Derivation of a macroscopic model for nutrient uptake by a single branch of hairy-roots

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ABSTRACT

In this article we study the process of nutrient uptake by a single root branch. We consider diffusion and active transport of nutrients dissolved in water. The uptake happens on the surface of thin root hairs distributed periodically and orthogonal to the root surface. Water velocity is defined by the Stokes equations. We derive a macroscopic model for nutrient uptake by a hairy root. The macroscopic model consists of a reaction-diffusion equation in the domain with hairs, and diffusion-convection equation in the domain without hairs. The macroscopic water velocity is described by the Stokes system in the domain without hairs, with no-slip condition on the boundary between domains with hairs and of free fluid.

Keywords: Homogenization, two-scale convergence, reaction-diffusion equations, flow in porous medium, partially perforated domain, Stokes equations.
AMS: 35B27, 74Q10, 74Q15, 35K57, 35K60, 76D07, 76M50.

1 Introduction

Hairy roots are roots genetically transformed by *Agrobacterium rhizogenes*. The resulting hairy root culture can be cultivated under sterile conditions in a hairy-root-reactor or in a flask. Hairy roots generally form numerous lateral branches and have a high growth rate. Hairy roots of Ophiorrhiza Mungos are currently gaining interest of pharmacologists, since a secondary product of their metabolism, camptothecin, is used in chemotherapie. An innovative approach for the production of pharmaceutical substances is the cultivation of hairy roots using a bioreactor. The roots can be cultivated for several weeks or months. During this time the pharmaceutical substances are constantly extracted from the bioreactor. In order to optimize biological processes in the bioreactor, especially the production of secondary metabolites that are valuable for pharmaceutical industry, it is necessary to understand the metabolism and growth of roots and to describe the transport processes through the roots network. Root growth and the creation of new branches depend on the supply of nutrients. To understand this process, we develop a mathematical model of nutrient transport and uptake.

Here, we consider the nutrient uptake by a single branch of the hairy-root. The surface of a root is covered with fine hairs. Hairs enlarge the surface of roots and, thus, increase the uptake of nutrients. For the flow processes the hairs sustain an obstacle due to their high density. In our model we consider water flow and diffusion of nutrient molecules dissolved in the water. The water velocity is defined by the Stokes equations. Substrates diffuse and are transported by the flow in the fluid part and are absorbed on the surface of the hairs. The scale of hairs is to small for numerical computation and therefore the derivation of a macroscopic model is required.

Thus, the aim of this work is to derive a macroscopic equation for nutrient uptake by a single branch of hairy roots, based on a microscopic description, using methods of asymptotic analysis (homogenization). Homogenization is a technique to pass from the microscopic model to a macroscopic model letting the proper scale parameter ε in the system tend to zero.

The model we propose is defined on a partially perforated domain. Fluid flow is defined by the Stokes equation, and nutrient concentration is modeled by a diffusion equation with the uptake reaction defined on the boundary of the microstructure. Thus, in the analysis we combine different techniques related to each part of the complete problem. Derivations of macroscopic models describing coupling of the diffusion and convection processes between cells, with diffusion in the cell or porous blocks through the reaction on the surface can be found in [3, 7, 16]. Homogenization of reaction-diffusion and reaction-diffusion-convection equations coupled with linear or nonlinear ordinary differential equations or with diffusion equations defined on the surface of cells was studied in [15, 23, 6, 14, 21]. The derivation of the macroscopic equations for the flow in partially perforated domains was considered in [12, 11, 13]. Homogenization of the elliptic equation in the partially perforated domain is shown in [9].

In order to define a macroscopic equation for the nutrient concentration we use the technique of two-scale convergence, which was introduced in [1] and [22] and extended to sequences of functions defined on surfaces in [2] and [23]. For the macroscopic model in the domain with hairs of constant length we obtain a reaction-diffusion equation with a reaction term related to the uptake process on the surface of hairs. As the macroscopic model for water velocity we obtain Stokes equations in the domain without hairs with no-slip condition on the boundary of the domain with hairs. A better approximation for the water velocity requires a construction of the boundary layer, see [13]. For our complicated geometry a boundary layer correction could be constructed only locally and will not be considered in this paper.

The paper is organized as follows. First, we present description of the considered geometry and the microscopic model. Then, we outline shortly known results on existence and uniqueness of solutions of the microscopic model. In section 3 we show a priori estimates for the water velocity and define macroscopic equations for the velocity field. In section 4 we prove a priori estimates for nutrient concentration and, after extension of the solutions from the porous domain to the whole domain, using there estimates, we show the convergence of solutions of the microscopic problem to the solutions of a macroscopic homogenized model.

2 Problem setting

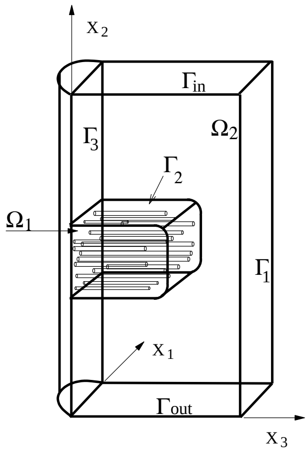
We consider a single root with hairs orthogonal to the root surface and distributed periodically. The nutrient uptake happens mostly on the hairs' surface.
Let $\Omega = (0,1) \times (0,M)^2$. For $0 < m_1 < m_2 < M$ and a smooth (C^2) function $G : \mathbb{R}^2 \to \mathbb{R}$ with $\sup_{x_1,x_2} |G| < M$ we define $\Omega_1 = \{(x_1, x_2) \in (0,1) \times (m_1, m_2), x_3 = G(x_1, x_2)\}$ and $\Omega_2 = \Omega \backslash \Omega_1$. For a mathematical formulation of the problem we define

- "Standard cell", $Z = [0,1]^2$, repeated periodically over $\mathbb{R}^2$, $Y_0 \subset Z$, an open subset with a smooth boundary $\Gamma = \partial Y_0$, $Y = Z \setminus \bar{Y_0}$, and ν the outer normal of Y.

- $Z^k = (Z + \sum_{i=1}^2 k_i e_i)$, $\quad Y_0^k = (Y_0 + \sum_{i=1}^2 k_i e_i)$, $\Gamma^k = \Gamma + \sum_{i=1}^2 k_i e_i$ for $k \in \mathbb{Z}^2$; $R^* = \cup \Gamma^k \times (0, L)$,

- $Q^\varepsilon = \cup\{\varepsilon Z^k | \varepsilon Z^k \subset \Omega_1 \cap \{x_3 = 0\}\}$, $R^\varepsilon = \cup\{\varepsilon \Gamma^k | \varepsilon Z^k \subset \Omega_1 \cap \{x_3 = 0\}\}$; $\Gamma^\varepsilon = \cup\{\varepsilon \Gamma^k \times (0,L) | \varepsilon Z^k \times (0,L) \subset \Omega_1\}$, L is the length of the hairs, $L \le \sup_{x_1,x_2} |G|$, $\varepsilon > 0$ is the ratio between the size of a cell and of the whole domain Ω_1.

- $\Omega_0^\epsilon = \cup\{\epsilon Y_0^k \times (0,L) | \epsilon Z^k \times (0,L) \subset \Omega_1\}$, $\Omega_1^\epsilon = \Omega_1 \setminus \Omega_0^\epsilon$ and $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2$.



We consider water flow and diffusion and an active transport of nutrients along a single root. The velocity of water flow is given by the Stokes equation

$$\begin{aligned}
&-\Delta u^\varepsilon + \nabla p^\varepsilon = 0 && \text{in } \Omega^\varepsilon,\\
&\text{div } u^\varepsilon = 0 && \text{in } \Omega^\varepsilon,\\
&p^\varepsilon = p_i,\ u^\varepsilon \times \nu = 0 && \text{on } \Gamma_{in},\\
&p^\varepsilon = p_o,\ u^\varepsilon \times \nu = 0 && \text{on } \Gamma_{out},\\
&u^\varepsilon = 0 && \text{on } \Gamma_1 \cup \Gamma_3,\\
&u^\varepsilon = 0 && \text{on } \Gamma^\varepsilon,\\
&u^\varepsilon,\ p^\varepsilon - \text{is } 1 - \text{periodic} && \text{in } x_1.
\end{aligned} \tag{1}$$

Remark. For the flat boundary div $u^\varepsilon = 0$ and $p^\varepsilon = p_i,\ u^\varepsilon \times \nu = 0$ on Γ_{in} is equivalent to $(\nabla u^\varepsilon - p^\varepsilon)\nu\nu = p_i$ and $u^\varepsilon \times \nu = 0$ on Γ_{in}.

Nutrient uptake takes place on the surface of the hairs.

$$
\begin{aligned}
&\partial_t c^\varepsilon - \nabla\cdot(D\nabla c^\varepsilon) + u^\epsilon \nabla c^\varepsilon = 0 && \text{in } (0,T)\times\Omega^\varepsilon,\\
&c^\varepsilon = c_D && \text{on } \Gamma_{in},\\
&(D\nabla c^\varepsilon - u^\varepsilon c^\varepsilon)\cdot\nu = 0 && \text{on } \Gamma_{out},\\
&\nabla c^\varepsilon\cdot\nu = 0 && \text{on } \Gamma_1\cup\Gamma_3,\\
&-D\nabla c^\varepsilon \nu^\varepsilon = \varepsilon f^\varepsilon(t,x,c^\varepsilon) && \text{on } \Gamma^\varepsilon,\\
&c^\varepsilon - \text{is } 1-\text{periodic} && \text{in } x_1,\\
&c^\varepsilon(0) = c_0 && \text{in } \Omega^\varepsilon,
\end{aligned}
\tag{2}
$$

where the uptake kinetic $f^\varepsilon(t,x,c^\varepsilon)$ can be modeled by a Michaelis-Menten kinetic

$$f(c^\varepsilon) = \frac{K_m c^\varepsilon}{K_n + c^\varepsilon}, \quad K_m > 0, K_n > 0.$$

The diffusion coefficient D^ε is defined in Ω_1 by a $Z-$ periodic function $D^\varepsilon_{i,j}(t,x) = D_{i,j}(t,x,\frac{x}{\varepsilon})$. The general reaction term is defined by a $Z-$ periodic function $f^\varepsilon(t,x,\xi) = f(t,\frac{x}{\varepsilon},\xi)$ defined on R^*.

We pose the following assumptions on the coefficients of the model.

Assumption 2.1 *1) The diffusion coefficient $D \in L^\infty((0,T)\times\Omega\times Z)^{3\times3}$ is uniformly elliptic: $D(t,x,y)\xi\xi \geq d_0|\xi|^2$, $d_0 > 0$, for $\xi\in\mathbb{R}^3$ and $\partial_t D \in L^\infty((0,T)\times\Omega\times Z)^{3\times3}$.*

2) The reaction term $f(t,y,\xi)$ is sublinear, Lipschitz continuous in ξ, differentiable in t, measurable in y, and positive for positive ξ, i.e. $f(t,y,\xi)\geq 0$ for $\xi\geq 0$.

3) The boundary condition $c_D \in H^1(0,T;H^2(\Omega))$, $c_D\in H^2(0,T;L^2(\Omega))$, c_D is periodic in x_1, the initial condition $c_0\in H^2(\Omega)$, $c_0|_{\partial\Omega} = c_D(0,x)$.

We define the spaces

$$
\begin{aligned}
V(\Omega^\varepsilon) &= \{v\in H^1(\Omega^\varepsilon),\, v=0 \text{ on } \Gamma_1\cup\Gamma_3,\, v\times\nu = 0 \text{ on } \Gamma_{in}\cup\Gamma_{out},\\
&\qquad v = 0 \text{ on } \Gamma^\varepsilon,\, v \text{ is periodic in } x_1\};\\
V_d(\Omega^\varepsilon) &= \{v\in V(\Omega^\varepsilon),\ \text{div } v = 0\};\\
W &= \{v\in H^1(\Omega^\varepsilon),\, v=0 \text{ on } \Gamma_{in}, v \text{ is periodic in } x_1\}.
\end{aligned}
$$

We start with a weak formulation of the microscopic model.

Definition 2.2 *A weak solution of (1), (2) is a triple of functions* $(u^\varepsilon,\, p^\varepsilon,\, c^\varepsilon)$ *such that*

$$u^\varepsilon \in V_d(\Omega^\varepsilon),\; p^\varepsilon \in L^2(\Omega^\varepsilon),$$

$$c^\varepsilon - c_D \in L^2(0,T;W),\; c^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \;\; \textit{and}$$

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon\, \nabla\phi\, dx - \int_{\Omega^\varepsilon} p^\varepsilon\;\, div\, \phi\, dx = -\int_{\Gamma_{in}} p_i\, \phi\cdot\nu\, d\sigma - \int_{\Gamma_{out}} p_o\, \phi\cdot\nu\, d\sigma, \tag{3}$$

$$\int_0^T\int_{\Omega^\varepsilon} \Big(\partial_t c^\varepsilon\, \psi + D^\varepsilon \nabla c^\varepsilon\, \nabla\psi - u^\varepsilon\, c^\varepsilon\, \nabla\psi\Big)\, dx\, dt = -\varepsilon \int_0^T\int_{\Gamma^\varepsilon} f^\varepsilon(t,x,c^\varepsilon)\psi\, d\sigma\, dt, \tag{4}$$

for all functions $\phi \in V(\Omega^\varepsilon)$ *and* $\psi \in L^2(0,T;W)$.

Theorem 2.3 *Let Assumption 2.1 be satisfied. Then, there exists a unique weak solution of the problem (1)–(2) such that* $u^\varepsilon \in V_d(\Omega^\varepsilon)\cap H^2(\Omega^\varepsilon_\delta)$, $p^\varepsilon \in L^2(\Omega^\varepsilon)\cap H^1(\Omega^\varepsilon_\delta)$, $c^\varepsilon - c_D \in L^2(0,T;W)$, $c^\epsilon \in H^1(0,T;L^2(\Omega^\epsilon))$, *where* $\Omega^\varepsilon_\delta$ *is the domain without the corners at the end of hairs.*

Proof. The existence of a solution of the Stokes equation with prescribed pressure on the boundary is shown in [8, 11]. Using Lax Milgram theorem and DeRham theorem a solution $u^\varepsilon \in V_d(\Omega^\varepsilon)$ and $p^\varepsilon \in L^2(\Omega^\varepsilon)$ is obtained. The solution is uniquely defined due to the boundary conditions for u^ε and p^ε. The regularity of the solution follows from the regularity for elliptic equations, regularity of the boundary of Ω_1 and boundary condition $u^\varepsilon = 0$ on $\Gamma_1\cup\Gamma_3$ (such boundary condition allows the extension of the solution across the root boundary by reflection). Thus the solution is in the space $H^2(\Omega^\varepsilon_\delta)\times H^1(\Omega^\varepsilon_\delta)$ ($\Omega^\varepsilon_\delta$ is the domain Ω^ε without corners at the end of hairs).
The existence of a solution, c^ε, of the parabolic equation can be shown using the existence of a solution for the problem with linear boundary conditions, see [17, 18]. Then, for $u^\varepsilon \in V_d\cap H^2(\Omega^\varepsilon_\delta)$ and $c^\varepsilon_{n-1} \in L^2(0,T;H^\beta(\Omega^\varepsilon))$, $1/2 \le \beta \le 1$, we obtain the solution of linear problem in $L^2(0,T;H^1(\Omega^\varepsilon))\cap H^1(0,T;L^2(\Omega^\varepsilon))$. Since, due to Lemma of Lions-Aubion, the embedding $L^2(0,T;H^1(\Omega^\varepsilon))\cap H^1(0,T;L^2(\Omega^\varepsilon)) \hookrightarrow L^2(0,T;H^\beta(\Omega^\varepsilon))$ for $1/2<\beta<1$ is compact, we conclude on the existence of a solution c^ε of problem (2).
To provide a priori estimates for c^ε we apply the following

$$\int_0^T\int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla(c^\varepsilon - c_D)\, dxdt \le \int_0^T\int_{\Gamma_{out}} u^\varepsilon (c^\varepsilon - c_D)^2\, d\sigma dt + \int_0^T\int_{\Omega^\varepsilon} u^\varepsilon c_D \nabla(c^\varepsilon - c_D)\, dxdt$$
$$\le \sup_{\Gamma_{out}} |u^\varepsilon| ||c^\varepsilon||_{L^2((0,T)\times\Omega^\varepsilon)} ||c^\varepsilon||_{L^2(0,T;H^1(\Omega^\varepsilon)} + \sup_{(0,T)\times\Omega^\varepsilon} |c_D| ||u^\varepsilon||_{L^2(\Omega^\varepsilon)} ||c^\varepsilon - c_D||_{L^2(0,T;H^1(\Omega^\varepsilon)}.$$

The uniqueness of the solution c^ε follows from the Lipschitz continuity of f and can be shown by considering the equation for the difference of two solutions c^ε_1 and c^ε_2. ■

3 Macroscopic equations for the fluid flow

We assume the following macroscopic model for the water flow

$$
\begin{aligned}
&-\Delta u^0 + \nabla \pi^0 = 0 && \text{in } \Omega_2,\\
&\text{div } u^0 = 0 && \text{in } \Omega_2,\\
&u^0 = 0 && \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3,2},\\
&u^0 \times \nu = 0,\ \pi^0 = p_i && \text{on } \Gamma_{in},\\
&u^0 \times \nu = 0,\ \pi^0 = p_o && \text{on } \Gamma_{out},\\
&u^0,\ \pi^0 - 1 - \text{periodic} && \text{in } x_1.
\end{aligned}
$$

Here $\Gamma_{3,2} = \Gamma_3 \cap \overline{\Omega}_2$. There exists a unique solution $u^0 \in V_d(\Omega_2) \cap H^2(\Omega_2)$, $\pi^0 \in H^1(\Omega_2)$, [11]. We extend u^0 by zero into Ω_1.

To show that u^0 is a macroscopic approximation of the microscopic velocity u^ε we need the following estimates in the porous medium.

Lemma 3.1 *Let $\phi \in H^1(\Omega_1^\varepsilon)$ be such that $\phi = 0$ on $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \setminus \partial\Omega_1$. Then, the following estimates hold*

$$
\begin{aligned}
||\phi||_{L^2(\Omega_1^\varepsilon)} &\le C\varepsilon ||\nabla\phi||_{L^2(\Omega_1^\varepsilon)},\\
||\phi||_{L^2(\Gamma_2)} &\le C\varepsilon^{1/2} ||\nabla\phi||_{L^2(\Omega_1^\varepsilon)}.
\end{aligned}
$$

Proof sketch. We consider first a unit cell Y. Zero boundary conditions on Γ yield the estimate

$$\int\limits_{Y\times(0,L)} |\phi(\bar{y}, y_3)|^2 dy \le C \int\limits_{Y\times(0,L)} |\nabla_{\bar{y}}\phi(\bar{y}, y_3)|^2 dy.$$

Then, for the scaling $\bar{x} = \varepsilon\bar{y}$, $x_3 = y_3$ we obtain

$$\int\limits_{\varepsilon Y\times(0,L)} |\phi(\frac{\bar{x}}{\varepsilon}, x_3)|^2 \frac{dx}{\varepsilon^2} \le C \int\limits_{\varepsilon Y\times(0,L)} \varepsilon^2 |\nabla_{\bar{x}}\phi(\frac{\bar{x}}{\varepsilon}, x_3)|^2 \frac{dx}{\varepsilon^2} \le C \int\limits_{\varepsilon Y\times(0,L)} \varepsilon^2 |\nabla\phi(\frac{\bar{x}}{\varepsilon}, x_3)|^2 \frac{dx}{\varepsilon^2}$$

and

$$\int\limits_{\Omega_1^\varepsilon} |\phi|^2\, dx \le C \sum_{i=1}^{N} \int\limits_{\varepsilon Y_i\times(0,L)} |\phi(\frac{\bar{x}}{\varepsilon}, x_3)|^2 d\bar{x}dx_3 \le C\varepsilon^2 \int\limits_{\Omega_1^\varepsilon} |\nabla\phi|^2\, dx.$$

For the estimate on the boundary we extend the function ϕ by zero into the whole Ω_1. Then, using the trace theorem for a function from $H^1(\Omega_1)$ we obtain

$$||\phi||_{L^2(\Gamma_2)} \le C ||\phi||^{1/2}_{L^2(\Omega_1)} ||\nabla\phi||^{1/2}_{L^2(\Omega_1)}.$$

Due to the estimate for $||\phi||_{L^2(\Omega_1^\varepsilon)}$ we obtain the second estimate of the lemma. ■

Now we can obtain estimates for $u^\varepsilon - u^0$.

Lemma 3.2 *For the solution of the Stokes problem we obtain the following a priori estimates*

$$\begin{aligned}
&||\nabla(u^\varepsilon - u^0)||_{L^2(\Omega^\varepsilon)^3} \le C\sqrt{\varepsilon},\\
&||u^\varepsilon||_{L^2(\Omega_1^\varepsilon)^3} \le C\varepsilon\sqrt{\varepsilon},\\
&||u^\varepsilon||_{L^2(\Gamma_2)^3} \le C\varepsilon,\\
&||u^\varepsilon - u^0||_{L^2(\Omega_2)^3} \le C\varepsilon,\\
&||p^\varepsilon - \pi^0||_{L^2(\Omega_2)} \le C\sqrt{\varepsilon},
\end{aligned}$$

where C is a constant independent from ε.

Proof. We consider the equation for the difference $u^\varepsilon - u^0$ and use the estimates in Lemma 3.1

$$\begin{aligned}
&\int\limits_{\Omega^\varepsilon} \nabla(u^\varepsilon - u^0)\nabla\phi\, dx - \int\limits_{\Omega^\varepsilon}(p^\varepsilon - \pi^0\chi(\Omega_2))\nabla\phi\, dx = \int\limits_{\Gamma_2}(\nabla u^0 - \pi^0)\,\nu\,\phi\, d\gamma\\
&\le \frac{1}{2}||\nabla u^0 - \pi^0||_{L^2(\Gamma_2)}||\phi||_{L^2(\Gamma_2)} \le C\varepsilon^{1/2}\Big(\int\limits_{\Gamma_2}|\nabla u^0 - \pi^0|^2\, d\gamma\Big)^{1/2}||\nabla\phi||_{L^2(\Omega_1^\varepsilon)}.
\end{aligned}$$

The estimate $\int_{\Gamma_2}|\nabla u^0 - \pi^0|^2 d\gamma \le C$ follows from the regularity of u^0 in the domain Ω_2. Then using div $u^\varepsilon = 0$ and div $u = 0$, Poincarès inequality and the trace inequality in Ω_1^ε yield

$$\begin{aligned}
||\nabla(u^\varepsilon - u^0)||_{L^2(\Omega_2)} &\le C\varepsilon^{1/2}, \qquad ||\nabla u^\varepsilon||_{L^2(\Omega_1^\varepsilon)} \le C\varepsilon^{1/2},\\
||u^\varepsilon||_{L^2(\Omega_1^\varepsilon)} &\le C\varepsilon^{3/2}, \qquad ||u^\varepsilon||_{L^2(\Gamma_2)} \le C\varepsilon.
\end{aligned}$$

To obtain the last two estimates in Lemma 3.2 we consider the equations for $w^\varepsilon = u^\varepsilon - u^0$ and $\pi^\varepsilon = p^\varepsilon - \pi^0$

$$\begin{aligned}
-\Delta w^\varepsilon + \nabla\pi^\varepsilon = 0 &\qquad \text{in } \Omega_2,\\
\text{div } w^\varepsilon = 0 &\qquad \text{in } \Omega_2,\\
w^\varepsilon = u^\varepsilon &\qquad \text{on } \Sigma = \Gamma_2,\\
w^\varepsilon = 0 &\qquad \text{on } \Gamma_1 \cup \Gamma_{3,2},\\
w^\varepsilon \times \nu = 0,\ \pi^\varepsilon = 0 &\qquad \text{on } \Gamma_{in} \cup \Gamma_{out},\\
w^\varepsilon,\, \pi^\varepsilon - \text{is } 1 - \text{periodic} &\qquad \text{in } x_1.
\end{aligned}$$

Now, we use the estimate for a very weak solution w^ε for the Stokes system, [5, 20]. We seek a solution $(w^\varepsilon, \pi^\varepsilon) \in L^2(\Omega_2) \times H^{-1}(\Omega_2)$ using the transposition method (for the definition of very weak solution see Appendix). Thus, we obtain

$$||w^\varepsilon||_{L^2(\Omega_2)} \leq C||u^\varepsilon||_{L^2(\Gamma_2)} \leq C\varepsilon.$$

The estimate for the pressure follows from the estimate for the velocity using Necas inequality

$$||\pi^\varepsilon||_{L^2(\Omega_2)} \leq C||\nabla \pi^\varepsilon||_{H^{-1}(\Omega_2)} \leq C\varepsilon^{1/2}. \qquad \blacksquare$$

4 Macroscopic equations for nutrient concentration

We will derive macroscopic equations for c^ε using tools of two-scale convergence. At first we prove a priori estimates for c^ε.

Lemma 4.1 *For the solution c^ε of the microscopic problem and $\varepsilon \leq d_0^2/4$, where d_0 is an upper bound for the matrix of diffusion coefficients, holds*

$$||c^\epsilon||_{L^\infty(0,T;L^2(\Omega^\epsilon))} + \|\nabla c^\epsilon||_{L^2(0,T;L^2(\Omega^\epsilon))} \leq C,$$
$$||\partial_t c^\epsilon\|_{L^\infty(0,T;L^2(\Omega^\epsilon))} + \|\partial_t \nabla c^\epsilon\|_{L^2(0,T;L^2(\Omega^\epsilon))} \leq C,$$

independent from ε.

Proof. We take $c^\varepsilon - c_D$ as a test function in equation (4) and obtain

$$\int_0^\tau \int_{\Omega^\varepsilon} \partial_t c^\varepsilon (c^\varepsilon - c_D)\, dxdt + \int_0^\tau \int_{\Omega^\varepsilon} D^\varepsilon \nabla c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt - \int_0^\tau \int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt$$
$$= -\varepsilon \int_0^\tau \int_{\Gamma^\varepsilon} f^\varepsilon(t, x, c^\varepsilon)(c^\varepsilon - c_D)\, d\sigma_x dt.$$

Now, we estimate the above integrals separately:

$$\int_0^\tau \int_{\Omega^\varepsilon} (D\nabla c^\varepsilon, \nabla c_D)\, dxdt \leq \delta \int_0^\tau \int_{\Omega^\varepsilon} |\nabla c^\varepsilon|^2\, dxdt + \frac{C}{\delta} \int_0^\tau \int_{\Omega^\varepsilon} |\nabla c_D|^2\, dxdt,$$
$$\int_0^\tau \int_{\Omega^\varepsilon} \partial_t c^\varepsilon c_D\, dxdt = \int_{\Omega^\varepsilon} \Big(c^\varepsilon(\tau) c_D(\tau) - c_0 c_D(0)\Big)\, dx - \int_0^\tau \int_{\Omega^\varepsilon} c^\varepsilon \partial_t c_D\, dxdt \leq$$
$$\int_{\Omega^\varepsilon} (\frac{1}{4}|c^\varepsilon(\tau)|^2 + 4|c_D(\tau)|^2 + \frac{1}{2}|c_0|^2 + \frac{1}{2}|c_D(0)|^2)\, dx + \frac{1}{2}\int_0^\tau \int_{\Omega^\varepsilon} \Big(|c^\varepsilon|^2 + |\partial_t c_D|^2\Big)\, dxdt.$$

For the convection term, using the estimate for $(u^\varepsilon - u^0)$ we obtain

$$\int_0^\tau \int_{\Omega^\varepsilon} u^\varepsilon c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt = \int_0^\tau \int_{\Omega^\varepsilon} (u^\varepsilon - u^0 + u^0) c^\varepsilon \nabla (c^\varepsilon - c_D)\, dxdt$$

$$\leq C\varepsilon^{1/2} ||c^\varepsilon||^2_{L^2(0,\tau;H^1(\Omega^\varepsilon))} + \sup_{\Omega_2} |u^0| \int_0^\tau \int_{\Omega^\varepsilon} (\frac{1}{\delta}|c^\varepsilon|^2 + \delta |\nabla (c^\varepsilon - c_D)|^2)\, dxdt.$$

To obtain estimates for the boundary integral we apply

$$||c^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \leq C||c^\varepsilon||^2_{L^2(\Omega^\varepsilon)} + \varepsilon^2 C ||\nabla c^\varepsilon||^2_{L^2(\Omega^\varepsilon)}.$$

It holds since by scaling we have

$$\varepsilon \int_{\Gamma^\epsilon} |c^\varepsilon|^2\, d\gamma \leq C\varepsilon \int_0^L \int_{R^\varepsilon} |c^\varepsilon|^2\, d\gamma \leq C \int_0^L \int_{Q^\varepsilon} \Big(|c^\varepsilon|^2 + \varepsilon^2 |\nabla c^\epsilon|^2 \Big)\, dx$$

$$\leq C \int_{\Omega^\epsilon} \Big(|c^\varepsilon|^2 + \varepsilon^2 |\nabla c^\varepsilon|^2 \Big)\, dx.$$

Then, we obtain that

$$\int_0^\tau \int_{\Gamma^\varepsilon} f^\varepsilon(t,x,c^\varepsilon)(c^\varepsilon - c_D)\, d\sigma_x dt \leq c_f \int_0^\tau \int_{\Gamma^\varepsilon} \Big(|c^\varepsilon|^2 + |c^\varepsilon||c_D| \Big) d\sigma_x dt$$

$$\leq C \int_0^\tau \int_{\Omega^\varepsilon} \Big(|c^\varepsilon|^2 + \varepsilon^2 |\nabla c^\varepsilon|^2 + |c_D|^2 + \varepsilon^2 |\nabla c_D|^2 \Big)\, dxdt.$$

Using the ellipticity assumption on D^ε, Gronwall inequality and Poincare inequality we obtain the first estimate in the lemma.
To obtain the estimate for time derivative we differentiate the equation with respect to t and use $\partial_t (c^\varepsilon - c_D)$ as a test function

$$\int_0^T \int_{\Omega^\varepsilon} \partial_t^2 c^\varepsilon \partial_t (c^\varepsilon - c_D)\, dxdt + \int_0^T \int_{\Omega^\varepsilon} D^\varepsilon \partial_t \nabla c^\varepsilon \partial_t \nabla (c^\varepsilon - c_D)\, dxdt$$

$$+ \int_0^T \int_{\Omega^\varepsilon} \partial_t D^\varepsilon \nabla c^\varepsilon \partial_t \nabla (c^\varepsilon - c_D)\, dxdt - \int_0^T \int_{\Omega^\varepsilon} u^\varepsilon \partial_t \nabla c^\varepsilon \partial_t \nabla (c^\varepsilon - c_D)\, dxdt$$

$$= -\varepsilon \int_0^T \int_{\Gamma^\varepsilon} \partial_\xi f^\varepsilon(t,x,c^\varepsilon) \partial_t c^\varepsilon \partial_t (c^\varepsilon - c_D)\, d\sigma_x dt - \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \partial_t f^\varepsilon(t,x,c^\varepsilon) \partial_t (c^\varepsilon - c_D)\, d\sigma_x dt.$$

Similar calculations as above yield estimates for the time derivative. Here, we apply

$$\int_0^\tau \int_{\Omega^\varepsilon} \partial_t^2 c^\varepsilon \partial_t c_D \, dxdt = \int_{\Omega^\varepsilon} \partial_t c^\varepsilon \partial_t c_D \, dx \Big|_0^\tau - \int_0^\tau \int_{\Omega^\varepsilon} \partial_t c^\varepsilon \partial_t^2 c_D \, dxdt \le \frac{1}{4} \int_{\Omega^\varepsilon} |\partial_t c^\varepsilon(\tau)|^2 \, dx +$$

$$4 \int_{\Omega^\varepsilon} |\partial_t c_D(\tau)|^2 \, dx + \frac{1}{2} \int_{\Omega^\varepsilon} (|\partial_t c^\varepsilon(0)|^2 + |\partial_t c_D(0)|^2) \, dx + \frac{1}{2} \int_0^\tau \int_{\Omega^\varepsilon} (|\partial_t c^\varepsilon|^2 + |\partial_t^2 c_D|^2) \, dxdt.$$

Due to the regularity assumption on c_0^ε and c_D, i.e. $c_D \in H^1(0,T;H^2(\Omega))$, $c_0 \in H^2(\Omega)$, and $c_0 = c_D(0)$, we obtain

$$\int_{\Omega^\varepsilon} (|\partial_t c^\varepsilon(0)|^2 + |\partial_t c_D(0)|^2) \, dx \le C\big(||c_0||_{H^2(\Omega)} + ||c_D||_{H^1(0,T;H^2(\Omega))}\big). \qquad \blacksquare$$

4.1 Convergence

Since c^ε is defined only on the domain Ω_1^ε we have to extend it into all Ω_1, see [4], [15], [23] for the proof.

Lemma 4.2 *1. For $c \in H^1(Y)$ there exists an extension $\tilde{c}$ from Y to Z, such that*

$$\|\tilde{c}\|_Z \le c_1 \|c\|_Y \quad \text{and} \quad \|\nabla \tilde{c}\|_Z \le c_2 \|\nabla c\|_Y.$$

2. For $c^\epsilon \in H^1(\Omega^\epsilon)$ there exists an extension $\tilde{c^\epsilon}$ from Ω^ϵ to Ω, such that

$$\|\tilde{c^\epsilon}\|_{H^1(\Omega)} \le c_3 \|c^\epsilon\|_{H^1(\Omega^\epsilon)}.$$

Remark 4.1 *For $c^\epsilon \in L^2(0,T;H^1(\Omega^\epsilon))$ we define*

$$\bar{c^\epsilon}(\cdot,t) := \tilde{c^\epsilon}(\cdot,t),$$

$c^\epsilon(\cdot,t) \in H^1(\Omega^\epsilon)$ for a.e. t. Since the extension operator is linear, $\bar{c^\epsilon} \in L^2(0,T;H^1(\Omega))$.

For the convergence on the boundary Γ^ε we use the following estimate

Lemma 4.3 *For a function $v^\varepsilon \in W^{\beta,2}(\Omega_1^\varepsilon)$, $\frac{1}{2} < \beta < 1$ one has the estimate*

$$\varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon|^2 \, d\sigma_x \le C \int_{\Omega_1^\varepsilon} |v^\varepsilon|^2 dx + C\varepsilon^{2\beta} \int_{\Omega_1^\varepsilon} \int_{Q^\varepsilon} \frac{|v^\varepsilon(\bar{x}_1, x_n) - v^\varepsilon(\bar{x}_2, x_n)|^2}{|\bar{x}_1 - \bar{x}_2|^{n-1+2\beta}} d\bar{x}_1 dx_2 \le ||v^\varepsilon||_{W^{\beta,2}(\Omega_1^\varepsilon)}.$$

Proof. For a function $v \in W^{\beta,2}(Y)$ we have from trace theorem

$$\int\limits_{\Gamma} |v|^2 \, d\sigma_{\bar{y}} \leq C \int\limits_{Y} |v|^2 d\bar{y} + C \int\limits_{Y} \int\limits_{Y} \frac{|v(y_1) - v(y_2)|^2}{|y_1 - y_2|^{n-1+2\beta}} d\bar{y}_1 d\bar{y}_2.$$

Now we apply the transformation $\bar{y} = \bar{x}/\varepsilon$ and obtain

$$\int\limits_{\varepsilon\Gamma_i} |v^\varepsilon|^2 \frac{d\sigma_{\bar{x}}}{\varepsilon^{n-2}} \leq C \int\limits_{\varepsilon Y_i} |v^\varepsilon|^2 \frac{d\bar{x}}{\varepsilon^{n-1}} + C \int\limits_{\varepsilon Y_i} \int\limits_{\varepsilon Y_i} \frac{|v^\varepsilon(\bar{x}_1, x_n) - v^\varepsilon(\bar{x}_2, x_n)|^2}{|\bar{x}_1 - \bar{x}_2|^{n-1+2\beta}} \varepsilon^{n-1+2\beta} \frac{d\bar{x}_1}{\varepsilon^{n-1}} \frac{d\bar{x}_2}{\varepsilon^{n-1}}.$$

Integrating the inequality over x_n, multiplying by ε^{n-1} and summing up over i from 1 to N, we obtain the estimate in the lemma. ∎

Thus, from the estimates for c^ε we obtain the following convergences

Lemma 4.4 *For c^ε the following convergence holds*

$$\begin{aligned}
c^\varepsilon &\to c && \textit{weakly in } L^2(0,T;H^1(\Omega)) \textit{ and weakly} - * \textit{ in } L^\infty(0,T;L^2(\Omega)\\
\partial_t c^\varepsilon &\to \partial_t c && \textit{weakly in } L^2(0,T;H^1(\Omega)) \textit{ and weakly} - * \textit{ in } L^\infty(0,T;L^2(\Omega),\\
c^\varepsilon &\to c && \textit{strongly in } L^2(0,T;W^{\beta,2}(\Omega)),\ \frac{1}{2} < \beta < 1,\\
c^\varepsilon &\to c && \textit{in two-scale sense in } \Omega_1,\\
\nabla c^\varepsilon &\to \nabla_x c + \nabla_y c_1 && \textit{in two-scale sense and } c_1 \in L^2((0,T) \times \Omega_1; H^1_{per}(Z)/\mathbb{R}),\\
\partial_t c^\varepsilon &\to \partial_t c && \textit{in two-scale sense in } \Omega_1.
\end{aligned}$$

And also

$$\lim_{\varepsilon \to 0} ||c^\epsilon - c||_{L^2((0,T)\times\Gamma^\varepsilon)} = 0.$$

Proof. From a priori estimates, Lemma 4.1, we obtain a weak convergence $c^\varepsilon \rightharpoonup c$ and $\partial_t c^\varepsilon \rightharpoonup \partial_t c$ in $L^2(0,T;H^1(\Omega))$, and weak-$*$ convergence in $L^\infty(0,T;L^2(\Omega)$.

To obtain strong convergence of c^ε in $L^2((0,T), W^{\beta,2}(\Omega))$, $\frac{1}{2} < \beta < 1$, we use the compact embedding of $W^{\beta,2}(\Omega)$ in $H^1(\Omega)$ and apply the Lions-Aubin Lemma, [19], with $B = W^{\beta,2}(\Omega)$. From Lemma 4.3 follows the inequality $\|c^\varepsilon\|^2_{\Gamma^\epsilon} \leq c_1 \|c^\varepsilon\|^2_{W^{\beta,2}(\Omega^\varepsilon)}$. Therefore, we obtain $\|c^\varepsilon - c\|_{L^2((0,T)\times\Gamma^\epsilon)} \leq c_1 \|c^\varepsilon - c\|^2_{L^2(0,T;W^{\beta,2}(\Omega^\epsilon))} \leq c_2 \|c^\varepsilon - c\|^2_{L^2(0,T;W^{\beta,2}(\Omega))} \to 0$ for $\varepsilon \to 0$.

Since c^ε, $\partial_t c^\varepsilon$ converges weakly to c, $\partial_t c$ in $L^2(0,T;H^1(\Omega))$, the compactness theorem (see Theorem 6.2 in Appendix) implies the two-scale convergence of c^ε and $\partial_t c^\varepsilon$ to the same functions c and $\partial_t c$, and existence of a function $c_1 \in L^2((0,T) \times \Omega)_1; H^1_{per}(Z)/\mathbb{R})$ such that, up to a subsequence, ∇c^ε two-scale converges to $\nabla_x c(x) + \nabla_y c_1(x,y)$. ∎

Now we can take the limit $\varepsilon \to 0$ and derive the macroscopic model for nutrient concentration.

Theorem 4.5 *The solutions of the microscopic problem c^ε converge to the solution of the following macroscopic problem*

$$
\begin{aligned}
\partial_t c_2 + u_0 \nabla c_2 - \nabla\cdot(D\nabla c_2) &= 0 && in\ \ (0,T)\times\Omega_2,\\
\partial_t c_1 - \nabla\cdot(D^{hom}\nabla c_1) + \frac{1}{|Y|}\int\limits_\Gamma f(t,y,c_1)\,d\sigma_y &= 0 && in\ \ (0,T)\times\Omega_1,\\
D^{hom}\nabla c_1\cdot\nu &= D\nabla c_2\cdot\nu && on\ \ \Gamma_2=\partial\Omega_1,\\
c_1 &= c_2 && on\ \ \Gamma_2,\\
c_2 &= c_D && on\ \ \Gamma_{in},\\
(D\nabla c_2 - u^0 c_2)\cdot\nu &= 0 && on\ \ \Gamma_{out},\\
\nabla c_2\cdot\nu &= 0 && on\ \ \Gamma_1\cup\Gamma_{3,2},\\
\nabla c_1\cdot\nu &= 0 && on\ \ \Gamma_{3,1},\\
c_1,\ c_2\ \ is\ 1-periodic & && in\ \ x_1,\\
c(0) &= c_0 && in\ \ \Omega,
\end{aligned}
$$

where $D^{hom}_{ij} = \frac{1}{|Y|}\sum_{k=1}^2\int_Y (D_{ij}(t,x,y) + D_{ik}(t,x,y)\partial_{y_k}s_j)\,dy$ *and* s_j *is the solution of the cell problem*

$$
-\nabla_y(D\,\nabla_y s_i) = \sum_{k=1}^{2}\partial_{y_k}D_{ki}\ \ in\ Y, \quad -D\frac{\partial s_i}{\partial\nu} = D_i\nu\ \ on\ \Gamma.
$$

Proof. We can rewrite the equation for c^ε in the form

$$
\int\limits_0^T\int\limits_{\Omega_2} c_t^\varepsilon\,\phi\,dx\,dt + \int\limits_0^T\int\limits_{\Omega_1^\varepsilon} c_t^\varepsilon\,\phi\,dx\,dt + \int\limits_0^T\int\limits_{\Omega_2} D\nabla c^\varepsilon\nabla\phi\,dxdt + \int\limits_0^T\int\limits_{\Omega_1^\varepsilon} D^\varepsilon\nabla c^\varepsilon\nabla\phi\,dxdt
$$
$$
-\int\limits_0^T\int\limits_{\Omega_2} u^\varepsilon\,c^\varepsilon\nabla\phi\,dx\,dt - \int\limits_0^T\int\limits_{\Omega_1^\varepsilon} u^\varepsilon\,c^\varepsilon\nabla\phi\,dx\,dt = -\varepsilon\int\limits_0^T\int\limits_{\Gamma_\varepsilon} f(c^\varepsilon)\,\phi\,d\sigma_x\,dt.
$$

Using as a test function $\phi\in C(0,T;C_0^\infty(\Omega_2))$, we obtain

$$
\int\limits_0^T\int\limits_{\Omega_2} c_t^\varepsilon\,\phi\,dxdt + \int\limits_0^T\int\limits_{\Omega_2} D\nabla c^\varepsilon\nabla\phi\,dxdt - \int\limits_0^T\int\limits_{\Omega_2} u^\varepsilon c^\varepsilon\nabla\phi\,dxdt = 0.
$$

The estimate $||u^\varepsilon - u^0||_{L^2(\Omega_2)} \le C\varepsilon$ or strong convergence of c^ε implies the convergence

$$
\int\limits_0^T\int\limits_{\Omega_2} u^\varepsilon c^\varepsilon\nabla\phi\,dxdt \to \int\limits_0^T\int\limits_{\Omega_2} u^0 c\nabla\phi\,dxdt.
$$

Thus, due to the weak convergence of c^ε in Ω_2 we obtain

$$\int_0^T\int_{\Omega_2} c_t\,\phi\,dxdt + \int_0^T\int_{\Omega_2} D\nabla c\nabla\phi\,dxdt - \int_0^T\int_{\Omega_2} u^0 c\nabla\phi\,dxdt = 0.$$

In Ω_1^ε we use the extension of function c^ε from Ω_1^ε to Ω_1 and the two-scale limit with a test function $\phi = \phi_1 + \varepsilon\phi_2$, $\phi_1 \in C((0,T); C_0^\infty(\Omega_1))$, $\phi_2 \in C((0,T); C_0^\infty(\Omega_1); C_{per}^\infty(Z))$ and obtain

$$\int_0^T\int_{\Omega_1} \chi_\varepsilon\, c_t^\varepsilon\,(\phi_1+\varepsilon\phi_2)dx\,dt \to \int_0^T\int_{\Omega_1} |Y|\,c_t\,\phi_1\,dx\,dt,$$

$$\int_0^T\int_{\Omega_1} \chi_\varepsilon\, u^\varepsilon\, c^\varepsilon\nabla(\phi_1+\varepsilon\phi_2)dx\,dt \to \int_0^T\int_{\Omega_1}\int_Y u\,c\,(\nabla\phi_1+\nabla_y\phi_2)\,dx\,dt\,dy$$

$$= |Y|\int_0^T\int_{\Omega_1} u\,c\,\nabla\phi_1\,dx\,dt = 0,$$

$$\int_0^T\int_{\Omega_1} \chi_\varepsilon\, D^\varepsilon\,\nabla c^\varepsilon\nabla(\phi_1+\varepsilon\phi_2)dxdt \to \int_0^T\int_{\Omega_1}\int_Y D\,(\nabla c+\nabla_y c_1)(\nabla\phi_1+\nabla_y\phi_2)\,dx\,dt\,dy.$$

Strong convergence of c^ε on Γ_ε and the Lipschitz continuity of f yield

$$\varepsilon\int_0^T\int_{\Gamma_\varepsilon} |(f^\varepsilon(t,x,c^\varepsilon) - f^\varepsilon(t,x,c))\phi(t,x,\frac{\bar{x}}{\varepsilon})|d\sigma_\varepsilon dxdt$$
$$\leq C_1||c^\varepsilon - c||_{L^2((0,T)\times\Gamma^\varepsilon)}||\phi||_{L^2((0,T)\times\Gamma^\varepsilon)} \leq \sigma(\varepsilon).$$

Thus, using the two-scale convergence of $f^\varepsilon(t,x,c)$ on Γ_ε we obtain for the boundary integral

$$\varepsilon\int_0^T\int_{\Gamma_\varepsilon} f^\varepsilon(t,x,c^\varepsilon)(\phi_1+\varepsilon\phi_2)d\sigma_x\,dt = \varepsilon\int_0^T\int_{\Gamma_\varepsilon}(f^\varepsilon(t,x,c^\varepsilon) - f^\varepsilon(t,x,c))(\phi_1+\varepsilon\phi_2)d\sigma_x\,dt$$

$$+\varepsilon\int_0^T\int_{\Gamma_\varepsilon} f^\varepsilon(t,x,c)(\phi_1+\varepsilon\phi_2)d\sigma_x\,dt \to \int_0^T\int_{\Omega_1}\int_\Gamma f(t,y,c)\,\phi_1\,d\sigma_y dx\,dt.$$

Then, the limit equation reads

$$\int_0^T \int_{\Omega_2} c_t\, \phi_1\, dx\, dt + \int_0^T \int_{\Omega_2} D\, \nabla c \nabla \phi_1\, dx\, dt - \int_0^T \int_{\Omega_2} u^0\, c\, \nabla \phi_1\, dx\, dt$$
$$+ \int_0^T \int_{\Omega_1} c_t\, \phi_1\, dx\, dt + \frac{1}{|Y|} \int_0^T \int_{\Omega_1} \int_Y D(\nabla c + \nabla_y c_1)(\nabla \phi_1 + \nabla_y \phi_2) dx\, dt\, dy$$
$$= -\frac{1}{|Y|} \int_0^T \int_{\Omega_1} \int_\Gamma f(t, y, c)\, d\sigma_y\, \phi_1\, dx\, dt.$$

To find an unknown function c_1 we choose in the last equation $\phi_1 = 0$ and obtain

$$\int_0^T \int_{\Omega_1} \int_Y D(\nabla c + \nabla_y c_1) \nabla_y \phi_2\, dx\, dt\, dy = 0.$$

From here follows that

$$c_1 = \sum_{k=1}^{3} s_k \nabla_{x_k} c,$$

where s_k are solutions of

$$\begin{aligned} -\nabla_y (D(t, y) \nabla s_k) &= \sum_{j=1}^{2} \partial_{y_j} D_{kj}(t, y), \\ -D \nabla s_k \cdot \nu &= \sum_{j=1}^{2} D_{kj}\, \nu_j. \end{aligned}$$

Then we obtain

$$\int_0^T \int_{\Omega_2} c_t\, \phi_1 dx\, dt + \int_0^T \int_{\Omega_2} D \nabla c \nabla \phi_1 dx\, dt + \int_0^T \int_{\Omega_2} u^0 \nabla c \phi_1 dx\, dt$$
$$+ \int_0^T \int_{\Omega_1} c_t\, \phi_1 dx\, dt + \int_0^T \int_{\Omega_1} D^{hom} \nabla c \nabla \phi_1 dx\, dt = -\frac{1}{|Y|} \int_0^T \int_{\Omega_1} \int_\Gamma f(t, y, c)\, d\sigma_y \phi_1 dx\, dt,$$

where $D_{ij}^{hom} = \frac{1}{|Y|} \sum_{k=1}^{2} \int_Y (D_{ij} + D_{ik} \partial_k s_j)\, dy$. We denote the concentration of nutrients in Ω_1 and Ω_2 by c_1 and c_2 respectively and obtain on the boundary $\partial\Omega_1$ in the weak sense the continuity condition $c^1 = c^2$ and $D \nabla c^1\, \nu = D^{hom} \nabla c^2\, \nu$ on $\partial\Omega_1$. ■

5 Conclusion.

We derived a macroscopic model for water transport and nutrients uptake by a single root branch. We found out that the uptake kinetics defined on the root hair surface are comming as reaction term in the macroscopic equation for nutrient concentration. The rigorous derivation of macroscopic model for a whole root system is possible only under a strong assumption on the geometry of the root network. Our macroscopic model for the uptake process verified the modeling of nutrients uptake process by whole root system as reaction-diffusion equation with reaction term, defined uptake process and depend on the root density.

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6 Appendix

Definition 6.1 *1. A sequence $\{v^\epsilon\} \subset L^2(\Omega)$ converges two-scale to $v \in L^2(\Omega \times Z)$ iff for any $\phi \in \mathcal{D}(\Omega, C^\infty_{per}(Z))$*

$$\lim_{\epsilon\to 0} \int_\Omega v^\epsilon(x)\phi(x, \frac{x}{\epsilon})\, dx = \int_\Omega \int_Z v(x,y)\phi(x,y)\, dxdy.$$

2. A sequence $\{v^\epsilon\} \subset L^2(\Gamma^\epsilon)$ converges two-scale to $v \in L^2(\Omega\times\Gamma)$ iff for $\psi \in \mathcal{D}(\Omega, C^\infty_{per}(\Gamma))$

$$\lim_{\epsilon\to 0} \epsilon \int_{\Gamma^\epsilon} v^\epsilon(x)\psi(x, \frac{x}{\epsilon}) d\gamma_x = \int_\Omega \int_\Gamma v(x,y)\psi(x,y) dx d\gamma_y.$$

Theorem 6.2 *1. Let $\{v_\epsilon\}$ be a bounded sequence in $L^2(\Lambda, H^1(\Omega))$, which converges weakly to a limit function $v \in L^2(\Lambda, H^1(\Omega))$. Then there exists $v_1 \in L^2(\Lambda \times \Omega, H^1_{per}(Z))$ such that, up to a subsequence, v_ϵ two-scale converges to v and ∇v_ϵ two-scale converges to $\nabla v(\lambda, x) + \nabla_y v_1(\lambda, x, y)$.*
2. Let $\{v_\epsilon\}$ and $\epsilon\nabla v_\epsilon$ be bounded sequences in $L^2(\Lambda \times \Omega))$. Then there exists $v_0 \in L^2(\Lambda \times \Omega, H^1_{per}(Z))$ such that, up to a subsequence, v_ϵ and $\epsilon\nabla v_\epsilon$ two-scale converge to $v_0(\lambda, x, y)$ and $\nabla_y v_0(\lambda, x, y)$ respectively.

Theorem 6.3 *From each bounded sequence $\{v^\epsilon\}$ in $L^2(\Lambda \times \Gamma^\epsilon)$ we can extract a subsequence, which two-scale converges to $v \in L^2(\Lambda \times \Omega \times \Gamma)$.*

For very weak solution we seek a solution $(w, \pi) \in L^2(\Omega_2) \times H^{-1}(\Omega_2)$ of

$$\begin{array}{ll}
-\Delta w + \nabla \pi = f & \text{in } \Omega_2, \\
\text{div } w = 0 & \text{in } \Omega_2, \\
w = \xi & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\
w \times \nu = \zeta_1, \;\; \pi = \pi_i & \text{on } \Gamma_{in}, \\
w \times \nu = \zeta_2, \;\; \pi = \pi_o & \text{on } \Gamma_{out}, \\
w, \pi \text{ is } 1-\text{periodic} & \text{in } x_1.
\end{array}$$

Let $\{\phi, q\}$ be given by

$$\begin{aligned}
&-\Delta\phi + \nabla q = g && \text{in } \Omega_2,\\
&\text{div } \phi = h && \text{in } \Omega_2,\\
&\phi = 0 && \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma,\\
&\phi \times \nu = 0,\ q = 0 && \text{on } \Gamma_{in} \cup \Gamma_{out},\\
&\phi,\ q \text{ is } 1-\text{periodic} && \text{in } x_1.
\end{aligned}$$

For $g \in L^2(\Omega_2)^3$, $h \in H = \{h \in H_0^1(\Omega_2), \int\limits_{\Omega_2} h = 0\}$ we have the solution $\phi \in H^2(\Omega_2)^2$, $q \in H^1(\Omega_2)$. Now we test the equations for w and π by ϕ and using $\int_{\Omega_2} w\nabla q\, dx = \int_\Gamma qI\nu w\, d\sigma$ obtain

$$\begin{aligned}
\int\limits_{\Omega_2} f\phi &= \int\limits_{\Omega_2} (-\Delta w + \nabla\pi)\phi\, dx = \int\limits_{\Omega_2} (-w\Delta\phi + w\nabla q - \pi \text{ div } \phi)\, dx\\
&+ \int\limits_{\Gamma} (\nabla\phi - qI)\nu w\, d\sigma - \int\limits_{\Gamma_{in}} (\zeta_1\nabla\phi\nu + \pi_i\phi\nu)\, d\sigma - \int\limits_{\Gamma_{out}} (\zeta_2\nabla\phi\nu + \pi_o\phi\nu)\, d\sigma.
\end{aligned}$$

We consider the linear continuous form $l : L^2(\Omega_2)^3 \times H \to \mathbb{R}$

$$\begin{aligned}
l(g,h) &= \langle f, \phi\rangle - \int\limits_{\Gamma} (\nabla\phi - qI)\nu\xi\, d\sigma\\
&\quad + \int\limits_{\Gamma_{in}} (\zeta_1\nabla\phi\nu + \phi_i\phi\nu)\, d\sigma + \int\limits_{\Gamma_{out}} (\zeta_2\nabla\phi\nu + \phi_o\phi\nu)\, d\sigma.
\end{aligned}$$

Definition 6.4 *We define (w, π) as a very weak solution if $(w, \pi) \in L^2(\Omega_2)^3 \times H^*$ and*

$$\int\limits_{\Omega_2} wg - \langle\pi, h\rangle_{H^*,H} = l(g,h) \quad \textit{for all} \quad (g,h) \in L^2(\Omega_2)^3 \times H.$$

Because of the linearity and continuity of l, the Riesz theorem implies the following

Proposition 6.5 ([5]) *There exists a unique very weak solution (w, π),*

$$\begin{aligned}
||w||_{L^2(\Omega_2)^3} &\leq C\Big(||f||_{L^2(\Omega_2)^3} + ||\xi||_{L^2(\Gamma_2)^3}\\
&\quad + ||\zeta_1||_{L^2(\Gamma_{in})} + ||\zeta_2||_{L^2(\Gamma_{out})} + ||\pi_i||_{L^2(\Gamma_{in})} + ||\pi_o||_{L^2(\Gamma_{out})}\Big).
\end{aligned}$$