

# Iterative $q$ -Difference Galois Theory

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# 1 Introduction

Initially, the Galois theory of  $q$ -difference equations was built for  $q$  unequal to a root of unity (see for instance [25]). This choice was made in order to avoid the increase of the field of constants to a transcendental field. However, P.A. Hendricks studied this problem in his PhD work under the supervision of M. van der Put (see [12]). In Chapter 6 he gave a notion of Galois groups for  $q$ -difference equations over  $\mathbb{C}(z)$  with  $q^m = 1$ . His idea was to compare the category  $Diff_{\mathbb{C}(z)}$  of  $q$ -difference modules over  $\mathbb{C}(z)$  with the category  $FMod_Z$  of modules over the ring  $\mathbb{C}(z^m)[t, t^{-1}]$ . He thus obtained an equivalence of categories and a fiber functor from  $Diff_{\mathbb{C}(z)}$  with values in the category  $Vect_{\mathbb{C}(z^m)}$  of vector spaces of finite dimension over  $\mathbb{C}(z^m)$ . However, in his case there is no unique Picard-Vessiot ring of a  $q$ -difference equation. This construction is also not totally satisfying because we do not want to have such transcendental base fields for Galois groups.

In the same matter, the question of the constant field for differential modules in positive characteristic has given rise to the construction of a new differential Galois theory in positive characteristic. The first work in this direction was made by H. Hasse and F.K. Schmidt [11], but it was only in 2000 when B.H. Matzat and M. van der Put set up a modern and systematic approach to this theory (see [17] and [16]). The main idea is to consider not only one derivation but a whole family of derivations, called *higher derivations* or *iterative derivations*. By defining the constants as the elements annihilated by the whole family of derivations, they succeeded in getting a good constant field, for instance  $\overline{\mathbb{F}_p}$  instead of  $\overline{\mathbb{F}_p}(z^p)$ . So they were able to give a complete description of the Picard-Vessiot theory of differential equations in positive characteristic and relate it to a Tannakian approach.

For  $q$ -difference theory, the problem is not the characteristic but the roots of unity. Inspired by the work of B.H. Matzat and M. van der Put, we consider in this paper a family of *iterative difference operators* instead of considering just one difference operator, and in this way we stop the increase of the constant field and succeed in setting up a Picard-Vessiot theory for  $q$ -difference equations where  $q$  is a root of unity. The theory we obtain is quite the exact translation of the iterative differential Galois theory developed by B.H. Matzat and M. van der Put to the  $q$ -difference world. This analogy between iterative differential Galois theory and iterative difference Galois theory could perhaps be explained in a more theoretical way, as it is done in the paper of Y. André [1] for classical theories. But for the moment, we will only focus on the construction of iterative  $q$ -difference Galois theory.

The interests of building such a theory are multiple. The first one is to fill in the gap in the classical  $q$ -difference Galois theory for roots of unity. The theory of iterative  $q$ -difference operators developed in this paper encompasses and extends the work of Singer and van der Put ([25]). For instance, this could provide a good functor of  $p$ -adic confluence from the world of  $q$ -difference to the world of differential equations, following the work of A. Pulita ([18]). However, instead of considering a  $(\sigma_q, \partial)$ -module at the roots of unity, (as

it is done in [18]), it will be more suitable to consider an iterative  $q$ -difference module. In a similar way, it will be very enlightening to build a confluence functor in characteristic  $p$  from iterative  $q$ -difference modules to iterative differential modules. Another goal of this theory will be to obtain an iterative  $q$ -difference version of the Grothendieck Conjecture following the work of L. Di Vizio [8] and the work of P.A. Hendricks [12]. In other words, we want to prove that our iterative  $q$ -difference groups are generated by the curvatures, which are linear operators in characteristic zero in the case of  $q$  a root of unity.

For the whole paper, we fix an algebraically closed field  $C$  and  $q \in C$  with  $q \neq 1$ . Let  $F = C(t)$  denote the field of rational functions over  $C$  and  $\sigma_q$  the automorphism of  $F$  which associates to a function  $f(z)$  the function  $f(qz)$ .

In the second section, we introduce the arithmetic basis of iterative  $q$ -difference algebra. In this section we work in all generality, i.e., we do not make any assumptions whether  $q$  is a root of unity or not. With this choice we want to emphasize the fact that we just generalize the Galois theory of  $q$ -difference of M.F Singer and M. van der Put ([25]). From the third section until the end of the paper, we will restrict ourselves to the case of  $q$  a primitive root of unity, where the most peculiar phenomena appear. In Section 3 we define the category of iterative  $q$ -difference modules and their relation with some specific category of projective systems. As in [16], the equivalence of categories yields a family of  $q$ -difference equations, related to the fact that an iterative  $q$ -difference operator is a family of maps. Such a family of equations can be regarded in two different ways, a general and a relative one using the projective system. Both formulations are used in later sections.

In Section 4, we build a Picard-Vessiot theory for iterative  $q$ -difference equations by using the classical theory as formulated for instance in [25]. To be a little more concrete, at the end of the section, we give a method to realize groups in dimension one as iterative  $q$ -difference Galois groups.

In Section 5, we adopt Kolchin's way of thinking and show how an iterative  $q$ -difference Galois group is formed by the  $C$ -points of an affine group-scheme. We also obtain the analogue of Kolchin's theorem for our theory and the usual Galois correspondence.

As a conclusion to this paper, we state an analogue of the Grothendieck-Katz conjecture for iterative  $q$ -difference Galois groups as in the work of L. Di Vizio.

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## 2 Iterative $q$ -difference rings

In considering an element  $q$  of a field  $C$  which may be a primitive root of unity and trying to construct a  $q$ -difference Galois theory, we have to deal with the problem that the field of constants of the usual  $q$ -difference operator extends to a transcendental field. To avoid this increase of the constants, we have to consider a more arithmetic approach, such as the one introduced by H. Hasse and F.K. Schmidt [11] for differential equations. Until the end

of this article, we let  $F = C(t)$  denote the field of rational functions over  $C$  and  $\sigma_q$  the  $q$ -difference operator of  $F$  defined as follows :  $\sigma_q(f(z)) := f(qz)$ .

## 2.1 $q$ -Arithmetic properties

In this paragraph, we just recall the most usual  $q$ -arithmetical objects.

**Definition 2.1** Let  $k \in \mathbb{Z}$ . Put  $[k]_q := \frac{q^k - 1}{q - 1}$ .

1. Let  $[k]_q!$  denote the element of  $C$  defined by  $[k]_q[k-1]_q \dots [1]_q$ . We will say that  $[k]_q!$  is the  $q$ -factorial of  $k$ .
2. Let  $\binom{r}{k}_q$  denote the element of  $C$  defined by  $\frac{[r]_q!}{[k]_q![(r-k)]_q!}$ . We will say that  $\binom{r}{k}_q$  is the  $q$ -binomial coefficient of  $r$  over  $k$ .
3.  $(1-t)_m := (1-t)(1-qt) \dots (1-q^{m-1}t)$ .

**Proposition 2.2** 1.  $\binom{r}{0}_q = \binom{r}{r}_q = 1$ .

2.  $\binom{0}{k}_q = 0$  if  $k \neq 0$  and  $\binom{0}{0}_q = 1$ .

3. Assume that  $q$  is a primitive  $n$ -th root of unity. Then for two integers  $a > b$ ,

$$\binom{an}{bn}_q = \binom{a}{b}. \quad (1)$$

4.  $\sum_{i+j=k, i \leq s, j \leq r} \binom{r}{j}_q \binom{s}{i}_q q^{i(r-j)} = \binom{r+s}{k}_q$  for all  $(k, r, s) \in \mathbb{N}^3$  with  $r + s \geq k$ .

### Proof of part 4

Let  $m \in \mathbb{N}$ . The function  $(1-t)_m$  of  $C(t)$  defined in 2.1 part 3 gives

$$(1-t)_m = \sum_{j=0}^m (-1)^j \binom{m}{j}_q q^{j(j-1)/2} t^j. \quad (2)$$

Because  $q^n = 1$  and  $n$  is the order of  $q$ , we have  $(1-t)_{an} = (1-t)_n^a$ . Using Equation (2), we obtain

$$\sum_{j=0}^{an} (-1)^j \binom{an}{j}_q q^{j(j-1)/2} t^j = \sum_{j=0}^a \binom{a}{j} (-1)^{nj} q^{n(n-1)j/2} t^{nj}.$$

By comparing the terms in  $t^{bn}$ , we obtain  $\binom{an}{bn}_q q^{\frac{bn(bn-1)}{2}} = \binom{a}{b} q^{b \frac{n(n-1)}{2}}$ .

### Proof of part 5

Let  $(k, r, s) \in \mathbb{N}^3$  with  $r + s \geq k$ . We have

$$(1-t)_{r+s} = (1-t)_r (1-q^r t)_s. \quad (3)$$

By comparing the terms in  $t^k$ , we obtain

$$(-1)^k q^{k(k-1)/2} \binom{r+s}{k}_q = \sum_{i+j=k, i \leq s, j \leq r} (-1)^{i+j} q^{k(k-1)/2} \binom{r}{j}_q \binom{s}{i}_q q^{i(r-j)}.$$

**Remark 2.3** If  $C$  is of characteristic  $p > 0$ , then for  $p^j > i$  we get from equation (1)  $\binom{np^j}{ni}_q = 0$ .

## 2.2 Iterative $q$ -difference ring

In this paragraph, we define the formal properties of the iterative  $q$ -difference operator. In the world of  $q$ -difference the analogue of the derivation  $\frac{d}{dt}$  is the operator  $\delta_q := \frac{\sigma_q - id}{(q-1)t}$  (see for instance [20]). Heuristically speaking, when  $q$  goes to 1,  $\delta_q$  goes to the usual derivation  $\frac{d}{dt}$ . Thus the main idea of our constructions is to deform the iterative derivations into iterative difference operators by replacing  $\partial$  by  $\delta_q$  and all the arithmetical factors occurring in their Definition 1.1 in [16] by  $q$ -analogues. The only change appears at the part 4 of Definition 2.4, where a twist by  $\sigma_q$  occurs.

**Definition 2.4** Let  $R$  be a  $q$ -difference extension of  $C(t)$  in the sense of [25], let  $\sigma_q$  be the  $q$ -difference operator of  $R$  and let  $\delta_R^* := (\delta_R^{(k)})_{k \in \mathbb{N}}$  be a collection of maps from  $R$  to  $R$ . The family  $\delta_R^*$  is called an **iterative  $q$ -difference operator** on  $R$ , if for all  $a, b \in R$  and all  $i, j, k \in \mathbb{N}$ , the following properties are satisfied

1.  $\delta_R^{(0)} = id$ ,
2.  $\delta_R^{(1)} = \frac{\sigma_q - id}{(q-1)t}$ ,
3.  $\delta_R^{(k)}(x + y) = \delta_R^{(k)}(x) + \delta_R^{(k)}(y)$ ,
4.  $\delta_R^{(k)}(ab) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a))\delta_R^{(i)}(b)$ ,
5.  $\delta_R^{(i)} \circ \delta_R^{(j)} = \binom{i+j}{i}_q \delta_R^{(i+j)}$ .

The set of iterative  $q$ -difference operators is denoted by  $ID_q(R)$ . For  $\delta_R^* \in ID_q(R)$ , the tuple  $(R, \delta_R^*)$  is called an **iterative  $q$ -difference ring** ( $ID_q$ -ring). We say that an element  $c$  of  $R$  is a constant if  $\delta_R^{(k)}(c) = 0$  for all  $k \in \mathbb{N}^*$ . We denote by  $C(R)$  the ring of constants of  $R$ .

**Remark 2.5** 1. If  $R$  is a ring, then  $C(R)$  is a ring. If  $R$  is a field, then  $C(R)$  is a field.

2. We have for all  $j, i \in \mathbb{N}$ ,

$$\sigma_q^j \delta_R^{(i)} = \frac{1}{q^{ji}} \delta_R^{(i)} \sigma_q^j. \quad (4)$$

**Proof**

In order to prove Equation (4), it is sufficient to prove it for  $j = 1$ , the general case obviously follows from this case.

For all  $k > 0$ , we have

$$\delta_R^{(k)}\left(t\frac{1}{t}\right) = 0 = \delta_R^{(k)}(t^{-1})t + \sigma_q(\delta_R^{(k-1)}(t^{-1})). \quad (5)$$

By part 5 of Definition 2.4, we get  $\delta_R^{(1)} \circ \delta_R^{(i)} = \delta_R^{(i)} \circ \delta_R^{(1)}$  for all  $i \in \mathbb{N}$ . Using part 2 and 4, we obtain that

$$\frac{\sigma_q - id}{t} \circ \delta_R^{(i)}(x) = \delta_R^{(i)} \circ \left(\frac{\sigma_q - id}{t}\right)(x) = \sum_{k=1}^i \sigma_q^k(\delta_R^{(i-k)}(\sigma_q - id)(x))\delta_R^{(k)}(t^{-1}) + \delta_R^{(i)}((\sigma_q - id)(x))t^{-1}$$

for all  $x \in R$  and  $i \in \mathbb{N}$ . By Equation (5), we get

$$\frac{\sigma_q}{t} \circ \delta_R^{(i)}(x) = \frac{-1}{t} \sigma_q \left[ \sum_{k=0}^{i-1} \sigma_q^k(\delta_R^{(i-1-k)}(\sigma_q - id)(x))\delta_R^{(k)}(t^{-1}) \right] + \frac{\delta_R^{(i)} \circ \sigma_q(x)}{t},$$

i.e.,

$$\frac{\sigma_q}{t} \circ \delta_R^{(i)}(x) = \frac{-1}{t} \sigma_q(\delta_R^{(i-1)} \circ \delta_R^{(1)}(x)) + \frac{\delta_R^{(i)} \circ \sigma_q(x)}{t}$$

that is,

$$\frac{\sigma_q}{t} \circ \delta_R^{(i)}(x) = -\frac{q-1}{t} \sigma_q \circ \left(\frac{q^i - 1}{q-1} \delta_R^{(i)}\right)(x) + \frac{\delta_R^{(i)} \circ \sigma_q(x)}{t}.$$

This last equation gives

$$\sigma_q \delta_R^{(i)}(x) = \frac{1}{q^i} \delta_R^{(i)} \sigma_q(x)$$

which concludes the proof.

**Remark 2.6 (Classical case)** *If  $q$  is not a root of unity then  $\delta_R^{(k)} = \frac{(\delta_R^{(1)})^k}{[k]_q!}$  and the iterative  $q$ -difference rings that we consider are the  $q$ -difference rings studied by M. van der Put and M.F Singer in [25].*

**Main example : The field of rational functions over  $C$**

**Definition 2.7** *Let  $k \in \mathbb{N}$ . Let  $\delta_q^{(k)}$  denote the additive map from  $C[t]$  to  $C[t]$  defined by  $\delta_q^{(k)}(\lambda t^r) := \lambda \binom{r}{k}_q t^{r-k}$ , for all  $r \in \mathbb{N}$ , and  $\lambda \in C$ . Using the formula  $\delta^{(k)}(ab) = \sum_{i+j=k} \sigma_q^i(\delta_q^{(j)}(a))\delta_q^{(i)}(b)$ , we extend  $\delta_q^{(k)}$  to  $F = C(t)$ .*

**Proposition 2.8** *The collection  $(\delta_q^{(k)})_{k \in \mathbb{N}}$  of maps from  $F$  to  $F$ , defined previously, satisfy*

1.  $\delta_q^{(0)} = id$ ,
2.  $\delta_q^{(1)} = \frac{\sigma_q - id}{(q-1)t}$ ,
3. for all  $k \in \mathbb{N}$ , the map  $\delta_q^{(k)}$  is additive,
4.  $\delta_q^{(i)} \circ \delta_q^{(j)} = \binom{i+j}{i}_q \delta_q^{(i+j)}$ .

**Proof**

By construction of  $(\delta_q^{(k)})_{k \in \mathbb{N}}$ , it is sufficient to prove that all the formulas hold upon evaluation on  $t^r$  with  $r \in \mathbb{N}$ .

1. Because  $\binom{k}{0}_q = 1$ , it is obvious that  $\delta_q^{(0)} = id$ .
2. For all  $r \in \mathbb{N}$ , we have  $\delta_q^{(1)}(t^r) = \binom{r}{1}_q t^{r-1} = \frac{q^r t^r - t^r}{(q-1)t} = \frac{\sigma_q - id}{(q-1)t}(t^r)$ .
3. Let  $r \in \mathbb{N}$ . We have

$$\delta_q^{(i)} \circ \delta_q^{(j)}(t^r) = \binom{r-j}{i}_q \binom{r}{j}_q t^r$$

and

$$\binom{r-j}{i}_q \binom{r}{j}_q = \binom{i+j}{i}_q \binom{r}{i+j}_q,$$

which gives

$$\delta_q^{(i)} \circ \delta_q^{(j)}(t^r) = \binom{i+j}{i}_q \delta_q^{(i+j)}(t^r).$$

**Proposition 2.9** *The field  $F = C(t)$  endowed with the collection of maps  $(\delta_q^{(k)})_{k \in \mathbb{N}}$  as in Definition 2.7 is an iterative  $q$ -difference field with  $\delta_q^{(n)}(t^n) = 1$  for all  $n \in \mathbb{N}$  and thus  $C(F) = C$ .*

**Tensor product of  $ID_q$ -rings**

**Lemma 2.10** *Let  $(R_1, \delta_{R_1}^*)$  and  $(R_2, \delta_{R_2}^*)$  be two iterative  $q$ -difference rings. We have*

$$\sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \sum_{i+j=k} \delta_{R_1}^{(j)}(a) \otimes \sigma_q^j(\delta_{R_2}^{(i)}(b)) \tag{6}$$

for all  $k \in \mathbb{N}$ ,  $(a, b) \in R_1 \times R_2$ .

**Proof**

The formula (6) is obviously true for  $k = 1$ , using the definition of  $\delta^{(1)}$ . If (6) holds for  $k$  and  $l$  in  $\mathbb{N}$ , we have

$$\binom{k+l}{k} \sum_{q, i+j=k+l} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \left( \sum_{r+s=l} \sigma_q^r(\delta_{R_1}^{(s)}) \otimes \delta_{R_2}^{(r)} \right) \left( \sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) \right)$$

that is

$$\left( \sum_{r+s=l} \sigma_q^r(\delta_{R_1}^{(s)}) \otimes \delta_{R_2}^{(r)} \right) \left( \sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) \right) = \left( \sum_{r+s=l} \delta_{R_1}^{(r)} \otimes \sigma_q^r(\delta_{R_2}^{(s)}) \right) \left( \sum_{i+j=k} \delta_{R_1}^{(j)}(a) \otimes \sigma_q^j(\delta_{R_2}^{(i)}(b)) \right)$$

and thus

$$\binom{k+l}{k} \sum_{q, i+j=k+l} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \binom{k+l}{k} \sum_{q, i+j=k+l} \delta_{R_1}^{(j)}(a) \otimes \sigma_q^j(\delta_{R_2}^{(i)}(b)).$$

Then, if  $\binom{k+l}{k}_q \neq 0$ , the formula (6) holds for  $k+l$ . If  $q$  is not a root of unity, we can conclude by induction.

Assume now that  $q^n = 1$ . It remains to show that Formula (6) holds for  $k \in n\mathbb{N}$ . We will first prove it for  $k = n$ .

Because  $\sum_{i+j=n} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \delta_{R_1}^{(n)}(a) \otimes b + a \otimes \delta_{R_2}^{(n)}(b) + \sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b)$ , the proof for  $k = n$  will be complete if we show that

$$\sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \sum_{i=1}^{n-1} \delta_{R_1}^{(i)}(a) \otimes \sigma_q^i(\delta_{R_2}^{(n-i)}(b)). \quad (7)$$

We have  $\delta^{(k)} = \frac{(\delta^{(1)})^k}{[k]_q!}$  and

$$(\delta^{(1)})^k = \frac{(-1)^k}{((q-1)t)^k} \sum_{j=0}^k (-1)^j \binom{k}{j}_{q^{-1}} q^{-\frac{j(j-1)}{2}} \sigma_q^j = \frac{1}{((q-1)t)^k} \sum_{j=0}^k a_{j,k} \sigma_q^j$$

for  $0 < k < n$  (see [8], Lemma 1.1.10). Then,

$$\sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \frac{1}{((q-1)t)^n} \sum_{l=1}^n \sum_{k=0}^l \sigma_q^l(a) \otimes \sigma_q^k(b) \left( \sum_{i=k, i \neq 0}^{i=l, i \neq n} \frac{a_{l-i, n-i} a_{k,i} q^{-i(n-i)}}{[n-i]_q! [i]_q!} \right). \quad (8)$$

If  $l \neq n$ ,  $k \neq 0$  and  $l \neq k$ , we have

$$\sum_{i=k, i \neq 0}^{i=l, i \neq n} \frac{a_{l-i, n-i} a_{k,i} q^{-i(n-i)}}{[n-i]_q! [i]_q!} = \frac{(-1)^{l+n} q^{-\frac{n(n-1)}{2}}}{[n-l]_{q^{-1}}! [k]_q! [l-k]_q!} \sum_{i=0}^{l-k} (-1)^i \binom{l-k}{i}_q q^{\frac{i(i-1)}{2}} = 0$$

(expand  $(1 - 1)_{l-k}$ ). If  $l = n$ , then

$$\sum_{i=k, i \neq 0}^{i=n, i \neq n} \frac{a_{n-i, n-i} a_{k, i} q^{-i(n-i)}}{[n-i]_q! [i]_q!} = \frac{(-1)^{n+k+1} q^{-\frac{n(n-1)}{2}}}{[k]_q! [n-k]_{q^{-1}}} = \sum_{i=0, i \neq 0}^{i=k, i \neq n} \frac{a_{k-i, n-i} a_{0, i} q^{-i(n-i)}}{[n-i]_q! [i]_q!}$$

(expand  $(1 - 1)_k$  and  $(1 - 1)_{n-k}$ ). Because  $\sigma_q^n = id$ , it follows that the equation (8) is symmetric in  $a$  and  $b$ . Thus the formula (7) holds and the equation (6) is true for  $k = n$ . For  $k = 2n$ , we have

$$\begin{aligned} \sum_{i+j=n} \sigma_q^i(\delta_{R_1}^{(j)}(a)) \otimes \delta_{R_2}^{(i)}(b) &= \delta_{R_1}^{(2n)}(a) \otimes b + a \otimes \delta_{R_2}^{(2n)}(b) + \\ &\sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(2n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b) + \delta_{R_1}^{(n)}(a) \otimes \delta_{R_2}^{(n)}(b) + \sum_{i=n+1}^{2n-1} \sigma_q^i(\delta_{R_1}^{(2n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b). \end{aligned}$$

Because  $\delta^{(2n-i)} = \delta^{(n-i)} \circ \delta^{(n)}$  for all  $i = 1, \dots, n-1$ , we obtain by (7)

$$\begin{aligned} \sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(2n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b) &= \sum_{i=1}^{n-1} \sigma_q^i(\delta_{R_1}^{(n-i)}(\delta_{R_1}^{(n)}(a))) \otimes \delta_{R_2}^{(i)}(b) = \sum_{i=1}^{n-1} \delta_{R_1}^{(n+i)}(a) \otimes \sigma_q^i(\delta_{R_2}^{(n-i)}(b)) \\ &= \sum_{i=n+1}^{2n-1} \delta_{R_1}^{(i)}(a) \otimes \sigma_q^i(\delta_{R_2}^{(2n-i)}(b)). \end{aligned}$$

We also have

$$\sum_{i=n+1}^{2n-1} \sigma_q^i(\delta_{R_1}^{(2n-i)}(a)) \otimes \delta_{R_2}^{(i)}(b) = \sum_{i=1}^{n-1} \delta_{R_1}^{(i)}(a) \otimes \sigma_q^i(\delta_{R_2}^{(2n-i)}(b)).$$

This concludes the proof for  $k = 2n$ . The same arguments gives the other cases.

**Proposition 2.11 (proposition, definition)** *Let  $(R_1, \delta_{R_1}^*)$  and  $(R_2, \delta_{R_2}^*)$  be two iterative  $q$ -difference rings. We define a collection of maps  $(\delta_{R_1 \otimes_F R_2}^{(k)})_{k \in \mathbb{N}}$  from  $R_1 \otimes_F R_2$  to  $R_1 \otimes_F R_2$  as follows :*

$$\delta_{R_1 \otimes_F R_2}^{(k)}(r_1 \otimes r_2) := \sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(r_1)) \otimes \delta_{R_2}^{(i)}(r_2) \quad \text{for all } k \in \mathbb{N}, r_1 \in R_1 \text{ and } r_2 \in R_2.$$

*Then  $(R_1 \otimes_F R_2, \delta_{R_1 \otimes_F R_2}^*)$  is an iterative  $q$ -difference ring.*

**Proof**

It is obvious that the family  $(\delta_{R_1 \otimes_F R_2}^{(k)})_{k \in \mathbb{N}}$  satisfies the three first parts of Definition 2.4. By Lemma 2.10 we have

$$\delta_{R_1 \otimes_F R_2}^{(k)}(r_1 \otimes r_2) = \sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(r_1)) \otimes \delta_{R_2}^{(i)}(r_2) = \sum_{i+j=k} \delta_{R_1}^{(j)}(r_1) \otimes \sigma_q^j(\delta_{R_2}^{(i)}(r_2))$$

for all  $k \in \mathbb{N}$ . Let  $(a, c) \in R_1^2$  and  $(b, d) \in R_2^2$ . We have

$$\delta_{R_1 \otimes R_2}^{(k)}((a \otimes b)(c \otimes d)) = \sum_{i+j=k} \sigma_q^i(\delta_{R_1}^{(j)}(ac)) \otimes \delta_{R_2}^{(j)}(bd),$$

$$\delta_{R_1 \otimes R_2}^{(k)}((a \otimes b)(c \otimes d)) = \sum_{i_1+i_2+j_1+j_2=k} \sigma_q^{i_1+i_2+j_1}(\delta_{R_1}^{(j_2)}(a)) \sigma_q^{i_1+i_2}(\delta_{R_1}^{(j_1)}(c)) \otimes \sigma_q^{i_1}(\delta_{R_2}^{(i_2)}(b)) \delta_{R_2}^{(i_2)}(d),$$

and thus,

$$\delta_{R_1 \otimes R_2}^{(k)}((a \otimes b)(c \otimes d)) = \sum_{i_1+j_2+i=k} \sigma_q^{i_1+i}(\delta_{R_1}^{(j_2)}(a)) \otimes \delta_{R_2}^{(i_1)}(d) (\sigma_q^{i_1}(\delta_{R_1 \otimes R_2}^{(i)}(c \otimes b))).$$

This gives

$$\delta_{R_1 \otimes R_2}^{(k)}((a \otimes b)(c \otimes d)) = \sum_{i_1+i_2+j_1+j_2=k} \sigma_q^{i_1+i_2+j_1}(\delta_{R_1}^{(j_2)}(a)) \sigma_q^{i_1}(\delta_{R_1}^{(i_2)}(c)) \otimes \sigma_q^{i_1+i_2}(\delta_{R_2}^{(j_1)}(b)) \delta_{R_2}^{(i_1)}(d),$$

and thus

$$\delta_{R_1 \otimes R_2}^{(k)}((a \otimes b)(c \otimes d)) = \sum_{i+j=k} \sigma_q^i(\delta_{R_1 \otimes R_2}^{(j)}(a \otimes b)) \delta_{R_1 \otimes R_2}^{(i)}(c \otimes d).$$

This is part 4 of Definition 2.4.

We now prove part 5. Let  $(k, l) \in \mathbb{N}^2$  and  $(a, b) \in R_1 \times R_2$ . We have

$$\delta_{R_1 \otimes R_2}^{(k)} \circ \delta_{R_1 \otimes R_2}^{(l)}(a \otimes b) = \sum_{i+j=l, i_1+j_1=k} q^{ij_1} \binom{j_1+j}{j_1}_q \binom{i_1+i}{i}_q \sigma_q^{i_1+i}(\delta_{R_1}^{(j_1+j)}(a)) \otimes \delta_{R_2}^{(i_1+i)}(b),$$

that is

$$\delta_{R_1 \otimes R_2}^{(k)} \circ \delta_{R_1 \otimes R_2}^{(l)}(a \otimes b) = \sum_{r+s=k+l} \sigma_q^r(\delta_{R_1}^{(s)}(a)) \otimes \delta_{R_2}^{(r)}(b) \left( \sum_{i+j=k, i \leq s, j \leq r} \binom{r}{j}_q \binom{s}{i}_q q^{i(r-j)} \right).$$

Using part 5 of Proposition 2.2, we obtain

$$\delta_{R_1 \otimes R_2}^{(k)} \circ \delta_{R_1 \otimes R_2}^{(l)}(a \otimes b) = \binom{k+l}{k}_q \sum_{r+s=k+l} \sigma_q^r(\delta_{R_1}^{(s)}(a)) \otimes \delta_{R_2}^{(r)}(b),$$

that is

$$\delta_{R_1 \otimes R_2}^{(k)} \circ \delta_{R_1 \otimes R_2}^{(l)}(a \otimes b) = \binom{k+l}{k}_q \delta_{R_1 \otimes R_2}^{(k+l)}(a \otimes b).$$

## 2.3 Twisted ring of formal power series

This paragraph is devoted to the relations between  $ID_q$ -rings and rings of formal power series. By encoding all properties of an iterative  $q$ -difference operator into twisted formal power series, Property 2.15 provides us with a very powerful tool for the proofs to come. This kind of twisted ring appears already in the work of Yves André (see [1]).

**Definition 2.12** *Let  $(R, \delta_R^*)$  be an iterative  $q$ -difference ring. The twisted ring  $R^{\sigma_q}[[T]]$  of formal series with coefficients in  $R$  is defined as follows : the additive structure of  $R^{\sigma_q}[[T]]$  is the same as the one of  $R[[T]]$ , the multiplicative structure is given by*

$$\lambda T^r * \mu T^k := \sigma_q^r(\mu) \lambda T^{r+k}$$

and extended by distributivity to  $R[[T]]$ .

We will denote by " ." the usual multiplication law on  $R[[T]]$ .

**Lemma 2.13** *The twisted ring  $(R^{\sigma_q}[[T]], +, *)$  as in Definition 2.12 is a non commutative ring with unity. We will denote by " ." the usual multiplication law on  $R[[T]]$ .*

### Proof

We have :

$$\lambda T^r * 1 = \lambda T^r * T^0 = \sigma_q^r(1) \lambda T^{r+0} = \lambda T^r = 1 * \lambda T^r = \sigma_q^0(\lambda) T^r = \lambda T^r.$$

Thus 1 is a neutral element for the twisted multiplication  $*$ .

Let us prove then, that  $*$  is associative.

$$\nu T^s * (\lambda T^r * \mu T^k) = \nu T^s * (\sigma_q^r(\mu) \lambda T^{r+k}) = \sigma_q^{r+s}(\mu) \sigma_q^s(\lambda) \nu T^{r+s+k}$$

and

$$(\nu T^s * \lambda T^r) * \mu T^k = (\sigma_q^s(\lambda) \mu T^{r+s}) * \mu T^k = \sigma_q^{r+s}(\mu) \sigma_q^s(\lambda) \nu T^{r+s+k}$$

give

$$\nu T^s * (\lambda T^r * \mu T^k) = (\nu T^s * \lambda T^r) * \mu T^k.$$

The product  $*$  is therefore associative.

Now, we want to introduce an iterative  $q$ -difference operator on  $(R^{\sigma_q}[[T]], +, \cdot)$ , that is to say a collection of maps  $\delta_T^*$  which satisfies all the properties of Definition 2.4.

First we need an automorphism  $\sigma_q$  on  $(R^{\sigma_q}[[T]], +, \cdot)$  such that  $(R^{\sigma_q}[[T]], +, \cdot)$  is a  $q$ -difference ring extension of  $F$ . We put  $\sigma_q(aT^i) := \sigma_q(a)q^i T^i$  for all  $i \in \mathbb{N}$  and  $a \in R$ . By extending this definition  $R$ -linearly,  $R^{\sigma_q}[[T]]$  becomes a  $q$ -difference ring extension of  $F$ . We put  $\delta_T^{(k)}(T^r) := \binom{r}{k}_q T^{r-k}$  for all  $(k, r) \in \mathbb{N}^2$  and extend this definition by  $R$ -linearity. Obviously  $(\delta_T^{(k)})_{k \in \mathbb{N}}$  is an iterative  $q$ -difference operator over  $(R^{\sigma_q}[[T]], +, \cdot)$  (see Definition 2.4).

**Definition 2.14** For all  $a \in R$ ,

$$\mathbf{T}_a(T) := \sum_{k \in \mathbb{N}} \delta_R^{(k)}(a) T^k.$$

is called the *q-iterative Taylor series* of  $a$ .

Two iterative  $q$ -difference operators  $\delta_R^*$  and  $\tilde{\delta}_R^*$  are called *equivalent* (viz  $\delta_R^* \sim \tilde{\delta}_R^*$ ), if there exist numbers  $n$  and  $m$  in  $\mathbb{N}$  such that  $\mathbf{T}_a(T^n) = \tilde{\mathbf{T}}_a(T^m)$  for all  $a \in R$ , where  $\tilde{\mathbf{T}}_a$  denotes the  $q$ -iterative Taylor series associated to  $\tilde{\delta}_R^*$ .

**Proposition 2.15** Let  $R$  be a  $q$ -difference ring extension of  $F$  and let  $\delta_R^* = (\delta_R^{(k)})_{k \in \mathbb{N}}$  be a sequence of maps from  $R$  to  $R$ . Let  $\delta_T^*$  be the iterative  $q$ -difference operator of  $(R^{\sigma_q}[[T]], +, \cdot)$  defined previously, and let  $\mathbf{I}$  denote the map

$$\mathbf{I} : \quad R^{\sigma_q}[[T]] \longrightarrow R, \quad \sum_{k \in \mathbb{N}} a_k T^k \longmapsto a_0.$$

Then  $\delta_R^*$  is an iterative  $q$ -difference operator for  $R$  if and only if

1.  $\mathbf{T}$  is a ring homomorphism from  $R$  to  $(R^{\sigma_q}[[T]], +, *)$ , with  $\mathbf{I} \circ \mathbf{T} = id_R$ ,
2.  $\delta_T^{(k)} \circ \mathbf{T} = \mathbf{T} \circ \delta_R^{(k)}$  for all  $k \in \mathbb{N}$ .

### Proof

The fact that  $\mathbf{T}$  is additive is equivalent to statement 3 in Definition 2.4. The compatibility of  $\mathbf{T}$  with the multiplication law in  $R$  and the twisted law  $*$  in  $R^{\sigma_q}[[T]]$ , in the case where  $\delta_R^*$  is an iterative  $q$ -difference operator comes from the equations

$$\mathbf{T}_{ab}(T) := \sum_{k \in \mathbb{N}} \delta_R^{(k)}(ab) T^k = \sum_{k \in \mathbb{N}} \left( \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a)) \delta_R^{(i)}(b) \right) T^k = \mathbf{T}_a(T) * \mathbf{T}_b(T).$$

The second property is equivalent to the property 5 of the same definition.

## 2.4 Iterative $q$ -difference morphisms and iterative $q$ -difference ideals

**Definition 2.16** Let  $(R, \delta_R^*)$  and  $(S, \delta_S^*)$  be two iterative  $q$ -difference rings. We say that a ring morphism  $\phi$  from  $R$  to  $S$  is an *iterative  $q$ -difference morphism* if and only if  $\delta_S^{(k)} \circ \phi = \phi \circ \delta_R^{(k)}$  for all  $k \in \mathbb{N}$ .

The set of all iterative  $q$ -difference morphisms from  $R$  to  $S$  is denoted by  $Hom_{ID_q}(R, S)$ . An iterative  $q$ -difference ideal  $I \subset R$  ( $ID_q$ -ideal) is an ideal of  $R$  stable by  $\delta_R^{(k)}$  for all  $k \in \mathbb{N}$ .

**Lemma 2.17** Let  $I$  be an  $ID_q$ -ideal of an iterative  $q$ -difference ring  $R$ , that is to say that  $I$  is stable under the action of  $\delta_R^*$ . Then the radical of  $I$  is a  $ID_q$ -ideal.

**Proof**

Assume that  $q$  is a  $n$ -th primitive root of unity. From  $\delta_R^{(1)} = \frac{\sigma_q - id}{(q-1)t}$ , we get

$$\sigma_q(a) = (q-1)t(\delta_R^{(1)}(a) - a), \text{ for all } a \in I.$$

This shows that  $\sigma_q(a) \in I$  for all  $a \in I$ . Thus  $I$  is a  $\sigma_q$ -ideal. Conversely, if  $I$  is a  $\sigma_q$ -ideal then it is a  $\delta_R^{(1)}$ -ideal. Now, let us consider  $a \in \sqrt{I}$ . There exists  $m \in \mathbb{N}$  such that  $a^m \in I$ . But,  $\sigma_q(a^m) = (\sigma_q(a))^m \in I$ . Thus  $\sigma_q(a) \in \sqrt{I}$ .

Now, we will prove by induction that for all  $i < n$ ,  $\delta_R^{(i)}$  stabilizes  $\sqrt{I}$ .

It is true for  $i = 1$ . If it is true for  $k < n - 1$ , then  $k < n$  and we have :

$$\delta_R^{(1)} \circ \delta_R^{(k-1)} = \binom{k}{1}_q \delta_R^{(k)}$$

where  $\binom{k}{1}_q \neq 0$  because  $k < n$ . We have that  $\delta_R^{(1)}$  and  $\delta_R^{(k-1)}$  stabilize  $\sqrt{I}$  (by first step and by inductive assumption). Thus  $\delta_R^{(k)}$  stabilizes  $\sqrt{I}$ . This concludes the proof by induction. It remains to consider the case where  $k = n$ . Let  $a \in \sqrt{I}$  and  $m \in \mathbb{N}$  such that  $a^m \in I$ . We have:

$$\delta_R^{(nm)}(a^m) = \sum_{i_1 + \dots + i_m = nm} \sigma_q^{i_2 + \dots + i_m}(\delta_R^{(i_1)}(a)) \dots \sigma_q^{i_m}(\delta_R^{(i_{m-1})}(a)) \delta_R^{(i_m)}(a). \quad (9)$$

Because  $\sigma_q^n = id$ , we can rewrite the equation (9) as follows  $\delta_R^{(nm)}(a^m) = (\delta_R^{(n)}(a))^m + B$  with

$$B = \sum_{i_1 + \dots + i_m = nm}^* \sigma_q^{i_2 + \dots + i_m}(\delta_R^{(i_1)}(a)) \dots \sigma_q^{i_m}(\delta_R^{(i_{m-1})}(a)) \delta_R^{(i_m)}(a)$$

where  $\sum_{i_1 + \dots + i_m = nm}^*$  means that we only consider the  $(i_1, \dots, i_m)$  such that there exists at least one  $j$  with  $i_j < n$ . We have already proved by induction that  $\sqrt{I}$  is stable by  $\sigma_q$  and by  $\delta_R^{(i)}$  for  $i < n$ . This implies that  $B \in \sqrt{I}$ . Then  $(\delta_R^{(n)}(a))^m$  belongs to  $\sqrt{I}$  since  $\delta_R^{(nm)}(a^m) \in I$  because  $I$  itself is an  $ID_q$ -ideal. It follows  $\delta_R^{(n)}(a) \in \sqrt{I}$ .

So we have proved that  $\sqrt{I}$  is stable under  $\delta_R^{(k)}$  for all  $k \leq n$ . Using the formula  $\delta_R^{(i)} \circ \delta_R^{(k-i)} = \binom{k}{i}_q \delta_R^{(k)}$  and an inductive proof, we easily show that  $\sqrt{I}$  is stable under  $\delta_R^{(k)}$  for all  $k \notin n\mathbb{N}$ . The proof for  $k \in n\mathbb{N}$  is an analogue of the case  $k = n$ . Therefore  $\sqrt{I}$  is an  $ID_q$ -ideal.

**Remark 2.18 (Classical case)** For  $q$  not equal to a root of unity, the proof of the previous lemma is more elementary (see Lemma 1.7 in [25]). The reason is that if  $I$  is a  $\sigma_q$ -ideal then its radical is obviously a  $\sigma_q$ -ideal because  $\sigma_q$  is an automorphism.

### 2.4.1 Extending iterative $q$ -difference operator

**Proposition 2.19** *Let  $R$  be an integral domain, and let  $S \subset R$  be a multiplicatively closed subset of  $R$  **stable under the action of  $\sigma_q$**  such that  $0 \notin S$ . Let  $\delta_R^*$  be an iterative  $q$ -difference operator on  $R$ . Then there exists a unique iterative  $q$ -difference operator  $\delta_{S^{-1}R}^*$  extending  $\delta_R^*$  to  $S^{-1}R$ .*

**Proof**

Because  $\delta_R^*$  is an iterative  $q$ -difference operator, the application  $\mathbf{T} : R \mapsto (R^{\sigma_q}[[T]], +, *)$  defined by  $a \mapsto \mathbf{T}_a(T)$  is a ring homomorphism (see 2.15). Since  $R$  is commutative, we have

$$\mathbf{T}_{ab}(T) = \mathbf{T}_a(T) * \mathbf{T}_b(T) = \mathbf{T}_b(T) * \mathbf{T}_a(T) \text{ for all } a, b \in R.$$

This allows us to define the quotient  $\frac{\mathbf{T}_a(T)}{\mathbf{T}_b(T)}^*$  of  $\mathbf{T}_a(T)$  by  $\mathbf{T}_b(T)$  with respect to the multiplication  $*$  for all  $(a, b) \in R \times R^*$ . Thereby, the map  $\mathbf{T}$  uniquely extends to a homomorphism  $\tilde{\mathbf{T}} : S^{-1}R \mapsto ((S^{-1}R)^{\sigma_q}[[T]], +, *)$  via  $\frac{a}{b} \mapsto \tilde{\mathbf{T}}_{\frac{a}{b}}(T) := \frac{\mathbf{T}_a(T)}{\mathbf{T}_b(T)}^*$ . Define  $\delta_{S^{-1}R}^{(k)}(\frac{a}{b})$  to be the coefficient of  $T^k$  in  $\tilde{\mathbf{T}}_{\frac{a}{b}}(T)$ . Then the collection of maps  $(\delta_{S^{-1}R}^{(k)})_{k \in \mathbb{N}}$  of  $S^{-1}R$  to itself satisfy conditions 1 and 2 of Proposition 2.15. Thus  $(\delta_{S^{-1}R}^{(k)})_{k \in \mathbb{N}}$  is an iterative  $q$ -difference operator for  $S^{-1}R$ . We also have

$$\tilde{\mathbf{T}}_{\delta_{S^{-1}R}^{(k)}(\frac{a}{b})}(T) = \delta_T^{(k)}(\tilde{\mathbf{T}}_a(T)) \text{ for all } a \in R, k \in \mathbb{N}.$$

The Taylor series associated to both sides of the previous equation extend uniquely to  $(S^{-1}R)^{\sigma_q}[[T]]$  and since they coincide on  $R^{\sigma_q}[[T]]$ , they have to be equal. Then

$$\tilde{\mathbf{T}}_{\delta_{S^{-1}R}^{(k)}(\frac{a}{b})}(T) = \delta_T^{(k)}(\tilde{\mathbf{T}}_a(T)) \text{ for all } a \in S^{-1}R, k \in \mathbb{N}.$$

By Proposition 2.15, we get that  $(\delta_{S^{-1}R}^{(k)})_{k \in \mathbb{N}}$  is an iterative  $q$ -difference operator of  $S^{-1}R$  which uniquely extends  $(\delta_R^{(k)})_{k \in \mathbb{N}}$ .

**Remark 2.20** *Let  $(R, \delta_R^*)$  be an integral iterative  $q$ -difference ring. It is obvious that the set  $S$  of non zero divisors of  $R$  is a multiplicatively closed set and moreover stable under the action of  $\sigma_q$ .*

**Remark 2.21** *In this paragraph we did not mention the possibilities of extending an iterative  $q$ -difference operator over a field  $K$  to a finitely generated separable field extension  $E/K$ . In fact, this problem appears already in the classical  $q$ -difference Galois theory : extending  $\sigma_q$  to an algebraic extension gives rise to uniqueness problems. Here is an example. Consider a difference field  $(K, \sigma_q)$ , where  $\sigma_q$  is the identity on some algebraically closed field  $C$  containing  $\mathbb{Q}$ ,  $K$  contains a solution  $y$  of  $\sigma(x) = cx$ , where  $c \in C$  is non-zero and is not a root of unity. Moreover assume that  $K$  does not contain the  $n$ -th roots of  $y$  for some  $n > 1$ . Consider the extension of  $K$  given by  $b^n = y$ . Then  $\sigma(b) = rb$ , where  $r^n = c$ . The possible choices for  $\sigma$  on  $K(b)$  depend on the choices of  $r$ , and there are  $n$  possibilities, which give rise to  $n$  non-isomorphic difference field extensions of  $K$ . But by chance, we will not have to handle such kind of extension till the end of the paper.*

## 2.5 The Wronskian determinant

In classical Galois theory of  $q$ -difference equations, there exists an analogue of the Wronskian called the  $q$ -Wronskian or the Casoratian. If we consider a  $\sigma_q$ -module  $\mathcal{M}$  over a field  $K$  and a family  $\mathcal{F} := \{y_1, \dots, y_m\}$  of elements of  $\mathcal{M}$ , we will define the  $q$ -Wronskian of the family  $\mathcal{F}$  as

$$W_q(y_1, \dots, y_m) := \det((\sigma_q^{i-1}(y_j))_{1 \leq i, j \leq m}).$$

The nullity of the  $q$ -Wronskian gives a criterion for linear independence of the  $y_i$ 's (see for instance [15]). But when  $q$  is a root of unity, the  $q$ -Wronskian could vanish for other reasons (for instance because  $\sigma_q^n = id$ ). Thus, we have to change the notion of  $q$ -Wronskian for iterative  $q$ -difference operators in order to get a similar criterion to the one in the classical theory.

**Theorem 2.22** *Let  $(K, \delta_K^*)$  be an iterative  $q$ -difference field with field of constants  $C$ . Then for any elements  $x_1, \dots, x_r$  of  $K$  linearly independent over  $C$ , the iterative Taylor series  $\mathbf{T}_{x_1}, \dots, \mathbf{T}_{x_r}$  are linearly independent over  $K$ .*

### Proof

This statement is obviously true for  $r = 1$ . We will proceed by induction on  $r$ . Let  $(H_r)$  be the hypothesis of induction, i.e., for any elements  $x_1, \dots, x_r$  of  $K$  linearly independent over  $C$ , the iterative Taylor series  $\mathbf{T}_{x_1}, \dots, \mathbf{T}_{x_r}$  are linearly independent over  $K$ . Suppose that  $(H_{r-1})$  is true and let  $x_1, \dots, x_r \in K$  be linearly independent over  $C$ . Assume that  $\mathbf{T}_{x_1}, \dots, \mathbf{T}_{x_r}$  are linearly dependent over  $K$ , i.e. :

$$\mathbf{T}_{x_r} = \sum_{j=1}^{r-1} a_j \mathbf{T}_{x_j}$$

where  $a_j \in K$  not all equal to zero. This relation implies that

$$\delta^{(k)}(x_r) = \sum_{j=1}^{r-1} a_j \delta^{(k)}(x_j) \text{ for all } k \in \mathbb{N} \quad (10)$$

We will prove that  $\sigma_q(a_j) = a_j$  for all  $1 \leq j \leq r-1$ . First of all, let us remark that if  $x_1, \dots, x_{r-1} \in K$  are linearly independent over  $C$  then  $\sigma_q(x_1), \dots, \sigma_q(x_{r-1}) \in K$  are linearly independent over  $C$ .

Because of  $\delta^{(1)} = \frac{\sigma_q - id}{(q-1)t}$  and from Equation (10), we have :

$$\sigma_q(\delta^{(k)}(x_r)) - \delta^{(k)}(x_r) = \sum_{j=1}^{r-1} a_j \sigma_q(\delta^{(k)}(x_j)) - \sum_{j=1}^{r-1} a_j \delta^{(k)}(x_j)$$

and

$$\sigma_q(\delta^{(k)}(x_r)) = \sum_{j=1}^{r-1} \sigma_q(a_j) \sigma_q(\delta^{(k)}(x_j)).$$

We also obtain that

$$\sum_{j=1}^{r-1} (\sigma_q(a_j) - a_j) \sigma_q(\delta^{(k)}(x_j)) = 0$$

for all  $k \in \mathbb{N}$ . Because  $\sigma_q(\delta^{(k)}(x_j)) = \frac{1}{q^k} \delta^{(k)}(\sigma_q(x_j))$ , we get

$$\sum_{j=1}^{r-1} (\sigma_q(a_j) - a_j) (\delta^{(k)}(\sigma_q(x_j))) = 0$$

for all  $k \in \mathbb{N}$ . This means that  $\sum_{j=1}^{r-1} (\sigma_q(a_j) - a_j) \mathbf{T}_{\sigma_q(x_j)} = 0$ . Since  $x_1, \dots, x_{r-1} \in K$  are linearly independent over  $C$ ,  $\sigma_q(x_1), \dots, \sigma_q(x_{r-1}) \in K$  are linearly independent over  $C$ . Thus we can apply the induction hypothesis ( $H_{r-1}$ ) to the set of elements  $\sigma_q(x_1), \dots, \sigma_q(x_{r-1})$  of  $K$  and so  $\sigma_q(a_j) = a_j$  for  $1 \leq j \leq r-1$  as desired.

For all  $k, i \in \mathbb{N}$ , we have

$$\binom{i+k}{k}_q \delta^{(i+k)}(x_r) = \delta^{(i)} \delta^{(k)}(x_r) = \sum_{j=1}^{r-1} \sum_{l=0}^i \sigma_q^{i-l}(\delta^{(l)}(a_j)) \binom{i+k-l}{k}_q \delta^{(i+k-l)}(x_j)$$

and

$$\binom{i+k}{k}_q \delta^{(i+k)}(x_r) = \binom{i+k}{k}_q \sum_{j=1}^{r-1} a_j \delta^{(k)}(x_j).$$

Because  $\sigma_q(a_j) = a_j$  for  $1 \leq j \leq r-1$ , the term for  $l=0$  on the right hand side is equal to the left hand side, thus

$$\sum_{j=1}^{r-1} \sum_{l=1}^i \sigma_q^{i-l}(\delta^{(l)}(a_j)) \binom{i+k-l}{k}_q \delta^{(i+k-l)}(x_j) = 0. \quad (11)$$

For  $i=1$ , we deduce from equation (11) that

$$\sum_j^{r-1} \delta^{(1)}(a_j) \delta^{(k)}(x_j) = 0.$$

By applying  $\delta^{(1)}$ , we obtain :

$$\sum_j^{r-1} \sigma_q(\delta^{(1)}(a_j)) \delta^{(1)}(\delta^{(k)}(x_j)) + \sum_j^{r-1} \delta^{(1)}(\delta^{(1)}(a_j)) \delta^{(k)}(x_j) = 0,$$

i.e., since  $\sigma_q^r \delta^{(s)} = \frac{1}{q^{rs}} \delta_R^{(s)} \sigma_q^r$  for all  $r, s \in \mathbb{N}$ , and the  $a_j$ 's are fixed by  $\sigma_q$ ,

$$\sum_j^{r-1} \frac{q(q^{k+1}-1)}{q-1} \delta^{(1)}(a_j) \delta^{(k+1)}(x_j) + \sum_j^{r-1} (q+1) (\delta^{(2)}(a_j)) \delta^{(k)}(x_j) = 0.$$

For  $i = 2$ , we deduce from equation (11) that

$$\sum_j^{r-1} \sigma_q(\delta^{(1)}(a_j)) \binom{k+l}{k}_q \delta^{(k+1)}(x_j) + \sum_j^{r-1} \delta^{(2)}(a_j) \delta^{(k)}(x_j) = 0.$$

By subtracting this from the equality above, we find :

$$\sum_j^{r-1} \delta^{(2)}(a_j) \delta^{(k)}(x_j) = 0.$$

By induction, the same arguments yields

$$\sum_j^{r-1} \delta^{(i)}(a_j) \delta^{(k)}(x_j) = 0 \quad \text{for } k \geq 0 \text{ and } i \geq 1.$$

This leads to

$$\sum_j^{r-1} \delta^{(i)}(a_j) \mathbf{T}_{x_j} = 0.$$

By hypothesis of induction  $(H_{r-1})$ , this implies that  $\delta^{(i)}(a_j) = 0$  for all  $i \geq 1$  and all  $1 \leq j \leq r-1$ . Hence all the  $a_j$ 's are constants and lie in  $C$ . But we have  $x_n = \sum_{j=1}^{r-1} a_j x_j$  (see equation (11) for  $k = 0$ ) and thus by assumption of  $C$ -linearly independence of  $x_1, \dots, x_r$ , we get that  $a_j = 0$  for all  $1 \leq j \leq r-1$ . This is the end of the proof.

**Corollary 2.23** *In the notation of Theorem 2.22, there exist numbers  $d_1, \dots, d_r \in \mathbb{N}$  such that*

$$\det((\delta^{(d_i)}(x_j))_{i,j=1}^r) \neq 0.$$

**Definition 2.24** *Let  $(K, \delta_K^*)$  be an  $ID_q$  field with  $C(K) = C$  and let  $x_1, \dots, x_r \in K$  be linearly independent over  $C$ . The smallest numbers  $d_1, \dots, d_r \in \mathbb{N}$  (in lexicographical order) such that  $\det((\delta^{(d_i)}(x_j))_{i,j=1}^r) \neq 0$  (which exist by Corollary 2.23) are called the **difference orders** of  $x_1, \dots, x_r$ . The determinant*

$$wr(x_1, \dots, x_r) := \det((\delta^{(d_i)}(x_j))_{i,j=1}^r)$$

*is called the **Wronskian determinant** of  $x_1, \dots, x_r$ .*

### 3 Iterative $q$ -difference modules

Until the end of this article, we will assume that  $q$  is a  $n$ -th primitive root of unity contained in an algebraically closed field  $C$ . But we do not make any assumption about the characteristic of the field  $C$ .

In Section 2, we have defined iterative  $q$ -difference rings. Following the classical way, we extend this concept to modules, in order to get a suitable notion of iterative  $q$ -difference equations associated to these modules.

**Definition 3.1** Let  $(R, \delta_R^*)$  be an iterative  $q$ -difference ring. Let  $M$  be a free  $R$ -module of finite type over  $R$ . We will say that  $(M, \delta_M^*)$  is an **iterative  $q$ -difference module** if there exists a family of maps  $\delta_M^* = (\delta_M^{(k)})_{k \in \mathbb{N}}$ , such that for all  $i, j, k \in \mathbb{N}$

1.  $\delta_M^{(0)} = id_M$ ,
2.  $\phi_M := (q-1)t\delta_M^{(1)} + id_M$  is a bijective map from  $M$  to  $M$ ,
3.  $\delta_M^{(k)}$  is an additive map from  $M$  to  $M$ ,
4.  $\delta_M^{(k)}(am) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a))\delta_M^{(i)}(m)$  for  $a \in R$  and  $m \in M$ ,
5.  $\delta_M^{(i)} \circ \delta_M^{(j)} = \binom{i+j}{i}_q \delta_M^{(i+j)}$ .

The set of all iterative  $q$ -difference modules over  $R$  is denoted by  $IDM_q(R)$ .

**Remark 3.2 (Classical case)** If  $q$  is not a root of unity, it is easy to see that  $\phi_M(am) = \sigma_q(a)\phi_M(m)$  for all  $a \in R$  and  $m \in M$ . Moreover,  $\delta_M^{(k)} = \frac{\delta_M^{(1)^k}{[k]_q!}$ . Thus, in the case where  $q$  is not a root of unity, an  $ID_q$ -module is nothing else than a  $q$ -difference module in the sense of [25].

As in 2.5, we easily show that we have for all  $j, i \in \mathbb{N}$ ,

$$\phi_M^j \delta_M^{(i)} = \frac{1}{q^{ji}} \delta_M^{(i)} \phi_M^j. \quad (12)$$

**Definition 3.3** Let  $(M, \delta_M^*)$  and  $(N, \delta_N^*)$  be two iterative  $q$ -difference modules over  $R$  and let  $\phi \in Hom_R(M, N)$ . We will say that  $\phi$  is an **iterative  $q$ -difference homomorphism** if  $\delta_N^{(k)} \circ \phi = \phi \circ \delta_M^{(k)}$  for all  $k \in \mathbb{N}$ .

**Definition 3.4** Let  $(R, \delta_R^*)$  be an iterative  $q$ -difference ring. Let  $(M, \delta_M^*)$  be an iterative  $q$ -difference module over  $R$ . The  $C(R)$ -module

$$V_M := \bigcap_{k \in \mathbb{N}} Ker(\delta_M^{(k)})$$

is called the **solution space** of the iterative  $q$ -difference module  $M$ . We will say that  $M$  is a **trivial** iterative  $q$ -difference module if  $M \simeq V_M \otimes_{C(R)} R$ .

**Theorem 3.5** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field. Let us denote by  $IDM_q(L)$  the category with objects the iterative  $q$ -difference modules over  $L$  and morphisms the iterative  $q$ -difference morphisms. Then  $IDM_q(L)$  is a neutral Tannakian category over  $C(L)$ . The unit object is  $(L, \delta_L^*)$ .

We refer to [16] for the fact that  $IDM_q(L)$  is an abelian category, the case for iterative differential modules being the same as the one of iterative  $q$ -difference modules. For  $M$  and  $N$  two objects of  $IDM_q(L)$ , we define the tensor product  $M \otimes N := M \otimes_L N$  by the usual tensor product as  $L$ -modules and turn it to an  $ID_q$ -module via

$$\delta_{M \otimes N}^{(k)}(x \otimes y) = \sum_{i+j=k} \phi_M^j(\delta_M^{(i)}(x)) \otimes \delta_N^{(j)}(y)$$

for all  $x \in M, y \in N$ . The proof that  $(\delta_{M \otimes N}^{(k)})_{k \in \mathbb{N}}$  is an iterative  $q$ -difference operator on  $M \otimes N$  is analogous to the proof of Proposition 2.11.

The dual of an object  $M$  of  $IDM_q(L)$  is then given by  $M^* = Hom_L(M, L)$  together with

$$\delta_{M^*}^{(k)}(f) = \sum_{i+j=k} (-1)^i q^{\frac{i(i+1)}{2}} \sigma_q^i(\delta_L^{(j)}) \circ f \circ \delta_M^{(i)} \circ \phi_M^{-i}$$

for all  $f \in M^*$ . The proof that  $(M, \delta_{M^*}^*)$  is an iterative  $q$ -difference module is left to the reader. We just recall that if  $(M, \phi_M)$  is a  $q$ -difference module in the sense of [25], then the  $M^*$  is endowed with a  $q$ -difference module structure via

$$\phi_{M^*}(f) := \sigma_q \circ f \circ \phi_M^{-1}.$$

The evaluation  $\epsilon : M \otimes M^* \rightarrow \mathbf{1}_{IDM_q(L)} = L$  sends  $x \otimes f$  to  $f(x)$ , and the coevaluation  $\eta : L \rightarrow M^* \otimes M$  is defined by mapping 1 to  $\sum_{i=1}^n x_i^* \otimes x_i$ , where  $\{x_i\}_{i=1}^n$  denotes an  $L$ -basis of  $M$  and  $\{x_i^*\}_{i=1}^n$  the associated dual basis of  $M^*$ . Note that the definition of  $\eta$  does not depend on the chosen basis. It remains to show that  $\epsilon$  and  $\eta$  are  $ID_q$ -homomorphism and that they satisfy  $(\epsilon \otimes id_M) \circ (id_M \otimes \eta) = id_M$  and  $(id_{M^*} \otimes \epsilon) \circ (\eta \otimes id_{M^*}) = id_{M^*}$  for all objects  $M$  of  $IDM_q(L)$ . We have

$$\begin{aligned} \epsilon \circ \delta_{M \otimes M^*}^{(k)}(x \otimes f) &= \epsilon \left( \sum_{i+j=k} \delta_M^{(i)}(x) \otimes \phi_{M^*}^j(\delta_{M^*}^{(j)}(f)) \right) = \sum_{i+j=k} \phi_{M^*}^i(\delta_{M^*}^{(j)}(f))(\delta_M^{(i)}(x)) \\ &= \sum_{i+j=k} \sum_{l=0}^j (-1)^l q^{l(l+1)/2} \sigma_q^{i+l}(\delta_L^{(j-l)}) \circ f \circ \delta_M^{(l)} \circ \phi_M^{-(i+l)}(\delta_M^{(i)}(x)) \\ &= \sum_{i+j=k} \sum_{l=0}^j (-1)^l q^{l(l+1)/2} \sigma_q^{l+i}(\delta_L^{(j-l)}) \circ f \circ q^{i(i+1)} \binom{i+l}{i}_q \delta_M^{(i+l)}(\phi_M^{-(i+l)}(x)) \end{aligned}$$

and thus

$$\epsilon \circ \delta_{M \otimes M^*}^{(k)}(x \otimes f) = \sum_{i_*+j_*=k} \sigma_q^{i_*}(\delta_L^{(j_*)}) \circ f \circ \delta_M^{(i_*)}(\phi_M^{-i_*}(x)) \left( \sum_{i=0}^{i_*} (-1)^i q^{i(i-1)/2} \binom{i_*}{i}_q \right).$$

By expanding  $(1-1)_{i_*}$ , we see that the inner sum equals zero if and only if  $i_* \neq 0$ . We thus get

$$\epsilon \circ \delta_{M \otimes M^*}^{(k)}(x \otimes f) = \delta_L^{(k)} f(x) = \delta_L^{(k)} \circ \epsilon(x \otimes f).$$

The proof for  $\eta$  is analogous.

Let  $x = \sum_{i=1}^n a_i x_i$  be in  $M$ , then  $(\epsilon \otimes id_M) \circ (id_M \otimes \eta)(x) = \epsilon \otimes id_M(x \otimes (\sum_{i=1}^n x_i^* \otimes x_i)) = \epsilon \otimes id_M(\sum_{i=1}^n (x \otimes x_i^*) \otimes x_i) = \sum_{i=1}^n x_i^*(x) \otimes x_i = \sum_{i=1}^n a_i x_i = x$ . Again, the second statement is proved analogously. Finally, we note that

$$End_{IDM_q(L)}(\mathbf{1}_{IDM_q(L)}) = End_{ID_q}(L) = C(L),$$

finishing the proof.

### 3.1 Iterative $q$ -difference modules and projective systems

In this paragraph, we will show that iterative  $q$ -difference operators and iterative derivations are closely related. First of all, let us consider the following proposition :

**Proposition 3.6** *Let  $q$  be a  $n$ -th root of unity. Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field and let  $(M, \delta_M^*)$  be an iterative  $q$ -difference module over  $L$ . Set  $L_0 = \bigcap_{j \in \mathbb{N}} Ker(\delta_L^{(j)})$  and  $M_0 = \bigcap_{j \in \mathbb{N}} Ker(\delta_M^{(j)})$ . Then  $(M_0, (\delta_M^{(nk)})_{k \in \mathbb{N}})$  is an iterative differential module over  $L_0$  (see [16]).*

Therefore, one could hope to get as in [16] some projective system deeply associated to our iterative  $q$ -difference module. But the problem is the following. In the case of characteristic zero we may regain all the iterative  $q$ -difference operators only with the knowledge of  $\delta_M^{(1)}$  and  $\delta_M^{(n)}$ . This is due to the formula  $(\delta_M^{(n)})^{n^{k-1}} = (n^{k-1})! \delta_M^{(n^k)}$  and to the fact that the family  $\{\delta_M^{(1)}, (\delta_M^{(n^k)})_{k \in \mathbb{N}}\}$  generates the iterative  $q$ -difference operator. But in positive characteristic, we have to consider the whole family  $\{\delta_M^{(1)}, (\delta_M^{(np^k)})_{k \in \mathbb{N}}\}$  to recover the iterative  $q$ -difference operator. Therefore, we can only obtain projective systems in positive characteristic. But, this fact is not a hindrance to the construction of iterative  $q$ -difference equations in characteristic 0.

As we have mentioned in the introduction, we will show that in positive characteristic, the category of iterative  $q$ -difference modules is closely related to the category of some specific projective systems. In this paragraph we obtain an equivalence between these two categories. This is a very nice tool because it allows us translate our computations from the non commutative world of iterative  $q$ -difference modules to the world of linear algebra, via the vector spaces associated to the projective systems.

This comparison between iterative differential modules and specific projective systems appears already in the work of B.H. Matzat and M. van der Put. But to obtain an equivalence of categories between the projective systems linked to iterative derivations and the ones associated to iterative  $q$ -difference, we need to have  $q^p = 1$  and this assumption makes no sense. A hope for realizing this equivalence will be perhaps to rebuild both theories over non-algebraically closed base rings, such as  $\mathbb{Z}/p^m\mathbb{Z}$  and try to reach the Witt vectors. But this is a future research topic.

However, it is very easy to obtain from a iterative  $q$ -difference module an iterative differential module (see 3.6).

### 3.1.1 Case of characteristic $p$

Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of characteristic  $p$  and let  $(M, \delta_M^*)$  be an iterative  $q$ -difference module over  $L$ . In positive characteristic, we have the exact analogue of the equivalence of categories obtained by Matzat in [16] Theorem 2.8.

Put  $L_1 = \text{Ker}(\delta_L^{(1)})$  and  $L_{k+1} = \cap_{0 \leq j < k} \text{Ker}(\delta_L^{(np^j)}) \cap L_1$  for  $k > 1$  and  $L_0 = L$ .

Put  $M_1 = \text{Ker}(\delta_M^{(1)})$ ,  $M_{k+1} = \cap_{0 \leq j < k} \text{Ker}(\delta_M^{(np^j)}) \cap M_1$  for all  $k > 1$  and  $M_0 = M$ .

**Proposition 3.7** 1.  $M_k$  is an  $L_k$ -vector space of finite dimension,

2.  $(M_k, \phi_k)_{k \in \mathbb{N}}$ , where  $\phi_k$  denotes the obvious injection from  $M_{k+1}$  to  $M_k$ , is a projective system,

3. the map  $\phi_k$  extends to an isomorphism of  $L_k$ -vector-spaces from  $M_{k+1} \otimes L_k$  to  $M_k$ .

#### Proof

The two first statements are obvious. Let us prove the third one. Let  $(m_1, \dots, m_s)$  be  $s$  elements of  $M_{k+1}$  linearly independent over  $L_{k+1}$ . Suppose that there are linearly dependent over  $L_k$  and let

$$\sum_{i=i_0, i \in \mathcal{I}} \lambda_i m_i = 0 \quad (13)$$

be a non trivial linear combination of the  $m_i$ 's over  $L_k$  where  $\mathcal{I}$  denotes a set of index of minimal length. Without loss of generality we may assume that  $\lambda_{i_0} = 1$ .

For  $np^k \leq j < np^{k+1}$ , apply  $\delta_M^{(j)}$  to Equation (13). We thus have, since  $m_i \in M_{k+1}$ ,

$$\sum_{i=i_0, i \in \mathcal{I}} \sum_{s=0}^j \sigma_q^s(\delta_L^{(j-s)}(\lambda_i)) \delta_M^{(s)}(m_i) = \sum_{i=i_0, i \in \mathcal{I}} \delta_L^{(j)}(\lambda_i) m_i = 0. \quad (14)$$

Substracting (17) from (16), we obtain

$$\sum_{i=i_0, i \in \mathcal{I}} (\delta_L^{(j)}(\lambda_i) - \lambda_i) m_i = 0. \quad (15)$$

By minimality, we have  $\delta_L^{(j)}(\lambda_i) - \lambda_i = 0$  for all  $i > i_0$  and for all  $np^k \leq j < np^{k+1}$ . That is to say that  $\lambda_i \in L_{k+1}$ . Because  $(m_1, \dots, m_s)$  are linearly independent over  $L_{k+1}$ , we get that  $\lambda_i = 0$  for all  $i$ . This is a contradiction. We thus have  $\dim_{L_{k+1}}(M_{k+1}) \leq \dim_{L_k}(M_k)$ . For all  $k \in \mathbb{N}$ , the application  $\delta_M^{(np^k)}$  is  $L_{k+2}$ -linear on  $M_{k+1}$  and  $(\delta_M^{(np^k)})^p = 0$ , so  $\dim_{L_{k+2}}(M_{k+2}) = \dim_{L_{k+2}}(\text{Ker}(\delta_M^{(np^k)})|_{M_{k+1}}) \geq \frac{1}{p} \dim_{L_{k+2}}(M_{k+1}) \geq \dim_{L_{k+1}}(M_{k+1})$ , where the last inequality comes from the fact that  $\delta_L^{(np^k)}$  is an  $L_{k+2}$ -linear endomorphism of  $L_{k+1}$  of order of nilpotence  $p$ .

But we also have  $(\delta_M^{(1)})^n = 0$ . Therefore, we have  $\dim_{L_1}(M_1) = \dim_{L_1}(\text{Ker}(\delta_M^{(1)})|_M) \geq \frac{1}{n} \dim_{L_1}(M) \geq \dim_L(M)$ , where the last inequality comes from the fact that  $\delta_L^{(1)}$  is an  $L_1$ -linear endomorphism of  $L$  of order of nilpotence  $n$  ( $q$  is a  $n$ -th primitive root of unity).

### 3.1.2 Case of characteristic 0

Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of zero characteristic and let  $(M, \delta_M^*)$  be an iterative  $q$ -difference module over  $L$ . Put, for all  $k \in \mathbb{N}^*$ ,  $L_k = \cap_{0 \leq j < k} \text{Ker}(\delta_L^{(n^j)})$ ,  $L'_0 = \text{Ker}(\delta_L^{(1)})$  and  $L_0 = L$ .

Put, for all  $k \in \mathbb{N}^*$ ,  $M_k = \cap_{0 \leq j < k} \text{Ker}(\delta_M^{(n^j)})$ ,  $M'_0 = \text{Ker}(\delta_M^{(1)})$  and  $M_0 = M$ .

**Proposition 3.8** 1.  $M_k$  is a  $L_k$ -vector space of finite dimension.

2.  $M_k = M_1$  for all  $k \geq 1$ .

3. Let  $\phi_1$  be the obvious injection from  $M_1$  to  $M'_0$ . Then the map  $\phi_1$  extends to a monomorphism of  $L'_0$ -vector-spaces from  $M_1 \otimes L'_0$  to  $M'_0$ .

4. Let  $\phi_0$  be the obvious injection from  $M'_0$  to  $M_0$ . Then the map  $\phi_0$  extends to an isomorphism of  $L$ -vector-spaces from  $M'_0 \otimes L$  to  $M_0$ .

#### Proof

The first statement is obvious. Because  $(\delta_M^{(n)})^{n^{k-1}} = (n^{k-1})! \delta_M^{(n^k)}$  for all  $k \geq 1$  (see part 4 of Proposition 2.2), we have  $M_k = M_1$  for all  $k \geq 1$ . The third statement is obvious.

We now prove the fourth statement. Let  $(m_1, \dots, m_s)$  be  $s$  elements of  $M'_0$  linearly independent over  $L'_0$ . Let us assume that they are linearly dependent over  $L$  and let us consider a non trivial linear combination of the  $m_i$ 's over  $L$  where  $\mathcal{I}$  denotes a set of index of minimal length :

$$\sum_{i=i_0, i \in \mathcal{I}} \lambda_i m_i = 0. \quad (16)$$

Without loss of generality we may assume that  $\lambda_{i_0} = 1$ .

Let us apply  $\delta_M^{(1)}$  to Equation (16). We then have, since  $m_i \in M'_0$ ,

$$\sum_{i=i_0, i \in \mathcal{I}} \sum_{s=0}^1 \sigma_q^s(\delta_L^{(1-s)}(\lambda_i)) \delta_M^{(s)}(m_i) = \sum_{i=i_0, i \in \mathcal{I}} \delta_L^{(1)}(\lambda_i) m_i = 0. \quad (17)$$

By subtracting (17) from (16), we obtain :

$$\sum_{i > i_0, i \in \mathcal{I}} (\delta_L^{(1)}(\lambda_i) - \lambda_i) m_i = 0. \quad (18)$$

By minimality, we have  $\delta_L^{(1)}(\lambda_i) - \lambda_i = 0$  for all  $i > i_0$ , that is to say that  $\lambda_i \in L'_0$ . Because  $(m_1, \dots, m_s)$  are linearly independent over  $L'_0$ , we get  $\lambda_i = 0$  for all  $i$ . This is a contradiction. We then have  $\dim_{L'_0}(M'_0) \leq \dim_L(M)$ .

Conversely, from  $(\delta_M^{(1)})^n = 0$  and  $(\delta_L^{(1)})^n = 0$  follows

$$\dim_{L'_0}(M'_0) = \dim_{L'_0}(\text{Ker}(\delta_M^{(1)}|_M)) \geq \frac{1}{n} \dim_{L'_0}(M) \geq \dim_L(M).$$

### 3.1.3 Equivalence of categories in the case of positive characteristic

In this paragraph, we keep the notation of Paragraph 3.1.1.

**Notation 3.9** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of characteristic  $p$ . Let us denote by  $Proj_q(L)$  the category of projective systems  $(N_k, \psi_k)_{k \in \mathbb{N}}$  over  $L$  with the properties:

1.  $N_k$  is an  $L_k$ -vector space of finite dimension and  $\psi_k$  is  $L_{k+1}$ -linear,
2. each  $\psi_k$  uniquely extends to an  $L_k$ -isomorphism

$$\tilde{\psi}_k : L_k \otimes_{L_{k+1}} M_{k+1} \longrightarrow M_k.$$

**Theorem 3.10** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of positive characteristic. Then the category  $Proj_q(L)$  is equivalent to the category  $IDM_q(L)$ .

#### Proof

We already saw in Proposition 3.7 how an object of  $IDM_q(L)$  leads to an object of  $Proj_q(L)$ . Conversely, let us consider  $(N_k, \psi_k)_{k \in \mathbb{N}}$  in the category  $Proj_q(L)$ . We will now construct its associated iterative  $q$ -difference module.

Put  $M_0 := N_0$  and define  $M_k := \psi_0 \circ \psi_1 \circ \dots \circ \psi_{k-1}(N_k)$ . Then  $M_k \subset M_{k+1} \subset \dots \subset M_0$ . Let  $B_k = \{b_1, \dots, b_m\}$  be an  $L_k$ -basis for  $M_k$ , then by property 2,  $B_k$  is an  $L$ -basis of  $M$ . Let  $x \in M$ , there exists  $(\lambda_i)_{i=1, \dots, m} \in L^m$  such that  $x = \sum_{i=1}^m \lambda_i b_i$ . Then, for all  $j < k$ , set

$$\delta_M^{(j)}(x) := \sum_{i=1}^m \delta_L^{(j)}(\lambda_i) b_i.$$

This is possible because we want  $B_k$  to lie in the kernel of  $\delta_M^{(j)}$  for  $j < k$ . Obviously the definition is independent of the choice of the basis. Therefore,  $(M_0, \delta_{M_0}^*)$  is an object  $IDM_q(L)$ .

Let us consider two objects  $\mathcal{M} := (M_k, \phi_k)_{k \in \mathbb{N}}$  and  $\mathcal{N} := (N_k, \psi_k)_{k \in \mathbb{N}}$  of  $Proj_q(L)$  and  $\alpha$  a morphism from  $\mathcal{M}$  to  $\mathcal{N}$  in the category  $Proj_q(L)$ , i.e.  $\alpha_k$  is  $L_k$  linear and the diagram

$$\begin{array}{ccc} M_k & \xrightarrow{\alpha_k} & N_k \\ \phi_k \uparrow & & \uparrow \psi_k \\ M_{k+1} & \xrightarrow{\alpha_{k+1}} & N_{k+1} \end{array}$$

is commutative. Then we have  $\delta_N^* \circ \alpha_0 = \alpha_0 \circ \delta_M^*$ . Also, with this property, it is then easy to verify that

$$Proj_q(L) \longrightarrow IDM_q(L)$$

$$(N_k, \psi_k) \longmapsto (M_0, \delta_{M_0}^*)$$

(with  $\delta_{M_0}^*$  as defined above) is in fact an equivalence of categories.

## 3.2 Iterative $q$ -difference equations

As we expect from standard  $q$ -difference Galois theory, any iterative  $q$ -difference module should give rise to an iterative  $q$ -difference equation consisting of a family of equations. Proposition 3.12 in the case of positive characteristic and Proposition 3.17 for characteristic zero show how to obtain this equation from a given  $ID_q$ -module.

### 3.2.1 Case of the characteristic $p$

**Proposition 3.11** *Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of characteristic  $p$ , and let  $(M, \delta_M^*)$  be an object of  $IDM_q(L)$ . Let us consider the canonical projective system  $(M_k, \phi_k)_{k \in \mathbb{N}}$  associated to  $M$ . For all  $k \in \mathbb{N}$ , let us choose an  $L_k$ -basis  $B_k$  of  $M_k$  with  $\phi_k B_k = B_{k+1}$  and let  $D_k \in GL_n(L_k)$  denote the matrix of  $\phi_k$  with respect to that basis, i.e.,  $B_k D_k = B_{k+1}$ .*

*Then, for any  $l \in \mathbb{N}$  and for any  $X \in L^n$ , we have :*

1.  $B_0 X = B_l X_l$  where  $X_l = D_{l-1}^{-1} \dots D_0^{-1} X$ ,
2.  $\delta_M^{(k)}(B_0 X) = B_l \delta_L^{(k)}(X_l)$  for  $k < l$ .

#### Proof

Part 1 is obvious by definition. Part 2 follows from

$$\delta_M^{(k)}(B_0 X) = \delta_M^{(k)}(B_l X_l) = B_l \delta_L^{(k)}(X_l) \text{ for } k < l.$$

**Proposition 3.12** *With the previous notation, and a basis  $B_0 = \{b_1, \dots, b_n\}$  of  $M$ , the following statements are equivalent*

1.  $B_0 \mathbf{y} = \sum_{i=1}^n y_i b_i \in V_M = \bigcap_{k \in \mathbb{N}} M_k$ .
2. For all  $l \in \mathbb{N}$ , we have  $\delta_L^{(1)}(\mathbf{y}_1) = 0$  and  $\delta_L^{(np^k)}(\mathbf{y}_l) = 0$  for  $0 \leq k < l$ , where  $\mathbf{y}_l = D_{l-1}^{-1} \dots D_0^{-1} \mathbf{y}$ .
- 3.

$$\delta_L^{(np^k)}(\mathbf{y}) = A_{k+1} \mathbf{y},$$

for all  $k \geq 0$  where  $A_{k+1} = \delta_L^{(np^k)}(D_0 \dots D_{k+1})(D_0 \dots D_{k+1})^{-1}$  and  $\delta_L^{(1)}(\mathbf{y}) = A_1 \mathbf{y}$  where  $A_1 = \delta_L^{(1)}(D_0 D_1)(D_0 D_1)^{-1}$ .

#### Proof

First, we show that statements 1 and 2 are equivalent :  $B_0 \mathbf{y} \in V_M$  if and only if  $\delta_M^{(k)}(B_0 \mathbf{y}) = 0$  for all  $k \in \mathbb{N}^*$ . The claim is obvious by using the equation

$$\delta_M^{(k)}(B_0 \mathbf{y}) = B_l \delta_L^{(k)}(\mathbf{y}_l)$$

which holds for  $k < l$  (see the previous proposition).  
 Finally, the equivalence of 2 and 3 is obtained using:

$$\delta_L^{(np^l)}(\mathbf{y}) = \delta_L^{(np^l)}(D_{l+1}\mathbf{y}_{l+2}) = \delta_L^{(np^l)}(D_0 \dots D_{l+1})\mathbf{y}_{l+2} + D_0 \dots D_{l+1}\delta_L^{(np^l)}(\mathbf{y}_{l+2}) = A_{l+1}\mathbf{y} + \delta_L^{(np^l)}(\mathbf{y}_{l+2})$$

and

$$\delta_L^{(1)}(\mathbf{y}) = \delta_L^{(1)}(D_1\mathbf{y}_2) = \delta_L^{(1)}(D_0 \dots D_1)\mathbf{y}_2 + D_0 \dots D_1\delta_L^{(1)}(\mathbf{y}_2) = A_1\mathbf{y} + \delta_L^{(1)}(\mathbf{y}_2).$$

**Definition 3.13** *The family of equations  $\{\delta_L^{(1)}(\mathbf{y}) = A_1\mathbf{y}, \delta_L^{(np^k)}(\mathbf{y}) = A_k\mathbf{y}\}_{k \geq 0}$  related to the  $IDM_q$ -module  $(M, \delta_M^*)$  by Proposition 3.12 is called an **iterative  $q$ -difference equation** ( $ID_qE$ ).*

We now give some examples of iterative  $q$ -difference equations over fields of positive characteristic.

**Example 3.14** *Let  $p$  be a prime number, let  $C = \overline{\mathbb{F}_p}$  be an algebraic closure of  $\mathbb{F}_p$  and let  $F = C(t)$  be the rational function field with coefficients in  $C$ . Let  $(a_l)_{l \geq 0}$  be a set of elements in  $C$ . Let  $M = Fb_1$ . Suppose that,  $D_{l+1} = (t^{a_l np^l}) \in Gl_1(F_{l+1})$  for  $l \in \mathbb{N}$  and  $D_1 = (1)$ . We have*

$$A_{k+1} = \delta_L^{(np^k)}(D_0 \dots D_{k+1})(D_0 \dots D_{k+1})^{-1} = \delta_L^{(n^k)}(t^{\sum_{i=0}^k a_i np^i})t^{-\sum_{i=0}^k a_i np^i} = \frac{a_k}{t^{np^k}}$$

because  $\left(\sum_{j=0}^k \frac{a_j np^j}{np^k}\right)_q = a_k$ . Hence  $\delta_M^{(np^k)}(y) = \frac{a_k}{t^{np^k}}y$  for all  $k \in \mathbb{N}$ .

**Example 3.15** *Let  $p$  be a prime number, let  $C = \overline{\mathbb{F}_p}$  be an algebraically closure of  $\mathbb{F}_p$  and let  $F = C(t)$  be the rational function field with coefficients in  $C$ . Let  $(a_l)_{l \geq 0}$  be a set of elements in  $C$ . Let  $M = Fb_1 \oplus Fb_2$ . Suppose that,*

$$D_{l+1} := \begin{pmatrix} 1 & a_l t^{np^l} \\ 0 & 1 \end{pmatrix} \text{ for all } l \in \mathbb{N}$$

and

$$D_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the formula (1), we obtain,

$$A_{k+1} = \begin{pmatrix} 1 & a_k \\ 0 & 1 \end{pmatrix}$$

and  $A_1 = 0$ . So, the associated  $ID_qE$  associated to  $M$  is

$$\delta^{(np^k)}(Y) = A_k Y = \begin{pmatrix} 0 & a_k \\ 0 & 0 \end{pmatrix} Y \text{ for all } k \in \mathbb{N}.$$

### 3.2.2 Case of characteristic zero

Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field of characteristic zero and let  $(M, \delta_M^*)$  be an iterative  $q$ -difference module over  $L$ . As in Paragraph 3.1.2,  $L_1 = \bigcap_{0 \leq j < 1} \text{Ker}(\delta_L^{(n^j)})$ ,  $L'_0 = \text{Ker}(\delta_L^{(1)})$ ,  $L_0 = L$ ,  $M_1 = \bigcap_{0 \leq j < 1} \text{Ker}(\delta_M^{(n^j)})$ ,  $M'_0 = \text{Ker}(\delta_M^{(1)})$  and  $M_0 = M$ .

**Notation 3.16** Let  $B'_0 = \{b'_1, \dots, b'_n\}$  (resp.  $B_0$ ) be a  $L'_0$ -basis of  $M'_0$  (resp. a  $L_0$ -basis of  $M_0$ ). Because of Proposition 3.8, we have  $M'_0 \otimes L \simeq M$ . Now let us denote by  $D_0 \in \text{Gl}_n(L'_0)$  the matrix of  $\phi_0$  with respect to the basis  $B'_0$  and  $B_0$ , i.e.,  $B_0 D_0 = B'_0$ . Let  $C_k$  be the matrix of  $\delta_M^{(k)}$  with respect to the basis  $B_0$  and let  $\Phi \in \text{Gl}_n(L)$  be the matrix of the action of  $\sigma_q$  with respect to the basis  $B_0$ . Set

1.  $A_0 := \text{Id}$ ,  $A_1 := -\Phi^{-1}C_1$ , and
2.  $A_k := -\Phi^{-k}(\sum_{j=0}^{k-1} \Phi^j C_{k-j} A_j)$  for  $k > 1$  inductively.

**Proposition 3.17** Using the previous notation, the following statements are equivalent:

1.  $B_0 \mathbf{y} = \sum_{i=1}^n y_i b_i \in V_M = \bigcap_{k \in \mathbb{N}} M_k$ .
2. For  $k \in \mathbb{N}$ , we have  $\delta_L^{(k)}(\mathbf{y}) = A_k \mathbf{y}$ , with  $A_k$  defined in Notation 3.16.

**Proof**

If  $B_0 \mathbf{y} = \sum_{i=1}^n y_i b_i \in V_M$ , then for all  $k \in \mathbb{N}$  we have  $\delta_M^{(k)}(B_0 \mathbf{y}) = 0$ . Let us first consider the case  $k = 1$ . We have

$$\delta_M^{(1)}(B_0 \mathbf{y}) = \sigma_q(B_0) \delta^{(1)}(\mathbf{y}) + \delta_M^{(1)}(B_0) \mathbf{y},$$

and thus

$$\delta^{(1)}(\mathbf{y}) = -\Phi^{-1}C_1.$$

We will proceed by induction and assume that we have  $\delta_L^{(j)}(\mathbf{y}) = A_j \mathbf{y}$  for all  $j < k$ . Then

$$\delta_M^{(k)}(B_0 \mathbf{y}) = 0 = \sum_{j=0}^k \sigma_q^j(\delta_M^{(k-j)}(B_0)) \delta^{(j)}(\mathbf{y}) = \sum_{j=0}^{k-1} B_0 \Phi^j C_{k-j} A_j \mathbf{y} + B_0 \Phi^k \delta^{(k)}(\mathbf{y}),$$

and thus

$$\delta^{(k)}(\mathbf{y}) = -\Phi^{-k} \left( \sum_{j=0}^{k-1} \Phi^j C_{k-j} A_j \right) \mathbf{y} = A_k \mathbf{y}.$$

Hence the first statement implies the second. By going through the computation backwards, we obtain the equivalence between the two statements.

**Definition 3.18** The family of equations  $\{\delta_L^{(k)}(\mathbf{y}) = A_k \mathbf{y}\}_{k \in \mathbb{N}}$  related to the  $IDM_q$ -module  $(M, \delta_M^*)$  by Proposition 3.17 is called an **iterative  $q$ -difference equation** ( $ID_q E$ ).

**Remark 3.19** We shall recall the introduction of Paragraph 3.1.2 : in the case of characteristic zero we may regain the iterative  $q$ -difference operator from the knowledge of  $\delta^{(1)}$  and  $\delta^{(n)}$ . Therefore, for the study of an iterative  $q$ -difference equation in zero characteristic, it is sufficient to know the iterative  $q$ -difference equations at level 1 and  $n$ .

**Remark 3.20** We keep using Notation 3.16 and we will show how to compute  $C_j$  by induction for  $j < n$ . We have by definition  $C_1 = \frac{\Phi - Id}{(q-1)t}$ . This implies

$$\frac{q^j - 1}{q - 1} \delta_M^{(j)}(B_0) = \delta_M^{(1)} \delta_M^{(j-1)}(B_0) = \delta_M^{(1)}(C_{j-1} B_0) = (\sigma_q(C_{j-1}) C_1 + \delta^{(1)}(C_{j-1})) B_0,$$

and thus

$$C_j = \frac{q - 1}{q^j - 1} (\sigma_q(C_{j-1}) C_1 + \delta^{(1)}(C_{j-1})). \quad (19)$$

**Example 3.21** Let  $L = \mathbb{C}(t)$  and let  $q$  be a  $n$ -th primitive root of unity. Let  $M = Fb_1$  be a rank one  $IDM_q(L)$ -module and suppose that  $\Phi(b_1) = b_1$ . Then an easy computation leads to  $C_j = 0$  for all  $1 \leq j < n$  and  $A_j = 0$  for  $1 \leq j < n$ . Now, let  $a_1$  be an integer and set  $C_n = \frac{a_1}{t^n}$ . Then  $A_n = \frac{a_1}{t^n}$ . By induction we get

$$A_{kn} = \frac{a_k}{t^{kn}}, \text{ for all } k \in \mathbb{N} \text{ where } a_{k+1} = \frac{1}{k+1} (ka_k + a_k a_1).$$

## 4 Iterative $q$ -difference Picard-Vessiot extensions

In this section, we develop a Picard-Vessiot theory for iterative  $q$ -difference equations. We build the Picard-Vessiot ring inspired by the usual construction, but we have to adapt our construction to a infinite set of variables, and thus some modifications are necessary.

### 4.1 Iterative Picard-Vessiot rings

**Notation 4.1** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field. If,

1. the characteristic of the constants field  $C$  of  $L$  is zero then let us denote by  $(k_C)$  the family  $(k)_{k \in \mathbb{N}}$ ,
2. the characteristic of the constants field  $C$  of  $L$  is positive equal to  $p$  then let us denote by  $(k_C)$  the family  $\{1, (np^k)_{k \in \mathbb{N}}\}$ .

**Remark 4.2 (Classical case)** As mentioned before, when  $q$  is not a root of unity, an iterative  $q$ -difference module is the same object as a  $q$ -difference module. Moreover, in this case the iterative  $q$ -difference equation is just obtained by considering the equation of level 1 and if there exists  $Y \in Gl_n(R)$  such that  $\delta_L^{(1)}(Y) = A_1 Y$  then for all  $k \in \mathbb{N}$  we have  $\delta_L^{(k_C)}(Y) = A_k Y$ . Thereby, when  $q$  is not a root of unity, an iterative  $q$ -difference equation is simply a  $q$ -difference equation in the sense of [25].

**Definition 4.3** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field, let  $(M, \delta_M^*)$  be an object of  $IDM_q(L)$ , and let  $\{\delta_L^{(kC)}(\mathbf{y}) = A_k \mathbf{y}\}_{k \in \mathbb{N}}$  be an **iterative  $q$ -difference equation** related to the  $IDM_q$ -module  $(M, \delta_M^*)$ , denoted by  $ID_q E(M)$ .

Let  $(R, \delta_R^*)$  be an iterative  $q$ -difference extension of  $(L, \delta_L^*)$ . A matrix  $Y \in Gl_n(R)$  is called a **fundamental solution matrix** for  $ID_q E(M)$  if  $\delta_R^{(kC)}(Y) = A_k Y$ , for all  $k \in \mathbb{N}$ .

The ring  $R$  is called an **iterative  $q$ -difference Picard-Vessiot ring** for  $ID_q E(M)$  ( $IPV_q$ -ring for short) if it fulfills the following conditions :

1.  $R$  is a simple  $ID_q$  ring (that means that  $R$  contains no proper iterative  $q$ -difference ideal ),
2.  $ID_q E(M)$  has a fundamental solution matrix  $Y$  with coefficients in  $R$ ,
3.  $R$  is generated by the coefficients of  $Y$  and  $\det(Y)^{-1}$ ,
4.  $C(R) = C(L)$ .

**Remark 4.4 (Classical case)** As in Remark 4.2, we easily see that if  $q$  is not a root of unity, the notion of an iterative Picard-Vessiot ring is exactly the same as the notion of Picard-Vessiot ring in the sense of Singer, van der Put ([25]).

**Proposition 4.5** Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field, with algebraically closed field of constants  $C(L)$ , and let  $R/L$  be a simple  $ID_q$ -ring. Then  $R$  is a reduced  $ID_q$ -ring. Moreover, if  $R$  is finitely generated over  $L$ , we have  $C(L) = C(E)$  where  $E$  denotes the localization of  $R$  by its set of non zeros divisors.

### Proof

The fact that  $R$  is a reduced  $ID_q$ -ring is a consequence of Lemma 2.17 where it is shown that if  $I$  is an  $ID_q$ -ideal the same is true for its radical. For the second statement, let us assume that  $R$  is finitely generated over  $L$ . Let  $c$  be a non zero constant of  $E$  and put  $J = \{a \in R | a.c \in R\}$ . First of all, because  $\delta_E^{(1)} = \frac{\sigma_q - id}{(q-1)t}$ , we have that  $\sigma_q^k(c) = c$  for all  $k \in \mathbb{N}$ . It is then quite clear that  $J$  is an  $ID_q$ -ideal of  $R$  because of  $\delta_R^{(k)}(a.c) = \sigma_q^k(c). \delta_R^{(k)}(a) = c. \delta_R^{(k)}(a)$  for all  $k \in \mathbb{N}$ . Since  $R$  is simple, and  $J$  is a non trivial  $ID_q$ -ideal, we have  $J = R$ , and thus  $1.c = c \in R$ . Suppose that  $c \notin C(L)$ . Thus for all  $d \in C(L)$  the ideal  $(c - d)R$  is a non trivial  $ID_q$ -ideal in  $R$  and also equal to  $R$ . This means that  $(c - d) \in R^*$  for all  $d \in C(L)$ . Let  $\phi_c : Spec(R) \mapsto \mathbb{A}_L^1$  be the morphism induced by

$$\phi : \quad L[T] \longrightarrow R, \quad T \longmapsto c.$$

Since  $Im(\phi_c) \cap \mathbb{A}_L^1(C(L))$  is empty,  $Im(\phi_c)$  does not contain any open subset of  $\mathbb{A}_L^1$ . Therefore the image of  $\phi_c$  in  $\mathbb{A}_L^1$  is finite and closed. This implies that  $c$  is algebraic over  $L$ . Let  $P \in L[X]$  be the minimal monic polynomial annihilating  $c$ . We have  $\delta_L^{(k)}(P(c)) = P^{\delta_L^{(k)}}(c) = 0$  where  $P^{\delta_L^{(k)}}$  denotes the element of  $L[X]$  obtained from  $P$  by applying  $\delta_L^{(k)}$  on the coefficients of  $P$ . By minimality of  $P$  we conclude that  $P \in C(L)[X]$ . Because  $C(L)$  is algebraically closed, we then have  $c \in C(L)$ . This is a contradiction!

**Proposition 4.6** *Let  $(L, \delta_L^*)$  be an  $ID_q$ -field and  $(R, \delta_R^*)$  be an  $ID_q$ -ring with  $q$ -difference operator extending the one given on  $L$ . Let  $Y$  and  $\tilde{Y}$  be two elements of  $Gl_n(R)$ , fundamental matrices of solutions for the  $ID_qE$ ,  $\delta_R^{(kC)}(\mathbf{y}) = A_k \mathbf{y}$ . Then, there exists a matrix  $P \in Gl_n(C(R))$  such that  $\tilde{Y} = YP$ . Moreover, if both  $L$  and  $R$  satisfy the conditions of Proposition 4.5 then  $P \in Gl_n(C(L))$ .*

**Proof**

It is obvious that there exists  $P \in Gl_n(R)$  such that  $\tilde{Y} = YP$ . We want to show by induction that for all  $k \in \mathbb{N}^*$ , we have  $\delta_R^{(k)}(P) = 0$ . For  $k = 1$  we obtain

$$\delta_R^{(1)}(\tilde{Y}) = \delta_R^{(1)}(Y)P + \sigma_q(Y)\delta_R^{(1)}(P) = A_1\tilde{Y} + \sigma_q(Y)\delta_R^{(1)}(P).$$

Thus,  $\delta_R^{(1)}(P) = 0$  (because  $\sigma_q$  is an automorphism of  $Gl_n(R)$ ). Using the formula

$$\delta_R^{(k)}(\tilde{Y}) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(Y))\delta_R^{(i)}(P),$$

we get by induction that  $\delta_R^{(k)}(P) = 0$  for all  $k \in \mathbb{N}^*$ . This implies that  $P \in Gl_n(C(R))$ .

**Theorem 4.7** *Let  $(L, \delta_L^*)$  be an iterative  $q$ -difference field with  $C(L)$  algebraically closed and let  $(M, \delta_M^*)$  be an object of  $IDM_q(L)$  with iterative  $q$ -difference equation  $\delta_L^{(kC)}(\mathbf{y}) = A_k \mathbf{y}$  ( $ID_qE(M)$ ). Then there exists an iterative  $q$ -difference Picard-Vessiot ring for the iterative  $q$ -difference equation which is unique up to iterative  $q$ -difference isomorphism.*

**Proof**

Let  $m$  be the dimension of  $M$  over  $L$  and set  $U = L[x_{(i,j)}, \det(x_{(i,j)})^{-1}]$ . The algebra  $U_0 := L[x_{(i,j)}]$  is given a structure of  $q$ -difference extension of  $L$  via  $\sigma_q(X) := \frac{A_1}{(q-1)t}X + X$  where  $X = (x_{(i,j)})_{(i,j)}$ . Because  $\sigma_q$  is a ring-automorphism, we have that the ideal  $S$  generated in  $U_0$  by  $\det(x_{i,j})$  is a  $\sigma_q$ -ideal and a multiplicatively closed set.  $U_0$  has a non trivial  $ID_q$ -structure via

$$\delta_{U_0}^* := \delta_P^{(kC)}(X) = A_k X, \text{ for all } k \in \mathbb{N}.$$

Because  $S$  satisfies the condition of Proposition 2.19, there exists a unique iterative  $q$ -difference operator  $\delta_{S^{-1}U_0}^*$  extending  $\delta_{U_0}^*$  on  $U = S^{-1}U_0$ . Let  $P \subset U$  be a maximal  $ID_q$ -ideal of  $U$ . Then  $R := U/P$  is a simple  $ID_q$ -ring and  $Y := \overline{X}$ , the image of  $X$  under the projection of  $U$  to  $R$ , is a fundamental solution matrix of  $ID_qE(M)$ . Moreover  $R/L$  is generated by the coefficients of  $Y$  and  $\det(Y)^{-1}$ . Thus  $R$  is an iterative  $q$ -difference Picard-Vessiot ring.

Assume that  $(R_1, \delta_{R_1}^*)$  and  $(R_2, \delta_{R_2}^*)$  are two iterative  $q$ -difference Picard-Vessiot rings for  $M$  with fundamental solution matrix  $Y_1$  (resp.  $Y_2$ ) in  $R_1$  (resp.  $R_2$ ). Put  $N = R_1 \otimes_L R_2$ .

As in Proposition 2.11 we endow  $N$  with an  $ID_q$ -structure. Let  $P \subset N$  be a maximal  $ID_q$ -ideal, then  $R' := N/P$  is a simple  $ID_q$  ring. The two maps :

$$\phi_1 : \quad R_1 \longrightarrow R', \quad r_1 \longmapsto \overline{(r_1 \otimes 1)}$$

and

$$\phi_2 : \quad R_2 \longrightarrow R', \quad r_2 \longmapsto \overline{(1 \otimes r_2)}.$$

induced by the natural inclusions are  $ID_q$ -monomorphisms, and  $\phi_1(Y_1)$  and  $\phi_2(Y_2)$  are two fundamental matrix solutions for  $M$  in  $R'$ . By Proposition 4.6, there exists  $P \in Gl_n(C(L))$  such that  $\phi_1(Y_1) = \phi_2(Y_2)P$  ( $C(L) = C(R_1) = C(R_2) = C(R')$ ), which implies that  $\phi_1(R_1) \simeq \phi_2(R_2)$ . This concludes the proof.

## 4.2 The iterative $q$ -difference Galois group

In this section, we will define the iterative  $q$ -difference Galois group associated to an iterative  $q$ -difference module. The way of describing such a group is the exact translation in the  $q$ -difference world of the work of A. Roescheisen (see [19]) in the case of iterative differential Galois theory. Until the end of this section,  $(L, \delta_L^*)$  will be an iterative  $q$ -difference field with algebraically closed field of constants  $C$ ,  $(R, \delta_R^*)$  an iterative  $q$ -difference Picard-Vessiot ring for the iterative  $q$ -difference equation  $\{\delta_L^{(kC)} Y = A_k Y, k \in \mathbb{N}\}$  defined over  $L$ .

**Notation 4.8** *Let  $S$  be a ring. We denote by  $Loc(S)$  its localization by its set of non-zero divisors.*

### 4.2.1 Functorial definition

First of all, let us remark that, given an algebra  $A$  over  $C$  and an iterative  $q$ -difference ring  $(S, \delta_S^*)$ , we define an iterative  $q$ -difference operator on  $S \otimes_C A$  by setting  $\delta_{S \otimes_C A}^{(k)}(s \otimes f) := \delta_S^{(k)}(s) \otimes f$  for all  $k \in \mathbb{N}$ . As in [19], we say that  $\delta_S^*$  is **extended trivially** to  $S \otimes_C A$ .

**Definition 4.9** *Let us define the functor*

$$\underline{Aut}(R/L) : \quad (Algebras/C) \longrightarrow (Groups), \quad A \longmapsto Aut_{ID_q}(R \otimes_C A / L \otimes_C A)$$

where  $\delta_R^*$  (resp.  $\delta_L^*$ ) is extended trivially to  $R \otimes_C A$  (resp.  $L \otimes_C A$ ).

In the following, we will show that the functor  $\underline{Aut}(R/L)$  is representable by a certain  $C$ -algebra of finite type and hence is an affine group-scheme of finite type over  $C$ .

**Lemma 4.10** *Let  $R$  be a simple  $ID_q$ -ring with  $C(R) = C$ , let  $A$  be a finitely generated  $C$ -algebra and  $R_A := R \otimes_C A$  with  $ID_q$ -structure trivially extended from  $R$ . Then there is a bijection*

$$\mathcal{I}(A) \longleftrightarrow \mathcal{I}_{ID_q}(R_A)$$

$$I \longmapsto R_A(1 \otimes_C I) = R \otimes_C I,$$

$$J \cap (1 \otimes_C A) \longleftarrow J$$

between the ideals of  $A$  and the  $ID_q$ -ideals of  $R_A$ .

**Proof**

Obviously, the two maps are well defined, and we only have to prove that they are inverse to each other.

1. We will prove that for  $I \in \mathcal{I}(A)$ , we have  $(R \otimes_C I) \cap (1 \otimes_C A) = I$ . It is obvious that  $I$  is contained in the ideal on the left side. Now let us consider a  $C$ -basis  $\{e_i | i \in \tilde{N}\}$  of  $I$ ; then  $R \otimes_C I$  is a free  $R$ -module with basis  $\{1 \otimes e_i | i \in \tilde{N}\}$  and an element  $f = \sum_{i \in \tilde{N}} r_i \otimes e_i \in R \otimes_C I$  is constant if and only if all the  $r_i$ 's are constants, i.e., if  $f \in I$ .
2. Conversely we have to prove that for  $J \in \mathcal{I}_{ID_q}(R_A)$ , we have  $R \otimes_C J \cap (1 \otimes_C A) = J$ . It is clear that  $J$  contains the ideal on the left side. Now, let  $\{e_i | i \in N\}$  a  $C$ -basis of  $A$ , where  $N$  denotes an index set. Then,  $\{1 \otimes e_i | i \in N\}$  is also a basis for the free  $R$ -module  $R_A$ .

For any subset  $N_0$  of  $N$  and  $i_0 \in N_0$ , let  $Ann_{N_0, i_0}$  be the ideal of all  $r \in R$  such that there exists an element  $g = \sum_{i \in N_0} s_i \otimes e_i \in J$  with  $s_{i_0} = r$ . Since the iterative  $q$ -difference operator of  $R_A$  acts trivially on  $A$  and  $J$  is an  $ID_q$ -ideal, it is clear that  $Ann_{N_0, i_0}$  is an  $ID_q$ -ideal. Because  $R$  is simple,  $Ann_{N_0, i_0}$  is equal to  $(0)$  or  $R$ .

Now, let  $N_0 \subset N$  be minimal for the property that  $Ann_{N_0, i_0} \neq (0)$  for at least one index  $i_0 \in N_0$  (minimal in the lattice of subsets). So there exists  $g = \sum_{i \in N_0} s_i \otimes e_i \in J$  with  $s_{i_0} = 1$  and by minimality of  $N_0$  we conclude that for all  $k \in \mathbb{N}^*$ ,  $\delta^{(k)}(g) = \sum_{i \in N_0, i \neq i_0} \delta_R^{(k)}(s_i) \otimes e_i = 0$ . This implies  $g \in J \cap (1 \otimes_C A)$ . Now let  $g = \sum_{i \in N} s_i \otimes e_i \in J$  be an arbitrary element and denote by  $N_1$  the set of indices  $i$  with  $s_i \neq 0$ . It follows from the definition that  $Ann_{N_1, i} \neq (0)$  for all  $i \in N_1$ . Hence there exists  $N_0 \subset N_1$  minimal as above,  $i_0 \in N_0$  and  $f = \sum_{i \in N_0} r_i \otimes e_i \in J \cap (1 \otimes_C A)$  with  $r_{i_0} = 1$ . By induction on the cardinality of  $N_1$ , we may assume that  $g - s_{i_0} f \in R \otimes_C J \cap (1 \otimes_C A) \subset J$ . Therefore  $g = g - s_{i_0} f + s_{i_0} f \in R \otimes_C J \cap (1 \otimes_C A)$  and hence  $R \otimes_C J \cap (1 \otimes_C A) = J$ .

**Proposition 4.11** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring associated to an iterative  $q$ -difference equation and let  $T$  be a  $ID_q$ -simple ring containing  $L$  with  $C(T) = C = C(L)$  such that there exists a fundamental matrix of solutions  $Y \in Gl_n(T)$ . Then*

there exists a finitely generated  $C$ -algebra  $U$  (with trivial  $ID_q$ -structure) and a  $T$ -linear  $ID_q$ -isomorphism

$$\gamma_T : T \otimes_L R \longrightarrow T \otimes_C U.$$

where the  $ID_q$ -structure is extended trivially to  $T \otimes_C U$ .  
(Actually  $U$  is isomorphic to the ring of constants of  $T \otimes_L R$ .)

**Proof**

$R$  is obtained as a quotient of  $L[X_{i,j}, (\det(X))^{-1}]$  with iterative  $q$ -difference operator given by  $\delta^{(k)}(X) = A_k X$  for all  $k \in \mathbb{N}$  by a maximal  $ID_q$ -ideal  $P \subset L[X_{i,j}, (\det(X))^{-1}]$ . We then define a  $T$ -linear homomorphism

$$\gamma_T : T \otimes_L L[X_{i,j}, \det(X)^{-1}] \longrightarrow T \otimes_C C[Z_{i,j}, \det(Z)^{-1}]$$

by  $X_{i,j} \mapsto \sum_{k=1}^n Y_{i,k} \otimes Z_{k,j}$ . The morphism  $\gamma_T$  is indeed a  $T$ -linear isomorphism and if we extend the  $ID_q$ -structure trivially to  $L[Z_{i,j}, (\det(Z))^{-1}]$ ,  $\gamma_T$  induces an  $ID_q$ -isomorphism. By the previous lemma, the  $ID_q$ -ideal  $\gamma_T(T \otimes P)$  is equal to  $T \otimes I$  for an ideal  $I \subset C[Z_{i,j}, (\det(Z))^{-1}]$ . So for  $U := C[Z_{i,j}, (\det(Z))^{-1}]/I$ ,  $\gamma_T$  induces an  $ID_q$ -isomorphism

$$\gamma_T : T \otimes_L R \longrightarrow T \otimes_C U.$$

**Theorem 4.12** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring. Then the group functor  $\underline{\text{Aut}}(R/L)$  is representable by the finitely generated  $C$ -algebra  $U = C(R \otimes_L R)$ , i.e.,  $\underline{\text{Aut}}(R/L)$  is an affine group-scheme of finite type over  $C$ .*

**Definition 4.13** *We call the affine group scheme  $\underline{\text{Aut}}(R/L)$  the **Galois group scheme**  $\underline{\text{Gal}}(R/L)$  of  $R$  over  $L$ .*

**Proof of theorem 4.12**

First we will show that for every  $C$ -algebra  $A$  any  $L_A$ -linear  $ID_q$ -homomorphism

$f : R_A \longrightarrow R_A$  is an isomorphism. The kernel of such a homomorphism  $f$  is an  $ID_q$ -ideal of  $R_A$ . So by Lemma 4.10, it is generated by constants, i.e., elements in  $1 \otimes A$ . But  $f$  is  $A$ -linear so its kernel is zero. If  $X \in Gl_n(R)$  is a fundamental solution matrix, then  $f(X) \in Gl_n(R_A)$  is also a fundamental solution matrix and so there exists a matrix  $D \in Gl_n(C_{R_A}) = Gl_n(A)$  such that  $X = f(X)D = f(XD)$ . Hence  $X_{i,j}, \det(X)^{-1} \in \text{Im}(f)$  and since  $R$  is generated by  $X_{i,j}, \det(X)^{-1}$  over  $L$ , the homomorphism  $f$  is also surjective. Using the isomorphism  $\gamma := \gamma_R$  of Proposition 4.11, for a  $C$ -algebra  $A$ , we obtain a chain of isomorphisms

$$\begin{aligned} \text{Aut}^{ID_q}(R_A/L_A) &= \text{Hom}_{L_A}^{ID_q}(R_A, R_A) \simeq \text{Hom}_{R_A}^{ID_q}(R_A \otimes_L R, R_A) \\ &\simeq \text{Hom}_{R_A}^{ID_q}(R_A \otimes_C U, R_A) \simeq \text{Hom}_C^{ID_q}(U, R_A) \simeq \text{Hom}_C(U, A). \end{aligned}$$

Hence  $U$  represents the functor  $\underline{\text{Aut}}(R/L)$ .

**Remark 4.14** *By taking a closer look on the isomorphisms in the previous proof, we see that the universal object  $id_U \in Hom_C(U, U)$  gives birth to the  $ID_q$ -automorphism  $\rho \otimes id_U : R \otimes_C U \longrightarrow R \otimes_C U$  where  $\rho = \gamma_R \circ (1 \otimes id_R) : R \longrightarrow R \otimes_L R \longrightarrow R \otimes_C U$ .*

**Corollary 4.15** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring over  $L$  and  $\mathcal{G} := \underline{Gal}(R/L)$  the Galois group scheme of  $R$ . Then  $Spec(R)$  is a  $\mathcal{G}_L$ -torsor.*

**Proof**

The isomorphism  $\gamma := \gamma_R$  of proposition 4.11, determines an isomorphism of schemes

$$Spec(\gamma) : Spec(R) \times_L \mathcal{G}_L = Spec(R) \times_C \mathcal{G} \longrightarrow Spec(R) \times_L Spec(R).$$

By the previous remark and  $R$ -linearity of  $\gamma$ , the composition of  $Spec(\gamma)$  with the projection on the second factor gives the action of  $\mathcal{G}_L$  on  $Spec(R)$  and the composition with the projection on the first factor equals the map  $Spec(R) \times_L \mathcal{G}_L \rightarrow Spec(R)$ . In other words,  $Spec(R)$  is a  $\mathcal{G}_L$ -torsor.

### 4.2.2 Galois correspondence

**Proposition 4.16 (Structure of the iterative  $q$ -difference ring)** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring over  $L$ . Then, there exist idempotents  $e_1, \dots, e_s \in R$  such that*

1.  $R = R_1 \oplus \dots \oplus R_s$  where  $R_i = e_i R$  and is a domain,
2. The direct sum  $E$  of the fraction fields of the  $R_i$ 's is an iterative  $q$ -difference ring.  $E$  is called the total iterative  $q$ -difference Picard-Vessiot extension of  $R$ .

**Proof**

Here, we give a partial analogue of Corollary 1.16 of [25]. We will thus follow the proof of Singer, van der Put. But because we work in any characteristic, it will be necessary to appeal to the book of Demazure, Gabriel ([7]) to assure smoothness.

Let  $\bar{L}$  be an algebraic closure of  $L$  and  $R = O(\mathcal{Z})$  for some  $\mathcal{G}_L$ -torsor  $\mathcal{Z}$ . Since  $\mathcal{G}_L(\bar{L})$  acts transitively on  $\mathcal{Z}(\bar{L})$ , this latter algebraic subset must be smooth ([7]). Therefore the  $L$ -irreducible components  $\mathcal{Z}_1, \dots, \mathcal{Z}_s$  must be disjoint. Thus  $O(\mathcal{Z})$  is equal to the product of the integral domains  $R_i = O(\mathcal{Z}_i)$ . Now let us consider the set  $S$  of non zero divisors in  $R$ . It is a multiplicatively closed set which does not contain 0, stable under the action of  $\sigma_q$ . By Proposition 2.19, the ring  $RS^{-1}$  is endowed with an iterative  $q$ -difference structure and it is obvious that  $RS^{-1} = \bigoplus_{i=1}^s Frac(R_i)$  where  $Frac(R_i)$  denotes the fraction field of  $R_i$ .

The next proposition shows that to be a torsor for an  $ID_q$ -simple ring means, roughly speaking, to be an iterative  $q$ -difference Picard-Vessiot ring.

**Proposition 4.17** *Let  $R/L$  be a simple  $ID_q$ -ring with algebraically closed field of constants  $C(R) = C$ . Further let  $\mathcal{G} \subset Gl_{n,C}$  be an affine group scheme over  $C$ . Assume that  $Spec(R)$  is a  $\mathcal{G}_L$ -torsor such that the corresponding isomorphism  $\gamma : R \otimes_L R \rightarrow R \otimes_C C[\mathcal{G}]$  is an  $ID_q$ -isomorphism. Then  $R$  is an iterative  $q$ -difference Picard-Vessiot ring over  $L$ .*

**Proof**

Since  $Spec(R)$  is a  $\mathcal{G}_L$ -torsor, the fiber product  $Spec(R) \times_{\mathcal{G}_L} Gl_{n,L}$  is a  $Gl_{n,L}$ -torsor. ( $Spec(R) \times_{\mathcal{G}_L} Gl_{n,L}$  is obtained as the quotient of the direct product by the  $\mathcal{G}_L$ -action given by  $(x, h).g := (xg, g^{-1}h)$  and is a right  $Gl_{n,L}$ -scheme acting on the second factor.) By Hilbert's Theorem 90, every  $Gl_{n,L}$ -torsor is trivial, i.e., we have an  $Gl_{n,L}$ -equivariant isomorphism

$$Spec(R) \times_{\mathcal{G}_L} Gl_{n,L} \longrightarrow Gl_{n,L}.$$

Then the closed embedding  $Spec(R) \longrightarrow Spec(R) \times_{\mathcal{G}_L} Gl_{n,L} \longrightarrow Gl_{n,L}$  leads to an epimorphism  $L[X_{i,j}, (\det(X))^{-1}] \longrightarrow R$ , which is  $\mathcal{G}_L$ -equivariant. Denote the image of  $X$  by  $Y$ . Then we obtain that the action of  $\mathcal{G}$  on  $Y$  is given by  $Y \mapsto Yg$  for any  $L$ -valued point  $g \in \mathcal{G}_L(L)$ . Since by assumption for every  $C$ -algebra  $A$  with trivial  $ID_q$ -structure, the action of  $\mathcal{G}(A)$  commutes with the iterative  $q$ -difference operator  $\delta^{(k)}(Y).Y^{-1}$  is  $\mathcal{G}$ -invariant for all  $k \in \mathbb{N}$ . So  $\delta^{(k)}(Y).Y^{-1} = A_k$  belongs to  $Gl_n(L)$  and  $Y$  is a fundamental solution matrix for the equation  $\{\delta^{(k)}(Y).Y^{-1}\}_{k \in \mathbb{N}}$ . Hence  $R$  is an  $ID_q$ -Picard-Vessiot ring.

In order to get a convenient Galois correspondence, we are obliged to define the notion of an invariant in a functorial way. Let  $S$  be a  $C$ -algebra and  $\mathcal{H}/C$  be a subgroup functor of the functor  $\underline{Aut}(S/C)$ , i.e., for every  $C$ -algebra  $A$ , the set  $\mathcal{H}(A)$  is a group acting on  $S_A$  and this action is functorial. An element  $s \in S$  is called invariant if for all  $A$ , the element  $s \otimes 1 \in S_A$  is invariant under  $\mathcal{H}(A)$ . The ring of invariants is denoted by  $S^{\mathcal{H}}$ . Let  $E = Loc(S)$  be the localization of  $S$  by all non zero-divisors. We call an element  $e = \frac{r}{s} \in E$  invariant under  $\mathcal{H}$ , if for each  $C$ -algebra  $A$  and all  $h \in \mathcal{H}(A)$ ,

$$h.(r \otimes 1).(s \otimes 1) = (r \otimes 1).h.(s \otimes 1).$$

$E^{\mathcal{H}}$  denotes the ring of invariants (for the independence of this definition of the choice of representation of  $e$  see [19]).

**Lemma 4.18** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring over  $L$ , let  $E$  denote its total iterative  $q$ -difference Picard-Vessiot extension and  $\mathcal{G} := \underline{Gal}(R/L)$  the Galois group scheme of  $R$ . Let  $\mathcal{H} \subset \mathcal{G}$  be a closed subgroup-scheme. Denote by  $\pi_{\mathcal{H}}^{\mathcal{G}} : C[\mathcal{G}] \longrightarrow C[\mathcal{H}]$  the epimorphism corresponding to the inclusion  $\mathcal{H} \hookrightarrow \mathcal{G}$ . Then an element of  $\frac{r}{s} \in E$  is invariant under the action of  $\mathcal{H}$  if and only if  $r \otimes s - s \otimes r$  is in the kernel of the map*

$$(id_R \otimes \pi_{\mathcal{H}}^{\mathcal{G}}) \circ \gamma : R \otimes_L R \longrightarrow R \otimes_C C[\mathcal{H}].$$

**Proof**

The following proposition is a special case of the Galois correspondence stated in Theorem 4.20.

**Proposition 4.19** *For every closed subgroup scheme  $\mathcal{H} \subset \mathcal{G}$ , the ring  $E^{\mathcal{H}}$  is an  $ID_q$ -ring in which every non zero divisor is a unit. Furthermore we have  $E^{\mathcal{H}} = L$  if and only if  $\mathcal{H} = \mathcal{G}$ .*

**Proof**

By the previous lemma, it is obvious that  $E^{\mathcal{H}}$  is an  $ID_q$ -ring in which every non-zero divisor is a unit. Next, let  $\frac{r}{s} \in E^{\mathcal{H}}$ . Then for all  $k \in \mathbb{N}$ , we have

$$\begin{aligned} & \delta^k(r \otimes s - s \otimes r) \cdot (s^k \otimes s^k) = \\ & \sum_{i_1+i_2+i_3=k} \sigma_q^{i_1+i_3}(\delta^{(i_2)}(\frac{r}{s}))s^k \sigma_q^{i_3}(\delta^{(i_1)}(s)) \otimes \delta^{(i_3)}(s)s^k - \delta^{(i_1)}(s)s^k \otimes \sigma_q^{i_1+i_3}(\delta^{(i_2)}(\frac{r}{s}))s^k \sigma_q^{i_1}(\delta^{(i_3)}(s)) \\ & = \sum_{i_1+i_2+i_3=k} (\sigma_q^{i_3}(\delta^{(i_1)}(s)) \otimes \delta^{(i_3)}(s))(\sigma_q^{i_1+i_3}(\delta^{(i_2)}(\frac{r}{s}))s^k \otimes s^k) - \\ & \quad \sum_{i_1+i_2+i_3=k} (\delta^{(i_1)}(s) \otimes \sigma_q^{i_1}(\delta^{(i_3)}(s)))(s^k \otimes \sigma_q^{i_1+i_3}(\delta^{(i_2)}(\frac{r}{s}))s^k) = \\ & \quad \sum_{i+j=k} (\delta^{(i)}(s \otimes s))(\sigma_q^i(\delta^{(j)}(\frac{r}{s}))s^k \otimes s^k - s^k \otimes \sigma_q^i(\delta^{(j)}(\frac{r}{s}))s^k). \end{aligned}$$

The left hand side lies in  $Ker(id_R \otimes \pi_{\mathcal{H}}^{\mathcal{G}})$ , since this kernel is an  $ID_q$ -ideal. So by induction, we get that  $(s \otimes s)(\delta^{(k)}(\frac{r}{s})s^k \otimes s^k - s^k \otimes \delta^{(k)}(\frac{r}{s})s^k) \in Ker(id_R \otimes \pi_{\mathcal{H}}^{\mathcal{G}})$  and hence  $\delta^{(k)}(\frac{r}{s}) \in E^{\mathcal{H}}$ . For the second statement : if  $\mathcal{H} = \mathcal{G}$ , then  $\pi_{\mathcal{H}}^{\mathcal{G}} = id_{C[\mathcal{G}]}$  and the considered kernel is trivial. Hence  $r \otimes s = s \otimes r \in R \otimes_L R$  is trivial for all  $\frac{r}{s} \in E^{\mathcal{G}}$ . Thus, there exists  $c \in L$  such that  $r = cs$ , i.e.,  $\frac{r}{s} = c \in L$ .

Assume  $\mathcal{H} \subsetneq \mathcal{G}$ . Since  $\mathcal{Z} = Spec(R)$  is a  $\mathcal{G}_L$ -torsor, the quotient scheme  $\mathcal{Z}/\mathcal{G}_L$  is equal to  $Spec(L)$ , in particular it is a scheme, and since  $\mathcal{G}_L$  and  $\mathcal{H}_L$  are affine,  $\mathcal{G}_L/\mathcal{H}_L$  also is a scheme. So by [13], I.5.16.(1),  $\mathcal{Z}/\mathcal{H}_L \simeq \mathcal{Z} \times^{\mathcal{G}_L} (\mathcal{G}_L/\mathcal{H}_L)$  is a scheme. According to Proposition 4.16,  $\mathcal{Z}$  is equal to the disjoint union of its irreducible components  $\{\mathcal{Z}_i\}_{i=1, \dots, s}$ . Let  $pr : \mathcal{Z} \rightarrow \mathcal{Z}/\mathcal{H}_L$  denote the canonical projection. Now let  $\bar{U} \subseteq \mathcal{Z}/\mathcal{H}_L$  be an affine open subset such that its inverse image  $\mathcal{U}$  by  $pr$  has a non empty intersection with all the  $\mathcal{Z}_i$ . We have a monomorphism  $pr_* : \mathcal{O}_{\mathcal{Z}/\mathcal{H}_L}(\bar{U}) \rightarrow \mathcal{O}_{\mathcal{Z}}(\mathcal{U})$  whose image is  $\mathcal{O}_{\mathcal{Z}}(\mathcal{U})^{\mathcal{H}}$ . By construction of  $\bar{U}$ , we have  $\mathcal{O}_{\mathcal{Z}}(\mathcal{U})^{\mathcal{H}} \subset E^{\mathcal{H}}$ . If  $E^{\mathcal{H}} = L$ , then also  $\mathcal{O}_{\mathcal{Z}}(\mathcal{U})^{\mathcal{H}} = L$ . So, for every affine open subset  $\bar{U} \subseteq \mathcal{Z}/\mathcal{H}_L$  such that its inverse image  $\mathcal{U}$  by  $pr$  has a non empty intersection with all the  $\mathcal{Z}_i$ , we have  $\mathcal{O}_{\mathcal{Z}/\mathcal{H}_L}(\bar{U}) = L$ , i.e.,  $\bar{U} \simeq Spec(L)$  is a single point. Hence  $\mathcal{Z}/\mathcal{H}_L = Spec(L)$ , which contradicts the assumption  $\mathcal{H} \subsetneq \mathcal{G}$ .

**Theorem 4.20 (Galois correspondence)** *Let  $R/L$  be an iterative  $q$ -difference Picard-Vessiot ring over  $L$ , let  $E$  denotes its total iterative  $q$ -difference Picard-Vessiot extension and let  $\mathcal{G} := \underline{Gal}(R/L)$  be the Galois group scheme of  $R$ .*

1. Then there is an antiisomorphism of lattices between:

$$\mathfrak{H} := \{\mathcal{H} | \mathcal{H} \subset \mathcal{G} \text{ closed subgroup scheme of } \mathcal{G}\}$$

and

$$\mathfrak{T} := \{T | L \subset T \subset E \text{ intermediate } ID_q\text{-ring s.t. any non zero divisor of } T \text{ is a unit of } T\}$$

given by  $\Psi : \mathfrak{H} \rightarrow \mathfrak{T}, \mathcal{H} \mapsto E^{\mathcal{H}}$  and  $\Phi : \mathfrak{T} \rightarrow \mathfrak{H}, T \mapsto \underline{Gal}(RT/T)$ .

2. If  $\mathcal{H} \subset \mathcal{G}$  is normal then  $R^{\mathcal{H}}$  is an iterative  $q$ -difference Picard-Vessiot ring over  $L$  and  $E^{\mathcal{H}}$  is its total iterative  $q$ -difference Picard-Vessiot extension; the Galois group scheme of  $R^{\mathcal{H}}$  over  $L$  is isomorphic to  $\mathcal{G}/\mathcal{H}$ .

3. For  $\mathcal{H} \in \mathfrak{H}$ , the extension  $E/E^{\mathcal{H}}$  is separable if and only if  $\mathcal{H}$  is reduced.

### Proof

1. Let  $T \in \mathfrak{T}$  be an intermediate  $ID_q$  ring such that any non zero divisor of  $T$  is a unit of  $T$ . Then the compositum  $RT \subset E$  is a  $ID_q$ -Picard-Vessiot ring over  $T$ . Furthermore, the canonical  $ID_q$ -epimorphism  $RT \otimes_C C[\mathcal{G}] \mapsto RT \otimes_T RT$  gives rise to an  $ID_q$ -epimorphism

$$RT \otimes_C C[\mathcal{G}] \xrightarrow{\gamma_{RT}^{-1}} RT \otimes LR \longrightarrow RT \otimes_T RT .$$

By Lemma 4.10, the kernel of this epimorphism is given by  $RT \otimes_C I$  for some ideal  $I \subset C[\mathcal{G}]$ . Denote by  $\mathcal{H}$  the closed sub-scheme of  $\mathcal{G}$  defined by  $I$ , then  $\gamma_{RT}$  induces a isomorphism

$$RT \otimes_T RT \simeq RT \otimes_C C[\mathcal{H}].$$

By construction, this isomorphism is the isomorphism for the base ring  $T$ , hence the sub-scheme  $\mathcal{H}$  equals the Galois group scheme  $\underline{Gal}(RT/T)$ . Thus  $\underline{Gal}(RT/T)$  is indeed a closed subgroup scheme of  $\mathcal{G}$ .

Now let us apply Proposition 4.19 to the extension  $E/T$ . It follows that  $E^{\underline{Gal}(RT/T)} = T$ , so  $\Psi \circ \Phi = id_{\mathfrak{T}}$ . On the other hand, for given  $\mathcal{H} \in \mathfrak{H}$  and  $T := E^{\mathcal{H}}$ , we get an  $ID_q$ -epimorphism  $RT \otimes_T RT \mapsto RT \otimes_C C[\mathcal{H}]$  induced by  $\gamma_{RT}$ . This embeds  $\mathcal{H}$  as a closed subgroup scheme in  $\underline{Gal}(RT/T)$ . But the localization  $Loc(RT)$  of  $RT$  by its set of non zero divisors is equal to  $E$ , so  $Loc(RT)^{\mathcal{H}} = E^{\mathcal{H}} = T$  and so by Proposition 4.19, we have  $\mathcal{H} = \underline{Gal}(RT/T)$ . Thereby  $\Phi \circ \Psi = id_{\mathfrak{H}}$ .

2. Let  $\mathcal{H} \subset \mathcal{G}$  be normal. The isomorphism  $\gamma$  is  $\mathcal{H}$ -equivariant and hence we get an  $ID_q$ -isomorphism

$$R \otimes_L R^{\mathcal{H}} \simeq R \otimes_C C[\mathcal{G}/\mathcal{H}].$$

Since  $R$  is normal,  $\mathcal{G}/\mathcal{H}$  is an affine group scheme with  $C[\mathcal{G}/\mathcal{H}] = C[\mathcal{G}]^{\mathcal{H}}$  ([7], III, Sec. 3, Thm. 5.6). Again by taking invariants the isomorphism above restricts to an isomorphism

$$R^{\mathcal{H}} \otimes_L R^{\mathcal{H}} \simeq R^{\mathcal{H}} \otimes_C C[\mathcal{G}/\mathcal{H}].$$

The ring  $R^{\mathcal{H}}$  is  $ID_q$ -simple, because for every  $ID_q$ -ideal  $P \subset R^{\mathcal{H}}$ , the ideal  $P.R \subset R$  is an  $ID_q$ -ideal, hence equals  $(0)$  or  $R$  and so  $P = (P.R)^{\mathcal{H}}$  is  $(0)$  or  $R^{\mathcal{H}}$ . Since  $L \subset R^{\mathcal{H}} \subset R$ , we also have  $C(R^{\mathcal{H}}) = C$ . So by proposition 4.17,  $R^{\mathcal{H}}$  is an  $ID_q$  Picard-Vessiot ring over  $L$  with Galois group scheme  $\mathcal{G}/\mathcal{H}$ . It remains to show that  $E^{\mathcal{H}} = \text{Loc}(R^{\mathcal{H}})$ .

Let  $\tilde{L} := \text{Loc}(R^{\mathcal{H}})$  and  $\tilde{\mathcal{G}} := \underline{\text{Gal}}(E/\tilde{L})$ . Then  $\mathcal{H}$  is a normal subgroup of  $\tilde{\mathcal{G}}$  and by the previous  $(R.\tilde{L})^{\mathcal{H}}$  is a  $\tilde{\mathcal{G}}/\mathcal{H}$ -torsor. But  $(R.\tilde{L})^{\mathcal{H}} = R^{\mathcal{H}}.\tilde{L} = \tilde{L}$ , so  $\tilde{\mathcal{G}} = \mathcal{H}$ , and hence  $E^{\mathcal{H}} = E^{\tilde{\mathcal{G}}} = \tilde{L} = \text{Loc}(R^{\mathcal{H}})$ .

3. Without loss of generality we may assume that  $\mathcal{H} = \mathcal{G}$ . Let us denote by  $\mathcal{G}_{red} \subset \mathcal{G}$  the closed reduced subgroup given by the nilradical ideal. Since  $\mathcal{G}_{red}$  is normal in  $\mathcal{G}$ , by the second statement  $\tilde{L} := \text{Loc}(R^{\mathcal{G}_{red}})$  is an  $ID_q$  Picard-Vessiot extension of  $L$  with Galois group scheme  $\underline{\text{Gal}}(\tilde{L}/L) = \mathcal{G}_{red}$ . But this group scheme is infinitesimal and so by [4], Cor. 1.12,  $\tilde{L}/L$  is purely inseparable. On the other hand, if  $E/L$  is inseparable and  $p = \text{char}(L)$ , then  $\tilde{L} := E \cap L^{\frac{1}{p}} \neq L$  is a finitely purely inseparable  $ID_q$ -ring extension of  $L$ . Since every such extension is an  $ID_q$ -Picard-Vessiot ring with an infinitesimal Galois group scheme,  $\mathcal{G}$  has a non reduced quotient and therefore  $\mathcal{G}$  is not reduced.

### 4.2.3 Examples of Galois groups

**The Galois group  $\mathbb{G}_m$  in characteristic  $p$**  Let us denote by  $C = \overline{\mathbb{F}_p}$  the algebraic closure of  $\mathbb{F}_p$ , where  $p$  is a prime number. Let  $F = C(t)$  be a rational function field with coefficients in  $C$ . Let  $(a_l)_{l \geq 0}$  be a set of elements in  $\mathbb{F}_p$ . Let  $M = Fb_1$  be the  $ID_q$ -module with corresponding  $ID_qE$ :

$$\delta_M^{(np^k)}(y) = \frac{a_k}{t^{np^k}} y$$

where  $k \in \mathbb{N}$  and

$$\delta_M^{(1)}(y) = \frac{y}{t}.$$

**Theorem 4.21** *Let  $M$  be as above with its associated  $ID_qE$ , and let  $\alpha = \sum_{l \geq 0} a_l p^l \in \mathbb{Q}_p$ . Then for an iterative Picard-Vessiot extension  $E/F$  for  $M$ , we have*

*$\underline{\text{Gal}}(E/F) \simeq \mathbb{Z}/m\mathbb{Z}$  for some  $m$  if  $\alpha \in \mathbb{Q}$  and  $\underline{\text{Gal}}(E/F) \simeq \mathbb{G}_m$  if  $\alpha \notin \mathbb{Q}$ .*

**Proof**

First of all, let us show that  $\underline{\text{Gal}}(E/F)$  is a subgroup of  $\mathbb{G}_m$ . Let  $y$  be a solution of the  $ID_q E$  associated to  $M$ , then  $E = F(y)$ . Let  $\tau \in \underline{\text{Gal}}(E/F)$  and  $l \in \mathbb{N}$ , we have  $\delta^{(np^l)}(\frac{\tau(y)}{y}) = 0$  and  $\delta^{(1)}(\frac{\tau(y)}{y}) = 0$ . Thus, there exist  $c \in C^*$  such that  $\tau(y) = cy$ . Therefore,  $\underline{\text{Gal}}(E/F) \subseteq \mathbb{G}_m$ .

Let us assume that  $\alpha = \frac{a}{m}$  where  $(a, m) \in \mathbb{Z} \times \mathbb{N}^*$ . Put  $z = t^{a/m}$ . Because  $z = t^\alpha$ , we have  $\delta^{(j)}(z) = 0$  if  $j \neq n^k$ . We have

$$\delta^{(np^k)}(z^m) = \sum_{i_1 + \dots + i_m = np^k} \sigma_q^{i_2 + \dots + i_m}(\delta^{(i_1)}(z)) \dots \sigma_q^{i_m}(\delta^{(i_{m-1})}(z)) \delta^{(i_m)}(z).$$

If one of the  $i_j$  is not equal to  $np^k$ , there exists  $i_l$  such that  $i_l \neq p^j$  for  $j \leq np^k$ . Then, an easy computation shows that for all  $k \in \mathbb{N}$ ,

$$\delta^{(np^k)}(z^m) = mz^{m-1} \delta^{(np^k)}(z).$$

It follows that

$$mz^{m-1} \delta^{(np^k)}(z) = \binom{a}{np^k}_q t^{a-np^k}.$$

By Proposition 2.2, we have  $\binom{a}{np^k}_q = ma_k$  and thus  $\delta_M^{(np^k)}(z) = \frac{a_k}{t^{np^k}} z$ . Because  $E = F(z)$  and  $z^m \in F$ , we get that  $\underline{\text{Gal}}(E/F)$  is a cyclic group.

Conversely, suppose that  $y$  is an algebraic solution of the  $ID_q E$  associated to  $M$ , then  $E = F(y)$  is algebraic over  $F$  and  $\underline{\text{Gal}}(E/F)(C) \subsetneq \mathbb{G}_m(C)$  is a cyclic group of order  $m$ . So there exist  $s \in \mathbb{Z}$  and  $(b_i)_{i \geq s}$  with  $b_s = 1$  such that  $y^m = \sum_{i \geq s} b_i t^i \in F$ . Thus,

$$my^{m-1} \delta^{(n)}(y) = y^m \frac{a_0}{t^n} = \delta^{(n)}(y^m) = \sum_{i \geq s} b_i \binom{i}{n}_q t^{i-n}.$$

By comparing the coefficient of  $t^l$ , we obtain

$$ma_0 = b_i \binom{i}{n}_q \text{ for all } i \geq s.$$

Since  $b_s = 1$  and because of the properties of  $q$ -binomials coefficients, we obtain

1.  $s = k_s n$  with  $k_s \in \mathbb{Z}$  and  $a_0 = \frac{k_s}{m}$ ,
2.  $b_i = 0$  for all  $i \not\equiv 0 \pmod n$ .

Induction using the higher iterative differences shows that  $b_i = 0$  for all  $i > s$  and hence that  $y^m = t^s$ . By an argument used in the first part of the proof it follows that  $\alpha = \frac{s}{m}$ .

**The Galois group  $\mathbb{G}_m$  in characteristic 0** Let  $L = \mathbb{C}(t)$  and let  $q$  be a  $n$ -th primitive root of unity. Let  $M = Fb_1$  be a rank one  $IDM_q(L)$ -module and suppose that  $\Phi(b_1) = b_1$ . Let  $a \in \mathbb{C}$ . Then, let us consider the  $ID_qE$  associated to  $M$ , that is  $\delta^{(1)}(\mathbf{y}) = 0$  and  $\delta^{(n)}(\mathbf{y}) = \frac{a}{nt^n}\mathbf{y}$ .

**Theorem 4.22** *Let  $M$  be as above with its associated  $ID_qE$ . Then for an iterative Picard-Vessiot extension  $E/F$  for  $M$ , we have*

*$\underline{\text{Gal}}(E/F)$  is finite cyclic if  $a \in \mathbb{Q}$  and  $\underline{\text{Gal}}(E/F) \simeq \mathbb{G}_m$  if  $a \notin \mathbb{Q}$ .*

**Proof**

First of all, let us show that  $\underline{\text{Gal}}(E/F)$  is a subgroup of  $\mathbb{G}_m$ . Let  $y$  be a solution of the  $ID_qE$  associated to  $M$ , then  $E = F(y)$ . Let  $\tau \in \underline{\text{Gal}}(E/F)$ . Then, we have

1.

$$\delta^{(1)}\left(\frac{\tau(y)}{y}\right) = \sigma_q\left(\frac{1}{y}\right)\tau(\delta^{(1)}y) + \delta^{(1)}\left(\frac{1}{y}\right)\tau(y) = 0, \quad (\delta^{(1)}(y) = 0),$$

2.

$$\delta^{(n)}\left(\frac{\tau(y)}{y}\right) = \left(\frac{1}{y}\right)\tau(\delta^{(n)}y) + \delta^{(n)}\left(\frac{1}{y}\right)\tau(y) = -\frac{a}{nt^n}\frac{\tau(y)}{y} + \frac{1}{y}\tau\left(\frac{a}{nt^n}y\right) = 0$$

Thus, there exist  $c \in C^*$  such that  $\tau(y) = cy$ . Therefore,  $\underline{\text{Gal}}(E/F) \leq \mathbb{G}_m$ .

Let us assume that  $a = \frac{nb}{m}$  where  $(b, m) \in \mathbb{Z} \times \mathbb{N}^*$ . Put  $z = t^{nb/m}$ . Because  $z = t^a$ , we have  $\delta^{(j)}(z) = 0$  if  $j \notin n\mathbb{N}$ . We have

$$\delta^{(n)}(z^m) = \sum_{i_1 + \dots + i_m = n} \sigma_q^{i_2 + \dots + i_m}(\delta^{(i_1)}(z)) \dots \sigma_q^{i_m}(\delta^{(i_{m-1})}(z)) \delta^{(i_m)}(z).$$

If one of the  $i_j$  is not equal to  $n$ , there exists  $i_l$  such that  $i_l \neq n$ . Then, an easy computation shows that

$$\delta^{(n)}(z^m) = mz^{m-1}\delta^{(n)}(z).$$

It follows that,

$$mz^{m-1}\delta^{(n)}(z) = \binom{nb}{n}_q t^{nb-n}.$$

By Proposition 2.2, we have  $\binom{nb}{n}_q = b = m\frac{a}{n}$  and thus  $\delta_M^{(n)}(z) = \frac{a}{nt^n}z$ . Thus  $E = F(z)$  and  $z^m \in F$ . It follows that  $\underline{\text{Gal}}(E/F)$  is a finite cyclic group.

Conversely, suppose that  $y$  is an algebraic solution of the  $ID_qE$  associated to  $M$ , then  $E = F(y)$  is algebraic over  $F$  and  $\underline{\text{Gal}}(E/F) \subsetneq \mathbb{G}_m$  is a cyclic group of order  $m$ . So there exist  $s \in \mathbb{Z}$  and  $(b_i)_{i \geq s}$  with  $b_s = 1$  such that  $y^m = \sum_{i \geq s} b_i t^i \in F$ . Thus,

$$my^{m-1}\delta^{(n)}(y) = y^m \frac{a}{nt^n} = \delta^{(n)}(y^m) = \sum_{i \geq s} b_i \binom{i}{n}_q t^{i-n}.$$

By comparing the coefficient of  $t^l$ , we obtain that  $\frac{a}{n} = b_i \binom{i}{n}_q$  for all  $i \geq s$ . Since  $b_s = 1$  and because of properties of the  $q$ -binomials coefficients, we get that:

1.  $s = k_s n$  with  $k_s \in \mathbb{N}$  and  $a = \frac{nk_s}{m}$ ;
2.  $b_i = 0$  for all  $i \neq 0 \pmod n$ .

Induction using the higher iterative difference shows that  $b_i = 0$  for all  $i > s$ . It follows that  $y^m = t^s$  and  $a = \frac{nk_s}{m}$ .

**The Galois group  $\mathbb{G}_a$  in positive characteristic** Let us denote by  $C = \overline{\mathbb{F}_p}$  the algebraic closure of  $\mathbb{F}_p$ , where  $p$  is a prime number. Let  $F = C(t)$  be a rational function field with coefficients in  $C$ . Let  $(a_l)_{l \geq 0}$  be a set of elements in  $\mathbb{F}_p$ . We choose  $q \in C$  a  $n$ -th primitive root of unity with  $n$  prime to  $p$ .

Let  $M = Fb_1 \oplus Fb_2$  be the  $ID_q$ -module with corresponding  $ID_q E$ :

$$\delta^{(np^k)}(Y) = A_k Y = \begin{pmatrix} 0 & a_k \\ 0 & 0 \end{pmatrix} Y$$

for  $k \in \mathbb{N}$ .

**Theorem 4.23** *Let  $M$  be as above with its associated  $ID_q E$ . Let  $\alpha = \sum_{l \geq 0} a_l p^l \in \mathbb{Q}_p$ . Then for an iterative Picard-Vessiot extension  $E/F$  for  $M$ , we have*

*$\underline{\text{Gal}}(E/F)$  is a finite subgroup of order  $r$  of  $\mathbb{G}_a$  if  $\alpha \in \mathbb{Q}$  and  $\underline{\text{Gal}}(E/F) \simeq \mathbb{G}_a$  if  $\alpha \notin \mathbb{Q}$ .*

For the proof, we need the following lemma.

**Lemma 4.24** *Let  $(a_l)_{l \geq 0}$  be a sequence of elements in  $\mathbb{F}_p$ . The following statements are equivalent :*

1. *The sequence  $(a_l)_{l \geq 0}$  is periodic from a certain rank;*
2.  *$g = \sum_{l \in \mathbb{N}} a_l t^{np^l} \in C((t))$  is separable algebraic over  $C(t)$ .*

**Proof**

see [16] p.30 and replace  $t$  by  $t^n$ .

**Proof of Theorem 4.23**

We start with the iterative differential equation,

$$\delta^{(np^k)}(Y) = A_k = \begin{pmatrix} 0 & a_k \\ 0 & 0 \end{pmatrix} Y$$

for  $k \in \mathbb{N}$ .

Writing  $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ , we find that  $\delta^{(k)}(y_2) = 0$  for all  $k \in \mathbb{N}$ , which implies  $y_2 \in C$ .

Using this result we obtain  $\delta^{(np^k)}(y_1) = a_k y_2$  for all  $k \in \mathbb{N}$  and  $\delta^{(1)}(y_1) = a_{-1} y_2$ . Thus, the formal solution  $y_1$  is equal to

$$y_1 = y_2 \left( \sum_{l \in \mathbb{N}} a_l t^{np^l} \right).$$

Then  $E = F(y_1, y_2) = F(y_1)$ , and for any  $\tau \in \underline{\text{Gal}}(E/F)$  we get

$$\delta^{(np^l)}(\tau(y_1) - y_1) = \tau(\delta^{(np^l)}(y_1)) - \delta^{(np^l)}(y_1) = \tau(y_2 a_l) - y_2 a_l = 0.$$

thus there exists  $d \in c$  such that  $\tau(y_1) = y_1 + d$ . Therefore  $\underline{\text{Gal}}(E/F)$  is a subgroup of  $\mathbb{G}_a$ .

Using Lemma 4.24, we obtain

1. the solution  $y_1$  is separable algebraic over  $F$  if  $\alpha \in \mathbb{Q}$  (the sequence  $(a_l)_{l \geq 0}$  is periodic from a certain index if and only if  $\alpha \in \mathbb{Q}$ ), so the Galois group is actually finite.
2. If  $\alpha \notin \mathbb{Q}$ , then  $y_1$  is transcendental over  $F$ , and hence  $E/F$  is purely transcendental of degree 1, showing that  $\underline{\text{Gal}}(E/F) \simeq \mathbb{G}_a$ .

**Remark 4.25** *These examples of iterative  $q$ -difference equations are obtained by  $q$ -deformation of the examples of B.H. Matzat in [16] example 2.14 and 2.15. The Galois groups obtained here are the same as those obtained by Matzat. The fact that simple Galois groups such as  $\mathbf{G}_m$  and  $\mathbf{G}_a$  do not degenerate by  $q$ -deformation give us a nice hope for confluence studies.*

## 5 An analogue of the Grothendieck-Katz conjecture

In this section, we state an analogue of the Grothendieck-Katz conjecture for iterative  $q$ -difference equations. In [8], L. Di Vizio proves this conjecture for  $q$ -difference equations with  $q$  non equal to a root of unity. Briefly, she shows that given a  $q$ -difference equation,  $\mathcal{L}y = 0$  with coefficients in  $\mathbb{Q}(t)$ , one can describe the behavior of the solutions of  $\mathcal{L}$  by considering the reduction of  $\mathcal{L}$  modulo the prime numbers.

Here is the iterative  $q$ -difference version of this conjecture.

Let  $K$  be a number field and  $\mathcal{O}_K$  the ring of integers of  $K$ . We choose an element  $q$  in  $K^*$ , non equal to 1. We denote by  $\Sigma_f$  the set of all finite places  $v$  of  $K$ . The uniformizer of the finite place  $v$  is denoted by  $\pi_v$  and  $||_v$  denotes the  $v$ -adic absolute value of  $K$ . We denote by  $p_v$  be the characteristic of the residue field  $k_v$  of  $\pi_v$ . For almost all finite places  $v$ , let  $\kappa_v$  be the multiplicative order of the image of  $q$  in  $k_v$ . If  $q$  is not a root of unity, let  $\pi_{q,v}$  be the integer power of  $\pi_v$  such that  $|\pi_{q,v}|_v = |1 - q^{\kappa_v}|_v$ . If  $q$  is a root of unity, let  $\pi_{q,v} = \pi_v$ .

**Conjecture 5.1** *Let  $(\mathcal{M}, \phi_M, \delta_M^*)$  be an iterative  $q$ -difference module defined over  $K(t)$ . The iterative  $q$ -difference module  $\mathcal{M}$  is isotrivial, i.e. becomes trivial after a finite base field extension if and only if for almost all finite places  $v$ , the reduction modulo  $\pi_{q,v}$  of  $\phi_M^{\kappa_v}$  is the identity and the one of  $(\delta_M^{\kappa_v})^{p_v}$  is equal to zero.*

If  $q$  is not a root of unity, Conjecture 5.1 is equivalent to Theorem 7.1.1 in [8] and when  $q$  goes to 1, we retrieve the classical Grothendieck's conjecture on  $p$ -curvatures, which predicts :

The differential equation  $Ly = 0$  with  $L \in \mathbb{Q}[\partial]$  has a full set of algebraic solutions if and only if for almost all primes  $p \in \mathbb{Z}$  the reduction modulo  $p$  of  $Ly = 0$  has a full set of solutions in  $\mathbf{F}_p(t)$  i.e. the  $p$ -curvature of  $L$  is equal to zero.

Here is an example where Conjecture 5.1 holds.

**Example 5.2 (Example 3.21)** *Let  $a \in K$ . Then, let us consider the  $ID_q E : \delta^{(1)}(\mathbf{y}) = 0$  and  $\delta^{(n)}(\mathbf{y}) = \frac{a}{nt^n} \mathbf{y}$ . Let  $v$  be a place of  $K$ . A simple calculation shows that the reduction of  $\delta_M^{(\kappa_v)^{p_v}}$  modulo  $\pi_{q,v}$  is equal to  $a(a-1)\dots(a-(p_v-1))$ . If, we assume that for almost all finite places  $v$ , the reduction modulo  $\pi_{q,v}$  of  $\phi_M^{\kappa_v}$  is the identity and the one of  $(\delta_M^{\kappa_v})^{p_v}$  is equal to zero, we get that for almost all finite places  $v$  there exists  $a_v \in \mathbb{Z}$  such that the valuation of  $a - a_v$  in  $\pi_v$  is strictly positive. By the Density Theorem of Chebotarev, we obtain that  $a \in \mathbb{Q}$ . We have proved in Theorem 4.22 that  $a \in \mathbb{Q}$  if and only if  $\mathcal{M}$  has a finite Galois group.*

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