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Signature Homology and Symmetric L-theory

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Abstract

Firstly, we prove the existence of an assembly map for the integral Novikov problem formulated in [Min04]. To achieve this we show that signature homology is a direct summand of Ranicki's symmetric L-theory and use the assembly map for symmetric L-theory.

Secondly, we construct a map from the bordism theory of PL-pseudomanifolds having a Poincaré duality in integral intersection homology to symmetric L-theory. We show that the homotopy cofibre of this map is an Eilenberg-MacLane space $K(\mathbb{Z}/2, 1)$. Thus, we obtain a geometric bordism description of symmetric L-theory.

Zunächst beweisen wir die Existenz einer Assemblyabbildung für das ganzzahlige Novikov Problem aus [Min04]. Um dies zu erreichen zeigen wir, das Signaturhomology ein direkter Summand von Ranickis symmetrischer L-Theorie ist. Nun können wir die Assemblyabbildung für symmetrische L-Theorie benutzen.

Weiterhin konstruieren wir eine Abbildung von der Bordismustheorie von PL-Pseudomannigfaltigkeiten, für die es eine Poincaré Dualität in ganzzahliger Schnitthomologie gibt, in die symmetrische L-Theorie. Wir zeigen, dass die Homotopiekofaser dieser Abbildung durch den Eilenberg-MacLane Raum $K(\mathbb{Z}/2, 1)$ gegeben ist. Auf diese Weise erhalten wir eine Beschreibung von symmetrischer L-Theorie als geometrischen Bordismus.

Introduction

The present work originated in the question if there is a construction of an assembly map for the integral Novikov problem formulated in [Min04]. Classically, the assembly map for the Novikov conjecture is a method to decide wether or not the conjecture holds for a given group. More precisely, let G be a discrete group, let M be an oriented closed smooth manifold of dimension n with $\pi_1(M) = G$ and $\alpha : M \to K(G,1)$ be a map. Then the Novikov conjecture for G predicts that the characteristic number

$$\mathrm{sig}_x(M,\alpha) := \langle L(M) \cup \alpha^*(x), [M] \rangle \in \mathbb{Q},$$

where $L(M)$ is the L-class of M, is homotopy invariant for all $x \in H^*(K(G,1);\mathbb{Q})$. That is, given another oriented closed smooth manifold N and an orientation preserving homotopy equivalence $f : N \to M$ we have

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N, \alpha \circ f).$$

Equivalently, the class

$$L_G(M,\alpha) = \alpha_*(L(M) \cap [M]) \in \bigoplus\nolimits_k H_{n-4k}(K(G,1);\mathbb{Q})$$

is homotopy invariant.

Let $\mathcal{S}_n(M)$ the set of isomorphisms classes of pairs (N,f), where N is a n-dimensional oriented closed smooth manifold and $f : N \to M$ an orientation preserving homotopy equivalence. Then the assembly map A is a map

$$A : \bigoplus\nolimits_k H_{n-4k}(K(G,1);\mathbb{Q}) \to L^n(\mathbb{Z}[G]) \otimes \mathbb{Q}$$

such that the composition

$$\begin{array}{rcl} \mathcal{S}_n(M) & \to & \bigoplus_k H_{n-4k}(K(G,1);\mathbb{Q}) \xrightarrow{A} L^n(\mathbb{Z}[G]) \otimes \mathbb{Q} \\ (N,f) & \mapsto & L_G(M,\alpha) - L_G(N,\alpha \circ f) \end{array}$$

is zero. Therefore, the Novikov conjecture for the group G follows from the injectivity of A. In fact, it is known that it is equivalent to the injectivity of A.

If we look for an integral refinement of the Novikov conjecture it is natural to look at signature homology defined in [Min04]. It's main properties are the existence of a natural transformation of multiplicative homology theories

$$u : \Omega^{SO} \to \mathrm{Sig}$$

and an isomorphism of graded rings

$$\mathrm{sig} : \mathrm{Sig}_* \to \mathbb{Z}[t],$$

where $\deg t = 4$, such that the following diagram commutes:

$$\begin{array}{ccc} \Omega_*^{SO} & \xrightarrow{u} & \mathrm{Sig}_* \\ & \searrow^{\mathrm{sig}} & \downarrow{\scriptstyle \mathrm{sig}} \\ & & \mathbb{Z}[t] \end{array}$$

Now, each closed oriented smooth manifold M of dimension n has a signature homology orientation class

$$[M]_{\mathrm{Sig}} := u([M, \mathrm{id}]) \in \mathrm{Sig}_n(M),$$

where $[M, \mathrm{id}] \in \Omega_n^{SO}(M)$ is the bordisms class of the identity. For $\pi_1(M) = G$, we say that $[M]_{\mathrm{Sig}}$ is homotopy invariant if for any map $\alpha : M \to K(G, 1)$ and for any other oriented manifold N together with an orientation preserving homotopy equivalence $f : N \to M$ we have

$$\alpha_*([M]_{\mathrm{Sig}}) = (\alpha \circ f)_*([N]_{\mathrm{Sig}}) \in \mathrm{Sig}_n(K(G, 1)).$$

If we take the tensor product with $\mathbb{Q}$ we have

$$\mathrm{Sig}_*(-) \otimes \mathbb{Q} \cong \bigoplus_{k=0}^{\infty} H_{*-4k}(-; \mathbb{Q}).$$

Furthermore, it can be shown that the signature homology orientation class reduces to

$$[M]_{\mathrm{Sig}} \otimes \mathbb{Q} = L(M) \cap [M] \in \bigoplus\nolimits_k H_{n-4k}(M; \mathbb{Q}).$$

Therefore, the Novikov conjecture for G is equivalent to the homotopy invariance of the rational signature homology fundamental class and an integral refinement would be the homotopy invariance of $[M]_{\mathrm{Sig}}$.

Integral Novikov problem. **(Kreck)** *Determine all discrete groups G for which the signature homology orientation class is homotopy invariant.*

Similarly to the rational case we are now looking for an assembly map whose injectivity would determine the answer to the integral Novikov problem for a given group G. That is, we are looking for a map

$$A : \mathrm{Sig}_n(K(G, 1)) \to L^n(\mathbb{Z}[G])$$

such that the composition

$$\begin{array}{rcl} \mathcal{S}_n(M) & \to & \mathrm{Sig}_n(K(G,1)) \xrightarrow{A} L^n(\mathbb{Z}[G]) \\ (N,f) & \mapsto & \alpha_*[M]_{\mathrm{Sig}} - (\alpha \circ f)_*[N]_{\mathrm{Sig}} \end{array}$$

is zero.

If we search the literature we quickly realize that there is another integral refinement of the Novikov conjecture which makes use of the symmetric L-theory of Ranicki in place of signature homology. It shares the property that each closed oriented smooth manifold has an L-theory orientation class which reduces to the Poincaré dual of the L-class after tensorizing with $\mathbb{Q}$.

Integral Novikov problem. (Ranicki) *Determine all discrete groups G for which the symmetric L-theory orientation class is homotopy invariant.*

Fortunately, in this setting Ranicki constructed an assembly map whose injectivity decides his integral Novikov problem. It is therefore obvious to ask how signature homology relates to symmetric L-theory. We will answer this question using the determination of the homotopy types of both theories. The result is that signature homology is a direct summand of symmetric L-theory.

It is important to note that there are finite groups for which the integral Novikov problem is known to be false. This explains why the term conjecture is replaced by problem in the integral setting.

Having answered this question the next step is to look for a geometric description of the assembly map for signature homology. This seems desireable since both the definition of symmetric L-theory and the assembly map make use of complicated simplicial methods which are not easily accessible.

While we fail to achieve this goal we will at least be able to reach a partial result which can be seen as a first step into this direction. Namely, we will show that symmetric L-theory can be described as bordism of certain spaces with singularities called IP-spaces, at least after passing to the 2-connected covers of both theories. IP-spaces are defined by the property that Poincaré duality holds for the intersection cohomology groups with integer coefficients.

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Contents

Chapter I

Sheaves

The purpose of this chapter is to provide background information on sheaf theory. We try to assemble all the bits and pieces which will be useful later. Therefore, the account will be very short and we will give proofs only for the facts which can not be found in the textbooks on sheaf theory.

1 Complexes of sheaves and the derived category of sheaves

The first section is devoted to the definition of sheaves and the various categories of sheaves. In particular, we will treat interesting subcategories of the derived category. Note that by a functor we always mean a covariant functor unless explicitly stated otherwise.

For a given topological space X let $\mathcal{X}$ be the category having the open sets of X as objects and inclusions as morphisms.

1.1 Definition. Let $\mathcal{A}$ be one of the abelian categories $\mathcal{A}b$ of abelian groups or $R - \mathcal{M}od$ of R-modules. Let X be a topological space.

(1) A *presheaf* P in $\mathcal{A}$ on X is a functor

$$P : \mathcal{X}^{op} \to \mathcal{A}.$$

For an open set U the elements of $P(U)$ are called the *sections* over U.

A *morphism* $f : P \to Q$ of presheaves P, Q is a natural transformation

$$f : P \to Q.$$

(2) For $x \in X$, P a presheaf. We define the *stalk* P_x at x by

$$P_x := \lim_{\rightarrow} P(U),$$

where the limit is taken over all neighbourhoods U of x.

(3) Let $U \subset X$ be an open set and $s \in P(U)$ a section over U. The *support* of s is the closed subset

$$\mathrm{supp}(s) := \{x \in U \mid s_x \neq 0 \in \mathbf{P}_x\}.$$

(4) A *sheaf* $\mathbf{P}$ in $\mathcal{A}$ on X is a presheaf such that for all open sets $U \subset X$ and all coverings $U \subset \bigcup U_i$ there is an exact sequence

$$\mathbf{P}(U) \to \prod_i \mathbf{P}(U_i) \xrightarrow{diff} \prod_{i,j} \mathbf{P}(U_i \cap U_j).$$

Presheaves form a category denoted by $PSh(X)$, let $Sh(X)$ denote the full subcategory consisting of all sheaves.

In $Sh(X)$ we have the following useful characterization of isomorphisms

1.2 Lemma. *A morphism $f : \mathbf{P} \to \mathbf{Q}$ is an isomorphism if and only if $f_x : \mathbf{P}_x \to \mathbf{Q}_x$ is an isomorphism for all $x \in X$.*

1.3 Proposition. *Both $PSh(X)$ and $Sh(X)$ are abelian categories. But $Sh(X)$ is not an abelian subcategory of $PSh(X)$. This is because the inclusion functor*

$$v : Sh(X) \to PSh(X)$$

is not right exact in general.

Although the inclusion functor v is not exact we at least have

1.4 Proposition. *The inclusion functor v admits a left adjoint*

$$\mathrm{sheaf} : PSh(X) \to Sh(X)$$

such that the adjoint morphism

$$\mathrm{sheaf} \circ v \to 1$$

is an isomorphism. sheaf *is called* sheafification. *It follows that v is left exact. Moreover,* sheaf *is an exact functor.*

1.5 Definition. Let $Ch(X)$ denote the category of cochain complexes of sheaves. Let $K(X)$ denote the corresponding homotopy category and $D(X)$ the derived category.

1.6 Proposition. *The derived category $D(X)$ exists in our universe.*

Proof. See [We94], 10.4.4. □

As usual we will denote by $Ch^b(X)$, $K^b(X)$ and $D^b(X)$ the full subcategories of $Ch(X)$, $K(X)$ and $D(X)$ of bounded complexes of sheaves.

1.7 Proposition. *Assume $\mathcal{A}$ is of finite injective dimension and X is of finite cohomological dimension. Let $\mathfrak{I}$ denote the full subcategory of $Sh(X)$ consisting of all injective sheaves. Then the inclusion functor*

$$K^b(\mathfrak{I}) \to K^b(X)$$

induces an equivalence of categories

$$K^b(\mathfrak{I}) \to D^b(X).$$

Proof. See [We94], 10.4.8. for the similar result for bounded below complexes, but without the finiteness assumptions. Now, the additional assumptions assure that the proof given there is still valid. □

1.8 Remark. Of course, the finiteness condition on $\mathcal{A}$ is satisfied for the category of abelian groups, for the category of R-modules the assumption means that R is of finite global dimension. We will always assume this and that X is of finite cohomological dimension.

We close this section with the following useful class of sheaves on pseudomanifolds.

1.9 Definition. Let X be a topological pseudomanifold of dimension n (for a definition see chapter II). A bounded complex of sheaves $\mathbf{A}^\bullet$ on X is called *constructible* if for all i and for all $0 \leq j \leq n$, $\mathbf{H}^i(\mathbf{A}^\bullet)_{|X_j - X_{j-1}}$ is locally constant and has finitely generated stalks. Let $Ch^b_c(X)$ resp. $D^b_c(X)$ be the full subcategory of $Ch^b(X)$ resp. $D^b(X)$ consisting of all constructible complexes of sheaves.

These sheaves have the following useful property.

1.10 Proposition. *Let $\mathbf{A}^\bullet$ be a constructible complex of sheaves on a compact pseudomanifold X. Then the hypercohomology groups $\mathbb{H}^k(X; \mathbf{A}^\bullet)$ are finitely generated for all k.*

2 Functors of sheaves

In this section we will collect the definitions of various functors on categories of sheaves and their relations. We will start with the functors associated to a map of spaces $f : X \to Y$.

2.1 Definition. The *direct image* functor $f_* : Sh(X) \to Sh(Y)$ is given by

$$\mathbf{P} \mapsto (U \mapsto \mathbf{P}(f^{-1}(U))).$$

It is right adjoint to the *inverse image* functor $f^* : Sh(Y) \to Sh(X)$ given by

$$\mathbf{P} \mapsto \text{sheaf}\left(U \mapsto \varinjlim_{V \supset f(U)} \mathbf{P}(V)\right).$$

The adjointness implies that f_* is left exact and f^* is right exact. In addition f^* is exact.

If f is an inclusion then f^* is called the restriction and denoted by $\mathbf{P}_{|X}$.

If we assume in addition that X and Y are locally compact then we have the *direct image with proper support* $f_! : Sh(X) \to Sh(Y)$ given by

$$\mathbf{P} \mapsto (U \mapsto \{s \in \mathbf{P}(U) \mid \operatorname{supp}(s) \hookrightarrow U \text{ is proper}\}) .$$

$f_!$ is left exact, however, $f_!$ is not an adjoint in general. Only after passing to the derived category, can a left adjoint always be obtained. However, if we assume that f is an inclusion of a locally closed subspace then a left adjoint $f^!$ can already be constructed on the sheaf level (see V.4 for details).

The cases of the collapse map $c : X \to pt$ and the inclusion of a point $i : x \hookrightarrow X$ are of special importance. Using the identification $Sh(pt) \cong \mathcal{A}$ we define:

2.2 Definition. (1) For $x \in X$, the *stalk functor* $(-)_x : Sh(X) \to \mathcal{A}$ is given by

$$\mathbf{P} \mapsto i^*\mathbf{P} = \mathbf{P}_x.$$

Its right adjoint is called the *skyscraper sheaf functor* i_*.

(2) The *global section functor* $\Gamma : Sh(X) \to \mathcal{A}$ is given by

$$\mathbf{P} \mapsto c_*\mathbf{P} = \mathbf{P}(X).$$

Its right adjoint is called the *constant sheaf functor* $c : \mathcal{A} \to Sh(X)$ and is given by

$$M \mapsto c^*(M) = \operatorname{sheaf}(U \mapsto M).$$

2.3 Definition. (1) The external Hom-functor Hom : $Sh(X)^{op} \times Sh(X) \to \mathcal{A}b$ is given by

$$(\mathbf{P}, \mathbf{Q}) \mapsto \operatorname{Hom}_{Sh(X)}(\mathbf{P}, \mathbf{Q}).$$

It is left exact in both variables.

(2) The internal Hom-functor **Hom** : $Sh(X)^{op} \times Sh(X) \to Sh(X)$ is given by

$$(\mathbf{P}, \mathbf{Q}) \mapsto (U \mapsto \operatorname{Hom}_{Sh(U)}(\mathbf{P}_{|U}, \mathbf{Q}_{|U})).$$

It is clear that we have the equality

$$\operatorname{Hom}_{Sh(X)}(\mathbf{P}, \mathbf{Q}) = \Gamma\mathbf{Hom}(\mathbf{P}, \mathbf{Q}).$$

The internal Hom-functor is left exact in both variables.

(3) The tensor product functor $- \otimes - : Sh(X) \times Sh(X) \to Sh(X)$ is given by

$$(\mathbf{P}, \mathbf{Q}) \mapsto \operatorname{sheaf}(U \mapsto \mathbf{P}(U) \otimes \mathbf{Q}(U)).$$

It is right exact in both variables.

(4) For a fixed sheaf $\mathbf{Q}$, we have that $- \otimes \mathbf{Q}$ is left adjoint to $\mathbf{Hom}(\mathbf{Q}, -)$

Since all the functors introduced above have some exactness property, we can pass to the derived category $D_b(X)$ and get derived functors Rf_*, f^*, $Rf_!$, $R\Gamma$, $R\mathrm{Hom}$, $R\mathbf{Hom}^\bullet$ and $\overset{L}{\otimes}$. Additionally, we also have a left adjoint $f^!$ to $Rf_!$. The functors $\mathbf{Hom}$ and $\otimes$ preserve constructibility and give rise to functors on Ch_c^b and D_c^b. The same holds for the functors f_*, f^*, $f_!$ and $f^!$ if we assume that f is a stratified map of pseudomanifolds.

Finally, we have the truncation functors

2.4 Definition. Let $\mathbf{A}^\bullet \in Ch(X)$ by a complex of sheaves, let $n \in \mathbb{Z}$. There are functors

$$\tau_{\leq n} : Ch(X) \to Ch(X)$$

and

$$\tau_{\geq n} : Ch(X) \to Ch(X)$$

given by

$$\mathbf{A}^\bullet \mapsto (\ldots \to \mathbf{A}^{n-2} \to \mathbf{A}^{n-1} \to \ker d^n \to 0 \to \ldots)$$

and

$$\mathbf{A}^\bullet \mapsto (\ldots 0 \to \operatorname{coker} d^{n-1} \to \mathbf{A}^{n+1} \to \mathbf{A}^{n+2} \to \ldots).$$

These are exact functors and preserve boundedness and constructibility. Therefore, they can be regarded as functors on all categories of complexes of sheaves introduced above.

Chapter II

Pseudomanifolds and Intersection homology

In this chapter we will treat a class of spaces with singularities, namely, the class of pseudomanifolds. These generalize the concept of a topological manifold in a suitable way. For an approach in the differential context see [Kr07]. We will study topological invariants for pseudomanifolds introduced by Goresky and MacPherson [GM83] called intersection homology. The major motivation for introducing these new invariants is a generalization of Poincaré duality to pseudomanifolds and the possibility to define characteristic classes for them. We will give only the proofs which contain important techniques we want to use later. All the missing arguments can be found again in [GM83] or in [Bo84].

1 Pseudomanifolds

From now on, let R be a noetherian ring of finite global dimension.

1.1 Definition. A *PL-space* X is a topological space together with a class $\mathcal{T}$ of locally finite triangulations which is closed under linear subdivision, and if T and T' are in $\mathcal{T}$, then so is their common linear refinement.

1.2 Definition. (1) A *stratified topological pseudomanifold* of dimension 0 is a countable set of points with the discrete topology. A stratified topological pseudomanifold of dimension n is a paracompact Hausdorff space X together with a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_{n-2} = X_{n-1} \subset X_n = X$$

such that

- $X_k - X_{k-1}$ are topological manifolds of dimension k, for all k, called the *pure* or *open strata*.

  - $X_n - X_{n-2}$ is dense in X.
  - For every $x \in X_k$ there is an open neighbourhood U of x in X and a compact pseudomanifold L of dimension $n-k-1$ such that $U \cong V \times \mathring{\text{cone}}(L)$ via a stratum preserving homeomorphism.

(2) A *stratified PL-pseudomanifold* is a PL-space X which is a topological pseudomanifold such that the filtration is given by closed PL-subspaces, all strata are PL-spaces and the local triviality is given by a strata preserving PL-isomorphism with L a PL-pseudomanifold. In this case L is called the *link* at x.

The closed subset X_{n-2} of a pseudomanifold X is called the *singular locus* of X and is also denoted by Σ.

1.3 Example. Every locally conelike topological stratifold is a stratified topological pseudomanifold.

2 PL-intersection homology

The intersection homology groups depend on a so called perversity function $\bar{p}$ which we now define.

2.1 Definition. A *perversity* $\bar{p}$ is a function $p : \mathbb{N} \to \mathbb{N}$ such that

- $p(2) = 0$.
- $p(k) \leq p(k+1) \leq p(k) + 1$.

2.2 Examples. There are several perversities which are of particular importance. The zero perversity $\bar{0} : n \mapsto 0$, the lower middle perversity $\bar{m} : n \mapsto \lfloor \frac{n-2}{2} \rfloor$, the upper middle perversity $\bar{n} : n \mapsto \lceil \frac{n-2}{2} \rceil$ and the top perversity $\bar{t} : n \mapsto n-2$.

Note that $\bar{m} + \bar{n} = \bar{t}$. Perversities with this property are called *complementary*.

Let $\bar{p}$ be a perversity. We define the PL-intersection chain complex of a PL-pseudomanifold X for perversity $\bar{p}$.

2.3 Definition. Let $C_\bullet(X)$ be the PL-chain complex of X. Set

$$IC_i^{\bar{p}}(X) := \{\xi \in C_i(X) \mid \begin{array}{l} \dim(|\xi| \cap X_{n-k}) \leq i - k + \bar{p}(k), \\ \dim(|\partial\xi| \cap X_{n-k}) \leq i - 1 - k + \bar{p}(k)\}. \end{array}$$

The conditions ensure that the boundary operators of $C_\bullet(X)$ induce boundary operators on $IC_\bullet^{\bar{p}}$. Thus, $IC_\bullet^{\bar{p}}$ forms a subcomplex of $C_\bullet$.

2.4 Definition. The *PL-intersection homology groups* for perversity $\bar{p}$ of a PL-pseudomanifold X are defined by

$$IH_i^{\bar{p}}(X) := H_i(IC_\bullet^{\bar{p}}).$$

3 Sheaf theoretic intersection homology

We now give a sheaf theoretic construction of intersection homology which applies to topological pseudomanifolds as well. We assume familiarity with the derived category of bounded complexes of sheaves $D^b(X)$ which will be heavily used in this subsection. For details we refer to [Bo84], [GM83] or [Ba06].

Given a n-dimensional topological pseudomanifold X we consider the following open subsets of X:

$$U_k := X - X_{n-k}$$

together with the obvious inclusions $U_k \xrightarrow{i_k} U_{k+1} \xleftarrow{j_k} X_{n-k} - X_{n-k-1}$.

We give an axiomatic characterization of intersection homology.

3.1 Definition. A constructible bounded complex of sheaves $\mathbf{A}^\bullet$ on X *satisfies the set of axioms* [AX1] if and only if

(1) $\mathbf{A}^\bullet_{|X-\Sigma} \cong \mathbf{R}_{X-\Sigma}[n]$, the constant sheaf on $X - \Sigma$.

(2) $\mathbf{H}^i(\mathbf{A}^\bullet) = 0$ for all $i < -n$.

(3) $\mathbf{H}^i(\mathbf{A}^\bullet_{|U_{k+1}}) = 0$ for all $i > \bar{p}(k) - n, k \geq 2$.

(4) The attaching map induces an isomorphism

$$\mathbf{H}^i(j_k^* \mathbf{A}^\bullet_{|U_{k+1}}) \to \mathbf{H}^i(j_k^* Ri_{k*} i_k^* \mathbf{A}^\bullet_{|U_{k+1}})$$

for $i \leq \bar{p}(k) - n$ and $k \geq 2$.

Using the Deligne construction one can show:

3.2 Proposition. *For a given topological pseudomanifold X there is a complex of sheaves $\mathbf{P}^\bullet$ satisfying the axioms [AX1]. This complex of sheaves is unique up to quasi-isomorphism.*

Proof. see [GM83]. We just give the definition of $\mathbf{P}^\bullet$, namely set

$$\mathbf{P}^\bullet := \tau_{\leq p(n)-n} Ri_{n*} \dots \tau_{\leq p(3)-n} Ri_{3*} \tau_{\leq p(2)-n} Ri_{2*} \mathbf{R}_{U_2}[n],$$

where $\tau_{\leq k}$ is the truncation functor. □

For an n-dimensional PL-pseudomanifold X the assignment

$$U \mapsto IC_i^{\bar{p}}(U) =: IC_{\bar{p}}^{-i}(U)$$

defines a complex of presheaves on X denoted by $\mathbf{IC}^\bullet_{\bar{p}}$. We summarize the properties of $\mathbf{IC}^\bullet_{\bar{p}}$

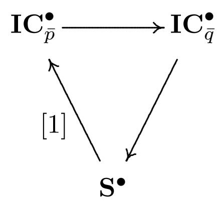
3.3 Proposition. *(1) $\mathbf{IC}^\bullet_{\bar{p}}$ is a complex of sheaves, called the* intersection chain sheaf for perversity $\bar{p}$.

(2) $\mathbf{IC}^\bullet_{\bar{p}}$ is soft.

(3) $\mathbf{IC}^\bullet_{\bar p}$ *satisfies axioms [AX1].*

It follows that for a PL-pseudomanifold the hypercohomology of the complex of sheaves $\mathbf{P}^\bullet_{\bar p}$ in 3.2 computes intersection homology of X. This is the motivation for defining intersection homology of a topological pseudomanifold in just this way. From now on, if there is no danger of confusion, we will always write $\mathbf{IC}^\bullet_{\bar p}$ for any complex of sheaves satisfying axioms [AX1].

We close this subsection by comparing different perversities. Suppose we are given two perversities $\bar p \le \bar q$, obviously there is a canonical morphism $\mathbf{IC}^\bullet_{\bar p} \to \mathbf{IC}^\bullet_{\bar q}$. We want to know under which conditions this is a quasi-isomorphism. Since $D^b_c(X)$ is a triangulated category, there is always a distinguished triangle



and a corresponding long exact sequence in hypercohomology. This sequence is called the *obstruction sequence for comparing $\bar p$ and $\bar q$.* We have the following result.

3.4 Proposition. *Suppose $p(c) = q(c)$ for all $c \neq k$, and $p(k)+1 = q(k)$. Then*

(1) $\operatorname{supp}\mathbf{H}^*(\mathbf{S}^\bullet) = \mathit{closure}(\operatorname{supp}\mathbf{H}^*(\mathbf{S}^\bullet) \cap (X_{n-k} - X_{n-k-1}))$, *where* supp *denotes the support of a sheaf.*

(2) *For $x \in X_{n-k} - X_{n-k-1}$ we have*

$$\mathbf{H}^i(\mathbf{S}^\bullet)_x = \begin{cases} 0 & \text{for all } i \neq q(k)-n \\ \mathbf{H}^i(\mathbf{IC}^\bullet_{\bar q})_x & \text{for } i = q(k)-n. \end{cases}$$

(3) *If we have $\mathbf{H}^{q(k)-n}(\mathbf{IC}^\bullet_{\bar q})_x = 0$ for all $x \in X_{n-k} - X_{n-k-1}$ then $\mathbf{IC}^\bullet_{\bar p} \to \mathbf{IC}^\bullet_{\bar q}$ is a quasi-isomorphism.*

4 The dualizing complex and Verdier duality

We recall two important tools from sheaf theory.

4.1 Definition. Let X be a topological pseudomanifold of dimension n. Fix a bounded injective resolution $R \to I^\bullet$ of R. Let $\mathbf{C}^\bullet(\mathbf{R}_X)$ denote the bounded canonical resolution of the constant sheaf $\mathbf{R}_X$ (see V.1 for more details). The *dualizing complex* is defined as the complex of sheaves

$$U \mapsto \operatorname{Hom}^\bullet(\Gamma_c(\mathbf{C}^\bullet(\mathbf{R}_X)_{|U}), I^\bullet).$$

The complex $\mathbb{D}_X^\bullet$ is quasi-isomorphic to the complex of sheaves of singular chains on X and its hypercohomology is equal to ordinary homology with closed support (i.e. Borel-Moore homology, cf. [GM83]). Furthermore [Bo84] shows

4.2 Proposition. *(1) The dualizing complex $\mathbb{D}_X$ is a complex of injectives.*

(2) The dualizing complex is constructible with respect to any topological stratification of X.

We can use the dualizing complex to define:

4.3 Definition. (1) We define a contravariant functor $\mathcal{D} : Ch_c^b(X) \to Ch_c^b(X)$ by

$$\mathcal{D} := \mathbf{Hom}^\bullet(-, \mathbb{D}_X^\bullet).$$

(2) Similarly, we define a contravariant functor $\mathcal{D} : D_c^b(X) \to D_c^b(X)$ by

$$\mathcal{D} := R\mathbf{Hom}^\bullet(-, \mathbb{D}_X^\bullet).$$

4.4 Remark. For our definition of the dualizing complex, we have chosen a fixed injective resolution $R \to I^\bullet$ of R by injectives. Clearly, other choices of resolutions lead to quasi-isomorphic complexes. Therefore, it is well defined up to isomorphism in $D_c^b(X)$, but not in $Ch_c^b(X)$.

Consequently, these different choices for $\mathbb{D}_X$ lead to different choices for the functors $\mathcal{D}$. However, we need one distinguished functor $\mathcal{D}$ on $Ch_c^b(X)$. This means we are forced to make a choice for the rest of this work.

The second tool is

4.5 Theorem. *(Verdier duality)*
Let $f : X \to Y$ be a continuous map between locally compact spaces. Let $\mathbf{A}^\bullet \in D^b(X)$ and $\mathbf{B}^\bullet \in D^b(Y)$. There is a canonical isomorphism in $D^b(Y)$:

$$Rf_* R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong R\mathbf{Hom}^\bullet(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet).$$

5 Maps from cohomology and to homology

First, we define what an R-orientation of a topological pseudomanifold is.

5.1 Definition. An R-orientation of X is the choice of a quasi-isomorphism

$$\mathbb{D}_{X-\Sigma}^\bullet \to \mathbf{R}_{X-\Sigma}[n].$$

5.2 Remark. If $\mathrm{char}(R) \neq 2$ then an R-orientation is equivalent to an orientation of $X - \Sigma$ in the usual topological sense. If $R = \mathbb{Z}/2$ then every X is $\mathbb{Z}/2$-orientable.

From now on, we suppose that X is R-orientable and we have chosen an orientation.

Let $j : \Sigma \to X$ be the inclusion of the singular locus and $i : X - \Sigma \to X$ be the inclusion of the non-singular part.

5.3 Proposition. *There exists a morphism $\phi : \mathbf{R}_X[n] \to \mathbb{D}^\bullet_X$ in $D^b_c(X)$ which lifts the orientation:*

$$\mathbf{R}_X[n] \to Ri_*\mathbf{R}_{X-\Sigma}[n] \to Ri_*\mathbb{D}^\bullet_{X-\Sigma}.$$

This morphism is unique in D^b_c.

This morphism is called the *cap product with the orientation class*. We will see that it will factorize over the complex of intersection chains. Let $\bar{p}$ be any perversity.

5.4 Proposition. *There is a unique factorization in $D^b_c(X)$ of the cap product with the orientation class*

$$\mathbf{R}_X[n] \to \mathbf{IC}^\bullet_{\bar{p}} \to \mathbb{D}^\bullet_X$$

such that $i^*\mathbf{R}_X[n] \to i^*\mathbf{IC}^\bullet_{\bar{p}}$ is the obvious morphism and $i^*\mathbf{IC}^\bullet_{\bar{p}} \to i^*\mathbb{D}^\bullet_X$ is given by the orientation.

6 The intersection pairing and Poincaré duality

Suppose $\bar{l} + \bar{m} \leq \bar{p}$ are perversities. We want to define a product morphism

$$\mathbf{IC}^\bullet_{\bar{l}} \overset{L}{\otimes} \mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{IC}^\bullet_{\bar{p}}[n].$$

Furthermore, we want this product to have nice properties. For example, the intersection pairings for different choices of $\bar{l}, \bar{m}$ and $\bar{p}$ should be compatible.

Before we start with the construction of the product, we give a criterion for lifting a morphism in $D^b_c(X)$. This result can be found in [GM83] and we will prove it in chapter V.

6.1 Lemma. *Let $\mathbf{A}^\bullet, \mathbf{B}^\bullet, \mathbf{C}^\bullet \in D^b(X)$ such that $\mathbf{H}^k(\mathbf{A}^\bullet) = 0$ for all $k \geq p+1$. Let $f : \mathbf{C}^\bullet \to \mathbf{B}^\bullet$ be a morphism of complexes such that $f^* : \mathbf{H}^k(\mathbf{C}^\bullet) \to \mathbf{H}^k(\mathbf{B}^\bullet)$ is an isomorphism for all $k \leq p$. Then the induced map*

$$\mathrm{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet) \to \mathrm{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$$

is an isomorphism. That is, every map $g : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ has a unique lift $\tilde{g} : \mathbf{A}^\bullet \to \mathbf{C}^\bullet$ such that $f\tilde{g} = g$.

Let $\mathbf{L}^\bullet, \mathbf{M}^\bullet$ and $\mathbf{P}^\bullet$ be the complexes of sheaves in 3.2 corresponding to $\bar{l}, \bar{m}$ and $\bar{p}$. We define the intersection pairing inductively over $U_k = X - X_{n-k}$ as follows:

On $U_2 = X - \Sigma$ we just take the multiplication

$$\mathbf{R}_{X-\Sigma}[n] \overset{L}{\otimes} \mathbf{R}_{X-\Sigma}[n] \to \mathbf{R}_{X-\Sigma}[n][n].$$

Now, suppose $\mu_k : \mathbf{L}_k^\bullet \overset{L}{\otimes} \mathbf{M}_k^\bullet \to \mathbf{P}_k^\bullet[n]$ is constructed. We have to define a morphism

$$(\tau_{\leq l(k)-n} Ri_{k*}\mathbf{L}_k^\bullet) \overset{L}{\otimes} (\tau_{\leq m(k)-n} Ri_{k*}\mathbf{M}_k^\bullet) \to \tau_{\leq p(k)-n} Ri_{k*}\mathbf{P}_k^\bullet.$$

The pairing μ_k induces a morphism

$$\begin{aligned}(\tau_{\leq l(k)-n} Ri_{k*}\mathbf{L}_k^\bullet) \overset{L}{\otimes} (\tau_{\leq m(k)-n} Ri_{k*}\mathbf{M}_k^\bullet) &\to Ri_{k*}\mathbf{L}_k^\bullet \overset{L}{\otimes} Ri_{k*}\mathbf{M}_k^* \\ &\to Ri_{k*}(\mathbf{L}_k^\bullet \overset{L}{\otimes} \mathbf{M}_k^\bullet) \\ &\to Ri_{k*}(\mathbf{P}_k^\bullet).\end{aligned}$$

Since the cohomology sheaf associated to

$$(\tau_{\leq l(k)-n} Ri_{k*}\mathbf{L}_k^\bullet) \overset{L}{\otimes} (\tau_{\leq m(k)-n} Ri_{k*}\mathbf{M}_k^\bullet)$$

vanishes in dimensions $j \geq l(k) + m(k) - 2n + 1$ we can apply the proposition and get a lift (in $D_c^b(X)$) to $\tau_{\leq p(k)-n} Ri_{k*}\mathbf{P}_k^\bullet[n]$. According to [GM83] the compatibility between the intersection pairings for different perversities, as well as the compatibility with the cup and cap products is easily checked.

We have an induced intersection pairing on the intersection homology groups

$$IH_i^{\bar{l}}(X) \otimes IH_j^{\bar{m}}(X) \to IH_{i+j-n}^{\bar{p}}(X).$$

This pairing is used to generalize Poincare duality to pseudomanifolds.

6.2 Definition. A pairing $\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet \to \mathbb{D}_X^\bullet[n]$ is called a *Verdier dual pairing* if it induces an isomorphism in $D_c^b(X)$,

$$\mathbf{A}^\bullet \to \mathcal{D}(\mathbf{B}^\bullet)[n].$$

Suppose for the rest of this section that R is a field. Suppose further that we are given complementary perversities $\bar{p}, \bar{q}$, i.e. $\bar{p} + \bar{q} = \bar{t}$. We have

6.3 Theorem. *The intersection pairing followed by the map to homology*

$$\mathbf{IC}_{\bar{p}}^\bullet \overset{L}{\otimes} \mathbf{IC}_{\bar{q}}^\bullet \to \mathbf{IC}_{\bar{t}}^\bullet \to \mathbb{D}_X^\bullet[n]$$

is a Verdier dual pairing.

Proof. The idea of the proof is to set

$$\mathbf{S}^\bullet := \mathcal{D}(\mathbf{IC}_{\bar{q}}^\bullet)[n]$$

and check the axioms for perversity $\bar{p}$. □

6.4 Corollary. *If X is compact, the pairing*

$$IH_*^{\bar{p}} \otimes IH_*^{\bar{q}} \to H_*(X) \to R$$

induces an isomorphism

$$IH_i^{\bar{p}}(X) \cong \mathrm{Hom}(IH_{n-i}^{\bar{q}}(X), R).$$

7 Self-duality of the intersection chain sheaf

Recall from 2.1 that $\bar{m}$ denotes the lower middle and $\bar{n}$ denotes the upper middle perversity. We want to give classes of PL-pseudomanifolds, for which $\mathbf{IC}^\bullet_{\bar{m}}$ is self-dual, that is PL-pseudomanifolds, for which

$$\mathbf{IC}^\bullet_{\bar{m}} \to \mathcal{D}(\mathbf{IC}^\bullet_{\bar{m}})[n]$$

is an isomorphism. First, we suppose again that R is a field. We have

7.1 Proposition. *Let X be a PL-pseudomanifold with only even codimensional strata. Then the canonical inclusion*

$$\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{IC}^\bullet_{\bar{n}}$$

is a quasi-isomorphism.

Proof. This follows directly from 3.4. □

We combine this result with Poincare duality.

7.2 Corollary. *If X has only even codimensional strata then*

$$\mathbf{IC}^\bullet_{\bar{m}} \to \mathcal{D}(\mathbf{IC}^\bullet_{\bar{m}})[n]$$

is an isomorphism in $D^b_c(X)$.

We want to find a larger class of spaces for which proposition 7.1 holds. This class was completely characterized by Siegel [Si83].

7.3 Definition. A PL-pseudomanifold X is called an *R-Witt space* if $IH^{\bar{m}}_l(L_x) = 0$ for all $x \in X_{n-2l-1} - X_{n-2l-2}$, where L_x is the link at x.

7.4 Proposition. *X is a R-Witt space if and only if the canonical morphism $\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{IC}^\bullet_{\bar{n}}$ is a quasi-isomorphism. Therefore, X is a R-Witt space if and only if $\mathbf{IC}^\bullet_{\bar{m}}$ is self-dual.*

Note, that R being a field is essential here. However, if one is interested in self-duality for $R = \mathbb{Z}$ there is the following class of spaces introduced by Goresky and Siegel in [GS83].

7.5 Definition. A stratified PL-pseudomanifold X is called an *IP-space* if

- $IH^{\bar{m}}_l(L_x) = 0$ for all $x \in X_{n-2l-1} - X_{n-2l-2}$,
- $IH^{\bar{m}}_{l-1}(L_x)$ is torsion free for all $x \in X_{n-2l} - X_{n-2l-1}$,

where L_x is again the link at x.

7.6 Proposition. *For $R = \mathbb{Z}$. X is an IP-space if and only if*

$$\mathbf{IC}^\bullet_{\bar{m}} \to \mathcal{D}(\mathbf{IC}^\bullet_{\bar{m}})[n]$$

is a quasi-isomorphism.

Proof. see [GS83]. □

This proposition implies Poincaré duality for IP-spaces, i.e. the linking pairing defined in [GS83] is non-degenerate. This explains the abbreviation IP, for Integral Poincaré.

8 The signature of Witt spaces and IP-spaces

Suppose we have a $4k$-dimensional $\mathbb{Q}$-Witt space X. Since $\mathbf{IC}^\bullet_{\bar{m}}$ is self-dual we can define the signature of X by taking the index of the pairing

$$IH^{\bar{m}}_{2k}(X) \otimes IH^{\bar{m}}_{2k}(X) \to \mathbb{Q}.$$

Similarly, for a $4k$-dimensional IP-space X we define the signature of X by taking the index of the pairing

$$IH^{\bar{m}}_{2k}(X)/\text{Torsion} \otimes IH^{\bar{m}}_{2k}/\text{Torsion} \to \mathbb{Z}$$

defined in [GS83].

8.1 Proposition. *Let X be a $4k$-dimensional $\mathbb{Q}$-Witt space or IP-space. The signature of X is a bordism invariant for Witt-bordism resp. IP-bordism.*

Proof. The proof is just the usual construction of an annihilating subspace of $IH^{\bar{m}}_{2k}(X)$ of half the rank (see [Si83]). □

Chapter III

Symmetric L-theory

In this chapter we will give the definitions of the symmetric L-groups and the symmetric $\mathbb{L}^{\cdot}$-spectrum. In order to do so we will first introduce the notion of algebraic bordism categories. These provide an axiomatic framework for defining symmetric structures and we can define the symmetric L-groups of such a bordism category.

As a first example we give the category of R-modules the structure of such a bordism category. Thus, we obtain the notion of a symmetric Poincaré complex and obtain the symmetric L-groups of the ring R as the symmetric L-groups of this bordism category.

Having defined the symmetric L-groups of a ring R we use a simplicial construction featuring $[k]$-ads to define the symmetric $\mathbb{L}R^{\cdot}$-spectrum. For the special case $R = \mathbb{Z}$ we will then determine the homotopy type of this spectrum.

In the last two sections of this chapter we will define symmetric Poincaré sheaves as a second example of an algebraic bordism category and relate these to symmetric Poincaré complexes via a simply connected assembly map.

All the definitions in this chapter are due to Ranicki except, that we take a cohomological approach. The books [Ra81] and [Ra92] serve as general references although we will use different sources as needed. We favoured the cohomological approach over the homological of Ranicki since it lends itself more easily to the sheaf theoretic construction of chapter V.

1 Algebraic bordism categories

In this section we start with the definition of algebraic bordism categories. They provide sufficient axioms for defining the L-groups.

1.1 Definition. Let $\mathcal{A}$ be an additive category.

(1) A subcategory $\mathcal{B} \subset Ch(\mathcal{A})$ is called *closed* if it is a full additive subcategory such that for each chain map $f : C^{\bullet} \to D^{\bullet}$ in $\mathcal{B}$ the algebraic mapping cone $\text{cone}^{\bullet}(f)$ lies in $\mathcal{B}$.

(2) Let $\mathcal{B} \subset Sh(\mathcal{A})$ be a closed subcategory. A chain complex $C^\bullet \in Ch(\mathcal{A})$ is called $\mathcal{B}$-*contractible* if $C^\bullet \in \mathcal{B}$. A chain map $f : C^\bullet \to D^\bullet$ is called a $\mathcal{B}$-*equivalence* if $\mathrm{cone}^\bullet(f)$ is $\mathcal{B}$-contractible.

1.2 Definition. An *algebraic bordism category* is an additive category $\mathcal{A}$ together with closed subcategories $\mathcal{B}, \mathcal{C} \subset Ch(\mathcal{A})$, a contravariant functor $T : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ and a natural transformation $e : 1 \to T^2$ such that the following axioms are fullfilled:

(1) $\mathrm{id}_{B^\bullet} : B^\bullet \to B^\bullet$ is a $\mathcal{C}$-equivalence for all $B^\bullet \in \mathcal{B}$.

(2) $T(e(A^\bullet)) \circ e(T(A^\bullet)) = \mathrm{id}_{T(A^\bullet)} : T(A^\bullet) \to T^3(A^\bullet) \to T(A^\bullet)$ for all $A^\bullet \in \mathcal{A}$.

(3) $e(B^\bullet) : B^\bullet \to T^2(B^\bullet)$ is a $\mathcal{C}$-equivalence for all $B^\bullet \in \mathcal{B}$.

The pair (T, e) is called a chain duality on $\mathcal{A}$. When speaking of an algebraic bordism category we will often specify only the triple $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ with the chain duality being understood.

The structure of an algebraic bordism category is sufficient to define symmetric L-groups. In particular there is an involution

$$\tau' : \mathrm{Hom}^\bullet(A^\bullet, T(A^\bullet)) \to \mathrm{Hom}^\bullet(A^\bullet, T(A^\bullet))$$

given by $\tau'(\varphi) := T(\varphi) \circ e(A^\bullet)$ for $\varphi \in \mathrm{Hom}^\bullet(A^\bullet, T(A^\bullet))$.

If in addtition the category $\mathcal{A}$ is enriched over itself we can also define a tensor product of two complexes $A^\bullet$ and $B^\bullet$ denoted by $A^\bullet \tilde{\otimes} B^\bullet$ in order to distinguish it from a possible internal tensor product in the category $Ch(\mathcal{A})$:

$$A^\bullet \tilde{\otimes} B^\bullet := T(\mathrm{Hom}^\bullet(A^\bullet, T(B^\bullet)))$$

Furthermore, we have an involution τ on $A^\bullet \tilde{\otimes} A^\bullet$ given by $T(\tau')$.

Let W be the standard free $\mathbb{Z}[\mathbb{Z}/2]$-module resolution of the trivial $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathbb{Z}$.

$$W : \ldots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2]$$

For a complex $A^\bullet$ we define a complex of abelian groups by

$$W^\% A^\bullet := \mathrm{Hom}^\bullet_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathrm{Hom}^\bullet(A^\bullet, TA^\bullet))$$

with cohomology groups denoted by $Q^n(A^\bullet) := H^n(W^\% A^\bullet)$. For a morphism of chain complexes $f : A^\bullet \to B^\bullet$ we get an induced morphism of chain complexes of abelian groups

$$f^\% : W^\% B^\bullet \to W^\% A^\bullet$$

and therefore an induced morphism

$$(f^\%)^* : Q^n(B^\bullet) \to Q^n(A^\bullet).$$

A $(-n)$-chain $\phi \in (W^{\%}A^\bullet)^{-n}$ is a collection of morphisms

$$\phi_s : A^{r+n+s} \to T(A)^r.$$

The boundary $d\phi \in (W^{\%}A^\bullet)^{-n+1}$ is the collection of morphims

$$(d\phi)_s = d\phi_s + (-1)^r \phi_s d + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) : A^{r+n+s-1} \to T(A)^r.$$

1.3 Definition. Let $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be an algebraic bordism category.

(1) An n-dimensional *symmetric complex* in Λ is a pair $(A^\bullet, \phi)$ with $A^\bullet$ $\mathcal{B}$-contractible and $\phi \in (W^{\%}A^\bullet)^{-n}$ a $(-n)$-cycle.

(2) An n-dimensional symmetric complex $(A^\bullet, \phi)$ is called *Poincaré* if ϕ_0 is a $\mathcal{C}$-equivalence.

(3) A *map* of n-dimensional symmetric complexes $f : (A^\bullet, \phi) \to (B^\bullet, \psi)$ is a chain map $f : A^\bullet \to B^\bullet$ such that $(f^{\%})^*([\psi]) = [\phi] \in Q^{-n}(A^\bullet)$.

(4) Given n-dimensional symmetric complexes $(A^\bullet, \phi), (B^\bullet, \psi)$ we define symmetric complexes by

$$-(A^\bullet, \phi) := (A^\bullet, -\phi)$$

and

$$(A^\bullet, \phi) \oplus (B^\bullet, \psi) := (A^\bullet \oplus B^\bullet, \phi \oplus \psi).$$

To prepare the definition of objects with boundary we consider a chain map $f : D^\bullet \to A^\bullet$. We denote

$$W^{\%}(f) := \mathrm{cone}^\bullet(f^{\%})$$

and

$$Q^n(f) := H^n(W^{\%}(f)).$$

A $-(n+1)$-chain $(\delta\phi, \phi) \in (W^{\%}(f))^{-(n+1)}$ is a $-n$-chain $\phi \in W^{\%}(A)^{-n}$ together with a collection of morphisms

$$(\delta\phi)_s : D^{r+n+s+1} \to T(D)^r.$$

The boundary $d(\delta\phi, \phi) \in (W^{\%}(f))^{-n}$ is the pair $(d\delta\phi, d\phi)$, where $d\phi$ is given as above and $d\delta\phi$ is the collection of morphisms $(d\delta\phi)_s : D^{r+n+s} \to T(D)^r$ given by

$$\begin{aligned}(d\delta\phi)_s = \; & d(\delta\phi)_s + (-1)^r(\delta\phi)_s d + (-1)^{n+s}((\delta\phi)_{s-1} + (-1)^s T(\delta\phi)_{s-1}) \\ & + (-1)^n T(f)\phi_s f.\end{aligned}$$

Given a $-(n+1)$-cycle $(\delta\phi, \phi) \in (W^{\%}(f))^{-(n+1)}$ we define a chain map

$$((\delta\phi)_0, \phi_0) := \begin{pmatrix} \delta\phi_0 \\ \phi_0 f \end{pmatrix} : D^{r+n+1} \to \mathrm{cone}^r(T(f)) = T(D)^r \oplus T(A)^{r+1}.$$

1.4 Definition. Let $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be an algebraic bordism category.

(1) An $(n+1)$-*dimensional symmetric pair in* Λ is a pair $(f : D^\bullet \to A^\bullet, (\delta\phi, \phi))$ where f is a chain map of $\mathcal{B}$-contractible chain complexes and $(\delta\phi, \phi) \in W^{\%}(f)$ a $-(n+1)$-cycle.

(2) An $(n+1)$-dimensional symmetric pair $(f : D^\bullet \to A^\bullet, (\delta\phi, \phi))$ is called *Poincaré* if $((\delta\phi)_0, \phi_0)$ is a $\mathcal{C}$-equivalence.

(3) A *map* of $(n+1)$-dimensional symmetric pairs $(\delta g, g, h) : (f : D^\bullet \to A^\bullet, (\delta\phi, \phi)) \to (f' : E^\bullet \to B^\bullet, (\delta\psi, \psi))$ is a pair of maps $(\delta g : D^\bullet \to E^\bullet, g : A^\bullet \to B^\bullet)$ and a homotopy h between $f' \circ (\delta g)$ and $g \circ f$ such that for the induced map $(\delta g, g)^*$ on $Q^{n+1}(f')$ we have

$$(\delta g, g, h)^*([\delta\psi, \psi]) = [\delta\phi, \phi] \in Q^{-(n+1)}(f).$$

(4) Given an $(n+1)$-dimensional symmetric Poincaré pair $(f : D^\bullet \to A^\bullet, (\delta\phi, \phi))$. We define its boundary to be the symmetric Poincaré complex

$$\partial(f : D^\bullet \to A^\bullet, (\delta\phi, \phi)) := (A^\bullet, \phi).$$

(5) A *cobordism* between n-dimensional symmetric Poincaré complexes $(A^\bullet, \phi)$ and $(B^\bullet, \psi)$ is an $(n+1)$-dimensional symmetric Poincaré pair $(f : D^\bullet \to C^\bullet, (\delta\zeta, \zeta))$ such that $(C^\bullet, \zeta) = (A^\bullet, \phi) \oplus -(B^\bullet, \psi)$.

1.5 Definition. Let $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be an algebraic bordism category. The n-*dimensional symmetric L-group* $L^n(\Lambda)$ is the cobordism group of n-dimensional symmetric Poincaré complexes in Λ.

2 Symmetric Poincaré Complexes

The first example of an algebraic bordism category is given by the category of cochain complexes of R-modules over a ring R: $\mathcal{A} = R - \mathcal{M}od$. The dualizing functor $T : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ is given by

$$T(P^\bullet) := P^* = \mathrm{Hom}^\bullet(P^\bullet, R).$$

The natural transformation $e : 1 \to T^2$ is the usual map

$$P \to P^{**}.$$

Let $\mathcal{B}$ be the full subcategory of $Ch(\mathcal{A})$ given by all bounded complexes of finitely generated projectives and $\mathcal{C}$ the full subcategory of all contractible bounded complexes.

The following result can be found in [Ra92].

2.1 Proposition. $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C}, T, e)$ *is an algebraic bordism category.*

For the rest of this work, whenever we speak of a symmetric (Poincaré) complex we will always mean a symmetric (Poincaré) complex in this algebraic bordism category. Whenever we speak of a symmetric (Poincaré) pair of complexes we will mean a symmetric (Poincaré) pair in this bordism category. We will denote the resulting bordism groups of n-dimensional Poincaré complexes in Λ by

$$L^n_p(R) := L^n(\Lambda).$$

We want give a more thorough treatment of this particular algebraic bordism category.

2.2 Definition. (1) A *homotopy equivalence* of symmetric Poincaré complexes is a map $f : (A^\bullet, \phi) \to (B^\bullet, \psi)$ such that $f : A^\bullet \to B^\bullet$ is a homotopy equivalence.

(2) A map of symmetric Poincaré pairs

$$(\delta g, g, h) : (f : D^\bullet \to A^\bullet, (\delta\phi, \phi)) \to (f' : E^\bullet \to B^\bullet, (\delta\psi, \psi))$$

is a homotopy equivalence if both δg and g are homotopy equivalences.

Having these notions, it is immediately clear that a Poincaré pair with contractible boundary is homotopy equivalent to a pair with zero boundary which itself is nothing but a Poincaré complex. Therefore, the homotopy equivalence classes of symmetric Poincaré complexes are in one-to-one correspondence with homotopy equivalence classes of symmetric Poincaré pairs with contractible boundary. But there is an even stronger result.

2.3 Proposition. [Ra80] *There is a one-to-one correspondence between homotopy equivalence classes of symmetric complexes $(A^\bullet, \phi)$ of dimension n and homotopy equivalence classes of symmetric Poincaré pairs of complexes $(f : D^\bullet \to B^\bullet, (\delta\psi, \psi))$ of dimension n.*

Proof. We will only give the constructions which give rise to the one-to-one correspondence. The remaining parts of the proof can be found in [Ra80],3.4.

Firstly, let $(A^\bullet, \phi)$ be an n-dimensional symmetric complex. Then an n-dimensional symmetric Poincaré pair of complexes $(f : D^\bullet \to B^\bullet, (0, \psi))$ is given by

$$D^\bullet := T(A^\bullet)[-n]$$

and

$$B^r := A^{r+1} \oplus T(A)^{r-n}$$

with differential

$$d = \begin{pmatrix} d & 0 \\ (-1)^r \phi_0 & d \end{pmatrix}.$$

That is, $B^\bullet = \mathrm{cone}^\bullet(\phi_0)$. The map f is given by the inclusion of the direct summand $D^\bullet$. We define a symmetric structure on $B^\bullet$ by

$$\psi_0 := \begin{pmatrix} (-1)^{r+n}\phi_1 & 1 \\ (-1)^{r(r+n-1)}e & 0 \end{pmatrix} : \mathrm{cone}^\bullet(\phi_0)^{r+n-1} \to T(\mathrm{cone}^\bullet(\phi_0))^r$$

and

$$\psi_s := \begin{pmatrix} (-1)^{r+n+s}\phi_{s+1} & 0 \\ 0 & 0 \end{pmatrix} : \mathrm{cone}^\bullet(\phi_0)^{r+n+s-1} \to T(\mathrm{cone}^\bullet(\phi_0))^r$$

An easy calculation shows that this is an n-dimensional symmetric Poincaré pair of complexes.

Now, let $(f : D^\bullet \to B^\bullet, (\delta\psi, \psi))$ be n-dimensional symmetric Poincaré Pair of complexes. Then the following construction, called the algebraic Thom construction, gives a symmetric complex $(A^\bullet, \phi)$:

$$A^\bullet := \mathrm{cone}(f)^\bullet[-1]$$

and

$$\phi_s := \begin{pmatrix} \delta\psi_s & 0 \\ (-1)^{r+n-1}\psi_s f & (-1)^{r+n+s}\psi_{s-1} \end{pmatrix} :$$
$$D^{r+n+s} \oplus B^{r+n+s-1} \to T(D)^r \oplus T(B)^{r+1}$$

□

There is also the important notion of algebraic surgery, which we will not describe here, since we are only interested in the homotopy equivalence part of the following proposition.

2.4 Proposition. [Ra80] *The equivalence relation of cobordism of n-dimensional symmetric Poincaré complexes is generated by homotopy equivalence and algebraic surgery.*

3 The symmetric $\mathbb{L}^\cdot$-spectrum

Let X be a PL-space. We look at the assignment

$$X \to L_p^n(\mathbb{Z}[\pi_1(X)]).$$

and ask if this gives rise to a generalized homology theory. Unfortunately, this is not the case. However there is a method of constructing a homology theory $\mathbb{L}^\cdot$ which admits a group homomorphism

$$A : \mathbb{L}^\cdot_*(X) \to L_p^*(\mathbb{Z}[\pi_1(X)])$$

that is an isomorphism for $X = pt$. The map A is the assembly map.

In order to construct this homology theory, we will first review some constructions in the world of symmetric Poincaré complexes. To make these and the following constructions easier to track we introduce some notations.

For the rest of this section we will abbreviate a symmetric Poincaré pair $(f : C^\bullet \to \partial C^\bullet, (\delta\phi, \phi))$ by C and its boundary by ∂C with the map f being written as $\pi_{\partial C}$ or $\pi_{\partial C}^C$. If the boundary is a direct sum we will indicate the composition of the boundary map and the projection to a direct summand by writing this direct summand as a subscript instead. Furthermore, the $\delta\phi$ will be denoted by $\phi(C)$ and ϕ by $\phi(\partial C)$. We will write $-C$ for the pair $(f : C \to \partial C, -(\delta\phi, \phi))$. A map of symmetric pairs will be denoted by $f : C \to D$ with the additional map $g : \partial C^{\bullet} \to \partial D^{\bullet}$ and the homotopy being understood. Additionally, we will consider all symmetric Poincaré complexes to be pairs with zero boundary. For the direct sum of two pairs $C \oplus D$ we will simply write $C + D$. Finally, we will write the sign $\simeq$ for the notion of homotopy equivalence.

The following definitions are dual to the definitions of gluing given in [Ra81].

3.1 Definition. (1) Suppose we are given two $(n+1)$-dimensional symmetric Poincaré pairs D, E such that $\partial D = A + -B$ and $\partial E = B + -C$. Then the *union of* D *and* E *along* B is defined to be the $(n+1)$-dimensional symmetric Poincaré pair $D +_B E$ with boundary $A + -C$ given by

$$(D +_B E)^r := D^r \oplus B^{r-1} \oplus E^r$$

with differential in degree r

$$d_{D+_B E} := \begin{pmatrix} d_D & 0 & 0 \\ (-1)^{r-1}\pi_B^D & d_B & (-1)^{r-1}\pi_B^E \\ 0 & 0 & d_E \end{pmatrix}$$

and boundary map in degree r

$$\pi_{A+-C} := \begin{pmatrix} \pi_A^D & 0 & 0 \\ 0 & 0 & \pi_C^E \end{pmatrix}.$$

The symmetric structure in degree $n + r + s + 1$ is given by

$$\phi(D +_B E)_s := \begin{pmatrix} \phi(D)_s & 0 & 0 \\ (-1)^{n+r}\phi(B)_s\pi_B^D & (-1)^{n+r+s+1}\phi(B)_{s-1} & 0 \\ 0 & (-1)^s T(\pi_B^E)\phi(B)_s & \phi(E)_s \end{pmatrix}.$$

(2) Suppose we are given two $(n+1)$-dimensional symmetric Poincaré pairs D, E such that $\partial D = A +_Z -B$ and $\partial E = B +_Z -C$ for n-dimensional symmetric Poincaré pairs A, B, C such that $\partial A = \partial B = \partial C = Z$. Then the *union of* D *and* E *along* B is defined to be the $(n+1)$ dimensional symmetric Poincaré pair $D +_B E$ with boundary $A +_Z -C$ given by the same formulas as above, except for the boundary map, which in degree r is given by

$$\pi_{A+_Z-B} := \begin{pmatrix} \pi_A & 0 & 0 \\ (-1)^{r+1} & \pi_Z^B & (-1)^{r+1} \\ 0 & 0 & \pi_B \end{pmatrix}$$

The following construction is a special case of the much more general theory of [LM06]. Namely, we will use the gluing constructions introduced above to construct a simplicial Ω-spectrum. Which represents the homology theory we are looking for. Let Δ_+^k denote the set of simplices of the standard k-simplex Δ^k.

3.2 Definition. A *pointed* $[k]$-*ad* M of dimension d is

(1) a collection
$$(M_\sigma, \sigma \in \Delta^k_+)$$
of symmetric Poincaré pairs of dimension $d - k + \dim \sigma$ such that $M_\emptyset = 0$. We write $\partial_i M_\sigma$ for $M_{d_i\sigma}$.

(2) a homotopy equivalence
$$f_\sigma : \sum_i (-1)^i \partial_i M_\sigma \to \partial M_\sigma$$
for each $\sigma \in \Delta^k_+$.

A *homotopy equivalence* of $[k]$-ads is a family of homotopy equivalences g_σ which are compatible with the boundary identifications.

Note that the second part of this definition only makes sense if it is understood inductively. That is, the Poincaré pairs over the 0-simplices are closed objects. The Poincaré pairs over the 1-simplices are glued together along the pairs over the 0-simplices according to the boundary relations in Δ^k and so on. For example, a pointed [0]-ad is simply a Poincaré complex. A pointed [1]-ad is a triple (D, A, B) with D a Poincaré pair and A, B Poincaré complexes such that $\partial D = A + -B$.

3.3 Definition. (1) Let $\mathbb{L}R^{\cdot}_k$ denote the augmented semi simplicial set with n-simplices given by the homotopy equivalence classes of pointed $(n-k)$-dimensional $[n]$-ads. The face maps
$$\partial_i : \mathbb{L}R^{\cdot}_{k,n+1} \to \mathbb{L}R^{\cdot}_{k,n}, \quad 0 \le i \le n+1$$
send a $[n+1]$-ad M to the $[n]$-ad $\partial_i M$.

(2) There is a map
$$t : \mathbb{L}R^{\cdot}_k \to \Omega\mathbb{L}R^{\cdot}_{k+1}$$
which is induced by the assignment that maps a $[k]$-ad M to the $[k+1]$-ad
$$t(M)_\sigma := \begin{cases} M_{\sigma - \{n+1\}} & \text{if } \{n+1\} \in \sigma \\ \emptyset & \text{else} \end{cases} .$$
See [RS71] for the precise definition of the loop space of a semi simplicial set.

These semi simplicial sets satisfy the following properties.

3.4 Proposition. *(1) $\mathbb{L}R^{\cdot}_k$ is a simplicial set.*

(2) $\mathbb{L}R^{\cdot}_k$ satisfies the Kan condition.

(3) The maps $t : \mathbb{L}R^{\cdot}_k \to \Omega\mathbb{L}R^{\cdot}_k$ give rise to an Ω-spectrum $\mathbb{L}R^{\cdot}$ of simplicial sets which coincides with the Ω-spectrum defined in [Ra92].

(4) $\pi_n(\mathbb{L}R^{\cdot}) = L^n(R)$.

Proof. (1) See [LM06].

(2) This follows from the existence of a cylinder constructed in [Ra92] and [LM06].

(3) See [LM06] for the fact that t is a bijection. That $\mathbb{L}R^{\cdot}$ coincides with the spectrum constructed in [Ra92] follows from the example 5.4 there and [LR87], §3.

(4) Again [LM06] or [Ra92].

□

We close this section with an explicit construction of the connected covers of $\mathbb{L}R^{\cdot}$ due to [Ra92].

3.5 Proposition. *The l-connected cover $\mathbb{L}R^{\cdot}\langle l\rangle$ of $\mathbb{L}R^{\cdot}$ is homotopy equivalent to the Ω-spectrum of augmented simplicial sets $(\mathbb{L}R^{\cdot}\langle l\rangle)_k$ which have as n-simplices all pointed $(n-k)$-dimensional $[n]$-ads M satisfying*

$$M_\sigma \text{ is contractible, if } \dim\sigma \le l + k - 1.$$

3.6 Definition. In the case of $R = \mathbb{Z}$ we simply write $\mathbb{L}^{\cdot}$ for the spectrum $\mathbb{L}\mathbb{Z}^{\cdot}\langle 0\rangle$.

4 The homotopy type of $\mathbb{L}^{\cdot}$

In this section we will determine the homotopy type of $\mathbb{L}^{\cdot}$. First we record for further use:

4.1 Theorem. *A MSO-module spectrum E is a generalized Eilenberg-MacLane spectrum after localizing at 2.*

Proof. See [Ta76] or [TW79]. □

4.2 Proposition. *(1)* $(\mathbb{L}^{\cdot})_{(2)} \simeq H\mathbb{Z}_{(2)}[t] \vee \Sigma(H\mathbb{Z}/2[t])$*, where t is of degree 4.*

(2) $\mathbb{L}^{\cdot} \otimes \mathbb{Z}[\frac{1}{2}] \simeq ko \otimes \mathbb{Z}[\frac{1}{2}]$*. Here, ko denote connected real K-theory.*

Proof. The first statement follows by the preceding theorem. The second is proved in [TW79]. □

5 Symmetric Poincaré Sheaves

The second example is given by the algebraic bordism category of constructible sheaves on a stratified topological pseudomanifold X. The weak chain duality will be given by dualizing with respect to the dualizing sheaf $\mathbb{D}_X^\bullet$ defined in II.4. By definition this is an internal operation, that is, the functor T is representable. We will use this to give a different description of a symmetric structure on a complex of sheaves via the natural tensor product of sheaves.

More precisely, let X be a topological stratified pseudomanifold. Let $\mathcal{A}$ denote the category of sheaves of R-modules on X. Recall that $\mathbb{D}_X^\bullet$ denotes the dualizing complex of sheaves on X. Define a functor $T : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ by

$$T(\mathbf{A}^\bullet) := \mathcal{D}(\mathbf{A}^\bullet) = \mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}_X^\bullet).$$

Then there is a natural transformation $e : 1 \to T^2$ which is induced by the map $S \to \mathrm{Hom}(\mathrm{Hom}(S,T),T)$ for R-modules S, T, generalizing the map of an R-module into its bidual (see [Bo84], V.8 for details).

Denote by $\mathcal{B}$ the full subcategory of $Ch(\mathcal{A})$ of constructible bounded complexes of Γ_c-acyclic sheaves and by $\mathcal{C}$ the full subcategory of $Ch(\mathcal{A})$ of exact bounded complexes of Γ_c-acyclic sheaves.

5.1 Proposition. $\Lambda(X) := (\mathcal{A}, \mathcal{B}, \mathcal{C}, T, e)$ *is an algebraic bordism category.*

Proof. The first axiom is trivial. The second is just decoding the definitions of T and e and the last one is proved in [Bo84], chapter V. □

5.2 Definition. For a topological stratified pseudomanifold X we define

$$L^n(X; R) := L^n(\Lambda(X)).$$

An n-dimensional symmetric (Poincaré) complex $(\mathbf{A}^\bullet, \phi)$ in $\Lambda(X)$ is called a *symmetric (Poincaré) sheaf on* X. Likewise, a symmetric (Poincaré) pair in $\Lambda(X)$ is called a *symmetric (Poincaré) pair of sheaves on* X. For $R = \mathbb{Z}$ we write just $L^n(X)$ instead of $L^n(X; \mathbb{Z})$.

The category of complexes of sheaves comes with an internal tensor product satisfying the adjointness formula

$$\mathbf{Hom}^\bullet(\mathbf{A}^\bullet \otimes \mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong \mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathrm{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet)).$$

Using this, we obtain

$$T(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet) \cong \mathbf{Hom}^\bullet(\mathbf{A}^\bullet, T\mathbf{A}^\bullet)$$

and

$$\mathbf{A}^\bullet \tilde{\otimes} \mathbf{A}^\bullet \cong T^2(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet).$$

Now, the tensor product $\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet$ comes with a natural involution τ induced by transposition of the factors. This induces an involution on $T(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet)$ given by $T(\tau)$. We have the following result relating the involutions $T(\tau)$ and τ' defined before.

5.3 Lemma. *The involution $T(\tau)$ coincides with τ' under the identification of $T(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet)$ with $\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, T\mathbf{A}^\bullet)$ above.*

Proof. After decoding the definitions of T and e, the assertion follows easily from the analogous statement for R-modules. □

5.4 Lemma. *A n-dimensional symmetric structure on a bounded constructible complex $\mathbf{A}^\bullet$ of Γ_c-acyclic sheaves is uniquely determined by a system of maps of degree $-s$*

$$\phi_s : \mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \to \mathbb{D}_X[-n]$$

satisfying $\phi_s = 0$ for $s < 0$ and

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0.$$

Proof. Let $\tilde{\phi}$ be a symmetric structure on $\mathbf{A}^\bullet$. That is, we have a system of maps

$$\tilde{\phi}_s : (\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet)^{r+n+s} \to \mathbb{D}_X^r$$

satisfying $\tilde{\phi}_s = 0$ for $s < 0$ and

$$d\tilde{\phi}_s + (-1)^r \tilde{\phi}_s d + (-1)^{n+s-1}\tilde{\phi}_{s-1} + (-1)^{n-1}\tilde{\phi}_{s-1}\tau = 0.$$

We define

$$\phi_s := (-1)^{\frac{r(r+1)}{2} + (s+1)(n+r+1)}\tilde{\phi}_s.$$

Direct (but lengthy) calculation shows that ϕ_s satisfies the relations of the statement and therefore uniquely determines $\tilde{\phi}$. □

5.5 Remark. If $\tilde{\phi}$ is a symmetric structure such that $\tilde{\phi} = d\tilde{\psi}$ for $\tilde{\psi} \in W^{\%}(\mathbf{A}^\bullet)^{-(n+1)}$ then $\tilde{\psi}$ is uniquely determined by a system of maps

$$\psi_s : \mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \to \mathbb{D}_X[n+1]$$

satisfying $\psi_s = 0$ for $s < 0$ and

$$d\psi_s + (-1)^{s-1}\psi_s d + (-1)^s \psi_{s-1}\tau + \psi_{s-1} = (-1)^{r+s-1}\phi_s.$$

With ϕ_s given as above. The correspondence is given by the same sign trick.

6 Simply connected assembly

In this section we will relate the algebraic bordism groups $L^n(R)$ and $L^n(X; R)$, for compact X, by constructing a map

$$A : L^n(X; R) \to L^n(R).$$

This map is called the simply connected assembly map in contrast to the universal assembly map

$$\tilde{A} : L^n(X; R) \to L^n(R[\pi_1(X)])$$

which takes the fundamental group into account and coincides with the first for simply connected spaces. However, we will not need the universal assembly here and therefore concentrate on the slightly easier construction of A.

In order to construct the map A we will first deal with some structure theorems on the derived category of R-modules. These are actually exercises in [KS90]. We consider the following full subcategories of $R - \mathcal{M}od$:

$R\text{-}\mathcal{M}od_f$	the category of all finitely generated R-modules
$R\text{-}\mathcal{M}od_{fp}$	the category of all finitely generated projective R-modules

The inclusion $R\text{-}\mathcal{M}od_{fp} \hookrightarrow R\text{-}\mathcal{M}od_f$ induces a functor

$$i : D^b(R\text{-}\mathcal{M}od_{fp}) \hookrightarrow D^b(R\text{-}\mathcal{M}od_f)$$

of the associated derived categories and we have:

6.1 Proposition. *The functor i is an equivalence of categories.*

Proof. Following [McL71],IV.4 , i is an equivalence if for each bounded complex $F^\bullet$ of finitely generated R-modules there is a bounded complex $P^\bullet$ of finitely generated projectives and a quasi-isomorphism $P^\bullet \to F^\bullet$. Since R has finite global dimension, this can be achieved by taking some bounded projective resolution. Note, that R being noetherian implies that for each surjection $P \to F$ with F and P finitely generated R-modules and P projective, the kernel is automatically finitely generated (see [La02],X.2). □

Furthermore, we have

6.2 Proposition. *Let $j : D^b(R\text{-}\mathcal{M}od_f) \to D^b_f(R\text{-}\mathcal{M}od)$ denote the inclusion functor of the bounded derived category of all finitely generated R-modules into the full subcategory $D^b_f(R\text{-}\mathcal{M}od)$ of $D^b(R\text{-}\mathcal{M}od)$ consisting of all complexes with finitely generated cohomology. Then j is an equivalence of categories.*

Proof. For the proof we need two basic properties of $R\text{-}\mathcal{M}od_f$. Firstly, since R is noetherian $R\text{-}\mathcal{M}od_f$ is a thick abelian subcategory of $R\text{-}\mathcal{M}od$. That is, given an exact sequence of R-modules

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

with M_1, M_2, M_4, M_5 finitely generated, then M_3 is finitely generated. Secondly, for a given surjection

$$M \to N$$

with N finitely generated there is a finitely generated R-module X and a map $X \to M$ such that the composition

$$X \to M \to N$$

is surjective. For the first property see [La02],X.2, again. The second property is obvious.

For a given complex $K^\bullet$ with finitely generated cohomology we have to construct a complex $F^\bullet$ of finitely generated modules and a quasi-isomorphism $f : F^\bullet \to K^\bullet$. Suppose inductively, that $f : F^k \to K^k$ has been constructed for $k > n$ such that $fd_F^k = d_K^k f$ and f induces a surjection $\ker d_F^{n+1} \to H^{n+1}(K^\bullet)$.

Let M be defined by the pullback square

$$\begin{array}{ccc} M & \longrightarrow & \ker d_F^{n+1} \\ \downarrow & & \downarrow \\ \operatorname{coker} d_K^{n-1} & \longrightarrow & \ker d_K^{n+1} \end{array}$$

and N be defined by the pullback square

$$\begin{array}{ccc} N & \longrightarrow & \ker d_F^{n+1} \\ \downarrow & & \downarrow \\ K^n & \longrightarrow & \ker d_K^{n+1} \end{array}$$

Then there is an exact sequence

$$0 \to H^n(K^\bullet) \to M \to \ker(d_F^{n+1} \to H^{n+1}(K^\bullet)) \to 0$$

which shows that M is finitely generated. Since the obvious map $N \to M$ is a surjection there is a finitely generated F^n and a map $F^n \to N$ such that

$$F^n \to N \to M$$

is a surjection. We define d_F^n by the composition

$$F^n \to N \to \ker d_F^{n+1} \hookrightarrow F^{n+1}$$

and $f : F^n \to K^n$ by the composition

$$F^n \to N \to K^n.$$

It is straightforward to check that $fd_F^n = d_K^n f$ and f induces a surjection

$$\ker d_F^n \to \ker d_K^n \to H^n(K^\bullet).$$

Moreover, it follows directly from the construction that f is a quasi-isomorphism. □

Combining these two results, we see that the forgetful functor gives an equivalence of categories

$$v : D^b(R - \mathcal{M}od_{fp}) \to D^b_f(R - \mathcal{M}od).$$

Consequently, there is a right adjoint

$$p : D^b_f(R - \mathcal{M}od) \to D^b(R - \mathcal{M}od)$$

such that the adjunction morphism $vp \to 1$ is an equivalence of functors. In other words, for every chain complex $K^\bullet$ with finitely generated cohomology this adjunction morphism gives a resolution

$$p(K^\bullet)^\bullet \to K^\bullet$$

by finitely generated projectives.

Furthermore, since the target of p is a category of complexes of projectives, p has a lift in the diagram

$$\begin{array}{ccc} K^b_f(R\text{-}\mathcal{M}od) & \dashrightarrow & K^b(R\text{-}\mathcal{M}od_{fp}) \\ \downarrow & & \downarrow \simeq \\ D^b_f(R\text{-}\mathcal{M}od) & \xrightarrow{p} & D^b(R\text{-}\mathcal{M}od_{fp}) \end{array}$$

to a functor

$$\tilde{p} : K^b_f(R\text{-}\mathcal{M}od) \to K^b(R\text{-}\mathcal{M}od_{fp}).$$

Which means that we have a functorial resolution after passing to the homotopy categories.

Now, let $(\mathbf{A}^\bullet, \phi)$ be an n-dimensional symmetric structure on X. Let $R \to I^\bullet$ be the injective resolution of R chosen in II.4 and $\mathbf{R}_X \to \mathbf{C}^\bullet(\mathbf{R}_X)$ be the bounded canonical resolution of the constant sheaf $\mathbf{R}_X$. Recall, that $\mathbb{D}_X$ is given by

$$U \mapsto \mathrm{Hom}^\bullet(\Gamma_c(\mathbf{C}^\bullet(\mathbf{R}_X)_{|U}), I^\bullet).$$

In [Bo84],V.7, it is shown that the functor $T = \mathbf{Hom}^\bullet(-, \mathbb{D}_X)$ restricted to $Ch^b_c(X)$ is isomorphic to

$$\mathbf{B}^\bullet \mapsto (U \mapsto \mathrm{Hom}^\bullet(\Gamma_c((\mathbf{B}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X))_{|U}), I^\bullet)$$

for all $\mathbf{B}^\bullet \in Ch^b_c(X)$. One can view this as a special form of Poincaré-Verdier duality on the sheaf level. It implies

$$\begin{aligned} \Gamma\mathbf{Hom}^\bullet(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet, \mathbb{D}_X[-n]) &\cong \mathrm{Hom}^\bullet(\Gamma_c(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X)), I^\bullet[-n]) \\ &= \mathrm{Hom}^\bullet(\Gamma(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X)), I^\bullet[-n]) \end{aligned}$$

where the equals sign holds because X is compact. Moreover, this isomorphism is $\mathbb{Z}/2$-equivariant, as can be easily seen from its definition in [Bo84]. Let v denote the forgetful functor $Sh(X) \to PSh(X)$, we use the adjunction morphism $1 \to v \circ \mathrm{sheaf}$ to construct a natural morphism

$$\Gamma\mathbf{A}^\bullet \otimes \Gamma\mathbf{A}^\bullet \to \Gamma(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet) \to \Gamma(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X)).$$

By assumption the complex of sheaves $\mathbf{A}^\bullet$ is constructible and therefore the cohomology of $\Gamma\mathbf{A}^\bullet$ is finitely generated in each dimension. Hence we can take the resolution

$$p(\Gamma\mathbf{A}^\bullet)^\bullet \to \Gamma\mathbf{A}^\bullet$$

constructed above and precompose to get a map

$$p(\Gamma\mathbf{A}^\bullet)^\bullet \otimes p(\Gamma\mathbf{A}^\bullet)^\bullet \to \Gamma(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X)).$$

This map induces a $\mathbb{Z}/2$-equivariant map

$$\tilde{p} : \Gamma\mathbf{Hom}^\bullet(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet, \mathbb{D}_X[-n]) \to \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet \otimes p(\Gamma\mathbf{A}^\bullet)^\bullet, I^\bullet).$$

On the other hand, since the tensor product of projectives is still projective we have a $\mathbb{Z}/2$-equivariant quasi-isomorphism

$$\mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet \otimes p(\Gamma\mathbf{A}^\bullet)^\bullet, R) \to \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet \otimes p(\Gamma\mathbf{A}^\bullet)^\bullet, I^\bullet).$$

Again, since

$$W : \ldots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2]$$

is a free resolution of $\mathbb{Z}$, we get an induced quasi-isomorphism

$$W^{\%}(p(\Gamma\mathbf{A}^\bullet)^\bullet) \to \mathrm{Hom}^\bullet_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet \otimes p(\Gamma\mathbf{A}^\bullet)^\bullet, I^\bullet).$$

We take the image $\tilde{p}^*(\phi)$ of $\phi \in W^{\%}(\mathbf{A}^\bullet)^{-n}$ and choose an element

$$p(\phi) \in W^{\%}(p(\Gamma\mathbf{A}^\bullet)^\bullet)^{-n}$$

whose cohomology class is mapped to the cohomology class of $\tilde{p}^*(\phi)$ under the above quasi-isomorphism. Note, that this choice does not have an effect since the identity map will be a homotopy equivalence of Poincaré complexes for any two of these choices.

6.3 Definition. There is an assignment A which assigns to every symmetric sheaf of dimension n a symmetric complex of the same dimension. It is given by

$$(\mathbf{A}^\bullet, \phi) \mapsto (p(\Gamma\mathbf{A}^\bullet)^\bullet, p(\phi)).$$

This assignment becomes functorial after passing to homotopy equivalence classes of symmetric complexes.

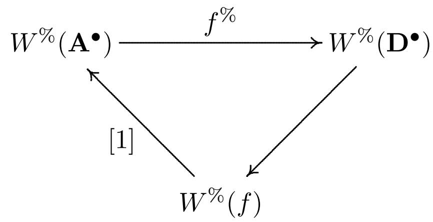
We want to have a similar construction for symmetric pairs of sheaves, which is compatible with the one given above.

Suppose $(f : \mathbf{D}^\bullet \to \mathbf{A}^\bullet, (\delta\phi, \phi))$ is an $(n+1)$-dimensional symmetric pair of sheaves. The first step is to notice that, since $p(\Gamma\mathbf{D}^\bullet)^\bullet \to \Gamma\mathbf{D}^\bullet$ and $p(\Gamma\mathbf{A}^\bullet)^\bullet \to \Gamma\mathbf{A}^\bullet$ are projective resolutions, there is a map

$$F : p(\Gamma\mathbf{D}^\bullet)^\bullet \to p(\Gamma\mathbf{A}^\bullet)^\bullet$$

which lifts $\Gamma f : \Gamma\mathbf{D}^\bullet \to \Gamma\mathbf{A}^\bullet$. This map constitutes the first part of a symmetric pair. Now, we must find a symmetric structure on F which bounds $p(\phi)$.

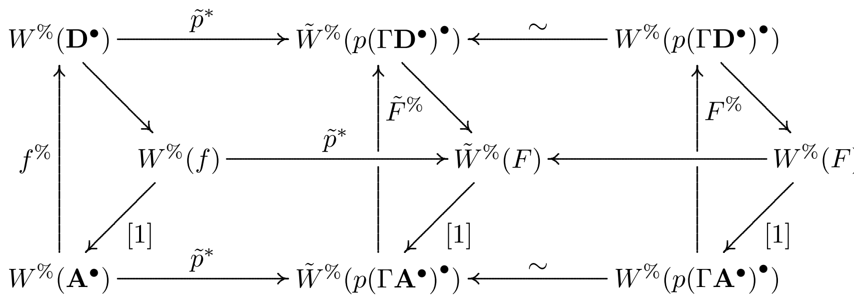
Since $W^{\%}(f)$ is defined as the mapping cone of $f^{\%}$ we get a distinguished triangle in the derived category of abelian groups $D(\mathcal{A}b)$:



We write $\tilde{W}^{\%}(X^\bullet)$ for the complex

$$\mathrm{Hom}^\bullet_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathrm{Hom}^\bullet(X^\bullet \otimes X^\bullet, I^\bullet),$$

where, as before, $I^\bullet$ is the fixed injective resolution of R. Then there is a diagram of distinguished triangles



By the five lemma, the map $W^{\%}(F) \to \tilde{W}^{\%}(F)$ is also a quasi-isomorphism. Hence, we can choose a representative $(\delta\psi, \psi) \in W^{\%}(F)^{n+1}$ of the cohomology class determined by $\tilde{p}^*(\delta\phi, \phi)$ under this quasi-isomorphism.

There is of course no reason to assume that the two choices we made fit together, that is $\psi = p(\phi)$. However, $[\psi] = [p(\phi)] \in Q^n(F)$ and the boundary $(p(\Gamma\mathbf{A}^\bullet)^\bullet, \psi)$ of $(F : p(\Gamma\mathbf{D}^\bullet)^\bullet \to p(\Gamma\mathbf{A}^\bullet)^\bullet, (\delta\psi, \psi))$ is homotopy equivalent to $(p(\Gamma\mathbf{A}^\bullet)^\bullet, p(\phi))$. Therefore, if no confusion is possible we will write $(p(\delta\phi), p(\phi))$ for $(\delta\psi, \psi)$.

6.4 Definition. There is an assignment A which assigns to every symmetric pair of sheaves of dimension $(n+1)$ a symmetric pair of complexes of the same dimension. It is given by

$$(f : \mathbf{D}^\bullet \to \mathbf{A}^\bullet, (\delta\phi, \phi)) \mapsto \left(F : p(\Gamma\mathbf{D}^\bullet)^\bullet \to p(\Gamma\mathbf{A}^\bullet)^\bullet, (p(\delta\phi), p(\phi))\right).$$

This assignment becomes functorial after passing to homotopy equivalence classes of symmetric pairs. Furthermore, we have

$$\partial A(f : \mathbf{D}^\bullet \to \mathbf{A}^\bullet, (\delta\phi, \phi)) \simeq A(\partial(f : \mathbf{D}^\bullet \to \mathbf{A}^\bullet, (\delta\phi, \phi))).$$

6.5 Lemma. *The assignments of 6.3 and 6.4 preserve the property of being Poincaré.*

Proof. We will prove only the first case, the second being similar.

Suppose $(\mathbf{A}^\bullet, \phi)$ is a symmetric Poincaré sheaf of dimension n. Let φ_0 denote the following composition

$$\begin{array}{rcl} p(\Gamma\mathbf{A}^\bullet)^\bullet & \to & \Gamma\mathbf{A}^\bullet \\ & \xrightarrow{\Gamma\phi_0} & \Gamma\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}_X[-n]) \\ & \cong & \mathrm{Hom}^\bullet(\Gamma(\mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{R}_X)), I^\bullet[-n]) \\ & \to & \mathrm{Hom}^\bullet(\Gamma\mathbf{A}^\bullet, I^\bullet[-n]) \\ & \to & \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet, I^\bullet[-n]) \end{array}$$

With the possible exception of $\Gamma\phi_0$, all the above maps are quasi-isomorphisms. However $\mathbf{A}^\bullet$ is a complex of Γ_c-acyclic and hence Γ-acyclic sheaves by compactness. Also, $\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}_X[-n])$ is a complex of flabby and hence Γ-acyclic sheaves. Now, Γ preserves quasi-isomorphisms between complexes of Γ-acyclic sheaves. Since ϕ_0 is a quasi-isomorphism by assumption we have that $\Gamma\phi_0$ is a quasi-isomorphism also. Therefore, the same holds for φ_0.

There is a commutative diagram

$$\begin{array}{ccc} p(\Gamma\mathbf{A}^\bullet)^\bullet & \xrightarrow{\varphi_0} & \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet, I^\bullet[-n]) \\ & \searrow^{p(\phi)_0} & \uparrow \sim \\ & & \mathrm{Hom}^\bullet(p(\Gamma\mathbf{A}^\bullet)^\bullet, \mathbb{Z}[-n]) \end{array}$$

This shows, that $p(\phi)_0$ is a quasi-isomorphism between projectives and hence a homotopy equivalence. $\square$

As a direct consequence we get

6.6 Proposition. *The assignment*

$$(\mathbf{A}^\bullet, \phi) \mapsto A(\mathbf{A}^\bullet, \phi)$$

induces a map

$$L^n(X; R) \to L^b(R)$$

again denoted by A. This map is the simply connected assembly map.

The simply connected assembly map has the following useful property.

6.7 Proposition. *Let $(\mathbf{A}^\bullet, \phi)$ and $(\mathbf{B}^\bullet, \psi)$ be two n-dimensional symmetric Poincaré sheaves on X. Let $f : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ be a map of symmetric Poincaré sheaves, that is $(f^{\%})^*([\psi]) = [\phi] \in Q^{-n}(\mathbf{A}^\bullet)$. Suppose further, that f is a quasi-isomorphism. Then $A(\mathbf{A}^\bullet, \phi) = A(\mathbf{B}^\bullet, \psi) \in L^n(R)$.*

Proof. We can complete the diagram of quasi-isomorphisms to a commutative diagram

$$\begin{array}{ccc} p(\Gamma\mathbf{A}^\bullet)^\bullet & \overset{F}{\dashrightarrow} & p(\Gamma\mathbf{B}^\bullet)^\bullet \\ \downarrow & & \downarrow \\ \Gamma\mathbf{A}^\bullet & \xrightarrow{\Gamma f} & \Gamma\mathbf{B}^\bullet \end{array}$$

by a quasi-isomorphism

$$F : p(\Gamma\mathbf{A}^\bullet)^\bullet \to p(\Gamma\mathbf{B}^\bullet)^\bullet.$$

By projectivity F is a homotopy equivalence. That $(F^{\%})^*([p(\psi)]) = [p(\phi)]$ follows easily. This implies the assertion. □

Chapter IV

Signature homology

In this chapter we will define signature homology and show that it is a direct summand of symmetric L-theory. As a result we obtain an assembly map for signature homology and thus answer the question from which this work originated.

1 Definition of signature homology

There are several ways to define signature homology and we choose the approach with the least prerequisites. For other ways of defining signature homology see [Min04].

According to Milnor, the coefficients of the unitary bordism ring are given by

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \ldots]$$

with $\deg x_i = 2i$. By taking connected sums with products of $\mathbb{C}P^2$'s it is easy to see that x_i can be represented by a unitary manifold M_i such that

$$\operatorname{sig}(M_i) = \begin{cases} 0 & \text{for } i \neq 2 \\ 1 & \text{for } i = 2 \end{cases} .$$

We define signature homology Sig as the unitary bordism theory with Baas-Sullivan singularities $\{M_i\}_{i\neq 2}$. We summarize some of its properties in the following proposition.

1.1 Proposition. *(1) There is an isomorphism*

$$\operatorname{sig} : \pi_*(Sig) \to \mathbb{Z}[t],$$

where $\deg(t) = 4$.

(2) There is a map of ring spectra

$$MSO \to Sig$$

such that the diagram

$$\begin{array}{ccc} \pi_*(MSO) & \longrightarrow & \pi_*(Sig) \\ & {}_{\text{sig}}\searrow \quad \swarrow_{\text{sig}} & \\ & \mathbb{Z}[t] & \end{array}$$

commutes.

(3)

$$Sig \otimes \mathbb{Z}[\frac{1}{2}] \simeq ko \otimes \mathbb{Z}[\frac{1}{2}]$$

(4)

$$Sig_{(2)} \simeq H\mathbb{Z}_{(2)}[t]$$

Proof. For (1)-(3) see [Min04], for (4) see [TW79] or [Min04]. □

2 Signature homology and symmetric L-theory

We will use the determination of the homotopy type of both signature homology and symmetric L-theory to show that Sig is a direct summand of $\mathbb{L}^{\cdot}$. Recall from [Su05] that there are two homotopy pullback squares which incorporate the results 1.1 and II.4.2.

$$\begin{array}{ccc} Sig & \longrightarrow & ko \otimes \mathbb{Z}[\frac{1}{2}] \\ \downarrow & & \downarrow \\ H\mathbb{Z}_{(2)}[t] & \longrightarrow & H\mathbb{Q}[t] \end{array} \qquad \begin{array}{ccc} \mathbb{L}^{\cdot} & \longrightarrow & ko \otimes \mathbb{Z}[\frac{1}{2}] \\ \downarrow & & \downarrow \\ H\mathbb{Z}_{(2)}[t] \vee \Sigma(H\mathbb{Z}/2[t]) & \longrightarrow & H\mathbb{Q}[t] \end{array}$$

By the universal property of the pullback, the obvious inclusion

$$H\mathbb{Z}_{(2)}[t] \to H\mathbb{Z}_{(2)}[t] \vee \Sigma(H\mathbb{Z}/2[t])$$

and projection

$$H\mathbb{Z}_{(2)}[t] \vee \Sigma(H\mathbb{Z}/2[t]) \to H\mathbb{Z}_{(2)}[t]$$

give rise to maps

$$i : Sig \to \mathbb{L}^{\cdot} \text{ and } p : \mathbb{L}^{\cdot} \to Sig$$

such that

$$pi \simeq \text{id}.$$

Therefore we have shown:

2.1 Proposition. *Signature homology Sig is a direct summand of symmetric L-theory $\mathbb{L}^{\cdot}$.*

Chapter V

The symmetric structure on $\mathbf{IC}^\bullet_{\bar{m}}$

In this chapter we will solve the problem of assigning an n-dimensional symmetric Poincaré complex to a closed IP-space X of the same dimension. Intuitively, one could be tempted to take the piecewise linear intersection chain complex and try to construct a symmetric structure for this complex. Unfortunately, the method of acyclic models used in the singular homology case to construct a symmetric structure can not be transferred to our case right away. The missing homotopy invariance of intersection homology is one reason for this. Moreover, it is not clear what the right choice of models should be.

This forces us to work in the world of sheaves and construct a symmetric Poincaré structure on the intersection homology sheaf of X. More precisely, we will construct a symmetric structure on $\mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(X)$ for a closed IP-space X of dimension n. By III.5.4 a symmetric structure on a complex of sheaves $\mathbf{A}^\bullet$ is given by a system of maps

$$\phi_s : \mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \to \mathbb{D}^\bullet_X[-n]$$

satisfying the relations

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0,$$

where τ denotes the transposition involution on $\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet$.

But there is a price we have to pay when working in the category of sheaves. Namely, we lose control over the size of the global sections of the intersection homology sheaf. In general, they will neither be finitely generated nor projective. However, we can repair this by using the simply connected assembly map of III.6.

The first step in the definition of the required family will be to choose a particular representative of the quasi-isomorphism class of $\mathbf{IC}^\bullet_{\bar{m}}$ which allows us to copy the symmetric construction of [Go84]. This will be done in the first section after recalling some more facts on sheaf theory.

Using these preliminaries and the results on relative homological algebra of appendix A we will give the construction of the symmetric structure on $\mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}$ in the third section and show how to derive a symmetric complex of finitely generated projectives.

We will also show that this symmetric structure is a certain sense uniquely determined by the property that it induces the cup product in cohomology. Finally, using uniqueness we will show how to define a symmetric pair of complexes of sheaves for a given topological pseudomanifold with collared boundary.

1 The complex of sheaves $\mathbf{P}^\bullet$

Firstly, we recall some more facts from sheaf theory. The proofs can be found in [Bre97] and hold for every topological space X.

1.1 Definition. Let $\mathbf{A}$ be a sheaf. The *canonical resolution* $\mathbf{C}^\bullet(\mathbf{A})$ is constructed as follows:

$$\mathbf{C}^0(\mathbf{A})(U) := \prod_{x\in U} \mathbf{A}_x,$$

the collection of possibly discontinuous sections over U. There is the obvious inclusion of all continuous sections inducing a monomorphism

$$\varepsilon : \mathbf{A} \to \mathbf{C}^0(\mathbf{A}).$$

We define $\mathbf{J}^1(\mathbf{A}) := \operatorname{coker} \varepsilon$ and inductively

$$\mathbf{C}^n(\mathbf{A}) := \mathbf{C}^0(\mathbf{J}^n(\mathbf{A})),$$

$$\mathbf{J}^{n+1}(\mathbf{A}) := \mathbf{J}^1(\mathbf{J}^n(\mathbf{A}))$$

such that

$$0 \to \mathbf{J}^n(\mathbf{A}) \to \mathbf{C}^n(\mathbf{A}) \to \mathbf{J}^{n+1}(\mathbf{A}) \to 0$$

is exact. Splicing these short exact sequences together we obtain the desired resolution $\mathbf{C}^\bullet(\mathbf{A})$.

The advantages of this resolution are summarized in the following proposition.

1.2 Proposition. *(1) The canonical resolution of $\mathbf{A}$ is a flabby resolution.*

(2) The canonical resolution is pointwise homotopically trivial, that is, for each $x \in X$ the corresponding cochain complex of stalks is contractible.

(3) For every sheaf $\mathbf{B}$ the tensor product $\mathbf{C}^\bullet(\mathbf{A}) \otimes \mathbf{B}$ is a resolution of $\mathbf{A} \otimes \mathbf{B}$.

(4) If $\mathbf{A}$ has torsion free stalks the same holds for $\mathbf{C}^n(\mathbf{A})$ and $\mathbf{J}^n(\mathbf{A})$ for all n.

(5) *If X is of finite cohomological dimension then there is a k such that $\mathbf{J}^k(\mathbf{A})$ is flabby for each sheaf $\mathbf{A}$. In this case we obtain a bounded canonical resolution which we will denote by $\mathbf{C}^\bullet(\mathbf{A})$ again.*

If we start with a complex of sheaves $\mathbf{A}^\bullet$ we have the obvious generalizations giving the canonical resolutions $\mathbf{C}^\bullet(\mathbf{A}^\bullet)$.

1.3 Lemma. *Let $\mathbf{A}$ be a sheaf with torsion free stalks. Let $\mathbf{B}^\bullet$ be a complex of sheaves. Then $\mathbf{B}^\bullet \otimes \mathbf{C}^\bullet(\mathbf{A})$ is a soft resolution of $\mathbf{B}^\bullet \otimes \mathbf{A}$.*

Proof. The only thing that needs to be proven is the softness of the resolution. But the tensor product $\mathbf{K} \otimes \mathbf{L}$ is soft for any $\mathbf{K}$ if $\mathbf{L}$ is soft with torsion free stalks. Since every flabby sheaf is soft we are done (see [Bre97], II.16.31). □

Now, using the canonical resolution of the constant sheaf $\mathbb{Z}_X$, we will fix a specific complex of sheaves in the quasi-isomorphism class of all complexes describing intersection homology. Recall, for an n-dimensional stratified topological pseudomanifold X we had $U_k := X - X_{n-k}$ and $i_k : U_k \to U_{k+1}$, the inclusion.

1.4 Definition. Let $\bar{p}$ denote a perversity and let $\mathbf{C}_k^\bullet$ denote the canonical resolution of the constant sheaf $\mathbb{Z}_{U_k}$ on U_k. We define

$$\mathbf{P}_3^\bullet := \tau_{\leq \bar{p}(2)-n}(i_2)_*(\mathbb{Z}_{U_2} \otimes \mathbf{C}_2^\bullet)$$

and then inductively

$$\mathbf{P}_{k+1}^\bullet := \tau_{\leq \bar{p}(k)-n}(i_k)_*(\mathbf{P}_k^\bullet \otimes \mathbf{C}_k^\bullet).$$

Then $\mathbf{P}_{\bar{p}}^\bullet := \mathbf{P}_{n+1}^\bullet \otimes \mathbf{C}_{n+1}^\bullet$.

We want to show that this complex of sheaves satisfies the axioms for intersection homology. We prepare the proof of this statement by a series of little lemmata which are actually exercises in [Bo84].

1.5 Lemma. *Let $f : X \to Y$ be a map of topological spaces. If $\mathbf{A}^\bullet$ is an exact complex of sheaves such that $\mathbf{A}^i$ is cohomologically trivial on all open subsets of X, for all i, then $f_*\mathbf{A}^\bullet$ is exact.*

Proof. We have

$$\mathbf{H}^k(f_*\mathbf{A}^\bullet)_y = \varinjlim H^k(\Gamma(U; f_*\mathbf{A}^\bullet)) = \varinjlim H^k(\Gamma(f^{-1}(U); \mathbf{A}^\bullet))$$

By assumption on $\mathbf{A}^i$ the hypercohomology spectral sequence is trivial and we get

$$\varinjlim H^k(\Gamma(f^{-1}(U); \mathbf{A}^\bullet)) = \varinjlim \mathbb{H}^k(f^{-1}(U); \mathbf{A}^\bullet)) = \varinjlim \mathbb{H}^k(f^{-1}(U); 0^\bullet) = 0,$$

since $\mathbf{A}^\bullet$ is exact. Here the limits are taken over all open neighbourhoods U of y. □

1.6 Lemma. *Let $f : X \to Y$ be a map. If $\mathbf{A}$ is cohomologically trivial on all open subsets of X, then $f_*\mathbf{A}$ is cohomologically trivial on all open subsets of Y.*

Proof. Let $\mathbf{A} \to \mathbf{I}^\bullet$ be an injective resolution of $\mathbf{A}$. Since f_* of an injective sheaf is injective the preceding lemma yields that $f_*\mathbf{I}^\bullet$ is an injective resolution of $f_*\mathbf{A}$. Therefore we conclude for all open U in Y

$$H^k(U; f_*\mathbf{A}) = H^k(\Gamma(U; f_*\mathbf{I}^\bullet)) = H^k(\Gamma(f^{-1}(U); \mathbf{I}^\bullet)) = H^k(f^{-1}(U); \mathbf{A}).$$

□

1.7 Lemma. *Let $f : X \to Y$ be a map. If $g : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ is a quasi-isomorphism then $f_*g : f_*\mathbf{A}^\bullet \to f_*\mathbf{B}^\bullet$ is a quasi-isomorphism provided that each $\mathbf{A}^i$ and $\mathbf{B}^i$ is cohomologically trivial on all open subsets of X.*

Proof. We have to show that the mapping cone of f_*g is exact. But $\mathbf{cone}^\bullet(f_*g) = f_*\mathbf{cone}^\bullet(g)$ and $\mathbf{cone}^\bullet(g)$ is cohomologically trivial in each degree. Therefore, the exactness follows from the exactness of $\mathbf{cone}^\bullet(g)$. □

1.8 Lemma. *Let $\mathbf{A}$ be a soft sheaf. Then $\mathbf{A}$ is cohomologically trivial on all open subsets of X.*

Proof. see [Bre97], II.16.1. □

We combine these results to get the following proposition.

1.9 Proposition. *Let $f : X \to Y$ be a map, $\mathbf{A}$ a soft sheaf on X. Then the canonical map $f_*\mathbf{A} \to Rf_*\mathbf{A}$ is a quasi-isomorphism.*

Turning our attention to the complex $\mathbf{P}^\bullet_{\bar{m}}$, we see:

1.10 Proposition. *(1) $\mathbf{P}^\bullet_{\bar{m}}$ is a complex of flat sheaves.*

(2) For any map $f : X \to Y$ between PL-pseudomanifolds $\mathbf{P}^\bullet_{\bar{m}}$ is f_*- and $f_!$-acyclic.

(3) $\mathcal{D}\mathbf{P}^\bullet_{\bar{m}} = \mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\bar{m}}, \mathbb{D}_X)$ is a complex of injectives.

Proof. (1) By [GeM02],III.4 this can be checked stalkwise. But for the stalks this is clear by construction.

(2) This follows since $\mathbf{P}^\bullet_{\bar{m}}$ is both soft by 1.3 and c-soft by [Bre97],II.9.18 and II.9.6.

(3) We have that the functor

$$\mathbf{Hom}(\mathbf{P}^k_{\bar{m}}, -) : Sh(X) \to Sh(X)$$

is right adjoint to

$$- \otimes \mathbf{P}^k_{\bar{m}} : Sh(X) \to Sh(X).$$

Since $\mathbf{P}^k_{\bar{m}}$ is flat, the latter is an exact functor. It follows from [We94],2.3.10 that $\mathbf{Hom}(\mathbf{P}^k_{\bar{m}}, -)$ preserves injectives which implies the assertion.

□

Furthermore we have:

1.11 Proposition. *Let X be a n-dimensional stratified topological pseudomanifold.*

(1) The complex of sheaves $\mathbf{P}^\bullet_{\bar{p}}[n]$ on X of definition 1.4 satisfies the axioms [AX1] of the intersection homology sheaves for perversity $\bar{p}$.

(2) If X is an IP-space. Then $\mathbf{P}^\bullet_{\bar{m}}$ is quasi-isomorphic to $\mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(X)$.

Proof. (1) Since both $(i_k)_*$ and $\tau_{\leq \bar{p}(k)-n}$ respect quasi-isomorphisms between sheaves which are cohomologically trivial on all open subsets of U_k the previous results imply that $\mathbf{P}^\bullet_{\bar{p}}$ is quasi-isomorphic to the Deligne sheaf.

(2) This now follows easily from the self duality of $\mathbf{IC}^\bullet_{\bar{m}}(X)$.

□

2 The symmetric structure on $\mathbf{P}^\bullet$

In [Bre97] Bredon gives a construction of a symmetric structure on the canonical resolution of the constant sheaf $\mathbb{Z}$ on any space. We will start with this structure on the non-singular part of a topological pseudomanifold. Following [Go84], we will extend this symmetric structure to a symmetric structure on $\mathbf{P}^\bullet_{\bar{m}}$. In contrast to [Go84] we will have to take care of signs, since we are working with integer coefficients.

We begin with a reformulation of the mentioned result of [Bre97] suited to our context.

2.1 Proposition. *Let X be any space and let $\phi : \mathbb{Z}_X \otimes \mathbb{Z}_X \to \mathbb{Z}_X$ be given by multiplication. There are maps*

$$\phi_s : \mathbf{C}^\bullet(\mathbb{Z}_X) \otimes \mathbf{C}^\bullet(\mathbb{Z}_X) \to \mathbf{C}^\bullet(\mathbb{Z}_X)$$

of degree $-s$ satisfying

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0,$$

and $\phi_s = 0$ for $s < 0$. Here τ denotes the transposition involution.

Proof. We will use the results of appendix A on relative homological algebra. We want to show that $\mathbf{C}^\bullet(\mathbb{Z}_X)\otimes\mathbf{C}^\bullet(\mathbb{Z}_X)$ is an E-resolution of $\mathbb{Z}_X\otimes\mathbb{Z}_X$ with E being the class of all pointwise split monomorphisms. By corollary A.6 we already know that $\mathbf{C}^\bullet(\mathbb{Z}_X)$ is an E-injective resolution of $\mathbb{Z}_X$. Furthermore, forming the tensor product with some flat sheaf preserves this property. Therefore,

$$\mathbb{Z}_X \otimes \mathbb{Z}_X \to \mathbf{C}^\bullet(\mathbb{Z}_X) \otimes \mathbb{Z}_X \to \mathbf{C}^\bullet(\mathbb{Z}_X) \otimes \mathbf{C}^\bullet(\mathbb{Z}_X)$$

is an E-injective resolution of $\mathbb{Z}_X \otimes \mathbb{Z}_X$. For the rest of this proof we will write $\mathbf{C}^\bullet$ for $\mathbf{C}^\bullet(\mathbb{Z}_X)$.

By theorem A.4 there is a morphism of resolutions

$$\phi_0 : \mathbf{C}^\bullet \otimes \mathbf{C}^\bullet \to \mathbf{C}^\bullet$$

extending the multiplication ϕ. It follows that

$$\phi_0(1-\tau) : \mathbf{C}^\bullet \otimes \mathbf{C}^\bullet \to \mathbf{C}^\bullet$$

extends $\phi(1-\tau) = 0$ and hence, is homotopically trivial. Again, by A.4 there is a null homotopy

$$\phi_1 : \mathbf{C}^\bullet \otimes \mathbf{C}^\bullet \to \mathbf{C}^\bullet$$

of degree -1, such that

$$\phi_0 - \phi_0\tau = d\phi_1 + \phi_1 d.$$

Multiplying the last equation with $(1+\tau)$ from the right shows that $\phi_1(1+\tau)$ anti-commutes with d. Therefore,

$$(-1)^r\phi_1(1+\tau) : (\mathbf{C}^\bullet \otimes \mathbf{C}^\bullet)^r \to \mathbf{C}^{r-1}$$

is a chain map of degree -1. Since it extends the zero map it must be homotopically trivial. In terms of ϕ_1 itself, this means there is a map

$$\phi_2 : \mathbf{C}^\bullet \otimes \mathbf{C}^\bullet \to \mathbf{C}^\bullet$$

of degree -2, such that

$$\phi_1 + \phi_1\tau = d\phi_2 - \phi_2 d.$$

Multiplying this equation by $(1-\tau)$ from the right shows that $\phi_2(1-\tau)$ is a chain map. Proceeding inductively, we get the desired statement. $\square$

2.2 Remark. It is clear from the preceding proof that we can also get a system of maps

$$\varphi_s : \mathbf{C}^\bullet(\mathbb{Z}_X) \otimes \mathbf{C}^\bullet(\mathbb{Z}_X) \to \mathbf{C}^\bullet(\mathbb{Z}_X)$$

of degree $-s$ satisfying

$$d\varphi_s + \varphi_s d + (-1)^{s+r-1}\varphi_{s-1}\tau - (-1)^r\varphi_{s-1} = 0,$$

for elements of degree r and $\varphi_s = 0$ for $s < 0$.

Our goal is to extend this construction to the intersection chain sheaf $\mathbf{P}^\bullet$ on an IP-space X. We start with the system of maps ϕ_s constructed above on the nonsingular part and extend this stratumwise to the remainder of X. The following construction is crucial in this step by step process.

Let $\phi_s : \mathbf{A}^\bullet \otimes \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ be a system of maps satisfying

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0,$$

and $\phi_s = 0$ for $s < 0$. We will define maps

$$h_s : (\mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbb{Z}_X)) \otimes (\mathbf{A}^\bullet \otimes \mathbf{C}^\bullet(\mathbb{Z}_X)) \to \mathbf{B}^\bullet \otimes \mathbf{C}^\bullet(\mathbb{Z}_X).$$

satisfying
$$dh_s + (-1)^{s-1} h_s d + (-1)^{s-1} h_{s-1}\tau - h_{s-1} = 0$$
and $h_s = 0$ for $s < 0$.

If φ_s denote the maps mentioned in remark 2.2, we define for each open subset $U \subset X$

$$h_s((a \otimes u) \otimes (b \otimes v)) := \sum_{i=0}^{s} (-1)^{kl+(s-1)(l+m)} \phi_i \tau^{s-i}(a \otimes b) \otimes \varphi_{s-i}\tau^{s-i}(u \otimes v),$$

for homogeneous elements $a, b \in \Gamma(U; \mathbf{A}^\bullet)$ and $u, v \in \Gamma(U; \mathbf{C}^\bullet(\mathbb{Z}_X))$ such that $\deg(a) = j, \deg(b) = k, \deg(u) = l, \deg(v) = m$.

2.3 Lemma. *The maps h_s just defined satisfy the required relations.*

Proof. Direct but lengthy calculation. □

2.4 Proposition. *Let X be an n-dimensional stratified topological pseudomanifold. There is a family of maps*

$$\phi_s : \mathbf{P}^\bullet_{\bar{m}} \otimes \mathbf{P}^\bullet_{\bar{m}} \to \mathbf{P}^\bullet_{\bar{t}}$$

of degree $-s$ satisfying

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0$$

and $\phi_s = 0$ for $s < 0$.

Proof. We denote by $\mathbf{P}^\bullet_k$ and $\mathbf{Q}^\bullet_k$ the intermediate steps in the construction of $\mathbf{P}^\bullet_{\bar{m}}$ and $\mathbf{P}^\bullet_{\bar{t}}$ respectively (see 1.4). Moreover, we denote $\mathbf{C}^\bullet(\mathbb{Z}_{U_k})$ by $\mathbf{C}^\bullet_k$. Suppose, we have constructed a system of maps

$$\phi_s : \mathbf{P}^\bullet_k \otimes \mathbf{P}^\bullet_k \to \mathbf{Q}^\bullet_k$$

satisfying the above relations. By the preceding construction we have a system

$$h_s : (\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \otimes (\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \to \mathbf{Q}^\bullet_k \otimes \mathbf{C}^\bullet_k$$

satisfying the relations. Since $(i_k)_*$ is an additive functor we can apply it to h_s to get a system

$$(i_k)_* h_s : (i_k)_*((\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \otimes (\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k)) \to (i_k)_*(\mathbf{Q}^\bullet_k \otimes \mathbf{C}^\bullet_k)$$

satisfying the relations.

Now recall, that if $Sh(X)$ denotes the category of sheaves on X and $PSh(X)$ the category of presheaves then the sheafification functor sheaf : $PSh(X) \to Sh(X)$ is left adjoint to the forgetful functor $v : Sh(X) \to PSh(X)$. Therefore, there is an adjunction morphism

$$1 \to v \circ \text{sheaf}.$$

Given any complex of sheaves $\mathbf{A}^\bullet$, we get a morphism of complexes in $PSh(X)$:

$$(i_k)_*\mathbf{A}^\bullet \otimes (i_k)_*\mathbf{A}^\bullet = (i_k)_*(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet) \to (i_k)_*\mathrm{sheaf}(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet)$$

Applying sheafification to this morphism, we see that there is a natural morphism of complexes in $Sh(X)$:

$$(i_k)_*\mathbf{A}^\bullet \otimes (i_k)_*\mathbf{A}^\bullet \to (i_k)_*(\mathbf{A}^\bullet \otimes \mathbf{A}^\bullet)$$

In our situation this yields a system of maps

$$\tilde{h}_s : (i_k)_*(\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \otimes (i_k)_*(\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \to (i_k)_*(\mathbf{Q}^\bullet_k \otimes \mathbf{C}^\bullet_k)$$

satisfying the relations. Consider the following commutative diagram

$$\begin{array}{ccc}
\mathbf{P}^\bullet_{k+1} \otimes \mathbf{P}^\bullet_{k+1} & \overset{\tilde{\phi}_s}{\dashrightarrow} & \mathbf{Q}^\bullet_{k+1} \\
\downarrow & \searrow^{\sigma_s} & \downarrow \\
(i_k)_*(\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) \otimes (i_k)_*(\mathbf{P}^\bullet_k \otimes \mathbf{C}^\bullet_k) & \xrightarrow{\tilde{h}_s} & (i_k)_*(\mathbf{Q}^\bullet_k \otimes \mathbf{C}^\bullet_k)
\end{array}$$

where the horizontal maps are given by inclusion of subcomplexes. For $s > 0$ it is obvious that the image of σ_s lies in the subcomplex $\mathbf{Q}^\bullet_{k+1} \subset (i_k)_*(\mathbf{Q}^\bullet_k \otimes \mathbf{C}^\bullet_k)$. For $s = 0$ the same follows from the fact that h_0 is a chain map and therefore maps $\ker d \otimes \ker d$ to $\ker d$. We conclude that the dotted arrow $\tilde{\phi}_s$ exists and satisfies the relations. □

2.5 Theorem. *Let X be an n-dimensional stratified topological pseudomanifold. There is a family of maps*

$$\phi_s : \mathbf{P}^\bullet_{\bar{m}} \otimes \mathbf{P}^\bullet_{\bar{m}} \to \mathbb{D}^\bullet_X[-n]$$

of degree $-s$ satisfying

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0$$

and $\phi_s = 0$ for $s < 0$.

Proof. Let

$$\mathbf{P}^\bullet_{\bar{t}} \cong \mathbf{IC}^\bullet_{\bar{t}}(X)[-n] \to \mathbb{D}_X[-n]$$

be the unique map in $D^b_c(X)$ factorizing the orientation map given by II.5.4. Since $\mathbb{D}_X[-n]$ is a comlex of injectives this can be represented by a map

$$\mathbf{P}^\bullet_{\bar{t}} \to \mathbb{D}_X[-n]$$

in $Ch^b_c(X)$. We fix such a map and compose the family constructed in the previous proposition with this map to get the desired result. □

2.6 Remark. Although we formulated this result for intersection homology with integer coefficients it directly generalizes to intersection homology with coefficients in an arbitrary noetherian ring R.

We will now use the simply connected assembly map to define:

2.7 Definition. Let X be a compact n-dimensional IP-space. We associate to X an n-dimensional symmetric cochain complex $\sigma(X)$ by

$$\sigma(X) := A(\mathbf{P}^\bullet, \phi).$$

We have

2.8 Proposition. *(1) Let X be an n-dimensional IP-space. Then the adjoint to ϕ_0*

$$\tilde{\phi}_0 : \mathbf{P}^\bullet_{\bar{m}} \to \mathcal{D}\mathbf{P}^\bullet_{\bar{m}}[-n]$$

is a quasi-isomorphism.

(2) If additionally X is compact, then $\sigma(X)$ is Poincaré.

Proof. (1) Since ϕ_0 induces the intersection homology cup product, this is just the self-duality of $\mathbf{P}^\bullet_{\bar{m}}$.

(2) The adjoint to $\alpha^*(\phi_0)$ is given by the composition

$$C^\bullet(X) \to \Gamma\mathbf{P}^\bullet_{\bar{m}} \xrightarrow{\tilde{\phi}_0} \Gamma\mathcal{D}\mathbf{P}^\bullet_{\bar{m}}[-n] = (\Gamma\mathbf{P}^\bullet_{\bar{m}})^*[-n] \to (C^\bullet(X))^*[-n].$$

But here, all morphisms are quasi-isomorphisms.

□

2.9 Remark. It even follows that

$$\tilde{\phi}_0 : \mathbf{P}^\bullet_{\bar{m}} \to \mathcal{D}\mathbf{P}^\bullet_{\bar{m}}[-n]$$

is a homotopy equivalence, since $\mathcal{D}\mathbf{P}^\bullet_{\bar{m}}$ is a complex of injectives

3 Uniqueness of the symmetric structure

In this subsection we will show that the symmetric structure on $\mathbf{P}^\bullet$ is essentially unique. In order to do this we will briefly recall some facts on the derived category from [GM83]. In particular, we will restate and prove lemma II.6.1.

3.1 Lemma. *Let $f : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ be a morphism of complexes of sheaves. Suppose $\mathbf{H}^i(\mathbf{A}^\bullet) = 0$ for $i > p$ and $\mathbf{H}^i(\mathbf{B}^\bullet) = 0$ for $i < p$. Then the canonical map*

$$\mathrm{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \mathrm{Hom}_{Sh(X)}(\mathbf{H}^p(\mathbf{A}^\bullet), \mathbf{H}^p(\mathbf{B}^\bullet))$$

is an isomorphism.

Proof. Up to quasi-isomorphism $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ look like

$$\begin{array}{ccccccccc} \ldots \to & \mathbf{A}^{p-1} & \xrightarrow{d^{p-1}} & \mathbf{A}^p & \to & 0 & \to \ldots \\ \ldots \to & 0 & \to & \mathbf{I}^p & \xrightarrow{d^p} & \mathbf{I}^{p+1} & \to \ldots \end{array}$$

with $\mathbf{I}^k$ injective. Now every morphism in $D^b(X)$ between these complexes corresponds to an actual homotopy class of morphisms. But this is nothing but a map

$$\mathbf{H}^p(\mathbf{A}^\bullet) = \operatorname{coker} d^{p-1} \to \ker d^p = \mathbf{H}^p(\mathbf{B}^\bullet).$$

□

3.2 Lemma. *Let $\mathbf{A}^\bullet, \mathbf{B}^\bullet, \mathbf{C}^\bullet \in D^b(X)$ such that $\mathbf{H}^k(\mathbf{A}^\bullet) = 0$ for all $k \geq p+1$. Let $f : \mathbf{C}^\bullet \to \mathbf{B}^\bullet$ be a morphism of complexes such that $f^* : \mathbf{H}^k(\mathbf{C}^\bullet) \to \mathbf{H}^k(\mathbf{B}^\bullet)$ is an isomorphism for all $k \leq p$. Then the induced map*

$$\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$$

is an isomorphism. In other words, every map $g : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ has a unique lift $\tilde{g} : \mathbf{A}^\bullet \to \mathbf{C}^\bullet$ such that $f\tilde{g} = g$.

Proof. We consider the mapping cone $\mathbf{M}^\bullet := \mathbf{cone}^\bullet(f)$ of f. By the long exact cohomology sequence we see that $\mathbf{H}^k(\mathbf{M}^\bullet) = 0$ for all $k < p$ and the induced map $\mathbf{H}^p(\mathbf{B}^\bullet) \to \mathbf{H}^p(\mathbf{M}^\bullet)$ is 0. Using the previous lemma, we see that

$$\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet) \cong \operatorname{Hom}_{Sh(X)}(\mathbf{H}^p(\mathbf{A}^\bullet), \mathbf{H}^p(\mathbf{M}^\bullet))$$

is the zero map and $\operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet[-1]) = 0$. Therefore, the statement follows from the long exact sequence

$$\ldots \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet[-1]) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{C}^\bullet)$$

$$\to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \to \operatorname{Hom}_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{M}^\bullet) \to \ldots$$

□

3.3 Theorem. *Let X be an IP-space of dimension n and let $U_2 := X - \Sigma$ denote the non-singular part of X. Let $\mathbf{P}^\bullet$ be the complex of sheaves quasi-isomorphic to $\mathbf{IC}^\bullet_{\bar{m}}$ constructed in the previous section. For*

$$Q^{-n}(\mathbf{P}^\bullet) = H^{-n}(\operatorname{Hom}^\bullet_{\mathbb{Z}[\mathbb{Z}/2]}(W, \Gamma\mathbf{Hom}^\bullet(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n])))$$

restriction to U_2 induces an isomorphism of abelian groups

$$\rho : Q^{-n}(\mathbf{P}^\bullet) \to \operatorname{Hom}_{Sh(U)}(\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2}, \mathbb{Z}_{U_2})^{\mathbb{Z}/2}.$$

where the $\mathbb{Z}/2$-action on $\operatorname{Hom}_{Sh(U)}(\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2}, \mathbb{Z}_{U_2})$ is given by the transposition involution.

Proof. Let $\phi \in Hom^{\bullet}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \Gamma\mathbf{Hom}^{\bullet}(\mathbf{P}^{\bullet} \otimes \mathbf{P}^{\bullet}, \mathbb{D}_X[-n]))$ represent an element in $Q^{-n}(\mathbf{P}^{\bullet})$. Following III.5.4 ϕ is given by a system of maps of degree $-s$

$$\phi_s : \mathbf{P}^{\bullet} \otimes \mathbf{P}^{\bullet} \to \mathbb{D}_X[-n]$$

satisfying

$$d\phi_s + (-1)^{s-1}\phi_s d + (-1)^{s-1}\phi_{s-1}\tau - \phi_{s-1} = 0$$

and $\phi_s = 0$ for $s < 0$. Let $j_2 : U_2 \hookrightarrow X$ denote the inclusion, then we have isomorphisms

$$\mathbf{H}^0(j_2^*\mathbf{P}^{\bullet}) \cong \mathbb{Z}_{U_2}$$

and

$$\mathbf{H}^0(j_2^*\mathbb{D}_X[-n]) \cong \mathbb{Z}_{U_2}$$

coming from the canonical resolutions of the constant sheaves on U_2 in 1.4.

Therefore, under the above identifications the restriction of ϕ_0 to U_2 induces a map

$$\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2} \to \mathbb{Z}_{U_2}$$

which commutes with the involution on $\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2}$ since ϕ_1 is a zero homotopy of $(\tau - 1)\phi_0$. We define $\rho([\phi])$ to be this map. Note that we used the algebraic Künneth theorem here.

We have to check that this assignment is well defined. Suppose $\phi = d\psi$. Again by III.5.4, ψ is given by a system of maps of degree $-s$

$$\psi_s : \mathbf{P}^{\bullet} \otimes \mathbf{P}^{\bullet} \to \mathbb{D}_X[-n-1]$$

such that $\psi_s = 0$ for $s < 0$ and

$$d\psi_s + (-1)^{s-1}\psi_s d + (-1)^s\psi_{s-1}\tau + \psi_{s-1} = (-1)^{r+s-1}\phi_s.$$

Now restricting ψ_0 to U_2 and preceding it by a sign $(-1)^{\deg+1}$ gives a zero homotopy of ϕ_0 restricted to U_2.

To see surjectivity of ρ, we simply observe that the construction of the symmetric structure on $\mathbf{P}^{\bullet}$ in the previous section can be modified to start with an arbitrary $\mathbb{Z}[\mathbb{Z}/2]$-equivariant map

$$\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2} \to \mathbb{Z}_{U_2}.$$

For injectivity, suppose we have $[\phi] \in Q^{-n}(\mathbf{P}^{\bullet})$ such that $\rho([\phi]) = 0$. By definition this means that

$$(\phi_{0|U_2})^* = 0 : \mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2} \to \mathbb{Z}_{U_2}.$$

Using 3.1 we have an isomorphism

$$\mathrm{Hom}_{D^b(U_2)}(j_2^*\mathbf{P}^{\bullet} \otimes j_2^*\mathbf{P}^{\bullet}, j_2^*\mathbb{D}_X[-n]) \to \mathrm{Hom}_{Sh(U_2)}(\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2}, \mathbb{Z}_{U_2}).$$

With $U_k = X - X_{n-k}$ and $i_k : U_k \to U_{k+1}$ respectively $j_k : U_k \to X$ the inclusions as before, we want to show that there is an isomorphism

$$\mathrm{Hom}_{D^b(U_{k+1})}(j_{k+1}^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet), j_{k+1}^*\mathbb{D}_X[-n]) \to \mathrm{Hom}_{D^b(U_k)}(j_k^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet), j_k^*\mathbb{D}_X[-n])$$

given by restriction.

We consider the adjunction morphism

$$j_{k+1}^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet) \to Ri_{k*}i_k^*j_{k+1}^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet) = Ri_{k*}j_k^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet)$$

coming from the adjoint pair (i_k^*, Ri_{k*}) which is the identity after restricting to U_k. In the same way we have a morphism

$$j_{k+1}^*\mathbb{D}_X[-n] \to Ri_{k*}j_k^*\mathbb{D}_X[-n].$$

which is the identity after restricting to U_k.

For a given morphism $f : j_k^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet) \to j_k^*\mathbb{D}_X[-n]$ in $D^b(U_k)$ we are looking for a lift in the diagram

$$\begin{array}{ccc} j_{k+1}^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet) & \overset{\tilde{f}}{\dashrightarrow} & j_{k+1}^*\mathbb{D}_X[-n] \\ \downarrow & & \downarrow \\ Ri_{k*}j_k^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet) & \xrightarrow{Ri_{k*}f} & Ri_{k*}j_k^*\mathbb{D}_X[-n] \end{array} \qquad (*)$$

To get this lift we want to apply lemma 3.2. According to [GM83], 1.11 the morphism $j_{k+1}^*\mathbb{D}_X[-n] \to Ri_{k*}j_k^*\mathbb{D}_X[-n]$ fits into a distinguished triangle in $D^b(U_{k+1})$

$$\begin{array}{ccc} j_{k+1}^*\mathbb{D}_X[-n] & \longrightarrow & Ri_{k*}j_k^*\mathbb{D}_X[-n] \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Rg_{k*}g_k^!j_{k+1}^*\mathbb{D}_X[-n+1] & \end{array}$$

where $g_k : X_{n-k} - X_{n-k-1} \to U_{k+1}$ denotes the inclusion. Since j_{k+1} is the inclusion of an open subset, we have $j_{k+1}^* = j_{k+1}^!$ and therefore

$$g_k^!j_{k+1}^*\mathbb{D}_X[-n] \cong \mathbb{D}_{X_{n-k}-X_{n-k-1}}[-n].$$

By [GM83], 1.12 we have

$$\mathbf{H}^r(Rg_{k*}g_k^!j_{k+1}^*\mathbb{D}_X[-n+1])_x = \begin{cases} \mathbb{Z}, & \text{for } r = k \text{ and } x \in X_{n-k} - X_{n-k-1} \\ 0, & \text{else.} \end{cases}$$

By the long exact cohomology sequence, this implies

$$\mathbf{H}^r(j_{k+1}^*\mathbb{D}_X[-n]) \to \mathbf{H}^r(Ri_{k*}j_k^*\mathbb{D}_X[-n])$$

is an isomorphism for $r \leq k-1$. However, $\mathbf{H}^r(j_{k+1}^*\mathbf{P}^\bullet \otimes j_{k+1}^*\mathbf{P}^\bullet) = 0$ for $r \geq k$ by the axioms for intersection chain sheaves [AX1] and the algebraic Künneth formula. Hence, by lemma 3.2 the lift $\tilde{f}$ in $(*)$ exists and is unique. If we apply i_k^* to the diagram $(*)$ we see that $i_k^*\tilde{f} = f$. Thus, i_k^* gives an isomorphism

$$\mathrm{Hom}_{D^b(U_{k+1})}(j_{k+1}^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet), j_{k+1}^*\mathbb{D}_X[-n]) \to \mathrm{Hom}_{D^b(U_k)}(j_k^*(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet), j_k^*\mathbb{D}_X[-n])$$

as claimed.

Now we compose these isomorphisms to get an isomorphism

$$\mathrm{Hom}_{D^b(X)}(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n]) \to \mathrm{Hom}_{Sh(U_2)}(\mathbb{Z}_{U_2} \otimes \mathbb{Z}_{U_2}, \mathbb{Z}_{U_2}).$$

but since $\mathbb{D}_X$ is injective we have

$$\mathrm{Hom}_{D^b(X)}(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n]) = \mathrm{Hom}_{K^b(X)}(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n]).$$

This means, that $(\phi_{0|U_2})^* = 0$ implies that ϕ_0 is homotopic to 0. Preceding a given zero homotopy by a sign $(-1)^{\deg+1}$, we get a map

$$\psi_0 : \mathbf{P}^\bullet \otimes \mathbf{P}^\bullet \to \mathbb{D}_X[-n-1].$$

with

$$(-1)^{r+1}\phi_0 = d\psi_0 - \psi_0 d : (\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet)^{n+r} \to \mathbb{D}_X^r.$$

We define

$$\tilde{\phi}_1 := (-1)^r \phi_1 : (\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet)^{n+r+1} \to \mathbb{D}_X^r$$

and compute

$$\begin{aligned} d\psi_0(1-\tau) - \psi_0(1-\tau)d &= (-1)^{r+1}\phi_0(1-\tau) \\ &= (-1)^{r+1}(d\phi_1 + \phi_1 d) = d\tilde{\phi}_1 - \tilde{\phi}_1 d \end{aligned}$$

Hence,

$$(\tilde{\phi}_1 - \psi_0(1-\tau)) : \mathbf{P}^\bullet \otimes \mathbf{P}^\bullet \to \mathbb{D}_X[-n-1]$$

is a chain map. Going through the previous discussion on $\mathrm{Hom}_{D^b(X)}(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n])$, we see that the same technique shows

$$\mathrm{Hom}_{D^b(X)}(\mathbf{P}^\bullet \otimes \mathbf{P}^\bullet, \mathbb{D}_X[-n-1]) = 0.$$

Therefore, this chain map is zero homotopic by a zero homotopy

$$\psi_1 : \mathbf{P}^\bullet \otimes \mathbf{P}^\bullet \to \mathbb{D}_X[-n-2]$$

with

$$\tilde{\phi}_1 - \psi_0(1-\tau) = d\psi_1 + \psi_1 d.$$

For ϕ_1 itself this means

$$(-1)^r\phi_1 = d\psi_1 + \psi_1 d - \psi_0\tau + \psi_0.$$

Again, we compute with $\tilde{\phi}_2 := (-1)^{r+1}\phi_2$:

$$d\psi_1(1+\tau) + \psi_1(1+\tau)d = (-1)^r\phi_1(1+\tau) = d\tilde{\phi}_2 + \tilde{\phi}_2 d.$$

Which in turn shows that $\tilde{\phi}_2 - \psi_1(1+\tau)$ is a chain map after preceding it with a sign $(-1)^{\deg}$, which is again zero homotopic, and so on ...

Summarizing these considerations, we get a system of maps

$$\psi_s : \mathbf{P}^\bullet \otimes \mathbf{P}^\bullet \to \mathbb{D}_X[-n-1]$$

of degree $-s$ which satisfy $\psi_s = 0$ for $s < 0$ and

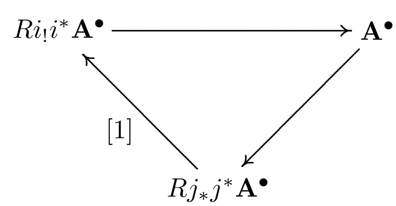
$$d\psi_s + (-1)^{s-1}\psi_s d + (-1)^s\psi_{s-1}\tau + \psi_{s-1} = (-1)^{r+s+1}\phi_s.$$

□

4 The symmetric structure for IP-spaces with boundary

We will conclude this chapter with the construction of a symmetric pair of complexes of sheaves for a given topological pseudomanifold with collared boundary $(Y, \partial Y)$ of dimension $n+1$.

We denote by $i : \mathrm{int}(Y) \hookrightarrow Y$ respectively $j : \partial Y \hookrightarrow Y$ the inclusions of the interior respectively boundary of Y. According to [GM83], 1.11 for every complex of sheaves $\mathbf{A}^\bullet$ there is a distinguished triangle in $D^b_c(Y)$



Where the map

$$Ri_!i^*\mathbf{A}^\bullet = Ri_!i^!\mathbf{A}^\bullet \to \mathbf{A}^\bullet$$

is the adjunction morphism coming from the adjoint pair of functors $(Ri_!, i^!)$ and

$$\mathbf{A}^\bullet \to Rj_*j^*\mathbf{A}^\bullet$$

is the adjunction morphism coming from the adjoint pair of functors (j^*, Rj_*).

In order to further investigate this triangle, we have to recall some facts on the involved functors. Note, that for an arbitrary map $f : X \to Z$ a right adjoint $f^! : Sh(Z) \to Sh(X)$ to $f_! : Sh(X) \to Sh(Z)$ may not exist (see [Ba06]). Such a functor can only be constructed in general after passing to derived categories. However, there is a construction of a right adjoint $f^!$ to $f_!$ on the sheaf level provides that f is an inclusion of a locally closed subspace. Following [Iv86], for a given sheaf $\mathbf{B}$ on Z we define the sheaf $\mathbf{B}^X$ on Z by

$$U \mapsto \mathbf{B}^X(U) := \{s \in \Gamma(U;\mathbf{B}) \mid \operatorname{supp}(s) \subset X\}.$$

We set $f^!\mathbf{B} := f^*\mathbf{B}^X$.

4.1 Lemma. *For sheaves* $\mathbf{E}$ *on* X *and* $\mathbf{F}$ *on* Z *there are isomorphisms*

$$\operatorname{Hom}_{Sh(X)}(\mathbf{E}, f^!\mathbf{F}) \cong \operatorname{Hom}_{Sh(Z)}(f_!\mathbf{E}, \mathbf{F}).$$

and

$$f_*\mathbf{Hom}(\mathbf{E}, f^!\mathbf{F}) \cong \mathbf{Hom}(f_!\mathbf{E}, \mathbf{F})$$

Proof. The first isomorphism is proved in [Iv86],II.6.6. The second isomorphism follows by considering the first simultaneously for all open subsets of X. □

4.2 Lemma. *The functor* $f_! : Sh(X) \to Sh(Z)$ *induces an equivalence of categories between* $Sh(X)$ *and the full subcategory of* $Sh(Z)$ *consisting of all sheaves* $\mathbf{E}$ *such that* $\mathbf{E}_x = 0$ *for* $x \in (Z - X)$. *The inverse functor is given by* f^*.

Proof. See [Iv86],II.6.4. □

Now, the adjunction morphism $f_!f^!\mathbf{B} \to \mathbf{B}$ is induced by the inclusions

$$\Gamma(U; f_!f^!\mathbf{B}) = \{s \in \Gamma(U;\mathbf{B}) \mid \operatorname{supp}(s) \subset X\} \hookrightarrow \Gamma(U;\mathbf{B}).$$

For $f = i : \operatorname{int}(Y) \hookrightarrow Y$ this morphism is part of an exact sequence (see [Iv86], II.6.11)

$$0 \to i_!i^!\mathbf{B} \to \mathbf{B} \to j_*j^*\mathbf{B} \to 0$$

where the last arrow is the adjunction morphism to the adjoint pair (j^*, j_*), with $j : \partial Y \to Y$ the inclusion. By passing to the derived category that short exact sequence transforms into the above distinguished triangle.

Now, we suppose $\mathbf{B} = i_*\mathbf{C}$ for some sheaf $\mathbf{C}$ on $\operatorname{int}(Y)$ and recall there is a canonical monomorphism $i_!\mathbf{C} \to i_*\mathbf{C}$ induced by the inclusions

$$\Gamma(U; i_!\mathbf{C}) = \{s \in \Gamma(i^{-1}(U);\mathbf{C}) \mid \operatorname{supp}(s) \text{ is closed in } U\} \hookrightarrow \Gamma(U; i_*\mathbf{C})$$

Note, that the support of a section is defined relative to the space the sheaf lives on. For example this means, that for a given open subset $U \subset Y$ the supports of an element in $\Gamma(U; i_*\mathbf{C})$ and the same element in $\Gamma(i^{-1}(U);\mathbf{C})$ are *not* equal in general when we consider it as a section over U or as a section over $i^{-1}(U)$.

Bearing this in mind we see that for $U \subset Y$ open

$$\begin{aligned}\Gamma(U; i_!i^!i_*\mathbf{C}) &= \{s \in \Gamma(i^{-1}(U);\mathbf{C}) \mid \operatorname{supp}(s) \subset i^{-1}(U) \text{ and is closed in } U\} \\ &= \Gamma(U; i_!\mathbf{C}).\end{aligned}$$

Therefore, the adjunction morphism

$$i_!\mathbf{C} = i_!i^!i_*\mathbf{C} \to i_*\mathbf{C}$$

coincides with the canonical monomorphism.

We apply these observations to our distinguished triangle for $\mathbf{A}^\bullet := Ri_*\mathbf{P}^\bullet_{\mathrm{int}(Y)}$, where $\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ denotes the complex of sheaves $\mathbf{P}^\bullet_{\bar{m}}$ on $\mathrm{int}(Y)$ constructed before. Similarly, we write $\mathbf{P}^\bullet_{\partial Y}$ for the complex $\mathbf{P}^\bullet_{\bar{m}}$ on ∂Y and so on. Since $\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ is both $i_!$ and i_*-acyclic, we get

$$\begin{array}{ccc} i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \longrightarrow & i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & j_*j^*i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \end{array}$$

where the upper morphism is given by the canonical monomorphism. Using [Ba02],4.4, we have isomorphisms in $D^b_c(Y)$

$$\begin{aligned} j_*j^*i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} &\cong j_*j^*i_*\mathbf{IC}^\bullet_{\mathrm{int}(Y)}[-n-1] \\ &\cong j_*\mathbf{IC}^\bullet_{\partial Y}[-n] \\ &\cong j_*\mathbf{P}^\bullet_{\partial Y}. \end{aligned}$$

This transforms our triangle into the very pleasant form

$$\begin{array}{ccc} i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \longrightarrow & i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow{\scriptstyle f} & \\ & j_*\mathbf{P}^\bullet_{\partial Y} & \end{array}$$

Our goal now is to put a symmetric structure on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$. For an IP-space Y we will then use this triangle to construct a symmetric Poincaré structure on $\mathbf{P}^\bullet_{\partial Y}$ which is part of a Poincaré pair. Using the uniqueness result 3.3 we will finally show that this symmetric Poincaré structure is isomorphic to the one constructed in 2.5.

We prepare the definition of the symmetric structure on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ by looking at the following isomorphism in $Ch(Y)$

$$\mathbf{Hom}^\bullet(i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} \otimes i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}, \mathbb{D}_Y) \cong i_*\mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)} \otimes \mathbf{P}^\bullet_{\mathrm{int}(Y)}, i^*\mathbb{D}_Y).$$

Fix a map $h : \mathbb{D}_{\mathrm{int}(Y)} \to i^*\mathbb{D}_Y = i^!\mathbb{D}_Y$ which represents the homotopy class of the isomorphism $i^!\mathbb{D}_Y \cong \mathbb{D}_{\mathrm{int}(Y)}$ in $D^b_c(\mathrm{int}(Y))$. Since the dualizing complex is injective and $i^!$ preserves injectives, we get an induced homotopy equivalence

$$\mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)} \otimes \mathbf{P}^\bullet_{\mathrm{int}(Y)}, \mathbb{D}_{\mathrm{int}(Y)}) \to \mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)} \otimes \mathbf{P}^\bullet_{\mathrm{int}(Y)}, i^*\mathbb{D}_Y).$$

This homotopy equivalence is clearly $\mathbb{Z}/2$-equivariant and therefore induces

$$Q^{-n-1}(i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}) \cong Q^{-n-1}(\mathbf{P}^\bullet_{\mathrm{int}(Y)}).$$

This shows that every symmetric structure on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ is uniquely determined by a symmetric structure on $\mathbf{P}^\bullet_{\mathrm{int}(Y)}$. At this point, it is important to remark that one will loose Poincaré duality even for IP-spaces since $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ is not quasi-isomorphic to its dual.

However, we are interested in a more explicit description of this induced symmetric structure on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$. Let $\phi \in W^{\%}(\mathbf{P}^\bullet_{\mathrm{int}(Y)})^{-n-1}$ be a symmetric structure. Decoding the above isomorphisms and homotopy equivalences and using 4.2, we see that the induced symmetric structure is given by the composition

$$\begin{array}{rcl} i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \xrightarrow{i_!\phi_s} & i_!\mathcal{D}\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \\ & \xrightarrow{h_*} & i_!\mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)}, i^*\mathbb{D}_Y[-n-1]) \\ & \hookrightarrow & i_*\mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)}, i^*\mathbb{D}_Y[-n-1]) \\ & \cong & \mathbf{Hom}^\bullet(i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}, \mathbb{D}_Y[-n-1]). \end{array}$$

It is also easy to see that this composition equals the following composition

$$\begin{array}{rcl} i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \hookrightarrow & i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} \\ & \xrightarrow{i_*\phi_s} & i_*\mathcal{D}\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \\ & \xrightarrow{h_*} & i_*\mathbf{Hom}^\bullet(\mathbf{P}^\bullet_{\mathrm{int}(Y)}, i^*\mathbb{D}_Y[-n-1]) \\ & \cong & \mathbf{Hom}^\bullet(i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}, \mathbb{D}_Y[-n-1]) \end{array}$$

which we denote by $(i_!\phi)_s$.

4.3 Definition. Let Y be a $(n+1)$-dimensional topological pseudomanifold with collared boundary. Let

$$\phi_s : \mathbf{P}^\bullet_{\mathrm{int}(Y)} \to \mathcal{D}\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1]$$

be the symmetric structure on $\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ constructed in 2.5. We define an $(n+1)$-dimensional symmetric structure $i_!\phi$ on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$ by the system of maps

$$(i_!\phi)_s : i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} \to \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1].$$

Given this $(n+1)$-dimensional symmetric structure $i_!\phi$ on $i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}$, we consider the distinguished triangle in $D^b_c(Y)$:

$$\begin{array}{ccc} i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} & \xrightarrow{(i_!\phi)_0} & \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \\ & \nwarrow \quad \swarrow & \\ & \mathbf{cone}^\bullet((i_!\phi)_0) & \end{array}$$

The following construction whose dual is due to [Ra80] gives a symmetric Poincaré pair.

Recall, that the mapping cone is given by

$$\mathbf{cone}^\bullet((i_!\phi)_0)^r := (i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)})^{r+1} \oplus (\mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)})^{r-n-1}$$

with differential

$$d = \begin{pmatrix} d & 0 \\ (-1)^{r+1}(i_!\phi)_0 & d \end{pmatrix} : \mathbf{cone}^\bullet((i_!\phi)_0)^r \to \mathbf{cone}^\bullet((i_!\phi)_0)^{r+1}.$$

The right map in the above distinguished triangle is the inclusion of the direct summand. Let

$$g : \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \to \mathbf{cone}^\bullet((i_!\phi)_0)$$

denote this map. Define a system of maps

$$\varphi_s : \mathbf{cone}^\bullet((i_!\phi)_0)^{r+n+s} \to (\mathcal{D}\mathbf{cone}^\bullet((i_!\phi)_0))^r$$

by

$$\varphi_0 := \begin{pmatrix} (-1)^{r+n}(i_!\phi)_1 & 1 \\ (-1)^{r(r+n+1)}e & 0 \end{pmatrix}$$

and

$$\varphi_s := \begin{pmatrix} (-1)^{r+n+s}(i_!\phi)_{s+1} & 0 \\ 0 & 0 \end{pmatrix}$$

for $s > 0$. Here e denotes the natural transformation $1 \to \mathcal{D}^2$ given in the definition of the algebraic bordism category $\Lambda(Y)$.

Easy but careful calculation reveals that

$$\Big(g : \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \to \mathbf{cone}^\bullet((i_!\phi)_0), (0, \varphi)\Big)$$

is an $(n+1)$-dimensional symmetric Poincaré pair.

4.4 Definition. Let Y be a compact $(n+1)$-dimensional stratified topological pseudomanifold with collared boundary ∂Y. We associate to $(Y, \partial Y)$ a symmetric Poincaré pair of complexes $\sigma(Y, \partial)$ by

$$\sigma(Y, \partial Y) := A\Big(g : \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1] \to \mathbf{cone}^\bullet((i_!\phi)_0), (0, \varphi)\Big).$$

We have

4.5 Proposition. *If Y is a compact IP-space with boundary, then*

$$\partial\sigma(Y, \partial Y) \simeq \sigma(\partial Y).$$

Where $\simeq$ denotes homotopy equivalence.

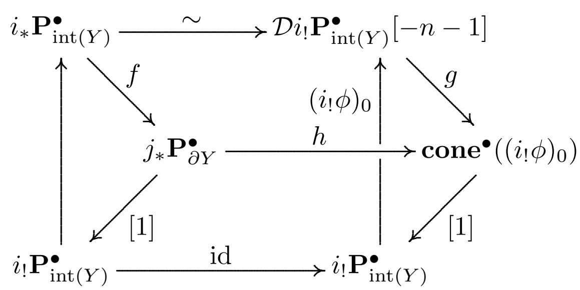
Proof. If Y is an IP-space, then

$$\phi_0 : \mathbf{P}^\bullet_{\mathrm{int}(Y)} \to \mathcal{D}\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1]$$

is a quasi-isomorphism. By it's construction we can factorize $(i_!\phi)_0$ as

$$i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)} \hookrightarrow i_*\mathbf{P}^\bullet_{\mathrm{int}(Y)} \to \mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1]$$

with the first map being the canonical inclusion. The second map is now a quasi-isomorphism and we have a morphism of distinguished triangles



By the 5-lemma, the morphism $h : j_*\mathbf{P}^\bullet_{\partial Y} \to \mathbf{cone}^\bullet((i_!\phi)_0)$ is a quasi-isomorphism. Therefore, we have

$$Q^{-n}(j_*\mathbf{P}^\bullet_{\partial Y}) \cong Q^{-n}(\mathbf{cone}^\bullet((i_!\phi)_0))$$

induced by h. But, the same argument as for $i_!\mathbf{P}^\bullet_{\mathrm{int}Y}$ shows

$$Q^{-n}(j_*\mathbf{P}^\bullet_{\partial Y}) \cong Q^{-n}(\mathbf{P}^\bullet_{\partial Y}).$$

By the uniqueness theorem 3.3 it is now enough to compute the action of φ_0 on cohomology over the non-singular part U_2 of ∂Y. Let $k : U_2 \to Y$ be the inclusion, then this action is given by

$$\mathbb{Z}_{U_2} \cong \mathbf{H}^0(k^*\mathbf{cone}^\bullet((i_!\phi)_0)) = \mathbf{H}^0(k^*\mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1])$$

$$\xrightarrow{(\varphi_0)^*_{|\mathcal{D}i_!\mathbf{P}^\bullet[-n-1]}} \mathbf{H}^0(k^*\mathcal{D}i_!\mathbf{P}^\bullet_{\mathrm{int}(Y)}[-n-1]) \cong \mathbb{Z}_{U_2}.$$

Meaning that the action is precisely given by the upper right entry of φ_0 which is 1. Hence, h is a map of Poincaré complexes and the assertion follows by III.6.7. □

The last result of this section not only shows, that the boundary of the symmetric structure on $(Y, \partial Y)$ is the symmetric structure on the boundary, but also that the symmetric structure behaves well under gluing.

4.6 Proposition. *Let $Z = Y +_A -X$ be a closed IP-space, such that X, Y are IP-spaces with $\partial X = \partial Y = A$. Then*

$$\sigma(Z) \simeq \sigma(Y, A) +_{\sigma(A)} -\sigma(X, A).$$

Proof. Note, that since the simply connected assembly comes from an additive functor on the homotopy categories of sheaves to the homotopy category of complexes it clearly respects mapping cones and therefore the gluing constructions in both algebraic bordism categories at hand. It is therefore sufficient to look at the gluing construction on the sheaf side.

If i^Y, i^X and $i^{\partial Y}$ denote the inclusions of $\mathrm{int}(Y)$, $\mathrm{int}(X)$ and ∂Y to Z, then it is easy to see that there is an isomorphism in $D_c^b(Z)$

$$\mathbf{P}^\bullet_Z \cong i^Y_* \mathbf{P}^\bullet_{\mathrm{int}(Y)} +_{i^{\partial Y}_* \mathbf{P}^\bullet_{\partial Y}} i^X_* \mathbf{P}^\bullet_{\mathrm{int}(X)}.$$

On Y the gluing construction coincides with the algebraic Thom construction giving back the symmetric structure on $\mathrm{int}(Y)$ we started with, and on X the gluing construction is dual to the algebraic Thom construction and hence gives the negative of the structure on $\mathrm{int}(X)$ we started with. □

4.7 Remark. The previous result directly generalizes to the situations in which we glue along parts of the boundary. Namely, we simply remove the parts of the boundary that are not glued. Although the proposition requires compact IP-spaces, the compactness is only needed to apply the simply connected assembly map. Therefore, we end up with the right Poincaré sheaf on the interior of the glued space and the generalization follows.

Chapter VI

Symmetric L-theory as geometric bordism

In this chapter we will use the results of the previous chapter to show that IP-spaces provide a geometric bordism description of symmetric L-theory.

1 Bordism of Integral Poincare spaces

We will show that bordism of singular IP-spaces form a generalized homology theory and give the coefficients of this theory.

1.1 Definition. (1) An oriented stratified pseudomanifold X of dimension n is called an *IP-space* if

- $IH_l^{\bar{m}}(L_x) = 0$ for all $x \in X_{n-2l-1} - X_{n-2l-2}$,
- $IH_{l-1}^{\bar{m}}(L_x)$ is torsion free for all $x \in X_{n-2l} - X_{n-2l-1}$,

where L_x is the link at x.

(2) An *IP-space with boundary* W of dimension n is a pair $(W, \partial W)$ of PL-spaces such that $W - \partial W$ and ∂W are IP-spaces of dimension n and $n-1$ respectively, together with a germ of collars $c : \partial W \times [0, \varepsilon) \to U$ such that c is an isomorphism of PL-spaces. ∂W is called the *boundary* of W.

We proceed as usual.

1.2 Definition. Let X be a topological space.

(1) A *singular IP-space* is a pair $(W, f : W \to Y)$ withe W an IP-space and f a continuous map.

(2) We define $\Omega_n^{IP}(X)$ to be the bordism group of compact n-dimensional IP-spaces.

In order to prove that Ω_*^{IP} is a homology theory we have to construct a boundary operator. Suppose $X = U \cup V$, with $U, V \subset X$ open and $U \cap V \neq \emptyset$. We have to construct a map

$$\partial : \Omega_n^{IP}(X) \to \Omega_{n-1}^{IP}(U \cap V).$$

Let $(Y, f : Y \to X)$ be a closed singular IP-space representing an element in $\Omega_n^{IP}(X)$. We define $A := f^{-1}(X - U)$ and $B := f^{-1}(X - V)$. Let $T : |K| \to Y$ be an admissible finite triangulation of Y. After taking linear subdivision we can assume that $T^{-1}(A)$ and $T^{-1}(B)$ are contained in closed disjoint subcomplexes A' and B' respectively. Now, take P to be a regular neighbourhood of A'. We can assume that A' is transversal to the stratification. This is automatic if there are subcomplexes K_i of K which triangulate the i-strata which we can assume, see [Hud69] chapter 3. Using [RS72] chapter 3, it follows easily that P can be given the structure of an IP-space with boundary. We define

$$\partial([Y, f]) := [T(\partial P), f_{|T(\partial P)}].$$

1.3 Lemma. $\partial : \Omega_n^{IP}(X) \to \Omega_{n-1}^{IP}(U \cap V)$ *constructed above is well defined.*

Proof. Suppose we have two representatives (Y, f) and (Y', f') of the same class. Then there is an IP-bordism (W, F) with boundary $\partial(W, F) = (Y, f) + (-Y', f')$. We apply the relative version of regular neighbourhood theory to W to get the desired bordism, see [RS72] chapter 4. □

The usual argument shows

1.4 Theorem. Ω_*^{IP} *together with the boundary operator constructed above is a homology theory.*

Proof. See [CF64] or [Kr07]. □

Pardon [Pa90] computed the coefficients of Ω_*^{IP}:

$$\Omega_n^{IP}(pt) \cong \begin{cases} \mathbb{Z} & \text{for } n \geq 0, n \equiv 0 \,(\text{mod } 4) \\ \mathbb{Z}/2 & \text{for } n \geq 5, n \equiv 1 \,(\text{mod } 4) \\ 0 & \text{else.} \end{cases}$$

The isomorphisms are given by the signature in dimensions $\equiv 0 \,(\text{mod } 4)$ and by the deRham invariant in dimensions $\equiv 1 \,(\text{mod } 4)$. We see that from dimension 2 on these coefficients coincide with the coefficients of the symmetric $\mathbb{L}^{\cdot}$-spectrum defined in chapter III. In the last section we will show, that after passing to the 2-connected covers, both theories will infact become isomorphic.

2 Bordism of IP-spaces and symmetric L-theory

Finally, we prove the following result:

2.1 Theorem. *Let IP denote the spectrum representing Ω^{IP}. Then there is a map of spectra*

$$IP \to \mathbb{L}^{\cdot}$$

such that

(1) the homotopy cofibre is $K(\mathbb{Z}/2, 1)$.

(2) the induced map

$$IP\langle 2\rangle \to \mathbb{L}^{\cdot}\langle 2\rangle$$

is a homotopy equivalence.

Proof. Let X be a space. Let $E(X)$ denote the Ω-spectrum of augmented simplicial sets $E(X)_k$ having as n-simplices all isomorphism classes of $(n-k)$-dimensional $[n]$-ads of singular IP-spaces (in X). Here the definition of $[k]$-ads is in total analogy to III.3.2. This gives rise to a continuous functor

$$E : \mathcal{T}op \to \Omega\text{-}\mathcal{S}pectra$$

and a map

$$X = Map(*, X) \to Map(E(*), E(X)).$$

If we write E for $E(*)$, we have the adjoint map

$$A : E \wedge X_+ \to E(X).$$

This map is called the assembly map and is defined in [Qu95]. By the comparison theorem for homology theories A is a homotopy equivalence, if the assignment

$$X \mapsto \pi_*(E(X))$$

is a homology theory. But

$$\pi_*(E(X)) = \Omega_*^{IP}(X)$$

and therefore we have

$$E \simeq IP \text{ and } E\langle 2\rangle \simeq IP\langle 2\rangle.$$

In analogy to section III.3 the 2-connected cover of E can be explicitly constructed as the Ω-spectrum of augmented simplicial sets $(E\langle 2\rangle)_k$ having n-simplices all $(n-k)$-dimensional pointed $[n]$-ads M of IP-spaces satisfying

$$M_\sigma = \emptyset, \text{ , if } \dim \sigma \leq l + k - 1.$$

From the last result in the previous chapter, we have that the assignment

$$(Y, \partial Y) \mapsto \sigma(Y, \partial Y),$$

where $(Y, \partial Y)$ is a compact IP-space with boundary, respects the gluing constructions. Therefore, it maps $[n]$-ads of IP-spaces to $[n]$-ads of symmetric Poincaré complexes. Thus we get maps of augmented simplicial sets

$$E_k \to \mathbb{L}^{\cdot}_k$$

and

$$(E\langle 2\rangle)_k \to (\mathbb{L}^{\cdot}\langle 2\rangle)_k$$

which clearly commute with the structure maps and give rise to a maps of spectra. The results now follow from the fact that by definition these maps preserve the signature and the deRham invariant. □

Appendix A

Relative homological algebra

We will first recall some easy facts from relative homological algebra which can be found in [DMO67] or [EM56]. We will need these to show that we are in a position to replace injective resolutions by canonical resolutions in every situation during the construction of the symmetric structure.

A.2 Definition. Let $\mathcal{A}$ be an abelian category.

(1) An object I in $\mathcal{A}$ is called *injective relative to a morphism* $e : B \to C$ in $\mathcal{A}$ if for every map $h : B \to I$ there is an extension $h' : C \to I$ such that $h = h'e$. That is, in the following diagram

$$\begin{array}{ccc} B & \xrightarrow{e} & C \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle h'} & \\ I & & \end{array}$$

the dotted arrow always exists and makes the diagram commute. We also say that I has the extension property for e.

(2) Let E be a class of morphisms in $\mathcal{A}$. An object I is called E-*injective* if it is injective relative to each morphism in E.

A.3 Definition. Let E be a class of morphisms in $\mathcal{A}$.

(1) A cochain complex $K^\bullet$ in $\mathcal{A}$ is called E-*acyclic* if for each r the morphism $\tilde{d}^{r+1} : \mathrm{coker}(d^r) \to K^{r+2}$ induced by d^{r+1} is in E.

(2) For B in $\mathcal{A}$, a complex $K^\bullet$ together with a map $\eta : B \to K^0$ such that $d^0\eta = 0$ is called a *cochain complex under* B. It is called an E-*resolution* of B if the complex

$$0 \to B \to K^0 \to K^1 \to \dots$$

is E-acyclic. We will denote such a complex under B by $B \xrightarrow{\eta} K^\bullet$.

(3) An E-resolution $B \to K^\bullet$ such that each K^r is E-injective is called an *E-injective resolution* of B.

A.4 Theorem. *Let E be a class of morphisms in $\mathcal{A}$. Let $f : B \to C$ be a morphism in $\mathcal{A}$. If $B \xrightarrow{\eta} K^\bullet$ is an E-resolution and $C \xrightarrow{\varepsilon} I^\bullet$ a cochain complex under C such that I^r is E-injective for all r, then there is a morphism of cochain complexes $F : K^\bullet \to I^\bullet$ such that*

$$\varepsilon f = F^0 \eta.$$

Furthermore, any two such cochain morphisms are cochain homotopic.

Proof. The proof is just the usual argument. We will denote the differential of $K^\bullet$ by d and the differential of $I^\bullet$ by δ. Since $B \to K^\bullet$ is E-acyclic η is in E. Therefore, we can extend $\varepsilon f : B \to I^0$ to $F^0 : K^0 \to I^0$ such that

$$\varepsilon f = F^0 \eta.$$

Suppose now that $F^r : K^r \to I^r$ has been constructed such that $F^r d^{r-1} = \delta^{r-1} F^{r-1}$. This condition guarantees that F^r induces a map

$$\tilde{F}^r : \operatorname{coker}(d^{r-1}) \to \operatorname{coker}(\delta^{r-1}).$$

By assumption $\tilde{d}^r : \operatorname{coker}(d^{r-1}) \to K^{r+1}$ is in E. Therefore, we can extend $\tilde{\delta}^r \tilde{F}^r$ to a morphism $F^{r+1} : K^{r+1} \to I^{r+1}$ such that

$$\tilde{\delta}^r \tilde{F}^r = F^{r+1} \tilde{d}^r.$$

This implies

$$\delta^r F^r = F^{r+1} d^r$$

and finishes the construction of F.

For the second assertion, it is enough to show that $f = 0$ implies that F is zero homotopic. Therefore, we have to construct morphisms

$$s^r : K^r \to I^{r-1}$$

such that

$$s^{r+1} d^r + s^r \delta^{r-1} = F^r.$$

We start with defining $s^0 := 0$. Since

$$F^0 d = \delta f = 0$$

we get an induced morphism $\tilde{F}^0 : \operatorname{coker}(\eta) \to I^0$. By assumption $\tilde{F}^0$ extends to a morphism $s^1 : K^1 \to I^0$ such that

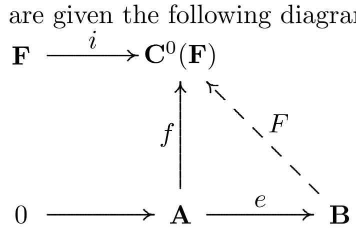
$$\tilde{F}^0 = s^1 \tilde{d}^0.$$

This implies
$$F^0 = s^1 d^0 = s^1 d^0 + \varepsilon s^0.$$
Now, suppose s^r is constructed such that
$$s^r d^{r-1} + \delta^{r-2} s^{r-1} = F^{r-1}.$$
For $G^r := F^r - \delta^{r-1} s^r$ we compute
$$G^r d^{r-1} = F^r d^{r-1} - \delta^{r-1} s^r d^{r-1} = 0.$$
Therefore, we get an induced morphism $\tilde{G}^r : \operatorname{coker}(d^{r-1}) \to I^r$. By assumption this can be extended to a morphism $s^{r+1} : K^{r+1} \to I^r$ such that
$$s^{r+1} \tilde{d}^r = \tilde{G}^r.$$
Which implies
$$s^{r+1} d^r = F^r - \delta^{r-1} s^r,$$
finishing the proof. □

Now we specialize to the situation we are interested in. Namely, let $\mathcal{A}$ be the category $Sh(X)$ of sheaves over some topological space X and let E be the class of all monomorphisms which are pointwise split. This means, a morphism $f : \mathbf{A} \to \mathbf{B}$ is in E if and only if f is a monomorphism and for each $x \in X$ the stalk sequence
$$0 \to A_x \xrightarrow{f_x} B_x$$
splits.

A.5 Proposition. *Let* $\mathbf{F}$ *be a sheaf. Then* $\mathbf{C}^0(\mathbf{F})$ *as constructed in VI.1.1 is* E*-injective.*

Proof. Suppose we are given the following diagram



with $e \in E$. We have to show, that there is a morphism $F : \mathbf{B} \to \mathbf{C}^0(\mathbf{F})$ making the diagram commute. If we consider this diagram stalkwise we have splittings $\mathrm{pr}_x : \mathbf{C}^0(\mathbf{F})_x \to \mathbf{F}_x$ and $\pi_x : \mathbf{B}_x \to \mathbf{A}_x$, where pr_x is the projection onto $\mathbf{F}_x$ coming from the product. We use these to define for an open set $U \subset X$ and $v \in \mathbf{B}(U)$:
$$F(v) := (\mathrm{pr}_x f_x \pi_x(v_x))_{x \in U} \in \prod_{x \in U} \mathbf{F}_x = \mathbf{C}^0(\mathbf{F})(U)$$

We have to show that $Fe = f$. For this, let $U \subset X$ be open and $w \in \mathbf{A}(U)$. Then we have

$$\begin{aligned}
(Fe)(w) &= \prod_{x \in U} \mathrm{pr}_x f_x \pi_x (e(w)_x) \\
&= \prod_{x \in U} \mathrm{pr}_x f_x \pi_x e_x (w_x) \\
&= \prod_{x \in U} \mathrm{pr}_x f_x (w_x) \\
&= f(w),
\end{aligned}$$

where the last equality follows from the universal property of the direct product. □

A.6 Corollary. *Let* $\mathbf{A}$ *be a sheaf. The canonical resolution* $\mathbf{A} \to \mathbf{C}^\bullet(\mathbf{A})$ *is an* E*-injective resolution.*

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