Control problems for nonlocal set evolutions

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Abstract

In this paper, we extend fundamental notions of control theory to evolving compact subsets of the Euclidean space.

Dispensing with any restriction of regularity, shapes can be interpreted as nonempty compact subsets of the Euclidean space $\mathbb{R}^N$. Their family $\mathcal{K}(\mathbb{R}^N)$, however, does not have any obvious linear structure, but in combination with the popular Pompeiu-Hausdorff distance $d\!\!l$, it is a metric space. Here Aubin's framework of morphological equations is used for extending ordinary differential equations beyond vector spaces, namely to the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.

Now various control problems are formulated for compact sets depending on time: open-loop, relaxed and closed-loop control problems – each of them with state constraints. Using the close relation to morphological inclusions with state constraints, we specify sufficient conditions for the existence of compact-valued solutions.

Contents

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1 Introduction

"Shapes and images are basically sets, not even smooth" as Aubin stated [3]. So whenever we want to investigate evolving shapes in full generality, we have to focus on subsets of the Euclidean space. In particular, these subsets should be only supposed to be nonempty and compact – but lacking any further assumptions about the regularity of their topological boundaries.

The main goal of this paper is to extend fundamental concepts of control theory to time-dependent compact subsets of the Euclidean space. Here the essential challenge results from the lacking vector space structure. Indeed, nonempty compact subsets of $\mathbb{R}^N$ do not have any obvious linear structure, but in combination with the well-known Pompeiu–Hausdorff distance $d\!l$, for example, they represent a metric space.

So the differential tools of classical control theory have to be extended step by step beyond the traditional border of vector spaces. For this purpose we continue a track initiated by Jean-Pierre Aubin in the 1990s: morphological equations and inclusions. They provide extensions of ordinary differential equations and differential inclusions respectively to the metric space $(\mathcal{K}(\mathbb{R}^N), d\!l)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu-Hausdorff distance.

In this paper, open-loop, relaxed and closed-loop control problems with state constraints are formulated for shapes, i.e. in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!l)$. A viability theorem presented by the author in [18] then lays the foundations for specifying conditions sufficient for the existence of their set-valued solutions.

Introducing nonrestrictive variations of compact sets in $\mathbb{R}^N$

Whenever a shape is to be optimized (in some sense), we require an appropriate form of "shape variations" for verifying if a compact set under consideration is a local minimizer or not. The so–called *velocity method* or *speed method* suggests an approach to hardly restrictive shape variations and, it has led Céa, Delfour, Zolésio and others to remarkable results about shape optimization (see e.g. [9, 11, 12, 28, 30] and references there). It is based on prescribing a vector field $v : \mathbb{R}^N \times [0, T] \longrightarrow \mathbb{R}^N$ such that the corresponding ordinary differential equation $\frac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot)$ induces a unique flow on $\mathbb{R}^N$. Indeed, supposing v to be sufficiently smooth, the Cauchy problem

$$\tfrac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot) \text{ in } [0, T], \qquad x(0) = x_0 \in \mathbb{R}^N$$

is always well–posed and, any compact initial set $K \subset \mathbb{R}^N$ is deformed to

$$\vartheta_v(t, K) \;:=\; \big\{\, x(t) \;\big|\; \exists\, x(\cdot) \in C^1([0, t], \mathbb{R}^N) : \; \tfrac{d}{dt}\, x(\cdot) = v(x(\cdot), \cdot) \text{ in } [0, t], \;\; x(0) \in K \big\}$$

after an arbitrary time $t \geq 0$. As a key advantage, this concept of set evolution does not require any regularity conditions on the compact set K or its topological boundary (but only on the vector field v). In a word, v can be interpreted as a "direction of deformation" in $(\mathcal{K}(\mathbb{R}^N), d\!l)$. So it is "possible to define directional derivatives and speak of shape gradient and shape Hessian with respect to the associated vector space of velocities. This second approach has been known in the literature as the *velocity method*" [11, Chapter 1, § 6]. (The 'first' approach mentioned there in [11] refers to perturbations of the identity map and applying techniques of differential geometry.)

Aubin seized this notion for extending ordinary differential equations to this metric space $(\mathcal{K}(\mathbb{R}^N), d\!l)$. The so–called *morphological equations* are sketched in [5] and then presented in [3, 4] in more detail. (They seem to be closer to ODEs in $\mathbb{R}^N$ than Panasyuk's similar concept of "quasidifferential equations" [23, 24, 25].)

The first aspect of generalization focuses on the "elementary set deformation" which are to describe the directions in $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$. Aubin suggested reachable sets of differential inclusions as a more general alternative to the velocity method. For any set–valued map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ and initial set $K \subset \mathbb{R}^N$ given, the so–called *reachable set* at time $t \geq 0$ is defined as

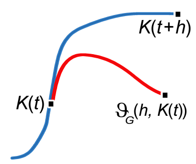
$$\begin{aligned} \vartheta_G(t, K) \quad := \quad \Big\{ x(t) \in \mathbb{R}^N \;\Big|\; & \exists\ x(\cdot) \in W^{1,1}([0,t],\, \mathbb{R}^N) :\ x(0) \in K, \\ & \tfrac{d}{d\tau}\, x(\tau) \in G(x(\tau)) \ \text{ for Lebesgue–almost every } \ \tau \in [0,t] \Big\}. \end{aligned}$$

In contrast to the velocity method, this kind of set deformation does not have to be reversible in time. (Geometrically speaking, "holes" of sets can disappear while expanding.) The well–known Theorem of Filippov ensures suitable properties of $[0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(t,K) \longmapsto \vartheta_G(t,K)$ if $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty compact values and is bounded Lipschitz continuous. In fact, the Relaxation Theorem of Filippov–Ważiewski (e.g. [2, § 2.4, Theorem 2]) implies no changes of reachable sets if each value of G is replaced by its convex hull. So we are always free to consider bounded Lipschitz continuous maps $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ whose nonempty compact values are convex in addition.

Differential inclusions with Lipschitz right–hand side
for specifying time derivatives of curves in $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$

The second key contribution of Aubin is a suggestion how to interprete such a set–valued map (or, strictly speaking, its reachable sets) as time derivative of a curve in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$.
Indeed, let $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ be a curve. A bounded Lipschitz set–valued map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact values represents a first–order approximation of $K(\cdot)$ at time $t \in [0,T[$ if

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!\!l\big(K(t+h),\ \vartheta_G(h,\, K(t))\big) \ = \ 0. \qquad (*)$$



Of course, such a map $G(\cdot)$ does not have to be unique and thus, *all* bounded Lipschitz maps with this property $(*)$ form the so–called *morphological mutation* $\overset{\circ}{K}(t)$ of $K(\cdot)$ at time $t \in [0,T[$. It is a subset of $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ denoting the set of all bounded Lipschitz maps $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact values. Correspondingly, $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ consists of all bounded Lipschitz maps $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact *and convex* values. $\overset{\circ}{K}(t) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ extends the time derivative to curves in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$.

Sets determine their own evolution: Morphological equations and inclusions

Ordinary differential equations are based on the fundamental notion of prescribing the time derivative of the wanted curve as a function of its current state and time. Now we are free to formulate the same problem for a set-valued curve in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$ as mutations are available:
For a function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ given, a Lipschitz continuous curve $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is called *solution* to the morphological equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ in $[0,T]$ if at Lebesgue-almost every time $t \in [0,T]$, the map $f(K(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the mutation $\overset{\circ}{K}(t)$ [3], i.e. by definition, the reachable set $\vartheta_{f(K(t))}(\cdot, K(t))$ satisfies

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!\!l\big(K(t+h),\ \vartheta_{f(K(t))}(h,\, K(t))\big) \ = \ 0.$$

At first glance, the term "equation" and the symbol $\ni$ might make a contradictory impression, but the mutation $\overset{\circ}{K}(t)$ has just been defined as *set* of all set-valued maps $G \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ sharing property $(*)$ above. (Strictly speaking, all these set-valued maps belong to the same equivalence class related with vanishing distances up to first order. In the following, however, we do not use the underlying equivalence relation explicitly because it does not provide additional insight, see [3, § 1.1] for more details.)

In this framework, Aubin extended the classical Theorem of Cauchy-Lipschitz (about existence and uniqueness of solutions) from ordinary differential equations to morphological equations as quoted in subsequent Theorem 2.7. Meanwhile also the counterpart of Peano's existence theorem has been proved and, Nagumo's classical result about existence of solutions satisfying state constraints has been verified for morphological equations as summarized in section 2.
So in a word, all relevant terms are now available for introducing control theory in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ of nonempty compact subsets of $\mathbb{R}^N$.

Considering ordinary differential equations and classical control theory in finite dimensions, differential inclusions and selection principles have played a key role. In this paper, we follow essentially the same track in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$. Indeed, the step from morphological equations to morphological inclusions is based on admitting more than just one set deformation for each state in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, i.e. the single-valued function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is replaced by a set-valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Correspondingly, a Lipschitz continuous curve $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is called *solution* to the morphological inclusion with $\mathcal{F}$ if at Lebesgue-almost every time $t \in [0, T]$, at least one map in $\mathcal{F}(K(t)) \subset \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ also belongs to the mutation $\overset{\circ}{K}(t)$, i.e. there exists a set-valued map $G \in \mathcal{F}(K(t)) \subset \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ satisfying
$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l\big(K(t+h),\ \vartheta_G(h,\ K(t))\big) \ = \ 0.$$

Reflecting this notion of a joint map in $\mathcal{F}(K(t))$ and $\overset{\circ}{K}(t) \subset \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, a morphological *inclusion* has to be written as *intersection* condition: $\quad \overset{\circ}{K}(\cdot) \ \cap \ \mathcal{F}(K(\cdot)) \ \neq \ \emptyset \quad$ (almost everywhere) in $[0, T]$.

Solutions to morphological inclusions are reachable sets with feedback

Consider a Lipschitz continuous solution $K(\cdot) : [0, T] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ to a morphological inclusion $\overset{\circ}{K}(\cdot) \ \cap \ \mathcal{F}(K(\cdot)) \ \neq \ \emptyset$ in $[0, T]$ with a given set-valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. The metric condition on $\overset{\circ}{K}(t)$ mentioned before has a geometric interpretation:
Indeed, for almost every $t \in [0, T]$, there exists a set-valued map $G_t \in \overset{\circ}{K}(t) \ \cap \ \mathcal{F}(K(t)) \subset \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ by definition. Let us extend $t \mapsto G_t \in \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ to the whole interval $[0, T]$ arbitrarily. Then, $\widetilde{G} : \mathbb{R}^N \times [0, T] \rightsquigarrow \mathbb{R}^N$, $(x, t) \mapsto G_t(x)$ is a set-valued map of both space and time and, we use it as right-hand side of a nonautonomous differential inclusion in $\mathbb{R}^N$, namely $x'(\cdot) \in \widetilde{G}(x(\cdot), \cdot)$ a.e. in $[0, T]$. Under appropriate assumptions about $\widetilde{G}$, its reachable set $\vartheta_{\widetilde{G}}(t, K(0)) \subset \mathbb{R}^N$ is nonempty compact at every $t \in [0, T]$ and, it even coincides with $K(t)$: $\quad K(t) \ = \ \vartheta_{\widetilde{G}}(t, K(0)) \quad$ for each $t \in [0, T]$
So $K(\cdot) : [0, T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ is characterized equivalently as reachable set of a *nonautonomous* differential inclusion in $\mathbb{R}^N$ whose set-valued right-hand side $\widetilde{G} : \mathbb{R}^N \times [0, T] \rightsquigarrow \mathbb{R}^N$ is induced by a selection of $\mathcal{F}(K(\cdot)) : [0, T] \rightsquigarrow \text{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ (see § 3.2 for more details). To the best of our knownledge, the detailed proof of this equivalence is given here for the first time.

In a word, each solution $K(\cdot) : [0, T] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ to a morphological equation or inclusion is directly related to reachable sets of a nonautonomous differential inclusion in $\mathbb{R}^N$ whose right–hand side depends on the wanted curve $K(\cdot)$. So this framework covers some types of *nonlocal set evolutions with feedback*.

Control problems for compact sets via morphological inclusions

Similarly to classical control theory in $\mathbb{R}^N$, a metric space (U, d_U) of control parameter and a single-valued function $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ of state and control are given. For each initial set $K(0) \in \mathcal{K}(\mathbb{R}^N)$, we are looking for a Lipschitz continuous curve $K(\cdot) : [0,T] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ solving the following nonautonomous morphological equation

$$\overset{\circ}{K}(t) \ni f(K(t), u(t)) \qquad \text{in } [0,T[$$

with a measurable control function $u(\cdot) : [0,T] \longrightarrow U$, i.e. by definition

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l \left(\vartheta_{f(K(t),u(t))}(h,\ K(t)),\ \ K(t+h)\right) \ = \ 0 \qquad \text{for almost every } t \in [0,T].$$

This is an open-loop control problem in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.
The existence of solutions is closely related to the corresponding morphological inclusion for which we take all admitted controls into consideration simultaneously. So we introduce the set-valued map

$$\mathcal{F}_U : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad K \mapsto \{f(K,u) \mid u \in U\} \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$$

and consider the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}_U(K(\cdot)) \neq \emptyset$ in $[0,T[$. In § 3.2, Proposition 3.3, sufficient conditions on U and f are formulated such that solutions to this morphological inclusion solve the morphological control problem and vice versa.
The step from inclusion to control problem requires the existence of a measurable control function and, it is concluded here from a well-known selection principle of Filippov whose Euclidean special case is usually applied to differential inclusions in $\mathbb{R}^N$ and classical control theory.

So all available results about morphological inclusions can be used for solving morphological control problems. In the following, a viability theorem presented by the author in [18] plays a key role. It concerns a morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ with state constraints $K(t) \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ at every time t and it is quoted in subsequent Theorem 2.13.
This viability theorem specifies sufficient conditions on $\mathcal{F}$ and the nonempty set $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ of constraints such that at least one solution $K(\cdot) : [0,1] \longrightarrow \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ starts at each initial set $K(0) \in \mathcal{V}$. So in § 3.3, the close relationship between morphological inclusions and control problems provides directly sufficient conditions on a morphological control system with state constraints for the existence of solutions (Proposition 3.7).
In § 3.4, essentially the same approach is then used for solving relaxed control problems in the morphological framework. They are based on replacing the metric space U of control parameters by the set of Borel probability measures on U (supplied with the linear Wasserstein metric). As immediate analytical benefit, we can weaken some conditions of convexity in Proposition 3.13.

The viability condition for morphological inclusions: "Admit a 'tangential' reachable set"

For differential inclusions in $\mathbb{R}^N$, the viability condition on a nonempty closed subset $V \subset \mathbb{R}^N$ is well-known [6]: Under appropriate assumptions about the set-valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, a solution $x(\cdot)$ of $x'(\cdot) \in F(x(\cdot))$ with all values in $V \subset \mathbb{R}^N$ starts at each point of V if and only if at every point $x \in V$, the set $F(x) \subset \mathbb{R}^N$ contains at least one vector v being "contingent" to V (in the sense of Bouligand), i.e.

$$\liminf_{h \downarrow 0} \ \tfrac{1}{h} \cdot \mathrm{dist}(x + h\, v, V) = 0.$$

As main result of [18], essentially the same viability condition – just formulated with reachable sets and in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ – is also sufficient for a morphological inclusion and any nonempty closed set of constraints $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$:

Under appropriate assumptions about the set-valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, a solution $K : [0,1] \longrightarrow \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ of $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ starts at each set $K(0) \in \mathcal{V}$ if for every set $K_0 \in \mathcal{V}$, at least one map $G \in \mathcal{F}(K_0) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is "contingent" to $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ in the following sense

$$\liminf_{h \downarrow 0} \ \tfrac{1}{h} \cdot \mathrm{dist}\,(\vartheta_G(h, K_0),\ \mathcal{V}) \ = \ 0. \qquad (**)$$

An example: Morphological control problems under "strong operability" constraints

In [17], Anne Gorre investigated morphological equations under the constraint that all evolving sets $K(t) \subset \mathbb{R}^N$ are contained in a fixed closed set $M \subset \mathbb{R}^N$. So the corresponding set of constraints is

$$\mathcal{V}_M \ := \ \big\{ K \in \mathcal{K}(\mathbb{R}^N) \ \big| \ K \subset M \big\}.$$

and Gorre coined the term "strongly operable in M". This type of constaint occurs, for example, when a robot is to walk or stand in a stable way (consider the projection of its highly sensitive center of gravity [17]) and when a bioreactor has to provide a suitable environment for a growing cell population. Gorre's exact characterization of "contingent to $\mathcal{V}_M$" is used here in combination with morphological inclusions and, so we obtain directly sufficient conditions for morphological control problems under strong operability constraints.

The step to closed-loop control problems for compact sets in $\mathbb{R}^N$

Consider morphological control problems with state constraints

$$\begin{cases} \overset{\circ}{K}(\cdot) \ \ni \ f(K(\cdot), u), \quad u \in U & \text{a.e. in } [0, T[\\ K(t) \ \in \ \mathcal{V} & \text{for every } t \in [0, T[. \end{cases}$$

The metric space (U, d_U) of control, function $f : \mathcal{K} \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and the closed set $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ of constraints are given. The morphological viability condition mentioned before indicates where candidates for a closed-loop control $u : \mathcal{V} \longrightarrow U$ can be found, namely among those controls $u \in U$ whose reachable sets $\vartheta_{f(K,u)}(\cdot, K)$ are "contingent" to $\mathcal{V}$ in the sense of condition $(**)$. This reflects the notion of *regulation maps* defined by Aubin for control problems in finite-dimensional vector spaces [6, § 6].

In § 3.5, we specify sufficient conditions on $U, f, \mathcal{V}$ such that Michael's famous selection theorem implies the existence of a continuous closed-loop control (Proposition 3.17).
Michael's selection theorem (quoted here in Proposition 3.18), however, focuses on lower semicontinuous set-valued maps. So we need information about the semicontinuity properties of these regulation maps.

In this regard, the classical results about finite-dimensional vector spaces serve as motivation again. The Clarke tangent cone $T_V^C(x) \subset \mathbb{R}^N$, $x \in V$, to a nonempty closed set $V \subset \mathbb{R}^N$ (alias circatangent set, see Definition A.1) is known to have closed graph whereas the Bouligand contingent cone to the same set does not have such a semicontinuity feature in general [7, 27]. Furthermore, Rockafellar characterized the interior of the convex Clarke tangent cone $T_V^C(x) \subset \mathbb{R}^N$ by a topological criterion leading to the so-called hypertangent cone ([26, Theorem 2], [10, § 2,4] and quoted here in App. B). So the set-valued map of hypertangent cones to a fixed set $V \subset \mathbb{R}^N$ is lower semicontinuous whenever all these cones are nonempty.

These two concepts, i.e. Clarke tangent cone and hypertangent cone to a given closed set, are extended to the morphological framework where the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ has replaced the Euclidean space. In Appendix A, we apply Aubin's definition of "circatangent transition set" [3, Definition 1.5.4] to $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ together with reachable sets of differential inclusions. The result proves to be a nonempty closed convex cone in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. In Appendix B, the so-called hypertangent transition set is introduced for a nonempty closed subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$. Its graph is identical to the interior of the graph of circatangent transition sets in $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.
In particular, this topological characterization proves to be very helpful for constructing closed-loop controls on the basis of Michael's selection principle (Proposition 3.17).

This introduction reflects the structure of the paper: § 2 gives a survey of morphological equations and inclusions. In particular, it provides all definitions of this framework and summarizes the essential theorems used subsequently. In § 3, we focus on morphological control problems and explain the link with morphological inclusions (§ 3.2). These results are then applied to open-loop control problems with state constraints (§ 3.3), relaxed control problems with state constraints (§ 3.4) and finally closed-loop control problems with state constraints (§ 3.5).

2 A brief guide to morphological equations and inclusions

2.1 Morphological equations of Aubin

Morphological equations provide typical geometric examples of so–called mutational equations. First presented in [5] and elaborated in [4, 3], mutational equations are to extend ordinary differential equations to a metric space (E, d). In a word, the key idea is to describe derivatives by means of continuous maps (called *transitions*) $\vartheta : [0,1] \times E \longrightarrow E$, $(h, x) \longmapsto \vartheta(h, x)$ instead of affine–linear maps $(h, x) \longmapsto x + h\, v$ (that are usually used in *vector spaces*). Strictly speaking, such a transition specifies the point $\vartheta(t, x) \in E$ to which any initial point $x \in E$ has been moved after time $t \in [0, 1]$. It can be interpreted as a first–order approximation of a curve $\xi : [0, T[\longrightarrow E$ at time $t \in [0, T[$ if

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\big(\xi(t+h),\ \ \vartheta(h, \xi(t))\big) \ = \ 0.$$

The so–called *morphological equations* apply this concept to the set $\mathcal{K}(\mathbb{R}^N)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu–Hausdorff distance $d\!\!l$,

$$\begin{aligned} d\!\!l(K_1, K_2) \quad &:= \quad \sup_{\substack{x \in K_1,\\ y \in K_2}} \ \Big\{ \mathrm{dist}(x, K_2),\ \ \mathrm{dist}(y, K_1) \Big\} \\ &= \quad \inf \ \ \big\{ \rho > 0 \,\big|\, K_1 \subset K_2 + \rho\, \mathbb{B}_1,\ K_2 \subset K_1 + \rho\, \mathbb{B}_1 \big\}. \end{aligned}$$

Here $\mathbb{B}_1$ always denotes the closed unit ball in $\mathbb{R}^N$, i.e. $\mathbb{B}_1 := \{x \in \mathbb{R}^N \mid |x| \le 1\}$. This is a very general starting point for geometric evolution problems as there are no a priori restrictions in regard to the regularity of sets and their boundaries. Motivated by the velocity method (often used in shape optimization, e.g. [9, 11, 12, 28, 30]), the flow along ordinary differential equations can lay the basis for transitions. Here, however, we follow a suggestion of Aubin (in [3, 4]) and consider a more general approach of evolutions instead: autonomous differential inclusions and their reachable sets.

Definition 2.1 ([3, Definition 3.7.1]) $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *consists of all set–valued maps* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *satisfying*

1. *F has nonempty compact values that are uniformly bounded in $\mathbb{R}^N$,*
2. *F is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance.*

$\mathrm{Lip}(M, \mathbb{R}^N)$ *consists of all bounded and Lipschitz continuous (single-valued) functions* $M \longrightarrow \mathbb{R}^N$.

Definition 2.2 *Choosing any set–valued map* $F : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, *the so–called* reachable set $\vartheta_F(t, K)$ *of the initial set* $K \in \mathcal{K}(\mathbb{R}^N)$ *at time* $t \in [0,T]$ *is defined as*

$$\vartheta_F(t,K) \;:=\; \Big\{ x(t) \in \mathbb{R}^N \;\Big|\; \exists\; x(\cdot) \in W^{1,1}([0,t],\, \mathbb{R}^N) :\; x(0) \in K, \; \tfrac{d}{d\tau}\, x(\tau) \in F(\tau, x(\tau)) \text{ for almost every } \tau \in [0,t] \Big\}$$

(*and correspondingly for* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and its autonomous differential inclusion*).

The special case of constant functions $F(\cdot) \equiv \{v\}$ (with an arbitrary vector $v \in \mathbb{R}^N$) leads to the Minkowski sum $\vartheta_F(t,K) = K + h \cdot v \subset \mathbb{R}^N$ and, for an initial set $K = \{x\}$ with just one element, in particular, we return to the familiar affine–linear map $(h,x) \longmapsto x + h \cdot v$ that has already been mentioned as motivation.

An essential contribution of Aubin was to specify appropriate continuity conditions on the maps $\vartheta : [0,1] \times E \longrightarrow E$, $(h,x) \longmapsto \vartheta(h,x)$ so that the familiar track of ordinary differential equations can be followed in a metric space (E,d). Here we quote his definition introduced in the monograph [3] (emphasizing the local features slightly more than his original version in [4]). Reachable sets of every set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ satisfy these conditions in the metric space $(\mathcal{K}(\mathbb{R}^N),\, d\!\!l)$:

Definition 2.3 ([3, Definition 1.1.2]) *Let (E,d) be a metric space. A map $\vartheta : [0,1] \times E \longrightarrow E$ is called* transition *on (E,d) if it satisfies the following conditions:*

1. $\vartheta(0,x) = x$ *for all* $x \in E$,
2. $\lim\limits_{h \downarrow 0} \frac{1}{h} \cdot d\left(\vartheta(t+h,x),\; \vartheta(h,\, \vartheta(t,x))\right) = 0$ *for all* $x \in E,\; t \in [0,1[$,
3. $\alpha(\vartheta) := \max\left(0,\; \sup\limits_{x \neq y}\; \limsup\limits_{h \downarrow 0}\; \frac{d(\vartheta(h,x),\, \vartheta(h,y)) \,-\, d(x,y)}{h \,\cdot\, d(x,y)}\right) < \infty$
4. $\beta(\vartheta) := \sup\limits_{x \in E}\; \limsup\limits_{h \downarrow 0}\; \frac{1}{h} \cdot d(x,\; \vartheta(h,x)) \;<\; \infty.$

For any two transitions $\vartheta_1, \vartheta_2 : [0,1] \times E \longrightarrow E$ on the same metric space (E,d), the transitional distance between ϑ_1 and ϑ_2 is defined by

$$d_\Lambda(\vartheta_1,\; \vartheta_2) \;:=\; \sup_{x \in E}\; \limsup_{h \downarrow 0}\; \frac{1}{h} \,\cdot\, d\left(\vartheta_1(h,x),\; \vartheta_2(h,x)\right).$$

Lemma 2.4 *For every set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, the map $\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$, $(h,K) \longmapsto \vartheta_F(h,K)$ of reachable sets (as introduced in Definition 2.2) is a well–defined transition on the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ according to Definition 2.3.*

To be more precise, the reachable sets satisfy for all initial sets $K, K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$, *set-valued maps* $F, G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *and times* $t, h \geq 0$

$$\begin{array}{rcl} \vartheta_F(0, K) &=& K, \\ \vartheta_F(t+h, K) &=& \vartheta_F(h,\ \vartheta_F(t, K)), \\ d\!\!\!l(\vartheta_F(h, K_1),\ \ \vartheta_F(h, K_2)) &\leq& d\!\!\!l(K_1, K_2) \cdot \quad e^{\mathrm{Lip}\, F \cdot h} \\ d\!\!\!l(\vartheta_F(h, K),\ \ \vartheta_G(h, K)) &\leq& d\!\!\!l_\infty(F, G)\ \cdot h\ \ e^{\mathrm{Lip}\, F \cdot h} \\ d\!\!\!l(\vartheta_F(t, K),\ \vartheta_F(t+h, K)) &\leq& \|F\|_\infty\ h \end{array}$$

with $\quad \|F\|_\infty \stackrel{\text{Def.}}{=} \sup_{x \in \mathbb{R}^N} \sup_{y \in F(x)} |y| \quad < \quad \infty$

$$d\!\!\!l_\infty(F, G) \stackrel{\text{Def.}}{=} \sup_{x \in \mathbb{R}^N} d\!\!\!l(F(x), G(x)) \quad < \quad \infty$$

and thus, $\ \alpha(\vartheta_F) \leq \mathrm{Lip}\, F, \quad \beta(\vartheta_F) \leq \|F\|_\infty, \quad d_\Lambda(\vartheta_F, \vartheta_G) \leq d\!\!\!l_\infty(F, G).$

In particular, $\quad d\!\!\!l\,(\vartheta_F(h, K_1),\ \vartheta_G(h, K_2)) \ \leq\ e^{\mathrm{Lip}\, F \cdot h}\ (d\!\!\!l(K_1, K_2)\ +\ h \cdot d\!\!\!l_\infty(F, G)).$

The proof is presented in [3, Proposition 3.7.3] – as a direct consequence of Filippov's Theorem (about solutions to differential inclusions in $\mathbb{R}^N$). In particular, this lemma justifies calling ϑ_F a *morphological transition* on $(\mathcal{K}(\mathbb{R}^N), d\!\!\!l)$ [3, Definition 3.7.2]. For the sake of simplicity, $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is sometimes identified with its morphological transition ϑ_F.
These reachable sets provide the tools for specifying (generalized) shape derivatives of a compact–valued tube $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$, i.e. a curve $K(\cdot) : [0, T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$. So the next step will be to solve equations prescribing an element of the morphological mutation.

Definition 2.5 ([3, Definition 3.7.9 (2)]) *For any compact–valued tube* $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$, *the* morphological mutation $\overset{\circ}{K}(t)$ *at time* $t \in [0, T[$ *consists of all set–valued maps* $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *satisfying*

$$\lim_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!\!l\,(\vartheta_F(h,\ K(t)),\ \ K(t+h)) \ = \ 0.$$

Definition 2.6 ([3, Definition 1.3.1, § 4.1)]) *For any given function* $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, *a compact–valued tube* $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ *is called solution to the morphological equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$

if 1. $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ *is Lipschitz continuous with respect to* $d\!\!\!l$ *and*

2. *for almost every* $t \in [0, T[,\ f(K(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *belongs to* $\overset{\circ}{K}(t)$
or, equivalently, $\quad \lim_{h \downarrow 0} \ \frac{1}{h} \cdot d\!\!\!l\,\big(\vartheta_{f(K(t))}(h,\ K(t)),\ \ K(t+h)\big) \ = \ 0.$

At first glance, the symbol $\ni$ here seems to be contradictory to the term "equation". The mutation $\overset{\circ}{K}(t)$, however, is defined as *subset* of *all* morphological transitions providing a first-order approximation of $K(t + \cdot)$ and so, the "right-hand side" $f(K(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ should be one of its elements. (In the classical framework of differentiable functions and vector spaces, the mutation consists of just one vector.)

As an essential result of [3, 4], the Cauchy–Lipschitz Theorem (about autonomous ordinary differential equations) has the following counterpart:

Theorem 2.7 ([3, Theorem 4.1.2]) *Suppose* $f : (\mathcal{K}(\mathbb{R}^N), d\!\!l) \longrightarrow \big(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N),\ d\!\!l_\infty\big)$ *to be* λ*–Lipschitz continuous and to satisfy* $M := \sup\limits_{K \in \mathcal{K}(\mathbb{R}^N)} \mathrm{Lip}\, f(K) < \infty$.
For every initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ *and time* $T \in]0, \infty[$, *there exists a unique solution* $K(\cdot) : [0, T[\leadsto \mathbb{R}^N$ *to the morphological equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ *with* $K(0) = K_0$.
Furthermore every Lipschitz compact–valued tube $Q : [0, T[\leadsto \mathbb{R}^N$ *with* $\overset{\circ}{Q}(t) \neq \emptyset$ *for every* $t \in [0, T[$ *satisfies the following estimate at each time* $t \in [0, T[$

$$d\!\!l(K(t), Q(t)) \ \le\ d\!\!l(K_0, Q(0)) \cdot e^{(M+\lambda)\, t} + \int_0^t e^{(M+\lambda)\,(t-s)} \cdot \inf_{G \in \overset{\circ}{Q}(s)} d\!\!l_\infty(f(Q(s)),\ G)\ ds.$$

In particular, the solution $K(\cdot)$ *depends on the initial set* K_0 *and the right–hand side* f *in a Lipschitz continuous way.*

Existence under (additional) state constraints proves to be a very interesting question for many applications. In the particular case of ordinary differential equations, Nagumo's Theorem gives a necessary and sufficient condition on the set of constraints $\mathcal{V}$ for existence of local solutions. It uses the contingent cone (in the sense of Bouligand) and has served as a key motivation for viability theory (see e.g. [6]).

Definition 2.8 ([6, Definition 1.1.3]) *Let* X *be a normed vector space,* $V \subset X$ *nonempty and* $x \in V$. *The* contingent cone *to* V *at* x *(in the sense of Bouligand) is*

$$T_V(x) \ :=\ \big\{ u \in X \ \big|\ \liminf_{h \downarrow 0}\ \tfrac{1}{h} \cdot \mathrm{dist}(x + h\, u,\ V) \ =\ 0 \big\}.$$

This classical definition of contingent cone in a vector space is now extended to the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ by using the morphological transitions of $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$:

Definition 2.9 ([3, Definition 1.5.2]) *For a nonempty subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *and any element* $K \in \mathcal{V}$,

$$T_{\mathcal{V}}(K) \ :=\ \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \ \Big|\ 0 = \liminf_{h \downarrow 0}\ \tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_F(h, K),\ \mathcal{V}\big) \Big\}$$

is called contingent transition set *of* $\mathcal{V}$ *at* K *(in the metric space* $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$*).*

Remark. Considering here the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ (instead of a normed vector space as in Definition 2.8) has an immediate consequence: By definition of the distance from a subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$,

$$\mathrm{dist}\big(\vartheta_F(h, K),\ \mathcal{V}\big) \ =\ \inf_{C \in \mathcal{V}}\ d\!\!l\big(\vartheta_F(h, K),\ C\big).$$

In particular, we cannot expect any trivial identities of the contingent cone to a compact subset $V \subset \mathbb{R}^N$ and the contingent transition set to $\mathcal{V} := \{V\} \subset \mathcal{K}(\mathbb{R}^N)$. Furthermore, some of the subsequent general results become definitely incorrect if the Pompeiu–Hausdorff distance $d\!\!l$ is replaced by the one–sided distance part called Pompeiu–Hausdorff excess (as defined in [3, § 3.2.1]).

Remark. The "geometric" background of reachable sets implies an additional property of morphological transitions in $T_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Indeed, for any $F \in T_{\mathcal{V}}(K)$, every map $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with $F(\cdot) = G(\cdot)$ in an open neighborhood of the compact set K is also contained in $T_{\mathcal{V}}(K)$ because $\vartheta_F(t, K) = \vartheta_G(t, K)$ for sufficiently small $t > 0$. So in other words, the criterion of $T_{\mathcal{V}}(K)$ depends only on an arbitrarily small neighborhood of the current set K. (The corresponding statement even holds for an open neighborhood of the boundary ∂K as a closer investigation of the boundaries $\partial \vartheta_F(t, K) \subset \vartheta_F(t, \partial K)$ reveals. These details, however, will not be used in the following.)

In fact, Nagumo's Theorem also holds for morphological equations:

Theorem 2.10 (Nagumo's theorem for morphological equations [3, Theorem 4.1.7]**)**
Suppose $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ to be nonempty and closed with respect to $d\!l$.
Let $f : (\mathcal{K}(\mathbb{R}^N), d\!l) \longrightarrow \big(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N),\ d\!l_\infty\big)$ be a continuous function satisfying

1. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)}$ Lip $f(M) < \infty$ (*uniform bound of Lipschitz constants*),
2. $\sup_{M \in \mathcal{K}(\mathbb{R}^N)} \|f(M)\|_\infty < \infty$ (*uniform bound of compact values*).

Then from any initial state $K_0 \in \mathcal{V}$ starts at least one Lipschitz solution $K(\cdot) : [0, T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ of $\overset{\circ}{K}(\cdot) \ni f(K(\cdot))$ viable in $\mathcal{V}$ (i.e. $K(t) \in \mathcal{V}$ for all t) if and only if $\mathcal{V}$ is a viability domain *of f in the sense of $f(M) \in T_{\mathcal{V}}(M)$ for each $M \in \mathcal{V}$.*

2.2 The step to morphological inclusions with state constraints: A viability theorem

In [18], sufficient conditions for the existence of viable solutions were presented for morphological *inclusions*, i.e. the single–valued function $f : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ of the right–hand side is replaced by a set–valued map $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. Correspondingly to Definition 2.6, we specify the solution to a morphological inclusion in the following way:
Definition 2.11 ([18, Definition 3.1]) *For any given function $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, a compact–valued tube $K(\cdot) : [0, T[\leadsto \mathbb{R}^N$ is called* solution *to the morphological inclusion*

$$\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$$

if 1. $K(\cdot) : [0, T[\leadsto \mathbb{R}^N$ is Lipschitz continuous with respect to $d\!l$ and
2. $\mathcal{F}(K(t)) \cap \overset{\circ}{K}(t) \neq \emptyset$ for almost every t, i.e. a set-valued map $G \in \mathcal{F}(K(t)) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to $\overset{\circ}{K}(t)$ or, equivalently, $\lim\limits_{h \downarrow 0} \frac{1}{h} \cdot d\!l\,(K(t+h),\ \vartheta_G(h,\ K(t))) = 0$,

In addition to $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, a constrained set $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ is now given.
We focus on the so-called *viability condition* demanding from each compact set $K \in \mathcal{V}$ that the value $\mathcal{F}(K)$ and the contingent transition set $T_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ have at least one morphological transition in common. Lacking a concrete counterpart of Aumann integral in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!l)$, the question of its necessity (for the existence of "in $\mathcal{V}$ viable" solutions) is more complicated than for differential inclusions in $\mathbb{R}^N$ and thus, we skip it here deliberately.

Convexity again comes into play, but we have to distinguish between (at least) two aspects: First, assuming $\mathcal{F}$ to have convex values in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and second, supposing each set-valued map $G \in \mathcal{F}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ (with $K \in \mathcal{K}(\mathbb{R}^N)$) to have convex values in $\mathbb{R}^N$. The latter, however, does not really provide a geometric restriction on morphological transitions. Indeed, the well–known Relaxation Theorem of Filippov–Ważiewski (e.g. [2, § 2.4, Theorem 2]) implies $\vartheta_G(t, K) = \vartheta_{\overline{co}\, G}(t, K)$ for every map $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, initial set $K \in \mathcal{K}(\mathbb{R}^N)$ and time $t \geq 0$. So we suppose the values of $\mathcal{F}$ to be in $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$:

Definition 2.12 ([18, Definition 3.4]) $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *denotes the set of all Lipschitz set–valued maps $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ whose (nonempty compact) values are convex in addition.*

Theorem 2.13 (Viability theorem for morphological inclusions [18, Theorem 3.5]**)**
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *a nonempty closed subset satisfying:*

1.) all values of $\mathcal{F}$ *are nonempty and convex (i.e. for any* $G_1, G_2 \in \mathcal{F}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $\lambda \in [0,1]$, *the set–valued map* $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1-\lambda) \cdot G_2(x)$ *also belongs to* $\mathcal{F}(K)$*)*

2.) $\sup\limits_{M \in \mathcal{K}(\mathbb{R}^N)} \; \sup\limits_{G \in \mathcal{F}(M)} \; \mathrm{Lip}\, G \; < \infty$ *(uniform bound on all Lipschitz constants of set-valued maps)*

$\sup\limits_{M \in \mathcal{K}(\mathbb{R}^N)} \; \sup\limits_{G \in \mathcal{F}(M)} \; \|G\|_\infty \; < \infty$ *(uniform bound on all compact set values)*

3.) the graph of $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*),*

4.) $T_{\mathcal{V}}(K) \cap \mathcal{F}(K) \neq \emptyset$ *for all* $K \in \mathcal{V}$.

Then for every initial set $K_0 \in \mathcal{V}$, *there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ *to the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \in \mathcal{V}$ *for all* $t \in [0,1]$.

Remark. In assumption (3.), the topology on $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is specified. A sequence $(G_n)_{n \in \mathbb{N}}$ in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is said to converge "locally uniformly" to $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ if for every nonempty compact set $M \subset \mathbb{R}^N$, $d\!\!l_\infty(G_n(\cdot)|_M, \, G(\cdot)|_M) \overset{\text{Def.}}{=} \sup\limits_{x \in M} d\!\!l(G_n(x), G(x)) \longrightarrow 0$ for $n \longrightarrow \infty$ using here the Pompeiu–Hausdorff distance $d\!\!l$ on $\mathcal{K}(\mathbb{R}^N)$.
Due to the uniform bounds in assumption (2.), the image $\mathcal{F}(\mathcal{K}(\mathbb{R}^N)) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is sequentially compact with respect to this topology (as proved in [18, Lemma 3.11]). So $\mathcal{F}$ is upper semicontinuous (in the sense of Bouligand and Kuratowski) according to [7, Proposition 1.4.8].

2.3 Set evolutions under strong operability constraints

Now Viability Theorem 2.13 is applied to a very special form of constraints:

$$\mathcal{V}_M := \left\{ K \in \mathcal{K}(\mathbb{R}^N) \;\middle|\; K \subset M \right\}$$

with some (arbitrarily fixed) nonempty closed subset $M \subset \mathbb{R}^N$. Anne Gorre coined the term "strongly operable in M" for this constraint [17]. Consequently, we obtain sufficient conditions on $M \subset \mathbb{R}^N$ and $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ for the existence of a Lipschitz continuous solution $K(\cdot) : [0,1] \longrightarrow \mathcal{K}(\mathbb{R}^N)$ satisfying

$$\begin{cases} \overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset & \text{a.e. in } [0,1] \\ \qquad\qquad K(t) \subset M & \text{for each } t \in [0,1] \end{cases}$$

Here we benefit from earlier results of Gorre [17] considering the corresponding problems with morphological *equations* (instead of inclusions). In a word, she proved $\mathcal{V}_M$ to be a closed subset of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and characterized their contingent transition sets completely by means of the tangential properties of the closed set $M \subset \mathbb{R}^N$. Then she applied Nagumo's theorem for morphological equations (quoted here in Theorem 2.10). Her characterization in Lemma 2.14 has been combined directly with Viability Theorem 2.13 in [18].

Lemma 2.14 ([17, Theorem 4.3]) *Let* $M \subset \mathbb{R}^N$ *be closed and nonempty. For every nonempty compact set* $K \in \mathcal{V}_M$ *(i.e.* $K \subset M$*) and each set–valued map* $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$*, the following two conditions are equivalent:*

1. $G \in T_{\mathcal{V}_M}(K)$*, i.e.* G *belongs to the contingent transition set of* $\mathcal{V}_M$ *at* K *(Definition 2.9).*
2. $G(x) \subset T_M(x)$ *for every* $x \in K$*, i.e.* $G(x)$ *is contained in Bouligand's contingent cone of* M *at each point* $x \in K \subset M$ *(Definition 2.8).* □

Theorem 2.15 (Set evolutions "strongly operable" in $M \subset \mathbb{R}^N$ [18, Theorem 4.5]**)**
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map and* $M \subset \mathbb{R}^N$ *a closed subset satisfying:*

1.) *all values of* $\mathcal{F}$ *are nonempty, convex (as in Theorem 2.13) and have the global bounds*
$$\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \ \sup_{G \in \mathcal{F}(K)} \big(\|G\|_\infty + \mathrm{Lip}\, G\big) < \infty,$$
2.) *the graph of* $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*),*
3.) *for any compact set* $K \subset M$*, there exists* $G \in \mathcal{F}(K)$ *with* $G(x) \subset T_M(x)$ *for every* $x \in K$*.*

Then for every nonempty compact set $K_0 \subset M$*, there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ *to the morphological inclusion* $\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset$ *with* $K(0) = K_0$ *and* $K(t) \subset M$ *for all* $t \in [0,1]$*.*

3 Morphological control problems

3.1 Formulation

Now a control parameter is to come into play. Indeed, the so-called control problems

$$\begin{cases} \frac{d}{dt}\, x(t) &= f(x(t), u) \\ u &\in U \end{cases} \tag{1}$$

have been studied thoroughly both in finite-dimensional and in infinite-dimensional vector spaces. Our contribution now is to formulate the corresponding problem in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ using the morphological framework for derivatives.

Definition 3.1 *Let (U, d_U) denote a metric space and $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ be given. $K : [0, T[\leadsto \mathbb{R}^N$ with values in $\mathcal{K}(\mathbb{R}^N)$ is called* solution *to the* morphological control problem

$$\begin{cases} \overset{\circ}{K}(\cdot) &\ni f(K(\cdot), u) \qquad \text{in } [0, T[\\ u &\in U \end{cases} \tag{2}$$

if there is a measurable function $u(\cdot) : [0, T[\longrightarrow U$ such that $K(\cdot)$ solves the nonautonomous morphological equation $\overset{\circ}{K}(t) \ni f(K(t), u(t))$ in $[0, T[$, i.e. satisfying

1. *$K(\cdot) : [0, T[\leadsto \mathbb{R}^N$ is Lipschitz continuous with respect to $d\!\!l$ and*
2. *for almost every $t \in [0, T[$, $f(K(t), u(t)) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to $\overset{\circ}{K}(t)$ or, equivalently,* $\displaystyle\lim_{h \downarrow 0} \frac{1}{h} \cdot d\!\!l\left(\vartheta_{f(K(t), u(t))}(h, K(t)),\ K(t+h)\right) = 0.$

Proposition 3.2 (Solutions as reachable sets) *Assume the metric space (U, d_U) to be complete and separable and, consider $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence. Suppose $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ to be continuous with $\sup_{\substack{M \in \mathcal{K}(\mathbb{R}^N) \\ u \in U}} (\|f(M, u)\|_\infty + \mathrm{Lip}\, f(M, u)) < \infty$. Let $K : [0, T[\leadsto \mathbb{R}^N$ be any compact-valued solution to the morphological control problem (2).*

Then there is a measurable function $u(\cdot) : [0, T[\longrightarrow U$ such that at every time $t \in [0, T[$, the compact set $K(t) \subset \mathbb{R}^N$ coincides with the reachable set $\vartheta_{f(K(\cdot), u(\cdot))}(t, K(0)) \subset \mathbb{R}^N$ of the nonautonomous differential inclusion $\frac{d}{d\tau}\, x(\tau) \in f(K(\tau), u(\tau))\,\big(x(\tau)\big) \subset \mathbb{R}^N$ a.e.

Proof. Due to Definition 3.1, $K(\cdot)$ is Lipschitz continuous with respect to Pompeiu-Hausdorff distance $d\!\!l$ and, there is a measurable function $u(\cdot) : [0, T[\longrightarrow U$ such that for almost every $t \in [0, T[$,

$$\lim_{h \downarrow 0} \frac{1}{h} \cdot d\!\!l\left(\vartheta_{f(K(t), u(t))}(h, K(t)),\ K(t+h)\right) = 0.$$

Filippov's Theorem in its generalized form (see e.g. [29, Theorem 2.4.3]) ensures the existence of solutions $x(\cdot) \in W^{1,1}([0, t], \mathbb{R}^N)$ to the *nonautonomous* differential inclusion $\frac{d}{d\tau}\, x(\tau) \in f(K(\tau), u(\tau))\,\big(x(\tau)\big)$ a.e. in $[0, t]$ (whose right-hand side is just measurable in time, but uniformly Lipschitz continuous in space). Moreover, the typical estimates hold which are well-known for autonomous differential inclusions and have already laid the foundations for Lemma 2.4.

So the reachable set $\vartheta_{f(K(\cdot), u(\cdot))}(t, K(0)) \subset \mathbb{R}^N$ is well-defined and compact for every $t \in [0, T[$ and, due to $B := \sup_{\substack{M \in \mathcal{K}(\mathbb{R}^N) \\ u \in U}} (\|f(M, u)\|_\infty + \mathrm{Lip}\, f(M, u)) < \infty$, the map $R : [0, T[\leadsto \mathbb{R}^N,\ t \mapsto \vartheta_{f(K(\cdot), u(\cdot))}(t, K(0))$

is B-Lipschitz continuous w.r.t. $d\!\!\!/$. Furthermore, [18, Corollary 3.14] ensures for almost every $t \in [0,T[$ that $f(K(t),u(t)) \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ belongs to its mutation $\overset{\circ}{R}(t)$.
As a consequence, the distance function $\delta : [0,T[\longrightarrow [0,\infty[, \quad t \longmapsto d\!\!\!/\,(R(t),K(t))$ is Lipschitz continuous with $\delta(0) = 0$ and satisfies at almost every time $t \in [0,T[$

$$\begin{aligned}
\limsup_{h\downarrow 0} \frac{\delta(t+h)-\delta(t)}{h} &= \limsup_{h\downarrow 0} \frac{1}{h} \cdot \Big(d\!\!\!/\big(R(t+h),\ K(t+h)\big) \qquad\qquad - d\!\!\!/\big(R(t),K(t)\big)\Big) \\
&\le \limsup_{h\downarrow 0} \frac{1}{h} \cdot \Big(d\!\!\!/\big(R(t+h),\ \vartheta_{f(K(t),u(t))}(h,\ R(t))\big) + \\
&\qquad\qquad d\!\!\!/\big(\vartheta_{f(K(t),u(t))}(h,\ R(t)),\ \vartheta_{f(K(t),u(t))}(h,\ K(t))\big) - d\!\!\!/\big(R(t),K(t)\big) + \\
&\qquad\qquad d\!\!\!/\big(\vartheta_{f(K(t),u(t))}(h,\ K(t)),\ K(t+h)\big)\Big) \\
&\le \ 0 \ + \ \limsup_{h\downarrow 0} \frac{1}{h} \cdot \delta(t)\ (e^{B\,h}-1) \ + \ 0 \\
&= B\ \delta(t).
\end{aligned}$$

So Gronwall's Lemma completes the proof: $\delta(\cdot) \equiv 0$. □

3.2 The link to morphological inclusions

In vector spaces, the close relationship between control problem (1) and the corresponding differential inclusion

$$\tfrac{d}{dt}\, x(t) \ \in \ \bigcup_{u\in U} f(x(t),u) \qquad \text{a.e.}$$

had been realized soon. A measurable selection provides the same link now for morphological inclusions. In a word, the classical techniques using appropriate measurable selections (which had been developed for differential inclusions in the Euclidean space) can also be used in the morphological framework because the transitions are in a complete separable metric space, i.e. here $\mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$.
So a main result of this section is the following equivalence:

Proposition 3.3 *Assume the metric space (U, d_U) to be complete and separable and, consider the set $\mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$ with the topology of locally uniform convergence. Let $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$ be a Carathéodory function (i.e. continuous in the first argument and measurable in the second one) satisfying*

$$\sup_{\substack{M\in\mathcal{K}(\mathbb{R}^N)\\ u\in U}} \big(\|f(M,u)\|_\infty + \mathrm{Lip}\, f(M,u)\big) \ < \ \infty.$$

Set $\mathcal{F}_U : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N),\ \ K \mapsto \{f(K,u) \mid u \in U\} \subset \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$.

A compact-valued tube $K(\cdot) : [0,T[\leadsto \mathbb{R}^N$ is solution to the morphological control problem

$$\begin{cases} \overset{\circ}{K}(\cdot) \ \ni \ f(K(\cdot),u) & \text{in } [0,T[\\ \quad u \ \in \ U \end{cases}$$

if and only if $K(\cdot)$ is solution to the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}_U(K(\cdot)) \neq \emptyset$ (in the sense of Definition 2.11).

Obviously, every morphological control problem leads to a morphological inclusion. So for proving Proposition 3.3, we require the inverse connection (i.e. from inclusion to control problem). In the literature about differential inclusions in vector spaces, it is usually based on a selection result that is said to go back to Filippov.

Lemma 3.4 (Filippov [7, Theorem 8.2.10]**)** *Consider a complete σ-finite measure space (Ω, A, μ), complete separable metric spaces X, Y and a measurable set-valued map $H : \Omega \rightsquigarrow X$ with closed nonempty images. Let $g : X \times \Omega \longrightarrow Y$ be a Carathéodory map (i.e. continuous in the first argument and measurable in the second one).*
Then for every measurable function $k : \Omega \longrightarrow Y$ satisfying

$$k(\omega) \in g(H(\omega), \omega) \quad \text{for } \mu\text{-almost all } \omega \in \Omega,$$

there exists a measurable selection $h(\cdot) : \Omega \longrightarrow X$ of $H(\cdot)$ such that

$$k(\omega) = g(h(\omega), \omega) \quad \text{for } \mu\text{-almost all } \omega \in \Omega.$$

So for applying Lemma 3.4 to morphological inclusions, we have to focus on two aspects: First, $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ as a separable metric space. Indeed, we usually supply $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence (as specified in the remark after Theorem 2.13). This topology can obviously be metrized by

$$d_{\mathrm{LIP}} : \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \longrightarrow [0,1], \quad (G, H) \longmapsto \sum_{j=1}^{\infty} 2^{-j} \frac{d\!\!l_\infty \left(G(\cdot)|_{\mathbb{B}_j(0)},\ H(\cdot)|_{\mathbb{B}_j(0)}\right)}{1 + d\!\!l_\infty \left(G(\cdot)|_{\mathbb{B}_j(0)},\ H(\cdot)|_{\mathbb{B}_j(0)}\right)}$$

with the abbreviation $d\!\!l_\infty \left(G(\cdot)|_{\mathbb{B}_j(0)},\ H(\cdot)|_{\mathbb{B}_j(0)}\right) \overset{\text{Def.}}{=} \sup_{x \in \mathbb{R}^N,\ |x| \le j} d\!\!l(G(x),\ H(x)) < \infty.$
Moreover, $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is separable with respect to d_{LIP} due to the (global) Lipschitz continuity of each of its set-valued maps and because both domains and values belong to the separable Euclidean space $\mathbb{R}^N$.
Second, we study measurability of the "derivatives" for any compact-valued solution $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$. Indeed for real-valued functions, it is well-known that Lipschitz continuity implies a Lebesgue-integrable weak derivative and, the latter coincides with the differential quotient at Lebesgue-almost every time (as a consequence of Rademacher's Theorem [27, Theorem 9.60]). In the morphological framework, however, the derivative is described as a subset of $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, namely the mutation (in the sense of Definition 2.5).

Proposition 3.5 (Measurability of compact mutation subsets)
For every threshold $B \in [0, \infty[$ and Lipschitz continuous tube $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ with values in $\mathcal{K}(\mathbb{R}^N)$, the following set-valued map of transitions

$$[0, T[\rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad t \mapsto \overset{\circ}{K}(t) \cap \{G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \mid \|G\|_\infty + \mathrm{Lip}\, G \le B\}$$

is Lebesgue-measurable.

Proof. For the sake of simplicity, we extend the Lipschitz map $K(\cdot) : [0, T[\rightsquigarrow \mathbb{R}^N$ continuously to $\mathbb{R}$ according to $K(s) := K(0)$ for $s < 0$ and $K(s) := \mathrm{Lim}_{t \uparrow T} K(t)$ for $s \ge T$.
The set $\mathcal{B} := \{G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \mid \|G\|_\infty + \mathrm{Lip}\, G \le B\} \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is compact with respect to d_{LIP} (i.e. locally uniform convergence in $\mathbb{R}^N$) as a consequence of Arzela–Ascoli's Theorem in metric spaces (see e.g. [16, Theorem 2]).
Furthermore set $\widehat{G} : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N,\ x \mapsto \mathbb{B}_{B+1}(0)$ as an auxiliary set-valued map not belonging to $\mathcal{B}$. ($\widehat{G}$ is just to ensure that all set-valued maps $[0, T[\rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ considered from now on have nonempty values. So the results of [7] about measurability can be applied directly.)

Now for each $m, n \in \mathbb{N}$, define the set-valued map $\mathcal{M}_{m,n} : [0,T] \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ in the following way: $\mathcal{M}_{m,n}(t)$ consists of $\widehat{G}$ and all maps $G \in \mathcal{B} \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$d\!\!l \,(\vartheta_G(h,\ K(t)),\ K(t+h)) \ \leq\ \tfrac{1}{m}\, h \qquad\qquad \text{for all } \ h \in [0, \tfrac{1}{n}].$$

The graph of $\mathcal{M}_{m,n}$ is closed. Indeed, let $\big((t_j, G_j)\big)_{j \in \mathbb{N}}$ be any convergent sequence in Graph $\mathcal{M}_{m,n} \subset [0,T] \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with the limit (t, G). If $G = \widehat{G}$, then we conclude $G_j = \widehat{G}$ for all large $j \in \mathbb{N}$. So we can restrict our considerations to $\{G_j, G \mid j \in \mathbb{N}\} \subset \mathcal{B}$ and in particular, for each $j \in \mathbb{N}$,

$$d\!\!l \left(\vartheta_{G_j}(h,\ K(t_j)),\ K(t_j+h)\right) \ \leq\ \tfrac{1}{m}\, h \qquad\qquad \text{for all } \ h \in [0, \tfrac{1}{n}].$$

Preceding Lemma 2.4 about reachable set of differential inclusions (applied to restrictions on a sufficiently large ball in $\mathbb{R}^N$) implies

$$d\!\!l \,(\vartheta_G(h,\ K(t)),\ K(t+h)) \ =\ \lim_{j\to\infty} d\!\!l \left(\vartheta_{G_j}(h,\ K(t_j)),\ K(t_j+h)\right) \ \leq\ \tfrac{1}{m}\, h \qquad \text{for all } \ h \in [0, \tfrac{1}{n}],$$

i.e. $G \in \mathcal{M}_{m,n}(t)$. Thus, Graph $\mathcal{M}_{m,n}$ is closed in $[0,T] \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

Furthermore, all values of $\mathcal{M}_{m,n}$ are nonempty, closed and contained in the compact subset $\mathcal{B} \cup \{\widehat{G}\} \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$. So due to [7, Proposition 1.4.8], $\mathcal{M}_{m,n} : [0,T] \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is upper semicontinuous (in the sense of Bouligand and Kuratowski). Finally, this property implies the measurability of $\mathcal{M}_{m,n}$ for each $m, n \in \mathbb{N}$ according to [7, Proposition 8.2.1].

Now we bridge the gap between the countable family $(\mathcal{M}_{m,n})_{m,n \in \mathbb{N}}$ of measurable set-valued maps and $[0,T[\, \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, $t \mapsto \overset{\circ}{K}(t) \cap \mathcal{B}$ considered in the claim: Due to the definition of $\mathcal{M}_{m,n}$,

$$\bigcup_{n \in \mathbb{N}} \ \mathcal{M}_{m,n}(t) \ \subset\ \left\{ G \in \mathcal{B} \ \middle|\ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l(\vartheta_G(h, K(t)),\ K(t{+}h)) \leq \tfrac{1}{m} \right\} \ \cup\ \{\widehat{G}\}$$
$$\bigcup_{n \in \mathbb{N}} \ \mathcal{M}_{m,n}(t) \ \supset\ \left\{ G \in \mathcal{B} \ \middle|\ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l(\vartheta_G(h, K(t)),\ K(t{+}h)) < \tfrac{1}{m} \right\} \ \cup\ \{\widehat{G}\}$$

Furthermore, Lemma 2.4 guarantees for every $G, H \in \mathcal{B}$ with $d\!\!l_\infty \left(G(\cdot)|_{\mathbb{B}_1(K(t))},\ H(\cdot)|_{\mathbb{B}_1(K(t))}\right) \ \leq\ \varepsilon$

$$d\!\!l \,(\vartheta_G(h, K(t)),\ \vartheta_H(h, K(t))) \ \leq\ \varepsilon\ h\ \ e^{B\,h} \qquad\qquad \text{for all } h \in [0, \tfrac{1}{B}]$$

and thus, we obtain for a sufficiently small radius $\widetilde{\varepsilon} > 0$ (depending on m, t) the inclusion (w.r.t. d_{LIP})

$$\mathbb{B}_{\widetilde{\varepsilon}}\Big(\mathcal{B} \cap \bigcup_{n \in \mathbb{N}} \ \mathcal{M}_{m,n}(t)\Big) \ \subset\ \left\{ G \in \mathcal{B} \ \middle|\ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l(\vartheta_G(h, K(t)),\ K(t{+}h)) \leq \tfrac{2}{m} \right\}$$

So the closure of the union on the left-hand side satisfies for every $t \in [0, T[$

$$\mathcal{B} \cap \overline{\bigcup_{n \in \mathbb{N}} \ \mathcal{M}_{m,n}(t)} \ \ \subset\ \left\{ G \in \mathcal{B} \ \middle|\ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l(\vartheta_G(h, K(t)),\ K(t{+}h)) \leq \tfrac{2}{m} \right\}.$$

We conclude (again) for each $t \in [0, T[$

$$\mathcal{B} \cap \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \in \mathbb{N}} \ \mathcal{M}_{m,n}(t)} \ \ =\ \left\{ G \in \mathcal{B} \ \middle|\ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\!\!l(\vartheta_G(h, K(t)),\ K(t{+}h)) \leq 0 \right\} \ \ =\ \overset{\circ}{K}(t) \cap \mathcal{B}.$$

Finally, [7, Theorem 8.2.4] ensures that the closure of a countable union and the countable intersection preserve measurability of set-valued maps $[0,T] \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ (see also [27, Proposition 14.11]). This completes the proof. □

Proof of Proposition 3.3. "$\Longleftarrow$" Let the compact-valued tube $K(\cdot) : [0, T[\, \rightsquigarrow \mathbb{R}^N$ be solution to the morphological inclusion $\overset{\circ}{K}(\cdot) \ \cap\ \mathcal{F}_U(K(\cdot)) \ \neq\ \emptyset$ (in the sense of Definition 2.11), i.e.

1. $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ is Lipschitz continuous with respect to $d\!\!\!l$ and
2. $\mathcal{F}_U(K(t)) \cap \overset{\circ}{K}(t) \neq \emptyset$ for almost every t, i.e. there is some $u \in U$ such that the set-valued map $f(K(t),u) \in \mathcal{F}_U(K(t)) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the mutation $\overset{\circ}{K}(t)$ or, equivalently,
$$\lim_{h \downarrow 0} \frac{1}{h} \cdot d\!\!\!l \left(K(t+h),\ \vartheta_{f(K(t),u)}(h,\ K(t))\right) = 0.$$

Setting $B := \sup_{\substack{M \in \mathcal{K}(\mathbb{R}^N) \\ u \in U}} \left(\|f(M,u)\|_\infty + \mathrm{Lip}\, f(M,u)\right) < \infty,$ the set-valued map
$$[0,T[\rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad t \mapsto \overset{\circ}{K}(t) \cap \{G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \mid \|G\|_\infty + \mathrm{Lip}\, G \leq B\}$$
is Lebesgue-measurable according to Proposition 3.5. Due to [7, Theorems 8.1.3, 8.2.4], the intersection
$$[0,T[\rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad t \mapsto \overset{\circ}{K}(t) \cap \mathcal{F}_U(K(t))$$
is also Lebesgue-measurable and thus has a measurable selection $k(\cdot) : [0,T] \longrightarrow \left(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), d_{\mathrm{LIP}}\right)$. Finally, Lemma 3.4 of Filippov provides a measurable selection $u(\cdot) : [0,T] \longrightarrow U$ of the constant map $H(\cdot) \equiv U : [0,T] \rightsquigarrow U$ such that $k(t) = f(K(t), u(t))$ for Lebesgue-almost every $t \in [0,T]$. □

Moreover, we obtain a representation as reachable set similarly to Proposition 3.2 about morphological control problems. It is the second key result in this section.

Proposition 3.6 (Solutions of morphological inclusions as reachable sets)
Let $\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be a set–valued map satisfying*

1.) all values of $\mathcal{F}$ *are nonempty and have the global bounds* $\sup_{K \in \mathcal{K}(\mathbb{R}^N)} \sup_{G \in \mathcal{F}(K)} \left(\|G\|_\infty + \mathrm{Lip}\, G\right) < \infty,$

2.) the graph of $\mathcal{F}$ *is closed (w.r.t. locally uniform convergence in* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$*).*

For every compact-valued solution $K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ *to the morphological inclusion*
$$\overset{\circ}{K}(\cdot) \cap \mathcal{F}(K(\cdot)) \neq \emptyset \qquad \text{in } [0,T[,$$
there exists a measurable selection $k(\cdot) : [0,T[\longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *of the composition* $\mathcal{F}(K(\cdot)) : [0,T[\rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *such that for Lebesgue-almost every* $t \in [0,T[$, *the set* $K(t)$ *coincides with the reachable set* $\vartheta_{k(\cdot)}(t, K(0)) \subset \mathbb{R}^N$ *of the nonautonomous differential inclusion* $\frac{d}{d\tau}\, x(\tau) \in k(\tau)\left(x(\tau)\right) \subset \mathbb{R}^N$ *a.e.*

Proof. As in the preceding proof of Proposition 3.3, there exists a measurable selection $k(\cdot) : [0,T] \longrightarrow \left(\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), d_{\mathrm{LIP}}\right)$ of the set-valued intersection
$$[0,T[\rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad t \mapsto \overset{\circ}{K}(t) \cap \mathcal{F}(K(t))$$
due to Proposition 3.5 and [7, Theorems 8.1.3, 8.2.4]. The right-hand side of the nonautonomous differential inclusion $\frac{d}{d\tau}\, x(\tau) \in k(\tau)\left(x(\tau)\right) \subset \mathbb{R}^N$ a.e. is measurable in time and uniformly Lipschitz continuous in space (due to assumption (1.)). So Filippov's Theorem about differential inclusions allows us to follow exactly the same track as for Proposition 3.2 (about morphological control problems). □

3.3 Application to control problems with state constraints

The relationship between morphological control problems and morphological inclusions opens the door to applying Viability Theorem 2.13. So we can now specify sufficient conditions on a morphological control problem with state constraints for having at least one viable solution:

Proposition 3.7 (Viability theorem for morphological control problems)
Assume the metric space (U, d_U) to be compact and separable and, consider the set $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence. Suppose for* $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and the nonempty closed subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$*:*

1.) *for any* $K \in \mathcal{K}(\mathbb{R}^N)$*, the set* $\{f(K,u) \mid u \in U\} \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *is convex, i.e. for any* $u_1, u_2 \in U$ *and* $\lambda \in [0,1]$*, there exists some* $u \in U$ *such that* $f(K,u) \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *is identical to the set–valued map* $\mathbb{R}^N \leadsto \mathbb{R}^N$, $x \mapsto \lambda \cdot f(K,u_1)(x) + (1-\lambda) \cdot f(K,u_2)(x)$ *(in the Minkowski sense).*

2.) $\sup_{\substack{K \in \mathcal{K}(\mathbb{R}^N) \\ u \in U}} \left(\|f(K,u)\|_\infty + \mathrm{Lip}\, f(K,u) \right) < \infty,$

3.) *f is continuous,*

4.) *for each* $K \in \mathcal{V}$*, there exists some* $u \in U$ *with* $f(K,u) \in T_{\mathcal{V}}(K)$*.*

Then for every initial set $K_0 \in \mathcal{V}$*, there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ *to the morphological control problem* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u)$, $u \in U$ *with* $K(0) = K_0$ *and* $K(t) \in \mathcal{V}$ *for all* $t \in [0,1]$*.*

Proof. Define the set-valued map $\mathcal{F}_U : \mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$, $K \mapsto \{f(K,u) \mid u \in U\}$.
Obviously, it has nonempty convex values due to assumption (1.). Moreover, the graph of $\mathcal{F}_U$ is a closed subset of $\mathcal{K}(\mathbb{R}^N) \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ because f is continuous and U is compact. So $\mathcal{F}_U$ satisfies the assumption of Viability Theorem 2.13 and thus, for every initial set $K_0 \in \mathcal{V}$, there exists a compact–valued Lipschitz continuous solution $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ to the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \mathcal{F}_U(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \in \mathcal{V}$ for all $t \in [0,1]$.
Due to Prop. 3.3, $K(\cdot)$ is solution to the morphological control problem $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u)$, $u \in U$. □

For a given closed subset $M \subset \mathbb{R}^N$, we conclude from Gorre's characterization in Lemma 2.14 directly:

Corollary 3.8 (Morphological control problems strongly operable in $M \subset \mathbb{R}^N$)
Assume the metric space (U, d_U) to be compact and separable and, consider the set $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence. Suppose for* $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and the nonempty closed subset* $M \subset \mathbb{R}^N$*:*

1.) *for any* $K \in \mathcal{K}(\mathbb{R}^N)$*, the set* $\{f(K,u) \mid u \in U\} \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *is convex (as in Proposition 3.7),*

2.) $\sup_{\substack{K \in \mathcal{K}(\mathbb{R}^N) \\ u \in U}} \left(\|f(K,u)\|_\infty + \mathrm{Lip}\, f(K,u) \right) < \infty,$

3.) *f is continuous,*

4.) *for each nonempty compact set* $K \subset M$*, there is* $u \in U$ *with* $f(K,u)(x) \subset T_M(x)$ *for all* $x \in K$*.*

Then for every nonempty compact subset $K_0 \subset M$*, there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ *to the morphological control problem* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u)$, $u \in U$ *with* $K(0) = K_0$ *and* $K(t) \subset M$ *for all* $t \in [0,1]$*.* □

3.4 Relaxed control problems with state constraints

Considering the morphological control problem

$$\begin{cases} \overset{\circ}{K}(\cdot) \ni f(K(\cdot), u) & \text{in } [0,T[\\ \quad u \in U \end{cases}$$

(and the statements in Proposition 3.7 or Corollary 3.8, for example), the convexity of $\{f(K,u) \,|\, u \in U\}$ $\subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is a hypothesis that can be difficult to verify.
For basically the same reason, the concept of "relaxed control" has been established for classical control problems in vector spaces. In a word, it is based on replacing the metric space U of control parameters by the set of Borel probability measures on U, from now on denoted by $\mathcal{P}(U)$.
The goal of this section is to adapt "relaxed controls" to the morphological framework.

Definition 3.9 *Let (U, d_U) be a metric space and consider $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence (metrized by d_{LIP} as in § 3.2). Suppose $g : U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ to be continuous. For any probability measure $\mu \in \mathcal{P}(U)$, the integral $\displaystyle\int_U g(u)\; d\mu(u)$ is defined as set-valued map by*

$$\int_U g(u)\; d\mu(u) : \; \mathbb{R}^N \rightsquigarrow \mathbb{R}^N, \quad x \mapsto \int_U g(u)(x)\; d\mu(u).$$

Remark. Using the notation of Definition 3.9, for each point $x \in \mathbb{R}^N$ fixed, the set-valued map $U \rightsquigarrow \mathbb{R}^N$, $u \mapsto g(u)(x)$ is compact-valued and continuous in the sense of Bouligand and Kuratowski. Thus the integral $\displaystyle\int_U g(u)(x)\; d\mu(u) \subset \mathbb{R}^N$ is well-defined in the sense of Aumann.

Definition 3.10 *Let (U, d_U) denote a metric space and $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ be given. $K : [0,T[\rightsquigarrow \mathbb{R}^N$ with values in $\mathcal{K}(\mathbb{R}^N)$ is called* solution *to the* morphological relaxed control problem

$$\begin{cases} \overset{\circ}{K}(\cdot) \ni f(K(\cdot), u) & \text{in } [0,T[\\ \quad u \in U \end{cases}$$

if there is a measurable function $\mu : [0,T[\longrightarrow \mathcal{P}(U), t \longmapsto \mu_t$ such that $K(\cdot)$ solves the nonautonomous morphological equation $\overset{\circ}{K}(t) \ni \displaystyle\int_U f(K(t), u)\; d\mu_t(u)$ in $[0,T[$, i.e. satisfying

1. *$K(\cdot) : [0,T[\rightsquigarrow \mathbb{R}^N$ is Lipschitz continuous with respect to $\mathrm{d\!l}$ and*
2. *for almost every $t \in [0,T[$, the closure $\overline{\displaystyle\int_U f(K(t), u)\; d\mu_t(u)} \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to $\overset{\circ}{K}(t)$.*

So the first question is now: Which effects do probability measures (on U) instead of U have on the corresponding set-valued map $\mathcal{F}_U : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$?

Proposition 3.11 *Assume the metric space (U, d_U) to be compact and separable.*
Consider the set $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence and the set $\mathcal{P}(U)$ of Borel probability measures on U with the topology of narrow convergence (i.e. the dual setting with continuous and thus bounded functions $U \longrightarrow \mathbb{R}$).

Let $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *be continuous with* $\sup_{\substack{K\in\mathcal{K}(\mathbb{R}^N)\\ u\in U}} \big(\|f(K,u)\|_\infty + \mathrm{Lip}\ f(K,u)\big) < \infty$ *and, set for each* $K \in \mathcal{K}(\mathbb{R}^N)$

$$\begin{aligned}\mathcal{F}_U(K) \;&:=\; \Big\{\ \overline{f(K,u)\qquad\qquad} \;\Big|\; u \in U\Big\},\\ \widetilde{\mathcal{F}}_U(K) \;&:=\; \Big\{ \int_U f(K,u)\ d\mu(u) \;\Big|\; \mu \in \mathcal{P}(U) \Big\}.\end{aligned}$$

Then,

1.) $\widetilde{\mathcal{F}}_U(\cdot)$ *is a set-valued map* $\mathcal{K}(\mathbb{R}^N) \leadsto \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *with* $\mathcal{F}_U(K) \subset \widetilde{\mathcal{F}}_U(K)$ *for every* $K \in \mathcal{K}(\mathbb{R}^N)$.
2.) $\widetilde{\mathcal{F}}_U(\cdot)$ *has closed convex values with* $\overline{co}\ \mathcal{F}_U(K) = \widetilde{\mathcal{F}}_U(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *for every* $K \in \mathcal{K}(\mathbb{R}^N)$.
3.) The graph of $\widetilde{\mathcal{F}}_U(\cdot)$ *is closed.*

The proof of this proposition uses some tools about Borel probability measures and Aumann integrals. It is postponed to the end of this section 3.4.
The main notion is now to consider $\mathcal{P}(U)$ as control set instead of U. For applying Proposition 3.3 about the relationship between control problem and morphological inclusion, however, the parameter space has to be metric. So we need the following lemma for obtaining the counterparts to Proposition 3.7 and Corollary 3.8. Proposition 3.13 and Corollary 3.14 are the main results of this section.

Lemma 3.12 ([1, §§ 5.1, 7.1])
Let $U \neq \emptyset$ *be a Polish space (i.e. complete and separable metric space) with a bounded metric* d_U.

Then the set $\mathcal{P}(U)$ *of Borel probability measures on* U *supplied with the topology of narrow convergence is metrizable and separable. An example for a suitable metric on* $\mathcal{P}(U)$ *is the linear Wasserstein distance (in its dual representation)*

$$d_{\mathcal{P}(U)}(\mu,\, \nu) \;:=\; \sup \Big\{ \int_U \psi\ d(\mu - \nu) \;\Big|\; \psi : U \longrightarrow \mathbb{R} \;\; 1\text{–}Lipschitz\ continuous \Big\}.$$

A subset $\mathcal{M} \subset \mathcal{P}(U)$ *is relatively compact in* $\mathcal{P}(U)$ *if and only if* $\mathcal{M}$ *is tight, i.e. for every* $\varepsilon > 0$, *there exists a compact subset* $C \subset U$ *with* $\mu(U \setminus C) \leq \varepsilon$ *for all* $\mu \in \mathcal{M}$ *(known as Prokhorov's Theorem).*

Proposition 3.13 (Viability theorem for morphological relaxed control problems)
Assume the metric space (U, d_U) *to be compact and separable. Consider the set* $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence and the set* $\mathcal{P}(U)$ *of Borel probability measures on* U *with the topology of narrow convergence.*
Suppose for $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ *and the nonempty closed subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$:

(i) $\sup_{\substack{K\in\mathcal{K}(\mathbb{R}^N)\\ u\in U}} \big(\|f(K,u)\|_\infty + \mathrm{Lip}\ f(K,u)\big) < \infty$,
(ii) f *is continuous,*
(iii) $T_{\mathcal{V}}(K) \cap \overline{co}\, \{f(K,u) \mid u \in U\} \neq \emptyset$ *for each* $K \in \mathcal{V}$.

Then for every initial set $K_0 \in \mathcal{V}$, *there exists a compact–valued Lipschitz continuous solution* $K(\cdot) : [0,1] \leadsto \mathbb{R}^N$ *to the morphological* relaxed *control problem* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), u),\ u \in U$ *(in the sense of Definition 3.10) with* $K(0) = K_0$ *and* $K(t) \in \mathcal{V}$ *for all* $t \in [0,1]$.

Proof. Considering $\big(\mathcal{P}(U),\ d_{\mathcal{P}(U)}\big)$ as metric parameter space instead of (U, d_U), the set-valued map

$$\widetilde{\mathcal{F}}_U : \mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N), \quad K \mapsto \Big\{ \overline{\int_U f(K,u)\ d\mu(u)} \ \Big|\ \mu \in \mathcal{P}(U) \Big\}$$

satisfies the assumptions of Viability Theorem 2.13 according to Proposition 3.11. So for each $K_0 \in \mathcal{V}$, there exists a compact–valued Lipschitz continuous solution $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ to the morphological inclusion $\overset{\circ}{K}(\cdot) \cap \widetilde{\mathcal{F}}_U(K(\cdot)) \neq \emptyset$ with $K(0) = K_0$ and $K(t) \in \mathcal{V}$ for all $t \in [0,1]$.
Finally Proposition 3.3 guarantees that $K(\cdot)$ is solution to the morphological control problem

$$\overset{\circ}{K}(\cdot) \ni \overline{\int_U f(K(\cdot),u)\ d\mu(u)}, \qquad \mu \in \mathcal{P}(U),$$

i.e. it solves the *relaxed* control problem. □

Corollary 3.14 (Morphological relaxed control problems strongly operable in $M \subset \mathbb{R}^N$)
Assume the metric space (U, d_U) to be compact and separable. Consider the set $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ with the topology of locally uniform convergence and the set $\mathcal{P}(U)$ of Borel probability measures on U with the topology of narrow convergence.
Suppose for $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ and the nonempty closed subset $M \subset \mathbb{R}^N$:

(*i*) $\sup_{\substack{K \in \mathcal{K}(\mathbb{R}^N)\\ u \in U}} \big(\|f(K,u)\|_\infty + \mathrm{Lip}\, f(K,u)\big) < \infty,$

(*ii*) *f is continuous,*

(*iii*) *for each compact $K \subset M$, there is a set-valued map $G \in \overline{co}\,\{f(K,u) \mid u \in U\} \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N,\mathbb{R}^N)$ satisfying $G(x) \subset T_M(x)$ for every $x \in K$.*

Then for every nonempty compact subset $K_0 \subset M$, there exists a compact–valued Lipschitz continuous solution $K(\cdot) : [0,1] \rightsquigarrow \mathbb{R}^N$ to the morphological relaxed *control problem $\overset{\circ}{K}(\cdot) \ni f(K(\cdot),u),\ u \in U$ (in the sense of Definition 3.10) with $K(0) = K_0$ and $K(t) \subset M$ for all $t \in [0,1]$.* □

Now we close this section with the proof of Proposition 3.11.

Proof of Proposition 3.11. (1.) As mentioned in the remark after Definition 3.9, the integral $\int_U f(K,u)\ d\mu(u)$ is a well-defined set-valued map $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ for each $K \in \mathcal{K}(\mathbb{R}^N)$, $u \in U$ and $\mu \in \mathcal{P}(U)$. Moreover, its closure is convex since all set-valued maps $f(K,u) \in \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$ have convex values and due to the general properties of Aumann integral (see e.g. [20, Theorem 2.1.17]).
Due to the assumption $B := \sup_{K,u}\big(\|f(K,u)\|_\infty + \mathrm{Lip}\, f(K,u)\big) < \infty$, all compact sets $f(K,u)(x)$ (with $K \in \mathcal{K}(\mathbb{R}^N), u \in U,\ x \in \mathbb{R}^N$) are contained in the closed convex ball $\{y \in \mathbb{R}^N \mid |y| \leq B\}$ and so are all values of the closures of $\int_U f(K,u)\ d\mu(u)$.

Finally we prove that $\overline{\int_U f(K,u)\ d\mu(u)} : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is B–Lipschitz continuous for each $K \in \mathcal{K}(\mathbb{R}^N)$.
For any $x_1, x_2 \in \mathbb{R}^N$, the inclusion $f(K,u)(x_1) \subset f(K,u)(x_2) + \mathbb{B}_{B\cdot|x_1-x_2|}(0) \subset \mathbb{R}^N$ holds for every $u \in U$ and we conclude from [7, Proposition 8.6.2]

$$\begin{aligned}\overline{\int_U f(K,u)(x_1)\ d\mu(u)} &\subset \overline{\int_U \big(f(K,u)(x_2) + \mathbb{B}_{B\cdot|x_1-x_2|}(0)\big)\ d\mu(u)}\\ &\subset \overline{\int_U f(K,u)(x_2)\ d\mu(u)} + \mathbb{B}_{B\cdot|x_1-x_2|}(0).\end{aligned}$$

(2.) The convexity of $\widetilde{\mathcal{F}}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ (with respect to pointwise convex combinations as in Theorem 2.13 (1)) results from the convexity of $\mathcal{P}(U)$. Furthermore, $\mathrm{co}\,\mathcal{F}(K) \subset \widetilde{\mathcal{F}}(K) \subset \overline{\mathrm{co}}\,\mathcal{F}(K)$ can be concluded easily from the fact that finite convex combinations of Dirac masses are dense in $\mathcal{P}(U)$ (since U is compact and separable [8, § 30]).

Now we prove that $\widetilde{\mathcal{F}}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed (with respect to locally uniform convergence) for every $K \in \mathcal{K}(\mathbb{R}^N)$. Indeed, let $(\mu_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{P}(U)$ such that

$$\overline{\int_U f(K,u)\ d\mu_n(u)} \ \stackrel{n\to\infty}{\longrightarrow}\ G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N) \qquad \text{locally uniformly in } \mathbb{R}^N.$$

As U is assumed to be compact, the sequence $(\mu_n)_{n\in\mathbb{N}}$ is tight and thus relatively compact in $\mathcal{P}(U)$ according to Lemma 3.12. So a subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$ converges narrowly to a measure $\mu_\infty \in \mathcal{P}(U)$. We want to verify for every $x \in \mathbb{R}^N$

$$\overline{\int_U f(K,u)(x)\ d\mu_\infty(u)} \ = \ G(x) \ \subset\ \mathbb{R}^N.$$

Indeed, the set-valued map $f(K,\cdot)(x) : U \rightsquigarrow \mathbb{R}^N$ is continuous with nonempty compact convex values. So both the closed integral in the recent claim and $G(x)$ are nonempty, compact and convex. For any vector $p \in \mathbb{R}^N$ and any measure $\nu \in \mathcal{P}(U)$, [7, Proposition 8.6.2] states

$$\sup\ \Big(p \cdot \overline{\int_U f(K,u)(x)\ d\nu(u)}\Big) \ = \ \int_U \sup\ (p \cdot f(K,u)(x))\ \ d\nu(u).$$

Here the single-valued function $\sup\ (p \cdot f(K,\cdot)(x)) : U \longrightarrow \mathbb{R}$ is continuous and bounded. So on the one hand we conclude from the narrow convergence $\mu_{n_j} \longrightarrow \mu_\infty$ for each $p \in \mathbb{R}^N$

$$\sup\ \Big(p \cdot \overline{\int_U f(K,u)(x)\ d\mu_{n_j}(u)}\Big) \ \stackrel{j\to\infty}{\longrightarrow}\ \sup\ \Big(p \cdot \overline{\int_U f(K,u)(x)\ d\mu_\infty(u)}\Big).$$

On the other hand the initial assumption of locally uniform convergence to $G(\cdot)$ implies for each $p \in \mathbb{R}^N$

$$\sup\ \Big(p \cdot \overline{\int_U f(K,u)(x)\ d\mu_{n_j}(u)}\Big) \ \stackrel{j\to\infty}{\longrightarrow}\ \sup\ (p \cdot G(x))\,.$$

So the two following convex sets coincide for every $x \in \mathbb{R}^N$

$$\overline{\int_U f(K,u)(x)\ d\mu_\infty(u)} \ = \ G(x) \ \subset\ \mathbb{R}^N.$$

Finally we have verified that $\widetilde{\mathcal{F}}(K) \subset \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed.

(3.) For proving that $\mathrm{Graph}\ \widetilde{\mathcal{F}} \subset \mathcal{K}(\mathbb{R}^N) \times \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is closed, let $(K_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$ be any sequences in $\mathcal{K}(\mathbb{R}^N)$ and $\mathcal{P}(U)$ respectively such that

$$\begin{array}{rcll} K_n & \stackrel{n\to\infty}{\longrightarrow} & K \in \mathcal{K}(\mathbb{R}^N) & \qquad \text{with respect to } d\!\!l, \\ \overline{\int_U f(K_n,u)\ d\mu_n(u)} & \stackrel{n\to\infty}{\longrightarrow} & G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) & \qquad \text{locally uniformly in } \mathbb{R}^N. \end{array}$$

Our goal is to verify $G \in \widetilde{\mathcal{F}}(K)$.
Due to the compactness of U, the set $\{\mu_n | n \in \mathbb{N}\} \subset \mathcal{P}(U)$ is tight and so there exists a subsequence (again denoted by) $(\mu_n)_{n\in\mathbb{N}}$ converging narrowly to some $\mu_\infty \in \mathcal{P}(U)$. In the proof of statement (2.), we have already drawn the conclusion for each $x \in \mathbb{R}^N$

$$\overline{\int_U f(K,u)(x)\ d\mu_n(u)} \ \stackrel{n\to\infty}{\longrightarrow}\ \overline{\int_U f(K,u)(x)\ d\mu_\infty(u)} \ \subset\ \mathbb{R}^N$$

So now it is sufficient to verify for each $x \in \mathbb{R}^N$

$$\overline{\int_U f(K_n, u)(x) \;\; d\mu(u)} \overset{n\to\infty}{\longrightarrow} \overline{\int_U f(K, u)(x) \quad d\mu(u)} \qquad \text{uniformly in } \mu \in \mathcal{P}(U)$$

since it ensures $\overline{\int_U f(K_n, u)(x) \;\; d\mu_n(u)} \overset{n\to\infty}{\longrightarrow} \overline{\int_U f(K, u)(x) \;\; d\mu_\infty(u)} \subset \mathbb{R}^N$ for each $x \in \mathbb{R}^N$.

Indeed, the continuous function $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ (between metric spaces) is uniformly continuous on the compact set $\{K, K_n \,|\, n \in \mathbb{N}\} \times U$. So evaluating the set-valued maps at a fixed point $x \in \mathbb{R}^N$ respectively, we obtain for each $\varepsilon > 0$ that some small radius $\delta = \delta(\varepsilon) > 0$ satisfies

$$d\!\!l(K_n, K) \;+\; d_U(u_1, u_2) \;\le\; \delta \quad \Longrightarrow \quad d\!\!l\big(f(K_n, u_1)(x),\;\; f(K, u_2)(x)\big) \;\le\; \varepsilon.$$

In particular, there is some $m = m(\varepsilon) \in \mathbb{N}$ with $d\!\!l\big(f(K_n, u)(x),\;\; f(K, u)(x)\big) \le \varepsilon$ for all $n \ge m, u \in U$. Since $f(K_n, u)(x)$ and $f(K, u)(x)$ are always compact convex subsets of $\mathbb{R}^N$, it implies for the closure of the Aumann integral with respect to any probability measure $\mu \in \mathcal{P}(U)$ [20, Theorem 2.1.17 (i)]

$$d\!\!l\Big(\overline{\int_U f(K_n, u) \;\; d\mu(u)},\;\; \overline{\int_U f(K, u) \;\; d\mu(u)}\Big) \;\le\; \varepsilon \qquad \text{for all } n \ge m(\varepsilon). \qquad \square$$

3.5 Closed control loops for problems with state constraints

In this section, we specify sufficient conditions on the control system and state constraints for the existence of a closed-loop control, i.e. a continuous function $u(\cdot) : \mathcal{V} \longrightarrow U$ is to provide a feedback law such that for any initial set $K_0 \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$, every Lipschitz solution $K(\cdot) : [0, T] \rightsquigarrow \mathbb{R}^N$ to the morphological equation

$$\begin{cases} \overset{\circ}{K}(\cdot) \;\ni\; f\,(K(\cdot),\; u(K(\cdot))) & \text{a.e. in } [0, T[\\ K(0) \;\in\; K_0 & \end{cases} \tag{3}$$

solves the morphological control problem with state constraints

$$\begin{cases} \overset{\circ}{K}(\cdot) \;\ni\; f(K(\cdot), u), \quad u \in U & \text{a.e. in } [0, T[\\ K(t) \;\in\; \mathcal{V} & \text{for every } t \in [0, T[. \end{cases} \tag{4}$$

Corresponding to Aubin's notion of *regulation maps* [6, § 6], Nagumo's Theorem 2.10 motivates to construct the wanted closed-loop control $u(\cdot) : \mathcal{V} \longrightarrow U$ as continuous selection of the set-valued map

$$\mathcal{V} \rightsquigarrow U, \quad K \mapsto \{u \in U \mid f(K, u) \in T_{\mathcal{V}}(K)\}$$

indicating "consistent" control parameters for preserving values in $\mathcal{V}$.

Applying Michael's famous selection theorem for lower semicontinuous functions [19],[7, Theorem 9.1.2], this approach has been developed for constrained control problems in the Euclidean space [6, § 6.6.1]. Our contribution now is to extend it to the morphological framework.

The key challenge is to specify appropriate subsets of the contingent transition set $T_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ so that "convenient" assumptions about them ensure the existence of a closed-loop control. For this purpose, we extend Clarke tangent cone and hypertangent cone to the morphological framework in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$. Their counterparts are called circatangent transition set $T^C_{\mathcal{V}}(K)$ and hypertangent transition set $H_{\mathcal{V}}(K)$ respectively and, several of their features are presented in Appendix A and B.

Definition 3.15 *For a nonempty subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ and any element $K \in \mathcal{V}$,*

$$T^C_{\mathcal{V}}(K) := \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; \forall\ h_n \downarrow 0,\ \ K_n \to K \ \ \textit{with}\ K_n \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N):\ \tfrac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_F(h_n, K_n),\ \mathcal{V}\big) \stackrel{n\to\infty}{\longrightarrow} 0 \Big\}$$

is called circatangent transition set *of $\mathcal{V}$ at K (in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$).*

Definition 3.16 *Consider the set $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence. For a nonempty subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ and any element $K \in \mathcal{V}$,*

$$H_{\mathcal{V}}(K) := \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; \exists\ \varepsilon > 0,\ \textit{neighborhood } U \textit{ of } F \quad \forall\ G \in U:\ \limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_G(h, M),\ \mathcal{V}\big) = 0 \textit{ uniformly in } M \in \mathcal{V} \cap \mathbb{B}_\varepsilon(K) \Big\}$$

is called hypertangent transition set *of $\mathcal{V}$ at K (in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$).*

Obviously, $H_{\mathcal{V}}(K) \subset T^C_{\mathcal{V}}(K) \subset T_{\mathcal{V}}(K)$ for every set $K \in \mathcal{V}$.
Corresponding to the Clarke tangent cone in the Euclidean space, the circatangent transition set $T^C_{\mathcal{V}}(K)$ is convex according to Corollary A.7. Furthermore, there is a close relation to the hypertangent transition set: Graph $H_{\mathcal{V}}(\cdot)$ is the interior of the graph of $T^C_{\mathcal{V}}(\cdot) : \mathcal{V} \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ due to Proposition B.7. So now we can formulate the main result of this section:

Proposition 3.17 (Closed-loop control for morphological equations) *Let U be a separable Banach space and, consider the set* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence. For a nonempty closed set $\mathcal{V} \subset (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and $f : \mathcal{K}(\mathbb{R}^N) \times U \longrightarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ suppose:*

(1.) f is continuous und bounded in the sense that $\displaystyle\sup_{\substack{M \in \mathcal{K}(\mathbb{R}^N)\\ u \in U}} \big(\|f(M,u)\|_\infty + \mathrm{Lip}\, f(M,u)\big) < \infty.$

(2.) $R^H : \mathcal{V} \rightsquigarrow U, \quad K \mapsto \{u \in U \mid f(K,u) \in H_{\mathcal{V}}(K)\}$ has nonempty convex values.

Then, the pointwise closure $\overline{R}^H : \mathcal{V} \rightsquigarrow U,\ K \mapsto \overline{R^H(K)}$ has a continuous selection $u(\cdot) : \mathcal{V} \longrightarrow U$. In particular, every Lipschitz continuous solution $K(\cdot) : [0,T] \rightsquigarrow \mathbb{R}^N$ to the morphological equation

$$\begin{cases} \overset{\circ}{K}(\cdot) \ni f\,(K(\cdot),\ u(K(\cdot))) & \text{a.e. in } [0,T[\\ K(0) \in K_0 \end{cases}$$

with initial set $K_0 \in \mathcal{V}$ is viable in $\mathcal{V}$, i.e. $K(t) \in \mathcal{V}$ for all $t \in [0,T]$.

In combination with Nagumo's theorem 2.10, Michael's celebrated selection theorem lays the analytical foundations. In particular, it motivates the choice of a Banach space for the controls (instead of a metric space as in earlier sections of this paper).

Proposition 3.18 (Michael [19],[7, Theorem 9.1.2]**)** *Let R be a lower semicontinuous set-valued map with nonempty closed convex values from a compact metric space X to a Banach space Y. Then R has a continuous selection, i.e. there exists a continuous single-valued function $r : X \longrightarrow Y$ with $r(x) \in R(x)$ for every $x \in X$.*

Proof of Proposition 3.17.
Following a track similar to [6, Proposition 6.3.2], we first verify the lower semicontinuity of

$$R^H : \mathcal{V} \rightsquigarrow U, \quad K \mapsto \{u \in U \mid f(K,u) \in H_{\mathcal{V}}(K)\}$$

(in the sense of Bouligand and Kuratowski). Indeed, choose $K \in \mathcal{V}$ and $u \in R^H(K)$ arbitrarily. Graph $H_{\mathcal{V}}$ is open in $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ as a direct consequence of Definition 3.16. So there is $r > 0$ with $\mathbb{B}_r(K) \times \mathbb{B}_r\big(f(K,u)\big) \subset$ Graph $H_{\mathcal{V}}$, i.e. $\mathbb{B}_r\big(f(K,u)\big) \subset H_{\mathcal{V}}(M)$ for all $M \in \mathbb{B}_r(K) \subset \mathcal{K}(\mathbb{R}^N)$. Finally the continuity of f provides a smaller radius $\rho \in]0, r[$ with $f(M,v) \in \mathbb{B}_r\big(f(K,u)\big) \subset H_{\mathcal{V}}(M)$ for all $v \in \mathbb{B}_\rho(u) \subset U$ and $M \in \mathbb{B}_\rho(K) \subset \mathcal{K}(\mathbb{R}^N)$.
In other words, the intersection of sets $R^H(M) \stackrel{\text{Def.}}{=} \{v \in U \mid f(M,v) \in H_{\mathcal{V}}(M)\}$ for all $M \in \mathbb{B}_\rho(K)$ contains the ball $\mathbb{B}_\rho(u) \subset U$ and thus, it is a neighborhood of $u \in R^H(K)$. So $R^H(\cdot) : \mathcal{V} \rightsquigarrow U$ is lower semicontinuous.
Now we consider the pointwise closure of R^H, i.e. $\overline{R}^H : \mathcal{V} \rightsquigarrow U$, $K \mapsto \overline{\{u \in U \mid f(K,u) \in H_{\mathcal{V}}(K)\}}$. Obviously, $\overline{R}^H(\cdot)$ has nonempty closed convex values in the Banach space U. Additionally, it inherits lower semicontinuity from $R^H(\cdot)$ as the topological criterion of lower semicontinuity (via neighborhoods) reveals easily. So for any compact ball $B \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$, Michael's Theorem (Proposition 3.18) provides a continuous selection $u_B : B \cap \mathcal{V} \longrightarrow U$ of the restriction $\overline{R}^H\big|_{B\cap\mathcal{V}} : B \cap \mathcal{V} \rightsquigarrow U$. Covering finally the metric space $\big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$ with countably many balls, a locally finite continuous partition of unit leads to the wanted continuous selection $u : \mathcal{V} \longrightarrow U$ of $\overline{R}^H : \mathcal{V} \rightsquigarrow U$ because all values of $\overline{R}^H$ are convex. $\square$

A Clarke tangent cone in the morphological framework: The circatangent transition set.

The invariance condition of Nagumo (in Theorem 2.10) has already served Aubin as motivation for extending the contingent cone $T_V(x)$ in a normed vector space to the morphological framework (see Definition 2.9 quoting [3, Definition 1.5.2]).
In this section, we seize the classical definition of Clarke tangent cone introduced by Frank H. Clarke in the seventies (see [10] for details) and extend it to the morphological framework. Following the alternative nomenclature of Aubin and Frankowska in [7, Definition 4.1.5 (2)], its counterpart will be called *circatangent transition set* – just because this term is shorter and fits to the established "contingent transition set". In [3, Definition 1.5.4], Aubin introduced circatangent transition sets in the more general framework of metric spaces and, Definition A.2 below is equivalent to the special case of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and morphological transitions. In [21], Murillo applied this concept to tuples of vectors and compact subsets in $\mathbb{R}^M \times \mathcal{K}(\mathbb{R}^N)$ and proved an asymptotic relationship between their contingent and circatangent transition set implying that the latter is closed [21, Theorem 4.6].
To the best of our knowledge, further features like convexity are extended from the Euclidean space to the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ in this paper for the first time.

Definition A.1 **(**[10, § 2.4],[7, § 4.1.3], [27, § 6.F]**)** *Let K be a nonempty subset of a normed vector space X and $x \in X$ belong to the closure of K. The* Clarke tangent cone *or* circatangent cone $T_K^C(x)$ *is defined (equivalently) by*

$$\begin{aligned} T_K^C(x) &:= \operatorname{Liminf}_{\substack{h \downarrow 0,\\ y \xrightarrow[K]{} x}} \frac{K-y}{h} \\ &= \Big\{ v \in X \;\Big|\; \forall\; h_n \downarrow 0,\;\; y_n \to x \;\; with \;\; y_n \in K: \;\; \operatorname{dist}\big(v,\; \tfrac{K-y_n}{h_n}\big) \stackrel{n\to\infty}{\longrightarrow} 0 \Big\} \\ &= \Big\{ v \in X \;\Big|\; \forall\; h_n \downarrow 0,\;\; y_n \to x \;\; with \;\; y_n \in K: \;\; \tfrac{1}{h_n} \cdot \operatorname{dist}\big(y_n + h_n \cdot v,\; K\big) \stackrel{n\to\infty}{\longrightarrow} 0 \Big\}. \end{aligned}$$

Definition A.2 *For a nonempty subset $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ and any element $K \in \mathcal{V}$,*

$$T_{\mathcal{V}}^C(K) := \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; \forall\; h_n \downarrow 0,\;\; K_n \to K \;\; with \;\; K_n \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N): \;\; \tfrac{1}{h_n} \cdot \operatorname{dist}\big(\vartheta_F(h_n, K_n),\; \mathcal{V}\big) \stackrel{n\to\infty}{\longrightarrow} 0 \Big\}$$

is called circatangent transition set *of $\mathcal{V}$ at K (in the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$).*

In fact, we do not have to restrict our considerations to arbitrary sequences $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$. An equivalent characterization of $T_{\mathcal{V}}^C(K)$ uses all sequences in $\mathcal{K}(\mathbb{R}^N)$ converging to K :

Lemma A.3 *For every nonempty closed subset $\mathcal{V} \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$ and $K \in \mathcal{V}$,*

$$T_{\mathcal{V}}^C(K) = \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; \forall\; h_n \downarrow 0,\;\; K_n \to K: \;\; \limsup_{n\longrightarrow\infty} \tfrac{1}{h_n} \cdot \Big(\operatorname{dist}\big(\vartheta_F(h_n, K_n),\; \mathcal{V}\big) \;-\; \operatorname{dist}\big(K_n,\; \mathcal{V}\big)\Big) \;\le\; 0 \Big\}$$

Proof. "$\supset$" is an obvious consequence of Definition A.2.

"$\subset$" For any $F \in T^C_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ choose the arbitrary sequences $(h_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ in $]0,\infty[$ and $\mathcal{K}(\mathbb{R}^N)$ respectively with $h_n \longrightarrow 0$, $d\!\!l(K_n, K) \longrightarrow 0$ for $n \longrightarrow \infty$. Since closed balls in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ are known to be compact, there exists a set $M_n \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ for each $n \in \mathbb{N}$ satisfying

$$d\!\!l(K_n, M_n) \;=\; \mathrm{dist}\big(K_n, \mathcal{V}\big) \;\longrightarrow 0\,.$$

$F \in T^C_{\mathcal{V}}(K)$ implies $\quad \frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_F(h_n, M_n),\ \mathcal{V}\big) \longrightarrow 0 \quad$ for $n \longrightarrow \infty$

and Lemma 2.4 ensures $\quad d\!\!l\big(\vartheta_F(h_n, K_n),\ \vartheta_F(h_n, M_n)\big) \;\le\; d\!\!l(K_n, M_n) \cdot e^{\mathrm{Lip}\, F \cdot h_n} \quad$ for each $n \in \mathbb{N}$.

Finally, we obtain

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty} \ \tfrac{1}{h_n} \cdot \Big(\mathrm{dist}\big(\vartheta_F(h_n, K_n),\ \mathcal{V}\big) - \mathrm{dist}\big(K_n,\ \mathcal{V}\big)\Big)\\
\le\ &\limsup_{n\longrightarrow\infty} \ \tfrac{1}{h_n} \cdot \Big(d\!\!l\big(\vartheta_F(h_n, K_n),\ \vartheta_F(h_n, M_n)\big) + \mathrm{dist}\big(\vartheta_F(h_n, M_n),\ \mathcal{V}\big) - d\!\!l(K_n, M_n)\Big)\\
\le\ &\limsup_{n\longrightarrow\infty} \ \Big(d\!\!l(K_n, M_n) \cdot \tfrac{e^{\mathrm{Lip}\, F \cdot h_n} - 1}{h_n} + \tfrac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_F(h_n, M_n),\ \mathcal{V}\big)\Big)\\
\le\ &0.
\end{aligned}$$

□

So far, the circatangent transition set has been characterized by two sequences providing the arbitrarily fixed link between "step size" $h_n > 0$ and neighboring sets $K_n \in \mathcal{K}(\mathbb{R}^N)$. The following condition proves to be equivalent and avoids countability as essential feature:

Lemma A.4 *Let $K \in \mathcal{K}(\mathbb{R}^N)$ be any element of the nonempty closed set $\mathcal{V} \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$. Then, a set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the circatangent transition set $T^C_{\mathcal{V}}(K)$ if and only if there is a function $\omega : [0,\infty[\longrightarrow [0,\infty[$ with $\lim_{\delta\to 0}\ \omega(\delta) = 0$ and*

$$\tfrac{1}{h} \cdot \big(\mathrm{dist}\big(\vartheta_F(h, M),\ \mathcal{V}\big) - \mathrm{dist}(M,\ \mathcal{V})\big) \;\le\; \omega\big(d\!\!l(M, K) + h\big) \qquad \textit{for all } h \in\,]0,1],\ M \in \mathcal{K}(\mathbb{R}^N).$$

Proof. "$\Longleftarrow$" is an immediate consequence of Lemma A.3.

"$\Longrightarrow$" The triangle inequality of $d\!\!l$ and Lemma 2.4 guarantee for all $h > 0$ and $M \in \mathcal{K}(\mathbb{R}^N)$

$$\mathrm{dist}\big(\vartheta_F(h, M),\ \mathcal{V}\big) - \mathrm{dist}(M,\ \mathcal{V}) \;\le\; d\!\!l\big(M,\ \vartheta_F(h, M)\big) \;\le\; \|F\|_\infty\, h.$$

So the auxiliary function $\omega : [0,\infty[\longrightarrow [0,\infty[$,

$$\omega(\delta) \;:=\; \sup\Big\{\tfrac{1}{h} \cdot \big(\mathrm{dist}\big(\vartheta_F(h, M),\ \mathcal{V}\big) - \mathrm{dist}(M,\ \mathcal{V})\big) \;\Big|\; M \in \mathcal{K}(\mathbb{R}^N),\ h \in]0,1],\ d\!\!l(M, K) + h \le \delta\Big\}$$

is well-defined and bounded for any set-valued map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

For $F \in T^C_{\mathcal{V}}(K)$, however, we still have to verify $\omega(\delta) \longrightarrow 0$ for $\delta \longrightarrow 0$.

If this asymptotic feature was not correct, there would exist some $\varepsilon > 0$ and sequences $(h_n)_{n\in\mathbb{N}}$, $(M_n)_{n\in\mathbb{N}}$ in $]0,1]$, $\mathcal{K}(\mathbb{R}^N)$ respectively satisfying for all $n \in \mathbb{N}$

$$d\!\!l(M_n, K) + h_n \le \tfrac{1}{n} \quad \text{and} \quad \tfrac{1}{h_n} \cdot \big(\mathrm{dist}\big(\vartheta_F(h_n, M_n),\ \mathcal{V}\big) - \mathrm{dist}(M_n,\ \mathcal{V})\big) \;\ge\; \varepsilon \;>\; 0.$$

Due to $h_n \downarrow 0$ and $M_n \longrightarrow K$, it would contradict the initial assumption $F \in T^C_{\mathcal{V}}(K)$ due to Lemma A.3.

□

As the next propositions reveal, this circatangent transition set shares some properties with the Clarke tangent cone in normed vector spaces. Indeed, it is a nonempty closed convex cone in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

Proposition A.5 *Let $K \in \mathcal{K}(\mathbb{R}^N)$ be any element of the nonempty closed set $\mathcal{V} \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$. Then,*

1. *the circatangent transition set $T^C_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ is a nonempty cone, i.e. for any $G \in T^C_{\mathcal{V}}(K)$ and $\lambda \geq 0$, the set–valued map $\mathbb{R}^N \leadsto \mathbb{R}^N$, $x \mapsto \lambda \cdot G(x)$ (in the Minkowski sense) also belongs to $T^C_{\mathcal{V}}(K)$.*

2. *for every threshold $B \in [0, \infty[$, the intersection $T^C_{\mathcal{V}}(K) \cap \{G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \mid \|G\|_\infty + \mathrm{Lip}\, G \leq B\}$ is closed in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with the topology of locally uniform convergence.*

Proof. (1.) Obviously, the constant set-valued map $G_0(\cdot) := \{0\} : \mathbb{R}^N \leadsto \mathbb{R}^N$ belongs to both $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and $T^C_{\mathcal{V}}(K)$ because $\vartheta_{G_0}(h, K) = K$ for every $K \in \mathcal{K}(\mathbb{R}^N)$ and $h \geq 0$. Thus, $T^C_{\mathcal{V}}(K) \neq \emptyset$. For proving the cone property, choose any $K \in \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$, $G \in T^C_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda > 0$. Moreover, let $(h_n)_{n \in \mathbb{N}}$ and $(K_n)_{n \in \mathbb{N}}$ be arbitrary sequences in $]0, \infty[$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ respectively with $h_n \longrightarrow 0$ and $d\!\!l(K_n, K) \longrightarrow 0$ (for $n \to \infty$).
Every solution $x(\cdot) \in W^{1,1}([0, h_n], \mathbb{R}^N)$ of $x'(\cdot) \in \lambda\, G(x(\cdot))$ induces a solution $y(\cdot) \in W^{1,1}([0, \frac{h_n}{\lambda}], \mathbb{R}^N)$ of $y'(\cdot) \in G(y(\cdot))$ (and vice versa) by time scaling, i.e. $x(t) = y(\lambda \cdot t)$. So, $\vartheta_{\lambda\, G}(h_n, K_n) = \vartheta_G(\frac{h_n}{\lambda}, K_n)$.
The assumption $G \in T^C_{\mathcal{V}}(K)$ guarantees now

$$\frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_{\lambda\, G}(h_n, K_n),\ \mathcal{V}\big) \;=\; \frac{1}{\lambda}\,\frac{\lambda}{h_n} \cdot \mathrm{dist}\big(\vartheta_G(\tfrac{h_n}{\lambda}, K_n),\ \mathcal{V}\big) \;\longrightarrow\; 0 \qquad \text{for } n \to \infty.$$

(2.) Let $(G^j)_{j \in \mathbb{N}}$ be a sequence in $T^C_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|G^j\|_\infty + \mathrm{Lip}\, G^j \leq B$ for each j and converging to $G(\cdot) \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ locally uniformly. Obviously, $\|G\|_\infty + \mathrm{Lip}\ G \leq B$ holds. Our aim is to verify $G \in T^C_{\mathcal{V}}(K)$.

Let $(h_n)_{n \in \mathbb{N}}$ and $(K_n)_{n \in \mathbb{N}}$ be any sequences in $]0, 1]$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ respectively with $h_n \longrightarrow 0$ and $d\!\!l(K_n, K) \longrightarrow 0$ (for $n \to \infty$). The last convergence implies that all K_n, $n \in \mathbb{N}$, and $K \in \mathcal{K}(\mathbb{R}^N)$ are contained in a ball $\mathbb{B}_R(0) \subset \mathbb{R}^N$ of sufficiently large radius $R < \infty$. So due to $\sup\limits_n\ h_n \leq 1$,

$$\bigcup_{j, n \in \mathbb{N}}\ \bigcup_{0 \leq t \leq h_n} \big(\vartheta_{G^j}(t, K_n) \,\cup\, \vartheta_G(t, K_n)\big) \;\subset\; \mathbb{B}_{R+B}(0) \;\subset\; \mathbb{R}^N.$$

On the basis of Lemma 2.4, we obtain the estimate for every $j, n \in \mathbb{N}$

$$\begin{aligned} \frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_G(h_n, K_n),\ \mathcal{V}\big) \;&\leq\; \frac{1}{h_n} \cdot d\!\!l\big(\vartheta_G(h_n, K_n),\ \vartheta_{G^j}(h_n, K_n)\big) \;+\; \frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_{G^j}(h_n, K_n),\ \mathcal{V}\big) \\ &\leq\; e^{B\, h_n} \cdot \sup_{|x| \leq R+B} d\!\!l(G(x),\ G^j(x)) \;+\; \frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_{G^j}(h_n, K_n),\ \mathcal{V}\big). \end{aligned}$$

For any $\varepsilon > 0$ given, we can fix $j \in \mathbb{N}$ sufficiently large with $\sup\limits_{|x| \leq R+B} d\!\!l(G(x),\ G^j(x)) < \varepsilon$ and, $G^j \in T^C_{\mathcal{V}}(K)$ guarantees $\limsup\limits_{n \longrightarrow \infty} \frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_G(h_n, K_n),\ \mathcal{V}\big) \;\leq\; \varepsilon$ with arbitrarily small $\varepsilon > 0$,
i.e. $\frac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_G(h_n, K_n),\ \mathcal{V}\big) \longrightarrow 0$ for $n \longrightarrow \infty$. □

Proposition A.6 *Let $K \in \mathcal{K}(\mathbb{R}^N)$ be any element of the nonempty closed set $\mathcal{V} \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$. Then, $T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is convex, i.e. for any $G_1, G_2 \in T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda \in [0, 1]$, the set–valued map $\mathbb{R}^N \leadsto \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1 - \lambda) \cdot G_2(x)$ (in the Minkowski sense) also belongs to the intersection $T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$.*

The proof of this convexity is based on parameterizing bounded set-valued maps with compact convex values and thus, it is postponed to the end of Appendix A. As a consequence, we obtain the convexity of the circatangent transition set rather easily:

Corollary A.7 *Let $K \in \mathcal{K}(\mathbb{R}^N)$ be any element of the nonempty closed set $\mathcal{V} \subset (\mathcal{K}(\mathbb{R}^N), d\!l)$. Then, $T^C_{\mathcal{V}}(K)$ is convex in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, i.e. for any $G_1, G_2 \in T^C_{\mathcal{V}}(K)$ and $\lambda \in [0,1]$, the set–valued map $\mathbb{R}^N \leadsto \mathbb{R}^N$, $x \mapsto \lambda \cdot G_1(x) + (1-\lambda) \cdot G_2(x)$ (in the Minkowski sense) also belongs to $T^C_{\mathcal{V}}(K)$.*

Proof. The well–known Relaxation Theorem of Filippov–Ważiewski (e.g. [2, § 2.4, Theorem 2]) implies $\vartheta_G(t, M) = \vartheta_{\overline{co}\, G}(t, M)$ for every map $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, initial set $M \in \mathcal{K}(\mathbb{R}^N)$ and time $t \geq 0$. As an immediate consequence, we conclude for any $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ that the pointwise convex hull $\overline{\mathrm{co}}\, G : \mathbb{R}^N \leadsto \mathbb{R}^N$ is contained in $T^C_{\mathcal{V}}(K)$ if and only if $G \in T^C_{\mathcal{V}}(K)$.
Choose now $G_1, G_2 \in T^C_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda \in [0,1]$ arbitrarily. Then, $\overline{\mathrm{co}}\, G_1$ and $\overline{\mathrm{co}}\, G_2$ are contained in $T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ and, Proposition A.6 implies

$$\lambda \cdot \overline{\mathrm{co}}\, G_1 + (1-\lambda) \cdot \overline{\mathrm{co}}\, G_2 \;\in\; T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N).$$

This last set-valued map is identical to the pointwise convex hull of $\lambda \cdot G_1 + (1-\lambda) \cdot G_2 : \mathbb{R}^N \leadsto \mathbb{R}^N$ because for any $M_1, M_2 \in \mathcal{K}(\mathbb{R}^N)$, the convex hulls satisfy $\overline{\mathrm{co}}\,(M_1 + M_2) \;=\; \overline{\mathrm{co}}\, M_1 \;+\; \overline{\mathrm{co}}\, M_2$.
Finally, $\overline{\mathrm{co}}\,(\lambda \cdot G_1 + (1-\lambda) \cdot G_2) \;\in\; T^C_{\mathcal{V}}(K)$ implies $\lambda \cdot G_1 + (1-\lambda) \cdot G_2 \;\in\; T^C_{\mathcal{V}}(K)$. □

Now we prepare the proof of Proposition A.6, i.e. the convexity of $T^C_{\mathcal{V}}(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ is to be verified. The following tools will be used:

Lemma A.8 (Parameterization of bounded maps, [7, Theorem 9.7.2]**)**
Consider a metric space X and a set–valued map $G : [a,b] \times X \leadsto \mathbb{R}^N$ satisfying
1. *G has nonempty compact convex values,*
2. *$G(\cdot, x) : [a,b] \leadsto \mathbb{R}^N$ is measurable for every $x \in X$,*
3. *there exists $k(\cdot) \in L^1([a,b])$ such that for every $t \in [a,b]$, the set–valued map $G(t, \cdot) : X \leadsto \mathbb{R}^N$ is $k(t)$–Lipschitz continuous.*

Then there exists a function $g : [a,b] \times X \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ (with $\mathbb{B}_1 \stackrel{\text{Def.}}{=} \{u \in \mathbb{R}^N : |u| \leq 1\}$) fulfilling
1. $\forall\ (t, x) \in [a,b] \times X :\quad G(t,x) \;=\; \big\{\, g(t,x,u) \;\big|\; u \in \mathbb{B}_1 \big\}$,
2. $\forall\ (x, u) \in X \times \mathbb{B}_1 :\quad g(\cdot, x, u) : [a,b] \longrightarrow \mathbb{R}^N$ *is measurable,*
3. $\forall\ (t, u) \in [a,b] \times \mathbb{B}_1 :\quad g(t, \cdot, u) : X \longrightarrow \mathbb{R}^N$ *is $c \cdot k(t)$–Lipschitz continuous*
4. $\forall\ t \in [a,b],\ x \in X,\ u, v \in \mathbb{B}_1 :\quad |g(t,x,u) - g(t,x,v)| \;\leq\; c\, \|G(t,x)\|_\infty\, |u - v|$

with a constant $c > 0$ independent of G.

Lemma A.9 *For every $\lambda \in]0,1[$, there exists $\mu \in L^1([0,1])$ satisfying*

$$\tfrac{1}{t} \cdot \int_0^t (\mu(s) - \lambda)\; ds \;\longrightarrow\; 0 \qquad (t \downarrow 0), \qquad \mu(\cdot) \in \{0,1\} \textit{ piecewise constant in }]0,1[.$$

Proof. $\mu(\cdot)$ is defined piecewise in each interval $\big[\frac{1}{\sqrt{n+1}},\ \frac{1}{\sqrt{n}}\big[$ $(n \in \mathbb{N})$.

Set $$\mu(t) := \begin{cases} 0 & \text{for} \quad \frac{1}{\sqrt{n+1}} \;\leq\; t \;<\; \frac{\lambda}{\sqrt{n+1}} + \frac{1-\lambda}{\sqrt{n}} \\ 1 & \text{for} \quad \frac{\lambda}{\sqrt{n+1}} + \frac{1-\lambda}{\sqrt{n}} \;\leq\; t \;<\; \frac{1}{\sqrt{n}} \end{cases} \qquad \text{for each } n \in \mathbb{N}.$$

Then, $$\int_{\frac{1}{\sqrt{n+1}}}^{\frac{1}{\sqrt{n}}} (\mu(s) - \lambda)\;\; ds \;=\; 0 \quad \text{and thus,} \int_0^{\frac{1}{\sqrt{n}}} (\mu(s) - \lambda)\;\; ds \;=\; 0.$$

Moreover, $$\int_{\frac{1}{\sqrt{n+1}}}^{\frac{1}{\sqrt{n}}} |\mu(s) - \lambda|\;\; ds \;=\; 2\,\lambda\,(1-\lambda)\,\Big(\tfrac{1}{\sqrt{n}} - \tfrac{1}{\sqrt{n+1}}\Big) \quad \text{implies}$$

$$\sup_{\frac{1}{\sqrt{n+1}} \leq t \leq \frac{1}{\sqrt{n}}} \;\tfrac{1}{t} \cdot \left|\int_0^t (\mu(s) - \lambda)\;\; ds\right| \;\leq\; \sqrt{n+1} \cdot \int_{\frac{1}{\sqrt{n+1}}}^{\frac{1}{\sqrt{n}}} |\mu(s) - \lambda|\;\; ds \;\stackrel{n \to \infty}{\longrightarrow}\; 0.$$ □

Proof of Proposition A.6. For any $G_1, G_2 \in T_{\mathcal{V}}^C(K) \cap \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$ and $\lambda \in \,]0,1[$, we have to verify that the set-valued map $G: \mathbb{R}^N \rightsquigarrow \mathbb{R}^N,\ x \mapsto \lambda \cdot G_1(x) + (1-\lambda) \cdot G_2(x)$ (in the Minkowski sense) also belongs to $T_{\mathcal{V}}^C(K)$. Indeed, G is obviously Lipschitz continuous with compact convex values and thus, $G \in \mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$. According to Lemma A.9, there exists a function $\mu \in L^1([0,1])$ satisfying

$$\tfrac{1}{t} \cdot \int_0^t (\mu(s) - \lambda)\ ds \longrightarrow 0 \qquad (t \downarrow 0), \qquad \mu(\cdot) \in \{0,1\} \text{ piecewise constant in }]0,1[.$$

Now we compare the evolution of an arbitrary set $M \in \mathcal{K}(\mathbb{R}^N)$ along the autonomous differential inclusion with the right-hand side

$$G: \mathbb{R}^N \rightsquigarrow \mathbb{R}^N, \qquad x \longmapsto \lambda \cdot G_1(x) + (1-\lambda) \cdot G_2(x)$$

and along the nonautonomous differential inclusion with the right-hand side

$$H: \mathbb{R}^N \times [0,1] \rightsquigarrow \mathbb{R}^N, \qquad (x,t) \longmapsto \mu(t) \cdot G_1(x) + (1-\mu(t)) \cdot G_2(x).$$

We verify $d\!\!l(\vartheta_G(t,M),\ \vartheta_H(t,M)) \le o(t)$ for $t \downarrow 0$ uniformly in M.

First, both set-valued functions $G_1(\cdot), G_2(\cdot)$ are parameterized on the basis of Lemma A.8 and, we obtain Lipschitz continuous functions

$$\begin{aligned} g_1 : \mathbb{R}^N \times \mathbb{B}_1 &\longrightarrow \mathbb{R}^N, \quad (x, u_1) \longmapsto g_1(x, u_1) \\ g_2 : \mathbb{R}^N \times \mathbb{B}_1 &\longrightarrow \mathbb{R}^N, \quad (x, u_2) \longmapsto g_2(x, u_2) \end{aligned}$$

with the closed unit ball $\mathbb{B}_1 \subset \mathbb{R}^N$ and satisfying $G_j(x) = \{ g_j(x, u_j) \mid u_j \in \mathbb{B}_1 \}$ $(j = 1, 2)$.
In particular, the functions

$$\begin{aligned} \mathbb{R}^N \times \mathbb{B}_1 \times \mathbb{B}_1 &\longrightarrow \mathbb{R}^N, \quad (x, u_1, u_2) \longmapsto \lambda \cdot g_1(x, u_1) + (1-\lambda) \cdot g_2(x, u_2) \\ \mathbb{R}^N \times [0,1] \times \mathbb{B}_1 \times \mathbb{B}_1 &\longrightarrow \mathbb{R}^N, \quad (x, t, u_1, u_2) \longmapsto \mu(t) \cdot g_1(x, u_1) + (1-\mu(t)) \cdot g_2(x, u_2) \end{aligned}$$

are Lipschitz continuous w.r.t. x, u_1, u_2 and parameterize the set-valued maps $G(\cdot), H(\cdot)$ respectively.

Let $x(\cdot) \in W^{1,1}([0,1], \mathbb{R}^N)$ denote any solution to the nonautonomous differential inclusion $x'(\cdot) \in H(x(\cdot), \cdot)$. Applying Lemma 3.4 of Filippov to this parameterization of the composed set-valued map H provides two measurable functions $k_1, k_2 : [0,1] \longrightarrow \mathbb{B}_1 \subset \mathbb{R}^N$ satisfying for almost every $t \in [0,1]$

$$x'(t) = \mu(t) \cdot g_1\big(x(t), k_1(t)\big) + (1-\mu(t)) \cdot g_2\big(x(t), k_2(t)\big) \in H(x(t), t).$$

Now fix $\varepsilon > 0$ arbitrarily. Considering the convolution of $k_j(\cdot)$ and a suitable smooth auxiliary function (with compact support), we obtain Lipschitz continuous approximations $\widetilde{k}_1, \widetilde{k}_2 : [0,1] \longrightarrow \mathbb{B}_1 \subset \mathbb{R}^N$ and a parameter $\widetilde{c}(\varepsilon) < \infty$ fulfilling

$$\int_0^1 \big(|k_1(t) - \widetilde{k}_1(t)| + |k_2(t) - \widetilde{k}_2(t)| \big)\ dt < \varepsilon, \qquad \mathrm{Lip}\,\widetilde{k}_1 + \mathrm{Lip}\,\widetilde{k}_2 \le \widetilde{c}(\varepsilon).$$

Each right-hand side of the differential equations

$$\begin{aligned} \widetilde{x}'(t) &= \mu(t) \cdot g_1\big(\widetilde{x}(t), \widetilde{k}_1(t)\big) + (1-\mu(t)) \cdot g_2\big(\widetilde{x}(t), \widetilde{k}_2(t)\big) \in H(\widetilde{x}(t), t) \\ \widetilde{y}'(t) &= \lambda \cdot g_1\big(\widetilde{y}(t), \widetilde{k}_1(t)\big) + (1-\lambda) \cdot g_2\big(\widetilde{y}(t), \widetilde{k}_2(t)\big) \in G(\widetilde{y}(t)) \end{aligned}$$

is measurable with respect to t and Lipschitz with respect to $\widetilde{x}(t),\ \widetilde{y}(t)$ and thus, unique solutions $\widetilde{x}(\cdot),\ \widetilde{y}(\cdot) \in W^{1,1}([0,1], \mathbb{R}^N)$ start at the joint initial point $\widetilde{x}(0) = \widetilde{y}(0) = x(0) =: x_0$. Firstly, $x(\cdot)$ and $\widetilde{x}(\cdot)$ always satisfy

$$\begin{aligned} \big|x(t) - \widetilde{x}(t)\big| &\le \int_0^t \Big(\big|g_1(x(s), k_1(s)) - g_1(\widetilde{x}(s), \widetilde{k}_1(s))\big| + \big|g_2(x(s), k_1(s)) - g_2(\widetilde{x}(s), \widetilde{k}_1(s))\big| \Big)\ ds \\ &\le \big(\mathrm{Lip}\,g_1 + \mathrm{Lip}\,g_2\big) \int_0^t \Big(\big|x(s) - \widetilde{x}(s)\big| + \big|k_1(s) - \widetilde{k}_1(s)\big| + \big|k_2(s) - \widetilde{k}_2(s)\big| \Big)\ ds \end{aligned}$$

and Gronwall's Lemma ensures $\big|x(t) - \widetilde{x}(t)\big| \le \mathrm{const}(\mathrm{Lip}\,g_1,\ \mathrm{Lip}\,g_2) \cdot \varepsilon\ t$ for every $t \in [0,1]$.

Secondly, we estimate the difference $\widetilde{x}(\cdot) - \widetilde{y}(\cdot)$

$$
\begin{aligned}
&|\widetilde{y}(t) - \widetilde{x}(t)| \\
= \quad & \Big| \int_0^t \Big(\lambda\, g_1\big|_{(\widetilde{y}(s),\, \widetilde{k}_1(s))} - \mu(s)\, g_1\big|_{(\widetilde{x}(s),\, \widetilde{k}_1(s))} + (1-\lambda)\, g_2\big|_{(\widetilde{y}(s), \widetilde{k}_2(s))} - (1-\mu(s))\, g_2\big|_{(\widetilde{x}(s), \widetilde{k}_2(s))} \Big)\, ds \Big| \\
\leq \quad & \Big| \int_0^t \Big((\lambda - \mu(s))\, g_1\big(\widetilde{y}(s),\, \widetilde{k}_1(s)\big) \qquad + (\mu(s) - \lambda)\, g_2\big(\widetilde{y}(s),\, \widetilde{k}_2(s)\big) \Big)\; ds \Big| \\
& + \int_0^t \mu(s) \cdot \mathrm{Lip}\, g_1 \cdot |\widetilde{x}(s) - \widetilde{y}(s)|\; ds \qquad + \int_0^t (1 - \mu(s)) \cdot \mathrm{Lip}\, g_2 \cdot |\widetilde{x}(s) - \widetilde{y}(s)|\; ds \\
\leq \quad & \Big| \int_0^t (\lambda - \mu(s)) \cdot \big(g_1(\widetilde{x}_0, \widetilde{k}_1(0)) - g_2(x_0, \widetilde{k}_1(0)) \big)\; ds \Big| \\
& + \int_0^t \big|\lambda - \mu(s)\big| \;\; (\mathrm{Lip}\, g_1 + \mathrm{Lip}\, g_2) \;\; \big(|\widetilde{y}(s) - x_0| + |\widetilde{k}_1(s) - \widetilde{k}_1(0)| + |\widetilde{k}_2(s) - \widetilde{k}_2(0)| \big)\, ds \\
& + \max\{\mathrm{Lip}\, g_1,\ \mathrm{Lip}\, g_2\} \;\cdot \int_0^t |\widetilde{x}(s) - \widetilde{y}(s)|\; ds \\
\leq \quad & c \;\cdot \Big(\Big| \int_0^t (\lambda - \mu(s))\;\, ds \Big| \;+\; \int_0^t (1 + 2\, \widetilde{c}(\varepsilon)) \cdot s\;\; ds \;+\; \int_0^t |\widetilde{x}(s) - \widetilde{y}(s)|\; ds \Big)
\end{aligned}
$$

with a positive constant c depending only on $G_1(\cdot)$, $G_2(\cdot)$ (and its finite Lipschitz constants).
Gronwall's Lemma ensures $|\widetilde{x}(t) - \widetilde{y}(t)| \leq o(t)$ for $t \downarrow 0$ uniformly with respect to the initial point x_0 (but in general *not* uniformly with respect to $\varepsilon > 0$).
Last, but not least, the triangle inequality provides a link between the given solution $x(\cdot)$ of $x'(\cdot) \in H(x(\cdot), \cdot)$ and the constructed solution $\widetilde{y}(\cdot)$ of $\widetilde{y}'(\cdot) \in G(\widetilde{y}(\cdot))$ after having fixed $\varepsilon > 0$ arbitrarily:
$\limsup_{t \downarrow 0} \; \frac{1}{t} \cdot \big| x(t) - \widetilde{y}(t) \big| \;\leq\; \mathrm{const}(G_1(\cdot), G_2(\cdot)) \cdot \varepsilon$ uniformly with respect to the initial point $x_0 \in \mathbb{R}^N$.
Thus, for any initial set $M \in \mathcal{K}(\mathbb{R}^N)$, the reachable sets satisfy

$$\mathrm{dist}\,(\vartheta_H(t, M),\;\; \vartheta_G(t, M)) \;\leq\; o(t) \qquad \text{for } t \downarrow 0 \qquad \text{uniformly in } M \in \mathcal{K}(\mathbb{R}^N).$$

The same uniform estimates holds for $\mathrm{dist}\,(\vartheta_G(t, M),\;\; \vartheta_H(t, M))$ since the preceding solutions $\widetilde{x}(\cdot)$ and $\widetilde{y}(\cdot)$ have required only the common "control parameters" $\widetilde{k}_1(\cdot)$, $\widetilde{k}_2(\cdot)$ and a joint initial point $x_0 \in \mathbb{R}^N$.
So we obtain $$d\!\!l\,(\vartheta_G(t, M),\;\; \vartheta_H(t, M)) \;\leq\; o(t) \qquad \text{for } t \downarrow 0 \qquad \text{uniformly in } M \in \mathcal{K}(\mathbb{R}^N).$$

Finally, we focus on the asymptotic features of $\vartheta_H(\cdot, \cdot)$ in regard to the circatangent transition set $T^C_{\mathcal{V}}(K)$, i.e. for any $\varepsilon > 0$, we verify the existence of a radius $r > 0$ such that all $h \in \,]0, r]$ and sets $M \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(M, K) \leq r$ satisfy

$$\mathrm{dist}\big(\vartheta_H(h, M),\, \mathcal{V}\big) \;-\; \mathrm{dist}(M,\, \mathcal{V}) \;\;\leq\;\; \varepsilon\, h.$$

As a consequence, for any sequences $h_n \downarrow 0$ and $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ converging to K

$$\tfrac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_H(h_n, K_n),\; \mathcal{V}\big) \longrightarrow 0 \qquad\qquad \text{for } n \longrightarrow \infty$$

and in combination with the uniform convergence mentioned before, we conclude

$$\tfrac{1}{h_n} \cdot \mathrm{dist}\big(\vartheta_G(h_n, K_n),\; \mathcal{V}\big) \longrightarrow 0 \qquad\qquad \text{for } n \longrightarrow \infty,$$

i.e. $G \in T^C_{\mathcal{V}}(K)$ due to Definition A.2.

Indeed, applying Lemma A.4 to $G_1, G_2 \in T^C_{\mathcal{V}}(K)$, we obtain a joint function $\omega : [0, \infty[\longrightarrow [0, \infty[$ satisfying $\lim_{\delta \to 0} \omega(\delta) = 0$, $\sup_{[0,\infty[} \omega(\cdot) < \infty$ and for all $j \in \{1, 2\}$, $h \in \,]0, 1]$, $M \in \mathcal{K}(\mathbb{R}^N)$

$$\tfrac{1}{h} \cdot \big(\mathrm{dist}\big(\vartheta_{G_j}(h, M),\, \mathcal{V}\big) \;-\; \mathrm{dist}(M,\, \mathcal{V}) \big) \;\;\leq\;\; \omega\big(d\!\!l(M, K) + h\big).$$

Fixing $\varepsilon > 0$ arbitrarily small, there exist a radius $R > 0$ with $\sup_{[0,R]} \omega(\cdot) \leq \varepsilon$ and additionally, some $r \in \,]0, \frac{R}{2}]$ such that $r \cdot \big(1 + \|G_1\|_\infty + \|G_2\|_\infty\big) \leq \frac{R}{2}$. Then, each $j \in \{1, 2\}$ and every $h \in \,]0, r]$,

$M \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(M,K) \le r$ satisfy

$$\begin{cases} d\!\!l\big(\vartheta_{G_j}(h,M),K\big) \;\le\; d\!\!l(M,K) \;+\; \|G_j\|_\infty\, h \;\le\; \frac{R}{2} \\ \operatorname{dist}\big(\vartheta_{G_j}(h,M),\,\mathcal{V}\big) \;-\; \operatorname{dist}(M,\,\mathcal{V}) \;\le\; \omega\big(d\!\!l(M,K)+h\big)\cdot h \quad \le\; \varepsilon\, h. \end{cases}$$

For drawing now conclusions about $\vartheta_H(h,M)$, we exploit the piecewise constant structure of auxiliary function $\mu(\cdot) : [0,1] \longrightarrow \{0,1\}$ (introduced in Lemma A.9). Indeed, there is a sequence $(t_k)_{k\in\mathbb{N}}$ tending to 0 monotonically such that $\mu(\cdot)$ is constant in every interval $[t_{k+1}, t_k[$, $k \in \mathbb{N}$.
Applying the last estimate in each of these intervals separately, we conclude for every $h \in\,]0,r]$, $M \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(M,K) \le r$ and sufficiently large $k \in \mathbb{N}$ with $t_{k+1} < h \le t_k$

$$\begin{array}{ll} \multicolumn{2}{l}{\operatorname{dist}\big(\vartheta_H(h,M),\,\mathcal{V}\big) \;-\; \operatorname{dist}(M,\,\mathcal{V})} \\ \le & \operatorname{dist}\big(\vartheta_H(h-t_{k+1}, \vartheta_H(t_{k+1},M)),\,\mathcal{V}\big) \;-\; \operatorname{dist}\big(\vartheta_H(t_{k+1},M),\,\mathcal{V}\big) \\ & +\; \operatorname{dist}\big(\vartheta_H(t_{k+1},M),\,\mathcal{V}\big) \;-\; \operatorname{dist}\big(\vartheta_H(t_{k+2},M),\,\mathcal{V}\big) \;\pm\; \ldots \;-\; \operatorname{dist}(M,\,\mathcal{V}) \\ \le & \varepsilon\cdot(h-t_{k+1}) \;+\; \varepsilon\cdot(t_{k+1}-t_{k+2}) \;+\; \ldots \\ \le & \varepsilon\cdot\; h. \end{array}$$

□

B The hypertangent transition set

For any closed subset of the Euclidean space, the interior of the Clarke tangent cone has been characterized by Rockafellar in 1979 [26]. Indeed,

Proposition B.1 (Rockafellar [26, Theorem 2], [27, Theorem 6.36]**)** *Let $K \subset \mathbb{R}^N$ be a closed set and $x \in K$. Then the interior of Clarke tangent cone to K at x satisfies*

$$\begin{aligned} T_K^C(x)^\circ \;&=\; \{v \in \mathbb{R}^N \mid \exists\, \varepsilon > 0 :\; (K \cap \mathbb{B}_\varepsilon(x)) \;+\;]0,\varepsilon[\,\cdot \mathbb{B}_\varepsilon(v) \subset K\} \\ &=\; \{v \in \mathbb{R}^N \mid \exists\, \varepsilon > 0 \;\; \forall\, y \in K \cap \mathbb{B}_\varepsilon(x),\; w \in \mathbb{B}_\varepsilon(v),\; \tau \in\,]0,\varepsilon[:\;\; y + \tau\cdot w \in K\} \end{aligned}$$

with $\mathbb{B}_\varepsilon(v)$ abbreviating the closed ball $\mathbb{B}_\varepsilon(v) := \{w \in \mathbb{R}^N \mid |w - v| \le \varepsilon\}$ and
U° denoting always the interior of a set U.

This equivalence serves as motivation for introducing the concept of "hypertangent cones":

Definition B.2 ([10, § 2,4]**)** *A vector v in a Banach space X is said to be* hypertangent *to the set $K \subset X$ at the point $x \in K$ if for some $\varepsilon > 0$, all vectors $y \in \mathbb{B}_\varepsilon(x) \cap K$, $w \in \mathbb{B}_\varepsilon(v) \subset X$ and real $t \in\,]0,\varepsilon[$ satisfy* $$y + t\cdot w \;\in\; K.$$

We now focus on a similar description in the morphological framework. To be more precise, we are going to specify subsets $H_{\mathcal{V}}(K) \subset \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$ of the circatangent transition sets $T_{\mathcal{V}}^C(K)$, $K \in \mathcal{V}$, whose graph $\mathcal{V} \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$, $K \mapsto H_{\mathcal{V}}(K)$ is identical to the interior of the graph of $T_{\mathcal{V}}^C(\cdot)$ in $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N,\mathbb{R}^N)$.

There is an essential difference between the vector space $\mathbb{R}^N$ and the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, however, preventing us from applying Definition B.2 directly. Indeed, considering the neighborhood of a vector $y + t\cdot v$ (with $y, v \in \mathbb{R}^N$, $t > 0$), *each* of its points can be represented as $y + t\cdot w$ with a "perturbed" vector w close to v, i.e. $w \in \mathbb{B}_\varepsilon(v)$. The corresponding statement does not hold for reachable sets of differential inclusions in general, i.e. for given $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, $K \in \mathcal{K}(\mathbb{R}^N)$, $t > 0$, *not every* compact set $M \subset \mathbb{R}^N$ with arbitrarily small Hausdorff distance from $\vartheta_F(t, K)$ can be represented as reachable set $\vartheta_{\widetilde{G}}(t, K)$ with some $\widetilde{G} \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ "close to" F. As a typical example, we can consider $M := \vartheta_F(t, K) \setminus \mathbb{B}_\varepsilon(x_0)^\circ \in \mathcal{K}(\mathbb{R}^N)$ with an interior point x_0 of $\vartheta_F(t, K)$ and sufficiently small $\varepsilon > 0$.

For this reason, we prefer a different approach to the interior of Graph $T^C_{\mathcal{V}}(\cdot)$, but seize the terminology of hypertangents:

Definition B.3 *Consider the set* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence. For a nonempty subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *and any element* $K \in \mathcal{V}$,

$$H_{\mathcal{V}}(K) := \Big\{ F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \;\Big|\; \exists\ \varepsilon > 0,\ \textit{neighborhood } U \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N) \textit{ of } F \quad \forall\ G \in U : \limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_G(h, M),\ \mathcal{V}\big) = 0 \textit{ uniformly in } M \in \mathcal{V} \cap \mathbb{B}_\varepsilon(K) \Big\}$$

is called hypertangent transition set *of* $\mathcal{V}$ *at* K *(in the metric space* $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$*).*

Lemma B.4 *Let* $K \in \mathcal{K}(\mathbb{R}^N)$ *be any element of the nonempty closed set* $\mathcal{V} \subset \big(\mathcal{K}(\mathbb{R}^N), d\!\!l\big)$. *Then, a set-valued map* $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *belongs to the hypertangent transition set* $H_{\mathcal{V}}(K)$ *if and only if there exist a radius* $\varepsilon > 0$ *and a neighborhood* $U \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *of* F *such that for each map* $G \in U$, *a modulus of continuity* $\omega : [0, 1] \longrightarrow [0, \infty[$ *(i.e.* $\lim_{\delta \to 0} \omega(\delta) = 0$*) satisfies*

$$\tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_G(h, M),\ \mathcal{V}\big) \;\le\; \omega(h) \qquad \textit{for all } h \in\,]0, 1],\ M \in \mathbb{B}_\varepsilon(K) \cap \mathcal{V} \subset \mathcal{K}(\mathbb{R}^N).$$

The proof follows the same track as for Lemma A.4 about the circatangent transition set. Furthermore, in combination with Lemma A.4, we conclude directly

Lemma B.5 *For every nonempty closed subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *and element* $K \in \mathcal{V}$, *the hypertangent transition set* $H_{\mathcal{V}}(K)$ *is contained in the interior of the circatangent transition set* $T^C_{\mathcal{V}}(K)$. □

For the same reason, we obtain an even more general result:

Lemma B.6 *Consider the set* $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *with the topology of locally uniform convergence. For every nonempty closed subset* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$, *the graph of hypertangent transition sets*

$$\mathcal{V} \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N), \quad K \mapsto H_{\mathcal{V}}(K)$$

is contained in the interior of the graph of $\mathcal{V} \rightsquigarrow \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, $K \mapsto T^C_{\mathcal{V}}(K)$. □

In fact, also the opposite inclusion holds and thus, we have a complete characterization of the interior of Graph $T^C_{\mathcal{V}}(\cdot)$ in $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$:

Proposition B.7 *Let* $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ *be nonempty and closed with respect to* $d\!\!l$. *Then,* Graph $H_{\mathcal{V}}(\cdot) \subset \mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ *is equal to the interior of* Graph $T^C_{\mathcal{V}}(\cdot)$ *in* $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.

Proof. Due to Lemma B.6, we just have to verify $F \in H_{\mathcal{V}}(K)$ for every set-valued map $F \in T^C_{\mathcal{V}}(K)$ such that (K, F) belongs to the interior of Graph $T^C_{\mathcal{V}}(\cdot)$ in $\mathcal{V} \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$.
There exist a radius $\rho > 0$ and a neighborhood $U \subset \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ of F (with respect to locally uniform convergence) such that all tuples $(M, G) \in \big(\mathcal{V} \cap \mathbb{B}_\rho(K)\big) \times U \subset \mathcal{K}(\mathbb{R}^N) \times \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ belong to Graph $T^C_{\mathcal{V}}(\cdot)$. For arbitrary $G \in U$, we now show indirectly

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot \mathrm{dist}\big(\vartheta_G(h, M),\ \mathcal{V}\big) \ = \ 0 \qquad \text{uniformly in } M \in \mathcal{V} \cap \mathbb{B}_\rho(K).$$

Otherwise there exist $\delta > 0$ and sequences $(h_n)_{n \in \mathbb{N}}$, $(M_n)_{n \in \mathbb{N}}$ in $]0, 1[$ and $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ respectively satisfying $\mathrm{dist}\big(\vartheta_G(h_n, M_n),\ \mathcal{V}\big) \ \geq \ \delta \cdot h_n$, $\quad 0 < h_n < \frac{1}{n}$, $\quad d\!\!l(M_n, K) \leq \rho$ for all $n \in \mathbb{N}$.
In the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, all bounded closed balls are compact. So there is a subsequence $(M_{n_j})_{j \in \mathbb{N}}$ converging to some $M \in \mathcal{V} \cap \mathbb{B}_\rho(K)$. Due to the choice of ρ and U, we obtain $G \in T^C_{\mathcal{V}}(M)$ in particular. This contradicts, however,

$$\begin{cases} \liminf\limits_{j \to \infty} \ \frac{1}{h_{n_j}} \cdot \mathrm{dist}\big(\vartheta_G(h_{n_j}, M_{n_j}),\ \mathcal{V}\big) & \geq \ \delta \ > \ 0 \\ \lim\limits_{j \to \infty} \ d\!\!l\big(M_{n_j},\ M\big) & = \ 0 \end{cases}$$

completing the indirect proof. □

Circatangent transition set $T^C_{\mathcal{V}}(K)$ and hypertangent transition set $H_{\mathcal{V}}(K)$ differ from each other in an essential feature: The condition on a map $F \in T^C_{\mathcal{V}}(K)$ depends on $\mathcal{V} \subset \mathcal{K}(\mathbb{R}^N)$ close to K, of course, but only on reachable sets of the set-valued map F. So in particular, it does not have any influence on this condition if we replace such a map $F \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ by its pointwise convex hull $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N,\ x \mapsto \overline{\mathrm{co}}\, F(x)$ – due to the Relaxation Theorem of Filippov-Ważiewski [2, § 2.4, Theorem 2]. The condition on $F \in H_{\mathcal{V}}(K)$, however, takes all set-valued maps $G \in \mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$ in a neighborhood of F into account. Considering the topology of locally uniform convergence in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$, the values of these neighboring set-valued maps G do not have to be convex even if F belongs to $\mathrm{LIP}_{\overline{co}}(\mathbb{R}^N, \mathbb{R}^N)$.

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