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On the motion of rigid bodies submerged in incompressible fluids

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Dedicated to my family

Zusammenfassung

Die mathematische Behandlung der Interaktion von Fluiden und Festkörpern ist noch voller ungelöster Probleme. In dieser Arbeit wird die Bewegung einer beliebigen Zahl von Körpern in einem beliebigen Fluid im drei-dimensionalen Raum untersucht (Partikelflüsse und grobe Suspensionen). Für dieses realistischen Szenario sind die hier erhaltenen Ergebnisse neu. Diese Arbeit liefert fast dieselben überraschenden Ergebnisse für drei Dimensionen wie sie von San Martín, Starovoitov, and Tucsnak (2003) in zwei Dimensionen bewiesen wurden. Im Einzelnen werden die folgenden Fragen untersucht.

- Wie modelliert man den Transport von vielen Körpern in einem inkompressiblen Fluid möglichst effizient?
- Wie modelliert man Kollisionen, und wie laufen diese ab?
- Wie konstruiert man ein Geschwindigkeitsfeld und die zugehörigen Bewegungen der eingebetteten Körper?

Im ersten Kapitel wird nachgewiesen, dass ebenfalls in drei Dimensionen die Bedingung der Verzerrungsfreiheit des Geschwindigkeitsfeldes einen starren Körper erhält. Dieses kann als unendliche Viskosität starrer Körper interpretiert werden. Hierdurch ist es möglich, ein einheitliches und globales Koordinatensystem zu verwenden, welches sowohl im Fluid, als auch in den Körpern verwendet werden kann. Die Äquivalenz dieser Beschreibungen wird in Lemma 1.1.3 zusammengefasst. Bisherige mathematische Beweise hierfür haben überraschenderweise nur den kommutativen zwei-dimensionalen Fall betrachtet.

Intuitiv erwartet man viele Kollisionen innerhalb eines Flusses voller Partikel. Daher überrascht ein Phänomen, welches im zweiten Kapitel gezeigt wird (Lemma 2.1.8). Alleine die Inkompressibilität und recht geringe Regularität des Geschwindigkeitsfeldes lassen im Wesentlichen eine Kollision fast immer mit relativer Geschwindigkeit Null ablaufen. Hierzu wird eine allgemeine Klasse von divergenzfreien Geschwindigkeitsfeldern untersucht. Es wird gezeigt, dass die Taylorreihe des Abstandes je zweier kollidierender Teilchen nur aus Nullen besteht soweit sie existiert. Diese a-priori Abschätzung gilt unabhängig vom Modell, welches das umgebende Fluid beschreibt. Eine Interpretation ist, dass der Fluidfilm zwischen Teilchen nicht abreißen kann. Eine Kraftübertragung mittels einer Randschicht ist hierdurch nicht ausgeschlossen. Überdies ist diese Klasse so groß, dass selbst die im dritten Kapitel konstruierte schwache Lösung der Navier-Stokes-Gleichungen in dieser liegt (Theorem 2.2.1).

Im dritten Kapitel wird in drei Dimensionen eine Lösung des Modells konstruiert, welches die Bewegung einer beliebiger Zahl von Teilchen in einem viskosen Fluid beschreibt (Theoreme 3.2.8 und 3.2.10). Dieses kommt ohne die üblichen künstliche Randschichten um die Teilchen aus (Notation 3.3.1). Außerdem benötigt dieses Approximationsverfahren dank des ersten Kapitels keine Referenzkoordinaten. Das hierdurch erhaltene Geschwindigkeitsfeld stellt eine globale Lösung dar, welche einen im Wesentlichen kollisionsfreien Transport beschreibt. Hierdurch werden bisherige Existenz-Resultate verallgemeinert. Diese lieferten die Existenz einer Lösung nur bis höchstens zum ersten Auftreten einer Kollision, oder mussten für die Konstruktion einer Lösung diese künstlich nach einer Kollision fortsetzen. Die hier konstruierte Lösung kann für kollidierende Körper im Falle einer höheren Regularität auf die Bewegung von Bällen an Stöcken eingegrenzt werden.

Abstract

Understanding the interaction of fluids and solids poses many unsolved and even unchallenged questions. Here, the motion of an arbitrary number of bodies is considered, which are immersed in an arbitrary incompressible fluid in three-dimensional space (particulate flows and thick suspensions). The obtained results are new within this realistic setting. The results almost match the surprising results of San Martín, Starovoitov, and Tucsna (2003) obtained in two dimensions. This work considers the following questions:

- How can the transport of many submerged bodies within an incompressible viscous fluid be modeled most efficiently?
- How can realistic collisions be incorporated into a model, and what happens during a collision?
- How can a particular velocity field and the associated transport of the bodies be constructed?

In the first chapter it is proved that even in three dimensions a rigid body motion is characterized by a vanishing symmetric gradient of the velocity field. This can be imagined as infinitely high viscosity of a rigid body. Furthermore, it yields a model for the mixture of solids and liquids that can be formulated without the use of a reference set. Instead, the common coordinate system of fluid dynamics is sufficient. The equivalence of the common approach of physics and the here used one is proved in Lemma 1.1.3. Surprisingly, a mathematical proof of this observation for the three-dimensional and therefore non-commutative case could not be found in the literature.

Intuitively, many collisions are expected in a particulate flow. Therefore, a phenomenon is observed in the second chapter which is surprising. Incompressibility and rather mild regularity assumptions prevent powerful collisions. To show this, a general class of incompressible velocity fields is considered. The main observation is that the Taylor series of the distance function of two colliding bodies vanishes up to an order that depends on the regularity of the velocity field. This a-priori estimate does not assume a particular model for the fluid the bodies are submerged in. An interpretation is that the fluid film does not break. A transmission of energy due to a boundary layer is hereby not effected. Furthermore, even the weak solution which is constructed in the third chapter is contained in this class (Theorem 2.2.1).

In the third chapter a procedure is presented to obtain a velocity field that describes the motion of bodies within a Newtonian fluid (Theorems 3.2.8 and 3.2.10). The solution is found without posing a security zone with artificial repellent forces around the submerged bodies or the boundary (Notation 3.3.1). Furthermore, this approximation scheme does not need reference coordinates. The hereby obtained global solution describes the almost always collision-free motion of an arbitrary number of submerged bodies. Previous existence results are generalized hereby. These results considered existence at most up to the first occurrence of a collision, or constructed a solution as artificial continuation in case of a collision. For solutions of higher regularity the here found velocity field can be narrowed to a motion of balls-on-sticks.

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.

There is another theory which states that this has already happened.

Douglas N. Adams (1998)

Introduction

This introduction is supposed to answer the following questions. Why this topic, what is the topic in the first place, and who might be interested in such results? What was done by me, and what was done before and by whom?

To answer these questions, the introduction consists of three sections: The mathematics starts in the third section on page vii. The second section contains some examples that need non-standard flow models. It presents these in the case of blood flow. Nevertheless, these and many more phenomena are observed in various areas besides hemodynamics. Often the nonstandard behavior of mixtures is even essential and is the very reason for their applicability. Examples are rheology modifiers in engineering, transport and mixing in food processing, or fluid flow in ink-jet printers. All too often, the nonstandard behavior of heterogeneous fluids is the cause of many unsolved problems and not understood behavior. Take for example sedimentation and transport in waste-water treatment-facilities, fluid transport in fuel cells, or again the ink transport in bubble-jet printers. To begin this introduction, the first section contains some thoughts on the usability of models more or less in general. It should serve as a warning concerning the usability of this work: I am interested in the limits of applicability of models, not in their applicability itself. Therefore, I consider a model which generalizes all other models in engineering and especially in hemodynamics. It is generalized in the sense that solutions to these models are special cases of the here considered one. I demonstrate that all solutions of this generalized model with minimal regularity have the probably unwanted feature of rather mild collisions.

Limits to Modeling

Personally, I do not believe that any model can describe any process that happens in the real world. The best it can do is to describe our anticipated picture of reality. Hence, a model is wrong if it does not coincide with our picture of reality — or will become so if we learn more about *life, universe and everything*. The philosophical interesting case that a model could be close to reality without being close to our picture thereof or could guide experimental research in new directions cannot be topic of mathematical research. Hence, being myself imprisoned in a reality distortion zone, I am looking for a model that comes closer to my anticipated picture of real flow and

perform a stress test concerning its durability.

My motivation is to understand the flow of real blood, and, to cut a long story short: I have no idea¹ what makes blood the *quite peculiar juice* it is, as Goethe's Mephisto calls it. Naturally, I had to restrict myself to some features of blood flow, which I first singled out following my personal taste, and then tried to understand. And, though I got stuck halfway, I found some personally rewarding insights into some of the models used in fluid mechanics and learned some mathematics on the journey. Thus, the true purpose of this work is to present those insights which I consider possibly interesting to others.

Where models come from

Prejudices simplify our life tremendously: They save us from further thinking that would delay further actions. They enable us to predict the future by knowing at most some isolated events we have (or someone else has) observed. Imagine life if you could not be sure of the basic laws of nature, as prejudices against nature are also called. Your great- . . .-great grandfather that met this rather juvenile sabertooth did not hesitate to run. He did not even think of this kitten as a possible exception to the rule *sabertooths eat humans* — probably this is the very reason why you can read this.

As a rule-of-thumb, biologists or people from medical sciences seem to enjoy the abundance of the different qualitative behaviors encountered in the life-sciences. In the mathematical sciences, however, diversity is often regarded as a lack of understanding and the absence of a comprehensive theory. Even though these two underlying attitudes seem to be mutually exclusive, in physics their combination has proved to be highly fertile and produced many *final* models. The term final is to be understood in the sense that we do not expect someone to produce a ground-breaking new model within the next days. If a final model is at hand, forecasts of future experiences are considered as permissible, even in pathological cases and without representative experiments.

Where models might go

It has to be admitted that the assumption that a final model exists – in the original sense of final – is pure imagination. Even the Navier-Stokes equations, which are nowadays accepted, were derived 1822 by Navier from a molecular model that mercifully was allowed to sink into oblivion. The equations had to be independently derived by Stokes by different means, see Galdi (2005). A simplified version of the Navier-Stokes equations is the final model of meteorologists to produce the weather forecasts of the next days, and nobody trusts local weather forecast for more than three days in advance. Nevertheless, it is not expected that a tremendously improved model will be proposed in the near future.

On a simpler scale the situation is more satisfactory: Newton's law of gravitation holds true for apples, nobody doubts that, but it also holds for mangos, papayas, and kiwi fruits though Newton most probably never had one of those. Hence, this model, together with most other physical laws and even theories, is *accepted*. Repeatable

¹The expression *I have no idea* should be understood in the sense that there is no unified model from which these phenomena can be derived in a mathematical sense. The engineering and life-science literature is full of ad-hoc models, derived from curve fitting of measured data.

experiments confirm a prototypic behavior of an idealized or averaged case that was predicted.

The situation in the biological sciences is quite the contrary. Every now and then seemingly established principles are abandoned or experience a fundamental change. Many areas in the biological sciences, especially those which are concerned with processes on a microscopic level, are still subject to extensive reconstruction.

Having said this, I admit that I do not believe that the model used here is even a candidate for the final model that describes the motion of erythrocytes in vessels of diameters starting from $10\mu m$ up to $400\mu m$. In my opinion it lacks the simplicity of a $F = ma$ or $U = RI$ -type law; it is too complicated and complex for numerical analysis; and it is even mathematically not understood. But, it is my sincere belief that it is much closer to our anticipated picture of the real world than any other model used so far.

Examples from hemodynamics

During my work I started to observe features that I assumed to be peculiarities of blood flow everywhere and on every occasion. My favorite observation is that it is utterly impossible to pick the pieces of an eggshell from scrambled-eggs mixtures with your fingertips. Though this might seem to be common knowledge, the eggshell observation is predicted by Lemma 2.1.8 on page 19. And the assumption of convexity, which yields a single contact point, seems to be strict, since with two forks you can pick the eggshell pieces. Nevertheless, though I present examples from blood flow, similar phenomena are observed for other suspensions too. I am especially interested how properties of a fluid change when particles are added. That this problem is important is demonstrated by many experimentalists' rheology-modifier articles and books.

Starting point

The starting point for the here presented results was the wish to understand blood flow. Hence, models of transport of mixtures of inhomogeneous and incompressible phases are of interest. Therefore, in this section the construction procedure of models used in hemodynamics is presented. Hemodynamics is the branch of medicine that considers the flow of blood. Furthermore, it is discussed, why such models are not satisfactory. On the other hand, in this section the key idea for the later introduced model is motivated in passing as well. So hemodynamical considerations are the origin for this work and were also the starting point for the research presented here. But, the features of blood flow presented here should be observed as what they are. Examples of the oddities of heterogeneous flows in general.

Classical model of hemodynamics and circumvention of a mathematical nuisance

Many different effects are perceived at vessel diameters of the range I am interested in. But often their names, their causes, or their side effects are confused or intermingled. To keep a clear nomenclature we stick to one of the original articles, namely Fåhræus and Lindqvist (1931), and add newer perspectives and names from the historic review

article of Goldsmith, Cokelet, and Gaehtgens (1989). The quoted physics can be found for example in Acheson (2005).

One of the oldest but still widely used model of blood flow was crafted by Jean Louis Marie Poiseuille (1797-1869). We consider a tube $\Omega = (0, L) \times B_{\mathbb{R}^2}(0, R)$ of length L and radius R , filled with a viscous fluid, and apply pressures p_{in} and p_{out} at the in- and outflow openings. Since the pipe is radially symmetric, and we are interested in an averaged case, we assume that the flow is radially symmetric and flow only occurs only in the direction of the pipe. These assumptions yield a special solution even to the full Navier-Stokes equations. In the case of a viscous fluid, the pressure gradient should be something like

$$\delta p := \frac{p_{\text{out}} - p_{\text{in}}}{L},$$

except for a unit vector pointing in the direction of the centerline of the tube. A multiple of δp is the normalized head loss, which is easy to measure. The flow rate through a cross section satisfies

$$q = \frac{\pi}{8} \frac{\delta p}{\mu} R^4,$$

which is again easy to measure, at least on average. Hence, for a given flow rate q and a pressure gradient δp , the viscosity of the fluid should be given by

$$\mu = \frac{\pi}{8} \frac{\delta p}{q} R^4.$$

Given now any substance that flows, we can define the *apparent viscosity* to be given by exactly this formula, i.e.

$$(1) \quad \mu_{\text{app}} := \frac{\pi}{8} \frac{\delta p}{q} R^4,$$

for a given flow rate and weighted pressure difference. The advantage of the right hand side of this formula is that it needs only easily measurable data. Similar formulas describe the flow that occurs due to exterior forces like gravitation instead of applied pressure differences. Because of such formulas we can calculate the viscosity of gases (H_2 has $10^{-4}P$), water ($10^{-2}P$), magma or lava (on average about 10^2 – 10^3P , but highly dependent on the temperature), glaciers (very roughly $10^{12}P$), granites (10^6 – $10^{26}P$), and even whole mountains. We use the unit P for poise (and cP for centipoise) to measure viscosity, though other units are more common today. The reason for our choice is the following. The viscosity of water is approximately $1cP$, i.e. all values can be reinterpreted as viscosity relative to water's viscosity. The viscosity of blood or plasma is of the same scale. The viscosity of water at room temperature is $1cP$ or 1 mPa s . The problem with these formulas is that they assume a certain model to be applicable to a particular flow, and assume very restrictive settings for this model.

Using this notion of apparent viscosity, engineers define a gas to be something of low viscosity, a liquid is characterized by a viscosity in the mid range, and something is rigid if its viscosity is extremely high. We will consider transport of rigid bodies within a viscous fluid as a two-fluid mixture to circumvent a classical problem of two natural coordinate systems. One system references a point in a moving solid

by its original position. The other fixes a point in a river and observes the river passing through. The result is, that in one way or the other both coordinate systems have to be taken care of, and the switch between them is singular of the worst thinkable degree. Modeling of touching bodies is prohibitive, since the point of contact has two possible origins in the past. The approach used here bypasses this problem by considering rigid bodies to be fluids of high viscosity. The rigidity is taken care of somewhere in the limit of viscosity to infinity. In this form, the lack of regularity and the lack of convergence estimates does not allow to follow this path in numerical analysis. But the same could be said of solutions to the homogeneous Navier-Stokes equations in three dimensional space for weak solutions which are the only ones known to exist for all time, independent of probably unrealistic smallness assumptions on the given data. The linearized rigidity equation $(D_{\text{sym}} \mathbf{u}) \mathbf{1}_{\Omega_{\text{solid}}} = 0$, see Lemma 1.1.3 on page 7, is sometimes incorporated into the conservation of momentum equation by a penalization scheme to allow numerical approximation. In this approach, the approximative stress tensor $(2\mu + n \mathbf{1}_{\Omega_{\text{solid}}^n}) D_{\text{sym}} \mathbf{u}^n$ is used. Here, \mathbf{u}^n is the corresponding approximative velocity field and $\mathbf{1}_{\Omega_{\text{solid}}^n}$ is the characteristic function of the approximative solid occupied domain. Sometimes this approach is considered as a formal Lagrange multiplier procedure for $n \in \mathbb{N}$. For such methods see Joseph (2005). We consider the limit $n \rightarrow \infty$ and show that such infinitely-viscous fluids actually represent rigid bodies.

Features of a realistic flow: blood flow

It seems that Fåhræus and Lindqvist (1931) were the first to observe that the above introduced notion of apparent viscosity is questionable, and to communicate this to the medical community. We mention some features of blood without claiming completeness, following (Fåhræus and Lindqvist, 1931) and (Goldsmith, Cokelet, and Gaetgens, 1989). Note that in the field of biorheology (or in case of general suspensions as well) these and other counter-intuitive behaviors of fluids are observed as well. Most of the classical examples are encountered in hemodynamics, which is the field of physiology that studies flow properties of blood.

Three phenomena of blood seem to be of special importance to be understood, if a model for blood is to be crafted. First of all, if blood is modeled as a homogeneous fluid, it seems to be shear thinning. This means that its apparent viscosity decreases with increasing shear rate. Secondly, at small vessel diameters the apparent viscosity is reduced, and reaches its minimum at approximately the size of the erythrocytes. In Figure 1, the original measurements that describe the Fåhræus-Lindqvist effect are shown. The third property is the nontrivial distribution of erythrocytes. They tend to accumulate at the centerline of the vessel. Hereby a skimmed layer of lower relative concentration of erythrocytes is observed (skimming-layer effect). The relative concentration of erythrocytes is called the hematocrit. Furthermore, erythrocytes are not distributed evenly in a branching system (Fåhræus effect). Finally, the flow behavior can drastically be changed due to diseases that deform the erythrocytes (Figure 2).

All these effects are due to the presence of cells, which form a different phase in the current. The cells need to be accepted as rheology modifiers that must not be neglected by modeling blood as homogeneous or even as intrinsically one-dimensional flow. The possibility of an uneven distribution of the different species forming the suspension blood is the major reason for the observed behavior. For a given particle

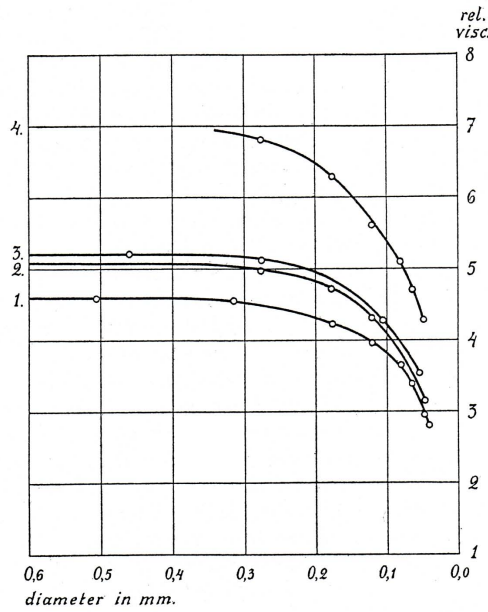


Figure 1: Original measurements of Fåhræus and Lindqvist (1931, Fig. 2). Viscosity is measured relative to the viscosity of water, i.e. unit is $1\text{cP} = 10^{-2}\text{P}$. Lines 1, 2 and 4 represent results obtained from blood samples of Torsten Lindqvist; line 3 is based on a sample of Robin Fåhræus.

concentration the flow depends mainly on the relative size of the skimming layer to the vessel diameter, i.e. approximately upon the ratio mean-erythrocyte-diameter to local average-vessel-diameter. To derive such observations mathematically, we need a model that allows such phenomena in the first place.

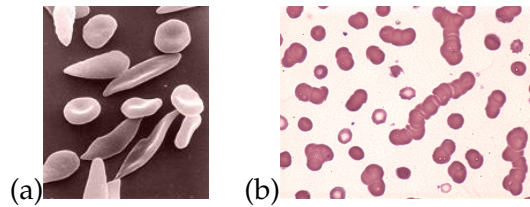


Figure 2: The red blood cells can deform due to diseases (a, sickle cell anemia), chemical processes or flow conditions (b, rouleaux formation); see (Wikipedia, 2007, sickle cell anemia) and (Maslak, 2004).

Besides the physical effects, many more effects are observed and need to be taken into consideration if a realistic model shall be crafted. Blood measurements are always highly biased. There are no measurements with untreated blood, simply because blood does not allow in vivo measurements, see (Rüger, 2001). All blood samples need to be treated chemically. Furthermore, even treated blood depends in highest order on the donor and even on the special sample, see for example the original measurements of Fåhræus and Lindqvist (1931, Fig. 2), reproduced in Figure 1. To complete the uncertainty in the data, the procedure of measurement of blood properties has a tremendous impact on the obtained results, which was the reason for R. Fåhræus to start his investigations, see (Goldsmith et al., 1989).

Modeling for the impatient and literature

Panta rhei and previous work

Many philosophers are involved in the context of *panta rhei*. Most often it is accredited to Heraklit, about 500 BC. Its literal meaning is: everything flows. We interpret it rather loosely and completely of topic as: everything behaves like a fluid.

Based on the conservation of mass, momentum $\mathbf{m} = \rho \mathbf{u}$, and volume, a general model is given. $\frac{D}{Dt} \rho = 0$ has to be satisfied for the mass density ρ . The momentum has to fulfill $\frac{D}{Dt} \mathbf{m} = \text{div } \mathbf{T}$, and the volume is constrained by $\text{div } \mathbf{u} = 0$. Here \mathbf{u} is the global velocity field. We used implicitly that the phases are immiscible. Here $\frac{D}{Dt}$ denotes the material derivative along streamlines of the flow. This general system needs to be closer defined by the choice of an appropriate stress tensor \mathbf{T} . We perceive rigidity as nothing but very high viscosity. Thus, rigid body motion in a viscous fluid is approximately a two-phase fluid motion, where surface tensions are negligible. Therefore, we model a mixture of different species as a limit of a sequence of models describing mixtures of immiscible liquids of different properties. We consider as limit-materials solids or liquids.

Based upon results from DiPerna and Lions (1989), P. L. Lions presented a theory in Lions (1996), which can be used to study and solve such systems; see for example Desjardins and Esteban (1999). Even more important, DiPerna and Lions provided compactness theorems that can be used to prove existence of solutions by considering limits of a sequence of penalized problems. Here we consider stress tensors of the form

$$(2) \quad \mathbf{T}_n = (2\mu + n \mathbf{1}_{\Omega_{\text{solid}}^n}) \mathbf{D}_{\text{sym}} \mathbf{u} - p \text{Id}$$

that are penalized versions of the original Newtonian fluid tensor

$$(3) \quad \mathbf{T} = 2\mu \mathbf{D}_{\text{sym}} \mathbf{u} - p \text{Id}.$$

The constant μ denotes the viscosity. More precisely it denotes the inverse of the so-called Reynolds number.

San Martín, Starovoitov, and Tucsnak (2003) showed that in two dimensions a velocity field that preserves a rigid body in motion can be approximated by letting (2) tend to (3). The precise meaning of this limit process is the topic the third chapter. A subset $\Omega_{\text{solid}}(t)$ occupied by a rigid body is approximated by a subset occupied by a fluid of very high viscosity. The penalization scheme was applied to a neighborhood of the evolved version of the reduced body. Hereby, there was no need to introduce an artificial repellent force around the submerged bodies. Formally, the repellent security zone is within the domain occupied by the body. Furthermore, their approach is even consistent in the sense that classical solutions satisfy this notion of approximative solutions. Using this approach San Martín et al. (2003) were able to construct a global solution to the Navier-Stokes equations that preserves submerged rigid bodies. In particular, they showed that the hereby obtained velocity field exists and transports submerged bodies almost always collision-free. This article was the first to my knowledge that considered global existence of solutions of flow and transport of more than one submerged body in a viscous, incompressible fluid. Here their approximation scheme is lifted from two to three dimensions.

The earlier work of Desjardins and Esteban (1999) showed only local existence in time. They showed that solutions exist in two dimensions up to the moment of a collision. In three dimensions a solution exists at most up to the moment of a collision, or up to the moment of a blow-up of the velocity gradient. Feireisl (2003a) studied a model for the motion of several rigid bodies submerged in a compressible viscous fluid. He showed global existence of a velocity field independent of the space dimension. In (Feireisl, 2003b) he considered incompressible flows. He introduced a continuation of a velocity field after an occurred collision. If two bodies should touch one another they are *defined* to stay together forever. Hereby he obtained a global in time weak solution at a high price (Feireisl, 2003b, page 427): “Such a condition may be viewed as a (very naive) form of the least energy principle.”

The common feature of the approaches in most of those references is that they completely break with the habit to use different coordinate systems for each submerged body. Having more than one submerged body makes it impossible to deal with collisions, because the contact point of two submerged bodies has potentially two different initial positions. The coordinate change gets singular in the moment of a collision. This is the reason why so many authors did not treat collisions. They only considered single submerged bodies in all space to study the properties of the flow, see for example Galdi and Vaidya (2001); Galdi et al. (2002). Numerical analysis can not cope with this problem up to today, see for example Bönisch (2006). Only engineers seem to have a model which circumvents this problem, see for example Joseph (2005) and references therein. But the problem of the so-called direct numerical simulation is that mathematics lacks far behind. Furthermore, the artificial repellent forces in a security zone at the boundaries can hardly be motivated.

Results

In this work the motion of an arbitrary number of bodies is considered, which are immersed in an arbitrary incompressible fluid in three dimensions. The motivation is to understand particulate flows and the rheology of thick suspensions. The obtained results are new for this realistic setting. In two dimensions San Martín, Starovoitov, and Tucsnak (2003) obtained similarly surprising results. In particular, the following questions are regarded.

- How can the transport of many submerged bodies within an incompressible viscous fluid be modeled most efficiently?
- How can realistic collisions be incorporated into a model, and what happens during a collision?
- How can a particular velocity field and the associated transport of the bodies be constructed?

Contents and answers to these questions

The above posed questions are answered in the three chapters that form the main part of this work.

In the first chapter we justify a model, which is closely related to the so called direct numerical simulations. We prove that it generalizes the classical approach to model solid-liquid mixtures even in three dimensions. Hereby we show that regular generalized solutions are classical ones. In two dimensions the proof hereof is much

simpler due to the commutativity of the group of rotations. In three dimensions the group of rotations is noncommutative. Therefore, the flow generated by a sum of rotations is in general not given as a simple superposition of the two flows, but as Lie-Trotter product. This approach demonstrates that it is sufficient to consider the standard coordinate system of fluid dynamics. Hereby, submerged rigid bodies can be imagined as fluids of infinitely high viscosity. This approach will be used in the remaining chapters.

In the second chapter, we consider the class of all incompressible viscosity fields that could preserve a submerged body. The definition of this class does not depend on any specific model for the conservation of momentum. For velocity fields of this class we demonstrate that collisions of bodies in incompressible fluids should not be observable for regular velocity fields in a spectacular way. Regular here means that we assume further smoothness properties of the solution. For example we obtain as corollary that the distance of submerged balls has a Taylor expansion which consists of zeros only. For weak solutions of the Navier-Stokes equations, we only need rather mild extra assumptions, like that their centers of mass should move continuously at the moment of a collision. Hereby we found that a phenomenon observed by San Martín, Starovoitov, and Tucsnak (2003) in two dimensions does occur in three dimensions as well. Collisions occur with relative velocity zero in a velocity field of the same regularity as typical weak solutions to the Navier-Stokes equations, see Corollary 2.2.2. If the distance is differentiable then all derivatives of the distance of two bodies that exist have to vanish. This includes for example the relative acceleration, which indicates the vanishing of direct body-body forces.

In the third chapter, we construct a weak solution as limit of solutions of approximative models. We lift the approximation scheme of San Martín, Starovoitov, and Tucsnak (2003) to three dimensions and demonstrate by a variant of the second chapter, that for such solutions collisions cannot occur. The main task in the third chapter is to prove that the limit velocity field satisfies the weak formulation of the conservation of momentum equation.

We close with considerations on the consequences which can be drawn for hemodynamics or modeling suspensions in general. Furthermore we consider numerical analysis of this approach and simulation of the motion of submerged solid bodies.

Literature

Some of the texts in the bibliography are more important than others and are often referred to. But others are better points to start a journey into the topic, and should be mentioned for readers without the background I assumed.

As standard functional analytic references I like (Werner, 2005) and (Yosida, 1995). The mathematical theory of dynamics and statics of homogeneous viscous fluids is presented in the books (Sohr, 2001) and (Ladyzhenskaya, 1969), whereas in (Lions, 1996) the heterogeneous incompressible case is described. The physicists' viewpoints are contained in detail in (Acheson, 2005) and in much condensed form in (Feynman et al., 1977a,b). The numerical perspective to fluid motion is presented in (Heywood et al., 1996), (Rannacher, 2006) and (Temam, 2001). Engineers have a vast amount of literature related to this topic, so a recommendation is difficult. As starting point maybe (Joseph, 2005) for direct numerical simulations or (White, 1999) for ad hoc calculations can be used.

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Modeling submerged bodies

1 Chapter

We consider a mixture of immiscible phases: one phase formed by solid particles, the other by an incompressible fluid. The model we consider is a continuous model that is used in engineering literature and does neither depend explicitly on the number of suspended particles, nor does the transport of solid particles need to be treated independently by a governing set of ordinary differential equations; see for example Joseph (2005). Our main contribution is the justification that this model truly generalizes the common approach. The approach we prefer is conceptionally simpler and easier to use for numerical analysis. It is justified because regular solutions of this generalized model are solutions to the standard one — even in three-dimensional space. In Lemma 1.1.3 this claim is proved as statement (iii) implies (i). Hereby, we show the applicability of this generalized model in three dimensions.

1.1 Why dimension matters

1.1.1 Different approaches to describe the motion of submerged particles

The prevailing way to describe the motion of a rigid body is to consider its motion relative to the position of its center of mass $x_c(t)$. This form of description can be rediscovered from the density-centered approach we use. The notations used for both approaches, and how the classical description can be obtained again from the density approach, will be illustrated in this subsection.

Let $\Omega_{\text{solid}}(t) \subseteq \mathbb{R}^n$ be a connected subset that is occupied by a single rigid body in motion at a certain time t . Then

$$M(t) := \int_{\Omega_{\text{solid}}(t)} \varrho(x, t) \, dx$$

gives its total mass whereas

$$x_c(t) := \frac{1}{|M(t)|} \int_{\Omega_{\text{solid}}(t)} x \varrho(x, t) \, dx$$

gives its center of mass. Here we denoted by $\varrho(x, t)$ the mass density at x at time t . The later position of an arbitrary particle, initially at $\xi \in \Omega_{\text{solid}}(0)$, is given by

$$(1.1.1) \quad \Xi(\xi, t) = Q(t) (\xi - x_c(0)) + x_c(t),$$

where $Q(t) \in \text{SO}(\mathbb{R}^n)$ is a rotation, i.e. $Q(t) \in \mathcal{L}(\mathbb{R}^n)$ satisfies $Q(t)Q(t)^* = \text{Id}$, $Q(t)^*Q(t) = \text{Id}$ and $\det Q(t) = 1$. If a global velocity field \mathbf{u} that transports the fluid particles according to $\partial_t \Xi(\xi, t) = \mathbf{u}(\Xi(\xi, t), t)$ is given, it necessarily has to satisfy

$$\mathbf{u}(\Xi(\xi, t), t) = \partial_t Q(t) (\xi - \mathbf{x}_c(0)) + \partial_t \mathbf{x}_c(t).$$

Hence, the velocity $\mathbf{u}(\mathbf{x}, t)$ at a position $\mathbf{x} \in \Omega_{\text{solid}}(t)$ can be formulated as

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \partial_t Q(t) Q(t)^* (\mathbf{x} - \mathbf{x}_c(t)) + \partial_t \mathbf{x}_c(t) \\ (1.1.2) \quad &=: R(t) \mathbf{x} + \mathbf{U}(t), \end{aligned}$$

for some $R(t) \in \text{so}(\mathbb{R}^n)$, i.e. $R(t) \in \mathcal{L}(\mathbb{R}^n)$ satisfies $R(t)^* = -R(t)$ and $\mathbf{U}(t) \in \mathbb{R}^3$. Especially, we obtain that the deformation (of the velocity field) within the flow satisfies for all t and $\mathbf{x} \in \Omega_{\text{solid}}(t)$

$$(1.1.3) \quad D_{\text{sym}} \mathbf{u}(\mathbf{x}, t) = \frac{1}{2} (R(t)^* + R(t)) = 0.$$

Thus, we obtain the three formulations of rigid body motion that can be found in the literature. The trajectory is of the form posed in (1.1.1), the velocity field is of the form (1.1.2), and the deformation (of the velocity field) satisfies (1.1.3). Although in physics and engineering these three criteria are used interchangeably and their equivalence is part of scientific folklore, only (1.1.1) implies (1.1.2) and (1.1.2) implies (1.1.3) can be easily checked. That (1.1.2) is necessary for (1.1.3) is a reincarnation of so called Cosserat's theorem from elasticity theory, but (1.1.2) implies (1.1.1) is not obvious in three dimensions and cannot be simply checked by applying the exponential due to an operational calculus which relies on the commutativity of the underlying algebra: The rotations in one and two-dimensional space are a commutative algebra; in higher dimensions they are not. Therefore, the only point of this section is to show that the statement used in engineering is correct and to provide an outline of a proof.

In the language of flows on manifolds we can interpret Ξ as a path on the manifold $M = \text{SE}(\mathbb{R}^n)$, which is given by matrices of the form

$$(1.1.4) \quad \begin{pmatrix} Q & \mathbf{x}_c \\ 0_{\mathbb{R}^n} & 1 \end{pmatrix},$$

and has as tangent space $TM = \text{se}(\mathbb{R}^n)$, which consists of matrices of the form

$$(1.1.5) \quad \begin{pmatrix} R & \mathbf{U} \\ 0_{\mathbb{R}^n} & 0 \end{pmatrix}.$$

In this formulation, the not surprising statement motivated so far is: For given $\Xi \in C^1(\mathbb{R}_+; M)$ there exists a velocity field $\mathbf{u} \in C(\mathbb{R}_+; TM)$ such that $\partial_t \Xi(t) = \mathbf{u}(t)$. The key observation here is the converse: Using the above introduced notation we obtain for a given velocity field $\mathbf{u} \in C(\mathbb{R}_+; TM)$ a path $\Xi \in C^1(\mathbb{R}_+; M)$ that satisfies

$$\partial_t \Xi(\xi, t) = \mathbf{u}(\Xi(\xi, t), t), \quad \Xi(\xi, 0) = \xi.$$

Slightly generalized, we use for modeling of suspensions the following. A velocity field $\mathbf{u} \in L_{\text{loc}}^1([0, T]; H_0^1(\Omega)^n)$ that satisfies $\mathbf{1}_{\Omega_{\text{solid}}(t)} D_{\text{sym}} \mathbf{u} = 0$ describes the motion of a rigid bodies in the classical sense (1.1.1). These bodies form the connected subsets of $\Omega_{\text{solid}}(t) = \Xi(\Omega_{\text{solid}}(0), t) \subseteq \mathbb{R}^n$.

1.1.2 Structural difference between two and higher dimensions

Considering the motion of rigid bodies in viscous fluids, we would end up with two different natural coordinate systems if we used the classical form of description: The motion of rigid bodies is usually described using a reference system which references the position at the starting time, whereas in fluid mechanics it is more common to assume a fixed position as a point-of-view and watch the fluid pass by.

To incorporate the rigidity constraint into the natural fluid-mechanical approaches it is important to characterize the preservation of the rigidity of a body by constraints on its velocity field alone. That the assumption $D_{\text{sym}} \mathbf{u} = 0$ might suffice can be motivated as application of Stone's Theorem. Provided the matrices $(R(s))_{0 \leq s \leq T}$ that define the above introduced velocity field $\mathbf{u}(\mathbf{x}, s) = R(s)\mathbf{x} + \mathbf{U}(t)$ commute, we can define for $0 \leq s, t \leq T$

$$(1.1.6) \quad A(t, s) := \int_s^t R(\tau) d\tau, \quad T(t, s) := e^{A(t, s)}$$

and obtain, using $A(t, s)^* = -A(s, t)$, the evolution by

$$(1.1.7) \quad T(t, s)^* = \left(e^{A(t, s)}\right)^* = e^{-A(s, t)} = T(s, t) = T(t, s)^{-1}.$$

That the hereby defined operator family $(T(t, s))_{s, t}$ actually solves the initial value problem needs already the commutativity of the family $(R(t))_t$. Otherwise $T(t, \tau)T(\tau, s) = T(t, s)$ for arbitrary s, t is difficult to prove.

By variation of constants, the solution Ξ of $\partial_t \Xi(\xi, t) = \mathbf{u}(\Xi(\xi, t), t)$ is of the form

$$(1.1.8) \quad \Xi(\xi, t) = T(t, 0)\xi + \int_0^t T(t, \tau) \mathbf{U}(\tau) d\tau.$$

Let $Q(t) := T(t, 0)$ and $x(t) := \int_0^t T(t, \tau) \mathbf{U}(\tau) d\tau$, then Ξ is a rigid body motion

$$\Xi(\xi, t) = Q(t)\xi + x(t),$$

since Q satisfies $Q(t)^* = Q(t)^{-1}$, $\det Q(t) = 1$, and $Q(0) = \text{Id}$. But there is no reason why $(R(s))_{0 \leq s \leq T}$ should commute in three dimensions, whereas in two dimensions every $R(t)$ is necessarily of the form

$$R(t) = r(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In three dimensions, solving $RT - TR = 0$ yields that only linear dependent, skew-symmetric matrices $R, T \in \mathcal{L}(\mathbb{R}^3)$ commute. But similar to the two-dimensional case, each skew-symmetric matrix $R \in \mathcal{L}(\mathbb{R}^3)$ is necessarily of the form

$$R = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Hence, every skew-symmetric matrix valued function $R \in C([0, T]; \mathcal{L}(\mathbb{R}^3))$ has a representation $R(t) = \omega_1(t)R_1 + \omega_2(t)R_2 + \omega_3(t)R_3$, where R_k is the generator of a rotation that preserves \mathbf{e}_k and $\omega_k \in C([0, T]; \mathbb{R})$. For example R_1 is given by

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

This means that in two dimensions the images of $R \in C([0, T]; \mathcal{L}(\mathbb{R}^2))$ necessarily commute and are a subset of the commutative sub-algebra of skew-symmetric linear mappings. In three or higher dimensions the images of $R \in C([0, T]; \mathcal{L}(\mathbb{R}^3))$ within $\mathcal{L}(\mathbb{R}^3)$ do not necessarily commute. For example $R_1 R_2 \neq R_2 R_1$. Hence, the representation of the solution used in equations (1.1.6) and (1.1.7) is not necessarily true.

The main difference between the commutative and the non-commutative case lies within the following formula: For bounded linear operators A and B , the property

$$(1.1.9) \quad e^{(A+B)} \stackrel{!}{=} e^A e^B$$

is valid if and only if A and B commute, i.e. for the general case (1.1.9) cannot be assumed. Furthermore, since a sum is at the foundation of the integral, the validity of the generalized case (1.1.7) of limits of sums is questionable. To circumvent this problem many approaches can be found in the literature. Although we use a more recent mathematical tools and a different language, in the next subsection we use an approach which goes back to Sophus Lie or at least to Hermann Weyl. Nevertheless, somewhere in the older literature the here demonstrated equivalence should have been proved already, but I just could not find a reference. That there exists a discrepancy between $e^{(A+B)}$ and $e^A e^B$ can be found often in the literature. It is again a result that dates back to Sophus Lie. The difference can be expressed as sums and products of the so called Lie-product $AB - BA$ that measures the degree of commutativity.

1.1.3 Perturbation of evolution equations

Since in two dimensions a rotation can only occur around an imagined axis perpendicular to the plane, we saw in the last subsection that the motion of rigid bodies in two dimensions is conceptionally simpler than the higher dimensional version. This section provides us with the tools and an approach to describe the resulting evolution.

The main idea of superimposed evolution processes is that the orbit of a point transported by velocity fields can be approximated by fast switching between the individual processes. For the linear autonomous case imagine three Cauchy problems:

$$\begin{aligned} \partial_t x(t) &= Ax(t), & x(0) &= x_0 \\ \partial_t x(t) &= Bx(t), & x(0) &= x_0 \\ \partial_t x(t) &= (A+B)x(t), & x(0) &= x_0. \end{aligned}$$

What is the relation between these problems? In general the simplest idea, namely

$$e^{(A+B)t} = e^{At} e^{Bt},$$

is wrong, as we have seen in the last subsection. But, we have an approximation

$$e^{(A+B)t} \approx \underbrace{e^{A \frac{t}{n}} e^{B \frac{t}{n}} \dots e^{A \frac{t}{n}} e^{B \frac{t}{n}}}_{2n \text{ terms}}.$$

At the limit $n \rightarrow \infty$ equality can be achieved.

In the nonautonomous case the idea is quite the same, just the notation is a little more tedious: The evolution-operator $T(t-s)$ from time s to time t , which only depends on the increment $t-s$ in the autonomous case, is replaced by $T(t,s)$: In the autonomous case we had $T_A(t,s) = e^{A(t-s)}$, $T_B(t,s) = e^{B(t-s)}$, and $T_{A+B}(t,s) = e^{(A+B)(t-s)}$. Hence, we expect

$$T_{A+B}(t,s) \approx \underbrace{T_A(t, t-\delta t) T_B(t, t-\delta t) \dots T_A(s+\delta t, s) T_B(s+\delta t, s)}_{2n \text{ terms}}$$

where $\delta t := \frac{t-s}{n}$. But for bounded problems, i.e. $A(t), B(t) \in \mathcal{L}(X)$, we can approximate these terms easily by using for example $T_A(t,s) \approx e^{A(s)(t-s)}$ as approximation, which we can explicitly estimate as an exponential series.

Let us come back to our problem. The main result of this section is that these ideas are useful in general. We are not going to prove this general claim, but the procedure to our problem. As motivation and illustration we recall some ideas from the general theory, but we refer for a complete presentation and the proofs to the books of Engel and Nagel (2000); Goldstein (1985); Pazy (1983).

In the nonautonomous case $\partial_t x(t) = A(t)x(t)$ posed on a Banach space X and $t \in \mathbb{R}$, its evolution family $(T(t,s))_{t \geq s > -\infty}$ can be transferred into an autonomous problem $\partial_t x(t) = \mathcal{A}x(t)$ on the Banach space $\mathcal{X} = C_0(\mathbb{R}; X)$ of asymptotically vanishing X -valued continuous functions¹ and a semigroup $(\mathcal{T}(t))_{t \geq 0}$ by

$$\mathcal{T}(t)f(s) := T(s, s-t)f(s-t).$$

Tough we do not use this semigroup, its approximation is the foundation of the proof of our application to the motion of rigid bodies. The generator of \mathcal{T} is the sum of a multiplication operator and a generator of a left-translation semigroup. We quote a corollary of these statements and collect them in the next theorem, which is taken almost literally from (Engel and Nagel, 2000, Example III.5.9), where we refer to for proof of this elementary case. The general case of hyperbolic evolution families is proved in (Nickel, 2000, Prop. 3.2).

Theorem 1.1.1 (Nonautonomous case of alternating direction scheme). *Let a Banach space X and a strongly continuous function $A \in C(\mathbb{R}; \mathcal{L}(X)_{st})$ be given. Then the nonautonomous Cauchy problem*

$$\begin{aligned} \partial_t x(t) &= A(t)x(t) \\ x(s) &= x_s \end{aligned}$$

has an evolution family $(U(t,s))_{s,t}$ of bounded operators that defines its unique solution by $x(t) = T(t,s)x_s$ and is given by

$$(1.1.10) \quad T(t,s)x = \lim_{N \rightarrow \infty} \prod_{k=1}^N \exp\left(\frac{t-s}{N} A\left(s + k \frac{t-s}{N}\right)\right)x,$$

where for all $x \in X$ the limit is uniform for s, t from compact subinterval of \mathbb{R} .

¹We denote the continuous functions with compact support by C_c . The closure of this space with respect to the supremum norm yields the space C_0 . A function $u \in C(\Omega)$ is asymptotically vanishing if for all $\varepsilon > 0$ a compact set $K \subseteq \Omega$ exists such that $|u(x)| \leq \varepsilon$ for all $x \in \Omega \setminus K$.

This formula provides us with an explicit characterization of the motion of a rigid body and enables us to prove the implication 2. to 1. of Lemma 1.1.3. The explicit characterization of the evolution $(T(t, s))_{0 \leq s, t \leq T}$ as limits of products of intermediate approximative evolutions can be directly generalized to the case of bounded time intervals $[0, T)$, provided the solution exists uniquely. More general assumptions that guarantee the existence of evolution systems and further approximation schemes for evolution systems generated by unbounded families $(A(t), D(A(t)))_{0 \leq t \leq T}$ are presented in (Pazy, 1983, Chapter 5). Essentially, (1.1.10) is a very special case of the scheme used in the proof of (Pazy, 1983, Chapter 5, Theorem 3.1).

Corollary 1.1.2 (Rigid Body Motion). *For $R \in C([0, T], \mathcal{L}(\mathbb{R}^n))$ such that $R(t)^* = -R(t)$ the evolution $(Q(t))_{t \geq 0}$ defined by the Cauchy Problem*

$$\begin{aligned}\partial_t \mathbf{x}(t) &= R(t) \mathbf{x}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned}$$

is unitary, $Q(t)^ = Q(t)^{-1}$. For our later interpretation this yields that Ξ , which describes the displacement of a rigid body within a rigid body velocity field \mathbf{u} , where $\mathbf{u}(\mathbf{x}, t) = R(t)\mathbf{x} + \mathbf{U}(t)$, is of the form $\Xi(\xi, t) = Q(t)\xi + \mathbf{x}(t)$ for some $Q \in C([0, T], \mathcal{L}(\mathbb{R}^n))$ that satisfies $Q(t)^* = Q(t)^{-1}$.*

Proof. The adjoint depends continuously on the matrix. Hence, using the previous theorem, standard properties of the adjoint, and the skew-symmetry of R_k 's, we obtain that $Q(t)^*$ satisfies

$$\begin{aligned}Q(t)^* &= \lim_{N \rightarrow \infty} \left(\prod_{k=1}^N \exp \left(\frac{t}{N} \left(\omega_1(k \frac{t}{N}) R_1 + \omega_2(k \frac{t}{N}) R_2 + \omega_3(k \frac{t}{N}) R_3 \right) \right) \right)^* \\ &= \lim_{N \rightarrow \infty} \prod_{k=N}^1 \exp \left(\frac{t}{N} \left(\omega_1(k \frac{t}{N}) R_1 + \omega_2(k \frac{t}{N}) R_2 + \omega_3(k \frac{t}{N}) R_3 \right) \right)^* \\ &= \lim_{N \rightarrow \infty} \prod_{k=N}^1 \exp \left(\frac{t}{N} \left(\omega_1(k \frac{t}{N}) R_1^* + \omega_2(k \frac{t}{N}) R_2^* + \omega_3(k \frac{t}{N}) R_3^* \right) \right)\end{aligned}$$

and is given by the representation formula

$$Q(t)^* = \lim_{N \rightarrow \infty} \prod_{k=N}^1 \exp \left(-\frac{t}{N} \left(\omega_1(k \frac{t}{N}) R_1 + \omega_2(k \frac{t}{N}) R_2 + \omega_3(k \frac{t}{N}) R_3 \right) \right).$$

To take care for the non-commutativity, the product has to be understood in the ordered sense, this is

$$\prod_{k=1}^N a_k = a_N \cdot \dots \cdot a_1 \quad \text{and} \quad \prod_{k=N}^1 a_k = a_1 \cdot \dots \cdot a_N.$$

Now $Q(t) Q(t)^* = \text{Id}$ and $Q(t)^* Q(t) = \text{Id}$ is a matter of evaluating middle terms. *q.e.d.*

We can state and prove now that all three characterizations of rigid body motion, which are used interchangeably in physics, are equivalent in the regular case.

Lemma 1.1.3 (Characterization of Rigid Body Motions). *Let Ξ be the path related to the flow \mathbf{u} in the above sense. Then, regardless of the space dimension, the following are equivalent:*

- (i) *The path Ξ describes the motion of a rigid body, i.e. $\Xi(\xi, t) = Q(t)\xi + \mathbf{x}(t)$, where $Q \in C([0, T], \mathcal{L}(\mathbb{R}^n))$, $Q(t)^* = Q(t)^{-1}$, $\det Q(t) = 1$, and $Q(0) = \text{Id}$.*
- (ii) *The field \mathbf{u} is the velocity field of a rigid body, i.e. $\mathbf{u}(\mathbf{x}, t) = R(t)\mathbf{x} + \mathbf{U}(t)$, where $R \in C([0, T], \mathcal{L}(\mathbb{R}^n))$ such that $R(t)^* = -R(t)$.*
- (iii) *The field \mathbf{u} is a rigid body velocity field, i.e. it satisfies $D_{\text{sym}} \mathbf{u}(t) = 0$.*

It should again be mentioned, that the equivalence of (ii) and (iii) is often found in the literature, as it is the main idea of the proof of Cosserat's theorem of elasticity theory (see Rannacher, 2006, Satz 1.5), which states that two solutions of a linear elasticity problem differ at most in a rigid body motion. That (i) implies (ii) is pretty easy. That (iii) implies (i) is exactly the direction we need to show to demonstrate that the herein used notion of generalized solution meaningfully generalizes the motion of a rigid body.

Proof. (i) \Rightarrow (ii): By definition of the particle path,

$$\begin{aligned} \mathbf{u}(\Xi(\xi, t), t) &= \partial_t \Xi(\xi, t) = \dot{Q}(t)\xi + \dot{\mathbf{x}}(t) = \dot{Q}(t)Q(t)^*(\Xi(\xi, t) - \mathbf{x}(t)) + \dot{\mathbf{x}}(t) \\ &= R(t)\Xi(\xi, t) + \mathbf{U}(t), \end{aligned}$$

for $R(t) := \dot{Q}(t)Q(t)^*$ and $\mathbf{U}(t) := -R(t)\mathbf{x}(t) + \dot{\mathbf{x}}(t)$. Now, $Q(t)Q(t)^* = \text{Id}$, which holds by assumption, implies $\dot{Q}(t)Q(t)^* = -Q(t)\dot{Q}(t)^*$ for all t . Hence, $R(t) = -R(t)^*$. Therefore, since all particle paths are at least locally defined, $\mathbf{u}(\mathbf{x}, t) = R(t)\mathbf{x} + \mathbf{U}(t)$.

(ii) \Rightarrow (iii): By assumption $R(t)$ is skew-symmetric, i.e. $R(t) + R(t)^* = 0$. Therefore $D_{\text{sym}} \mathbf{u} = 0$ by

$$(1.1.11) \quad D_{\text{sym}} \mathbf{u} = \frac{1}{2} (D\mathbf{u} + D\mathbf{u}^*) = \frac{1}{2} (R(t) + R(t)^*).$$

(iii) \Rightarrow (ii): Analogously to the proof of Cosserat's Theorem, see for example (Temam, 1985, Lemma I.1.1), we show at first $D^2 \mathbf{u} \equiv 0$: By assumption $\partial_k \mathbf{u}^l = -\partial_l \mathbf{u}^k$ for all $k, l = 1, \dots, N$. Hence, exchanging derivatives yields for all i, j, k

$$2\partial_i \partial_j \mathbf{u}^k = -\partial_i \partial_k \mathbf{u}^j - \partial_j \partial_k \mathbf{u}^i = -\partial_k (\partial_i \mathbf{u}^j + \partial_j \mathbf{u}^i) = 0,$$

or $D^2 \mathbf{u} \equiv 0$. Therefore, $\mathbf{u}(\mathbf{x}, t) = R(t)\mathbf{x} + \mathbf{U}(t)$. Reading (1.1.11) backwards, $D_{\text{sym}} \mathbf{u} \equiv 0$ yields the skew symmetry of $R(t)$.

(ii) \Rightarrow (i): We are only interested in formulations which are integrated over compact sets of the form $[0, t]$, where $t < T$. Hence, the regularity $L^1_{\text{loc}}([0, T])$ can be replaced by $L^1([0, T^*])$ for arbitrary $0 < T^* < T$. To shorten the notation, let $T = T^*$ be possible. Since we only need the continuous case we formulate it as such. If $R \in C([0, T], \mathcal{L}(\mathbb{R}^n))$ and $\mathbf{U} \in C([0, T], \mathbb{R}^n)$ then $(T(t, s))_{0 \leq t, s \leq T}$ exists uniquely by Picard's Theorem and $\Xi(\xi, t)$ is defined by the variation of constants formula (1.1.8) for $0 \leq t < T$ and $\xi \in \Omega$. By definition,

$$\Xi(\xi, t) = \xi + \int_0^t \mathbf{u}(\Xi(\xi, \tau), \tau) d\tau = \underbrace{\left(\xi + \int_0^t R(\tau)\Xi(\xi, \tau) d\tau \right)}_{=: Q(\xi, t)} + \int_0^t \mathbf{U}(\tau) d\tau.$$

To prove our claim, $Q(\xi, t)$ should at least depend linearly on ξ . This is equivalent to $D_\xi^2 Q(\xi, t) = 0$ in the distribution sense, which we need to show: By construction we have

$$D \Xi(\xi, t) = \text{Id} + \int_0^t R(\tau) D \Xi(\xi, \tau) d\tau.$$

and

$$\partial_{\xi^k} D \Xi(\xi, t) = \int_0^t R(\tau) \partial_{\xi^k} D \Xi(\xi, \tau) d\tau.$$

The last equation is a linear ordinary differential equation to the initial value 0 in mild formulation, which yields $\partial_{\xi^k} D \Xi(\xi, t) = 0$. Therefore $Q(t) := D_\xi \Xi(\xi, t)$ is independent of ξ and Q satisfies

$$Q(t) = \text{Id} + \int_0^t R(\tau) Q(\tau) d\tau,$$

or, in strong formulation,

$$(1.1.12) \quad \begin{cases} \partial_t Q(t) = R(t) Q(t), \\ Q(0) = \text{Id}. \end{cases}$$

By Liouville's theorem,

$$\begin{cases} \partial_t \det Q(t) = \text{Trace } R(t) \det Q(t), \\ \det Q(0) = 1. \end{cases}$$

Due to the skew symmetry of $R(t)$, $\text{Trace } R(t) = 0$. Hence, we obtain $\det Q(t) \equiv 1$.

To this point, the considerations which preceded this lemma combined with the last idea are close to standard. The last, and most involved, step is to prove that actually $Q(t) \in \text{SO}(\mathbb{R}^3)$. But this is what we've proved in Corollary 1.1.2, so the claimed equivalence is proved. *q.e.d.*

Remark 1.1.4 (Eulerian Angles). In the engineering literature the so called Eulerian description of rotations is quite popular, because it can be used to calculate and describe any given rotation. The major drawback is that the angles are not independent of one another. This is due to the very same non-commutativity that caused the problems above. To use the Eulerian angles we first need to fix the axes of rotation and the order in which they are considered. For details we refer to Landau and Lifšic (1967). So we only describe the ideas of the Eulerian angles and comment that these are just a particular way to think about the characterization (i) of Lemma 1.1.3.

Euler's theorem on rotations states that every rotation $Q \in \text{SO}(\mathbb{R}^3)$ can be written as a product $Q = Q_3(\phi_3)Q_2(\phi_2)Q_1(\phi_1)$, where $Q_k(\phi_k)$ is a rotation of angle ϕ_k that preserves a certain fixed vector e_k , where $\{e_k\}_{k=1,2,3}$ spans all of \mathbb{R}^3 . Sometimes such vectors are chosen relative to the rigid body as axes of symmetry, this is they depend implicitly on Q . Using our notation, for any given $Q \in \text{SO}(\mathbb{R}^3)$ there exist such ϕ_k , but they depend on the special order of the product. Especially, it is in general false to assume

$$Q = Q_3(\phi_3)Q_2(\phi_2)Q_1(\phi_1) \stackrel{?}{=} Q_1(\phi_1)Q_2(\phi_2)Q_3(\phi_3),$$

which would yield characterization (i). Furthermore, Euler's angle theorem assumes that $Q \in \text{SO}(\mathbb{R}^3)$, a claim we want to prove. It is nothing but a different parametrization.

1.2 Modeling submerged bodies

The introduction provided us with the necessary notions to formulate a model which describes the transport of mixtures of rigid bodies and incompressible fluids. In this section this model is introduced and motivated. The very same or similar models can be found in Desjardins and Esteban (1999); Joseph (2005); San Martín et al. (2003). This model can be considered as a limit case of an heterogeneous mixture of viscous fluids as considered in DiPerna and Lions (1989); Lions (1996).

1.2.1 Conserved quantities

The momentum density $\mathbf{m}(\mathbf{x}, t) = \varrho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ should satisfy the *Conservation of Momentum* or *Newton's law*

$$(1.2.1) \quad \partial_t \mathbf{m} + \operatorname{div}(\mathbf{m} \otimes \mathbf{u}) = \operatorname{div} \mathbf{T} + \varrho \mathbf{g},$$

where \mathbf{T} denotes the stress tensor and external forces \mathbf{g} . Since we are only modeling mixing phenomena and are not interested in chemical effects, the mass density ϱ should satisfy the *Conservation of Mass*

$$(1.2.2) \quad \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

and the *Conservation of Volume* or *Incompressibility*

$$(1.2.3) \quad \operatorname{div} \mathbf{u} = 0$$

should be satisfied. The derivation of these equations can be found in most text books on fluid mechanics, see (Chorin and Marsden, 2000; Lions, 1996; Rannacher, 2006).

The conservation of mass and volume of each subset $V \subseteq \Omega$ yields for the transported set $V(t) := \{\Xi(\mathbf{x}, t) | \mathbf{x} \in V\}$

$$\begin{aligned} \partial_t \mathbf{1}_{V(t)}(\mathbf{x}) + \operatorname{div}(\mathbf{1}_{V(t)}(\mathbf{x})\mathbf{u}(\mathbf{x}, t)) &= 0 \\ \mathbf{1}_{V(0)}(\mathbf{x}) &= \mathbf{1}_V(\mathbf{x}). \end{aligned}$$

In terms of the connected subsets Ω_{solid}^i of the solid phase, we obtain that

$$\partial_t \mathbf{1}_{\Omega_{\text{solid}}^i(t)} + \operatorname{div}(\mathbf{1}_{\Omega_{\text{solid}}^i(t)}\mathbf{u}) = 0$$

for initial data defined by the initial decomposition into solid and fluid phases. Hence, using that \mathbf{u} is solenoidal, we obtain that the characteristic functions $\chi^i = \mathbf{1}_{\Omega_{\text{solid}}^i}$ and the mass density satisfy

$$\partial_t \chi^i + \mathbf{u} \cdot \operatorname{grad} \chi^i = 0$$

which we will add to the standard notion of a weak solution.

Hence, our notion of a weak solution consists of the following parts: conservation of momentum $\varrho \mathbf{u}$, conservation of mass by transport of ϱ , and transport of each sub-species χ^i .

1.2.2 Weak Solutions

We are now prepared to formulate the notion of a weak solution that generalizes the above conservation principles. We use effectively the same notion as (San Martín et al., 2003, Def. 1), although in three instead of two dimensions. This notion is the foundation for our existence proof in the third chapter. This chapter mainly serves as motivation and justification for the compatibility condition $\chi D_{\text{sym}} \phi = 0$.

The notion of a weak solution introduced now actually is a result of the existence proof of the last chapter. If a test function ϕ satisfies $\chi D_{\text{sym}} \phi = 0$ then it itself is naturally a candidate for a velocity field that describes the motion of rigid bodies. These bodies are contained in the connected subsets Ω_{solid}^i of the support of the characteristic function χ . But, as we will see in the third chapter, the motion of two sets Ω_{solid}^i and Ω_{solid}^j seems to be less restricted in three dimensions than in two dimensions. Suppose Ω_{solid}^i and Ω_{solid}^j have a common boundary point. In two dimensions they necessarily move like one rigid body. In three dimensions one degree of freedom is left for smooth velocity fields. These bodies can still move like balls-on-sticks. See Lemma 2.1.8 for the distance and Lemma 3.4.2, Remark 3.4.3 and Proposition 3.4.6 for the restriction of the relative rotations. Although we do not apply these estimates to test functions, considerations similar to the proof of Lemma 3.6.5 yield that the test functions need to be further restricted. Since within open and connected set a rigid body motion is uniquely defined, see Lemma 3.4.1, this extra degree of freedom is controlled by assuming $D_{\text{sym}} \phi = 0$ in an open neighborhood of the rigid bodies.

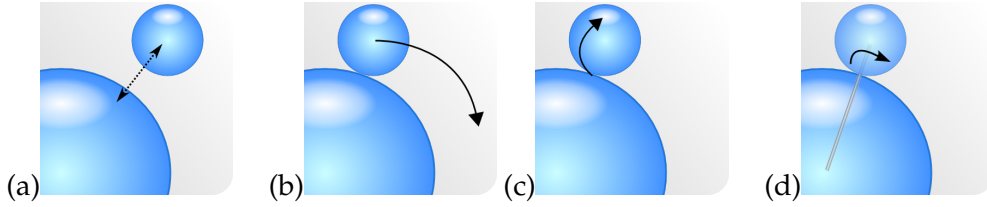


Figure 1.2.1: Without posing a particular model, a solenoidal velocity field that preserves submerged bodies is in three dimensions less restricted than in two dimensions. The motions depicted in (a), (b) and (c) are controlled by higher regularity of the velocity field. The purely three dimensional degree of freedom given by a rotation that preserves the normal of the common tangential plane is not restricted. A balls on a stick motion (d) is still possible.

Let the initial mass density satisfy $\rho_0 \in L^\infty(\Omega)$. Let the initial velocity field be a solenoidal square integrable field $u_0 \in L^2_\sigma(\Omega)$. Let $\Omega_{\text{solid}}^i \subseteq \Omega$ for $i = 1, \dots, N$ be disjoint, open and smooth sets with characteristic functions $\chi^i(0)$ and let $\Omega_{\text{solid}} = \cup_{i=1}^N \Omega_{\text{solid}}^i$ be the subset occupied by solids. The space-time cylinder is denoted by $Q_T := \Omega \times [0, T]$ and the derivative along a path Ξ of the flowing material by $\frac{D}{Dt}$, the so called *material derivative*. For a vector valued function $\phi = [\phi_l]_l$ it is given by

$$\frac{D}{Dt} \phi := \partial_t \phi + \text{grad } \phi u = \left[\partial_t \phi_l + \sum_{k=1}^3 u_k \partial_{x_k} \phi_l \right]_l$$

and for a scalar valued function ψ by

$$\frac{D}{Dt}\psi := \partial_t \psi + \text{grad } \psi \cdot \mathbf{u} = \partial_t \psi + \sum_{k=1}^3 u_k \partial_{x_k} \psi.$$

Definition 1.2.1 (Weak Solution). The functions \mathbf{u} , ϱ , χ^i form a weak solution of the above formulated conservation principles if they satisfy the following.

$$(1.2.4) \quad \mathbf{u} \in L^\infty(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; H_{0,\sigma}^1(\Omega)),$$

$$(1.2.5) \quad \varrho \in L^\infty(0, T; L^\infty(\Omega)),$$

$$(1.2.6) \quad \chi^i \in C^{0+\frac{1}{p}}(0, T; L^p(\Omega)) \text{ for all } 1 \leq p < \infty \text{ and } \chi^i(\mathbf{x}, t) \in \{0, 1\}$$

$$(1.2.7) \quad \chi := \sum_{i=1}^N \chi^i \text{ satisfies } \chi(\mathbf{x}, t) \in \{0, 1\}$$

and these functions satisfy

$$(1.2.8) \quad \int_{Q_T} \varrho \mathbf{u} \cdot \frac{D}{Dt} \boldsymbol{\phi} = - \int_{\Omega} \mathbf{m}_0(\mathbf{x}) \boldsymbol{\phi}(\mathbf{x}, 0) d\mathbf{x} + \int_{Q_T} 2\mu D_{\text{sym}} \mathbf{u} : D_{\text{sym}} \boldsymbol{\phi} - \int_{Q_T} \varrho f \cdot \boldsymbol{\phi}$$

for all $\boldsymbol{\phi} \in \mathcal{D}([0, T]; \mathcal{D}_\sigma(\Omega))$ that are compatible in the sense

$$(1.2.9) \quad D_{\text{sym}} \boldsymbol{\phi}(\mathbf{x}, t) = 0$$

for all t and \mathbf{x} in an open neighborhood of $\text{supp } \chi(t)$. Furthermore,

$$(1.2.10) \quad \int_{Q_T} \varrho \cdot \frac{D}{Dt} \psi = - \int_{\Omega} \varrho_0(\mathbf{x}) \psi(\mathbf{x}, 0) d\mathbf{x}$$

$$(1.2.11) \quad \int_{Q_T} \chi^i \cdot \frac{D}{Dt} \psi = - \int_{\Omega} \chi_0^i(\mathbf{x}) \psi(\mathbf{x}, 0) d\mathbf{x}$$

for all $\psi \in C^1(0, T; C^1(\Omega))$ that satisfy $\psi(T) = 0$.

Remark 1.2.2 (Compatibility within an open neighborhood). Considering the above definition, all of the above weak formulations are standard in this setting besides one that is easily overlooked. The assumption $D_{\text{sym}} \boldsymbol{\phi}(\mathbf{x}, t) = 0$ for all t and \mathbf{x} in an open neighborhood of $\text{supp } \chi(t)$ is crucial. The natural weak formulation would have been to pose the compatibility condition only for \mathbf{x} from $\text{supp } \chi(t)$ and not to include an open neighborhood. In any dimension, $D_{\text{sym}} \boldsymbol{\phi}(\mathbf{x}, t) = 0$ within an open set restricts $\boldsymbol{\phi}(t)$ to the form $R(t)\mathbf{x} + \mathbf{U}(t)$ for $R(t)$ and $\mathbf{U}(t)$ fixed within a connected subset. See for example Lemma 1.1.3.

In two dimensions these two definitions are equivalent, see San Martín, Starovoitov, and Tucsna (2003, Prop. 4.1.) and Lemma 3.2.1. In three dimensions the natural definition is most probably less restrictive than the above one in the sense that $R(t)$ could have extra degrees of freedom: The balls-on-sticks motion we later will not be able to avoid. Thus, admitting only a smaller space of test functions yields less restriction on the solutions. Effectively, only the projection onto the subspace spanned by these test functions is controled. In Chapter 2 we will see that the relative velocity

field is restricted even for weak solutions in the above sense, especially it is restricted for the test functions. In Chapter 3, see for example Remark 3.4.3, we will see, that we can only control two of the three possible axes of rotation. That is the reason we deviate here from San Martín et al. (2003) and use this formulation following Feireisl (2003b). If a-priori it is known that no collisions occur these definitions coincide, see Proposition 3.4.6.

Velocity fields of collisions

We consider a mixture of immiscible phases and prove a-priori estimates. For global velocity fields of a certain regularity these estimates necessarily have to be satisfied. These estimates are valid independent of the particular model that describes the transport of rigid bodies within mixtures of incompressible phases. As corollaries we demonstrate that rigid bodies which are submerged in an incompressible fluid do not collide in three dimensions in a spectacular way if the velocity field is slightly more regular than a weak solution of the Navier-Stokes equations. In two dimensions this phenomenon was observed by San Martín et al. (2003). We generalize and extend their $H^{1,2}$ -estimates to $H^{s,p}$ -solutions in three dimensions in Lemma 2.1.8. Especially, we show that neither strong nor classical solutions of any model which aims at modeling transport of rigid bodies submerged in an incompressible phase can capture the concept of collisions in the sense of impetuous body-body interaction. The terms of a Taylor expansion of the distance function will necessarily vanish up to an order defined by s and p , see Theorems 2.2.1 and the Corollaries 2.2.2 and 2.2.3. Nevertheless, a transmission of momentum from one body to another can still take place due to the presence of a boundary layer, whose influence we do not study. Hence, our results can be interpreted in the form that there always remains a thin film of fluid between submerged bodies.

2.1 Estimates that govern the interaction of rigid bodies

In this section we prove that two rigid bodies that touch in a point in time have almost always and will almost always move like two bodies on a stick. This result is due to the rigidity of the velocity field within the rigid bodies and the incompressibility of each of the phases. Although we often refer to the balls-on-sticks motions of chemical models, the following idea might be easier to digest. Meat-balls on a stick in sauce stay on the stick independent of the type of sauce.

2.1.1 Sobolev type spaces of fractional order and properties

To prove the claimed statements, we need a couple of auxiliary lemmas. These need some notations, which we only mention. We refer to Adams (1975) for Sobolev spaces, to Kato (1995) for forms, and to Pazy (1983) or Engel and Nagel (2000)

for fractional powers of sectorial operators. All Laplacians or Stokes operators are sectorial. Therefore, we only consider one-side form-bounded, selfadjoint operators. Since the set Ω , that later contains all other sets, is bounded and we always consider functions which vanish on a part of the boundary of each connected component of a subset, the involved operators are even strictly positive in the form sense. For the more general cases of unbounded sets or operators which are not strictly positive in the form sense see Sohr (2001). For generalized Kato-Friedrich extensions see Kato (1995). For similar Poincaré estimates see Adams (1975).

Notation 2.1.1. Let $V \subset \mathbb{R}^n$ be an open, bounded, and connected set and let $\Gamma \subset \bar{V}$ be a closed submanifold of dimension $n - 1$. A particular choice would be $\Gamma = \partial V$. The space of test functions which vanish on Γ is denoted by

$$\mathcal{D}(\bar{V} \setminus \Gamma) := \left\{ \phi \in C^\infty(\mathbb{R}^n) \mid \text{supp } \phi \subset \bar{V} \setminus \Gamma \text{ and } \text{supp } \phi \text{ is compact} \right\},$$

the once differentiable functions which vanish on Γ are defined as closure

$$H_0^{1,2}(\bar{V} \setminus \Gamma) := \overline{\mathcal{D}(\bar{V} \setminus \Gamma)}^{H^{1,2}}$$

of these test functions

The fractional versions are introduced using the functional calculus related to the Laplacian. This can be seen as a generalized version of the $H^{s,2}(\mathbb{R}^n)$ spaces that are sometimes introduced via the Fourier transform, which is nothing but the unitary mapping that defines the functional calculus of the Laplacian on \mathbb{R}^n .

Notation 2.1.2 (Fractional Sobolev Spaces). Let $-\Delta$ denote the Laplace operator to homogeneous Dirichlet boundary values on Γ , this is formally $u = 0$ on the boundary Γ . This operator is defined on

$$D(-\Delta) = \left\{ H_0^{1,2}(\bar{V} \setminus \Gamma) \mid -\Delta u \in L^2 \right\}.$$

in the associated operator sense to the form defined by

$$\int \text{grad } u \cdot \text{grad } v.$$

It is non-negative and selfadjoint. Therefore, the operator has a functional calculus which allows us to define the abstract Sobolev spaces

$$H_0^{s,2}(\bar{V} \setminus \Gamma) := D\left(\sqrt{-\Delta}^s\right)$$

for $s > 0$.

We only need two special versions of these spaces. We consider functions $u \in H^{s,2}(\Omega)$ that vanish in the above sense on a common submanifold within Ω , and functions that vanish on the boundary. For these spaces an estimate of Poincaré-type is valid.

Lemma 2.1.3 (Poincaré Inequality in $H^{s,2}$ spaces). Let $V \subset \mathbb{R}^n$ be an open, bounded and connected set and $\Gamma \subset \bar{V}$ a closed submanifold of dimension at least $n - 1$. Let the maximal

distance of a point in V to a point in Γ be denoted by $d := \sup_{x \in V} \text{dist}(x, \Gamma) < \infty$ and let $s \geq 0$. Then all $u \in H_0^{s,2}(\bar{V} \setminus \Gamma)$ satisfy

$$(2.1.1) \quad \|u\|_{L^2(V)} \leq cd^s \|D^s u\|_{L^2(V)},$$

where c can be chosen independent of $0 \leq s \leq 1$ and V . For domains of finite width that lie between two hyperplanes of distance d , the constant c can be chosen as 1.

Sketch of proof. An integration along a path that connect an arbitrary point and a point of the set Γ , on which every function vanishes, yields an estimate for the function value in terms of the derivative. The minimal path length is bounded by d . Integration and an application of Hölder inequality yields the $s = 1$ case. See for example (Adams, 1975, 6.26). The spectral mapping principle $\sigma(f(A)) = f(\sigma(A))$ yields the claimed estimate by using the definition of $\sqrt{-\Delta}^s$. *q.e.d.*

2.1.2 Taylor expansions

We need a theorem from an introductory course analysis that characterizes the differentiability and the derivatives of a function at once.

Theorem 2.1.4 (Taylor expansion and polynomial approximation). *Let $k \in \mathbb{N}$ and a function $h : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ be given. If h is k -times differentiable at $t = 0$ then there exists a polynomial function $P \in \mathbb{R}[t]$ of degree at most k that satisfies $|h(t) - P(t)| = o(t^k)$. Furthermore, this polynomial is unique and is representable in the form $P(t) = \sum_{l=0}^k p_l \frac{t^l}{l!}$ for $p_l = (\partial_t^l h)(0)$. Especially we obtain: If a function $h : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ satisfies $|h(t)| = o(t^s)$ for some $s \in \mathbb{R}_+$, then for all integers $0 \leq l \leq s$ necessarily $(\partial_t^l h)(0) = 0$ is satisfied.*

It is worth noting that $|h(t)| = o(t^s)$ for all $s \in \mathbb{R}_+$ does not imply that h vanishes in a neighborhood; $h(t) = e^{-\frac{1}{t^2}}$ defines one of the standard examples.

2.1.3 Some auxiliary integrals

Some technical results are used in the proof of the main theorem of this section. To straighten the proof, we collect and prove them in the following auxiliary lemma.

Auxiliary Lemma 2.1.5. *Choose local coordinates such that 0 is the common point of a sphere of radius δ and its tangential plane. In appropriately chosen local coordinates this tangential plane is $\mathbb{R}^2 \times \{0\}$. Hence, a point ξ on the sphere has distance $\pm c(r)$ to this surface, where $r = |(\xi_1, \xi_2)|$ denotes the distance from the point of contact within the tangential plane, the surface point is parametrized as $\xi = (\xi_1, \xi_2, \xi_3)$, and the height is given by $c(r) = \pm\delta - \sqrt{\delta^2 - r^2}$. It satisfies*

$$(2.1.2) \quad \frac{r^2}{2\delta} \leq c(r) \leq \frac{r^2}{\delta}$$

for all $r \leq \delta$. Furthermore, since the ball is convex, the straight line connecting $(0, 0, 0)$ to $(\xi_1, \xi_2, c(r))$ lies within the ball, i.e.

$$(2.1.3) \quad s \frac{c(r)}{r} \geq c(s)$$

for all $0 \leq s \leq r \leq \delta$

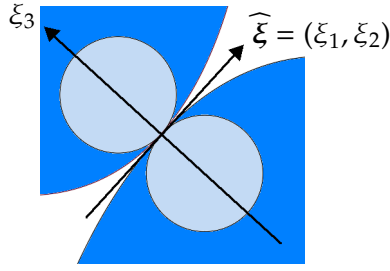


Figure 2.1.1: In local coordinates the touching of two balls is prototypic for the touching of two convex sets.

Proof. The form of $c(r)$ is an application of Pythagoras' law. Denoting by $\theta := \frac{r}{\delta} \leq 1$ for $r \leq \delta$ the relative radius, we prove the three inequalities in terms of θ :

Evaluating the squares yields that $1 - \theta^2 \leq \left(1 - \frac{1}{2}\theta^2\right)^2$ is obvious. Applying the square root and the definition of θ and $c(r)$ yields the first inequality of (2.1.2).

Since $\theta^2(\theta^2 - 1) \leq 0$ we obtain from

$$(1 - \theta^2)^2 = 1 - \theta^2 + \theta^2(\theta^2 - 1) \leq 1 - \theta^2$$

by applying the square root and reordering of the terms $1 - \sqrt{1 - \theta^2} \leq \theta^2$, which is equivalent to the second inequality of (2.1.2).

To prove the third inequality (2.1.3), it is sufficient to show that

$$h(s) := s \frac{c(r)}{r} - c(s)$$

defines a function h which is non-negative for $0 \leq s \leq r \leq \delta$. Since $h(0) = 0 = h(r)$ and

$$\frac{\partial^2}{\partial s^2} h(s) = -\frac{\delta^2 + 2s^2}{(\delta^2 - s^2)^{\frac{3}{2}}} \leq 0$$

for $s \leq r$, we obtain $h(s) \geq 0$.

q.e.d.

2.1.4 A-priori estimates for the minimal interparticle distance

The first auxiliary lemma we use quite often, yields upper and lower bounds for the interaction zone of two locally strictly convex rigid bodies. On the one hand it generalizes Lemma 2.1.3 to L^p spaces, on the other hand it uses the special structure of the interaction zone to obtain sharper estimates.

Lemma 2.1.6 (Poincaré estimate). *For $0 < r < \delta$ denote by $c(r) := \delta - \sqrt{\delta^2 - r^2}$ again the distance of a sphere of radius δ to its tangential plane. The domain C_r is defined by*

$$C_r := \left\{ \xi = (\widehat{\xi}, \xi_3)^t \in \mathbb{R}^3 \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) \leq \xi_3 \leq c(r) \right\},$$

Its lower part is denoted by

$$(2.1.4) \quad \Gamma_r := \left\{ \xi = (\widehat{\xi}, \xi_3)^t \in \mathbb{R}^3 \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) = \xi_3 \right\}$$

Then for $p \geq 2$ all $v \in H^{s,p}(C_r)$ that vanish on the part Γ_r of the boundary satisfy the Poincaré-type estimate

$$(2.1.5) \quad \|v\|_{L^1(C_r)} \leq c_\delta r^4 r^{2s-\frac{4}{p}} \|D^s v\|_{L^p(C_r)},$$

where the factor c_δ depends on the maximal radius δ and the differentiability level s and p . In detail,

$$c_\delta = \left(\frac{2}{\delta}\right)^{s+\frac{p-1}{p}} \pi^{\frac{p-1}{p}}$$

defines an upper bound of this constant. Especially, we obtain $\|v\|_{L^1(C_r)} = o\left(r^4 r^{2s-\frac{4}{p}}\right)$.

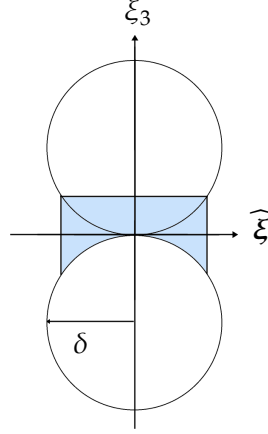


Figure 2.1.2: The prototypic interaction zone of two rigid bodies with locally strictly convex boundary is a sand-glass type domain.

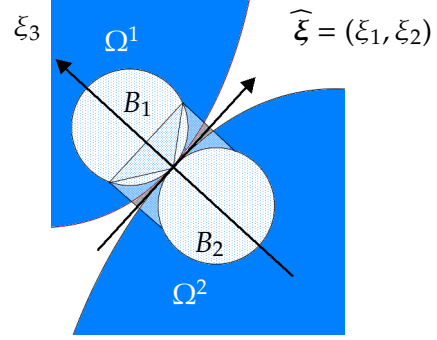


Figure 2.1.3: The two-dimensional section of two convex sets which touch with local coordinate system, shrinking cone domains C_r^1 , and extended domains C_r .

Proof. (a) The volume of the sand glass is bounded by the volume of the tube, which is its convex hull. Hence $c(r) \leq \frac{r^2}{\delta}$ yields

$$|C_r| \leq 2c(r)\pi r^2 \leq \frac{2\pi}{\delta} r^4.$$

Hölder's inequality applied to sets of finite measure yields $\|v\|_{L^1(C_r)} \leq |C_r|^{\frac{1}{2}} \|v\|_{L^2(C_r)}$ and hereby

$$(2.1.6) \quad \|v\|_{L^1(C_r)} \leq \sqrt{\frac{2\pi}{\delta}} r^2 \|v\|_{L^2(C_r)}.$$

(b) Since the Laplacian is nonnegative and selfadjoint, we can apply the spectral mapping theorem and Poincaré's inequality $\|v\|_{L^2} \leq d \|Dv\|_{L^2}$, see (Adams, 1975, 6.26) for a proof, where d is the maximal distance to a submanifold on which v vanishes, and obtain $\|v\|_2 \leq d^s \|D^s v\|_2$. Thus, using that d is bounded by $2c(r) \leq 2\frac{r^2}{\delta}$ in our case, we get

$$(2.1.7) \quad \|v\|_{L^2(C_r)} \leq \left(\frac{2}{\delta}\right)^s r^{2s} \|D^s v\|_{L^2(C_r)}.$$

(c) Applying again Hölder's inequality and the monotony of the integral yields

$$\begin{aligned}
 \|D^s v\|_{L^2(C_r)} &\leq |C_r|^{\frac{1}{2}-\frac{1}{p}} \|D^s v\|_{L^p(C_r)} \\
 (2.1.8) \qquad &\leq \left(\frac{2\pi}{\delta} r^4\right)^{\frac{1}{2}-\frac{1}{p}} \|D^s v\|_{L^p(C_r)}
 \end{aligned}$$

for all $0 < r \leq \delta$. Applying now (2.1.7), (2.1.6) and (2.1.8) sequentially yields

$$(2.1.9) \qquad \|v\|_{L^1(C_r)} \leq c_\delta \left(r^4 r^{2s-\frac{4}{p}}\right) \|D^s v\|_{L^p(C_r)},$$

for the predicted constant c_δ .

(d) Since we assumed $D^s v \in L^p(C_r)$ for one and therefore all small r , the measure $|D^s v|^p dx$ is absolutely continuous with respect to dx on C_δ , at least starting from some r . Hence, the convergence of $|C_r|$ to zero for r to zero yields that $\|D^s v\|_{L^p(C_r)}$ converges to zero as well. This justifies the little o and finishes the proof. *q.e.d.*

Ultimately, we want to show that two rigid bodies move like one body if they touch each other. This is equivalent to the statement that one body has no motion relative to the other. In a first step we show that there is neither lift-off nor penetration if we consider the motion of rigid bodies, i.e. we do not assume incompressibility of the surrounding phase. Nevertheless, to prove the three dimensional case a higher regularity of the motion close to the touching point is needed, whereas in two dimensions any weak solution is regular enough.

Proposition 2.1.7 (Regularity of centers of mass). *Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded domain. Let $B(t) \subseteq \Omega$, for $0 \leq t \leq T$, be subsets with common characteristic function $\mathbf{1}_B : [0, T] \rightarrow \{0, 1\}$ and centers of mass which are denoted by*

$$x_c(t) := \frac{1}{|B(t)|} \int_{B(t)} x \, dx.$$

- (a) *If $\mathbf{1}_{B_k}$ is of class $C([0, T]; L^p(\Omega))$ then the center of mass x_c is of class $C([0, T]; \mathbb{R}^n)$.*
- (b) *If $\mathbf{1}_{B_k}$ is of class $C^\alpha((0, T); L^p(\Omega))$ for some $0 < \alpha < 1$ then the center of mass x_c is of class $C^\alpha((0, T); \mathbb{R}^n)$.*

Proof. Applying Hölder's inequality yields for the offset in time

$$\begin{aligned}
 |x_c(t+\tau) - x_c(t)| &= \frac{1}{|B(t)|} \left| \int_{B(t+\tau) \cup B(t)} (\mathbf{1}_{B(t+\tau)} - \mathbf{1}_{B(t)}) x \, dx \right| \\
 &\leq \frac{1}{|B(t)|} \left(\int_{B(t+\tau) \cup B(t)} |x|^q \, dx \right)^{\frac{1}{q}} \|\mathbf{1}_{B(t+\tau)} - \mathbf{1}_{B(t)}\|_{L^p(\Omega)}
 \end{aligned}$$

and yields hereby continuity. The Hölder continuity allows the extra estimate

$$\leq c(\mathbf{1}_B) \left(\int_{B(t+\tau) \cup B(t)} |x|^q \, dx \right)^{\frac{1}{q}} \tau^\alpha,$$

which shows the Hölder continuity of the center of mass. For both we used that the middle term is bounded by

$$c(\Omega) := \left(\int_{\Omega} |x|^q dx \right)^{\frac{1}{q}} < \infty.$$

Hence, the centers of mass depend continuously on time in the same way the characteristic functions do. *q.e.d.*

2.1.5 Estimates for the development of the interparticle distance

Lemma 2.1.8 (Evolution of collisions). *Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded domain, let u be a given velocity field of regularity class $L^\varsigma(0, T; H_0^{s,p}(\Omega)^3)$, for some $s \geq 0$, that transports a mixture of rigid and other phases. Define the value S_{\max} by*

$$(2.1.10) \quad S_{\max} := \begin{cases} \frac{2p}{4+p(1-2s)} \frac{\varsigma-1}{\varsigma} & \text{if } 2s - \frac{4}{p} < 1 \\ \infty & \text{if } 2s - \frac{4}{p} \geq 1. \end{cases}$$

Let $\Omega^1(t)$ and $\Omega^2(t)$ denote two disjoint rigid phases within Ω , i.e. $D_{\text{sym}} u(x, t) = 0$ in $\Omega^1(t) \cup \Omega^2(t)$ for $t \in (0, T)$, and Ω^1 and Ω^2 touch at time $t_0 \in (0, T)$ in a point in which the boundaries are (locally) strictly convex and C^2 . If their distance $\text{gap}(t) := \text{dist}(\Omega^1(t), \Omega^2(t))$ is continuous at the moment t_0 then it satisfies the following:

- (a) The distance necessarily satisfies the estimate $\text{gap}(t) = o(|t - t_0|^{S_{\max}})$.
- (b) If the distance function is k -times differentiable in t_0 , where $k \in \mathbb{N}$, then all derivatives in t_0 up to the order $\min\{k, S_{\max}\}$ necessarily vanish.
- (c) In particular, if $S_{\max} \geq 1$ and the distance function is continuous at the moment of a collision then it is necessarily differentiable at this moment with vanishing derivative.

Remark 2.1.9. The n -dimensional version of S_{\max} is

$$\frac{2p}{(n+1) + p(1-2s)} \frac{\varsigma-1}{\varsigma},$$

which indicates that we need higher regularity in higher dimensions, but we only prove the new three dimensional case. For weak solutions of the Navier-Stokes equations $S_{\max} = 2$ in two dimension and $S_{\max} = 1$ in three dimensions.

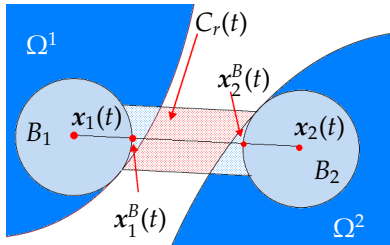


Figure 2.1.4: Two dimensional version depicting further transport after possible touching and the cylinder of radius r .

Proof. A Sobolev spaces of higher order of derivatives is contained in a space of smaller order. Furthermore, by boundedness of the domain or Sobolev-embeddings

we change p . Hence, it is sufficient to consider $2s - \frac{4}{p} < 1$. By increasing again the chosen smaller s up to the limit case $\frac{2}{p} + \frac{1}{2}$ the corresponding order passes any given convergence rate.

Denote the common point at time t_0 by x_M , $\{x_M\} = \overline{\Omega^1(t_0)} \cap \overline{\Omega^2(t_0)}$, and the two balls within these sets that touch at x_M at this point in time by B_1 and B_2 , this is

$$\begin{aligned} B_1(t_0) &= B^o(0, \delta) + x_M - \delta v_{\Gamma_1}(x_M) \subseteq \Omega^1 \\ B_2(t_0) &= B^o(0, \delta) + x_M - \delta v_{\Gamma_2}(x_M) \subseteq \Omega^2. \end{aligned}$$

More precisely, they are given by $B_k(t) := (\text{supp } \chi_k(t))^o$, where the characteristic function satisfies

$$\begin{aligned} \partial_t \chi_k + \text{div}(\chi_k u) &= 0 \\ \chi_k(t_0) &= \mathbf{1}_{B_k(t_0)}. \end{aligned}$$

We prefer the notion $\mathbf{1}_{B_k}$ instead of χ_k , though the first is defined by the second.

The centers of the balls B_1 and B_2 are denoted by x_1 and x_2 . They are given as centers of mass and satisfy

$$x_k(t) := \frac{1}{|B_k(t)|} \int_{B_k(t)} x \, dx$$

These balls and points are transported within the velocity field as is pictured in Figure 2.1.4. The positions at a time t are denoted by $B_1(t)$, $B_2(t)$, $x_1(t)$, and $x_2(t)$. The points given by the intersection of the straight line connecting $x_1(t)$ and $x_2(t)$ and the boundaries of the balls $B_1(t)$ and $B_2(t)$ are given by

$$(2.1.11) \quad x_1^B(t) := x_1(t) + \delta \frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|}$$

$$(2.1.12) \quad x_2^B(t) := x_2(t) - \delta \frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|}$$

The main contents of this proof lies within the justification of the following estimation. For some $S \in \mathbb{R}_+$,

$$(2.1.13) \quad 0 \leq \text{dist}(\Omega^1(t), \Omega^2(t)) \leq \text{dist}(B_1(t), B_2(t)) \leq |x_1^B(t) - x_2^B(t)|$$

$$(2.1.14) \quad = |x_1(t) - x_2(t)| - 2\delta =: h(t)$$

$$(2.1.15) \quad \stackrel{!}{=} o(|t - t_0|^S)$$

is true. Due to Taylor's expansion, the last inequality states that for a differentiable distance function $\text{gap}(t) = \text{dist}(\Omega^1(t), \Omega^2(t))$ all derivatives up to the order S vanish, provided they exist. If $s < \frac{1}{2} + \frac{2}{p}$, then we will see that the maximal S is given as claimed above. Otherwise, it is true for all $S \in \mathbb{R}_+$, this is, the distance function and all its derivatives that exist vanish. Since this is the difficult part, we save it for last the last steps.

1. The inequalities of the first line are rather obvious: they hold by definition of the distance of sets and the fact that $B_k(t)$ is a subset of $\Omega^k(t)$. The last is true, since B_k was at time t_0 a subset of Ω^k , and therefore is for all time $t \in (0, T)$ by the properties of solutions of the transport equation. This proves (2.1.13).

2. The equality (2.1.14) is proved by

$$\begin{aligned}
 |x_2^B(t) - x_1^B(t)|^2 &= \left| x_2(t) - x_1(t) - 2\delta \frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|} \right|^2 \\
 &= |x_2(t) - x_1(t)|^2 - 2(2\delta) |x_2(t) - x_1(t)| + 4\delta^2 \\
 &= (|x_2(t) - x_1(t)| - 2\delta)^2.
 \end{aligned}$$

Furthermore, reformulating this equation yields $|x_2(t) - x_1(t)| \geq 2\delta > 0$. This last estimate is true, since by construction $x_1^B(t)$ and $x_2^B(t)$ are points of the straight line connecting $x_1(t)$ to $x_2(t)$.

The equality (2.1.15) is proved in the remaining steps:

3. The volume of subsets is preserved, this yields $|B_k(t)| = |B_k(0)|$ for all times t , since the flow is solenoidal.

4. We introduce a new coordinate system $\xi = (\widehat{\xi}, \xi_3)$ that is fixed in $x_1^B(t)$; $\widehat{\xi}$ describes the coordinate plane perpendicular to the straight line connecting $x_1(t)$ and $x_2(t)$, and ξ_3 is the coordinate that is oriented in normal direction towards $x_2(t)$. In this coordinate system, it is easier to define sets and structures.

Especially important is the normal vector

$$e_3(t) := \frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|},$$

that appeared implicitly already in the definition of x_k^B . It is normalized, that is $e_3(t) \cdot e_3(t) \equiv 1$ for all t . Hence it satisfies at least formally $e_3(t) \cdot \partial_t e_3(t) \equiv 0$. Therefore,

$$(2.1.16) \quad x_2(t) - x_1(t) = (h(t) + 2\delta) e_3(t)$$

yields, either by formal calculation or as distributional derivatives in $\mathcal{D}^*([0, T))$,

$$\begin{aligned}
 \partial_t h(t) &= \partial_t ((x_2(t) - x_1(t)) \cdot e_3(t)) \\
 &= \partial_t (x_2(t) - x_1(t)) \cdot e_3(t) + (x_2(t) - x_1(t)) \cdot \partial_t e_3(t) \\
 &= (\partial_t x_2(t) - \partial_t x_1(t)) \cdot e_3(t) + (h(t) + 2\delta) e_3(t) \cdot \partial_t e_3(t) \\
 &= (\partial_t x_2(t) - \partial_t x_1(t)) \cdot e_3(t).
 \end{aligned}$$

5. Since u is within $\Omega^k(t)$ a rigid-velocity field, the velocity has for every point in time and parameter $k = 1, 2$ a representation

$$(2.1.17) \quad u(x, t) = R_k(t) (x - x_k(t)) + U_k(t) \quad \text{for } x \in \Omega^k(t),$$

where $R_k(t) \in \mathcal{L}(\mathbb{R}^3)$ are appropriate skew-symmetric matrices and $U_k(t) \in \mathbb{R}^3$ are vectors. We define the motion relative to Ω^1 by

$$v(x, t) := u(x, t) - R_1(t)(x - x_1(t)) - U_1(t) \quad \text{for } x \in \Omega.$$

Hence, for $x \in \Omega^1(t)$ we obtain $v(x, t) = 0$, and for $x \in \Omega^2(t)$

$$(2.1.18) \quad v(x, t) = R_2(t) (x - x_2(t)) - R_1(t) (x - x_1(t)) + U_2(t) - U_1(t).$$

Using (2.1.17) formally or by calculation of the distributional derivatives in $\mathcal{D}^*([0, T))$ of $x_k(t)$, as defined by (2.1.11, 2.1.12), we get

$$\partial_t x_k(t) = u(x_k(t), t) = U_k(t).$$

Hence, $\partial_t x_2(t) - \partial_t x_1(t) = \mathbf{U}_2(t) - \mathbf{U}_1(t)$ and the above calculations of $\partial_t h$ yield

$$\partial_t h(t) = (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \mathbf{e}_3(t).$$

6. We define now special sets, shown in Figure 2.1.4, that will be needed. To simplify their definition, we'll use the above introduced coordinate system and set, using $\xi = (\widehat{\xi}, \xi_3) = \xi(x, t)$,

$$C_r(t) := \left\{ x \in \Omega \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) \leq \xi_3 \leq h(t) + c(|\widehat{\xi}|) \right\}.$$

The three parts of the boundary $\partial C_r(t)$ are given by

$$\begin{aligned} \Gamma_r^0(t) &:= \left\{ x \in \Omega \mid |\widehat{\xi}| = r \text{ and } -c(|\widehat{\xi}|) \leq \xi_3 \leq h(t) + c(|\widehat{\xi}|) \right\}, \\ \Gamma_r^1(t) &:= \left\{ x \in \Omega \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) = \xi_3 \right\}, \\ \Gamma_r^2(t) &:= \left\{ x \in \Omega \mid |\widehat{\xi}| \leq r \text{ and } \xi_3 = h(t) + c(|\widehat{\xi}|) \right\}. \end{aligned}$$

7. Two properties of $\Gamma_r^2(t)$ are used later: First, $\Gamma_r^2(t)$ is equivalently characterized as $S_{\mathbb{R}^3}(x_2(t), \delta) \cap C_r(t)$, and, using (2.1.16) therefore yields, that every point x of $\Gamma_r^2(t)$ satisfies

$$(2.1.19) \quad x := x_2(t) + \delta \nu_{\Gamma_r^2(t)}(x) := x_1(t) + (h(t) + 2\delta) \mathbf{e}_3(t) + \delta \nu_{\Gamma_r^2(t)}(x),$$

where ν , which depends on space and time, denotes the outer normal vector with respect to one of the balls. Then the surface integral $\int_{\Gamma_r^2(t)} \nu \, dx$ is a multiple of $\mathbf{e}_3(t)$, as can be seen by integration on a segment of the sphere $S_{\mathbb{R}^3}(x_2(t), \delta)$:

$$(2.1.20) \quad \int_{\Gamma_r^2(t)} \nu \, dx = \int_0^r \int_0^{2\pi} \begin{bmatrix} -\frac{\sigma^2 \cos \theta}{\delta - \sqrt{\delta^2 - \sigma^2}} \\ \frac{\sigma^2 \sin \theta}{\delta - \sqrt{\delta^2 - \sigma^2}} \\ \sigma \end{bmatrix} d\theta \, d\sigma = \pi r^2 \mathbf{e}_3(t).$$

8. By construction¹, $v(t) = 0$ on $\Gamma_r^1(t)$ and $v(t) \in H_\sigma^s(C_r(t))$. Hence,

$$\begin{aligned} 0 &= \int_{C_r(t)} \operatorname{div} v(t) = \int_{\partial C_r(t)} v(t) \cdot \nu \\ &= \int_{\Gamma_r^0(t)} v(t) \cdot \nu + \int_{\Gamma_r^1(t)} v(t) \cdot \nu + \int_{\Gamma_r^2(t)} v(t) \cdot \nu, \end{aligned}$$

or

$$\begin{aligned} \int_{\Gamma_r^0(t)} v(t) \cdot \nu &= - \int_{\Gamma_r^2(t)} v(t) \cdot \nu \\ &= - \int_{\Gamma_r^2(t)} \left(R_2(t)(x - x_2(t)) - R_1(t)(x - x_1(t)) + \mathbf{U}_2(t) - \mathbf{U}_1(t) \right) \cdot \nu_x \, dx \end{aligned}$$

¹Our notation is at this point a little misleading: $\int_M f = \int_M f(x) \, dx$ always denotes integration with respect to the natural measure of M . So, if M is $n-1$ dimensional, it is the $n-1$ -dimensional Lebesgue-measure; if M is n dimensional, the n -dimensional version, and, furthermore, dx denotes integration with respect to the space variables of the function v , that depends also on time. Furthermore, for a function like v depending on x and t , we write $v(t)$ instead of $v(\cdot, t)$.

We'll consider this integral in three steps: We use the characterization (2.1.19) and the skew-symmetry of $R_k(t)$ to see that

$$\int_{\Gamma_r^2(t)} R_2(t) (\mathbf{x} - \mathbf{x}_2(t)) \cdot \mathbf{v}_x \, d\mathbf{x} = \delta \int_{\Gamma_r^2(t)} R_2(t) \mathbf{v} \cdot \mathbf{v} = 0,$$

that

$$\begin{aligned} \int_{\Gamma_r^2(t)} R_1(t) (\mathbf{x} - \mathbf{x}_1(t)) \cdot \mathbf{v}_x \, d\mathbf{x} &= \int_{\Gamma_r^2(t)} R_1(t) \left((h(t) + 2\delta) \mathbf{e}_3(t) + \delta \mathbf{v}_{\Gamma_r^2(t)}(\mathbf{x}) \right) \cdot \mathbf{v}_x \, d\mathbf{x} \\ &= (h(t) + 2\delta) R_1(t) \mathbf{e}_3(t) \cdot \int_{\Gamma_r^2(t)} \mathbf{v} + \delta \int_{\Gamma_r^2(t)} R_1(t) \mathbf{v} \cdot \mathbf{v} \\ &= \pi r^2 (h(t) + 2\delta) R_1(t) \mathbf{e}_3(t) \cdot \mathbf{e}_3(t) + \delta \int_{\Gamma_r^2(t)} R_1(t) \mathbf{v} \cdot \mathbf{v} = 0, \end{aligned}$$

and that

$$\begin{aligned} \int_{\Gamma_r^2(t)} (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \mathbf{v} &= (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \int_{\Gamma_r^2(t)} \mathbf{v} \\ &= \pi r^2 (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \mathbf{e}_3(t). \end{aligned}$$

Hence, the combination of the these considerations yields

$$(2.1.21) \quad \int_{\Gamma_r^0(t)} \mathbf{v}(t) \cdot \mathbf{v} = -\pi r^2 (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \mathbf{e}_3(t).$$

This equation actually is true for all $r < \delta$.

9. By definition of the points $\mathbf{x}_k(t)$ the last considerations yield

$$\begin{aligned} \pi r^2 \partial_t h(t) &= \pi r^2 (\partial_t \mathbf{x}_2(t) - \partial_t \mathbf{x}_1(t)) \cdot \mathbf{e}_3(t) \\ &= \pi r^2 (\mathbf{U}_2(t) - \mathbf{U}_1(t)) \cdot \mathbf{e}_3(t) = \int_{\Gamma_r^0(t)} \mathbf{v}(t) \cdot \mathbf{v}, \end{aligned}$$

or, after integration and application of Hölder's inequality

$$\begin{aligned} \frac{1}{3} \pi r^3 |\partial_t h(t)| &\leq \int_0^r \int_{\Gamma_\sigma^0(t)} |\mathbf{v}(\mathbf{x}, t)| \, d\mathbf{x} \, d\sigma \leq \int_{C_r(t)} |\mathbf{v}(\mathbf{x}, t)| \, d\mathbf{x} \\ &\leq |C_r(t)|^{\frac{1}{2}} \|\mathbf{v}(t)\|_{L^2(C_r(t))} \\ (2.1.22) \quad &\leq |C_r(t)|^{\frac{1}{2}} (h(t) + 2c(r))^s \left\| \mathbf{D}^s \mathbf{v}(t) \right\|_{L^2(C_r(t))} \end{aligned}$$

$$(2.1.23) \quad \leq |C_r(t)|^{\frac{p-1}{p}} (h(t) + 2c(r))^s \left\| \mathbf{D}^s \mathbf{v}(t) \right\|_{L^p(C_r(t))}.$$

Since $C_r(t)$ is a subset of the full cylinder, we obtain $|C_r(t)| \leq \pi r^2 (h(t) + 2c(r))$. Furthermore, integrability over the set $C_\delta(t)$ and absolute continuity yield $\left\| \mathbf{D}^s \mathbf{v}(t) \right\|_{L^p(C_r(t))} = o(r^0)$. Therefore, for every $0 < r < \delta$

$$(2.1.24) \quad \frac{1}{3} \pi r^3 |\partial_t h(t)| \leq \pi^{\frac{p-1}{p}} r^{2\frac{p-1}{p}} (h(t) + 2c(r))^{s+\frac{p-1}{p}} \left\| \mathbf{D}^s \mathbf{v} \right\|_{L^p(C_r(t))}.$$

or, equivalently,

$$(2.1.25) \quad |\partial_t h(t)| \leq C r^{-\frac{p+2}{p}} (h(t) + 2c(r))^{s+\frac{p-1}{p}} \|D^s \mathbf{v}\|_{L^p(C_r(t))}.$$

By assumption, the two sets touch at time t_0 , i.e. $h(t_0) = 0$. Hence, for all t close to t_0 continuity of h yields $h(t) < \delta^2$. Therefore, since (2.1.25) is valid for any r , we may use for every t a special radius, namely $r = \sqrt{h(t)}$ for all t close to t_0 and obtain

$$|\partial_t h(t)| \leq \underbrace{C \|D^s \mathbf{v}(t)\|_{L^p(C_r(t))}}_{=: a_r(t)} h(t)^{s+\frac{p-4}{2p}}.$$

10. We can assume that the characteristics don't leave Ω , for example by posing the homogeneous Dirichlet boundary condition $\mathbf{u} = 0$ or any other condition that guarantees $C_r(t) \subseteq \Omega$ for all t close to t_0 . Since we defined $a_r(t) = C \|D^s \mathbf{v}(t)\|_{L^p(C_\delta(t))}$ we obtain from $\mathbf{u} \in L^\zeta(t_0 - \varepsilon, t_0 + \varepsilon; H^{s,p}(\Omega))$ the integrable upper bound

$$a_r(t) \leq c |C_r(t)|^{\frac{1}{p}} \|D^s \mathbf{v}(t)\|_{L^p(\Omega)}$$

and that a_r satisfies $\int_{t_0}^t a_r(\sigma)^\zeta d\sigma \rightarrow 0$ for t to t_0 and for $r \rightarrow 0$.

11. So we obtain, without loss of generality for $t \geq t_0$ using $h(t_0) = 0$ that

$$\begin{aligned} |h(t)|^{1-\frac{(2s+1)p-4}{2p}} &= |h(\sigma)|^{1-\frac{(2s+1)p-4}{2p}} \Big|_{\sigma=t_0}^{\sigma=t} \\ &= \int_{t_0}^t \left(1 - \frac{(2s+1)p-4}{2p}\right) |h(\sigma)|^{-\frac{(2s+1)p-4}{2p}} \partial_\sigma h(\sigma) d\sigma \\ &\leq \left(1 - \frac{(2s+1)p-4}{2p}\right) \int_{t_0}^t a_r(\sigma) d\sigma \\ &\leq \left(1 - \frac{(2s+1)p-4}{2p}\right) (t-t_0)^{\frac{\zeta-1}{\zeta}} \left(\int_{t_0}^t a_r(\sigma)^\zeta d\sigma\right)^{\frac{1}{\zeta}} = o\left(|t-t_0|^{\frac{\zeta-1}{\zeta}}\right) \end{aligned}$$

or

$$(2.1.26) \quad |h(t)| = o\left(|t-t_0|^{\frac{2p}{4+p(1-2s)} \frac{\zeta-1}{\zeta}}\right).$$

Hence, a Taylor expansion up to the integer order $k \leq \frac{2p}{4+p(1-2s)} \frac{\zeta-1}{\zeta}$ necessarily vanishes. Therefore all derivatives $\partial_t^k h$ must vanish in t_0 up to this order. The claims (a) and (b) directly follow from this, since $0 \leq \text{gap}(t) \leq h(t)$ by (2.1.14):

$$(2.1.27) \quad |\text{gap}(t)| = o\left(|t-t_0|^{\frac{2p}{4+p(1-2s)} \frac{\zeta-1}{\zeta}}\right).$$

12. We used continuity of the distance $\text{gap}(t)$ to prove the above estimates, but differentiability with vanishing derivative results from

$$\frac{\text{gap}(t) - \text{gap}(t_0)}{t - t_0} = \frac{\text{gap}(t)}{t - t_0} = o\left(|t-t_0|^{s_{\max}-1}\right),$$

which proves (c).

q.e.d.

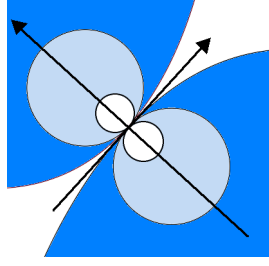


Figure 2.1.5: Iteration of the proof for different radii δ yields that no rotations are possible that do not preserve normal to the common tangential plane

Remark 2.1.10. (a) We actually proved a more general estimate. For any choice of touching balls $B_1(t) = B(x_1(t), \delta)$ and $B_2(t) = B(x_2(t), \delta)$ of radius δ , we have

$$(2.1.28) \quad \text{dist}(B(x_1(t), \delta) - B(x_2(t), \delta)) = o\left(|t - t_0|^{\frac{2p}{4+p(1-2s)} - \frac{\epsilon-1}{\epsilon}}\right),$$

if δ is small enough to be a touching ball to the boundary of a rigid body filled domain. This is just a reformulation of equation (2.1.26) due to equation (2.1.14), but yields already estimates on the relative rotations. We are not pursuing this path, since in the next chapter better estimates will be proved. Hence, we only sketch the idea here. The balls contained in on common set all move as parts of a rigid body. Hence, the distance of two arbitrary points is constant. Especially, these two balls have no relative rotation. The distance between the two bigger balls as well as the distance between the two smaller balls are of order

$$o\left(|t - t_0|^{\frac{2p}{4+p(1-2s)} - \frac{\epsilon-1}{\epsilon}}\right).$$

Hence, a relative rotation of the two small balls relative to the axis connecting the two bigger balls is not plausible.

(b) Boundedness of the containing set Ω provides many estimates that otherwise need to be assumed. If for some $\varepsilon > 0$ we knew that u is locally in time essentially bounded in space and time, we would obtain $C_r(t) \subseteq \Omega$ for all t close to t_0 that satisfy

$$|t - t_0| < \frac{1}{c} \text{dist}(C_r(t_0), \partial\Omega).$$

In physics often classical solution are considered which either vanish rapidly at infinity or are assumed to have finite speed of propagation. Hence, such solutions satisfy the L^∞ -estimate assumed above. Therefore, these solutions satisfy in a neighborhood of the interaction zone the above used assumptions.

2.2 Motion of bodies in incompressible fluids

2.2.1 Conservation of energy and weak solutions

Researchers working with weak solutions to the Navier-Stokes equations believe that at least an energy-inequality of the form

$$(2.2.1) \quad \frac{1}{2} \int_{\Omega} \varrho(t) |u(t)|^2 + \mu \int_0^t \int_{\Omega} |\text{grad } u|^2 \leq \frac{1}{2} \int_{\Omega} \varrho(0) |u(0)|^2 + \int_0^t \int_{\Omega} \varrho g \cdot u$$

should be satisfied for a realistic velocity field \mathbf{u} , where \mathbf{u} , ϱ and \mathbf{g} depend on space and time. Hence, velocity fields are only considered as possible candidates for a solution if they are at least from the class

$$(2.2.2) \quad L^2_{\text{loc}}([0, T]; H^{1,2}(\Omega)^n) \cap L^\infty_{\text{loc}}([0, T]; L^2(\Omega)^n).$$

Nevertheless, not even the velocity of the already highly idealized Poiseuille's flow that even satisfies the Navier-Stokes equations ($\mu > 0$) and the Euler equations ($\mu = 0$) everywhere in the classical sense, is contained in this class if the pipe is unbounded and the flow through a cross section is nonzero. Vice versa, it is unknown if every weak solution of the class of equation (2.2.2) satisfies the energy estimate (2.2.1). Hence, one needs to show that the energy estimate is satisfied for a constructed weak solution.

Furthermore, it is reasonable to hope that at least the centers of mass of the rigid bodies in motion move continuously in time. For balls $\Omega^1_{\text{solid}}(t) = B(\mathbf{x}_c^1(t), r^1)$ and $\Omega^2_{\text{solid}}(t) = B(\mathbf{x}_c^2(t), r^2)$ this is enough to prove that the distance function of these is continuous in time, since

$$\text{dist}(\Omega^1_{\text{solid}}(t), \Omega^2_{\text{solid}}(t)) = |\mathbf{x}_c^1(t) - \mathbf{x}_c^2(t)| - r^1 - r^2.$$

Under these rather mild assumptions we obtain that balls do collide at most in the least spectacular form:

Theorem 2.2.1. *Let $\mathbf{u} \in L^2_{\text{loc}}([0, T], H^{1,2}_{0,\sigma}(\Omega)^3) \cap L^\infty_{\text{loc}}([0, T], L^2(\Omega)^3)$ be a velocity field that is supposed to model the transport of two rigid bodies within a bounded set or to model the motion of a rigid body within a bounded set. Let $\Omega^1_{\text{solid}}(t)$ be one of these bodies and $\Omega^2_{\text{solid}}(t)$ be either another body or constantly the exterior set $\mathbb{R}^3 \setminus \overline{\Omega}$. Suppose*

$$\text{gap}(t) := \text{dist}(\Omega^1_{\text{solid}}(t), \Omega^2_{\text{solid}}(t))$$

is continuous at the moment t_0 and $\text{gap}(t_0) = 0$, i.e. $\Omega^1_{\text{solid}}(t_0)$ and $\Omega^2_{\text{solid}}(t_0)$ collide. Let the common boundary $\overline{\Omega^1_{\text{solid}}(t_0)} \cap \overline{\Omega^2_{\text{solid}}(t_0)}$ contain at least one point at which both sets are strictly convex. Then

$$\text{gap}(t) = o(|t - t_0|)$$

Especially we obtain that $\partial_t \text{gap}(t_0)$ exists and equals 0.

Proof. The regularity assumptions formulated for a weak solution and Lemma 2.1.8 yield that $S_{\max} = 1$ in three dimensions. Hence, $\text{gap}(t) := o(|t - t_0|)$ yields that the first derivative exists by definition of the first derivative,

$$\left| \frac{\text{gap}(t) - \text{gap}(t_0)}{t - t_0} - 0 \right| = o(1).$$

Which yields the claims.

q.e.d.

2.2.2 Velocity fields that describe the transport almost everywhere

Our notation of strong solution is a weak solution with enough regularity such that the differential equations can be given a point-wise meaning. For solutions of Navier-Stokes equations these assumptions are the following, which yield continuous solutions $\Xi \in C([0, T]; L^p(\Omega)^3)$ of

$$\begin{aligned}\partial_t \Xi(\xi, s; t) &= \mathbf{u}(\Xi(\xi, s; t), t) \\ \Xi(\xi, s; s) &= \xi\end{aligned}$$

for $s, t \in (0, T)$ and $\xi \in \Omega$. Hence our assumption on continuity is satisfied and yields the following corollary.

Corollary 2.2.2. *Let \mathbf{u} be a velocity field that is supposed to model the transport of rigid bodies within an incompressible fluid weak solution that has the regularity of a strong solution. In particular \mathbf{u} satisfies*

$$\begin{aligned}\mathbf{u} &\in L_{loc}^\infty([0, T]; H_{0,\sigma}^1(\Omega)) \cap L_{loc}^2([0, T], H^{2,2}(\Omega)^3) \\ \partial_t \mathbf{u} &\in L_{loc}^2([0, T], L_\sigma^2(\Omega)^3).\end{aligned}$$

Then all temporal derivatives of the distance between two bodies necessarily have to vanish at the moment they collide provided they exist.

Proof. The regularity assumptions posed on a strong solution and Lemma 2.1.8 yield that $S_{\max} = \infty$. Hence, for all $S \geq 1$ the estimate $\text{gap}(t) := o(|t - t_0|^S)$ is valid, as claimed. Especially we see, that the first derivative exists, as for weak solutions, by definition of the first derivative, since

$$\left| \frac{\text{gap}(t) - \text{gap}(t_0)}{t - t_0} - 0 \right| = o(1).$$

Equivalently, we can formulate this as a Taylor expansion: Let $S \in \mathbb{N}$ then

$$\text{gap}(t) = \sum_{k=0}^S 0 \frac{(t - t_0)^k}{k!} + o(|t - t_0|^S).$$

which yields the derivatives of higher order, provided they exist.

q.e.d.

2.2.3 Classical description of motion bodies

Especially in computer graphics an again different description of the motion of submerged bodies is sometimes used. On page 2 we introduced the elements of $\text{se}(\mathbb{R}^3)$ in equation (1.1.5) for a single body in motion. For N bodies the idea is iterated. The possible tangential vectors of a configuration of N rigid bodies suspended in an

incompressible fluid can be described in the form

$$\partial_t \mathbf{X}(t) = \begin{pmatrix} \begin{matrix} 0 & \omega_3^1(t) & -\omega_2^1(t) & U_1^1(t) \\ -\omega_3^1(t) & 0 & \omega_1^1(t) & U_2^1(t) \\ \omega_2^1(t) & \omega_1^1(t) & 0 & U_3^1(t) \\ 0 & 0 & 0 & 0 \end{matrix} & 0_{\mathbb{R}^{4 \times 4}} & \dots \\ \vdots & \ddots & \ddots \\ \begin{matrix} 0 & \omega_3^N(t) & -\omega_2^N(t) & U_1^N(t) \\ -\omega_3^N(t) & 0 & \omega_1^N(t) & U_2^N(t) \\ \omega_2^N(t) & \omega_1^N(t) & 0 & U_3^N(t) \\ 0 & 0 & 0 & 0 \end{matrix} & 0_{\mathbb{R}^{4 \times 4}} & \dots \end{pmatrix},$$

which has in three dimensions at most $6N$ -degrees of freedom.

Corollary 2.2.3. *Let \mathbf{u} be a classical solution of any model that describes the transport of rigid bodies suspended in an incompressible fluid, then the particles touch each other at a moment in time if and only if they do so initially. Especially the above introduced $\mathbf{X}(t)$ does not change its degrees of freedom in time.*

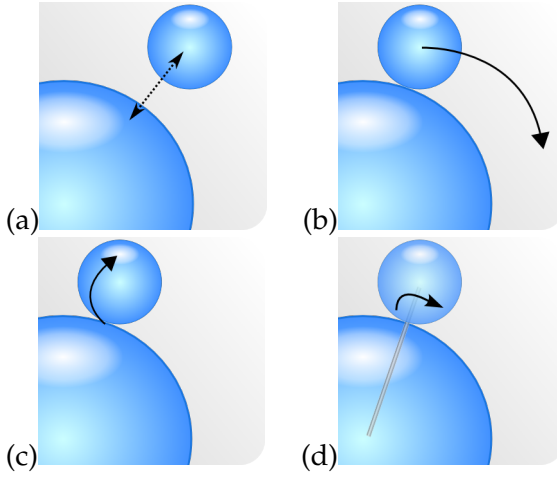


Figure 2.2.1: Rolling on top of the common tangential space could be observable: For weak solutions with continuous motion of the submerged bodies we excluded the motion depicted in (a). Nevertheless, the motions in (b),(c), and (d) could still be observable.

Posed in terms of the evolution $(X(t))_{t \geq 0}$ we obtain that $X(t)$ stays on a submanifold that describes the motion of bodies which at any point in time have the same number of common boundary points. For example to describe the flow of two submerged bodies that touch at a certain time, i.e. with 3 degrees of freedom for the common \mathbf{U} and one for the relative rotation $\omega_3 e_3$, four variables are sufficient for all t . Similarly for all pairs of submerged bodies that share a common boundary point once in time/for all times. By iteration we obtain a balls-on-sticks motion similar to the molecular models of chemistry.

Existence of solutions

We consider a model that describes the motion of several rigid particles which are submerged in an incompressible fluid. Our weak formulation neglects surface forces and enables us to construct a global solution. We achieve this by extending a construction procedure of San Martín, Starovoitov, and Tucsnak (2003) from two to three dimensions. The solution is obtained as limit of a sequence of solutions of approximative models. These approximative models describe binary fluids where the viscosity of one of the species tends to infinity with increasing model index. The hereby obtained solution describes an almost always collisions-free motion.

3.1 Introduction

We construct our solution by following the steps of the existence proof of San Martín et al. (2003). We only deviate if results need to be adapted, notations collide with other notations in three dimensions or are less compatible with our tool box reference Sohr (2001). In the introduction we introduce the way we name variables, functions, and fields. Their used variants are introduced later as needed. Furthermore, we describe the setup of this chapter.

3.1.1 Notations and Setup

Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set that represents the domain occupied by the fluid and N rigid bodies. We denote by $\Omega_{\text{fluid}}(t)$ the domain occupied by the fluid and by $\Omega_{\text{solid}}^i(t)$, $i = 1, \dots, N$, the domains occupied by the rigid bodies at time t . We set $\Omega_{\text{solid}} = \cup_{i=1}^N \Omega_{\text{solid}}^i$ for the solid filled domain and denote the external set as $\Omega_{\text{solid}}^0 := \mathbb{R}^3 \setminus \overline{\Omega}$, which represents a solid body not in motion.

The full system of equations that model the motion of rigid bodies submerged in a viscous Newtonian fluid can be written as follows. For all time within the fluid filled subset $\Omega_{\text{fluid}}(t)$

$$(3.1.1) \quad \varrho_{\text{fluid}} (\partial_t \mathbf{u} + \text{grad } \mathbf{u} \mathbf{u}) = \text{div} (2\mu D_{\text{sym}} \mathbf{u}) - \text{grad } p + \varrho_{\text{fluid}} \mathbf{g}$$

$$(3.1.2) \quad \text{div } \mathbf{u} = 0$$

$$(3.1.3) \quad \partial_t \varrho + \text{div} (\varrho \mathbf{u}) = 0$$

should be satisfied. We couple fluid and solid velocity by assuming continuity of the velocity fields on the common boundaries $\partial\Omega_{\text{solid}}^i(t)$. Within the solids $x \in \Omega_{\text{solid}}^i(t)$, $i = 1, \dots, N$,

$$(3.1.4) \quad M^i \partial_t^2 x_c^i(t) = - \int_{\partial\Omega_{\text{solid}}^i(t)} \mathbf{T} \nu + \int_{\Omega_{\text{solid}}^i(t)} \varrho g,$$

$$(3.1.5) \quad J^i \partial_t \omega^i(t), = - \int_{\partial\Omega_{\text{solid}}^i(t)} \mathbf{T} \nu \times (x - x_c^i(t)) dx + \int_{\Omega_{\text{solid}}^i(t)} \varrho g \times (x - x_c^i(t)) dx$$

should be satisfied. As initial values we assume that $u(x, 0) = u_0(x)$ and

$$(3.1.6) \quad \varrho(x, 0) = \varrho_{\text{fluid}} \mathbf{1}_{\Omega_{\text{fluid}}(0)}(x) + \varrho_{\text{solid}} \mathbf{1}_{\Omega_{\text{solid}}(0)}(x)$$

for $x \in \Omega$ are given. Furthermore, the initial positions $\Omega_{\text{solid}}^i(0) = \Omega_{\text{solid}}^{i0}$ and therefore $\Omega_{\text{solid}}(0) = \cup_{i=1}^N \Omega_{\text{solid}}^i(0)$ are given. Furthermore the equations for the motion of the centers of mass are implicitly given by

$$x_c^i(0) = x_c^{i0}, \quad \partial_t x_c^i(0) = v^{i0}, \quad \partial_t \omega^i(0) = \omega^{i0}.$$

We do not model non-trivial surface forces. Hence, coupling of weak and strong formulation is obtained through integration by parts. The hereby obtained surface integrals represent forces due to non-trivial surface forces and can be neglected.

The variables x and ξ always denote spatial positions. By t we denote time. The unknown functions all depend on space and time and are given as follows. The velocity of the fluid is denoted by u . The density distribution of the fluid is $\varrho_{\text{fluid}} \mathbf{1}_{\Omega_{\text{fluid}}}$, where $\varrho_{\text{fluid}} > 0$ is the scalar value of the mass-density of the fluid. Similarly, the density distribution of the solids is $\varrho_{\text{solid}} \mathbf{1}_{\Omega_{\text{solid}}}$, where $\varrho_{\text{solid}} > 0$ is the scalar value of the mass-density of the solids. We actually do not use that ϱ_{solid} is constant. For example for an arbitrary function ϱ_{solid} , which is bounded above and below, we obtain that $\varrho(0)$ is above and below initially. A property which is preserved by Equation (3.1.6) for the transported version $\varrho(t)$. The centers of mass of the solids $\Omega_{\text{solid}}^i(t)$ are denoted as $x_c^i(t)$. Their angular velocity is denoted as $\omega^i(t)$.

We denote by g external forces per unit mass, by $\mu > 0$ the viscosity or more precisely the inverse of the Reynold's number, by M^i the total mass of Ω_{solid}^i , by $J^i(t)$ the inertial momentum of $\Omega_{\text{solid}}^i(t)$, by $\mathbf{T} = 2\mu D_{\text{sym}} u - p \text{Id}$ the Newtonian stress tensor of the fluid, and by p the fluid pressure.

Notations of integrals are abbreviated in a way which should be meaningful from the context. We often write $\int_{\Omega} f$ instead of $\int_{\Omega} f(x) dx$ or $\int_{\Omega} f(t)$ instead of $\int_{\Omega} f(x, t) dx$ for time-dependent functions.

3.1.2 Contents

We prove existence in the following steps. These steps form next sections.

2. Notation and presentation of main results
3. Description of the main steps of the existence proof and the approximation scheme
4. Properties of the space $K(\chi)$ of velocity fields that preserve a rigid body at $\text{supp } \chi$ for a characteristic function χ . This Hilbert space is given by

$$K(\chi) := \{u \in H_0^1(\Omega)^3 \mid \text{div } u = 0, \chi D_{\text{sym}} u = 0\}.$$

5. Solving the transport equations using the method of DiPerna and Lions.
6. Estimates and convergence properties of the sequence of approximative solutions.
7. Compactness of the sequence of approximative solutions and strong convergence of a subsequence in $L^2(0, T; L^2(\Omega)^3)$.
8. Conduct of the existence proof and considerations on collisions for the constructed solution.

3.2 Notation and presentation of main results

Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set with C^2 -boundary $\partial\Omega$. For a vector valued function $\mathbf{u} = [u_k]_k \in L^2(\Omega)^3$ we denote its symmetric part of the gradient by

$$D_{\text{sym}} \mathbf{u} = \frac{1}{2} [u_{k,l} + u_{l,k}]_{kl},$$

where we used the shorthands $u_{k,l} = \partial_{x_l} u_k$, $k, l = 1, 2, 3$. In our setting with Dirichlet boundary conditions the symmetric derivative and the divergence can always be calculated in the distributional sense. As function spaces we use the spaces of solenoidal functions

$$\begin{aligned} \mathcal{D}_\sigma(\Omega) &:= \left\{ \boldsymbol{\phi} \in \mathcal{D}(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0 \right\} \\ L_\sigma^2(\Omega) &:= \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega)^3 \\ H_{0,\sigma}^1(\Omega) &:= \text{the closure of } \mathcal{D}_\sigma(\Omega) \text{ in } H^1(\Omega)^3. \end{aligned}$$

The subscript σ distinguishes solenoidal function spaces from their classical counterparts. The H -spaces are always obtained as closure of spaces, whereas the W -spaces are obtained by restriction from distributional spaces. In general many counter-intuitive behaviors are observed. In our setting of bounded, regular sets and $p < \infty$ the strong and weak approaches yield identical spaces, see Sohr (2001) for details. Hence, we have

$$\begin{aligned} L_\sigma^2(\Omega) &= \left\{ \mathbf{u} \in L^2(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, (\mathbf{u} \cdot \boldsymbol{\nu}) \in H^{-\frac{1}{2}}(\partial\Omega) \right\} \\ H_{0,\sigma}^1(\Omega) &= \left\{ \mathbf{u} \in H^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, \mathbf{u} = 0 \in H^{\frac{1}{2}}(\partial\Omega) \right\} \\ &= \left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0 \right\}. \end{aligned}$$

Furthermore we define for $1 \leq p < \infty$

$$\begin{aligned} H_\sigma^{1,p}(\Omega) &:= \left\{ \mathbf{u} \in H^{1,p}(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0 \right\} \\ H_{0,\sigma}^{1,p}(\Omega) &:= \left\{ \mathbf{u} \in H_0^{1,p}(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0 \right\}. \end{aligned}$$

For a characteristic function χ we denote the space of compatible velocities by

$$(3.2.1) \quad K(\chi) = \left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, \chi D_{\text{sym}} \mathbf{u} = 0 \right\},$$

and observe the following elementary properties.

Lemma 3.2.1. *The space $K(\chi)$ has the following properties:*

(a) The set $K(\chi)$ is a closed subspace of $H_{0,\sigma}^1(\Omega)$.

(b) For every connected subset of the support $\text{supp } \chi$ there exists a skew symmetric matrix $R \in \mathcal{L}(\mathbb{R}^3)$ and a vector $U \in \mathbb{R}^3$ such that $\mathbf{u}(\mathbf{x}) = R\mathbf{x} + U$ for all $\mathbf{x} \in U$. Since every skew symmetric matrix in \mathbb{R}^3 is defined by a vector $\boldsymbol{\omega} \in \mathbb{R}^3$, we obtain equivalently $\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega} \times \mathbf{x} + U$ for all $\mathbf{x} \in U$.

Proof. The proof of (a) follows from the continuity of the bounded linear operator $\chi D_{\text{sym}} \in \mathcal{L}(H_0^1(\Omega)^3; L^2(\Omega)^{3 \times 3})$. The second claim, (b), in this form will be used quite often and will be proved later on again. The idea is the central idea of the proof an idea from elasticity theory: if $D_{\text{sym}} \mathbf{u} = 0$, then for all $k, l, m = 1, 2, 3$ we obtain

$$2u_{k,lm} = -(u_{l,mk} + u_{m,lk}) = -(u_{l,m} + u_{m,l})_k = 0.$$

Hence, $D^2 \mathbf{u} = 0$ or $\mathbf{u}(\mathbf{x}) = R\mathbf{x} + U$. Using $R + R^* = 2 D_{\text{sym}} \mathbf{u} = 0$ yields skew symmetry. The expression $\boldsymbol{\omega} \times \mathbf{x} + U$ is just a reformulation. *q.e.d.*

By C^2 -regularity we obtain the following.

Proposition 3.2.2. *There exists a minimal $\delta > 0$ such that for all $i = 0, \dots, N$, i.e. including the external domain, for all $\mathbf{x} \in \Omega_{\text{solid}}^i(0)$ an open ball $B(\boldsymbol{\xi}, \delta) \subset \Omega_{\text{solid}}^i(0)$ exists such that $\mathbf{x} \in B(\boldsymbol{\xi}, \delta)$.*

Notation 3.2.3. We fix this value of δ of the last proposition throughout this work. Furthermore, we introduce for $\tau > 0$ the τ -neighborhood of a set $G \subseteq \mathbb{R}^3$ by

$$(3.2.2) \quad G^\tau := \{\mathbf{x} \in \mathbb{R}^3 \mid \text{dist}(\mathbf{x}, G) < \tau\}$$

and especially for the above mentioned δ the δ -neighborhood

$$(3.2.3) \quad G^{\text{ext}} := \{\mathbf{x} \in \mathbb{R}^3 \mid \text{dist}(\mathbf{x}, G) < \delta\}$$

and the δ -kernel

$$(3.2.4) \quad G^{\text{int}} := \{\mathbf{x} \in G \mid B(\mathbf{x}, \delta) \subseteq G\}$$

It can be easily seen that for open sets G that satisfy the above proposition

$$G = (G^{\text{ext}})^{\text{int}} = (G^{\text{int}})^{\text{ext}}.$$

Our standard convolution kernel for the regularization is denoted by ϑ_ε , and, although other functions have the needed properties as well, we use

$$(3.2.5) \quad \vartheta(\mathbf{x}) := \begin{cases} c \cdot e^{-\frac{1}{1-|\mathbf{x}|^2}} & \text{for } |\mathbf{x}| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathbf{x} \in \mathbb{R}^n$, where the constant c is chosen such that ϑ is normalized, this is

$$c = \left(\int_{B_{\mathbb{R}^n}} e^{-\frac{1}{1-|\mathbf{x}|^2}} d\mathbf{x} \right)^{-1}.$$

For arbitrary $\varepsilon > 0$ we define the rescaled version, ϑ_ε , by $\vartheta_\varepsilon(\mathbf{x}) = \varepsilon^{-n} \vartheta(\varepsilon^{-1} \mathbf{x})$. The essential properties for these mollifier-kernels are their non-negativity, $\vartheta_\varepsilon(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, their normalization, $\int_{\mathbb{R}^n} \vartheta_\varepsilon(\mathbf{x}) d\mathbf{x} = 1$, their independence with respect to rotations, $\vartheta_\varepsilon(\mathbf{x}) = \vartheta_\varepsilon(Q\mathbf{x})$ for all $Q \in O(\mathbb{R}^n)$, and their compact support, here $\text{supp } \vartheta_\varepsilon = B_{\mathbb{R}^n}(0, \varepsilon)$.

Notation 3.2.4. For $f \in L^1_{\text{loc}}(\mathbb{R}^3)$ we denote by \bar{f} its regularization

$$(3.2.6) \quad \bar{f} := \vartheta_\delta * f = \int_{\mathbb{R}^3} \vartheta(x - y) f(y) dy.$$

Similarly, if $f \in L^1_{\text{loc}}(\Omega)$ we extend it by zero to all of \mathbb{R}^3 . Hereby we define $\bar{f} \in C^\infty(\mathbb{R}^3)$. The vector valued case is obtained coordinate wise. We obtain regularized versions \bar{u} of a velocity field and approximations \bar{u}^n hereof, which have support within Ω^{ext} , as should be kept in mind for our later definition of an approximative solution.

One key observation for this regularization is that it does not change the property to preserve a submerged rigid body. At least within a δ -kernel of the originally occupied subset a rigid body is not sheared, as can be seen by applying the derivative under the integral. To be able to refer to this observation by number, we formulate it as remark.

Remark 3.2.5. If u is a rigid velocity field in G , then \bar{u} is a rigid velocity field in G^{int} .

Notation 3.2.6. We denote by $\chi^i_{\text{solid}}(t) := \mathbf{1}_{\Omega^i_{\text{solid}}(t)}$ the characteristic functions of the individual solid bodies $\Omega^i_{\text{solid}}(t)$ and by $\chi_{\text{solid}} = \sum_{i=1}^N \chi^i_{\text{solid}}$ the characteristic function of the domain occupied by these bodies. Furthermore, we denote $\zeta^i_{\text{solid}}(t) := \mathbf{1}_{\Omega^i_{\text{solid}}(t)^{\text{int}}}$ the characteristic function of the δ -kernel of $\Omega^i_{\text{solid}}(t)$ and $\zeta_{\text{solid}} = \sum_{i=1}^N \zeta^i_{\text{solid}}$ the δ -kernel of Ω_{solid} . For $T > 0$ we denote by Q_T the space-time cylinder $\Omega \times [0, T]$. For an arbitrary characteristic function ξ on Q_T we denote

$$L^p(0, T; K(\xi)) := \left\{ u \in L^p(0, T; H^1_{0,\sigma}(\Omega)) \mid u(t) \in K(\xi(\cdot, t)) \text{ for a.a. } t \right\}$$

in the sense $\xi D_{\text{sym}} u = 0$ in $L^p(0, T; L^2(\Omega)^{3 \times 3})$.

We now define what we call a weak solution. The weak formulation of the conservation of momentum is identical to the definition used in Feireisl (2003b). This is due to the nature of the test functions. In the case of a collisions, the assumption $\chi D_{\text{sym}} \phi = 0$ does not control relative rotations of the two colliding bodies completely. This is a major difference to the article by San Martín et al. (2003) upon which this whole work is based upon and where the other weak formulations are taken from.

Definition 3.2.7 (Weak Solution). Let $\varrho_0 \in L^\infty(\Omega)$ be the initial mass distribution, $u_0 \in L^2_\sigma(\Omega)$ be an initial velocity and $m_0(x) = \varrho_0(x)u_0(x)$ initial momentum, $\Omega^i_{\text{solid}}(0) \subseteq \Omega$ for $i = 1, \dots, N$ be disjoint, open and smooth (C^2) sets with characteristic functions χ^i . Let $\Omega_{\text{solid}}(0) = \cup_{i=1}^N \Omega^i_{\text{solid}}(0)$ be the subset occupied initially by solids.

Let $u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega))$, $\varrho \in L^\infty(0, T; L^\infty(\Omega))$, $\chi^i \in C^{0+\frac{1}{p}}(0, T; L^p(\Omega))$ for all $1 \leq p < \infty$ and $\chi^i(x, t) \in \{0, 1\}$, and $\chi := \sum_{i=1}^N \chi^i$ satisfies $\chi(x, t) \in \{0, 1\}$, i.e. it is a characteristic function or equivalently the sets Ω^i_{solid} are disjoint. Then these functions form a weak solution if they satisfy the following.

$$(3.2.7) \quad \begin{aligned} \int_{Q_T} \varrho u \cdot (\partial_t \phi + \text{grad } \phi u) &= - \int_{\Omega} m_0(x) \phi(x, 0) dx \\ &+ \int_{Q_T} 2\mu D_{\text{sym}} u : D_{\text{sym}} \phi - \int_{Q_T} \varrho g \cdot \phi \end{aligned}$$

for all $\phi \in \mathcal{D}([0, T]; \mathcal{D}_\sigma(\Omega))$ that are compatible in the sense

$$(3.2.8) \quad D_{\text{sym}} \phi(\mathbf{x}, t) = 0$$

for all t and \mathbf{x} in an open neighborhood of $\text{supp } \chi(t)$. Furthermore,

$$(3.2.9) \quad \int_{Q_T} \varrho \cdot (\partial_t \psi + \text{grad } \psi \mathbf{u}) = - \int_{\Omega} \varrho_0(\mathbf{x}) \psi(\mathbf{x}, 0) \, d\mathbf{x}$$

$$(3.2.10) \quad \int_{Q_T} \chi^i \cdot \frac{D}{Dt} (\partial_t \psi + \text{grad } \psi \mathbf{u}) = - \int_{\Omega} \chi^i(\mathbf{x}, 0) \psi(\mathbf{x}, 0) \, d\mathbf{x}$$

for all $\psi \in C^1(0, T; C^1(\Omega))$ that satisfy $\psi(T) = 0$.

Theorem 3.2.8 (Existence). *Let $\varrho_0 \in L^\infty$ satisfy $\varrho_0(\mathbf{x}) > \underline{\varrho} > 0$ for some constant $\underline{\varrho}$. Let $\mathbf{g} \in L^2(0, T; L^2(\Omega)^3)$. Then there exists at least one weak solution to the above equations. Furthermore, this solution satisfies an energy estimate*

$$(3.2.11) \quad \frac{1}{2} \int_{\Omega} \varrho |\mathbf{u}(t)|^2 \, d\mathbf{x} + \int_{Q_T} 2\mu |D_{\text{sym}} \mathbf{u}| \, d\mathbf{x} \, dt \leq C \left(\frac{1}{2} \int_{\Omega} \varrho_0 |\mathbf{u}_0|^2 \, d\mathbf{x} + \|\mathbf{g}\|_{L^2(Q_T)} \right)$$

for some constant $C > 0$. Furthermore, there exist affine isometries $\Xi_{\text{solid}}^i(\cdot, s; t)$ on \mathbb{R}^3 that satisfy in particular

$$\Omega_{\text{solid}}^i(t) = \Xi_{\text{solid}}^i(\Omega_{\text{solid}}^i(s), s; t)$$

for all $s, t \in [0, T]$ and all $i = 1, \dots, N$. These mappings are Lipschitz-continuous in s, t .

Remark 3.2.9. By definition of a solution, $\chi_{\text{solid}} = \sum_{i=0}^N \chi_{\text{solid}}^i$ is a characteristic function, i.e. the sets Ω_{solid}^i may never overlap, but since the sets Ω_{solid}^i are open, this does not exclude touching. This property is due to the conservation of mass and volume: the volume of the support of $\chi^i(t)$ and of every sum $\chi^{i_1}(t) + \dots + \chi^{i_l}(t)$

Theorem 3.2.10 (Collisions). *The weak solution \mathbf{u} and the minimal gap*

$$\text{gap}(t) := \min \left\{ \text{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) \mid i, j = 0, \dots, N; i \neq j \right\}$$

between the submerged bodies and between bodies and outer boundary satisfy the following.

- (a) Let $E := \{t \in [0, T] \mid \text{gap}(t) = 0\}$. If $t_0 \in E$ exists and for some $0 \leq i, j \leq N$ with $i \neq j$, the sets $\Omega_{\text{solid}}^i(t_0)$ and $\Omega_{\text{solid}}^j(t_0)$ have a common boundary point then the velocity field \mathbf{u} at $\Omega_{\text{solid}}^i(t_0) \cup \Omega_{\text{solid}}^j(t_0)$ consists of two rigid body motions $\mathbf{u}(\mathbf{x}, t) = R^i(t)\mathbf{x} + \mathbf{U}(t)$ at $\Omega_{\text{solid}}^i(t_0)$ and $\mathbf{u}(\mathbf{x}, t) = R^j(t)\mathbf{x} + \mathbf{U}(t)$ at $\Omega_{\text{solid}}^j(t_0)$ for a single $\mathbf{U} \in \mathbb{R}^3$ and skew symmetric $R^l(t) \in \mathcal{L}(\mathbb{R}^3)$, $l = i, j$.
- (b) If $\text{gap}(t_0) = 0$ then the two-body distance satisfies necessarily

$$\text{gap}_{ij}(t) := \text{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) = o(|t - t_0|),$$

where $0 \leq i, j \leq N$ are those values $i \neq j$ for that this minimum is achieved. Especially, gap_{ij} is differentiable at t_0 with $\partial_t \text{gap}_{ij}(t_0) = 0$.

3.3 Main steps of the existence proof

In this section we construct the approximative solutions as solutions of heterogeneous Newtonian flow models. The key idea is to track the evolution of the δ -kernel of the subsets of higher viscosity and to use the δ -neighborhood of the evolved version of the δ -kernel to penalize the term $D_{\text{sym}} \mathbf{u}$. Furthermore, the transport is considered within a regularized flow $\bar{\mathbf{u}}$.

3.3.1 Approximative solutions

We assume as regularity of approximative solutions the standard regularity properties that might be expected of weak solutions of a heterogeneous flow. The key difference to most approximation schemes in numerical analysis is that a classical solution indeed is an approximative solution. Therefore, this notation of a approximative solution is consistent. It is identical to the two dimensional formulation posed by San Martín et al. (2003).

Notation 3.3.1 (Approximative Solution). For $n \in \mathbb{N}$ and $i = 1, \dots, N$ we consider approximative velocity fields $\mathbf{u}^n \in L^\infty(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; H_{0,\sigma}^1(\Omega))$, approximative densities $\varrho^n \in L^\infty(0, T; L^\infty(\Omega))$, approximative characteristic functions χ^{in} of the submerged bodies and ζ^{in} of their δ -kernels both of class $C^{0+\frac{1}{p}}(0, T; L^p(\Omega))$ for all $1 \leq p < \infty$, where $\chi^{in}(x, t) \in \{0, 1\}$, where $\chi^n := \sum_{i=1}^N \chi^{in}$ satisfies $\chi^n(x, t) \in \{0, 1\}$, and $\zeta^n := \sum_{i=1}^N \zeta^{in}$ is coupled to χ by $\text{supp } \zeta^n = (\text{supp } \chi^n)^{\text{int}}$. Furthermore, the following equations have to be satisfied.

$$(3.3.1) \quad \int_{Q_T} \varrho^n \mathbf{u}^n \cdot (\partial_t \phi + \text{grad } \phi \mathbf{u}^n) = - \int_{\Omega} m_0(x) \phi(x, 0) dx - \int_{Q_T} \varrho g \cdot \phi + \int_{Q_T} (2\mu + n\chi^n) D_{\text{sym}} \mathbf{u}^n : D_{\text{sym}} \phi$$

for all $\phi \in \mathcal{D}([0, T]; \mathcal{D}_\sigma(\Omega))$ and

$$(3.3.2) \quad \int_{Q_T} \varrho^n \cdot (\partial_t \psi + \text{grad } \psi \mathbf{u}^n) = - \int_{\Omega} \varrho_0(x) \psi(x, 0) dx$$

$$(3.3.3) \quad \int_{Q_T^{\text{ext}}} \zeta^{in} \cdot (\partial_t \eta + \text{grad } \eta \bar{\mathbf{u}}^n) = - \int_{\Omega^{\text{ext}}} \zeta_0^i(x) \eta(x, 0) dx$$

for all $\psi \in C^1(0, T; C^1(\Omega))$ that satisfy $\psi(T) = 0$ and all $\eta \in C^1(0, T; C^1(\Omega^{\text{ext}}))$ that satisfy $\eta(T) = 0$.

The approximative nature is due to two relaxations of the assumptions of a weak solution: On the one hand, (3.3.3) and ζ^{in} describes the transport of a δ -kernel of the i th submerged solid body and the position χ^{in} of this body is found as δ -neighborhood of the support of this set. On the other hand, (3.3.1) allows that $D_{\text{sym}} \mathbf{u}^n$ might be different from zero within the support of χ^{in} , but to high values hereof are penalized.

3.3.2 Existence of approximative solutions

The notation of approximative solutions is modeled exactly to fit into the framework of heterogeneous fluid flow in the sense of Lions (1996) with accompanying densities.

For such fluid models the procedure of DiPerna and Lions (1989) and Lions (1996) provides existence of solutions which yields the following results, as demonstrated later.

Theorem 3.3.2 (Existence of approximative solutions and a-priori bounds). *Solutions exist and satisfy the following a-priori bound uniformly in $n \in \mathbb{N}$: There exists a $C > 0$ that satisfies*

$$(3.3.4) \quad \begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho^n |\mathbf{u}^n(t)|^2 \, d\mathbf{x} + \int_{Q_T} (2\mu + n\chi^n) |\mathbf{D}_{sym} \mathbf{u}^n|^2 \, d\mathbf{x} \, dt \\ & \leq C \left(\frac{1}{2} \int_{\Omega} \varrho_0 |\mathbf{u}_0|^2 \, d\mathbf{x} + \|g\|_{L^2(Q_T)}^2 \right), \end{aligned}$$

this is the sequence $(\mathbf{u}^n)_n$ is uniformly bounded in $L^2(0, T; H_{0,\sigma}^1(\Omega)) \cap L^\infty(0, T; L_\sigma^2(\Omega))$. Furthermore, $\varrho^n(\mathbf{x}, t) > \underline{\varrho} > 0$ for a.a. $\mathbf{x} \in \Omega$ and $t \in [0, T]$, and for all $1 \leq p \leq \infty$ the associated densities satisfy $\|\varrho^n(t)\|_{L^p(\Omega)} = \|\varrho_0\|_{L^p(\Omega)}$ and for all $i = 1, \dots, N$ and $\|\zeta^{in}(t)\|_{L^p(\Omega)} = \|\zeta_0^i\|_{L^p(\Omega)}$ for characteristic functions ζ^{in} . Hence, these solutions satisfy uniformly

$$(3.3.5) \quad \|\varrho^n\|_{L^\infty(0,T;L^p(\Omega))} = \|\varrho_0\|_{L^p(\Omega)}$$

and for all $i = 1, \dots, N$

$$(3.3.6) \quad \|\zeta^{in}\|_{L^\infty(0,T;L^p(\Omega))} = \|\zeta_0^i\|_{L^p(\Omega)}.$$

By standard compactness arguments the equations (3.3.4), (3.3.5) and (3.3.6) yield converging subsequences, which we will denote again as the original sequences: $(\mathbf{u}^n)_{n \in \mathbb{N}}$, $(\varrho^n)_{n \in \mathbb{N}}$, $(\zeta^n)_{n \in \mathbb{N}}$ satisfy

$$(3.3.7) \quad \mathbf{u}^n \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0, T; H_{0,\sigma}^1(\Omega)) \text{ and weakly}^* \text{ in } L^\infty(0, T; L_\sigma^2(\Omega))$$

and by properties of the mollification

$$(3.3.8) \quad \bar{\mathbf{u}}^n \rightharpoonup \bar{\mathbf{u}} \text{ weakly in } L^2(0, T; H_0^1(\Omega^{\text{ext}})^3) \text{ and weakly}^* \text{ in } L^\infty(0, T; L_\sigma^2(\Omega^{\text{ext}}))$$

and using that $\bar{\mathbf{u}} \in \mathcal{D}(\Omega^{\text{ext}})$ we even know that $(\bar{\mathbf{u}}^n)_n \subseteq L^\infty(0, T; C^2(\Omega^{\text{ext}}))$ is uniformly bounded. Furthermore

$$(3.3.9) \quad \varrho^n \rightharpoonup \varrho \text{ weakly}^* \text{ in } L^\infty(0, T; L^\infty(\Omega))$$

and for all $i = 1, \dots, N$

$$(3.3.10) \quad \zeta^{in} \rightharpoonup \zeta^i \text{ weakly}^* \text{ in } L^\infty(0, T; L^\infty(\Omega^{\text{ext}})).$$

Something we'll need as an extra, compared to the two dimensional case, is that the uniform boundedness in $L^2(0, T; H_{0,\sigma}^1(\Omega)) \cap L^\infty(0, T; L_\sigma^2(\Omega))$ of the sequence $(\mathbf{u}^n)_{n \in \mathbb{N}}$ implies uniform boundedness in $L^{\frac{8}{3}}(0, T; L^4(\Omega))^3$ and therefore we can choose the subsequence such that

$$(3.3.11) \quad \mathbf{u}^n \rightharpoonup \mathbf{u} \text{ weakly in } L^{\frac{8}{3}}(0, T; L^4(\Omega))^3$$

Remark 3.3.3. In three dimensions boundedness of a function v in $L^2(0, T; H^{1,2}(\Omega)^3)$ and in $L^\infty(0, T; L^2(\Omega)^3)$ yields boundedness in $L^{\frac{8}{3}}(0, T; L^4(\Omega)^3)$.

Proof. Standard Sobolev embedding results yield that $H_0^{1,2}(\Omega) \subseteq L^6(\Omega)$ is a continuous embedding in three dimensions. Therefore, the embedding

$$L^2(0, T; H_0^{1,2}(\Omega)^3) \subseteq L^2(0, T; L^6(\Omega)^3)$$

is continuous.

Furthermore, for all $v \in L^{p_1}(0, T; L^{p_2}(\Omega)) \cap L^{q_1}(0, T; L^{q_2}(\Omega))$, for all $0 \leq \theta \leq 1$ and $\frac{1}{r_i} = \theta \frac{1}{p_i} + (1 - \theta) \frac{1}{q_i}$ we have $v \in L^{r_1}(0, T; L^{r_2}(\Omega))$. In our case we will use $\theta = \frac{1}{4}$ and obtain from the formal convex combinations

$$\begin{aligned} \frac{1}{r_1} &= \theta \frac{1}{\infty} + (1 - \theta) \frac{1}{2} = \frac{3}{8} \\ \frac{1}{r_2} &= \theta \frac{1}{2} + (1 - \theta) \frac{1}{6} = \frac{1}{4} \end{aligned}$$

the values $r_1 = \frac{8}{3}$ and $r_2 = 4$.

The convexity result of Lebesgue spaces can be seen as follows. If one of the indices p_i or q_i is ∞ the Hölder estimate is obvious. For the most technical case we consider

$$\begin{aligned} \int_0^T \left(\int_\Omega |v|^{r_2} \right)^{\frac{r_1}{r_2}} &= \int_0^T \left(\int_\Omega |v|^{\theta r_2} |v|^{(1-\theta)r_2} \right)^{\frac{r_1}{r_2}} \\ &\leq \int_0^T \left(\left(\int_\Omega |v|^{\theta r_2 \frac{p_2}{\theta r_2}} \right)^{\frac{\theta r_2}{p_2}} \left(\int_\Omega |v|^{(1-\theta)r_2 \frac{q_2}{(1-\theta)r_2}} \right)^{\frac{(1-\theta)r_2}{q_2}} \right)^{\frac{r_1}{r_2}} \\ &= \int_0^T \left(\left(\int_\Omega |v|^{p_2} \right)^{\frac{1}{p_2} \theta r_1} \left(\int_\Omega |v|^{q_2} \right)^{\frac{1}{q_2} (1-\theta)r_1} \right) \\ &= \int_0^T \left(\|v(t)\|_{L^{p_2}}^{\theta r_1} \|v(t)\|_{L^{q_2}}^{(1-\theta)r_1} \right) dt \\ &\leq \left(\int_0^T \|v(t)\|_{L^{p_2}}^{\theta r_1 \frac{p_1}{\theta r_1}} dt \right)^{\frac{\theta r_1}{p_1}} \left(\int_0^T \|v(t)\|_{L^{q_2}}^{(1-\theta)r_1 \frac{q_1}{(1-\theta)r_1}} dt \right)^{\frac{(1-\theta)r_1}{q_1}} \end{aligned}$$

and obtain $\|v\|_{L^{r_1}L^{r_2}} \leq \|v\|_{L^{p_1}L^{p_2}}^\theta \|v\|_{L^{q_1}L^{q_2}}^{1-\theta}$. Thus we find that $v \in L^{\frac{8}{3}}([0, T^*], L^4(\Omega)^3)$ in our case. *q.e.d.*

We denote by $\chi^i(t)$ the characteristic function of $(\text{supp } \zeta^i(t))^{\text{ext}}$ and by $\chi := \sum_{i=1}^N \chi^i$. Hereby we obtain functions u , ϱ , χ which are the candidate of our solution to the original problem. Hence, the only purpose of the last pages is to show, that these functions really are a solution. The key ingredients are the following two claims.

Proposition 3.3.4 (Approximative solutions approximate solutions). *The limits u , ϱ , χ of a subsequence of the approximative solutions satisfy all the assumptions on a weak solution in the above defined sense, at least besides the conservation of momentum equation (3.2.7), which we will consider later.*

To prove that the conservation of momentum equation in the weak sense is satisfied as well is rather involved and the following proposition turns out to be a major step towards the proof of Theorem 3.2.8.

Proposition 3.3.5 (Approximative solutions approximate solutions).

A subsequence of $(\mathbf{u}^n)_{n \in \mathbb{N}}$ converges strongly to \mathbf{u} in $L^2(0, T; L^2(\Omega)^3)$.

With these propositions satisfied we obtain the following steps of the proof which we will conduct:

- (a) Results from DiPerna and Lions (1989) and Lions (1996) yield that after extracting subsequences

$$\begin{aligned} \mathbf{u}^n &\longrightarrow \mathbf{u} \text{ strongly in } L^2(0, T; L^2_\sigma(\Omega)) \\ \varrho^n &\longrightarrow \varrho \text{ strongly in } C(0, T; L^p(\Omega)) \text{ for all } 1 \leq p < \infty \end{aligned}$$

- (b) An approximation property of the spaces that preserve rigid bodies shows that \mathbf{u} , ϱ , χ satisfies the conservation of momentum equation

$$\begin{aligned} \int_{Q_T} \varrho \mathbf{u} \cdot (\partial_t \boldsymbol{\phi} + \text{grad } \boldsymbol{\phi} \mathbf{u}) &= - \int_{\Omega} m_0(x) \boldsymbol{\phi}(x, 0) \, dx \\ &+ \int_{Q_T} 2\mu D_{\text{sym}} \mathbf{u} : D_{\text{sym}} \boldsymbol{\phi} - \int_{Q_T} \varrho f \cdot \boldsymbol{\phi} \end{aligned}$$

it is sufficient to check this only for all those $\boldsymbol{\phi} \in \mathcal{D}([0, T]; \mathcal{D}_\sigma(\Omega))$ that satisfy $D_{\text{sym}} \boldsymbol{\phi} = 0$ in a τ -neighborhood thereof.

- (c) The Hölder-continuity of the characteristic function χ is then a direct consequence of the Hölder continuity of the functions Ξ^i .
- (d) The claimed energy inequality (3.2.11) is due to the energy inequality (3.3.4) of the approximate solutions and the lower semi-continuity of the involved terms.

The precise versions of the above statements and the details of this existence proof are the contents of the remaining sections.

3.3.3 Reference Table of notations

Although the notion of approximative solutions is straight forward, the proof of the convergence of a subsequence is involved since in the definition of the converging subsequences for the velocity, the densities, the the characteristic functions of the approximative solid domain and its different connected subsets a lot of notation and many subsequences are involved. To give the reader a possibility to check a notation, we list the involved terms.

Term	Usage	First appearance
\mathbf{u}^n	Approximative velocity fields	Theorem 3.3.2
$\overline{\mathbf{u}^n}$	Regularized version of approximative velocity fields	Equation 3.3.8
\mathbf{u}	Limit of the approximative velocity fields	Equation 3.3.11
$\mathbf{u}_{\text{solid}}^i$	Rigid velocity field for Ω_{solid}^i close to the approximative velocity fields	Lemma 3.6.5
$\mathbf{u}_{\text{solid}}^i$	Rigid velocity field for Ω_{solid}^i within limit of the approximative velocity fields	Equation (3.5.21)
ζ^n	Transported version of the characteristic function of δ -kernel of the solid domain within the velocity field $\overline{\mathbf{u}^n}$	Theorem 3.3.2

Term	Usage	First appearance
ζ^{in}	Transported version of the characteristic function of δ -kernel of the i th connected subdomain of the solid domain within the velocity field $\bar{\mathbf{u}}^n$	Theorem 3.3.2
ζ	Transported version of the characteristic function of δ -kernel of the solid domain within the limit field $\bar{\mathbf{u}}$	Equation (3.3.10)
ζ^i	Transported version of the characteristic function of δ -kernel of the i th connected subdomain of the solid domain within the velocity field $\bar{\mathbf{u}}$	Equation (3.3.10)
Ξ^n	Transport due to the approximative velocity fields \mathbf{u}^n	Equation (3.5.9)
Ξ	Transport due to the limit velocity field \mathbf{u}	Equation (3.5.10)
Ξ_{solid}^i	Transport due to the rigid velocity field part of $\mathbf{u}_{\text{solid}}^i$ extended to all of \mathbb{R}^3 .	Equation (3.5.22)
ϱ^n	Solution of the conservation of mass equation within the velocity field \mathbf{u}^n	Theorem 3.3.2
ϱ	Limit of the functions ϱ^n	Equation (3.3.9)
χ^n	Characteristic function of the δ -neighborhood of the support of ζ^n	Notation 3.3.1
χ^{in}	Characteristic function of the δ -neighborhood of the support of ζ^{in}	Notation 3.3.1
χ	Characteristic function of the δ -neighborhood of the support of ζ	Implicitly on page 36
χ^i	Characteristic function of the δ -neighborhood of the support of ζ^i	Implicitly on page 36

3.4 Properties of the space of rigidity preserving velocity field

In this section we provide the needed estimates to deal with the three dimensional problem. In the first subsection we generalize the estimates of San Martín et al. (2003) from two to three dimensions¹, to general differentiability classes and to L^p -integrability.

3.4.1 Characterization of rigidity preserving velocity fields

We need many properties of submerged bodies. In this section we consider further properties of momentary velocity fields that preserve submerged rigid bodies. Throughout, we assume that these bodies are bounded and have C^2 -boundaries which are locally convex. We denote these disjoint bodies as $\bar{\Omega}^i$ and by χ^i the corresponding characteristic functions of the internal sets for $i = 1, \dots, N$. Especially we obtain that the minimal distance between rigid bodies

$$\text{gap} = \min \left\{ \text{dist}(\Omega^i, \Omega^j) \mid i, j = 0, \dots, k; i \neq j \right\}$$

be greater than zero and

$$K(\chi) = \bigcap_{i=1}^N K(\chi^i).$$

¹Actually, we generalize most of the estimates to n dimensions, since our proofs do not depend on the dimension n anymore, but we need only $n = 3$ later on.

We now want to observe that for each of the subsets Ω^i there exists a unique rigid velocity field in the classical sense, i.e. one of the form $\omega^i \times x + \mathbf{U}^i$. To show these we consider the following results which we quote from Temam (1985).

Let $W \subseteq \mathbb{R}^3$ be an arbitrary set of the same regularity as the sets Ω^i and the set Ω . Let $1 < p < \infty$. Then we set

$$(3.4.1) \quad \text{LD}^p(W) := \left\{ \mathbf{u} \in L^p(W)^3 \mid \text{D}_{\text{sym}} \mathbf{u} \in L^p(W)^{3 \times 3} \right\}$$

and obtain a Banach space if endowed with the norm $\|\cdot\|_{\text{LD}^p}$ given by

$$(3.4.2) \quad \|\mathbf{u}\|_{\text{LD}^p}^p := \|\mathbf{u}\|_{L^p(W)}^p + \|\text{D}_{\text{sym}} \mathbf{u}\|_{L^p(W)}^p.$$

See for example (Temam, 1985, equation (1.68)).

Let W satisfy the cone condition and let $1 < p < \infty$. Then there exist a constant M_2 depending only on W such that *Korn's inequality* is satisfied:

$$(3.4.3) \quad \|\text{D} \mathbf{u}\|_{L^p(W)}^p \leq M_2 \left(\|\mathbf{u}\|_{L^p(W)}^p + \|\text{D}_{\text{sym}} \mathbf{u}\|_{L^p(W)}^p \right)$$

for all $\mathbf{u} \in \text{LD}^p(W)$. See for example (Temam, 1985, equation (1.70)). For $p = 1$ these statements are not correct. The reverse statement

$$(3.4.4) \quad M_1 \|\text{D}_{\text{sym}} \mathbf{u}\|_{L^p(W)} \leq \|\text{D} \mathbf{u}\|_{L^p(W)}$$

for some $M_1 > 0$ follows from the triangle inequality applied to the L^p -norm and the finite number of terms. Hence, $\text{LD}^p(W) = W^{1,p}(W)$ and their norms are equivalent for all $1 < p < \infty$:

$$(3.4.5) \quad c_1 \|\mathbf{u}\|_{W^{1,p}(W)} \leq \|\mathbf{u}\|_{\text{LD}^p(W)} \leq c_2 \|\mathbf{u}\|_{W^{1,p}(W)}$$

for two constants c_1, c_2 which will later be absorbed into the generic constant c .

An often used observation, which can be proved directly, is that the kernel of the symmetric gradient consist of affine functions with skew symmetric linear part, see (Temam, 1985, Lemma I.1.1):

Lemma 3.4.1 (Kernel of D_{sym}). *Let W be connected and $\mathbf{u} \in \mathcal{D}(W)^3$ be in the kernel of the symmetric gradient in the distribution sense, i.e. $\text{D}_{\text{sym}} \mathbf{u} = 0 \in \mathcal{D}^*(W)^3$, then there exists a skew symmetric matrix $R \in \mathcal{L}(\mathbb{R}^3)$ and a vector $\mathbf{U} \in \mathbb{R}^3$ such that $\mathbf{u}(\mathbf{x}) = R\mathbf{x} + \mathbf{U}$. In three dimensions equivalently there exists an $\omega \in \mathbb{R}^3$ such that $\mathbf{u}(\mathbf{x}) = \omega \times \mathbf{x} + \mathbf{U}$.*

This observation yields that the kernel of the bounded linear operator D_{sym} on $H^{1,p}(W)^3$ to $L^p(W)^{3 \times 3}$ is the closed subspace

$$(3.4.6) \quad \text{LD}_0(W) := \left\{ \mathbf{u} \in H^{1,p}(W)^3 \mid \text{D}_{\text{sym}} \mathbf{u} = 0 \right\}$$

of $H^{1,p}(W)^3$ and for connected and bounded W is of the form

$$(3.4.7) \quad \text{LD}_0(W) = \left\{ \mathbf{u} \in H^{1,p}(W)^3 \mid \mathbf{u}(\mathbf{x}) = R\mathbf{x} + \mathbf{U}, R \in \mathcal{L}(\mathbb{R}^n), R^* + R = 0, \mathbf{U} \in \mathbb{R}^n \right\}$$

Similarly we obtain for each connected subset a representation of the form $\mathbf{u}(\mathbf{x}) = R\mathbf{x} + \mathbf{U}$.

One of the key approximation features of these spaces is

$$(3.4.8) \quad \inf \left\{ \|u - v\|_{H^{1,p}(W)} \mid v \in LD_0(W) \right\} \leq c \|D_{\text{sym}} u\|_{L^p(W)}.$$

For another generic constant c we can therefore assert: For every $u \in H^{1,p}(W)$ there exists at least one $v \in LD_0(W)$ such that

$$(3.4.9) \quad \|u - v\|_{H^{1,p}(W)} \leq c \|D_{\text{sym}} u\|_{L^p(W)}$$

for a constant c independent of u and v . The proof of (3.4.8) consists of the observations that on $H^{1,p}(W)/\mathbb{R}$, this is up to a constant, an inequality of the form

$$\|D u\|_{L^p(W)} \leq M'_2 \|D_{\text{sym}} u\|_{L^p(W)}$$

is valid, which yields that $\|D_{\text{sym}} \cdot\|_{L^p(W)}$ defines an equivalent norm. Since the constants \mathbb{R}^3 form as $u \equiv U$ a subspace of LD_0 , we obtain (3.4.8).

The last statements will be used much later, but the introduced expressions can now be used to observe that if two bodies should touch, they can only move together as two bodies on a common axis, which we imagine as motion of *balls on sticks*.

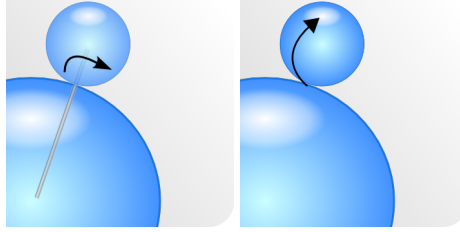


Figure 3.4.1: Connected balls can only roll on top of each other. For higher regularity the relative velocity field is one of a rotation that preserves the normal to the common tangential plane.

Lemma 3.4.2 (Momentary Case). *Let $p \geq 2$ and $u \in H^{s,p}(\Omega)^3$ be the momentary velocity field describing the motion of a mixture of rigid and other incompressible phases. Let Ω^1 and Ω^2 be two disjoint rigid phases within Ω , i.e. $D_{\text{sym}} u = 0$ in $\Omega^1 \cup \Omega^2$.*

Should they touch each other in a point where Ω^1 and Ω^2 are (locally) strictly convex and the boundary is C^2 , then the velocity field v of the motion of Ω^1 relative to Ω^2 has to satisfy the following:

If $s \geq \frac{2}{p}$ then the velocity field is of the form $v(\xi) = \omega \times \xi$ in the local coordinates ξ for some $\omega \in \mathbb{R}^3$. If $s \geq \frac{2}{p} + 1$ we even find $\omega = \omega_3 e_3$ for some $\omega_3 \in \mathbb{R}$. This is v describes the rolling on top of the common tangential plane.

Remark 3.4.3. (a) For our later reference we prove two estimates for velocity fields v , that are velocities relative to the velocity field within Ω^2 . These are of the form $v(\xi) = R\xi + U$ in Ω^1 , vanish in Ω^2 , and satisfy

$$(3.4.10) \quad |U| \leq |R| r \sqrt{1 + \frac{r^2}{\delta^2}} + c_\delta r^{2s-\frac{4}{p}} \|D^s v\|_{L^p(C_r)}$$

and for $U = 0$ then

$$(3.4.11) \quad \sqrt{\omega_1^2 + \omega_2^2} \leq \tilde{c}_\delta r^{2(s-1)-\frac{4}{p}} \|D^s v\|_{L^p(C_r)}.$$

The domain C_r is a circular cylinder centered at a point independent of r . Its height is the distance the maximal distance of spheres from the tangential plane. Actually, we consider a truncated version of C_r , see Figure 3.4.2. The important observation is that for $r \rightarrow 0$ the left hand sides vanish if the above assumptions are fulfilled.

(b) The assumption $s \geq \frac{2}{p} + 1$ is not satisfied for common weak solutions ($s = 1$ and $p = 2$). But, it is satisfied for an arbitrary test function ϕ that is compatible in the sense $D_{\text{sym}} \phi = 0$ in $\Omega^1 \cup \Omega^2$. Hence, such test functions have less degrees of freedom than those for two not connected bodies ($\mathbf{U} = 0, \omega_1 = \omega_2 = 0$). But they have an extra degree of freedom if compared to functions that satisfy $D_{\text{sym}} \phi = 0$ even in an open neighborhood of $\Omega^1 \cup \Omega^2$, since ω_3 is not restricted.

Proof. By assumption, there exists a point $x_M \in \partial\Omega^1 \cap \partial\Omega^2$. Locality does not influence the following since Ω^k can be restricted to a neighborhood $B^o(x_M, R)$ of x_M for a sufficiently small R . Furthermore, due to the convexity of each Ω^k , a radius $\delta > 0$ and two open balls

$$\begin{aligned} B_1 &:= B^o(0, \delta) + x_M - \delta v_{\Gamma_1}(x_M) \subseteq \Omega^1 \\ B_2 &:= B^o(0, \delta) + x_M - \delta v_{\Gamma_2}(x_M) \subseteq \Omega^2 \end{aligned}$$

exist within the sets Ω^1 and Ω^2 , respectively, such that $\partial B_1 \cap \partial B_2 = \{x_M\}$.

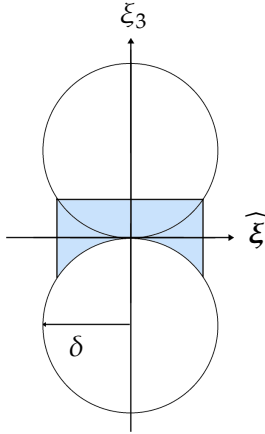


Figure 3.4.2: The prototypic interaction zone of two rigid bodies with locally strictly convex boundary is a sand-glass type domain.

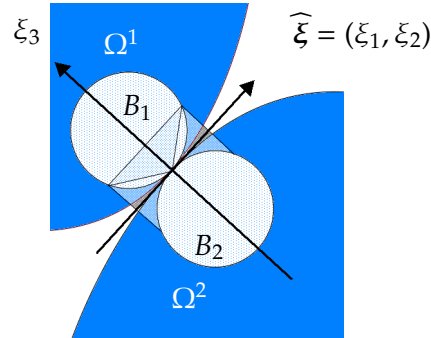


Figure 3.4.3: The two-dimensional section of two convex sets which touch with local coordinate system, shrinking cone domains C_r^1 , and extended domains C_r .

Touching of these C^2 -sets enables us to choose unified local coordinates $\widehat{\xi} = (\xi_1, \xi_2)^t$ of the common tangential manifold at x_M of these sets, and to denote by ξ_3 the projection on the (outer) normal of B_2 . This situation is depicted in Figure 3.4.3.

By assumption $D_{\text{sym}} u = 0$ in Ω^1 and Ω^2 . Hence, by the representation of rigid body motions, two vectors $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{R}^3$ and two skew-symmetric mappings $R_1, R_2 \in \mathcal{L}(\mathbb{R}^3)$ exist, such that

$$(3.4.12) \quad u(\xi) = \begin{cases} \mathbf{U}_1 + R_1 \xi & \text{for } \xi \in \Omega^1 \\ \mathbf{U}_2 + R_2 \xi & \text{for } \xi \in \Omega^2. \end{cases}$$

Hence, the *motion relative to* Ω^2 is given by

$$(3.4.13) \quad v(\xi) := u(\xi) - U_2 - R_2 \xi \quad \text{for } \xi \in \Omega$$

(a) We will prove $U_1 = U_2$ by an appropriate choice of a sequence $(C_r^1)_r$ of subdomains of Ω^1 and domains $(C_r)_r$: The height of the semi-sphere above a point $\widehat{\xi}$ of the tangential plane with distance $r = |\widehat{\xi}|$ from 0 is given by $c(r) = \delta - \sqrt{\delta^2 - r^2}$. For $0 \leq r \leq \delta$ we denote by

$$C_r^1 := \left\{ \xi = (\widehat{\xi}, \xi_3)^t \in \mathbb{R}^3 \mid |\widehat{\xi}| \leq r \text{ and } \frac{c(r)}{r} |\widehat{\xi}| \leq \xi_3 \leq c(r) \right\}$$

a family of inverted circular cones with top perpendicular to the normal and end in zero. The cone lies within B_1 , therefore $C_r^1 \subseteq B_1 \subseteq \Omega^1$. Hence, $v(\xi) = U + R\xi$ for all $\xi \in C_r^1$, where $U := U_1 - U_2 \in \mathbb{R}^3$ and $R := R_1 - R_2 \in \mathcal{L}(\mathbb{R}^3)$. So we obtain

$$(3.4.14) \quad \int_{C_r^1} v \, d\xi = U \int_{C_r^1} d\xi + R \int_{C_r^1} \xi \, d\xi$$

Dividing by $|C_r^1|$ and solving for U yields

$$(3.4.15) \quad |U| \leq \frac{1}{|C_r^1|} \int_{C_r^1} |v| \, d\xi + \frac{1}{|C_r^1|} |R| \left| \int_{C_r^1} \xi \, d\xi \right|$$

To prove $U = 0$ we'll show that each term on the right hand side converges to zero as r gets smaller.

(b) The second term on the right hand is bounded by

$$\frac{1}{|C_r^1|} \left| \int_{C_r^1} \xi \, d\xi \right| \leq \max_{\xi \in C_r^1} |\xi| = \sqrt{r^2 + c(r)^2} \leq r \sqrt{1 + \frac{r^2}{\delta^2}}.$$

Hence, for small r

$$(3.4.16) \quad \frac{1}{|C_r^1|} \left| \int_{C_r^1} \xi \, d\xi \right| = O(r).$$

(c) The first term on the right hand side goes to zero: To prove this, we consider v as a function on the sets

$$C_r := \left\{ \xi = (\widehat{\xi}, \xi_3) \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) \leq \xi_3 \leq c(r) \right\},$$

that are obtained if the cones are extended to the next sphere and have roughly the form of the container of a sand glass. By construction $v(\xi) = 0$ for all $\xi \in \Omega^2$. Hence, v vanishes on the lower part

$$(3.4.17) \quad \Gamma_r := \left\{ \xi = (\widehat{\xi}, \xi_3) \mid |\widehat{\xi}| \leq r \text{ and } -c(|\widehat{\xi}|) = \xi_3 \right\}$$

of the boundary of C_r . By assumption on u and construction of v , we obtain $v \in H^{s,p}(C_\delta)^3$. Since the cone C_r^1 is as subset of the sand glass domain C_r , we obtain

$$\|v\|_{L^1(C_r^1)} \leq \|v\|_{L^1(C_r)}.$$

The generalized Poincaré inequality 2.1.6 allows us to estimate

$$(3.4.18) \quad \|v\|_{L^1(C_r^1)} \leq c_\delta \left(r^4 r^{2s-\frac{4}{p}} \right) \|D^s v\|_{L^p(C_r)}.$$

Together with the estimate

$$|C_r^1| = \frac{1}{3} \pi r^2 c(r) \geq \frac{\pi}{3\delta} r^4$$

of the volume of C_r^1 , we obtain from

$$\frac{1}{|C_r^1|} \int_{C_r^1} |v| d\xi \leq \frac{3\delta}{\pi} c_\delta r^{2s-\frac{4}{p}} \|D^s v\|_{L^p(C_r)}$$

the estimate

$$(3.4.19) \quad \frac{1}{|C_r^1|} \int_{C_r^1} |v| d\xi = o\left(r^{2s-\frac{4}{p}}\right)$$

for $r \rightarrow 0$. By equations (3.4.16) and (3.4.19) the right hand side of (3.4.15) and therefore \mathbf{U} satisfies

$$(3.4.20) \quad |\mathbf{U}| = O(r) + o\left(r^{2s-\frac{4}{p}}\right).$$

Thus, the constant velocity \mathbf{U} has to be identical zero if $s \geq \frac{2}{p}$.

(d) We have shown so far that $v(\xi) = R\xi$. In the remaining steps we show that R can only be a rotation that preserves the $\mathbb{R}e_3$ axes: Let R be given in the form

$$(3.4.21) \quad R = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & \omega_1 & 0 \end{pmatrix},$$

this is we chose a representation such that

$$R\xi = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \omega_3 \xi_2 - \omega_2 \xi_3 \\ \omega_1 \xi_3 - \omega_3 \xi_1 \\ \omega_2 \xi_1 - \omega_1 \xi_2 \end{pmatrix} = \boldsymbol{\omega} \times \xi$$

for $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^t$. The center of mass of a circular cone is located at a fourth of the height above the base of the cone. Therefore, due to our coordinate system,

$$\frac{1}{|C_r^1|} \int_{C_r^1} \xi d\xi = \frac{3}{4} c(r) e_3.$$

The remaining part of the velocity field has to satisfy

$$\begin{aligned} \left| R \int_{C_r^1} \xi d\xi \right| &= \frac{1}{4} \pi r^2 c(r)^2 \sqrt{\omega_1^2 + \omega_2^2} \\ &\geq \tilde{c}_\delta r^6 \sqrt{\omega_1^2 + \omega_2^2}. \end{aligned}$$

Hence, assuming $\mathbf{U} = 0$, using equation (3.4.14) and estimate (3.4.18) yields

$$\sqrt{\omega_1^2 + \omega_2^2} \leq \tilde{c}_\delta r^{2(s-1)-\frac{4}{p}} \|D^s v\|_{L^p(C_r)},$$

which implies $\omega_1 = \omega_2 = 0$ for $s \geq \frac{2}{p} + 1$ which shows the final claim.

q.e.d.

Remark 3.4.4. We spared the reader from the n -dimensional case, but it might be interesting to see why the two dimensional case seems to need less regularity. The critical inequality (3.4.19) in n dimensions is

$$\frac{1}{|C_r^1|} \int_{C_r^1} |v| d\xi = o\left(r^{2s - \frac{n+1}{p}}\right).$$

Therefore, in two dimensions less regularity is needed than in higher dimensions. But independent of the considered dimension, the regularity of the characteristic function of the bodies translates into regularity of the motion of its center of mass.

One corollary we can draw right away from the last Lemma 3.4.2 is the case of one body that occupies $\Omega^1 \subseteq \Omega$ and the exterior domain $\Omega^0 := \mathbb{R}^3 \setminus \overline{\Omega^1}$, which is obtained for the a-priori known velocity field $\mathbf{u} = 0$ in Ω^0 :

Corollary 3.4.5 (Momentary Case, Collisions with rigid wall). *Let $p \geq 2$ and $\mathbf{u} \in H_0^{s,p}(\Omega)^3$ be the momentary velocity field describing the motion of rigid body Ω^1 within the domain Ω , i.e. $D_{\text{sym}} \mathbf{u} = 0$ in Ω^1 . Suppose Ω^1 touches the boundary of Ω in a point where Ω^1 and Ω^0 are (locally) strictly convex and the boundary is C^2 , then the velocity field \mathbf{v} of the motion of Ω^1 has to satisfy the following:*

If $s \geq \frac{2}{p}$ then the velocity field is of the form $\mathbf{u}(\xi) = \boldsymbol{\omega} \times \xi$ in the local coordinates ξ for some $\boldsymbol{\omega} \in \mathbb{R}^3$. If $s \geq \frac{2}{p} + 1$ then $\boldsymbol{\omega} = \omega_3 \mathbf{e}_3 \in \mathbb{R}^3$ where \mathbf{e}_3 is the normal to the boundary and \mathbf{e}_1 and \mathbf{e}_2 span the tangential plane for some $\omega_3 \in \mathbb{R}$. Hence, the velocity field describes the rolling on top of the boundary.

3.4.2 Approximation of rigidity preserving velocity fields

To prove existence of a solution we generalize the approximation procedure of San Martín, Starovoitov, and Tucsnak (2003) to three dimensions. The importance of this approach lies in its selection strategy. A velocity field which preserves rigid bodies can be approximated by velocity fields that preserves rigidity even in a neighborhood of the original body. In two dimensions this approximation is possible in H^1 . In three dimensions H^1 convergence takes place only except for an arbitrary small neighborhood around the touching point, but global L^2 convergence is possible which suffices to conduct proof.

For $i = 1, \dots, N$ let $\Omega^i \subseteq \Omega$ be open, disjoint and connected subsets, $\Omega^0 := \mathbb{R}^3 \setminus \overline{\Omega}$ be the exterior domain of Ω , $\chi^i := \mathbf{1}_{\Omega^i}$ be the characteristic functions of these sets, $\chi := \sum_{i=1}^k \chi^i$ the characteristic function of the rigid bodies, and

$$K(\chi) := \left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{u} = 0 \text{ and } \chi D_{\text{sym}} \mathbf{u} = 0 \text{ in } \Omega \right\}.$$

Let $\Omega^{i\tau} := \cup_{x \in \Omega_{\text{solid}}^i} B(x, \tau)$ be the open τ -neighborhood of the set Ω^i , the characteristic functions of these sets are $\chi^{i\tau} := \mathbf{1}_{\Omega^{i\tau}}$, and $\chi^\tau := \sum_{i=1}^k \chi^{i\tau}$ is the characteristic function of the τ -neighborhood of the domain occupied by the submerged rigid bodies. We denote by

$$K_\tau(\chi) := \left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{u} = 0 \text{ and } \chi^\tau D_{\text{sym}} \mathbf{u} = 0 \text{ in } \Omega \right\}$$

the space of all possible (momentary) velocity fields that preserve the volume and a τ -neighborhood of the rigid bodies. Now we can state the proposition that allows us to restrict testing of the weak formulation to functions which preserves

τ -neighborhoods. Although we will later apply it to $\phi \in K(\chi)$, we formulate it for \mathbf{u} to simplify reading and the difference to the function ψ that play an important role in the next proof.

Proposition 3.4.6 (Approximation of rigid velocity fields). *Let χ be collision-free, that means the connected sub-domains of $\text{supp } \chi(t)$ have a distance greater than zero. For any $\mathbf{u} \in K(\chi)$ there exists for all small $\tau > 0$ an $\mathbf{u}_\tau \in K_\tau(\chi)$ such that the sequence $(\mathbf{u}_\tau)_{\tau>0}$ converges to \mathbf{u} in $H^1(\Omega)^3$. Especially, the same is true locally within subsets of Ω that contain two moving rigid bodies that do not collide.*

If χ is not collision-free, then $(\mathbf{u}_\tau)_{\tau>0}$ converges to \mathbf{u} in $L^2(\Omega)^3$ and in H^1 except for a τ -ball around collisions. This is not necessarily on the set

$$\Omega \setminus \bigcup_{i \neq j} B\left(\overline{\Omega_{\text{solid}}^i} \cap \overline{\Omega_{\text{solid}}^j}, \tau\right),$$

of total measure that is bounded by $c \frac{4}{3} N \pi \tau^3$ and tends to zero for smaller values of τ .

Remark 3.4.7 (Differences between two and three dimensions). (a) The proof of this proposition tries to mimic the proof of (San Martín et al., 2003, Prop. 4.3) of the identical statement in two dimensional spaces as far as possible. The two dimensional version uses the so called stream function approach, where a velocity field $\mathbf{u} = (u_1, u_2) \in H^1(\Omega)^2$ has a representation

$$\mathbf{u} = \text{curl } \psi = \begin{pmatrix} \partial_2 \psi \\ -\partial_1 \psi \end{pmatrix}$$

for some scalar valued $\psi \in H^2(\Omega)$. If $D_{\text{sym}} \mathbf{u} = 0$ in an open and connected subset Ω_{solid}^i , then the velocity field \mathbf{u} is necessarily of the form

$$\mathbf{u}_{\text{solid}}^i(\mathbf{x}) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix},$$

which is valid for all $\mathbf{x} \in \Omega_{\text{solid}}^i$ of a connected subset of $\Omega_{\text{solid}} = \{\mathbf{x} \in \Omega \mid D_{\text{sym}} \mathbf{u}(\mathbf{x}) = 0\}$. In this case ψ_{solid}^i is uniquely defined up to a constant $\psi_{\text{solid}}^i(0) \in \mathbb{R}$ by

$$\psi_{\text{solid}}^i(\mathbf{x}) = \frac{1}{2} \omega |\mathbf{x}|^2 + \begin{pmatrix} -u_2(0) \\ u_1(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \psi_{\text{solid}}^i(0).$$

The constant $\psi_{\text{solid}}^i(0)$ is defined by the assumption $\psi_{\text{solid}}^i(\mathbf{x}) = \psi(\mathbf{x})$ for $\mathbf{x} \in \Omega_{\text{solid}}^i$. Let $\psi_{\text{solid}} \in H^2(\Omega)$ equal ψ_{solid}^i in a 2τ -neighborhood of Ω_{solid} and η_τ be an appropriately chosen approximation of the characteristic function of the fluid domain. Then an approximation ψ_τ of ψ is obtained by

$$\psi_\tau := \psi_{\text{solid}} + \eta_\tau \cdot (\psi - \psi_{\text{solid}}) \longrightarrow \psi \text{ strongly in } H^2(\Omega)^2.$$

This approach assumes the existence of such a function $\psi_{\text{solid}} \in H^2(\Omega)$. In three dimensions the corresponding ψ_{solid} is only defined in a neighborhood of Ω_{solid}^i and η_τ is an approximation of the characteristic function of the solid occupied domain. The switch between these two approximations can be seen for example in (3.4.25), but the approximation scheme is necessarily inverted in (3.4.24) of the following proof: $\psi_\tau := \psi + \eta_\tau \cdot (\psi_{\text{solid}} - \psi)$

(b) But a second observation is restrictive. In two dimensions the above proposition is valid without the assumption that no collisions occur. Thus, test functions satisfy all regularity assumptions needed to apply the estimates stated in Remark 3.4.3. Hence, these functions describe the motion of balls-on-sticks. Testing with such functions therefore does not control a rotation that preserves the normal to the shared tangential plane.

Proof. Let Ω_{solid}^i denote the connected subsets of

$$\Omega_{\text{solid}} = \{x \in \Omega \mid D_{\text{sym}} u(x) = 0\}.$$

Assume at first that no collisions occur, denote the minimal distance by

$$\delta := \min \left\{ \text{dist}(\Omega_{\text{solid}}^i, \Omega_{\text{solid}}^j) \mid i, j = 0, \dots, k; i \neq j \right\},$$

and let $\tau_0 := \frac{1}{6}\delta$. For a given $u \in K(\chi)$ there exists a $\psi \in H^2(\Omega)^3 \cap H_0^1(\Omega)^3$ that satisfies $u := \text{curl } \psi \in H^1(\Omega)^3$, since $\text{div } u = 0$ by assumption. It is found as solution to $-\Delta \psi = \text{curl } u$, which needs to be satisfied almost everywhere.

For each connected subset and $x \in \Omega_{\text{solid}}^i$ the velocity field u satisfies $u(x) = u_{\text{solid}}^i(x)$, where the velocity of the rigid body is given by

$$(3.4.22) \quad u_{\text{solid}}^i(x) = \omega^i \times x + \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} 0 & \omega_3^i & -\omega_2^i \\ -\omega_3^i & 0 & \omega_1^i \\ \omega_2^i & -\omega_1^i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} u_1^i \\ u_2^i \\ u_3^i \end{pmatrix}.$$

as is defined for all $x \in \mathbb{R}^3$, where $\omega^i = \text{curl } u$ in Ω_{solid}^i . Note that $u_j^i = u_{\text{solid } j}^i(0)$ defines these constants uniquely. The assumption $\text{curl } \psi_{\text{solid}}^i(x) = u_{\text{solid}}^i(x)$ for each i and $x \in \Omega_{\text{solid}}^i$ yields that $\psi(x) = \psi_{\text{solid}}^i(x)$, where the function ψ_{solid}^i is of the form

$$(3.4.23) \quad \psi_{\text{solid}}^i(x) := \begin{pmatrix} \frac{1}{2}\omega_1^i(x_2^2 + x_3^2) \\ \frac{1}{2}\omega_2^i(x_1^2 + x_3^2) \\ \frac{1}{2}\omega_3^i(x_1^2 + x_2^2) \end{pmatrix} + \begin{pmatrix} a_1^i & a_2^i + u_3^i & a_3^i - u_2^i \\ a_2^i & a_4^i & a_5^i + u_1^i \\ a_3^i & a_5^i & a_6^i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \psi_1^i(0) \\ \psi_2^i(0) \\ \psi_3^i(0) \end{pmatrix},$$

where $a_1, \dots, a_6, \psi_1^i(0), \psi_2^i(0), \psi_3^i(0)$ are constants, which are uniquely defined by the assumption $\psi(x) = \psi_{\text{solid}}^i(x)$ on the open nonempty set Ω_{solid}^i . We extend the rigid velocity fields at first to an $\frac{5}{2}\tau_0$ neighborhood of $\Omega_{\text{solid}} = \bigcup_{i=0}^k \Omega_{\text{solid}}^i$ and beyond by zero and obtain ψ_{solid} , defined by

$$\psi_{\text{solid}}(x) := \begin{cases} \psi_{\text{solid}}^i(x) & \text{for } \text{dist}(x, \Omega_{\text{solid}}^i) \leq \frac{5}{2}\tau_0, \\ 0 & \text{for } \text{dist}(x, \Omega_{\text{solid}}^i) > \frac{5}{2}\tau_0, \end{cases}$$

which is in $H^2(\Omega_{\text{solid}}^{\frac{5}{2}\tau_0})$. Therefore, the difference satisfies $\psi_{\text{solid}} - \psi \in H^2(\Omega_{\text{solid}}^{\frac{5}{2}\tau_0})$. We want to construct an approximation ψ_τ such that

$$(3.4.24) \quad \psi_\tau := \psi + \eta_\tau \cdot (\psi_{\text{solid}} - \psi) \longrightarrow \psi \text{ strongly in } H^2(\Omega)^3,$$

where η_τ is an approximation to the characteristic function of Ω_{solid} . By construction $u_\tau := \text{curl } \psi_\tau$ is an approximation of u in $H^1(\Omega)^3$ and satisfies $D_{\text{sym}} u_\tau = 0$ in a τ -neighborhood $\Omega_{\text{solid}}^\tau$ of Ω_{solid} .

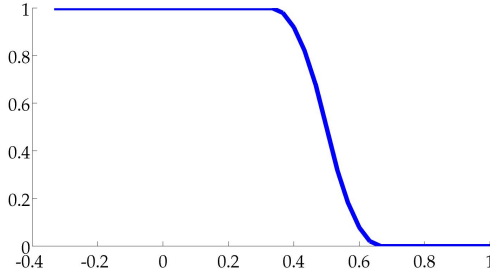


Figure 3.4.4: The rigid velocity field is extended up to a distance of $\tau = \frac{1}{3}$, is connected to the velocity of the fluid within the distance τ to 2τ , here $(\frac{1}{3}, \frac{2}{3})$, for minimal interparticle and particle wall distance $\delta = 2$.

To extend the rigid function ψ_{solid}^i to a τ -neighborhood of Ω_{solid}^i such that outside of a 2τ -neighborhood we reobtain the original ψ , we define initially

$$\tilde{\eta}_\tau(s) := \begin{cases} 1 & \text{for } s \leq \tau, \\ 1 - 2(\tau^{-1}s - 1)^2 & \text{for } \tau < s \leq \frac{3}{2}\tau, \\ 2(2 - \tau^{-1}s)^2 & \text{for } \frac{3}{2}\tau < s < 2\tau, \\ 0 & \text{for } s > 2\tau, \end{cases}$$

and choose as approximation to the characteristic function of Ω_{solid}

$$(3.4.25) \quad \eta_\tau(x) := \sum_{i=0}^k \tilde{\eta}_\tau(\text{dist}(x, \Omega_{\text{solid}}^i)) = 1 - \prod_{i=0}^k (1 - \tilde{\eta}_\tau(\text{dist}(x, \Omega_{\text{solid}}^i))),$$

where $x \in \mathbb{R}^3$. As approximation to the original ψ we define ψ_τ by

$$(3.4.26) \quad \begin{aligned} \psi_\tau(x) &:= \psi(x) + \eta_\tau(x) (\psi_{\text{solid}}(x) - \psi(x)) \\ &= \psi(x) + \sum_{i=0}^k \tilde{\eta}_\tau(\text{dist}(x, \Omega_{\text{solid}}^i)) (\psi_{\text{solid}}^i(x) - \psi(x)). \end{aligned}$$

We want to show, that $\psi_\tau \rightarrow \psi$ in $H^2(\Omega)^3$ or equivalently that

$$\eta_\tau(\psi_{\text{solid}} - \psi) \rightarrow 0 \text{ in } H^2(\Omega)^3.$$

We observe the following properties of the involved functions, which we formulate using the Landau symbols *capital-O* and *small-o*.

- Since $\psi_{\text{solid}} - \psi = 0$ within $\Omega_{\text{solid}}^\tau$ and $\eta_\tau = 0$ outside of $\Omega_{\text{solid}}^{2\tau}$, the support of $\eta_\tau(\psi_{\text{solid}} - \psi)$ is given by the closure of

$$U_{2\tau} := \left\{ x \in \Omega \mid 0 < \min_{i=0, \dots, k} \text{dist}(x, \Omega_{\text{solid}}^i) < 2\tau \right\}.$$

By assumption on the regularity of Ω_{solid} we have $|U_{2\tau}| = O(\tau)$ for small τ . Note that only for $\tau \rightarrow \infty$ it is of order $O(\tau^3)$, but we cannot use this estimate.

- Since $\psi_{\text{solid}} - \psi \in H^2(\Omega_{\text{solid}}^{2\tau_0})^3$ is independent of τ , especially

$$\|\text{grad}^2(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} \rightarrow 0 \text{ for } \tau \rightarrow 0,$$

and $\psi_{\text{solid}} - \psi$ vanishes identically on the set Ω_{solid} , Poincaré's inequality yields that

$$\|\psi_{\text{solid}} - \psi\|_{L^2(U_{2\tau})} \leq \tau^2 \|\text{grad}^2(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} = o(\tau^2) \text{ for } \tau \rightarrow 0.$$

and that for $j = 1, 2, 3$ that the first order derivatives satisfy

$$\|\partial_j(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} \leq \tau \|\text{grad}^2(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} = o(\tau) \text{ for } \tau \rightarrow 0.$$

and for $j, l = 1, 2, 3$ the second order derivatives satisfy

$$\|\partial_{jl}(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} \leq \|\text{grad}^2(\psi_{\text{solid}} - \psi)\|_{L^2(U_{2\tau})} = o(1).$$

- Since $0 \leq \eta_\tau(s) \leq 1$ for all s and all τ yields that the functions $\tilde{\eta}_\tau$ and η_τ are both uniformly bounded in τ . Therefore $\|\eta_\tau\|_{L^\infty(U_{2\tau})} = 1$.
- By regularity assumption of the sets Ω_{solid}^i , the first derivative of $\tilde{\eta}_\tau$ and therefore the first derivatives of η_τ are uniformly bounded of order $O(\tau^{-1})$ for small τ . Similarly, the second derivatives are uniformly bounded of order $O(\tau^{-2})$ for small τ .

Applying all those convergence properties plus a lot of times Hölder's inequality (102-times), yields that $\|\eta_\tau(\psi_{\text{solid}} - \psi)\|_{H^2(U_{2\tau})} = o(1)$ for $\tau \rightarrow 0$. Since by construction $(\eta_\tau(\psi_{\text{solid}} - \psi))$ vanishes outside of $U_{2\tau}$, we obtain $\|\eta_\tau(\psi_{\text{solid}} - \psi)\|_{H^2(\Omega)} = o(1)$ for $\tau \rightarrow 0$. Therefore, ψ_τ is an approximation of ψ in $H^2(\Omega)^3$, we obtain an approximation \mathbf{u}_τ of \mathbf{u} by

$$\mathbf{u}_\tau := \text{curl } \psi_\tau \longrightarrow \text{curl } \psi = \mathbf{u} \text{ strongly in } H^1(\Omega)^3,$$

and \mathbf{u}_τ satisfies $\text{D}_{\text{sym}} \mathbf{u}_\tau = 0$ in a τ -neighborhood of each Ω_{solid}^i for all $\tau < \frac{1}{3}\delta$. Hence, we found the approximation of \mathbf{u} as claimed.

In the case of collisions the above considerations stay valid besides in a neighborhood of collisions. Furthermore, with respect to the L^2 -norm jumps on a subset of measure $n - 1$ are not observable. By (3.4.22) the velocity $\mathbf{u}_{\text{solid}}^i$ is fixed on Ω_{solid}^i , similarly the velocity $\mathbf{u}_{\text{solid}}^j$ on a touching domain Ω_{solid}^j . By estimate (3.4.10) even the relative velocity vanishes. In the τ -neighborhood of total measure bounded by $\frac{4}{3}\pi\tau^3$ of a collision point we interpolate between ψ_{solid}^i and ψ_{solid}^j in the above introduced sense. The resulting pasted version between the bodies Ω_{solid}^i and Ω_{solid}^j , $\tilde{\psi}$, is then pasted to the global ψ , similarly to equation (3.4.26),

$$\psi_\tau(\mathbf{x}) := \psi(\mathbf{x}) + \eta_\tau(\mathbf{x})(\tilde{\psi}(\mathbf{x}) - \psi(\mathbf{x})).$$

We obtain that for τ to zero we obtain convergence in $H^1(\Omega)$ and therefore of $\mathbf{u}_\tau := \text{curl } \psi_\tau$ to \mathbf{u} . This finishes the proof. q.e.d.

Corollary 3.4.8. *Without collisions, the spaces $K(\chi)$ and $\overline{\bigcup_{\tau>0} K_\tau(\chi)}$ coincide. Hence, a weak solution in the sense of Definition 3.2.7 satisfies the weak form of the conservation of momentum equation even for test functions ϕ that satisfy $\text{D}_{\text{sym}} \phi = 0$ on $\text{supp } \chi$.*

Proof. By definition we have for each $\tau > 0$ that $K_\tau(\chi) \subseteq K(\chi)$. Hence, $\bigcup_{\tau>0} K_\tau(\chi) \subseteq K(\chi)$. By construction $K(\chi)$ is a closed subspace of $H_0^1(\Omega)^3$. Thus,

$$\overline{\bigcup_{\tau>0} K_\tau(\chi)} \subseteq K(\chi)$$

is satisfied. By Proposition 3.4.6 this inclusion is dense. *q.e.d.*

3.5 Compactness of the density fields and transport equations

3.5.1 Compactness results of R. J. DiPerna and P.-L. Lions

We quote from DiPerna and Lions (1989) and Lions (1996) the following theorems, but adapt the notation to our needs. Their theory of renormalized solutons yields a very general existence theorem, if the characteristic functions X of the underlying velocity field or its regularization does not leave the set Ω . Therefore, for the following the assumption $v = 0$ on $\partial\Omega$, which is hidden in the subscript 0 is crucial. Similar results are valid for the all-space or periodic case; in- and out-flow problems are out of reach.

The following theorem will applied the cases $U = \Omega$ and $U = \Omega^{\text{ext}}$ and v equal u , u^n and \bar{u}_n .

Theorem 3.5.1 (Solution to transport equations). *Let $U \subseteq \mathbb{R}^3$, let $v \in L^2(0, T; H_0^1(U))$, with $\operatorname{div} v = 0$, and let $\kappa_0 \in L^\infty(U)$ be non-negative. Then there exists a unique $\kappa \in L^\infty(0, T; L^\infty(U) \cap C([0, T]; L^1(U)))$ that satisfies*

$$(3.5.1) \quad \partial_t \kappa + \operatorname{div}(\kappa v) = 0 \quad \text{in } \mathcal{D}([0, T] \times U)^*,$$

$$(3.5.2) \quad \kappa(x, 0) = \kappa_0(x) \text{ in } L^\infty(U)$$

in the weak sense, by which we mean

$$\int_{U_T} \kappa(x, t) (\partial_t \eta + \operatorname{grad} \eta \cdot \bar{v}) \, dx \, dt = - \int_U \kappa_0(x) \eta(x, 0) \, dx$$

is satisfied for all $\eta \in C^1(0, T; C^1(U))$ that satisfy $\eta(T) = 0$. Furthermore, for all $a < b \in \mathbb{R}$ we have that the volume

$$\left| \{x \in \Omega \mid a \leq \kappa(x, t) \leq b\} \right|$$

is independent of t , which yields that for all $1 \leq p \leq \infty$ the norm $\|\kappa(t)\|_{L^p} = \|\kappa_0\|_{L^p}$ is independent of t , and that $\kappa(t)$ is a characteristic function if and only if κ_0 is a characteristic function.

The proof of this statement can be found in (Lions, 1996, Thm. 2.1).

Theorem 3.5.2 (Convergence of solutions of transport equations). *For a bounded set $U \subseteq \mathbb{R}^3$ let a sequence of velocities $(v^n)_n \subseteq L^2(0, T; H_0^1(U))$, with $\operatorname{div} v^n = 0$, and a sequence of initial values $(\kappa_0^n)_n \subseteq L^\infty(U)$ be given. Let $\kappa^n \in L^\infty(0, T; L^\infty(U) \cap C([0, T]; L^1(U)))$ denote the corresponding weak solutions which exist according to Theorem 3.5.1. Furthermore, assume that there exists $v \in L^2(0, T; H_0^1(U))$ and non-negative $\kappa_0 \in L^\infty(U)$ such that*

$$v^n \rightharpoonup v \quad \text{weakly in } L^2(0, T; H_{0,\sigma}^1(U))$$

$$\kappa_0^n \rightarrow \kappa_0 \quad \text{strongly in } L^1(U).$$

If the sequence $(\kappa^n)_n$ is uniformly bounded in $L^\infty(0, T; L^\infty(U))$ then it converges strongly in $C([0, T]; L^1(U))$ for all $1 \leq p < \infty$ to the unique weak solution $\kappa \in L^\infty(0, T; L^\infty(U)) \cap C([0, T]; L^1(U))$ of

$$\begin{aligned} \partial_t \kappa + \operatorname{div}(\kappa v) &= 0 && \text{in } \mathcal{D}([0, T] \times U)^*, \\ \kappa(x, 0) &= \kappa_0(x) && \text{in } L^\infty(U). \end{aligned}$$

The proof of this theorem can be found in (Lions, 1996, Thm. 2.4.(1), Rem. 2.4).

3.5.2 Solving the transport equations

In this subsection we apply the theorems of Lions to our setting and observe, due to our choice of approximative solutions and their properties on page 36, equation (3.3.7), (3.3.8), the identical initial values we obtain the following:

Lemma 3.5.3 (Existence of weak solutions). *For all $1 \leq p < \infty$ the approximative densities ϱ^n and approximative characteristic functions of the δ -kernels of the solid domains ζ^{in} satisfy*

$$(3.5.3) \quad \varrho^n \longrightarrow \varrho \text{ strongly in } C([0, T]; L^p(\Omega))$$

$$(3.5.4) \quad \zeta^{in} \longrightarrow \zeta^i \text{ strongly in } L^p(0, T; L^p(\Omega^{ext})),$$

where ϱ and ζ^i satisfy

$$(3.5.5) \quad \int_{Q_T} \varrho \cdot (\partial_t \psi + \operatorname{grad} \psi \cdot u) = - \int_{\Omega} \varrho_0(x) \psi(x, 0) \, dx$$

$$(3.5.6) \quad \int_{Q_T^{ext}} \zeta^i \cdot (\partial_t \eta + \operatorname{grad} \eta \cdot \bar{u}) = - \int_{\Omega^{ext}} \zeta_0^i(x) \eta(x, 0) \, dx$$

for all $\psi \in C^1(0, T; C^1(\Omega))$ that satisfy $\psi(T) = 0$ and all $\eta \in C^1(0, T; C^1(\Omega^{ext}))$ that satisfy $\eta(T) = 0$.

From this lemma we obtain directly the convergence of the functions χ^{in} , which we defined to be the characteristic functions of the δ -neighborhood of $\operatorname{supp} \zeta^{in}$. To see this we need some properties of the symmetric difference of sets: We will denote by

$$U \ominus V := (U \setminus V) \cup (V \setminus U)$$

the symmetric difference of two sets $U, V \subseteq \mathbb{R}^3$. Then a sequence $(U^n)_n$, where $U^n \subseteq \mathbb{R}^3$ is said to converge to $U \subseteq \mathbb{R}^3$ if

$$\int_{\mathbb{R}^3} |\mathbf{1}_{U^n} - \mathbf{1}_U| = |U^n \ominus U| \rightarrow 0.$$

Since Ω^{ext} is bounded, all these characteristic functions are in all $L^p(\Omega^{ext})$ spaces if $U, U^n \subseteq \Omega$ is assumed. Hence, if $U^n \rightarrow U$ then $U^{n, ext} \rightarrow U^{ext}$ in this sense. Applying this statement to the supports yields the following corollary.

Corollary 3.5.4. *For all $1 \leq p < \infty$ the approximative characteristic functions χ^{in} of the solid domains, i.e. the characteristic functions of the δ -neighborhood of $\operatorname{supp} \zeta^{in}$ and $\operatorname{supp} \zeta^n$ satisfy*

$$(3.5.7) \quad \chi^{in} \longrightarrow \chi^i \text{ strongly in } L^p(0, T; L^p(\Omega^{ext})),$$

$$(3.5.8) \quad \chi^n \longrightarrow \chi \text{ strongly in } L^p(0, T; L^p(\Omega^{ext})).$$

We now want to consider the convergence of the characteristic functions ζ^{in} and χ^{in} in more detail. Since these are defined with respect to the smoothed velocity fields \bar{u}^n , they could be obtained alternatively via the method of characteristics.

Notation 3.5.5 (Approximative Characteristics). By assumption, see e.g. (3.3.8), the sequence $(\bar{u}^n)_n \subseteq L^2(0, T; H_0^1(\Omega^{ext})^3) \cap L^\infty(0, T; L_\sigma^2(\Omega^{ext}))$ is uniformly bounded and satisfies

$$\bar{u}^n \longrightarrow \bar{u} \text{ weakly in } L^2(0, T; H_0^1(\Omega^{ext})^3) \text{ and weakly}^* \text{ in } L^\infty(0, T; L_\sigma^2(\Omega^{ext})).$$

By construction¹ $\bar{u}^n(\cdot, t), \bar{u} \in C_0^2(\Omega^{ext})^3$ and $\bar{u}^n(x, \cdot), \bar{u} \in L^\infty(0, T; \mathbb{R}^3)$, and, what is more, (\bar{u}^n) is bounded in $L^\infty(0, T; C^2(\Omega^{ext}))$ since it approximates \bar{u} .

We denote the characteristics with respect to these regularized flow fields by Ξ^n and Ξ , respectively, this they satisfy

$$(3.5.9) \quad \begin{aligned} \partial_t \Xi^n(\xi, s; t) &= \bar{u}^n(\Xi^n(\xi, s; t), t) \\ \Xi^n(\xi, s; s) &= \xi \end{aligned}$$

and

$$(3.5.10) \quad \begin{aligned} \partial_t \Xi(\xi, s; t) &= \bar{u}(\Xi(\xi, s; t), t) \\ \Xi(\xi, s; s) &= \xi \end{aligned}$$

for all $0 \leq s, t \leq T$ and all $\xi \in \Omega^{ext}$. These initial value problems have unique solutions, see e.g. Amann (1995) for the theory of classical solutions and McShane (1947) for mild solutions, i.e. they satisfy

$$\Xi^n(\xi, s; t) = \xi + \int_s^t \bar{u}^n(\Xi^n(\xi, s; \tau), \tau) d\tau$$

and

$$\Xi(\xi, s; t) = \xi + \int_s^t \bar{u}(\Xi(\xi, s; \tau), \tau) d\tau.$$

It should be noted, that the space $W^{1,\infty}$ consists of the Lipschitz continuous functions and we are interested in absolutely continuous functions in time, i.e. functions which are integral to function which is integrable on compact sets of the form $[0, t]$ for $0 < t < T$. With these notions, we obtain the following Lemma on solutions of ordinary differential equations:

Lemma 3.5.6 (A-priori estimates for the characteristics). *Let $(\Xi^n)_n$ be the sequence of characteristics corresponding to the regularized approximative velocity fields \bar{u}^n and Ξ be characteristic corresponding to the limit \bar{u} of these fields.*

(a) *The family of coordinate changes of a given $\xi \in \Omega^{ext}$, i.e.*

$$\Xi^n(\cdot, s; t) : \Omega^{ext} \rightarrow \Omega^{ext} : \xi \mapsto \Xi^n(\xi, s; t)$$

is uniformly bounded in $C^2(\Omega^{ext})^3$ uniformly in starting time $s \in [0, T]$, terminal time $t \in [0, T]$ and sequence index n .

¹By $C_0^2(\Omega)$ we denote the the Banach space of functions that twice differentiable vanish on the boundary. This is a closed subspace of $C_c^2(\mathbb{R}^3)$, the space of twice differentiable functions with compact support, which is never a Banach space.

(b) For all n and all $\xi \in \Omega^{\text{ext}}$ the family of trajectories

$$\Xi^n(\xi, s; \cdot) : [0, T] \rightarrow \Omega^{\text{ext}} : t \mapsto \Xi^n(\xi, s; t)$$

is uniformly bounded in $W^{1,\infty}([0, T]; \mathbb{R})^3$ with respect to the initial time s , initial position ξ and n . Furthermore, the family of initial positions for a given ξ , i.e. the inverse mappings

$$\Xi^n(\xi, \cdot; t) : [0, T] \rightarrow \Omega^{\text{ext}} : s \mapsto \Xi^n(\xi, s; t)$$

is uniformly bounded in $W^{1,\infty}([0, T]; \mathbb{R})^3$ with respect to the time t , terminal position ξ and n .

(c) For all n the characteristics preserve the volume, i.e. $\det D \Xi^n(\xi, s; t) = 1$ for all $\xi \in \Omega^{\text{ext}}$, $s, t \in [0, T]$, and $n \in \mathbb{N}$.

(d) Selecting again a subsequence of the indices n , we can assume that for all $0 \leq \alpha < 1$ the sequence $(\Xi^n)_n$ converges strongly to Ξ in $C^\alpha([0, T] \times [0, T]; C^1(\Omega^{\text{ext}})^3)$.

Proof. These statements follow from standard theory of ordinary differential equations and the known a-priori regularity estimates and dependance on parameter theorems, see e.g. (McShane, 1947, IX.69) and DiPerna and Lions (1989). Hence, we only need to show that these estimates are valid uniformly.

(a) Since all functions \mathbf{u}^n vanish outside Ω^{ext} by construction, $\Xi^n(\xi, s; t) \in \Omega^{\text{ext}}$ for all s, t and all $\xi \in \Omega^{\text{ext}}$. Since Ω and therefore Ω^{ext} are bounded, we obtain that $\Xi^n(\xi, s; t)$ is uniformly bounded for all s, t and all $\xi \in \Omega^{\text{ext}}$. Furthermore, by differentiability of the velocity fields we obtain that $\Xi^n(\xi, s; t)$ depends twice continuously differentiable on ξ . Therefore, taking once and twice the derivatives of (3.5.9) yields, using for vectors subscripts of the form X_j as coordinate j -index, using $X_{,k}$ as shorthand for derivatives $\partial_{\xi_k} X$ or $\partial_{x_k} X$, respectively, and using the summation convention on iterated indices, that for $j, k, l = 1, 2, 3$ the characteristics satisfy

$$\begin{aligned} \partial_t \Xi_j^n(\xi, s; t) &= \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t), \\ \Xi_j^n(\xi, s; s) &= \xi_j, \end{aligned}$$

their first derivatives satisfy

$$\begin{aligned} \partial_t \Xi_{j,k}^n(\xi, s; t) &= \partial_m \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) \Xi_{m,k}^n(\xi, s; t), \\ \Xi_{j,k}^n(\xi, s; s) &= \delta_{jk}, \end{aligned}$$

and the second derivatives satisfy

$$\begin{aligned} \partial_t \Xi_{j,kl}^n(\xi, s; t) &= \partial_{m_1 m_2} \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) \Xi_{m_1,k}^n(\xi, s; t) \Xi_{m_2,l}^n(\xi, s; t) \\ &\quad + \partial_{m_1} \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) \Xi_{m_1,kl}^n(\xi, s; t) \\ \Xi_{j,kl}^n(\xi, s; s) &= 0. \end{aligned}$$

Since the sequence $(\overline{\mathbf{u}}^n)$ is uniformly bounded in $L^\infty(0, T; C^2(\Omega^{\text{ext}}))$, as an approximation of $\overline{\mathbf{u}} \in L^\infty(0, T; C^2(\Omega^{\text{ext}}))$, we obtain that $\Xi_j^n(\xi, s; t)$, $\Xi_{j,k}^n(\xi, s; t)$ and $\Xi_{j,kl}^n$ are uniformly bounded in $L^\infty([0, T]^2; C^2(\Omega^{\text{ext}}))$. This yields (a).

(b) As was observed already in (a), $\Xi^n(\xi, s; t) \in \Omega^{\text{ext}}$ for all ξ, s and t yields, due to the boundedness of Ω^{ext} , that $\Xi^n(\xi, \cdot; \cdot) \in L^\infty([0, T]^2; \mathbb{R}^3)$ uniformly in ξ . Thus, we only need to consider the derivatives. By construction $\partial_t \Xi_j^n(\xi, s; t) = \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t)$. Therefore it is uniformly bounded, since $(\overline{\mathbf{u}}^n)_n \subseteq L^\infty(0, T; C^2(\Omega^{\text{ext}}))$ is uniformly bounded. The estimate in s follows from the following observation: $\partial_s \Xi_j^n(\xi, s; t)$ satisfies

$$\begin{aligned} \partial_t (\partial_s \Xi_{j,k}^n(\xi, s; t)) &= D \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) (\partial_s \Xi_{m,k}^n(\xi, s; t)), \\ \partial_s \Xi_{j,k}^n(\xi, s; s) &= -\overline{\mathbf{u}}_j^n(\xi, s). \end{aligned}$$

And again, uniform boundedness of $(\overline{\mathbf{u}}^n)_n \subseteq L^\infty(0, T; C^2(\Omega^{\text{ext}}))$ yields uniform boundedness of $\partial_s \Xi_{j,k}^n(\xi, s; t)$. Which proves (b).

(c) Liouville's Theorem, see e.g. Amann (1995), allows us to deduce from

$$\begin{aligned} \partial_t \Xi_{j,k}^n(\xi, s; t) &= \partial_m \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) \Xi_{m,k}^n(\xi, s; t) \\ \Xi_{j,k}^n(\xi, s; s) &= \delta_{jk}. \end{aligned}$$

and from $\text{Trace } D \overline{\mathbf{u}}_j^n = \text{div } \overline{\mathbf{u}}^n = 0$ that

$$\begin{aligned} \partial_t \det D \Xi^n(\xi, s; t) &= \text{Trace } D \overline{\mathbf{u}}_j^n(\Xi^n(\xi, s; t), t) \det D \Xi^n(\xi, s; t) = 0 \\ \det D \Xi^n(\xi, s; s) &= 1, \end{aligned}$$

which yields $\det D \Xi^n(\xi, s; t) = 1$ for all s, t and ξ .

(d) Due to the compactness of the embedding

$$W^{1,\infty}([0, T]^2; C^2(\Omega)^3) \subseteq C^\alpha([0, T]^2; C^1(\Omega)^3)$$

and the above estimates, we can select a strongly converging subsequence of $(\Xi^n)_n$ and obtain a limit in $C^\alpha([0, T]^2; C^1(\Omega)^3)$. Hence, we only need to observe that this limit solves the characteristic equation that defines Ξ . Since two locally integrable functions f and g are equal almost everywhere if they are equal as densities of measures, i.e. if $\int_U f = \int_U g$ for all measurable sets U , we consider the integrated mild formulation of (3.5.9) for $U \subseteq \Omega^{\text{ext}}$,

$$(3.5.11) \quad \int_U \Xi^n(\xi, s; t) d\xi = \int_U \xi d\xi + \int_s^t \int_U \overline{\mathbf{u}}^n(\Xi^n(\xi, s; \tau), \tau) d\xi d\tau,$$

and observe that all terms are essentially uniformly bounded. The sequence Ξ^n converges point wise, the sequence $(\overline{\mathbf{u}}^n)_n$ can after possibly selecting again a subsequence, be assumed to converge almost everywhere. Hence, by Lebesgue's dominated convergence theorem, the $C^\alpha([0, T]^2; C^1(\Omega)^3)$ -limit of $(\Xi^n)_n$ satisfies

$$(3.5.12) \quad \int_U \Xi(\xi, s; t) d\xi = \int_U \xi d\xi + \int_s^t \int_U \overline{\mathbf{u}}(\Xi(\xi, s; \tau), \tau) d\xi d\tau,$$

which is the integrated mild formulation (3.5.10), i.e. these two limits coincide. *q.e.d.*

The solutions which were constructed using the existence theory of DiPerna and Lions via solving the transport equations, yield the same solution, as is obtained using the method of characteristics, in this sense these paths to the solutions are consistent:

Corollary 3.5.7 (Consistency with method of characteristics). *The characteristic functions of the transported approximative submerged bodies within the approximative velocity fields are consistent with the characteristic functions of this velocity field, i.e. χ^{in} and Ξ^n to the approximative velocity field \mathbf{u}^n and χ^i and Ξ to the limit velocity field \mathbf{u} satisfy*

$$(3.5.13) \quad \zeta^{in}(\mathbf{x}, t) = \zeta^{in}(\Xi^n(\mathbf{x}, t; 0), 0)$$

$$(3.5.14) \quad \zeta^i(\mathbf{x}, t) = \zeta^i(\Xi(\mathbf{x}, t; 0), 0).$$

for all s, t and all $\mathbf{x} \in \Omega^{\text{ext}}$ and $i = 0, \dots, N$.

Proof. We follow line of thoughts similar to the last proof, this is we consider the weak formulations of the defining differential equations and their limit behavior: The function ζ^{in} was found as the unique solution that satisfies

$$(3.5.15) \quad \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\mathbf{x}, t) \left(\partial_t \eta(\mathbf{x}, t) + \text{grad } \eta(\mathbf{x}, t) \overline{\mathbf{u}^n}(\mathbf{x}, t) \right) d\mathbf{x} dt = - \int_{\Omega^{\text{ext}}} \zeta^{in}(\mathbf{x}, 0) \eta(\mathbf{x}, 0) d\mathbf{x}$$

for all $\eta \in C^1(0, T; C^1(\Omega^{\text{ext}}))$ that satisfy $\eta(T) = 0$. But $\zeta^{in}(\Xi^n(\mathbf{x}, t; 0), 0)$ satisfies, using the coordinate change $\mathbf{x} = \Xi^n(\xi, 0; t)$, $\det D \Xi^n(\xi, 0; t) = 1$ and integration by parts, also the same equation:

$$\begin{aligned} & \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\Xi^n(\mathbf{x}, t; 0), 0) \left(\partial_t \eta(\mathbf{x}, t) + \text{grad } \eta(\mathbf{x}, t) \overline{\mathbf{u}^n}(\mathbf{x}, t) \right) d\mathbf{x} dt \\ &= \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\xi, 0) \left(\partial_t \eta(\Xi^n(\xi, 0; t), t) + \text{grad } \eta(\Xi^n(\xi, 0; t), t) \overline{\mathbf{u}^n}(\Xi^n(\xi, 0; t), t) \right) d\xi dt \\ &= \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\xi, 0) \frac{d}{dt} (\partial_t \eta(\Xi^n(\xi, 0; t), t)) d\xi dt \end{aligned}$$

which yields that

$$\begin{aligned} & \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\Xi^n(\mathbf{x}, t; 0), 0) \left(\partial_t \eta(\mathbf{x}, t) + \text{grad } \eta(\mathbf{x}, t) \overline{\mathbf{u}^n}(\mathbf{x}, t) \right) d\mathbf{x} dt \\ &= - \int_{\Omega^{\text{ext}}} \zeta^{in}(\xi, 0) \eta(\xi, 0) d\xi \end{aligned}$$

for all $\eta \in C^1(0, T; C^1(\Omega^{\text{ext}}))$ that satisfy $\eta(T) = 0$. Hence, (3.5.13) is satisfied.

The case of ζ^i is different, since it was defined as limit in equation (3.3.10). Therefore, it is not obvious that the method of characteristics, incorporated in Ξ and the candidate ζ^i of a solution to the transport equation for \mathbf{u} are compatible. We obtain this as limiting case of the ζ^{in} considerations: We chose the common initial data ζ_0^i

for all ζ^{in} and obtain from Lebesgue's dominated convergence theorem, that

$$\begin{aligned} & \int_{\Omega_T^{\text{ext}}} \zeta^i(\Xi(x, t; 0), 0) (\partial_t \eta(x, t) + \text{grad } \eta(x, t) \bar{u}(x, t)) \, dx \, dt \\ &= \lim_n \int_{\Omega_T^{\text{ext}}} \zeta^{in}(\Xi^n(x, t; 0), 0) (\partial_t \eta(x, t) + \text{grad } \eta(x, t) \bar{u}^n(x, t)) \, dx \, dt \\ &= - \int_{\Omega^{\text{ext}}} \zeta_0^i(\xi) \eta(\xi, 0) \, d\xi \end{aligned}$$

for all $\eta \in C^1(0, T; C^1(\Omega^{\text{ext}}))$ that satisfy $\eta(T) = 0$. Hence, (3.5.14) is satisfied as well, which finishes the proof. *q.e.d.*

Now we have all the ingredients to prove that \mathbf{u} , ϱ , χ satisfy the requirements of a weak solution as defined in definition 3.2.7, besides the conservation of momentum equation (3.2.7):

Proof of Proposition 3.3.4. We collect the following statements to prove equations (3.2.10) and (3.2.9):

(a) The energy estimate (3.3.4) provides us with the estimate

$$(3.5.16) \quad 0 \leq n \int_{Q_T} \chi^{in} |\mathbf{D}_{\text{sym}} \mathbf{u}^n|^2 \, dx \, dt \leq C(\varrho_0, \mathbf{u}_0, g)$$

for all $i = 0, \dots, N$ and all n . Hence, $(\chi^{in} |\mathbf{D}_{\text{sym}} \mathbf{u}^n|^2)_n$ converges strongly in $L^1(Q_T)$ to zero. Hence, a subsequence $(\chi^{in} \mathbf{D}_{\text{sym}} \mathbf{u}^n)_n$ converges almost everywhere in Q_T to zero.

(b) By equation (3.3.7) the sequence $(\mathbf{D} \mathbf{u}^n)_n$ converges weakly in $L^2(Q_T)^{3 \times 3}$ to $\mathbf{D} \mathbf{u}$ this is

$$(3.5.17) \quad \int_{Q_T} \mathbf{D} \mathbf{u}^n : \Phi \longrightarrow \int_{Q_T} \mathbf{D} \mathbf{u} : \Phi$$

for all $\Phi \in L^2(Q_T)^{3 \times 3}$. Therefore, at first for all those $\Phi \in L^2(Q_T)^{3 \times 3}$, which are symmetric $\Phi^t = \Phi$ we have

$$(3.5.18) \quad \int_{Q_T} \mathbf{D}_{\text{sym}} \mathbf{u}^n : \Phi = \int_{Q_T} \mathbf{D} \mathbf{u}^n : \Phi \longrightarrow \int_{Q_T} \mathbf{D} \mathbf{u} : \Phi = \int_{Q_T} \mathbf{D}_{\text{sym}} \mathbf{u} : \Phi,$$

and hereby for all Φ

$$(3.5.19) \quad \int_{Q_T} \mathbf{D}_{\text{sym}} \mathbf{u}^n : \Phi \longrightarrow \int_{Q_T} \mathbf{D}_{\text{sym}} \mathbf{u} : \Phi,$$

where we used twice that the product of a symmetric and a skew symmetric matrix vanishes. Hence $(\mathbf{D}_{\text{sym}} \mathbf{u}^n)_n$ converges weakly in $L^2(Q_T)^{3 \times 3}$ and is bounded in $L^2(Q_T)^{3 \times 3}$.

(c) By equation (3.5.7) for all $1 \leq p < \infty$ the sequence $(\chi^{in})_n$ converges strongly in $L^p(Q_T)$ to χ^i . Hence, $(\chi^{in} D_{\text{sym}} \mathbf{u}^n)_n$ converges weakly in $L^1(Q_T)^{3 \times 3}$ to $\chi^i D_{\text{sym}} \mathbf{u}$ since

$$\begin{aligned} & \left| \int_{Q_T} (\chi^{in} D_{\text{sym}} \mathbf{u}^n - \chi^i D_{\text{sym}} \mathbf{u}) : \Phi \right| \\ & \leq \| \chi^{in} - \chi^i \|_{L^2(Q_T)} \| D_{\text{sym}} \mathbf{u}^n \|_{L^2(Q_T)} \| \Phi \|_{L^\infty(Q_T)} + \left| \int_{Q_T} (D_{\text{sym}} \mathbf{u}^n - D_{\text{sym}} \mathbf{u}) : (\chi^i \Phi) \right| \\ & \longrightarrow 0 \end{aligned}$$

for $n \rightarrow \infty$ and all $\Phi \in L^\infty(Q_T)^{3 \times 3}$.

(d) By (a) the sequence $(\chi^{in} D_{\text{sym}} \mathbf{u}^n)_n$ converges almost everywhere to zero; by (c) this sequence converges weakly to $\chi^i D_{\text{sym}} \mathbf{u}$. Thus, $\chi^i D_{\text{sym}} \mathbf{u} = 0$ almost everywhere, by applying Egorov's theorem, for details see (Alt, 2006, U6.1).

(e) Now we are able to consider conservation of volume equation (3.2.10) for individual submerged bodies: Setting now

$$(3.5.20) \quad \Omega_{\text{solid}}^i(t) := \text{supp } \chi^i(\cdot, t) = (\text{supp } \zeta^i(\cdot, t))^{\text{ext}} = \Xi^i(\Omega_{\text{solid}}^i(0)^{\text{int}}, 0; t)^{\text{ext}}$$

we see, using (d), that

$$D_{\text{sym}} \mathbf{u}(x, t) = 0 \text{ for almost all } \Omega_{\text{solid}}^i(t).$$

By Lemma 3.5.6 $\Xi \in C^\alpha([0, T] \times [0, T]; C^1(\Omega^{\text{ext}})^3)$, yields that $\Omega_{\text{solid}}^i(0)$ stays connected. Hence, there exist for each i and t rigid velocity fields $\mathbf{u}_{\text{solid}}^i(x, t) = R^i(t)x + \mathbf{U}^i(t)$, and

$$(3.5.21) \quad \mathbf{u}(x, t) = \mathbf{u}_{\text{solid}}^i(x, t) \text{ for } x \in \Omega_{\text{solid}}^i(t).$$

Let the transport within this global rigid velocity field be denoted by Ξ_{solid}^i , this is

$$(3.5.22) \quad \begin{aligned} \partial_t \Xi_{\text{solid}}^i(x, s; t) &= \mathbf{u}_{\text{solid}}(\Xi_{\text{solid}}^i(x, s; t), t) \text{ for } s, t \in [0, T] \\ \Xi_{\text{solid}}^i(x, s; s) &= \mathbf{u}_{\text{solid}}(\Xi_{\text{solid}}^i(x, s; t), t) \text{ for } s \in [0, T] \text{ and } x \in \mathbb{R}^3. \end{aligned}$$

For points within the solids the two velocity fields coincide by definition. Therefore, the trajectories coincide as well:

$$\Xi_{\text{solid}}^i(\xi, 0; t) = \Xi(\xi, 0; t) \text{ for } \xi \in \Omega_{\text{solid}}^i(t), t \in [0, T].$$

Hence, the characteristic functions of the δ -kernels satisfy

$$(3.5.23) \quad \zeta^i(x, t) = \zeta^i(\Xi_{\text{solid}}^i(x, t; 0), 0) \text{ for } x \in \Omega_{\text{solid}}^i(t), t \in [0, T].$$

Contrary to the flow \mathbf{u} the flows $\mathbf{u}_{\text{solid}}^i$ preserve globally rigid bodies by construction and especially the preserve the δ -neighborhoods. Therefore equation (3.5.23) yields the equivalent for the characteristic functions χ^i :

$$\chi^i(x, t) = \chi^i(\Xi_{\text{solid}}^i(x, t; 0), 0) \text{ for } x \in \Omega_{\text{solid}}^i(t), t \in [0, T],$$

or in weak differential formulation

$$\int_{Q_T^{\text{ext}}} \chi^i (\partial_t \eta + \text{div}(\mathbf{u}_{\text{solid}}^i \chi^i)) = - \int_{\Omega} \chi^i(\cdot, 0) \eta(\cdot, 0) \text{ for all } \eta \in \mathcal{D}([0, T] \times \Omega^{\text{ext}})^*.$$

By equation (3.5.21) the functions $\chi^i \mathbf{u}_{\text{solid}}^i$ and $\chi^i \mathbf{u}$ coincide within Ω^{ext} , and outside Ω the limit velocity $\mathbf{u} \in L^2(0, T; H_0^1(\Omega)^3)$ can be assumed to vanish. Hence, the restricted version of the last equation is valid:

$$\int_{Q_T} \chi^i (\partial_t \eta + \operatorname{div}(\mathbf{u} \chi^i)) = - \int_{\Omega} \chi^i(\cdot, 0) \eta(\cdot, 0) \quad \text{for all } \eta \in \mathcal{D}([0, T] \times \Omega)^*.$$

But this is exactly equation (3.2.10).

(f) The conservation of mass equation (3.2.9) now follows from the following considerations: Adding up the functions χ^i yields that $\chi = \sum_i \chi^i$ satisfies

$$\int_{Q_T} \chi (\partial_t \eta + \operatorname{div}(\mathbf{u} \chi)) = - \int_{\Omega} \chi(\cdot, 0) \eta(\cdot, 0) \quad \text{for all } \eta \in \mathcal{D}([0, T] \times \Omega)^*$$

as well. But $\chi(\cdot, 0)$ is a characteristic function. Hence, by properties of solutions of transport equations χ is a characteristic function as well. Therefore, for $i \neq j$ the characteristic functions satisfy $\chi^i + \chi^j = 0$ almost everywhere or equivalently the open sets $\Omega_{\text{solid}}^i(t)$ and $\Omega_{\text{solid}}^j(t)$ are disjoint. That ϱ satisfies (3.2.9) is a result of Lion's theorem 3.5.2, which finishes the proof of Proposition 3.3.4. *q.e.d.*

3.6 Estimates and convergence properties of approximative solutions

At first we want to show that the approximative domains occupied by the solid bodies, i.e.

$$\Omega_{\text{solid}}^{\text{in}}(t) := \operatorname{supp} \chi^{\text{in}}(t)^\circ = (\operatorname{supp} \zeta^{\text{in}}(t)^\tau)^\circ$$

and the domains defined by the limit of the characteristic functions

$$\Omega_{\text{solid}}^i(t) := \operatorname{supp} \chi^i(t)^\circ = (\operatorname{supp} \zeta^i(t)^\tau)^\circ$$

are arbitrary close to each other uniformly in t and i for big enough n . We will deduce this from the uniform convergence of the approximative trajectories Ξ^n to Ξ . The main difficulty lies therefore in the definition of $\Omega_{\text{solid}}^{\text{in}}$ and Ω_{solid}^i as δ -neighborhoods of the transported version of the δ -kernel of the starting position $\Omega_{\text{solid}}^i(0)$. We introduced in notation 3.2.3 on page 32 the τ -neighborhood G^τ of a set G . For the special choice $\tau = \delta$ we introduced the external set of G as G^{ext} and its $\tau = \delta$ -kernel G^{int} . The value δ is related to the maximal curvature of the boundaries of the submerged solids and the boundary of the containing set.

Lemma 3.6.1 (Convergence of approximate solid domains). *For any $\tau > 0$ there exists a n_0 such that for all $n \geq n_0$ the supports of all characteristic functions χ^{in} of the approximative rigid domains and the support of χ^i are within a τ -neighborhood of each other, as introduced in equation (3.2.2), this is*

$$(3.6.1) \quad \Omega_{\text{solid}}^{\text{in}}(t) \subseteq \Omega_{\text{solid}}^i(t)^\tau \quad \text{and} \quad \Omega_{\text{solid}}^i(t) \subseteq \Omega_{\text{solid}}^{\text{in}}(t)^\tau$$

for all $t \in [0, T]$ and all $i = 1, \dots, N$. Hence, the same inclusion holds true for the unions of these sets, this is

$$(3.6.2) \quad \Omega_{\text{solid}}^n(t) \subseteq \Omega_{\text{solid}}(t)^\tau \quad \text{and} \quad \Omega_{\text{solid}}(t) \subseteq \Omega_{\text{solid}}^n(t)^\tau$$

for all $t \in [0, T]$.

Proof. We found in Lemma 3.5.6.(d) that $(\Xi^n)_n$ converges strongly to Ξ in $C^\alpha([0, T] \times [0, T]; C^1(\Omega^{\text{ext}})^3)$. This yields especially uniform convergence in $C([0, T] \times [0, T]; C(\Omega^{\text{ext}})^3)$. Hence, the sets

$$\begin{aligned}\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}} &= \Xi^n(\Omega_{\text{solid}}^i(0)^{\text{int}}, 0; t) \\ \Omega_{\text{solid}}^i(t)^{\text{int}} &= \Xi(\Omega_{\text{solid}}^i(0)^{\text{int}}, 0; t)\end{aligned}$$

are close to each other for all $t \in [0, T]$ in the sense that for all $\tau > 0$ there exists a $n_0 > 0$ such that

$$\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}} \subseteq \left(\Omega_{\text{solid}}^i(t)^{\text{int}}\right)^\tau \quad \text{and} \quad \Omega_{\text{solid}}^i(t)^{\text{int}} \subseteq \left(\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}}\right)^\tau$$

uniformly for all $t \in [0, T]$ and $n > n_0$ and $i = 1, \dots, N$. For the particular choice $\tau = \delta$ we the τ -neighborhood of the δ -kernel is the definition of our sets. Hence, the statement follows from the above together with the following considerations on internal sets, external sets, and neighborhoods:

$$\begin{aligned}\Omega_{\text{solid}}^{\text{in}}(t) &= \left(\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}}\right)^{\text{ext}} \subseteq \left(\left(\Omega_{\text{solid}}^i(t)^{\text{int}}\right)^\tau\right)^{\text{ext}} = \left(\Omega_{\text{solid}}^i(t)^{\text{int}}\right)^{\tau+\delta} \\ &= \left(\Omega_{\text{solid}}^i(t)^{\text{int}}\right)^{\tau+\delta} = \Omega_{\text{solid}}^i(t)^\tau\end{aligned}$$

and

$$\begin{aligned}\Omega_{\text{solid}}^i(t) &= \left(\Omega_{\text{solid}}^i(t)^{\text{int}}\right)^{\text{ext}} \subseteq \left(\left(\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}}\right)^\tau\right)^{\text{ext}} = \left(\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}}\right)^{\tau+\delta} \\ &= \left(\Omega_{\text{solid}}^{\text{in}}(t)^{\text{int}}\right)^{\tau+\delta} = \Omega_{\text{solid}}^{\text{in}}(t)^\tau,\end{aligned}$$

which finishes the proof of equation (3.6.1).

q.e.d.

We now present and prove three propositions which are needed to show that the approximative integrated energies converge, this is we show in Proposition 3.7.2 that $\int_{Q_T} \varrho^n |\mathbf{u}^n|^2$ converges to $\int_{Q_T} \varrho |\mathbf{u}|^2$, which will yield that $\int_{Q_T} |\mathbf{u}^n|^2$ converges to $\int_{Q_T} |\mathbf{u}|^2$ for the proof of Proposition 3.3.5 on page 72. The missing objective, all these results lead to, is still to show that ϱ and \mathbf{u} satisfy the conservation of momentum equation in weak form.

In equation (3.2.1) we introduced the closed subspace

$$K(\chi) = \left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, \chi D_{\text{sym}} \mathbf{u} = 0 \right\},$$

of velocity fields that preserve rigid bodies at $\operatorname{supp} \chi$. For \mathbf{u} to be a solution, we would like to have that \mathbf{u} is compatible with χ in the sense $\chi D_{\text{sym}} \mathbf{u} = 0$ or in the weak sense that restricts the test functions hereby.

Let $P(\chi(t))$ denote the projection in $H_0^1(\Omega)^3$ to the closed subspace $K(\chi)$ then the error can be measured as $P(\chi(\cdot)) \mathbf{u} - \mathbf{u} \in L^2(0, T; H_0^1(\Omega)^3)$ and being equal to zero within this space means that \mathbf{u} preserve an rigid body at χ . In a bigger space $L^2(0, T; H_0^s(\Omega)^3)$, for $0 \leq s < 1$ in a approximate sense even in a neighborhood close to $\operatorname{supp} \chi$. To make this statement precise we introduce some notation.

Notation 3.6.2. For a characteristic function denote the closure of $K(\chi)$ in $H^s(\Omega)^3$ by

$$(3.6.3) \quad K^s(\chi) := \overline{\left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, \chi D_{\text{sym}} \mathbf{u} = 0 \right\}}^{H^s},$$

which is a closed subspace of

$$(3.6.4) \quad H_{0,\sigma}^s(\Omega) = \overline{\left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0 \right\}}^{H^s}$$

for $0 \leq s \leq 1$. For a given characteristic function we can also consider the τ -neighborhood of the support of χ and define χ^τ to be the characteristic function thereof. Then we set

$$(3.6.5) \quad K_\tau^s(\chi) := \overline{\left\{ \mathbf{u} \in H_0^1(\Omega)^3 \mid \operatorname{div} \boldsymbol{\phi} = 0, \chi^\tau \operatorname{D}_{\text{sym}} \mathbf{u} = 0 \right\}}^{H^s}.$$

Again, hereby closed subspaces of $H_0^s(\Omega)^3$ are defined. They satisfy the following embedding chain

$$(3.6.6) \quad K_\tau^1(\chi) \subseteq K_\tau^s(\chi) \subseteq K_{\tau_0}^s(\chi) \subseteq K_{\tau_0}^0(\chi)$$

for all $0 \leq s \leq 1$ and all $0 < \tau_0 \leq \tau$. This chain is inherited from the embedding properties of the sobolev spaces H^s and their definition (3.6.3). We denote the projection onto these subspaces of H^s by $P_\tau^s(\chi)$ as well as their embedding into a bigger space according to the chain in (3.6.6). For time dependent characteristic functions we use $P_\tau^s(\chi(t))$ for the projection on $K_\tau^s(\chi(t))$. The important observation is that the test functions are from a space that which is possibly not dense in the space the sought velocity fields is from. We will see that in case of no collisions the test functions determine are dense. In the case of collisions the relative rotations of the two colliding bodies will not necessarily be restricted. Our weak formulation only considers the projected part of the velocity field.

The next steps consider approximation of \mathbf{u} by $P^s(\chi(\cdot)) \mathbf{u}$ in $L^2(0, T; L^2(\Omega)^3)$ or \mathbf{u}^n by $P^s(\chi(\cdot)) \mathbf{u}^n$ in $L^2(0, T; L^2(\Omega)^3)$. We need these properties later to prove convergence of the energies in Proposition 3.7.2. For these, we need some auxiliary observations. Furthermore, we prove a characterization of the velocity field as almost balls-on-sticks like in passing.

We are interested in the size of the set of moments in time, where $\Omega_{\text{solid}}^i(t)$ and $\Omega_{\text{solid}}^j(t)$ get close to each other without a common point. For this we define for every distance $0 < \tau < 1$

$$(3.6.7) \quad E_\tau^{ij} := \left\{ t \in [0, T] \mid 0 < \operatorname{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) < \tau \right\}$$

for all $i, j = 0, \dots, N$. The set of all those moments is denoted by

$$(3.6.8) \quad E_\tau := \bigcup_{i,j=0}^N E_\tau^{ij}.$$

Now the following observation states that there are almost now points in time, where the solid bodies come close to each other, since the Lebesgue measure of the set E_τ goes to zero for small τ :

Proposition 3.6.3. *The above defined sequence $(E_\tau)_{\tau>0}$ satisfies $\lim_{\tau \rightarrow 0} |E_\tau| = 0$.*

Proof. The motion of the rigid bodies is continuous, since each rigid body motion Ξ_{solid}^i is continuous and even $\Xi \in C^\alpha([0, T]^2; C^1(\Omega^{\text{ext}})^3)$. Hence, the distances

$$d^{ij}(t) := \text{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) = \text{dist}(\Xi_{\text{solid}}^i(\Omega_{\text{solid}}^i(0), 0; t), \Xi_{\text{solid}}^j(\Omega_{\text{solid}}^j(0), 0; t))$$

define continuous functions d^{ij} . The sets $E_\tau^{ij} = \{t \in [0, T] \mid d^{ij}(t) \in (0, \tau)\}$ and E_τ are therefore measurable. Furthermore they satisfy $E_\tau \subseteq E_{\tau_0}$ for $\tau < \tau_0$ and $\cap E_\tau = \emptyset$. By monotony properties of the Lebesgue measure we obtain

$$\lim_{\tau \rightarrow 0} |E_\tau| = \left| \bigcap_{0 < \tau < 1} E_\tau \right| = |\emptyset| = 0$$

as claimed. *q.e.d.*

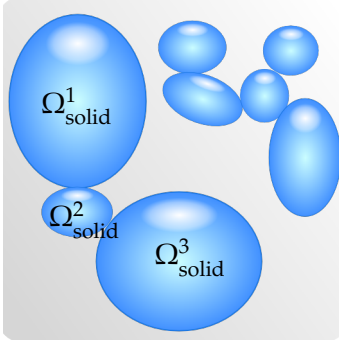


Figure 3.6.1: Separation of the moving bodies Ω_{solid}^i into interaction groups. Here two groups can be seen: One group consists of Ω_{solid}^1 , Ω_{solid}^2 and Ω_{solid}^3 ; the other group of the remaining five bodies. These groups have the property that for all $\tau > 0$ the τ -neighborhood of their union is connected and every of its points is center to an open ball of radius τ which is completely contained in this τ -neighborhood. Here we have two groups, this is $M = 2$. The first group J_1 consists of the $N_1 = 3$ elements $\{1, 2, 3\}$, the second group J_2 consists of $N_2 = 5$ elements.

Now we will consider the possible collisions that might still happen. We will see that there structure is not arbitrary. For this we need to separate the bodies for each time $t \in [0, T]$ into groups.

Notation 3.6.4. For all $t \in [0, T]$ we separate the indices $0, 1, \dots, N$ of the solid domains into $M(t) \in \{1, \dots, N+1\}$ groups $J_k = \{i_1, \dots, i_{N_k(t)}\}$ of $N_k(t)$ elements. A groups consists of the maximal number of those indices for that every τ -neighborhood of the union of the sets $\cup_{i \in J_k} \Omega_{\text{solid}}^i(t)$ for every $\tau > 0$ is connected; see Figure 3.6.1. If being in contact with is considered as a transitive property then a group consists of all those bodies which are in contact with each other. For such a group we denote the common characteristic function as

$$\chi^{J_k} = \sum_{i \in J_k} \chi^i,$$

and

$$\Omega_{\text{solid}}^{J_k}(t) = \bigcup_{i \in J_k} \Omega_{\text{solid}}^i(t).$$

Furthermore we use for the whole union of all limit sets

$$\Omega_{\text{solid}}(t) := \bigcup_{i=0}^N \Omega_{\text{solid}}^i(t)$$

and by abuse of notation for the union of all approximative sets at step n

$$\Omega_{\text{solid}}^n(t) := \bigcup_{i=0}^N \Omega_{\text{solid}}^{in}(t).$$

That this set $\Omega_{\text{solid}}^n(t)$ of the union of approximative sets and the limit i -submerged body $\Omega_{\text{solid}}^i(t)$ could collide for the n th body or i th should cause no confusion, since we will never use a notation such as $\Omega_{\text{solid}}^3 0(t)$, which is not uniquely defined.

Now we consider the momentary situation of the approximative velocities and observe, that for almost all moments in time a rigid velocity field is arbitrarily close.

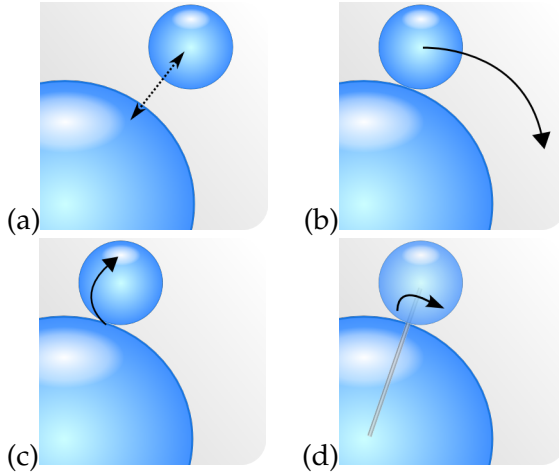


Figure 3.6.2: Rolling on top of the common tangential space could be observable only for plain weak solutions whereas the motion (a) is excluded already. If our solution u can be shown by different means to be from $H^{s,p}$ in space with $s \geq \frac{2}{p}$ then the motion (b) is excluded; if $s \geq 1 + \frac{4}{p}$ then even (c) is excluded. A balls on a stick motion is still possible.

Lemma 3.6.5 (Group velocities). *For almost every point in time $t \in [0, T]$ there exists a subsequence of the sequence $(u^n(t))_n$ such that for almost all $t \in [0, T]$ a corresponding sequence of rigid velocity fields $(u_{\text{solid}}^{in}(t))_n$, $i = 0, \dots, N$ satisfies*

$$\lim_{n \rightarrow \infty} \|u^n(t) - u_{\text{solid}}^{in}(t)\|_{W^{1,p}(\Omega_{\text{solid}}^i(t))} = 0$$

for all $1 \leq p < 2$ and $i = 1, \dots, N$.

Let $u(t) \in H^{s,p}(\Omega)^3$ for all t close to a collision point in time t_0 for some $s > 1$ and $p \geq 1$ then we can characterize the velocity fields within a group even further:

If p can be chosen such that $s \geq \frac{2}{p}$ for all t close to t_0 then these velocity fields $u_{\text{solid}}^{in}(t)$ of a group can be chosen to have the same constant part $U^i(t)$ and differ at most in a velocity field of a rotation of the form $Rx = \omega \times x$ for some skew symmetric R or $\omega \in \mathbb{R}^3$.

If p can be chosen such the velocity field satisfies $s \geq 1 + \frac{2}{p}$ for all t close to t_0 then all velocity fields $u_{\text{solid}}^{in}(t)$ of a group can be chosen to have the same constant part $U^i(t)$ and differ at most in a velocity field of a rotation of the form $\omega_3 e_3 \times x$ for some $\omega_3 \in \mathbb{R}$ this is it is a velocity that preserves the normal to the common tangential planes. This motion describes the motion of balls on sticks as experienced in the ball-stick-models of chemistry.

Proof. For arbitrary $\varepsilon > 0$ we want to show that there exists an $n_0 = n_0(\varepsilon)$ such that for a subsequence

$$\|u^n(t) - u_{\text{solid}}^{in}(t)\|_{W^{1,p}(\Omega_{\text{solid}}^i(t))} \leq \varepsilon$$

for all $1 \leq p < 2$ and $i = 1, \dots, N$ and all $n \geq n_0$, all $t \in M_\varepsilon$ where $|[0, T] \setminus M_\varepsilon| < \varepsilon$, $\mathbf{u}_{\text{solid}}^{i,n}(\mathbf{x}, t) = R^{in}(t)\mathbf{x} + \mathbf{U}^{in}(t)$ and $\mathbf{U}^{in}(t)$ and $R^{in}(t)$ can be chosen identically or at least to have as much coordinates in common as possible for all members of a group $J_i(t)$.

In what follows, we will use Hölder's inequality quite often for the same constants: Let $\tilde{p} = \frac{2}{2-p}$ and $\tilde{q} = \frac{2}{p}$ then \tilde{p} and \tilde{q} satisfy $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$. All Hölder derived estimates use these values. Furthermore, c will denote a generic constant which may depend on the given initial data and length T of the time interval.

(a) The energy estimate (3.3.4) yields that

$$(3.6.9) \quad \int_0^T \int_{\Omega} (2\mu + n\chi^n(\mathbf{x}, t)) |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^2 d\mathbf{x} dt \leq C \left(\frac{1}{2} \int_{\Omega} \varrho_0 |\mathbf{u}_0|^2 d\mathbf{x} + \|\mathbf{g}\|_{L^2(Q_T)}^2 \right),$$

and hereby that

$$F^n(t) := \int_{\Omega} \chi^n(\mathbf{x}, t) |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^2 d\mathbf{x} = \|\mathbf{D}_{\text{sym}} \mathbf{u}^n(t)\|_{L^2(\Omega_{\text{solid}}^n(t))}^2$$

defines a sequence $(F^n)_n \subseteq L^1(0, T)$ such that $(nF^n)_n$ is uniformly bounded in $L^1(0, T)$ by a constant c which yields that $F^n \rightarrow 0$ strongly in $L^1(0, T)$ due to $\|F^n\|_{L^1(0, T)} \leq c \frac{1}{n}$. Furthermore, the same equation (3.6.9) yields that the terms

$$G^n(t) := \int_{\Omega} |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^2 d\mathbf{x} dt = \|\mathbf{D}_{\text{sym}} \mathbf{u}^n(t)\|_{L^2(\Omega)}^2$$

are uniformly bounded as functions $(G^n)_n \subseteq L^1(0, T)$, this is $\|G^n\|_{L^1(0, T)} \leq c$.

(b) By Lemma 3.6.1 we obtain $|\Omega_{\text{solid}}(t) \setminus \Omega_{\text{solid}}^n(t)| \leq c\tau(n) \rightarrow 0$ for $n \rightarrow \infty$ uniformly in t . Since the flows are divergence free we know that the volume does not change in time. Hence, $|\Omega_{\text{solid}}^n(t)| = |\Omega_{\text{solid}}^n(0)|$ and therefore these volumes are uniformly bounded, this is $|\Omega_{\text{solid}}^n(t)| \leq c$.

(c) Using the notations of (a), the estimates of (b), and Hölder's inequality we obtain for $1 \leq p < 2$

$$(3.6.10) \quad \begin{aligned} & \int_{\Omega_{\text{solid}}(t)} |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^p d\mathbf{x} \\ & \leq \underbrace{\int_{\Omega_{\text{solid}}^n(t)} |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^p d\mathbf{x} + \int_{\Omega_{\text{solid}}(t) \setminus \Omega_{\text{solid}}^n(t)} |\mathbf{D}_{\text{sym}} \mathbf{u}^n(\mathbf{x}, t)|^p d\mathbf{x}}_{=: A^n(t)} \\ & \leq |\Omega_{\text{solid}}^n(t)|^{\frac{2-p}{2}} \|\mathbf{D}_{\text{sym}} \mathbf{u}^n\|_{L^2(\Omega_{\text{solid}}^n(t))}^p + |\Omega_{\text{solid}}(t) \setminus \Omega_{\text{solid}}^n(t)|^{\frac{2-p}{2}} \|\mathbf{D}_{\text{sym}} \mathbf{u}^n\|_{L^2(\Omega)}^p \\ & \leq cF^n(t)^{\frac{p}{2}} + c\tau(n)^{\frac{2-p}{2}} G^n(t)^{\frac{p}{2}} \end{aligned}$$

Thus, we obtain again by Hölder's inequality and the estimates of (a)

$$\begin{aligned} \int_0^T A^n(t) dt & \leq c \int_0^T F_n(t)^{\frac{p}{2}} dt + c\tau(n)^{\frac{2-p}{2}} \int_0^T G^n(t)^{\frac{p}{2}} dt \\ & \leq cT^{\frac{2-p}{2}} \left(\int_0^T F^n(t) dt \right)^{\frac{p}{2}} + cT^{\frac{2-p}{2}} \tau(n)^{\frac{2-p}{2}} \left(\int_0^T G^n(t) dt \right)^{\frac{p}{2}} \\ & \leq c \frac{1}{n} + c\tau(n)^{\frac{2-p}{2}} \rightarrow 0 \end{aligned}$$

for $n \rightarrow \infty$. Hence, $(A^n)_n \subseteq L^1(0, T)$ converges strongly to zero. Thus, a subsequence, again denoted as $(A^n)_n$, converges almost everywhere to zero as well. By Egorov's Theorem there exists a set $M_\varepsilon \subseteq [0, T]$ such that

$$(3.6.11) \quad A^n \longrightarrow 0 \text{ uniformly for } t \in M_\varepsilon$$

$$(3.6.12) \quad |[0, T] \setminus M_\varepsilon| < \varepsilon.$$

Applying again the energy estimate and the boundedness of ϱ^n above, we can assume that $\|\mathbf{u}^n(t)\|_{L^2(\Omega)}$ is uniformly bounded. Furthermore, we just obtained in passing a third estimate:

$$\begin{aligned} \|\mathbf{u}^n(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} &\leq c \left(\|\mathbf{u}^n(t)\|_{L^p(\Omega_{\text{solid}}^i(t))} + \|\mathbf{D}_{\text{sym}} \mathbf{u}^n(t)\|_{L^p(\Omega_{\text{solid}}^i(t))} \right) \\ &\leq c \left(\|\mathbf{u}^n(t)\|_{L^2(\Omega)} + \|\mathbf{D}_{\text{sym}} \mathbf{u}^n(t)\|_{L^p(\Omega_{\text{solid}}^i(t))} \right) \\ &\leq c + cA^n(t)^{\frac{1}{p}} \end{aligned}$$

Hence, for all $1 \leq p \leq 2$ all $t \in M_\varepsilon$, all $0 < \varepsilon < \varepsilon_0$ and all $n \geq n(\varepsilon_0)$ we can assume

$$(3.6.13) \quad \|\mathbf{u}^n(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i)} \leq \frac{1}{\varepsilon}$$

and consider only such ε and n in the remaining part of the proof.

(d) It would be nice to have $\zeta^{in}(t) \mathbf{D}_{\text{sym}} \mathbf{u}^n(t) = 0$, this is that $\mathbf{u}^n(t)$ is within each $\text{supp } \zeta^{in}(t)$ a rigid body motion. But from our construction we can only observe that $\chi^{in}(t) \mathbf{D}_{\text{sym}} \mathbf{u}^n(t)$ is bounded uniformly in $L^p(\Omega)$ for all $1 \leq p \leq 2$ and t : Since $\Omega_{\text{solid}}^i(t)$ is connected, there exists by (3.4.9) for every moment $t \in M_\varepsilon$ a rigid body velocity field $\mathbf{u}_{\text{solid}}^{in}(t) \in \text{LD}_0$ such that

$$(3.6.14) \quad \|\mathbf{u}^n(t) - \mathbf{u}_{\text{solid}}^{in}(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} \leq c \|\mathbf{D}_{\text{sym}} \mathbf{u}^n\|_{L^p(\Omega_{\text{solid}}^i(t))} \leq cA^n(t)^{\frac{1}{p}},$$

where the last inequality is due to (3.6.10). By equation (3.6.11) we just found that we can choose $\mathbf{u}_{\text{solid}}^{in}(t)$ arbitrarily close to $\mathbf{u}^n(t)$ in $H^{1,p}(\Omega_{\text{solid}}^i(t))$ and this uniformly in t . We can assume that $\mathbf{u}_{\text{solid}}^{in}(\mathbf{x}, t) = R^{in}(t)\mathbf{x} + \mathbf{U}^{in}(t)$ is defined globally for every $\mathbf{x} \in \mathbb{R}^3$. Therefore, we obtain from equation (3.6.14)

$$\begin{aligned} \|\mathbf{u}_{\text{solid}}^{in}(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} &\leq \|\mathbf{u}^n(t) - \mathbf{u}_{\text{solid}}^{in}(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} + \|\mathbf{u}^n(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} \\ &\leq cA^n(t) + \frac{1}{\varepsilon}. \end{aligned}$$

Thus, for large enough n

$$(3.6.15) \quad \|\mathbf{u}_{\text{solid}}^{in}(t)\|_{H^{1,p}(\Omega_{\text{solid}}^i(t))} \leq \frac{2}{\varepsilon} \quad \text{for all } t \in M_\varepsilon \text{ and all } n \geq n_0$$

is satisfied.

(e) For a given $i \in \{0, \dots, N\}$ two cases are possible: Either $\Omega_{\text{solid}}^i(t)$ is the only member of its group, or there exist other members. Now equation (3.6.14) proves that

$$\lim_{n \rightarrow \infty} \|\mathbf{u}^n(t) - \mathbf{u}_{\text{solid}}^{in}(t)\|_{W^{1,p}(\Omega_{\text{solid}}^i(t))} = 0.$$

If $\Omega_{\text{solid}}^i(t)$ is the only member of its group, we proved all claims of the lemma. Therefore, we only have to consider the case of more than one member in a group in the rest of this proof. For all numbers of members, we proved the first claim of the lemma so far.

(f) One of the conceptional difficulties is, that it is possible that two members $\Omega_{\text{solid}}^i(t)$ and $\Omega_{\text{solid}}^j(t)$ have no common boundary point, see for example the case of Ω_{solid}^1 and Ω_{solid}^3 in Figure 3.6.1, on page 61. Since only a finite number of submerged bodies is considered, we can iterate in finite steps through all pairs connecting two bodies of a group. Hence, we can assume $\text{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) = 0$ for two indices $i \neq j$ from zero to N , this is we include the outer set $\Omega^0 := \mathbb{R}^3 \setminus \overline{\Omega}$ into our considerations.

We denote the relative velocity fields of $\Omega_{\text{solid}}^i(t)$ and $\Omega_{\text{solid}}^j(t)$ to their rigid body velocity fields by

$$(3.6.16) \quad \mathbf{v}^{l,n}(\mathbf{x}, t) := \mathbf{u}^n(\mathbf{x}, t) - \mathbf{u}_{\text{solid}}^{l,n}(\mathbf{x}, t) \quad \text{for all } \mathbf{x} \in \Omega,$$

for $l = i, j$, and obtain as new formulation of equation (3.6.14)

$$(3.6.17) \quad \|\mathbf{v}^{l,n}(t)\|_{H^{1,p}(\Omega_{\text{solid}}^l(t))} \leq c \|\mathbf{D}_{\text{sym}} \mathbf{u}^n\|_{L^p(\Omega_{\text{solid}}^l(t))} \leq c A^n(t)^{\frac{1}{p}}.$$

By construction, we can consider $\mathbf{v}^{l,n}(t)$ as function in $H_{\sigma}^{s,p}(\Omega)$, this is we do not claim zero boundary values. Furthermore, we introduce $\mathbf{v}^n \in H_{\sigma}^{s,p}(\Omega)$, $R^n(t) \in \mathcal{L}(\mathbb{R}^3)$, and $\mathbf{U}^n(t) \in \mathbb{R}^3$ by

$$(3.6.18) \quad \mathbf{v}^n(\mathbf{x}, t) := \mathbf{v}^{j,n}(\mathbf{x}, t) - \mathbf{v}^{i,n}(\mathbf{x}, t)$$

and

$$(3.6.19) \quad R^n(t)\mathbf{x} + \mathbf{U}^n(t) := (R^{in}(t) - R^{jn}(t))\mathbf{x} + (\mathbf{U}^{in}(t) - \mathbf{U}^{jn}(t))$$

$$(3.6.20) \quad = \mathbf{u}_{\text{solid}}^{i,n}(\mathbf{x}, t) - \mathbf{u}_{\text{solid}}^{j,n}(\mathbf{x}, t)$$

and observe that by equation (3.6.16)

$$(3.6.21) \quad \mathbf{v}^n(\mathbf{x}, t) = R^n(t)\mathbf{x} + \mathbf{U}^n(t).$$

Hence, to prove that the distance between the bodies does not change, we need to show that the relative velocity $\mathbf{U}^n(t)$ vanishes; to prove that the bodies actually are fixed at the common point, we need to show that all components of the skew symmetric matrix $R^n(t)$ that describe rotations with respect to tangential vectors vanish. We will prove this by reducing it to the same estimates we used in the proof of Lemma 2.1.8 on page 19.

(g) By the bound (3.6.15) we have

$$|R^{ln}(t)| |\Omega_{\text{solid}}^l(t)|^{\frac{1}{p}} = \|\text{grad } \mathbf{u}_{\text{solid}}^{l,n}(t)\|_{L^p(\Omega_{\text{solid}}^l(t))} \leq \frac{c}{\varepsilon}$$

and hereby

$$|\mathbf{U}^{ln}(t)| |\Omega_{\text{solid}}^l(t)|^{\frac{1}{p}} = \|\mathbf{u}_{\text{solid}}^{l,n}(t) - R^{ln}(t) \cdot\|_{L^p(\Omega_{\text{solid}}^l(t))} |\Omega_{\text{solid}}^l(t)|^{\frac{1}{p}} \leq \frac{c}{\varepsilon}$$

for all $t \in M_\varepsilon$ and all $n \geq n_0$ and $l = i, j$. Thus, we obtain $|R^n(t)| \leq \frac{\varepsilon}{\varepsilon}$ and $|\mathbf{U}^n(t)| \leq \frac{\varepsilon}{\varepsilon}$ for all $t \in M_\varepsilon$ and all $n \geq n_0$ which yields

$$\|\mathbf{v}^n(t)\|_{H^{s,p}(\Omega)} \leq \frac{c(s,p)}{\varepsilon},$$

where the constant $c(s,p)$ depends on the particular choice of s and p , but can be chosen independently of n and ε . By equation (3.6.21) the function $\mathbf{v}^n(t)$ is affine and by assumption the set Ω is bounded. We can choose arbitrary s and p .

(h) The steps which led in the proof of Lemma 3.4.2 from equation (3.4.13) on page 43 to equation (3.4.20) shall now be applied to

$$(3.6.22) \quad \mathbf{U}^n(t) = \mathbf{v}^n(x, t) - R^n(t)x \quad \text{for all } x \in \Omega$$

with the appropriate notational changes; hereby we would obtain for all $t \in M_\varepsilon$ and all $n \geq n_0$

$$(3.6.23) \quad |\mathbf{U}^n(t)| \leq \frac{c(s,p)}{\varepsilon} o\left(r^{2s-\frac{4}{p}}\right) + \frac{c}{\varepsilon} O(r)$$

for all $r < \delta$ small enough. The problem is that the key ingredient of the above referred to proof was a Poincaré inequality which needs $\mathbf{v}^n(x, t)$ to vanish on at least on one of the sets $\Omega_{\text{solid}}^l(t)$. Hence, we consider a perturbed version that does not change the values on the set $\Omega_{\text{solid}}^j(t)$ to keep the information on $\mathbf{U}^n(t)$ and $R^n(t)$ and is bounded outside of $\Omega_{\text{solid}}^j(t)$: Let $\widetilde{\mathbf{v}}^{jn}$ be an extension of $\mathbf{u}^n(x, t) - \mathbf{u}_{\text{solid}}^{ln}(x, t)$ from $\Omega_{\text{solid}}^j(t)$ to Ω such that it differs from \mathbf{v}^{jn} at least somewhere outside of $\Omega_{\text{solid}}^j(t)$. Furthermore, such an extension can be chosen for fixed s and p such that

$$(3.6.24) \quad \|\widetilde{\mathbf{v}}^{jn}\|_{H^{s,p}(\Omega)} \leq c \|\mathbf{v}^{jn}\|_{H^{s,p}(\Omega_{\text{solid}}^j(t))}.$$

Now we introduce the perturbed version of \mathbf{v}^n by

$$(3.6.25) \quad \widetilde{\mathbf{v}}^n(x, t) := \widetilde{\mathbf{v}}^{jn}(x, t) - \mathbf{v}^{in}(x, t),$$

which is used instead of (3.6.18), and observe that on the set $\Omega_{\text{solid}}^j(t)$ the functions $\widetilde{\mathbf{v}}^n$ and \mathbf{v}^n coincide and the equation corresponding to (3.4.13) is

$$(3.6.26) \quad (\widetilde{\mathbf{v}}^n(x, t) - \mathbf{v}^n(x, t)) = -\widetilde{\mathbf{v}}^n(x, t) + \mathbf{U}^n(t) + R^n(t)x,$$

where the left hand side vanishes on $\Omega_{\text{solid}}^j(t)$. Now the above steps and the boundedness of $\widetilde{\mathbf{v}}^{jn}$ by \mathbf{v}^{jn} by equation (3.6.24) yields for an again different generic constant

$$(3.6.27) \quad |\mathbf{U}^n(t)| \leq \frac{c(s,p)}{\varepsilon} o\left(r^{2s-\frac{4}{p}}\right) + \frac{c}{\varepsilon} O(r),$$

where we only needed the boundedness of $\Omega_{\text{solid}}^j(t)$. Since we did not use boundedness of $\Omega_{\text{solid}}^i(t)$ these considerations include the case of the external set $\Omega_{\text{solid}}^i(t) = \Omega^0$. Especially we obtain that the limes superior satisfies due to the estimate

$$\overline{\lim}_{n \rightarrow \infty} |\mathbf{U}^n(t)| \leq \frac{c(s,p)}{\varepsilon} o\left(r^{2s-\frac{4}{p}}\right) + \frac{c}{\varepsilon} O(r)$$

the same bounds. Since r is arbitrary we obtain for all $p \geq \frac{2}{s}$

$$\lim_{n \rightarrow \infty} |\mathbf{U}^n(t)| = 0.$$

Thus, due to the limits in (c), we obtain that for all $p \in [\frac{2}{s}, 2)$ that we can chose all $\mathbf{u}_{\text{solid}}^{i,n}(t)$ of the same group to have a common $\mathbf{U}^n(t)$ that satisfies

$$\lim_{n \rightarrow \infty} \|\mathbf{u}^n(t) - \mathbf{u}_{\text{solid}}^{i,n}(t)\|_{W^{1,p}(\Omega_{\text{solid}}^i(t))} = 0,$$

which proves the second claim.

(i) Assume now that $s \geq \frac{2}{p} + 1$. We apply again a convergence estimate formulated in Remark 3.4.3. By equation (3.4.11)

$$\sqrt{\omega_1^n + \omega_2^n} \leq c r^{2(s-1) - \frac{4}{p}}$$

for a representation of R^n as $R^n(t)h = \omega^n \times h$. Hence, we obtain that the relative rotations that do not preserve the normal to the common Which finishes the proof. *q.e.d.*

Proposition 3.6.6. *The functions \mathbf{u} and χ , defined above, satisfy for $0 < s < 1$*

$$(3.6.28) \quad \lim_{\tau \rightarrow 0} \|P_\tau^s(\chi(\cdot))\mathbf{u} - \mathbf{u}\|_{L^1(0,T;L^2(\Omega))} = 0.$$

Proof. The idea of the proof is to apply Lebesgue's dominated convergence theorem to the sequence $(f_\tau)_{\tau>0}$ defined by

$$f_\tau : t \mapsto \|P_\tau^s(\chi(t))\mathbf{u}(t) - \mathbf{u}(t)\|_{L^2(\Omega)}.$$

and to show even strong convergence in $L^2(0,T;\mathbb{R})$. This yields convergence in $L^1(0,T;\mathbb{R})$. Since P_τ^s is a projection, we have $0 \leq f_\tau(t) \leq c \|\mathbf{u}(t)\|_{L^2(\Omega)}$. Hence, the sequence $(f_\tau)_{\tau>0}$ is bounded in $L^2(0,T;\mathbb{R})$. Therefore, we only need to show that $\lim_{\tau \rightarrow 0} f_\tau(t) = 0$ for almost all points in time t .

Let \mathbf{u} be a particular realization of its equivalence class in $L^2(0,T;H_0^1(\Omega)^3)$ and $s \in (0,1)$. For almost all $t \in [0,T]$ we found $\chi(t) D_{\text{sym}} \mathbf{u}(t) = 0$, see step (d) of the proof of Corollary 3.5.7. Assume that $\chi(t) D_{\text{sym}} \mathbf{u}(t) = 0$ for this particular t . By Proposition 3.4.6 there exists a sequence (\mathbf{u}_τ) such that $\mathbf{u}_\tau \in K_\tau^0(\chi(t))$ $\|\mathbf{u}_\tau - \mathbf{u}(t)\|_{L^2(\Omega)} \rightarrow 0$. Hence,

$$\|P_\tau^s(\chi(t))\mathbf{u}(t) - \mathbf{u}(t)\|_{L^2(\Omega)} \leq \|\mathbf{u}_\tau - \mathbf{u}(t)\|_{L^2(\Omega)} \longrightarrow 0$$

for τ to zero. This finishes the proof of (3.6.28) by an application of the dominated convergence theorem. *q.e.d.*

Remark 3.6.7. In two dimensions one can show that the sequence $(E_\tau)_{\tau>0}$ and the so far chosen subsequences satisfy for for all $0 \leq s < 1$

$$\lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^2([0,T];H_{0,\sigma}^s(\Omega))} = 0$$

In three dimensions this seems to be impossible. But the later presented proof of the convergence of the energies, Proposition 3.7.2, only relies on the convergence in $L^2(0,T;L^2(\Omega))$, which is considered now, but needs some prior considerations.

Proposition 3.6.8. *Let $(E_\tau)_{\tau>0}$ be defined by (3.6.8). Then for all $0 < s < 1$ we have*

$$\lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^1([0,T];L^2(\Omega))} = 0$$

Proof. (a) Suppose we find that for arbitrary $\varepsilon > 0$ there exists a $\tau_0 > 0$ such that for all $\tau < \tau_0$

$$(3.6.29) \quad |E_\tau| \leq \varepsilon$$

$$(3.6.30) \quad \lim_{n \rightarrow \infty} \|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^2([0,T] \setminus E_\tau; L^2(\Omega))} \leq \varepsilon$$

$$(3.6.31) \quad \|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^2([0,T]; L^2(\Omega))} \leq c$$

are satisfied. Then

$$\begin{aligned} & \|\mathbf{u}^n - P_\tau^s(\chi(\cdot))\mathbf{u}^n\|_{L^1 L^2} \\ &= \left(\int_{[0,T] \setminus E_\tau} \|\mathbf{u}^n - P_\tau^s(\chi(\cdot))\mathbf{u}^n\|_{L^2(\Omega)} + \int_{E_\tau} \|\mathbf{u}^n - P_\tau^s(\chi(\cdot))\mathbf{u}^n\|_{L^2(\Omega)} \right) \\ &\leq \sqrt{T} \|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^2([0,T] \setminus E_\tau; L^2(\Omega))} + |E_\tau|^{\frac{1}{2}} \left(\int_{E_\tau} \|\mathbf{u}^n - P_\tau^s(\chi(\cdot))\mathbf{u}^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

yields an ε -estimate for the convergence and hereby convergence. Due to Proposition 3.6.3 equation (3.6.29) is satisfied for all small $\tau < \tau_0$. Equation (3.6.31) is due to boundedness of the operators P_τ^s and the uniform boundedness of the sequence $(\mathbf{u}^n)_n$. Hence, only need to prove equation (3.6.30). We show that $\|P_\tau^s(\chi(\cdot))\mathbf{u}^n(\cdot) - \mathbf{u}^n(\cdot)\|_{H_{0,\sigma}^s(\Omega)}$ converges almost everywhere in $[0, T] \setminus E_\tau$ to zero and is uniformly bounded in $L^{\frac{2}{s}}(0, T)$. Therefore, it converges in particular in $L^{\frac{2}{s}}(0, T)$ strongly to zero and therefore in $L^2(0, T)$ as well.

(b) Interpolation of $H_{0,\sigma}^s(\Omega)$ -spaces yields for parameter $0 < s < 1$ that for any $\mathbf{w} \in H_{0,\sigma}^s(\Omega)$

$$\|\mathbf{w}\|_{H_{0,\sigma}^s(\Omega)} \leq \|\mathbf{w}\|_{L^2(\Omega)}^{1-s} \|\mathbf{w}\|_{H^1(\Omega)}^s$$

and hereby

$$(3.6.32) \quad \|\mathbf{w}\|_{H_{0,\sigma}^{\frac{2}{s}}(\Omega)}^{\frac{2}{s}} \leq \|\mathbf{w}\|_{L^2(\Omega)}^{\frac{2-2s}{s}} \|\mathbf{w}\|_{H^1(\Omega)}^2$$

is satisfied. Due to the energy estimate (3.3.4) we obtain that the sequence $(\mathbf{u}^n)_n$ is uniformly bounded in $L^2(0, T; H_{0,\sigma}^1(\Omega))$ and in $L^\infty(0, T; L_v^2(\Omega))$. Thus, the sequence $(P_\tau^s(\chi(t))\mathbf{u}^n(t) - \mathbf{u}^n(t))$ satisfies

$$\begin{aligned} \int_0^T \|P_\tau^s(\chi(t))\mathbf{u}^n(t) - \mathbf{u}^n(t)\|_{H_{0,\sigma}^{\frac{2}{s}}(\Omega)}^{\frac{2}{s}} dt &\leq c \int_0^T \|\mathbf{u}^n(t)\|_{H_{0,\sigma}^{\frac{2}{s}}(\Omega)}^{\frac{2}{s}} dt \\ &\leq c \int_0^T \|\mathbf{u}\|_{L^2(\Omega)}^{\frac{2-2s}{s}} \|\mathbf{u}\|_{H^1(\Omega)}^2 dt \\ &\leq c \int_0^T \|\mathbf{u}\|_{H^1(\Omega)}^2 dt \leq c, \end{aligned}$$

where c denotes again a generic constant. This yields boundedness of

$$\left(\|P_\tau^s(\chi(\cdot))\mathbf{u}^n(\cdot) - \mathbf{u}^n(\cdot)\|_{H_{0,\sigma}^s(\Omega)} \right) \subseteq L^{\frac{2}{s}}(0, T)$$

(c) Let from now on $t \in [0, T] \setminus E_\tau$. A projection yields the best approximation of a given point within a given set. Hence, any other choice is not worse. This yields

$$\|P_\tau^s(\chi(\cdot))\mathbf{u}^n - \mathbf{u}^n\|_{L^2([0, T] \setminus E_\tau; L^2(\Omega))} \leq \|v^{n\tau} - \mathbf{u}^n\|_{L^2([0, T] \setminus E_\tau; L^2(\Omega))}$$

for any choice $v^{n\tau} \in K_\tau^s(\Omega_{\text{solid}}(t))$. Therefore by monotonicity of the spaces $K_\tau^s(\Omega_{\text{solid}}(t))$ in τ it is sufficient to find a $\tau_0 > 0$ and a sequence $(v^{n\tau})_n$ for parameter such that

$$\lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \|v^{n\tau} - \mathbf{u}^n\|_{L^2([0, T] \setminus E_\tau; L^2(\Omega))} = 0$$

for almost all $t \in [0, T] \setminus E_{\tau_0}$. Hence, to prove the convergence estimate we need to construct such a sequence $(v^{n\tau})$.

(d) By Lemma 3.6.1 for any $\gamma > 0$ there exists a n_0 such that for all $n \geq n_0$

$$\Omega_{\text{solid}}^{\text{in}}(t) \subseteq \Omega_{\text{solid}}^i(t)^\gamma \quad \text{and} \quad \Omega_{\text{solid}}^i(t) \subseteq \Omega_{\text{solid}}^{\text{in}}(t)^\gamma$$

for all $t \in [0, T]$ and all $i = 1, \dots, N$. Especially by possibly increasing n_0 we can obtain

$$|\Omega_{\text{solid}}^i(t)^\tau \setminus \Omega_{\text{solid}}^{\text{in}}(t)| < \tau_0 \quad \text{and} \quad |\Omega_{\text{solid}}^{\text{in}}(t)^\tau \setminus \Omega_{\text{solid}}^i(t)| < \tau_0$$

due to the geometry of the involved sets by choosing $\gamma < c \tau_0 |\partial \Omega_{\text{solid}}^i(0)|$. Let $t \in [0, T] \setminus E_{\tau_0}$ and let $\text{dist}(\Omega_{\text{solid}}^i(t), \Omega_{\text{solid}}^j(t)) > \tau_0$. Due to Proposition 3.4.6 there exists a $v^{n\tau_0} \in K_{\tau_0}^s(\chi^{\text{in}}(t))$ which is arbitrarily close in $L^2(\Omega)$. Hence, we can assume

$$\|v^{n\tau_0} - \mathbf{u}^n(t)\|_{L^2(\Omega)} \leq \varepsilon$$

Since $K_{\tau_0}^s(\chi^{\text{in}}(t)) \subseteq K_\tau^s(\chi^{\text{in}}(t))$ we have

$$\|P_\tau^s(\chi(t))\mathbf{u}^n(t) - \mathbf{u}^n(t)\|_{L^2(\Omega)} \leq \|P_{\tau_0}^s(\chi(t))\mathbf{u}^n(t) - \mathbf{u}^n(t)\|_{L^2(\Omega)}.$$

Thus, by possible reducing τ_0 and increasing n_0 once more, we obtain that there exists an τ_0 such that

$$\lim_{n \rightarrow \infty} \|P_\tau^s(\chi(t))\mathbf{u}^n(t) - \mathbf{u}^n(t)\|_{L^2(\Omega)} \leq \varepsilon$$

for all $\tau \leq \tau_0$. Applying now (b) and (d) to (a) finishes the proof. *q.e.d.*

3.7 Strong convergence of the approximative velocities

Proposition 3.7.1 (Convergence of projected energies). *For any $0 < s < 1$ there exists a minimal distance $\tau_0 > 0$ such that the densities $(\varrho^n)_n$ and ϱ , and the velocities $(\mathbf{u}^n)_n$ and \mathbf{u} satisfy*

$$(3.7.1) \quad \lim_n \int_{Q_T} \varrho^n \mathbf{u}^n \cdot P_\tau^s(\chi(\cdot))\mathbf{u}^n = \int_{Q_T} \varrho \mathbf{u} \cdot P_\tau^s(\chi(\cdot))\mathbf{u}$$

for any $0 < \tau < \tau_0$

Proof. Fix the parameter s be given.

(a) On the one hand we observe that for any $\tau > 0$ due to Lemma 3.6.1 there exists a $n_0 > 0$ such that

$$\Omega_{\text{solid}}^n(t) \subseteq \Omega_{\text{solid}}(t)^{\frac{\tau}{2}}$$

for all $t \in [0, T]$. On the other hand, by Lemma 3.5.6, the trajectories

$$\Xi(\xi, 0; \cdot) : [0, T] \rightarrow \Omega : \xi \mapsto \Xi(\xi, 0, t)$$

are Lipschitz continuous uniformly in ξ . For sufficiently small time steppings $\theta = \frac{T}{N}$ and a given number $N \in \mathbb{N}$, we observe that

$$\Omega_{\text{solid}}(t)^{\frac{\tau}{2}} \subseteq \Omega_{\text{solid}}(k\theta)^{\tau} \quad \text{and} \quad \Omega_{\text{solid}}(k\theta)^{\frac{\tau}{2}} \subseteq \Omega_{\text{solid}}(t)^{\tau}$$

for all $t \in [(k-1)\theta, k\theta]$, $k = 1, \dots, N$. Especially we obtain that

$$(3.7.2) \quad P_{\frac{\tau}{2}}^0(\chi(k\tau)) P_{\tau}^s(\chi(t)) = P_{\tau}^s(\chi(t))$$

for all $t \in [(k-1)\theta, k\theta]$, $k = 1, \dots, N$, and all $s \geq 0$, this is the projection on the space $K_{\tau}^s(\chi(t))$ is not altered by the projection on the bigger space $K_{\frac{\tau}{2}}^0(\chi(k\tau))$. For our later use it should be noted that $P_{\frac{\tau}{2}}^0(\chi(k\tau))$ is a self adjoint projection.

(b) Let $I_k := [(k-1)\theta, k\theta]$ denote the above used time intervals. We now want to estimate the derivative $\frac{\partial}{\partial t} P_{\frac{\tau}{2}}(\chi(t))(\varrho^n \mathbf{u}^n)$ similarly to the standard estimates for the derivatives in one of the construction of solutions to the Navier-Stokes equations: We want to apply Lions-Aubin Lemma and derive estimates for functions and their derivatives. We consider equation (3.3.1), i.e.

$$\begin{aligned} \int_{Q_T} \varrho^n \mathbf{u}^n \cdot \partial_t \boldsymbol{\phi} &= - \int_{Q_T} \varrho^n \mathbf{u}^n \cdot \text{grad } \boldsymbol{\phi} \mathbf{u}^n - \int_{\Omega} m_0(x) \boldsymbol{\phi}(x, 0) \, dx \\ &\quad + \int_{Q_T} (2\mu + n\chi^n) D_{\text{sym}} \mathbf{u} : D_{\text{sym}} \boldsymbol{\phi} - \int_{Q_T} \varrho f \cdot \boldsymbol{\phi}, \end{aligned}$$

but consider only test functions $\boldsymbol{\phi}$ such that $\boldsymbol{\phi}(t) = 0$ for $t \in [0, T] \setminus I_k$, $\boldsymbol{\phi}(t) \in H_{0,\sigma}^1(\Omega)$ and $D_{\text{sym}} \boldsymbol{\phi}(t) \chi(k\theta) = 0$ for all $t \in I_k$, i.e. $\boldsymbol{\phi}(t) \in K_{\frac{\tau}{2}}(\chi(k\theta))$. Then boundedness in $L^{\frac{8}{3}}(0, T; L^4(\Omega)^3)$ of (\mathbf{u}^n) and of $(\varrho^n)_n$ in $L^{\infty}(0, T; L^{\infty}(\Omega))$, see page 36, combined with standard estimates as elaborated in (Temam, 2001, Thm. 3.3, p.201), yield that

$$\left| \int_{Q_T} \varrho^n \mathbf{u}^n \cdot \partial_t \boldsymbol{\phi} \right| \leq c \|\boldsymbol{\phi}\|_{L^4(I_k; H_{0,\sigma}^1(\Omega))}$$

This is boundedness of $\frac{\partial}{\partial t} P_{\frac{\tau}{2}}(\chi(t))(\varrho^n \mathbf{u}^n)$ in $L^{\frac{4}{3}}(I_k; K_{\frac{\tau}{2}}(\chi(k\theta))^*)$, where $K_{\frac{\tau}{2}}(\chi(k\theta))^*$ is the dual space of $K_{\frac{\tau}{2}}(\chi(k\theta))$.

(c) The energy estimate (3.3.4) and the uniform boundedness estimate of the densities (3.3.5) now yield due to

$$\|\varrho^n \mathbf{u}^n\|_{L^2(Q_T)}^2 \leq \|\varrho^n\|_{L^{\infty}(Q_T)} \int_{Q_T} \varrho^n |\mathbf{u}^n|^2$$

that the sequence $(\varrho^n \mathbf{u}^n)_n$ is bounded in $L^2(I_k; L^2(\Omega))$. Hence, $\left(P_{\frac{\tau}{2}}^0(\chi(t))(\varrho^n \mathbf{u}^n) \right)_n$ is bounded in $L^2(I_k; K_{\frac{\tau}{2}}^0(\chi(k\theta)))$.

To check the assumption of the Lions-Aubin Lemma, see (Showalter, 1997, Pro. III.1.3, p. 106), we set $\mathbf{w}^n := P_{\frac{\tau}{2}}^0(\chi(t))(\varrho^n \mathbf{u}^n)$ and observe that $E_0 := K_{\frac{\tau}{2}}^0(\chi(k\theta))$ is compactly embedded in $E := K_{\frac{\tau}{2}}^s(\chi(k\theta))^*$, which is continuously embedded into the space $E_1 := K_{\frac{\tau}{2}}^1(\chi(k\theta))^*$. Hence, $(\mathbf{w}^n)_n$ is bounded in $L^2(I_k; E_0)$ and $(\frac{\partial}{\partial t} \mathbf{w}^n)_n$ is bounded in $L^{\frac{4}{3}}(I_k; E_1)$. Therefore, $(\mathbf{w}^n)_n$ is relatively compact sequence in $L^2(I_k; E)$. By Lemma 3.5.3 the sequence $(\varrho^n)_n$ converges strongly in $L^\infty(I_k; L^\infty(\Omega))$ and by equation (3.3.7) the sequence $(\mathbf{u}^n)_n$ converges weakly in $L^2(I_k; L^2(\Omega))$. Therefore the sequence $(\varrho^n \mathbf{u}^n)_n$ converges weakly in $L^2(I_k; L^2(\Omega))$ and $P_{\frac{\tau}{2}}^0$, as continuous projection on $L^2(\Omega)^3$, is compact if combined with the embedding of E_0 into E . Hence, $(\mathbf{w}^n)_n$ converges strongly in $L^2(I_k; E)$.

(d) By (3.3.7) the sequence $(\mathbf{u}^n)_n$ converges weakly in $L^2(0, T; H_{0,\sigma}^1(\Omega))$, which implies weak convergence of $(P_\tau^s(\chi(\cdot))\mathbf{u}^n)_n$ in $L^2(0, T; K_{\frac{\tau}{2}}^s(\chi(k\theta)))$ for all $0 \leq s < 1$.

(e) Due to the above considerations, we can split the time integral on the several intervals I_k and obtain

$$\begin{aligned} \lim_n \int_{Q_T} \varrho^n \mathbf{u}^n \cdot P_\tau^s(\chi(\cdot))\mathbf{u}^n &= \lim_n \sum_{k=1}^N \int_{I_k} \langle \varrho^n \mathbf{u}^n, P_\tau^s(\chi(\cdot))\mathbf{u}^n \rangle_{L^2(\Omega) \times L^2(\Omega)} \\ &= \lim_n \sum_{k=1}^N \int_{I_k} \left\langle \varrho^n \mathbf{u}^n, P_{\frac{\tau}{2}}^0(\chi(k\tau)) P_\tau^s(\chi(\cdot))\mathbf{u}^n \right\rangle_{L^2(\Omega) \times L^2(\Omega)} \\ &= \lim_n \sum_{k=1}^N \int_{I_k} \left\langle P_{\frac{\tau}{2}}^0(\chi(k\tau))(\varrho^n \mathbf{u}^n), P_\tau^s(\chi(\cdot))\mathbf{u}^n \right\rangle_{L^2(\Omega) \times L^2(\Omega)} \\ &= \lim_n \sum_{k=1}^N \int_{I_k} \left\langle P_{\frac{\tau}{2}}^0(\chi(k\tau))(\varrho^n \mathbf{u}^n), P_\tau^s(\chi(\cdot))\mathbf{u}^n \right\rangle_{\left(K_{\frac{\tau}{2}}^s(\chi(k\theta))\right)^* \times K_{\frac{\tau}{2}}^s(\chi(k\theta))}. \end{aligned}$$

The weak and strong sequences convergences now yield

$$\begin{aligned} \lim_n \int_{Q_T} \varrho^n \mathbf{u}^n \cdot P_\tau^s(\chi(\cdot))\mathbf{u}^n &= \sum_{k=1}^N \int_{I_k} \left\langle P_{\frac{\tau}{2}}^0(\chi(k\tau))(\varrho \mathbf{u}), P_\tau^s(\chi(\cdot))\mathbf{u} \right\rangle_{\left(K_{\frac{\tau}{2}}^s(\chi(k\theta))\right)^* \times K_{\frac{\tau}{2}}^s(\chi(k\theta))} \\ &= \sum_{k=1}^N \int_{I_k} \langle \varrho \mathbf{u}, P_\tau^s(\chi(\cdot))\mathbf{u} \rangle_{L^2(\Omega) \times L^2(\Omega)} = \int_{Q_T} \varrho \mathbf{u} \cdot P_\tau^s(\chi(\cdot))\mathbf{u}. \end{aligned}$$

Therefore, the claim is proved.

q.e.d.

Proposition 3.7.2 (Convergence of integrated energies).

The densities $(\varrho^n)_n$ and ϱ , and the velocities $(\mathbf{u}^n)_n$ and \mathbf{u} satisfy

$$(3.7.3) \quad \lim_n \int_{Q_T} \varrho^n |\mathbf{u}^n|^2 = \int_{Q_T} \varrho |\mathbf{u}|^2.$$

Proof. Considering the difference of the terms of the left and the right hand side of

equation (3.7.3) we add a helpful zero and obtain

$$(3.7.4) \quad \begin{aligned} \int_{Q_T} \varrho^n |\mathbf{u}^n|^2 - \int_{Q_T} \varrho |\mathbf{u}|^2 &= \int_{Q_T} \varrho^n \mathbf{u}^n P_\tau^s(\chi(\cdot)) \mathbf{u}^n - \varrho \mathbf{u} P_\tau^s(\chi(\cdot)) \mathbf{u} \\ &\quad + \int_{Q_T} \varrho^n \mathbf{u}^n (\mathbf{u}^n - P_\tau^s(\chi(\cdot)) \mathbf{u}^n) + \varrho \mathbf{u} (P_\tau^s(\chi(\cdot)) \mathbf{u} - \mathbf{u}). \end{aligned}$$

The first line of the right hand side tends to zero due to Proposition 3.7.1. The terms of the second line will be estimated separately. Uniform boundedness of the sequences $(\varrho^n)_n$ in $L^\infty(0, T; L^\infty(\Omega))$ and $(\mathbf{u}^n)_n$ in $L^\infty(0, T; L^2(\Omega)^3)$, respectively, and Hölder's inequality yield

$$\left| \int_{Q_T} \varrho^n \mathbf{u}^n (\mathbf{u}^n - P_\tau^s(\chi(\cdot)) \mathbf{u}^n) \right| \leq \|\varrho^n\|_{L^\infty L^\infty} \|\mathbf{u}^n\|_{L^\infty L^2} \|\mathbf{u}^n - P_\tau^s(\chi(\cdot)) \mathbf{u}^n\|_{L^1 L^2}$$

By Proposition 3.6.8 the right-hand-side and therefore the first term converges to zero.

Boundedness of ϱ in $L^\infty(0, T; L^\infty(\Omega))$ and \mathbf{u} in $L^\infty(0, T; L^2(\Omega)^3)$, respectively, similar calculations, and Proposition 3.6.6 applied to

$$\left| \int_{Q_T} \varrho \mathbf{u} (P_\tau^s(\chi(\cdot)) \mathbf{u} - \mathbf{u}) \right|$$

yield that this term is arbitrary small as well. Hence, for large n all terms on the left hand side of Equation (3.7.4) tends to zero. Hereby, the proof is finished. *q.e.d.*

Finally, we are able to prove the last prerequisite of the existence proof.

Proof of Proposition 3.3.5.: We have to show the strong convergence of the sequence $(\mathbf{u}^n)_n$ in $L^2(0, T; L^2(\Omega))$ to \mathbf{u} . For this we consider at first

$$\begin{aligned} \left| \int_{Q_T} \varrho (|\mathbf{u}^n|^2 - |\mathbf{u}|^2) \right| &\leq \left| \int_{Q_T} \varrho^n |\mathbf{u}^n|^2 - \varrho |\mathbf{u}|^2 \right| + \left| \int_{Q_T} (\varrho^n - \varrho) |\mathbf{u}^n|^2 \right| \\ &\leq \left| \int_{Q_T} \varrho^n |\mathbf{u}^n|^2 - \int_{Q_T} \varrho |\mathbf{u}|^2 \right| + \|\varrho^n - \varrho\|_{L^4 L^2} \|\mathbf{u}^n\|_{L^{\frac{4}{3}} L^2}^2. \end{aligned}$$

The sequence $(|\mathbf{u}^n|^2)_n$ is bounded in $L^{\frac{4}{3}}(0, T; L^2(\Omega))$, since the sequence $(\mathbf{u}^n)_n$ is bounded in $L^{\frac{8}{3}}(0, T; L^4(\Omega)^3)$ by equation (3.3.11). By Lemma 3.5.3 the sequence $(\varrho^n - \varrho)_n$ converges to zero in $C([0, T]; L^p(\Omega))$ for all $1 \leq p < \infty$, and therefore especially in $L^4(0, T; L^2(\Omega))$ since $T < \infty$. The first term on the right hand side converges by Proposition 3.7.2 to zero. Thus, we see

$$(3.7.5) \quad \lim_n \left| \int_{Q_T} \varrho (|\mathbf{u}^n|^2 - |\mathbf{u}|^2) \right| = 0.$$

Using that $1 < \frac{1}{\varrho}$ we obtain

$$\begin{aligned} \|\mathbf{u}^n - \mathbf{u}\|_{L^2 L^2}^2 &= \int_{Q_T} |\mathbf{u}^n - \mathbf{u}|^2 = \int_{Q_T} (|\mathbf{u}^n|^2 - |\mathbf{u}|^2) + 2 \int_{Q_T} \mathbf{u} (\mathbf{u} - \mathbf{u}^n) \\ &\leq \frac{1}{\varrho} \left| \int_{Q_T} \varrho (|\mathbf{u}^n|^2 - |\mathbf{u}|^2) \right| + 2 \int_{Q_T} \mathbf{u} (\mathbf{u} - \mathbf{u}^n). \end{aligned}$$

By equation (3.7.5) the first term on the right hand side converges to zero. By construction the sequence $(\mathbf{u}^n)_n$ converges weakly to \mathbf{u} in $L^2(0, T; L^2(\Omega))$. Therefore, the second term on the right hand side goes to zero as well. Hence, we obtain

$$\lim_n \|\mathbf{u}^n - \mathbf{u}\|_{L^2 L^2}^2 = 0,$$

which proves the claimed strong convergence in $L^2(0, T; L^2(\Omega))$ and finishes the proof of the proposition. *q.e.d.*

3.8 Existence of a solution and structure of collisions

Proof of Existence Theorem 3.2.8. As we stated on page 38, the missing part of the existence proof is that the limit velocity \mathbf{u} satisfies the conservation of momentum equation which shall be proved now. By Proposition 3.3.5 and Lemma 3.5.3 we can deduce immediately that

$$(3.8.1) \quad \mathbf{u}^n \longrightarrow \mathbf{u} \quad \text{strongly in } L^2(0, T; L^2_\sigma(\Omega)),$$

$$(3.8.2) \quad \varrho^n \longrightarrow \varrho \quad \text{strongly in } L^8(0, T; L^4(\Omega)),$$

and by choice of the subsequence we know that

$$(3.8.3) \quad \mathbf{u}^n \longrightarrow \mathbf{u} \quad \text{weakly in } L^{\frac{8}{3}}(0, T; L^4(\Omega)^3)$$

$$(3.8.4) \quad D_{\text{sym}} \mathbf{u}^n \longrightarrow D_{\text{sym}} \mathbf{u} \quad \text{weakly in } L^2(0, T; L^2(\Omega)^{3 \times 3}).$$

To show that \mathbf{u} and ϱ satisfy the weak form of the conservation of momentum equation, we need to show that

$$(3.8.5) \quad \begin{aligned} \int_{Q_T} \varrho \mathbf{u} \cdot (\partial_t \boldsymbol{\phi} + \text{grad } \boldsymbol{\phi} \mathbf{u}) &= - \int_{\Omega} m_0(x) \boldsymbol{\phi}(x, 0) \, dx \\ &+ \int_{Q_T} 2\mu D_{\text{sym}} \mathbf{u} : D_{\text{sym}} \boldsymbol{\phi} - \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\phi} \end{aligned}$$

is satisfied for all $\boldsymbol{\phi} \in \mathcal{D}([0, T]; \mathcal{D}_\sigma(\Omega))$ that satisfy $D_{\text{sym}} \boldsymbol{\phi} = 0$ in an arbitrary open neighborhood of $\text{supp } \chi$. To prove this, it is sufficient to consider only those $\boldsymbol{\phi}$ that satisfy $\chi^\tau D_{\text{sym}} \boldsymbol{\phi} = 0$, this is $D_{\text{sym}} \boldsymbol{\phi}(x, t) = 0$ for all t and $x \in \Omega_{\text{solid}}^i(0)^\tau$, the τ -neighborhood of Ω_{solid}^i . For sufficiently small τ , any open neighborhood is included in such a neighborhood. Fix now such a $\boldsymbol{\phi}$.

Due to Lemma 3.6.1, there even exists a n_0 such that $\chi^n D_{\text{sym}} \boldsymbol{\phi} = 0$ for all $n \geq n_0$. Hence, the approximative solutions already satisfy

$$(3.8.6) \quad \int_{Q_T} \chi^n D_{\text{sym}} \mathbf{u}^n : D_{\text{sym}} \boldsymbol{\phi} = 0$$

for all $n \geq n_0$ without going to the limit.

The convergence properties (3.8.1), (3.8.2), (3.8.3) and (3.8.4) now yield that we can pass to the limit of the remaining terms of the approximative form of the conservation

of momentum from page 35, this is in equation

$$\begin{aligned} \int_{Q_T} \varrho^n \mathbf{u}^n \cdot (\partial_t \phi + \text{grad } \phi \mathbf{u}^n) &= - \int_{\Omega} m_0(x) \phi(x, 0) dx \\ &+ \int_{Q_T} 2\mu D_{\text{sym}} \mathbf{u}^n : D_{\text{sym}} \phi - \int_{Q_T} \varrho f \cdot \phi \end{aligned}$$

where equation (3.8.6) was already applied. We obtain that \mathbf{u} and ϱ satisfy equation (3.8.5) at least for all ϕ such that $D_{\text{sym}} \phi = 0$ in a τ -neighborhood of Ω_{solid}^i . Hence, (3.8.5) is valid also for all ϕ that satisfy $D_{\text{sym}} \phi = 0$ in an open neighborhood. If χ is collision-free, Proposition 3.4.6 and Corollary 3.4.8 yield that these test functions arbitrarily well approximate test functions that satisfy $\chi D_{\text{sym}} \phi = 0$, what we did not claim but corresponds to the two-dimensional formulation of a weak solution.

The last open claim is the regularity of the characteristic functions. In Definition 3.2.7 we demanded that the characteristic function χ^i should be in $C^{0+\frac{1}{p}}(0, T; L^p(\Omega))$ for all $1 \leq p < \infty$.

For each index $i = 1, \dots, N$ the characteristic function $\chi^i \in C([0, T]; L^p(\Omega))$ for all $1 \leq p < \infty$ and $\Omega_{\text{solid}}^i(t) = \Xi_{\text{solid}}^i(\Omega_{\text{solid}}^i(s), s; t)$ by definition. Furthermore, Ξ_{solid}^i is Lipschitz-continuous in s and t , due to Lemma 3.5.6 and the definition of Ξ_{solid}^i , yields that $|\Xi_{\text{solid}}^i(\mathbf{x}, s; t) - \mathbf{x}| \leq L|t - s|$. Hence, $\Omega_{\text{solid}}^i(t) \subseteq \Omega_{\text{solid}}^i(s)^\tau$ and $\Omega_{\text{solid}}^i(s) \subseteq \Omega_{\text{solid}}^i(t)^\tau$ for $\tau = L|t - s|$ yields

$$\begin{aligned} \int_{\Omega} |\chi^i(\mathbf{x}, t) - \chi^i(\mathbf{x}, s)|^p d\mathbf{x} &= |\Omega_{\text{solid}}^i(t) \setminus \Omega_{\text{solid}}^i(s)| + |\Omega_{\text{solid}}^i(s) \setminus \Omega_{\text{solid}}^i(t)| \\ &\leq |\Omega_{\text{solid}}^i(s)^\tau \setminus \Omega_{\text{solid}}^i(s)| + |\Omega_{\text{solid}}^i(t)^\tau \setminus \Omega_{\text{solid}}^i(t)| \\ &\leq c\tau (|\partial\Omega_{\text{solid}}^i(s)| + |\partial\Omega_{\text{solid}}^i(t)|) \\ &\leq c |\partial\Omega_{\text{solid}}^i(0)| |t - s|, \end{aligned}$$

which implies

$$\|\chi^i(t) - \chi^i(s)\|_{L^p(\Omega)} \leq c |t - s|^{\frac{1}{p}}$$

for all $1 \leq p < \infty$. Therefore, every statement of this chapter is finally proved. *q.e.d.*

After finishing the proof of all statements, we want to check the collision phenomenon, which we found in the last chapter, for the here obtained solution. In particular for the here approximated solution we are able to describe the relative motion of two colliding bodies even further: If we knew that the velocity fields has slightly higher regularity than a standard weak solution then rolling on top of the common tangential plane is excluded.

We recall Lemma 2.1.8 from page 19 that describes the motion for weak solutions. We formulated it in the version satisfied by the here constructed solution.

Lemma 3.8.1 (Evolution of collisions). *Let $\Omega \subseteq \mathbb{R}^3$ be an open and bounded domain. Let $\mathbf{u} \in L^2(0, T; H^{1/2}(\Omega)^3)$ be the velocity field that transports a mixture of rigid and other phases. If $\Omega_1(t)$ and $\Omega_2(t)$ are two disjoint rigid phases within Ω , i.e. $D_{\text{sym}} \mathbf{u}(\mathbf{x}, t) = 0$ in $\Omega_1(t) \cup \Omega_2(t)$ for $t \in (0, T)$, and Ω_1 and Ω_2 touch at time $t_0 \in (0, T)$ in a point in which the boundaries are (locally) strictly convex and C^2 . Let $\text{gap}(t) := \text{dist}(\Omega_1(t), \Omega_2(t))$ denote their distance then it satisfies the following.*

- (a) *The distance necessarily satisfies the estimate $\text{gap}(t) = o(|t - t_0|)$.*
- (b) *In particular, the distance function is necessarily differentiable at this moment with vanishing derivative.*

Hence, we obtain for our here constructed solution the predicted phenomenon. Collisions might occur, but are almost always far from being fascinating. Furthermore, for this solution we have a more precise description of the relative motion if further regularity is assumed.

Conclusions and future work

4.1 Understanding particulate flows and blood

Suppose you want to model blood or any other mixture of solids and liquids where a high number of submerged bodies occurs. If the extension of a body is observable, these heterogeneous flows are often referred to as particulate flows or, for smaller bodies, as thick suspensions. To demonstrate why the homogeneous models so far are wanting, let us consider a highly idealized case that motivates the scale of modeling error so far.

Let c be a given volume fraction of a volume $V = L\pi R^2$ of a pipe of length L and diameter R . Submerged solid balls of radius r have volume $v = \frac{4}{3}\pi r^3$. Therefore, approximately

$$N = c \frac{3LR^2}{4r^3}$$

bodies have to be contained in the flow. The total surface in contact with the fluid is given by the surface of the submerged balls plus the surface of the outer boundary, this is

$$A = 3\pi LR^2 \frac{c}{r} \pi + 2L\pi R,$$

where $A_1 := 3\pi LR^2 \frac{c}{r}$ is the total surface of the balls. Since the derivation of Newtonian fluid does not capture boundary effects, a larger surface indicates a larger modeling error. Hereby, the modeling error increases in bigger vessels and the paradigm that blood may be modeled as Newtonian incompressible fluid can hardly be defended. Especially, not in bigger vessels.

That the boundary layers are important can also be seen by different considerations. The average distance between submerged bodies is

$$\tau = \frac{(1-c)V}{A_1} = \frac{1-c}{c} \frac{r}{3}.$$

This value is independent of the total volume.

Since the balls represent the minimal surface for a given volume v , in real world scenarios the effective surface could be even bigger whereas the average distance could be smaller. The scale should be correct.

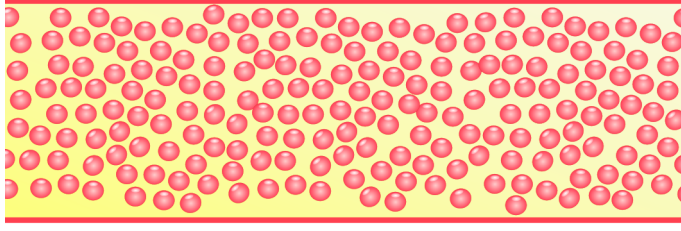


Figure 4.1.1: Modeling particulate flows or thick suspensions poses many unsolved and sometimes even unchallenged problems. Some are inherited from homogeneous fluids, some are new. Surface effects are more important, due to the much increased surface. Phenomenologically a non-Newtonian behavior is observed. This poses a high complexity for numerical analysis and even simulations to capture the influence of the two phases. And many more new challenges are encountered. The formulation of in- and outflow is still difficult in this setting. Especially, the in- and out-flow of submerged bodies needs extra treatments.

To obtain a feeling for the case of blood consider the following. Assume a hematocrit of $c = 0.5$. So the mean distance is $\tau = \frac{1}{12}r$, where r is the mean erythrocyte radius. The disk shaped red blood cells have an average volume of $v = 105\mu\text{m}^3$. Thus, setting $r = 2.9\mu\text{m}$ yields approximately the same concentration. In general, concentrations beyond 25% have an average distance τ below body radius. So for blood we are surely within the region where individual bodies, and therefore boundary layers, cannot be neglected.

These are the reasons why I advocate for using heterogeneous fluid models to model the transport of erythrocytes. Whether it is to be modeled as Newtonian or non-Newtonian is most probably not the most important question. In my opinion, the plasma should be modeled as Newtonian fluid, not because it is more appropriate, but because it might be simpler to start with. If features of the flow in bigger vessels are of interest, I recommend not to trust any numerical calculations or engineering simulations. In the true sense of understanding, this flow is far beyond understanding. See the introduction for a discussion. Any calculation should be checked by experiments, although in case of blood this is hardly possible. Hopefully, one day it is possible to upscale the heterogeneous flow model I recommend for smaller vessels to bigger length scales while keeping the scale of the velocity fixed. Hereby a homogeneous, possibly now non-Newtonian, flow model could be obtained that can be defended by this derivation as acceptable. But this is a conclusion for mathematics, not for engineers. Homogenization of free boundary problems seems to be difficult. Nevertheless, fluids like blood evade in-vivo experiments and are of obvious necessity to be understood. To model and simulate numerically the flow through stenoses or stents, or to understand sedimentation and forming of restenoses, are only some examples.

Since here the embedded surfaces are modeled, the here used model has the possibility to be closer to reality than every model used in hemodynamics so far. All previous models used in medicine are based on the assumption that a certain flow profile can be regarded as prototypic. As simplest example we mentioned Poiseuille's flow. But even the more complex stratified flows are parallel to the centerline. These consist of iterated Poiseuille profiles with continuous velocity fields.

Hence, these are only phenomenologically closer to the anticipated flow profiles of blood. Here, no such assumption is necessary. Nevertheless, I do not believe that the here favored model can be motivated in the medical communities. It is just not appropriate for training of medical personnel. Only engineers dared to apply their, often phenomenological, models of particulate flows to blood flow in the medical communities.

Hence, for understanding particulate flows the here used model is hardly applicable by engineers, let alone even less mathematically trained users of models. Thus, these groups, which definitely need more appropriate models, have to be content with the prospect of the advent of better models.

4.2 Simulation and numerical analysis

Also in engineering simulations, the possibility of collisions of bodies needs special attention. According to Joseph (2005, Chapter III, page 11f), four strategies to deal with collisions are used. They all define *a security zone around the particle such that when the gap between particles is smaller than the security zone, a repelling force is activated*. The drawback of this approach is that it needs extra modeling of these repelling forces, and that most of these models *keep the particles farther apart than they ought to be*. In principle, these approaches present a purely artificial boundary layer model which has no equivalent in the real world. They are for the sole purpose of numerical feasibility. Admittedly, the persuasive power of the produced images and movies is impressive.

Our approach has neither of these features. We do not have the questionable boundary layer assumption, nor do we have the comforting situation to present nice images and movies . . . so far. Hence, the production of these is future work.

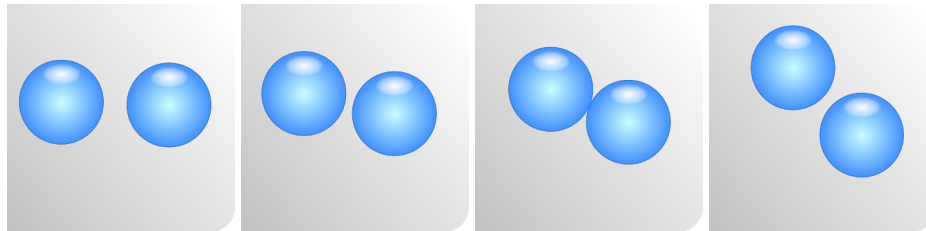


Figure 4.2.1: Collisions are prohibitive in a reference coordinate system. The fluid domain changes in time and can even involve topological changes. The contact point has two different past versions and two different future versions. Thus, no bijection between reference and real-world domain can exist, let alone a diffeomorphisms.

Most numerical analysts are aware that the two natural coordinate systems used in fluid mechanics and elasticity, respectively, are incompatible. In elasticity theory a reference to the undeformed body or another reference system seems to be the most natural description to understand the forces due to deformation. This is possible, since the particular position of the body in space is of no importance. In standard fluid mechanics such an approach is futile due to the complexity of the flow profiles. Besides this, it is useless. Standard models are based on the deformation of the velocity field and not on the past formation of the material.

We recommend to track the δ -kernel of the submerged bodies instead of the original body (Notation 3.3.1). Using a level set approach, this is straight forward. The level set function of the δ -kernel is given as shifted level-set function of the submerged bodies. Hence, they pose no new difficulties for a level-set approach. Hence, the only new part of our notion of an approximative solution should be easily implementable into a fluid solver that tracks binary fluid flows using level-sets. The major benefit of this approach is that it works without extra assumption on the velocity field in the boundary layer around the submerged bodies. Actually, it even reduces the assumptions on the velocity field in the sense that only the δ -kernel of the submerged bodies is supposed to be rigid. Hereby a hypothetically existing solution can satisfy our notion of an approximative solution. This approach therefore is consistent.

Nevertheless, all approaches to simulation of flow problems have a common skeleton in the cupboard, namely uniqueness of the solution of the Navier-Stokes. We selected just too many times subsequences of all sequences $(\mathbf{u}^n)_n, (\zeta^n)_n, (\varrho^n)_n, \dots$ with different features. This is not feasible for a numerical approach. Furthermore, numerical stability of weak solutions seems to be unknown even in case of uniqueness. To obtain convincing pictures, more-or-less heuristic stabilization methods have to be used. This is not related to here considered model. It is common to all fluid models.

We proved existence of a weak solution that satisfies an energy inequality (Theorem 3.2.8). It is not known if every weak solution satisfies such an inequality or if even a conservation of energy equation is satisfied. We considered a model without the need to implement heuristic and non-physical boundary layer equations that keep the bodies apart (Notation 3.3.1). The applied procedure to obtain a solution can directly be implemented and should be as good or bad as any other model derived from Navier-Stokes. There is no need to select subsequences if the solution is unique, as always has to be assumed in this area. So, the here advocated model is numerically only as bad as the standard Navier-Stokes equations, needs to be stabilized heuristically, but does not introduce artificial forces which are hidden somewhere in if-then clauses in the source code and influence the outcome mathematically unrestrained. Hence, it should be simpler to produce convincing pictures with the here favored approach.

4.3 Analysis

Of utmost importance for all estimates in this work was the assumption of incompressibility. The crucial observation for these estimates was that the flow through the boundary of an imaginary contact cylinder between two approaching balls has to balance the volume that is squeezed out by the balls that enter the imagined cylinder, captured in Equation (2.1.21). Therefore, the most natural idea is to drop the incompressibility assumption. But without this assumption a different equation is needed to close the system of equations. The choice of this extra equation seems to be a different open problem. For particular selections really ridiculous solutions are possible. Feireisl (2003a) demonstrated that a lead ball which is fixed in a containing bigger sphere at a single point at the top yields a stationary solution if the sphere is filled with a compressible fluid. Hence, true compressibility might not be the path to take.

Realistic fluids are not incompressible in a strict sense. Water is labeled as incompressible because it is difficult to compress, but it is simple to decompress. Doubling the pressure from atmospheric pressure to a water depth of ten meter does no real change. Reducing the pressure to half of an atmosphere changes quite a lot since evaporation and cavitation due to moving objects can not be ignored anymore. And even at standard pressure, evaporation and cavitation occurs¹. Hence a kind of delayed incompressibility equation might be more realistic. For example in numerical simulations other approaches have shown to be useful. Allowing the fluid to be slightly compressible leads to convincing pictures and simulations as well. The open question here still is what is modeled by these ad-hoc assumptions.

The next open problem is that the here used model actually is still too difficult. For any given computer power, the numbers of submerged bodies can be increased till a brute force calculation is not feasible any more. In the case of blood or other particulate flows the number of bodies is prohibitive. The only chance is to use a simpler model that yields on average the same results. Similarly to the case of flow through sand, where Stokes flow in the pores can be replaced by a Darcy law on the domain without sand, a homogenized model would solve the blood-flow problem definitely. Imagining a periodic flow through a pipe that is loaded with bodies, the here considered sequence of binary fluid models (the approximative models of Notation 3.3.1) are more accessible to homogenization approaches than the classical multi-coordinate-models. A hereby homogenized model would yield a ground flow for a particulate flow. It could serve as level-zero approximation of a hierarchical mathematical approximation scheme. In the used terms of Notation 3.3.1, first the number N of bodies tends to infinity, then the rigidity increment n .

Applications that use mixtures of different species are plenty. Naturally blood is an example, but also waste water treatment and even glacier melting and motion are applications. Sedimentation and accumulation of different phases within waste water can be easily imagined. But the melting of glaciers is also a surprising application. The glaciers of Greenland melt faster than any model predicts. Hereby they produce unpredictable amounts of sweet water that influence the global conveyor belt, which for example transports warmth to Europe, in unknown strength. The here used approximative model can simply be adapted to produce a melting phenomenon with increasing pressure. If viscosity is decreasing with increasing pressure, we obtain a new melting model that can produce melting in the middle of a solid. Since forming of new cavities filled with water within glaciers are observed, they could be studied with this adapted model. We just need to use an approximative tensor $2\mu(p) D_{\text{sym}} \mathbf{u}^n$, where the viscosity $\mu(p)$ decreases to the viscosity μ_{Fluid} of the fluid for increasing absolute value $|p|$ of the pressure.

We obtained in three dimensions a formulation which is almost equivalent to the two-dimensional version of San Martín et al. (2003). Our theorems allow collisions of different bodies or bodies with the outer boundary only with vanishing relative velocity and, if the second derivative exists, with vanishing relative acceleration (Lemma 2.1.8 and Theorem 3.2.10). This is independent of the model used for the

¹A hopefully convincing demonstration that works especially well in old houses or university departments is the so called water hammer phenomenon. Rapidly shut the faucet that controls the inflow to a device that needs momentarily a large volume of water. A washing machine is a good candidate. If the plumbing was done well, you hear nothing. Otherwise, water pipes may break. Water is not incompressible.

fluid. Only incompressibility and regularity of the global velocity field was used. Hence, contrary to the opinion expressed by San Martín et al. (2003, Remark 2.3), my opinion is that this phenomenon is not due to a deficiency of the notion of weak solution. It is due to the non-applicability of the incompressibility assumption. The solution we constructed can be described more precisely. If a minimally higher regularity is assumed, not only the distance of two bodies can be considered, but their relative rotation can be described as well. For strong solutions only a motion of balls-on-sticks is possible.

Therefore, we extended the local existence result of Desjardins and Esteban (1999), which proved existence at most up to the first occurrence of a collision. We proposed a global weak solution that deals with collisions. Feireisl (2003b) considered incompressible flows, but used a continuation of a velocity field after an occurred collision. We motivated that the approximative models can be used by themselves to study mixtures of fluids with an arbitrary number of submerged bodies and that these approximative flow models can be used for different purposes as well. The sequence of approximative models and the limit model can be used to study blood flow and other particulate flows. They provide the opportunity to homogenize particulate flow and to bring suspensions into the reach of computability which goes beyond convincing pictures and impressive movies.

4.4 Conclusion

I started with the statement that this work considers three of the many fundamental questions, and actually I solved these three. But this only posed many new fundamental questions. Hence, I even enlarged the amount of incomprehension concerning the motion of bodies within a three-dimensional viscous fluid. The urgent need to get an idea on what drives nature on the one hand and fundamental mathematical problems on the other hand makes up the challenging endeavor fluid mechanics still poses to all sciences. So, though this is the end of this work, the quest will go on.

Acknowledgments

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Bibliography

- David J. Acheson. *Elementary Fluid Dynamics*. Clarendon Press, Oxford, 2005.
- Douglas Noël Adams. *The Ultimate Hitchhiker's Guide To The Galaxy*. Wings, 1998.
- Robert A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
- Hans Wilhelm Alt. *Lineare Funktionalanalysis*. Springer-Verlag, Berlin, 5. edition, 2006.
- Herbert Amann. *Gewöhnliche Differentialgleichungen*. Walter de Gruyter, Berlin, 2. edition, 1995.
- Sebastian Bönisch. *Adaptive Finite Element Methods for Rigid Particulate Flow Problems*. PhD thesis, Univeristy of Heidelberg, 2006.
- Alexandre J. Chorin and Jerrold E. Marsden. *A Mathematical Introduction to Fluid Mechanics*. Springer-Verlag, Berlin, 3rd ed. 1993. corr. 4th printing edition, 2000.
- Benoit Desjardins and M. J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.*, 146:59 – 71, 1999.
- R. J. DiPerna and Pierre-Louis Lions. Ordinary differential equation, transport theory and Sobolev spaces. *Invent. Math.*, 98:511 – 547, 1989.
- Klaus-Jochen Engel and Rainer Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, 2000.
- Robin Fåhræus and Torsten Lindqvist. The viscosity of the blood in capillary tubes. *American Journal of Physiology*, 96(3):562 – 567, 1931.
- Eduard Feireisl. On the motion of rigid bodies in a viscous compressible fluid. *Arch. Ration. Mech. Anal.*, 167(4):281–308, 2003a. ISSN 0003-9527.
- Eduard Feireisl. On the motion of rigid bodies in a viscous incompressible fluid. *J. Evol. Equ.*, 3(3):419–441, 2003b. ISSN 1424-3199. Dedicated to Philippe Bénilan.
- Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Vol. I*. Addison-Wesley, Reading, Mass., 1977a.

- Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Vol. II*. Addison-Wesley, Reading, Mass., 1977b.
- Giovanni P. Galdi. *Lecture Notes of: An Introduction to the Navier-Stokes Initial-Boundary Value Problem*, 2005.
- Giovanni P. Galdi and Ashwin Vaidya. Translational steady fall of symmetric bodies in a navier–stokes liquid, with application to particle sedimentation. *Journal of mathematical fluid mechanics*, 3:183 – 211, 2001.
- Giovanni P. Galdi, Ashwin Vaidya, Milan Pokorný, Daniel D. Joseph, and Jimmy Feng. Orientation of symmetric bodies falling in a second-order liquid at nonzero Reynolds number. *Mathematical Models and Methods in Applied Sciences*, 12(11): 1653–1690, 2002.
- Harry L. Goldsmith, Giles R. Cokelet, and Peter Gahtgens. Robin Fåhræus 1888–1968: evolution of his concepts in cardiovascular physiology. *American Journal of Physiology*, 257(3, 2):H1005 – H1015, 1989.
- Jerome A. Goldstein. *Semigroups of linear operators and applications*. Oxford University Press, New York, 1985.
- John G. Heywood, Rolf Rannacher, and Stefan Turek. Artificial boundary conditions and flux and pressure conditions for the incompressible navier-stokes equations. *International Journal for Numerical Methods in Fluids*, 22:325 – 352, 1996.
- Daniel D. Joseph. *Interrogations of Direct Numerical Simulation of Solid-Liquid Flows*. eFluids, 2005. URL <http://www.efluids.com/efluids/books/joseph.htm>.
- Tosio Kato. *Perturbation Theory for Linear Operators, 2nd-Edition*. Classics in Mathematics/ Grundlehren der mathem. Wissenschaften 132. Springer-Verlag, Berlin, 1995.
- Olga A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York, 2. edition, 1969.
- Lev Davidovič Landau and Evgenij Michajlovič Lifšic. *Mechanik*, volume 1 of *Lehrbuch der theoretischen Physik*. Akademie-Verlag, Berlin, 5 edition, 1967.
- Pierre-Louis Lions. *Mathematical Topics in Fluid Mechanics 1, Incompressible Models*. Oxford Lecture Series in Mathematics and its Application. Clarendon Press, Oxford, 1996.
- Peter Maslak, July 2004. URL <http://ashimagebank.hematologylibrary.org/cgi/content/full/2004/0712/101153#F2>.
- Edward J. McShane. *Princeton mathematical series*. Princeton University Press, 1947.
- Gregor Nickel. Evolution semigroups and product formulas for nonautonomous cauchy problems. *Math. Nachr.*, 212:101 – 116, 2000.
- Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer-Verlag, New York, 1983.

- Rolf Rannacher. *Lecture Notes of: Numerische Methoden der Kontinuumsmechanik*. University of Heidelberg, 2006. URL <http://numerik.uni-hd.de/~lehre/notes/>.
- Dina Rüger. *Vergleichende Untersuchungen zur Orientierung von Blutzellen in lebenden Mikrogefäßen*, MD thesis, Rheinisch-Westfälischen Technischen Hochschule Aachen, Medizinische Fakultät, August 2001. URL <http://deposit.ddb.de/cgi-bin/dokserv?idn=964507692>.
- Jorge Alonso San Martín, Victor Starovoitov, and Marius Tucsnak. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Archive for Rational Mechanics and Analysis*, 161(2):113 – 147, 2003.
- Ralph Edwin Showalter. *Monotone Operators in Banach Space and Nonlinear Differential Equations*. Mathematical Surveys and Monographs 49. American Mathematical Society, 1997.
- Hermann Sohr. *The Navier-Stokes Equations: An elementary functional analytic approach*. Birkhäuser Verlag, Basel, 2001.
- Roger Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, Providence, 2001.
- Roger Temam. *Mathematical Problems in Plasticity*. Gauthier-Villars, 1985.
- Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, 5. edition, 2005.
- Frank M White. *Fluid mechanics*. WCB/McGraw-Hill, Boston, Mass., 4th ed edition, 1999. ISBN 0070697167.
- Wikipedia. English Edition of Wikipedia. The free encyclopedia that anyone can edit., 2007. URL en.wikipedia.org.
- Kôsaku Yosida. *Functional Analysis*, volume 123 of *Classics in Mathematics/Grundlehren der mathem. Wissenschaften*. Springer-Verlag, Berlin, 6. edition, 1995.