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# Algorithmic Randomness

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## Zusammenfassung

Wir betrachten algorithmische Zufälligkeit im Cantorraum  $\mathcal{C}$  der unendlichen Binärfolgen. Durch ein algorithmisches Zufälligkeitskonzept wird eine Menge von Elementen von  $\mathcal{C}$  bestimmt, denen jeweils die Eigenschaft zugeordnet wird, zufällig zu sein. Solche Konzepte werden unter Verwendung von verschiedenen berechenbarkeitstheoretischen Begriffen definiert und gehen im Wesentlichen auf die folgenden drei intuitiven Anforderungen an zufällige Folgen zurück: Die Anfangsstücke einer zufälligen Folge sollen effektiv inkomprimierbar sein, keine zufällige Folge soll in einer effektiven Nullmenge von Folgen mit einer “Ausnahmeeigenschaft” enthalten sein, und schließlich soll für ein Wettspiel, in welchem die Bits einer Folge nacheinander geraten werden, bei einer zufälligen Folge keine effektive Strategie dem Spieler unbeschränkt viel Kapital verschaffen. Für verschiedene Formalisierungen dieser Anforderungen werden jeweils Versionen von Kolmogorov-Komplexität, Tests und Martingalen verwendet. Wird einer dieser drei Begriffe in der Definition eines Zufälligkeitskonzepts benutzt, so stellt sich generell die Frage nach grundlegenden äquivalenten Definitionen, in denen die jeweils anderen beiden Begriffe verwendet werden. Diese Frage blieb für das zentrale Konzept der berechenbaren Zufälligkeit, welches von Schnorr unter Verwendung von Martingalen eingeführt worden war, lange unbeantwortet.

Wir geben in dieser Arbeit eine Charakterisierung der berechenbaren Zufälligkeit unter Verwendung von Tests an, wobei wir die von uns eingeführten *beschränkten Tests* benutzen. Unser Ergebnis wurde unabhängig von der zuvor von Downey, Griffiths und LaForte angegebenen Testcharakterisierung der berechenbaren Zufälligkeit durch die von ihnen eingeführten *abgestuften Tests* erzielt.

Gestützt auf beschränkte Tests definieren wir *beschränkte Maschinen* und mit diesen eine Version der Kolmogorov-Komplexität, mit deren Hilfe wir eine weitere Charakterisierung der berechenbaren Zufälligkeit beweisen. Auf Grund dieses Ergebnisses ist es möglich, wie in analogen Fällen interessante Lowness- und Trivialitätseigenschaften einzuführen, die grob gesagt “Anti-Zufälligkeitseigenschaften” sind. Wir definieren und untersuchen die Begriffe *Lowness für beschränkte Maschinen* und *beschränkte Trivialität*. Mit Hilfe eines Satzes von Nies lässt sich zeigen, dass nur die berechenbaren Folgen low für beschränkte Maschinen sind. Ferner zeigen wir neben interessanten Eigenschaften der beschränkten Maschinen, dass die beschränkt trivialen Folgen K-trivial sind. Des Weiteren definieren wir *Lowness für berechenbare Maschinen*, einen Lowness-Begriff im Kontext der Schnorr-Zufälligkeit. Wir beweisen, dass eine Folge genau dann low für berechenbare Maschinen ist, wenn sie computably traceable ist.

Nach einem zentralen Satz, den Gács und Kučera unabhängig voneinander bewiesen haben, ist jede Folge effektiv aus einer geeigneten Martin-Löf zufälligen Folge dekodierbar. Wir geben einen etwas einfacheren Beweis dieses Satzes an, wobei wir eine zufällige Folge mit der geforderten Eigenschaft dadurch konstruieren, dass wir gegen geeignete Martingale diagonalisieren. Mit Hilfe einer Variante

jener Konstruktion beweisen wir, dass eine berechenbar zufällige Folge existiert, die schwach truth-table autoreduzierbar ist. Ferner zeigen wir, dass eine Folge genau dann aufzählbar selbstreduzierbar ist, wenn die entsprechende reelle Zahl aufzählbar ist.

Schließlich untersuchen wir Zusammenhänge zwischen dem Lebesguemaß und effektiven Maßen auf  $\mathcal{C}$ . Wir beweisen die folgende Erweiterung eines Ergebnisses von Book, Lutz und Wagner: Eine gegen endliche Varianten abgeschlossene Vereinigung von  $\Pi_1^0$ -Klassen hat genau dann Lebesguemaß null, wenn sie keine Kurtz-zufällige Folge enthält. Wir zeigen jedoch, dass sogar eine  $\Sigma_2^0$ -Klasse mit Lebesguemaß null keine Kurtz-Nullklasse zu sein braucht. Anschließend wenden wir uns Almost-Klassen zu und beweisen unter anderem, dass bezüglich einer beschränkten Reduzierbarkeit jede Almost-Klasse berechenbare Packing-Dimension null hat.

## Abstract

We consider algorithmic randomness in the Cantor space  $\mathcal{C}$  of the infinite binary sequences. By an algorithmic randomness concept one specifies a set of elements of  $\mathcal{C}$ , each of which is assigned the property of being random. Miscellaneous notions from computability theory are used in the definitions of randomness concepts that are essentially rooted in the following three intuitive randomness requirements: the initial segments of a random sequence should be effectively incompressible, no random sequence should be an element of an effective measure null set containing sequences with an “exceptional property”, and finally, considering betting games, in which the bits of a sequence are guessed successively, there should be no effective betting strategy that helps a player win an unbounded amount of capital on a random sequence. For various formalizations of these requirements one uses versions of Kolmogorov complexity, of tests, and of martingales, respectively. In case any of these notions is used in the definition of a randomness concept, one may ask in general for fundamental equivalent definitions in terms of the respective other two notions. This was a long-standing open question w.r.t. computable randomness, a central concept that had been introduced by Schnorr via martingales.

In this thesis, we introduce *bounded tests* that we use to give a characterization of computable randomness in terms of tests. Our result was obtained independently of the prior test characterization of computable randomness due to Downey, Griffiths, and LaForte, who defined *graded tests* for their result.

Based on bounded tests, we define *bounded machines* which give rise to a version of Kolmogorov complexity that we use to prove another characterization of computable randomness. This result, as in analog situations, allows for the introduction of interesting lowness and triviality properties that are, roughly speaking, “anti-randomness” properties. We define and study the notions *lowness for bounded machines* and *bounded triviality*. Using a theorem due to Nies, it can be shown that only the computable sequences are low for bounded machines. Further we show some interesting properties of bounded machines, and we demonstrate that every boundedly trivial sequence is K-trivial. Furthermore we define *lowness for computable machines*, a lowness notion in the setting of Schnorr randomness. We prove that a sequence is low for computable machines if and only if it is computably traceable.

Gács and independently Kučera proved a central theorem which states that every sequence is effectively decodable from a suitable Martin-Löf random sequence. We present a somewhat easier proof of this theorem, where we construct a sequence with the required property by diagonalizing against appropriate martingales. By a variant of that construction we prove that there exists a computably random sequence that is weak truth-table autoreducible. Further, we show that a sequence is computably enumerable self-reducible if and only if its associated real is computably enumerable.

Finally we investigate interrelations between the Lebesgue measure and effective measures  $\mathcal{C}$ . We prove the following extension of a result due to Book, Lutz, and Wagner: A union of  $\Pi_1^0$ -classes that is closed under finite variations has Lebesgue measure zero if and only if it contains no Kurtz random real. However we demonstrate that even a  $\Sigma_2^0$ -class with Lebesgue measure zero need not be a Kurtz null class. Turning to Almost classes, we show among other things that every Almost class with respect to a bounded reducibility has computable packing dimension zero.



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*Ovaj rad posvećujem ocu Dragoljubu, sestri Jeleni,  
i preminuloj majci Anici.*



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# Chapter 1

## Introduction

### 1.1 Algorithmic Randomness

The theory of algorithmic randomness has evolved from the quest for an axiomatic foundation of probability theory. In 1919 von Mises [48] introduced *Kollektivs*, a concept designated to capture an intuitive, general notion of randomness and of probability. Since the work of Wald [50] and Church [10] it has become common practice to represent those *Kollektivs* that serve as a basis for algorithmic randomness concepts by infinite binary sequences which satisfy a certain “randomness” property as specified by von Mises. We call an infinite binary sequence  $X = X(0)X(1)\dots$  *random in the sense of von Mises* if the following condition is satisfied. For  $i = 0, 1$  there is a real  $p_i$  such that for any infinite subsequence  $X(k_0)X(k_1)\dots$  which is chosen according to an “admissible selection procedure”, the limit

$$\lim_n \frac{|\{m < n : X(k_m) = i\}|}{n} \quad (1.1)$$

exists and is equal to  $p_i$ . We stress that in particular, a selection procedure resulting in  $k_n = n$  for all  $n$  is admissible. Needless to say, von Mises interprets the value  $p_i$  as the probability of  $i$ .

Von Mises [48] did not specify the meaning of “admissible selection procedure” rigorously, which was met with criticism. Based on a more precise formulation that von Mises [49] gave in a monograph in 1931, we interpret a *selection procedure* as a total function  $f$  from the set of finite binary strings to  $\{0, 1\}$  such that the mapping  $f$  composed with  $n \mapsto X(0)\dots X(n-1)$  yields the characteristic function of  $\{k_0, k_1, \dots\}$ , where  $k_0, k_1, \dots$  is an increasing sequence of numbers (resulting in the subsequence  $X(k_0)X(k_1)\dots$ ).

Now according to von Mises, a selection procedure is admissible if it is given by a “rule”. But there remains a gap as von Mises did not define the term “rule”. Note that by a straightforward argument, in the nontrivial case  $p_i \neq 0, 1$  there exists no random sequence in the sense of von Mises if one imposes no restrictions on the selection procedures at all. Wald [50] proved the existence of random sequences in the sense of von Mises for any countable set of admissible selection procedures. Subsequently Church [10] suggested that an admissible selection procedure “should be represented mathematically, not as a function, or even as a definition of a function, but as an effective algorithm for the values of a function”. Here Church refers to an effectivity notion in the sense of the three equivalent concepts  $\lambda$ -definability,  $\mu$ -recursiveness, and (Turing) computability.

The contributions of Wald and Church now add up to rigorously defined randomness concepts, which are often referred to as (instances of) *stochasticity*. For the definitions of the stochasticity notions we confine ourselves to the most prominent case of  $p_0 = p_1 = 1/2$ . Taking up the work of Wald, we say that for any countable set  $F$  of selection procedures, an infinite binary sequence  $X(0)X(1)\dots$  is *F-stochastic* if for  $i = 0, 1$  and for any selection procedure  $f \in F$  resulting in an infinite subsequence  $X(k_0)X(k_1)\dots$ , the limit (1.1) exists and is equal to  $1/2$ . In the special case suggested by Church, where  $F$  is the set of computable functions, the *F-stochastic* sequences are called *computably stochastic*.

Ville [47] showed the inadequacy of stochasticity as a formalization of any reasonable intuitive notion of randomness in 1939, six years after Kolmogorov’s ground-breaking measure theoretical foundation of probability theory [24]. In Kolmogorov’s framework, we consider the Lebesgue measure on the Cantor space  $2^\omega$  of all infinite binary sequences, where by Lebesgue measure we mean the probability measure which is equal to the infinite product measure resulting from fair coin tossing. Now with probability 0, a sequence  $X = X(0)X(1)\dots \in 2^\omega$  satisfies

$$(\forall n \in \omega) \frac{|\{m < n : X(m) = 1\}|}{n} \geq \frac{1}{2}, \quad (1.2)$$

which is e.g. a consequence of the law of the iterated logarithm. Certainly any randomness concept  $C$  should be such that no  $C$ -random sequence  $X$  satisfies (1.2). However Ville [47] constructed for any countable set of selection procedures  $F$  an *F-stochastic* sequence  $X$  such that (1.2) is satisfied.

The next important algorithmic randomness concept, that did away with the just described inadequacy of stochasticity, was introduced by Martin-Löf in 1966. A *probability-one law* is an assertion which, for some property  $P$ ,

states that the class of infinite binary sequences satisfying  $P$  has Lebesgue measure 1. If a sequence satisfies  $P$  then it *obeys* the probability-one law, otherwise it *violates* the law. While stochastic sequences are defined in a way such that they obey the strong law of large numbers, Ville’s example showed that there is a stochastic sequence  $X$  which violates the law of the iterated logarithm. Now Martin-Löf calls a sequence random if it obeys a certain class of effective probability-one laws (including, for example, the law of the iterated logarithm). In this sense, one describes a random sequence by saying that it is “typical”.

While Martin-Löf’s definition of randomness is still the most prominent one today, it has also been criticized, most notably by Schnorr who has proposed further randomness concepts. Some of Schnorr’s main contributions were obtained in the framework of Ville’s investigations on martingales, which are generalizations of selection procedures. Martingales are the basis for the betting game approach to algorithmic randomness, which requires that a random sequence be “unpredictable”. In a nutshell, a sequence  $X$  is unpredictable if in the limit one cannot accumulate an unbounded amount of capital when playing a betting game where one bets successively on the single bits of  $X$ . Schnorr considered effective versions of martingales and used these e.g. to give a characterization of the random sequences in the sense of Martin-Löf.

Besides the requirements that a random sequence be “typical” and “unpredictable”, there is a third main approach which requires, roughly, that the initial segments of a random sequence be “incompressible”. These approaches along with the above-mentioned contributions of Martin-Löf and of Schnorr are reviewed briefly for further use in Chapter 2. For a comprehensive account we refer e.g. to Ambos-Spies and Kučera [1] and to the monograph of Li and Vitányi [29]. The forthcoming monographs of Downey and Hirschfeldt [15] and of Nies [33] are extensive presentations of algorithmic randomness in the realm of computability theory.

## 1.2 Definitions and Notation

We review some standard definitions and notation. For details see e.g. Odifreddi [36] and Soare [43].

**Numbers, sets, sequences, classes, and quantifiers.** We let  $\omega$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  denote the set of natural numbers, of rational numbers, and of real numbers, respectively. Furthermore, we will use  $\mathbb{Q}^+$  and  $\mathbb{R}^+$  to denote the

set of positive rationals and of positive reals, respectively. The term  $2^\omega$  denotes the Cantor space, i.e., the class of the countably infinite binary sequences equipped with the product topology which is derived from the discrete topology on  $\{0,1\}$ . An element of  $2^\omega$  is called a *sequence* and is usually denoted by one of the letters  $X$ ,  $Y$ , and  $Z$ . A sequence  $X = X(0)X(1)\dots$  is identified with the set  $A \subseteq \omega$  where  $n \in A$  iff  $X(n) = 1$ . We shall use  $0^\omega$  to denote the sequence that consists only of 0s. If not stated otherwise, *number* means natural number, *set* means set of numbers, and *class* means set of sequences. The relative complement of a class  $\mathcal{C}$  in  $2^\omega$  is denoted by  $\bar{\mathcal{C}}$ .

We let  $q_0, q_1, \dots$  denote the nonnegative rationals, ordered according to an effective representation. A computable sequence of rational numbers is a sequence  $q_{i_0}, q_{i_1}, \dots$  where  $k \mapsto i_k$  is a computable function.

The quantifiers  $\exists, \forall, \exists^\infty$ , and  $\forall^\infty$  denote respectively: there exists, for all, there exist infinitely many, and for almost every.

**Strings, finite sets of strings, initial segments  $X \upharpoonright n$ , cones  $[\sigma]$ .** A *string* is a finite binary sequence, usually denoted by  $\rho, \sigma$ , or  $\tau$ . We denote the set of all strings by  $2^{<\omega}$ , and the empty string by  $\varepsilon$ . We write  $|\sigma|$  for the length of a string  $\sigma$ . The  $(i+1)$ st bit of a string  $\sigma$  is denoted by  $\sigma(i)$ , so we have  $\sigma = \sigma(0)\dots\sigma(|\sigma|-1)$ . We identify numbers and strings by the following bijection from  $\omega$  to  $2^{<\omega}$ . Let  $\sigma_0, \sigma_1, \dots$  be the sequence of all strings ordered by the length-lexicographical ordering. Then a number  $n$  is mapped to the  $n$ th string  $\sigma_n$ . If  $V = \{\sigma_{i_1}, \dots, \sigma_{i_k}\}$  is some finite set of strings then  $2^{i_1} + \dots + 2^{i_k}$  is the *canonical index of  $V$* . We let  $D_m$  denote the finite set of the strings with the canonical index  $m$ , where  $D_0$  denotes the empty set.

For any strings  $\sigma$  and  $\tau$ , the term  $\sigma\tau$  denotes the concatenation of  $\sigma$  and  $\tau$ , and  $\sigma \preceq \tau$  means that  $\sigma$  is a prefix of  $\tau$ . In this case, if  $\sigma \neq \tau$ , then  $\sigma$  is a *proper prefix* of  $\tau$ . Further, if  $X$  is a sequence then  $\sigma X$  is the sequence  $Y$ , where  $Y(i) = \sigma(i)$  if  $i < |\sigma|$  and  $Y(i) = X(i - |\sigma|)$  otherwise. Similarly,  $\sigma \preceq X$  means that  $\sigma(i) = X(i)$  for all  $i < |\sigma|$ . The *initial segment  $X \upharpoonright n$  of a sequence  $X$  of length  $n$*  is the string  $\sigma$  of length  $n$  such that  $\sigma \preceq X$ .

We call the basic open classes of  $2^\omega$  *cones*. I.e., a cone is a class  $\mathcal{C} \subseteq 2^\omega$  such that there is a string  $\sigma$  with  $\mathcal{C} = \{X \in 2^\omega : \sigma \preceq X\}$ . In this case,  $\mathcal{C}$  is called the *cone generated by  $\sigma$*  and is denoted by  $[\sigma]$ . If  $A$  is a set of strings then let  $[A] = \cup_{\sigma \in A} [\sigma]$ .

A set of strings  $A \subseteq 2^{<\omega}$  is called *prefix-free* if for any two distinct strings  $\sigma, \tau \in A$ , we have that  $\sigma \not\preceq \tau$ .



**Lebesgue measure  $\mu$ .** Throughout this thesis, the letter  $\mu$  denotes the Lebesgue measure on  $2^\omega$ , where by Lebesgue measure we mean the probability measure which is equal to the countably infinite product measure of the uniform probability measure on  $\{0, 1\}$ . For  $\sigma$  a string and  $A$  a set of strings, we write  $\mu[\sigma]$  and  $\mu[A]$  instead of  $\mu([\sigma])$  and  $\mu([A])$ , respectively.

**Partial computable functions, computably enumerable sets.** We fix a standard listing  $\{\varphi_e\}_{e \in \omega}$  of the partial computable functions from  $\omega$  to  $\omega$ . Accordingly,  $W_0, W_1, \dots$  is a standard listing of the computably enumerable (c.e.) sets, where by definition, each  $W_e$  is the domain  $\text{dom } \varphi_e$  of  $\varphi_e$ . If the  $e$ th Turing machine on input  $x$  halts after less than  $s$  computation steps and outputs  $y$  then we write  $\varphi_{e,s}(x) = y$  provided that  $x, y, e < s$ . If such a  $y$  exists, i.e., if  $\varphi_{e,s}(x)$  converges, then we write  $\varphi_{e,s}(x) \downarrow$ ; otherwise  $\varphi_{e,s}(x) \uparrow$  denotes that  $\varphi_{e,s}(x)$  diverges. For every  $e, s$  let  $W_{e,s} = \text{dom } \varphi_{e,s}$ .

For every  $e, x$  we write  $\varphi_e(x) \downarrow$  if there is an  $s$  such that  $\varphi_{e,s}(x) \downarrow$ . Let  $\varphi, \psi$  be partial computable functions. Then for any  $x$ ,  $\varphi(x) \simeq \psi(x)$  means that  $\varphi(x) \downarrow \Leftrightarrow \psi(x) \downarrow$ , and that  $\varphi(x) = \psi(x)$  in case  $\varphi$  and  $\psi$  both converge on  $x$ . Furthermore,  $\varphi \simeq \psi$  means that  $\varphi(x) \simeq \psi(x)$  for all  $x \in \omega$ .

By the identification of numbers and strings, partial computable functions are also viewed as functions from strings to strings, and accordingly, the c.e. sets are viewed as sets of strings as well.

**Reducibilities.** Recall that the partial computable functions can be relativized to any sequence  $X$ . Roughly, an oracle Turing machine  $M$  with oracle  $X$  works like an ordinary Turing machine except that it has an additional input tape on which the characteristic sequence of  $X$  is written. During the computation,  $M$  may ask a *query*  $n \in \omega$ , i.e.,  $M$  reads the oracle tape at position  $n$  and checks whether  $X(n) = 0$  or  $X(n) = 1$ . Let  $Q(M, X, x, s)$  be the set of queries occurring during the first  $s$  computation steps of the computation of  $M$  on input  $x$  with oracle  $X$ . Similarly, we let  $Q(M, X, x)$  be the set of queries occurring during the entire computation of  $M$ . If  $M$  on input  $x$  with oracle  $X$  outputs  $y$  then we write  $M(X, x) = y$  or  $M^X(x) = y$ .

We fix a standard listing  $M_0, M_1, \dots$  of all oracle Turing machines. The  $e$ th machine  $M_e$  computes a partial functional  $\Phi_e: 2^\omega \times \omega \rightarrow \omega$  that we call the *eth Turing functional*. As above we may also write  $\Phi_e^X(x) = y$  instead of  $\Phi_e(X, x) = y$ . Similar to above, we write  $\Phi_{e,s}^X(x) = y$  if  $x, y, e < s$  and if the  $e$ th oracle machine  $M_e$  with oracle  $X$  on input  $x$  outputs  $y$  in less than  $s$  computation steps. Further, we let  $W_{e,s}^X = \text{dom } \Phi_{e,s}^X$  and  $W_e^X =$

$\text{dom } \Phi_e^X$ . The *use functions of a machine  $M_e$  with oracle  $X$*  are the functions  $u_{M_e, s}^X$  ( $s \in \omega$ ) and  $u_{M_e}^X$ , defined as follows. We let  $u_{M_e, s}^X(x)$  and  $u_{M_e}^X(x)$  be the maximum number in  $\{0\} \cup Q(M_e, X, x, s)$  and  $\{0\} \cup Q(M_e, X, x)$ , respectively. Note that for all  $X, e, x, y, s$  we have  $u_{M_e, s}^X(x) < s$ .

A sequence  $X$  is *Turing-reducible* to a sequence  $Y$  if there is an oracle Turing machine  $M$  such that  $M(Y, x) = X(x)$  for all  $x$ . The definition of *truth-table-reducibility* is basically the same, except that in addition we require that  $M$  is total, i.e., for all oracles  $Z$  and for all inputs  $x$ , the computation of  $M(Z, x)$  eventually terminates. By a result due to Nerode and to Trakhtenbrot [36, Proposition III.3.2], for any  $\{0, 1\}$ -valued total oracle Turing machine there is an equivalent one that is again total and queries its oracle nonadaptively (i.e.,  $M$  computes a list of queries that are asked simultaneously and after receiving the answers,  $M$  is not allowed to access the oracle again). A sequence  $X$  is *weak truth-table-reducible* to a sequence  $Y$  if  $X$  is Turing-reducible to  $Y$  by an oracle Turing machine  $M$  and if there is a computable function  $g$  such that for every oracle  $X$  the use of  $M$  with oracle  $X$  is bounded by  $g$ . A sequence  $X$  is *computably enumerable in* a sequence  $Y$  if there is an oracle Turing machine  $M$  such that  $M(Y, x) = 1$  in case  $x \in X$  and  $M(Y, x)$  is undefined otherwise. For  $r$  in  $\{\text{tt}, \text{wtt}, \text{T}, \text{c.e.}\}$ , we say  $X$  is  $r$ -reducible to  $Y$ , or  $X \leq_r Y$  for short, if  $X$  is reducible to  $Y$  with respect to truth-table, weak truth-table, Turing, or computably enumerable reducibility, respectively. By the above it is immediate that

$$X \leq_{\text{tt}} Y \implies X \leq_{\text{wtt}} Y \implies X \leq_{\text{T}} Y \implies X \leq_{\text{c.e.}} Y ,$$

and in fact it can be shown that all these implications are strict.

We shall also consider reductions of a sequence to itself. Of course, reducing a sequence to itself is trivial if there are no further restrictions on the oracle Turing machine performing the reduction. This leads to the concepts of *autoreducibility* and *self-reducibility*.

A sequence is *T-autoreducible* if it can be reduced to itself by an oracle Turing machine that is not allowed to query the oracle at the current input, and a sequence is *T-self-reducible* if it can be reduced to itself by an oracle Turing machine that may only query the oracle at places strictly less than the current input. For reducibilities other than Turing reducibility, the concepts of auto- and self-reducibility are defined in the same manner. E.g., a sequence is *wtt-autoreducible* if it is T-autoreducible by an oracle Turing machine with a computable bound on its use, and a sequence  $X$  is *c.e.-self-reducible* if there is an oracle Turing machine that on input  $x$  queries its

oracle only at places  $z < x$  and such that  $M(X, x) = 1$  in case  $x \in X$  and, otherwise,  $M(X, x)$  is undefined.

**$\Sigma_n^0$ -classes,  $\Pi_n^0$ -classes.** A class  $\mathcal{C} \subseteq 2^\omega$  is a  $\Sigma_1^0$ -class if  $\mathcal{C} = [W]$  for some computably enumerable set  $W$ . Moreover, if  $e \in \omega$  is such that  $\mathcal{C} = [W_e]$  then  $e$  is called an *index of  $\mathcal{C}$* . The complement of a  $\Sigma_1^0$ -class  $\mathcal{C}$  is called a  $\Pi_1^0$ -class, and it has the same indices as  $\mathcal{C}$ .  $\Sigma_1^0$ - and  $\Pi_1^0$ -classes are also called “effectively open” and “effectively closed”, respectively. In general, the complement of a  $\Sigma_n^0$ -class  $\mathcal{C}$  is called a  $\Pi_n^0$ -class and it has the same indices as  $\mathcal{C}$ . Further, a class  $\mathcal{C} \subseteq 2^\omega$  is a  $\Sigma_{n+1}^0$ -class if it is an *effective union* of  $\Pi_n^0$ -classes, i.e., if  $\mathcal{C}$  is a union of  $\Pi_n^0$ -classes  $\mathcal{D}_0, \mathcal{D}_1, \dots$  such that there is a computable function  $g$  which maps each  $n$  to an index of  $\mathcal{D}_n$ . In this situation, an index of the function  $g$  is called an *index of  $\mathcal{C}$* .

### 1.3 Thesis Outline and Bibliographical Notes

Throughout this thesis, “sequence” means infinite binary sequence, unless stated otherwise.

*Chapter 2.* We recapitulate the definitions of Martin-Löf randomness, of computable randomness, and of Schnorr randomness. Along the way, we review the three main approaches to algorithmic randomness. These rely on effective measures, betting games, and incompressibility, which in turn are based on the respective notions of tests, martingales, and Kolmogorov complexity.

*Chapter 3.* Martin-Löf randomness is defined in terms of tests [30]. Two central results in the theory of algorithmic randomness were the characterizations of Martin-Löf randomness in terms of martingales and in terms of Kolmogorov complexity due to Schnorr. Computable randomness was introduced by Schnorr [41] using martingales. We give a positive answer to a question of Ambos-Spies and Kučera, who have asked whether computable randomness can be characterized in terms of tests [1, Open Problem 2.6]. Namely, we introduce *bounded Martin-Löf tests* and we prove that a sequence is computably random if and only if it withstands every bounded Martin-Löf test. We note that a solution to the question of Ambos-Spies and Kučera was obtained independently and earlier by Downey, Griffiths, and LaForte [12] (see the bibliographical notes below).

*Chapter 4.* Inspired by the test characterization of computable randomness in terms of bounded Martin-Löf tests, we define a version of Kolmogorov complexity by introducing *bounded machines*. That version is used

to characterize computable randomness in a similar fashion as Martin-Löf randomness was characterized via Kolmogorov complexity by Schnorr. The latter result allowed for a study of certain lowness notions in the Martin-Löf setting (*lowness for K* and *K-triviality*). Based on our characterization result, we investigate the corresponding lowness notions w.r.t. computable randomness. Namely, we prove some results on *lowness for bounded machines* and *bounded triviality*. As a chronological remark, we note that the results in this chapter were obtained after the results in Chapter 5.

*Chapter 5.* Downey and Griffiths [13] introduced *computable machines* and gave a characterization of Schnorr randomness via a version of Kolmogorov complexity which they defined in terms of computable machines. They initiated the study of *Schnorr triviality* as an analog of K-triviality (see above). In this chapter, we define *lowness for bounded machines* as an analog of lowness for K. We prove that a sequence is low for computable machines if and only if it is computably traceable. Consequently, by results of Terwijn and Zambella [45], and of Kjos-Hanssen, Nies, and Stephan [23], lowness for bounded machines is equivalent to lowness for Schnorr randomness and to lowness for Schnorr tests.

*Chapter 6.* For every sequence  $X$  there is a Martin-Löf random sequence  $Y$  such that  $X$  is effectively decodable from  $Y$ . This is a central result obtained independently by Gács [21] and Kučera [25]. Using ideas from their proofs, that are formulated in terms of tests, we present a somewhat simpler proof in terms of martingales. More precisely, a sequence  $Y$  as above is constructed by diagonalizing against appropriate martingales. By a variant of that construction, we prove that there exists a computably random sequence that is weak truth-table autoreducible. Further, we show that a sequence is computably enumerable self-reducible if and only if its associated real is computably enumerable.

*Chapter 7.* Book, Lutz, and Wagner [4] show that any union of  $\Pi_1^0$ -classes that is closed under finite variation is a Lebesgue null class if and only if it contains no Martin-Löf random sequence. We extend this result by showing that any union of  $\Pi_1^0$ -classes that is closed under finite variation is a Lebesgue null class if and only if it contains no Kurtz random sequence, where Kurtz randomness is another randomness concept that is weaker than Martin-Löf randomness. For any randomness concept  $C$ , we consider  $C$ -null classes, which are effective versions of Lebesgue null classes (on the Cantor space  $2^\omega$ ). While for  $\Sigma_2^0$ -classes being a Schnorr null class is equivalent to being a Lebesgue null class, we show that the corresponding assertion for “Kurtz null class” instead of “Schnorr null class” is false. Finally, we prove

two results on “Almost” classes. We demonstrate that every Almost class with respect to a bounded reducibility has computable packing dimension zero. We further show, given a bounded reducibility  $R$  that is upwards closed under finite variation, that a sequence is contained in the respective Almost class if and only if it is computable and not  $R$ -deep.

### **Bibliographical Notes**

The first characterization of computable randomness in terms of tests is due to Downey, Griffiths, and LaForte [12]. Later and independently, the characterization result presented in Chapter 3 was obtained together with Merkle and Slaman. It is part of an article [32] that was published in the journal *Theory of Computing Systems*.

Chapter 5 is joint work with Downey, Greenberg, and Nies, and will be published in the proceedings of the conference *Computational Prospects of Infinity* [11].

The results of Chapter 6, which were obtained together with Merkle, were published in the *Journal of Symbolic Logic* [31].

The material presented in Chapter 7 is joint work with Merkle.



## Chapter 2

# Fundamental Randomness Notions

The three main approaches to define a concept of an algorithmically random sequence rely on effective measures, betting games, and incompressibility. In this chapter, we recapitulate these approaches as well as the respective fundamental notions of tests, martingales, and Kolmogorov complexity. After recalling the definition of Martin-Löf randomness in terms of tests, we review characterizations of Martin-Löf randomness via martingales and via Kolmogorov complexity. These fundamental results due to Schnorr show that Martin-Löf randomness is a robust notion. In addition to Martin-Löf randomness, we review computable randomness and Schnorr randomness, two central concepts which were introduced by Schnorr.

### 2.1 Tests and Randomness due to Martin-Löf

The most prominent randomness concept has been introduced by Martin-Löf [30] using measure theory. As pointed out in Section 1.1, stochasticity has been considered an inadequate randomness concept because of a construction due to Ville, which shows that for any countable set  $F$  of selection procedures there is an  $F$ -stochastic sequence  $X$  which violates the law of the iterated logarithm. Now Martin-Löf's approach was to identify the random sequences with those sequences which obey all probability-one laws of a certain collection (that contains more than just the strong law of large numbers). In this sense, a random sequence is often called "typical". Clearly, no sequence can obey *all* probability-one laws since for each sequence  $X$ , the class  $2^\omega \setminus \{X\}$  has Lebesgue measure 1. Martin-Löf introduced an effective

version of probability-one laws, or rather an effective version of null classes, nowadays called *Martin-Löf null classes*, that are interpreted as the classes of sequences which violate some effectively given probability-one law. Consequently, a sequence is random in the sense of Martin-Löf if it does not belong to any Martin-Löf null classes.

The basis for the definition of effective null classes is the following classical characterization. A class  $\mathcal{C}$  is a Lebesgue null class if and only if there is a sequence of classes  $\mathcal{A}_0, \mathcal{A}_1, \dots$  such that each  $\mathcal{A}_n$  is an open covering of  $\mathcal{C}$ , i.e., a union of basic open classes with  $\mathcal{C} \subseteq \mathcal{A}_n$ , such that  $\mu(\mathcal{A}_n) \leq 2^{-n}$ . We note w.r.t. this classical characterization, that instead of the sequence  $2^{-n}$  for the measure bounds one may choose any sequence that converges to 0.

In what follows, we recall the definition of Martin-Löf's randomness concept which is nowadays referred to as Martin-Löf randomness.

As stated in section 1.2,  $W_0, W_1, \dots$  denotes a standard listing of the computably enumerable sets. Recall that a *uniformly computably enumerable sequence of sets* is a sequence of sets  $A_0, A_1, \dots$  such that there is a computable function  $g$  with  $A_n = W_{g(n)}$  for all  $n$ .

**Definition 2.1** (Martin-Löf [30]). (i) A Martin-Löf test is a uniformly computably enumerable sequence of sets  $A_0, A_1, \dots$  such that  $\mu[A_n] \leq 2^{-n}$  for each  $n$ .

(ii) A class of reals  $\mathcal{C}$  is a Martin-Löf null class if there is a Martin-Löf test  $A_0, A_1, \dots$  such that  $\mathcal{C} \subseteq \bigcap_n A_n$ . In this case, we say that  $\mathcal{C}$  is covered by  $A_0, A_1, \dots$  and if, moreover,  $\mathcal{C}$  is the singleton  $\{X\}$  then we also say that  $X$  is covered by  $A_0, A_1, \dots$ .

(iii) A sequence  $X$  is Martin-Löf random if  $X$  is not covered by any Martin-Löf test.

**Convention.** We shall use the following bit of terminology. A real  $X$  *withstands* a Martin-Löf test  $A_0, A_1, \dots$  if  $X$  is not covered by  $A_0, A_1, \dots$ .

Obviously, a sequence is Martin-Löf random if and only if it does not belong to any Martin-Löf null class.

The measure bounds  $2^0, 2^{-1}, \dots$  in the first item of Definition 2.1 could be replaced by any other suitable sequence of measure bounds without changing the resulting concepts as we show next.

**Remark 2.2.** (i) As stated in Section 1.2,  $q_0, q_1, \dots$  is an (effective) sequence of the nonnegative rational numbers. For any computable function



$f: \omega \rightarrow \omega$  where  $\lim_n q_{f(n)} = 0$ , we call a sequence of reals  $r_0, r_1, \dots$  *effectively  $f$ -converging with limit  $r$*  if  $r = \lim_n r_n$  and for all  $n$ ,  $|r - r_n| \leq q_{f(n)}$ . Now suppose  $r_0, r_1, \dots$  is effectively  $f$ -converging with limit  $r$  and  $g$  is any computable function with  $\lim_n q_{g(n)} = 0$ . Then it is not hard to see that one can compute effectively a sequence of indices  $n_0, n_1, \dots$  such that the subsequence  $r_{n_0}, r_{n_1}, \dots$  is effectively  $g$ -converging with limit  $r$ .

(ii) By the above, note that instead of the measure bounds  $2^0, 2^{-1}, \dots$  in the first item of Definition 2.1 we could choose  $q_{f(0)}, q_{f(1)}, \dots$  for any computable function  $f$  such that  $\lim_n q_{f(n)} = 0$ , without changing the null class and the randomness concept.  $\triangleleft$

To explain the term “test”, we note that Martin-Löf motivates his definition of randomness by statistical tests for randomness, which rely on probability-one laws. We give a sketch of these ideas following the original account of Martin-Löf [30]. First we discuss tests which accept or reject strings, and later we shall extend the arguments to acceptance and rejection of sequences. Fix a probability-one law, say the strong law of large numbers, and consider a test which rejects (a string) if the ratio of 1s differs “too much from  $1/2$ ”. We restrict our attention to “levels of significance”  $\delta = 1/2, 1/4, 1/8, \dots$ . Again, this choice is arbitrary (see Remark 2.2). Now the test may be given by the following “prescription”: Given a string  $\sigma$  of length  $m$ ,

“reject the hypothesis of randomness on the level  $\delta = 2^{-n}$  provided

$$\left| 2 \left( \sum_{i=0 \dots m-1} \sigma(i) \right) - m \right| > f(n, m),$$

where  $f$  is determined by the requirement that the number of strings of length  $m$  for which the inequality holds should be  $\leq 2^{m-n}$ . Further, it should not be possible to diminish  $f$  without violating this condition.” (Cited with slight adaptations from [30].)

To each level  $\delta = 2^{-n}$ , the set of strings  $U_n$  for which the hypothesis of randomness is rejected is called a “critical region”. Martin-Löf requires that a test have “nested” critical regions, i.e.,  $U_k \subseteq U_n$  for all  $k \geq n$ .

Now we turn our attention to sequences. Fix a probability-one law and let  $\mathcal{N}$  denote the (Lebesgue null) class of all sequences violating that law. Martin-Löf considers a (classical) open covering  $\mathcal{U}_n$  of  $\mathcal{N}$  with, say,  $\mu(\mathcal{U}_n) \leq 2^{-n}$  and a set  $U_n = \{\sigma \in 2^{<\omega} : [\sigma] \subseteq \mathcal{U}_n\}$ . Then  $U_n$  is interpreted as the critical region of a test on the level  $\delta = 2^{-n}$ . Martin-Löf argues that

a uniformly c.e. sequence of nested critical regions as above is the right formalization of the tests used in statistics. We note that, as in Definition 2.1, one can abandon the requirement that the critical regions be nested without changing the null class concept.

It is readily verified that the concept of a Martin-Löf null class is an effective version of the classical concept of a Lebesgue null class. In particular, every Martin-Löf null class has Lebesgue measure zero. Then by  $\sigma$ -additivity of measures, the union  $\mathcal{N}$  of all Martin-Löf null classes also has Lebesgue measure zero. A remarkable fact about  $\mathcal{N}$  is stated in the following theorem, analogs of which are not true in general for other randomness concepts that are defined or characterized by a stricter test notion.

**Theorem 2.3** (Martin-Löf [30]). *There is a universal Martin-Löf test, i.e., there is a Martin-Löf test  $U_0, U_1, \dots$  such that for each class  $\mathcal{C} \subseteq 2^\omega$ , we have that  $\mathcal{C}$  is a Martin-Löf null class if and only if  $\mathcal{C}$  is covered by  $U_0, U_1, \dots$*

*Proof.* First we give an effective list of all Martin-Löf tests. To this end, we uniformly enumerate sets  $A_{k,n}$  with  $k, n \in \omega$  as follows. At stage  $s$ , enumerate for every  $k, n < s$  with  $\varphi_{k,s}(n) \downarrow$  all elements of  $W_{\varphi_{k(n),s}} \setminus W_{\varphi_{k(n),s-1}}$  into  $A_{k,n}$  provided that  $\mu[W_{\varphi_{k(n),s}}] \leq 2^{-n}$ . We denote the approximation of  $A_{k,n}$  at the end of stage  $s$  by  $A_{k,n}^s$ . Now the  $n$ th component of the list's  $k$ th Martin-Löf test  $A_{k,0}, A_{k,1}, \dots$  is defined as a union of finite sets  $A_{k,n} = \cup_s A_{k,n}^s$ . One can easily verify that via the above construction, exactly the Martin-Löf tests are listed in an effective way. Let

$$U_n = \bigcup_k A_{k,k+n+1}.$$

$U_0, U_1, \dots$  is a Martin-Löf test because

$$\mu[U_n] \leq \sum_k \mu[A_{k,k+n+1}] \leq \sum_k 2^{-(k+n+1)} \leq 2^{-n}.$$

To complete the proof we note that by construction, if a class  $\mathcal{C}$  is covered by the  $k$ th Martin-Löf test of our list, i.e., if  $\mathcal{C} \subseteq \cap_n [A_{k,n}]$ , then  $\mathcal{C}$  is covered by  $U_0, U_1, \dots$ , too.  $\square$

## 2.2 Martingales

One of the major approaches to randomness is via betting games (see Subsection 2.2.1). Betting games are based on betting strategies which can be

coded into martingales. The concept of a martingale was, in a more general context, introduced by Levy. A characterization of Lebesgue null classes in terms of martingales was obtained by Ville [47]. Schnorr [40, 41, 42] considered effective versions of martingales and, in particular, gave a characterization of Martin-Löf randomness in terms of subcomputable martingales (see Definition 2.12 and Theorem 2.13).

**Definition 2.4.** A martingale is a function  $d: 2^{<\omega} \rightarrow \mathbb{R}^+ \cup \{0\}$  such that for all  $\sigma \in 2^{<\omega}$ ,

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}. \quad (2.1)$$

Equation (2.1) is called the *fairness condition*, which is motivated by the subsequent discussion on betting games.

### 2.2.1 Betting Games

We give the following description of a betting game. A player successively places bets on the individual bits of an unknown sequence  $X \in 2^\omega$ . The betting proceeds in rounds  $i = 1, 2, \dots$ . During round  $i$ , the player receives as input the length  $i - 1$  prefix of  $X$  and then, first, decides whether to bet on the  $i$ th bit being 0 or 1 and, second, determines the stake that shall be bet. The stake might be any fraction between 0 and 1 of the capital accumulated so far, i.e., in particular, the player is not allowed to incur debts. Formally, a player can be identified with a *betting strategy*

$$b: 2^{<\omega} \rightarrow [-1, 1]$$

where on input  $\sigma$  the absolute value of  $b(\sigma)$  is the fraction of the current capital that shall be at stake. Further, if  $b(\sigma)$  is negative then the bet is placed on the next bit being 0. Otherwise, i.e., if  $b(\sigma)$  is nonnegative then the bet is placed on the next bit being 1. We note that betting strategies are generalizations of the selection procedures discussed in Section 1.1 (see e.g. [1]).

We call the betting game fair because of how the capital is calculated after each round, which we shall show next. This will also explain the term “fairness condition” for (2.1). The player starts with strictly positive, finite capital  $d_b(\varepsilon)$ . At the end of each round, in case the current guess has been correct, the capital is increased by this round’s stake and, otherwise, is decreased by the same amount. So given a betting strategy  $b$  and the initial capital, we can inductively determine the corresponding *payoff function*  $d_b$

by applying the equations

$$d_b(\sigma 0) = d_b(\sigma) - b(\sigma) \cdot d_b(\sigma)$$

and

$$d_b(\sigma 1) = d_b(\sigma) + b(\sigma) \cdot d_b(\sigma).$$

Intuitively speaking, the payoff  $d_b(\sigma)$  is the capital the player accumulates until the end of round  $|\sigma|$  by betting on a sequence that has the string  $\sigma$  as a prefix. Adding up the two equations, one gets that the payoff function  $d_b$  is a martingale.

Conversely, any martingale  $d$  determines an initial capital  $d(\varepsilon)$  and a betting function  $b$  (where we let  $b(\sigma) = 0$  in case  $d(\sigma) = 0$ ). By the preceding discussion it follows for games as described above that for any martingale there is an equivalent betting strategy and vice versa.

### 2.2.2 Succeeding Martingales and Randomness

Considering some arbitrary class  $\mathcal{B}$  of betting strategies, a common intuition would be to call a sequence  $X$  random with respect to  $\mathcal{B}$  iff in the limit, one cannot accumulate an unbounded amount of capital when playing with any strategy  $b \in \mathcal{B}$  against  $X$ . In this sense, one also says that a random sequence has to be “unpredictable”. Since during the game the capital is described by a martingale and since there is a 1-1 correspondence between betting strategies and martingales, one usually considers certain classes of martingales to define randomness concepts and omits an explicit consideration of the underlying betting strategies.

**Definition 2.5.** *A martingale  $d$  succeeds on a sequence  $X \in 2^\omega$  if*

$$\limsup_{n \rightarrow \infty} d(X \upharpoonright n) = \infty.$$

*A martingale  $d$  succeeds on a class  $\mathcal{C} \subseteq 2^\omega$  if  $d$  succeeds on every sequence in  $\mathcal{C}$ .*

As described above we shall choose a class of admissible martingales  $\mathcal{M}$  and call a sequence  $X$  random with respect to  $\mathcal{M}$  if no martingale  $d \in \mathcal{M}$  succeeds on  $X$ . Clearly, we cannot choose  $\mathcal{M}$  to be the class of all martingales since for any sequence  $X$  there is a trivial betting strategy which doubles its capital in every round when applied to a game against  $X$ . To get algorithmic randomness concepts we shall consider (several) classes of *effective* martingales.

As we shall see below the Martin-Löf random sequences are exactly those sequences on which no *subcomputable* martingale succeeds. But first we note some useful properties of martingales and we recall the notion of a computably enumerable real.

**Remark 2.6.** If  $d$  is a martingale,  $\sigma$  a string, and  $n$  a natural number, then

$$d(\sigma) = \frac{1}{2^n} \sum_{\tau \in \{0,1\}^n} d(\sigma\tau). \quad (2.2)$$

This can be proved with an easy inductive argument that uses the fairness condition (2.1). Conversely, (2.1) is a special case of (2.2) where  $n = 1$ .  $\triangleleft$

**Remark 2.7.** For every set of strings  $A$  there is a subset  $B \subseteq A$  such that  $B$  is prefix-free and  $[A] = [B]$ . Such a set  $B$  consists of all strings in  $A$  that do not have a proper prefix in  $A$ .  $\triangleleft$

We will state an effective version of the above remark in Proposition 4.2.

**Lemma 2.8** (Ville [47]). *Let  $d$  be a martingale.*

(i) *For any prefix-free set  $A \subseteq 2^{<\omega}$  and any string  $\sigma \in 2^{<\omega}$ ,*

$$\sum_{\{\tau \in A: \sigma \preceq \tau\}} 2^{-|\tau|} d(\tau) \leq 2^{-|\sigma|} d(\sigma). \quad (2.3)$$

(ii) *Let  $\text{Succ}^n(d) = \{\sigma : d(\sigma) \geq n\}$  for any  $n \in \omega$ . Then*

$$\mu[\text{Succ}^n(d)] \leq d(\varepsilon) \frac{1}{n}. \quad (2.4)$$

Inequality (2.4) is called *Kolmogorov's inequality*.

*Proof.* (i) It suffices to consider finite sets  $A$  only. Let  $m = \max\{|\tau| : \tau \in A\}$ . Then, by applying (2.2) in the first and last line below,

$$\begin{aligned} \sum_{\{\tau \in A: \sigma \preceq \tau\}} 2^{|\sigma| - |\tau|} d(\tau) &\leq \sum_{\{\tau \in A: \sigma \preceq \tau\}} \sum_{\xi \in \{0,1\}^{m-|\tau|}} 2^{|\sigma| - |\tau| - (m-|\tau|)} d(\tau\xi) \\ &= \sum_{\{\rho \in \{0,1\}^{m-|\sigma|} : \exists \tau \in A (\sigma \preceq \tau \preceq \sigma\rho)\}} 2^{|\sigma| - m} d(\sigma\rho) \\ &\leq \sum_{\rho \in \{0,1\}^{m-|\sigma|}} 2^{-(m-|\sigma|)} d(\sigma\rho) \\ &= d(\sigma). \end{aligned}$$

(ii) By Remark 2.7, let  $A \subseteq \text{Succ}^n(d)$  be prefix-free such that  $[A] = [\text{Succ}^n(d)]$ . Then by (i),

$$n\mu[A] = n \sum_{\sigma \in A} 2^{-|\sigma|} \leq \sum_{\sigma \in A} 2^{-|\sigma|} d(\sigma) \leq d(\varepsilon).$$

□

**Definition 2.9.** (i) *A computably enumerable real, c.e. real for short, is a real that is the limit of a nondecreasing computable sequence of rational numbers.*

(ii) *A uniformly c.e. sequence of reals is a sequence of reals  $r_0, r_1, \dots$  such that there is a computable function  $f$  in two arguments which satisfies the following condition: for each  $n$ , the sequence  $q_{f(n,0)}, q_{f(n,1)}, \dots$  witnesses that  $r_n$  is a c.e. real.*

In the literature, sequences are also called *sets* or *reals*. The former alternative is due to the identification of sequences and sets of natural numbers as discussed in Section 1.2. On the other hand, one can view a sequence as a binary expansion of a real in the unit interval  $[0, 1)$  and vice versa, where in order to get a bijection, one considers e.g. only sequences with infinitely many 0s. In this sense, we can assign properties that we have defined w.r.t. sequences, like “being Martin-Löf random”, to reals (in the unit interval), too.

Note that the measures of the components of a Martin-Löf test are a uniformly c.e. sequence of reals. To make a brief digression on uniformly c.e. sequences of reals, we note without proof the following result on universal Martin-Löf tests and its converse below.

**Theorem 2.10** (Kučera and Slaman [26]). *If  $U_0, U_1, \dots$  is a universal Martin-Löf test then  $\mu[U_0], \mu[U_1], \dots$  is a uniformly c.e. sequence of Martin-Löf random reals.*

**Theorem 2.11** (Merkle, Mihailović, and Slaman [32]). *Let  $r_0, r_1, \dots$  be a uniformly c.e. sequence of Martin-Löf random reals with  $r_n \leq 2^{-n}$  for every  $n$ . Then there is a universal Martin-Löf test  $U_0, U_1, \dots$  such that for each  $n$ ,  $\mu[U_n] = r_n$ .*

Now we turn to the martingale characterization of Martin-Löf randomness.

**Definition 2.12** (Schnorr [41]). *A subcomputable martingale is a martingale  $d$  such that  $d(\varepsilon), d(0), d(1), d(00), \dots$  is a uniformly c.e. sequence of reals.*

**Theorem 2.13** (Schnorr [41]). *A class  $\mathcal{C} \subseteq 2^\omega$  is a Martin-Löf null class if and only if there is a subcomputable martingale that succeeds on  $\mathcal{C}$ . In particular, a sequence is Martin-Löf random if and only if no subcomputable martingale succeeds on it.*

*Proof.* Suppose that  $d$  is a subcomputable martingale that succeeds on a class  $\mathcal{C}$ . We may assume that the initial capital  $d(\varepsilon)$  of  $d$  is less than or equal to 1. For every  $n$ , let  $A_n = \text{Succ}^{2^n}(d)$  where the sets  $\text{Succ}^0(d), \text{Succ}^1(d), \dots$  are defined in Lemma 2.8 (ii), i.e., let

$$A_n = \text{Succ}^{2^n}(d) = \{\sigma \in 2^{<\omega} : d(\sigma) \geq 2^n\}. \quad (2.5)$$

We claim that the sequence  $A_0, A_1, \dots$  is a Martin-Löf test. Indeed, it is obviously uniformly c.e. and by Kolmogorov's inequality (2.4),  $\mu[A_n] \leq 2^{-n}$  for each  $n$ . Moreover,  $A_0, A_1, \dots$  covers  $\mathcal{C}$  because, by construction, a sequence  $X$  is covered by  $A_0, A_1, \dots$  if and only if  $d$  succeeds on  $X$ .

For the converse direction, assume that  $\mathcal{C}$  is a Martin-Löf null class. For every string  $\sigma$ , let  $d_\sigma$  be the martingale with initial capital  $2^{-|\sigma|}$  that doubles along  $\sigma$ , i.e.,  $d_\sigma$  has the value  $2^{|\tau|-|\sigma|}$  on any prefix  $\tau$  of  $\sigma$ , the value 1 on any extension of  $\sigma$ , and the value 0 otherwise. If we pick any Martin-Löf test  $A_0, A_1, \dots$  which covers  $\mathcal{C}$ , then  $d$  defined by

$$d(\xi) = \sum_{\{\sigma : \sigma \in \cup_n A_n\}} d_\sigma(\xi)$$

is a subcomputable martingale that succeeds on  $\mathcal{C}$ . □

**Corollary 2.14.** *There is a universal subcomputable martingale, i.e., there is a subcomputable martingale which succeeds on all sequences which are not Martin-Löf random.*

*Proof.* In order to obtain a martingale as desired, it suffices to apply the construction from the proof of Theorem 2.13 to the Martin-Löf null class of all sequences that are not Martin-Löf random. □

The following will turn out useful in constructions of random sequences in Chapter 6.

**Definition 2.15.** A martingale  $d$  succeeds on a sequence  $X$  by unbounded limit inferior if

$$\liminf_{n \rightarrow \infty} d(X \upharpoonright n) = \infty.$$

**Remark 2.16.** There is a subcomputable martingale  $d$  that succeeds by unbounded limit inferior on every sequence that is not Martin-Löf random.

For a proof, it suffices to note that the universal subcomputable martingale as constructed in the proof of Corollary 2.14 already has the desired property.  $\triangleleft$

## 2.3 Kolmogorov Complexity

We briefly review prefix-free Kolmogorov Complexity as a basis for the incompressibility approach to algorithmic randomness.

**Definition 2.17.** (i) A prefix-free machine is a Turing machine  $M$  such that the domain of  $M$  is a prefix-free set.

(ii) Given a prefix-free machine  $M$ , the  $M$ -complexity  $K_M(\sigma)$  of a string  $\sigma$  is defined by

$$K_M(\sigma) = \min \{ |\tau| : M(\tau) = \sigma \},$$

where  $K_M(\sigma) = \infty$  in case  $\sigma$  is not in the range of  $M$ .

**Theorem 2.18.** There is a universal prefix-free machine  $U$ , i.e., for all prefix-free machines  $M$ ,

$$(\exists c \in \omega)(\forall \sigma \in 2^{<\omega}) K_U(\sigma) \leq K_M(\sigma) + c. \quad (2.6)$$

Such a machine  $U$  is for example given by the following specification:  $U$  converges on an input  $\rho$  if and only if  $\rho = 1^e 0\tau$  for some  $e \in \omega$  and  $\tau \in 2^{<\omega}$  such that  $M = \varphi_e$  converges on  $\tau$ . In this case  $U$  outputs  $M(\tau)$ . In particular, we have

$$(\forall e \in \omega)(\forall \tau \in 2^{<\omega}) U(1^e 0\tau) \simeq \varphi_e(\tau). \quad (2.7)$$

**Definition 2.19.** We fix a universal machine  $U$  as above and define the prefix-free Kolmogorov complexity  $K$  of a string  $\sigma$  by  $K(\sigma) = K_U(\sigma)$ .

Prefix-free Kolmogorov complexity is a version of the “plain Kolmogorov complexity”. This concept is defined similarly where one abandons the requirement that the considered machines be prefix-free. Plain Kolmogorov



complexity was introduced in the 1960s pairwise independently by Chaitin, Kolmogorov, and Solomonoff. In particular, they proved a result for the plain complexity notion which Theorem 2.18 is an analog of. The history of the development of plain Kolmogorov complexity is rather convoluted, see Li and Vitányi [29, Sect. 1.13] who give a detailed account of that matter. Prefix-free Kolmogorov complexity was introduced independently by Levin [28], Gács [20], and Chaitin [8] in 1974 and 1975, see also Li and Vitányi [29, Sect. 3.10].

Note that if  $M$  is a prefix-free machine, then for the *halting probability*  $\mu[\text{dom } M]$  of  $M$  we have  $\mu[\text{dom } M] = \sum \{2^{-|\sigma|} : \sigma \in \text{dom } M\} \leq 1$ . Furthermore, observe that  $\mu[\text{dom } M]$  is a c.e. real. The halting probability of any *universal* prefix-free machine is a natural example of a *Martin-Löf random* c.e. real. Such a real is called a *Chaitin's  $\Omega$  number*.

The idea underlying the approach to algorithmic randomness via Kolmogorov complexity can be briefly described as follows. Informally, it is required that the initial segments of a random sequence should not be “significantly compressible”, i.e., their Kolmogorov complexity should not be “significantly” low. It is a remarkable result that by a suitable incompressibility requirement we can characterize the Martin-Löf random sequences.

**Theorem 2.20** (Schnorr<sup>1</sup>). *A sequence  $X$  is Martin-Löf random if and only if*

$$(\exists c \in \omega)(\forall n \in \omega) K(X \upharpoonright n) \geq n - c. \quad (2.8)$$

By variations of (2.8) one obtains characterizations of other randomness concepts, see Downey and Hirschfeldt [15] for a comprehensive account. E.g., one could change the complexity notion by considering different classes of machines. Furthermore, the constant  $c$  might be replaced by a suitable function of the length  $n$ , and finally one may alter the requirement “for all lengths” to “for infinitely many lengths”.

For future reference we present a proof of the ‘only if’ part of Theorem 2.20, omitting a proof of the more difficult ‘if direction’. (For a proof of the ‘if direction’ see for example [29].) So for all  $m \in \omega$ , let

$$A_m = \{\sigma \in 2^{<\omega} : K(\sigma) \leq |\sigma| - m\}. \quad (2.9)$$

---

<sup>1</sup>Li and Vitányi [29, Section 3.10] report on the following: In the original submission of [8], Chaitin proposed to call a sequence  $X$  random if  $X$  satisfies (2.8). Acting as a referee of that paper, Schnorr showed the equivalence to Martin-Löf randomness.

Note that the sequence  $(A_m)_{m \in \omega}$  is uniformly c.e. and for each  $m$ ,

$$\mu[A_m] \leq \sum_{\sigma \in A_m} 2^{-|\sigma|} \leq 2^{-m},$$

where the last inequality holds because  $M$  is a prefix-free machine. Consequently,  $(A_m)_{m \in \omega}$  is a Martin-Löf test. Now assume that for a real  $X$ , (2.8) does not hold. Then for all  $m$  there is an  $n$  such that  $K(X \upharpoonright n) < n - m$ . Hence  $X$  is covered by the Martin-Löf test  $(A_m)_{m \in \omega}$ . This concludes the proof of the ‘only if’ part of the theorem.  $\square$

## 2.4 Two Concepts due to Schnorr

In a series of papers [38, 39, 40, 41] and a monograph [42] Schnorr indicated possible deficiencies of Martin-Löf randomness and argued in favor of a randomness concept which is weaker than Martin-Löf randomness. The concept he introduced is referred to as Schnorr randomness today. Aside from this notion, Schnorr [41] defined an intermediate concept which is nowadays known as computable randomness.

### 2.4.1 Computable Randomness

Schnorr [41] raises an objection to Martin-Löf randomness as follows. By Theorem 2.13, the Martin-Löf random sequences are exactly those sequences on which no subcomputable martingale succeeds. Now Schnorr considers martingales that we shall refer to as *supercomputable martingales*. While the range of subcomputable martingales is uniformly approximable from below, the supercomputable martingales are defined similarly by requiring that the range be uniformly approximable from *above*. Schnorr [41] calls a sequence “(2)-random” if no supercomputable martingale succeeds on it and he shows that this randomness concept is not equivalent to Martin-Löf randomness. He argues that this asymmetry is a deficiency of Martin-Löf randomness, and with computable randomness he aims at “developing a concept of randomness based on martingales whose algorithmic structure is symmetrical”.

After defining two versions of computable martingales we review Schnorr’s result [42] by which both versions are equivalent, i.e., give rise to the same null classes.

For the following definitions see Remark 2.2.

**Definition 2.21.** (i) An effectively converging sequence of rationals with limit  $q$  is a sequence of rationals  $q_{i_1}, q_{i_2}, \dots$  such that  $q = \lim_n q_{i_n}$  and for all  $n$ ,  $|q - q_{i_n}| \leq 2^{-n}$ .

(ii) A computable real is a real  $r$  such that there is a computable function  $f = \varphi_e$  with  $q_{f(0)}, q_{f(1)}, \dots$  being an effectively converging sequence of rationals with limit  $r$ . In this case  $r$  is computable via  $f$ .

(iii) A uniformly computable sequence of reals is a sequence of reals  $r_0, r_1, \dots$  such that there is a computable function  $g$  with  $r_n$  being computable via  $\varphi_{g(n)}$  for each  $n$ .

**Definition 2.22.** (i) An  $\mathbb{R}$ -computable martingale is a martingale  $d$  such that  $d(\varepsilon), d(0), d(1), d(00), \dots$  is a uniformly computable sequence of reals.

(ii) A  $\mathbb{Q}$ -computable martingale is a martingale  $d$  such that there is a computable function  $h$  with  $d(\sigma) = q_{h(\sigma)}$  for each string  $\sigma$ .

Observe that, trivially, a martingale  $d$  is  $\mathbb{R}$ -computable if and only if there is a computable function  $D: 2^{<\omega} \rightarrow \omega$  such that  $d(\sigma)$  is computable via  $\varphi_{D(\sigma)}$  for each string  $\sigma$ .

**Proposition 2.23** (Schnorr [42]). For every  $\mathbb{R}$ -computable martingale  $d$  there is a  $\mathbb{Q}$ -computable martingale  $\tilde{d}$  such that for all sequences  $X$ ,  $d$  succeeds on  $X$  if and only if  $\tilde{d}$  succeeds on  $X$ .

*Proof* (cf. [33]). Consider the function  $f(\sigma) = d(\sigma 0) - d(\sigma)$ . Note that  $d(\sigma 1) - d(\sigma) = -f(\sigma)$  and so

$$d(\sigma b) = d(\sigma) + (-1)^b f(\sigma) \quad \text{for } b = 0, 1.$$

Consequently for all strings  $\sigma$ ,

$$d(\sigma) = d(\varepsilon) + \sum_{n=0}^{|\sigma|-1} (-1)^{\sigma(n)} f(\sigma \upharpoonright n).$$

Let  $D$  be a computable function such that each value  $d(\sigma)$  is computable via  $\varphi_{D(\sigma)}$ . Define a function  $\tilde{f}$  by

$$\tilde{f}(\sigma) = q_{\varphi_{D(\sigma 0)}(|\sigma|+4)} + q_{\varphi_{D(\sigma)}(|\sigma|+4)}.$$

Then

$$\left| f(\sigma) - \tilde{f}(\sigma) \right| \leq 2 \cdot 2^{-(|\sigma|+4)} = 2^{-|\sigma|-3}.$$

Define  $\tilde{d}$  inductively as follows. Choose  $\tilde{d}(\varepsilon)$  such that  $d(\varepsilon) + 1/4 \leq \tilde{d}(\varepsilon) \leq d(\varepsilon) + 3/4$ . Then

$$\left| d(\varepsilon) + \frac{1}{2} - \tilde{d}(\varepsilon) \right| \leq \frac{1}{4}.$$

Now suppose  $\tilde{d}$  is already defined on a string  $\sigma$ . Then let

$$\tilde{d}(\sigma b) = \tilde{d}(\sigma) + (-1)^b \tilde{f}(\sigma) \quad \text{for } b = 0, 1.$$

Obviously,  $\tilde{d}$  satisfies the fairness condition (2.1). Furthermore,

$$\begin{aligned} & \left| d(\sigma) + \frac{1}{2} - \tilde{d}(\sigma) \right| \\ & \leq \left| d(\varepsilon) + \frac{1}{2} - \tilde{d}(\varepsilon) \right| + \left| \sum_{n=0}^{|\sigma|-1} (-1)^{\sigma(n+1)} \left( f(\sigma \upharpoonright n) - \tilde{f}(\sigma \upharpoonright n) \right) \right| \\ & \leq \frac{1}{4} + 2^{-3} \sum_{n=0}^{|\sigma|-1} 2^{-n} \\ & \leq \frac{1}{2}. \end{aligned}$$

It follows that  $d(\sigma) \leq \tilde{d}(\sigma) \leq d(\sigma) + 1$  for all strings  $\sigma$ , and so the  $\mathbb{Q}$ -computable martingale  $\tilde{d}$  succeeds on exactly those sequences which  $d$  succeeds on.  $\square$

**Convention.** We let *computable martingale* be short for  $\mathbb{Q}$ -computable martingale.

**Definition 2.24.** (i) A *computable null class* is a class  $\mathcal{C}$  such that there is a *computable martingale* which succeeds on  $\mathcal{C}$ .

(ii) A *computably random sequence* is a sequence  $X$  such that no *computable martingale* succeeds on  $X$ .

The following remark is an analog of Remark 2.16 and will also be useful in the constructions of random sequences in Chapter 6.

**Remark 2.25.** For every computable martingale there is another computable martingale  $d$  that succeeds on exactly the same sequences as the first martingale such that  $d$  succeeds by unbounded limit inferior on any sequence on which it succeeds at all. The construction of the martingale  $d$  is well-known and works, intuitively speaking, by putting aside one unit of capital every time the capital reaches a certain threshold, while from then on using the remainder of the capital in order to bet according to the initial martingale.  $\triangleleft$

## 2.4.2 Schnorr Randomness

In his arguments against Martin-Löf randomness and in favor of Schnorr randomness, Schnorr refers to the approaches via measure theory and via betting games. As to the measure theory approach, Schnorr claims that Martin-Löf tests cannot be interpreted as *effective* tests,

“[...] because for a Martin-Löf test  $A_0, A_1, \dots$  the values

$$\mu([A_n] \cap [\sigma]) 2^{|\sigma|} \quad (\sigma \in 2^{<\omega}, n \in \omega) \quad (2.10)$$

cannot be computed effectively in general. The above value is the probability that a sequence starting with  $\sigma$  is contained in the open neighborhood  $[A_n]$  of  $\bigcap_n [A_n]$ . It reveals to what extent [...]  $\sigma$  complies with the probability-one law which corresponds to the null class covered by  $(A_n)_{n \in \omega}$ .”<sup>2</sup>

To overcome the possible disadvantage Schnorr described, he proposed a notion of null classes with stronger constructivity properties. What he introduced as “total rekursive Sequentialtests” and “total rekursive Nullmengen” is nowadays referred to “Schnorr tests” and “Schnorr null classes”. Schnorr [38] remarked that his null class concept corresponds to the null class concept that had been studied by Brouwer [5] in the context of intuitionism.

**Definition 2.26** (Schnorr [38]). (i) A Schnorr test is a Martin-Löf test  $A_0, A_1, \dots$  such that  $\mu[A_0], \mu[A_1], \dots$  is a uniformly computable sequence of reals.

(ii) A class of sequences is a Schnorr null class if there is a Schnorr test which covers it.

(iii) A sequence is Schnorr random if it withstands every Schnorr test.

We note that often Schnorr tests are defined by requiring that the Martin-Löf test  $A_0, A_1, \dots$  satisfies  $\mu[A_n] = 2^{-n}$  for each  $n$ . Both definitions give rise to the same null classes, though.

Note that if  $(A_n)_n$  is a Schnorr test then all the values in (2.10) are uniformly computable. Indeed, fix  $\sigma$  and  $n$ , and consider an approximation  $\cup_s A_{n,s}$  of  $A_n$  by finite sets  $A_{n,s}$ . To get an approximation of  $\mu([A_n] \cap [\sigma]) 2^{|\sigma|}$  to within  $2^{-k}$  let  $s_0$  be the least  $s$  such that

$$\mu[A_n] - \mu[A_{n,s}] \leq 2^{-|\sigma|-k}.$$

---

<sup>2</sup>Taken from [42] and translated with adaptations to our terminology and notation. See also the discussion on tests below Definition 2.1.

Then  $\mu([A_{n,s_0}] \cap [\sigma]) 2^{|\sigma|}$  approximates  $\mu([A_n] \cap [\sigma]) 2^{|\sigma|}$  to within  $2^{-k}$ .

Aside from Martin-Löf randomness, Schnorr argued that computable randomness is not adequate, either. He argued that in order to recognize a sequence  $X$  as nonrandom based on the martingale approach it does not suffice that for a computable martingale  $d$  the sequence  $\{d(X \upharpoonright n) : n \in \omega\}$  is unbounded. Because “the growth of this sequence could be so slow that it is not recognizable to an observer who disposes only of effective methods” (cited from [42]). These remarks lead to the following definition.

**Definition 2.27** (Schnorr [40]). *A computable martingale  $d$  succeeds i.o.-strongly on a sequence  $X$  if there is a computable nondecreasing unbounded function  $h$  such that for infinitely many  $n$ ,  $d(X \upharpoonright n) > h(n)$ .*

It turns out that the Schnorr random sequences are exactly those sequences on which no computable martingale succeeds i.o.-strongly.

**Theorem 2.28** (Schnorr [40]). *A sequence  $X$  is Schnorr random if and only if no computable martingale succeeds i.o.-strongly on  $X$ .*

Finally, Downey and Griffiths gave a characterization of Schnorr randomness via incompressibility. To this end, they introduced *computable machines*, and showed a characterization of the Schnorr random sequences which is similar to the corresponding characterization of the Martin-Löf random sequences (Theorem 2.20).

**Definition 2.29** (Downey and Griffiths [13]). *A computable machine is a prefix-free machine  $M$  such that the halting probability of  $M$ ,  $\sum_{\sigma \in \text{dom } M} 2^{-|\sigma|}$ , is a computable real.*

**Theorem 2.30** (Downey and Griffiths [13]). *A sequence  $X$  is Schnorr random if and only if for all computable machines  $M$ ,*

$$(\exists c \in \omega)(\forall n \in \omega) K_M(X \upharpoonright n) \geq n - c.$$

Note that in contrast to Theorem 2.20, a quantification over computable machines is needed in Theorem 2.30 because there exists no “universal computable machine”.

A version of the following technical remark will be applied in Chapter 5.

**Remark 2.31.** Let  $M = M_e$  be a computable machine. Then there is a strictly increasing, computable function  $f$  such that

$$\mu[W_{e,f(0)}], \mu[W_{e,f(1)}], \dots$$

is a nondecreasing, effectively converging sequence of rationals with limit  $\mu[\text{dom } M]$ . Indeed, suppose that  $h$  is a computable function such that  $q_{h(0)}, q_{h(1)}, \dots$  is an effectively converging sequence of rationals with limit  $\mu[\text{dom } M]$ . Then a function  $f$  as above can be defined for example as follows, beginning with  $f(0) = 0$ . For each  $n > 0$ , let  $f(n)$  be the least  $s > f(n-1)$  such that  $\mu[W_{e,s}]$  is at least  $q_{h(n+2)} - 2^{-(n+1)}$ . It is easily verified that  $f$  is as required.  $\triangleleft$

## 2.5 Interrelations

In the following two theorems we summarize some of the characterization results cited above.

**Theorem 2.32** (Schnorr [41]). *For every sequence  $X$  the following are equivalent:*

- (i)  *$X$  is Martin-Löf random, i.e.,  $X$  withstands a universal Martin-Löf test.*
- (ii) *No (universal) subcomputable martingale succeeds on  $X$ .*
- (iii)  $(\exists c \in \omega)(\forall n \in \omega) K(X \upharpoonright n) \geq n - c$ .

**Theorem 2.33** (Schnorr [40]; Downey and Griffiths [13]). *For every sequence  $X$  the following are equivalent:*

- (i)  *$X$  is Schnorr random, i.e.,  $X$  withstands every Schnorr test.*
- (ii) *No computable martingale succeeds i.o.-strongly on  $X$ .*
- (iii) *For every computable machine  $M$ ,*

$$(\exists c \in \omega)(\forall n \in \omega) K_M(X \upharpoonright n) \geq n - c.$$

Similar to the above characterizations of Martin-Löf and Schnorr randomness we will obtain a “test characterization” and a “machine characterization” of computable randomness in Chapters 3 and 4, respectively.

Note that the following implications are an immediate consequence of the above theorems:

$$X \text{ Martin-Löf random} \Rightarrow X \text{ computably random} \Rightarrow X \text{ Schnorr random.}$$

Further, the first implication cannot be reversed as shown by Schnorr [41], and the second one cannot be reversed as shown by Wang [52]. For the

following theorem recall that a sequence  $X$  (interpreted as a set) is *high* if  $X' \geq_T \emptyset''$ , where  $'$  denotes the jump operator.

**Theorem 2.34** (Nies, Stephan, and Terwijn [35]). *For every sequence  $X$  the following are equivalent:*

- (i)  $X$  is high.
- (ii)  $\exists Y \equiv_T X$ ,  $Y$  is computably random but not Martin-Löf random.
- (iii)  $\exists Z \equiv_T X$ ,  $Z$  is Schnorr random but not computably random.

*Furthermore, the same equivalence holds if one considers c.e. reals.*

Referring to personal communication, Nies, Stephan, and Terwijn [35] remark that the fact that Schnorr and computable randomness can be separated by c.e. reals was independently proven by Downey and Griffiths.



## Chapter 3

# A Test Characterization of Computable Randomness

For both Martin-Löf and Schnorr randomness, which are defined via Martin-Löf and Schnorr tests, respectively, there are characterizations in terms of martingales and Kolmogorov complexity as shown in Theorems 2.32 and 2.33. In this chapter, we consider computable randomness, which is defined via martingales and we ask for a characterization in terms of (some suitably restricted class of) Martin-Löf tests.

In the first section, we review related work of Downey, Griffiths, and LaForte and we introduce bounded Martin-Löf tests.

Subsequently, we show in Section 3.2 that computable null classes are exactly those classes which are covered by bounded Martin-Löf tests. As a consequence, a sequence is computably random if and only if it withstands every bounded Martin-Löf test, which gives a positive answer to a question of Ambos-Spies and Kučera, who have asked whether computable randomness can be characterized in terms of Martin-Löf tests [1, Open Problem 2.6].

### 3.1 Related Work and Bounded Martin-Löf Tests

The first characterization of computable randomness in terms of tests is due to Downey, Griffiths, and LaForte [12], who introduced *computably graded Martin-Löf tests* and showed that a sequence is computably random if and only if it withstands all computably graded Martin-Löf tests. Later and independently, Merkle, Mihailović, and Slaman [32] found a similar characterization result which is formulated in different terms, though. They

introduced bounded Martin-Löf tests and proved a characterization of computable randomness via bounded Martin-Löf tests. Below we shall present the latter result, but first we review the characterization due to Downey, Griffiths, and LaForte [12].

**Definition 3.1** (Downey, Griffiths, and LaForte [12]). *A Martin-Löf test  $(A_n)_{n \in \omega}$  is computably graded if there is a computable function  $f : 2^{<\omega} \times \omega \rightarrow \mathbb{R}$  such that, for any  $n \in \omega$ ,  $\sigma \in 2^{<\omega}$ , and any prefix-free set of strings  $\{\sigma_i\}_{i \leq I}$  with  $\cup_{i=0}^I [\sigma_i] \subseteq [\tau]$ , the following conditions are satisfied:*

- (i)  $\mu([A_n] \cap [\sigma]) \leq f(\sigma, n)$ ;
- (ii)  $\sum_{i=0}^I f(\sigma_i) \leq 2^{-n}$ ;
- (iii)  $\sum_{i=0}^I f(\sigma_i) \leq f(\tau, n)$ .

**Theorem 3.2** (Downey, Griffiths, and LaForte [12]). *A sequence is computably random if and only if it withstands all computably graded Martin-Löf tests.*

Now we turn to the definition of *bounded Martin-Löf tests* [32].

**Definition 3.3.** *A martingale  $d$  has the effective savings property if there is a computable function  $f : 2^{<\omega} \rightarrow \mathbb{Q}^+ \cup \{0\}$  such that*

- (i)  $f(\sigma) \leq d(\sigma)$  for all strings  $\sigma$ ,
- (ii)  $f$  is nondecreasing, i.e., if  $\sigma \preceq \tau$  then  $f(\sigma) \leq f(\tau)$ ,
- (iii) for any sequence  $X$ ,  $d$  succeeds on  $X$  if and only if  $f$  is unbounded on the initial segments of  $X$ .

**Remark 3.4.** For every computable martingale  $d$  there is a computable martingale  $\tilde{d}$  with initial capital  $\tilde{d}(\varepsilon) = 1$  such that

- $\tilde{d}$  succeeds on exactly the same sequences as  $d$  and
- $\tilde{d}$  has the effective savings property.

Note that a martingale  $\tilde{d}$  as required can be constructed by using the construction method that is sketched in Remark 2.25.  $\triangleleft$

**Definition 3.5.** (i) *A mass distribution on Cantor space is a mapping  $\nu : 2^{<\omega} \rightarrow \mathbb{R}$  such that for any string  $\sigma$ ,  $\nu(\sigma) = \nu(\sigma 0) + \nu(\sigma 1)$ .*

- (ii) A mass distribution  $\nu$  is computable if it is rational-valued and there is a computable function  $D$  such that  $\nu(\sigma) = q_{D(\sigma)}$  for each string  $\sigma$ .
- (iii) A probability distribution (on Cantor space) is a mass distribution  $\nu$  where  $\nu(\varepsilon) = 1$ .

Formally, mass distributions and martingales are quite similar concepts (see Schnorr [40]) where the additivity condition  $\nu(\sigma) = \nu(\sigma 0) + \nu(\sigma 1)$  corresponds to the fairness condition (2.1). Observe that given a mass distribution  $\nu$ , the function  $\sigma \mapsto 2^{|\sigma|}\nu(\sigma)$  is a martingale with initial capital  $\nu(\varepsilon)$  and conversely, given a martingale  $d$ , the function  $\sigma \mapsto d(\sigma)/2^{|\sigma|}$  is a mass distribution.

**Definition 3.6.** A sequence  $(A_n)_{n \in \omega}$  of sets of strings is a bounded Martin-Löf test if it is uniformly computably enumerable and if there is a computable probability distribution  $\nu$  such that for any  $n \in \omega$  and for any string  $\sigma$ ,

$$\mu([A_n] \cap [\sigma]) \leq \frac{\nu(\sigma)}{2^n}. \quad (3.1)$$

To verify that a bounded Martin-Löf test is a Martin-Löf test indeed, simply let  $\sigma$  in (3.1) be the empty string.

Consider the values in (2.10), each of which Schnorr interprets as probability that a sequence with initial segment  $\sigma$  is contained in  $[A_n]$ . These conditional probabilities are uniformly computable for Schnorr tests, while for bounded Martin-Löf tests they have the following property. Given a Martin-Löf test  $A_0, A_1, \dots$  which is bounded via  $\nu$ , if we let  $d(\sigma) = 2^{|\sigma|}\nu(\sigma)$  for all strings  $\sigma$  then by the above discussion,  $d$  is a martingale. Consequently,  $d_n = 2^{-n}d$  is a martingale for each  $n$ . Consider for every  $n$  the function  $\pi_n$  defined by

$$\pi_n(\sigma) = \mu([A_n] \cap [\sigma])2^{|\sigma|},$$

where these values are the conditional probabilities in (2.10). It is easy to see that for all  $n \in \omega$ ,  $\sigma \in 2^{<\omega}$

$$\pi_n(\sigma) = \frac{1}{2}\pi_n(\sigma 0) + \frac{1}{2}\pi_n(\sigma 1),$$

hence each  $\pi_n$  is a martingale. By (3.1), every  $\pi_n$  is uniformly bounded from above by the computable martingales  $d_n$ .

## 3.2 Computable Randomness via Bounded Martin-Löf Tests

**Theorem 3.7.** *A class of sequences is a computable null class if and only if it is covered by a bounded Martin-Löf test. In particular, a sequence is computably random if and only if it withstands every bounded Martin-Löf test.*

*Proof.* First assume we are given a computable null class  $\mathcal{C} \subseteq 2^\omega$ . By Remark 3.4, pick a computable martingale  $d$  with initial capital 1 which succeeds on  $\mathcal{C}$  and has the effective savings property via some computable, nondecreasing function  $f$ . In order to obtain a bounded Martin-Löf test  $(A_n)_{n \in \omega}$  via some probability distribution  $\nu$  as required, let for all  $n$

$$A_n = \{ \sigma \in 2^{<\omega} : f(\sigma) \geq 2^n \}$$

(see (2.5)). Note that the sequence  $(A_n)_{n \in \omega}$  is uniformly computably enumerable. Consider the sets  $\text{Succ}^0(d), \text{Succ}^1(d), \dots$  defined in Lemma 2.8 (ii). Then obviously for each  $n$ ,  $A_n \subseteq \text{Succ}^{2^n}(d)$ . By Kolmogorov's inequality,  $\mu[A_n] \leq 2^{-n}$  and consequently,  $(A_n)_{n \in \omega}$  is a Martin-Löf test. In order to prove that  $\mathcal{C}$  is covered by  $(A_n)_{n \in \omega}$ , fix any sequence  $X$  in  $\mathcal{C}$ . Then  $d$  succeeds on  $X$  and, in particular,  $f$  is unbounded on the initial segments of  $X$ ; hence for all  $n \in \omega$  there is some prefix of  $X$  in  $A_n$  and  $X$  is contained in the intersection of the  $[A_n]$ .

In order to prove that the Martin-Löf test  $(A_n)_{n \in \omega}$  is bounded, let for every string  $\sigma$

$$\nu(\sigma) = \frac{d(\sigma)}{2^{|\sigma|}}.$$

The mapping  $\nu$  is a probability distribution, which follows immediately from the fairness condition (2.1). Fix any index  $n \in \omega$  and string  $\sigma$ . We have to show that (3.1) holds. First assume that  $\sigma$  has some prefix  $\sigma_0$  in  $A_n$ . In this case (3.1) holds because by construction and choice of  $f$ , we have

$$2^n \leq f(\sigma_0) \leq f(\sigma) \leq d(\sigma)$$

and hence

$$\mu([A_n] \cap [\sigma]) = \mu[\sigma] = 2^{-|\sigma|} = \frac{\nu(\sigma)}{d(\sigma)} \leq \frac{\nu(\sigma)}{2^n}. \quad (3.2)$$

Next consider the case where  $\sigma$  does not have a prefix in  $A_n$ . Let  $A_n^\sigma$  be a prefix-free subset of  $\{ \tau \in A_n : \sigma \preceq \tau \}$  such that  $[A_n^\sigma] = [\{ \tau \in A_n : \sigma \preceq \tau \}]$

(see Remark 2.7). Then (3.1) holds because

$$\mu([\sigma] \cap [A_n]) = \sum_{\tau \in A_n^\sigma} 2^{-|\tau|} \leq \sum_{\tau \in A_n^\sigma} \frac{\nu(\tau)}{2^n} \leq \frac{\nu(\sigma)}{2^n},$$

where the latter two inequalities hold by (3.2) and by Lemma 2.8 (i), respectively.

For the converse direction, assume that we are given a Martin-Löf test  $(A_n)_{n \in \omega}$  which is bounded via some probability distribution  $\nu$ . By the discussion following Definition 3.5, the function  $\sigma \mapsto \nu(\sigma)2^{|\sigma|}$  is a computable martingale which succeeds on any sequence in  $\mathcal{C}$  because by assumption any such sequence is contained in the intersection of the  $[A_n]$ , i.e., has prefixes in all the  $A_n$ , where for all strings  $\sigma$  in  $A_n$  we have  $\nu(\sigma)2^{|\sigma|} \geq 2^n$  according to (3.1).  $\square$



## Chapter 4

# Computable Randomness and Lowness Properties

We define a version of Kolmogorov complexity by introducing *bounded machines*, which is inspired by the test characterization of computable randomness via bounded Martin-Löf tests (Theorem 3.7). Using this result we show a machine characterization of computable randomness. In other words, we prove a characterization of the computably random sequences as to their initial segment complexities, similar to Theorems 2.20 and 2.30 for Martin-Löf and Schnorr randomness, respectively. More precisely, we show in Theorem 4.5 that a sequence  $X$  is computably random if and only if for every bounded machine  $M$ ,

$$(\exists c \in \omega) (\forall n \in \omega) K_M(X \upharpoonright n) \geq n - c.$$

Such machine characterizations regarding other randomness notions allowed for a study of certain lowness properties. We mention lowness for  $K$  and  $K$ -triviality in the setting of Martin-Löf randomness; for Schnorr randomness the analog concepts are lowness for computable machines (cf. Chapter 5) and Schnorr triviality. In Sections 4.2 and 4.3, we define and study analog notions in the setting of computable randomness: *lowness for bounded machines* and *bounded triviality*. We argue that lowness for bounded machines implies lowness for computable randomness. By a result due to Nies [34], a sequence is low for computable randomness if and only if it is computable. Hence, a sequence is low for bounded machines if and only if it is computable (Theorem 4.9).

Turning to bounded triviality, we observe some properties of bounded machines which may seem surprising. It turns out that there is a bounded

machine that has for each string  $0^n$  a program that is not longer than the shortest program for  $0^n$  with respect to a universal (unbounded) prefix-free machine. Further, bounded machines behave quite differently from both prefix-free machines and computable machines with respect to different representations of natural numbers (see Remarks 4.11 and 4.17). These observations suggest to study another notion: *weakly boundedly trivial* sequences. We show that these sequences form a superclass of the boundedly trivial sequences and a subclass of the K-trivials (Theorem 4.19).

## 4.1 A Characterization of Computable Randomness in terms of Bounded Machines

Inspired by the bounded tests defined in Section 3.1, we introduce bounded machines and we show a machine characterization of computable randomness in terms of bounded machines. As discussed in Section 3.1, Downey, Griffiths, and LaForte [12] gave a first test characterization of computable randomness in terms of graded tests. We remark that in the introductory section of [12] Downey, Griffiths, and LaForte note that their test characterization of computable randomness “could be turned into a machine one also”, yet without elaborating this further.

The following lemma is a standard tool, that can be applied in a variety of situations; for examples of applications that are similar to ours see the machine characterizations of Schnorr and of Kurtz randomness by Downey and Griffiths [13] and by Downey, Griffiths, and Reid [14], respectively. A proof of Lemma 4.1 can be found for instance in the forthcoming monograph of Downey and Hirschfeldt [15]. The latter authors attribute the lemma in its effective version (which we use) to Levin and Chaitin.

**Lemma 4.1** (Kraft-Chaitin Theorem). *Let  $\langle d_0, \sigma_0 \rangle, \langle d_1, \sigma_1 \rangle, \dots$  be a computably enumerable set of “axioms”, where an axiom denotes a pair  $\langle d, \sigma \rangle$  consisting of a number  $d$  (called length) and a string  $\sigma$ . Furthermore suppose that  $\sum_i 2^{-d_i} \leq 1$  (in this case the list of axioms is called a Kraft-Chaitin set). Then there is a prefix-free machine  $M$  with  $\text{dom } M = \{\tau_0, \tau_1, \dots\}$  such that  $|\tau_i| = d_i$  and  $M(\tau_i) = \sigma_i$  for each  $i$ .*

For further use, we note some well-known facts in the following proposition.

**Proposition 4.2.** *One can pass effectively from any Martin-Löf test  $(A_n)_{n \in \omega}$  to a Martin-Löf test  $(B_n)_{n \in \omega}$  such that for each  $n$ ,*



- the component  $B_n$  is prefix-free,
- $|\sigma| \geq n$  for all  $\sigma \in B_n$ ,
- and  $[A_n] = [B_n]$ .

*Proof.* First note that we may assume that each  $A_n$  contains no string of length less than  $n$ . (Otherwise, instead of enumerating a string  $\sigma$  of length less than  $n$  into  $A_n$ , put all strings  $\sigma' \succ \sigma$  of length  $n$  into  $A_n$ .) Now it suffices to argue that there is an effective procedure which, given an index  $e$  of a c.e. set  $A = W_e$  containing only strings of length  $n$  or greater, enumerates a prefix-free set  $B = \cup_s B_s$  such that  $[A] = [B]$ . We may assume that  $|W_{n,s+1} - W_{n,s}| \leq 1$  for all  $n, s$ . Let  $B_0 = W_{e,0}$ . If, for  $s > 0$ ,  $W_{e,s} - W_{e,s-1} = \emptyset$  then simply let  $B_s = B_{s-1}$ . Otherwise let  $\sigma$  denote the single element in  $W_{e,s} - W_{e,s-1}$  and consider the following three cases. In the first case  $\sigma$  is incomparable to any element of  $B_{s-1}$ ; then let  $B_s = B_{s-1} \cup \{\sigma\}$ . We have the second case if there is an element of  $B_{s-1}$  which is a prefix of  $\sigma$ ; here we let  $B_s = B_{s-1}$ . For the remaining case consider the maximal length  $\ell$  of the (finitely many) strings in  $B_{s-1}$  and let  $B_s = B_{s-1} \cup \{\tau \in 2^\ell : \tau \succeq \sigma \text{ and no string in } B_{s-1} \text{ is a prefix of } \tau\}$ .  $\square$

**Remark 4.3.** It is straightforward that if  $(A_n)_{n \in \omega}$  is a Martin-Löf test which is bounded via a computable probability distribution  $\nu$ , then the test  $(B_n)_{n \in \omega}$  constructed in Proposition 4.2 is also bounded via  $\nu$ .  $\triangleleft$

If  $M$  is a prefix-free machine, let  $S_n^M$  for  $n \in \omega$  denote the set containing every string whose length exceeds its  $M$ -complexity by at least  $n$ , i.e.,

$$S_n^M = \{\sigma \in 2^{<\omega} : K_M(\sigma) \leq |\sigma| - n\}.$$

Note that the sets  $S_n^M$  are similar to the components of the test which is constructed to characterize Martin-Löf random sequences by their initial segment complexity, see (2.9). Observe that for any given  $M$ , the sets  $S_n^M$  are uniformly computably enumerable. Further,

$$\mu[S_n^M] \leq \sum_{\sigma \in S_n^M} \mu[\sigma] = \sum_{\sigma \in S_n^M} 2^{-|\sigma|} \leq \frac{1}{2^n}$$

because  $M$  is a prefix-free machine.

**Definition 4.4.** A bounded machine is a prefix-free machine  $M$  such that there is a computable probability distribution  $\nu$  satisfying

$$(\forall \sigma \in 2^{<\omega}) (\forall n \in \omega) \left[ \mu\left([S_n^M] \cap [\sigma]\right) \leq \frac{\nu(\sigma)}{2^n} \right]. \quad (4.1)$$

In other words, a prefix-free machine  $M$  is a bounded machine iff the family of the sets  $S_n^M = \{\sigma \in 2^{<\omega} : K_M(\sigma) \leq |\sigma| - n\}$  is a bounded Martin-Löf test.

**Theorem 4.5.** *A sequence  $X$  is computably random if and only if for each bounded machine  $M$ ,*

$$(\exists c \in \omega) (\forall n \in \omega) [K_M(X \upharpoonright n) \geq n - c]. \quad (4.2)$$

*Proof.* We show that a sequence  $X$  withstands all bounded Martin-Löf tests if and only if (4.2) holds for every bounded machine. For the *If* part, suppose that  $(A_n)_n$  is a Martin-Löf test which is bounded via a computable probability distribution  $\nu$  such that  $X \in \bigcap_n [A_n]$ . By Proposition 4.2 and Remark 4.3 we may assume that each  $A_n$  is a prefix-free set  $\bigcup_{i \in I_n} \{\sigma_{n,i}\}$ , where  $I_n$  is some (finite or infinite) set of numbers, such that  $|\sigma_{n,i}| \geq n$  for each  $i \in I_n$ . We enumerate a list of axioms which, by the Kraft-Chaitin Theorem, ensures that there is a prefix-free machine  $M$  that, for each  $n$  and  $i$ , outputs  $\sigma_{2(n+1),i}$  on some input of length  $|\sigma_{2(n+1),i}| - n$ . Indeed, the enumerated list of axioms  $\{(|\sigma_{2(n+1),i}| - n, \sigma_{2(n+1),i}) : n \in \omega, i \in I_n\}$  is a Kraft-Chaitin set because

$$\begin{aligned} \sum_{n \in \omega, i \in I_n} 2^{-(|\sigma_{2n+2,i}| - n)} &= \sum_n (2^n \sum_i 2^{-|\sigma_{2n+2,i}|}) \\ &= \sum_n 2^n \mu[A_{2n+2}] \\ &\leq \sum_n 2^n 2^{-(2n+2)} \leq \frac{1}{2}. \end{aligned}$$

To show that the machine  $M$  is bounded note that for each  $n$ , we have

$$S_n^M = \bigcup_{k \geq 1} \bigcup_{i \in I_n} \{\sigma_{2(n+k),i}\}$$

and thus for each  $\sigma$  and  $n$ ,

$$[S_n^M] \cap [\sigma] = \bigcup_{k \geq 1} \bigcup_{i \in I_n} [\sigma_{2(n+k),i}] \cap [\sigma] = \bigcup_{k \geq 1} [A_{2(n+k)}] \cap [\sigma].$$

Hence

$$\begin{aligned} \mu([S_n^M] \cap [\sigma]) &\leq \sum_{k \geq 1} \mu([A_{2(n+k)}] \cap [\sigma]) \leq \sum_{k \geq 1} \frac{\nu(\sigma)}{2^{2(n+k)}} \\ &\leq \frac{\nu(\sigma)}{2^{2n}} \sum_{k \geq 1} 2^{-2k} \leq \frac{\nu(\sigma)}{2^n}. \end{aligned}$$

This shows that (4.1) is satisfied, so  $M$  is bounded. In order to argue that (4.2) does not hold let a number  $c$  be given. By choice of  $X$  there is a (unique)  $i$  such that  $X \in [\sigma_{2^{c+2}, i}]$ . If  $n = |\sigma_{2^{c+2}, i}|$  then by construction we have  $K_M(X \upharpoonright n) \leq n - (c + 1)$ , hence (4.2) does not hold.

For the converse direction, suppose  $M$  is a bounded machine via a computable probability distribution  $\nu$  such that  $(\forall c)(\exists n)[K_M(X \upharpoonright n) < n - c]$ . We define a bounded Martin-Löf test  $(A_k)_k$  which covers  $X$  by letting  $A_k$  be equal to  $S_k^M$  for each  $k$ . Indeed, if  $\sigma \in 2^{<\omega}$  then, by (4.1),

$$\mu([A_k] \cap [\sigma]) = \mu([S_k^M] \cap [\sigma]) \leq \frac{\nu(\sigma)}{2^k}$$

for each  $k$ , hence  $(A_k)_k$  is a bounded Martin-Löf test. Furthermore, by hypothesis, for all  $k$  there exists an  $n$  such that  $X \upharpoonright n \in S_k^M$ , and thus  $X \in \cap_k [A_k]$ .  $\square$

We remark that there is no *universal bounded machine*, i.e., there is no bounded machine  $M$  such that for all bounded machines  $N$ ,

$$(\exists c \in \omega) (\forall \sigma \in 2^{<\omega}) K_M(\sigma) \leq K_N(\sigma) + O(1).$$

If such a universal bounded machine existed, one could obtain a *universal computable martingale* from it, i.e., a computable martingale that succeeds on all sequences which are not computably random. It is known, though, that no universal computable martingale exists. Hence, contrary to the Martin-Löf case (Theorem 2.20) but similar to the Schnorr case with computable machines (Theorem 2.30), a quantification over bounded machines is necessary in Theorem 4.5.

## 4.2 Lowness for Bounded Machines

Recall from the Section 2.3 that  $U$  denotes the universal prefix-free machine with respect to which the prefix-free Kolmogorov complexity  $K$  is defined. If  $U$  may use an oracle  $X$ , we get a relativized version  $K^X$  of  $K$  (see Subsection 4.2.1 for definitions). A sequence  $X$  is called *low for*  $K$  if  $K(\sigma) \leq K^X(\sigma) + O(1)$  for every string  $\sigma$ . In a central set of results, Nies and Hirschfeldt prove the equivalence of lowness for  $K$  with further lowness properties (see Theorem 4.12).

Lowness for computable machines as an analog of lowness for  $K$  in the setting of Schnorr randomness was studied by Downey, Greenberg, Mihailović, and Nies [11] (see Chapter 5).

In Subsection 4.2.2 we investigate lowness for bounded machines.

### 4.2.1 Relativized Kolmogorov Complexity

Similar to Definition 2.17, we say that an oracle Turing machine  $M$  is *prefix-free* if for all oracles  $X \in 2^\omega$ , the domain of  $M^X$  is a prefix-free set. In this case let the  $M^X$ -complexity of a string  $\sigma$  be

$$K_M^X(\sigma) = \min \{|\tau| : M^X(\tau) = \sigma\}.$$

We define a *universal prefix-free oracle machine*  $U$  by requiring, similar to (2.7), that

$$(\forall e \in \omega)(\forall X \in 2^\omega)(\forall \tau \in 2^{<\omega}) U^X(1^e 0 \tau) \simeq \Phi_e(X, \tau).$$

Further, on inputs which do not have the form  $1^e 0 \tau$  as above,  $U$  will diverge for all oracles. We call the machine  $U$  universal because for every prefix-free oracle machine  $M$ ,

$$(\exists c \in \omega)(\forall X \in 2^\omega)(\forall \sigma \in 2^{<\omega}) K_U^X(\sigma) \leq K_M^X(\sigma) + c$$

(see (2.6)). Note that the *coding constant*  $c$  does not depend on the oracle. Now we fix a universal prefix-free oracle machine  $U$  as above and we define the *prefix-free Kolmogorov complexity*  $K^X(\sigma)$  relative to an oracle  $X$  of a string  $\sigma$  by  $K^X(\sigma) = K_U^X(\sigma)$ .

### 4.2.2 No Noncomputable Sequence is Low for Bounded Machines

If  $X$  is a sequence, then an  $X$ -bounded machine is a prefix-free oracle Turing machine  $M$  such that there is an  $X$ -computable probability distribution  $\nu$  with

$$(\forall \sigma \in 2^{<\omega})(\forall n \in \omega) \left[ \mu \left( \left[ S_n^{M^X} \right] \cap [\sigma] \right) \leq \frac{\nu(\sigma)}{2^n} \right].$$

**Definition 4.6.** *A sequence  $X$  is low for bounded machines if for all  $X$ -bounded machines  $M$  there is a bounded machine  $N$  such that for all  $\sigma \in 2^{<\omega}$ ,  $K_N(\sigma) \leq K_M^X(\sigma) + O(1)$ .*

Given an  $X$ -bounded machine  $M$ , it is not true in general that  $M$  is  $Y$ -bounded for every oracle  $Y$ . Indeed, suppose  $M$  is divergent on all inputs given any oracle  $X$  with  $X(0) = 1$ ; on the other hand, if  $Y(0) = 0$  then let  $M^Y$  behave like the universal prefix-free machine  $U$ . While for all oracles  $X$  of the former type,  $M$  is trivially  $X$ -bounded, we have that  $M^Y$  is not  $Y$ -bounded for  $Y = \emptyset$ . For if otherwise, Martin-Löf randomness and computable randomness would be equivalent by Theorems 2.20 and 4.5. Nevertheless we have the following proposition.

**Proposition 4.7.** *For every sequence  $X$  and for every  $X$ -bounded machine  $M$  there is an oracle Turing machine  $\widetilde{M}$  such that  $M^X \simeq \widetilde{M}^X$  and  $\widetilde{M}$  is a  $Y$ -bounded machine for every oracle  $Y$ .*

*Proof sketch.* Let  $M$  be a machine which is  $X$ -bounded via an  $X$ -computable probability distribution  $\nu$ . We may assume that  $M^Y$  is prefix-free for every oracle  $Y$ . Let  $N$  be an oracle machine such that  $N$  with oracle  $X$  computes  $\nu$ . Let  $\nu^Y$  denote the partial function computed by the machine  $N$  with oracle  $Y$ . Now for any oracle  $Y$ , the machine  $\widetilde{M}^Y$  on input  $x$  executes the following instructions. At stage  $t$ , wait for  $\nu^Y$  to converge on all strings  $\sigma$  of length  $t$ . Then check if all of the following conditions are satisfied:

- In case  $t = 0$ :  $\nu^Y(\varepsilon) = 1$ .
- In case  $t > 0$ , for all  $\tau$  of length  $t - 1$ :  $\nu^Y(\tau) = \nu^Y(\tau 0) + \nu^Y(\tau 1)$ .
- Let  $S_n^{M_t^Y} = \{\tau \in 2^{<\omega} : \text{there is a } \rho, |\rho| \leq |\tau| - n \text{ such that } M^Y \text{ on input } \rho \text{ outputs } \tau \text{ in at most } t \text{ steps}\}$ . Then

$$(\forall \sigma, |\sigma| \leq t) (\forall n \leq t) \mu \left( \left[ S_n^{M_t^Y} \right] \cap [\sigma] \right) \leq \frac{\nu^Y(\sigma)}{2^n}.$$

If the preceding conditions are not all satisfied then let the computation diverge. Else simulate  $t$  many steps of the computation of  $M^Y$  on input  $x$ . If this simulated computation converges then output  $M^Y(x)$  and halt, else move to stage  $t + 1$ .  $\square$

In connection with Theorems 4.8 and 4.9 we recall the definitions of  $X$ -computably random sequences and of lowness for computable randomness. A sequence is  *$X$ -computably random* if no  $X$ -computable martingale succeeds on it.  $X$  is called *low for computable randomness* if each computably random sequence is  $X$ -computably random.

**Theorem 4.8** (Nies [34]). *A sequence is low for computable randomness if and only if it is computable.*

**Theorem 4.9.** *A sequence is low for bounded machines if and only if it is computable.*

*Proof Idea.* We can verify that Theorem 4.5 relativizes, where we make use of a relativized version of the Kraft-Chaitin Theorem (Lemma 4.1). So a sequence  $Z$  is  $X$ -computably random if and only if for each  $X$ -bounded machine  $M$ , for all  $n$ ,  $K_M^X(Z \upharpoonright n) \geq n - O(1)$ . Therefore, if a sequence is low for bounded machines then it is low for computable randomness. The proof is completed by applying Theorem 4.8.  $\square$

### 4.3 Triviality

The prefix-free Kolmogorov complexity  $K$  was defined on strings but it is also common to write  $K(n)$  for numbers  $n$ . Here we identify as usual the  $n$ th string  $\sigma_n$  in the length-lexicographic ordering of all strings with the number  $n$ , and so we have  $K(n) = K(\sigma_n)$ .

However, there are several ways to define  $K(n)$ , depending on the way one wishes to represent numbers by strings. Another possibility is for example to consider  $K(0^n)$ . The two possibilities do not differ substantially since

$$K(n) \leq K(0^n) + O(1) \leq K(n) + O(1), \quad (4.3)$$

and we do not care about additive constants here. For other representations one gets similar relations which is a consequence of the following well-known lemma.

**Lemma 4.10.** *If  $g$  is a computable function from strings to strings then  $K(g(\sigma)) \leq K(\sigma) + O(1)$  for each string  $\sigma$ .*

**Remark 4.11.** It is easy to verify that the following analog of Lemma 4.10 for computable machines is true: If  $g$  is a computable function from strings to strings and if  $M$  is a computable machine, then there is a computable machine  $N$  such that  $K_N(g(\sigma)) \leq K_M(\sigma) + O(1)$  for each string  $\sigma$ . Hence the choice of the representation of numbers does not matter in the computable machine setting, either. For further reference, we mention that in particular we have the following analog of (4.3).

For each computable machine  $M$  there is a computable machine  $N$  such that  $(\forall n \in \omega) K_N(n) \leq K_M(0^n) + O(1)$ . (4.4)

◁

By definition, a sequence  $X$  is *K-trivial* if for all  $n$ ,  $K(X \upharpoonright n) \leq K(n) + O(1)$ . I.e., all initial segments of  $X$  have minimal prefix-free complexity (within an additive constant). The class of K-trivials was introduced by Chaitin [9]. A central set of results in the theory of algorithmic randomness were proved by Nies and Hirschfeldt. They show the equivalence of a number of “anti-randomness” properties, including the property of being K-trivial. To define another such property called *lowness for Martin-Löf randomness*, one considers Martin-Löf tests relative to an oracle. Namely, a sequence of sets  $A_0, A_1, \dots$  is called a *Martin-Löf test relative to an oracle  $X$*  if there is a computable function  $g$  such that for each  $n$ ,  $A_n = W_{g(n)}^X$  and  $\mu[A_n] \leq 2^{-n}$ . Further, a sequence  $Y$  is *Martin-Löf random relative to an oracle  $X$*  if  $Y$

withstands every Martin-Löf test relative to  $X$ , i.e., if for every Martin-Löf test  $A_0^X, A_1^X, \dots$  relative to  $X$ , we have that  $Y \notin \bigcap_n [A_n^X]$ . Now a sequence  $X$  is called *low for Martin-Löf randomness* if each Martin-Löf random sequence is Martin-Löf random relative to  $X$ .

**Theorem 4.12** (Nies and Hirschfeldt [34]). *For every sequence  $X \in 2^\omega$ , the following are equivalent:*

- $X$  is low for  $K$ .
- $X$  is  $K$ -trivial.
- $X$  is low for Martin-Löf randomness.

We briefly mention  $K$ -reducibility. A sequence  $X$  is  *$K$ -reducible* to a sequence  $Y$  if for all  $n$ ,  $K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1)$ ; in this case we write  $X \leq_K Y$ . Then  $X$  is  $K$ -trivial iff  $X \leq_K 0^\omega$ .

An analog of  $K$ -triviality in the case of Schnorr randomness was introduced by Downey and Griffiths [13]. They define *Schnorr reducibility* as follows. A sequence  $X$  is Schnorr reducible to a sequence  $Y$  if for every computable machine  $M$  there is a computable machine  $N$  such that  $K_N(X \upharpoonright n) \leq K_M(Y \upharpoonright n) + O(1)$ . Now a sequence  $X$  is *Schnorr trivial* if  $X$  is Schnorr reducible to  $0^\omega$ . For results on Schnorr triviality see [12, 13, 19]. We study an analog triviality concept based on bounded machines.

**Definition 4.13.** (i) *A sequence  $X$  is boundedly reducible to a sequence  $Y$  if for every bounded machine  $M$  there is a bounded machine  $N$  such that*

$$(\exists c \in \omega)(\forall n \in \omega) [K_N(X \upharpoonright n) \leq K_M(Y \upharpoonright n) + c].$$

*In this case we write  $X \leq_{\text{bnd}} Y$ .*

(ii) *A sequence  $X$  is boundedly trivial if  $X \leq_{\text{bnd}} 0^\omega$ .*

As the following proposition shows, there is a characterization of bounded triviality where the prefix-free Kolmogorov complexity  $K$  is involved. As for the proof, we recall that the letter  $U$  denotes the universal prefix-free machine relative to which the prefix-free Kolmogorov complexity  $K = K_U$  is defined.

**Proposition 4.14.** (i) *There is a bounded machine  $U_b$  such that*

$$K_{U_b}(0^n) \leq K(n) + O(1).$$

(ii) A sequence  $X$  is boundedly trivial if and only if there is a bounded machine  $N$  such that

$$(\exists c \in \omega)(\forall n \in \omega) [K_N(X \upharpoonright n) \leq K(n) + c].$$

*Proof.* Clearly, (ii) follows from (i). To prove (i), we define a prefix-free machine  $U_b$  by letting  $U_b(\tau) = 0^{|\tau|}$  for all  $\tau \in \text{dom}(U)$ . Obviously,  $U_b$  is bounded via the probability measure  $\nu$  which is given by  $\nu(0^n) = 1$  for each  $n$ . By construction, we have  $K_{U_b}(0^n) \leq K(0^n) \leq K(n) + O(1)$  for all  $n$ .  $\square$

**Corollary 4.15.** *Every computable sequence is boundedly trivial.*

*Proof.* Suppose  $X$  is computable. Similar to the proof of Proposition 4.14, we define a bounded machine  $N$  as follows. For all  $\tau \in \text{dom}(U)$ , determine the number  $n$  such that  $U(\tau)$  is the  $n$ th string in  $2^{<\omega}$ , and let  $N(\tau) = X \upharpoonright n$ . Clearly,  $N$  is bounded via the probability distribution  $\nu$  given by  $\nu(\sigma) = 1$  for all  $\sigma \prec X$ , and we have  $K_N(X \upharpoonright n) \leq K(n) + O(1)$ .  $\square$

The next corollary follows from Proposition 4.14 (ii) but will also be an immediate consequence of Theorem 4.19.

**Corollary 4.16.** *Every boundedly trivial sequence is  $K$ -trivial.*

**Remark 4.17.** There exists no bounded machine  $M$  such that

$$K_M(n) \leq K(n) + O(1).$$

Otherwise, by Theorem 4.5 we would have that a sequence is computably random if and only if it is Martin-Löf random. Hence there is no bounded machine  $N$  such that  $K_N(n) \leq K_{U_b}(0^n) + O(1)$  for all  $n$ , where  $U_b$  denotes the bounded machine from Proposition 4.14 (i). It follows that an analog of (4.4) is not true for bounded machines, and therefore an analog of Lemma 4.10 for bounded machines is also false (see Remark 4.11).  $\triangleleft$

The above remark suggests the following definition.

**Definition 4.18.** *A sequence  $X$  is weakly boundedly trivial if for every bounded machine  $M$  there is a bounded machine  $N$  such that*

$$(\exists c \in \omega)(\forall n \in \omega) [K_N(X \upharpoonright n) \leq K_M(n) + c].$$

In the following theorem, an interesting relation to the  $K$ -trivial sequences is established.



**Theorem 4.19.** (i) *Every boundedly trivial sequence is weakly boundedly trivial.*

(ii) *Every weakly boundedly trivial sequence is K-trivial.*

**Lemma 4.20** (folklore). *For every sequence  $X$ , if  $a_0, a_1, \dots$  is an increasing and unbounded computable sequence of numbers, and if  $K(X \upharpoonright a_n) \leq K(a_n) + O(1)$ , then  $X$  is K-trivial.*

*Proof.* We define a prefix-free machine  $M$  as follows. If  $M$  finds a  $\tau \in \text{dom}(U)$ , it checks whether there is an index  $n$  such that  $a_n = |U(\tau)|$ . If there exists such an  $n$ , which by hypothesis is not greater than  $a_n$ , the machine  $M$  outputs  $U(\tau) \upharpoonright n$ . It follows that for all  $n$

$$K(X \upharpoonright n) \leq K(X \upharpoonright a_n) + O(1) \leq K(a_n) + O(1) \leq K(n) + O(1).$$

□

*Proof of Theorem 4.19.* Item (i) is an immediate consequence of Proposition 4.14 (ii). We prove item (ii). Considering our standard representation of numbers by strings, where the  $n$ th string  $\sigma_n$  in the length-lexicographic ordering is identified with the number  $n$ , we let  $a_0, a_1, a_2, a_3 \dots$  denote the numbers that correspond to the strings  $\lambda, 0, 0^2, 0^3 \dots$ . Now let  $U_b$  denote the machine from Proposition 4.14 (i). If  $X$  is a weakly boundedly trivial sequence, then there is a bounded machine  $N$  such that for all  $n$ ,  $K_N(X \upharpoonright n) \leq K_{U_b}(n) + O(1)$ . So we have

$$K(X \upharpoonright a_n) \leq K_N(X \upharpoonright a_n) + O(1) \leq K_{U_b}(a_n) + O(1) \leq K(a_n) + O(1).$$

Hence by Lemma 4.20,  $X$  is K-trivial. □



## Chapter 5

# Schnorr Randomness: Lowness for Computable Machines

By Theorems 4.5 and 2.30, computably random sequences and Schnorr random sequences can be characterized w.r.t. their initial segment complexities via bounded and computable machines, respectively. While in Section 4.2, a lowness notion for bounded machines was investigated, we define in this chapter a lowness notion for computable machines. We show that the sequences which are low for computable machines are exactly the computably traceable sequences. Thus by known results, lowness for computable machines is equivalent to other lowness notions with respect to Schnorr randomness. Namely, a sequence  $X$  is low for computable machines iff  $X$  is low for Schnorr tests iff  $X$  is low for Schnorr randomness. The latter two properties and the definition of computable traceability are reviewed in Section 1, where we also introduce lowness for computable machines. In Section 2, we prove the above mentioned coincidence of the class of sequences which are low for computable machines and the class of computably traceable sequences.

### 5.1 Lowness Notions for Schnorr Randomness

Below Remark 4.11, we reviewed the definition of lowness for Martin-Löf randomness via relativizations of Martin-Löf tests and of Martin-Löf randomness. In what follows, we recapitulate similar lowness notions for Schnorr randomness, and we introduce lowness for computable machines.

In the following definition we use a straightforward relativized version of uniformly computable sequences of reals (see Definition 2.21).

**Definition 5.1.** *Let  $X \in 2^\omega$ .*

- (i) *A Schnorr test relative to  $X$  is a Martin-Löf test  $A_0^X, A_1^X, \dots$  relative to  $X$  such that  $\mu[A_0^X], \mu[A_1^X], \dots$  is a uniformly  $X$ -computable sequence of reals.*
- (ii) *A sequence is Schnorr random relative to  $X$  if it withstands every Schnorr test relative to  $X$ .*

**Definition 5.2.** *A sequence  $X$  is low for Schnorr tests if for every Schnorr test  $A_0^X, A_1^X, \dots$  relative to  $X$  there is a Schnorr test  $B_0, B_1, \dots$  such that  $\bigcap_n [A_n^X] \subseteq \bigcap_n [B_n]$ .*

Recall from Subsection 1.2 that for each  $n \in \omega$ ,  $D_n$  denotes the finite set whose canonical index is  $n$ .

**Definition 5.3** (Terwijn and Zambella [45]). *A sequence  $X \in 2^\omega$  is computably traceable if there is a computable function  $h$  such that the following condition is satisfied. For all functions  $g \leq_T X$ , there is a computable function  $r$  such that for the finite sets  $D_{r(n)}$  we have that  $|D_{r(n)}| \leq h(n)$  and  $g(n) \in D_{r(n)}$ .*

We remark that, as noticed by Terwijn and Zambella, if  $X$  is computably traceable then for the witnessing function  $h$  we can choose any computable, nondecreasing and unbounded function.

**Theorem 5.4** (Terwijn and Zambella [45]). *A sequence  $X$  is low for Schnorr tests if and only if  $X$  is computably traceable.*

We note that while all K-trivials are  $\Delta_2^0$  by a result of Chaitin [9], the computably traceable sequences are all of hyperimmune-free degree, and there are  $2^{\aleph_0}$  many of them.

**Definition 5.5.** *A sequence  $X$  is low for Schnorr randomness if each Schnorr random sequence is Schnorr random relative to  $X$ .*

It is not hard to see that if a sequence is low for Schnorr tests then it is also low for Schnorr randomness. Whether the converse also holds was an open question of Ambos-Spies and Kučera [1]. It was answered in the affirmative by Kjos-Hanssen, Nies, and Stephan.

**Theorem 5.6** (Kjos-Hanssen, Nies, and Stephan [23]). *A sequence is low for Schnorr randomness if and only if it is low for Schnorr tests.*

Note that if we consider an analog notion of lowness for Martin-Löf tests then the equivalence of that notion to lowness for Martin-Löf randomness is easy to prove. One direction is trivial as above in the Schnorr case. The other direction is a consequence of the following fact which is an analog of Theorem 2.3: Given some sequence  $X$  there is a Martin-Löf test  $U_0^X, U_1^X, \dots$  relative to  $X$  which is universal for all Martin-Löf tests relative to  $X$ , i.e., if  $A_0^X, A_1^X, \dots$  is a Martin-Löf test relative to  $X$  then  $\bigcap_n [A_n^X] \subseteq \bigcap_n [U_n^X]$ .

We shall introduce another lowness notion for Schnorr randomness which is an analog of K-triviality in the setting of Martin-Löf randomness. For any  $X \in 2^\omega$ , an  $X$ -computable machine is a prefix-free Turing machine  $M$  such that  $\mu[\text{dom } M]$  is an  $X$ -computable real.

**Definition 5.7.** *A sequence  $X \in 2^\omega$  is low for computable machines if for all  $X$ -computable machines  $M$  there is a computable machine  $N$  such that for all  $n$ ,*

$$K_N(n) \leq K_M^X(n) + O(1).$$

Similar to the reasoning before Proposition 4.7, we can argue that an  $X$ -computable machine  $M$  need not be  $Y$ -computable for all oracles  $Y$ . However we have the following proposition, the proof of which uses a straightforward relativized version of Remark 2.31.

**Proposition 5.8.** *For every sequence  $X$  and for every  $X$ -computable machine  $M$  there is an oracle Turing machine  $\widetilde{M}$  such that  $M^X \simeq \widetilde{M}^X$  and  $\widetilde{M}$  is a  $Y$ -computable machine for every oracle  $Y$ .*

*Proof sketch.* Let  $X$  be a sequence and let  $M = M_e$  be an  $X$ -computable machine. We may assume that for every oracle  $Y$ ,  $M^Y$  is prefix-free, i.e.,  $W_e^Y$  is a prefix-free set. We define the oracle Turing machine  $\widetilde{M}$  as follows. Let  $F : 2^\omega \times \omega \rightarrow \omega$  be a partial computable functional with  $F(Y, 0) = 0$  for all  $Y \in 2^\omega$  such that for every  $n > 0$ ,  $F(X, n)$  is defined and greater than  $F(X, n - 1)$ , and

$$\mu[\text{dom } M^X] - \mu[W_{e, F(X, n)}^X] \leq 2^{-n}.$$

Now for any oracle  $Y$ , the machine  $\widetilde{M}^Y$  on input  $x$  executes the following instructions. At stage  $n > 0$ , first wait for  $F(Y, n)$  to converge. If  $F(Y, n) <$

$F(Y, n - 1)$  then let the computation diverge. Else continue as follows. If the condition

$$(\forall m < n) \mu \left[ W_{e, F(Y, n)}^Y \right] - \mu \left[ W_{e, F(Y, m)}^Y \right] \leq 2^{-m} \quad (5.1)$$

is satisfied then do the following: If  $x \in W_{e, F(Y, n)}^Y$  then output  $M^Y(x)$  and stop, else move to stage  $n + 1$ . (Note that  $x < F(Y, n)$  whenever  $x \in W_{e, F(Y, n)}^Y$ .) On the other hand, if (5.1) is not satisfied then let the computation diverge. Note that the construction is uniform in  $M, F$  but not in  $M$  alone.  $\square$

Recall that a sequence  $X$  is *Schnorr trivial* if for every computable machine  $M$  there is a computable machine  $N$  such that for all  $n$ ,

$$K_N(X \upharpoonright n) \leq K_M(n) + O(1)$$

(see page 43). This notion was initially explored by Downey and Griffiths [13] and Downey, Griffiths and LaForte [12], who showed that this class does not coincide with the sequences that are low for Schnorr randomness. For instance, there are Turing complete Schnorr trivial sequences.

In the next section we show that unlike the situation for triviality, the coincidence of the sequences low for Martin-Löf randomness and the low for  $K$  ones carries over to the Schnorr case (cf. Theorem 4.12).

## 5.2 Equality of Three Lowness Classes

**Theorem 5.9.** *A sequence  $X$  is low for computable machines if and only if  $X$  is computably traceable.*

**Corollary 5.10** (Franklin [19]). *If a sequence is low for Schnorr randomness then it is Schnorr trivial.*

*Proof.* Let  $N$  be a computable machine. Let  $L$  be an  $X$ -computable machine such that for all  $n$ ,  $K_L^X(X \upharpoonright n) = K_N(n)$  (for all  $x$ , if  $N(x) = n$  then let  $L(x) = X \upharpoonright n$ .) Then there is some computable machine  $M$  such that for all  $x$ ,  $K_M(x) \leq K_L^X(x) + O(1)$ ;  $M$  is as required to witness that  $X$  is Schnorr trivial.  $\square$

To argue that the *Only If* direction of Theorem 5.9 is true, we note that a relativized version of the Kraft-Chaitin Theorem (Lemma 4.1) can be used to show that Theorem 2.30 relativizes. Namely, we have that  $Y$  is

$X$ -Schnorr random if and only if for all  $X$ -computable machines  $M$  and for all  $n \in \omega$ ,

$$K_M^X(Y \upharpoonright n) \geq n - O(1).$$

Therefore, every sequence  $X$  that is low for computable machines is low for Schnorr randomness, and by Theorems 5.4 and 5.6 it follows further that  $X$  is low for Schnorr tests and thus is computably traceable.

The following remark is straightforward.

**Remark 5.11.** If we enumerate a Kraft-Chaitin set  $\langle d_0, \sigma_0 \rangle, \langle d_1, \sigma_1 \rangle, \dots$  such that  $\sum_i 2^{-d_i}$  is a computable real, then the machine produced by the Kraft-Chaitin Theorem (Lemma 4.1) is computable.  $\triangleleft$

*Proof of the If direction of Theorem 5.9.* Let  $X$  be a computably traceable sequence and let  $h$  be a computable function as in Definition 5.3. Fix a computable, decreasing sequence of positive rationals  $p_0, p_1, \dots$  such that  $\sum_{n \in \omega} h(n)p_n$  is finite; moreover, we want the convergence to be quick, say for every  $m \in \omega$ ,

$$\sum_{n \geq m} h(n)p_n < 2^{-m}. \quad (5.2)$$

Let  $M = M_e$  be an  $X$ -computable machine and let  $f$  be a strictly increasing,  $X$ -computable function such that  $\mu [W_{e,f(n)}^X]$  approximates  $\mu [\text{dom } M^X]$  to within  $2^{-n}$ . For each  $n \in \omega$ , let  $k_n$  be the least number such that  $2^{-k_n} < p_n$ , and let  $t_n$  be equal to  $f(k_n)$ . Consequently, we have

$$\mu [\text{dom } M^X] - \mu [W_{e,t_n}^X] < p_n.$$

We let  $W = W_{e,t_0}^X$  and for  $n \in \omega$ , we let  $V_n = W_{e,t_{n+1}}^X \setminus W_{e,t_n}^X$ . Further, define partial functions  $\varphi$  and  $\psi_n$  ( $n \in \omega$ ) by

$$\varphi = \{(\tau, \sigma) : \tau \in W \ \& \ M^X(\tau) = \sigma\}$$

and

$$\psi_n = \{(\tau, \sigma) : \tau \in V_n \ \& \ M^X(\tau) = \sigma\}.$$

The domain  $V_n$  of each function  $\psi_n$  is finite, and so  $\psi_n$  has a natural number code that we denote by  $g(n)$ . Now the sequence  $(t_n)_n$  is  $X$ -computable, and so the sequences  $(V_n)_n$  and  $(\psi_n)_n$ , and the function  $g$  are  $X$ -computable, too. We note further that for all  $n \in \omega$ ,  $\mu [\text{dom } \psi_n] < p_n$ .

By hypothesis on  $X$ , there is a sequence of finite sets  $(F_n)_n$  such that the following conditions are satisfied:

- There is a computable function  $r$  such that  $F_n = D_{r(n)}$  for all  $n$ ,
- $|F_n| \leq h(n)$  for each  $n \in \omega$ ,
- and  $g(n) \in F_n$  for each  $n \in \omega$ .

By cancelling elements, we may assume that for every  $n \in \omega$ , each element of  $F(n)$  is (the code for) a finite function  $\psi$  such that

$$\text{dom } \psi \text{ is prefix-free and } \mu[\text{dom } \psi] < p_n. \quad (5.3)$$

Enumerate a Kraft-Chaitin set  $L$  as follows. Let  $\langle d, \sigma \rangle \in L$  if there is some  $\tau$  such that  $|\tau| = d$ , and one of the following holds:

- $(\tau, \sigma) \in \varphi$ ,
- for some  $n$  and for some  $\psi \in F_n$ ,  $(\tau, \sigma) \in \psi$ .

The set  $L$  is computably enumerable. Further, the sum of the axiom lengths  $\sum_{\langle d, \sigma \rangle \in L} 2^{-d}$  is a computable real since by (5.2) and (5.3), we have that for any  $m$ ,

$$\sum \{2^{-|\tau|} : (\exists n \geq m)(\exists \psi \in F_n)[\tau \in \text{dom } \psi]\} \leq \sum_{n \geq m} h(n)p_n \leq 2^{-m}.$$

By Remark 5.11 we get a computable machine  $N$  such that for some constant  $c$ , if  $\langle d, \sigma \rangle \in L$ , then  $K_N(\sigma) \leq d + c$ . On the other hand, it follows from the construction that  $\langle K_M^X(\sigma), \sigma \rangle \in L$  whenever  $\sigma$  is in the range of  $M^X$ . Thus  $N$  is as required.  $\square$



## Chapter 6

# Oracle Power versus Randomness

To construct a Martin-Löf random sequence  $Y$ , one may consider a universal Martin-Löf test  $U_0, U_1, \dots$  and build the sequence  $Y$  such that it is not contained in the intersection of the cones  $[U_n]$ . Along the way of constructing  $Y$ , one may be able to make sure that  $Y$  has some additional property. This approach was used by Gács [21] and Kučera [25], who obtained independently the following celebrated result. For every sequence  $X$  there is a Martin-Löf random sequence  $Y$  such that  $X$  is Turing reducible to  $Y$ . The proofs of Gács and Kučera are quite different from each other, but interestingly, while their results are stated for Turing reducibility, the reductions constructed in both proofs are indeed already weak truth-table (wtt-) reductions [44, Section 6.1].

In Section 1, we present a comparatively simple proof of the above result that every sequence  $X$  wtt-reduces to a Martin-Löf random sequence  $Y$  by making use of the martingale characterization of Martin-Löf randomness. Namely, our construction works by diagonalizing against a universal sub-computable martingale. Building on the latter construction idea, we obtain in a second construction a more efficient coding of  $X$  into  $Y$ . Namely, we arrange that not more than  $m + o(m)$  bits of  $Y$  are needed in order to code the first  $m$  bits of  $X$ . This result and the corresponding construction are implicit in the work of Gács, while our account in terms of martingales is again less involved than the original one in terms of Martin-Löf tests.

By a variant of our basic construction, we obtain in Section 2 a computably random sequence that is weak truth-table autoreducible. Further, we observe that there is a Martin-Löf random sequence that is computably

enumerable self-reducible. The existence of such a sequence follows by the fact that a sequence is computably enumerable self-reducible if and only if it is computably enumerable and by the known fact that the leftmost real in the complement of any given component of a universal Martin-Löf test is computably enumerable and Martin-Löf random.

The mentioned results on auto- and selfreducibility do not extend to slightly less powerful reducibilities. More precisely, no computably random sequence is truth-table autoreducible and no Martin-Löf random sequence is Turing-autoreducible. The latter assertion is due to Trakhtenbrot [46], while both assertions can be obtained as corollaries to work of Ebert, Merkle, and Vollmer [17], who demonstrate that such autoreductions are not even possible in the more liberal setting where one just requires that in the limit the reducing machine computes the correct value for a constant nonzero fraction of all places, while signalling ignorance about the correct value for the other places.

## 6.1 Every Sequence is Reducible to a Martin-Löf Random Sequence

The following technical remark is crucial to the constructions of random sequences in this and the next section.

**Remark 6.1.** Given a rational  $p > 1$  and a natural number  $k$ , we can compute a length  $\ell(p, k)$  such that for any martingale  $d$  and any string  $\sigma$ ,

$$\left| \left\{ \tau \in \{0, 1\}^{\ell(p, k)} : d(\sigma\tau) \leq pd(\sigma) \right\} \right| \geq k.$$

That is, for any martingale  $d$  and for any interval  $I$  of length  $\ell(p, k)$  there are (at least)  $k$  assignments  $\tau$  on  $I$  on any of which  $d$  increases its capital by at most a factor of  $p$  while betting on  $I$ , no matter how the restriction  $\sigma$  of the unknown sequence to the places to the left of  $I$  looks like.

For a proof, observe that by the generalized fairness condition (2.2) and by Kolmogorov's inequality (2.4)

$$\frac{|\{\tau \in \{0, 1\}^\ell : d(\sigma\tau) > pd(\sigma)\}|}{2^\ell} < \frac{1}{p}.$$

By  $p > 1$ , we have  $1 - 1/p > 0$ , hence it suffices to choose  $\ell(p, k)$  so large that  $(1 - 1/p)2^{\ell(p, k)}$  is at least  $k$ , i.e., it suffices to let

$$\ell(p, k) \geq \log \frac{k}{1 - \frac{1}{p}} = \log \frac{kp}{p - 1} = \log k + \log p - \log(p - 1). \quad (6.1)$$

**Theorem 6.2** (Gács [21], Kučera [25]). *Every sequence is wtt-reducible to a Martin-Löf random sequence.*

*Proof.* Fix a decreasing sequence  $p_0, p_1, \dots$  of rationals with  $p_i > 1$  for all  $i$  such that the sequence  $q_0, q_1, \dots$  converges where

$$q_s = \prod_{i \leq s} p_i .$$

In addition, assume that given  $i$  we can compute an appropriate representation of  $p_i$ . For  $s = 0, 1, \dots$ , let  $\ell_s = \ell(p_s, 2)$ , where  $\ell(\cdot, \cdot)$  is the function from Remark 6.1. Partition the natural numbers into consecutive intervals  $I_0, I_1, \dots$  of length  $\ell_0, \ell_1, \dots$ , respectively. For further use note that by choice of the  $\ell_s$ , for any string  $\sigma$  and any martingale  $d$ , there are at least two strings  $\tau$  of length  $\ell_s$  such that

$$d(\sigma\tau) \leq p_s d(\sigma) . \tag{6.2}$$

Let  $X$  be any sequence. We construct a sequence  $Y$  to which  $X$  is wtt-reducible, where the construction is done in stages  $s = 0, 1, \dots$ . During stage  $s$  we specify the restriction of  $Y$  to  $I_s$ . We ensure that  $Y$  is Martin-Löf random as follows. According to Remark 2.16, fix a subcomputable martingale  $d$  that succeeds by unbounded limit inferior on all sequences that are not Martin-Löf random. Observe that by appropriately normalizing  $d$  we can assume  $d(\varepsilon) < 1$ . At stage  $s$ , call a string  $\tau$  of length  $\ell_s$  an *admissible extension* in case  $s = 0$  if  $d(\tau) \leq q_0$  and in case  $s > 0$  if

$$d(\sigma\tau) \leq q_s \quad \text{where } \sigma = Y|(I_0 \cup \dots \cup I_{s-1}) .$$

During each stage  $s$ , we let  $Y|I_s$  be equal to some admissible extension. Since the  $q_s$  are bounded this implies that  $d$  does not succeed on  $Y$  by unbounded limit superior, hence  $Y$  is Martin-Löf random.

We will argue in a minute that at each stage there are at least two admissible extensions. Assuming the latter, the sequence  $X$  can be coded into  $Y$  as follows. During stage  $s$  let  $Y|I_s$  be equal to the greatest admissible extension in case  $s$  is in  $X$ , and let  $Y|I_s$  be equal to the least admissible extension otherwise. An oracle Turing machine  $M$  that wtt-reduces  $X$  to  $Y$  works as follows. On input  $s$ ,  $M$  queries its oracle in order to obtain the restrictions  $\sigma_s$  and  $\tau_s$  of the oracle to the sets  $I_0 \cup \dots \cup I_{s-1}$  and  $I_s$ , respectively. Then  $M$  runs two subroutines in parallel. Subroutine 0 simulates in parallel enumerations of  $d(\sigma_s\tau)$  for all  $\tau < \tau_s$  and terminates if the simulation shows that  $d(\sigma_s\tau) > q_s$  for all these  $\tau$ , i.e., Subroutine 0 terminates

if the simulation shows that no such  $\tau$  is an admissible extension of  $\sigma_s$ . Subroutine 1 does the same for all  $\tau > \tau_s$ .

In case Subroutine  $i$  terminates before Subroutine  $1 - i$ , then  $M$  outputs  $i$ . By construction, with oracle  $Y$  for every  $s$  exactly one of the subroutines terminates and  $M$  computes  $X(s)$  correctly.

It remains to show that at each stage there are at least two possible extensions to choose from. For stage  $s = 0$ , this follows by  $d(\varepsilon) < 1$  and the choice of  $I_0$ . For any stage  $s > 0$  assume by induction that the restriction  $\sigma_s$  of  $Y$  to the intervals  $I_0$  through  $I_{s-1}$  could be defined by choosing admissible extensions at the previous stages and that hence we have  $d(\sigma_s) \leq q_{s-1}$ . Then by (6.2) there are at least two strings  $\tau$  of length  $l_s$  where

$$d(\sigma_s \tau) \leq p_s d(\sigma_s) \leq p_s q_{s-1} = q_s ,$$

i.e., at stage  $s$  there are at least two admissible extensions.  $\square$

**Remark 6.3.** Let  $r_0, r_1, \dots$  be a sequence of nonnegative reals, let  $p_i = 1 + r_i$  and let  $q_s = \prod_{i \leq s} p_i$ . Then the sequence  $q_0, q_1, \dots$  converges if and only if the sum  $\sum r_i$  converges (see, for example, Apostol [2, Theorem 8.52]).  $\triangleleft$

By Remark 6.3, in the proof of Theorem 6.2 we could for example choose  $p_i$  to be equal to  $1 + (i + 1)^{-2}$ . For this choice, by (6.1), we then have  $\ell_i \geq \log(i + 1)$ , i.e., in the limit we use more and more bits of  $Y$  in order to code a single bit of  $X$ . The next remark shows that with the current construction this cannot be avoided by choosing a different sequence  $p_0, p_1, \dots$

**Remark 6.4.** The construction in the proof of Theorem 6.2 requires in the limit an unbounded number of bits of  $Y$  in order to code a single bit of  $X$ .

In the proof of Theorem 6.2, a single bit  $X(i)$  has been coded into  $\ell_i$  bits of  $Y$ , where by construction and (6.1), the number  $\ell_i$  was chosen to be at least

$$\ell(p_i, 2) \geq 1 + \log p_i - \log(p_i - 1) .$$

Furthermore, the construction required that the nondecreasing sequence  $q_0, q_1, \dots$ , where  $q_s = \prod_{i \leq s} p_i$ , is bounded and hence converges. By Remark 6.3, this implies that the sequence of the values  $p_i - 1$  goes to 0, and thus the values of  $-\log(p_i - 1)$  and the  $\ell_i$  go to infinity.  $\triangleleft$

Next we give a slightly more involved construction that allows to code an arbitrary sequence  $X$  into a Martin-Löf random sequence  $Y$  such that in the limit in order to code the first  $m$  bits of  $X$  only  $m + o(m)$  bits of  $Y$  are required. This result and the corresponding construction are implicit in

the work of Gács [21] and, in particular, the procedure used to define the strings  $\tau_i$  is due to him. However, the account in terms of martingales is again less involved than the original one in terms of Martin-Löf tests [7, 21].

**Definition 6.5.** *A sequence  $X$  is wtt-reducible to a sequence  $Y$  with vanishing relative redundancy if  $X$  is wtt-reducible to  $Y$  by a Turing machine  $M$  such that the use of  $M$  is bounded by a nondecreasing computable function  $g$  where*

$$\limsup_{x \rightarrow \infty} \frac{g(x)}{x} \leq 1.$$

**Theorem 6.6.** *Every sequence is wtt-reducible to a Martin-Löf random sequence with vanishing relative redundancy.*

*Proof.* We assume that we are given a sequence  $X$  and construct a Martin-Löf random sequence  $Y$  such that  $X$  wtt-reduces to  $Y$  with vanishing relative redundancy. The construction of  $Y$  is similar to the one used in the proof of Theorem 6.2, however, instead of coding single bits of  $X$  individually into intervals of  $Y$ , now we partition the natural numbers into consecutive intervals  $J_0, J_1, \dots$  of appropriate lengths  $m_0, m_1, \dots$  and code the restriction of  $X$  to  $J_s$  in one pass. The coding works again by extending the already constructed part of  $Y$  by an appropriate admissible extension, where now we have to require that there is an admissible extension for each of the  $2^{m_s}$  possible assignments on  $J_s$ .

For the moment, let  $p_0, p_1, \dots$  and  $q_0, q_1, \dots$  be any sequences that satisfy the specifications given in the proof of Theorem 6.2. Recall the definition of the function  $\ell(\cdot, \cdot)$  from Remark 6.1 and partition the natural numbers into consecutive intervals  $I_0, I_1, \dots$  where interval  $I_s$  has length

$$\ell_s = \ell(p_s, 2^{m_s}).$$

By Remark 2.16, choose a universal subcomputable martingale  $d$  that succeeds by unbounded limit inferior on any sequence that is not Martin-Löf random; as before we can assume  $d(\varepsilon) < 1$ . Fix a computable function  $\tilde{d}(\cdot, \cdot)$  witnessing that  $d$  is subcomputable, i.e., for all strings  $\sigma$ , the sequence  $\tilde{d}(\sigma, 0), \tilde{d}(\sigma, 1), \dots$  is nondecreasing and converges to  $d(\sigma)$ .

The construction of the Martin-Löf random sequence  $Y$ , to which  $X$  is wtt-reducible, is done in stages  $s = 0, 1, \dots$ . During stage  $s$ , we let the restriction of  $Y$  to  $I_s$  be equal to an *admissible extension*, where admissible extension is defined as in the proof of Theorem 6.2, i.e., a string  $\tau$  is an admissible extension of the already constructed prefix  $\sigma$  of  $Y$  if  $d(\sigma\tau) \leq q_s$ . Again, we can argue that by choosing an admissible extension at each stage,

the sequence  $Y$  will be Martin-Löf random. Furthermore, as before, an easy induction argument shows that by choice of the interval lengths  $\ell_s$ , during the construction at each stage  $s$ , there are at least  $2^{m_s}$  admissible extensions.

At stage  $s$ , let  $\sigma_s$  denote the restriction of  $Y$  to  $I_0 \cup \dots \cup I_{s-1}$ . We proceed in substages  $t = 0, 1, \dots$ , where during substage  $t$  we define  $2^{m_s}$  strings

$$\tilde{\tau}_1(t), \tilde{\tau}_2(t), \dots, \tilde{\tau}_{2^{m_s}}(t)$$

of length  $\ell_s$ . At substage 0, we let  $\tilde{\tau}_i(0)$  through  $\tilde{\tau}_{2^{m_s}}(0)$  be equal to the  $2^{m_s}$  least strings of length  $\ell_s$ . At any substage  $t > 0$ , for  $i = 1, \dots, 2^{m_s}$  we successively define  $\tilde{\tau}_i(t)$  where we let

$$\tilde{\tau}_i(t) = \tilde{\tau}_i(t-1) \quad \text{in case} \quad \tilde{d}(\sigma_s \tilde{\tau}_i(t-1), t) \leq q_s.$$

Otherwise, i.e., in case the approximation  $\tilde{d}(\cdot, t)$  to  $d$  reveals that  $\tilde{\tau}_i(t-1)$  is not admissible, we let  $\tilde{\tau}_i(t)$  be equal to the least unused string of length  $\ell_s$ , where a string is unused if it differs from all strings  $\tilde{\tau}_{i'}(t')$  that have been defined so far during stage  $s$ .

For all  $i$ , the sequence  $\tilde{\tau}_i(\cdot)$  does not contain two distinct admissible extensions, because by construction

$$\text{if } \tilde{\tau}_i(t) \text{ is defined and admissible, then } \tilde{\tau}_i(t) = \tilde{\tau}_i(t+1) = \dots \quad (6.3)$$

Suppose that eventually the construction reaches a point where there are no unused strings left. Then in particular the at least  $2^{m_s}$  admissible extensions have already been used, hence these strings must have appeared in pairwise different sequences  $\tilde{\tau}_i(\cdot)$ . Consequently, in each such sequence an admissible extension has appeared, thus by (6.3), from this point on there will be no attempt to assign an unused string to any  $\tilde{\tau}_i(t)$ . In summary, the  $\tilde{\tau}_i(t)$  are all defined.

Next we argue that each sequence  $\tilde{\tau}_i(\cdot)$  converges to an admissible extension  $\tau_i$ . By (6.3), it suffices to show that in each such sequence eventually some admissible extension appears. So fix  $i$ . By construction and because all  $\tilde{\tau}_i(t)$  are defined, for any substage  $t$  such that  $\tilde{\tau}_i(t)$  is not admissible, there is a substage  $t' > t$  where  $\tilde{\tau}_i(t')$  is set equal to the least unused string. The latter cannot happen more often than there are strings of length  $m_s$ , hence eventually an admissible extension must appear in the sequence  $\tilde{\tau}_i(t)$ .

In order to code the restriction  $X|_{J_s}$  of the sequence  $X$  to the interval  $J_s$  into the sequence  $Y$ , determine  $i$  such that  $X|_{J_s}$  is the  $i$ th string in the lexicographic ordering of all strings of length  $m_s$ , then let  $Y|_{I_s}$  be equal to  $\tau_i$ . Observe in this connection that a straightforward induction on  $t$

shows that the strings  $\tilde{\tau}_1(t)$  through  $\tilde{\tau}_{2^{m_s}}(t)$  and hence also their limits  $\tau_1$  through  $\tau_{2^{m_s}}$  are mutually distinct.

The following oracle Turing machine  $M$  wtt-reduces  $X$  to  $Y$ . On input  $x$ , the machine computes the index  $s = s(x)$  such that the interval  $J_s$  contains  $x$ . Then  $M$  queries its oracle in order to obtain the restrictions  $\sigma$  and  $\tau$  of the oracle to the sets  $I_0 \cup \dots \cup I_{s-1}$  and  $I_s$ , respectively. The Turing machine successively simulates the substages  $t = 0, 1, \dots$  of stage  $s$  in order to compute the strings  $\tilde{\tau}_1(t), \dots, \tilde{\tau}_{2^{m_s}}(t)$ . If the oracle is indeed  $Y$ , then  $\tau$  must eventually appear among the computed strings, i.e.,  $\tau = \tilde{\tau}_i(t)$  for some  $i$  and  $t$ . Due to the way  $X$  has been coded into  $Y$ , this means that the restriction of  $X$  to  $J_s$  is equal to the  $i$ th string in the lexicographic ordering of all strings of length  $m_s$ , hence  $M$  can simply look up the bit  $X(x)$  in the latter string.

It remains to show that we can arrange that the reduction from  $X$  to  $Y$  has vanishing relative redundancy. On input  $x$ , the Turing machine  $M$  queries the restriction of the oracle to the sets  $I_0$  through  $I_{s(x)}$  where  $s(x)$  is the index such that the interval  $J_{s(x)}$  contains  $x$ . Therefore, the use of  $M$  on input  $x$  is bounded by the nondecreasing computable function

$$g(x) = \ell_0 + \dots + \ell_{s(x)}. \quad (6.4)$$

For all  $s$ , let  $z_s$  be the least number in the interval  $J_s$ . Note that on each interval  $J_s$ , the function  $x \mapsto g(x)/x$  attains its maximum at  $z_s$ , hence

$$r := \limsup_{x \rightarrow \infty} \frac{g(x)}{x} = \limsup_{s \rightarrow \infty} \frac{g(z_s)}{z_s}. \quad (6.5)$$

Next we argue that the sequence  $p_0, p_1, \dots$  and the  $m_s$  and  $\ell_s$  can be chosen such that  $r = 1$ . First, let  $p_s = 1 + (s+1)^{-2}$  for all  $s \geq 0$ . Then by Remark 6.3 the sequence  $q_0, q_1, \dots$  converges as required. By Remark 6.1, we can assume that for all  $s > 0$ ,

$$\ell_s \leq m_s + O(\log s). \quad (6.6)$$

For any  $s \geq 1$  we have  $z_s = m_0 + \dots + m_{s-1}$  and  $s(z_s) = s$ , hence by (6.4) and (6.6) we obtain

$$\frac{g(z_s)}{z_s} = \frac{\ell_0 + \dots + \ell_s}{m_0 + \dots + m_{s-1}} \leq 1 + \frac{m_s}{m_0 + \dots + m_{s-1}} + \frac{O(s \log s)}{m_0 + \dots + m_{s-1}}.$$

Therefore, if we choose for example  $m_s = s+1$ , it follows by (6.5) that  $r = 1$  and the constructed reduction has vanishing relative redundancy.  $\square$

It appears to us that compared to the original proofs by Gács and Kučera, the proofs of Theorems 6.2 and 6.6 are somewhat more intuitive and require less technical machinery and that this is mainly due to the fact that the latter proofs work by diagonalizing against a universal martingale, whereas the former ones are formulated in terms of Martin-Löf tests. However, the ideas underlying the original and the current proofs are essentially the same; in particular, the procedure for defining the strings  $\tilde{\tau}_i(t)$  in the proof of Theorem 6.6 is taken from Gács. Note in this connection that, similar to the original proofs given by Gács and Kučera, the oracle Turing machines we have constructed in order to wtt-reduce a given sequence  $X$  to a Martin-Löf random sequence  $Y$  do not depend on the sequence  $X$ , i.e., there is a single machine that wtt-reduces any given sequence to some Martin-Löf random sequence.

Hertling [7, 22] investigates general assumptions on a class  $\mathcal{C}$  that imply that the result of Gács and Kučera as stated in Theorems 6.2 and 6.6 holds with  $\mathcal{C}$  in place of the class of Martin-Löf random sequences. He introduces concepts of effectively growing Cantor classes and proves along the lines of Gács' work [21] that the result of Gács and Kučera goes through for any class  $\mathcal{C}$  that is constructively closed and contains an effectively growing Cantor class of appropriate type. For ease of reference, we sketch in Remark 6.8 a proof of his result that uses our terminology and is based on the proof of Theorem 6.6. Before, we state in Remark 6.7 a straightforward characterization of the concept of effectively closed class. Recall from Section 1.2 that a subclass  $\mathcal{C}$  of Cantor space is *effectively closed* (or a  $\Pi_1^0$ -class) if  $\mathcal{C}$  is the complement of a class of the form  $[W]$  where the set  $W$  is computably enumerable. For the scope of Remarks 6.7 and 6.8 and with an arbitrary class  $\mathcal{C}$  understood, say a string  $\sigma$  is an *admissible prefix* if it is a prefix of a sequence in  $\mathcal{C}$ .

**Remark 6.7.** Let  $\mathcal{C}$  be any class. Then  $\mathcal{C}$  is effectively closed if and only if the two following conditions are satisfied.

- (i) The set of strings that are not admissible prefixes is computably enumerable.
- (ii) Any sequence that extends infinitely many admissible prefixes is already in  $\mathcal{C}$ .

First assume that  $\mathcal{C}$  is effectively closed and let  $W$  be a computably enumerable set such that  $\mathcal{C}$  is equal to the complement of  $[W]$ . Then a



string  $\sigma$  is not admissible if and only if the cones  $[\rho]$  with  $\rho$  in  $W$  cover the cone above  $\sigma$ . In the latter situation, by compactness of Cantor space [36], the cone above  $\sigma$  is already covered by finitely many of these cones. Hence by enumerating  $W$  we will eventually detect all strings  $\sigma$  that are not admissible and (i) follows. In order to show (ii), it suffices to observe that any sequence not in  $\mathcal{C}$  has a prefix  $\rho$  in  $W$  and hence extends only finitely many admissible prefixes.

Next, assume that  $\mathcal{C}$  satisfies (i) and (ii) and let  $V$  be the set of strings that are not admissible prefixes. Then  $V$  is computably enumerable by (i) and  $\mathcal{C}$  is equal to the complement of  $[V]$  by (ii), hence  $\mathcal{C}$  is effectively closed.

◁

**Remark 6.8.** Let  $\mathcal{C}$  be an effectively closed class and assume that there are computable sequences  $\ell_0, \ell_1, \dots$  and  $m_0, m_1, \dots$  of nonzero natural numbers such that

- (iii) for every  $s$ , any admissible prefix of length  $\ell_0 + \dots + \ell_{s-1}$  can be extended in  $2^{m_s}$  different ways to an admissible prefix of length  $\ell_0 + \dots + \ell_s$ .

Then any sequence  $X$  is wtt-reducible to a sequence  $Y$  in  $\mathcal{C}$ , where the reduction can be chosen such that for any  $s$ , the prefix of  $X$  of length  $m_0 + \dots + m_s$  can be computed from the prefix of  $Y$  of length  $\ell_0 + \dots + \ell_s$ . (Condition (iii) is essentially the same as Hertling's condition on effectively growing Cantor classes.)

We omit the details of the proof, which is very similar to the corresponding part of the proof of Theorem 6.6. The proof exploits that by Remark 6.7 the effectively closed class  $\mathcal{C}$  satisfies (i) and (ii). The sequence  $Y$  is again constructed in stages, where during stage  $s$  we extend an admissible prefix of length  $\ell_{s-1}$  to an admissible prefix of length  $\ell_s$ , hence by (ii) the constructed sequence  $Y$  is indeed in  $\mathcal{C}$ . Furthermore, we can argue that by (i) and (iii) the procedure that computes the  $\tilde{\tau}_i(t)$  can be defined as before. ◁

In the proofs of Theorem 6.2 and 6.6 an analogue of assumption (iii) has been obtained from Remark 6.1 on martingales. While Gács uses a similar argument formulated in terms of measure, the approach of Kučera is different. In his proof, the interval lengths  $\ell_s$  are not specified in advance but are computed by an inductive process, which exploits an interesting technical lemma [25, Lemma 8]. The lemma asserts that there is a computable, rational-valued function  $b$  such that  $\mu[\mathcal{C}_\sigma] > b(\sigma) > 0$  whenever the class  $\mathcal{C}_\sigma$  has nonzero measure, where  $\mathcal{C}_\sigma$  is the intersection of the cone  $[\sigma]$

with the complement of any fixed class  $[W_{g(i)}]$  from some specific universal Martin-Löf test.

## 6.2 Self- and Autoreductions of Random Sequences

Recall from Section 1.2 the concepts of self- and autoreducibility. The bits of a sequence that is self- or autoreducible depend on each other in an effective way and one might be tempted to assume that in the case of a random sequence such dependencies cannot exist. However, for example we obtain autoreducible random sequences if we consider a concept of reducibility where the underlying model of computation is powerful enough to simply compute a random sequence. This indicates that when asking whether random sequences may be autoreducible we have to be more specific about the types of random sequence and reducibility. In what follows, we investigate the question of how powerful a model of computation is required in order to be able to autoreduce Martin-Löf random and computably random sequences.

First, in the proof of Theorem 6.9, we use techniques similar to the ones employed in the proof of Theorem 6.2 in order to construct a computably random sequence that is wtt-autoreducible. Subsequently, in Corollary 6.12, we argue that there are Martin-Löf random sequences that are c.e.-selfreducible. Finally, in Remark 6.14, we observe that the two latter results cannot be extended to slightly less powerful reducibilities because it is known that computably random sequences cannot be tt-autoreducible and that Martin-Löf random sequences cannot be T-autoreducible.

**Theorem 6.9.** *There is a sequence that is computably random and wtt-autoreducible.*

*Proof.* A sequence  $Y$  as required can be obtained by a construction similar to the one used in the proof of Theorem 6.2. Choose  $p_0, p_1, \dots$  and  $q_0, q_1, \dots$  as in that proof and let  $r_s = (p_s - 1)/2$ . Again, partition the natural numbers into consecutive intervals  $I_0, I_1, \dots$ , however now interval  $I_s$  has length

$$\ell_s = \ell(1 + r_s, 3).$$

Let  $d_0, d_1, \dots$  be an appropriate effective enumeration of all partial computable functions from  $2^{<\omega}$  to the rational numbers and let

$$E = \{e: d_e \text{ is a (total) martingale with initial capital } d_e(\varepsilon) = 1\}.$$

For the sake of simplicity we assume that 0 is in  $E$ . Furthermore, let

$$\bar{d}_s = \sum_{\{e \in E: e \leq s\}} c_e d_e \quad \text{where} \quad c_e = \frac{r_e}{2^{l_0 + \dots + l_e}}.$$

The sequence  $Y$  is constructed in stages  $s = 0, 1, \dots$  where during stage  $s$  we specify the restriction of  $Y$  to the interval  $I_s$ . At stage  $s$  call a string  $\tau$  of length  $\ell_s$  an *admissible extension* if  $s = 0$  or if  $s > 0$  and we have

$$\bar{d}_{s-1}(\sigma\tau) \leq (1 + r_s) \bar{d}_{s-1}(\sigma) \quad \text{where} \quad \tau = Y|I_0 \cup \dots \cup I_{s-1}.$$

Again at every stage  $s$  we will let  $Y|I_s$  be equal to some admissible extension and we argue that this way the sequence  $Y$  automatically becomes computably random. For a proof of the latter it suffices to show that for all  $s$  we have

$$\bar{d}_s(Y|I_0 \cup \dots \cup I_s) \leq q_s. \tag{6.7}$$

If there were some  $d_j$  that succeeds on  $Y$ , then by Remark 2.25 there would be some  $d_i$  that succeeds on  $Y$  by unbounded limit inferior. But the latter contradicts (6.7) because the  $q_i$  are bounded and because  $\bar{d}_s \geq c_i d_i$  for  $s \geq i$ .

Inequality (6.7) follows by an inductive argument. For  $s = 0$  we have

$$\bar{d}_0(Y|I_0) = c_0 d_0(Y|I_0) \leq c_0 2^{\ell_0} = r_0 \leq p_0 = q_0.$$

In the induction step, let  $\sigma$  and  $\tau$  be the restriction of  $Y$  to  $I_0 \cup \dots \cup I_{s-1}$  and to  $I_s$ , respectively. By the definition of admissible extension and by the induction hypothesis, we have

$$\bar{d}_{s-1}(\sigma\tau) \leq (1 + r_s) \bar{d}_{s-1}(\sigma) \leq (1 + r_s) q_{s-1}.$$

By definition, the values of  $\bar{d}_{s-1}(\sigma\tau)$  and of  $\bar{d}_s(\sigma\tau)$  are the same in case  $s$  is not in  $E$ , while otherwise they differ by

$$c_s d_s(\sigma\tau) \leq c_s 2^{|\sigma\tau|} \leq r_s,$$

where the inequalities follow because a martingale can at most double at each step and by the definition of  $c_s$ . In summary, we have

$$\bar{d}_s(\sigma\tau) \leq (1 + r_s) q_{s-1} + r_s \leq (1 + 2r_s) q_{s-1} = p_s q_{s-1} = q_s.$$

It remains to show that we can arrange that  $Y$  is wtt-autoreducible. At stage  $s$ , let  $(\tau_0, \tau_1)$  be the least pair of admissible extensions such that  $\tau_0$  and  $\tau_1$  differ at least at two places. Then let the restriction of  $Y$  to  $I_s$

be equal to  $\tau_0$  in case  $s \notin E$  and be equal to  $\tau_1$  otherwise. Observe that there is always such a pair because by choice of  $\ell_s$  there are at least three admissible extensions, hence there are at least two admissible extensions that differ in at least two distinct places. (Indeed, given any three mutually distinct strings  $\rho, \rho', \rho''$  of the same length, then if  $\rho$  and  $\rho'$  differ only at one place,  $\rho''$  must differ from  $\rho'$  at some other place, hence  $\rho'$  and  $\rho''$  or  $\rho$  and  $\rho''$  differ in at least two places.)

The sequence  $Y$  is wtt-autoreducible by an oracle Turing machine  $M$  that works as follows. For simplicity, we describe the behavior of  $M$  for the case where its oracle is indeed  $Y$  and omit the straightforward considerations for other oracles; anyway it should be clear from the description that  $M$  is of wtt-type.

On input  $x$ , first  $M$  determines the index  $s$  such that  $x$  is in  $I_s$ . Then  $M$  queries the oracle at all places in  $I_0 \cup \dots \cup I_s$  except at  $x$ ; this way  $M$  obtains in particular the restrictions  $\sigma_0, \dots, \sigma_{s-1}$  of  $Y$  to  $I_0, \dots, I_{s-1}$ , respectively, and, up to one bit, the restriction  $\tau$  of  $Y$  to  $I_s$ . Next  $M$  successively computes  $E(j)$  for  $j = 1, \dots, s-1$ ; this can be done because given the  $\sigma_i$  and the values  $E(0)$  through  $E(j-1)$ , it is possible to compute the admissible strings and the strings  $\tau_0$  and  $\tau_1$  of stage  $j$ , and by comparing the two latter strings to  $\tau_j$  one can then compute  $E(j)$ . Finally,  $M$  determines  $Y(x)$  by computing the strings  $\tau_0$  and  $\tau_1$  of stage  $s$  and by comparing them to the known part of  $\tau$ . The last step exploits that  $\tau_0$  and  $\tau_1$  differ at least at two places and thus differ on  $I_s \setminus \{x\}$ .  $\square$

Next we argue that there is a Martin-Löf random sequence that is c.e.-self-reducible. In order to demonstrate this assertion, it suffices to review the known fact that there are computably enumerable reals that are Martin-Löf random and to observe that a sequence is c.e.-self-reducible if and only if its associated real is computably enumerable.

The *real associated with* a sequence  $X$  is  $0.b_0b_1\dots$  where  $b_i = X(i)$ . A real is called *Martin-Löf random* if it is associated to a Martin-Löf random sequence.

In Remark 6.10, we review the well-known fact that there are reals that are c.e. and Martin-Löf random. Note that a real has the two latter properties if and only if it is a Chaitin  $\Omega$  number, i.e., is equal to the halting probability of some universal prefix-free Turing machine — see page 21. For a proof of this equivalence and for references see Calude [6], where the equivalence is attributed to work of Calude, Hertling, Khoussainov, and Wang, of Chaitin, of Kučera and Slaman, and of Solovay.

**Remark 6.10.** There is a c.e. real that is Martin-Löf random.

For a proof, consider any component  $[W_{g(i)}]$  of a universal Martin-Löf test and let  $Y$  be the leftmost, i.e., lexicographically least, sequence in the complement of this component. Note that the sequence  $Y$  and thus also its associated real are Martin-Löf random. Furthermore, let  $\rho^s = \rho_1^s \dots \rho_s^s$  be the lexicographically least string  $\rho$  of length  $s$  such that the cone  $[\rho]$  is not contained in the union of the cones  $[\sigma]$  over the first  $s$  strings  $\sigma$  that are enumerated into  $W$ . Then it can be shown that  $Y$  is the limit of the nondecreasing computable sequence formed by the rationals  $0.\rho_1^s \dots \rho_s^s$ , hence the real associated to  $Y$  is c.e.  $\triangleleft$

**Proposition 6.11.** *A sequence is c.e.-self-reducible if and only if its associated real is computably enumerable.*

*Proof.* Fix any sequence  $Y = b_0b_1\dots$ . The equivalence asserted in the proposition is immediate in case the sequence of bits of  $Y$  is eventually constant, i.e., if  $b_j = b_{j+1} = \dots$  for some  $j$ . So assume otherwise.

First let  $Y$  be c.e.-self-reducible by an oracle Turing machine  $M$ . We define inductively a computable sequence  $Y_0, Y_1, \dots$  of rational numbers that converges nondecreasingly to  $Y$  and where  $Y_s$  can be written in the form

$$Y_s = 0.b_0^s \dots b_s^s, \quad b_j^i \in \{0, 1\}.$$

Let  $M_s(Z, x)$  be the approximation to  $M(Z, x)$  obtained by running  $M$  for  $s$  steps on input  $x$  and oracle  $Z$ ; i.e.,  $M_s(Z, x) = M(Z, x)$  if  $M$  terminates within  $s$  computation steps, and  $M_s(Z, x)$  is undefined otherwise. For a start, let  $b_0^0 = 0$ , i.e.,  $Y_0 = 0$ . In order to define  $Y_s$  for  $s > 0$ , we distinguish two cases. In case for some  $j < s$ , we have

$$b_j^{s-1} = 0 \quad \text{and} \quad M_s(b_0^{s-1} \dots b_{j-1}^{s-1} 0^\omega, j) = 1,$$

then let  $j_s$  be the least such  $j$  and let

$$Y_s = 0.b_0^{s-1} \dots b_{j_s-1}^{s-1} 10^{s-j_s}.$$

In case there is no such  $j$ , let

$$Y_s = 0.b_0^{s-1} \dots b_{s-1}^{s-1} 0.$$

By construction, the sequence  $Y_0, Y_1, \dots$  is nondecreasing. Furthermore, an easy induction argument shows that  $b_0^s b_1^s \dots$  converges pointwise to  $b_0 b_1 \dots$  as  $s$  goes to infinity, and consequently the  $Y_s$  converge to  $Y$ .

Next assume that the real  $Y = 0.b_0b_1\dots$  is computably enumerable. Let  $Y_0, Y_1, \dots$  be a computable sequence of rationals that converges nondecreasingly to  $Y$ . Then  $Y$  is c.e.-self-reducible by an oracle Turing machine  $M$  that works as follows. On input  $s$ ,  $M$  queries its oracle in order to obtain the length  $s$  prefix  $c_0\dots c_{s-1}$  of the oracle. Then  $M$  checks successively for  $i = 0, 1, \dots$  whether

$$Y_i > 0.c_0\dots c_{s-1}1; \tag{6.8}$$

if eventually such an index  $i$  is found,  $M$  outputs 1 while otherwise, if there is no such  $i$ ,  $M$  does not terminate.

Now suppose that  $M$  is applied to oracle  $Y$  and any input  $s$ . If  $b_s = 0$ , then (6.8) is false for all  $i$ , hence  $M$  does not terminate. On the other hand, if  $b_s = 1$  then  $Y$  is strictly larger than the righthand side of (6.8) because by case assumption there is some  $j > s$  such that  $b_j = 1$ . Hence (6.8) is true for almost all  $i$  and  $M$  eventually outputs 1.  $\square$

By Remark 6.10 and Proposition 6.11, the following corollary is now immediate.

**Corollary 6.12.** *There is a sequence that is Martin-Löf random and c.e.-self-reducible.*

Remark 6.13 gives an alternate direct proof of Corollary 6.12, which is derived from the proof of Theorem 6.2.

**Remark 6.13.** In the proof of Theorem 6.2, we have constructed a Martin-Löf random sequence where bit  $X(i)$  of the given sequence  $X$  has been coded into interval  $I_i$  by choosing either the least or the greatest admissible extension. If we adjust the construction such that in each interval simply the least admissible extension is chosen, we obtain a sequence that is Martin-Löf random and c.e.-self-reducible.

The construction in the proof of Theorem 6.2 yields a Martin-Löf random sequence in case the chosen extensions are always admissible. Thus it suffices to show that the sequence  $Y$  that is obtained by always choosing the least admissible extension is c.e.-self-reducible. A machine  $M$  witnessing that  $Y$  is c.e.-self-reducible works as follows. On input  $x$ , first  $M$  queries its oracle at all places strictly less than  $x$  and receives as answer the length  $x$  prefix  $\rho_x$  of its oracle. Then  $M$  computes the index  $s$  such that  $x$  is in the interval  $I_s$ , and lets  $\sigma_s$  be the prefix of  $\rho_x$  of length  $\ell_0 + \dots + \ell_{s-1}$ . Note that  $Y(x) = 1$  if and only if during stage  $s$  of the construction there has been no admissible extension  $\tau$  such that  $\sigma_s\tau$  extends  $\rho_x0$  and recall that an extension  $\tau$  is admissible if  $d(\sigma_s\tau) \leq q_s$ . So  $M$  may simply try to prove  $d(\sigma_s\tau) > q_s$  for

all  $\tau$  where  $\sigma_s\tau$  extends  $\rho_x0$  by approximating  $d$  from below, then outputting a 1 in case of success.  $\triangleleft$

By Theorem 6.9 and Corollary 6.12, there are computably random sequences that are wtt-autoreducible and Martin-Löf random sequences that are c.e.-selfreducible. By the following remark, these results do not extend to the less powerful T-reducibility and tt-reducibility, respectively, i.e., no computably random sequence is tt-autoreducible [17] and no Martin-Löf random sequence is T-autoreducible [17, 46].

**Remark 6.14.** Consider the following, more liberal variant of T-autoreducibility. A sequence  $Z$  is infinitely often (i.o.) T-autoreducible if there is an oracle Turing machine that on input  $x$  eventually outputs either the correct value  $Z(x)$  or a special symbol that signals ignorance about the correct value; in addition, the correct value is computed for infinitely many inputs. The concept of i.o. tt-autoreducibility is defined accordingly, i.e., we require in addition that the machine performing the reduction is total.

Ebert [16] showed that every Martin-Löf random sequence is i.o. tt-autoreducible. By results of Ebert, Merkle, and Vollmer [17], any Martin-Löf random sequence can be i.o.-tt-autoreduced such that the fraction of correctly computed places up to input  $x$  exceeds  $r(x)$  where  $r$  is any given computable rational-valued function that goes nonascendingly to 0; on the other hand, no Martin-Löf random sequence  $Y$  is i.o. T-autoreducible in such a way that in the limit the fraction of places where  $Y(x)$  is computed correctly is a nonzero constant and the latter assertion remains true with Martin-Löf random and i.o. T-autoreducible replaced by computably random and i.o. tt-autoreducible. In particular, no Martin-Löf random sequence is T-autoreducible and no computably random sequence is tt-autoreducible.  $\triangleleft$





## Chapter 7

# Lebesgue Measure versus Effective Measures

By definition, each Martin-Löf null class is a Lebesgue null class. Book, Lutz, and Wagner [4] show that for  $\Pi_1^0$ -classes the converse direction also holds. This result is used in [4] to show that any union of  $\Pi_1^0$ -classes that is closed under finite variation is a Lebesgue null class if and only if it contains no Martin-Löf random sequence.

In the first section, we recall the definition of Kurtz null classes and of Kurtz random sequences, among other things. In Section 2, we extend the results of Book, Lutz, and Wagner [4] cited above by showing that “Martin-Löf null class” and “Martin-Löf random sequence” can be replaced by “Kurtz null class” and “Kurtz random sequence”, respectively. While for  $\Sigma_2^0$ -classes being a Schnorr null class is equivalent to being a Lebesgue null class, we show that the corresponding assertion for “Kurtz null class” instead of “Schnorr null class” is false.

In the last section, we demonstrate that every Almost class with respect to a bounded reducibility  $R$  has computable packing dimension zero. Furthermore we show, given a bounded reducibility  $R$  which is upwards closed under finite variation, that a sequence is contained in the respective Almost class if and only if it is computable and not  $R$ -deep.

## 7.1 Introduction

### Languages and Reducibilities

In this chapter, a sequence  $X$  is occasionally viewed as a *language*  $L$ , i.e., as a set of strings, where the  $n$ th string  $\sigma_n$  in the length-lexicographical ordering of all strings is contained in  $L$  iff  $X(n) = 1$ . As usual, let  $M_0, M_1, \dots$  be a standard listing of all oracle Turing machines. If the  $e$ th machine with oracle  $X$  computes a total  $\{0, 1\}$ -valued function then  $L(M_e^X)$  denotes the language accepted by  $M_e^X$ , where  $\sigma \in L(M_e^X)$  iff  $M_e^X(\sigma) = 1$ . In case  $X$  is the empty set, we also write  $L(M_e)$  instead of  $L(M_e^X)$ .

A class  $\mathcal{C} \subseteq 2^\omega$  is *computably presentable* if  $\mathcal{C} = \emptyset$  or if there is a computable function  $g$  such that  $M_{g(n)}$  is total for every  $n \in \omega$  and  $\mathcal{C} = \bigcup_{n \in \omega} \{L(M_{g(n)})\}$ . In the latter case  $g$  is called a *computable presentation* of  $\mathcal{C}$ .

A reducibility in the language setting is a binary relation on  $2^\omega$  such that there is a set  $I \subseteq \omega$  satisfying the following condition:  $X \leq_R Y$  iff there is an  $i \in I$  such that  $X = L(M_i^Y)$ . Moreover, a reducibility is *bounded* if there is a computable function  $g$  with  $g(\omega) = I$  and if for every oracle  $X$  and for every  $i \in I$ , the machine  $M_i^X$  is total. In this case  $g$  is called a *computable presentation* of  $\leq_R$ .

Given a reducibility  $R$ , we shall consider the *upper  $R$ -cone*  $R^{-1}(X)$  of a sequence  $X$  which is defined by  $R^{-1}(X) = \{Y : X \leq_R Y\}$ . A sequence  $X$  is a finite variation of a sequence  $Y$ , written  $X =^* Y$ , if  $X$  and  $Y$  differ at most at finitely many places. A reducibility  $R$  is *upwards closed under finite variation* if for all sequences  $X, Y_0$ , and  $Y_1$  with  $Y_0 =^* Y_1$  we have that  $X \leq_R Y_0$  if and only if  $X \leq_R Y_1$ . Similarly, a reducibility  $R$  is *downwards closed under finite variation* if for all sequences  $X_0, X_1$ , and  $Y$  with  $X_0 =^* X_1$  we have that  $X_0 \leq_R Y$  if and only if  $X_1 \leq_R Y$ .

### Kurtz Randomness

Similar to Martin-Löf, Kurtz required that a random sequence obey all effective probability-one laws. Yet the basis for Kurtz's definition of randomness is not to specify effective null classes, which contain sequences violating probability-one laws. With his randomness concept, Kurtz rather directly considers those sequences which *obey* probability-one laws.

**Definition 7.1** (Kurtz [27]). A Kurtz test is a  $\Sigma_1^0$ -class which has Lebesgue measure 1. A sequence  $X$  is Kurtz random if for all Kurtz tests  $\mathcal{U}$ ,  $X \in \mathcal{U}$ . We let  $\mathcal{KR}$  denote the class of the Kurtz random sequences.

We remark that Kurtz referred to the above concept as “weak randomness”.

Kurtz randomness can be characterized in terms of tests covering null classes, similar to the randomness concepts we have considered before.

**Definition 7.2** (Wang [51]). A Kurtz null test is a sequence of finite sets  $(V_n)_{n \in \omega}$  such that the following properties hold:

- (i)  $\sum_{\sigma \in V_n} 2^{-|\sigma|} \leq 2^{-n}$  for all  $n$ .
- (ii) There is a computable function  $g$  such that  $g(n)$  is the canonical index of  $V_n$  for each  $n$ .

In this case  $(V_n)_{n \in \omega}$  is the Kurtz null test given by  $g$ .

**Theorem 7.3** (Wang [51]). A sequence is Kurtz random if and only if it withstands every Kurtz null test.

Finally, there is also a martingale characterization of Kurtz randomness.

**Definition 7.4.** A computable martingale  $d$  succeeds strongly on a sequence  $X$  if there is a computable nondecreasing unbounded function  $h$  such that for all but finitely many  $n$ ,  $d(X \upharpoonright n) > h(n)$ .

**Theorem 7.5** (Wang [51]). A sequence  $X$  is Kurtz random if and only if no computable martingale succeeds strongly on  $X$ .

Note that a Kurtz null test is a Martin-Löf test. Thus we say that a class  $\mathcal{C} \subseteq 2^\omega$  is a *Kurtz null class* if it is covered by a Kurtz null test.

**Notation.** If  $\mathcal{C}$  is a Kurtz null class then we write  $\mu_{\text{Kurtz}}(\mathcal{C}) = 0$ .

## Some Fundamental Theorems

**Theorem 7.6** (Lebesgue Density Theorem). Let  $\mathcal{C} \subseteq 2^\omega$  be a measurable class with  $\mu(\mathcal{C}) > 0$ . Then for every nonnegative rational  $\epsilon < 1$  there is a string  $\sigma$  such that  $\mu(\mathcal{C} \cap [\sigma])2^{|\sigma|} \geq \epsilon$ .

Below we shall make use of a standard technique where the Lebesgue Density Theorem is used in order to obtain a “probability amplification”.

A class  $\mathcal{C} \subseteq 2^\omega$  is *closed under finite variation* if for any  $X \in \mathcal{C}$  the following holds true:

$$(\forall Y \in 2^\omega) \left[ X =^* Y \implies Y \in \mathcal{C} \right].$$

**Theorem 7.7** (Kolmogorov’s 0-1-Law). *If  $\mathcal{C}$  is a measurable class of sequences which is closed under finite variation then either  $\mu(\mathcal{C}) = 0$  or  $\mu(\mathcal{C}) = 1$ .*

**Definition 7.8.** *Let  $\sigma$  be a finite partial 0-1-valued function.*

- (i) *The  $\sigma$ -patch  $X_{(\sigma)}$  of a sequence  $X$  is the sequence  $Y$  which agrees with  $\sigma$  on arguments in  $\text{dom } \sigma$  and agrees with  $X$ , otherwise.*
- (ii) *The  $\sigma$ -patch  $\Gamma_{(\sigma)}$  of a functional  $\Gamma$  is defined by  $\Gamma_{(\sigma)}(X) = \Gamma(X_{(\sigma)})$ .*

**Theorem 7.9** (Sacks [37]). *For every noncomputable sequence  $X$  the upper Turing-cone  $\text{T}^{-1}(X)$  is a Lebesgue null class.*

In fact, the preceding theorem holds for any reducibility. To argue that the latter assertion is true, suppose that  $R$  is a reducibility and  $X$  is a sequence such that  $\mu(R^{-1}(X)) > 0$ . Then by  $\sigma$ -additivity of  $\mu$ , there is an  $i$  such that for  $\Gamma = M_i$  we have  $\mu(\{Y : X = \Gamma^Y\}) > 0$ . By the Lebesgue density theorem there is a partial 0-1-valued function  $\sigma$  with finite domain such that for all  $n$ ,  $\mu(\{Y : X(n) = \Gamma_{(\sigma)}^Y(n)\}) > 3/4$ , and consequently,  $X$  is computable.

## 7.2 Results on Null Classes

Above we introduced the term  $\mu_{\text{Kurtz}}(\mathcal{C}) = 0$  to denote that a class  $\mathcal{C}$  is a Kurtz null class. Similarly, we denote Schnorr, computable, and Martin-Löf null classes by substituting  $\mu_{\text{Kurtz}}$  above by  $\mu_{\text{Schnorr}}$ ,  $\mu_{\text{comp}}$ , and  $\mu_{\text{ML}}$ , respectively.

It is straightforward from the various characterizations of Schnorr randomness and Kurtz randomness that the former implies the latter. We also note that obviously every Kurtz null class is a Schnorr null class. Hence for any measurable class  $\mathcal{C} \subseteq 2^\omega$ ,

$$\begin{aligned} \mu_{\text{Kurtz}}(\mathcal{C}) = 0 &\implies \mu_{\text{Schnorr}}(\mathcal{C}) = 0 \implies \mu_{\text{comp}}(\mathcal{C}) = 0 \implies \\ &\mu_{\text{ML}}(\mathcal{C}) = 0 \implies \mu(\mathcal{C}) = 0, \end{aligned}$$

where in general none of the implications can be reversed. Suitable choices for  $\mathcal{C}$  such that the last implication can be reversed, are presented in the following lemma due to Book, Lutz, and Wagner.

**Lemma 7.10** (Book, Lutz, and Wagner [4]). *If  $\mathcal{C}$  is a  $\Pi_1^0$ -class with  $\mu(\mathcal{C}) = 0$  then  $\mathcal{C}$  is a Martin-Löf null class.*

It is easy to modify the proof of Lemma 7.10 to get a proof of the following stronger result.

**Lemma 7.11.** *If  $\mathcal{C}$  is  $\Pi_1^0$ -class with  $\mu(\mathcal{C}) = 0$ , then  $\mathcal{C}$  is a Kurtz null class.*

*Proof.* Let  $\mathcal{C}$  be a  $\Pi_1^0$ -class with  $\mu(\mathcal{C}) = 0$ . Then by definition there is a computably enumerable set  $A = W_i$  such that  $\mathcal{C} = \overline{[A]}$ . Let  $f$  be a computable function such that  $A = \text{rng } f$  and let  $A_j = \{f(0), \dots, f(j)\}$ . Define for each  $n$ ,

$$j(n) = \min\{j: \mu[A_j] \geq 1 - 2^{-n}\}.$$

The function  $j(\cdot)$  is total because by hypothesis on  $\mathcal{C}$  we have that  $\mu[A] = 1$ ; furthermore,  $j(\cdot)$  is obviously computable. To define a Kurtz null test  $V_0, V_1 \dots$ , which covers  $\mathcal{C}$ , consider for each  $n$  the maximal length  $m(n)$  of the strings contained in  $A_{j(n)}$ . Now let

$$V_n = \{\sigma \in 2^{m(n)}: \sigma \text{ has no prefix in } A_{j(n)}\}$$

for each  $n$ . It is straightforward to define a computable function  $g$  such that for all  $n$ ,  $g(n)$  is the canonical index for  $V_n$ . Furthermore it is immediate from the construction that  $\mu[V_n] \leq 2^{-n}$  and  $\mathcal{C} \subseteq [V_n]$  for every  $n$ .  $\square$

**Corollary 7.12** (of the proof of Lemma 7.11). *From an index  $i$  where  $\mathcal{C} = \overline{[W_i]}$  is a Lebesgue null class, one can compute an index  $k$  such that the Kurtz null test given by  $g = \varphi_k$  covers  $\mathcal{C}$ .*

The following theorem is a straightforward strengthening of a result due to Book, Lutz, and Wagner [4] which is based on Lemma 7.10. They show, in particular, that for each union  $\mathcal{C}$  of  $\Pi_1^0$ -classes which is closed under finite variation  $\mu(\mathcal{C}) = 1$  is equivalent to  $\mathcal{C}$  containing a Martin-Löf random sequence. The theorem below is based on Lemma 7.11 whence we get a strengthening from Martin-Löf randomness to Kurtz randomness.

**Theorem 7.13.** (i) *Let  $\mathcal{C}$  be a union of  $\Pi_1^0$ -classes that is closed under finite variation. Then*

$$\mu(\mathcal{C}) = 1 \iff \mathcal{C} \cap \mathcal{KR} \neq \emptyset.$$

(ii) Let  $\mathcal{C}$  be an intersection of  $\Sigma_1^0$ -classes that is closed under finite variation. Then

$$\mu(\mathcal{C}) = 1 \iff \mathcal{KR} \subseteq \mathcal{C}.$$

*Proof.* “ $\Rightarrow$ ” for (i) and “ $\Leftarrow$ ” for (ii) are immediate from  $\mu(\mathcal{KR}) = 1$ . For the converse direction of (i), assume  $\mu(\mathcal{C}) < 1$ . By hypothesis on  $\mathcal{C}$  we can apply the Kolmogorov 0-1-Law, so  $\mu(\mathcal{C}) = 0$ . By Lemma 7.11,  $\mathcal{C}$  is a union of Kurtz null classes, hence  $\mathcal{C}$  has empty intersection with  $\mathcal{KR}$ . To show “ $\Rightarrow$ ” for (ii), assume  $\mathcal{KR} \not\subseteq \mathcal{C}$ , whence  $\overline{\mathcal{C}} \cap \mathcal{KR} \neq \emptyset$ . Note that  $\overline{\mathcal{C}}$  is a union of  $\Pi_1^0$ -classes as in the hypothesis of (i), so it follows that  $\mu(\mathcal{C}) = 0$ .  $\square$

By Kolmogorov’s 0-1-Law, item (i) of the above theorem implies the following: a union  $\mathcal{C}$  of  $\Pi_1^0$ -class which is closed under finite variation has Lebesgue measure zero if and only if  $\mathcal{C}$  contains no Kurtz random sequence. While the latter is the same as saying that for each sequence  $X \in \mathcal{C}$  there is some Kurtz null test which covers  $X$ , it does not mean that  $\mu_{\text{Kurtz}}(\mathcal{C}) = 0$  in general. If we weaken the latter condition to  $\mu_{\text{Schnorr}}(\mathcal{C}) = 0$  then this new condition is equivalent to  $\mu(\mathcal{C}) = 0$  for certain unions  $\mathcal{C}$  of  $\Pi_1^0$ -classes, namely for  $\Sigma_2^0$ -classes.

**Theorem 7.14.** *For every  $\Sigma_2^0$ -class  $\mathcal{C}$ ,*

$$\mu(\mathcal{C}) = 0 \iff \mu_{\text{Schnorr}}(\mathcal{C}) = 0. \quad (7.1)$$

The theorem can be proved by using Lemma 7.11 and by observing that an effective union of  $\Pi_1^0$ -Schnorr null classes is a Schnorr null class.

As we show next, the assertion of Theorem 7.14 becomes false if we replace  $\mu_{\text{Schnorr}}$  by  $\mu_{\text{Kurtz}}$  in (7.1).

**Theorem 7.15.** *There is a  $\Sigma_2^0$ -class which is a Lebesgue null class (and hence a Schnorr null class) but not a Kurtz null class.*

*Proof.* We construct a  $\Sigma_2^0$ -class  $\mathcal{C} = \bigcup_{i \in \omega} \overline{[W_{h(i)}]}$  where each  $\Pi_1^0$ -class  $\overline{[W_{h(i)}]}$  either is the empty set or consists of exactly one computable sequence such that we meet all of the following requirements  $\mathcal{R}_e$  by which we diagonalize against (a superclass of) the Kurtz null tests.

$$\begin{aligned} \mathcal{R}_e : & \text{ If } (\exists x \in \omega) \left[ x \geq 1 \quad \& \quad \varphi_e(x) \downarrow \quad \& \quad \mu[D_{\varphi_e(x)}] \leq 2^{-x} \right] \\ & \text{ then } (\exists i \in \omega) \left[ \overline{[W_{h(i)}]} \cap [D_{\varphi_e(x)}] = \emptyset \right]. \end{aligned}$$

The construction is done in stages  $s = 0, 1, \dots$  where at the end of stage  $s$  we define  $h(s)$ . Say that a requirement  $\mathcal{R}_e$  at stage  $s$  with  $e \leq s$  *requires attention* if we have not yet met  $\mathcal{R}_e$  and if there is an  $x$  such that

$$1 \leq x \leq s \quad \& \quad \varphi_{e,s}(x) \downarrow \quad \& \quad \mu [D_{\varphi_{e,s}(x)}] \leq 2^{-x}. \quad (7.2)$$

At stage  $s$ , if there is no  $\mathcal{R}_e$  requiring attention then let  $h(s)$  be equal to some index such that  $[W_{h(s)}] = 2^\omega$ . Otherwise, consider the least  $e$  such that  $\mathcal{R}_e$  requires attention and meet  $\mathcal{R}_e$  as follows. Fix the least  $x$  such that (7.2) is satisfied. Let  $h(s)$  be such that  $\overline{[W_{h(s)}]}$  contains only the computable sequence  $X$  which is lexicographically least in the complement of  $[D_{\varphi_{e,s}(x)}]$ . Obviously, by the above procedure we meet all requirements and moreover, the constructed  $\Sigma_2^0$ -class has Lebesgue measure 0.  $\square$

Book, Lutz, and Wagner [4] show that given a bounded reducibility  $\leq_R$ , the upper  $\leq_R$ -cone of a computable sequence is a union of  $\Pi_1^0$ -classes. In fact, as is implicit in their proof, the following stronger result is true.

**Lemma 7.16.** *Suppose  $R$  is a bounded reducibility and  $Y$  is a computable sequence. Then the upper  $\leq_R$ -cone of  $Y$  is a  $\Sigma_2^0$ -class.*

*Proof.* Let  $g$  be a computable presentation of  $R$  and for each  $i$ , let

$$R_i^{-1}(Y) = \{X : Y = L(M_{g(i)}^X)\}.$$

Then  $R^{-1}(Y) = \bigcup_{i \in \omega} R_i^{-1}(Y)$ , and we shall prove that each  $R_i^{-1}(Y)$  is a  $\Pi_1^0$ -class. Equivalently, we show that each complement  $\mathcal{C}_i = 2^\omega - R_i^{-1}(Y)$  is a  $\Sigma_1^0$ -class. Let  $m_i^X$  denote the use function of  $M_i^X$  for any  $X \in 2^\omega$  and  $i \in \omega$ . We define a partial computable function  $h_i$  on pairs of strings as follows. For any strings  $\sigma$  and  $\tau$ ,

$$\text{let } h_i(\sigma, \tau) = \tau \quad \text{if } Y(\sigma) \neq M_{g(i)}^{\tau 0^\omega}(\sigma) \text{ and } m_{g(i)}^{\tau 0^\omega}(\sigma) \leq |\tau|.$$

Otherwise let  $h_i(\sigma, \tau)$  be undefined. For every  $X \in 2^\omega$ , we obviously have

$$X \in \mathcal{C}_i \iff (\exists n) X \upharpoonright n \in \text{rng } h_i \iff X \in [\text{rng } h_i].$$

Thus  $\mathcal{C}_i$  is a  $\Sigma_1^0$ -class for each  $i$ . Furthermore, it is obvious from the above definition of the functions  $h_i$  that indices of these functions can be produced in an effective way. Hence  $R^{-1}(Y) = \bigcup_{i \in \omega} R_i^{-1}(Y)$  is a  $\Sigma_2^0$ -class.  $\square$

### 7.3 Almost Classes and Computable Depth

By definition, the Almost class with respect to some reducibility  $R$  is the class of sequences  $X$  such that the upper  $R$ -cone of  $X$  has Lebesgue measure 1. Book, Lutz, and Wagner [4] use Almost classes to give characterizations of complexity classes in terms of Martin-Löf random sequences. We show that Almost classes w.r.t. bounded reducibilities have computable packing dimension 0. This result is obtained by showing that every Almost class w.r.t. a bounded reducibility is contained in a computably presentable class, and by showing that every computably presentable class has computable packing dimension 0.

Subsequently we consider computably deep sequences which were introduced by Fenner, Lutz, Mayordomo, and Reardon [18]. For any bounded reducibility that is upwards closed under finite variation, we show that a sequence is contained in the respective Almost class if and only if it is computable and not  $R$ -deep.

**Definition 7.17.** *Let  $R$  be a reducibility. Then the Almost class with respect to  $R$  is  $\text{ALMOST}_R = \{X : \mu(\{Y : X \leq_R Y\}) = 1\}$ .*

We note that every sequence in  $\text{ALMOST}_R$  is computable. This is an immediate consequence of the generalized version of Sacks' theorem, as discussed below Theorem 7.9.

**Theorem 7.18.** (i) *For every bounded reducibility  $\leq_R$ ,  $\text{ALMOST}_R$  is contained in a computably presentable class.*

(ii) *Let  $\leq_R$  be a bounded reducibility that is upwards and downwards closed under finite variation. Then  $\text{ALMOST}_R$  is a computably presentable class.*

*Proof.* For a proof of (i), let  $g$  be a computable presentation of  $\leq_R$  and suppose we have a bijective function from  $\omega \times 2^{<\omega}$  to  $\omega$  that is a natural coding which takes each pair consisting of a number and a string to a code number. For each  $s \in \omega$ , we define a sequence  $X_s$  as follows. Suppose  $s$  codes a pair  $(e, \sigma)$  consisting of a number  $e$  and a string  $\sigma$ . In order to determine  $X_s(n)$  for an  $n \in \omega$ , we check whether for all  $m \leq n$  there is a bit  $b_{s,m}$  such that for  $\Gamma = M_{g(e)}$  we have

$$\mu\left(\left\{Y : b_{s,m} = \Gamma_{(\sigma)}^Y(m)\right\}\right) > \frac{3}{4}, \quad (7.3)$$



where  $\Gamma_{(\sigma)}$  is the  $\sigma$ -patch of  $\Gamma$ . If this property is satisfied then we let  $X_s(n) = b_{s,n}$ , else  $X_s(n) = 0$ . Now consider some computable function  $h$  which takes each  $s$  to an index of a machine that computes the characteristic function of  $X_s$ . We claim that  $h$  is a computable presentation of a class  $\mathcal{C}$  which contains  $\text{ALMOST}_R$ . Indeed, suppose  $X \in \text{ALMOST}_R$ . Then by  $\sigma$ -additivity of  $\mu$ , there is an  $\varepsilon > 0$  and an  $i$  such that we have  $\mu(\{Y : X = L(M_{g(i)}^Y)\}) > \varepsilon$ . By the Lebesgue Density Theorem there is a string  $\sigma$  such that for  $\Gamma = M_{g(i)}$  we have that  $\mu(\{Y : X = L(\Gamma_{(\sigma)}^Y)\}) > 3/4$ . Hence by construction,  $X$  is an element of the sequence  $X_0, X_1, \dots$  and thus  $X$  is contained in  $\mathcal{C}$ .

To prove (ii), first note that  $\text{ALMOST}_R$  is closed under finite variation because  $\leq_R$  is downwards closed under finite variation. It suffices to change the definition of the sequences  $X_0, X_1, \dots$  in the proof of (i) as follows.

Suppose  $Z \in \text{ALMOST}_R$ . Then for any  $s$  coding some pair  $(e, \sigma)$ , and for any  $n$ , check whether for all  $m \leq n$  there is a bit  $b_{s,m}$  such that for  $\Gamma = M_{g(e)}$  we have

$$\mu \left( \bigcap_{m \leq n} \{Y : b_{s,m} = \Gamma_{(\sigma)}^Y(m)\} \right) > \frac{3}{4}.$$

If this condition is satisfied then let  $X_s(n) = b_{s,n}$  else  $X_s(n) = Z(n)$ .

Now consider the resulting class  $\mathcal{C}$  as in the proof of (i). We claim that  $\mathcal{C} = \text{ALMOST}_R$ . As in the previous proof, we can argue that  $\mathcal{C} \supseteq \text{ALMOST}_R$ . Conversely, suppose  $X \in \mathcal{C}$ . If  $\mu(R^{-1}(X)) = 0$  then for any  $(e, \sigma)$ , by continuity of the reductions, from some number on  $X$  will contain exactly the same elements as  $Z$ , so  $X$  and  $Z$  differ at most at finitely many places. Since  $\text{ALMOST}_R$  is closed under finite variation, it follows that  $X \in \text{ALMOST}_R$ . On the other hand, suppose  $\mu(R^{-1}(X)) > 0$ . Now  $R^{-1}(X)$  is closed under finite variation because by hypothesis,  $\leq_R$  is upwards closed under finite variation. Hence by Kolmogorov's 0-1-Law we get that  $\mu(R^{-1}(X)) = 1$ , i.e.,  $X \in \text{ALMOST}_R$ . It follows  $\mathcal{C} = \text{ALMOST}_R$  as desired.  $\square$

In the next two theorems we show measure and dimension properties of computably presentable classes.

**Theorem 7.19.** *There exists a computably presentable class which is not a Kurtz null class.*

The theorem follows easily from the proof of Theorem 7.15. Just observe that the classes  $\overline{[W_{h(s)}]}$  form a computably presentable class.

For the next theorem we recall the definition of computable packing dimension that was introduced by Athreya, Hitchcock, Lutz, and Mayor-domo [3].

**Definition 7.20.** (i) For any  $s \geq 0$ , we say that a computable martingale  $d$  is strongly  $s$ -successful on a class  $\mathcal{C} \subseteq 2^\omega$  if for each sequence  $X \in \mathcal{C}$ ,

$$(\forall^\infty n) d(X \upharpoonright n) \geq 2^{(1-s)n}. \quad (7.4)$$

(ii) The computable packing dimension of a class  $\mathcal{C}$  is defined by

$$\dim_{\mathbb{P}}^{\text{comp}} \mathcal{C} = \inf\{s: \text{some computable martingale } d \\ \text{is strongly } s\text{-successful on } \mathcal{C}\}.$$

**Theorem 7.21.** Every computably presentable class has computable packing dimension 0.

*Proof.* Let  $g$  be a computable presentation of some arbitrary computably presentable class  $\mathcal{C} \subseteq 2^\omega$  and pick some  $s > 0$ . We present a computable betting strategy  $b$  such that the corresponding computable martingale  $d$  with initial capital  $d(\varepsilon) = 1$  is  $s$ -successful on  $\mathcal{C}$ . During the game, the strategy  $b$  computes a nondecreasing sequence of numbers  $k_0 = 0, k_1, k_2, \dots$  and places bets as follows. On input  $\sigma$  with  $|\sigma| = n > 0$ ,  $k_n$  is set to the least  $k$  such that

$$k_{n-1} \leq k \leq n \quad \& \quad L(M_{g(k)}) \upharpoonright n = \sigma,$$

if such a  $k$  exists. Otherwise,  $k_n$  is set equal to  $k_{n-1}$ . Now we specify the bets. If the input is the empty string, then no bet is placed. Otherwise, the total amount of  $d(\sigma) - 2^{-(n+1)}$  is placed on a bet on the next bit being equal to  $L(M_{g(k_n)})(n)$ . Note that the resulting martingale  $d$  is total on all sequences. If the betting strategy is applied to a sequence  $X \in \mathcal{C}$ , then the sequence  $k_0, k_1, \dots$  will eventually not change any more. Say for some  $i_0$  we have that  $k_{i_0} = k_j$  for all  $j \geq i_0$ . Then  $d$  is increased by the respective stakes on all inputs of length at least  $i_0$ , and thus we have that (7.4) is satisfied for all  $s > 0$ .  $\square$

The following theorem is an immediate consequence of Theorems 7.18 and 7.21.

**Theorem 7.22.** For every bounded reducibility  $\leq_R$ ,  $\text{ALMOST}_R$  has computable packing dimension 0.

Now we turn to computably deep sequences as defined by Fenner, Lutz, Mayordomo, and Reardon [18].

**Definition 7.23.** *Let  $\leq_R$  be a bounded reducibility. A sequence  $X$  is computably  $R$ -deep if  $\mu_{\text{comp}}(R^{-1}(X)) = 0$ .*

**Theorem 7.24.** *Let  $\leq_R$  be a bounded reducibility which is upwards closed under finite variation. Then a sequence  $X$  is contained in  $\text{ALMOST}_R$  if and only if  $X$  is computable and not  $R$ -deep.*

*Proof.* First we note that  $\text{ALMOST}_R$  is contained in the class of computable sequences by the discussion following Definition 7.17. By Lemma 7.16, the upper  $\leq_R$ -cone of a computable sequence  $X$  is a  $\Sigma_2^0$ -class. Hence by Theorem 7.14 a computable sequence  $X$  is computably  $R$ -deep if and only if the upper  $\leq_R$ -cone of  $X$  has Lebesgue measure 0. Now the upper  $\leq_R$ -cone of  $X$  is closed under finite variation since by hypothesis,  $\leq_R$  is upwards closed under finite variation. Thus by Kolmogorov's 0-1-Law a computable sequence  $Y$  is not computably  $R$ -deep if and only if  $Y \in \text{ALMOST}_R$ .  $\square$



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