$.

Since the local equation of an A_{2k+1}-singularity is

$$y_1^2 + y_2^2 + y_3^2 + y_4^{2k+2} = 0,$$

there exists a hyperplane H in $\mathbb{C}\mathrm{P}^4$, intersecting V transversely, such that $p \in H$ and $(V \cap H, p)$ is a singularity of type A_1. Without lose of generality, we may suppose H is defined by the equation $x_0 = 0$. Then the linear system is

$$\mathscr{S}_k = \{h \in \bar{S}_{2d-k-5} |\ h(p) = 0\}.$$

Since $H^j(\mathbb{C}\mathrm{P}^{2m}; \mathbb{C}) \to H^j(V; \mathbb{C})$ is a monomorphism for all $j \leq 2(m-1)$, $\dim H^4(V; \mathbb{C}) = 1 + \dim H^4_0(V)$. Then according to this proposition

$$\dim H_2(M; \mathbb{C}) = \dim H^4(V; \mathbb{C}) = 1 + \mathrm{def}(\mathscr{S}_k) = 1 + 1 - \mathrm{codim}\mathscr{S}_k.$$

It is clear that $\mathrm{codim}\mathscr{S}_k = 0$ if $d < (k+5)/2$ and $\mathrm{codim}\mathscr{S}_k = 1$ if $d \geq (k+5)/2$. ■

2.2 Classification of Certain 6-Manifolds

As we have seen in last section, the smooth part of a 3-dimensional hypersurface with a unique A_{2k+1}-singularity is a simply-connected smooth 6-manifold with boundary diffeomorphic to $S^2 \times S^3$, and fulfilling certain conditions on cohomology and characteristic classes. This leads us to consider the following classification problem.

Let M^6 be a 6-dimensional oriented smooth manifold fulfilling the following conditions (A):

1. M is simply-connected;
2. the boundary of M is diffeomorphic to $S^2 \times S^3$;
3. $H_2(M)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, $H_2(M, \partial M)$ is isomorphic to $\mathbb{Z}$ and the trilinear form

$$\begin{array}{ccc} H^2(M, \partial M) \times H^2(M, \partial M) \times H^2(M, \partial M) & \longrightarrow & \mathbb{Z} \\ (x, y, z) & \mapsto & \langle x \cup y \cup z, [M, \partial M] \rangle \end{array}$$

   is nontrivial.

Our goal is to classify such manifolds upto orientation-preserving diffeomorphisms.

Let us consider the invariants of such manifolds. First of all, we have a short exact sequence

$$0 \to H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to 0.$$

Secondly, we have characteristic classes, namely, the Euler characteristic $\chi(M) \in \mathbb{Z}$, the first Pontrjagin class $p_1(M) \in H^4(M)$, and the second Stiefel-Whitney class $w_2(M)$. A generator x of $H_2(M, \partial M)$ is called *preferred* if the Kronecker dual of x, $x^* \in H^2(M, \partial M)$, satisfies that $\langle x^* \cup x^* \cup x^*, [M, \partial M]\rangle$ is positive. Clearly the preferred generator of $H_2(M, \partial M)$ is uniquely determined by this property and depends on the orientation of M. A bilinear form which reflects the cohomology multiplication of $H^*(M)$ can be defined as:

$$\begin{aligned} q : H^2(M) \times H^2(M) &\longrightarrow \mathbb{Z} \\ (u, v) &\mapsto \langle u \cup v \cup x^*, [M, \partial M]\rangle. \end{aligned}$$

Now we can formulate the classification of such manifolds upto orientation-preserving diffeomorphisms via these invariants.

Theorem 2.5. *The Euler characteristic $\chi(M)$, the Poincaré dual of the first Pontrjagin class $Dp_1(M)$, the second Stiefel-Whitney class $w_2(M)$, the preferred generator $x \in H_2(M, \partial M)$, the short exact sequence*

$$0 \to H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to 0$$

and the bilinear form $q : H^2(M) \times H^2(M) \to \mathbb{Z}$ form a complete system of invariants of oriented diffeomorphism type of M; i.e., there is an orientation-preserving diffeomorphism between M_0 and M_1 if and only if $\chi(M_0) = \chi(M_1)$ and there exists an isomorphism Φ between the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\partial M_0) & \longrightarrow & H_2(M_0) & \longrightarrow & H_2(M, \partial M_0) & \longrightarrow & 0 \\ & & \Phi \downarrow & & \Phi \downarrow & & \Phi \downarrow & & \\ 0 & \longrightarrow & H_2(\partial M_1) & \longrightarrow & H_2(M_1) & \longrightarrow & H_2(M, \partial M_1) & \longrightarrow & 0 \end{array}$$

s.t. $\Phi(Dp_1(M_0)) = Dp_1(M_1)$, $\Phi(x_0) = x_1$, and the dual of Φ, $\Phi^* : H^2(M_1) \to H^2(M_0)$, is an isometry between the bilinear forms on $H^2(M_0)$ and $H^2(M_1)$, and $\Phi^* w_2(M_1) = w_2(M_0)$.

We will prove this theorem by the modified surgery theory introduced in section 1.4. This theory converts the classification problem of manifolds into the problems of determining some bordism classes and certain obstruction. We will first prove the theorem for spin manifolds, and then show that with some trivial modifications, the proof is also valid for nonspin manifolds.

2.2.1 spin case

Let M be an oriented spin manifold, fulfilling the conditions (A) given at the beginning of this section. In order to apply the modified surgery theory, we first need to determine the normal 2-type to M. Consider the fiberation

$$\xi : B = (\mathbb{C}\mathrm{P}^\infty)^2 \times B\mathrm{Spin} \xrightarrow{\eta \times p} BO \times BO \xrightarrow{\oplus} BO,$$

where $p : B\mathrm{Spin} \to BO$ is the canonical projection, $\eta : (\mathbb{C}\mathrm{P}^\infty)^2 \to BO$ is the classifying map of a trivial complex line bundle over $(\mathbb{C}\mathrm{P}^\infty)^2$, and $\oplus$ is the H-space structure of BO given by the Whitney sum of the universal vector bundles. Because M is a simply-connected spin manifold, there is a unique classifying map $M \to B\mathrm{Spin}$ of the spin structure on the stable normal bundle νM. By choosing an isomorphism $H_2(M) \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}$ we get a map $M \to K(\mathbb{Z} \oplus \mathbb{Z}, 2) = (\mathbb{C}\mathrm{P}^\infty)^2$, which induces the given isomorphism. Put these two maps together, we get a map

$$M \xrightarrow{\bar{\nu}} B\mathrm{Spin} \times (\mathbb{C}\mathrm{P}^\infty)^2 = B,$$

which is clearly a lift of the normal Gauss map $\nu : M \to BO$:

$$\begin{array}{ccc} & & B \\ & \overset{\bar{\nu}}{\nearrow} & \Big\downarrow \xi \\ M & \xrightarrow{\quad \nu \quad} & BO \end{array}$$

Since $\pi_2(B\mathrm{Spin}) = \pi_3(B\mathrm{Spin}) = 0$, $\pi_3((\mathbb{C}\mathrm{P}^\infty)^2) = 0$, we conclude that (B, ξ) is the normal 2-type of M and $\bar{\nu}$ is a normal 2-smoothing.

Now Let M_i (i=0, 1) be as above, with the same Euler characteristic, $\bar{\nu}_i : M_i \to B$ be the normal 2-smoothing of M_i defined as above and

$$f : \partial M_0 \to \partial M_1$$

be an orientation-preserving diffeomorphism compatible with the normal 2-smoothings. Let $N_f = M_0 \cup_f (-M_1)$, then $\bar{\nu}_0$ and $\bar{\nu}_1$ fit together to give a map

$$\bar{\nu}_f = \bar{\nu}_0 \cup \bar{\nu}_1 : N_f \to B\mathrm{Spin} \times (\mathbb{C}\mathrm{P}^\infty)^2.$$

According to theorem 1.24 we have

Lemma 2.6. *f extends to an orientation-preserving diffeomorphism*

$$F : M_0 \to M_1$$

compatible with the normal structures if and only if

$$[N_f, \bar{\nu}_f] = 0 \in \Omega_6(B;\xi).$$

Since η is the classifying map of a trivial bundle, the bordism group $\Omega_6(B;\xi)$ is just the spin bordism group $\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$, which is defined as the bordism group of maps from closed spin 6-manifolds to $(\mathbb{C}\mathrm{P}^\infty)^2$.

Therefore, due to the above lemma, in order to prove theorem 2.5, it suffices to show that there exists an orientation-preserving diffeomorphism $f : \partial M_0 \to \partial M_1$, s.t. $(N_f, \bar{\nu}_f)$ is null-bordant in $\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. We compute the bordism invariants of this bordism group at first.

Lemma 2.7. *There is an injective homomorphism*

$$\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2) \longrightarrow H_2((\mathbb{C}\mathrm{P}^\infty)^2) \oplus H_6((\mathbb{C}\mathrm{P}^\infty)^2).$$

$$[Y,h] \quad \mapsto \quad (h_* Dp_1(Y), h_*[Y])$$

Proof. From the naturality of p_1 it is easy to see that the map is a well-defined homomorphism. Now consider the Atiyah-Hirzebruch spectral sequence for $\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. We have $E^2_{p,q} = H_p((\mathbb{C}\mathrm{P}^\infty)^2; \Omega_q^{\mathrm{spin}})$. Since $E^2_{2,4}$, $E^2_{4,2}$ and $E^2_{6,0}$ are the only non-zero terms on the segment $p+q=6$, so are $E^\infty_{2,4}$, $E^\infty_{4,2}$ and $E^\infty_{6,0}$. Then we have two exact sequences:

$$0 \to G \to \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2) \longrightarrow E^\infty_{6,0} \subset H_6((\mathbb{C}\mathrm{P}^\infty)^2)$$

$$[Y,h] \quad \mapsto \quad h_*[M]$$

and

$$0 \to E^\infty_{2,4} \to G \to E^\infty_{4,2} \to 0.$$

The proof of the lemma is based on the following calculations:

1. $E^{\infty}_{2,4} = E^2_{2,4} = H_2((\mathbb{C}\mathrm{P}^{\infty})^2; \Omega_4^{\mathrm{spin}})$, since $E^4_{6,1}$ is a finite group and thus $d_4 : E^4_{6,1} \to E^4_{2,4}$ is trivial.

2. The differential

$$d_2 : E^2_{6,1} = H_6((\mathbb{C}\mathrm{P}^{\infty})^2; \mathbb{Z}/2) \to E^2_{4,2} = H_4((\mathbb{C}\mathrm{P}^{\infty})^2; \mathbb{Z}/2)$$

is dual to the Steenrod square

$$\mathrm{Sq}^2 : H^4((\mathbb{C}\mathrm{P}^{\infty})^2; \mathbb{Z}/2) \to H^6((\mathbb{C}\mathrm{P}^{\infty})^2; \mathbb{Z}/2).$$

Therefore $E^{\infty}_{4,2} = E^3_{4,2} = \mathrm{Coker} d_2 \cong (\mathbb{Z}/2)^2$.

3. The generators of $E^{\infty}_{2,4} = H_2(\mathbb{C}\mathrm{P}^{\infty})^2) \otimes \Omega_4^{\mathrm{spin}} \cong \mathbb{Z} \oplus \mathbb{Z}$ are given as follows: let K be the Kummer surface, which is a generator of Ω_4^{spin}. For $j = 1, 2$, consider

$$\phi_j : K \times \mathbb{C}\mathrm{P}^1 \xrightarrow{pr_2} \mathbb{C}\mathrm{P}^1 \xrightarrow{i} \mathbb{C}\mathrm{P}^{\infty}_j \longrightarrow (\mathbb{C}\mathrm{P}^{\infty})^2,$$

then $E^{\infty}_{2,4}$ is generated by $[K \times \mathbb{C}\mathrm{P}^1, \phi_j]$. Let x_j be the canonical generator of $H^2(\mathbb{C}\mathrm{P}^{\infty}_j)$, then the bordism number

$$\langle \phi_j^*(x_j) \cup p_1(K \times \mathbb{C}\mathrm{P}^1), [K \times \mathbb{C}\mathrm{P}^1] \rangle = \langle p_1(K), [K] \rangle = 48.$$

4. The sequence $0 \to E^{\infty}_{2,4} \to G \to E^{\infty}_{4,2} \to 0$ is nonsplitting. This can be seen as follows: let $V(3)$ be a smooth hypersurface in $\mathbb{C}\mathrm{P}^4$ of degree 3, then $V(3)$ is spin. For $j = 1, 2$, consider

$$g_j : N_j = 3\mathbb{C}\mathrm{P}^3 - V(3) \xrightarrow{i} \mathbb{C}\mathrm{P}^{\infty}_j \longrightarrow (\mathbb{C}\mathrm{P}^{\infty})^2.$$

It is seen that $[N_j, g_j] \in G$ and the bordism number

$$\langle g_j^*(x_j) \cup p_1(N_j), [N_j] \rangle = 24.$$

Therefore the above sequence is nonsplitting.

■

Now fix an orientation of $S^2 \times S^3$ and choose an orientation-reversing diffeomorphism $\varphi : \partial M_0 \to S^2 \times S^3$, then the map

$$S^2 \times S^3 \xrightarrow{\varphi^{-1}} \partial M_0 \subset M_0 \xrightarrow{\bar{\nu}_0} B \xrightarrow{\mathrm{pr}_1} (\mathbb{C}\mathrm{P}^\infty)^2$$

extends to a map $S^2 \times D^4 \to (\mathbb{C}\mathrm{P}^\infty)^2$ uniquely upto homotopy relative to the boundary. Let $Y_0 = M_0 \cup_\varphi (S^2 \times D^4)$ and

$$h_0 : Y_0 = M_0 \cup_\varphi (S^2 \times D^4) \to (\mathbb{C}\mathrm{P}^\infty)^2$$

be the union of the corresponding maps on M_0 and $S^2 \times D^4$. Since there is a unique spin structure on $S^2 \times D^4$, we get an element $[Y_0, h_0]$ in $\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. (Here we choose the orientation of $S^2 \times D^4$ so that it induces the fixed orientation of $S^2 \times S^3$.) Do the same construction for the orientation-reversing diffeomorphism $\partial M_1 \xrightarrow{f^{-1}} \partial M_0 \xrightarrow{\varphi} S^2 \times S^3$ we obtain

$$Y_1 := M_1 \cup_{\varphi \circ f^{-1}} S^2 \times D^4 \xrightarrow{h_1} (\mathbb{C}\mathrm{P}^\infty)^2$$

and $[Y_1, h_1] \in \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. It is clear from the construction that

$$[N_f, \bar{\nu}_f] = [Y_0, h_0] - [Y_1, h_1] \in \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2).$$

Now we study the bordism class $[Y_i, h_i]$. (For simplicity, we omit the subscription i in the following discussion.) Y is an oriented simply-connected 6-manifold with $H_2(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the map $h : Y \to (\mathbb{C}\mathrm{P}^\infty)^2$ induces an isomorphism on H_2. According to [W1], the diffeomorphism type of Y is determined by the Euler characteristic $\chi(Y)$, (here $\chi(Y) = \chi(M) + 2$), the Poincaré dual of the first Pontrjagin class $Dp_1(Y)$ and the trilinear form

$$\begin{array}{ccc} H^2(Y) \times H^2(Y) \times H^2(Y) & \xrightarrow{\quad\mu\quad} & \mathbb{Z}. \\ (a, b, c) & \mapsto & \langle a \cup b \cup c, [Y] \rangle \end{array}$$

It is seen from the Mayer-Vietoris sequence that the inclusion $j : M \to Y$ induces an isomorphism on H_2. We identify $H_2(M)$ and $H_2(Y)$ using this isomorphism. Choose a basis of $H_2(M)$, $\{e_1, e_2\}$, s.t. e_1 is the image of a generator of $H_2(\partial M)$ under the inclusion $H_2(\partial M) \to H_2(M)$, and e_2 maps to the preferred generator of $H_2(M, \partial M)$ under the projection $H_2(M) \to H_2(M, \partial M)$:

$$\begin{array}{ccccc} H_2(\partial M) & \longrightarrow & H_2(M) & \longrightarrow & H_2(M, \partial M) \\ 1 & \mapsto & e_1,\ e_2 & \mapsto & x \end{array}$$

Under this basis, the invariants of Y can be expressed as follows $(\star)$:

- $\chi(Y) = \chi(M) + 2$
- $Dp_1(Y) = p \cdot e_1 + Dp_1(M) \cdot e_2$, for some $p \in \mathbb{Z}$
- $\mu(e_1^*, e_1^*, e_1^*) = \lambda$, for some $\lambda \in \mathbb{Z}$
- the restriction of μ on $H^2(Y) \times H^2(Y) \times \mathbb{Z} \cdot e_2^*$

Here $Dp_1(M)$ is understood as an integer under the isomorphism $H_2(M, \partial M) \cong \mathbb{Z}$ given by the preferred generator, $e_i^* \in H^2(Y)$ is the Kronecker dual of e_i. According to [W1], there is a relation between p and λ, namely, $p \equiv 4\lambda \pmod{24}$. Clearly the restriction of μ on $H^2(Y) \times H^2(Y) \times \mathbb{Z} \cdot e_2^*$ is equivalent to the bilinear form q.

Concerning the relation between these invariants and the bordism class $[Y, h]$, we have the following lemma.

Lemma 2.8. *Let Y^6 be a spin 6-manifold, $h : Y \to (\mathbb{C}\mathrm{P}^\infty)^2$ be a map inducing an isomorphism on H_2. Then the bordiam class $[Y, h] \in \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$ is determined by the Poincaré dual of the first Pontrjagin class $Dp_1(Y)$ and the trilinear form*

$$\begin{array}{ccc} H^2(Y) \times H^2(Y) \times H^2(Y) & \xrightarrow{\quad\mu\quad} & \mathbb{Z}. \\ (a, b, c) & \mapsto & \langle a \cup b \cup c, [Y] \rangle \end{array}$$

Proof. According to lemma 2.7, the bordism class $[Y, h]$ is determined by $Dp_1(Y)$ and $h_*[Y]$. Therefore we only need to show that the trilinear form μ determines $h_*[Y] \in H_6((\mathbb{C}\mathrm{P}^\infty)^2)$. Since $H^6((\mathbb{C}\mathrm{P}^\infty)^2)$ is generated by elements of the form $u \cup v \cup w$, $u, v, w \in H^2((\mathbb{C}\mathrm{P}^\infty)^2)$, $h_*[Y]$ is determined by the evaluation

$$\langle u \cup v \cup w, h_*[Y] \rangle = \langle h^*(u) \cup h^*(v) \cup h^*(w), [Y] \rangle.$$

h_* is an isomorphism, so is h^*, hence the evaluation is equivalent to the trilinear form μ. ∎

If we identify $H^2(M)$ and $H^2(Y)$ via j^*, then μ can be viewed as a trilinear form on $H^2(M)$. Actually, a big part of this trilinear form is determined by the bilinear form q on $H^2(M)$. More precisely, we have

Lemma 2.9. *The restriction of μ on $H^2(M) \times H^2(M) \times H^2(M, \partial M)$ is equivalent to the bilinear form $q : H^2(M) \times H^2(M) \to \mathbb{Z}$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccccc}
H^2(Y) & \times & H^2(Y) & \times & H^2(Y) & \xrightarrow{\mu} & \mathbb{Z} \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow & & \| \\
H^2(M) & \times & H^2(M) & \times & H^2(M) & \xrightarrow{\mu} & \mathbb{Z} \\
\| & & \| & & \bigcup & & \| \\
H^2(M) & \times & H^2(M) & \times & H^2(M, \partial M) & \longrightarrow & \mathbb{Z}
\end{array}$$

The map on the bottom is $(u, v, w) \mapsto \langle u \cup v \cup w, [M, \partial M] \rangle$, for $u, v \in H^2(M)$, $w \in H^2(M, \partial M)$. This is equivalent to the bilinear form q. ∎

Lemma 2.10. *Let $Y = M \cup_\varphi (S^2 \times D^4)$, h be as above. Then there exists an orientation-preserving diffeomorphism $g : S^2 \times S^3 \to S^2 \times S^3$ s.t. for $Y' = M \cup_{g \circ \varphi} S^2 \times D^4$, we have $p = \lambda = 0$ in $(\star)$.*

Proof. Let $P = (-(S^2 \times D^4)) \cup_g (S^2 \times D^4)$ for some orientation-preserving diffeomorphism $g : S^2 \times S^3 \to S^2 \times S^3$, then P is a 6-dimensional, simply-connected spin manifold with $H_2(P) \cong \mathbb{Z}$. Define a map $k : P \to (\mathbb{C}\mathrm{P}^\infty)^2$ as follows:

On the first copy of $S^2 \times D^4$, k is the extension of

$$S^2 \times S^3 \xrightarrow{\varphi^{-1}} \partial M \subset M \xrightarrow{\bar{\nu}} B \xrightarrow{pr_1} (\mathbb{C}\mathrm{P}^\infty)^2,$$

and on the second copy of $S^2 \times D^4$ k is the extension of

$$S^2 \times S^3 \xrightarrow{g} S^2 \times S^3 \xrightarrow{\varphi^{-1}} \partial M \subset M \xrightarrow{\bar{\nu}} B \xrightarrow{pr_1} (\mathbb{C}\mathrm{P}^\infty)^2.$$

Then from the construction, it is seen that

$$k_* : H_2(P) \xrightarrow{\cong} h_*(\mathbb{Z} \cdot e_1) \subset H_2((\mathbb{C}\mathrm{P}^\infty)^2)$$

and

$$[Y, h] + [P, k] = [Y', h'] \in \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2).$$

Therefore, $h_* Dp_1(Y) + k_* Dp_1(P) = h'_* Dp_1(Y')$ and

$$\begin{aligned} & \mu'(e_1^*, e_1^*, e_1^*) \\ = \ & \langle e_1^* \cup e_1^* \cup e_1^*, [Y'] \rangle \\ = \ & \langle (h'^*)^{-1}(e_1^* \cup e_1^* \cup e_1^*), h'_*[Y'] \rangle \\ = \ & \langle (h^*)^{-1}(e_1^* \cup e_1^* \cup e_1^*), h_*[Y] \rangle + \langle (h^*)^{-1}(e_1^* \cup e_1^* \cup e_1^*), k_*[P] \rangle \\ = \ & \mu(e_1^*, e_1^*, e_1^*) + \langle u^* \cup u^* \cup u^*, [P] \rangle \end{aligned}$$

where $u^* = k^*((h^*)^{-1}(e_1^*)) \in H^2(P)$ is a generator.

According to the classification result of [W1], for any $b, c \in \mathbb{Z}$, there exists an orientation-preserving diffeomorphism $g : S^2 \times S^3 \to S^2 \times S^3$ s.t. $P = (-(S^2 \times D^4)) \cup_g (S^2 \times D^4)$ satisfies

$$\begin{cases} Dp_1(P) & = \ 4b \cdot u \\ \langle u^* \cup u^* \cup u^*, [P] \rangle & = \ 6c + b \end{cases}$$

where $u \in H_2(P)$ is the Kronecker dual of u^*. Choosing

$$\begin{cases} b & = \ -p/4 \\ c & = \ (p - 4\lambda)/24 \end{cases}$$

we get a corresponding g. For the corresponding Y' we see that

$$\begin{cases} Dp_1(Y') & = \ p \cdot e_1 + Dp_1(M) \cdot e_2 - p \cdot e_1 = Dp_1(M) \cdot e_2 \\ \mu'(e_1^*, e_1^*, e_1^*) & = \ 0 \end{cases}$$

This proves the lemma. ■

Proof of Theorem 2.5. Suppose that M_0 and M_1 satisfy the assumptions. We choose normal 2-smoothings $\bar{\nu}_i$ $(i = 1, 2)$, compatible with Φ, i.e. the isomorphism between $H_2(M_0)$ and $H_2(M_1)$ induced by $\bar{\nu}_0$ and $\bar{\nu}_1$ coinsides with

Φ. Lemma 2.10 ensures us to choose diffeomorphisms $\varphi_i : \partial M_i \to S^2 \times S^3$ $(i = 1, 2)$, s.t. $Y_i = M_i \cup_{\varphi_i} (S^2 \times D^4)$ satisfies

$$\begin{cases} Dp_1(Y_i) &= Dp_1(M_i) \cdot e_2^i \\ \mu(e_1^{i*}, e_1^{i*}, e_1^{i*}) &= 0 \end{cases}$$

where $\{e_1^0,\ e_2^0\}$ is a basis of $H_2(M_0)$ as before and $\{e_1^1 = \Phi(e_1^0),\ e_2^1 = \Phi(e_2^0)\}$ is a basis of $H_2(M_1)$. Let f be the composition

$$\partial M_0 \xrightarrow{\varphi_0} S^2 \times S^3 \xrightarrow{\varphi_1^{-1}} \partial M_1,$$

then the isomorphism between $H_2(\partial M_0)$ and $H_2(\partial M_1)$ induced by f coinsides with Φ. Thus f is compatible with the normal 2-smoothings since Φ is an isomorphism of the short exact sequences. We claim that $[Y_0, h_0] = [Y_1, h_1] \in \Omega_6^{\text{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. Once we have shown this, since $[N_f, \bar{\nu_f}] = [Y_0, h_0] - [Y_1, h_1]$, lemma 2.6 implies that M_0 is diffeomorphic to M_1. The diffeomorphism induces the same isomorphism on homology as Φ does, hence is compatible with the B-structures.

Since $\Phi : H_2(M_0, \partial M_0) \to H_2(M_1, \partial M_1)$ maps $Dp_1(M_0)$ to $Dp_1(M_1)$ and $\Phi^* : H^2(M_1) \to H^2(M_0)$ is an isometry of the bilinear forms, it follows that $\Phi : H_2(Y_0) \to H_2(Y_1)$ maps $Dp_1(Y_0)$ to $Dp_1(Y_1)$ and $\Phi^* : H^2(Y_1) \to H^2(Y_0)$ preserves the trilinear form. By lemma 2.8, $[Y_0, h_0] = [Y_1, h_1] \in \Omega_6^{\text{spin}}((\mathbb{C}\mathrm{P}^\infty)^2)$. This finishes the proof of theorem 2.5 in the spin case. ■

2.2.2 nonspin case

In this subsection we prove theorem 2.5 for nonspin manifolds. The proof is essentially the same as in last subsection, but we need to make some modification of the technical details.

First of all, in this case, the normal 2-type of M is described as follows: Consider the fibration

$$\xi : B = (\mathbb{C}\mathrm{P}^\infty)^2 \times B\mathrm{Spin} \xrightarrow{\eta \times p} BO \times BO \xrightarrow{\oplus} BO,$$

where $p : B\mathrm{Spin} \to BO$ is the canonical projection, $\eta : (\mathbb{C}\mathrm{P}^\infty)^2 \to BO$ is the classifying map of the complex line bundle $\mathrm{pr}_1^*(H)$, where H is the canonical line bundle over $\mathbb{C}\mathrm{P}^\infty$ and $\mathrm{pr}_1 : (\mathbb{C}\mathrm{P}^\infty)^2 \to \mathbb{C}\mathrm{P}^\infty$ is the projection to the first factor. Let $\bar{\nu}_1 : M \to (\mathbb{C}\mathrm{P}^\infty)^2$ be a map which induces an isomorphism on H_2 and s.t. $\bar{\nu}_1^*((1,0)) \equiv w_2(\nu M) \pmod 2$. (This is the case since M is nonspin.) Then $w_2(\nu M - \bar{\nu}_1^*(\eta)) = 0$ and therefore there is a (unique) lift $\bar{\nu}_2 : M \to B\mathrm{Spin}$ classifying $\nu M - \bar{\nu}_1^*(\eta)$. Let $\bar{\nu} = \bar{\nu}_1 \times \bar{\nu}_2 : M \to B$, then $\bar{\nu}$ is a normal 2-smoothing of M in the normal 2-type B.

As in last subsection, we can form the manifold $Y = M \cup_\varphi (S^2 \times D^4)$ and get an element in the corresponding bordism group. Lemma 2.6 still holds for this normal structure. Now the bordism group $\Omega_6(B;\xi)$ is identified with the twisted spin bordism group $\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2;\eta)$, which is defined as the bordism group of maps f from closed 6-manifolds X to $(\mathbb{C}\mathrm{P}^\infty)^2$, together with a spin structure on $f^*(\eta) \oplus \nu X$. There is an isomorphism

$$\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2;\eta) \cong \widetilde{\Omega}_8^{\mathrm{spin}}(\mathrm{Th}(\eta)),$$

where $\mathrm{Th}(\eta)$ is the Thom space of the complex line bundle η over $(\mathbb{C}\mathrm{P}^\infty)^2$. For the invariants of this bordism group we have a lemma analogous to lemma 2.7.

Lemma 2.11. *There is an injective homomorphism*

$$\Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2;\eta) \longrightarrow H_2((\mathbb{C}\mathrm{P}^\infty)^2) \oplus H_6((\mathbb{C}\mathrm{P}^\infty)^2).$$

$$[Y,h] \quad \mapsto \quad (h_* Dp_1(\tau Y \oplus h^*(\eta)), h_*[Y])$$

Proof. Consider the Atiyah-Hirzubruch spectral sequence for $\widetilde{\Omega}_8^{\mathrm{spin}}(\mathrm{Th}(\eta))$. $E^2_{p,q} = \widetilde{H}_p(\mathrm{Th}(\eta);\Omega_q^{\mathrm{spin}})$. Since $E^2_{4,4}$, $E^2_{6,2}$ and $E^2_{8,0}$ are the only non-zero terms on the segment $p+q=8$, so are $E^\infty_{4,4}$, $E^\infty_{6,2}$ and $E^\infty_{8,0}$. We have two exact sequences:

$$0 \to G \to \widetilde{\Omega}_8^{\mathrm{spin}}(\mathrm{Th}(\eta)) \to \widetilde{H}_8(\mathrm{Th}(\eta))$$

and

$$0 \to E^\infty_{4,4} \to G \to E^\infty_{6,2} \to 0.$$

The proof is based on the following calculations:

1. $E^\infty_{4,4} = E^2_{4,4} = H_4(\mathrm{Th}(\eta);\Omega_4^{\mathrm{spin}})$, since $E^2_{8,1}$ is a finite group and thus $d_4 : E^4_{8,1} \to E^4_{4,4}$ is trivial.

2. The differential

$$d_2 : E^2_{8,1} = H_8(\mathrm{Th}(\eta);\mathbb{Z}/2) \to E^2_{6,2} = H_6(\mathrm{Th}(\eta);\mathbb{Z}/2)$$

is dual to the Steenrod square

$$\mathrm{Sq}^2 : H^6(\mathrm{Th}(\eta); \mathbb{Z}/2) \to H^8(\mathrm{Th}(\eta); \mathbb{Z}/2).$$

Passing to the base space, the corresponding map is

$$H^4((\mathbb{C}\mathrm{P}^\infty)^2; \mathbb{Z}/2) \to H^6((\mathbb{C}\mathrm{P}^\infty)^2; \mathbb{Z}/2), \ x \mapsto w_2 \cup x + \mathrm{Sq}^2 x$$

where w_2 is the second Stiefel-Whitney class of η. Therefore

$$E^\infty_{6,2} = E^3_{6,2} = \mathrm{Coker} d_2 \cong (\mathbb{Z}/2)^2.$$

3. The generators of $E^\infty_{4,4} = H_4(\mathrm{Th}(\eta)) \otimes \Omega^{\mathrm{spin}}_4$ are given as follows: note that $\eta = H \times \mathbf{0}$ where H the canonical line bundle and $\mathbf{0}$ the 0-dimensional vector bundle over $\mathbb{C}\mathrm{P}^\infty$, thus

$$\mathrm{Th}(\eta) = \mathrm{Th}(H) \wedge \mathrm{Th}(\mathbf{0}) = \mathbb{C}\mathrm{P}^\infty_1 \wedge (\mathbb{C}\mathrm{P}^\infty_2)^+.$$

Therefore $H_4(\mathrm{Th}(\eta)) = H_4(\mathbb{C}\mathrm{P}^\infty_1) \oplus H_2(\mathbb{C}\mathrm{P}^\infty_1) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_2)$. Let

$$h_1 : S^2 \times S^2 \to \mathbb{C}\mathrm{P}^\infty_1 \wedge (\mathbb{C}\mathrm{P}^\infty_2)^+ = \mathrm{Th}(\eta)$$

represent a generator of $H_2(\mathbb{C}\mathrm{P}^\infty_1) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_2)$ and

$$h_2 : S^2 \times S^2 \to \mathbb{C}\mathrm{P}^\infty_1 \times \mathbb{C}\mathrm{P}^\infty_1 \xrightarrow{\otimes} \mathbb{C}\mathrm{P}^\infty_1 \subset \mathrm{Th}(\eta)$$

represent the twice of a generator of $H_4(\mathbb{C}\mathrm{P}^\infty)$, where $\otimes$ is induced by the tensor product of line bundles. (Note that a computation by the Atiyah-Hirzebruch spectral sequence shows that the image of $\Omega^{\mathrm{spin}}_4(\mathbb{C}\mathrm{P}^\infty) \to H_4(\mathbb{C}\mathrm{P}^\infty)$ has index 2, thus it is impossible to realize the generator of $H_4(\mathbb{C}\mathrm{P}^\infty)$ by a spin 4-manifold.) Let K be the Kummer surface, which is a generator of Ω^{spin}_4. Condiser

$$\phi_1 : K \times S^2 \times S^2 \to S^2 \times S^2 \xrightarrow{h_1} \mathrm{Th}(\eta)$$

and

$$\phi_2 : K \times S^2 \times S^2 \to S^2 \times S^2 \xrightarrow{h_2} \mathrm{Th}(\eta).$$

Then $E^\infty_{4,4}$ is generated by $[\phi_1, K \times S^2 \times S^2]$ and $\frac{1}{2}[\phi_2, K \times S^2 \times S^2]$.

From the above calculations, it is seen that an element $[M, f] \in \widetilde{\Omega}_8^{\mathrm{spin}}(\mathrm{Th}(\eta))$ is determined by $f_*[M] \in H_8(\mathrm{Th}(\eta))$ and $f_* Dp_1(M) \in H_4(\mathrm{Th}(\eta))$. Passing to the base space, an element $[Y, h] \in \Omega_6^{\mathrm{spin}}((\mathbb{C}\mathrm{P}^\infty)^2; \eta)$ is determined by $h_*[Y]$ and $h_* Dp_1(\tau Y \oplus h^*(\eta))$. ■

Since

$$\begin{aligned} h_* Dp_1(\tau Y \oplus h^*(\eta)) &= h_* Dp_1(Y) + h_* Dp_1(h^*\eta) \\ &= h_* Dp_1(Y) + h_* Dh^* p_1(\eta) \\ &= h_* Dp_1(Y) + p_1(\eta) \cap h_*([Y]), \end{aligned}$$

the bordism class is determined by $h_* Dp_1(Y)$ and $h_*([Y])$, as in last subsection.

Finally note that lemma 2.10 is also true in this case since the relation $p \equiv 4\lambda \pmod{24}$ in $(\star)$ still holds (c.f. [Zh]).

It is easy to check that after these modification the proof in last subsection is valid for nonspin M.

2.3 Proofs of the Main Results

After the preparations in the above sections we are ready to prove the main results of the dissertation in this section.

Theorem 2.12. *For $i = 1, 2$, let $V_i \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface of degree d_i with a unique singularity $p_i \in V_i$, such that the link of p_i is diffeomorphic to $S^2 \times S^3$. Suppose that the second homology group with of the nonsingular part of V_i is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and that d_i is square-free. Let μ_i be the Milnor number of p_i. Then V_1 and V_2 are diffeomorphic as stratifolds if and only if $d_1 = d_2$ and $\mu_1 = \mu_2$.*

Proof. Let $V \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface fulfilling the assumptions. Let M be the smooth part of V. Then M fulfills the conditions (A) in section 2.2. Consider the short exact sequence

$$0 \longrightarrow H_2(\partial M) \xrightarrow{i} H_2(M) \xrightarrow{j} H_2(M, \partial M) \longrightarrow 0.$$

Let $y \in H_2(\partial M) \cong \mathbb{Z}$ be a generator and $x \in H_2(M, \partial M) \cong \mathbb{Z}$ be the preferred generator. Let $b = i(y)$ and $a \in j^{-1}(x)$, then $\{a, b\}$ is a basis of

$H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let a^*, b^* be the Kronecker dual of a, b, then $\{a^*, b^*\}$ is a basis of $H^2(M)$. Remember we have a Gysin sequence

$$0 \longrightarrow H^3(U) \xrightarrow{R} H^2(V^*) \xrightarrow{\delta} H^4(\mathbb{C}\mathrm{P}^4 - p)$$

and we know that $H^3(U) \cong \mathbb{Z}$. (See the paragraph before lemma 2.2.) Let $u \in H^3(U)$ be a generator, if we identify $H^2(V^*)$ and $H^2(M)$ by the inclusion, then the primitive element $R(u)$ can be written as $R(u) = ma^* + nb^*$ for some coprime $m, n \in \mathbb{Z}$. According to lemma 2.2, and the fact that $q(a^*, a^*) = a^* \cup a^* \cup x^* = d$, we have

$$\begin{cases} 0 = q(R(u), a^*) = md + n \cdot q(b^*, a^*) \\ 0 = q(R(u), b^*) = m \cdot q(a^*, b^*) + n \cdot q(b^*, b^*) \end{cases}$$

This implies

$$\begin{cases} q(a^*, b^*) = -md/n \in \mathbb{Z} \\ q(b^*, b^*) = m^2 d/(n^2) \in \mathbb{Z} \end{cases}$$

Since m and n are coprime, d is divisible by n^2. If d is square-free, then n has to be 1 and thus $R(u) = ma^* + b^*$. We can perform a basis change in $H_2(M)$ by replacing a by $a + mb$. Under this new basis we have $R(u) = b^*$ and the bilinear form q is represented by the matrix

$$\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, by lemma 2.1, $p_1(M)$ and $\chi(M)$ are determined by the degree d and the Milnor number μ of p. M is spin if and only if d is odd. Note that $i^*(w_2(M)) = 0$ since ∂M is spin. Thus if d is even, $w_2(M)$ is the image of the generator of $H^2(M, \partial M; \mathbb{Z}/2)$.

Now let V_1 and V_2 be hypersurfaces fulfilling the conditions in this theorem, with degree d_i and Milnor number μ_i. Let M_1 (resp. M_2) be the smooth part of V_1 (resp. V_2). Then in order to show that V_1 and V_2 are diffeomorphic as parameterized stratifolds with isolated singularities, it suffices to show that M_1 is diffeomorphic to M_2. Let $\{a_1, b_1\}$ (resp. $\{a_2, b_2\}$) be the basis of $H_2(M_1)$ (resp. $H_2(M_2)$) chosen as above. If $d_1 = d_2$ and $\mu_1 = \mu_2$, then $\chi(M_1) = \chi(M_2)$. Let Φ be the (unique) isomorphism of the corresponding short exact sequences, which maps a_1 to a_2 and b_1 to b_2. Thus Dp_1, w_2 and q

are all preserved by Φ^*. Therefore according to theorem 2.5, M_1 is diffeomorphic to M_2. Conversely, if V_1 and V_2 are diffeomorphic as stratifolds, then the smooth part M_1 and M_2 are diffeomorphic, thus $d_1 = d_2$ and $\mu_1 = \mu_2$. ■

Corollary 2.13. *For $i = 1, 2$, let $V_i \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface of degree d_i with a unique singularity of type A_{2k_i+1} ($k_i \geq 0$). Suppose that $d_i < (k_i+5)/2$ and is square-free. Then V_1 and V_2 are diffeomorphic as stratifolds if and only if $d_1 = d_2$ and $k_1 = k_2$.*

For $i = 1, 2$, let $V_i \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface of degree d_i with a unique singularity of type A_{2k_i}, then V_1 and V_2 are diffeomorphic as stratifolds if and only if $d_1 = d_2$ and $k_1 = k_2$.

Proof. Let $V \subset \mathbb{C}\mathrm{P}^4$ be a hypersurface of degree d with a unique singularity of type A_{2k+1} ($k \geq 0$), then the Milnor number μ of this singularity is $2k+1$, and according to proposition 1.19, the boundary of the smooth part M is diffeomorphic to $S^2 \times S^3$. If $d < (k+5)/2$, according to lemma 2.3, $H_2(M)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Thus the corollary is a consequence of theorem 2.12.

If $V \subset \mathbb{C}\mathrm{P}^4$ is a hypersurface of degree d with a unique singularity of type A_{2k}, then according to 1.19, the boundary of the smooth part M is diffeomorphic to S^5. In this case $H_2(M) \cong H_2(M, \partial M) \cong H_2(V) \cong \mathbb{Z}$. Let $N = M \cup_{\partial M} D^6$, then the classification of M is equivalent to the classification of N. N is a simply-connected 6-dimensional closed manifold with $H_2(N) \cong \mathbb{Z}$, thus according to the classification results of [W1] and [Zh], the Euler characteristic $\chi(N)$, the Pontrjagin class $p_1(N)$, the Stielfel-Whitney class $w_2(N)$, and the cubic form

$$H^2(N) \times H^2(N) \times H^2(N) \longrightarrow \mathbb{Z}$$

are complete invariants of N. From lemma 2.1, it is seen that all these invariants are determined by d and k. ■

Chapter 3

Examples

In section 3.1 we develope a recognition principle to judge if a given singularity on a hypersurface in $\mathbb{C}\mathrm{P}^4$ is of type A_k. Then in section 3.2 we construct two families of cubic hypersurfaces in $\mathbb{C}\mathrm{P}^4$ with a unique singularity of type A_5. As an application of the main results we see that they are diffeomorphic as stratifolds.

3.1 Criterion for Singularities

In the section we establish a criterion to determine the normal type for an isolated singularity. We follow the definitions in [AGV1].

Definition 3.1. *A polynomial $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be* ***quasihomogeneous*** *of degree d with weights $(\alpha_1, \cdots, \alpha_n)$ if for any $\lambda > 0$ we have*

$$f(\lambda^{\alpha_1} x_1, \cdots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \cdots, x_n).$$

A quasihomogeneous polynomial is said to be ***non-degenerate*** *if 0 is an isolated singularity of f.*

For example, the normal form for A_k-singularity

$$z_1^{k+1} + z_2^2 + z_3^2 + z_4^2$$

is a non-degenerate quasihomogeneous polynomial of weights $(\frac{1}{k+1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and degree 1.

Definition 3.2. *A polynomial $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be **semi-quasihomogenous** of degree d with weights $(\alpha_1, \cdots, \alpha_n)$ if it is of the form $f = f_0 + g$, where f_0 is a non-degenerate quasihomogeneous polynomial of degree d with weights $(\alpha_1, \cdots, \alpha_n)$ and g is a polynomial of degree greater than d according to these weights.*

In [B-W] a principle is given to judge whether an isolated singulariy on a complex surface is of type A_k. This principle can be generalised to dimension 3 without essential difficulty.

Lemma 3.3. *If $f(z_1, z_2, z_3, z_4)$ is a semiquasihomogeneous polynomial of degree 1 with weights $(\frac{1}{2m}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then by a change of co-ordinates we can reduce the degree 1 part to the normal form for A_{2m-1}-singularity given above and the resulting function will remain semiquasihomogeneous.*

Proof. There are 10 monomials of degree 1 with weights $(\frac{1}{2m}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$:

$$z_1^{2m},\ z_1^m z_2,\ z_1^m z_3,\ z_1^m z_4,\ z_2^2,\ z_2 z_3,\ z_2 z_4,\ z_3^2,\ z_3 z_4,\ z_4^2.$$

Thus the degree 1 part of the polynomial is a linear combination of these monomials:

$$a_1 z_1^{2m} + (a_2 z_2 + a_3 z_3 + a_4 z_4) z_1^m + (a_5 z_2^2 + a_6 z_2 z_3 + a_7 z_2 z_4 + a_8 z_3^2 + a_9 z_3 z_4 + a_{10} z_4^2)$$

If $a_2, a_3, a_4 \neq 0$, then let $w_2 = a_2 z_2 + a_3 z_3 + a_4 z_4$. Since 0 is an isolated singularity, by a change of co-ordinates, we can reduce the quadratic form of z_2, z_3, z_4 to the normal form $z_2^2 + z_3^2 + z_4^2$. The resulting polynomial then is

$$\begin{aligned} & a_1 z_1^{2m} + (a_2 z_2 + a_3 z_3 + a_4 z_4) z_1^m + z_2^2 + z_3^2 + z_4^2 \\ = \ & (a_1 - \tfrac{a_2^2 + a_3^2 + a_4^2}{4}) z_1^{2m} + (\tfrac{a_2}{2} z_1^m + z_2)^2 + (\tfrac{a_3}{2} z_1^m + z_3)^2 + (\tfrac{a_4}{2} z_1^m + z_4)^2 \end{aligned}$$

Then let $w_i = \frac{a_i}{2} z_1^m + z_i$ for $i = 2, 3, 4$. A final re-scaling will reduce the degree 1 part to the standard form and one easily checks that the other terms will still have weights > 1. ■

Lemma 3.4. *For the normal form for A_k-singularity, $f = z_1^{k+1} + z_2^2 + z_3^2 + z_4^2$, the local ring of f, i.e., the ring of formal power series factored by the Jacobian ideal of f, $\mathcal{O}/Jf$ has a monomial basis over the complex numbers, each element of which has weight < 1.*

Remark. In the terminology of [AGV1], f is said to have no **superdiagonal elements**.

Proof. This follows by direct computation. The Jacobian ideal of the normal form for A_k-singularity has generators z_1^k, z_2, z_3, z_4, so $1, z_1, \ldots, z_1^{k-1}$ is a monomial basis of the local ring $\mathcal{O}/Jf$. The weights are respectively $0, \frac{1}{k+1}, \ldots, \frac{k-1}{k+1}$. ∎

Lemma 3.5 (recognition principle)**.** *If $f(z_1, \ldots, z_4)$ is as in lemma 3.3, then f is equivalent to the normal form $z_1^{k+1} + z_2^2 + z_3^2 + z_4^2$.*

Proof. By lemma 3.3 we can reduce f to the normal form plus terms of degree > 1. According to [AGV1, page 194, Theorem], a further change of co-ordinates will reduce f to $z_1^{k+1} + z_2^2 + z_3^2 + z_4^2 + \sum_1^s c_i e_i$, where $c_i \in \mathbb{C}$ and $e_1, \ldots, e_s$ are superdiagonal elements. By lemma 3.4 above $s = 0$ and the assertion is proved. ∎

3.2 Constructions

In [B-W] the cubic surfaces in $\mathbb{C}\mathrm{P}^3$ with isolated singularities have been classified. In this section we construct, from cubic surfaces in $\mathbb{C}\mathrm{P}^3$, two families of cubic hypersurfaces in $\mathbb{C}\mathrm{P}^4$ with a unique A_5-singularity.

Lemma 3.6. *Let $V_0 \subset \mathbb{C}\mathrm{P}^3$ be a cubic surface defined by $F_0(x_0, x_1, x_2, x_3)$ such that $p_0 = [1, 0, 0, 0]$ is a unique singularity of V_0. Suppose that the affine curves $F_0(0, 1, x_2, x_3)$, $F_0(0, x_1, 1, x_3)$, $F_0(0, x_1, x_2, 1)$ are all smooth. Let $F = F_0 + x_0 x_4^2$, then F defines a cubic hypersurface $V \subset \mathbb{C}\mathrm{P}^4$ with $p = [1, 0, 0, 0, 0]$ a unique singularity of V and p is the stabilization of p_0.*

Proof. On the co-ordinate chart $x_0 = 1$, the affine equation of V is

$$F_0(1, x_1, x_2, x_3) + x_4^2 = 0.$$

It is easy to see that $(0, 0, 0, 0)$ is the unique singularity, which is the stabilization of p_0. The smoothness of the curves $F_0(0, 1, x_2, x_3)$, $F_0(0, x_1, 1, x_3)$ and $F_0(0, x_1, x_2, 1)$ ensures that there are no singularities on other co-ordinate charts. ∎

Example. Now consider the family of hypersurfaces defined by

$$F(x_0, x_1, x_2, x_3, x_4) = x_0(x_4^2 + x_1x_2) + x_1x_3(x_1 + ax_3) + bx_2^3, (a \neq 0, b \neq 0)$$

According to the classification of cubic surfaces in [B-W], the polynomials

$$F_0 = x_0x_1x_2 + x_1x_3(x_1 + ax_3) + bx_2^3, \quad (a \neq 0, b \neq 0)$$

define a family of cubic surfaces with a unique singularity at $[1, 0, 0, 0]$ of type A_5. It is easily checked that F_0 fulfills the conditions in the above lemma. Therefore F defines a family of hypersurfaces of degree 3 in $\mathbb{C}\mathrm{P}^4$ with $p = [1, 0, 0, 0, 0]$ a unique singularity of type A_5. (This can be seen also by the recognition principle in last section. Namely, the polynomial

$$\begin{aligned} & F(1, x_1, x_2, x_3, x_4) \\ = \; & x_1x_2 + x_1x_3(x_1 + ax_3) + bx_2^3 + x_4^2 \\ = \; & x_1(x_2 + x_3(x_1 + ax_3)) + bx_2^3 + x_4^2 \\ = \; & x_1x_2' + b(x_2' - x_3(x_1 + ax_3))^3 + x_4^2 \end{aligned}$$

is semiquasihomogeneous of x_1, x_2', x_3, x_4 of degree 1 according to the weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2})$.)

Analogously we have the following construction.

Lemma 3.7. *Let $V_0 \subset \mathbb{C}\mathrm{P}^3$ be a cubic surface defined by $F_0(x_0, x_1, x_2, x_3)$ such that $p_0 = [1, 0, 0, 0]$ is a unique singularity of V_0 of type A_1. Suppose that the affine curve $F_0(1, 0, x_2, x_3)$ has a unique singularity at $(0, 0)$ and that the affine curves $F_0(x_0, 0, 1, x_3)$, $F_0(x_0, 0, x_2, 1)$ are smooth. Let $F = F_0 + x_1x_4^2$, then F defines a cubic hypersurface $V \subset \mathbb{C}\mathrm{P}^4$ with $p = [1, 0, 0, 0, 0]$ a unique singularity of type A_5.*

Proof. According to [B-W], F_0 is of the form $F_0 = x_0(x_2^2 - x_1x_3) + f_3(x_0, \ldots, x_3)$ where f_3 is of degree 3. Let $x_3' = x_3 - x_4^2$, then $F(1, x_1, x_2, x_3', x_4)$ is semiquasihomogeneous of x_1, x_2, x_3', x_4 of weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6})$. Then according to the recognition principle, $[1, 0, 0, 0, 0]$ is an A_5-singularity. ∎

Example. Consider the polynomials

$$F_0 = x_0(x_2^2 - x_1x_3) + ax_1^3 + bx_3^3, \quad (a \neq 0, b \neq 0).$$

According to [B-W], these polynomials define a family of cubic surfaces with a unique singularity at $[1,0,0,0]$ of type A_1. It is easily checked that F_0 fulfills the conditions in the above lemma. Therefore the polynomials

$$F = x_0(x_2^2 - x_1x_3) + ax_1^3 + bx_3^3 + x_1x_4^2, \quad (a \neq 0, b \neq 0)$$

define a family of hypersurfaces of degree 3 in $\mathbb{C}\mathrm{P}^4$ with $p = [1,0,0,0,0]$ a unique singularity of type A_5.

Remark. According to corollary 2.13, the hypersurfaces in these two families are all diffeomorphic as stratifolds. However, it is not clear to the author whether we can deform one to the other. At least, there is no obvious deformation.

Appendix

2-Dimensional Complete Intersections

In this appendix we will discuss the topological classification of complex 2-dimensional complete intersections with an isolated singularity. We will use a classification result of simply-connected 4-manifolds with connected boundary, due to Vogel, and then try to compute the invariants appearing in Vogel's classification for 2-dimensional complete intersections.

1.1 Classification of 4-Manifolds with Boundary

Let M be a simply-connected compact smooth 4-manifold with connected boundary. By Poincaré duality and the universal coefficient theorem, $H_2(M)$ is torsion free and M is a Moore space of type $M(H_2(M), 2)$. We have the following natural invariants of M.

First of all, there is an exact sequence

$$H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \to 0,$$

and there is an intersection form on $H_2(M)$, which is symmetric bilinear.

Secondly, for a spin structure σ on ∂M, we have an element $[(\partial M, \sigma), i]$ in $\Omega_3^{\text{spin}}(M)$. A computation by the Atiyah-Hirzebruch spectral sequence shows that

$$\Omega_3^{\text{spin}}(M) \cong \Omega_1^{\text{spin}} \otimes H_2(M) \cong \text{Hom}(H_2(M, \partial M), \mathbb{Z}/2).$$

Thus we obtain a map

$$\gamma : \mathrm{Spin}(\partial M) \to \mathrm{Hom}(H_2(M, \partial M), \mathbb{Z}/2).$$

where $\mathrm{Spin}(\partial M)$ denotes the set of equivalent classes of spin structures on ∂M. Note that $H^1(\partial M; \mathbb{Z}/2)$ acts on $\mathrm{Spin}(\partial M)$ freely and transitively.

Furthermore, the torsion subgroup T of $H_1(\partial M)$ is endowed with a non-singular linking pairing

$$\lambda : T \otimes T \to \mathbb{Q}/\mathbb{Z}.$$

And for any spin structure σ on ∂M there is a quadratic refinement μ_σ of λ:

$$\mu_\sigma : T \to \mathbb{Q}/2\mathbb{Z}.$$

λ and μ_σ are subjected to the following relations:

1. $\mu_\sigma(x + y) = \mu_\sigma(x) + \mu_\sigma(y) + 2\lambda(x, y)$, for $x, y \in T$;
2. $\mu_\sigma(x) \equiv \lambda(x, x) \pmod{\mathbb{Z}}$, for $x \in T$;
3. $\mu_{u\cdot\sigma}(x) = \mu_\sigma(x) + < u, x >$, for $x \in T$, $u \in H^1(\partial M; \mathbb{Z}/2)$.

Let X be a compact 3-manifold, L be a finitely generated free abelian group, $f : L \to H_1(X)$ be an epimorphism and T be the torsion subgroup of $H_1(X)$. Let $A(L, f)$ be the set of pairs (b, γ), where b is a $\mathbb{Z}$-valued symmetric bilinear form over $K = \mathrm{Ker} f$ (b extends to a $\mathbb{Q}$-valued symmetric bilinear form over $K' = f^{-1}(T)$) and $\gamma : \mathrm{Spin}(X) \to \mathrm{Hom}(L, \mathbb{Z}/2)$ is a map such that the following conditions are fulfilled:

1. $\lambda(f(x), f(y)) + b(x, y) \equiv 0 \pmod{\mathbb{Z}}$ for $x, y \in K' = f^{-1}(T)$;
2. $\mu_\sigma(f(x)) + b(x, x) \equiv \gamma_\sigma(x) \pmod{2\mathbb{Z}}$, for $x \in K'$, $\sigma \in \mathrm{Spin}(X)$;
3. $\mu_{u\cdot\sigma}(x) = \mu_\sigma(x) + < u, f(x) >$, for $x \in L$, $\sigma \in \mathrm{Spin}(X)$, $u \in H^1(X; \mathbb{Z}/2)$.

Theorem ([Vo]). *Let (b, γ) be an element of $A(L, f)$, b being non-singular. Then there exists a compact simply-connected topological 4-manifold M with boundary X, unique upto to homeomorphism, and satisfying the following condistions:*

1. $H^2(M) = L$;

2. *f is the composition $H^2(M) \to H^2(X) \to H_1(X)$;*

3. *the intersection form over $H_2(M)$ is $g^*(b)$, where g is the surjection from $H_2(M)$ to K induced by the composition*

$$H_2(M) \to H_2(M, X) \to H^2(M) = L;$$

4. *if σ is a spin structure on X, then the inclusion map $X \to M$ gives the linear map γ_σ via the isomorphism $\Omega_3^{\text{spin}}(M) \to Hom(L, \mathbb{Z}/2)$.*

1.2 Computation of the Invariants

In this section we apply several known results to compute the invariants stated in last section for complete intersections with an isolated singularity.

Let $V \subset \mathbb{C}\mathrm{P}^N$ be a complete intersection of complex dimension 2. Let $q \in V$ be a unique singularity. Denote the smooth part of V be M. Then by Lefschetz theorem, $\pi_1(V)$ is trivial. $\partial M = \mathrm{Link}(q)$ is a connected 3-manifold. Furthermore, there is a lattice homomorphism

$$\varphi_V : L_q \longrightarrow \bar{L}$$

such that

$$H_2(V) \cong \mathbb{Z} \oplus \mathrm{coker}\varphi_V \quad H_3(V) \cong \ker\varphi_V \quad (\star)$$

where L_q is the Milnor lattice of the singularity q and $\bar{L}$ is a non-degenerate lattice. (c.f. [D1])

Now let's consider the fundamental group and the homology groups of M. First of all, by Van-Kampen theorem, $\pi_1(V) \cong \pi_1(M)/\pi_1(\partial M)$, since V is simply-connected, this implies that $i_* : \pi_1(\partial M) \to \pi_1(M)$ is a surjection. Secondly, by Poincaré duality and $(\star)$ we see that

$$H_1(M) \cong H^3(M, \partial M) \cong H^3(V) \cong \ker\varphi_V \oplus \mathrm{tors}(\mathrm{coker}\varphi_V).$$

Now assume that the Milnor lattice of q is non-degenerate, then since $\bar{L}$ is also non-degenerate, $\ker\varphi_V$ is trivial. To have an estimation on $\mathrm{tors}(\mathrm{coker}\varphi_V)$ we have the following general consideration: for a nondegenerate lattice A, we define a set

$$T(A) = \{\mathrm{tors}(N/A) | N \text{ is a supralattice of } A\}$$

here a supralattice means that there is an embedding $A \hookrightarrow N$ of lattices. Now φ_V is an embedding, therefore tors(cokerφ_V) $\in T(L_q)$. For a given lattice A, the calculation of $T(A)$ is purely arithmetic and for Milnor lattices of simple singularities we have the following results:

- $T(A_k) = \{\mathbb{Z}/n \mid n^2|k+1 \text{ and } k(k+1)/n^2 \text{ is even}\}$
- $T(D_k) = \begin{cases} \{0, \mathbb{Z}/2\} & k \equiv 0 \pmod 8 \\ \{0\} & \text{else} \end{cases}$
- $T(E_6) = T(E_7) = T(E_8) = \{0\}$

It is seen that for many integers k, $T(A_k) = \{0\}$. (At least for those k, $k+1$ is square-free.) Now assume that q is of type A_k and $T(A_k) = \{0\}$. This implies tors(cokerφ_V)= 0, therefore $H_1(M) = 0$. In this situation, $\partial M = \text{Link}(q)$ is just the lens space $L(k+1, k)$ with fundamental group $\pi_1(\partial M) \cong \mathbb{Z}/(k+1)$. $\pi_1(M)$ is a quotient of $\pi_1(\partial M)$, hence abelian. Therefore $\pi_1(M)$ is an abelian group with trivial abelianization, hence is trivial. In this case the classification of [Vo] is ready to apply.

Remark. It is interesting to ask what $\pi_1(M)$ might be for other simple singularities q with trivial $T(L_q)$. We still have $H_1(M) = 0$ and now $\pi_1(\partial M) = G$ is a (nonabelian) finite normal subgroup of $SU(2)$. Now the question is, which quotient group of such a G has trivial abelianization?

From now on we assume that q is an A_k-singularity with $T(A_k) = \{0\}$. Then M is a simply-connected 4-manifold with boundary $\partial M \cong L(k+1, k)$. There is a short exact sequence

$$0 \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \to 0.$$

The intersection form on $H_2(M)$ is equivalent to the lattice

$$H^2(V) \times H^2(V) \xrightarrow{\cup} \mathbb{Z}.$$

There is a concrete description of the latter (c.f. [D1]). In general, let V be a complete intersection of complex dimension n (n even) with a unique singularity q. As said before, there is a lattice homomorphism $\varphi_V : L_q \to \bar{L}$. Let V_0 be a small smooth deformation of V, E be a linear subspace of codimension $n/2$ in the ambient projective space, transverse to V_0. Denote $h = [E \cap V_0] \in H_n(V_0)$, then we have $\bar{L} \cong h^{\perp} \subset H_n(V_0)$. Let $I = \mathrm{Im}\varphi_V \subset \bar{L} \subset H_n(V_0)$ be the image of φ_V, then we have

$$H^n(V) \cong I^{\perp} \subset H_n(V_0), \quad H_n(V) \cong H_n(V_0)/I.$$

The former isomorphism is a lattice isomorphism.

Remark. As mentioned in section 1.3, topological invariants of $H_n(V_0)$, such as the rank, the signature and the parity, can be calculated explicitly from the multidegree $\mathbf{d}$ of V_0.

Now we come back to the case $n = 2$. The lattice on $H^2(V)$ is determined by the embedding $I = L_q \hookrightarrow H_2(V_0)$. Furthermore, since $H_2(V_0)/I \cong H_2(V) \cong H^2(M)$ is torsion free, this embedding is primitive. For primitive embeddings, we have the following theorem due to Nikulin [Ni]

Theorem (Uniqueness of a Primitive Embedding)**.** *Let $i : M \to N$ be a primitive embedding of an even nondegenerate lattice M of signature (m_+, m_-) into an even nondegenerate lattice N of signature (n_+, n_-). Then this embedding is unique up to an automorphism of N provided the following conditions are satisfied:*

1. $n_+ \geq m_+$, $n_- \geq m_-$;
2. $rk(N) - rk(M) \geq l(D(M)) + 2$, *where $l(F)$ denotes the minimal number of generators of a finite abelian group F and $D(M)$ denotes the discriminant of the lattice M.*

For the Milnor lattice of 2-dimensional A_k-singularities, it is known that L_{A_k} is a negative definite even lattice of rank k and the discriminant $D(L_{A_k})$ is isomorphic to $\mathbb{Z}/(k+1)$, thus $l(D(L_{A_k})) = 1$.

For simplicity, in the rest part of this appendix we concentrate our computation on surfaces $V \subset \mathbb{C}\mathrm{P}^3$ of degree d. According to the computation in section 1.3, we see that the lattice $H_2(V_0)$ is even if and only if d is even, the rank of $H_2(V_0)$ is $d(d^2 - 4d + 6) - 2$ and the signature is $d(4 - d^2)/3$. Thus the conditions in Nikulin's theorem is now

$$\begin{cases} d(d^2 - 4d + 6) - 5 - k \geq 0 \\ d(2d^2 - 6d + 7) - 3 - 3k \geq 0 \\ d \text{ is even} \end{cases}$$

For (d,k) fulfilling these inequalities, the embedding $L_{A_k} \hookrightarrow H_2(V_0)$ is unique and therefore the intersection form on $H_2(M)$ is uniquely determined by (d,k).

Example. For $k \leq 6$, $T(A_k) = \{0\}$. In this range for all $d \geq 4$, even, the inequalities hold.

In order to apply Vogel's theorem to get a classification result, we still need to consider the map

$$\gamma : \mathrm{Spin}(V) \to \mathrm{Hom}(H_2(M, \partial M), \mathbb{Z}/2).$$

At the moment the behavior of this map is not completely clear to the author, but at least for some special cases, it is quite simple. Recall that this map is defined as follows: for a given spin structure σ on ∂M, the inclusion $i : \partial M \to M$ gives an element

$$[(\partial M, \sigma), i] \in \Omega_3^{\mathrm{spin}}(M) \cong \Omega_1^{\mathrm{spin}} \otimes H_2(M) \cong \mathrm{Hom}(H_2(M, \partial M), \mathbb{Z}/2).$$

Now $\partial M \cong \mathrm{Link}(A_k) \cong L(k+1,k)$. If k is even, then $H^1(\partial M; \mathbb{Z}/2) = 0$ and there is a unique spin structure on ∂M. Furthermor, if d is even, M is spin. Then this spin structure extends to a spin structure on M, therefore $\gamma = 0$.

We have thus seen that if $V \subset \mathbb{C}\mathrm{P}^3$ is a complex surface of degree d with a unique singularity which is of type A_k, under the assumption that k is even, d is even, $T(A_k) = \{0\}$ and

$$\begin{cases} d(d^2 - 4d + 6) - 5 & \geq & k \\ d(2d^2 - 6d + 7) - 3 & \geq & 3k \end{cases}$$

all the invariants in Vogel's classification are determined by the pair (d,k).

Let V, V' be as above, M, M' be the smooth parts. We have short exact sequences

$$0 \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \to 0$$

$$0 \to H_2(M') \to H_2(M', \partial M') \to H_1(\partial M') \to 0.$$

By the above computation, it is seen that the the symmetric bilinear forms on $H_2(M)$ and $H_2(M')$ are isometric, so are the $\mathbb{Q}$-valued symmetric bilinear forms on $H_2(M, \partial M)$ and $H_2(M', \partial M')$. Now ∂M and $\partial M'$ are both lens spaces $L(k+1, k)$. We choose an diffeomorphism $\varphi : \partial M \to \partial M'$, then φ induces an isometry of the linking forms, $\varphi_* : H_1(\partial M) \to H_1(\partial M')$. Now we can apply a theorem of Nikulin [Ni] and claim that φ_* extends to an isometry of the short exact sequences. Acctually what we need is a weaker version of Nikulin's theorem:

Theorem ([Ni], Theorem 1.14.2)**.** *Let L be an even, indefinite lattice such that $rk(L) > l(D(L)) + 2$, then each isometry of the linking form of L is induced by an isometry of L.*

Now $H_2(M) \cong I^\perp \subset H_2(V_0)$, $H_2(M)$ is even since $H_2(V_0)$ is even. If the rank of the negatively definite part of $H_2(V_0)$ is strictly larger than the rank of I, (note that $I \cong L_{A_k}$ is negatively definite) then $H_2(M)$ is indefinite. After these considerations we can draw the conclusion:

Theorem 1.1. *Let $V \subset \mathbb{CP}^3$ be a complex surface of degree d with a unique singularity which is of type A_k, under the assumption that k is even, d is even, $T(A_k) = \{0\}$ and*

$$\begin{cases} d(d^2 - 4d + 6) - 5 & \geq & k \\ d(2d^2 - 6d + 7) - 3 & > & 3k \end{cases}$$

the hemeomorphism type of V is completely determined by (d, k).

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