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Dihedral Flavor Symmetries

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Zusammenfassung

Diese Dissertation behandelt die Möglichkeit den Flavorsektor des Standardmodells der Teilchenphysik (mit Neutrinomassen), das heißt die Massen und Mischungsmatrizen der Fermionen, mithilfe einer diskreten, nicht-abelschen Flavorsymmetrie zu beschreiben. Insbesondere werden massenunabhängige Texturen untersucht, bei denen ein oder mehrere Mischungswinkel allein durch die Gruppentheorie bestimmt werden. Zu diesem Zweck wird eine systematische Analyse einer großen Klasse von diskreten Symmetrien, der Diedergruppen, durchgeführt. Der Ursprung von massenunabhängigen Texturen, die aus solchen Symmetrien entstehen, wird untersucht und es wird gezeigt, dass solche Strukturen auf natürliche Weise aus der Minimierung von skalaren Potentialen folgt, wobei die Skalare Eichsingletts, sogenannte Flavons, sind, die nur unter der Flavorsymmetrie nicht-trivial transformieren. Aus diesen Vorgaben werden zwei Modelle konstruiert, eines beschreibt die Leptonen und beruht auf der Gruppe D_4 , das andere die Quarks und verwendet die Gruppe D_{14} . Im zweiten Modell ist es das Element V_{ud} der Quark-Mischungsmatrix - im wesentlichen der Cabibbo-Winkel - das in erster Näherung allein durch die Gruppentheorie vorhergesagt wird. Abschließend wird die Möglichkeit diskutiert, die diskrete Flavorgruppe als Untergruppe einer kontinuierlichen Eichsymmetrie zu beschreiben und es wird gezeigt, dass dafür die ursprüngliche Eichsymmetrie von verhältnismäßig großen Darstellungen gebrochen werden muss.

Abstract

This thesis deals with the possibility of describing the flavor sector of the Standard Model of Particle Physics (with neutrino masses), that is the fermion masses and mixing matrices, with a discrete, non-abelian flavor symmetry. In particular, mass independent textures are considered, where one or several of the mixing angles are determined by group theory alone and are independent of the fermion masses. To this end a systematic analysis of a large class of discrete symmetries, the dihedral groups, is analyzed. Mass independent textures originating from such symmetries are described and it is shown that such structures arise naturally from the minimization of scalar potentials, where the scalars are gauge singlet flavons transforming non-trivially only under the flavor group. Two models are constructed from this input, one describing leptons, based on the group D_4 , the other describing quarks and employing the symmetry D_{14} . In the latter model it is the quark mixing matrix element V_{ud} - basically the Cabibbo angle - which is at leading order predicted from group theory. Finally, discrete flavor groups are discussed as subgroups of a continuous gauge symmetry and it is shown that this implies that the original gauge symmetry is broken by fairly large representations.

Lark on the moon, singing -
sweet song
of non-attachment.

Bashō

Contents

1	Introduction	9
1.1	Flavor in the Standard Model	10
1.2	Neutrino Masses and Leptonic Mixing	14
2	Non-Abelian Discrete Flavor Symmetries	19
2.1	Theoretical Approaches to the Flavor Sector	19
2.2	Properties of Discrete Non-Abelian Flavor Symmetries	22
2.3	The D_4 Scaling Model - a Worked-Out Example	23
2.3.1	Phenomenological Considerations - Scaling	23
2.3.2	The Model	24
3	Dihedral Flavor Groups	27
3.1	Introduction to Dihedral Groups	27
3.1.1	Single-Valued Dihedral Groups	27
3.1.2	Double-Valued Dihedral groups	28
3.2	Non-Trivial Subgroups	29
3.3	Breaking Chains	32
3.4	Mass Matrices	33
3.4.1	General Results	34
3.4.2	Conventions and Notation	36
3.4.3	Three Singlet Structure	37
3.4.4	Two Doublet Structure	40
3.4.5	Mass Matrices in D'_n	46
3.4.6	Majorana Mass Matrices	47
3.5	Diagonalization and Mixing Matrices	49
4	Subgroup Mismatch in D_n	57
4.1	General Considerations	57
4.1.1	Derivation of the Non-Trivial Mixing Angle	57
4.1.2	Comparison with Phenomenology	60
4.2	A D_4 Example from the Literature	62
4.3	D_7 and the Cabibbo Angle - a Worked-Out Example	63
4.3.1	Quark Sector of the D_7 Model	63
4.3.2	Higgs Sector of the D_7 Model	64
5	Flavons and Flapon Potentials	67
5.1	Flapon Potentials in the SM	68
5.1.1	One Doublet	69
5.1.2	Several Doublets	71
5.2	Flapon Potentials in the MSSM	72

5.2.1	What Changes in Supersymmetry?	72
5.2.2	One Doublet	74
5.2.3	R-Symmetry and Driving Fields	76
5.2.4	Several Doublets	77
5.2.5	Further Aspects of Flavon Models	80
6	Putting It All Together:	
	Two Worked-Out Examples	83
6.1	D_4 in the Leptonic Sector	83
6.1.1	Lepton Masses and Mixing	84
6.1.2	Phenomenology	86
6.1.3	Flavon Superpotential	89
6.1.4	Next-to-Leading Order Corrections	90
6.2	D_{14} in the Quark Sector	93
6.2.1	Quark Masses and Mixing	95
6.2.2	Flavon Superpotential	101
6.3	Conclusions and Outlook	104
7	Outlook: Where Could Such a Symmetry Come From?	107
8	Summary and Conclusions	119
	Acknowledgements	123
A	Glossary	125
A.1	Theory of Discrete Groups	125
A.2	Abbreviations	127
B	Mathematical Details of Dihedral Groups	129
B.1	Kronecker Products	129
B.2	Clebsch Gordan Coefficients	130
B.2.1	...for D_n	130
B.2.2	...for D'_n	131
B.3	Real Representation Matrices for D_n	132
C	Results and Tables	135
C.1	Decomposition under Subgroups	135
C.2	Breaking Chains for D'_n	140
C.3	Possible Forms of V from Subgroup Mismatch	140
C.4	D_7 Higgs Potential	141
C.5	D_{14} Flavon Potential at Next-to-Leading Order	143
	Bibliography	150

Chapter 1

Introduction

Who ordered that? Isidor Isaac Rabi's exclamation upon the discovery of the muon is still unanswered, more than fifty years later. And to top it off: Whoever ordered the muon found that he/she liked it and decided to order a lot more.

Physicists may not have been able to answer the (possibly unanswerable) question concerning the fundamental reason for the existence of the muon. At least the discovery of a full set of three generations, that is three copies of each fundamental fermion occurring in ordinary atomic matter (including the neutrino), along with the precise experimental determination of the mass hierarchies and the mixing among these generations, have allowed us to at least formulate the question more precisely. The fundamental one remains: Why are there three generations of fermions? The most satisfying answer we can give to date is that three generations is the minimal number needed to allow for CP violation, a discovery for which the 2008 Nobel Prize in physics was awarded to Makoto Kobayashi and Toshihide Maskawa.

But even if we humbly accept that three generations of fermions exist, we still need to explain how the observed masses and mixings come to be. It is the answer to this question to which this doctoral thesis will attempt to contribute a small part.

As the title of this thesis already suggests, our approach to this question is based on symmetries. The symmetry principle is one of the foundations of the Standard Model of Particle Physics (SM). The symmetries of space-time and the internal gauge symmetries constrain the allowed terms in the Standard Model Lagrangian in such a way that only a small number of free parameters is left to be determined from experiment. The success of this approach is spectacular. And it is interesting that the SM has its weakest points where the symmetry principle is faced with its limitations. For one, the electroweak gauge symmetry is broken in nature. This breaking is performed in the SM through the Higgs Mechanism, which leaves many questions unanswered, both experimentally (the Higgs boson is the only particle in the SM which has not been observed in experiment) and theoretically (most notably the hierarchy problem).

There is also one aspect of the Standard Model which remains virtually untouched by the symmetry principle altogether: the flavor sector, that is the masses of and the mixing among the three fermion generations. This sector contains by far the majority of the free parameters in the SM Lagrangian and can be considered the part of the model that most resembles a semi-empirical hodgepodge, rather than a fundamental theory. Physicists have thus been looking for a symmetry governing flavor for quite a while. In this thesis we join this search by analyzing a large class of possible symmetries, the dihedral groups, with respect to their suitability as symmetries of flavor.

To go about this, we will start by reviewing the Standard Model of Particle Physics (SM), as far as it deals with flavor physics, in section 1.1. We review the origin of masses and mixing, which are determined by a large number of free parameters in the SM. We go on to discuss neutrino masses and mixing in section 1.2.

Chapter 2 is then devoted to presenting the general approach of this thesis towards the flavor puzzle. The experimental results presented in chapter 1 are analyzed in section 2.1. Some of the observed regularities cry out for an explanation, and we highlight some theoretical approaches towards explaining them, before, in section 2.2, presenting our method of choice, augmenting the SM by a discrete non-abelian flavor symmetry G_f and thereby constraining the free flavor parameters of the SM and finding deeper reasons for non-trivial correlations among them. To further illustrate this general description, we present a worked-out flavor symmetry model in section 2.3. From these considerations, we are led to a systematic approach to the problem of finding both a suitable group G_f and the transformation properties of the involved fields, based mainly on the existence of residual subgroups which remain exact after spontaneous breaking of the flavor symmetry.

We perform such a systematic analysis for the set of single-valued and double-valued dihedral groups in chapter 3. This analysis reveals a very interesting case of subgroup mismatch, discussed in chapter 4, first from a general perspective (section 4.1), then through the discussion of an instance, where it has been implicitly used in the literature (section 4.2), and finally along the lines of a worked-out example, in which this subgroup mismatch leads to the prediction of the Cabibbo angle (section 4.3). In chapter 5, we discuss how to separate the scales of electroweak and flavor symmetry breaking, leading to models with G_f broken at a high scale by gauge singlet flavon fields. The dynamics involved in this breaking are discussed in a general manner both for the SM (section 5.1) and for the Minimal Supersymmetric Standard Model (MSSM), easily the most popular candidate for physics beyond the SM (section 5.2). We obtain the result that the conservation of the subgroups necessary for the subgroup mismatch, can be obtained as a prediction from minimizing very general scalar potentials invariant under G_f . All the elements gathered in the preceding sections are then put together in two exemplary models in chapter 6. The first model is based on the group D_4 and uses the subgroup mismatch to predict maximal atmospheric mixing in the lepton sector (section 6.1), the second one uses D_{14} for a prediction of the Cabibbo angle (section 6.2).

We finish with an outlook on the possible origin of a flavor symmetry G_f in chapter 7, focussing on the possibility that the discrete flavor group may be a subgroup of a spontaneously broken gauge symmetry, and finally offer our conclusions in chapter 8. A glossary of abbreviations and group theory terms is given in appendix A, while additional mathematical information on the dihedral groups can be found in appendix B and several bulky results have been moved to appendix C. Parts of this thesis have already been published in [1–5].

1.1 Flavor in the Standard Model

The SM contains 45 Weyl fermions, all of which transform non-trivially under the gauge group of the SM, $SU(3)_c \times SU(2)_L \times U(1)_Y$. They can be divided into three generations of 15 Weyl fermions each. The transformation properties under the gauge group repeat themselves in each generation and are given, for the first generation, in table 1.1. Only the names change for the second generation, with (c, s, ν_μ, μ) instead of (u, d, ν_e, e), and (t, b, ν_τ, τ) for the third generation. When talking about all three generations, we will often alternatively write $u_i^{(c)}$, $d_i^{(c)}$, ν_α , e_α and e_i^c with $i = 1, 2, 3$ and $\alpha = e, \mu, \tau$, since for left-handed leptons the numerical indices are reserved for the mass eigenstates, as we will discuss below. We will also use the notation Q_i and l_α for the quark and lepton $SU(2)_L$ doublets, respectively.

Fermion	$SU(3)_c$	$SU(2)_L$	I_3	$U(1)_Y$
$\begin{pmatrix} u \\ d \end{pmatrix}$	$\mathbf{\underline{3}}$	$\mathbf{\underline{2}}$	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{3}$
u^c	$\mathbf{\overline{3}}$	$\mathbf{\underline{1}}$	0	$-\frac{4}{3}$
d^c	$\mathbf{\overline{3}}$	$\mathbf{\underline{1}}$	0	$\frac{2}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}$	$\mathbf{\underline{1}}$	$\mathbf{\underline{2}}$	$\frac{1}{2}$ $-\frac{1}{2}$	-1
e^c	$\mathbf{\underline{1}}$	$\mathbf{\underline{1}}$	0	2

Table 1.1: The transformation properties of the first generation of SM fermions under the SM gauge group in the Grand Unified Theory (GUT) notation, where all fermions are written as left-handed Weyl spinors. The $SU(3)_c$ multiplet structure has been suppressed.

None of these fermions can have a gauge invariant mass term in the Lagrangian. Mass is generated by Yukawa couplings to the Higgs scalars $H = \begin{pmatrix} H_0 \\ H^- \end{pmatrix}$ or their conjugate $\epsilon H^* = \begin{pmatrix} -H^+ \\ H_0^* \end{pmatrix}$, with transformation properties $(\mathbf{1}, \mathbf{2}, \mp 1)$ under the SM gauge group:

$$-\mathcal{L}_Y = Y_{ij}^u (u_i u_j^c H_0^* + d_i u_j^c H^+) + Y_{ij}^d (d_i d_j^c H_0 - u_i d_j^c H^-) + Y_{\alpha j}^e (e_\alpha e_j^c H_0 - \nu_\alpha e_j^c H^-) + h.c. \quad (1.1)$$

Note that we have in this work adopted the basis in which the left-handed fermions are associated with the rows of the Yukawa coupling matrices, the left-handed conjugate fermions with the columns.

Due to its multiply named potential ("Mexican hat", "wine bottle")

$$V = -\mu^2 (|H^0|^2 + |H^-|^2) + \lambda (|H^0|^2 + |H^-|^2)^2, \quad (1.2)$$

with $\mu, \lambda > 0$, the Higgs boson obtains a non-vanishing vacuum expectation value (VEV) $\begin{pmatrix} v_{\text{wk}} \\ 0 \end{pmatrix}$, $v_{\text{wk}} = \sqrt{\frac{\mu^2}{2\lambda}}$, breaking the SM gauge group down to $SU(3)_c \times U(1)_{em}$. The transformation properties of the SM fields under the residual $U(1)_{em}$, i.e. the electric charge, is given by the Gell-Mann Nishijima formula

$$Q = I_3 + \frac{Y}{2}. \quad (1.3)$$

At the same time, the Higgs VEV gives Dirac masses to all SM fermions, except for the neutrinos, with mass matrices

$$(M_L)_{ij} = v_{\text{wk}} Y_{ij}^L, \quad (1.4)$$

where L is u , d or e . In the latter case mixed Greek and Latin indices should be used. If, as we have conceded, we accept for the time being that there are three generations, we are faced with the question of how to determine the elements of the mass matrices, as both the Higgs VEV and the Yukawa couplings are free parameters in the SM. The first question is of course, how they can be determined experimentally, that is first of all, in which observables they appear.

The overall scale of the fermion masses v_{wk} can directly be calculated from the Fermi constant G_F , as determined in measurements of the muon lifetime. This is because the Higgs VEV also gives mass to the electroweak gauge bosons, to which it couples in a gauge invariant way,

i.e. without the large number of free parameters encountered in its coupling to fermions. The electroweak scale v_{wk} is thus determined to be $(2\sqrt{2}G_F)^{-\frac{1}{2}} \approx 174 \text{ GeV}$. Its origin is to be found in the precise dynamics of electroweak symmetry breaking, with which we will not be concerned in this thesis.

With the overall scale given, the question is how to measure the dimensionless Yukawa couplings. We can never measure the mass matrices directly. We can however measure the masses, which correspond to the square roots of the eigenvalues of the squared mass matrices MM^\dagger or $M^\dagger M$, that is the singular values of M . For the charged leptons and the top quark the determination of the masses is pure kinematics. All other quarks will hadronize before decaying (if they decay at all), and the actual (current) quark masses need to be extracted from measurable hadronic properties, by the means of Quantum Chromodynamics (QCD) calculations on the lattice or effective field theories, such as chiral perturbation theory, heavy quark effective theory or non-relativistic QCD. The fermion masses depend on the momentum scale, due to renormalization group running. We have chosen to give all fermion masses at the scale of the Z boson mass, $M_Z \approx 91.2 \text{ GeV}$. One observes a large hierarchy between the masses of the three generations [6]:

$$\begin{aligned} m_u(M_Z) &= 1.27_{-0.42}^{+0.50} \text{ MeV}, & m_c(M_Z) &= 0.619_{-0.084}^{+0.084} \text{ GeV}, & m_t(M_Z) &= 171.7_{-3.0}^{+3.0} \text{ GeV}, \\ m_d(M_Z) &= 2.90_{-1.19}^{+1.24} \text{ MeV}, & m_s(M_Z) &= 55_{-15}^{+16} \text{ MeV}, & m_b(M_Z) &= 2.89_{-0.09}^{+0.09} \text{ GeV}, \\ m_e(M_Z) &= 0.486570161_{-0.000000042}^{+0.000000042} \text{ MeV}, & m_\mu(M_Z) &= 102.7181359_{-0.00000092}^{+0.00000092} \text{ MeV}, \\ m_\tau(M_Z) &= 1746.24_{-0.19}^{+0.20} \text{ MeV}. \end{aligned} \tag{1.5}$$

One can already see from the precision of these numbers what a difficult task the extraction of quark masses is, compared to determining lepton masses.

For a full knowledge of the Yukawa couplings, the eigenvectors of the fermion mass matrices are needed. These can partially be observed in the weak interaction, due to the fact that the weak basis, i.e. which pairs of quarks form a doublet under $SU(2)_L$, no longer coincides with the mass bases, the bases in which the Yukawa couplings and hence the fermion mass matrices are diagonal. Due to the universality of the weak interaction, one is free to choose the weak basis and one usually takes it to coincide with the mass basis for the up-type quarks, such that all mixing originates from the down-type quark sector. The mass basis d'_j is then related to the weak basis d_i by a unitary transformation U_d . To be more precise, if

$$U_d^\dagger M_d M_d^\dagger U_d = \text{diag}(m_d^2, m_s^2, m_b^2), \tag{1.6}$$

that is

$$U_d^\dagger M_d U_{d^c} = \text{diag}(m_d, m_s, m_b), \tag{1.7}$$

where U_{d^c} is an unobservable unitary transformation of the left-handed conjugate down quarks, then $d_i = (U_d^*)_{ij} d'_j$. We can thus reformulate the charged current interaction term in the Lagrangian, expressed using the weak eigenstates u_i and d_i ,

$$-\frac{g}{\sqrt{2}} [u_i^* \bar{\sigma}^\mu d_i] W_\mu^+ + \text{h.c.}, \tag{1.8}$$

in terms of the mass eigenstates u'_i and d'_i ,

$$-\frac{g}{\sqrt{2}} \left[(u'_i)^* \bar{\sigma}^\mu (V_{CKM})_{ij} d'_j \right] W_\mu^+ + \text{h.c.}, \tag{1.9}$$

where $\bar{\sigma}^\mu$ is a four vector of two-by-two matrices given by $(\mathbb{1}, -\vec{\sigma})$, with $\vec{\sigma}$ the three-vector of Pauli matrices. We see that the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix is

equal to U_d^* . In general, as in the models presented later on, the weak basis need not be fixed by the up quark mass basis. If the two bases are not identical, one needs to introduce a second unitary transformation U_u , with

$$U_u^\dagger M_u M_u^\dagger U_u = \text{diag}(m_u^2, m_c^2, m_t^2) \quad (1.10)$$

and consequently $u_i = U_u^* u'_i$, and the CKM quark mixing matrix is given by the general formula

$$V_{CKM} = U_u^T U_d^* = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (1.11)$$

The elements of the CKM matrix are measurable in processes involving the weak interaction. They contain all the observable information on the eigenvectors of the quark Yukawa matrices. The CKM matrix is usually parameterized in the following way:

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (1.12)$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. The mixing angle among the first two generations θ_{12} is also called the Cabibbo angle θ_C . All mixing angles are restricted to lie in the first quadrant, such that all sines and cosines are positive, while the Kobayashi-Maskawa (KM) phase δ may lie between 0 and 2π . An alternate parameterization of the CKM matrix, which focusses more strongly on the hierarchy among the mixing matrix elements, is the Wolfenstein parameterization,

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \quad (1.13)$$

The two parameterizations are related by

$$s_{12} = \lambda, \quad s_{23} = A\lambda^2, \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta). \quad (1.14)$$

The Wolfenstein parameterization is mainly characterized by the small expansion parameter λ , which is, as can be inferred from comparing the two parameterizations, equivalent to $\sin \theta_C$. The absolute values of the CKM matrix elements can be extracted from measurements of many processes, mainly involving decay processes (of hadrons or the top quark), but also meson mixing or deep inelastic neutrino scattering. They can then further be constrained by imposing certain relations among the matrix elements, arising from unitarity. The fit results determined in this way are [7]

$$|V_{CKM}| = \begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415_{-0.0011}^{+0.0010} \\ 0.00874_{-0.00037}^{+0.00026} & 0.0407 \pm 0.0010 & 0.999133_{-0.000043}^{+0.000044} \end{pmatrix}. \quad (1.15)$$

While the magnitudes of the CKM elements are parameterization independent, the phases are not. Instead of giving the phase, one therefore usually uses a parameterization independent measure of CP violation, the Jarlskog invariant J_{CP} , given by

$$J_{CP} \sum_{m,n} \epsilon_{ikm} \epsilon_{jln} = \text{Im}[V_{ij} V_{kl} V_{il}^* V_{kj}^*]. \quad (1.16)$$

It can be extracted from measurements of CP violation and asymmetry in meson mixing and decays. The current experimental value is [7]

$$J_{CP} = (3.05_{-0.20}^{+0.19}) \times 10^{-5}. \quad (1.17)$$

This is then all the information that can experimentally be obtained on the Yukawa matrices in the SM. It already gives us a large number of free parameters in the flavor sector: 9 masses, 3 mixing angles and one CP violating phase (disregarding the Higgs VEV, which is not really a part of the flavor sector) for a total of 13 free parameters, easily the majority of free parameters in the SM. In the SM the eigenvectors of the charged lepton mass matrix are not even partially observable: due to the mass degeneracy of the neutrinos (all three have vanishing mass) they have no defined mass basis. One is thus free to choose the weak basis to correspond with the charged lepton mass basis and no leptonic mixing shows up in the weak interaction. Experiments in the last decade have however been able to observe leptonic mixing and thus proven that neutrinos do have (non-degenerate) masses. This has not been incorporated into a new Standard Model, since the exact nature of neutrino mass generation, in particular whether they are Dirac or Majorana fermions, remains undetermined. For this reason, we discuss neutrino masses and leptonic mixing in a separate section, the next one.

1.2 Neutrino Masses and Leptonic Mixing

Starting its measurements in 1970, the Homestake experiment was the first to establish the solar neutrino problem [8], that is a flux of electron neutrinos from the sun that was significantly smaller than expected from the Standard Solar Model. By the early 90s, the experiments KamiokaNDE II and IMB, both experiments actually designed to measure proton decay, also showed first hints towards a deficit in the flux of atmospheric muon neutrinos [9, 10]. In 1998 then, Super-KamiokaNDE provided the first evidence of atmospheric neutrino oscillations [11]. This was followed in 2001 by direct evidence of solar neutrino oscillations in the SNO experiment [12], where the electron neutrinos missing in the solar neutrino flux were detected as neutrinos of other flavors.

Neutrino oscillations can only occur if neutrinos have non-degenerate masses, i.e. if at least two of the three neutrino masses are non-vanishing. In this case a neutrino mass basis is defined by the neutrino mass matrix and can no longer be chosen to coincide with the mass basis of the charged leptons and the weak basis, regardless of the actual mechanism of neutrino mass generation. We thus have to introduce a unitary transformation U_ν relating the neutrino mass basis and the weak basis, i.e. $\nu_\alpha = (U_\nu^*)_{\alpha i} \nu_i$, where the ν_i with $i = 1, 2, 3$ are the neutrino mass eigenstates. In general neutrinos with Latin indices denote mass eigenstates. This again effects the charged current weak interaction, which then reads

$$-\frac{g}{\sqrt{2}} [e_\alpha^* \bar{\sigma}^\mu (V_{PMNS})_{\alpha i} \nu_i] W_\mu^- + \text{h.c.}, \quad (1.18)$$

where the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $V_{PMNS} = U_\nu^*$ is the leptonic analog of the CKM matrix. Note however, that it is defined in a complementary fashion, as it is the weak isospin $I_3 = -\frac{1}{2}$ field that is chosen to have its mass basis correspond to the weak basis, in contrast to the quark sector. In the leptonic sector, we will also later be departing from convention and define a weak basis that is neither the charged lepton nor the neutrino mass basis. In this case another unitary transformation is introduced, $e_\alpha = (U_e^*)_{\alpha i} e_i$, with e_i the charged lepton mass basis, which is determined by

$$U_e^\dagger M_e M_e^\dagger U_e = \text{diag}(m_e^2, m_\mu^2, m_\tau^2). \quad (1.19)$$

The general formula for the PMNS matrix is then

$$V_{PMNS} = U_e^T U_\nu^* = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix}. \quad (1.20)$$

We still need to calculate U_ν . For this we need to know the neutrino mass matrix, and hence need to make some statement on the origin of neutrino mass. The first question we need to answer is whether the neutrino has a Majorana or a Dirac mass. Dirac masses are conceptually simpler, since they are completely analogous to the masses of the other SM fermions. We need to introduce three additional Weyl fermions ν_α^c (two are actually sufficient, since one of the neutrinos can be massless) transforming as total singlets under the SM gauge group. We can then write down additional Yukawa couplings

$$-\mathcal{L}_D = Y_{\alpha\beta}^\nu (\nu_\alpha \nu_\beta^c H_0^* + e_\alpha \nu_\beta^c H^+) + h.c., \quad (1.21)$$

which lead to a neutrino mass matrix

$$M_\nu = v_{\text{wk}} Y^\nu \quad (1.22)$$

and a neutrino diagonalization matrix U_ν defined by

$$U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = \text{diag}(m_1^2, m_2^2, m_3^2). \quad (1.23)$$

However, one has to take into account that the charge-conjugated left-handed neutrinos are total SM singlets, and thus a direct (Majorana) mass term is allowed for them. In the case of Dirac neutrinos, this mass term needs to be forbidden, for example by imposing conservation of lepton number. If it does exist it takes the form

$$-\mathcal{L}_M = \frac{1}{2} \nu_\alpha^c (M_{\nu^c})_{\alpha\beta} \nu_\beta^c + h.c. \quad (1.24)$$

Since the elements of M_{ν^c} are not related to the electroweak scale, they will be large in general. We can then write down an effective theory, in which the heavy Majorana neutrinos, which are basically identical with the ν^c , are integrated out. Only three light Majorana neutrinos remain in the theory: These are then the left-handed neutrinos, ignoring small admixtures. Their Majorana mass matrix is given by

$$M_\nu = -v_{\text{wk}}^2 (Y^\nu)^T M_{\nu^c}^{-1} Y^\nu. \quad (1.25)$$

Since we integrate out the ν^c , we could just take the agnostic approach of writing down only an effective operator to begin with. The general effective, non-renormalizable operator giving a Majorana mass to the left-handed neutrinos is

$$\frac{1}{\Lambda} \tilde{Y}_{\alpha\beta}^\nu (\nu_\alpha H_0^* + e_\alpha H^+) (\nu_\beta H_0^* + e_\beta H^+), \quad (1.26)$$

where Λ here and in the following always denotes the high energy scale at which an effective operator is generated. Equation (1.25) is then only one, albeit very plausible, possible origin for the light neutrino Majorana mass matrix. If the light neutrino Majorana mass matrix is generated in this way, one speaks of a Type I Seesaw model [13–15]. Another popular possibility for generating a Majorana mass matrix M_ν is the introduction of a scalar $SU(2)_L$ triplet which acquires a small VEV. Combining this with a Type I Seesaw model leads to a mixed or Type II Seesaw model [16]. In any case low energy phenomenology is determined solely by M_ν . For Majorana neutrinos, the neutrino diagonalization matrix is then determined by

$$U_\nu^\dagger M_\nu U_\nu^* = \text{diag}(m_1, m_2, m_3). \quad (1.27)$$

The parameterization of the resulting PMNS matrix depends on whether neutrinos are Dirac or Majorana particles. For Dirac neutrinos, the analogy to the quark case is complete, and we can parameterize V_{PMNS} in the same way we parameterized V_{CKM} in equation (1.12). Since we now have two sets of mixing angles, we will sometimes write θ_{ij}^q and θ_{ij}^l for the mixing angles of quarks and leptons, respectively. For Majorana neutrinos we have two phases, which cannot be removed by a rephasing of the fields, for the simple reason that we have less fields to rephase than in the quark sector. Even if we introduce left-handed conjugate neutrinos in a Type I Seesaw setup, rephasing them has no effect on the light neutrino mass matrix. We thus parameterize the PMNS matrix as

$$V_{PMNS} = \tilde{V} \left(\theta_{12}^l, \theta_{23}^l, \theta_{13}^l, \delta^l \right) \cdot \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.28)$$

where \tilde{V} is parameterized in the same way as the CKM matrix, and ϕ_1 and ϕ_2 are the Majorana phases. They lie in the interval $[0, \pi)$.

Information on the singular values and the eigenvectors of M_ν can be obtained from experiment, regardless of the nature of neutrinos, be it Majorana or Dirac. As opposed to the quark case, where mixing matrix elements are measured mainly in decay processes and masses are extracted from the properties of bound states, information on the neutrino mass matrix is obtained from oscillation experiments. Two properties can be measured: The mixing angles between generations, which appear in the standard parameterization of V_{PMNS} , and the differences of the squared neutrino masses.

The leading solar oscillation parameters are $\Delta m_{21}^2 = m_2^2 - m_1^2$ and correspondingly $\sin^2 \theta_{12}$. They are measured in oscillation experiments with solar neutrinos, such as SNO, and in terrestrial experiments using reactor antineutrinos, such as KamLAND [17]. The leading atmospheric oscillation parameters are $\Delta m_{31}^2 = m_3^2 - m_1^2$ and $\sin^2 \theta_{23}$, obtained initially in oscillation experiments with atmospheric neutrinos, such as SuperKamiokaNDE, but also in terrestrial long baseline experiments observing the disappearance of muon neutrinos, such as MINOS [18]. The third mixing angle is measured mainly in experiments with reactor neutrinos, but with a shorter source-detector distance than in KamLAND, and is therefore sometimes called the reactor angle or the CHOOZ angle, after one of the main experiments of this type [19]. The best-fit values as well as the allowed 1σ and 3σ ranges of all the oscillation parameters are

$$\begin{aligned} \sin^2 \theta_{12} &= 0.304_{-0.016, -0.054}^{+0.022, +0.066}, \\ \Delta m_{21}^2 &= \left(7.65_{-0.20, -0.60}^{+0.23, +0.69} \right) \times 10^{-5} \text{ eV}^2, \\ \sin^2 \theta_{23} &= 0.50_{-0.06, -0.14}^{+0.07, +0.17}, \\ |\Delta m_{31}^2| &= \left(2.40_{-0.11, -0.33}^{+0.12, +0.35} \right) \times 10^{-3} \text{ eV}^2, \\ \sin^2 \theta_{13} &= 0.01_{-0.011}^{+0.016} (\leq 0.056). \end{aligned} \quad (1.29)$$

These numbers are all taken from a recent global analysis [20]. In contrast to the quark sector, some essential parameters are still entirely unknown. To begin with, the sign of Δm_{31}^2 has not been measured, and therefore both $m_3 > m_{1,2}$ (normal hierarchy) and $m_3 < m_{1,2}$ (inverted hierarchy) are still possible. Also, only mass differences are known so far, the absolute mass scale of neutrinos remains undetermined. Lower limits can obviously be obtained from the squared mass differences themselves. An upper bound on the effective electron neutrino mass,

$$m_\beta = \sqrt{\sum_k |(V_{PMNS})_{ek}|^2 m_k^2} < 2.3 \text{ eV} \quad (1.30)$$

at 95 % C.L., was obtained in the Mainz experiment by measuring the kinematics of tritium beta decay [21]. The KATRIN experiment will improve this bound by almost an order of magnitude, using the same general principle [22]. Since relic neutrinos act as hot dark matter (HDM) in the universe, bounds on the energy density of HDM translate into a bound for the sum of neutrino masses. The exact bound depends on what other astrophysical and cosmological data are considered and what models are used for dark energy. A fairly conservative estimate is [23]

$$\sum_j m_j \leq 0.80 \text{ eV} \quad (1.31)$$

at 95 % C.L. Finally, neutrinoless double beta decay not only provides a measure of the absolute neutrino mass scale, but, if observed, provides an unambiguous proof of the Majorana nature of the neutrino. The current upper limit on the effective mass measured in this process comes from the Heidelberg-Moscow experiment [24]:

$$|m_{ee}| = \left| \sum_k (V_{PMNS})_{ek}^2 m_k \right| < (0.35 - 1.2) \text{ eV} \quad (1.32)$$

at 90 % C.L., where the range is due to uncertainties from the calculation of the involved nuclear matrix elements [7]. Claims to evidence for neutrinoless double beta decay [25] are controversial and have yet to be confirmed. In any case, experiments of the next generation, such as GERDA, will improve this limit to about 0.1 eV or measure a non-vanishing m_{ee} [26]. There are also experiments, coming up or already taking data, that will improve the measurements of the oscillation parameters, for example Borexino, which started taking its data on solar neutrinos in 2008 [27], or Double Chooz [28], which will improve the limit on $\sin \theta_{13}$.

We finally mention, that the number of light neutrinos, that also interact weakly, is limited to three by the measurement of the Z boson decay width at LEP [7]. This does not however exclude the possibility of sterile neutrinos, which could mix with regular neutrinos after electroweak symmetry breaking. Measurements by the LSND experiment seemed to indicate oscillations involving a large squared mass splitting [29], which could be explained by the introduction of at least one sterile neutrino. The subsequent MiniBoone experiment however could not confirm these specific findings [30]. There is thus currently no conclusive evidence towards the existence of more than the three known neutrino species.

We have collected in this first chapter the data and observations currently available to guide us towards something like a model of flavor. In the next section, we begin by putting this data into perspective, pointing towards regularities and structures which such a model might be or should be able to explain.

Chapter 2

Non-Abelian Discrete Flavor Symmetries

2.1 Theoretical Approaches to the Flavor Sector

Until the discovery of neutrino oscillations and the measurement of the leptonic mixing angles, the most striking feature of the parameters of the flavor sector was certainly the hierarchy among the masses of the three generations. Early models of the flavor sector thus tend to focus strongly on this aspect. For the quarks for example we have

$$\begin{aligned} m_u : m_c : m_t &\sim \lambda^8 : \lambda^4 : 1, \\ m_d : m_s : m_b &\sim \lambda^4 : \lambda^2 : 1, \\ m_b : m_t &\sim \lambda^2 : 1, \end{aligned} \tag{2.1}$$

where $\lambda \sim 0.2$ is the small expansion parameter suggested by the Wolfenstein parameterization of the CKM matrix. We will be using it as a generic measure for small expansion parameters throughout this thesis. To explain this hierarchy, one needs a theory that determines the magnitude of the Yukawa couplings in the SM Lagrangian.

The simplest ansatz is to set some of the Yukawa couplings, and thereby some of the mass matrix elements, to zero. This is referred to as choosing certain textures for the Yukawa matrices. An overview of phenomenologically viable textures is given in [31]. As these texture zeroes are not motivated by symmetries or some other principle, they are theoretically not very appealing. Texture studies however can give guidance as to what Yukawa matrix structures are favorable and should be reproduced by more involved models. Such studies have therefore also been performed for Majorana [32] and Dirac [33] neutrinos.

An explanation for the mass hierarchies from a symmetry was provided by Froggatt and Nielsen (FN) [34]. The additional symmetry is a $U(1)_{FN}$. It is a horizontal symmetry, meaning that under this symmetry the different generations transform differently, as opposed to all the known symmetries of the SM, which are then correspondingly called vertical. Considering for example the down quark sector, one can assign charges q_i^L to the Q_i and q_i^R to the d_i^c , where all charges are either positive integers or zero. Then the only non-zero Yukawa couplings Y_{ij}^d are those for which $q_i^L = q_j^R = 0$, as long as the $U(1)_{FN}$ is an exact symmetry. If this FN symmetry is however broken spontaneously by the VEV of a gauge singlet scalar θ with $U(1)_{FN}$ charge of -1, the additional Yukawa couplings are generated by effective, non-renormalizable operators

$$\tilde{Y}_{ij} Q_i d_j^c H \left(\frac{\theta}{\Lambda} \right)^{q_i^L + q_j^R}, \tag{2.2}$$

where the couplings \tilde{Y}_{ij} are assumed to be of order 1, while $\frac{\langle\theta\rangle}{\Lambda}$ must be a small parameter $\epsilon \ll 1$. We then obtain a hierarchy among the low energy effective Yukawa couplings $Y_{ij} = \tilde{Y}_{ij}\epsilon^{q_i^L+q_j^R}$ if we choose the $U(1)_{FN}$ charges accordingly. This translates directly into a hierarchy of fermion masses. We use the FN mechanism in the models of this thesis to (partially) generate the quark mass hierarchy. An alternative approach is to explain the mass hierarchy using wave function overlap and localization in extra dimensions [35]. As none of the models presented in this thesis makes use of extra dimensions, we will not be using such a mechanism. Apart from the fermion mass hierarchy, other regularities have been noted. Two examples are the Koide lepton mass relation [36]

$$\sqrt{m_e + m_\mu + m_\tau} = \sqrt{\frac{2}{3}} (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}), \quad (2.3)$$

and the Georgi-Jarlskog relations, combining charged lepton and down-type quark masses [37]

$$m_b = m_\tau, \quad 3m_s = m_\mu, \quad m_d = 3m_e. \quad (2.4)$$

As these two relations rely on the exact values of the masses, rather than on a more general hierarchy, they are in general only true at one particular energy scale. For the Georgi-Jarlskog relations this is a high energy scale, around the scale at which one expects the unification of coupling constants in a Grand Unified Theory (GUT). A GUT, in which charged leptons and down-type quarks are unified in a single multiplet, such as $SU(5)$, is also the adequate theoretical framework to obtain this relation. The Koide mass relation on the other hand is true for the pole masses of the leptons, i.e. the actual masses obtained from kinematics. No attempt at deriving the Koide mass relation from a symmetry has been entirely successful, as all models need to artificially eliminate allowed terms in the Lagrangian [38].

Taking mass hierarchy as the fundamental attribute of the SM flavor sector, the quark mixing angles arise as a byproduct. There exist several phenomenologically successful expressions which relate mixing angles with quark mass ratios and can be obtained from specific mass matrix textures [39–43], for example the Gatto-Sartori-Tonin (GST) [39] relation

$$\theta_C \approx \sqrt{\frac{m_d}{m_s}}, \quad (2.5)$$

which explains the smallness of the mixing among the first two generations by the mass hierarchy among them. The full GST relation includes another contribution from the up-type quarks, which is however subleading, as the mass hierarchy is more pronounced in that sector, cf. equation (2.1).

The measurement of the large leptonic mixing angles and neutrino masses substantially changed the flavor picture. Even though we do not yet know the absolute neutrino mass scale, the mass squared differences already tell us that the neutrino mass hierarchy is by no means as pronounced as for the charged fermions¹. On the other hand, two of the mixing angles are large. Even more, the measured leptonic mixing angles are all compatible with quite special values, $\sin^2\theta_{23} = \frac{1}{2}$, $\sin^2\theta_{12} = \frac{1}{3}$ and $\sin^2\theta_{13} = 0$. If these values were exact, the leptonic mixing matrix would take the form [44]

$$V_{PMNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (2.6)$$

¹The lightest neutrino can be massless, so that the relative hierarchy between the lightest and the next-to-lightest neutrino can be arbitrarily large. The relative hierarchy between the two heavier neutrinos, however, is limited by the squared mass differences.

where we have suppressed possible Majorana phases. This special form of the mixing matrix is called the Harrison-Perkins-Scott (HPS) mixing matrix or the tri-bimaximal mixing (TBM) matrix and begs explanation. Even if we allow for $\sin^2 \theta_{12} \neq \frac{1}{3}$, we are still left with the striking fact of one maximal and one vanishing mixing angle, with a third mixing angle large but non-maximal. The corresponding mixing pattern (of which tri-bimaximal mixing is a special case) is hence called bi-large. Since the interchange of the second and the third row (corresponding to the indices μ and τ) only results in sign changes, a setup with such a mixing matrix is also called μ - τ -symmetric.

These mixing angles do not look as if they arise from fermion mass ratios. We rather expect that these mixing angles arise from symmetries of the mass matrices that are independent of the eigenvalues, so-called mass independent textures [45]. This also leads to a re-evaluation of the quark mixing angles. On the one hand, there may be a non-trivial relation with the leptonic mixing angles, such as

$$\theta_{12}^l + \theta_C \approx \frac{\pi}{4}, \quad (2.7)$$

which is referred to as Quark-Lepton Complementarity (QLC) [46]. On the other hand, the Cabibbo angle may also be described through a specific value, rather than through order of magnitude estimations [2, 47, 48]:

$$\theta_C \approx \frac{\pi}{14}. \quad (2.8)$$

This implies that quark mixing may also be described by mass independent textures, and we use this relation later on in this thesis. New theoretical approaches needed to be found, more suited to predicting mixing patterns than hierarchical masses. Symmetries are again the method of choice, but it needs to be reconsidered what kind of symmetries should be used. Abelian symmetries, such as the $U(1)_{FN}$, have only one-dimensional representations. They are well suited for describing mass hierarchies: Each generation is treated separately and assigned a charge determining its place in the hierarchy. Small mixings arise as a byproduct. In predicting specific mixing patterns among generations however, it is the relationship among the generations which becomes more important than the specific properties of each generation by itself. We therefore need to move to higher-dimensional representations, in which two or three generations are unified. These only exist in non-abelian groups.

This leaves the question of whether the horizontal flavor group should be continuous or discrete. As the dimension of the irreducible representations (irreps) of continuous groups grows quite rapidly with the size of the group, only very few groups are really able to accommodate the three fermion generations without introducing new fermionic degrees of freedom. Taking into account that² $U(2) \cong SU(2) \times U(1)$ and $U(3) \cong SU(3) \times U(1)$, we have only three such groups, all of which have been employed as flavor symmetries: $SU(2)$ [50, 51], $SO(3)$ [52, 53] and $SU(3)$ [54, 55], where $SU(2)$ and $SO(3)$ are of course very closely related, the first being the double covering group of the second. Although these models easily predict large mixing, they have problems predicting specific values for the mixing angles: A very elaborate vacuum alignment is needed to reproduce TBM, for example. This can easily be understood from the fact that in a continuous flavor symmetry no particular direction in flavor space is singled out - we are free in our choice of axes. This is different in discrete flavor groups - here we have a finite number of elements mapped by a finite number of representations to a finite number of representation matrices with finitely many eigenvectors. Thus, a certain number of directions in flavor space are singled out, like the axes of a crystal. This will play an important role in the

²Actually, in both cases a cyclic group needs to be modded out, just as the SM gauge group is really $SU(3) \times SU(2) \times U(1)/Z_6$ [49]. This is however irrelevant for our purposes.

analysis of chapter 3. But first we will describe some general properties of non-abelian discrete flavor symmetries.

2.2 Properties of Discrete Non-Abelian Flavor Symmetries

When designing a model of flavor using a horizontal symmetry group G_f , the first task is obviously to pick a group. We have already elaborated on why we want to choose G_f non-abelian and discrete. What further properties should G_f have? We want to accommodate the three known fermion generations in irreps of G_f and also want to make use of the non-abelianity of G_f , that is we want to unify at least two generations in an irrep. If we do not want to introduce new fermionic degrees of freedom, this means that we must demand that G_f has at least one two- or three-dimensional representation. This is already quite a strong restriction, although it is by no means as strong as in the case of continuous groups, where it limited us to three groups. In the discrete case, there is still quite a large (in fact infinite) set of groups to choose from, and we begin by giving a brief overview of groups that have been used as flavor groups in the literature.

The most popular choice for G_f is probably A_4 [56–62], the group of even permutations of four distinct objects or equivalently the symmetry group of the tetrahedron. The reason for its popularity stems from the fact that it is ideally suited for predicting tri-bimaximal mixing. We give more details on the mechanism responsible for this, when we consider the breaking of G_f below. For now, we also mention that the double-valued group of A_4 , T' , has been employed to extend a model predicting TBM for leptons to the quark sector [63]. Also recent studies point towards the fact that the special properties of A_4 are no coincidence, but are related to the fact that A_4 is a maximal subgroup of S_4 , the group of permutations of four distinct objects. In fact it can be shown that, under certain restrictions, predicting TBM specifically requires G_f to be (or contain) S_4 [64], which has also been studied as a flavor symmetry in its own right [65, 66]. Of the other alternating and permutation groups, only very few are eligible candidates for a flavor symmetry. $A_3 \cong Z_3$ is abelian, while the A_n with $n \geq 6$ have no more two- or three-dimensional irreps. This leaves A_5 , which has rarely been studied, but may be of interest for describing the solar angle with the golden ratio [67]. The situation is similar for the permutation groups: $S_2 \cong Z_2$ is abelian, the S_n with $n \geq 5$ have no small irreps, leaving S_3 which is isomorphic to the dihedral group D_3 and will thus be included in our study in the next chapter. It is the smallest non-abelian group and has thus been very popular as a flavor symmetry [68–79]. There are however sequences of groups, where the groups corresponding to large integers are also viable flavor symmetry candidates, most notably the dihedral symmetries, D_n and their double-valued counterparts the D'_n . They have been used extensively in the literature [33, 80–97], most existing models however stick to a small n . We will discuss the dihedral groups extensively in section 3. Other interesting sequences of groups are subgroups of $SU(3)$, the groups $\Delta(3n^2)$ and $\Delta(6n^2)$, which all have a representation content compatible with being a flavor symmetry. Their mathematical structure has been studied in [98, 99], a more phenomenological study along the lines of the analysis presented in this thesis for the dihedral groups however does not yet exist. These subgroups of $SU(3)$ have gained some popularity as flavor groups [100–102]. There are also other groups, which have been used as flavor groups, but are not part of a neat sequence, such as for example the groups T_7 or $\Sigma(81)$ [103–105].

We see that there exists quite a large selection of flavor groups. Before focussing on a specific subset, the dihedral groups, we will continue by discussing the general properties of models incorporating a flavor symmetry. After choosing a group, representations need to be assigned to the SM fermions and the charge-conjugated neutrinos ν^c , if they are present in the model. Just as for the FN symmetry, this will force most, if not all, of the Yukawa couplings Y_{ij}^L to be zero. To obtain phenomenologically viable mixing patterns, we need to break G_f spontaneously

with scalar VEVs. There are in principle two ways to do this: One is to use gauge singlet scalars, as in the FN case. We will discuss this possibility in detail in section 5. For now we will focus on the case where the flavor symmetry is broken along with the electroweak symmetry. This means that we need a flavor-charged Higgs boson, or rather several of them. The framework is then that of a multi-Higgs doublet model with Higgs bosons transforming non-trivially under G_f . Such multi-Higgs doublet models tend to lead to phenomenological problems, and we will be discussing them. However, as most general results apply equally well to both modes of flavor symmetry breaking, and since the first models presented in this thesis break G_f at the electroweak scale, it seems sensible to concentrate on these models for the time being.

When constructing a full model, we need to consider two sectors of the Lagrangian: We begin by writing down the allowed Yukawa couplings, considering the transformation properties of the fermions and the Higgs bosons. We then consider the Higgs potential and minimize it. The resulting VEV configuration, together with the Yukawa couplings, gives the fermion mass matrices. If our flavor group commutes with the standard model gauge group, and in particular with $SU(2)_L$, as we will assume throughout this thesis, then we set, without loss of generality, the G_f basis to be identical with the weak basis. The unitary matrices diagonalizing the squared mass matrices then give the quark and lepton mixing matrices, according to equations (1.11) and (1.20); the eigenvalues of the squared mass matrices give the squared masses, taking into account the regular changes for Majorana neutrinos. The results can then be compared with phenomenology or taken as predictions of the model.

This is of course an excessive oversimplification. In general the results we want to produce will not just drop out of our model, after choosing symmetry group and representation content. Some additional engineering will be necessary in general. This is often performed by extending the flavor symmetry group to $G_f \times Z_n$, i.e. by introducing an auxiliary abelian symmetry. Both fermions and scalars transform non-trivially under Z_n . We can in this way divide our fermions into several sectors, depending on their Z_n charge. For example, considering leptons, we can choose charges in such a way that one set of scalar VEVs contributes only to the charged lepton mass matrix, while another set of scalars, with a different Z_n charge, contributes only to the neutrino mass matrix, at least at leading order. This is particularly interesting if the two different sets of scalars acquire different VEV structures. For instance, take A_4 and two sets of scalars $(\varphi_1^{l,\nu}, \varphi_2^{l,\nu}, \varphi_3^{l,\nu})$ transforming under the three-dimensional representation of that group: In models predicting TBM from A_4 it is now a crucial ingredient that the scalars φ_i^{ν} coupling to neutrinos acquire a VEV proportional to $(1, 1, 1)$, while the scalars φ_i^l coupling to charged leptons acquire a VEV proportional to $(1, 0, 0)$.

This can also be understood in another way: The scalar VEVs coupling to neutrinos conserve a Z_2 subgroup of A_4 , while the scalar VEVs coupling to charged leptons conserve a Z_3 subgroup. It is this mismatch of subgroups which is responsible for creating the mass independent texture and thus the special mixing pattern. This will be the guiding principle in our general analysis of the dihedral groups in chapter 3. But first, we will take a look at a flavor symmetry model, to see some of the general principles described in this section in action.

2.3 The D_4 Scaling Model - a Worked-Out Example

2.3.1 Phenomenological Considerations - Scaling

Before we begin presenting the model, some comments on its phenomenological objectives are in order. The model we present concerns only leptons. Its main goal is obtaining a Majorana mass matrix for the light neutrinos that obeys the scaling ansatz [106]. This scaling hypothesis demands that the ratio $\frac{(M_\nu)_{\alpha\beta}}{(M_\nu)_{\alpha\gamma}}$ is independent of the flavor α , i.e.

Field	$D_4 \times Z_2$ quantum number
ϕ_2	$\underline{\mathbf{1}}^+$
e^c, ν_e^c	$\underline{\mathbf{1}}^-$
l_e, ν_μ^c	$\underline{\mathbf{3}}^+$
ν_τ^c, ϕ_1	$\underline{\mathbf{3}}^-$
ϕ_3	$\underline{\mathbf{4}}^-$
$\begin{pmatrix} l_\mu \\ l_\tau \end{pmatrix}, \begin{pmatrix} \phi_4 \\ \phi_5 \end{pmatrix}$	$\underline{\mathbf{2}}^+$
$\begin{pmatrix} \mu^c \\ \tau^c \end{pmatrix}$	$\underline{\mathbf{2}}^-$

Table 2.1: Transformation properties under $D_4 \times Z_2$ of the particle content of the D_4 scaling model.

$$\frac{(M_\nu)_{e\beta}}{(M_\nu)_{e\gamma}} = \frac{(M_\nu)_{\mu\beta}}{(M_\nu)_{\mu\gamma}} = \frac{(M_\nu)_{\tau\beta}}{(M_\nu)_{\tau\gamma}} = c \text{ for fixed } \beta \text{ and } \gamma, \quad (2.9)$$

in the basis, where the charged lepton mass matrix is diagonal. There are three possibilities and the only one allowed by phenomenology is the one with $\beta = \mu$ and $\gamma = \tau$. The resulting mass matrix reads

$$\mathcal{M}_\nu = \begin{pmatrix} A & B & B/c \\ B & D & D/c \\ B/c & D/c & D/c^2 \end{pmatrix}. \quad (2.10)$$

The most important phenomenological prediction of scaling is that equation (2.10) leads to an inverted hierarchy with $m_3 = 0$ and $U_{e3} = 0$. Atmospheric neutrino mixing is governed by the scaling factor c and the relation $\tan^2 \theta_{23} = 1/c^2$, i.e. is in general non-maximal for $c \neq 1$.

We note that from $m_3 = 0$ it follows that only the difference of the Majorana phases is physical. It appears in the neutrinoless double beta decay effective mass m_{ee} . This parameter takes a very simple form for the inverted hierarchy with $m_3 = \theta_{13} = 0$, if we neglect the solar mass splitting:

$$|m_{ee}| = \sqrt{|\Delta m_{31}^2|} \sqrt{1 - \sin^2 2\theta_{12} \sin^2(\phi_1 - \phi_2)}. \quad (2.11)$$

The range of m_{ee} lies for the best-fit parameters from equation (1.29) between 0.019 and 0.049 eV, while at $1(3)\sigma$ it ranges between 0.017 and 0.050 eV (0.012 and 0.052 eV). This would make an observation of neutrinoless double beta decay conceivable in the future [107]. Further phenomenological implications of the scaling ansatz are discussed in [1]. We now continue by presenting a specific model using the flavor symmetry $D_4 \times Z_2$ that predicts scaling.

2.3.2 The Model

We work in the framework of a Type I Seesaw mechanism and employ a $D_4 \times Z_2$ flavor symmetry to generate scaling. This model only describes the lepton sector: Apart from the regular conjugated neutrinos $\nu_{e,\mu,\tau}^c$, the conjugated charged leptons e^c, μ^c, τ^c and the lepton doublets $l_{e,\mu,\tau}$, we introduce five Higgs doublets $\phi_{1,2,3,4,5}$. We have thus, as a first step, determined G_f as well as the particle content. The exact assignment of representations under $D_4 \times Z_2$ is shown in

table 2.1. The superscripts $+$, $-$ refer to the transformation properties under Z_2 , the boldface numbers to the D_4 representations. For the mathematical details of the D_4 group, we refer the reader to section 3.1.1. From the assignment in Table 2.1 the following Yukawa Lagrangian is obtained:

$$\begin{aligned}
-\mathcal{L} = & k_1 l_e e^c \phi_1 + l_\mu \mu^c (k_2 \phi_1 - k_3 \phi_3) + l_\tau \tau^c (k_2 \phi_1 + k_3 \phi_3) \\
& + h_1 l_e \nu_e^c \phi_1 + h_2 l_e \nu_\mu^c \phi_2 + h_3 (l_\mu \nu_\mu^c \phi_4 + l_\tau \nu_\mu^c \phi_5) \\
& + \frac{1}{2} (\nu_e^c \nu_e^c M_1 + \nu_\mu^c \nu_\mu^c M_2 + \nu_\tau^c \nu_\tau^c M_3) + h.c.
\end{aligned} \tag{2.12}$$

From now on, as in this Lagrangian, we will drop the explicit decomposition into $SU(2)_L$ component fields. Assuming all five Higgs fields acquire a non-vanishing VEV, the neutrino Dirac mass matrix can be written as

$$\begin{pmatrix} a e^{i\varphi} & b & 0 \\ 0 & d & 0 \\ 0 & e & 0 \end{pmatrix} v_{\text{wk}} , \tag{2.13}$$

and both the charged lepton mass matrix and the Majorana mass matrix of the charge-conjugated neutrinos are diagonal. We can always make the entries of these two diagonal matrices real and positive by rephasing the charge-conjugated charged leptons and neutrinos. This leaves us the freedom, to rephase the l_α to remove all phases from the neutrino Dirac mass matrix, except for the one complex phase included in the matrix of equation (2.13). There is thus only one complex phase in the model and the entries of the Dirac mass matrix are given by

$$a = |h_1 \langle \phi_1 \rangle| , b = |h_2 \langle \phi_2 \rangle| , d = |h_3 \langle \phi_4 \rangle| , e = |h_3 \langle \phi_5 \rangle|. \tag{2.14}$$

We can then calculate the light neutrino Majorana mass matrix using the Type I Seesaw formula to obtain

$$M_\nu = - \frac{v_{\text{wk}}^2}{M_2} \begin{pmatrix} \frac{M_2}{M_1} a^2 e^{2i\varphi} + b^2 & b d & b e \\ b d & d^2 & d e \\ b e & d e & e^2 \end{pmatrix} . \tag{2.15}$$

Note that the third heavy neutrino mass M_3 does not appear in M_ν : the third charge-conjugated neutrino does not take part in the seesaw because all its Dirac Yukawa couplings vanish. The low energy mass matrix obeys scaling with $c = d/e$. We therefore have $m_3 = U_{e3} = 0$ and $\tan^2 \theta_{23} = e^2/d^2$ and the effective mass governing neutrinoless double beta decay is

$$m_{ee} = \frac{v_{\text{wk}}^2}{M_2} \left| \frac{M_2}{M_1} a^2 e^{2i\varphi} + b^2 \right| = \sqrt{\Delta m_{31}^2} \sqrt{1 - \sin^2 2\theta_{12} \sin^2(\phi_1 - \phi_2)} . \tag{2.16}$$

We now turn to the scalar, i.e. the flavor symmetry breaking, sector. First note that multi-Higgs models, such as the one analyzed here, typically predict flavor changing neutral currents (FCNCs) and Lepton Flavor Violation (LFV) in the charged lepton sector at dangerous levels. In this model the charged lepton Yukawa matrices for all Higgs bosons are diagonal³. This

³Note that this is a stronger condition than demanding that the charged lepton mass matrix be diagonal, as cancellations may occur.

means that tree-level LFV is entirely forbidden and processes such as $\mu \rightarrow e\gamma$ are suppressed either by the Glashow-Iliopoulos-Maiani (GIM) mechanism (for light Majorana neutrinos in the loop) or by the masses of the heavy charge-conjugated neutrinos. This is discussed in detail in [108]. Since this model focusses only on the lepton sector, no statement can be made on FCNCs, which could violate limits stemming from rare meson decays. The only remaining phenomenological constraint are limits from direct searches. The analysis of the scalar masses in a very similar model ⁴ in [1] shows that it is generally possible to choose the couplings in the potential in such a way that the masses of all physical scalars are large enough to avoid such limits. This is no longer necessarily true for models with more scalar states, and we will return to these problems in section 4.3.

What remains to be considered is the VEV structure of the five Higgs bosons. Most importantly we observe that the scaling parameter c is related to the scalar VEVs through

$$c = \left| \frac{\langle \phi_4 \rangle}{\langle \phi_5 \rangle} \right|. \quad (2.17)$$

We thus see how the mixing pattern is determined not only by our initial choice of representation content, but also by the structure of the VEVs of higher-dimensional representations of G_f . This is very similar to the A_4 case, where it was the VEV structure of the three-dimensional representations which led to TBM. In A_4 models the required structures conserve non-trivial subgroups of the original flavor group. As we will see in the next chapter, a Z_2 symmetry of D_4 is conserved for $c = 1$ that is for $|\langle \phi_4 \rangle| = |\langle \phi_5 \rangle|$. This leads to a $\mu - \tau$ -symmetric model, with maximal atmospheric mixing, which seems to indicate that conserved subgroups may be a generic feature of models where mass independent textures predict specific mixing angles. It is thus an interesting project to take a flavor group, determine its possible subgroups, the VEV structures that conserve it and the resulting mixing. Such an analysis can give more general criteria as to which flavor group to use, a pressing question considering the large amount of models available.

We could of course perform such a systematic discussion for D_4 alone, however it is much more economical to perform a general analysis for the entire set of dihedral groups D_n and their double-valued counterparts D'_n . They are predestined for such an analysis, as they do not contain representations larger than two-dimensional, independent of the index n , and hence can all theoretically be used as flavor groups, without having to add additional fermions to the SM. We will perform just such a systematic analysis in the following chapter.

⁴The Yukawa Lagrangian is identical, there will only be slight differences in the potential.

Chapter 3

Dihedral Flavor Groups

3.1 Introduction to Dihedral Groups

3.1.1 Single-Valued Dihedral Groups

The dihedral group D_n is the symmetry group of the regular two-sided polygon with n corners and n edges (often called n -gon). Two-sided means that the symmetry transformations not only include rotations (these make up the cyclic group Z_n), but also reflections, which correspond to flipping the two sides of the polygon. In fact, it is this property which gives the dihedral groups their name, $\delta\iota\epsilon\delta\rho\sigma$ meaning two-sided in ancient Greek.

Since reflections and rotations do not commute in general, all D_n groups are non-abelian apart from D_1 ($\cong Z_2$), which is the symmetry of the two-sided point and therefore does not include rotations, and D_2 ($\cong Z_2 \times Z_2$), the symmetry group of the two-sided line, which only contains rotations by 180° , which commute with reflections.

The rotations are generated by the element A, the reflections by B. These two generators then fulfill the following relations:

$$A^n = \mathbb{1} \quad , \quad B^2 = \mathbb{1} \quad , \quad ABA = B . \quad (3.1)$$

A given element of D_n can then be written as A^m or as BA^m , with $m = 0, 1, \dots, n-1$, making for a total of $2n$ elements. We will go into the subgroup structure in much detail later on, but one can already see that any D_n will contain a Z_n subgroup containing only the rotations, generated by A. This abelian subgroup will have n elements, so by a well-known lemma of representation theory [109], any irreducible representation of D_n will have a dimension less or equal than $\frac{2n}{n} = 2$.

The number of one-dimensional representations is strictly limited. Since B is of order 2, it must always be represented by 1 or -1 . The commutation relation between A and B then directly implies that A must also be represented by 1 or -1 . If the group index n is odd, then A is of odd order and can only be represented by 1, leaving us with two one-dimensional representations, the trivial one, denoted by $\mathbf{1}_1$, for which $A = B = 1$, and the non-trivial one, denoted by $\mathbf{1}_2$, for which $A = 1, B = -1$. If the index n is even, we get two additional non-trivial singlet representations, $\mathbf{1}_3$ with $A = -1, B = 1$, and $\mathbf{1}_4$ with $A = B = -1$. All other irreducible representations are two-dimensional and their number can be directly inferred from the fundamental equation of representation theory

$$|D_n| = 2n = \sum_i (\dim \rho_i)^2, \quad (3.2)$$

with the sum going over all irreducible representations ρ_i . For an even n this gives $(\frac{n}{2} - 1)$ two-dimensional representations and for n being odd D_n has $\frac{n-1}{2}$ two-dimensional representations.

We will denote the two-dimensional representations with $\underline{2}_j$, where the index $j = 1, \dots, \frac{n-1}{2}$ for D_n with n odd and $j = 1, \dots, \frac{n}{2} - 1$ for D_n with n even.

As opposed to the case of one-dimensional representations, there is no unique matrix representation for two-dimensional representations. We will in the following be using complex matrix representations, where the generators of the two-dimensional representations are [109]

$$A = \begin{pmatrix} e^{(\frac{2\pi i}{n})j} & 0 \\ 0 & e^{-(\frac{2\pi i}{n})j} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.3)$$

with $j = 1, \dots, \frac{n}{2} - 1$ for n even and $j = 1, \dots, \frac{n-1}{2}$ for n odd. Note that even though we have chosen complex matrices to represent the generators for the two-dimensional representations, the trace of the representation matrix for any given group element (and thereby the character of the representation) will be a real number. One can in fact use real representation matrices, as is often done in the literature, which shows that the two-dimensional representations of D_n are real. We will discuss the most common, explicitly real set of representation matrices in appendix B.3. For now, we mention that, since the representations are real, there exists a unitary matrix U which links the generators to their complex conjugates: $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For any $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

transforming under a two-dimensional representation, the combination $U \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix} = \begin{pmatrix} a_2^* \\ a_1^* \end{pmatrix}$ transforms as the same two-dimensional representation.

For completeness it should be mentioned that one often finds the notation D_{2n} instead of D_n , where the group index denotes the order of the group instead of the order of A , the generator of rotations.

3.1.2 Double-Valued Dihedral groups

The groups D'_n are the double-valued counterparts of the groups D_n . Double-valued means that a rotation by 360° is equivalent to the product of two reflections but not to the identity. Only a rotation by 720° or equivalently the product of four reflections gives the unit element. The relation between D'_n and D_n is thus similar to that between the Lie groups $SU(2)$ and $SO(3)$. Therefore, of the groups D'_n only D'_1 is abelian, since it only contains rotations by 360° , which are equivalent to two reflections. D'_1 is thus isomorphic to the double-valued group of reflections, Z_4 . All other double-valued dihedral groups have two generators, which fulfill the relations

$$A^n = R, \quad B^2 = R, \quad R^2 = \mathbb{1}, \quad ABA = B. \quad (3.4)$$

R commutes with all elements of D'_n . Any element of D'_n can then be written as A^m or as BA^m , with $m = 0, 1, \dots, 2n-1$, giving a total of $4n$ elements. They will also contain a maximal abelian subgroup containing only the rotations, which in this case is Z_{2n} . This abelian subgroup has $2n$ elements, so that again the dimension of irreducible representations is limited to be no larger than $\frac{4n}{2n} = 2$.

As B is now of order 4, it can be mapped by a one-dimensional representation to $1, -1, -i$ or i , while A can again only be represented by 1 or -1 due to the commutation relation. We then have the additional constraint $A^n = B^2$. For an even group index n , this means that B cannot be imaginary, and we have the same four one-dimensional representations we had in the single-valued case, that is all one-dimensional representations are real. For an odd n , the choice of B uniquely determines A , leaving us again with four one-dimensional representations, with $\underline{1}_1$ and $\underline{1}_2$ being defined as for an even n (and thereby real), but with the representations $\underline{1}_3$ and $\underline{1}_4$ now being complex conjugated to each other, the generators being represented by

$A = -1$, $B = -i$ and $A = -1$, $B = i$, respectively.

We conclude that the double-valued dihedral groups always have 4 one-dimensional representations, and thus $n - 1$ two-dimensional representations. For the representation $\underline{2}_j$, we choose generator matrices having a similar form as for D_n [110, 111]:

$$A = \begin{pmatrix} e^{(\frac{\pi i}{n})j} & 0 \\ 0 & e^{-(\frac{\pi i}{n})j} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.5)$$

for j even, and

$$A = \begin{pmatrix} e^{(\frac{\pi i}{n})j} & 0 \\ 0 & e^{-(\frac{\pi i}{n})j} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (3.6)$$

for j odd, $j=0,1,\dots,n-1$. All two-dimensional representations will again have a real character, however only the two-dimensional representations $\underline{2}_j$ with j even are real, i.e. not only their characters are real, but there also exists a set of real representation matrices. In contrast to this, the representations $\underline{2}_j$ with j odd are pseudo-real, i.e. their characters are real, but one cannot find a set of representation matrices which are also real. Therefore no alternative set of real representation matrices exists.

To obtain the group D'_n from the group D_n one has to add the pseudo-real and complex representations. Therefore the real representations are usually called even, while the pseudo-real and complex ones are named odd representations [110], in the sense that R is $\mathbb{1}$ in case of an even representation and $-\mathbb{1}$ for an odd one. Comparing the generators of even and odd representations one recognizes that the generator B contains an extra factor i for odd representations.

Since also for D'_n all two-dimensional representations are real or pseudo-real, i.e. not complex, there exists a similarity transformation U between the representation matrices and their complex conjugates. If the index j of the representation $\underline{2}_j$ is even, U is the same as for the representations of D_n . For j being odd, $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ such that $\begin{pmatrix} -a_2^* \\ a_1^* \end{pmatrix} = U \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix}$ transforms in the same way as $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{2}_j$ with j odd.

A final word on notation: The simplest non-abelian double-valued dihedral group, D'_2 , is also called the quaternion group. Hence, one often finds the notation Q_{2n} or Q_{4n} instead of D'_n , i.e. taking the group index to be the order of the generator of rotations or the order of the group itself. In fact the group index n in D'_n indeed has no direct interpretation from the group structure, but is chosen to make the connection between a double-valued group and its single-valued counterpart clearer.

3.2 Non-Trivial Subgroups

We now already know enough about the dihedral groups to determine the subgroups of a general D_n or D'_n group, using the generators given in section 3.1. We already mentioned in the last chapter that these subgroups are intimately related to the VEV structure of scalars transforming non-trivially under the flavor group G_f . It will thus be most instructive to describe the method of calculating subgroups by considering the physics behind it.

When a scalar, transforming under a non-trivial irreducible representation of G_f , acquires a VEV, G_f will be broken, because there will be elements of the group that do not leave the VEV invariant. G_f is thus no longer a symmetry of the vacuum of our theory. However in general there may be elements of G_f that do leave the VEV invariant; these elements then comprise

the subgroup left unbroken by the scalar VEV. We go through the irreducible representations one by one, assuming there exists a scalar transforming under this representation and acquiring a VEV. We then check which elements of G_f leave such a VEV invariant. These elements form a group, whose group structure has to be determined. This is straightforward, as all subgroups turn out to be either dihedral or cyclic.

For the one-dimensional representations we only need to look at which elements of the group are represented by a 1 - these elements then form the conserved subgroup. For the two-dimensional representations the conserved group may depend on the VEV structure, i.e. on the relation between the two VEV components of the doublet. We thus need to determine the eigenvalues and eigenvectors of the group elements. All group elements which have the same eigenvector corresponding to the eigenvalue of 1 form a subgroup and this group is conserved by a VEV proportional to this common eigenvector.

For this calculation, we only have to consider two general matrices, the first one being

$$A^m = \begin{pmatrix} e^{(\frac{4\pi im}{G})j} & 0 \\ 0 & e^{-(\frac{4\pi im}{G})j} \end{pmatrix}, \quad (3.7)$$

where G is the order of G_f ($2n$ for single-valued, $4n$ for double-valued groups) and $0 \leq m \leq \frac{G}{2} - 1$ an integer. This matrix only has an eigenvalue of 1 if $\frac{2mj}{G}$ is an integer. Since $m < \frac{G}{2}$, this means that the greatest common divisor (gcd) of j and $\frac{G}{2}$ must be greater than one¹. In this case $\underline{2}_j$ is an unfaithful representation which maps $A^{\frac{G}{2g(\frac{G}{2}, j)}}$ to the unit matrix, where we have used the shorthand notation $g(x, y, \dots)$ for the greatest common divisor of the integers x, y, \dots . The subgroup conserved by the VEV of such an unfaithful representation will always contain all group elements represented by the unit matrix and will consequently have $A^{\frac{G}{2g(\frac{G}{2}, j)}}$ as a generator.

The other general matrix we need to consider is

$$BA^m = \begin{pmatrix} 0 & e^{(\frac{4\pi im}{G})j} \\ e^{-(\frac{4\pi im}{G})j} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & ie^{(\frac{4\pi im}{G})j} \\ ie^{-(\frac{4\pi im}{G})j} & 0 \end{pmatrix}, \quad (3.8)$$

where m again lies between 0 and $\frac{G}{2} - 1$. The second matrix is for representations of D'_n with odd index j . It has no eigenvalues of 1. Thus we observe that for D'_n the structure of the doublet VEVs with odd index j is irrelevant for the conserved subgroup. The first matrix on the other hand, always has an eigenvalue of 1, corresponding to the eigenvector

$$\begin{pmatrix} e^{-\frac{4\pi ijm}{G}} \\ 1 \end{pmatrix}. \quad (3.9)$$

We thus have for a given two-dimensional representation a class of subgroups, parameterized by m , where one of the generators of the subgroup will be BA^m .

To make sure that we have determined all subgroups, we need to consider possible combinations of two or more scalars, transforming under different representations. Further subgroups will necessarily be subgroups of the subgroups determined above. Since all of the subgroups encountered so far are either dihedral or cyclic, we know that all further subgroups will also be either dihedral or cyclic. As it turns out, we need at most 2 different representations to reach any possible subgroup of our original D_n or D'_n .

We have listed all subgroups of a general group D_n , as well as the minimal scalar VEV content needed to break D_n down to that group. We will frequently be using the name of the

¹In the original paper [2], we incorrectly assumed that j itself had to divide $\frac{G}{2}$. This has been corrected in the results given in this thesis. An erratum is in preparation.

representation to mean the scalar transforming under that representation, whenever there is no ambiguity. For two-dimensional representations the structure of the scalar VEV is important, so we need to differentiate: We denote an arbitrary VEV by $\langle \underline{\mathbf{2}}_j \rangle$, while a VEV proportional to the eigenvector of equation (3.9) will be denoted by $\langle \underline{\mathbf{2}}_j \rangle'$. We get the following results for D_n groups:

$$\begin{aligned}
D_n & \xrightarrow{\langle \underline{\mathbf{1}}_1 \rangle} D_n \\
D_n & \xrightarrow{\langle \underline{\mathbf{1}}_2 \rangle} Z_n = \langle A \rangle \\
D_n & \xrightarrow{\langle \underline{\mathbf{1}}_3 \rangle} D_{\frac{n}{2}} = \langle A^2, B \rangle \\
D_n & \xrightarrow{\langle \underline{\mathbf{1}}_4 \rangle} D_{\frac{n}{2}} = \langle A^2, BA \rangle \\
D_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle} Z_{g(n,j)} = \langle A^{\frac{n}{g(n,j)}} \rangle \quad (g(n,j) > 1) \\
D_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle} \text{nothing} \quad (g(n,j) = 1) \\
D_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle'} D_{g(n,j)} = \langle A^{\frac{n}{g(n,j)}}, BA^m \rangle \quad (g(n,j) > 1; m = 0, 1, \dots, \frac{n}{g(n,j)} - 1) \\
D_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle'} Z_2 = \langle BA^m \rangle \quad (g(n,j) = 1; m = 0, 1, \dots, n-1) \\
D_n & \xrightarrow{\langle \underline{\mathbf{1}}_2 \rangle + \langle \underline{\mathbf{1}}_3 \rangle} Z_{\frac{n}{2}} = \langle A^2 \rangle \quad (\text{one can also use } \langle \underline{\mathbf{1}}_2 \rangle + \langle \underline{\mathbf{1}}_4 \rangle \text{ or } \langle \underline{\mathbf{1}}_3 \rangle + \langle \underline{\mathbf{1}}_4 \rangle)
\end{aligned}$$

and for D'_n groups we get:

$$\begin{aligned}
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_1 \rangle} D'_n \\
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_2 \rangle} Z_{2n} = \langle A \rangle \\
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_3 \rangle \text{ or } \langle \underline{\mathbf{1}}_4 \rangle} Z_n = \langle A^2 \rangle \quad (\text{for odd } n) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_3 \rangle} D'_{\frac{n}{2}} = \langle A^2, B \rangle \quad (\text{for even } n) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_4 \rangle} D'_{\frac{n}{2}} = \langle A^2, BA \rangle \quad (\text{for even } n) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle} Z_{g(2n,j)} = \langle A^{\frac{2n}{g(2n,j)}} \rangle \quad (g(2n,j) > 2) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle} Z_2 = \langle A^n \rangle \quad (g(2n,j) = 2) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle} \text{nothing} \quad (g(2n,j) = 1) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle'} D'_{\frac{g(2n,j)}{2}} = \langle A^{\frac{2n}{g(2n,j)}}, BA^m \rangle \quad (g(2n,j) > 2 \text{ and even}; m = 0, 1, \dots, \frac{2n}{g(2n,j)} - 1) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{2}}_j \rangle'} Z_4 = \langle BA^m \rangle \quad (g(2n,j) = 2; m = 0, 1, \dots, n-1) \\
D'_n & \xrightarrow{\langle \underline{\mathbf{1}}_2 \rangle + \langle \underline{\mathbf{1}}_3 \rangle} Z_n = \langle A^2 \rangle \quad (\text{for even } n; \text{ one can also use } \langle \underline{\mathbf{1}}_2 \rangle + \langle \underline{\mathbf{1}}_4 \rangle \text{ or } \langle \underline{\mathbf{1}}_3 \rangle + \langle \underline{\mathbf{1}}_4 \rangle)
\end{aligned}$$

Some of these results can also be found in references [112] and [113].

We now know the minimal VEV structure needed to break G_f down to a given subgroup. The maximal VEV structure preserving that subgroup is then achieved by allowing VEVs for all representations which have at least one component transforming trivially under the subgroup in question. Therefore we list the transformation properties of the representations of our original dihedral group under a given subgroup. These can be found by expressing the generators of the subgroup in terms of the generators A and B of G_f . As a general feature we remark that two-dimensional representations become reducible in several cases. The complete list of decompositions is given in Appendix C.1 along with the maximal VEV structure.

3.3 Breaking Chains

We have seen that a dihedral group will in general have several non-trivial subgroups. It may be interesting to consider models, where the breaking of the flavor symmetry happens in several steps. This can be realized by having scalars transforming under different representations acquiring their VEVs at several distinct symmetry breaking scales, leading to a stepwise reduction of the flavor symmetry. Such an approach can be useful for explaining the observed hierarchy of fermion masses. Again this discussion can be performed in a very general way, considering all possible breaking patterns of a dihedral group, where the symmetry breaking happens in an arbitrary number of steps.

As the amount of breakings is arbitrary in theory, it is more economical to give only the building blocks for such a breaking chain. For each building block, we assume that we have already broken down to one of the subgroups of D_n or D'_n determined in the last section. We then consider how adding an additional VEV will further reduce the symmetry.

For the subgroups of D_n we only differentiate between two types. The first type are dihedral subgroups $D_q = \langle A^{\frac{n}{q}}, BA^m \rangle$. This includes the groups $D_{\frac{n}{2}}$ as well as the abelian group $Z_2 = \langle BA^{m1} \rangle$, which is isomorphic to D_1 . These subgroups are further parameterized by the integer m which appears in their generators. It is restricted to lie between 0 and $\frac{n}{q} - 1$ and thereby, along with the index q , uniquely defines the dihedral subgroup in question. The other type are the cyclic subgroups $Z_q = \langle A^{\frac{n}{q}} \rangle$. For even n these may also include a $Z_2 = \langle A^{\frac{n}{2}} \rangle$. In this section, this is the group we mean by Z_2 , the other Z_2 's will be denoted by D_1 . In the following sections, when we discuss the subgroups in more detail, we will always give the subgroup generators, to make clear which group we mean. This would however make the breaking chain building blocks rather cumbersome. Finally, note that the group Z_1 is trivial, i.e. its conservation corresponds to a total breaking of G_f .

We also use the notation m_j for the phase factor m in the VEV $\langle \mathbf{2}_j \rangle'$. Its relation to the subgroup index m is of crucial importance in the building blocks given below.

We find two paths in the breaking sequences, one along the dihedral groups and one along the cyclic groups, in the minimal case eliminating one prime divisor of the order in each step. At any point in the sequence we can step over from the dihedral to the cyclic path (from which there is of course no turning back). The cyclic path ends at some Z_q , with q being the smallest prime factor of n , while the dihedral path will end at D_1 . Note that if a representation is allowed a VEV under the subgroup in question, it will of course not break the symmetry any further. These cases are however still listed below for completeness. The building blocks for D_n are:

$$\begin{array}{ll}
D_q & \xrightarrow{\langle \mathbf{1}_2 \rangle} Z_q \\
Z_q & \xrightarrow{\langle \mathbf{1}_2 \rangle} Z_q \\
D_q & \xrightarrow{\langle \mathbf{1}_3 \rangle} D_{g(\frac{n}{2}, q)} \quad (m \text{ even for } \mathbf{1}_3 \text{ and } m \text{ odd for } \mathbf{1}_4) \\
D_q & \xrightarrow{\langle \mathbf{1}_3 \rangle} Z_{g(\frac{n}{2}, q)} \quad (m \text{ odd for } \mathbf{1}_3 \text{ and } m \text{ even for } \mathbf{1}_4) \\
Z_q & \xrightarrow{\langle \mathbf{1}_3 \rangle} Z_{g(\frac{n}{2}, q)} \\
D_q & \xrightarrow{\langle \mathbf{2}_j \rangle} Z_{g(q, j)} \\
Z_q & \xrightarrow{\langle \mathbf{2}_j \rangle} Z_{g(q, j)} \\
D_q & \xrightarrow{\langle \mathbf{2}_j \rangle'} D_{g(q, j)} \quad (m_j \bmod \frac{n}{q} = m)
\end{array}$$

$$\begin{aligned}
D_q & \xrightarrow{\langle \mathbf{2}_j \rangle'} Z_{g(q,j)} \quad (m_j \bmod \frac{n}{q} \neq m) \\
Z_q & \xrightarrow{\langle \mathbf{2}_j \rangle'} Z_{g(q,j)}
\end{aligned}$$

The corresponding results for D'_n are given in appendix C.2. The restrictions on the phase factors m_j occur if the direction in which we have broken is important. Several breaking directions occur if we deal with distinct subgroups showing the same structure - for example in D_n we have two $D_{\frac{n}{2}}$ subgroups: $\langle A^2, B \rangle$ and $\langle A^2, BA \rangle$. The Z_2 subgroup generated by BA^{m_j} is only a subgroup of the first in case of m_j even and only a subgroup of the second, if m_j is odd. Hence we need to impose restrictions on the phase factor m_j .

Finally, note that in any of the chains given above we can interchange $\mathbf{1}_3$ and $\mathbf{1}_4$ and again receive a viable breaking sequence. This may cause some of the requirements to change, as shown along with the breaking chain building blocks.

We finally mention that breaking chains of dihedral flavor symmetries have been used to explain the fermion mass hierarchy. In [81] D'_3 is broken down to Z_6 and then further down, while [82] breaks a D_6 flavor symmetry down to Z_2 and then fully breaks it.

3.4 Mass Matrices

The breaking chains discussed in the last chapter may enable us to get a handle on the hierarchy of the fermion masses. However, our interest in discrete flavor symmetries is mainly due to their ability to describe the mixing among the fermions of different generations. To determine what kind of mixing patterns can be obtained from imposing a flavor symmetry G_f with non-trivial conserved subgroups, we need to determine the mass matrices which can arise in such a setup. The mixing matrices are then obtained in the next section from diagonalizing these mass matrices. We will first be discussing Dirac mass matrices, then comment on Majorana mass matrices.

We assume that G_f is broken by the VEVs of multiple flavor-charged $SU(2)_L$ doublets, i.e. by copies of the SM Higgs boson. These VEVs then break G_f down to one of its non-trivial subgroups, determined in section 3.2. To then determine the resulting Dirac mass matrices M which are generated when G_f is broken to one of these subgroups, we need to decide on two things: The G_f transformation properties of the Weyl fermions coupling to form the Dirac mass term and the representations present in the scalar sector.

The transformation properties of the fermions are not limited by our choice of conserved subgroup. We will only impose one limitation: We do not want all fermions involved in the mass term to transform under one-dimensional representations of G_f , since the resulting mass matrix structures could then also be obtained from an abelian G_f . We will therefore not allow for this possibility when considering Dirac mass terms, leaving us with three general options for the transformation properties of the fermions. The first two possibilities are

$$L \sim (\mathbf{1}_{\mathbf{i}_1}, \mathbf{1}_{\mathbf{i}_2}, \mathbf{1}_{\mathbf{i}_3}), \quad L^c \sim (\mathbf{1}_{\mathbf{j}}, \mathbf{2}_{\mathbf{k}}) \quad \text{and} \quad L \sim (\mathbf{1}_{\mathbf{j}}, \mathbf{2}_{\mathbf{k}}), \quad L^c \sim (\mathbf{1}_{\mathbf{i}_1}, \mathbf{1}_{\mathbf{i}_2}, \mathbf{1}_{\mathbf{i}_3}),$$

where L denotes the three left-handed fermions involved in the mass terms (i.e. the Q_i or the l_α) and L^c denotes the three left-handed conjugate fermions (i.e. the u_i^c, d_i^c, e_i^c or, in the case of Dirac neutrinos, ν_α^c). We call these assignments the left-handed and right-handed three singlet structure, respectively. The mass matrices of the two cases are related by simple transposition, which corresponds to switching the transformation properties of left-handed and left-handed conjugate fermions. For our study of mass matrices, we can thus limit ourselves to one of these cases and only discuss the mass matrices for the left-handed three singlet structure (which

we then simply refer to as three singlet structure). As soon as mixing comes into play, it is important how the left-handed fermions transform, and we need to distinguish between the two in section 3.5.

The last possibility for assigning the fermions to representations of G_f is

$$L \sim (\mathbf{1}_j, \mathbf{2}_j), \quad L^c \sim (\mathbf{1}_1, \mathbf{2}_k),$$

which we call the two doublet structure.

The mass matrices arise when the Higgs bosons acquire a VEV. We do not choose their transformation properties, as we did for the fermion fields. Instead, when determining the mass matrix generated by breaking down to a certain subgroup, we reference tables C.1 to C.4 in appendix C to determine which representations are allowed a VEV, while keeping the subgroup intact. We then assume that our model contains a Higgs boson for each of these possible representations. In this way the mass matrix structure is entirely determined by the properties of G_f and its subgroup, and not by our choice of scalar fields.

To determine the mass matrices we then assume that all these Higgs bosons acquire a VEV, with a structure conserving the relevant subgroup. We can then easily eliminate Higgs bosons from our model, by setting their VEVs to zero in the mass matrix.

All the group theoretical tools we need, i.e. the Kronecker products and Clebsch Gordan coefficients, are given in appendix B. We can thus go ahead and determine the possible mass matrices, where a mass matrix is determined by two choices: the choice of subgroup and the choice of fermion representations.

Note that we will restrict ourselves in a fairly major way: We do not consider mass matrices with a zero determinant, that is with at least one zero eigenvalue. This is motivated by phenomenology, except when considering neutrinos. After all, it is by no means experimentally ruled out that there exists a massless neutrino. In general the uncertainty concerning the mechanism of neutrino mass generation is the most significant unknown in this general discussion of mass matrices. Apart from the non-zero determinant, we will further limit us by, for the moment, not considering Type I Seesaw structures, where the light neutrino mass matrix arises from the interplay of a Dirac and a Majorana mass matrix. We will thus only be considering two cases: The case of pure Dirac neutrinos is obviously covered by our general discussion of Dirac mass matrices. The other possibility is a pure Majorana mass term, which we will discuss in section 3.4.6.

Before that, we give our general results in section 3.4.1, then, in section 3.4.2, we give the conventions and notation we use in sections 3.4.3 and 3.4.4, where we discuss the three singlet and two doublet structures, respectively, for single-valued groups D_n . The resulting mass matrices will be discussed subgroup by subgroup. In section 3.4.5 we discuss why no new Dirac mass matrix structures appear for double-valued groups, before moving on to Majorana mass matrices in section 3.4.6.

3.4.1 General Results

We encounter a very limited number of distinct Dirac mass matrix structures: In total only five distinct structures are possible. We display them for down-type fermions (down-type quarks and charged leptons). We have to make this specification, since our Higgs fields are always assumed to transform as the SM Higgs doublet. In the SM the field H itself couples to the down-type fermions, while its conjugate ϵH^* is coupled to up-type ones. This difference is relevant here, since we have used complex generators and will also have explicit phases in our VEVs. We discuss the changes for up-type fermions, i.e. up-type quarks and Dirac neutrinos, in section 3.4.2.

The first possible structure we obtain is a diagonal mass matrix

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}. \quad (3.10)$$

The second type is a semi-diagonal mass matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix}, \quad (3.11)$$

where the squared mass matrix MM^\dagger has eigenvalues $|A|^2$, $|B|^2$ and $|C|^2$. We also encounter a block matrix structure

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & D & E \end{pmatrix}, \quad (3.12)$$

for which the squared mass matrix has the characteristic polynomial $(\lambda - |A|^2)(\lambda^2 - \lambda(|B|^2 + |C|^2 + |D|^2 + |E|^2) + (|B|^2 + |C|^2)(|D|^2 + |E|^2) - |BD^* + CE^*|^2)$. These three structures are so common that we will not give them explicitly each time. Instead, we will give the type of matrix, followed by a listing of the non-zero entries, where we use the notation and nomenclature introduced here. Note that in some cases the entries in the 2-3-submatrix are further correlated, e.g. two elements therein may be equal.

Finally, we find two structures which appear only once, both times for the subgroups $Z_2 = \langle BA^m \rangle$. One has one texture zero and the general structure

$$\begin{pmatrix} 0 & C & -Ce^{-i\phi k} \\ A & D & De^{-i\phi k} \\ B & E & Ee^{-i\phi k} \end{pmatrix}, \quad (3.13)$$

with the group theoretical phase

$$\phi = \frac{2\pi m}{n}. \quad (3.14)$$

In this case the squared mass matrix has the characteristic polynomial $(\lambda - 2|C|^2)(\lambda^2 - \lambda(|A|^2 + |B|^2 + 2(|D|^2 + |E|^2)) + (|A|^2 + 2|D|^2)(|B|^2 + 2|E|^2) - |AB^* + 2DE^*|^2)$. This mass matrix only shows up for a three singlet structure. The other mass matrix structure has no texture zeros and is of the form

$$\begin{pmatrix} A & C & Ce^{-i\phi k} \\ B & D & E \\ Be^{-i\phi j} & Ee^{i(k-j)\phi} & De^{-i(j+k)\phi} \end{pmatrix}, \quad (3.15)$$

where the squared mass matrix has the characteristic polynomial $(\lambda - |D - Ee^{ik\phi}|^2)(\lambda^2 - (|A|^2 + 2|B|^2 + 2|C|^2 + |D + Ee^{ik\phi}|^2)\lambda + (|A|^2 + 2|C|^2)(2|B|^2 + |D + Ee^{ik\phi}|^2) - 2|AB^* + C(D^* + E^*e^{-ik\phi})|^2)$. This mass matrix only appears with a two doublet structure. We will discuss further relevant properties of all five mass matrices in section 3.5.

We additionally find one case, where all matrix elements are distinct and non-zero. We do not consider it, as it corresponds to a smaller symmetry being fully broken: D_n breaks down to Z_q , where for all two-dimensional representations $\mathbf{2}_j$ showing up in the model j is a multiple of q , i.e. $j = c_j q$, c_j an integer. We can then replace the original D_n symmetry by a $D_{\frac{n}{q}}$ and all the representations $\mathbf{2}_j$ by $\mathbf{2}_{c_j}$, as they are in fact all unfaithful representations of the original D_n symmetry. Breaking the original symmetry down to Z_q then corresponds to fully breaking the

smaller symmetry². As we want to consider conserved subgroups, we dismiss this case, when it shows up.

A simple example is the case of a D_8 symmetry, where only the representation $\underline{\mathbf{2}}_2$ is used, while there are no fields transforming as $\underline{\mathbf{2}}_1$ or $\underline{\mathbf{2}}_3$. An unstructured VEV of a Higgs field transforming as $\underline{\mathbf{2}}_2$ breaks D_8 down to $Z_2 = \langle A^4 \rangle$. This model is equivalent to one, where the flavor group D_4 is fully broken by the VEV of a Higgs boson transforming as $\underline{\mathbf{2}}_1$. Note that such a case still allows for some non-trivial correlations among the mass matrix elements, through the ratio of the VEVs of the two doublet components (recall the scaling factor of section 2.3). Nevertheless, such correlations are then not determined by a preserved subgroup, but only by the fact that we use a non-abelian symmetry.

3.4.2 Conventions and Notation

In the following, κ 's and α 's denote Yukawa couplings, $\langle \phi_i \rangle$ denotes the VEV of the Higgs field transforming as $\underline{\mathbf{1}}_i$. For the VEVs of Higgs fields transforming as $\underline{\mathbf{2}}_j$ we have two possibilities: Either they are allowed to acquire an arbitrary VEV, in which case we denote the VEV by

$$\begin{pmatrix} \langle \psi_j^1 \rangle \\ \langle \psi_j^2 \rangle \end{pmatrix},$$

or they are allowed only a certain VEV structure. As discussed in section 3.2 there is only one such structure. We write the VEV of Higgs fields transforming as $\underline{\mathbf{2}}_j$ and acquiring this VEV structure as

$$\langle \psi_j \rangle = \begin{pmatrix} e^{\frac{-2\pi i j m}{n}} \\ 1 \end{pmatrix}.$$

Note that the VEV structure determines the relative phase between the two doublet components, but not the overall phase. We are thus in general free to decide which component we want to include the phase factor in. We will on several occasions want to make use of this freedom, to simplify the appearance of our mass matrix. For example, if we have a left-handed fermion transforming as $\underline{\mathbf{1}}_3$ or $\underline{\mathbf{1}}_4$ and the left-handed conjugate fermions transforming as $\underline{\mathbf{2}}_j$, we need a Higgs boson transforming as $\underline{\mathbf{2}}_{(\frac{n}{2}-j)}$ to form an invariant Yukawa coupling (as can be seen from the Kronecker products of appendix B.1). We then write the VEVs of these Higgs fields as

$$\langle \psi_{\frac{n}{2}-j} \rangle = \begin{pmatrix} (-1)^m \\ e^{\frac{-2\pi i j m}{n}} \end{pmatrix}.$$

In this way, only the phase factor $e^{\frac{-2\pi i j m}{n}}$ shows up in the mass matrix and not its complex conjugate. Similarly the VEV of Higgs fields transforming as $\underline{\mathbf{2}}_{(n-(j+k))}$ is written as

$$\langle \psi_{n-(j+k)} \rangle = \begin{pmatrix} 1 \\ e^{\frac{-2\pi i (j+k) m}{n}} \end{pmatrix},$$

so that it contains the same phase as that of the Higgs fields transforming as $\underline{\mathbf{2}}_{j+k}$.

As mentioned in section 3.4.1, our mass matrices are given for down-type fermions. To get the

²To be more precise: In all cases the representations of the Higgs fields have the property that their index is divisible by q , but this is not necessarily true for the representations under which the fermions transform. However, one can then always find some other representations for the fermions which reproduce exactly the same matrix structure and which have the property that their index is also divisible by q such that the case can be reduced to a smaller symmetry which is fully broken.

corresponding mass matrices for up-type fermions a few changes have to be implemented, as all VEVs need to be complex conjugated. This corresponds to the following substitutions:

$$\begin{aligned}
\langle \phi_i \rangle &\rightarrow \langle \phi_i \rangle^*, \\
\langle \psi_j^1 \rangle &\rightarrow \langle \psi_j^2 \rangle^* \quad \text{and} \quad \langle \psi_j^2 \rangle \rightarrow \langle \psi_j^1 \rangle^*, \\
\text{or } \langle \psi_j \rangle &\rightarrow \langle \psi_j \rangle^* e^{\frac{2\pi i j m}{n}}.
\end{aligned} \tag{3.16}$$

Finally, note that we have in general left all minus signs and phases in the mass matrices, even if they can be trivially rotated away.

3.4.3 Three Singlet Structure

To get a mass matrix with a non-zero determinant in the three singlet structure, at least one two-dimensional representation has to get a VEV, otherwise the second and third column of the mass matrix will be zero. This means we can ignore all subgroups, where no two-dimensional representation is allowed a VEV, leaving us with three subgroups to consider: $Z_2 = \langle \mathbf{BA}^m \rangle$, D_q and Z_q . The two possible two-dimensional representations that can show up are $\underline{\mathbf{2}}_{\mathbf{k}}$ and $\underline{\mathbf{2}}_{(\frac{n}{2}-\mathbf{k})}$ coming from the product of $\underline{\mathbf{2}}_{\mathbf{k}}$ with a one-dimensional representation.

We first consider the subgroup, where all two-dimensional representations acquire a VEV, that is $Z_2 = \langle \mathbf{BA}^m \rangle$.

$$Z_2 = \langle \mathbf{BA}^m \rangle$$

Due to the large number of possible combinations of one-dimensional representations, we shall only give rows as building blocks for a mass matrix. We group them according to the index j , i.e. according to the transformation properties of the first generation of left-handed conjugate fermions. We give the row vectors for the p -th row of the mass matrix, depending on the index i_p , i.e. the transformation properties of the p -th generation of left-handed fermions. We assume that m is even. To get to an odd m , we need to switch $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ and ϕ_3 and ϕ_4 , as can be inferred from table C.1.

$$\begin{aligned}
&j = 1, i_p = 1 \\
&(\kappa_p \langle \phi_1 \rangle, \quad \alpha_p \langle \psi_k \rangle, \quad \alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 2, 4, i_p = 1 \\
&(0, \quad \alpha_p \langle \psi_k \rangle, \quad \alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 3, i_p = 1 \\
&(\kappa_p \langle \phi_3 \rangle, \quad \alpha_p \langle \psi_k \rangle, \quad \alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 1, 3, i_p = 2 \\
&(0, \quad \alpha_p \langle \psi_k \rangle, \quad -\alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 2, i_p = 2 \\
&(\kappa_p \langle \phi_1 \rangle, \quad \alpha_p \langle \psi_k \rangle, \quad -\alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 4, i_p = 2 \\
&(\kappa_p \langle \phi_3 \rangle, \quad \alpha_p \langle \psi_k \rangle, \quad -\alpha_p \langle \psi_k \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 1, i_p = 3 \\
&(\kappa_p \langle \phi_3 \rangle, \quad \alpha_p \langle \psi_{\frac{n}{2}-k} \rangle, \quad \alpha_p \langle \psi_{\frac{n}{2}-k} \rangle e^{\frac{-2\pi i k m}{n}}) \\
&j = 2, 4, i_p = 3 \\
&(0, \quad \alpha_p \langle \psi_{\frac{n}{2}-k} \rangle, \quad \alpha_p \langle \psi_{\frac{n}{2}-k} \rangle e^{\frac{-2\pi i k m}{n}})
\end{aligned}$$

$$\begin{aligned}
& j = 3, i_p = 3 \\
& (\kappa_p < \phi_1 >, \quad \alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{\frac{-2\pi i k m}{n}}) \\
& j = 1, 3, i_p = 4 \\
& (0, \quad -\alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{\frac{-2\pi i k m}{n}}) \\
& j = 2, i_p = 4 \\
& (\kappa_p < \phi_3 >, \quad -\alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{\frac{-2\pi i k m}{n}}) \\
& j = 4, i_p = 4 \\
& (\kappa_p < \phi_1 >, \quad -\alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{\frac{-2\pi i k m}{n}})
\end{aligned}$$

These building blocks can be reproduced with the Clebsch Gordan coefficients given in appendix B.2.1, which are an essential ingredient for our discussion of the mass matrix structures in general.

The apparently large number of possible combinations actually give only one general option, a mass matrix with one zero entry. Mass matrices with more than one zero entry or no zero entry give a determinant of zero, and so will not be considered here. This is due to the fact that there is a correlation between the entry in the first column and the relative sign of the entries in the second and third column. A non-zero entry in the first column necessarily implies either a relative sign between the entries in the other two columns (for $j=2$ and $j=4$) or no relative sign (for $j=1$ and $j=3$) - we cannot however have both cases in the same mass matrix. To see what a typical mass matrix looks like, consider an example. Let $j = i_2 = 1$, $i_1 = 2$ and $i_3 = 3$. The mass matrix then reads

$$\begin{pmatrix}
0 & \alpha_1 < \psi_k > & -\alpha_1 < \psi_k > e^{\frac{-2\pi i k m}{n}} \\
\kappa_2 < \phi_1 > & \alpha_2 < \psi_k > & \alpha_2 < \psi_k > e^{\frac{-2\pi i k m}{n}} \\
\kappa_3 < \phi_3 > & \alpha_3 < \psi_{\frac{n}{2}-k} > & \alpha_3 < \psi_{\frac{n}{2}-k} > e^{\frac{-2\pi i k m}{n}}
\end{pmatrix}. \quad (3.17)$$

The scalar representations and the position of the zero (within the first column) can vary, depending on our assignment of fermion representations, but the general structure will always be the one texture zero structure of equation (3.13).

$$D_q = \langle \mathbf{A}^{\frac{n}{q}}, \mathbf{B}\mathbf{A}^m \rangle$$

We continue with those cases, where not all two-dimensional representations can acquire a VEV. For the subgroups D_q , we only need to consider the case where $\frac{n}{q}$ is odd. As the only two-dimensional representation for the fermions is $\mathbf{2}_k$, the only relevant two-dimensional representations for Higgs fields are $\mathbf{2}_k$ and $\mathbf{2}_{(\frac{n}{2}-k)}$. These are the only representations that arise when coupling $\mathbf{2}_k$ with a one-dimensional representation.

Now we can see from table C.2 that the Higgs bosons transforming under these representations can only acquire a VEV if q divides k and q divides $\frac{n}{2} - k$, respectively. If $\frac{n}{q}$ is even, these two conditions are equivalent, as $\frac{n}{q}$ being even implies that q divides $\frac{n}{2}$. Therefore, either neither Higgs field can receive a VEV, and we end up having the second and third columns equal to zero, or both Higgs fields can receive a VEV. In this second case we can write $n = 2c_n q$, $k = c_k q$ and conversely $\frac{n}{2} - k = (c_n - c_k)q$. We have only unfaithful representations, and are effectively dealing with a flavor group D_{2c_n} which is then broken down to $Z_2 = \langle \mathbf{B}\mathbf{A}^m \rangle$.

If, however $\frac{n}{q}$ is odd, the two conditions are mutually exclusive (except where $q=1$, but D_1 is the same as $Z_2 = \langle \mathbf{B}\mathbf{A}^m \rangle$), and we consider them separately below. We do not need to consider the case where neither of the two conditions is fulfilled, as then no Higgs field transforming under a two-dimensional representation will acquire a VEV and we are again left with a mass matrix containing two zero column vectors.

As in section 3.4.3, we consider the structure of the p -th row, depending on i_p , j and k . The simplest entry is the M_{p1} element. We have, since only ϕ_1 can acquire a VEV, that $M_{p1} = \kappa_p < \phi_1 >$ if $i_p = j$ or $M_{p1} = 0$ otherwise. We give the other entries as building block row vectors $(M_{p2} \ M_{p3})$. They are, for q dividing k ,

$$\begin{aligned} i_p = 1 : & \quad (\alpha_p < \psi_k >, \quad \alpha_p < \psi_k > e^{-\frac{2\pi i k m}{n}}) \\ i_p = 2 : & \quad (\alpha_p < \psi_k >, \quad -\alpha_p < \psi_k > e^{-\frac{2\pi i k m}{n}}) \\ i_p = 3, 4 : & \quad (0, \quad 0) \end{aligned}$$

and for q dividing $(\frac{n}{2} - k)$

$$\begin{aligned} i_p = 1, 2 : & \quad (0, \quad 0) \\ i_p = 3 : & \quad ((-1)^m \alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{-\frac{2\pi i k m}{n}}) \\ i_p = 4 : & \quad ((-1)^{m+1} \alpha_p < \psi_{\frac{n}{2}-k} >, \quad \alpha_p < \psi_{\frac{n}{2}-k} > e^{-\frac{2\pi i k m}{n}}) \end{aligned}$$

The multiple possibilities can again be reduced to one general form. This is again due to the fact that we have a correlation between the element in the first column and those in the second and third columns. Only those rows, for which $i_p = j$ can have a non-zero element in the first column. So if we choose the element in the first column to be non-zero, we have also determined whether the elements in the second and third column are zero. We must have at least one zero entry in the second and third column, otherwise we are only breaking a smaller G_f down to its subgroup $Z_2 = \langle \text{BA}^m \rangle$, a case we have already discussed. So, we need to choose j in such a way that zero elements in the second and third column are anti-correlated with a zero element in the first column, otherwise we would end up with a zero row vector. This means that $j = 3$ or $j = 4$ for q dividing k , while if q divides $(\frac{n}{2} - k)$ we must choose $j = 1$ or $j = 2$. We can then not have another row where $i_p = j$, because there again the elements in the second and third column would be zero and two row vectors would be linearly dependent. So exactly one of the i_p must be equal to j , say i_1 . And we can say even more: To ensure a non-zero determinant, i_2 and i_3 must be unequal, since otherwise the second and third row vector will be linearly dependent. We can thus write the particle content more exactly as $L \sim (\mathbf{1}_{i_1}, \mathbf{1}_{i_2}, \mathbf{1}_{i_3})$ and $L^c \sim (\mathbf{1}_{i_1}, \mathbf{2}_k)$, with all three one-dimensional representations mutually inequivalent. All mass matrices will then be of the block form of equation (3.12). We give as an example the entries for $i_1 = 3$, $i_2 = 1$, $i_3 = 2$ and q dividing k :

$$\begin{aligned} A = \kappa_1 < \phi_1 >, \quad B = \alpha_2 < \psi_k >, \quad C = \alpha_2 < \psi_k > e^{-\frac{2\pi i k m}{n}}, \\ D = \alpha_3 < \psi_k >, \quad E = -\alpha_3 < \psi_k > e^{-\frac{2\pi i k m}{n}}. \end{aligned} \quad (3.18)$$

$$\mathbf{Z}_q = \langle \mathbf{A}^{\frac{n}{q}} \rangle$$

The major difference between the subgroups D_q and Z_q is that for the latter all doublets will acquire arbitrary VEVs, and hence the mass matrices will exhibit less symmetry. As for the D_q subgroups, we only have to consider $\frac{n}{q}$ odd: As can be inferred from table C.1, all one-dimensional representations can acquire a VEV if $\frac{n}{q}$ is even. This would leave us with only two possibilities: Either again none of the relevant Higgs bosons that transform under a two-dimensional representation of the flavor group can acquire a VEV (if q does not divide k), or we are faced with the case where all Higgs fields relevant for Yukawa terms (that is both those transforming under one-dimensional and under two-dimensional representations) can acquire an arbitrary VEV - this corresponds to the case discussed in section 3.4.1 of a smaller flavor symmetry being fully broken.

So we can again set $\frac{n}{q}$ to be odd and start by giving the elements in the first column. They are $M_{p1} = \kappa_p < \phi_1 >$ if $i_p = j$, $M_{p1} = \kappa_p < \phi_2 >$ if $\mathbf{1}_{i_p} \times \mathbf{1}_j = \mathbf{1}_2$ or $M_{p1} = 0$ otherwise, whereas

the elements in the second and third column are, for q dividing k ,

$$\begin{aligned} i_p = 1: & (\alpha_p \langle \psi_k^2 \rangle, \alpha_p \langle \psi_k^1 \rangle) \\ i_p = 2: & (\alpha_p \langle \psi_k^2 \rangle, -\alpha_p \langle \psi_k^1 \rangle) \\ i_p = 3, 4: & (0, 0) \end{aligned}$$

and for q dividing $(\frac{n}{2}-k)$

$$\begin{aligned} i_p = 1, 2: & (0, 0) \\ i_p = 3: & (\alpha_p \langle \psi_{\frac{n}{2}-k}^1 \rangle, \alpha_p \langle \psi_{\frac{n}{2}-k}^2 \rangle) \\ i_p = 4: & (\alpha_p \langle \psi_{\frac{n}{2}-k}^1 \rangle, -\alpha_p \langle \psi_{\frac{n}{2}-k}^2 \rangle) \end{aligned}$$

We can reduce the large number of possible combinations. First we set j , which must either be in $\{1, 2\}$ or in $\{3, 4\}$. If we want the M_{p1} element to be non-zero, then i_p must be in the same set as j and we also know whether the elements in the second and third column are zero or not. If we now choose j in such a way that a non-zero element in the first column implies non-zero elements in the second and third column (and thereby a zero element in the first column implies a zero row vector), we are left to choose between a mass matrix with nine distinct non-zero elements and a mass matrix with at least one zero row vector and thereby a zero determinant. We therefore need to choose j in such a way that a non-zero element in the first column implies a zero in the second and third column, i.e. $j = 3$ or $j = 4$ if q divides k , $j = 1$ or $j = 2$ if q divides $\frac{n}{2} - k$. If we however choose two elements in the first column to be non-zero, then those two row vectors will be linearly dependent, and we will have a zero determinant. So, we need to choose one i_p in the same set as j , while the other two must lie outside that set - and, again, they cannot be equal, to ensure linear independence of the corresponding row vectors. The general structure will then always be a block matrix. We give as an example the entries for the case where $j = 3$, $i_1 = 4$, $i_2 = 1$, $i_3 = 2$ and q divides k :

$$\begin{aligned} A &= \kappa_1 \langle \phi_2 \rangle, \quad B = \alpha_2 \langle \psi_k^2 \rangle, \quad C = \alpha_2 \langle \psi_k^1 \rangle, \\ D &= \alpha_3 \langle \psi_k^2 \rangle, \quad E = -\alpha_3 \langle \psi_k^1 \rangle. \end{aligned} \tag{3.19}$$

3.4.4 Two Doublet Structure

In our discussion of the two doublet structure we frequently use an additional index, p , given by $\mathbf{1}_i \times \mathbf{1}_1 = \mathbf{1}_p$. We will assume $j \geq k$, that is the index of the two-dimensional representation under which the left-handed fermions transform is larger than the representation index for the left-handed conjugate fermions. This can be done without loss of generality, since the corresponding mass matrix structures for $j \leq k$ can simply be obtained by transposing the mass matrices given in this section.

For the two doublet structure we will discuss all possible subgroups, as they all give viable mass matrices. Another difference compared to the three singlet structure is that the mass matrices given in this section are also potential candidates for Majorana mass matrices, if we impose the conditions $j = k$ and $i = 1$. If a mass matrix can also be used as a Majorana mass matrix, we will mention this and briefly note which Yukawa couplings need to be equal in that case and which terms drop out due to anti-symmetry. A further discussion of Majorana mass matrices is then performed in section 3.4.6.

$$\mathbf{Z}_n = \langle \mathbf{A} \rangle$$

We read off from table C.1 that only $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ can get a VEV when conserving a \mathbf{Z}_n subgroup. This limits our freedom in choosing representations for the fermions: The two

doublets must couple to form a $\mathbf{1}_1$ or a $\mathbf{1}_2$, otherwise the second and third row vectors of the mass matrix will be zero. This imposes the condition $j = k$. Also, we need $p = 1$ or 2 , otherwise the first row vector will turn out to be zero. These restrictions leave us with a semi-diagonal mass matrix structure with entries

$$A = \kappa_1 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \phi_1 \rangle + \kappa_3 \langle \phi_2 \rangle, \quad C = \kappa_2 \langle \phi_1 \rangle - \kappa_3 \langle \phi_2 \rangle. \quad (3.20)$$

If $p = 1$ this is also a possible structure for a Majorana mass matrix. In this case the anti-symmetric part, i.e. the terms containing $\langle \phi_2 \rangle$, drop out.

$$\mathbf{Z}_{\frac{n}{2}} = \langle \mathbf{A}^2 \rangle$$

From table C.1 we infer that this subgroup only allows one-dimensional representations to acquire a VEV. So, we need the product of the two doublets to contain at least one one-dimensional representation. We are thereby left with three possibilities:

Case (i): $j = k$, $j + k \neq \frac{n}{2}$ gives a semi-diagonal matrix.

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_1 \rangle + \kappa_2 \langle \phi_2 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle - \kappa_2 \langle \phi_2 \rangle. \quad (3.21)$$

Case (ii): $j + k = \frac{n}{2}$, $j = k = \frac{n}{4}$ gives a block structure.

$$\begin{aligned} A &= \kappa_5 \langle \phi_p \rangle, \quad B = \kappa_3 \langle \phi_3 \rangle + \kappa_4 \langle \phi_4 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle + \kappa_2 \langle \phi_2 \rangle, \\ D &= \kappa_1 \langle \phi_1 \rangle - \kappa_2 \langle \phi_2 \rangle, \quad E = \kappa_3 \langle \phi_3 \rangle - \kappa_4 \langle \phi_4 \rangle. \end{aligned} \quad (3.22)$$

Case (iii): $j \neq k$, $j + k = \frac{n}{2}$ gives a diagonal structure.

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle + \kappa_2 \langle \phi_4 \rangle, \quad C = \kappa_1 \langle \phi_3 \rangle - \kappa_2 \langle \phi_4 \rangle. \quad (3.23)$$

Cases (i) and (ii) are also a possibility for Majorana mass matrices. In this case the anti-symmetric terms containing $\langle \phi_2 \rangle$ drop out.

$$\mathbf{Z}_q = \langle \mathbf{A}^{\frac{n}{q}} \rangle$$

This subgroup requires quite an amount of case differentiation, since we want to make the discussion as general as possible, and allow all possible relations between q and the other indices of the model. We will first discuss the case where $\frac{n}{q}$ is even, and then, at the end of this subsection, discuss the slight changes induced by $\frac{n}{q}$ being odd.

As an ordering principle in our discussion, we have taken the structure of the resulting mass matrix, as only the three characteristic types discussed in section 3.4.1 will show up.

Most of the conditions deal with the question which two-dimensional representations are allowed a VEV. This translates directly into deciding whether q divides the index of that representation. The two-dimensional representations which can show up are $\mathbf{2}_j$ and $\mathbf{2}_k$, $\mathbf{2}_{(\frac{n}{2}-j)}$ and $\mathbf{2}_{(\frac{n}{2}-k)}$ from the coupling of two-dimensional with one-dimensional representations and $\mathbf{2}_{j+k}$, $\mathbf{2}_{(n-(j+k))}$ and $\mathbf{2}_{j-k}$ from the coupling of the two two-dimensional representations, cf. appendix B.1. We will only give mass matrices for the case where $\mathbf{2}_j \times \mathbf{2}_k$ contains $\mathbf{2}_{j+k}$. Mass matrices for the case where it contains $\mathbf{2}_{(n-(j+k))}$ can be obtained by replacing ψ_{j+k} by $\psi_{n-(j+k)}$ and then switching the components of the doublet.

As q must divide n for Z_q to be a subgroup of D_n , q dividing $j + k$ and q dividing $n - (j + k)$ are equivalent. As already noted, q dividing j is equivalent to q dividing $\frac{n}{2} - j$, if $\frac{n}{q}$ is even, which we assume for this discussion.

To ensure a non-zero determinant, q must at least divide either $(j - k)$ (This includes the case

	j	k	j+k	j-k	Structure
q divides					Det[M]=0
q divides			×		Diagonal
q divides				×	Semi-diagonal
q divides	×				Det[M]=0
q divides		×			Det[M]=0
q divides			×	×	Block
q divides	×	×	×	×	Full

Table 3.1: Index relations and corresponding mass matrix structure. For a conserved subgroup Z_q or D_q , one determines which of the relevant doublet indices j , k , $(j+k)$ and $(j-k)$ are divided by q and can then read off the resulting mass matrix structure.

$j=k$, as all numbers divide zero, corresponding to the fact that $\mathbf{1}_1$ and $\mathbf{1}_2$ can get a VEV for an arbitrary q) or $(j+k)$: If not, the two by two submatrix in the lower right-hand corner of the mass matrix will be zero. This implies directly that q must divide either both j and k , or neither of the two. If q divides j and k however, then it also divides $j+k$ and $j-k$ - hence all relevant two-dimensional representations can acquire an arbitrary VEV and all one-dimensional representations can acquire a VEV anyway. As this leads to the case where the mass matrix contains nine distinct entries, we disregard it. We summarize our findings in table 3.1 and discuss the different cases in detail below.

Diagonal Matrix This structure appears in the following case: q must divide $(j+k)$ but q does not divide $(j-k)$, j or k . Note that this case is not possible for $q=2$, since the sum and the difference of two numbers are either both odd or both even, nor is it possible for $j=k$. For $j+k \neq \frac{n}{2}$ this gives

$$A = \kappa_1 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \psi_{j+k}^2 \rangle, \quad C = \kappa_2 \langle \psi_{j+k}^1 \rangle. \quad (3.24)$$

If $j+k = \frac{n}{2}$ the mass matrix entries are

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle + \kappa_2 \langle \phi_4 \rangle, \quad C = \kappa_1 \langle \phi_3 \rangle - \kappa_2 \langle \phi_4 \rangle. \quad (3.25)$$

Semi-diagonal Matrix This structure shows up, if q divides $(j-k)$, but not $(j+k)$, j or k . Again, this is not possible if $q=2$, nor is it possible for $j+k = \frac{n}{2}$, since this contradicts the conditions $q \nmid (j+k) = \frac{n}{2}$ and $\frac{n}{q}$ being even. This leaves two cases: For $j \neq k$ the mass matrix entries are

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \psi_{j-k}^2 \rangle, \quad C = \kappa_1 \langle \psi_{j-k}^1 \rangle, \quad (3.26)$$

while for $j=k$ we get

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_1 \rangle + \kappa_2 \langle \phi_2 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle - \kappa_2 \langle \phi_2 \rangle, \quad (3.27)$$

which is a candidate for a Majorana mass matrix if we omit the anti-symmetric terms.

Block Matrix This structure shows up, if q divides $(j-k)$ and $(j+k)$, but not j and k . This forces q to be even, as q must divide $2j = (j-k) + (j+k)$ while not dividing j , that is, a factor of 2 is relevant for making a number divisible by q . In case $j+k = \frac{n}{2}$, $j \neq k$ we get

$$\begin{aligned} A &= \kappa_4 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle + \kappa_2 \langle \phi_4 \rangle, \quad C = \kappa_3 \langle \psi_{j-k}^2 \rangle, \\ D &= \kappa_3 \langle \psi_{j-k}^1 \rangle, \quad E = \kappa_1 \langle \phi_3 \rangle - \kappa_2 \langle \phi_4 \rangle. \end{aligned} \quad (3.28)$$

In case $j + k \neq \frac{n}{2}$ and $j \neq k$, we get

$$\begin{aligned} A &= \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \psi_{j+k}^2 \rangle, \quad C = \kappa_1 \langle \psi_{j-k}^2 \rangle, \\ D &= \kappa_1 \langle \psi_{j-k}^1 \rangle, \quad E = \kappa_2 \langle \psi_{j+k}^1 \rangle, \end{aligned} \quad (3.29)$$

and if $j+k \neq \frac{n}{2}$ and $j = k$, we get

$$\begin{aligned} A &= \kappa_4 \langle \phi_p \rangle, \quad B = \kappa_3 \langle \psi_{2j}^2 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle + \kappa_2 \langle \phi_2 \rangle, \\ D &= \kappa_1 \langle \phi_1 \rangle - \kappa_2 \langle \phi_2 \rangle, \quad E = \kappa_3 \langle \psi_{2j}^1 \rangle. \end{aligned} \quad (3.30)$$

Finally, concerning the case $j = k$, $j + k = \frac{n}{2}$: this leads us to the conditions $q \mid \frac{n}{2}$ and $q \nmid \frac{n}{4} = j = k$, which implies either $q = \frac{n}{2}$ (already covered) or q even and $\frac{n}{4}$ odd. The mass matrices are the same in both cases, with entries:

$$\begin{aligned} A &= \kappa_5 \langle \phi_p \rangle, \quad B = \kappa_3 \langle \phi_3 \rangle + \kappa_4 \langle \phi_4 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle + \kappa_2 \langle \phi_2 \rangle, \\ D &= \kappa_1 \langle \phi_1 \rangle - \kappa_2 \langle \phi_2 \rangle, \quad E = \kappa_3 \langle \phi_3 \rangle - \kappa_4 \langle \phi_4 \rangle. \end{aligned} \quad (3.31)$$

For $\frac{n}{q}$ even, q either divides both j and $(\frac{n}{2} - j)$ or neither, and we need no further case differentiation.

If however $\frac{n}{q}$ is odd, we need to pay closer attention. In case i and l are both in $\{1, 2\}$, the discussion is as above, since $\underline{\mathbf{2}}(\frac{n}{2}-j)$ and $\underline{\mathbf{2}}(\frac{n}{2}-k)$ will not show up as the representation of a Higgs field. In case i and l are both in $\{3, 4\}$ we can also use the above discussion if we substitute j by $(\frac{n}{2} - j)$ and k by $(\frac{n}{2} - k)$ in the conditions, which changes nothing concerning the divisibility of sums and differences, as q will always divide n .

Further changes occur due to the fact that $\phi_{3,4} \sim \underline{\mathbf{1}}_{\mathbf{3,4}}$ are not allowed a VEV anymore, see table C.1. This is relevant for the last remaining case: We drop for a moment the condition that $j \geq k$ and instead impose the condition $i \in \{1, 2\}$ and $l \in \{3, 4\}$ (as opposed to the other way around). This means that M_{11} will be zero, since $p = 3$ or 4 and ϕ_p is then not allowed a VEV due to $\frac{n}{q}$ being odd. If we now want to avoid having a zero column or row vector in our mass matrix, both $\psi_{\frac{n}{2}-j}$ and ψ_k must acquire a VEV, i.e. q must divide k and $(\frac{n}{2} - j)$. By assumption however, q does not divide $\frac{n}{2}$, which leads us straight to the conclusion that q does not divide $(j+k)$ or $|j-k|$, thereby leaving the two-by-two matrix in the lower right-hand corner of the mass matrix zero, which results in a vanishing determinant.

$Z_2 = \langle \mathbf{B} \mathbf{A}^m \rangle$

As this structure also strongly depends on the one-dimensional representations under which the fermions transform, we have reduced it entirely to building blocks, to avoid having to deal with too many subcases. As we can read off table C.1, all Higgs bosons transforming under a two-dimensional representation will acquire a structured VEV. The only constraints that arise are therefore due to the Higgs bosons transforming under one-dimensional representations. This means that the M_{11} entry is of special interest. We will first write down the general structure and then use this to explain why M_{11} has to be non-zero to ensure a non-zero determinant:

$$\begin{pmatrix} \kappa_3 w & \kappa_4 X_1 & \kappa_4 Y_1 e^{-\frac{2\pi i k m}{n}} \\ \kappa_5 X_2 & \kappa_1 u & \kappa_2 v \\ \kappa_5 Y_2 e^{-\frac{2\pi i j m}{n}} & \kappa_2 v e^{-\frac{2\pi i (j-k)m}{n}} & \kappa_1 u e^{-\frac{2\pi i (j+k)m}{n}} \end{pmatrix}, \quad (3.32)$$

where:

$u = \langle \phi_3 \rangle$ if $j + k = \frac{n}{2}$, m even

	i = 1	i = 2	i = 3	i = 4
X_1	$\langle \psi_k \rangle$	$\langle \psi_k \rangle$	$(-1)^m \langle \psi_{\frac{n}{2}-k} \rangle$	$(-1)^{m+1} \langle \psi_{\frac{n}{2}-k} \rangle$
Y_1	$\langle \psi_k \rangle$	$-\langle \psi_k \rangle$	$\langle \psi_{\frac{n}{2}-k} \rangle$	$\langle \psi_{\frac{n}{2}-k} \rangle$

Table 3.2: Building blocks for matrix of two doublet structure under the subgroups $Z_2 = \langle \text{BA}^m \rangle$

$u = \langle \phi_4 \rangle$ if $j + k = \frac{n}{2}$, m odd

$u = \langle \psi_{\min[j+k, n-j-k]} \rangle$ if $j + k \neq \frac{n}{2}$

$v = \langle \phi_1 \rangle$ if $j = k$

$v = \langle \psi_{j-k} \rangle$ if $j \neq k$

$w = \langle \phi_p \rangle$ where $p=1,3$ if m even, $p=1,4$ if m odd

Note that some of the phase factors degenerate to signs, in case of $j=k$ or $j+k=\frac{n}{2}$. X_i and Y_i depend on the transformation properties of the doublets and singlets involved, i.e. X_1 and Y_1 depend on i and k , X_2 and Y_2 depend on j and l . Building blocks for X_1 and Y_1 are given in table 3.2. The same table can be used for X_2 and Y_2 , substituting l for i and j for k .

For the discussion of the M_{11} element let us assume that m is even; the reasoning for an odd m will be analogous. $w = 0$ implies that $p=2$ or 4 , that is $i=1$ or 3 while $l=2$ or 4 . Switching i and l is also possible - the reasoning is again analogous. If we now consult the table for the X and Y entries, we see that this implies a relative minus sign between the X_2 and Y_2 entries, while there is no relative minus sign between the X_1 and Y_1 entries, so the sum of the second and third row vector is proportional to the first row vector. This is true also with non-trivial phases. Therefore $w \neq 0$ must hold.

If $j = k$ and $i = 1$, the matrix can be made symmetric by imposing $\kappa_4 = \kappa_5$ and can then also show up as a Majorana mass matrix.

$D_{\frac{n}{2}} = \langle \mathbf{A}^2, \mathbf{B} \rangle$ and $D_{\frac{n}{2}} = \langle \mathbf{A}^2, \mathbf{BA} \rangle$

The discussion for these subgroups is very similar to that of the subgroup $Z_{\frac{n}{2}}$, as only one-dimensional representations can receive a VEV, in this case, however, not all of them (see table C.2). We obtain the mass matrices for $D_{\frac{n}{2}} = \langle \mathbf{A}^2, \mathbf{B} \rangle$ by simply eliminating all terms containing ϕ_2 and ϕ_4 from the mass matrices for $Z_{\frac{n}{2}}$, making sure that we do not end up with a zero determinant. To prevent this, $\mathbf{1}_p$ must be allowed a VEV, that is p must be 1 or 3. We can then distinguish between the following three sub-cases as for $Z_{\frac{n}{2}}$:

Case (i): $j = k$ gives a semi-diagonal structure.

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_1 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle \quad (3.33)$$

Case (ii): $j = k, j + k = \frac{n}{2}$ gives a block structure.

$$\begin{aligned} A &= \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \phi_3 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle, \\ D &= \kappa_1 \langle \phi_1 \rangle, \quad E = \kappa_2 \langle \phi_3 \rangle \end{aligned} \quad (3.34)$$

Case (iii): $j \neq k, j + k = \frac{n}{2}$ gives a diagonal structure.

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle, \quad C = \kappa_1 \langle \phi_3 \rangle \quad (3.35)$$

Note that in cases (i) and (iii) we get two degenerate eigenvalues. We can get the mass matrices generated by breaking to $D_{\frac{n}{2}} = \langle \mathbf{A}^2, \mathbf{BA} \rangle$ by substituting ϕ_3 by ϕ_4 in the matrices above.

In this case p must either be 1 or 4 and there is also a relative minus sign between the two occurrences of $\langle \phi_4 \rangle$. Case (i) and (ii) are applicable for Majorana mass matrices - in this case $p=1$ since i must be equal to l .

$$D_q = \langle \mathbf{A}^{\frac{n}{q}}, \mathbf{B}\mathbf{A}^m \rangle$$

This case is very similar to the Z_q case. The two differences are: a) Less one-dimensional representations are allowed a VEV and b) the two-dimensional representations are only allowed a structured VEV (see table C.2). The first difference implies restrictions on the index p - if the entry in the upper left-hand corner is zero, the determinant is zero too. So, we need to impose the condition that $\mathbf{1}_p$ is allowed a VEV which depends on the evenness and oddness of m and $\frac{n}{q}$, see table C.2.

The second difference means replacing the arbitrary doublet VEVs by two components with the same absolute value and fixed relative phase. The case where we have nine independent entries thereby does not show up here - instead the full matrix structure now corresponds to breaking a smaller G_f down to $Z_2 = \langle \mathbf{B}\mathbf{A}^m \rangle$. We briefly comment on this at the end of this subsection.

We perform the entire discussion assuming an even m . This means p has to be either 1 or 3 ($\frac{n}{q}$ even). To get the mass matrices for an odd m replace ϕ_3 by ϕ_4 . The relative signs that occur in this case are encoded in factors of $(-1)^m$, which are therefore left in, even though the rest of the discussion concerns only even m .

As for the case of Z_q there are no major differences between $\frac{n}{q}$ being even or odd, except that we need to impose the condition $p = 1$, if $\frac{n}{q}$ is odd, as then only $\mathbf{1}_1$ is allowed a VEV. Furthermore, if i and l are in $\{3, 4\}$, one must also replace j and k by $(\frac{n}{2} - j)$ and $(\frac{n}{2} - k)$ in the conditions, respectively. The ordering we use is the same as in section 3.4.4, that is the mass matrices are classified according to their structure.

Diagonal Matrix For this structure we must have q dividing $(j+k)$ but q not dividing $(j-k)$, j or k . This is neither possible for $q=2$ nor for $j = k$. For $j+k \neq \frac{n}{2}$ this gives

$$A = \kappa_1 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \psi_{j+k} \rangle, \quad C = \kappa_2 \langle \psi_{j+k} \rangle e^{-\frac{2\pi im(j+k)}{n}}. \quad (3.36)$$

If $j+k = \frac{n}{2}$ the mass matrix entries are

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle, \quad C = (-1)^m \kappa_1 \langle \phi_3 \rangle. \quad (3.37)$$

Note that for both of these matrices the squared mass matrix MM^\dagger has two degenerate eigenvalues.

Semi-diagonal Matrix For this structure q must divide $(j-k)$, but not $(j+k)$, j or k . This is neither possible if $q = 2$, nor is it possible for $j+k = \frac{n}{2}$. We are left with two cases: If $j \neq k$ we end up with

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \psi_{j-k} \rangle, \quad C = \kappa_1 \langle \psi_{j-k} \rangle e^{-\frac{2\pi im(j-k)}{n}}, \quad (3.38)$$

while for $j = k$ we get

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_1 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle, \quad (3.39)$$

which is a candidate for a Majorana mass matrix if $i = 1$. Both these matrices give degenerate eigenvalues in the squared mass matrix.

Block Matrix This structure shows up, for q dividing $(j - k)$ and $(j + k)$, but not j and k . This forces q to be even. For $j + k = \frac{n}{2}$, $j \neq k$ we get

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_1 \langle \phi_3 \rangle, \quad C = \kappa_3 \langle \psi_{j-k} \rangle, \quad (3.40)$$

$$D = \kappa_3 \langle \psi_{j-k} \rangle e^{-\frac{2\pi im(j-k)}{n}}, \quad E = (-1)^m \kappa_1 \langle \phi_3 \rangle. \quad (3.41)$$

In case $j + k \neq \frac{n}{2}$ and $j \neq k$, we get

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \psi_{j+k} \rangle, \quad C = \kappa_1 \langle \psi_{j-k} \rangle, \\ D = \kappa_1 \langle \psi_{j-k} \rangle e^{-\frac{2\pi im(j-k)}{n}}, \quad E = \kappa_2 \langle \psi_{j+k} \rangle e^{-\frac{2\pi im(j+k)}{n}}. \quad (3.42)$$

If $j + k \neq \frac{n}{2}$ and $j = k$, we get

$$A = \kappa_2 \langle \phi_p \rangle, \quad B = \kappa_3 \langle \psi_{2j} \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle, \\ D = \kappa_1 \langle \phi_1 \rangle, \quad E = \kappa_3 \langle \psi_{2j} \rangle e^{-\frac{4\pi imj}{n}}, \quad (3.43)$$

and if $j + k = \frac{n}{2}$ and $j = k = \frac{n}{4}$, that is for q even and $\frac{n}{4}$ odd, we get

$$A = \kappa_3 \langle \phi_p \rangle, \quad B = \kappa_2 \langle \phi_3 \rangle, \quad C = \kappa_1 \langle \phi_1 \rangle, \\ D = \kappa_1 \langle \phi_1 \rangle, \quad E = (-1)^m \kappa_2 \langle \phi_3 \rangle. \quad (3.44)$$

Full Matrix If q divides all relevant indices, that is if it divides j and k and thereby automatically divides $(j - k)$ and $(j + k)$, we are actually breaking a smaller G_f down to its subgroup $Z_2 = \langle BA^m \rangle$, as discussed in section 3.4.3. This is because $Z_2 = \langle BA^m \rangle$ is equivalent to D_1 , where the above conditions are automatically fulfilled, as 1 divides any integer. We therefore do not need to consider this case.

3.4.5 Mass Matrices in D'_n

The double-valued dihedral groups have subgroup-conserving VEV structures that are very similar to those of their single-valued counterparts. A general correspondence between subgroups of D_n and D'_n can be established, by looking at the allowed VEVs in tables C.1 to C.4:

D_n	Z_n	$Z_{\frac{n}{2}}$	Z_q	D_2	$D_{\frac{n}{2}}$	D_q	$Z_{(q=2)}$	Z_2
D'_n	Z_{2n}	Z_n	Z_q	Z_4	$D'_{\frac{n}{2}}$	$D'_{\frac{q}{2}}$	Z_2	—

As one can see, each subgroup of a single-valued dihedral group has a counterpart. One can thus expect that no new mass matrix structures appear for double-valued dihedral groups and a detailed analysis, as performed for single-valued groups in the last two sections, shows that this is indeed the case. But rather than giving the, rather uninteresting, details of this analysis here, we will instead give a short, but complete, heuristic explanation for the absence of new structures.

The only relevant difference between single- and double-valued groups is the existence of odd representations. Therefore, at least one odd representation should show up in our model in order to find possible new mass matrix structures.

For scalars transforming under odd two-dimensional representations, we find that they do not get a structured VEV. In case they are allowed to get a VEV, the even two-dimensional representations are also allowed arbitrary VEVs. This is only possible for Z_q subgroups, as can be seen from table C.3. We thus have the same situation as for the Z_q subgroups of single-valued groups: All doublets whose index is divided by q can get arbitrary VEVs. The fact that some of them are odd and some are even is only reflected in additional signs due to differences in

the Clebsch Gordan coefficients (see appendix B.2.2). These can however be absorbed into the VEVs, as these are arbitrary anyway, and which doublets are odd and which are even becomes irrelevant. We might as well be in a single-valued group D_{2n} instead of using D'_n .

Similarly, for the odd one-dimensional representations, i.e. $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ of D'_n with n odd, only two subgroups allow non-vanishing VEVs, namely Z_n and Z_q for $\frac{2n}{q}$ being even. In both cases $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ simultaneously get a VEV. This again means that the sign changes in the Clebsch Gordan coefficients can be trivially absorbed.

One should further note that, if at least one of the fermion representations is an odd two-dimensional representation, this results in the 3rd column being multiplied by -1 for the mass matrix structures of equations (3.10), (3.11) and (3.12). These additional signs can be rotated away by redefining the left-handed conjugate fermions. We would need the mass matrix structure of equation (3.13) or (3.15) for this sign change to have phenomenological consequences. These mass matrix structures however only appear for the subgroup $Z_2 = \langle \text{BA}^m \rangle$, which has no counterpart for double-valued groups, so that these two structures do not arise for double-valued groups.

Strictly speaking, we encounter the mass matrix structures of equations (3.13) and (3.15) for double-valued groups, but all such cases can be reduced to a single-valued group. In the simplest case we have used only even representations for the fermions. Then it is clear that one could also have used a single-valued group right from the start. Another case occurs, if all the fermions transform as odd representations. Then the Higgs fields have to transform as even ones. One cannot simply reduce this case to a single-valued group, as one does not find odd representations in D_n groups. However, one always finds some equivalent assignment for the fermions using only even representations which leads to the same mass matrix structure and which can then be reduced to the case of a single-valued group. Furthermore, one finds cases in which the mass matrix is allowed to have arbitrary entries, but there exists no smaller symmetry of the original group which is fully broken. This is the same case as already mentioned for the D_n groups, if the unbroken subgroup is Z_q , and we can again find equivalent assignments for the fermions which result in the same matrix structure and which indeed correspond to a smaller group being fully broken.

We thus conclude that no new Dirac mass matrix structures appear for double-valued groups. Still, when transferring our results for D_n to D'_n with odd n , one needs to keep in mind that the representations $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ are complex (conjugated). This implies that $\underline{\mathbf{1}}_3 \times \underline{\mathbf{1}}_4 = \underline{\mathbf{1}}_1$, so we have to replace $\langle \phi_3 \rangle$ in the mass matrices by $\langle \phi_4 \rangle$, and vice versa, even where they show up only implicitly as $\langle \phi_p \rangle$. This also leads to differences, when switching to up-type mass matrices: ϕ_4^* transforms as $\underline{\mathbf{1}}_3$. So if we encounter a $\langle \phi_3 \rangle$ in the down-type mass matrix, we need a $\langle \phi_4 \rangle^*$ for the up-type mass matrix. For odd two-dimensional representations an additional minus sign is introduced along with the second component of the VEV when switching to the up-type mass matrix, due to the matrix U introduced in section 3.1.2. All these changes do not lead to new structures.

The remainder of this thesis will thus mainly be concerned with the single-valued dihedral groups D_n . This is not to say that the double-valued groups are entirely useless as flavor symmetries. However, their special properties are not reflected in their subgroup structure and therefore are not relevant in our ansatz of using conserved subgroups to obtain mass independent textures.

3.4.6 Majorana Mass Matrices

Majorana mass matrices correspond to Yukawa couplings involving two identical fermions, i.e. $L = L^c$. When determining the pure Majorana mass matrices for left-handed neutrinos, the $SU(2)_L$ doublets of the Dirac mass matrices need to be replaced by $SU(2)_L$ triplets, which naturally does not change the flavor structure, or by the product of two flavor-charged SM

Higgs fields. This does not change the structure either: The product of two VEVs cannot break a subgroup that is conserved by the two of them separately. Hence no additional terms in the mass matrix can arise. Also, since a total singlet under G_f is always allowed a VEV, we can construct any term in the mass matrix that is allowed when coupling to one scalar only. Our results for Majorana mass matrices can of course also be used for the mass matrices of the pure Majorana mass term of left-handed conjugate neutrinos, in which case the Higgs fields are replaced by gauge singlet scalars and the VEVs of total singlets can be replaced by direct mass terms. In any case, the fact that we couple identical fermions, forces Majorana mass matrices to be symmetric.

As already mentioned at the beginning of this section on mass matrices, we keep our constraint of a non-vanishing determinant, although this is no longer phenomenologically motivated for neutrino Majorana masses, in order to keep the discussion manageable.

In section 3.4.4 we always mentioned when and how Majorana mass terms can show up if the Majorana fermions transform under one two- and one one-dimensional representation of G_f , as these correspond to mass matrices of the two doublet type. For D'_n , we have to mention that in case of odd two-dimensional representations the terms containing $\langle \phi_1 \rangle$ are anti-symmetric. So, whenever we remark in 3.4.4 that the anti-symmetric terms containing $\langle \phi_2 \rangle$ drop out, it is instead the terms containing $\langle \phi_1 \rangle$ that will drop out. This does not lead to new structures compared to the Dirac case.

We should however for completeness discuss another kind of structure: By allowing the three singlet structure for the charged lepton mass matrix, we have allowed for the possibility that the left-handed leptons transform as three one-dimensional representations of G_f . For a pure Majorana mass term for left-handed neutrinos this would mean that all fermions in the Yukawa term transform under one-dimensional representations, a case we have not considered so far.

For D_n with n arbitrary and for D'_n with n even we know that, in addition to the Majorana mass matrix being symmetric, all diagonal entries will be non-zero, as two identical one-dimensional representations always couple to form a trivial representation.

We first discuss the case of n even for D_n and D'_n . Looking at tables C.1 to C.4 we see that depending on the preserved subgroup the following one-dimensional representations can get a VEV: only $\underline{\mathbf{1}}_1$, $\underline{\mathbf{1}}_n$ and $\underline{\mathbf{1}}_i$ with $i \neq 1$ or all one-dimensional representations. Especially, the case in which three representations get a VEV is excluded. Concerning the assignment of the fermions we can distinguish the following three cases: either all three generations transform in the same way or two of them transform as the same representation or all three transform as different representations. If the three generations are assigned to $(\underline{\mathbf{1}}_i, \underline{\mathbf{1}}_i, \underline{\mathbf{1}}_i)$, all mass matrix entries are non-zero, i.e. the Majorana mass matrix is a general symmetric matrix with 6 independent parameters, as $\underline{\mathbf{1}}_i \times \underline{\mathbf{1}}_i = \underline{\mathbf{1}}_1$ holds. For the assignment $(\underline{\mathbf{1}}_{i_1}, \underline{\mathbf{1}}_{i_1}, \underline{\mathbf{1}}_{i_2})$ there exist two possible structures: either the matrix has a block structure or it has 6 independent entries. In the first case one has to ensure that $\underline{\mathbf{1}}_{i_1} \times \underline{\mathbf{1}}_{i_2}$ is not allowed a VEV by the preserved subgroup, while in the second case $\underline{\mathbf{1}}_{i_1} \times \underline{\mathbf{1}}_{i_2}$ should also acquire a VEV. In the last case, all fermions transform under different one-dimensional representations, which allows apart from the block and the arbitrary structure the possibility of having a matrix with non-vanishing entries on the diagonal only. The case only occurs, if the preserved subgroup only allows $\underline{\mathbf{1}}_1$ to acquire a VEV and therefore the flavor symmetry is not broken in the Majorana mass sector.

For the case of D_n with n odd, we only have two one-dimensional representations to choose from. Therefore at least two generations of fermions have to transform under the same representation, forbidding the structure with non-vanishing entries only on the diagonal.

For D'_n with n odd, structures not found above could only arise from fermion assignments involving the representations $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$, as $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ are complex and hence $\underline{\mathbf{1}}_3 \times \underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4 \times \underline{\mathbf{1}}_4 = \underline{\mathbf{1}}_2$. This means if $\underline{\mathbf{1}}_2$ is not allowed a VEV, we can have zero elements on the diagonal. However, we find that if $\underline{\mathbf{1}}_2$ is not allowed a VEV, only the trivial representation

$\underline{\mathbf{1}}_1$ is allowed a VEV for n odd, i.e. the dihedral symmetry is unbroken in the Majorana mass sector. The structure arising from $L \sim (\underline{\mathbf{1}}_{i_1}, \underline{\mathbf{1}}_{i_2}, \underline{\mathbf{1}}_{i_3})$ with $i_1 \in \{1, 2\}$ and $i_2, i_3 \in \{3, 4\}$, $i_2 \neq i_3$ is then the same as in case of the two doublet assignment, if the two odd one-dimensional representations $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ are replaced by the doublet, i.e. it is a semi-diagonal matrix with the obvious restriction that C equals B .

3.5 Diagonalization and Mixing Matrices

What we are now interested in most is the phenomenological implications of the mass matrices determined in the last section, i.e. their eigenvalues and the matrices that diagonalize them. We want to allow for, indeed will need, the possibility of breaking to different subgroups in different sectors, that is the scalar VEVs generating, for example, the up-type quark mass matrix will conserve a different subgroup than the VEVs generating the down-type quark mass matrix. Because of this, we will start by only considering the mixing generated in the up (neutrino) or the down (charged lepton) sector. We then discuss the combination of both, i.e. the possible CKM and PMNS mixing matrices, at the end of the section.

Even though the diagonalization matrices we present in the following only describe the mixing arising in one sector, we still attempt to give them using the familiar parameterization of the CKM-matrix for the mixing angles, to increase readability. We will not be able to put them in the standard parameterization for the phases, as this may need rephasing, which can only be done once the mixing from the up and the down sector has been combined.

We will again mainly be considering Dirac mass matrices, commenting on modifications for Majorana mass matrices as we go along. For a given mass matrix M , only the mixing of the left-handed fermions will be observable in the weak interaction. This mixing is given by the the diagonalization matrix U_L which is calculated by demanding that $U_L^\dagger M M^\dagger U_L$ be diagonal. For Majorana fermions the diagonalization matrix will also contain additional phases, which will strongly depend on the phases of the elements of M , i.e. will be mostly independent of the underlying group structure. Since the possible Majorana phases of neutrinos have not been measured, they will anyhow not serve as a criterion for finding viable models, and will not be discussed in this section.

For these reasons the relevant matrix structure is not actually that of the mass matrix itself, but rather of the squared mass matrix MM^\dagger . As shown in the last section, we only have a limited selection of mass matrices. When considering the squared mass matrices, the selection is even more limited, and we can reduce our discussion to four general types, which we will discuss one by one.

Note that, as in the last section, we do not discuss cases which differ only in a permutation of the fermion generations. Even if these give different diagonalization matrices, the permutation will cancel, when considering the net mixing of left-handed fermions. The question of assigning eigenvalues is somewhat more tricky. We will in general just use one eigenvalue assignment and will give that assignment explicitly. Exchanging the assignment of eigenvalues is then equivalent to exchanging columns in the diagonalization matrix. For the CKM and PMNS mixing matrices this means that switching eigenvalue assignments in the up sector corresponds to exchanging rows, while switching eigenvalues in the down sector corresponds to exchanging columns. The assignment of eigenvalues we use is almost entirely arbitrary. The only guiding principle we have used is that the diagonalization matrix should not have zeroes on the diagonal, i.e. should not look too much like a permutation matrix. Also, if one eigenvalue is larger than another one independent of parameter values, we have assigned the larger one to a higher generation than the smaller one. An additional arbitrariness could also arise, if two eigenvalues are degenerate by chance, i.e. only due to our numerical choice of parameters and not because

of the structure of the mass matrix. But, since no mass degenerate fermion generations are known to exist, we will not dwell on this possibility any further.

We have grouped the discussion according to the four types of squared mass matrices. For each type, we will at first give the general form, along with the diagonalization matrix and the eigenvalues. We will then give further details in tabular form. This further information directly relates to the observations made in the last section: We give a list of all mass matrix structures which can lead to a given squared mass matrix structure and show how the parameters of the mass matrix and the squared mass matrix are related.

A word on notation: We have used the abbreviations $c_\theta = \cos \theta$ and $s_\theta = \sin \theta$. As in the last section, the entries in the mass matrices will be denoted by capital letters from the beginning of the alphabet and are complex in general. The entries in the squared mass matrices will be denoted by lower case letters from the beginning of the alphabet, and will be real in general: All phases will be given explicitly. Signs need not necessarily be absorbed in these phases, we will thus always mention whether parameters are allowed to be negative. For the phases we continue to use $\phi = \frac{2\pi m}{n}$. When giving the general structure of the mass matrices, we will place the phases where they would be in a down-type mass matrix. In most cases, the resulting diagonalization matrix is exactly the same (no complex conjugation needed) if we consider the diagonalization of the corresponding up-type mass matrix. If there are any differences, these will be noted where they appear. It now becomes important for the three singlet structure, whether it is the left-handed or the left-handed conjugate fermions which transform as three one-dimensional representations, that is, whether we have a left-handed or a right-handed three singlet structure. When referring to the indices of the two-dimensional representations, we will be using j to denote the index of the representation under which the left-handed fermions transform, and k for the index of the representation under which the left-handed conjugate fermions transform. This is the same notation we have been using for the two-doublet structure throughout. The one-dimensional representations, when needed, are given explicitly and we do not use letters as indices for them in this section.

No mixing The simplest squared mass matrix is simply diagonal. It does not lead to any mixing, i.e. is diagonalized by the identity matrix, and has the general form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \quad (3.45)$$

Its eigenvalues are a (1st generation), b (2nd generation) and c (3rd generation), which are all three positive. Further information is given in table 3.3.

One maximal mixing angle The simplest non-trivial diagonalization matrix we encounter has one maximal and two zero mixing angles. It is of the form

$$U_{\max} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{e^{ij\phi}}{\sqrt{2}} & -\frac{e^{ij\phi}}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.46)$$

and is generated by the squared mass matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & ce^{i\phi j} \\ 0 & ce^{-i\phi j} & b \end{pmatrix}, \quad (3.47)$$

M	a	b	c
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$	$ A ^2$	$ B ^2$	$ C ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & Be^{-i\phi(j+k)} \end{pmatrix}$	$ A ^2$	$ B ^2$	$ B ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix}$	$ A ^2$	$ B ^2$	$ C ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & Be^{-i\phi(j-k)} & 0 \end{pmatrix}$	$ A ^2$	$ B ^2$	$ B ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & Be^{-ik\phi} \\ 0 & -C & Ce^{-ik\phi} \end{pmatrix}$	$ A ^2$	$2 B ^2$	$2 C ^2$

Table 3.3: Mass matrices and models leading to no mixing. From left to right we give the mass matrix along with the expressions for the parameters of the squared mass matrix in terms of those of the mass matrix.

which has the eigenvalues a (1st generation), $b - c$ (2nd generation) and $b + c$ (3rd generation). a and b are always positive, but c need not be, in which case of course the eigenvalue assignment would have to be changed. All further information is given in table 3.4. Note that the first and second mass matrices are the down- and up-type variants of the same model. We list them separately, as there is a slight difference in the eigenvalues.

One free mixing angle The next simplest diagonalization matrix has one free and two zero mixing angles and is of the general form

$$U_{\text{free}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta e^{i\beta} \\ 0 & -s_\theta e^{-i\beta} & c_\theta \end{pmatrix}. \quad (3.48)$$

It is generated by the squared mass matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & de^{i\beta} \\ 0 & de^{-i\beta} & c \end{pmatrix}, \quad (3.49)$$

which has eigenvalues a (1st generation) and $\frac{1}{2}(b + c \pm \sqrt{(b - c)^2 + 4d^2})$ (with the minus sign for the 2nd, the plus sign for the 3rd generation). All parameters are positive. The mixing angle is given by

$$\tan(2\theta) = \frac{2d}{c - b}, \quad (3.50)$$

M	a	$b + c$	$b - c$
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & Ce^{-i\phi(j-k)} & Be^{-i\phi(j+k)} \end{pmatrix}$	$ A ^2$	$ B + Ce^{i\phi k} ^2$	$ B - Ce^{i\phi k} ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & Be^{i\phi(j+k)} & Ce^{i\phi(j-k)} \\ 0 & C & B \end{pmatrix}$	$ A ^2$	$ B + Ce^{-i\phi k} ^2$	$ B - Ce^{-i\phi k} ^2$
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & -C \\ 0 & Be^{-i\phi j} & Ce^{-i\phi j} \end{pmatrix}$	$ A ^2$	$2 B ^2$	$2 C ^2$

Table 3.4: Mass matrices and models generating one maximal mixing angle. For details see caption of table 3.3.

and is thus defined to lie in the interval $[0, \frac{\pi}{2}]$. All further information can be found in table 3.5.

One free, one maximal mixing angle The most complicated structure is a diagonalization matrix with one free, one maximal and one zero mixing angle. Let us first take a look at the squared matrix structure which gives such a diagonalization matrix:

$$\begin{pmatrix} a & de^{i\beta} & de^{i(\beta+j\phi)} \\ de^{-i\beta} & b & ce^{ij\phi} \\ de^{-i(\beta+j\phi)} & ce^{-ij\phi} & b \end{pmatrix}. \quad (3.51)$$

This squared mass matrix has the eigenvalues $(b - c)$ and $\frac{1}{2}(a + b + c \pm \sqrt{(a - b - c)^2 + 8d^2})$. All parameters are positive except for c which could also be negative. We will be discussing all possible eigenvalue assignments in the next chapter, for now we assign the eigenvalue $(b - c)$ to the third generation, the other two eigenvalues to the first and second, where the eigenvalue with the plus sign naturally belongs to the second generation. The diagonalization matrix is then

$$U_{\text{fm}} = \begin{pmatrix} c_\theta e^{i\beta} & s_\theta e^{i\beta} & 0 \\ -\frac{s_\theta}{\sqrt{2}} e^{i\phi j} & \frac{c_\theta}{\sqrt{2}} e^{i\phi j} & -\frac{1}{\sqrt{2}} e^{i\phi j} \\ -\frac{s_\theta}{\sqrt{2}} & \frac{c_\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (3.52)$$

In any case the free mixing angle is given by

$$\tan(2\theta) = \frac{2\sqrt{2}d}{b + c - a} \quad (3.53)$$

and lies in the closed interval from 0 to $\frac{\pi}{2}$. All further information is given in tables 3.6 and 3.7. The first table gives those values which differ for up- and down-type mass matrices, while the second one gives those values which are the same (and hence shows only the down-type mass matrices).

M	a	b	c	d	β
$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & D & E \end{pmatrix}$	$ A ^2$	$ B ^2 + C ^2$	$ D ^2 + E ^2$	$ BD^* + CE^* $	$\arg[BD^* + CE^*]$
$\begin{pmatrix} 0 & C & -Ce^{-i\phi k} \\ A & D & De^{-i\phi k} \\ B & E & Ee^{-i\phi k} \end{pmatrix}$	$2 C ^2$	$ A ^2 + 2 D ^2$	$ B ^2 + 2 E ^2$	$ AB^* + 2DE^* $	$\arg[AB^* + 2DE^*]$

Table 3.5: Mass matrices and models generating one free mixing angle. For details see caption of table 3.3. The argument function $\arg[x + iy]$ of a complex number is $\arctan \frac{y}{x}$.

M	d	c	β
$\begin{pmatrix} 0 & A & B \\ C & D & E \\ -Ce^{-i\phi} & De^{-i\phi} & Ee^{-i\phi} \end{pmatrix}$	$ AD^* + BE^* $	$ D ^2 + E ^2 - C ^2$	$\arg[AD^* + BE^*]$
$\begin{pmatrix} 0 & A & B \\ Ce^{i\phi} & De^{i\phi} & Ee^{i\phi} \\ -C & D & E \end{pmatrix}$	$ AD^* + BE^* $	$ D ^2 + E ^2 - C ^2$	$\arg[AD^* + BE^*] - j\phi$
$\begin{pmatrix} A & C & Ce^{-i\phi k} \\ B & D & E \\ Be^{-i\phi} & Ee^{-i(\phi+k)\phi} & De^{-i(\phi+k)\phi} \end{pmatrix}$	$ AB^* + C(D^* + E^*e^{-ik\phi}) $	$ B ^2 + 2\text{Re}[DE^*e^{-ik\phi}]$	$\arg[AB^* + C(D^* + E^*e^{-ik\phi})]$
$\begin{pmatrix} A & Ce^{i\phi k} & C \\ Be^{i\phi} & De^{i(\phi+k)\phi} & Ee^{-i(k-j)\phi} \\ B & E & D \end{pmatrix}$	$ AB^* + C(D^* + E^*e^{ik\phi}) $	$ B ^2 + 2\text{Re}[DE^*e^{ik\phi}]$	$\arg[AB^* + C(D^* + E^*e^{ik\phi})] - j\phi$

Table 3.6: Mass matrices leading to one free and one maximal mixing angle - parameters which differ in the up and down cases. For details see caption of table 3.3. The argument function $\arg[x + iy]$ of a complex number is $\arctan \frac{y}{x}$.

M	a	b
$\begin{pmatrix} 0 & A & B \\ C & D & E \\ -Ce^{-ij\phi} & De^{-ij\phi} & Ee^{-ij\phi} \end{pmatrix}$	$ A ^2 + B ^2$	$ D ^2 + E ^2 + C ^2$
$\begin{pmatrix} A & C & Ce^{-i\phi k} \\ B & D & E \\ Be^{-ij\phi} & Ee^{-i(j-k)\phi} & De^{-i(j+k)\phi} \end{pmatrix}$	$ A ^2 + 2 C ^2$	$ B ^2 + D ^2 + E ^2$

Table 3.7: Mass matrices and models generating one free and one maximal mixing angle - parameters which are the same in the up and down cases. For details see caption of table 3.3.

Matrix	left-handed 3 singlet	right-handed 3 singlet	2 doublet
$\mathbb{1}$	D_q	-	$Z_{\frac{n}{2}}, Z_q, Z_n, D_{\frac{n}{2}}, D_q$
U_{\max}	-	D_q	$D_{\frac{n}{2}}, D_q$
U_{free}	Z_q, Z_2	Z_q	$Z_{\frac{n}{2}}, Z_q$
U_{fm}	-	Z_2	Z_2

Table 3.8: Relation between diagonalization matrices and conserved subgroups. The column gives the transformation properties of the fermions, the row gives the diagonalization matrix. One can then read off the possible conserved subgroups. Z_2 always means $Z_2 = \langle \text{BA}^m \rangle$.

Looking back at our discussion of Dirac mass matrices in the previous chapter, and comparing the structures encountered there with the tables in this chapter, we find that we have indeed covered all possible cases, keeping the assumption that a non-trivial subgroup of our original dihedral flavor symmetry is preserved and demanding a non-zero determinant. We obtain four kinds of diagonalization matrices. In table 3.8 we give the connection between conserved subgroup and diagonalization matrix. The table should be read in the following way: We fix the diagonalization matrix we want to obtain and the transformation properties of the fermions. We can then read off the possible conserved D_n subgroups.

CKM and PMNS matrices What possibilities do we now have to construct the CKM or PMNS mixing matrices from the diagonalization matrices determined in this section? A mixing matrix will be of the form $U_i^T U_j^*$, with $U_{i,j}$ one of the four diagonalization matrices. We will mainly be concerned with mixing angles in the following, i.e. with the moduli of the mixing matrix elements, as the phases depend strongly on parameterization and in the lepton sector also on the origin of neutrino mass.

We recall what was the original purpose of our analysis of the D_n subgroups: We wanted to find mass independent textures, i.e. mixing angles which are only determined by the structure of the mass matrix and are independent of exact numerical values. We have so far encountered maximal mixing angles (meaning diagonalization matrix elements with an absolute value of $1/\sqrt{2}$) and vanishing mixing angles (corresponding to diagonalization matrix elements with absolute values 1 and 0). However, when considering the possible combinations of diagonalization matrices, we observe an interesting thing: The maximal and vanishing mixing angles are not the only ones that can be predicted by group theory alone, there is a third possibility. If, for example, different subgroups $Z_2 = \langle \text{BA}^{m_1} \rangle$ and $Z_2 = \langle \text{BA}^{m_2} \rangle$ with $m_1 \neq m_2$ are conserved in different sectors we predict for the absolute value of one element of the mixing matrix

$$\left| \cos \left(\frac{\pi(m_1 - m_2)j}{n} \right) \right|, \quad (3.54)$$

where j is the index of the irrep $\mathbf{2}_j$ under which the left-handed fermions involved in the mixing transform. We will discuss this effect and how it arises in detail in the following chapter.

Chapter 4

Subgroup Mismatch in D_n

We have observed at the end of the last chapter that, in addition to vanishing and maximal mixing angles, we have a third possibility for predicting an element of the quark or lepton mixing matrix from group theory alone. This is not entirely unheard of, as models implementing an A_4 flavor symmetry are actually able to predict all elements of the PMNS matrix and so reproduce the HPS mixing matrix, as discussed in section 2.2. Still, the D_n case is highly interesting and may be of relevance for both quarks and leptons, as we shall see in this chapter. We will discuss how the matrix element is generated by subgroup mismatch in section 4.1. This mechanism has been used in the literature to predict maximal atmospheric mixing, without however analyzing its origin in D_n group theory. We give the details in section 4.2 and then go on to present a model in which this mechanism is used to predict the Cabibbo angle in the quark sector in section 4.3.

4.1 General Considerations

4.1.1 Derivation of the Non-Trivial Mixing Angle

The desired subgroup mismatch can only arise if the two involved sectors each conserve a subgroup with $BA^{m_{1,2}}$ as generator. The parameters m_1 and m_2 must be different, otherwise the group theoretically determined mixing matrix element is simply 1 as can be seen from equation (3.54). A D_n group has only three types of subgroups that have an element of the form BA^m as generator, the groups Z_2 , D_j and $D_{\frac{n}{2}}$ (where m is set to be 0 or 1). In fact if we realize that $Z_2 \cong D_1$, we can reformulate the condition: To obtain a non-trivial, purely group theoretical prediction for an element of the CKM or PMNS matrix, we need to demand that distinct dihedral subgroups are conserved in the two involved sectors and that the intersection of their elements should not contain an element of the form BA^m .

We further see from the analysis of section 3.5 that we need to have a diagonalization matrix of the form U_{\max} or U_{fm} in both sectors, otherwise the subgroup index m_i will not show up in the diagonalization matrix U_i . This immediately disqualifies all models where the left-handed fermions transform as three one-dimensional representations of G_f , and is in accordance with the connection between subgroups and diagonalization matrices given in table 3.8. Both diagonalization matrices U_{\max} and U_{fm} arise from squared mass matrices with a very similar form, in fact the squared mass matrix of equation (3.47) arises from that of equation (3.51) in the limit $d = 0$. This corresponds to taking the limit $\theta = 0$ in U_{fm} , as can be seen from equation (3.53), and indeed U_{\max} and U_{fm} are equal in this limit. This can also be understood from the conserved subgroups: A diagonalization matrix U_{fm} corresponds to a conserved subgroup Z_2 , while U_{\max} corresponds to a larger dihedral subgroup which contains Z_2 as a subgroup. A conserved Z_2 allows a VEV for all two-dimensional representations, while the larger dihedral subgroup

will allow a VEV only for certain (or no) two-dimensional representations, leading to $d = 0$. If d were allowed to be non-vanishing even for the larger dihedral subgroup, we would encounter the case, mentioned in section 3.4.4, of having only unfaithful representations and effectively only conserving the Z_2 subgroup of a smaller original dihedral flavor symmetry.

We will thus, without loss of generality, for our general discussion only consider the case of distinct Z_2 subgroups conserved in both sectors, keeping in mind that we can arrive at larger conserved subgroups taking the limits described above.

Our explicit case is thus conserved subgroups $Z_2 = \langle BA^{m_1} \rangle$ and $Z_2 = \langle BA^{m_2} \rangle$ in sectors 1 and 2 (which can mean up- and down-type quarks or charged leptons and neutrinos). If we additionally impose that the left-handed fermions transform under one one-dimensional and one two-dimensional representation $\mathbf{2}_j$, we have specified the unique case giving us a diagonalization matrix U_{fm} (up to permutation of columns - we will come to this in a moment) in both sectors. To make the further discussion more transparent we first factorize the diagonalization matrix U_i which is of the form U_{fm} in the following way:

$$U_i = \begin{pmatrix} e^{i\beta_i} & 0 & 0 \\ 0 & e^{i\phi_{ij}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos \theta_i & \sin \theta_i & 0 \\ -\sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.1)$$

or, in an abbreviated form

$$U_i = \begin{pmatrix} e^{i\beta_i} & 0 & 0 \\ 0 & e^{i\phi_{ij}} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_{23} \left(\frac{\pi}{4} \right) R_{12} (\theta_i). \quad (4.2)$$

As we did for the subgroup index m_i , we label all parameters in the sector i ($i = 1, 2$) with the corresponding index, while otherwise keeping the notation of section 3.5. The free mixing angles from the two sectors are thus denoted by θ_1 and θ_2 . The definitions of $R_{23} \left(\frac{\pi}{4} \right)$ and $R_{12} (\theta_i)$ follow from comparing equations (4.1) and (4.2). We then obtain the CKM or PMNS matrix as

$$V = (U_1)^T (U_2)^* = R_{12}^T (\theta_1) R_{23}^T \left(\frac{\pi}{4} \right) \begin{pmatrix} e^{i(\beta_1 - \beta_2)} & 0 & 0 \\ 0 & e^{i(\phi_1 - \phi_2)j} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_{23} \left(\frac{\pi}{4} \right) R_{23} (\theta_2). \quad (4.3)$$

As one can clearly see, if now $\phi_1 = \phi_2$, that is $m_1 = m_2$ and the same Z_2 subgroup is conserved in both sectors, the diagonal phase matrix in the middle has only a non-trivial (11)-entry and thus commutes with $R_{23} \left(\frac{\pi}{4} \right)$. We then have the multiplication $R_{23}^T \left(\frac{\pi}{4} \right) R_{23} \left(\frac{\pi}{4} \right)$ which gives the identity, since $R_{23} \left(\frac{\pi}{4} \right)$ is an orthogonal matrix. All that is then left in V is one non-vanishing mixing angle θ_{12} fulfilling

$$|\cos \theta_{12}| = |e^{i(\beta_1 - \beta_2)} \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2|. \quad (4.4)$$

We thereby see the relevance of the fact that we conserve different subgroups in the two sectors: If not the maximal mixing angles from the two sectors show up with the same phase and simply cancel. If however $m_1 \neq m_2$, we obtain

$$V = \frac{1}{2} R_{12}^T (\theta_1) \begin{pmatrix} e^{i(\beta_1 - \beta_2)} & 0 & 0 \\ 0 & 1 + e^{ij(\phi_1 - \phi_2)} & 1 - e^{ij(\phi_1 - \phi_2)} \\ 0 & 1 - e^{ij(\phi_1 - \phi_2)} & 1 + e^{ij(\phi_1 - \phi_2)} \end{pmatrix} R_{12} (\theta_2). \quad (4.5)$$

We thus get an additional rotation by an angle given by

$$|\cos \theta| = \frac{1}{2} \left| 1 + e^{ij(\phi_1 - \phi_2)} \right| = \left| \cos \left(\frac{\pi(m_1 - m_2)j}{n} \right) \right|. \quad (4.6)$$

This angle is now entirely determined by group theoretical factors and not by arbitrarily tunable free parameters. After multiplication with the two rotation matrices on the left and on the right, we end up with one element of the mixing matrix V which is entirely determined by group theory. In our case this will be V_{33} , whose absolute value is then given to be

$$|V_{33}| = \left| \cos \left(\frac{\pi(m_1 - m_2)j}{n} \right) \right|, \quad (4.7)$$

as already announced. Comparing this with the measured values of the elements of the CKM and PMNS matrices can now give us a clear indication on how to construct our model using a dihedral flavor symmetry, especially on how to pick n . But first we need to understand, why it was V_{33} and not some other mixing matrix element which ended up being determined by group theory.

For this we need to return to the question of assigning eigenvalues. We first note that U_i has a column vector

$$\vec{v}_i = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} e^{i\phi_i j} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.8)$$

This is the eigenvector of the squared mass matrix corresponding to the eigenvalue $(b_i - c_i)$. In our exemplary calculation above, we implicitly assigned this eigenvalue to the third generation in both sectors, as this was the convention adopted in section 3.5. The group theoretically determined element V_{33} then arises from the Hermitian scalar product of the eigenvectors from both sectors, i.e. $V_{33} = \vec{v}_1^* \cdot \vec{v}_2$. If however we assign $(b_1 - c_1)$ to the a th generation and $(b_2 - c_2)$ to the b th, it will naturally be V_{ab} which is determined by group theory and whose absolute value will be given by equation (3.54). For the case of conserved Z_2 subgroups the ordering of the other two eigenvalues is determined, and we have no further freedom in assigning eigenvalues, at least in the quark and charged lepton sectors. For neutrinos we do not know whether the third generation is lighter or heavier than the other two, so that there are additional possibilities. These however only correspond to a redefinition of θ_ν , which is anyway a free parameter. The three different forms of the diagonalization matrix U_{fm} are then

$$\begin{aligned} U_{\text{fm}} &= \begin{pmatrix} c_\theta e^{i\beta} & s_\theta e^{i\beta} & 0 \\ -\frac{s_\theta}{\sqrt{2}} e^{i\phi j} & \frac{c_\theta}{\sqrt{2}} e^{i\phi j} & -\frac{1}{\sqrt{2}} e^{i\phi j} \\ -\frac{s_\theta}{\sqrt{2}} & \frac{c_\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \\ U'_{\text{fm}} &= \begin{pmatrix} c_\theta e^{i\beta} & 0 & s_\theta e^{i\beta} \\ -\frac{s_\theta}{\sqrt{2}} e^{i\phi j} & -\frac{1}{\sqrt{2}} e^{i\phi j} & \frac{c_\theta}{\sqrt{2}} e^{i\phi j} \\ -\frac{s_\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{c_\theta}{\sqrt{2}} \end{pmatrix}, \\ U''_{\text{fm}} &= \begin{pmatrix} 0 & c_\theta e^{i\beta} & s_\theta e^{i\beta} \\ -\frac{1}{\sqrt{2}} e^{i\phi j} & -\frac{s_\theta}{\sqrt{2}} e^{i\phi j} & \frac{c_\theta}{\sqrt{2}} e^{i\phi j} \\ \frac{1}{\sqrt{2}} & -\frac{s_\theta}{\sqrt{2}} & \frac{c_\theta}{\sqrt{2}} \end{pmatrix}. \end{aligned} \quad (4.9)$$

Combining them leads to nine distinct possibilities for the mixing matrix, which are displayed in appendix C.3. We can now go ahead and compare the expression of equation (3.54) with the experimentally determined values for all CKM and PMNS matrix elements, and find which

ones it is best suited to describe.

4.1.2 Comparison with Phenomenology

We start by considering the quark sector. We will restrict ourselves to values of n smaller than 30, since larger values of n correspond to larger groups. Enforcing $n < 30$ leads to a group order smaller than 60 (cf. section 3.1.1) which seems to be reasonable. With a larger n we could tune the group theoretically determined mixing matrix element to arbitrary precision, but smaller groups are more easily handleable and a model of flavor should anyway be in some sense minimal. Also, we expect small corrections to this element and thus should allow for small deviations from our leading order prediction.

We immediately see that this means that the elements in the third row and column of V_{CKM} are too small or too close to 1 (for the case of V_{tb}) to be described in this way. However the elements of the 1-2 sub-block, i.e. $|V_{ud}|$, $|V_{us}|$, $|V_{cd}|$ and $|V_{cs}|$, can be quite readily approximated by $|\cos(\frac{\pi}{n}(m_u - m_d)j)|$. We can accommodate the CKM matrix element $|V_{us}|$ by $\cos(\frac{3\pi}{7}) \approx 0.2225$, i.e. by taking $n = 7$ and for example $j = 3$, $m_u = 1$ and $m_d = 0$. As $|V_{cd}| \approx |V_{us}|$ holds to good accuracy, also $|V_{cd}|$ can be described well by $\cos(\frac{3\pi}{7})$. This favors a flavor group D_7 . Furthermore $|V_{ud}| \approx |V_{cs}|$ can be approximated well as $\cos(\frac{\pi}{14}) \approx 0.9749$ which points towards the flavor group D_{14} . These are the best possible approximations for $n \leq 30$ given the best fit values of equation (1.15).

Note that, even if $|V_{us}|$ is taken to be $\cos(\frac{3\pi}{7})$, there is no unique solution which flavor symmetry has to be used and to which subgroup it has to be broken, since for example taking $j = 1$, $m_u = 3$, $m_d = 0$ and $n = 7$ leads to $|\cos(\frac{\pi}{n}(m_u - m_d)j)| = |\cos(\frac{3\pi}{7})|$ as well as $j = 3$, $m_u = 1$, $m_d = 0$ and $n = 7$ and also $j = 1$, $m_u = 6$, $m_d = 0$ and $n = 14$. As $|\cos(\frac{4\pi}{7})|$ equals $|\cos(\frac{3\pi}{7})|$, this allows us to deduce further possible values for j , m_u , m_d and n , such as $j = 1$, $m_u = 0$, $m_d = 4$ and $n = 7$.

We will study the cases $|V_{us}|$ and $|V_{cd}|$ equal to $\cos(\frac{3\pi}{7})$ and $|V_{ud}|$ and $|V_{cs}|$ equal to $\cos(\frac{\pi}{14})$ in greater detail and thereby check whether we can always adjust the two other mixing angles θ_{13}^q and θ_{23}^q with the free angles θ_u and θ_d and also the Jarlskog invariant J_{CP} with the difference of the two phases β_u and β_d . In this analysis there are only three free parameters in the mixing matrix: $\theta_{u,d}$ and $\alpha = \beta_u - \beta_d$, since only this phase difference will appear in V_{CKM} . We use these to fit the other two quark mixing angles θ_{13}^q and θ_{23}^q as well as the CP violation J_{CP} .

The forms of V presented in appendix C.3 show that two of the elements $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$ and $|V_{ts}|$ are determined by $\cos(\theta_{u,d})$ in each of the four different cases. As these elements are small, the free angles θ_u and θ_d are restricted to be $\theta_{u,d} \approx \frac{\pi}{2}$. Therefore $\theta_{u,d}$ is expanded around $\frac{\pi}{2}$, $\theta_{u,d} = \frac{\pi}{2} - \epsilon_{u,d}$, $\epsilon_{u,d} > 0$. The resulting four CKM matrices are (up to the first order in $\epsilon_{u,d}$)

$$|V_{CKM}^{11}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{3\pi}{7}) \epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i}) \epsilon_d - 2 e^{i\alpha} \epsilon_u| \\ \cos(\frac{3\pi}{7}) \epsilon_u & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i}) \epsilon_u - 2 e^{i\alpha} \epsilon_d| & 1 \end{pmatrix}, \quad (4.10)$$

$$|V_{CKM}^{12}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) \epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i}) \epsilon_d - 2 e^{i\alpha} \epsilon_u| \\ \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i}) \epsilon_u - 2 e^{i\alpha} \epsilon_d| & \cos(\frac{\pi}{14}) \epsilon_u & 1 \end{pmatrix}, \quad (4.11)$$

$$|V_{CKM}^{21}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i}) \epsilon_d - 2 e^{i\alpha} \epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{\pi}{14}) \epsilon_d \\ \cos(\frac{\pi}{14}) \epsilon_u & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i}) \epsilon_u - 2 e^{i\alpha} \epsilon_d| & 1 \end{pmatrix}, \quad (4.12)$$

$$|V_{CKM}^{22}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i}) \epsilon_d - 2 e^{i\alpha} \epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) \epsilon_d \\ \frac{1}{2} |(1 + e^{\frac{\pi}{7}i}) \epsilon_u - 2 e^{i\alpha} \epsilon_d| & \cos(\frac{3\pi}{7}) \epsilon_u & 1 \end{pmatrix}. \quad (4.13)$$

Element (ij)	Possible cosines
(21)	$\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(22)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(23)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(31)	$\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(32)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(33)	$\cos(\frac{2\pi}{7}) (\approx 0.6235), \cos(\frac{3\pi}{14}) (\approx 0.7818)$

Table 4.1: Possibilities for the group theoretically determined element in V_{PMNS} . Note that, e.g. $\cos(\frac{3\pi}{7})$ equals $\cos(\frac{6\pi}{14})$, i.e. it could also be reproduced in the group D_{14} with $j = 1$ and $m_l - m_\nu = 6$ and not only in D_7 with $j = 1$ and $m_l - m_\nu = 3$. Furthermore, for example, $\cos(\frac{4\pi}{7})$ is also included implicitly in the list, as $|\cos(\frac{4\pi}{7})| = |\cos(\frac{3\pi}{7})|$.

Without loss of generality we have set the representation index j to 1, the group theoretical phase ϕ_u to zero ($m_u = 0$) and the phase ϕ_d to $\frac{2\pi}{14}$ ($m_d = 1, n = 14$) for equations (4.10) and (4.13), while we take it to be $\frac{6\pi}{7}$ ($m_d = 3, n = 7$) for equations (4.11) and (4.12). Furthermore one can calculate J_{CP} in this case:

$$\begin{aligned} J_{CP}^{11} &= \frac{1}{8} \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{\pi}{14}\right) \sin(2\theta_d) \sin(2\theta_u) \sin\left(\frac{\pi}{14} - \alpha\right) \\ &\approx \frac{1}{2} \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{\pi}{14}\right) \sin\left(\frac{\pi}{14} - \alpha\right) \epsilon_u \epsilon_d. \end{aligned}$$

The value of J_{CP} belonging to V_{CKM}^{22} , i.e. J_{CP}^{22} , is of the same form as J_{CP}^{11} . For V_{CKM}^{12} and V_{CKM}^{21} one finds

$$\begin{aligned} J_{CP}^{12} = J_{CP}^{21} &= -\frac{1}{8} \sin\left(\frac{6\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right) \sin(2\theta_d) \sin(2\theta_u) \sin\left(\frac{3\pi}{7} - \alpha\right) \\ &\approx -\frac{1}{2} \sin\left(\frac{6\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right) \sin\left(\frac{3\pi}{7} - \alpha\right) \epsilon_u \epsilon_d. \end{aligned}$$

A numerical computation was performed in [3], and it could be shown that in all four cases all data can be fitted with reasonable accuracy. Anyway, one expects further small corrections due to higher-dimensional operators and possibly explicit breaking of the residual subgroups.

A similar analysis as done in the case of V_{CKM} can also be carried out for the lepton mixing matrix V_{PMNS} . As the entries of V_{PMNS} are not as strongly restricted by experiments, there are several more possibilities to accommodate the various matrix elements regarding the choice of the group index n , and the values m_l, m_ν and j . We will thus restrict ourselves to two cases: First, as we intend to build a model which includes quarks as well as leptons, we stick to the values $n = 7, n = 14$, which fit the CKM matrix elements of the 1-2 sub-block best, if we restrict ourselves to small n . We will then in the next section deal with another special case, where this mechanism (for $n=4$) has been implicitly used in the existing literature.

Fixing $n = 7$ or $n = 14$, we check element by element of V_{PMNS} whether we can put it into the form $|\cos(\frac{l\pi}{7})|$ where $l = 0, 1, 2, \dots, 6$ or $|\cos(\frac{l\pi}{14})|$ with $l = 0, 1, 2, \dots, 13$. Taking the 90% C.L. bounds on the absolute values of the PMNS matrix elements from [114], all elements of the second and third row can be approximated by a cosine of the form $|\cos(\frac{l\pi}{7})|$ ($l = 0, 1, 2, \dots, 6$) or $|\cos(\frac{l\pi}{14})|$ ($l = 0, 1, 2, \dots, 13$), omitting the trivial possibility that the (13) element can be approximated by 0.

Again, a numerical fit of the mixing angles was performed in [3], assuming that the neutrinos are Dirac particles and that they have the same ordering as the other fermions, i.e. the neutrino mass spectrum is normally ordered. Here it was found that, as opposed to the quark case, there are some cosines listed in table 4.1 for which no fit with $\chi^2 < 1$ could be found. In all these cases the value of the fixed V_{PMNS} element lies almost outside the allowed ranges. For the (23) element, for example, the possible cosine is $\cos(\frac{2\pi}{7}) \approx 0.6235$ which is quite close to the lower bound (0.61) of the allowed range. However it could be shown that in general it is possible to treat the lepton mixings in the same way as the ones of the quarks in a D_7 or D_{14} flavor symmetry. In section 4.3 we will however only be constructing an explicit model for the quark sector. This is also due to the fact that for the lepton sector a different application of the D_n subgroup mismatch is more promising. This will be discussed in the next section.

4.2 A D_4 Example from the Literature

Before discussing the D_4 model in which the subgroup mismatch described in this section is used, we first mention some other examples, where non-trivial subgroups of dihedral flavor symmetries are conserved and explicitly mentioned. [74] and [83] mention the possibility of breaking their dihedral flavor symmetry down to a non-trivial subgroup. [74] uses a D_3 flavor symmetry, which is then broken down to Z_2 to achieve maximal atmospheric lepton mixing and $\theta_{13}^l = 0$. [83] uses a D_5 symmetry. If this is broken down to Z_2 , maximal mixing can be achieved. However, without separating the Yukawa sectors or explicitly breaking the residual symmetry, the resulting maximal mixing angles in the neutrino and charged lepton sector will cancel in the leptonic mixing matrix.

The most interesting example for our purposes however is given in [84], where a D_4 flavor symmetry is used. Before going into mathematical detail, it should be noted that the authors use a different set of (real) generators for D_4 , which makes the comparison less straightforward. The relation between this basis and ours is discussed in appendix B.3.

To obtain the distinct conserved subgroups an additional $Z_2^{(aux)}$ is used for separating the three different Yukawa sectors: the charged lepton sector, the Dirac neutrino sector and the Majorana neutrino sector. The transformation properties of the fermions are chosen to give a two doublet structure in all three sectors. Then one breaks down to a different subgroup in each sector: $D_2 = \langle A^2, B \rangle$ is conserved in the charged lepton sector. D_4 is conserved in the Dirac neutrino sector, while it is broken down to $Z_2 = \langle BA^3 \rangle$ in the Majorana neutrino sector. We will not go into the intricacies of the Type I Seesaw here. One can see that the net symmetry conserved in the neutrino sector is $Z_2 = \langle BA^3 \rangle$ and this is all we really need for our analysis.

One should note that this model does not include scalars transforming under all representations allowed a VEV by the conserved subgroup. To be more specific, a gauge singlet transforming as $\underline{1}_4$ is absent from the model, leading to two texture zeros in the right-handed neutrino mass matrix. They effect the relations between the eigenvalues of the light neutrino mass matrix, i.e. they enforce a normal hierarchy, but do not change our general symmetry discussion.

With $D_2 = \langle A^2, B \rangle$ conserved in the lepton sector and $Z_2 = \langle BA^3 \rangle$ conserved in the lepton sector, one can already see, from our discussion in section 4.1, what will happen: One element of V_{PMNS} will be determined by group theory alone. In this case it will be $V_{\tau 3}$ and is given to be

$$|V_{\tau 3}| = \left| \cos \left(\frac{\pi(0-3) \cdot 1}{4} \right) \right| = \frac{1}{\sqrt{2}} \quad (4.14)$$

But more than this: It is a D_2 that is conserved in the charged lepton sector, not a Z_2 , so, when comparing with V^{33} in appendix C.3, we should set $\theta_1 = \theta_l = 0$. This means that the entire third column of V_{PMNS} is determined by group theory, and this model predicts maximal atmospheric mixing and vanishing θ_{13}^l . [85] goes on to discuss soft breaking of the D_4 in the

scalar potential. More precisely, the Z_2 subgroup conserved in the neutrino sector is broken explicitly. Thus, the resulting doublet VEV no longer conserves Z_2 , which can lead to sizable deviations from a maximal θ_{23}^l , while leaving $\theta_{13}^l = 0$.

It is important to note that in this model the appearance of the maximal mixing angle is crucially dependent on $n = 4$. A similar result is reproduced by a D_3 flavor symmetry in reference [71]. But although the result is similar, it is not the same mechanism that is being used, as is already obvious from the fact that 4 does not divide the group index n . In this case, the authors work in the same group basis we have used, so a comparison is more straightforward. Still it needs to be taken into account that they present their Dirac mass matrices in the basis $\bar{L}R$, i.e. using the Dirac basis, while our results are always given in the basis LL^c . The $D_3(\cong S_3)$ flavor symmetry is again joined by a Z_2 symmetry, which creates the same three sectors. D_3 is then broken to $Z_3 = \langle A \rangle$ in the charged lepton sector. In the Dirac neutrino sector the entire flavor symmetry is conserved, while it is broken down to a Z_2 in the Majorana neutrino sector - in this case all three equivalent subgroups $\langle B \rangle$, $\langle BA \rangle$ and $\langle BA^2 \rangle$ are mentioned. Now all mixing is generated in the neutrino sector, where the conserved Z_2 leads to one maximal and one free mixing angle, as discussed in section 3.5. Here the maximal scalar field content is allowed: The additional parameter which arises from having the full scalar field content does not affect the structure of the neutrino mass matrix obtained from the Type I Seesaw formula, can however affect the neutrino mass hierarchy, as in this case the normal hierarchy is not necessarily predicted. The interesting thing to realize however, is that the maximal mixing angle predicted in this model has quite a different origin from that in the D_4 model, and does not arise from the subgroup mismatch we have discussed.

In any case, we have presented an existing, worked-out example, of how the subgroup mismatch can lead to maximal atmospheric mixing in the lepton sector. We now proceed to the presentation of an explicit model where this subgroup mismatch is used to predict the Cabibbo angle in the quark sector.

4.3 D_7 and the Cabibbo Angle - a Worked-Out Example

After having shown in section 4.1.2 that one element of V_{CKM} can be explained in terms of group theoretical indices, we want to go a step further and construct a viable model at least for the quark sector which includes this issue. The model is viable, if we find a numerical solution which accommodates not only the mixing parameters contained in V_{CKM} , but also the quark masses. Due to the strong hierarchy among the quarks this is a non-trivial task, although the number of parameters in the mass matrices M_u and M_d will exceed the number of observables. Furthermore we have to show that a Higgs potential exists allowing us to realize the desired VEV structure. In the simplest case we assume that all Higgs fields are $SU(2)_L$ doublets as the Higgs field in the SM.

4.3.1 Quark Sector of the D_7 Model

Here we present a model using the three singlet structure and the dihedral group D_7 . Both of them are here only chosen for simplicity: A further model using the two doublet structure is presented in [3], but it needs more scalar degrees of freedom. And D_7 is chosen as flavor symmetry, since it is smaller than D_{14} . As we will see later, D_{14} has some further advantages that actually make it the more attractive symmetry. For now we are however left with the possibility of either determining $|V_{us}|$ or $|V_{cd}|$ in terms of group theoretical quantities as $\cos(\frac{3\pi}{7})$. We assign the quarks to

$$Q_1 \sim \mathbf{1}_1, \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \mathbf{2}_1, \quad u_1^c, d_1^c \sim \mathbf{1}_2, \quad u_{2,3}^c, d_{2,3}^c \sim \mathbf{1}_1 \quad (4.15)$$

under D_7 . We stick to our assumption of maximal relevant scalar field content, i.e. we assume that the theory contains Higgs doublet fields transforming as $\mathbf{1}_1$ and $\mathbf{2}_1$, which we call H_s and $H_{1,2}$. As the relation between the mixing parameters of V_{CKM} and the group theoretical indices only arises, if the flavor symmetry D_7 is broken down to a subgroup $Z_2 = \langle \text{BA}^{m_u} \rangle$ by fields which couple to up quarks, while it is broken down to $Z_2 = \langle \text{BA}^{m_d} \rangle$ with $m_d \neq m_u$ by fields coupling to down quarks, we need an extra symmetry to perform this separation. In the SM this can be achieved by a $Z_2^{(aux)}$ symmetry:

$$d_i^c \rightarrow -d_i^c \quad \text{and} \quad H_s^d \rightarrow -H_s^d, \quad H_i^d \rightarrow -H_i^d, \quad (4.16)$$

while all other fields Q_i , u_i^c , H_s^u and $H_{1,2}^u$ are invariant under $Z_2^{(aux)}$. So, we deal with six Higgs fields coupling to the fermions, $H_s^u \sim (\mathbf{1}_1, +1)$, $H_{1,2}^u \sim (\mathbf{2}_1, +1)$ and $H_s^d \sim (\mathbf{1}_1, -1)$, $H_{1,2}^d \sim (\mathbf{2}_1, -1)$ under $D_7 \times Z_2^{(aux)}$. The mass matrices are of the form

$$M_u = \begin{pmatrix} 0 & y_1^u \langle H_s^u \rangle^* & y_2^u \langle H_s^u \rangle^* \\ y_3^u \langle H_1^u \rangle^* & y_4^u \langle H_1^u \rangle^* & y_5^u \langle H_1^u \rangle^* \\ -y_3^u \langle H_2^u \rangle^* & y_4^u \langle H_2^u \rangle^* & y_5^u \langle H_2^u \rangle^* \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} 0 & y_1^d \langle H_s^d \rangle & y_2^d \langle H_s^d \rangle \\ y_3^d \langle H_2^d \rangle & y_4^d \langle H_2^d \rangle & y_5^d \langle H_2^d \rangle \\ -y_3^d \langle H_1^d \rangle & y_4^d \langle H_1^d \rangle & y_5^d \langle H_1^d \rangle \end{pmatrix},$$

where $y_i^{u,d}$ denote Yukawa couplings. The VEV structure is taken to be

$$\langle H_s^{d,u} \rangle > 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle H_1^u \rangle = v_u e^{-\frac{3\pi i}{7}} \quad \text{and} \quad \langle H_2^u \rangle = v_u e^{\frac{3\pi i}{7}},$$

with $v_d > 0$ and $v_u > 0$. The VEVs are set to be real apart from the phase $\pm \frac{3\pi}{7}$ which is necessary for the correct breaking to the desired subgroup of D_7 . The mass matrices then take the form given in the second and first lines of table 3.6, respectively, where the parameters A , B , ... can be written in terms of Yukawa couplings and VEVs:

$$A_u = y_1^u \langle H_s^u \rangle, \quad B_u = y_2^u \langle H_s^u \rangle, \quad C_u = y_3^u v_u e^{-\frac{3\pi i}{7}}, \quad D_u = y_4^u v_u e^{-\frac{3\pi i}{7}}, \quad E_u = y_5^u v_u e^{-\frac{3\pi i}{7}}, \\ A_d = y_1^d \langle H_s^d \rangle, \quad B_d = y_2^d \langle H_s^d \rangle, \quad C_d = y_3^d v_d, \quad D_d = y_4^d v_d, \quad E_d = y_5^d v_d,$$

together with $\phi_u = \frac{6\pi}{7}$ ($m_u = 3$), $\phi_d = 0$ ($m_d = 0$) and $j = 1$, as the left-handed quark doublets of the second and third generation transform as $\mathbf{2}_1$. The preserved Z_2 subgroups are generated by BA^3 and B in the up and the down quark sector, respectively.

We now assign the mass eigenvalues $\sqrt{2}|C_d|$ and $\sqrt{2}|C_u|$ to the strange and the up quark, respectively. They correspond to the decisive eigenvalue $(b-c)$ in the language of section 4.1, so that $|V_{us}|$ is then given by $\cos(\frac{3\pi}{7}) = 0.2225$. A numerical analysis is performed in [3], to fit the other mixing matrix elements and the masses. The resultant Yukawa couplings show a strong hierarchy. This can be alleviated by introducing an additional FN symmetry, which leads to all Yukawa couplings being of natural size. This then constitutes a proof of principle that the subgroup mismatch mechanism can indeed lead to a prediction of the Cabibbo angle in a realistic model. However, things get less realistic and less elegant once we take a closer look at the Higgs sector, which we will do in the next section.

4.3.2 Higgs Sector of the D_7 Model

We first construct the three Higgs doublet potential for the Higgs fields $H_s \sim \mathbf{1}_1$ and $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim \mathbf{2}_1$, which couple to the up quarks. The potential has the form¹

¹Note that σ_2 is complex, but it can be made real by appropriate redefinition of the field H_s , for example.

$$\begin{aligned}
V_3(H_s, H_i) &= -\mu_s^2 H_s^\dagger H_s - \mu_D^2 \sum_{i=1}^2 H_i^\dagger H_i + \lambda_s (H_s^\dagger H_s)^2 + \lambda_1 \left(\sum_{i=1}^2 H_i^\dagger H_i \right)^2 \\
&+ \lambda_2 \left(H_1^\dagger H_1 - H_2^\dagger H_2 \right)^2 + \lambda_3 |H_1^\dagger H_2|^2 \\
&+ \sigma_1 (H_s^\dagger H_s) \left(\sum_{i=1}^2 H_i^\dagger H_i \right) + \{ \sigma_2 (H_s^\dagger H_1) (H_s^\dagger H_2) + \text{h.c.} \} + \sigma_3 \sum_{i=1}^2 |H_s^\dagger H_i|^2.
\end{aligned} \tag{4.17}$$

As already shown in [83], this potential has an additional $U(1)$ symmetry, i.e. there exists a further $U(1)$ symmetry in the potential apart from the $U(1)_Y$ gauge symmetry. This further symmetry is necessarily broken by our desired VEV structure such that a massless Goldstone boson appears in the Higgs spectrum which is not eaten by a gauge boson. This problem cannot be solved by taking into account the whole potential for all six Higgs fields, since even if the terms coupling the fields H_s^u , $H_{1,2}^u$ and H_s^d , $H_{1,2}^d$ are included, we find an accidental $U(1)$ symmetry in the potential. Therefore we have to enlarge the Higgs sector by further Higgs fields in order to create new D_7 invariant couplings which break this accidental symmetry explicitly. We find that this can be done in the simplest way by adding two Higgs fields transforming as $\mathbf{2}_2$ under D_7 . Due to their transformation properties they do not directly couple to the fermions. We decided to add two such fields to the three Higgs fields which couple to the down quarks. Therefore the model contains eight Higgs doublet fields in total, three of them couple to up and three of them to down quarks, while the other two are needed for a viable Higgs sector:

$$\begin{aligned}
H_s^u &\sim (\mathbf{1}_1, +1), \quad \begin{pmatrix} H_1^u \\ H_2^u \end{pmatrix} \sim (\mathbf{2}_1, +1), \\
H_s^d &\sim (\mathbf{1}_1, -1), \quad \begin{pmatrix} H_1^d \\ H_2^d \end{pmatrix} \sim (\mathbf{2}_1, -1) \quad \text{and} \quad \begin{pmatrix} \chi_1^d \\ \chi_2^d \end{pmatrix} \sim (\mathbf{2}_2, -1)
\end{aligned} \tag{4.18}$$

under $D_7 \times Z_2^{(aux)}$. The complete potential consists of three parts,

$$V = V_u + V_d + V_{mixed}, \tag{4.19}$$

where V_u denotes the part of the potential which only contains Higgs fields coupling to the up quarks, V_d contains the five Higgs fields which have a non-vanishing $Z_2^{(aux)}$ charge (three of them give masses to the down quarks), while V_{mixed} consists of all other terms. The explicit form of the potential is given in appendix C.4.

The VEV structure of the fields $H_s^{d,u}$ and $H_{1,2}^{d,u}$ is determined by our desire to break down to two distinct Z_2 subgroups in the up and the down quark sector :

$$\langle H_s^{d,u} \rangle > 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle H_1^u \rangle = v_u e^{-\frac{3\pi i}{7}} \quad \text{and} \quad \langle H_2^u \rangle = v_u e^{\frac{3\pi i}{7}},$$

with $v_d > 0$ and $v_u > 0$. In contrast to this, the VEV structure of the fields $\chi_{1,2}^d$ is not fixed in this way. However, in order to preserve the Z_2 subgroup generated by B not only through the VEVs of the fields H_s^d and $H_{1,2}^d$, but also by the VEVs of the fields $\chi_{1,2}^d$, $\langle \chi_1^d \rangle = \langle \chi_2^d \rangle > 0$ will be assumed in the following.

We proceed in the following way in order to find a minimum of this potential which allows for our choice of VEVs: First we treat V_u and V_d separately to find a viable solution for these two parts of the potential. Note that we can allow all parameters in the potential V_d to be real, as the VEVs of the corresponding Higgs fields are also real. Since V_u suffers from the above mentioned accidental $U(1)$ symmetry, we find a fourth massless particle in the Higgs mass

spectrum. In a second step we add as many terms as necessary from V_{mixed} to get a minimum of the whole potential V which does not have more than the usual three Goldstone bosons. It turns out that it is sufficient to take into account three terms in addition to V_u and V_d to get a viable solution. The terms are of the form

$$\kappa_2 \left(H_s^{u\dagger} H_s^d \right)^2 + \kappa_5 \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^d \right)^2 + \kappa_{19} \left(H_s^{u\dagger} H_s^d \right) \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^d \right) + \text{h.c.} \subset V_{mixed}.$$

Note that we take all VEVs to have the same absolute value, since this considerably simplifies the search for a numerical solution, as a fine-tuning of the parameters in the Higgs potential is avoided. However, in principle other solutions should also be possible, e.g. the fact that the up quarks are much heavier than the down quarks could be explained by assuming that the VEVs of the fields $H_s^u, H_{1,2}^u$ are larger than those of the fields $H_s^d, H_{1,2}^d$.

The rest of the discussion of the potential is delegated to appendix C.4 where we present a numerical solution for the parameters of the Higgs potential and the resulting Higgs masses. We discuss here only the two most important points, which will be guiding us in our further discussion. First of all: It can be shown that the subgroup conserving VEV is indeed a possible solution minimizing the Higgs potential. However, it can by no means be considered as a prediction of the Higgs potential, as it depends strongly on our choice of couplings. We could have hoped that the D_7 invariance of the potential might favor a group theoretically non-trivial minimum, but this is not the case. This can be traced back to two origins: First of all, the $SU(2)_L$ invariance is by far the dominating attribute of the potential, and tends to drown out effects of the D_7 invariance. Secondly, we want to conserve two distinct subgroups, but the two sets of Higgs bosons are not cleanly separated in the potential, as our $Z_2^{(aux)}$ symmetry is too small to accomplish this. We could think of enlarging the auxiliary symmetry, but we actually need the mixing terms in the potential to prevent the appearance of too many Goldstone bosons. The accidental symmetries leading to these Goldstone bosons can again be understood as a consequence of the $SU(2)_L$ invariance.

We also encounter an additional more immediate problem: The resulting Higgs masses are in between 50 and 500 GeV. Some of these values are already excluded by direct searches [7]. Additional constraints will arise from FCNCs, as they do in the MSSM, where low energy flavor observables can be used to constrain the mass of the charged Higgs (see, for example, [115]). There are two reasons for the too low Higgs masses: On the one hand V_u contains an accidental symmetry and on the other hand all mass parameters of the potential are chosen to be of natural order, i.e. to be around the electroweak scale. Additionally, all quartic couplings of the potential must be perturbative. Due to the low scalar masses, this model cannot be considered to be fully realistic. One might think of introducing explicit soft D_7 breaking terms, to lift the scalar masses.

However, there exists the possibility to solve all of the problems discussed, by supersymmetrizing the model and disentangling the scales of flavor and electroweak symmetry breaking. This then means that, while the electroweak symmetry is still broken by one or more Higgs bosons, the flavor symmetry is broken at a high scale by flavons, flavor-charged gauge singlets. Both models presented in this chapter, the D_7 model as well as the D_4 model discussed in section 4.2, can be extended in this way. The possibility of supersymmetrizing the D_4 model was in fact already mentioned in the original paper [84]. But before we modify both models in this way, the next chapter is devoted to more general aspects of the implementation of such a scheme for dihedral flavor symmetries.

Chapter 5

Flavons and Flavon Potentials

We have found that a potential of flavor-charged $SU(2)_L$ doublet scalars tends to give us a lot of freedom as to the structure of the scalar VEVs that minimize the potential. In particular, we can adjust the parameters in the potential in such a way that the scalar VEVs conserve a non-trivial subgroup of our dihedral flavor group, but this is by no means a prediction of the model. The conservation of a non-trivial subgroup is in that case only a mathematical construct, bearing no direct relevance to the symmetry-breaking dynamics.

In very simple cases, a VEV structure conserving a subgroup can in fact arise naturally from the extremization of a Higgs doublet potential. Consider the most simple phenomenologically viable D_3 invariant potential for $SU(2)_L$ doublet scalars, which is a three Higgs potential with the three scalar fields transforming as $SU(2)_L$ doublets and as $\mathbf{2}$ and $\mathbf{1}_1$ under D_3 . Without the additional singlet the potential would exhibit an accidental $U(1)$ symmetry, apart from the gauge symmetries. If we use the real D_n representation matrices discussed in appendix B.3, we can solve the extremization conditions under the assumption of real parameters and also real VEVs. In this way all allowed VEV configurations correspond to the specific VEV structure of equation (B.5). This subgroup-conserving VEV structure is the only one that can extremize the potential while breaking the flavor symmetry, if one assumes real parameters in the potential and no correlations among those parameters.

In contrast to this, one finds that for the simplest $SU(2)_L$ doublet potential in D_4 no such explicit statement can be made. The above VEV structure is still allowed, however the $SU(2)_L$ structure of the potential tends to obscure the D_n structure. Additionally, the inclusion of a large number of flavor-charged Higgs bosons can lead to problems with phenomenology: In the D_7 model of section 4.3 we had two neutral scalars lighter than the current lower bound on the SM Higgs of around 114 GeV, and even if these bounds can be evaded, such as in the D_4 model of section 2.3, we could easily run into problems with LFV and FCNCs. It can thus be said that a large number of Higgs-like scalars around the electroweak scale is disfavored by phenomenology.

Both problems can be solved simultaneously by disentangling the scales of electroweak and flavor symmetry breaking. This means that we no longer assume that an enlarged, flavor-charged SM Higgs sector is also responsible for the flavor symmetry breaking. Instead, we introduce a new type of scalar, responsible only for the breaking of the flavor symmetry. Such scalars are called flavons. These flavons will be SM gauge singlets, and thereby only charged under the flavor symmetry. Since their VEVs are then not associated with electroweak symmetry breaking, they are not constrained by the weak vector boson masses. We can break our flavor symmetry at a scale different from the electroweak scale and no light scalars, apart from the SM Higgs, are necessarily predicted in the theory. Problems with LFV and FCNCs may return, when we supersymmetrize, through the structure of the supersymmetry (SUSY) soft breaking terms. This so-called SUSY flavor problem is however a generic problem of supersymmetry and not

directly related to the breaking of a flavor symmetry.

When replacing flavor-charged Higgs bosons with flavons, the first question is of course: Were all our considerations so far useless? The answer is no. To understand this, we consider how the masses of the SM fermions are generated in a scenario employing flavons. The SM fermions will still transform non-trivially under the flavor symmetry. The term

$$Y_{ij}L_iL_j^cH, \quad (5.1)$$

where H is the lone SM Higgs boson, is then only allowed if $Y_{ij}L_iL_j^c$ is a singlet under the full, unbroken flavor group. Such a term may be an option for generating masses for the third generation, especially for the top quark, which is the only fermion in the SM with a mass of the order of the electroweak scale. Such terms alone, however, will not be able to reproduce the full flavor phenomenology. After all, as we have discussed at length, it is precisely the structure of the flavor symmetry breaking which can lead to the observed mixing angles. We thus need to consider in our Yukawa Lagrangian non-renormalizable terms of the form

$$Y_{ijk}L_iL_j^cH\frac{\varphi_k}{\Lambda}, \quad (5.2)$$

where the φ_k denote the flavons and Λ is the (high) scale at which these interactions are generated, either by integrating out heavy vector-like fermions or heavy Higgs scalars. If we then substitute the VEV $\langle\phi_i\rangle$ of a scalar charged both under $SU(2)_L$ and G_f , as used in section 3, by the product $\langle H\rangle\frac{\langle\varphi_i\rangle}{\Lambda}$, which transforms in the same way under both groups, all the results from section 3 can be directly transferred to a flavon model. The only obvious exception are of course direct (with respect to the flavor symmetry) mass terms, such as the one given in equation (5.1). Here we have two options: We can just substitute the G_f singlet of section 3 with the SM Higgs boson or we can forbid the direct mass term with an additional abelian symmetry (as we are using in all models), and substitute the original singlet scalar by the product of the SM Higgs and a singlet flavon, charged only under this abelian symmetry. The second approach does not modify the structure of the mass matrices at all, while the first approach only introduces a natural hierarchy among the mass matrix elements, which may actually be useful to generate realistic fermion masses without fine-tuning. We will be using both approaches later on.

The general conclusion is that the use of flavons for breaking G_f does not invalidate the results obtained so far, once the obvious modifications have been made. On the other hand, the flavon potentials show a lot more of the structure of the dihedral flavor group, and as we will show in the following, the subgroup conservation can in fact, under very general assumptions, be considered a dynamical prediction. Note that we have, by introducing the cutoff scale Λ , adopted an effective field theory approach. We will therefore, in the potentials and superpotentials we discuss in the next two sections, take into account corrections from operators suppressed by higher powers of Λ , where it becomes necessary.

We begin in section 5.1 by discussing D_n invariant flavon potentials in the SM: Such approaches have been used in the literature, for example in [86] and also in [84], where flavons give mass to the heavy charge-conjugated neutrinos. We also want to see how generic the prediction of subgroup conservation really is. We discuss the main problem of such an approach and how it can be alleviated by supersymmetrizing and then go on to discuss flavon potentials in the MSSM in section 5.2.

5.1 Flavon Potentials in the SM

When introducing flavons into the SM we need to think about, for the first time, whether we are dealing with real or complex scalars. Up until now, we did not need to worry about this: Since

the scalars we used so far transformed under a complex representation of $SU(2)_L \times U(1)_Y$, we were forced to take them to be complex. The flavons we are now dealing with are gauge singlets, and all representations of D_n are real, so the minimal choice is to take real flavons. Complex flavons could be separated into two scalars transforming separately under D_n . The reality of the representation can of course be ruined by introducing an additional abelian symmetry larger than Z_2 , but we will not be considering this in the following, since the realistic models we will build later on will be supersymmetric anyway.

We encounter one small difficulty, when switching to real scalars: We have always been using complex generators so far and we do not want to abandon them at this point, as they have proved a lot easier to handle when dealing with the conserved subgroups. To be able to use our complex generators, while using a real scalar, we need to introduce a 'fake-complex' scalar doublet $\begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}$ [71]. This property (that the lower doublet component is the complex conjugate of the upper one) is not changed by the action of any D_n element, so this is indeed a representation of D_n . The resulting doublet scalar has two real degrees of freedom, as we would expect for a real scalar transforming under a two-dimensional representation of D_n . Details on how this fake-complex scalar is related to the real doublet obtained using real generators can be found in appendix B.3.

5.1.1 One Doublet

We begin with the simplest case: We take one doublet

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \varphi \begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma} \end{pmatrix} \quad (5.3)$$

transforming as the $\mathbf{2}_j$ of some D_n . We specifically set $j = 1$ for our calculations, but the results apply equally well for any other faithful representation. We then consider a scalar potential containing this doublet and other scalars transforming under one-dimensional representations of D_n . We begin by assuming n is odd. There are then only two other scalars we need to consider, ψ_1 and ψ_2 , transforming as $\mathbf{1}_1$ and $\mathbf{1}_2$, respectively. We then find that there are two distinct terms containing only the doublet field which transform as a total singlet under D_n . Either the one which arises from two doublets coupling to a singlet,

$$\varphi_1 \varphi_2 = \varphi^2, \quad (5.4)$$

or the term arising if we couple n doublets together, where none of them are coupled to pairwise singlets,

$$\varphi_1^n + \varphi_2^n = 2\varphi^n \cos n\gamma. \quad (5.5)$$

These can then couple to the singlet terms containing only one-dimensional representations, ψ_1 and ψ_2^2 .

On the other hand, there is only one term containing only the doublet field and transforming as $\mathbf{1}_2$ under D_n , since the term coupling two doublets to form $\mathbf{1}_2$ (the analog of equation (5.4)) is antisymmetric and vanishes identically if we are coupling the same field. We can however couple n doublets to ψ_2 :

$$(\varphi_1^n - \varphi_2^n)\psi_2 = 2i\varphi^n \psi_2 \sin n\gamma. \quad (5.6)$$

It is the phase γ which is relevant for subgroup conservation. This phase only shows up in equations (5.5) and (5.6), so we only need to consider them, when minimizing the potential

with respect to γ . In particular, the terms proportional to φ^2 , such as a doublet mass term, do not contain this phase. For $n = 3$ only three terms in the potential are relevant:

$$(\mu + \lambda_1 \psi_1) \varphi^3 \cos 3\gamma + \lambda_2 \varphi^3 \psi_2 \sin 3\gamma. \quad (5.7)$$

If ψ_2 gets a VEV, this does not allow for any prediction of γ independent of the values of the real parameters μ , λ_1 and λ_2 . However, if we go to $n = 5$ or larger, the terms containing γ only show up in non-renormalizable corrections to the potential. In this way, the term containing the sine is suppressed, since it only shows up one order after the term containing the cosine, i.e. the minimization of the potential with respect to γ depends in leading order only on the term with the cosine.

If the coefficient multiplying the cosine in the potential (what was $\mu \langle \varphi \rangle^3$ for D_3 , where the coupling has positive mass dimension, will be $\langle \varphi \rangle^n \frac{\lambda}{\Lambda^{n-4}}$ in case $n \geq 5$) is positive, then a minimization of the potential, where $\langle \varphi \rangle$ is already determined by the renormalizable operators, leads to $\gamma = \frac{m\pi}{n}$ with m an odd integer. Conversely, if the prefactor is negative, then $\gamma = \frac{m\pi}{n}$ with m an even integer or zero. The relative phase between the two doublet components is then given as

$$2\gamma = \frac{2\pi m}{n}, \quad (5.8)$$

where the sign of the coefficient determines whether m is odd or even. We have therefore found a simple example, where the conservation of a subgroup arises as a prediction of the potential. In this case the subgroup is, for a doublet transforming as $\mathbf{2}_1$, $Z_2 = \langle \text{BA}^m \rangle$. We further observe that the conservation of this subgroup does not allow a VEV for the scalar ψ_2 transforming as $\mathbf{1}_2$ (cf. table C.1), which concurs with our observation: If ψ_2 acquires a VEV, the Z_2 subgroup will be broken, when considering higher order terms in the Lagrangian.

Moving on to the case where n is even, we find that we can write down two more non-vanishing terms involving the doublet:

$$\varphi_1^{\frac{n}{2}} \pm \varphi_2^{\frac{n}{2}}, \quad (5.9)$$

transforming as $\mathbf{1}_3$ and $\mathbf{1}_4$, respectively. If we do not include scalars transforming under these representations they are irrelevant, and the discussion is the same as for an odd n . If we do include such a scalar, then the phase γ already shows up in terms of order $\frac{n}{2} + 1$, i.e.

$$\lambda \psi_r (\varphi_1^{\frac{n}{2}} \pm \varphi_2^{\frac{n}{2}}), \quad (5.10)$$

with $r=3$ or 4 and ψ_r being the flavon transforming as $\mathbf{1}_r$. Let us take for concreteness $r=3$. Then equation (5.10) is of the form

$$2\lambda \psi_3 \varphi^{\frac{n}{2}} \cos \frac{n\gamma}{2}. \quad (5.11)$$

For $n = 4$ we again have two relevant terms in the Lagrangian:

$$\mu \psi_3 \varphi^2 \cos 2\gamma + \lambda \varphi^4 \cos 4\gamma. \quad (5.12)$$

These do allow for a subgroup conserving solution, but it is not the only possible one. However, for $n > 4$, only the term containing ψ_3 will be relevant for the minimization of the potential at leading order: Its mass dimension is smaller than that of the second term, which will thus be suppressed by additional powers of $\frac{1}{\Lambda}$. For $\frac{\lambda}{\Lambda^{\frac{n}{2}-3}} \langle \psi_3 \rangle \langle \varphi \rangle^{\frac{n}{2}}$ positive we get $\gamma = \frac{2\pi m'}{n}$ with m' an odd integer, while for $\frac{\lambda}{\Lambda^{\frac{n}{2}-3}} \langle \psi_3 \rangle \langle \varphi \rangle^{\frac{n}{2}}$ negative we get the same result with m' even. In both cases the relative phase between the two doublet components will be

$$2\gamma = \frac{4\pi m'}{n} = \frac{2\pi m}{n}, \quad (5.13)$$

with $m = 2m'$ even, regardless of any signs, i.e. it will conserve the same subgroup as the VEV of ψ_3 . An analogous reasoning holds for $r=4$, where m is then forced to be odd. Introducing both scalars ψ_3 and ψ_4 , and allowing VEVs for them, ruins the subgroup conservation, as it should. So, as for the case of even n , we generally predict a VEV conserving a Z_2 subgroup. For the case of an odd n , we could not predict whether the subgroup parameter m was odd or even, as this depended on the vacuum expectation value of φ . For even n , the subgroup parameter m is however determined by the scalar content of our model, making dihedral groups with n even more attractive for model building.

5.1.2 Several Doublets

We might prefer the couplings that contain the relative phase of doublet components, and thereby lead to subgroup conservation, to show up at the renormalizable level. We assume an odd n to begin with. We observed that the phase γ appears in operators where we have coupled n doublets, with none of them coupling to pairwise singlets. We can thus introduce the relative phase at the renormalizable level, if we introduce several doublets, whose doublet representation indices j can add up to n in a renormalizable operator. This is of course trivial for the case $n = 3$, since $1+1+1=3$ and the Z_2 subgroup is automatically conserved at the renormalizable level, as long as we disallow a VEV for a scalar transforming as $\underline{\mathbf{1}}_2$ or simply do not introduce such a field at all. As a non-trivial example we discuss the case of $n = 7$.

We consider the potential with the two doublets $\underline{\mathbf{2}}_1$ and $\underline{\mathbf{2}}_3$. This is sufficient, as we can add doublet indices up to n at the renormalizable level ($3+3+1=7$) without needing to introduce a $\underline{\mathbf{2}}_2^1$. We write the two doublets in our potential as

$$\varphi_j \begin{pmatrix} e^{i\gamma_j} \\ e^{-i\gamma_j} \end{pmatrix}, \quad (5.14)$$

with $j=1,3$. The potential will then contain two renormalizable terms containing the two phases:

$$\lambda_1 \varphi_1^3 \varphi_3 \cos(3\gamma_1 - \gamma_3) + (\mu + \lambda_2 \psi_1) \varphi_1 \varphi_3^2 \cos(\gamma_1 + 2\gamma_3). \quad (5.15)$$

As we have two terms and two independent phases, we can minimize them separately. From the minimization of the first term we obtain

$$3\gamma_1 - \gamma_3 = \pi m_1, \quad (5.16)$$

where m_1 is even or odd, depending on whether the prefactor $\lambda \langle \varphi_1 \rangle^3 \langle \varphi_3 \rangle$ is positive or negative. Inserting the result for γ_3 in the other minimization condition,

$$\gamma_1 + 2\gamma_3 = \pi m_2, \quad (5.17)$$

yields

$$\gamma_1 = \frac{(m_2 + 2m_1)\pi}{7}, \quad (5.18)$$

where m_2 is even or odd, depending on the sign of $\mu \langle \varphi_1 \rangle \langle \varphi_3 \rangle^2$. For the other phase we obtain

¹The addition of such a doublet would actually cause problems, since we would have two further ways to add up to n ($3+2+2$ and $2+2+2+1$), and could no longer uniquely predict the relative phases of the doublet VEVs.

$$\gamma_3 = \frac{3(m_2 + 2m_1)\pi}{7} - m_1\pi. \quad (5.19)$$

The second summand in the phase is irrelevant. For the relative phases that determine subgroup conservation, we get

$$2\gamma_1 = \frac{2m\pi}{7}, \quad (5.20)$$

$$2\gamma_3 = 3\frac{2m\pi}{7}. \quad (5.21)$$

We find that the potential again predicts the conservation of a subgroup $Z_2 = \langle \text{BA}^m \rangle$, with

$$m = m_2 + 2m_1, \quad (5.22)$$

as both doublet VEVs fulfill the necessary condition. In this specific case, whether m is odd or even, is determined only by m_2 , as m_1 shows up with a factor of two. This is however not a generic effect, as can be seen if one calculates the alternative case with $\mathbf{2}_2$ instead of $\mathbf{2}_3$.

For most n (as for $n = 7$, if we include all three doublets) this method fails, since we have more terms containing phases in the Lagrangian, than we have phases, and hence we cannot minimize term by term, as we did above. The subgroup conserving VEV is still allowed, but its uniqueness can no longer be shown.

In a very general setup, such as the one we have chosen, showing the uniqueness of a solution is difficult in general. It is therefore not these ambiguities, caused by too many couplings in the Lagrangian, that cause us to skip the case of an even n for now and move on to supersymmetrization. We have shown above that the prediction of conserved subgroups appears to be a generic feature of D_n invariant potentials (once all the $SU(2)_L$ clutter is removed), and hopefully dispelled any doubts that this feature arises only from supersymmetric potential engineering. The main reason for supersymmetrizing (apart from all the other arguments in favor of supersymmetry, such as the hierarchy problem, gauge coupling unification or dark matter) is the existence of a troublesome term in the SM flavon potential that we have not mentioned so far.

Even though we have managed to disentangle the electroweak and the the flavor symmetry breaking scale, we have not entirely decoupled the two scalar sectors: The potential contains terms coupling the squared SM Higgs and squared flavons, e.g.

$$(HH^\dagger)\varphi^2. \quad (5.23)$$

These will lead to corrections to the Higgs mass on the order of $\langle \varphi \rangle$. This may seem a trivial concern in a scenario where the hierarchy problem is unsolved. But, as we will see in the following, these couplings can naturally be avoided in a supersymmetric framework, which is a major reason why, for the remainder of this section, we work in the framework of the MSSM.

5.2 Flavon Potentials in the MSSM

5.2.1 What Changes in Supersymmetry?

Supersymmetry (SUSY) is at this time an integral part of the graduate curriculum, and introductions to supersymmetry are readily available (a deservedly popular one is [116]), so I do not feel that it is necessary to give a general introduction to the MSSM at this point. In the following, we will only look at several aspects of supersymmetry which are germane to our discussion of flavon potentials.

The first important difference concerns the complexity of the scalars. In supersymmetry, a scalar is no longer just that, but is part of a supersymmetric multiplet. For the case of a scalar, this will always be a chiral multiplet. This multiplet will contain an equal amount of fermionic and bosonic degrees of freedom. Since we need two degrees of freedom to construct a Weyl spinor, the bosonic part of a chiral multiplet will necessarily contain two scalar degrees of freedom, which can be written as one complex scalar. This means that, in contrast to the SM case, we do not have the minimal option of introducing only two scalar degrees of freedom to make up a two-dimensional representation of D_n . For this reason, we will be working only with actual complex scalars (as opposed to the fake complex ones considered above) in the following. Supersymmetrization also means that the introduction of gauge singlet, flavor-charged flavons necessitates the introduction of a Weyl spinor transforming in the same way, a so-called flavino. We will not be concerned with these, as they do not play a role for the scalar potential.

The scalar potential in supersymmetry consists of two main contributions: One of these, the D-terms, only arise for scalars which transform non-trivially under some gauge group. Since this is not the case for our flavon fields, they play no role in our discussion. The flavon potential will thus only consist of F-terms. These can easily be calculated with the introduction of the superpotential, an analytic, third-order polynomial in the chiral superfields of the theory:

$$w = \mu_{ij}\varphi_i\varphi_j + \lambda_{ijk}\varphi_i\varphi_j\varphi_k. \quad (5.24)$$

This superpotential must be invariant under all symmetries of the theory. Since we will not be considering fields which are total singlets under all symmetries (even if we introduce a field which is a singlet under the SM gauge groups and G_f , it transforms non-trivially under some auxiliary abelian symmetry), we do not need to consider linear terms. These are only relevant for supersymmetry breaking, which should anyway happen at a scale below that of flavor symmetry breaking. In any case, the actual scalar potential is given by

$$V = \sum_i F_i F_i^* = \sum_i \frac{\partial w}{\partial \varphi_i} \frac{\partial w^*}{\partial \varphi_i^*}. \quad (5.25)$$

Since this is a sum of squared absolute values, the potential can never be negative. In fact if the minimum is not at $V = 0$, supersymmetry will be spontaneously broken. Supersymmetry, if it exists, is of course broken in nature. It will be broken in a hidden sector and the breaking is only communicated to the visible sector, which includes the SM fields as well as the flavons, through effective soft SUSY breaking terms. These should lie at a scale of about 1 TeV to solve the hierarchy problem, while our flavor symmetry will be broken at a higher scale. We can thus neglect the effects of SUSY breaking, and, when minimizing our flavon potentials, always assume a supersymmetric minimum. Making this assumption, that is taking the supersymmetric limit, means that finding the field configuration which minimizes the scalar potential reduces to solving the equations

$$F_i = \frac{\partial w}{\partial \varphi_i} = 0 \quad (5.26)$$

for all scalar degrees of freedom φ_i in our model.

Since the superpotential is only a third degree polynomial, we have managed to eliminate the troubling coupling between the Higgs and the flavons: We need at least two Higgs fields, to construct a gauge singlet, and then have only one more field to which to couple, which needs to be a flavor singlet in itself. Of course we may need a flavor singlet which transforms non-trivially under an additional abelian group. If the Higgs field also transforms non-trivially under this group, we may be able to construct a total singlet out of three fields. We will return to this more subtle point later on.

Before returning to our discussion of flavon potentials, another word is in order concerning the Higgs boson in the MSSM. Due to the analyticity of the superpotential, the complex conjugated Higgs field ϵH^* cannot show up in the MSSM superpotential, as it does in the SM Yukawa couplings giving mass to the up-type quarks and neutrinos. Hence, we need to introduce two Higgs fields with opposite hypercharge, h_d and h_u , the first giving mass to the down-type quarks and charged leptons, the second giving mass to the up-type quarks and possibly the neutrinos. This is also necessary for anomaly cancellation, as the Higgs superfields will contain fermionic components, the Higgsinos. The gauge singlet combination of two Higgs fields to which a singlet flavon could couple is thus $h_u h_d$.

Both Higgs scalars acquire VEVs, related by

$$\langle h_u \rangle^2 + \langle h_d \rangle^2 = v_{wk}^2, \quad (5.27)$$

whereas their ratio

$$\frac{\langle h_u \rangle}{\langle h_d \rangle} = \frac{v_u}{v_d} = \tan \beta \quad (5.28)$$

is a free parameter in the MSSM and can only be restricted by phenomenology. Often a large $\tan \beta \approx m_t/m_b \approx 60$ is assumed to explain why the top quark mass is so much larger than that of the bottom quark and the τ . In the models presented in chapter 6, however, the top quark is the only fermion which acquires mass from a renormalizable operator. Its large mass can thus be explained without invoking a large $\tan \beta$.

5.2.2 One Doublet

We can leave aside the MSSM Higgs fields for the moment, and as for the SM case, consider the simplest possible field content, that is only one doublet $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, transforming under a faithful two-dimensional representation of D_n , which we take to be $\mathbf{2}_1$ for concreteness. As in the SM, we will then in general need to go to non-renormalizable operators, to see whether a subgroup is conserved when minimizing the potential. We again begin with n odd. The most general flavon superpotential containing only the complex doublet is, up to terms of dimension n ,

$$w_f = \mu(\varphi_1 \varphi_2) + \sum_{t=2}^{\frac{n-1}{2}} \frac{\lambda_t}{\Lambda^{(2t-3)}} (\varphi_1 \varphi_2)^t + \frac{\lambda_n}{\Lambda^{n-3}} (\varphi_1^n + \varphi_2^n), \quad (5.29)$$

where μ has a mass dimension of 1, while all the λ_t are dimensionless. Λ is of course again the scale at which these interactions are generated. It will be the mass scale of some heavy messenger fields which communicate the flavor symmetry breaking to the MSSM fields. To find the supersymmetric minimum, we need to solve the following two equations:

$$\frac{\partial w_f}{\partial \varphi_1} = \mu \varphi_2 + \sum_{t=2}^{\frac{n-1}{2}} \frac{\lambda_t}{\Lambda^{(2t-3)}} t \varphi_1^{(t-1)} \varphi_2^t + n \frac{\lambda_n}{\Lambda^{n-3}} \varphi_1^{(n-1)} = 0, \quad (5.30)$$

$$\frac{\partial w_f}{\partial \varphi_2} = \mu \varphi_1 + \sum_{t=2}^{\frac{n-1}{2}} \frac{\lambda_t}{\Lambda^{(2t-3)}} t \varphi_1^t \varphi_2^{t-1} + n \frac{\lambda_n}{\Lambda^{n-3}} \varphi_2^{(n-1)} = 0, \quad (5.31)$$

where we have used the fields instead of the field VEVs, so as not to clutter up the equations, which we will generally do in the following where it is unambiguous. We can multiply the first

equation by φ_1 and the second equation by φ_2 . Subtracting the two resulting equations from each other, we find that all terms vanish except for those proportional to λ_n and we obtain (for non-vanishing λ_n)

$$\varphi_1^n = \varphi_2^n, \quad (5.32)$$

i.e. only subgroup-conserving VEVs. There appears however to be no possibility to determine whether the phase factor m will be odd or even. The reasoning is equivalent for an even n if we include no scalars transforming as non-trivial one-dimensional representations. We recall from the SM discussion that including a flavon ψ_r transforming as $\mathbf{1}_r$, $r=3$ or 4 , changes this. If we include such a flavon, the terms determining the VEV structure already show up in operators of dimension $(\frac{n}{2} - 1)$. The superpotential up to this order is

$$w_f = \mu_1(\varphi_1\varphi_2) + \mu_2\psi_r^2 + \sum_{s=2}^{\lceil \frac{n}{4} \rceil} \sum_{t=0}^s \frac{\lambda_{st}}{\Lambda^{(2s-3)}} (\varphi_1\varphi_2)^t \psi_r^{2(s-t)} + \frac{\lambda}{\Lambda^{(\frac{n}{2}-2)}} \psi_r (\varphi_1^{\frac{n}{2}} \pm \varphi_2^{\frac{n}{2}}) \quad (5.33)$$

where the ceiling function $\lceil x \rceil$ denotes the smallest integer greater than or equal to x , i.e. $\lceil \frac{n}{4} \rceil$ is $\frac{n}{4}$ if n is divisible by 4 and $\frac{n+2}{4}$ if not. We will also later be using the floor function $\lfloor x \rfloor$ which denotes the greatest integer smaller than or equal to x . Choosing $r=3$ gives the plus sign in front of the last term, while $r=4$ gives a minus sign.

We now have to solve a system of three equations, of which we only need to consider the first two. This is sufficient for our purposes, if we assume that the system allows for a non-trivial solution at all.

$$\frac{\partial w_f}{\partial \varphi_1} = \mu_1\varphi_2 + \sum_{s=2}^{\lceil \frac{n}{4} \rceil} \sum_{t=1}^s \frac{\lambda_{st}}{\Lambda^{(2s-3)}} t \varphi_1^{(t-1)} \varphi_2^t \psi_r^{2(s-t)} + \frac{n}{2} \frac{\lambda}{\Lambda^{(\frac{n}{2}-2)}} \psi_r \varphi_1^{(\frac{n}{2}-1)} = 0 \quad (5.34)$$

$$\frac{\partial w_f}{\partial \varphi_1} = \mu_1\varphi_1 + \sum_{s=2}^{\lceil \frac{n}{4} \rceil} \sum_{t=1}^s \frac{\lambda_{st}}{\Lambda^{(2s-3)}} t \varphi_1^t \varphi_2^{(t-1)} \psi_r^{2(s-t)} \pm \frac{n}{2} \frac{\lambda}{\Lambda^{(\frac{n}{2}-2)}} \psi_r \varphi_2^{(\frac{n}{2}-1)} = 0 \quad (5.35)$$

We can again multiply by φ_1 and φ_2 respectively, subtract the equations from each other and thereby eliminate all terms except for those proportional to λ . If we then also assume that ψ_r gets a non-zero VEV (if not we have the same situation as for odd n), and no parameters in the potential vanish, we obtain

$$\varphi_1^{\frac{n}{2}} = \pm \varphi_2^{\frac{n}{2}}, \quad (5.36)$$

which is exactly the condition we need for a VEV configuration conserving the same subgroup as the one-dimensional representation, i.e. by the choice of our field content we can determine, whether the phase factor m is even or odd. Note that all these considerations are independent of the lower order terms, i.e. the VEV structure is only decided at this order of corrections.

We have retrieved the results we obtained in the SM: Flavon potentials in general favor VEVs conserving subgroups, and for an even n we can influence the subgroup parameter m by choosing which one-dimensional representations to include in our potential. As in the SM case, we need to go to operators with a large mass dimension to obtain these results. To get these results at the renormalizable level we again need to include several doublet flavons. This already made things more complicated in the SM, and we can already anticipate that things will get worse for complex scalars. We therefore do a slight detour at this point, to introduce the $U(1)_R$ symmetry and the driving field mechanism, which at the same time will get rid of the problem of Higgs-flavon coupling once and for all.

5.2.3 R-Symmetry and Driving Fields

An R-symmetry denotes a symmetry under which different components of a superfield transform differently. Most commonly an R-parity, that is a Z_2 symmetry, is assumed. For vector superfields containing the regular gauge bosons, it is always those gauge bosons which transform trivially, while their fermionic counterparts, the gauginos, are odd under R-parity. For a chiral superfield, one can choose which component (the scalar or the fermion) is even and which one odd. In the MSSM one chooses the R-parity assignments in such a way that all SM particles are even under R-parity and all superpartners are odd. In this way, the lightest supersymmetric particle (which is really an imprecise way of saying R-odd particle) is stable since it cannot decay to SM particles, and is therefore a good candidate for dark matter. Additionally (and this was the original motivation) R-parity forbids certain terms in the MSSM superpotential which can lead to proton decay and lepton number violation.

This R-parity can now be considered as a subgroup of a larger $U(1)_R$ symmetry. We can assign each chiral superfield an integer R-charge q under this symmetry group. The transformation properties of the component fields are then

$$A \rightarrow e^{iq\alpha} A, \quad (5.37)$$

$$\psi \rightarrow e^{i(q-1)\alpha} \psi, \quad (5.38)$$

with A and ψ the scalar and fermionic components of the chiral superfield, respectively, and $\alpha \in [0, 2\pi)$. We see that if we restrict α to 0 and π , we retrieve R-parity, where q being even or odd determines how the component fields transform.

Since this symmetry does not commute with supersymmetry, the supersymmetry generators and the elements of superspace also transform non-trivially under this R-symmetry. We will not go into these details of the supersymmetry algebra at this point. For our purposes it is only important that the terms in the superpotential are not allowed to be singlets of $U(1)_R$ but need to have an R-charge of 2 (when considering only the Z_2 subgroup, this of course is again a singlet).

To obtain the correct R-parity as a subgroup, we need to assign an R-charge of $q = 1$ to chiral superfields containing SM fermions and an R-charge of $q = 0$ to the chiral superfields containing the Higgs scalars. In this way, the term in the superpotential which leads to the SM Yukawa couplings,

$$Y_{ij} L_i L_j^c H, \quad (5.39)$$

has an R-charge of 2 and is allowed. Since we want to couple further flavons to this term, cf. equation (5.2), to obtain flavor structure in the mass matrices, the R-charge assignment for the flavon fields needs to be zero. To then obtain a non-vanishing superpotential for the flavons, we need to introduce an additional type of field with R-charge 2. These fields are also gauge singlets and in general transform non-trivially under G_f , that is they only differ from the flavons through their R-charge. These so-called driving fields were first used for a flavor symmetry model in [59]. We denote them by the same Greek letter as their flavon counterpart, but with a zero superscript, e.g. ψ_3^0 for a driving field transforming as a $\mathbf{1}_3$. These driving fields do not couple to the SM fermions, whose mass matrix structure is still only determined by the flavon VEVs. The driving fields are then only relevant for obtaining the VEV structure for the flavons, which is now calculated by solving the equations

$$\frac{\partial w}{\partial \varphi_i^0} = 0. \quad (5.40)$$

What about the VEVs of the driving fields? They should better be zero. We have managed to eliminate couplings between the MSSM Higgs fields and the flavons, since such a coupling term would have an R-charge of zero. However the Higgs fields can still couple to a driving field, that is a singlet under G_f . In any given model, one therefore needs to double check whether the driving fields indeed do not acquire a VEV.

At some point $U(1)_R$ has to be broken, since we need the so-called μ -term, $\mu H_u H_d$, in the superpotential to end up with electroweak symmetry breaking. However μ should be of the order of the electroweak scale, and hence a driving field VEV is not an option. We will not be concerned with the origin of the μ -term in the following and will ignore the Higgs field entirely in the following discussion. The squared Higgs VEV may of course show up in the F-term obtained by differentiating the superpotential with respect to a flavor singlet driving field, but it is much smaller than the scales involved in our flavor symmetry breaking and can therefore be safely neglected, just as we neglect the soft SUSY breaking terms.

5.2.4 Several Doublets

In what follows we will always employ the driving field mechanism discussed above. Since the superpotential, as opposed to the scalar potential of the SM, only contains terms up to order 3 in the scalar fields, we do not have as many possibilities to add doublet indices up to the group index n . We therefore choose a different approach here: We introduce all possible scalar doublets, that is a doublet flavon $\begin{pmatrix} \varphi_{j1} \\ \varphi_{j2} \end{pmatrix}$ and a doublet driving field $\begin{pmatrix} \varphi_{j1}^0 \\ \varphi_{j2}^0 \end{pmatrix}$ for each representation $\mathbf{2}_j$ of our group D_n . Once we have discussed this general case, we will consider how to reduce this large field content, while still making subgroup conservation a prediction of the potential. As always we begin with the simpler case of n odd, in which case the most general superpotential is

$$\begin{aligned}
w_f &= \sum_{s=1}^{\frac{n-1}{2}} \mu_s (\varphi_{s1} \varphi_{s2}^0 + \varphi_{s2} \varphi_{s1}^0) \\
&+ \sum_{s=1}^{\frac{n-3}{2}} \sum_{t=s}^{\frac{n-1}{2}-s} \alpha_{st} (\varphi_{s1} \varphi_{t1} \varphi_{(s+t)2}^0 + \varphi_{s2} \varphi_{t2} \varphi_{(s+t)1}^0) \\
&+ \sum_{s=1}^{\frac{n-3}{2}} \sum_{t=s+1}^{\frac{n-1}{2}} \beta_{st} (\varphi_{s1} \varphi_{t2} \varphi_{(t-s)1}^0 + \varphi_{s2} \varphi_{t1} \varphi_{(t-s)2}^0) \\
&+ \sum_{s=1}^{\frac{n-1}{2}} \sum_{\substack{t = \frac{n+1}{2} - s \\ t \geq s}}^{\frac{n-1}{2}} \gamma_{st} (\varphi_{s1} \varphi_{t1} \varphi_{(n-s-t)1}^0 + \varphi_{s2} \varphi_{t2} \varphi_{(n-s-t)2}^0). \tag{5.41}
\end{aligned}$$

To find a supersymmetric minimum of the potential, we compute the partial derivatives:

$$\frac{\partial w_f}{\partial \varphi_{j1}^0} = \mu_j \varphi_{j2} + \sum_{s=1}^{\lfloor \frac{j}{2} \rfloor} \alpha_{s(j-s)} \varphi_{s2} \varphi_{(j-s)2} + \sum_{s=1}^{\frac{n-1}{2}-j} \beta_{s(j+s)} \varphi_{s1} \varphi_{(j+s)2} + \sum_{s=\frac{n+1}{2}-j}^{\lfloor \frac{n-j}{2} \rfloor} \gamma_{s(n-s-j)} \varphi_{s1} \varphi_{(n-s-j)1} = 0. \tag{5.42}$$

For the derivative with respect to the second component of the j th driving doublet, we have the above equation with all indices of one replaced by indices of two and vice versa. We then

multiply $\frac{\partial w_f}{\partial \varphi_{j1}^0}$ and $\frac{\partial w_f}{\partial \varphi_{j2}^0}$ by φ_{j1} and φ_{j2} respectively, and subtract the two equations from each other. This leads to the following equation:

$$\begin{aligned}
0 &= \sum_{s=1}^{\lfloor \frac{j}{2} \rfloor} \alpha_{s(j-s)} (\varphi_{s2} \varphi_{j1} \varphi_{(j-s)2} - \varphi_{s1} \varphi_{j2} \varphi_{(j-s)1}) \\
&+ \sum_{s=1}^{\frac{n-1}{2}-j} \beta_{s(j+s)} (\varphi_{s1} \varphi_{j1} \varphi_{(j+s)2} - \varphi_{s2} \varphi_{j2} \varphi_{(j+s)1}) \\
&+ \sum_{s=\frac{n+1}{2}-j}^{\lfloor \frac{n-j}{2} \rfloor} \gamma_{s(n-s-j)} (\varphi_{s1} \varphi_{j1} \varphi_{(n-s-j)1} - \varphi_{s2} \varphi_{j2} \varphi_{(n-s-j)2}). \tag{5.43}
\end{aligned}$$

These equations can become arbitrarily complicated for large n . They can however easily be solved, by assuming that they should be satisfied for arbitrary values of the coefficients α , β and γ , that is all summands should vanish independently. This is of course by no means a proof of the uniqueness of the solution. However, it is definitely a very special solution, since it is fully independent of the parameters in the potential. In the realistic models discussed later on, we will always be able to show that the subgroup-conserving VEV is indeed a unique solution, after we have chosen the particle content based on our more general considerations here. Also, in a full model, some terms will be absent due to the presence of additional abelian symmetries. In any case, minimizing the potential in this way leaves us with two general types of equations. The first type stems from the terms in the first two sums of equation (5.43):

$$\varphi_{j1} \varphi_{k1} \varphi_{(j+k)2} = \varphi_{j2} \varphi_{k2} \varphi_{(j+k)1}. \tag{5.44}$$

This equation must hold for arbitrary j and k with $(j+k) \leq \frac{n-1}{2}$. The other set of equations reads

$$\varphi_{j1} \varphi_{k1} \varphi_{(n-j-k)1} = \varphi_{j2} \varphi_{k2} \varphi_{(n-j-k)2} \tag{5.45}$$

and is derived from the last sum of equation (5.43). It has to hold for arbitrary j and k obeying $(n-j-k) \leq \frac{n-1}{2}$. We solve the equations iteratively. Setting $j = k = 1$ we obtain from equation (5.44):

$$\left(\frac{\varphi_{11}}{\varphi_{12}} \right)^2 = \frac{\varphi_{21}}{\varphi_{22}}, \tag{5.46}$$

and iterating:

$$\left(\frac{\varphi_{11}}{\varphi_{12}} \right)^j = \frac{\varphi_{j1}}{\varphi_{j2}} \tag{5.47}$$

for all j (note that this trivially holds for the case of $n = 3$, where the first set of equations does not exist, as there is only one doublet). Inserting this in equation (5.45) we obtain

$$\varphi_{11}^n = \varphi_{12}^n. \tag{5.48}$$

Combining the last two equations, we obtain

$$\varphi_{j1} = e^{-\frac{2\pi m j}{n}} \varphi_{j2}, \tag{5.49}$$

m an integer between 0 and $n-1$. Thus, all VEVs necessarily conserve a subgroup $Z_2 = \langle \text{BA}^m \rangle$, and the subgroup is the same for all doublets, even though m is undetermined.

We have up until now considered the maximal field content. We can however observe that we only need a fraction of the equations to obtain our minimization conditions. Hence it should be possible to obtain subgroup-conserving VEVs with a reduced field content. We first observe that we really only require one of the equations (5.45), hence we only need one term in the potential, where the indices of the field add up to n . So, we only have to introduce all doublets (as flavons and driving fields) up to $\underline{2}_{\lceil \frac{n}{3} \rceil}$ (This makes no difference for $n \leq 7$).

We can further minimize the model, if we observe that, of the equations (5.44), we actually only require those with $j = 1$. So apart from the driving fields which are necessary to get equations such as equation (5.45), we can dump all of them, except for the one transforming as $\underline{2}_1$. To summarize: When conserving a Z_2 symmetry in D_n with n odd, we need to introduce all flavons up to $\underline{2}_{\lceil \frac{n}{3} \rceil}$. As driving fields we definitely should introduce φ_1^0 (or some other doublet, whose index does not divide n), in addition if 3 divides n we introduce $\varphi_{\frac{n}{3}}^0$, if 3 divides $n+1$ we introduce $\varphi_{\frac{n+1}{3}}^0$ and $\varphi_{\frac{n-2}{3}}^0$ and if 3 divides $n+2$ we introduce $\varphi_{\frac{n+2}{3}}^0$ and either $\varphi_{\frac{n-1}{3}}^0$ or $\varphi_{\frac{n-4}{3}}^0$ for the minimal potential. This is obviously not a unique solution. In general, we might also need specific flavon doublets to generate the desired mass matrices. But one can see that, even though we used all two-dimensional representations for our general considerations, a given model may contain a lot less scalar fields and still give the correct vacuum alignment. This may then also make it possible to solve the minimization equations without any further assumptions, as we shall see later in our worked-out models.

Before ending our general discussion of flavon potentials, we consider the superpotential invariant under a D_n symmetry with n even. As before, if we do not introduce any fields transforming as non-trivial singlets, the discussion is pretty much equivalent to the one for odd n . Adding fields transforming as $\underline{1}_3$ or $\underline{1}_4$ however provides the possibility of determining whether m is odd or even and hence is interesting for model building. We write down the most general superpotential with doublet fields up to indices of $\lceil \frac{n}{4} \rceil$ (taking the lesson we learned from studying the case of n odd) and a singlet ψ_r transforming as $\underline{1}_r$, with $r=3,4$:

$$\begin{aligned}
w_f &= \sum_{s=1}^{\lceil \frac{n}{4} \rceil} \mu_s (\varphi_{s1} \varphi_{s2}^0 + \varphi_{s2} \varphi_{s1}^0) \\
&+ \sum_{s=1}^{\lceil \frac{n}{4} \rceil - 1} \sum_{t=s}^{\lceil \frac{n}{4} \rceil - s} \alpha_{st} (\varphi_{s1} \varphi_{t1} \varphi_{(s+t)2}^0 + \varphi_{s2} \varphi_{t2} \varphi_{(s+t)1}^0) \\
&+ \sum_{s=1}^{\lceil \frac{n}{4} \rceil - 1} \sum_{t=s+1}^{\lceil \frac{n}{4} \rceil} \beta_{st} (\varphi_{s1} \varphi_{t2} \varphi_{(t-s)1}^0 + \varphi_{s2} \varphi_{t1} \varphi_{(t-s)2}^0) \\
&+ \gamma (\varphi_{\lceil \frac{n}{4} \rceil 1} \varphi_{\lfloor \frac{n}{4} \rfloor 1}^0 \pm \varphi_{\lceil \frac{n}{4} \rceil 2} \varphi_{\lfloor \frac{n}{4} \rfloor 2}^0) \psi_r \\
&+ \delta (\varphi_{\lfloor \frac{n}{4} \rfloor 1} \varphi_{\lceil \frac{n}{4} \rceil 1}^0 \pm \varphi_{\lfloor \frac{n}{4} \rfloor 2} \varphi_{\lceil \frac{n}{4} \rceil 2}^0) \psi_r.
\end{aligned} \tag{5.50}$$

Through a similar calculation as above, we obtain equation (5.47) and instead of equation (5.48) we get

$$\varphi_{11}^{\frac{n}{2}} = \pm \varphi_{12}^{\frac{n}{2}}, \tag{5.51}$$

giving VEVs conserving $Z_2 = \langle \text{BA}^m \rangle$, with m even if $r=3$ and m odd if $r=4$. We have already minimized the number of fields by only letting the doublet indices run to $\lceil \frac{n}{4} \rceil$. As for an odd

n , one can further reduce the number of driving fields: In addition to the driving field φ_1^0 , we need a driving field $\varphi_{\frac{n}{4}}^0$ if 4 divides n or two driving fields $\varphi_{\frac{n+2}{4}}^0$ and $\varphi_{\frac{n-2}{4}}^0$ otherwise, to obtain the last two terms in the superpotential.

Instead of introducing a flavon ψ_r , we could include a driving field ψ_r^0 . Both of them should not be present, as long as the mass term is not disallowed by an abelian symmetry. Replacing the flavon by the driving field is equivalent to replacing the last two terms in the superpotential of equation (5.50) with just one coupling

$$\gamma(\varphi_{\lceil \frac{n}{4} \rceil 1} \varphi_{\lfloor \frac{n}{4} \rfloor 1} \pm \varphi_{\lceil \frac{n}{4} \rceil 2} \varphi_{\lfloor \frac{n}{4} \rfloor 2}) \psi_r^0, \quad (5.52)$$

which directly leads to the equation

$$\varphi_{\lceil \frac{n}{4} \rceil 1} \varphi_{\lfloor \frac{n}{4} \rfloor 1} = \mp \varphi_{\lceil \frac{n}{4} \rceil 2} \varphi_{\lfloor \frac{n}{4} \rfloor 2}, \quad (5.53)$$

so the connection between m being even or odd and our choice of r is switched, compared to a flavon transforming as $\underline{\mathbf{1}}_r$. This does not contradict the overall conservation of the subgroup, as the driving field should not get a VEV. For $n = 4$ this is in fact the only equation we need to ensure the correct vacuum alignment, and we will thus later be using a driving field transforming as $\underline{\mathbf{1}}_4$ in our D_4 model of section 6.1.

We have shown that the flavon VEV alignment leading to subgroup conservation also appears naturally in a supersymmetric framework with driving fields, and that the scalar field content can be reduced to a manageable level. The challenge is now to design models, where this natural alignment is not obscured by other components of the model, such as was the case for the potential of flavor charged Higgs bosons. Before we move on to the actual model building, we briefly discuss some further aspect of models with flavons.

5.2.5 Further Aspects of Flavon Models

Next-to-Leading Order Corrections In full models, such as the ones presented in the next chapter, we want to conserve different subgroups in different sectors. Just as in the D_7 model of section 4.3, this can be achieved by introducing an abelian symmetry. This abelian symmetry separates the flavons in (at least) two sectors, one coupling to up-type and one to down-type fermions. The VEVs of the two flavon sectors conserve different subgroups and the subgroup mismatch in the mass and diagonalization matrices ensues. The two types of flavons need only differ in their transformation properties under the abelian symmetry. Complex conjugation for the up-type scalars, which was necessary in multi-Higgs models due to the $SU(2)_L$ structure, is not required in flavon models.

But we do not want to separate the flavons only with respect to the Yukawa couplings. For this a Z_2 would be sufficient. We saw in the last section that the couplings in the flavon potential tend to lead to the same subgroup being conserved by the VEVs of all flavons. Hence, we would like our abelian symmetry to separate the two sectors in the flavon potential as well. The minimal symmetry that can accomplish this is a Z_3 , as the superpotential is a third-order polynomial. If one set of flavons (and driving fields) transforms trivially under Z_3 while the other sector acquires a non-trivial phase $\omega_3 = e^{\frac{2\pi i}{3}}$, no coupling between both sectors is allowed in the renormalizable part of the superpotential. For those fields transforming non-trivially under Z_3 , the coupling of one flavon and one driving field to form a quadratic mass term is also forbidden. These were however anyway not essential for the vacuum alignment.

Generating fermion masses through non-renormalizable couplings to the flavons implies that we can no longer move the cutoff scale Λ arbitrarily high. Contributions to both the Yukawa and the flavon superpotential from higher order operators are thus subleading, but by no means entirely negligible. What effect do these corrections have?

We begin by assuming that all flavon VEVs which are non-vanishing at leading order (LO) are of the same order of magnitude. This is a fairly natural assumption, as many of them will be linked through the minimization of the scalar potential. We will see later on, how this tendency is reinforced when taking into account next-to-leading order (NLO) corrections. With this assumption, we have a natural expansion parameter in our model, namely

$$\epsilon \sim \left(\frac{\text{VEV}}{\Lambda} \right), \quad (5.54)$$

which can be given in terms of the generic Wolfenstein expansion parameter λ . Corrections to LO phenomenology now arise in two ways. First of all, at NLO, flavons from one sector can couple to the SM fields from another. For concreteness, say the left-handed conjugate down quarks transform as ω_3^2 under Z_3 , while the left-handed quark doublets and the left-handed conjugate up quarks transform trivially. This will actually be the case for the D_{14} model of section 6.2. We then have two types of flavons, the φ_i^u transforming trivially under Z_3 , contributing only to the up quark Yukawa couplings at leading order, and the φ_i^d , acquiring a phase ω_3 under Z_3 and thus coupling only to the down quarks at LO. Then, depending on the exact D_n structure, the following term is allowed at NLO:

$$Q_i d_j^c h_d Y_{ijkl} \frac{\varphi_k^u \varphi_l^d}{\Lambda}. \quad (5.55)$$

As the VEVs of φ_k^u and φ_l^d preserve different subgroups, this term disturbs the subgroup-conserving structure of the down quark mass matrix, which is thus corrected at order ϵ .

Another correction arises from the flavon potential. There will be two sets of driving fields φ^{0u} and φ^{0d} with the same respective Z_3 transformation properties as their flavon counterparts. Then terms such as

$$\frac{1}{\Lambda} \lambda_{ijkl} \varphi_i^d \varphi_j^d \varphi_k^d \varphi_l^{0u} \quad \text{and} \quad \frac{1}{\Lambda} \lambda'_{ijkl} \varphi_i^d \varphi_j^d \varphi_k^u \varphi_l^{0d} \quad (5.56)$$

are allowed in the flavon superpotential at NLO. The separation of the two sectors in the scalar potential is thus not perfect and there will be NLO corrections to the F-terms. We thus need to perturb the subgroup conserving VEV structures obtained at LO, e.g.

$$\langle \varphi_i^u \rangle = \langle \varphi_i^u \rangle_0 + \delta \langle \varphi_i^u \rangle. \quad (5.57)$$

The natural size of these corrections is

$$\delta \text{VEV} \sim \mathcal{O} \left(\frac{\text{VEV}}{\Lambda} \right) \cdot \text{VEV} \sim \epsilon \cdot \text{VEV} \quad (5.58)$$

but this needs to be calculated explicitly in a given model. The shifted VEVs will then, at NLO, no longer be subgroup-conserving, and again the subgroup-conserving structure of the quark mass matrices are corrected at order ϵ . If we had not assumed in the beginning that all flavon VEVs are of the same order, flavon VEVs which were small at LO could receive large corrections at NLO. Our assumption is thus the most stable one. Both types of NLO corrections, from the Yukawa and from the flavon superpotential, will be taken into account in the models discussed in the next chapter.

Flavon Mass Spectrum and Flat Directions Another aspect of flavon models merits further study: Although we have moved the scale of flavor symmetry breaking up, eliminating potential problems of a multi-Higgs model such as Lepton Flavor Violation, our flavons and

flavinos (as well as the driving fields and their superpartners, which one could conceivably call drivinos) are of course physical fields and we will briefly consider the actually observed spectrum. In the supersymmetric limit the masses of scalars and fermions are of course equal. The flavon mass matrix is given by

$$M_{ij}^2 = \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \Big|_{\varphi=\langle \varphi \rangle}, \quad (5.59)$$

where the indices i, j run over all flavons and driving fields, counting real and imaginary parts separately, and the scalar potential is given by equation (5.25). If the flavons and driving fields are left with a non-vanishing mass after flavor symmetry breaking, this will generically be of the order of the flavon VEVs and therefore phenomenologically uninteresting. However it is quite likely that due to the special VEV structure at the potential minimum, some of the flavons will remain massless.

This is not a problem however: Aside from the fact that both flavons and flavinos can obtain additional soft masses through supersymmetry breaking effects, which might not even respect the flavor symmetry, these apparently massless modes will also in general disappear when taking into account next to leading order effects. These flavons will therefore be lighter than the others, but not massless.

If, for a given mode, not only the mass term vanishes at LO, but also the cubic and the quartic term, i.e. if the potential has a flat direction, this may be of interest for inflationary scenarios [117]. Where it is feasible, we thus check for such almost flat directions in the flavon potential.

Chapter 6

Putting It All Together: Two Worked-Out Examples

We present two discrete non-abelian flavor symmetry models, putting together all the elements we have assembled in the last chapters. Both of them use the subgroup mismatch discussed in chapter 4 and symmetry breaking with flavons and driving fields as discussed in chapter 5. We first, in section 6.1, present a supersymmetrized version of the D_4 model described in section 4.2 and then, in section 6.2, present a supersymmetric model based on D_{14} predicting the Cabibbo angle.

6.1 D_4 in the Leptonic Sector

We construct a supersymmetrized version of the model presented by Grimus and Lavoura in [84], which we already discussed in section 4.2. In [86] it has already been attempted to build a D_4 model in which the Higgs doublets transforming non-trivially under the flavor group are replaced by flavons. Since this model is non-supersymmetric the vacuum alignment problem is not straightforward to solve and indeed one has to require that one of the quartic couplings in the potential vanishes. However, such an assumption will not be stable against corrections and has to be considered as a severe tuning. In a second version of this model [87] which is supersymmetric, the potential is not studied such that the question of the vacuum alignment also remains open. Using the insights on flavon potentials gained in the last chapter, the model presented here solves that problem.

As in the original model, this model predicts θ_{23} maximal and $\theta_{13} = 0$ in the lepton sector. To achieve this, we augment the MSSM by the flavor symmetry $D_4 \times Z_5$. The auxiliary group needed to separate the charged lepton and the neutrino sector is somewhat larger than the minimal Z_3 , to avoid ending up with charged lepton and neutrino masses of the same order. In any case, for a flavon model the auxiliary symmetry would need to be larger than the Z_2 employed in [84].

Our model contains the three left-handed lepton doublets l_α , the three left-handed conjugate charged leptons e_i^c , the MSSM Higgs doublets $h_{u,d}$ and two sets of flavons $\{\chi_e, \varphi_e\}$ and $\{\chi_\nu, \varphi_\nu, \psi_{1,2}\}$ which break D_4 in the charged lepton and the neutrino sector, respectively. The transformation properties of these fields are collected in table 6.1. We take the fermion fields to transform as in [84], with the difference that we do not explicitly introduce left-handed conjugate neutrinos. The transformation properties of the flavons are chosen according to the maximal scalar field content allowed by the subgroups we want to conserve, D_2 in the charged lepton sector and Z_2 for neutrinos.

Field	l_e	$l_{\mu,\tau}$	e_1^c	$e_{2,3}^c$	h_u	h_d	χ_e	φ_e	χ_ν	φ_ν	$\psi_{1,2}$
D_4	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_5$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_3$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{2}}_5$
Z_5	ω_5	ω_5	1	1	ω_5^3	ω_5	ω_5^3	ω_5^3	ω_5^2	ω_5^2	ω_5^2

Table 6.1: Particle content of the D_4 model. l_α denotes the three left-handed lepton $SU(2)_L$ doublets, e_i^c are the left-handed conjugate charged leptons and $h_{u,d}$ are the MSSM Higgs doublets. The flavons χ_e , φ_e , χ_ν , φ_ν and $\psi_{1,2}$ only transform under $D_4 \times Z_5$. The phase factor ω_5 is $e^{\frac{2\pi i}{5}}$.

6.1.1 Lepton Masses and Mixing

The invariance of the charged lepton and neutrino mass terms under the flavor group $D_4 \times Z_5$ requires the presence of at least one flavon. Thus, charged lepton masses are generated by non-renormalizable operators only. As mentioned in section 5.2.1, this allows for a small τ mass compared to the top quark mass without relying on a large value of $\tan \beta$. The neutrinos receive Majorana masses through the dimension-5 operator $lh_u lh_u / \Lambda$ which can be made invariant under the flavor group by coupling to a flavon. The part of the superpotential giving lepton masses reads at LO

$$\begin{aligned}
w_l = & y_1^e \chi_e l_1 e_1^c \frac{h_d}{\Lambda} + y_2^e \chi_e (l_2 e_3^c + l_3 e_2^c) \frac{h_d}{\Lambda} + y_3^e \varphi_e (l_2 e_2^c - l_3 e_3^c) \frac{h_d}{\Lambda} \\
& + y_1 \chi_\nu l_1 l_1 \frac{h_u^2}{\Lambda^2} + y_2 l_1 (l_2 \psi_2 + l_3 \psi_1) \frac{h_u^2}{\Lambda^2} + y_2 (l_2 \psi_2 + l_3 \psi_1) l_1 \frac{h_u^2}{\Lambda^2} + y_3 \varphi_\nu (l_2 l_2 + l_3 l_3) \frac{h_u^2}{\Lambda^2} \\
& + y_4 \chi_\nu (l_2 l_3 + l_3 l_2) \frac{h_u^2}{\Lambda^2}.
\end{aligned} \tag{6.1}$$

For the moment we simply assume that the flavons χ_e and φ_e acquire the VEVs

$$\langle \varphi_e \rangle = u_e \quad \text{and} \quad \langle \chi_e \rangle = w_e. \tag{6.2}$$

As discussed in chapter 3 these VEVs break D_4 down to D_2 generated by A^2 and BA in the charged lepton sector. The VEVs of the flavons coupling only to neutrinos at LO, are of the form

$$\langle \varphi_\nu \rangle = u, \quad \langle \chi_\nu \rangle = w, \quad \begin{pmatrix} \langle \psi_1 \rangle \\ \langle \psi_2 \rangle \end{pmatrix} = v \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{6.3}$$

and therefore leave a Z_2 subgroup, generated by B , unbroken. As will be discussed in section 6.1.3, the vacuum structure in equation (6.2) and equation (6.3) is a natural result of the minimization of the flavon potential. We obtain the following fermion mass matrices, when inserting the flavon VEVs and $\langle h_{u,d} \rangle = v_{u,d}$:

$$M_l = \frac{v_d}{\Lambda} \begin{pmatrix} y_1^e w_e & 0 & 0 \\ 0 & y_3^e u_e & y_2^e w_e \\ 0 & y_2^e w_e & -y_3^e u_e \end{pmatrix} \quad \text{and} \quad M_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1 w & y_2 v & y_2 v \\ y_2 v & y_3 u & y_4 w \\ y_2 v & y_4 w & y_3 u \end{pmatrix}. \tag{6.4}$$

The matrix $M_l M_l^\dagger$ is diagonalized through the unitary matrix U_l , which acts on the left-handed charged lepton fields and is given by

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4}/\sqrt{2} & e^{-i\pi/4}/\sqrt{2} \\ 0 & e^{-i\pi/4}/\sqrt{2} & e^{i\pi/4}/\sqrt{2} \end{pmatrix}. \tag{6.5}$$

Note that this corresponds to transforming to the basis of real generators used in [84]. This is discussed in further detail in appendix B.3. For the masses of the charged leptons we find

$$m_e = \frac{v_d}{\Lambda} |y_1^e w_e|, m_\mu = \frac{v_d}{\Lambda} |y_3^e u_e + i y_2^e w_e| \quad \text{and} \quad m_\tau = \frac{v_d}{\Lambda} |y_3^e u_e - i y_2^e w_e|. \quad (6.6)$$

In order to arrive at non-degenerate masses for the μ and the τ lepton either $y_3^e u_e$ or $y_2^e w_e$ has to be non-real, indicating CP violation in the Yukawa couplings and/or flavon VEVs. For m_τ being around 2 GeV (cf. equation (1.5)) we find that for small $\tan \beta$ - corresponding to v_d of the order of 100 GeV - the ratio of the flavon VEVs u_e and w_e over the cutoff scale Λ should fulfill

$$\frac{u_e}{\Lambda}, \frac{w_e}{\Lambda} \sim \lambda^2 \approx 0.04, \quad (6.7)$$

where $\lambda = \sin \theta_C$ is the expansion parameter used in the Wolfenstein parameterization of the CKM matrix, cf. section 1.1. The smallness of the ratio m_e/m_τ is in this model only explained by the assumption of a small enough coupling y_1^e . Similarly, m_μ/m_τ enforces a certain cancellation between the two contributions $y_3^e u_e$ and $i y_2^e w_e$ in m_μ . In [84] these problems have been solved by the assumption that the electron couples to a Higgs field different from those coupling to the μ and the τ lepton and by an additional symmetry which leads to $m_\mu = 0$, if it is unbroken. The neutrino mass matrix in the charged lepton mass basis reads (indicated by a prime ('))

$$M'_\nu = U_l^\dagger M_\nu U_l^* = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1 w & y_2 v & y_2 v \\ y_2 v & y_4 w & y_3 u \\ y_2 v & y_3 u & y_4 w \end{pmatrix}. \quad (6.8)$$

M'_ν is still $\mu - \tau$ symmetric, just as M_ν was. This is due to the subgroup mismatch mechanism discussed in chapter 4. It then immediately follows that the lepton mixing angle θ_{13} vanishes and θ_{23} is maximal. The solar mixing angle θ_{12} is not predicted, but in general expected to be large. Also the Majorana phases $\phi_{1,2}$ are not constrained. The lepton mixing matrix is of the form

$$V_{PMNS} = \text{diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}) \cdot \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\frac{\sin \theta_{12}}{\sqrt{2}} & \frac{\cos \theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin \theta_{12}}{\sqrt{2}} & \frac{\cos \theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}). \quad (6.9)$$

The Majorana phases $\phi_{1,2}$ can be extracted from V_{PMNS} by bringing it into the standard form given in equation (1.28). Assuming that all flavon VEVs are of the same size, the estimate in equation (6.7) also holds for the VEVs of the flavons χ_ν , φ_ν and $\psi_{1,2}$. For small $\tan \beta$ a light neutrino mass scale between $\sqrt{|\Delta m_{31}^2|} \approx 0.05$ eV and 1 eV fixes the range of the cutoff scale Λ to be

$$4 \cdot 10^{11} \text{ GeV} \lesssim \Lambda \lesssim 8 \cdot 10^{12} \text{ GeV}. \quad (6.10)$$

As shown in section 6.1.3, we can assume that CP is only spontaneously violated in this model by imaginary VEVs w_e and w of χ_e and χ_ν . Thus, apart from w_e and w all other parameters, i.e. couplings and VEVs, are real in the following. According to equation (6.6) an imaginary w_e allows the μ and the τ lepton mass to be non-degenerate. In the neutrino sector only the VEV w of χ_ν is imaginary, whereas all other entries in M'_ν are real, so that the matrix in equation (6.8) can be written as

$$M'_\nu = \frac{v_u^2}{\Lambda} \frac{v}{\Lambda} \begin{pmatrix} i s & t & t \\ t & i x & z \\ t & z & i x \end{pmatrix}, \quad (6.11)$$

where we define the real parameters

$$s = y_1 \frac{\text{Im}(w)}{v}, \quad t = y_2, \quad x = y_4 \frac{\text{Im}(w)}{v} \quad \text{and} \quad z = y_3 \frac{u}{v}. \quad (6.12)$$

6.1.2 Phenomenology

In the following we analyze the phenomenology of this model. For the eigenvalues of $M'_\nu M_\nu^\dagger$ we find

$$m_{2,1}^2 = \frac{1}{2} \left(\frac{v_u^2}{\Lambda} \right)^2 \left(\frac{v}{\Lambda} \right)^2 \left[s^2 + 4t^2 + x^2 + z^2 \pm \sqrt{(s-x)^2(8t^2 + (s+x)^2) + 2(4t^2 + x^2 - s^2)z^2 + z^4} \right]$$

and $m_3^2 = \left(\frac{v_u^2}{\Lambda} \right)^2 \left(\frac{v}{\Lambda} \right)^2 (x^2 + z^2).$ (6.13)

This assignment of the eigenvalues is unambiguous, since $m_2^2 > m_1^2$ is experimentally known and the eigenvalue corresponding to the eigenvector $(0, 1, -1)^T$ can only be m_3^2 . The solar mixing angle θ_{12} is found to depend on s, t, x and z in the following way:

$$\tan(2\theta_{12}) = \frac{2\sqrt{2}|t|\sqrt{(s-x)^2 + z^2}}{x^2 + z^2 - s^2}, \quad (6.14)$$

which corresponds to equation (3.53). Before discussing the general case with unconstrained parameters s, t, x and z we comment on the special case in which z vanishes, since then the model contains only three real parameters which can be determined by the three experimental quantities $\Delta m_{21}^2, |\Delta m_{31}^2|$ and θ_{12} . According to equation (6.12) either y_3 or u have to vanish for $z = 0$ to hold. Assuming that y_3 is zero however has to be regarded as fine-tuning. In contrast to that, a vanishing VEV u can be explained either through the absence of the flavon φ_ν from the model or through a flavon potential which only allows configurations with $u = 0$ as minima. The neutrino mass m_3 is then proportional to $|x|$. From equation (6.13) and equation (6.14) we can derive for $z = 0$

$$m_3^2 = -\frac{1}{4} \frac{\cos^4 \theta_{12}}{\sin^2 \theta_{12}} \frac{(\Delta m_{21}^2 + \Delta m_{31}^2 (\tan^4 \theta_{12} - 1))^2}{\Delta m_{31}^2 (1 + \tan^2 \theta_{12}) - \Delta m_{21}^2}. \quad (6.15)$$

Neglecting the solar mass squared difference we can simplify this expression to

$$m_3^2 \approx -\Delta m_{31}^2 \cot^2 2\theta_{12}. \quad (6.16)$$

Equation (6.16) shows that $\Delta m_{31}^2 < 0$, i.e. the neutrinos have to have an inverted hierarchy. Similar results can also be found in [118]. A relation analogous to equation (6.15) can be found for $|m_{ee}|$. Note that $|m_{ee}|$ is proportional to $|s|$ due to equation (6.11) and can be written in terms of $m_3, \tan \theta_{12}$ and the mass squared differences as

$$|m_{ee}|^2 = m_3^2 \frac{(\Delta m_{21}^2 (1 - 2 \tan^2 \theta_{12}) + \Delta m_{31}^2 (\tan^4 \theta_{12} - 1))^2}{(\Delta m_{21}^2 + \Delta m_{31}^2 (\tan^4 \theta_{12} - 1))^2}. \quad (6.17)$$

In the limit of vanishing solar mass splitting we find

$$|m_{ee}| \approx m_3. \quad (6.18)$$

Taking the best-fit values given in equation (1.29) we obtain $s \approx 0.02075, t \approx 0.03502, x \approx 0.02146$ for $v_u \approx 100 \text{ GeV}, \Lambda \approx 4 \cdot 10^{11} \text{ GeV}$ and $v/\Lambda \approx \lambda^2 \approx 0.04$. Actually we find four solutions which all lead to the same absolute values, but to different signs for s, t and x , with

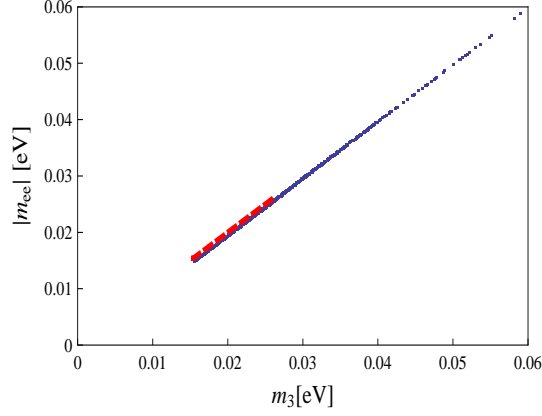


Figure 6.1: $|m_{ee}|$ plotted against m_3 for $z \neq 0$. The dashed red line indicates the results for $z = 0$. Mass squared differences and the solar mixing angle are in the allowed 2σ ranges [20]. As one can see, $|m_{ee}|$ and m_3 have nearly the same value. Additionally, one finds that m_3 has a lower bound around 0.015 eV. For $z = 0$ we also find an upper bound on m_3 .

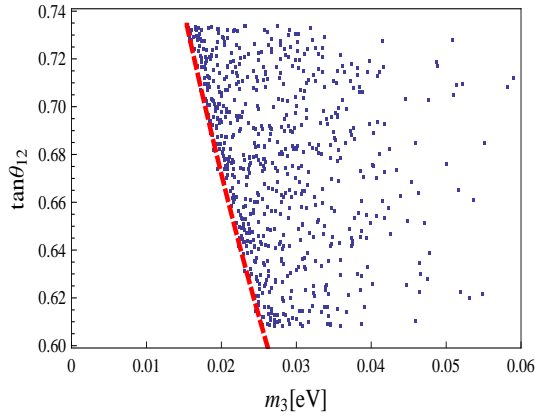


Figure 6.2: $\tan \theta_{12}$ plotted against m_3 for non-vanishing z . Again the dashed red line indicates $z = 0$ (assuming the best-fit value for the atmospheric mass squared difference) and gives a lower bound for $z \neq 0$. Apart from that the results for $\tan \theta_{12}$ are only constrained by the requirement that they are within the experimental 2σ ranges [20], $0.61 \lesssim \tan \theta_{12} \lesssim 0.73$.

the constraint that s and x have the same sign. The neutrino masses are $m_1 \approx 0.05348$ eV, $m_2 \approx 0.05419$ eV and $m_3 \approx 0.02146$ eV. Their sum $\sum m_i \approx 0.1291$ eV lies below the upper bound required from cosmological data (equation 1.31). $|m_{ee}|$ equals 0.02075 eV which could conceivably be detectable in the future [107]. The two Majorana phases $\phi_{1,2}$ are $\phi_1 = \pi/2$ and $\phi_2 = 0$. For tritium β decay we find $m_\beta \approx 0.05370$ eV which is substantially smaller than the expected sensitivity of the KATRIN experiment.

Turning to the general case with $z \neq 0$ we first observe that also in this case the light neutrinos have to have an inverted hierarchy. To see this let us assume that the matrix in equation (6.11) would allow the neutrinos to be normally ordered, i.e. $m_3 > m_1$ as well as $m_3 > m_2$. From $m_3^2 - m_2^2 > 0$ then follows

$$x^2 + z^2 - s^2 - 4t^2 - \sqrt{(s-x)^2(8t^2 + (s+x)^2) + 2(4t^2 + x^2 - s^2)z^2 + z^4} > 0. \quad (6.19)$$

From this we can deduce

$$x^2 + z^2 > s^2 + 4t^2 \quad \text{and} \quad 16t^2(t^2 + x(s-x) - z^2) > 0, \quad (6.20)$$

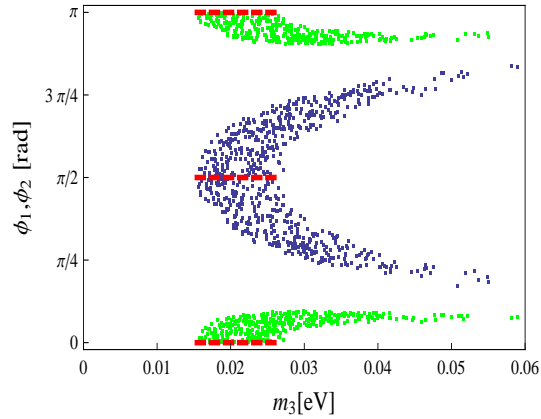


Figure 6.3: The Majorana phases ϕ_1 (blue) and ϕ_2 (green) plotted against the lightest neutrino mass m_3 for non-vanishing z . The values for $z = 0$, $\phi_1 = \frac{\pi}{2}$, $\phi_2 = 0$, are displayed by dashed red lines. Notice that the results for $z \neq 0$ are centered around these values. The measured quantities, Δm_{21}^2 , $|\Delta m_{31}^2|$ and θ_{12} , are within the 2σ ranges [20].

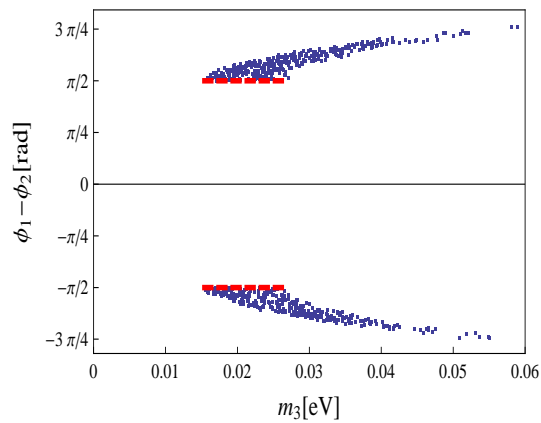


Figure 6.4: Phase difference $\phi_1 - \phi_2$ against m_3 for $z \neq 0$. The case $z = 0$, $|\phi_1 - \phi_2| = \pi/2$, is given by the dashed red lines. As one can see, $|\phi_1 - \phi_2|$ is restricted to the interval $[\pi/2, 3\pi/4]$ for $m_3 \lesssim 0.06$ eV. Its deviation from $\pi/2$ increases with increasing m_3 . Again, the mass squared differences and θ_{12} are within the experimentally allowed 2σ ranges [20].

where the first inequality follows from the fact that the square root in equation (6.19) is always positive and the second inequality is obtained by bringing the square root to the right hand side of equation (6.19) and then squaring both sides, which is allowed since both are positive. Rearranging the first inequality of equation (6.20) and taking $t \neq 0$ (otherwise θ_{12} is zero) for the second one, we get

$$x^2 - s^2 > 4t^2 - z^2 \quad \text{and} \quad t^2 - z^2 > x(x - s). \quad (6.21)$$

The sum of these inequalities leads to

$$s(x - s) > 3t^2 > 0. \quad (6.22)$$

From equation (6.22) we see that s and x have the same sign, while $x^2 > s^2$, hence $x(x - s) > s(x - s)$. Together with the second inequality of equation (6.21) this implies

$$t^2 - z^2 > s(x - s). \quad (6.23)$$

Combining the last two inequalities, we find $t^2 - z^2 > 3t^2$, an obvious contradiction. Thus, the neutrinos cannot be normally ordered as assumed by $m_3^2 > m_2^2$. Instead we always have $m_2^2 > m_3^2$ which is only possible in case of an inverted hierarchy. Note that it is a priori not clear that also m_1 is larger than m_3 , since the size of the mass squared differences has to be tuned so that $\Delta m_{21}^2 \ll |\Delta m_{31}^2|$. In fact, Δm_{21}^2 is given by

$$\Delta m_{21}^2 = \left(\frac{v_u^2}{\Lambda}\right)^2 \left(\frac{v}{\Lambda}\right)^2 \sqrt{(s-x)^2(8t^2 + (s+x)^2) + 2(4t^2 + x^2 - s^2)z^2 + z^4}. \quad (6.24)$$

It vanishes, if $z = 0$ and $s = x$. Thus, $\Delta m_{21}^2 \ll |\Delta m_{31}^2|$ holds, if these equalities are nearly met. As noted, the vanishing of z can be made a natural result of the model. The near equality $s \approx x$ however has to be regarded as a certain tuning of the couplings y_1 and y_4 , see equation (6.12). We study the general case $z \neq 0$ with a numerical analysis. To fix the light neutrino mass scale we adjust the resulting solar mass squared difference to its best-fit value. At the same time the atmospheric mass squared difference and the mixing angle θ_{12} have to be within the allowed 2σ ranges [20]. First, we note that our numerical results confirm that z has to be in general smaller than the parameters s , t and x and that s and x have to have nearly the same value. In figure 6.1 we plotted $|m_{ee}|$ against the lightest neutrino mass m_3 .

As one can see, the approximate equality of $|m_{ee}|$ and m_3 , deduced for $z = 0$ in equation (6.18), still holds for $z \neq 0$. The dashed red line is the result for $z = 0$. One finds that m_3 has a minimal value around 0.015 eV, i.e. m_3 cannot vanish, and for $z = 0$ it also has a maximal value around 0.027 eV. These two bounds can be found as well by using equation (6.16). The non-vanishing of $m_3 \approx |m_{ee}|$ agrees with the findings in the literature that $|m_{ee}|$ is required to be larger than about 0.01 eV, if neutrinos follow an inverted hierarchy [119, 120]. Figure 6.2 shows that the relation in equation (6.16), which is fulfilled to a good accuracy for $z = 0$, gives a lower bound for $z \neq 0$ in the $\tan \theta_{12}$ - m_3 plane and no further constraints on the solar mixing angle can be derived. Note that we used the best-fit value of the atmospheric mass squared difference for the dashed red line in figure 6.2. Finally, we plot the Majorana phases ϕ_1 and ϕ_2 in figure 6.3 against the lightest neutrino mass m_3 . As one can see, the phase ϕ_1 (blue) varies between $\pi/8$ and $7\pi/8$, while ϕ_2 (green) either lies in the interval $[0, \pi/8]$ or $[7\pi/8, \pi]$ for small values of m_3 , i.e. $m_3 \lesssim 0.06$ eV. The dashed red lines indicate again the value of ϕ_1 and ϕ_2 achieved in the limit $z = 0$. As the difference $\phi_1 - \phi_2$ of the two Majorana phases is the only quantity which can be realistically determined by future experiments through

$$|m_{ee}| = |m_1 \cos^2 \theta_{12} e^{2i(\phi_1 - \phi_2)} + m_2 \sin^2 \theta_{12}|, \quad (6.25)$$

we also plot $\phi_1 - \phi_2$ against m_3 in figure 6.4. This plot shows that the phase difference has to lie in the rather narrow ranges $[-3\pi/4, -\pi/2]$ or $[\pi/2, 3\pi/4]$ for small values of m_3 . As one can see, the deviations from $|\phi_1 - \phi_2| = \pi/2$ ($z = 0$ case) become larger for larger values of m_3 .

6.1.3 Flavon Superpotential

In the following we discuss the flavon superpotential and show that the VEV structure assumed in (equation (6.2) and) equation (6.3) naturally arises, as does the spontaneous CP violation. This is a fairly simple specific example of the general potentials discussed in chapter 5. Since we are dealing with smallest possible dihedral group with an even n , we do not even need any driving fields transforming as two-dimensional representations. All the driving fields needed for constructing the potential are $\chi_e^0 \sim (\mathbf{1}_1, \omega^4)$, $\sigma^0 \sim (\mathbf{1}_4, \omega)$ and $\chi_\nu^0 \sim (\mathbf{1}_1, \omega)$, given with their transformation properties under (D_4, Z_5) . The renormalizable $D_4 \times Z_5$ invariant superpotential for flavons and driving fields reads

$$w_f = a \chi_e^0 \chi_e^2 + b \chi_e^0 \varphi_e^2 + c \sigma^0 (\psi_1^2 - \psi_2^2) + d \chi_\nu^0 \psi_1 \psi_2 + e \chi_\nu^0 \varphi_\nu^2 + f \chi_\nu^0 \chi_\nu^2. \quad (6.26)$$

Assuming that the flavons acquire their VEVs in the supersymmetric limit we can use the F-terms of the driving fields to determine the vacuum structure of the flavons. The equations

$$\frac{\partial w_f}{\partial \chi_e^0} = a \chi_e^2 + b \varphi_e^2 = 0, \quad (6.27a)$$

$$\frac{\partial w_f}{\partial \sigma^0} = c(\psi_1^2 - \psi_2^2) = 0, \quad (6.27b)$$

$$\frac{\partial w_f}{\partial \chi_\nu^0} = d \psi_1 \psi_2 + e \varphi_\nu^2 + f \chi_\nu^2 = 0, \quad (6.27c)$$

result in

$$\langle \chi_e \rangle = \pm i \sqrt{\frac{b}{a}} \langle \varphi_e \rangle, \quad \langle \psi_1 \rangle = \pm \langle \psi_2 \rangle, \quad \langle \chi_\nu \rangle = \pm i \sqrt{\frac{d \langle \psi_1 \rangle \langle \psi_2 \rangle + e \langle \varphi_\nu \rangle^2}{f}} \quad (6.28)$$

which can be re-written as

$$w_e = \pm i \sqrt{\frac{b}{a}} u_e, \quad \langle \psi_1 \rangle = \pm v, \quad w = \pm i \sqrt{\frac{d \langle \psi_1 \rangle \langle \psi_2 \rangle + e u^2}{f}}. \quad (6.29)$$

Note that the VEVs $\langle \varphi_e \rangle = u_e$, $\langle \psi_2 \rangle = v$ and $\langle \varphi_\nu \rangle = u$ are unconstrained by the potential. Note further that the three signs in equations (6.28) and (6.29) can be chosen independently of each other. For the discussion of the preserved subgroup structure it is anyway only relevant whether $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ or $\langle \psi_1 \rangle = -\langle \psi_2 \rangle$. For $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ as used in equation (6.3) we conserve a subgroup Z_2 of D_4 generated by B, whereas the relation $\langle \psi_1 \rangle = -\langle \psi_2 \rangle$ indicates that the Z_2 subgroup generated by BA^2 is left unbroken. This Z_2 group is also not a subgroup of the D_2 group conserved in the charged lepton sector. Thus, the subgroups of the charged lepton and the neutrino sector will be properly mismatched in both cases. We will only consider the case of $\langle \psi_1 \rangle = \langle \psi_2 \rangle = v$. Equation (6.29) shows then that the VEVs w_e and w necessarily have to be imaginary, so that CP is spontaneously violated, if the parameters a, \dots, f and the VEVs u_e , v and u are chosen as positive.

Only vanishing VEVs are allowed for the driving fields and this can be shown very easily in this model. We only need to consider the three F-terms

$$\frac{\partial w_f}{\partial \chi_e} = 2a \chi_e \chi_e^0 = 0, \quad (6.30)$$

$$\frac{\partial w_f}{\partial \varphi_\nu} = 2e \varphi_\nu \chi_\nu^0 = 0, \quad (6.31)$$

$$\frac{\partial w_f}{\partial \psi_1} = 2c \psi_1 \sigma^0 + d \psi_2 \chi_\nu^0 = 0, \quad (6.32)$$

to see that, if the parameters a, \dots, f and the flavon VEVs are non-zero, as it is in our case, the driving fields cannot acquire a VEV. Finally, note that we find flat directions in this potential in the case of spontaneous CP violation under discussion here.

6.1.4 Next-to-Leading Order Corrections

In order to determine how our results are corrected at NLO, we take into account the effects of operators which are suppressed by one more power of the cutoff scale Λ compared to the LO. It turns out that, due to the Z_5 symmetry, there are actually no contributions to the fermion

masses from two-flavon insertions. Hence, the only NLO corrections we need to consider are those of the flavon superpotential, which lead to a shift in the flavon VEVs parameterized as

$$\langle \chi_e \rangle = w_e + \delta w_e, \quad \langle \chi_\nu \rangle = w + \delta w \quad \text{and} \quad \langle \psi_1 \rangle = v + \delta v. \quad (6.33)$$

The VEVs $\langle \varphi_e \rangle = u_e$, $\langle \varphi_\nu \rangle = u$ and $\langle \psi_2 \rangle = v$ which are not determined at LO remain unconstrained also at NLO. As will be discussed in section 6.1.4, the shifts δw and δw_e are in general complex, whereas the shift δv in the VEV $\langle \psi_1 \rangle$ is real for this type of spontaneous CP violation.

Fermion Masses

The VEV shifts induce corrections to the lepton mass matrices given in equation (6.4) when the shifted VEVs are inserted into the LO Yukawa couplings of equation (6.1). In case of the charged lepton masses only the VEV of χ_e is shifted. Such a shift is however not relevant, since it can be absorbed into the Yukawa couplings y_1^e and y_2^e . These then become complex which however does not affect our results. In particular, U_l is still given by equation (6.5). The form of the neutrino mass matrix is changed through the shifts of the VEVs into

$$M_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1(w + \delta w) & y_2 v & y_2(v + \delta v) \\ y_2 v & y_3 u & y_4(w + \delta w) \\ y_2(v + \delta v) & y_4(w + \delta w) & y_3 u \end{pmatrix}. \quad (6.34)$$

Note that δw cannot be simply absorbed into w , since δw is complex, whereas w is purely imaginary. In the charged lepton mass basis the matrix in equation (6.34) reads

$$M'_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1(w + \delta w) & y_2(v + e^{i\pi/4}\delta v/\sqrt{2}) & y_2(v + e^{-i\pi/4}\delta v/\sqrt{2}) \\ y_2(v + e^{i\pi/4}\delta v/\sqrt{2}) & y_4(w + \delta w) & y_3 u \\ y_2(v + e^{-i\pi/4}\delta v/\sqrt{2}) & y_3 u & y_4(w + \delta w) \end{pmatrix}. \quad (6.35)$$

To evaluate the shifts in the neutrino masses and to discuss the deviations of the mixing angles from their LO values, especially θ_{13} from zero and θ_{23} from maximal, we parameterize the Majorana neutrino mass matrix as

$$M'_\nu = \frac{v_u^2}{\Lambda} \frac{v}{\Lambda} \begin{pmatrix} i s (1 + \alpha \epsilon) & t (1 + e^{i\pi/4} \epsilon) & t (1 + e^{-i\pi/4} \epsilon) \\ t (1 + e^{i\pi/4} \epsilon) & i x (1 + \alpha \epsilon) & z \\ t (1 + e^{-i\pi/4} \epsilon) & z & i x (1 + \alpha \epsilon) \end{pmatrix}, \quad (6.36)$$

with s , t , x and z as given in equation (6.12) and

$$\alpha \epsilon = \frac{\delta w}{w}, \quad \alpha = \alpha_r + i \alpha_i \quad \text{and} \quad \epsilon = \frac{1}{\sqrt{2}} \frac{\delta v}{v} \approx \lambda^2 \approx 0.04, \quad (6.37)$$

where we assume that ϵ is positive. The neutrino masses and mixing parameters resulting from equation (6.36) can then be calculated in an expansion in the small parameter ϵ . We observe that the mass shift of m_3^2 would vanish for δw being zero. Its explicit form is

$$(m_3^{\text{NLO}})^2 = (m_3^{\text{LO}})^2 + 2 \left(\frac{v_u^2}{\Lambda} \right)^2 \left(\frac{v}{\Lambda} \right)^2 x (\alpha_r x + \alpha_i z) \epsilon, \quad (6.38)$$

with $(m_3^{\text{LO}})^2$ given in equation (6.13). Similarly, the masses m_1^2 and m_2^2 undergo shifts proportional to ϵ . A simple expression can however only be found for the sum $m_1^2 + m_2^2$:

$$(m_1^{\text{NLO}})^2 + (m_2^{\text{NLO}})^2 = (m_1^{\text{LO}})^2 + (m_2^{\text{LO}})^2 + 2 \left(\frac{v_u^2}{\Lambda} \right)^2 \left(\frac{v}{\Lambda} \right)^2 (2\sqrt{2}t^2 + \alpha_r(s^2 + x^2) - \alpha_i x z) \epsilon \quad (6.39)$$

$(m_{1,2}^{\text{LO}})^2$ can be found in equation (6.13). The mixing angle θ_{13} no longer vanishes and we find

$$\sin \theta_{13} \approx \left| \frac{t x}{t^2 + (s-x)x - z^2} \right| \epsilon. \quad (6.40)$$

For θ_{23} we get

$$\tan \theta_{23} \approx 1 + \sqrt{2} \frac{x z}{t^2 + (s-x)x - z^2} \epsilon. \quad (6.41)$$

The deviation from maximal atmospheric mixing can also be expressed through

$$|\cos 2\theta_{23}| \approx \sqrt{2} \left| \frac{x z}{t^2 + (s-x)x - z^2} \right| \epsilon \approx \sqrt{2} \left| \frac{z}{t} \right| \sin \theta_{13}. \quad (6.42)$$

From both formulae one can deduce that in the case $z = 0$ the corrections to maximal atmospheric mixing are not of the order ϵ , but only arise at $\mathcal{O}(\epsilon^2)$. Contrary to this θ_{13} still receives corrections of order ϵ , if $z = 0$. The solar mixing angle θ_{12} which is not fixed to a precise value in this model also gets corrections of order ϵ . We note that the smallness of $|s-x|$ and z , required by the smallness of Δm_{21}^2 , might lead to a disturbance of the expansion in the parameter ϵ .

A correlation between $\cos 2\theta_{23}$ and $\sin \theta_{13}$ depending only on physical quantities, Δm_{ij}^2 , ..., and not on the parameters of the model, s, t, \dots , can be obtained by an analytic consideration which is done analogously to the study performed in [71]. Clearly, the matrix in equation (6.35) is no longer $\mu - \tau$ symmetric, however we find the following remnants of this symmetry:

$$(M'_\nu)_{e\mu} = (M'_\nu)_{e\tau}^* \quad \text{and} \quad (M'_\nu)_{\mu\mu} = (M'_\nu)_{\tau\tau}. \quad (6.43)$$

Equation (6.43) shows that $\mu - \tau$ symmetry is only broken by phases, but not by the absolute values of the matrix elements. This leads to

$$\begin{aligned} 0 &= |(M'_\nu)_{e\mu}|^2 + |(M'_\nu)_{\mu\mu}|^2 - |(M'_\nu)_{e\tau}|^2 - |(M'_\nu)_{\tau\tau}|^2 \\ 0 &= (M'_\nu M'_\nu^\dagger)_{\mu\mu} - (M'_\nu M'_\nu^\dagger)_{\tau\tau} = \sum_{j=1}^3 m_j^2 (|(U_{MNS})_{\mu j}|^2 - |(U_{MNS})_{\tau j}|^2) \\ 0 &= \left((\sin^2 \theta_{12} - \sin^2 \theta_{13} \cos^2 \theta_{12}) m_1^2 + (\cos^2 \theta_{12} - \sin^2 \theta_{13} \sin^2 \theta_{12}) m_2^2 - \cos^2 \theta_{13} m_3^2 \right) \cos(2\theta_{23}) \\ &\quad - \Delta m_{21}^2 \sin 2\theta_{12} \sin 2\theta_{23} \cos \delta \sin \theta_{13}. \end{aligned} \quad (6.44)$$

Since $\sin \theta_{13} \sim \mathcal{O}(\epsilon)$ and $\cos 2\theta_{23} \sim \mathcal{O}(\epsilon)$ is already known, we can linearize this equation and obtain, using best-fit values for the physical quantities and the fact that neutrinos have an inverted hierarchy in this model,

$$\cos 2\theta_{23} \approx - \frac{\Delta m_{21}^2 \sin 2\theta_{12}}{\Delta m_{32}^2 + \Delta m_{21}^2 \sin^2 \theta_{12}} \cos \delta \sin \theta_{13} \approx 0.03 \cos \delta \sin \theta_{13}. \quad (6.45)$$

Equation (6.45) can be used to estimate the largest possible deviation from maximal mixing. For $\sin \theta_{13}$ being at its 2σ limit of 0.2 and $|\cos \delta| = 1$, $|\cos 2\theta_{23}|$ still has to be less than 6×10^{-3} which is well within the 1σ error. Finally, we note that equation (6.45) must be consistent with equation (6.42) and thus we again find that z has to be small.

Flavon Superpotential

The corrections to the flavon superpotential stem from terms involving one driving field and three flavons. We find

$$\begin{aligned} \Delta w_f &= \frac{k_1}{\Lambda} \chi_e^0 \chi_\nu^3 + \frac{k_2}{\Lambda} \chi_e^0 \chi_\nu \varphi_\nu^2 + \frac{k_3}{\Lambda} \chi_e^0 \chi_\nu \psi_1 \psi_2 + \frac{k_4}{\Lambda} \chi_e^0 \varphi_\nu (\psi_1^2 + \psi_2^2) \\ &\quad + \frac{k_5}{\Lambda} \sigma^0 \varphi_e \chi_e^2 + \frac{k_6}{\Lambda} \sigma^0 \varphi_e^3 + \frac{k_7}{\Lambda} \chi_\nu^0 \chi_e^3 + \frac{k_8}{\Lambda} \chi_\nu^0 \chi_e \varphi_e^2. \end{aligned} \quad (6.46)$$

Assuming that CP is only spontaneously violated forces all k_i to be real. We calculate the F-terms of $w_f + \Delta w_f$ for the driving fields, using that the VEVs can be parameterized as

$$\langle \chi_e \rangle = w_e + \delta w_e, \quad \langle \chi_\nu \rangle = w + \delta w \quad \text{and} \quad \langle \psi_1 \rangle = v + \delta v. \quad (6.47)$$

The VEVs $\langle \varphi_e \rangle = u_e$, $\langle \varphi_\nu \rangle = u$ and $\langle \psi_2 \rangle = v$ are not determined at LO. For our calculation of the VEV shifts, we keep only terms containing up to one VEV shift or the suppression factor $1/\Lambda$, but not both. As the VEV shifts we calculate turn out to be small, this calculation is self-consistent. The F-terms then lead to

$$2 a w_e \delta w_e + \frac{1}{\Lambda} (k_1 w^3 + k_2 u^2 w + k_3 v^2 w + 2 k_4 u v^2) = 0, \quad (6.48a)$$

$$2 c v \delta v + \frac{u_e}{\Lambda} (k_5 w_e^2 + k_6 u_e^2) = 0, \quad (6.48b)$$

$$d v \delta v + 2 f w \delta w + \frac{w_e}{\Lambda} (k_7 w_e^2 + k_8 u_e^2) = 0. \quad (6.48c)$$

Here we have chosen the solutions with + in equation (6.29). The explicit form of the shifts reads

$$\delta v = -\frac{1}{2 c v \Lambda} \frac{u_e}{\Lambda} (k_5 w_e^2 + k_6 u_e^2), \quad (6.49a)$$

$$\delta w = \frac{1}{4 c f} \frac{1}{w \Lambda} (d (k_5 w_e^2 + k_6 u_e^2) u_e - 2 c (k_7 w_e^2 + k_8 u_e^2) w_e), \quad (6.49b)$$

$$\delta w_e = -\frac{1}{2 a} \frac{1}{w_e \Lambda} (k_1 w^3 + k_2 u^2 w + k_3 v^2 w + 2 k_4 u v^2). \quad (6.49c)$$

As one can see, for our type of spontaneous CP violation δv is real, whereas δw_e and δw turn out to be complex in general. As can be read off from equation (6.49) all shifts are generically of the natural size, i.e. $\frac{\delta \text{VEV}}{\text{VEV}} \sim \lambda^2$. Finally, note that the free parameters $\langle \varphi_e \rangle = u_e$, $\langle \varphi_\nu \rangle = u$ and $\langle \psi_2 \rangle = v$ are still undetermined.

This ends our discussion of the D_4 model for the moment. Before we consider what we have learned from this model, we first discuss a second model using D_{14} model for the quark sector, in the next section.

6.2 D_{14} in the Quark Sector

We construct a supersymmetric model using flavons and driving fields in which an element of the CKM matrix is predicted by group theory, as discussed in section 4.1.2. We presented a non-supersymmetric model with the same aim in section 4.3, based on the flavor group D_7 . However, we found in chapter 5 that the necessary vacuum alignment can be more easily achieved in models where the dihedral flavor symmetry has an even index n . In section 4.1.2, we found that D_7 is not the only symmetry that is adequate for predicting a CKM matrix element, but that D_{14} can be used to predict $|V_{ud}|$ or $|V_{cs}|$. We thus construct our supersymmetric model using D_{14} as a flavor symmetry. The auxiliary, abelian symmetry used to separate sectors is taken to be the minimal Z_3 . As we only consider the quark sector, we should also be able to describe the mass hierarchy, for which purpose we introduce an additional $U(1)_{FN}$ under which only the left-handed conjugate quarks transform.

As for the D_7 model of section 4.3, we choose the fermion transformation properties such that the mass matrices have a right-handed three singlet structure. The left-handed quarks Q_1 and Q_2 are unified into the D_{14} doublet $\mathbf{2}_1$, denoted by Q_D , while the third generation of left-handed quarks Q_3 , the left-handed conjugate up-type quark t^c , and the left-handed conjugate down-type quark s^c , transform trivially under D_{14} , i.e. as $\mathbf{1}_1$. The remaining two generations of

Field	Q_D	Q_3	u^c	c^c	t^c	d^c	s^c	b^c	$h_{u,d}$
D_{14}	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_1$
Z_3	1	1	1	1	1	ω_3^2	ω_3^2	ω_3^2	1
$U(1)_{FN}$	0	0	2	0	0	1	1	0	0
Field	$\psi_{1,2}^u$	$\chi_{1,2}^u$	$\xi_{1,2}^u$	η^u	$\psi_{1,2}^d$	$\chi_{1,2}^d$	$\xi_{1,2}^d$	η^d	σ
D_{14}	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{2}}_4$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{2}}_4$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_1$
Z_3	1	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3
$U(1)_{FN}$	0	0	0	0	0	0	0	0	0

Table 6.2: Particle Content of the D_{14} Model. Here we display the transformation properties of fermions and flavons under the flavor group $D_{14} \times Z_3 \times U(1)_{FN}$. The symmetry Z_3 separates the up and down quark sector. The left-handed quark doublets are denoted by $Q_D = Q_{1,2}$, $Q_1 = (u, d)^T$, $Q_2 = (c, s)^T$, $Q_3 = (t, b)^T$ and the left-handed conjugate quarks by u^c , c^c , t^c and d^c , s^c , b^c . The flavon fields indexed by a u give masses to the up quarks only, at LO. Similarly, the fields which carry an index d (and the field σ) couple only to down quarks at LO. Additionally, we assume the existence of a field θ which is a gauge singlet transforming trivially under $D_{14} \times Z_3$. It is responsible for the spontaneous breaking of the $U(1)_{FN}$ symmetry. Without loss of generality its charge under $U(1)_{FN}$ can be chosen as -1 . Note that ω_3 is a non-trivial third root of unity, $\omega_3 = e^{\frac{2\pi i}{3}}$.

left-handed conjugate fields, i.e. c^c and u^c in the up quark and d^c and b^c in the down quark sector, are assigned to the one-dimensional representations $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$. The fact that the transformation properties of the left-handed conjugate down quark fields are permuted compared to those of the left-handed conjugate up quark fields is merely due to the desire to arrive at a down quark mass matrix \mathcal{M}_d which has a large (33) entry, cf. equation (6.59). However, since this is just a permutation of the left-handed conjugate fields it is neither relevant for quark masses nor for mixing. We assign a trivial Z_3 charge to left-handed quarks and left-handed conjugate up quarks, while the left-handed conjugate down quarks transform as ω_3^2 under Z_3 . The MSSM Higgs doublets h_u and h_d do not transform under $D_{14} \times Z_3$.

For the flavon content, we abandon our principle of including all flavon representations which can appear in the Yukawa couplings. By eliminating possible flavons, we create zeroes in the quark mass matrices, which are only filled by higher order corrections. Together with the FN mechanism this creates the necessary hierarchy in the mass matrix elements. We have two flavon sectors: The flavons which give masses to the up quarks at LO transform trivially under Z_3 , while the flavon fields responsible for the down quark masses at LO acquire a phase ω under Z_3 . In both sectors we have flavons transforming according to the doublets $\underline{\mathbf{2}}_1$, $\underline{\mathbf{2}}_2$ and $\underline{\mathbf{2}}_4$. Not all of these flavons couple to the SM fermions at LO, but they are needed for the vacuum alignment, as discussed in section 5.2.4. We have a flavon transforming as $\underline{\mathbf{1}}_3$ in the up sector and a flavon transforming as $\underline{\mathbf{1}}_4$ in the down sector, to arrive at the desired subgroup mismatch. Since the left-handed conjugate down quarks transform non-trivially under Z_3 , whereas h_d does not transform under Z_3 , the bottom quark does not acquire a mass at the renormalizable level, unlike the top quark. We thus need to introduce an additional D_{14} singlet flavon σ in the down sector, which generates the bottom quark mass. The top quark is then the only fermion acquiring mass at the renormalizable level, and we can assume a small $\tan \beta$.

We finally need to assign FN charges. Without the FN symmetry, the strange mass would be generated at the same order as the bottom mass. To suppress it we assign a charge of $+1$ under $U(1)_{FN}$ to the left-handed conjugate down-type quark s^c . We also need to suppress the masses of the first generation quarks, for which we assign non-vanishing FN charges to both u^c and d^c . The transformation properties of the quarks and flavons under $D_{14} \times Z_3 \times U(1)_{FN}$ are summarized in table 6.2.

6.2.1 Quark Masses and Mixing

As discussed in section 5.2.5, the separation of the two sectors will not hold to all orders. Consequently, the residual symmetries will be broken, and our group theoretical prediction of the mixing matrix element(s) will receive corrections. In the leptonic case, this could nicely be divided into LO contributions, which respect the residual symmetries, and NLO contributions, which don't. Things are not as simple for our D_{14} quark model, as the levels of the contributions conserving the residual symmetries and of the contributions that break them are mixed due to additional effects needed to ensure the quark mass hierarchy. First of all, we have two small expansion parameters: ϵ which is as before the ratio of the flavon VEVs breaking $D_{14} \times Z_3$ over the cutoff scale Λ and t which is the ratio $\frac{\langle \theta \rangle}{\Lambda}$, where θ is the scalar field whose VEV breaks the FN symmetry, which we do not call flavon, reserving the term for those fields that break $D_{14} \times Z_3$. We assume these two small parameters to be of the same order in the following, i.e. we assume that all flavon VEVs and the VEV of the FN field are of the same order of magnitude. Additionally, not all terms allowed by the residual symmetries arise at the order of one-flavon insertions, as we do not include all scalar VEVs allowed by these symmetries. To structure our discussion of quark masses and mixing we thus distinguish between contributions to the mass matrices that respect the residual symmetries and those that break them.

The VEVs of the flavons transforming trivially under Z_3 , i.e. those with an index u , will conserve a subgroup $Z_2 = \langle \text{BA}^{m_u} \rangle$ at LO, while the flavons transforming non-trivially under Z_3 , i.e. those with an index d (and σ), will conserve a subgroup $Z_2 = \langle \text{BA}^{m_d} \rangle$. The subgroup-conserving contributions then come from those terms in the Yukawa superpotential in which up-type quarks couple only to flavons transforming trivially under Z_3 and down-type quarks only couple to flavons transforming non-trivially. The leading subgroup-conserving terms in the up sector will thus be those with one or two flavons, while in the down sector only the terms with one flavon will give the leading subgroup-conserving contribution. We first discuss these contributions, then go on to discuss the full mass matrices including higher order contributions, which break the residual symmetries. Our results in the subgroup-conserving limit turn out to be stable up to corrections of order ϵ , thus legitimizing this approach.

Subgroup-Conserving Limit

In the up quark sector the only renormalizable coupling generates the top quark mass:

$$Q_3 t^c h_u, \quad (6.50)$$

where we omit, as we will always do in the following, all Yukawa couplings. These multiply each operator separately and are taken to be complex numbers with an absolute value of order 1. The other elements of the second column as well as the (32) element of the up quark mass matrix M_u arise at the one flavon level through the terms

$$\frac{1}{\Lambda} (Q_D \psi^u) t^c h_u \quad (6.51)$$

and

$$\frac{1}{\Lambda} Q_3 (c^c \eta^u) h_u, \quad (6.52)$$

respectively. We do not specifically write down the D_{14} contractions in doublet component fields, but rather group together all those fields which are contracted to form a D_{14} singlet in a round bracket. This notation is unambiguous - one would need at least four flavons to have two possibilities for contraction. The elements of the 1-2 sub-block of M_u are generated

by two-flavon insertions. The elements of the first column of this sub-block originate from the terms

$$\frac{\theta^2}{\Lambda^4}(Q_D u^c \chi^u \xi^u) h_u + \frac{\theta^2}{\Lambda^4}(Q_D u^c (\xi^u)^2) h_u + \frac{\theta^2}{\Lambda^4}(Q_D \psi^u \eta^u u^c) h_u, \quad (6.53)$$

while the terms

$$\frac{1}{\Lambda^2}(Q_D c^c \chi^u \xi^u) h_u + \frac{1}{\Lambda^2}(Q_D c^c (\xi^u)^2) h_u + \frac{1}{\Lambda^2}(Q_D c^c \psi^u \eta^u) h_u \quad (6.54)$$

are responsible for the elements in the second column. For all elements we only give the dominant contributions. The elements in the third column all receive contributions from two-flavon terms. As they respect the residual Z_2 , they can however be absorbed into the existing couplings. Only the (31) element of M_u vanishes at this level. If we then insert the $Z_2 = \langle \text{BA}^{m_u} \rangle$ conserving doublet VEVs

$$\begin{pmatrix} \langle \psi_1^u \rangle \\ \langle \psi_2^u \rangle \end{pmatrix} = v^u \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \chi_1^u \rangle \\ \langle \chi_2^u \rangle \end{pmatrix} = w^u \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \xi_1^u \rangle \\ \langle \xi_2^u \rangle \end{pmatrix} = z^u \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6.55)$$

together with $\langle \eta_u \rangle \neq 0$, we obtain for the up quark mass matrix

$$M_u = \begin{pmatrix} -\alpha_1^u \epsilon^2 t^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\ \alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\ 0 & \alpha_4^u \epsilon & y_t \end{pmatrix} \langle h_u \rangle, \quad (6.56)$$

where we have set $m_u = 0$, which can as always be done without loss of generality, since it is only the phase difference which is observable in the subgroup mismatch. All couplings α_i^u and y_t are complex. Apart from an exchange of the first and third row this corresponds exactly to the matrix given in the first line of table 3.6. This exchange of rows is just a permutation of the left-handed quarks, which has no effect on phenomenology. We thus see that the omission of certain flavons does not change the structure of our mass matrix, as the texture zeroes induced by their omission are removed by subgroup-conserving two-flavon insertions. The only net effect at this level is a hierarchy among the mass matrix elements.

We can discuss the down quark mass matrix M_d in a similar fashion, but only taking into account terms with one flavon, of which there are three:

$$\frac{\theta}{\Lambda^2}(Q_D \psi^d) s^c h_d, \quad \frac{\theta}{\Lambda^2} Q_3 s^c \sigma h_d \quad \text{and} \quad \frac{1}{\Lambda} Q_3 (b^c \eta^d) h_d, \quad (6.57)$$

where the first one is responsible for the (33) entry, the second one for the (32) entry and the third one gives the dominant contribution to the (12) and (22) elements of M_d . The VEVs of the flavons in the down sector conserve the residual subgroup $Z_2 = \langle \text{BA}^{m_d} \rangle$ and are thus given by

$$\begin{pmatrix} \langle \psi_1^d \rangle \\ \langle \psi_2^d \rangle \end{pmatrix} = v^d \begin{pmatrix} e^{-\frac{\pi i m_d}{7}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \chi_1^d \rangle \\ \langle \chi_2^d \rangle \end{pmatrix} = w^d \begin{pmatrix} e^{-\frac{\pi i m_d}{7}} \\ e^{\frac{\pi i m_d}{7}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \langle \xi_1^d \rangle \\ \langle \xi_2^d \rangle \end{pmatrix} = z^d \begin{pmatrix} e^{-\frac{2\pi i m_d}{7}} \\ e^{\frac{2\pi i m_d}{7}} \end{pmatrix}, \quad (6.58)$$

with $\langle \eta^d \rangle$ and $\langle \sigma \rangle$ both non-zero. From our discussion in section 4.1.2 we infer that after setting $m_u = 0$, we need to set $m_d = 1$ or $m_d = 13$ to obtain a phenomenologically viable result for $|V_{ud}|$. Note that, while we will show in the next section how the VEV configurations of equation (6.58) arise naturally from the flavon superpotential, the exact value of the parameter m_d , beyond its being an odd integer between 1 and 13, cannot be determined. As seen in chapter 5 this is

a generic problem. We thus simply have to choose m_d to be minimal or maximal, and, for simplicity and without loss of generality, we will work in the following with $m_d = 1$, keeping in mind that there are three other possible values for the Cabibbo angle which could equally well be predicted in this model, but are in conflict with experiment. Inserting the flavon VEVs, the down quark mass matrix is then

$$M_d = \begin{pmatrix} 0 & \alpha_1^d t \epsilon & 0 \\ 0 & \alpha_1^d e^{-\pi i/7} t \epsilon & 0 \\ 0 & \alpha_2^d t \epsilon & y_b \epsilon \end{pmatrix} \langle h_d \rangle, \quad (6.59)$$

where again all couplings α_i^d and y_b are complex. Comparing this with the matrix given in the first line of table 3.6 (and again performing the unphysical exchange of rows one and three), we see how the absence of some of the allowed flavons leads to texture zeroes in the mass matrix. This was different in the up quark sector, where these zeroes disappeared when taking into account two-flavon insertions. In the down quark sector however, the next subgroup-conserving contribution, which would get rid of these zeroes, occurs at the level of four-flavon insertions. It is thus strongly suppressed with respect to the two-flavon insertions that violate the residual symmetries and will thus be discussed in the next section. The prediction of the Cabibbo angle from the subgroup mismatch, however, is stable, i.e. is only corrected at order ϵ , as shown in the next section. At first however, we determine the size of ϵ itself.

For this we consider the quark masses. We discuss the exact dependence on the mass matrix parameters only after we have taken subgroup-breaking terms into account. However, we can already see at this point how the quark mass hierarchy emerges:

$$\begin{aligned} m_u : m_c : m_t &\sim \epsilon^4 : \epsilon^2 : 1, \\ m_d : m_s : m_b &\sim 0 : \epsilon : 1, \\ m_b : m_t &\sim \epsilon : 1, \end{aligned} \quad (6.60)$$

where we have used $\epsilon \sim t$. The third equation holds for small $\tan \beta$. Comparing this with equation (2.1), we find that $\epsilon \approx \lambda^2 \approx 0.04$. In this case the hierarchy of quark masses is reproduced, except for the down quark mass which vanishes at this level but is generated by subgroup-breaking two-flavon insertions. We can also calculate the CKM matrix, and obtain

$$|V_{CKM}| = \begin{pmatrix} |\cos(\frac{\pi}{14})| & |\sin(\frac{\pi}{14})| & 0 \\ |\sin(\frac{\pi}{14})| & |\cos(\frac{\pi}{14})| & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{O}(\epsilon^4) & \mathcal{O}(\epsilon^2) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon^2) \end{pmatrix}. \quad (6.61)$$

As we expected from the construction of the model, the value of $|V_{ud}|$ is exact at this level and is only corrected by subgroup-breaking effects. These corrections are thus necessary to move $|V_{ud}|$ into the experimentally allowed range. We further observe that having ϵ as our expansion parameter, as opposed to $\lambda \sim \sqrt{\epsilon}$ as in the Wolfenstein parameterization, makes it impossible to predict the elements $|V_{td}|$ and $|V_{ub}|$, which are both $\sim \lambda^3$, at the correct order of magnitude: At this level, $|V_{td}|$ is slightly too large and $|V_{ub}|$ is slightly too small. For the measure of CP violation in the quark sector, J_{CP} , we find it to be of order ϵ^3 at this level and thus of the correct order of magnitude.

Full Results

We continue by considering how these results are corrected when taking into account subgroup-breaking contributions. This leads to the introduction of new terms in our parameterization of the mass matrices. The parameterization is then exact in the sense that all higher order

corrections can be absorbed in one of the parameters. As we did for the symmetry-conserving terms, we show the operators which give the dominant contribution to the terms violating the symmetry. There are now two possible contributions: One is from Yukawa coupling terms in the superpotential containing more flavons than considered in the symmetry-conserving limit. In the up quark sector this means three-flavon insertions, in the down quark sector two-flavon insertions. Another contribution is from shifts in the flavon VEVs due to mixing of the two sectors in the flavon potential at NLO. Our parameterization of these complex VEV shifts is given in equation (6.96). Their exact form is discussed in section 6.2.2 and appendix C.5. As can be inferred from the detailed calculations given in appendix C.5, the generic size of the shifts is $\epsilon \cdot \text{VEV}$ for all VEVs being of the order $\epsilon \Lambda$, as we would have expected. Because of this, we can easily compare VEV shift contributions with two or three flavon insertions. There are thus no ambiguities as to what are the leading contributions for each subgroup breaking parameter in the mass matrix.

We start with the (11) and (21) elements of the up quark mass matrix. They receive subgroup-breaking corrections from the operators

$$\begin{aligned}
& \frac{\theta^2}{\Lambda^4} ((Q_D u^c \delta \chi^u \xi^u) h_u + (Q_D u^c \chi^u \delta \xi^u)) h_u + \frac{\theta^2}{\Lambda^4} (Q_D u^c \xi^u \delta \xi^u) h_u + \frac{\theta^2}{\Lambda^4} (Q_D \delta \psi^u \eta^u u^c) h_u \\
& + \frac{\theta^2}{\Lambda^5} (Q_D \psi^d \chi^d) (u^c \eta^d) h_u + \frac{\theta^2}{\Lambda^5} (Q_D u^c (\chi^d)^3) h_u + \frac{\theta^2}{\Lambda^5} (Q_D u^c (\psi^d)^2 \xi^d) h_u + \frac{\theta^2}{\Lambda^5} (Q_D u^c \chi^d (\xi^d)^2) h_u \\
& + \frac{\theta^2}{\Lambda^5} (Q_D u^c (\chi^d)^2 \xi^d) h_u + \frac{\theta^2}{\Lambda^5} (Q_D u^c \chi^d \xi^d) \sigma h_u + \frac{\theta^2}{\Lambda^5} (Q_D u^c (\xi^d)^2) \sigma h_u + \frac{\theta^2}{\Lambda^5} (Q_D \psi^d) (u^c \eta^d) \sigma h_u,
\end{aligned} \tag{6.62}$$

where a notation such as $\delta \chi^u$ means that the operator which was already considered as part of the LO symmetry-conserving effects contributes also to the symmetry-breaking term, when first-order VEV corrections to the VEV of χ^u are considered. There exist of course also three-flavon insertions involving only flavons from the up quark sector. They are subgroup-conserving at leading order and can therefore be fully absorbed into the existing couplings. The corrections to the (12) and (22) elements are dominantly generated by

$$\begin{aligned}
& \frac{1}{\Lambda^2} ((Q_D c^c \delta \chi^u \xi^u) + (Q_D c^c \chi^u \delta \xi^u)) h_u + \frac{1}{\Lambda^2} (Q_D c^c \xi^u \delta \xi^u) h_u + \frac{1}{\Lambda^2} (Q_D c^c \delta \psi^u \eta^u) h_u \\
& \frac{1}{\Lambda^3} (Q_D c^c \psi^d \chi^d \eta^d) h_u + \frac{1}{\Lambda^3} (Q_D c^c (\chi^d)^3) h_u + \frac{1}{\Lambda^3} (Q_D c^c (\psi^d)^2 \xi^d) h_u + \frac{1}{\Lambda^3} (Q_D c^c \chi^d (\xi^d)^2) h_u \\
& + \frac{1}{\Lambda^3} (Q_D c^c (\chi^d)^2 \xi^d) h_u + \frac{1}{\Lambda^3} (Q_D c^c \chi^d \xi^d) \sigma h_u + \frac{1}{\Lambda^3} (Q_D c^c (\xi^d)^2) \sigma h_u + \frac{1}{\Lambda^3} (Q_D c^c \psi^d \eta^d) \sigma h_u,
\end{aligned} \tag{6.63}$$

while the (13) and (23) elements are dominantly corrected only by

$$\frac{1}{\Lambda} (Q_D \delta \psi^u) t^c h_u, \tag{6.64}$$

since the two-flavon insertions are subgroup-conserving, i.e. we only need to consider the symmetry-breaking arising from the VEV shifts at this order. Subgroup-breaking contributions from three flavon insertions are subdominant. As the (32) and (33) elements are already described by general complex parameters in the subgroup-conserving limit, and are not directly correlated with other mass matrix entries, all higher order corrections to them are subdominant and can be absorbed in existing couplings. This leaves only the (31) element, which was zero in the limit of a conserved subgroup and is then dominantly generated by the following three-flavon operators:

$$\begin{aligned}
& \frac{\theta^2}{\Lambda^5} Q_3(u^c \eta^d) \sigma^2 h_u + \frac{\theta^2}{\Lambda^5} Q_3(u^c \eta^d) (\psi^d)^2 h_u + \frac{\theta^2}{\Lambda^5} Q_3(u^c \eta^d) (\chi^d)^2 h_u + \frac{\theta^2}{\Lambda^5} Q_3(u^c \eta^d) (\xi^d)^2 h_u \\
& + \frac{\theta^2}{\Lambda^5} Q_3(u^c \eta^d) (\eta^d)^2 h_u + \frac{\theta^2}{\Lambda^5} Q_3(u^c \psi^d \chi^d \xi^d) h_u + \frac{\theta^2}{\Lambda^5} Q_3(u^c \psi^d (\xi^d)^2) h_u.
\end{aligned} \tag{6.65}$$

Note that there are symmetry-conserving couplings of the same order, from the coupling to three up-type flavons, such as

$$\frac{\theta^2}{\Lambda^5} Q_3 u^c (\psi_1^u \chi_1^u \xi_1^u - \psi_2^u \chi_2^u \xi_2^u), \tag{6.66}$$

which we have written with explicit doublet components, to make clear how these contributions vanish at this order, due to the symmetry-preserving VEV structure. They only contribute subdominantly at the next order, through VEV shifts. The exact up quark mass matrix can then be parameterized as

$$M_u = \begin{pmatrix} t^2 (-\alpha_1^u \epsilon^2 + \beta_1^u \epsilon^3) & \alpha_2^u \epsilon^2 + \beta_2^u \epsilon^3 & \alpha_3^u \epsilon + \beta_3^u \epsilon^2 \\ \alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\ \beta_4^u t^2 \epsilon^3 & \alpha_4^u \epsilon & y_t \end{pmatrix} \langle h_u \rangle, \tag{6.67}$$

where we have defined, without loss of generality, the couplings in the first two rows in such a way that the subgroup breaking corrections only appear in the first row. This necessitates a redefinition of the couplings $\alpha_{1,2,3}^u$, so that they only correspond to their counterparts from equation (6.56) at leading order. The same holds true for α_4^u and y_t , which also need to absorb contributions from subgroup-breaking terms. As in the subgroup-conserving limit, all couplings α_i^u , β_i^u and y_t are complex in general.

Turning to the down sector, the (11) and (21) mass matrix elements are dominantly generated by the same terms in the superpotential. These are two-flavon insertions and hence symmetry-breaking in the down sector, thus $|M_d|_{11} \neq |M_d|_{12}$. The relevant terms in the superpotential are

$$\begin{aligned}
& \frac{\theta}{\Lambda^3} (Q_D d^c \xi^d \chi^u) h_d + \frac{\theta}{\Lambda^3} (Q_D d^c \chi^d \xi^u) h_d + \frac{\theta}{\Lambda^3} (Q_D d^c \xi^d \xi^u) h_d + \frac{\theta}{\Lambda^3} (Q_D \psi^d) (d^c \eta^u) h_d \\
& + \frac{\theta}{\Lambda^3} (Q_D d^c \eta^d \psi^u) h_d.
\end{aligned} \tag{6.68}$$

The same is true for the (13) and (23) elements which are dominantly generated by

$$\begin{aligned}
& \frac{1}{\Lambda^2} (Q_D b^c \xi^d \chi^u) h_d + \frac{1}{\Lambda^2} (Q_D b^c \chi^d \xi^u) h_d + \frac{1}{\Lambda^2} (Q_D b^c \xi^d \xi^u) h_d + \frac{1}{\Lambda^2} (Q_D b^c \psi^d \eta^u) h_d \\
& + \frac{1}{\Lambda^2} (Q_D \psi^u) (b^c \eta^d) h_d,
\end{aligned} \tag{6.69}$$

while the (12) and (22) elements were already non-zero at the subgroup-conserving level. They are dominantly corrected by both VEV shifts and two-flavon insertions:

$$\begin{aligned}
& \frac{\theta}{\Lambda^2} (Q_D \delta \psi^d) s^c h_d \\
& + \frac{\theta}{\Lambda^3} (Q_D \chi^d \psi^u) s^c h_d + \frac{\theta}{\Lambda^3} (Q_D \psi^d \chi^u) s^c h_d + \frac{\theta}{\Lambda^3} (Q_D \psi^u) s^c \sigma h_d.
\end{aligned} \tag{6.70}$$

As for M_u we can again absorb the subgroup-breaking terms for the (32) and (33) elements into the existing couplings, which leaves us to discuss the (31) entry, which was vanishing in the subgroup-conserving limit. It is dominantly generated by one single operator,

$$\frac{\theta}{\Lambda^3} Q_3(d^c \eta^u) \sigma h_d. \quad (6.71)$$

The most general down quark mass matrix parameterization for our model is then

$$M_d = \begin{pmatrix} \beta_1^d t \epsilon^2 & t(\alpha_1^d \epsilon + \beta_4^d \epsilon^2) & \beta_5^d \epsilon^2 \\ \beta_2^d t \epsilon^2 & \alpha_1^d e^{-\pi i/7} t \epsilon & \beta_6^d \epsilon^2 \\ \beta_3^d t \epsilon^2 & \alpha_2^d t \epsilon & y_b \epsilon \end{pmatrix} \langle h_d \rangle. \quad (6.72)$$

This looks like subgroup conserving terms plus subgroup breaking corrections. However, as for the up quarks, couplings need to be redefined, for example to move the subgroup breaking contributions of the first two rows into the first row alone. Thus, the couplings only correspond to their counterparts from equation (6.59) at leading order. All couplings α_i^d , β_i^d and y_b are again complex.

We can rephase the left-handed conjugate quark fields, which has no effect on CP violation in the CKM matrix. Through this rephasing, we can make all the elements in the third row real, that is β_4^u , α_4^u , y_t , β_3^d , α_2^d and y_b . This somewhat simplifies our final formulae for the quark masses, which are given by (again using $t \approx \epsilon$)

$$m_u^2 = 2|\alpha_1^u|^2 \langle h_u \rangle^2 \epsilon^8 + \mathcal{O}(\epsilon^9), \quad (6.73)$$

$$m_c^2 = 2 \frac{|\alpha_3^u \alpha_4^u - y_t \alpha_2^u|^2}{y_t^2} \langle h_u \rangle^2 \epsilon^4 + \mathcal{O}(\epsilon^5), \quad (6.74)$$

$$m_t^2 = y_t^2 \langle h_u \rangle^2 + \mathcal{O}(\epsilon^2), \quad (6.75)$$

$$m_d^2 = \frac{1}{2} |\beta_1^d - \beta_2^d e^{i\pi/7}|^2 \langle h_d \rangle^2 \epsilon^6 + \mathcal{O}(\epsilon^7), \quad (6.76)$$

$$m_s^2 = 2|\alpha_1^d|^2 \langle h_d \rangle^2 \epsilon^4 + \mathcal{O}(\epsilon^5), \quad (6.77)$$

$$m_b^2 = y_b^2 \langle h_d \rangle^2 \epsilon^2 + \mathcal{O}(\epsilon^4). \quad (6.78)$$

Thus, taking into account the subdominant contributions discussed in this section, we can reproduce the full quark mass hierarchy, including a non-vanishing mass for the down quark. The full results for the elements of the CKM mixing matrix are

$$|V_{ud}| = \cos\left(\frac{\pi}{14}\right) + \mathcal{O}(\epsilon), \quad |V_{cs}| = \cos\left(\frac{\pi}{14}\right) + \mathcal{O}(\epsilon), \quad (6.79)$$

$$|V_{us}| = \sin\left(\frac{\pi}{14}\right) + \mathcal{O}(\epsilon), \quad |V_{cd}| = \sin\left(\frac{\pi}{14}\right) + \mathcal{O}(\epsilon), \quad (6.80)$$

$$|V_{cb}| = \frac{\epsilon}{\sqrt{2}} \left| \frac{\beta_5^d + \beta_6^d}{y_b} - \frac{2\alpha_3^u}{y_t} \right| + \mathcal{O}(\epsilon^2), \quad |V_{ts}| = \frac{\epsilon}{\sqrt{2}} \left| \frac{\beta_5^d + \beta_6^d e^{i\pi/7}}{y_b} - \frac{\alpha_3^u (1 + e^{i\pi/7})}{y_t} \right| + \mathcal{O}(\epsilon^2) \quad (6.81)$$

$$|V_{ub}| = \frac{\epsilon}{\sqrt{2}} \left| \frac{\beta_5^d - \beta_6^d}{y_b} \right| + \mathcal{O}(\epsilon^2), \quad |V_{td}| = \frac{\epsilon}{\sqrt{2}} \left| \frac{\beta_5^d - \beta_6^d e^{i\pi/7}}{y_b} - \frac{\alpha_3^u (1 - e^{i\pi/7})}{y_t} \right| + \mathcal{O}(\epsilon^2), \quad (6.82)$$

$$|V_{tb}| = 1 + \mathcal{O}(\epsilon^2). \quad (6.83)$$

The small corrections to $|V_{ud}|$ can safely move it into the experimentally allowed range, which it missed only marginally in the symmetry conserving limit. All other mixing matrix elements can also be accommodated, as has been confirmed by a numerical check, although a slight tuning is

Field	$\psi_{1,2}^{0u}$	$\varphi_{1,2}^{0u}$	$\rho_{1,2}^{0u}$	$\psi_{1,2}^{0d}$	$\varphi_{1,2}^{0d}$	$\rho_{1,2}^{0d}$
D_{14}	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_3$	$\underline{\mathbf{2}}_5$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_3$	$\underline{\mathbf{2}}_5$
Z_3	1	1	1	ω	ω	ω

Table 6.3: Driving Fields of the D_{14} Model. The transformation properties of the driving fields under the flavor symmetry D_{14} and under Z_3 . None of the driving fields is charged under $U(1)_{FN}$. The fields indexed with a u (d) drive the VEVs of the flavons giving masses to the up (down) quarks.

necessary to suppress $|V_{td}|$ and $|V_{ub}|$, which are now both of order $\epsilon \sim \lambda^2$. Finally, the Jarlskog invariant is

$$J_{CP} = \frac{\epsilon^2 \sin \frac{\pi}{7}}{4y_b^2 y_t} \left(2y_b \text{Re} \left((\alpha_3^u)^* (\beta_5^d - \beta_6^d) \right) - y_t \left(|\beta_5^d|^2 - |\beta_6^d|^2 \right) \right) + \mathcal{O}(\epsilon^3). \quad (6.84)$$

Subgroup-breaking corrections give the dominant contributions to J_{CP} , so that it is now of order ϵ^2 , which is slightly too large. A small J_{CP} can however be accommodated along with the absolute values of the mixing matrix elements. This is aided by the fact that already the factor $\frac{1}{4} \sin \frac{\pi}{7} \sim 0.11 \sim \frac{\lambda}{2}$ leads to a suppression.

6.2.2 Flavon Superpotential

Leading Order Potential

Turning to the discussion of the flavon potential w_f , it is time to introduce the necessary driving fields. From our discussion in section 5.2.4 we know that one minimal set of driving fields consists of the representations $\underline{\mathbf{2}}_1$, $\underline{\mathbf{2}}_3$ and $\underline{\mathbf{2}}_4$ for both sectors. This set of driving fields does indeed lead to the correct VEV alignment at LO, for this model we however prefer to replace the driving field transforming as $\underline{\mathbf{2}}_4$ with a field transforming as $\underline{\mathbf{2}}_5$. The reason for this is that this will slightly reduce the amount of terms in the NLO flavon superpotential. As far as vacuum alignment at LO is concerned, these two choices are equivalent. Our notation for the driving fields, along with their exact transformation properties, is given in table 6.3.

For the discussion of the flavon superpotential, we can return to the simple separation into LO renormalizable results and NLO corrections. The flavon superpotential at the renormalizable level consists of two parts,

$$w_f = w_{f,u} + w_{f,d}, \quad (6.85)$$

where $w_{f,u}$ gives rise to the alignment of the flavons only coupling to up quarks at LO, and $w_{f,d}$ to the alignment of the flavons coupling to down quarks. We take all parameters in w_f to be real, i.e. we restrict ourselves to the case of spontaneous CP violation in the flavon sector. $w_{f,u}$ reads

$$\begin{aligned} w_{f,u} = & M_\psi^u (\psi_1^u \psi_2^{0u} + \psi_2^u \psi_1^{0u}) + a_u (\psi_1^u \chi_1^u \varphi_2^{0u} + \psi_2^u \chi_2^u \varphi_1^{0u}) + b_u (\psi_1^u \chi_2^u \psi_1^{0u} + \psi_2^u \chi_1^u \psi_2^{0u}) \\ & + c_u (\psi_1^u \xi_2^u \varphi_1^{0u} + \psi_2^u \xi_1^u \varphi_2^{0u}) + d_u \eta^u (\xi_1^u \varphi_1^{0u} + \xi_2^u \varphi_2^{0u}) + e_u (\psi_1^u \xi_1^u \rho_2^{0u} + \psi_2^u \xi_2^u \rho_1^{0u}) \\ & + f_u \eta^u (\chi_1^u \rho_1^{0u} + \chi_2^u \rho_2^{0u}). \end{aligned} \quad (6.86)$$

The conditions for the vacuum alignment are given by the F-terms

$$\begin{aligned}
\frac{\partial w_{f,u}}{\partial \psi_1^{0u}} &= M_\psi^u \psi_2^u + b_u \psi_1^u \chi_2^u = 0, \\
\frac{\partial w_{f,u}}{\partial \psi_2^{0u}} &= M_\psi^u \psi_1^u + b_u \psi_2^u \chi_1^u = 0, \\
\frac{\partial w_{f,u}}{\partial \varphi_1^{0u}} &= a_u \psi_2^u \chi_2^u + c_u \psi_1^u \xi_2^u + d_u \eta^u \xi_1^u = 0, \\
\frac{\partial w_{f,u}}{\partial \varphi_2^{0u}} &= a_u \psi_1^u \chi_1^u + c_u \psi_2^u \xi_1^u + d_u \eta^u \xi_2^u = 0, \\
\frac{\partial w_{f,u}}{\partial \rho_1^{0u}} &= e_u \psi_2^u \xi_2^u + f_u \eta^u \chi_1^u = 0, \\
\frac{\partial w_{f,u}}{\partial \rho_2^{0u}} &= e_u \psi_1^u \xi_1^u + f_u \eta^u \chi_2^u = 0.
\end{aligned} \tag{6.87}$$

If we assume that none of the parameters in the potential vanishes and ψ_1^u acquires a non-zero VEV, we arrive at

$$\begin{pmatrix} \langle \psi_1^u \rangle \\ \langle \psi_2^u \rangle \end{pmatrix} = v^u \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \chi_1^u \rangle \\ \langle \chi_2^u \rangle \end{pmatrix} = w^u \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \langle \xi_1^u \rangle \\ \langle \xi_2^u \rangle \end{pmatrix} = z^u \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.88}$$

with

$$w^u = -\frac{M_\psi^u}{b_u}, \quad z^u = \frac{w^u}{2d_u e_u} \left(c_u f_u \pm \sqrt{4a_u d_u e_u f_u + (c_u f_u)^2} \right) \quad \text{and} \quad \langle \eta^u \rangle = -\frac{e_u}{f_u} \frac{v^u z^u}{w^u} \tag{6.89}$$

as the unique solution, where we have set a possible relative phase of the doublet components to zero as discussed in section 6.2.1. Analogously, the flavon superpotential which drives the vacuum alignment of the fields $\psi_{1,2}^d$, $\chi_{1,2}^d$, $\xi_{1,2}^d$, η^d and σ is given by

$$\begin{aligned}
w_{f,d} &= m_\psi^d \sigma \left(\psi_1^d \psi_2^{0d} + \psi_2^d \psi_1^{0d} \right) + a_d \left(\psi_1^d \chi_1^d \varphi_2^{0d} + \psi_2^d \chi_2^d \varphi_1^{0d} \right) + b_d \left(\psi_1^d \chi_2^d \psi_1^{0d} + \psi_2^d \chi_1^d \psi_2^{0d} \right) \\
&+ c_d \left(\psi_1^d \xi_2^d \varphi_1^{0d} + \psi_2^d \xi_1^d \varphi_2^{0d} \right) + d_d \eta^d \left(\xi_1^d \varphi_1^{0d} - \xi_2^d \varphi_2^{0d} \right) + e_d \left(\psi_1^d \xi_1^d \rho_2^{0d} + \psi_2^d \xi_2^d \rho_1^{0d} \right) \\
&+ f_d \eta^d \left(\chi_1^d \rho_1^{0d} - \chi_2^d \rho_2^{0d} \right).
\end{aligned} \tag{6.90}$$

From the F-terms of the driving fields $\psi_{1,2}^{0d}$, $\varphi_{1,2}^{0d}$ and $\rho_{1,2}^{0d}$ we derive the following equations:

$$\begin{aligned}
\frac{\partial w_{f,d}}{\partial \psi_1^{0d}} &= m_\psi^d \sigma \psi_2^d + b_d \psi_1^d \chi_2^d = 0, \\
\frac{\partial w_{f,d}}{\partial \psi_2^{0d}} &= m_\psi^d \sigma \psi_1^d + b_d \psi_2^d \chi_1^d = 0, \\
\frac{\partial w_{f,d}}{\partial \varphi_1^{0d}} &= a_d \psi_2^d \chi_2^d + c_d \psi_1^d \xi_2^d + d_d \eta^d \xi_1^d = 0, \\
\frac{\partial w_{f,d}}{\partial \varphi_2^{0d}} &= a_d \psi_1^d \chi_1^d + c_d \psi_2^d \xi_1^d - d_d \eta^d \xi_2^d = 0, \\
\frac{\partial w_{f,d}}{\partial \rho_1^{0d}} &= e_d \psi_2^d \xi_2^d + f_d \eta^d \chi_1^d = 0, \\
\frac{\partial w_{f,d}}{\partial \rho_2^{0d}} &= e_d \psi_1^d \xi_1^d - f_d \eta^d \chi_2^d = 0.
\end{aligned} \tag{6.91}$$

These are uniquely solved by the vacuum structure

$$\begin{pmatrix} \langle \psi_1^d \rangle \\ \langle \psi_2^d \rangle \end{pmatrix} = v^d \begin{pmatrix} e^{-\frac{\pi i m_d}{7}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \chi_1^d \rangle \\ \langle \chi_2^d \rangle \end{pmatrix} = w^d \begin{pmatrix} e^{-\frac{\pi i m_d}{7}} \\ e^{\frac{\pi i m_d}{7}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \langle \xi_1^d \rangle \\ \langle \xi_2^d \rangle \end{pmatrix} = z^d \begin{pmatrix} e^{-\frac{2\pi i m_d}{7}} \\ e^{\frac{2\pi i m_d}{7}} \end{pmatrix} \quad (6.92)$$

with

$$w^d = -\frac{m_\psi^d x}{b_d} \quad \text{and} \quad z^d = \frac{w^d}{2d_d e_d} \left(c_d f_d \pm \sqrt{4a_d d_d e_d f_d + (c_d f_d)^2} \right). \quad (6.93)$$

The VEVs of the two singlets σ and η^d are

$$\langle \sigma \rangle = x \quad \text{and} \quad \langle \eta^d \rangle = e^{-\frac{4\pi i m_d}{7}} \frac{e_d v^d z^d}{f_d w^d}. \quad (6.94)$$

Thereby, we assume that again none of the parameters in the potential vanishes and that the two fields ψ_1^d and σ get a non-vanishing VEV. The parameter m_d is an odd integer in the range $\{1, \dots, 13\}$. Due to this it is clear that the Z_2 subgroups preserved in the up and down quark sector cannot be the same and that the mixing angle arising through this mismatch is always non-trivial. However, we are unfortunately not able to predict the precise value of the parameter m_d , so that we have to choose the vacuum structure with $m_d = 1$ (or $m_d = 13$) by hand in order to predict the Cabibbo angle correctly.

Apart from this freedom, we can also choose the signs in front of the square roots in both up and down sector. All sign choices are entirely uncorrelated. We also have a total of three free dimensionful parameters, x , v^u and v^d . These are the three VEVs which we had to assume to be non-zero, to make the above solutions unique. Along with the mass parameter M_ψ^u , the only dimensionful parameter in the LO flavon superpotential, we have four dimensionful parameters which determine the scale of the flavon VEVs. Nothing forces them to be of the same order, but we have made this assumption in the last section, because, as discussed in section 5.2.5, only then will our LO VEV configuration be stable with respect to NLO corrections.

We finally remark that, given the flavon VEV structure, the driving fields are not allowed to get a VEV.

Next-to-leading order potential

In the NLO flavon potential we take into account all terms containing three flavons and one driving field. Due to the Z_3 symmetry two types of three-flavon combinations can couple to a driving field with an index u , namely either all three flavons transform trivially under Z_3 or all three transform non-trivially. For a driving field with a d index, two of the three flavons also transform non-trivially under Z_3 and the third one necessarily has to transform trivially. The NLO corrections to the flavon superpotential can be written as

$$\Delta w_f = \Delta w_{f,u} + \Delta w_{f,d}, \quad (6.95)$$

where the terms of $\Delta w_{f,u}$ ($\Delta w_{f,d}$) are responsible for the shifts of the flavons uncharged (charged) under the Z_3 symmetry. The exact form of the terms is given in appendix C.5. For the shifts of the VEVs we choose the following convention:

$$\begin{aligned}
\langle \psi_2^u \rangle &= v^u + \delta v^u, \quad \langle \chi_i^u \rangle = w^u + \delta w_i^u, \quad \langle \xi_i^u \rangle = z^u + \delta z_i^u, \quad \langle \eta^u \rangle = -\frac{e_u}{f_u} \frac{v^u z^u}{w^u} + \delta \eta^u \\
\langle \psi_2^d \rangle &= v^d + \delta v^d, \quad \langle \chi_1^d \rangle = e^{-\frac{\pi i m_d}{7}} (w^d + \delta w_1^d), \quad \langle \chi_2^d \rangle = e^{\frac{\pi i m_d}{7}} (w^d + \delta w_2^d), \\
\langle \xi_1^d \rangle &= e^{-\frac{2\pi i m_d}{7}} (z^d + \delta z_1^d), \quad \langle \xi_2^d \rangle = e^{\frac{2\pi i m_d}{7}} (z^d + \delta z_2^d) \quad \text{and} \quad \langle \eta^d \rangle = e^{-\frac{4\pi i m_d}{7}} \left(\frac{e_d}{f_d} \frac{v^d z^d}{w^d} + \delta \eta^d \right),
\end{aligned} \tag{6.96}$$

while

$$\langle \psi_1^u \rangle = v^u, \quad \langle \psi_1^d \rangle = v^d e^{-\frac{\pi i m_d}{7}} \quad \text{and} \quad \langle \sigma \rangle = x \tag{6.97}$$

remain as free parameters which might be fixed through the NLO corrections, but actually are not. As can be read off from the detailed calculations given in appendix C.5, the generic size of the shifts is

$$\delta \text{VEV} \sim \mathcal{O}\left(\frac{\text{VEV}}{\Lambda}\right) \cdot \text{VEV} \sim \epsilon \cdot \text{VEV}. \tag{6.98}$$

All shifts are also complex in general, with no preferred phase. This is all the information needed to reproduce the results on quark masses and mixing from the last section. All further details are given in the appendix.

6.3 Conclusions and Outlook

We have constructed two models incorporating all the elements of dihedral model building gathered in chapters 3, 4 and 5. One of these models uses the symmetry D_4 to describe the lepton sector, the other one uses D_{14} in the quark sector. The obvious question is: Should we not have one unique symmetry describing the entire flavor sector, especially in view of the fact that we might want to unify quarks and leptons in a Grand Unified Theory? The reason for having constructed two separate models is of course first of all practical one. We began our general survey of dihedral groups looking for mass independent textures which are able to describe and predict the mixing of the three fermion generations. We found that dihedral flavor symmetries can, through the mismatch of conserved subgroups discussed in chapter 4, lead to mass matrix structures in which one element of the fermionic mixing matrix is predicted by group theory alone. This then leads to considering dihedral groups D_n whose index n is a multiple of 4, to describe maximal atmospheric mixing in the leptonic sector, while the Cabibbo mixing angle in the quark sector points towards groups whose index n is divided by 7. Taking into account the vacuum alignment necessary to attain this subgroup mismatch, which is natural in setups breaking the flavor symmetry with gauge singlet flavons as discussed in chapter 5, leads us to favor D_n groups with an even index n . This does not change the smallest group adequate for the description of leptons, but leads us to consider D_{14} as the smallest viable group for the quark sector.

One can thus see, how the different structures of quark and lepton mixing favor different flavor groups. This is quite a generic problem: Models which use A_4 or S_4 to describe tri-bimaximal lepton mixing will typically face problems in predicting large enough mixing for quarks. The advantage of the two models presented here, is that at least quark and lepton mixing have the same group theoretical origin: In both cases, the largest mixing angle is predicted to be approximately $\frac{\pi}{n}$. The possibility of combining the two approaches, using a larger dihedral symmetry is therefore certainly worth studying. A natural candidate is D_{28} , as 28 is the least

common multiple of 4 and 14. The discussion of such a model is beyond the scope of this thesis, but we will conclude this chapter by comparing the D_4 and D_{14} models according to several points which may be important for the construction of such a model.

The first question we can ask is how unique the prediction of the largest mixing angle is in both models. In both cases the scalar VEV structure is not entirely uniquely defined. This especially concerns the relative phase between the two components of a doublet VEV. In the D_4 model, there is only one doublet flavon - the VEVs of its two components can differ by a relative minus sign or not, both solutions minimize the flavon potential, cf. equation (6.28). In the D_{14} model, we have, after setting the phase factor m_u in the up sector to zero, the freedom to choose the phase factor in the down sector to be an odd integer between 1 and 13. Different choices can lead to different mixing angles, a fact which we do not observe in the D_4 model. This is however only because of the smallness of the flavor group: We only have two possibilities for choosing the conserved subgroup in the neutrino sector of the D_4 model, $Z_2 = \langle B \rangle$ and $Z_2 = \langle BA^2 \rangle$. These two subgroups give the same phenomenology. We observe a similar thing for D_{14} , where choosing $m_d = 1$ or $m_d = 13$, i.e. conserving the subgroups $Z_2 = \langle BA \rangle$ or $Z_2 = \langle BA^{13} \rangle$, leads to the same phenomenology. But since D_{14} is a larger group, we have further choices and these then lead to a different prediction for the Cabibbo angle. For larger dihedral groups, a new mechanism would thus be desirable, with which such degeneracies in the flavon potential can be lifted. It may be of importance here that in the D_{14} model it is the maximal and minimal choice of m (or rather of the difference $m_d - m_u$ as we have set $m_u = 0$) which gives the correct prediction for the mixing angle.

Another question is the origin and magnitude of those mixing angles which are not determined by group theory alone. In the D_4 model we have one zero angle, which is also predicted from group theory, more specifically from the fact that a D_2 subgroup is conserved in the charged lepton sector, rather than a Z_2 . The third, solar, mixing angle is naturally large, as we introduce no further symmetry in the neutrino mass matrix which could lead to a hierarchy among the matrix elements. In the D_{14} quark model on the other hand, the other two mixing angles are small but non-vanishing. This leads to a different approach in the description of the mixing angles: The smallness of those mixing angles not predicted by group theory alone, θ_{23}^q and θ_{13}^q , arises from the fact that our model predicts a hierarchy among the quark mass matrix elements, partly due to the absence of flavons which could couple to fermions at LO, partly due to an additional FN symmetry. So, although we generate the largest mixing angle for both quarks and leptons in the same manner, the difference among the two mixing patterns is still manifest in the way we generate the two smaller mixing angles. Our approach to this problem depends strongly on future, more precise measurements of the neutrino mixing angles. The main question is the size of the reactor mixing angle θ_{13}^l : After all the 3σ bound on $\sin \theta_{13}^l$ is currently of the order of λ . The reactor mixing angle may thus be of the order of the Cabibbo angle, it may be of the order of one of the other quark mixing angles, or it may be even smaller.

We finally come to a point which is of great importance, when considering the compatibility of a flavor symmetry model with Grand Unification. It is the question of assigning flavor group representations to the fermions. Here one can observe a major difference between our two models: While the D_4 model has mass matrices of the two doublet structure, in the D_{14} model the quark representations are assigned in such a way that they give the (right-handed) three singlet structure. This becomes necessary because we want to accommodate the quark mass hierarchy in the latter model. Assigning the left-handed conjugate quarks to three different one-dimensional representation gives us, among other things, the freedom of assigning FN charges to each generation separately. Such a diverse set of representations becomes problematic when faced with the restrictions imposed by Grand Unification. Such restrictions occur since, if GUT and flavor symmetry commute, fermions united in one representation of the GUT must transform in the same way under G_f . How this affects our choice of representations in detail,

depends on the actual Grand Unified group.

The remainder of this thesis will however not be dealing with this (interesting) subject, but with another aspect of flavored model building which arises when considering the interplay between flavor symmetries and high scale, i.e. GUT or Planck scale, physics: The question of the high energy origin of a discrete non-abelian flavor symmetry. This question is made more pressing by indications that a global symmetry can never be fundamental because it will be broken by quantum gravity effects at the Planck scale [121]. The last chapter of this thesis is thus devoted to a discussion of what might lie behind a discrete flavor symmetry, manifesting itself in fermion masses and mixing patterns at low energies. In particular we discuss the possibility of embedding G_f into a continuous, possibly gauged, symmetry, and the problems one will face in such an approach.

Chapter 7

Outlook: Where Could Such a Symmetry Come From?

When discussing the origin of a discrete non-abelian flavor symmetry, a natural impulse is of course to assume that the flavor symmetry is connected to one of the symmetries already extensively used in quantum field theory, that is the flavor symmetry could be either a space-time or an internal gauge symmetry. Connecting a flavor symmetry to the symmetries of space-time necessitates an extension of space-time itself. Thus flavor symmetries have been connected with discrete symmetries arising in compactified extra dimensions, with [122] or without [60] string theory.

In this chapter we will be considering the other possibility, i.e. that a discrete flavor symmetry is indeed the conserved residual subgroup of a spontaneously broken gauged flavor symmetry. The idea of embedding a discrete flavor symmetry in a larger continuous group has been discussed in the literature, for example in [65, 113]. However no complete models exist, in the sense that the discrete flavor symmetry is only motivated by a possible underlying continuous symmetry, but the Lagrangian used for phenomenological considerations is only invariant under the discrete group, i.e. the continuous group is explicitly broken such as for example in [123]. This leads to restrictions on representations and to correlations between Yukawa couplings, not only through group theoretical compatibility, but also through demanding anomaly freedom for an underlying gauge symmetry [124, 125]. This does not however solve the problem of the underlying symmetry breaking dynamics.

To obtain a complete model, one needs to determine the scalar representations that break the gauge symmetry as well as their VEV structure. In general the necessary representations are well known, and are in fact mentioned as a motivation in some discrete flavor symmetry models. The necessary VEV structures have also been partially discussed [113], however often not in a flavor context [126–128]. Here the reasons for the absence of complete symmetry breaking models become clear: In general large and unwieldy representations are needed to break down to a phenomenologically interesting discrete subgroup. It is the aim of this chapter to show that in fact the most well-known and easily handleable representations lead only to the conservation of a very limited amount of non-abelian discrete subgroups. In fact, the only non-abelian discrete subgroup we can obtain with the small representations discussed in this section is D'_2 , which, as can be inferred from the discussion of chapter 3, does not have a rich enough structure to predict mass independent textures from conserved subgroups.

We want to break a hypothetical continuous flavor symmetry (gauged or not is irrelevant for this discussion) at a high energy scale. This flavor symmetry should commute with the SM and transform the three generations of fermions into each other. If we limit ourselves to three generations, we only need to consider the groups $SU(2)$ and $SU(3)$ as all other semi-simple Lie groups do not have two- or three-dimensional representations. $SO(3)$ need not be discussed

separately, as an $SO(3)$ gauge theory can simply be considered as an $SU(2)$ theory with a limited representation content. For $SU(2)$ the fermions will transform as $\underline{2} + \underline{1}$ or $\underline{3}$. The relevant Kronecker products are thus

$$\begin{aligned}\underline{2} \times \underline{2} &= \underline{1} + \underline{3}, \\ \underline{2} \times \underline{3} &= \underline{2} + \underline{4}, \\ \underline{3} \times \underline{3} &= \underline{1} + \underline{3} + \underline{5}.\end{aligned}\tag{7.1}$$

For a flavor symmetry $SU(3)$ the possible representations for three fermion generations are $\underline{3}$ and $\bar{\underline{3}}$, with Kronecker products

$$\begin{aligned}\underline{3} \times \underline{3} &= \bar{\underline{3}} + \underline{6}, \\ \bar{\underline{3}} \times \bar{\underline{3}} &= \underline{3} + \bar{\underline{6}}, \\ \underline{3} \times \bar{\underline{3}} &= \underline{1} + \underline{8}.\end{aligned}\tag{7.2}$$

We then in the following discuss the breaking of these flavor symmetries by small representations, where smallness means dimension smaller than five for $SU(2)$ and smaller than eight for $SU(3)$. This choice of representations is motivated by the fact that these are the representations that can couple to fermions at LO, as can be read off the Kronecker products above. These are also the representations whose VEVs can easily be discussed using linear algebra. We discuss how the continuous symmetry is broken by VEVs of scalars transforming under these representations and show that no non-abelian discrete symmetries, apart from D'_2 , can be conserved with these representations alone, and thus that one generically needs larger representations and group theory beyond simple linear algebra to model such a scenario. In fact, D'_2 itself only arises as the double-valued group of the abelian D_2 if we break $SU(2)$ with the unfaithful five-dimensional representation, which is also a representation of $SO(3)$.

To determine whether a certain VEV structure conserves a subgroup of the flavor symmetry $SU(2)$ or $SU(3)$, we do the same we did for the discrete dihedral symmetries in section 3.2: We test which elements of the flavor symmetry leave the VEV invariant. We will assume the minimal scalar content for any representation, i.e. real scalars for real representations, complex scalars for pseudo-real and complex representations. We then check for each representation, which subgroups the VEV of a scalar field transforming under this representation can conserve. We also consider combinations of VEVs, but only where such a combination could lead to a non-abelian subgroup. We begin by discussing those representations which can be written with one index, i.e. which can be written as vectors in our linear algebra treatment, then continue with those representations with two indices, i.e. matrix representations. Here we begin with the most familiar (from the SM gluons), the adjoint representation of $SU(3)$, then continue with the very similar $\underline{5}$ of $SO(3)$. The least common representations, the $\underline{6}$ of $SU(3)$ and the $\underline{4}$ of $SU(2)$ end the chapter.

Breaking $SU(2)$ with a doublet In the two-dimensional representation of $SU(2)$ the group elements are mapped onto the 2×2 unitary matrices with unit determinant. Thereby each group element has two eigenvalues λ_1 and λ_2 . They must obey the constraint that $\lambda_1 \lambda_2 = 1$, as the product of the eigenvalues is just the determinant. Hence if one of the eigenvalues is 1, then so is the other one. The only 2×2 matrix with two eigenvalues of 1 is obviously the unit matrix. Hence, the identity element is the only element of the group that can leave a doublet VEV invariant.

We conclude that the VEV of a scalar transforming as a doublet of $SU(2)$ always breaks the entire group. This of course does not change if we add further scalars of any sort.

Breaking $SU(2)$ with a triplet The triplet is the fundamental representation of $SO(3)$ and an unfaithful representation of $SU(2)$. The group elements are mapped onto the 3×3 orthogonal matrices with unit determinant. These can be thought of as rotations in three-dimensional Euclidean space. If such a rotation leaves a vector invariant, the vector must be parallel (or, obviously, antiparallel) to the axis of rotation. Hence any given triplet VEV will conserve the subgroup formed by the rotations around the axis defined by the VEV. Thus the VEV of any triplet will break $SU(2)$ down to $\text{Spin}(2)$, the double covering of $SO(2)$, which is in fact isomorphic to $SO(2)$ and $U(1)$.

Note that there is an $SO(2)$ for each possible axis, i.e. infinitely many $SO(2)$'s that are all mutually disjoint (up to the identity element). This means that if we introduce two triplets, their VEVs will either be linearly dependent or not. If they are linearly dependent they break to the same subgroup. If they are linearly independent they break to disjoint subgroups, hence fully breaking $SO(3)$. As the triplet is an unfaithful representation of $SU(2)$, we will always conserve a subgroup Z_2 under which all components of the triplet transform trivially, while both components of the doublet transform non-trivially.

We conclude that if we use three-dimensional representations to break $SU(2)$, we either leave invariant a $U(1) \cong SO(2)$ symmetry or a Z_2 . In particular no non-abelian subgroups can be conserved. We therefore do not need to consider combining a triplet VEV with a VEV of a different representation.

Breaking $SU(3)$ with a triplet The argumentation for $SU(3)$ is in fact very similar to the one for $SU(2)$ broken by a triplet. As the intuitive geometric derivation used above is not so readily applied in the complex three-dimensional Euclidean space, we give a more elaborate proof using linear algebra. This derivation (with the obvious modifications) can also be applied to $SU(2)$.

In the three-dimensional representation of $SU(3)$ the group elements are mapped onto the 3×3 unitary matrices with unit determinant. Therefore each element will have three eigenvalues λ_1 , λ_2 and λ_3 . If one of these eigenvalues, say λ_1 , is 1 (i.e. if the element is able to be part of some conserved subgroup), then the other two eigenvalues have to fulfill $\lambda_2\lambda_3 = 1$, since the matrix has a unit determinant. This means that if λ_2 is also equal to 1, then $\lambda_3 = 1$ as well. That is, the only element with more than one eigenvalue equal to 1 is the identity element, the only element with three 1 eigenvalues.

This means that each element which is not the identity will have at most one eigenvector corresponding to an eigenvalue of 1. For a simple example, the matrix

$$\begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & e^{-i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.3)$$

will have the eigenvector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (7.4)$$

corresponding to an eigenvalue of 1. As no direction in three-dimensional complex space is favored, there will exist for each complex 3-vector non-trivial group elements having this vector as an eigenvector with eigenvalue 1. These elements form the subgroup conserved by a VEV

proportional to that eigenvector. As each non-trivial element has at most one such eigenvector, these subgroups will all be disjoint.

What is the subgroup conserved by such a VEV? We can already guess that it will be $SU(2)$, but we will motivate this conclusion a bit better in the following. Consider the group of elements that leave invariant a vector \vec{v} . We then make a unitary similarity transformation

$$U \rightarrow U' = \begin{pmatrix} \vec{x} & \vec{y} & \vec{v} \end{pmatrix}^\dagger U \begin{pmatrix} \vec{x} & \vec{y} & \vec{v} \end{pmatrix}, \quad (7.5)$$

where U is an element of the group and \vec{x} and \vec{y} are arbitrary mutually orthogonal vectors that are also orthogonal to \vec{v} . We obtain

$$U' = \begin{pmatrix} U'_{2 \times 2} & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.6)$$

As U' is unitary by itself and also has unit determinant, we see that the three-dimensional representation reduces to the two-dimensional plus the one-dimensional representation of $SU(2)$. Since all the $SU(2)$ subgroups are disjoint, introducing two or more triplet scalars either breaks to an $SU(2)$ (in case their VEVs are linearly dependent) or breaks the entire $SU(3)$ group (if they are not).

What about the anti-triplets? The arguments are the same as for the triplets, if we consider them separately, as the two representations can only be distinguished if they show up together. But even if we introduce scalars transforming as triplets and scalars transforming as anti-triplets, we do not find any new subgroups: The reasoning is the same as above, each scalar VEV breaks to a specific $SU(2)$ and they are all disjoint. The only thing we observe is that if we introduce a scalar triplet and a scalar anti-triplet they break to the same $SU(2)$ if the VEV of the triplet is proportional to the complex conjugated VEV of the anti-triplet. If this is not the case, they break to disjoint $SU(2)$'s, i.e. they fully break $SU(3)$.

We conclude that an arbitrary collection of scalar triplets and anti-triplets either conserves an $SU(2)$ subgroup of our original $SU(3)$ symmetry or fully breaks that symmetry.

Breaking $SU(3)$ with the adjoint representation For discussing the breaking of a continuous group with matrix representations, we begin with the eight-dimensional adjoint representations of $SU(3)$, as it is probably the best known. We can write the VEV of a scalar transforming under the adjoint representation of $SU(3)$ as a Hermitian 3×3 traceless matrix V . It then transforms under $SU(3)$ in the following way:

$$V \rightarrow V' = UVU^\dagger, \quad (7.7)$$

where U is a special, unitary matrix. As V is traceless, we need to consider two distinct cases: Either V has three distinct eigenvalues, or it has two degenerate eigenvalues λ , the third eigenvalue being -2λ . The only possible VEV with three degenerate eigenvalues is the zero matrix, i.e. a vanishing VEV, which naturally does not break $SU(3)$.

We first consider the case of a V with three distinct eigenvalues. We are looking for the subgroup of $SU(3)$ formed by those elements U which leave V invariant, i.e. for which $V = V'$. This set is just the set of all matrices U that commute with V . What does it mean if U commutes with V ? Let \vec{v}_i be the eigenvector V associated with the eigenvalue λ_i , which we have assumed to be nondegenerate. Then

$$V(U\vec{v}_i) = U(V\vec{v}_i) = \lambda_i(U\vec{v}_i). \quad (7.8)$$

Hence $U\vec{v}_i$ is also an eigenvector of V with eigenvalue λ_i . As this eigenvalue is non-degenerate $U\vec{v}_i$ must linearly depend on \vec{v}_i . Therefore \vec{v}_i is also an eigenvector of U . This holds for all three

eigenvectors of V . We can thereby again reformulate the subgroup conserved by this VEV: It is the set of all U having the same set of eigenvectors as V . The most general form for an element of this group is then

$$U = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} \end{pmatrix} \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}^\dagger. \quad (7.9)$$

This representation is clearly unitarily equivalent to a diagonal representation, i.e. it reduces to three representations of $U(1)$. As α and β are however independent, there are actually two distinct $U(1)$ groups. Therefore an adjoint VEV with three distinct eigenvalues breaks $SU(3)$ down to $U(1) \times U(1)$. In particular such a VEV can never conserve a non-abelian subgroup of $SU(3)$ and we do not need to consider it any further.

We now proceed to VEVs V having two degenerate eigenvalues. The eigenvectors of V are now no longer uniquely defined. If \vec{v}_1 and \vec{v}_2 are two orthonormal eigenvectors of V corresponding to the same eigenvalue, we can find an arbitrary orthonormal basis of the corresponding eigenspace as $a\vec{v}_1 + b\vec{v}_2$ and $-\bar{b}\vec{v}_1 + \bar{a}\vec{v}_2$, with a and b two complex numbers obeying $|a|^2 + |b|^2 = 1$. We can therefore write any matrix in $SU(3)$ that commutes with V in the following form:

$$\begin{pmatrix} (a\vec{v}_1 + b\vec{v}_2) & (-\bar{b}\vec{v}_1 + \bar{a}\vec{v}_2) & \vec{v}_3 \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} \end{pmatrix} \begin{pmatrix} (a\vec{v}_1 + b\vec{v}_2) & (-\bar{b}\vec{v}_1 + \bar{a}\vec{v}_2) & \vec{v}_3 \end{pmatrix}^\dagger \quad (7.10)$$

To reduce this representation, we do a unitary equivalence transformation by multiplying on the right by

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \quad (7.11)$$

and on the left with the Hermitian conjugate. The resulting matrix is

$$\begin{pmatrix} |a|^2 e^{i\alpha} + |b|^2 e^{i\beta} & \bar{a}\bar{b}(e^{i\alpha} - e^{i\beta}) & 0 \\ \bar{a}\bar{b}(e^{i\alpha} - e^{i\beta}) & |a|^2 e^{i\beta} + |b|^2 e^{i\alpha} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} \end{pmatrix}. \quad (7.12)$$

We now show that this is a representation of $SU(2) \times U(1)$. To do this we factorize the matrix:

$$\begin{pmatrix} e^{i\frac{(\alpha+\beta)}{2}} & 0 & 0 \\ 0 & e^{i\frac{(\alpha+\beta)}{2}} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} \end{pmatrix} \begin{pmatrix} |a|^2 e^{i\frac{(\alpha-\beta)}{2}} + |b|^2 e^{-i\frac{\alpha-\beta}{2}} & 2iab^* \sin\left(\frac{\alpha-\beta}{2}\right) & 0 \\ 2ia^*b \sin\left(\frac{\alpha-\beta}{2}\right) & |a|^2 e^{-i\frac{(\alpha-\beta)}{2}} + |b|^2 e^{i\frac{\alpha-\beta}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.13)$$

These two matrices commute. The first matrix is the representation of $U(1)$, with the first two generations transforming in the same way, and the third with double and opposite charge. If we identify

$$x = |a|^2 e^{i\frac{(\alpha-\beta)}{2}} + |b|^2 e^{-i\frac{\alpha-\beta}{2}}, \quad (7.14)$$

$$y = 2iab^* \sin\left(\frac{\alpha-\beta}{2}\right), \quad (7.15)$$

and observe that $|x|^2 + |y|^2 = 1$, we see that the second matrix furnishes a representation of $SU(2)$, under which the first two generations form a doublet and the third generation is a singlet.

We also note that we have the correct number of free parameters: The absolute value of a (or b), the phase of ab^* and the phase difference $(\alpha - \beta)$.

We consider the case of two adjoint VEVs, where both VEVs have degenerate eigenvalues. First of all, their non-degenerate eigenvalues could correspond to the same eigenvector. In this case, they will naturally break to the same subgroup. Then we could have the case, where the non-degenerate eigenvalue of the second VEV corresponds to an eigenvector lying in the eigenspace of the degenerate eigenvalue of the first VEV. This, in a way, singles out a basis of that eigenspace and thereby coincides with the VEV of an octet with three distinct eigenvalues, i.e. conserves a subgroup $U(1) \times U(1)$. Finally, if there is no relation between the eigenvectors of the two VEVs, we only conserve the subgroup Z_3 , corresponding to the three third roots of unity, which can never be broken by adjoint scalars.

Finally, combining a degenerate adjoint VEV with a triplet VEV, we find three possibilities: First, the triplet VEV can coincide with the non-degenerate eigenvector. In this case $e^{-i(\alpha+\beta)}$ must be equal to 1 and we break down to the same $SU(2)$ conserved by the triplet VEV alone. If the triplet VEV lies in the degenerate eigenspace, we break the $SU(2)$ conserved by the octet VEV and are left with only a residual $U(1)$. If, finally, the triplet VEV is not an eigenvector of the adjoint VEV we again break the entire group.

Thus, the only new non-abelian subgroup of $SU(3)$ we can conserve with the VEV of a scalar transforming under the adjoint representation is the subgroup $SU(2) \times U(1)$ if the VEV has two degenerate eigenvalues.

Breaking $SU(2)$ with the five-dimensional representation The calculations here are very similar to those of the last section, so we will try and be brief. The VEV V of a scalar transforming under the five-dimensional representation can be written as a 3×3 traceless, real symmetric matrix. It transforms under $SU(2)$ as

$$V \rightarrow V' = OVO^T, \quad (7.16)$$

with O a special orthogonal matrix. Again the question of invariance can be reduced to a question of commutation and hence coincident eigenspaces. For a VEV with nondegenerate eigenvalues the general form of elements in the conserved subgroup is

$$O = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^m & 0 \\ 0 & 0 & (-1)^{n+m} \end{pmatrix} \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}^T, \quad (7.17)$$

with n, m integers (as V is symmetric it can be diagonalized by a real orthogonal matrix, hence O will have only real eigenvectors and therefore only real eigenvalues). After a similarity transformation this is a representation of $Z_2 \times Z_2 \cong D_2$. However, as we are actually breaking $SU(2)$ with an unfaithful representation, we are actually conserving the double-valued group D'_2 . The $SU(2)$ doublet will transform as a doublet in this group as well, while the triplet, as can be seen from the matrix above, decomposes into the three non-trivial one-dimensional representations. As we saw in section 3.2, D'_2 has no non-abelian subgroups, so we do not need to consider a combination of this VEV with others.

We move on to considering VEVs V with two degenerate eigenvalues. The elements of the conserved subgroup must still have \vec{v}_3 as an eigenvector with a real eigenvalue. There are however two possibilities to do this. One is to assign the eigenvalue 1 to \vec{v}_3 . These are all elements having \vec{v}_3 as axis of rotation. They form the subgroup $SO(2)$. We also have those elements, where the eigenvalue -1 is assigned to \vec{v}_3 . These are of the form

$$\begin{pmatrix} (c\vec{v}_1 + s\vec{v}_2) & (-s\vec{v}_1 + c\vec{v}_2) & \vec{v}_3 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^{n+1} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} (c\vec{v}_1 + s\vec{v}_2) & (-s\vec{v}_1 + c\vec{v}_2) & \vec{v}_3 \end{pmatrix}^T, \quad (7.18)$$

where s and c are now the sine and cosine, respectively, of some undefined angle. Unitarily transforming by multiplying on the right by

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \quad (7.19)$$

and on the left by its transpose, we end up with

$$\begin{pmatrix} (-1)^n(c^2 - s^2) & 2cs(-1)^n & 0 \\ 2cs(-1)^n & (-1)^n(s^2 - c^2) & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (7.20)$$

As we know that the above matrix must still have a unit determinant, we know that the upper left 2×2 matrix must have determinant -1 and must also be orthogonal. Combining the two sets of elements, we find that our representation is reducible, reducing to the defining representation of $O(2)$ and the one-dimensional representation, where each element is mapped onto its determinant. As our original group was $SU(2)$, we are actually breaking to the double covering of $O(2)$, which is $\text{Pin}(2)$. Combining several such VEVs, they can coincide in the non-degenerate eigenvector, in which case $\text{Pin}(2)$ is conserved, the non-degenerate eigenvector of one can lie in the degenerate eigenspace of the other, in which case the conserved subgroup is D'_2 , or their eigenbases could be unrelated, in which case only Z_2 is conserved.

There are thus only two non-abelian groups which can be the residual subgroup of $SU(2)$ after breaking with the VEV of a five-dimensional representation: The group D'_2 for non-degenerate eigenvalues and the group $\text{Pin}(2)$ for degenerate eigenvalues. Some of these results can also be found in [113].

Breaking $SU(3)$ with the six-dimensional representation Writing the VEV of the six-dimensional representation as a complex, symmetric 3×3 matrix V , it transforms under $SU(3)$ in the following way:

$$V \rightarrow V' = UVU^T. \quad (7.21)$$

Demanding invariance can then be rewritten as the condition

$$UV = VU^*. \quad (7.22)$$

We now note that V need not necessarily be diagonalizable. However, since V is complex and symmetric can be written in the form

$$W^T V W = V_{diag}, \quad (7.23)$$

with W unitary [129]. We can write W as $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$. The \vec{w}_i are then singular vectors of V obeying the relation

$$V\vec{w}_i = \sigma_i \vec{w}_i^*, \quad (7.24)$$

with σ_i the diagonal elements of V_{diag} , i.e. the singular values of V . The condition of equation (7.22) then leads to

$$V(U^* \vec{w}_i) = UV\vec{w}_i = \sigma_i U\vec{w}_i^* = \sigma_i (U^* \vec{w}_i)^*. \quad (7.25)$$

If V has three distinct singular values, this means that all \vec{w}_i need to be eigenvectors of U^* . Also, the corresponding eigenvalue of U^* needs to be real. Therefore the discussion is the same as for the quintuplet of $SO(3)$: The conserved subgroup is D_2 . If V has two degenerate eigenvalues, then U^* should act on the corresponding singular space with only real coefficients, that is it should be block-diagonalizable to give an orthogonal 2×2 submatrix. The conserved subgroup is then $O(2)$. As V need not be traceless, we encounter the additional case of three degenerate singular values. Here U^* needs to act on all singular vectors with real coefficients, so the conserved subgroup in this case is $SO(3)$. Of these subgroups only the last two are non-abelian and need to be considered in combination with other VEVs.

We demanded above that the eigenvalues of U need to be real. This condition stems from equation (7.24): If \vec{w}_i obeys that relation, then $\alpha\vec{w}_i$ only obeys the same relation if α is real. Or of course if σ_i is zero. Thereby VEVs with zero eigenvalues are algebraically special: The group elements preserving this VEV can have complex eigenvalues corresponding to the singular vectors of V with singular value 0. A special unitary matrix cannot have only one non-real eigenvalues. Hence the case of interest is a VEV with two zero eigenvalues. The group elements preserving this VEV are of the same form as those of equation (7.10), with the additional condition that $e^{-i(\alpha+\beta)}$, the eigenvalue corresponding to the non-zero singular value, must be real. We can therefore substitute the parameter $\alpha + \beta$ by the integer parameter m defined by $\alpha + \beta = m\pi$ and write the group elements conserving V in the form

$$\begin{pmatrix} i^m & 0 & 0 \\ 0 & i^m & 0 \\ 0 & 0 & (-1)^m \end{pmatrix} \begin{pmatrix} |a|^2 e^{i\frac{\alpha-\beta}{2}} + |b|^2 e^{-i\frac{\alpha-\beta}{2}} & 2iab^* \sin\left(\frac{\alpha-\beta}{2}\right) & 0 \\ 2ia^*b \sin\left(\frac{\alpha-\beta}{2}\right) & |a|^2 e^{-i\frac{\alpha-\beta}{2}} + |b|^2 e^{+i\frac{\alpha-\beta}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.26)$$

The conserved subgroup is therefore $SU(2) \times Z_4$, where the first two generations form a doublet of $SU(2)$ and a faithful representation of Z_4 , the third generation a singlet of $SU(2)$ and an unfaithful, non-trivial representation of Z_4 .

What if we combine two six-dimensional VEVs? If they coincide in all three singular vectors, the subgroup is determined by the VEV with less degenerate eigenvalues. If they have only one singular vector in common, we break to the subgroup of elements having two degenerate real eigenvalues, that is Z_2 . If they have no singular vectors in common, we break $SU(3)$ fully. Zero eigenvalues are only relevant if the VEVs coincide in all three singular vectors anyway and the zero eigenvalues correspond to the same eigenspace. In this case the full subgroup $SU(2) \times Z_4$ is conserved.

We thus have three non-abelian groups that can be conserved by a sextet VEV, $O(2)$ for two degenerate singular values, $SO(3)$ for three degenerate singular values, and $SU(2) \times Z_4$ for two zero eigenvalues. We now need to consider combinations of these three cases with the other VEVs we have discussed so far, triplets and octets.

What if both a $\underline{\mathbf{6}}$ and a triplet acquire a VEV? If the triplet VEV is not a singular vector of the $\underline{\mathbf{6}}$, then $SU(3)$ is fully broken. What if it is a singular vector? If V has two degenerate singular values, the triplet can correspond to the non-degenerate singular value. In this case, the determinant of the 2×2 submatrix is fixed to be one, and the conserved subgroup is $SO(2)$ or $U(1)$. If the triplet VEV corresponds to a degenerate singular value, the degeneracy becomes irrelevant and the subgroup is Z_2 . If V has three degenerate singular values, the triplet, which is in the defining representation of $SO(3)$, breaks that subgroup in the usual way down to $U(1)$ or fully breaks it, if the real and imaginary parts of the triplet VEV are not parallel. If we combine a V with two zero eigenvalues with a triplet VEV, we again have two possibilities: The triplet

VEV can correspond to the non-zero singular value. In this case m is fixed to be 0 or 2, and we break down to $SU(2)$ (the former Z_4 element can just be multiplied into the $SU(2)$ element, without changing the determinant). If the triplet VEV is an eigenvector of V corresponding to a zero eigenvalue, we need to take a closer look. The $SU(2)$ element in equation (7.26) has eigenvalues $e^{\pm i\frac{(\alpha-\beta)}{2}}$. So, without loss of generality, we must now demand $(i)^m e^{i\frac{(\alpha-\beta)}{2}} = 1$. The resulting element then has in addition two eigenvalues of -1 , corresponding to fixed vectors. The conserved subgroup is then Z_2 .

We proceed to combining a VEV of a $\underline{\mathbf{6}}$ with an adjoint VEV. The adjoint VEV must have two degenerate eigenvalues, as only then do we have the possibility of conserving a non-abelian subgroup. If there does not exist a basis of singular vectors for the sextet VEV, that is also a basis of eigenvectors for the octet VEV, $SU(3)$ will be fully broken. In particular, the eigenvector of the octet VEV corresponding to the non-degenerate eigenvalue, $v_{\vec{3}}$, must always be a singular vector of the sextet VEV V . The discussion is then similar to that for the triplet, with the triplet VEV replaced by $v_{\vec{3}}$.

If V has two degenerate singular values, $v_{\vec{3}}$ can correspond to the non-degenerate singular value. In this case nothing changes and the conserved subgroup is $O(2)$. If $v_{\vec{3}}$ corresponds to a degenerate singular value, the degeneracy becomes irrelevant and the subgroup is D_2 . If V has three degenerate singular values, one of the degeneracies becomes irrelevant and we break down to $O(2)$. Finally, considering the case of a sextet VEV V with two zero eigenvalues, we again have two possibilities: $v_{\vec{3}}$ can correspond to the non-zero singular value. In this case nothing changes, and $SU(2) \times Z_4$ is still the conserved subgroup. If $v_{\vec{3}}$ is an eigenvector of V corresponding to a zero eigenvalue, a specific basis is singled out for the elements of the conserved subgroup. It is thus only determined by the possible eigenvalues, and cannot be non-abelian. In this case it will be $U(1) \times Z_2$. Thus, no new non-abelian subgroups can be attained by combining the VEVs of these different $SU(3)$ representations.

Breaking $SU(2)$ with the four-dimensional representation We finally deal with the most complicated of the matrix representations, the $\underline{\mathbf{4}}$ of $SU(2)$. As it arises from the product of a vector and a spinor, it can be written as a 3×2 complex matrix, with one spinor index and one vector index. There must be further constraints, as such a matrix has 6 complex degrees of freedom. To find them, we take a look at the Clebsch Gordan coefficients.

Writing the $\underline{\mathbf{4}}$ as a matrix, it acts on a spinor and transforms it into a vector. As the Clebsch Gordan coefficients are normally given in spherical coordinates we start with these, later switching back to Cartesian coordinates, where the scalar product of two vectors is simply matrix multiplication. In spherical coordinates, we can give the four degrees of freedom of the $\underline{\mathbf{4}}$ as ϕ_1 ($m=\frac{3}{2}$), ϕ_2 ($m=\frac{1}{2}$), ϕ_3 ($m=-\frac{1}{2}$) and ϕ_4 ($m=-\frac{3}{2}$). Correspondingly we write the two components of the spinor we want to transform into a vector as ψ_1 ($m=\frac{1}{2}$) and ψ_2 ($m=-\frac{1}{2}$). Using the Clebsch Gordan coefficients for $SU(2)$ [7] we find that they combine in the following way to form a vector:

$$\frac{1}{2}(\sqrt{3}\phi_1\psi_2 - \phi_2\psi_1) \quad (m = 1), \quad (7.27)$$

$$\frac{1}{\sqrt{2}}(\phi_2\psi_2 - \phi_3\psi_1) \quad (m = 0), \quad (7.28)$$

$$\frac{1}{2}(\phi_3\psi_2 - \sqrt{3}\phi_4\psi_1) \quad (m = -1). \quad (7.29)$$

Switching to Cartesian coordinates, this corresponds to a vector

$$\begin{pmatrix} \frac{1}{2\sqrt{2}}[(\phi_2 - \sqrt{3}\phi_4)\psi_1 + (\phi_3 - \sqrt{3}\phi_1)\psi_2] \\ \frac{i}{2\sqrt{2}}[-(\phi_2 + \sqrt{3}\phi_4)\psi_1 + (\phi_3 + \sqrt{3}\phi_1)\psi_2] \\ \frac{1}{\sqrt{2}}(\phi_2\psi_2 - \phi_3\psi_1) \end{pmatrix}. \quad (7.30)$$

This vector arises from multiplying a spinor by the following matrix:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2}(\phi_2 - \sqrt{3}\phi_4) & \frac{1}{2}(\phi_3 - \sqrt{3}\phi_1) \\ -\frac{i}{2}(\phi_2 + \sqrt{3}\phi_4) & \frac{i}{2}(\phi_3 + \sqrt{3}\phi_1) \\ -\phi_3 & \phi_2 \end{pmatrix}, \quad (7.31)$$

or, in a more sensible parameterization

$$V = \begin{pmatrix} a & b \\ c & d \\ -b + id & a + ic \end{pmatrix}, \quad (7.32)$$

with a, b, c and d complex. This is then the most general form for the VEV of a $\underline{4}$. It transforms in the following way:

$$V \rightarrow V' = OVU^\dagger, \quad (7.33)$$

as it has one vector and one spinor index. O and U are of course not independent but describe a rotation of the same magnitude around the same axis. It can be checked by explicit calculation that V' can be parameterized in the same way as V for an arbitrary rotation.

Again we reformulate the condition of invariance as a condition on the eigensystems. We first observe that we can deduce from equation (7.33) the following two equations:

$$VV^\dagger = OVV^\dagger O^T \quad (7.34)$$

$$V^\dagger V = UV^\dagger VU^\dagger, \quad (7.35)$$

from which we immediately deduce that the eigenvectors of VV^\dagger (i.e. the left singular vectors of V , denoted by \vec{u}_i) must also be eigenvectors of O (with the usual ambiguities for degenerate singular and eigenvalues) and the right singular vectors of V , denoted by \vec{w}_i , must be eigenvectors of U . Using this knowledge, we go back to equation (7.33). We find that

$$VU\vec{w}_i = V\mu_i\vec{w}_i = \sigma_i\mu_i\vec{u}_i, \quad (7.36)$$

for $i = 1, 2$, μ_i the eigenvalues of U and σ_i the singular values of V . We also have

$$VU\vec{w}_i = OV\vec{w}_i = O\sigma_i\vec{u}_i = \lambda_i\sigma_i\vec{u}_i, \quad (7.37)$$

with λ_i the eigenvalues of O . From the last two equations we can deduce that $\lambda_i = \mu_i$. As O and V are rotations by the same angle θ , their eigenvalues are $e^{\pm i\theta}$ and 1 for O and $e^{\pm i\frac{\theta}{2}}$ for U . How can they be made to coincide? Apart from the trivial case of both being the unit matrix, we are only left with the possibility of identifying the exponential eigenvalues, which is only possible for $\theta = \pm\frac{4\pi}{3}$. The final left singular vector \vec{u}_3 is then the eigenvector of O corresponding to the eigenvalue 1, i.e. it defines the axis of rotation. If it is real, we then break to all rotations around that axis, with the angles given above. This is a Z_3 subgroup of $SU(2)$. If the axis is complex (and real and imaginary parts are not linearly dependent), no such elements exist and $SU(2)$ is fully broken.

Is the subgroup expanded if V has degenerate singular values? We first take the case $\sigma = \sigma_1 =$

$\sigma_2 \neq 0$. \vec{u}_3 is still an eigenvector of O , the \vec{u}_i and \vec{w}_i however need not be eigenvectors of O and U , respectively. Rather we have

$$VU\vec{w}_i = V(\alpha_i\vec{w}_1 + \beta_i\vec{w}_2) = \sigma(\alpha_i\vec{u}_1 + \beta_i\vec{u}_2) \quad (7.38)$$

$$VU\vec{w}_i = OV\vec{w}_i = \sigma O\vec{u}_i = \sigma(\alpha'_i\vec{u}_1 + \beta'_i\vec{u}_2), \quad (7.39)$$

from which we can immediately infer that $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. This means that again their eigenvalues need to coincide, and we again break to Z_3 or nothing. Finally, we need to consider the case, where one of the non-trivial singular values is zero, say $\sigma_2 = 0$. In this case O and U need only coincide in one eigenvalue, but this condition is already strong enough to constrain the elements in the same way, i.e. giving Z_3 as the conserved subgroup. We thus find that a VEV for the four-dimensional representation of $SU(2)$ can never lead to the conservation of a non-abelian subgroup, and we thus do not need to combine it with any other VEVs.

The results of this chapter can be summarized in one sentence: The only non-abelian, discrete subgroup that can be conserved by VEVs of the small representations (dimension less than five and eight, respectively) of $SU(2)$ and $SU(3)$ is the group D'_2 , which is too small to reproduce the mass independent textures described in this thesis. Finding the origin of a discrete flavor symmetry, thus either requires more involved group theory for the continuous group, or a different ansatz, in which the discrete group itself is fundamental. But that, as they say, is another story...

Chapter 8

Summary and Conclusions

What have we learned? We started from the assumption that the large amount of free parameters in flavor physics can be described with the help of a spontaneously broken discrete, non-abelian flavor symmetry. This choice of symmetry puts the mixing matrices, as opposed to the fermion mass hierarchies, in the center of attention. The question is then: How far can one get with such an assumption? Should we expect all mixing angles in the quark and the lepton sector to arise naturally from our flavor symmetry? This does not seem to be conceivable. The question should rather be: Which aspects of the fermion mixing patterns can be described? To address this question in a systematic manner, we have taken a class of discrete symmetries, the dihedral groups D_n and their double-valued counterparts D'_n , and have determined what they are capable of in this respect. Here our guiding principle was: What a flavor symmetry can predict, depends on how it is broken. To be more specific: The symmetries of the mass matrices, which then lead to predictions for the mixing angles independent of the exact numerical values of the parameters, should be residual subgroups of the original flavor symmetry.

We thus first determined all possible subgroups of a dihedral flavor symmetry and which scalar VEVs are compatible with their being the residual subgroup after spontaneous symmetry breaking. This then led us to a determination of possible mass and mixing matrices. *Possible* is of course not entirely well-defined, unless some further assumptions are put in. These will partly be motivated by physics considerations but also by calculational feasibility. In our case, we limited ourselves to mass matrices with a non-zero determinant and also demanded that at least two generations of left-handed or left-handed conjugate fermions are unified in a two-dimensional representation. Given these assumptions, the possible mass matrices could be limited to five different types, the corresponding diagonalization matrices could even be described by only four distinct structures. This was then all the input necessary to determine the possible mixing matrices.

Of these mixing matrices we were mainly interested in those where certain mixing angles, or rather certain elements of the mixing matrix, are described solely by group theoretical parameters, such as the group order, the index of a representation or a parameter distinguishing between several types of isomorphic subgroups. We found three kinds of such angles: The first two are the extremal angles, i.e. maximal and vanishing mixing. They appear quite naturally in flavor symmetry models, as the $\mu - \tau$ symmetric mass matrix structures from which they arise are fairly simple. A third kind of mass independent angle arises, which can be considered a signature of a dihedral symmetry. Due to mismatch between the subgroups conserved in the two sectors involved in the mixing, one element of the mixing matrix can be described by the formula

$$\left| \cos \left(\frac{\pi(m_1 - m_2)j}{n} \right) \right|, \quad (8.1)$$

where j is a representation index, $2n$ is the order of the dihedral symmetry and m_1 and m_2 parameterize the two conserved subgroups. After comparing this formula with phenomenology and analyzing several existing models with dihedral flavor symmetries, we showed that this subgroup mismatch mechanism could be relevant for describing the Cabibbo angle on the one hand, the maximal atmospheric neutrino mixing on the other hand. Although maximal atmospheric mixing can be described well by a $\mu - \tau$ symmetric neutrino mass matrix and a diagonal charged lepton mass matrix, the ansatz with the subgroup mismatch is different inasmuch as the origin of the largest mixing angles in the quark and lepton sectors can be described in a similar way.

Before implementing this mechanism in an actual model, it had to be checked whether the conservation of subgroups is just a mathematical oddity, or if the necessary VEV configurations occur naturally, when minimizing D_n invariant scalar potentials. And indeed under very general assumptions - in particular the scalars breaking the flavor symmetry should be gauge singlet flavons - this is the case and even the mismatch of subgroups can be partially explained if the group index n is even. The most promising approach uses supersymmetry and additional scalars, the driving fields.

We then had all the necessary ingredients to construct two models, one for leptons with the flavor symmetry D_4 and one for quarks with the flavor symmetry D_{14} . For the leptonic model a phenomenological analysis was performed to obtain predictions regarding the unknown aspects of the neutrino sector. For quarks the phenomenology is more fully known, and we focussed more on other aspects, such as the interplay of the discrete symmetry and a Froggatt-Nielsen symmetry, which work together to reproduce the mass hierarchy among quark generations, while leaving the prediction for the Cabibbo angle intact. In both models NLO corrections to our leading group theoretical predictions were calculated. Finally, similarities and differences between the two models were discussed, in view of the fact that the final goal should of course be a model, in which quarks and leptons are described simultaneously. Such a model, which would also ideally include a unification of quarks and leptons with respect to the gauge symmetries, i.e. a GUT, is certainly one of the goals for which the calculations in this thesis have hopefully laid the necessary foundations. Are there also more general lessons on how to build on the work presented in this thesis?

To answer this question, we consider our original approach. It can be considered to be more of a top-down approach: instead of starting with a specific mixing pattern inspired by phenomenology and then finding a symmetry group which can reproduce this pattern, we took the symmetry group as our starting point and then proceeded to show which symmetry patterns can be produced by it. We have shown that such an analysis can be performed for a set of discrete groups and can give direct input for model building. It is thus an interesting prospect to analyze other (classes of) groups in a similar fashion. In particular the $SU(3)$ subgroups $\Delta(3n^2)$ and $\Delta(6n^2)$ are prime candidates for such a classification according to flavor aspects.

On the other hand, this thesis can hopefully also offer some input towards the interplay of discrete flavor symmetries with other extensions of the Standard Model. This interaction can occur in two directions: First of all, SM extensions can lead to constraints on the flavor symmetry. We have already mentioned the typical example, where the unification of several SM fermions in a GUT representation can lead to restrictions on our choice of flavor group representations. Such restrictions can easily be implemented as further constraints in the framework given by this thesis. Model building with these constraints however is far more challenging. There does appear to be some tension between the GUT and flavor symmetry approaches, and it remains to be seen how this can best be resolved. The other possibility is that the flavor symmetry influences the SM extension: Both models we presented towards the end of this thesis are supersymmetric. It is thus interesting to ask, what further repercussions a flavor symmetry has in a supersymmetric model. After all, the MSSM has an extended flavor sector due to the

sleptons and squarks, and these sfermions will also have to transform non-trivially under our flavor symmetry. This may be of interest for the structure of the soft SUSY breaking terms, which can then also be discussed along the lines of the mass matrix structures described in this thesis.

Of course a high energy extension of the SM can even lead to an explanation of the flavor symmetry itself. We have discussed the possibility of the emergence of a discrete flavor symmetry from the high scale breaking of a continuous gauge symmetry and could show that this requires rather intricate breaking schemes, with large and unwieldy representations. It may thus be more favorable to discuss different origins for flavor symmetries, such as extra dimensions. The discrete flavor symmetry could then be considered to be of a geometric origin, favoring dihedral groups, which have a straightforward geometrical interpretation. Time will tell which of these directions will be the most fruitful. Experiments in neutrino physics will teach us more about the mixing patterns, in particular how close to maximal atmospheric mixing really is and whether $\theta_{13} = 0$ can be considered a good approximation. At the same time results from the Large Hadron Collider will provide information on the general framework in which to build a flavor symmetry: Will we be working with supersymmetry, as we did in the main models of this thesis, or will some other extension of the Standard Model prevail? These aspects need to be taken into account in our search for a symmetry of flavor. And someday we may actually know who ordered the muon.

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Appendix A

Glossary

A.1 Theory of Discrete Groups

Caveat: The definitions given below do not meet the requirements of absolute mathematical precision. However all definitions are correct with regard to the instances appearing in this thesis.

abelian A group G is abelian if it fulfills a further axiom:

- For all $a, b \in G$, $a \circ b = b \circ a$

character The character of a representation is a mapping from the group G to the complex numbers. An element of the group is mapped to the trace of its representation matrix. The character is said to be real, if all elements of the group are mapped to real numbers, otherwise it is complex. As the trace is an invariant, the characters of equivalent representations are identical.

Clebsch Gordan coefficients The representation matrices of the Kronecker product of two representations R_1 and R_2 of a group G act on the vector space given by the tensor product of the two vector spaces on which the representation matrices of R_1 and R_2 act. This vector space can be decomposed into invariant subspaces transforming under irreducible representations of G . The decomposition is given by the Clebsch Gordan coefficients.

complex representation A representation is complex if its character is complex.

continuous A group is continuous if its elements can be parameterized by one or more continuous parameters. It will thus have infinitely many elements and is not discrete.

cyclic A cyclic group is a group with only one generator. A cyclic group will always be abelian. The cyclic group of order n is denoted by Z_n .

dimension of a representation The dimension of a representation R is the dimension of the vector space on which the representation matrices act. If R maps the group elements to $d \times d$ matrices, the dimension of the representation is d . An irreducible representation is usually denoted by its dimension, a d -dimensional representation is written as \underline{d} . If there are several representations of the same dimension, there will be additional indices.

discrete A group is discrete if the set of distinct elements is finite.

equivalent Two representations R_1 and R_2 of a group G are (unitarily) equivalent if there exists a unique (unitary) matrix U such that $UR_1(a)U^{-1} = R_2(a)$ for all $a \in G$.

faithful A representation is faithful if it is injective, that is if each distinct group element is associated with a different matrix. Unfaithful representations will always map more than one element to the unit matrix.

generator The smallest subset of elements of a group G , for which each element of G can be written as a product of elements of the subgroup, is said to generate G . The elements of this subset are called generators. There may be several such smallest subsets and thus the generators are not uniquely defined. The number of generators however is uniquely defined.

group A group G is a set of elements along with a binary operation (group multiplication) $\circ : G \times G \rightarrow G$ fulfilling the group axioms:

- For all $a, b, c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$ (Associativity)
- There exists a unit element e for which $e \circ a = a \circ e = a$, for all $a \in G$
- For each element a there is an inverse element a^{-1} fulfilling $a \circ a^{-1} = a^{-1} \circ a = e$

irreducible representation (irrep) A representation is irreducible if it is not equivalent to a representation, where all representation matrices are block diagonal. For a reducible representation on the other hand, one can always, through an equivalence transformation, decompose the vector space, on which the representation matrices act, into invariant subspaces.

isomorphic Two groups are isomorphic if there exists a bijective mapping between them that conserves group multiplication. The two groups can then be considered identical for all our practical purposes, they are only defined differently.

Kronecker product The Kronecker product of two representations R_1 and R_2 of a group G is again a representation, $R_1 \times R_2$. Its dimension is the product of the dimensions of R_1 and R_2 and the representation matrices are the Kronecker matrix products of the representation matrices of R_1 and R_2 . The Kronecker product of two representations is usually written as its decomposition into irreducible representations.

maximal subgroup A subgroup H of a group G is called maximal, if there is no proper subgroup of G that contains H as a proper subgroup.

order (of a group) The order of the set of elements, i.e. the number of distinct elements of the group.

order (of an element) The smallest integer r for which $a^r = e$ is the order of the element a .

pseudo-real representation A representation is pseudo-real if its character is real, but it is not equivalent to a representation whose representation matrices have only real entries.

real representation A representation is real if it is equivalent to a representation whose representation matrices have only real entries. The character of such a representation will always be real.

representation A d -dimensional representation is a mapping from the group G to the invertible $d \times d$ matrices, where the matrix entries are real or complex numbers. The mapping conserves group multiplication, i.e. for a representation R and $a, b \in G$, we have for the representation matrices $R(a)R(b) = R(a \circ b)$, where the representation matrices are multiplied by usual matrix multiplication.

subgroup A subgroup H of a group G is a subset of the set of elements of G that is closed under group multiplication, i.e. for $a, b \in H$, $a \circ b$ is also in H . Each group G has an improper subgroup, the group G itself, and a trivial subgroup, the group containing only the unit element.

A.2 Abbreviations

- CKM = Cabibbo-Kobayashi-Maskawa (quark mixing matrix)
- FCNC = Flavor-Changing Neutral Current
- FN = Froggatt-Nielsen ($U(1)$ flavor symmetry)
- gcd = greatest common divisor
- GIM = Glashow-Iliopoulos-Maiani (mechanism for suppression of flavor change)
- GST = Gatto-Sartori-Tonin (relation between Cabibbo angle and quark masses)
- GUT = Grand Unified Theory
- HDM = Hot Dark Matter
- HPS = Harrison-Perkins-Scott (special form of lepton mixing matrix, equivalent to TBM)
- LFV = Lepton Flavor Violation
- LO = leading order
- MSSM = Minimal Supersymmetric Standard Model
- NLO = next-to-leading order
- PMNS = Pontecorvo-Maki-Nakagawa-Sakata (lepton mixing matrix)
- QCD = Quantum Chromodynamics
- QLC = Quark-Lepton Complementarity
- SM = Standard Model of Particle Physics
- SUSY = Supersymmetry
- TBM = tri-bimaximal mixing
- VEV = Vacuum Expectation Value

Appendix B

Mathematical Details of Dihedral Groups

B.1 Kronecker Products

The products of the one-dimensional representations of D_n are:

\times	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$
$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$
$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_3$
$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$
$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_1$

where the representations $\underline{\mathbf{1}}_{3,4}$ only exist in groups D_n with an even index n . The products $\underline{\mathbf{1}}_i \times \underline{\mathbf{2}}_j$ transform as

$$\underline{\mathbf{1}}_{1,2} \times \underline{\mathbf{2}}_j = \underline{\mathbf{2}}_j$$

and for n even there are also

$$\underline{\mathbf{1}}_{3,4} \times \underline{\mathbf{2}}_j = \underline{\mathbf{2}}_k \quad \text{with } k = \frac{n}{2} - j.$$

The products $\underline{\mathbf{2}}_i \times \underline{\mathbf{2}}_i$ are of the form $\underline{\mathbf{1}}_1 + \underline{\mathbf{1}}_2 + \underline{\mathbf{2}}_j$ with $j = \min(2i, n - 2i)$. In case the group D_n has an index n which is divisible by four one also finds the structure $\underline{\mathbf{2}}_i \times \underline{\mathbf{2}}_i = \sum_{j=1}^4 \underline{\mathbf{1}}_j$ for $i = \frac{n}{4}$.

The mixed products $\underline{\mathbf{2}}_i \times \underline{\mathbf{2}}_j$ can have two structures: For $i + j \neq \frac{n}{2}$ we have $\underline{\mathbf{2}}_i \times \underline{\mathbf{2}}_j = \underline{\mathbf{2}}_k + \underline{\mathbf{2}}_l$ with $k = |i - j|$ and $l = \min(i + j, n - (i + j))$, while for $i + j = \frac{n}{2}$ we have $\underline{\mathbf{2}}_i \times \underline{\mathbf{2}}_j = \underline{\mathbf{1}}_3 + \underline{\mathbf{1}}_4 + \underline{\mathbf{2}}_k$ with $k = |i - j|$.

For D'_n with n even the one-dimensional representations have the same product structure as for D_n while for n being odd they are:

\times	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$
$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$
$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_3$
$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_1$
$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_2$

due to the fact that the two one-dimensional representations $\underline{\mathbf{1}}_3$ and $\underline{\mathbf{1}}_4$ are complex conjugated to each other. The rest of the formulae for the different product structures are the same as in the case of D_{2n} , i.e. in each formula above which is given for D_n one has to replace n by $2n$. The Kronecker products can also be found in [130].

B.2 Clebsch Gordan Coefficients

We give the Clebsch Gordan coefficients in the following way: We start from two multiplets, $(a_1, a_2, \dots, a_{d_1})^T$ and $(b_1, b_2, \dots, b_{d_2})^T$, transforming under a d_1 - and a d_2 -dimensional representation of G_f , respectively. The Kronecker product of these representations contains a d_3 -dimensional representation. A multiplet $(c_1, c_2, \dots, c_{d_3})^T$ transforming under this representation can then be constructed from d_3 $d_1 \times d_2$ matrices M_i through

$$c_i = (a_1, a_2, \dots, a_{d_1}) M_i \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_{d_2} \end{pmatrix}.$$

These matrices M_i are the Clebsch Gordan coefficients, and we present them in a column vector of matrices.

B.2.1 ...for D_n

For $\mathbf{1}_i \times \mathbf{1}_j = \mathbf{1}_k$ the Clebsch Gordan coefficient is trivially one. For $\mathbf{1}_i \times \mathbf{2}_j$ the Clebsch Gordan coefficients are

$$\text{for } i = 1: \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_j \text{ and for } i = 2: \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_j$$

If the index n of D_n is even, the group has two further one-dimensional representations $\mathbf{1}_{3,4}$ whose products with $\mathbf{2}_j$ are of the form

$$\text{for } i = 3: \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{\frac{n}{2}-j} \text{ and for } i = 4: \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{\frac{n}{2}-j}$$

For the products $\mathbf{2}_i \times \mathbf{2}_i$ the covariant combinations are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \mathbf{1}_1 \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \mathbf{1}_2$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{2i} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{n-2i}$$

If the index n of D_n is even and $i = \frac{n}{4}$, there is a second possibility: $\mathbf{2}_i \times \mathbf{2}_i = \sum_{j=1}^4 \mathbf{1}_j$. The Clebsch Gordan coefficients are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \mathbf{1}_1 \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \mathbf{1}_2 \quad ,$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \mathbf{1}_3 \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \mathbf{1}_4$$

For the products $\mathbf{2}_i \times \mathbf{2}_j$ with $i \neq j$ there are the two structures $\mathbf{2}_i \times \mathbf{2}_j = \mathbf{2}_k + \mathbf{2}_l$ with $k = |i - j|$ and $l = \min(i + j, n - (i + j))$ or $\mathbf{2}_i \times \mathbf{2}_j = \mathbf{1}_3 + \mathbf{1}_4 + \mathbf{2}_k$ with $k = |i - j|$, the latter structure

occurring for $i + j = \frac{n}{2}$. The Clebsch Gordan coefficients for $\underline{2}_i \times \underline{2}_j = \underline{2}_k + \underline{2}_l$ are

$$\left(\begin{array}{c} \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{i-j} \text{ or } \left(\begin{array}{c} \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{j-i}$$

and

$$\left(\begin{array}{c} \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \end{array} \right) \sim \underline{2}_{i+j} \text{ or } \left(\begin{array}{c} \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \\ \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{n-(i+j)}$$

For the structure $\underline{2}_i \times \underline{2}_j = \underline{1}_3 + \underline{1}_4 + \underline{2}_k$ with $i + j = \frac{n}{2}$ the Clebsch Gordan coefficients are

$$\left(\begin{array}{c} 1 & 0 \\ 0 & 1 \end{array} \right) \sim \underline{1}_3 \text{ , } \left(\begin{array}{c} 1 & 0 \\ 0 & -1 \end{array} \right) \sim \underline{1}_4$$

and

$$\left(\begin{array}{c} \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{i-j} \text{ or } \left(\begin{array}{c} \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{j-i}$$

B.2.2 ...for D'_n

For n even, the Clebsch Gordan coefficients for the products $\underline{1}_i \times \underline{2}_j = \underline{2}_k$ are the same as in the case of D_{2n} , i.e. for $i = 3, 4$ the condition for k is $j + k = n$ instead of $\frac{n}{2}$.

If n is odd, the same holds for j odd whereas for j even the Clebsch Gordan coefficients of the products $\underline{1}_3 \times \underline{2}_j$ and $\underline{1}_4 \times \underline{2}_j$ have to be interchanged.

The Clebsch Gordan coefficients for the products $\underline{2}_i \times \underline{2}_i$ are the same as for D_{2n} , if i is even. Similarly, if n is even, nothing changes for $\underline{2}_i \times \underline{2}_j$ with $i \neq j$ as long as i, j are both even or one is even and one is odd. For n being odd the only difference is that, if the product is of the form $\underline{2}_i \times \underline{2}_j = \underline{1}_3 + \underline{1}_4 + \underline{2}_k$, the Clebsch Gordan coefficients for the covariant combinations transforming as $\underline{1}_3$ and $\underline{1}_4$ are interchanged.

We thus only need to give explicit Clebsch Gordan coefficients for the case where both two-dimensional representations are odd. For the the products $\underline{2}_i \times \underline{2}_i = \underline{1}_1 + \underline{1}_2 + \underline{2}_j$ with $j = \min(2i, 2n - 2i)$ and i odd, one finds

$$\left(\begin{array}{c} 0 & 1 \\ -1 & 0 \end{array} \right) \sim \underline{1}_1, \quad \left(\begin{array}{c} 0 & 1 \\ 1 & 0 \end{array} \right) \sim \underline{1}_2$$

and

$$\left(\begin{array}{c} \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \\ \left(\begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array} \right) \end{array} \right) \sim \underline{2}_{2i} \text{ or } \left(\begin{array}{c} \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \\ \left(\begin{array}{cc} -1 & 0 \\ 0 & 0 \end{array} \right) \end{array} \right) \sim \underline{2}_{2n-2i}$$

If $i = \frac{n}{2}$ and odd (n must even but not divisible by 4), one has $\underline{2}_i \times \underline{2}_i = \sum_{j=1}^4 \underline{1}_j$. The Clebsch Gordan coefficients are

$$\left(\begin{array}{c} 0 & 1 \\ -1 & 0 \end{array} \right) \sim \underline{1}_1 \text{ , } \left(\begin{array}{c} 0 & 1 \\ 1 & 0 \end{array} \right) \sim \underline{1}_2 \text{ ,}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \mathbf{1}_3 \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \mathbf{1}_4$$

Finally, the product $\mathbf{2}_i \times \mathbf{2}_j$ for i, j being odd is either $\mathbf{2}_k + \mathbf{2}_l$ with $k = |i - j|$ and $l = \min(i + j, 2n - (i + j))$ or, if $i + j = n$, $\mathbf{1}_3 + \mathbf{1}_4 + \mathbf{2}_k$ with $k = |i - j|$. The Clebsch Gordan coefficients in the first case are

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{i-j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{j-i}$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{i+j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{2n - (i+j)}$$

In the second case the Clebsch Gordan coefficients are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \mathbf{1}_3 \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \mathbf{1}_4$$

and

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{i-j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \mathbf{2}_{j-i}$$

B.3 Real Representation Matrices for D_n

As representation matrices are defined only up to similarity transformations, one will frequently find two-dimensional representation matrices of D_n that differ significantly from ours. A common convention has representation matrices which are real. This has the advantage that the reality of the two-dimensional irreps is made explicit. We then have that for real representations $\begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix}$ transforms as the same two-dimensional representation, and no additional unitary transformation is needed. However the structural discussion performed in this thesis is more tedious using real matrices, which is why we have opted for using the complex generators. Still, to make our results more easily comparable to existing models, we describe the alternate representation using real matrices and explicitly show its equivalence to the representation we have used.

For a dihedral group D_n and a representation $\mathbf{2}_j$, the generators A and B can be represented by the real matrices

$$A = \begin{pmatrix} \cos\left(\frac{2\pi j}{n}\right) & -\sin\left(\frac{2\pi j}{n}\right) \\ \sin\left(\frac{2\pi j}{n}\right) & \cos\left(\frac{2\pi j}{n}\right) \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B.1})$$

In [84], instead $g = B$ and $h = BA^3$ are used as generators, the representation however is the same as the one presented here. It does not depend on which two elements are designated as generators. We need to show that this representation is equivalent to the one we have been using. The explicit similarity transformation is given by

$$U^\dagger G_r U = G_c, \quad (\text{B.2})$$

where G_r is a representation matrix in the basis of real generators, G_c is a representation matrix in the basis of complex generators used throughout this thesis and the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}. \quad (\text{B.3})$$

We thus have that if $\begin{pmatrix} a \\ b \end{pmatrix}$ transforms as a two-dimensional representation using complex generators then

$$U \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i(a+b) \\ (a-b) \end{pmatrix} \quad (\text{B.4})$$

transforms as a two-dimensional representation using the real generators. In particular the special VEV structure given by equation (3.9) translates to

$$\begin{pmatrix} -\cot \frac{m\pi j}{n} \\ 1 \end{pmatrix}, \quad (\text{B.5})$$

which already shows, why calculations in the basis of the real generators become more involved: Instead of just having a relative phase between doublet components, the two components actually get different absolute values.

This basis change corresponds to the basis change in the model of section 6.1, when switching to the primed basis in which the charged lepton mass matrix is diagonal. The charged lepton mass matrix is

$$M_l = \frac{v_d}{\Lambda} \begin{pmatrix} y_1^e w_e & 0 & 0 \\ 0 & y_3^e u_e & y_2^e w_e \\ 0 & y_2^e w_e & -y_3^e u_e \end{pmatrix}, \quad (\text{B.6})$$

and consequently $U_l^\dagger M_l U_{e^c}$ is a diagonal matrix with positive entries, where

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4}/\sqrt{2} & e^{-i\pi/4}/\sqrt{2} \\ 0 & e^{-i\pi/4}/\sqrt{2} & e^{i\pi/4}/\sqrt{2} \end{pmatrix}. \quad (\text{B.7})$$

and U_{e^c} equals $U_l^* P$ where P is a diagonal phase matrix. Going to the basis in which charged leptons are diagonal thus corresponds to the transformations $l \rightarrow U_l^T l$ and $e^c \rightarrow U_{e^c}^\dagger e^c = P^* U_l^T e^c$. Thus, apart from an unphysical rephasing of the left-handed conjugate fields induced by P , both fields are transformed in the same way. If we now rewrite

$$U_l^T = U_l = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & e^{-3i\pi/4} \end{pmatrix}, \quad (\text{B.8})$$

with U as in equation (B.3), one can see that this transformation consists of a rephasing and then a transformation to the basis of real generators, as the second and third generations transform as a doublet of D_4 . This then coincides with the basis chosen in [84]. The identifications for the singlets are the following: $\mathbf{1}_{++}$ corresponds to $\mathbf{1}_1$, $\mathbf{1}_{+-}$ to $\mathbf{1}_3$, $\mathbf{1}_{-+}$ to $\mathbf{1}_2$ and $\mathbf{1}_{--}$ to $\mathbf{1}_4$. In [84] the charged lepton mass matrix is diagonal without any further transformation and $\theta_{13} = 0$ and θ_{23} maximal can be directly read off from the neutrino mass matrix. This is the same in our case, if we go into the primed basis, see M'_ν in equation (6.8).

We finally comment on the fake-complex scalars used in section 5.1. If we have a real scalar transforming as a doublet of D_n , we can write it, using complex generators, as

$$\begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}. \quad (\text{B.9})$$

Moving to the basis of real generators with the unitary transformation U , we see that this indeed corresponds to the real doublet representation, since

$$U \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} = i\sqrt{2} \begin{pmatrix} \text{Re}(\varphi) \\ \text{Im}(\varphi^*) \end{pmatrix}. \quad (\text{B.10})$$

The imaginary prefactor is irrelevant, as it is not affected by transformations. The important thing is that we have a doublet where both components are independent real variables.

Appendix C

Results and Tables

C.1 Decomposition under Subgroups

The decomposition of representations of D_n and D'_n under their subgroups is given in tables C.1, C.2, C.3 and C.4. We have used the following non-standard convention for the representation of Z_n : The representation $\mathbf{1}_k$ transforms as $e^{\frac{2\pi i}{n}(k)}$, so that $\mathbf{1}_0$ denotes the trivial representation and $\mathbf{1}_{(n+k)} = \mathbf{1}_k$.

We will denote the components of the two-dimensional representation $\mathbf{2}_k$ by

$$\mathbf{2}_k \sim \begin{pmatrix} a_k \\ b_k \end{pmatrix}.$$

For two-dimensional representations under dihedral subgroups, we find in the tables the general identification $\mathbf{2}_k \sim \mathbf{2}_k$. However, dihedral subgroups will have less two-dimensional representations than the original group, so we need to make the following identifications, if the dihedral subgroup has no representation $\mathbf{2}_k$:

In D_q , q even:

$$(e^{\frac{\pi imq}{n}} a_{\frac{q}{2}} + b_{\frac{q}{2}}) \sim \mathbf{1}_3, \quad (-e^{\frac{\pi imq}{n}} a_{\frac{q}{2}} + b_{\frac{q}{2}}) \sim \mathbf{1}_4, \quad (\text{C.1})$$

$$(e^{\frac{2\pi imq}{n}} a_q + b_q) \sim \mathbf{1}_1, \quad (-e^{\frac{2\pi imq}{n}} a_q + b_q) \sim \mathbf{1}_2, \quad (\text{C.2})$$

$$\begin{pmatrix} e^{\frac{2\pi imk}{n}} a_k \\ b_k \end{pmatrix} \sim \mathbf{2}_k \quad (\text{if } k < \frac{q}{2}), \quad (\text{C.3})$$

$$\begin{pmatrix} b_k \\ e^{\frac{2\pi imk}{n}} a_k \end{pmatrix} \sim \mathbf{2}_{q-k} \quad (\text{if } q > k > \frac{q}{2}). \quad (\text{C.4})$$

In D_q , q odd:

$$(e^{\frac{2\pi imq}{n}} a_q + b_q) \sim \mathbf{1}_1, \quad (-e^{\frac{2\pi imq}{n}} a_q + b_q) \sim \mathbf{1}_2, \quad (\text{C.5})$$

$$\begin{pmatrix} e^{\frac{2\pi imk}{n}} a_k \\ b_k \end{pmatrix} \sim \mathbf{2}_k \quad (\text{if } k \leq \frac{q-1}{2}), \quad (\text{C.6})$$

$$\begin{pmatrix} b_k \\ e^{\frac{2\pi imk}{n}} a_k \end{pmatrix} \sim \mathbf{2}_{q-k} \quad (\text{if } q > k > \frac{q-1}{2}). \quad (\text{C.7})$$

If $k > q$, $\mathbf{2}_k \sim \mathbf{2}_{(k \bmod q)}$. If q divides k , then $\mathbf{2}_k$ transforms just as $\mathbf{2}_q$, i.e. as $\mathbf{1}_1 + \mathbf{1}_2$. For D'_q one can make the same identifications as for D_q , q even, if one makes the substitutions $q \rightarrow 2q$ and $n \rightarrow 2n$. For q dividing k , $\mathbf{2}_{2k}$ then transforms as $\mathbf{2}_{2q}$, i.e. as $\mathbf{1}_1 + \mathbf{1}_2$.

$D_n \rightarrow$	Subgroup	VEV allowed?
$Z_n = \langle A \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_{\frac{n}{2}}$	
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_{\frac{n}{2}}$	
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{(n-k)}$	
$Z_q = \langle A^{\frac{n}{q}} \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_{\frac{n}{2}}$	if $\frac{n}{q}$ even
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_{\frac{n}{2}}$	if $\frac{n}{q}$ even
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{q-k}$	if $q \mid k$
$Z_2 = \langle BA^m \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_1$	
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_m$	if m even
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_{m+1}$	if m odd
$\underline{\mathbf{2}}_k \rightarrow$	$(e^{\frac{2\pi i m k}{n}} a_k + b_k) \sim \underline{\mathbf{1}}_0, (-e^{\frac{2\pi i m k}{n}} a_k + b_k) \sim \underline{\mathbf{1}}_1$	$\begin{pmatrix} e^{-\frac{2\pi i k m}{n}} \\ 1 \end{pmatrix}$
$Z_{\frac{n}{2}} = \langle A^2 \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{(\frac{n}{2}-k)}$	

Table C.1: Transformation properties of the representations of a dihedral group under its abelian subgroups, as determined in section 3.2. The rightmost column shows whether a representation has a component which transforms trivially under the subgroup, i.e. if a scalar field transforming under this representation can acquire a VEV, while conserving this subgroup. If only a specific VEV structure is allowed, it is given explicitly, otherwise an arbitrary VEV is allowed.

D_n	→	Subgroup	VEV allowed?
$D_{\frac{n}{2}} = \langle A^2, B \rangle$			
$\underline{\mathbf{1}}_1$	→	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2$	→	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3$	→	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_4$	→	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{2}}_k$	→	$\underline{\mathbf{2}}_k$	
$D_{\frac{n}{2}} = \langle A^2, BA \rangle$			
$\underline{\mathbf{1}}_1$	→	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2$	→	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3$	→	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_4$	→	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{2}}_k$	→	$\underline{\mathbf{2}}_k$	
$D_q = \langle A^{\frac{n}{q}}, BA^m \rangle$			
$\underline{\mathbf{1}}_1$	→	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2$	→	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3$	→	$\underline{\mathbf{1}}_1 \left(\frac{n}{q} \text{ even}, m \text{ even} \right)$ $\underline{\mathbf{1}}_2 \left(\frac{n}{q} \text{ even}, m \text{ odd} \right)$ $\underline{\mathbf{1}}_3 \left(\frac{n}{q} \text{ odd}, m \text{ even} \right)$ $\underline{\mathbf{1}}_4 \left(\frac{n}{q} \text{ odd}, m \text{ odd} \right)$	yes
$\underline{\mathbf{1}}_4$	→	$\underline{\mathbf{1}}_1 \left(\frac{n}{q} \text{ even}, m \text{ odd} \right)$ $\underline{\mathbf{1}}_2 \left(\frac{n}{q} \text{ even}, m \text{ even} \right)$ $\underline{\mathbf{1}}_3 \left(\frac{n}{q} \text{ odd}, m \text{ odd} \right)$ $\underline{\mathbf{1}}_4 \left(\frac{n}{q} \text{ odd}, m \text{ even} \right)$	yes
$\underline{\mathbf{2}}_k$	→	$\underline{\mathbf{2}}_k$	$\begin{pmatrix} e^{-\frac{2\pi i k m}{n}} \\ 1 \end{pmatrix}$ (if $q \mid k$)

Table C.2: Transformation properties of the representations of a dihedral group under its non-abelian subgroups. For further details see caption of table C.1.

$D'_n \rightarrow$	Subgroup	VEV allowed?
$Z_{2n} = \langle A \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_n$	
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_n$	
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{(2n-k)}$	
$Z_n = \langle A^2 \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{(n-k)}$	
$Z_q = \langle A^{\frac{2n}{q}} \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_n$	if $\frac{2n}{q}$ even
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_n$	if $\frac{2n}{q}$ even
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_{q-k}$	if $q \mid k$ (k can be odd)
$Z_4 = \langle BA^m \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_0 (n \text{ even}, m \text{ even})$ $\underline{\mathbf{1}}_1 (n \text{ odd}, m \text{ odd})$ $\underline{\mathbf{1}}_2 (n \text{ even}, m \text{ odd})$ $\underline{\mathbf{1}}_3 (n \text{ odd}, m \text{ even})$	yes
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_2 (n \text{ even}, m \text{ even})$ $\underline{\mathbf{1}}_3 (n \text{ odd}, m \text{ odd})$ $\underline{\mathbf{1}}_0 (n \text{ even}, m \text{ odd})$ $\underline{\mathbf{1}}_1 (n \text{ odd}, m \text{ even})$	yes
$\underline{\mathbf{2}}_k \rightarrow$	$(e^{\frac{\pi imk}{n}} a_k + b_k) \sim \underline{\mathbf{1}}_{0,1}, (-e^{\frac{\pi imk}{n}} a_k + b_k) \sim \underline{\mathbf{1}}_{2,3}$	$\begin{pmatrix} e^{-\frac{\pi ikm}{n}} \\ 1 \end{pmatrix}$ (if k even)
$Z_2 = \langle A^n \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_0$	yes
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_n$	if n even
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_n$	if n even
$\underline{\mathbf{2}}_k \rightarrow$	$a_k \sim \underline{\mathbf{1}}_k, b_k \sim \underline{\mathbf{1}}_k$	if k even

Table C.3: Transformation properties of the representations of a double-valued dihedral group under its abelian subgroups. For the decomposition of the two-dimensional D'_n representations under its subgroup Z_4 one has to mention that $\underline{\mathbf{2}}_k$ for k even splits up into $\underline{\mathbf{1}}_0$ and $\underline{\mathbf{1}}_2$ under Z_4 , while for k being odd the representations are $\underline{\mathbf{1}}_1$ and $\underline{\mathbf{1}}_3$. For further details see caption of table C.1.

$D'_n \rightarrow$	Subgroup	VEV allowed?
$D'_{\frac{n}{2}} = \langle A^2, B \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{2}}_k \rightarrow$	$\underline{\mathbf{2}}_k$	
$D'_{\frac{n}{2}} = \langle A^2, BA \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{2}}_k \rightarrow$	$\underline{\mathbf{2}}_k$	
$D'_q = \langle A^{\frac{n}{q}}, BA^m \rangle$		
$\underline{\mathbf{1}}_1 \rightarrow$	$\underline{\mathbf{1}}_1$	yes
$\underline{\mathbf{1}}_2 \rightarrow$	$\underline{\mathbf{1}}_2$	
$\underline{\mathbf{1}}_3 \rightarrow$	$\underline{\mathbf{1}}_1 \left(\frac{n}{q} \text{ even}, m \text{ even}\right)$ $\underline{\mathbf{1}}_2 \left(\frac{n}{q} \text{ even}, m \text{ odd}\right)$ $\underline{\mathbf{1}}_3 \left(\frac{n}{q} \text{ odd}, m \text{ even}\right)$ $\underline{\mathbf{1}}_4 \left(\frac{n}{q} \text{ odd}, m \text{ odd}\right)$	yes
$\underline{\mathbf{1}}_4 \rightarrow$	$\underline{\mathbf{1}}_1 \left(\frac{n}{q} \text{ even}, m \text{ odd}\right)$ $\underline{\mathbf{1}}_2 \left(\frac{n}{q} \text{ even}, m \text{ even}\right)$ $\underline{\mathbf{1}}_3 \left(\frac{n}{q} \text{ odd}, m \text{ odd}\right)$ $\underline{\mathbf{1}}_4 \left(\frac{n}{q} \text{ odd}, m \text{ even}\right)$	yes
$\underline{\mathbf{2}}_k \rightarrow$	$\underline{\mathbf{2}}_k$	$\left(e^{\frac{-\pi i k m}{n}} \right)$ (if $q \mid k$), k even

Table C.4: Transformation properties of the representations of a double-valued dihedral group under its non-abelian subgroups. For further details see caption of table C.1.

C.2 Breaking Chains for D'_n

We give the possible building blocks for breaking sequences of a double-valued dihedral group D'_n . The breaking sequences for D_n along with a general discussion of the conventions used is given in section 3.3.

For the subgroups of D'_n we again differentiate between only two types. First, the dihedral subgroups $D'_q = \langle A^{\frac{n}{q}}, \text{BA}^m \rangle$. Again, this will include abelian groups $Z_4 = \langle \text{BA}^{m_1} \rangle$, as these groups are isomorphic to D'_1 . Second, the cyclic subgroups $Z_q = \langle A^{\frac{2n}{q}} \rangle$. A Z_4 is to be interpreted as this second type, the other Z_4 's will be denoted by D'_1 .

$$\begin{array}{ll}
D'_q & \xrightarrow{\langle \underline{1}_2 \rangle} Z_{2q} \\
Z_q & \xrightarrow{\langle \underline{1}_2 \rangle} Z_q \\
D'_q & \xrightarrow{\langle \underline{1}_3 \rangle} D'_{\frac{g(n,2q)}{2}} \quad (n \text{ even; } m \text{ even for } \underline{1}_3 \text{ and } m \text{ odd for } \underline{1}_4) \\
D'_q & \xrightarrow{\langle \underline{1}_3 \rangle} Z_{g(n,2q)} \quad (n \text{ odd or } m \text{ odd for } \underline{1}_3 \text{ or } m \text{ even for } \underline{1}_4) \\
Z_q & \xrightarrow{\langle \underline{1}_3 \rangle} Z_{g(n,q)} \\
D'_q & \xrightarrow{\langle \underline{2}_j \rangle} Z_{g(2q,j)} \\
Z_q & \xrightarrow{\langle \underline{2}_j \rangle} Z_{g(q,j)} \\
D'_q & \xrightarrow{\langle \underline{2}_j \rangle'} D'_{\frac{g(2q,j)}{2}} \quad (g(2q,j) \text{ even, } m_j \bmod \frac{n}{q} = m) \\
D'_q & \xrightarrow{\langle \underline{2}_j \rangle'} Z_{g(2q,j)} \quad (g(2q,j) \text{ odd or } m_j \bmod \frac{n}{q} \neq m) \\
Z_q & \xrightarrow{\langle \underline{2}_j \rangle'} Z_{g(q,j)}
\end{array}$$

C.3 Possible Forms of V from Subgroup Mismatch

As discussed in section 4.1, for conserved Z_2 subgroups there exist three possible diagonalization matrices in each sector (up and down sector, charged lepton and neutrino sector), according to the three possible identifications of the eigenvalue $b_i - c_i$. Out of these one can form nine possible mixing matrices $V^{ab} = U_1^T U_2^*$ with $a, b = 1, 2, 3$ and $U_i \in \{U_{\text{fm}}, U'_{\text{fm}}, U''_{\text{fm}}\}$ where U_i depends on the group theoretical phase ϕ_i (the index m_i) and contains the mixing angle θ_i and the free phase β_i . They all have the property that one of their matrix elements, namely the element (ab) , is completely determined by group theory, i.e. by the index n of the dihedral group, by the index j of the two-dimensional representation $\underline{2}_j$ under which two of the three generations of $SU(2)_L$ doublets transform and by the breaking directions described by m_1 and m_2 in the two different sectors. In the following we abbreviate $\beta_1 - \beta_2$ with α , $\sin(\theta_i)$ with s_i and $\cos(\theta_i)$ with c_i .

$$\begin{aligned}
V^{11} &= \frac{1}{2} \begin{pmatrix} 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V^{12} &= \frac{1}{2} \begin{pmatrix} (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V^{13} &= \frac{1}{2} \begin{pmatrix} (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \\ (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
V^{21} &= \frac{1}{2} \begin{pmatrix} -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V^{22} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V^{23} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \end{pmatrix} \\
V^{31} &= \frac{1}{2} \begin{pmatrix} -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \\ 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \end{pmatrix} \\
V^{32} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \end{pmatrix} \\
V^{33} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \end{pmatrix}
\end{aligned}$$

C.4 D_7 Higgs Potential

We discuss the Higgs potential of the D_7 model presented in section 4.3. The potential for the three Higgs fields H_s^u and $H_{1,2}^u$, which couple only to up quarks, i.e. are even under the additional $Z_2^{(aux)}$ symmetry, is of the same form as V_3 in equation (4.17). As mentioned there, it has an accidental $U(1)$ symmetry. We have in addition five Higgs fields which are odd under the extra $Z_2^{(aux)}$. These are H_s^d , $H_{1,2}^d$ and $\chi_{1,2}^d$. The most general potential for these five scalar fields is

$$\begin{aligned}
V_d &= -(\mu_s^d)^2 H_s^{d\dagger} H_s^d - (\mu_D^d)^2 \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) - (\tilde{\mu}_D^d)^2 \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
&+ \lambda_s^d (H_s^{d\dagger} H_s^d)^2 + \lambda_1^d \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right)^2 + \tilde{\lambda}_1^d \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right)^2 + \lambda_2^d (H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d)^2 + \tilde{\lambda}_2^d (\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d)^2 \\
&+ \lambda_3^d |H_1^{d\dagger} H_2^d|^2 + \tilde{\lambda}_3^d |\chi_1^{d\dagger} \chi_2^d|^2 + \sigma_1^d (H_s^{d\dagger} H_s^d) \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\sigma}_1^d (H_s^{d\dagger} H_s^d) \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
&+ \{ \sigma_2^d (H_s^{d\dagger} H_1^d) (H_s^{d\dagger} H_2^d) + \text{h.c.} \} + \{ \tilde{\sigma}_2^d (H_s^{d\dagger} \chi_1^d) (H_s^{d\dagger} \chi_2^d) + \text{h.c.} \} + \sigma_3^d \left(\sum_{i=1}^2 |H_s^{d\dagger} H_i^d|^2 \right) + \tilde{\sigma}_3^d \left(\sum_{i=1}^2 |H_s^{d\dagger} \chi_i^d|^2 \right) \\
&+ \tau_1^d \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) + \tau_2^d (H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d) (\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d) \\
&+ \{ \tau_3^d (H_1^{d\dagger} \chi_1^d) (H_2^{d\dagger} \chi_2^d) + \text{h.c.} \} + \tau_4^d \left(\sum_{i=1}^2 |H_i^{d\dagger} \chi_i^d|^2 \right) + \{ \tau_5^d (H_1^{d\dagger} \chi_2^d) (H_2^{d\dagger} \chi_1^d) + \text{h.c.} \} + \tau_6^d (|H_1^{d\dagger} \chi_2^d|^2 + |H_2^{d\dagger} \chi_1^d|^2) \\
&+ \{ \tau_7^d \{ (H_2^{d\dagger} \chi_1^d) (\chi_2^{d\dagger} \chi_1^d) + (H_1^{d\dagger} \chi_2^d) (\chi_1^{d\dagger} \chi_2^d) \} + \text{h.c.} \} \\
&+ \{ \omega_1^d \{ (H_s^{d\dagger} H_1^d) (H_2^{d\dagger} \chi_2^d) + (H_s^{d\dagger} H_2^d) (H_1^{d\dagger} \chi_1^d) \} + \text{h.c.} \} + \{ \omega_2^d \{ (H_s^{d\dagger} H_1^d) (\chi_1^{d\dagger} H_1^d) + (H_s^{d\dagger} H_2^d) (\chi_2^{d\dagger} H_2^d) \} + \text{h.c.} \} \\
&+ \{ \omega_3^d \{ (H_s^{d\dagger} \chi_1^d) (H_1^{d\dagger} H_2^d) + (H_s^{d\dagger} \chi_2^d) (H_2^{d\dagger} H_1^d) \} + \text{h.c.} \}.
\end{aligned} \tag{C.8}$$

This five Higgs potential is free from accidental symmetries. However, the combined potential $V_u + V_d$ has an accidental $SU(2) \times U(1) \times U(1)$ symmetry. It is broken explicitly by mixing terms which couple the Higgs fields $H_{s,1,2}^u$ and $H_{s,1,2}^d$, $\chi_{1,2}^d$. The following potential V_{mixed} contains all such terms which are invariant under the symmetry $D_7 \times Z_2^{(aux)}$:

$$\begin{aligned}
V_{mixed} = & \kappa_1(H_s^{u\dagger} H_s^u)(H_s^{d\dagger} H_s^d) + \{\kappa_2(H_s^{u\dagger} H_s^d)^2 + \text{h.c.}\} + \kappa_3|H_s^{u\dagger} H_s^d|^2 \\
& + \kappa_4 \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\kappa}_4 \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \{\kappa_5 \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^d \right)^2 + \text{h.c.}\} + \kappa_6 |H_1^{u\dagger} H_1^d + H_2^{u\dagger} H_2^d|^2 \\
& + \kappa_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u)(H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d) + \tilde{\kappa}_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u)(\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d) \\
& + \{\kappa_8 (H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d)^2 + \text{h.c.}\} + \{\tilde{\kappa}_{[5-8]} (H_1^{u\dagger} \chi_1^d)(H_2^{u\dagger} \chi_2^d) + \text{h.c.}\} \\
& + \kappa_9 |H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d|^2 + \tilde{\kappa}_{[6+9]} (|H_1^{u\dagger} \chi_1^d|^2 + |H_2^{u\dagger} \chi_2^d|^2) + \kappa_{10} \{(H_2^{u\dagger} H_1^u)(H_1^{d\dagger} H_2^d) + \text{h.c.}\} \\
& + \{\kappa_{11} (H_2^{u\dagger} H_1^d)(H_1^{u\dagger} H_2^d) + \text{h.c.}\} + \{\tilde{\kappa}_{11} (H_1^{u\dagger} \chi_2^d)(H_2^{u\dagger} \chi_1^d) + \text{h.c.}\} + \kappa_{12} (|H_2^{u\dagger} H_1^d|^2 + |H_1^{u\dagger} H_2^d|^2) \\
& + \tilde{\kappa}_{12} (|H_1^{u\dagger} \chi_2^d|^2 + |H_2^{u\dagger} \chi_1^d|^2) + \kappa_{13} (H_s^{u\dagger} H_s^u) \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\kappa}_{13} (H_s^{u\dagger} H_s^u) \left(\sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \{\kappa_{14} (H_s^{u\dagger} H_1^d)(H_s^{u\dagger} H_2^d) + \text{h.c.}\} + \{\tilde{\kappa}_{14} (H_s^{u\dagger} \chi_1^d)(H_s^{u\dagger} \chi_2^d) + \text{h.c.}\} + \kappa_{15} (|H_s^{u\dagger} H_1^d|^2 + |H_s^{u\dagger} H_2^d|^2) \\
& + \tilde{\kappa}_{15} (|H_s^{u\dagger} \chi_1^d|^2 + |H_s^{u\dagger} \chi_2^d|^2) + \kappa_{16} (H_s^{d\dagger} H_s^d) \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) + \{\kappa_{17} (H_s^{d\dagger} H_1^u)(H_s^{d\dagger} H_2^u) + \text{h.c.}\} \\
& + \kappa_{18} \left(\sum_{i=1}^2 |H_s^{d\dagger} H_i^u|^2 \right) + \{\kappa_{19} (H_s^{u\dagger} H_s^d) \left(\sum_{i=1}^2 H_i^{u\dagger} H_i^d \right) + \text{h.c.}\} \\
& + \{\kappa_{20} (H_s^{u\dagger} H_s^d) \left(\sum_{i=1}^2 H_i^{d\dagger} H_i^u \right) + \text{h.c.}\} + \{\kappa_{21} \{(H_s^{u\dagger} H_1^u)(H_s^{d\dagger} H_2^d) + (H_s^{u\dagger} H_2^u)(H_s^{d\dagger} H_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{22} \{(H_s^{u\dagger} H_1^d)(H_s^{d\dagger} H_2^u) + (H_s^{u\dagger} H_2^d)(H_s^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{23} \{(H_s^{u\dagger} H_1^u)(H_1^{d\dagger} H_s^d) + (H_s^{u\dagger} H_2^u)(H_2^{d\dagger} H_s^d)\} + \text{h.c.}\} \\
& + \{\kappa_{24} \{(H_s^{u\dagger} H_1^d)(H_1^{u\dagger} H_s^d) + (H_s^{u\dagger} H_2^d)(H_2^{u\dagger} H_s^d)\} + \text{h.c.}\} \\
& + \{\kappa_{25} \{(H_s^{d\dagger} H_1^u)(H_2^{u\dagger} \chi_2^d) + (H_s^{d\dagger} H_2^u)(H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{26} \{(H_s^{d\dagger} H_1^u)(\chi_1^{d\dagger} H_1^u) + (H_s^{d\dagger} H_2^u)(\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} \\
& + \{\kappa_{27} \{(H_s^{d\dagger} \chi_1^d)(H_1^{u\dagger} H_2^u) + (H_s^{d\dagger} \chi_2^d)(H_2^{u\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{28} \{(H_s^{u\dagger} H_1^d)(H_2^{u\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^d)(H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{29} \{(H_s^{u\dagger} H_1^d)(\chi_1^{d\dagger} H_1^u) + (H_s^{u\dagger} H_2^d)(\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} \\
& + \{\kappa_{30} \{(H_s^{u\dagger} \chi_1^d)(H_1^{d\dagger} H_2^u) + (H_s^{u\dagger} \chi_2^d)(H_2^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{31} \{(H_s^{u\dagger} \chi_1^d)(H_1^{u\dagger} H_2^d) + (H_s^{u\dagger} \chi_2^d)(H_2^{u\dagger} H_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{32} \{(H_s^{u\dagger} H_1^u)(H_2^{d\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^u)(H_1^{d\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{33} \{(H_s^{u\dagger} H_1^u)(\chi_1^{d\dagger} H_1^d) + (H_s^{u\dagger} H_2^u)(\chi_2^{d\dagger} H_2^d)\} + \text{h.c.}\}.
\end{aligned} \tag{C.9}$$

In our numerical analysis we restricted ourselves to the inclusion of a minimal number of terms from V_{mixed} which break all accidental symmetries such that only those three Higgs particles remain massless which are eaten by the W^\pm and Z bosons. As explained in section 4.3, the three terms κ_2 , κ_5 and κ_{19} are sufficient.

We want to obtain the following VEV configuration:

$$\begin{aligned}
\langle H_s^{d,u} \rangle &= 61.5 \text{ GeV}, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = \langle \chi_1^d \rangle = \langle \chi_2^d \rangle = 61.5 \text{ GeV}, \quad \langle H_1^u \rangle = 61.5 e^{-\frac{3\pi i}{7}} \text{ GeV} \\
\text{and } \langle H_2^u \rangle &= 61.5 e^{\frac{3\pi i}{7}} \text{ GeV},
\end{aligned}$$

which allows all parameters in the potential V_d to be real, as all fields H_s^d , $H_{1,2}^d$ and $\chi_{1,2}^d$ have real VEVs. Furthermore we can remove the phase of σ_2^u such that we are left with only three

complex parameters stemming from V_{mixed} .

The mass parameters are set to be of the electroweak scale:

$$\mu_s^u = 100 \text{ GeV} , \quad \mu_D^u = 200 \text{ GeV} , \quad \mu_s^d = 100 \text{ GeV} , \quad \mu_D^d = 200 \text{ GeV} \quad \text{and} \quad \tilde{\mu}_D^d = 150 \text{ GeV} .$$

One possible setup of parameters is then:

For V_u :

$$\lambda_s^u = 0.959337 , \quad \lambda_1^u = 2.52548 , \quad \lambda_2^u = 0.374967 , \quad \lambda_3^u = -0.588842 , \quad \sigma_1^u = 1.62353 , \\ \sigma_2^u = -0.276964 , \quad \sigma_3^u = -0.283914 .$$

For V_d :

$$\lambda_s^d = 1.70438 , \quad \lambda_1^d = 3.76598 , \quad \tilde{\lambda}_1^d = 1.47549 , \quad \lambda_2^d = -0.344036 , \quad \tilde{\lambda}_2^d = -0.185157 , \\ \lambda_3^d = -0.304589 , \quad \tilde{\lambda}_3^d = -0.733236 , \quad \sigma_1^d = 0.22429 , \quad \tilde{\sigma}_1^d = 4.6792 , \quad \sigma_2^d = -0.87457 , \\ \tilde{\sigma}_2^d = -2.0284 , \quad \sigma_3^d = 0.961454 , \quad \tilde{\sigma}_3^d = 0.649984 , \quad \tau_1^d = 2.96557 , \quad \tau_2^d = 1.22903 , \\ \tau_3^d = -2.02133 , \quad \tau_4^d = -1.22242 , \quad \tau_5^d = -2.31577 , \quad \tau_6^d = 2.38236 , \quad \tau_7^d = -0.660102 , \\ \omega_1^d = 0.452165 , \quad \omega_2^d = -2.112 , \quad \omega_3^d = -1.63452 .$$

And for the three complex non-zero couplings from V_{mixed} :

$$\kappa_2 = -0.638073 + i 0.0277608 , \quad \kappa_5 = 0.312782 + i 0.140162 , \quad \kappa_{19} = -0.278402 - i 0.124756 .$$

Note that all parameters have absolute values smaller than 5 and hence they are still in the perturbative regime. With these parameter values we obtain the desired VEV structure. The Higgs masses are then

$$513 \text{ GeV} , \quad 499 \text{ GeV} , \quad 426 \text{ GeV} , \quad 414 \text{ GeV} , \quad 386 \text{ GeV} , \quad 365 \text{ GeV} , \quad 321 \text{ GeV} , \quad 266 \text{ GeV} , \quad 246 \text{ GeV} , \\ 227 \text{ GeV} , \quad 178 \text{ GeV} , \quad 159 \text{ GeV} , \quad 134 \text{ GeV} , \quad 81 \text{ GeV} \quad \text{and} \quad 55 \text{ GeV}$$

for the neutral scalars. Due to the explicit CP violation in the potential we can no longer distinguish between scalars and pseudo-scalars. For the charged scalar fields we get

$$367 \text{ GeV} , \quad 333 \text{ GeV} , \quad 294 \text{ GeV} , \quad 261 \text{ GeV} , \quad 145 \text{ GeV} , \quad 115 \text{ GeV} \quad \text{and} \quad 55 \text{ GeV} .$$

There will therefore in general be scalar states too light to pass the constraints coming from direct searches.

C.5 D_{14} Flavon Potential at Next-to-Leading Order

In this appendix we discuss the form of the VEV shifts induced by the NLO corrections of the flavon superpotential in the D_{14} model of section 6.2. These corrections can be written as

$$\Delta w_f = \Delta w_{f,u} + \Delta w_{f,d} .$$

The shifts of the VEVs given in equation (6.96) are of the form

$$\langle \psi_2^u \rangle = v^u + \delta v^u , \quad \langle \chi_i^u \rangle = w^u + \delta w_i^u , \quad \langle \xi_i^u \rangle = z^u + \delta z_i^u , \quad \langle \eta^u \rangle = -\frac{e_u v^u z^u}{f_u w^u} + \delta \eta^u , \\ \langle \psi_2^d \rangle = v^d + \delta v^d , \quad \langle \chi_1^d \rangle = e^{-\frac{\pi i m_d}{7}} \left(w^d + \delta w_1^d \right) , \quad \langle \chi_2^d \rangle = e^{\frac{\pi i m_d}{7}} \left(w^d + \delta w_2^d \right) , \\ \langle \xi_1^d \rangle = e^{-\frac{2\pi i m_d}{7}} \left(z^d + \delta z_1^d \right) , \quad \langle \xi_2^d \rangle = e^{\frac{2\pi i m_d}{7}} \left(z^d + \delta z_2^d \right) \quad \text{and} \quad \langle \eta^d \rangle = e^{-\frac{4\pi i m_d}{7}} \left(\frac{e_d v^d z^d}{f_d w^d} + \delta \eta^d \right) ,$$

with v^d , v^u and x remaining unchanged, since they are free parameters. For the actual calculation of the shifts we choose a plus sign in z^u and z^d in front of the square root, see equations (6.89) and (6.93). The NLO corrections to the flavon potential of the set of fields $\{\psi_{1,2}^u, \chi_{1,2}^u, \xi_{1,2}^u, \eta^u\}$ are given by

$$\Delta w_{f,u} = \frac{1}{\Lambda} \left(\sum_{k=1}^{16} r_k^u I_k^{R,u} + \sum_{k=1}^{11} s_k^u I_k^{S,u} + \sum_{k=1}^{12} t_k^u I_k^{T,u} \right). \quad (\text{C.10})$$

The invariants $I_k^{R,u}$ read

$$\begin{aligned} I_1^{R,u} &= \sigma^2 (\psi_1^d \psi_2^{0u} + \psi_2^d \psi_1^{0u}) & I_9^{R,u} &= (\psi_1^d \psi_2^{0u} + \psi_2^d \psi_1^{0u}) (\eta^d)^2 \\ I_2^{R,u} &= \sigma (\psi_1^d \chi_2^d \psi_1^{0u} + \psi_2^d \chi_1^d \psi_2^{0u}) & I_{10}^{R,u} &= (\psi_1^u \psi_2^{0u} + \psi_2^u \psi_1^{0u}) (\eta^u)^2 \\ I_3^{R,u} &= (\psi_1^d \psi_2^{0u} + \psi_2^d \psi_1^{0u}) (\psi_1^d \psi_2^d) & I_{11}^{R,u} &= (\psi_1^d \chi_1^d \xi_2^d \psi_1^{0u} + \psi_2^d \chi_2^d \xi_1^d \psi_2^{0u}) \\ I_4^{R,u} &= (\psi_1^u \psi_2^{0u} + \psi_2^u \psi_1^{0u}) (\psi_1^u \psi_2^u) & I_{12}^{R,u} &= (\psi_1^u \chi_1^u \xi_2^u \psi_1^{0u} + \psi_2^u \chi_2^u \xi_1^u \psi_2^{0u}) \\ I_5^{R,u} &= (\psi_1^d \psi_2^{0u} + \psi_2^d \psi_1^{0u}) (\chi_1^d \chi_2^d) & I_{13}^{R,u} &= \eta^d (\chi_1^d \xi_1^d \psi_1^{0u} - \chi_2^d \xi_2^d \psi_2^{0u}) \\ I_6^{R,u} &= (\psi_1^u \psi_2^{0u} + \psi_2^u \psi_1^{0u}) (\chi_1^u \chi_2^u) & I_{14}^{R,u} &= \eta^u (\chi_1^u \xi_1^u \psi_1^{0u} + \chi_2^u \xi_2^u \psi_2^{0u}) \\ I_7^{R,u} &= (\psi_1^d \psi_2^{0u} + \psi_2^d \psi_1^{0u}) (\xi_1^d \xi_2^d) & I_{15}^{R,u} &= \eta^d \left((\xi_1^d)^2 \psi_2^{0u} - (\xi_2^d)^2 \psi_1^{0u} \right) \\ I_8^{R,u} &= (\psi_1^u \psi_2^{0u} + \psi_2^u \psi_1^{0u}) (\xi_1^u \xi_2^u) & I_{16}^{R,u} &= \eta^u \left((\xi_1^u)^2 \psi_2^{0u} + (\xi_2^u)^2 \psi_1^{0u} \right). \end{aligned} \quad (\text{C.11})$$

For the $I_k^{S,u}$ we find

$$\begin{aligned} I_1^{S,u} &= \sigma (\psi_1^d \chi_1^d \varphi_2^{0u} + \psi_2^d \chi_2^d \varphi_1^{0u}) & I_7^{S,u} &= (\psi_1^u \chi_2^u \xi_1^u \varphi_2^{0u} + \psi_2^u \chi_1^u \xi_2^u \varphi_1^{0u}) \\ I_2^{S,u} &= \sigma (\psi_1^d \xi_2^d \varphi_1^{0u} + \psi_2^d \xi_1^d \varphi_2^{0u}) & I_8^{S,u} &= \left((\chi_1^d)^2 \psi_2^{0u} + (\chi_2^d)^2 \psi_1^{0u} \right) \\ I_3^{S,u} &= \sigma \eta^d (\xi_1^d \varphi_1^{0u} - \xi_2^d \varphi_2^{0u}) & I_9^{S,u} &= \left((\chi_1^u)^2 \psi_2^{0u} + (\chi_2^u)^2 \psi_1^{0u} \right) \\ I_4^{S,u} &= \left((\psi_1^d)^3 \varphi_2^{0u} + (\psi_2^d)^3 \varphi_1^{0u} \right) & I_{10}^{S,u} &= \eta^d \left((\chi_1^d)^2 \varphi_1^{0u} - (\chi_2^d)^2 \varphi_2^{0u} \right) \\ I_5^{S,u} &= \left((\psi_1^u)^3 \varphi_2^{0u} + (\psi_2^u)^3 \varphi_1^{0u} \right) & I_{11}^{S,u} &= \eta^u \left((\chi_1^u)^2 \varphi_1^{0u} + (\chi_2^u)^2 \varphi_2^{0u} \right) \\ I_6^{S,u} &= (\psi_1^d \chi_2^d \xi_1^d \varphi_2^{0u} + \psi_2^d \chi_1^d \xi_2^d \varphi_1^{0u}) \end{aligned} \quad (\text{C.12})$$

and for the $I_k^{T,u}$

$$\begin{aligned} I_1^{T,u} &= \sigma (\psi_1^d \xi_1^d \rho_2^{0u} + \psi_2^d \xi_2^d \rho_1^{0u}) & I_7^{T,u} &= (\chi_1^d \xi_1^d \psi_2^d \rho_2^{0u} + \chi_2^d \xi_2^d \psi_1^d \rho_1^{0u}) \\ I_2^{T,u} &= \sigma \eta^d (\chi_1^d \rho_1^{0u} - \chi_2^d \rho_2^{0u}) & I_8^{T,u} &= (\chi_1^u \xi_1^u \psi_2^u \rho_2^{0u} + \chi_2^u \xi_2^u \psi_1^u \rho_1^{0u}) \\ I_3^{T,u} &= \eta^d \left((\psi_1^d)^2 \rho_1^{0u} - (\psi_2^d)^2 \rho_2^{0u} \right) & I_9^{T,u} &= \left((\xi_1^d)^2 \psi_1^d \rho_1^{0u} + (\xi_2^d)^2 \psi_2^d \rho_2^{0u} \right) \\ I_4^{T,u} &= \eta^u \left((\psi_1^u)^2 \rho_1^{0u} + (\psi_2^u)^2 \rho_2^{0u} \right) & I_{10}^{T,u} &= \left((\xi_1^u)^2 \psi_1^u \rho_1^{0u} + (\xi_2^u)^2 \psi_2^u \rho_2^{0u} \right) \\ I_5^{T,u} &= \left(\psi_1^d (\chi_1^d)^2 \rho_2^{0u} + \psi_2^d (\chi_2^d)^2 \rho_1^{0u} \right) & I_{11}^{T,u} &= \eta^d (\chi_2^d \xi_1^d \rho_1^{0u} - \chi_1^d \xi_2^d \rho_2^{0u}) \\ I_6^{T,u} &= \left(\psi_1^u (\chi_1^u)^2 \rho_2^{0u} + \psi_2^u (\chi_2^u)^2 \rho_1^{0u} \right) & I_{12}^{T,u} &= \eta^u (\chi_2^u \xi_1^u \rho_1^{0u} + \chi_1^u \xi_2^u \rho_2^{0u}). \end{aligned} \quad (\text{C.13})$$

In the flavon sector of fields with non-trivial Z_3 charge, the non-renormalizable terms are of the form

$$\Delta w_{f,d} = \frac{1}{\Lambda} \left(\sum_{k=1}^{21} r_k^d I_k^{R,d} + \sum_{k=1}^{14} s_k^d I_k^{S,d} + \sum_{k=1}^{16} t_k^d I_k^{T,d} \right). \quad (\text{C.14})$$

The invariants $I_k^{R,d}$ are the following

$$\begin{aligned}
I_1^{R,d} &= \sigma^2 (\psi_1^u \psi_2^{0d} + \psi_2^u \psi_1^{0d}) & I_{12}^{R,d} &= (\psi_1^d \psi_2^{0d} - \psi_2^d \psi_1^{0d}) \eta^u \eta^d \\
I_2^{R,d} &= \sigma (\psi_1^d \chi_2^u \psi_1^{0d} + \psi_2^d \chi_1^u \psi_2^{0d}) & I_{13}^{R,d} &= (\psi_1^u \psi_2^{0d} + \psi_2^u \psi_1^{0d}) (\eta^d)^2 \\
I_3^{R,d} &= \sigma (\psi_1^u \chi_2^d \psi_1^{0d} + \psi_2^u \chi_1^d \psi_2^{0d}) & I_{14}^{R,d} &= (\psi_1^u \chi_1^d \xi_2^d \psi_1^{0d} + \psi_2^u \chi_2^d \xi_1^d \psi_2^{0d}) \\
I_4^{R,d} &= \left((\psi_1^d)^2 \psi_2^u \psi_2^{0d} + (\psi_2^d)^2 \psi_1^u \psi_1^{0d} \right) & I_{15}^{R,d} &= (\psi_1^d \chi_1^u \xi_2^d \psi_1^{0d} + \psi_2^d \chi_2^u \xi_1^d \psi_2^{0d}) \\
I_5^{R,d} &= (\psi_1^u \psi_2^{0d} + \psi_2^u \psi_1^{0d}) (\psi_1^d \psi_2^d) & I_{16}^{R,d} &= (\psi_1^d \chi_1^d \xi_2^u \psi_1^{0d} + \psi_2^d \chi_2^d \xi_1^u \psi_2^{0d}) \\
I_6^{R,d} &= (\psi_1^d \psi_2^{0d} \chi_1^d \chi_2^u + \psi_2^d \psi_1^{0d} \chi_2^d \chi_1^u) & I_{17}^{R,d} &= \eta^d (\chi_1^u \xi_1^d \psi_1^{0d} - \chi_2^u \xi_2^d \psi_2^{0d}) \\
I_7^{R,d} &= (\psi_1^d \psi_2^{0d} \chi_2^d \chi_1^u + \psi_2^d \psi_1^{0d} \chi_1^d \chi_2^u) & I_{18}^{R,d} &= \eta^d (\chi_1^d \xi_1^u \psi_1^{0d} - \chi_2^d \xi_2^u \psi_2^{0d}) \\
I_8^{R,d} &= (\psi_1^u \psi_2^{0d} + \psi_2^u \psi_1^{0d}) (\chi_1^d \chi_2^d) & I_{19}^{R,d} &= \eta^u (\chi_1^d \xi_1^d \psi_1^{0d} + \chi_2^d \xi_2^d \psi_2^{0d}) \\
I_9^{R,d} &= (\psi_1^d \psi_2^{0d} \xi_1^d \xi_2^u + \psi_2^d \psi_1^{0d} \xi_2^d \xi_1^u) & I_{20}^{R,d} &= \eta^d (\xi_1^d \xi_1^u \psi_2^{0d} - \xi_2^d \xi_2^u \psi_1^{0d}) \\
I_{10}^{R,d} &= (\psi_1^d \psi_2^{0d} \xi_2^d \xi_1^u + \psi_2^d \psi_1^{0d} \xi_1^d \xi_2^u) & I_{21}^{R,d} &= \eta^u \left((\xi_1^d)^2 \psi_2^{0d} + (\xi_2^d)^2 \psi_1^{0d} \right) \\
I_{11}^{R,d} &= (\psi_1^u \psi_2^{0d} + \psi_2^u \psi_1^{0d}) (\xi_1^d \xi_2^d) .
\end{aligned} \tag{C.15}$$

The second set reads

$$\begin{aligned}
I_1^{S,d} &= \sigma (\psi_1^u \chi_1^d \varphi_2^{0d} + \psi_2^u \chi_2^d \varphi_1^{0d}) & I_8^{S,d} &= (\psi_1^u \chi_2^d \xi_1^d \varphi_2^{0d} + \psi_2^u \chi_1^d \xi_2^d \varphi_1^{0d}) \\
I_2^{S,d} &= \sigma (\psi_1^d \chi_1^u \varphi_2^{0d} + \psi_2^d \chi_2^u \varphi_1^{0d}) & I_9^{S,d} &= (\psi_1^d \chi_2^u \xi_1^d \varphi_2^{0d} + \psi_2^d \chi_1^u \xi_2^d \varphi_1^{0d}) \\
I_3^{S,d} &= \sigma (\psi_1^u \xi_2^d \varphi_1^{0d} + \psi_2^u \xi_1^d \varphi_2^{0d}) & I_{10}^{S,d} &= (\psi_1^d \chi_2^d \xi_1^u \varphi_2^{0d} + \psi_2^d \chi_1^d \xi_2^u \varphi_1^{0d}) \\
I_4^{S,d} &= \sigma (\psi_1^d \xi_2^u \varphi_1^{0d} + \psi_2^d \xi_1^u \varphi_2^{0d}) & I_{11}^{S,d} &= \left((\chi_1^d)^2 \psi_2^u \varphi_2^{0d} + (\chi_2^d)^2 \psi_1^u \varphi_1^{0d} \right) \\
I_5^{S,d} &= \sigma \eta^d (\xi_1^u \varphi_1^{0d} - \xi_2^u \varphi_2^{0d}) & I_{12}^{S,d} &= (\chi_1^d \chi_1^u \psi_2^d \varphi_2^{0d} + \chi_2^d \chi_2^u \psi_1^d \varphi_1^{0d}) \\
I_6^{S,d} &= \sigma \eta^u (\xi_1^d \varphi_1^{0d} + \xi_2^d \varphi_2^{0d}) & I_{13}^{S,d} &= \eta^d (\chi_1^d \chi_1^u \varphi_1^{0d} - \chi_2^d \chi_2^u \varphi_2^{0d}) \\
I_7^{S,d} &= \left((\psi_1^d)^2 \psi_1^u \varphi_2^{0d} + (\psi_2^d)^2 \psi_2^u \varphi_1^{0d} \right) & I_{14}^{S,d} &= \eta^u \left((\chi_1^d)^2 \varphi_1^{0d} + (\chi_2^d)^2 \varphi_2^{0d} \right)
\end{aligned} \tag{C.16}$$

and finally the $I_k^{T,d}$ are given by

$$\begin{aligned}
I_1^{T,d} &= \sigma (\psi_1^u \xi_1^d \rho_2^{0d} + \psi_2^u \xi_2^d \rho_1^{0d}) & I_9^{T,d} &= (\chi_1^u \xi_1^d \psi_2^d \rho_2^{0d} + \chi_2^u \xi_2^d \psi_1^d \rho_1^{0d}) \\
I_2^{T,d} &= \sigma (\psi_1^d \xi_1^u \rho_2^{0d} + \psi_2^d \xi_2^u \rho_1^{0d}) & I_{10}^{T,d} &= (\chi_1^d \xi_1^u \psi_2^d \rho_2^{0d} + \chi_2^d \xi_2^u \psi_1^d \rho_1^{0d}) \\
I_3^{T,d} &= \sigma \eta^u (\chi_1^d \rho_1^{0d} + \chi_2^d \rho_2^{0d}) & I_{11}^{T,d} &= (\chi_1^d \xi_1^d \psi_2^u \rho_2^{0d} + \chi_2^d \xi_2^d \psi_1^u \rho_1^{0d}) \\
I_4^{T,d} &= \sigma \eta^d (\chi_1^u \rho_1^{0d} - \chi_2^u \rho_2^{0d}) & I_{12}^{T,d} &= \left((\xi_1^d)^2 \psi_1^u \rho_1^{0d} + (\xi_2^d)^2 \psi_2^u \rho_2^{0d} \right) \\
I_5^{T,d} &= \eta^d (\psi_1^d \psi_1^u \rho_1^{0d} - \psi_2^d \psi_2^u \rho_2^{0d}) & I_{13}^{T,d} &= (\xi_1^u \xi_1^d \psi_1^d \rho_1^{0d} + \xi_2^u \xi_2^d \psi_2^d \rho_2^{0d}) \\
I_6^{T,d} &= \eta^u \left((\psi_1^d)^2 \rho_1^{0d} + (\psi_2^d)^2 \rho_2^{0d} \right) & I_{14}^{T,d} &= \eta^d (\chi_2^u \xi_1^d \rho_1^{0d} - \chi_1^u \xi_2^d \rho_2^{0d}) \\
I_7^{T,d} &= \left(\psi_1^u (\chi_1^d)^2 \rho_2^{0d} + \psi_2^u (\chi_2^d)^2 \rho_1^{0d} \right) & I_{15}^{T,d} &= \eta^d (\chi_2^d \xi_1^u \rho_1^{0d} - \chi_1^d \xi_2^u \rho_2^{0d}) \\
I_8^{T,d} &= (\psi_1^d \chi_1^u \chi_1^d \rho_2^{0d} + \psi_2^d \chi_2^u \chi_2^d \rho_1^{0d}) & I_{16}^{T,d} &= \eta^u (\chi_2^d \xi_1^d \rho_1^{0d} + \chi_1^d \xi_2^d \rho_2^{0d}) .
\end{aligned} \tag{C.17}$$

With these corrections to the superpotential we can calculate the shifts which correct the flavon VEVs. We take the parameterization given in equation (6.96) and plug this into the F-terms arising from the corrected superpotential. We then calculate the VEV shifts in the same way as for the D_4 model in section 6.2.2, i.e. we linearize the equations in the small quantities δVEV and $1/\Lambda$. We can then derive the following equations for the shifts of the flavons with index u

from the F-terms of the driving fields ψ_i^{0u} , φ_i^{0u} and ρ_i^{0u} :

$$\begin{aligned}
\frac{\partial(w_{f,u} + \Delta w_{f,u})}{\partial\psi_1^{0u}} &= b_u (v^u \delta w_2^u - w^u \delta v^u) + \frac{1}{\Lambda} \left\{ r_1^u x^2 v^d + r_2^u x v^d w^d \right. \\
&+ (v^d)^3 e^{-\frac{\pi i m_d}{7}} \left[r_3^u - r_9^u \left(\frac{e_d z^d}{f_d w^d} \right)^2 \right] + (v^u)^3 \left[r_4^u + r_{10}^u \left(\frac{e_u z^u}{f_u w^u} \right)^2 \right] + r_5^u (w^d)^2 v^d + r_6^u (w^u)^2 v^u \\
&+ v^d (z^d)^2 \left[r_7^u - r_{13}^u \frac{e_d}{f_d} - r_{15}^u \left(\frac{e_d z^d}{f_d w^d} \right) \right] + v^u (z^u)^2 \left[r_8^u - r_{14}^u \frac{e_u}{f_u} - r_{16}^u \left(\frac{e_u z^u}{f_u w^u} \right) \right] \\
&\left. + r_{11}^u v^d w^d z^d + r_{12}^u v^u w^u z^u \right\} = 0 \tag{C.18}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,u} + \Delta w_{f,u})}{\partial\psi_2^{0u}} &= b_u (v^u \delta w_1^u + w^u \delta v^u) + \frac{1}{\Lambda} \left\{ r_1^u e^{-\frac{\pi i m_d}{7}} x^2 v^d + r_2^u e^{-\frac{\pi i m_d}{7}} x v^d w^d \right. \\
&+ (v^d)^3 e^{-\frac{2\pi i m_d}{7}} \left[r_3^u - r_9^u \left(\frac{e_d z^d}{f_d w^d} \right)^2 \right] + (v^u)^3 \left[r_4^u + r_{10}^u \left(\frac{e_u z^u}{f_u w^u} \right)^2 \right] + r_5^u e^{-\frac{\pi i m_d}{7}} (w^d)^2 v^d \\
&+ r_6^u (w^u)^2 v^u + v^d (z^d)^2 e^{-\frac{\pi i m_d}{7}} \left[r_7^u - r_{13}^u \frac{e_d}{f_d} - r_{15}^u \left(\frac{e_d z^d}{f_d w^d} \right) \right] \\
&\left. + v^u (z^u)^2 \left[r_8^u - r_{14}^u \frac{e_u}{f_u} - r_{16}^u \left(\frac{e_u z^u}{f_u w^u} \right) \right] + r_{11}^u e^{-\frac{\pi i m_d}{7}} v^d w^d z^d + r_{12}^u v^u w^u z^u \right\} = 0 \tag{C.19}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,u} + \Delta w_{f,u})}{\partial\varphi_1^{0u}} &= a_u (v^u \delta w_2^u + w^u \delta v^u) + c_u v^u \delta z_2^u + d_u z^u \left[\delta \eta^u - \left(\frac{e_u v^u}{f_u w^u} \right) \delta z_1^u \right] \\
&+ \frac{1}{\Lambda} \left\{ e^{\frac{\pi i m_d}{7}} s_1^u x v^d w^d + e^{\frac{\pi i m_d}{7}} x v^d z^d \left[s_2^u - s_3^u \left(\frac{e_d z^d}{f_d w^d} \right) \right] + s_4^u (v^d)^3 + s_5^u (v^u)^3 \right. \\
&\left. + v^d w^d z^d e^{\frac{\pi i m_d}{7}} \left(s_6^u - s_{10}^u \frac{e_d}{f_d} \right) + v^u w^u z^u \left(s_7^u - s_{11}^u \frac{e_u}{f_u} \right) + e^{\frac{\pi i m_d}{7}} s_8^u v^d (w^d)^2 + s_9^u v^u (w^u)^2 \right\} = 0 \tag{C.20}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,u} + \Delta w_{f,u})}{\partial\varphi_2^{0u}} &= a_u v^u \delta w_1^u + c_u (v^u \delta z_1^u + z^u \delta v^u) + d_u z^u \left[\delta \eta^u - \left(\frac{e_u v^u}{f_u w^u} \right) \delta z_2^u \right] \\
&+ \frac{1}{\Lambda} \left\{ e^{-\frac{2\pi i m_d}{7}} s_1^u x v^d w^d + e^{-\frac{2\pi i m_d}{7}} x v^d z^d \left[s_2^u - s_3^u \left(\frac{e_d z^d}{f_d w^d} \right) \right] + s_4^u e^{-\frac{3\pi i m_d}{7}} (v^d)^3 + s_5^u (v^u)^3 \right. \\
&\left. + v^d w^d z^d e^{-\frac{2\pi i m_d}{7}} \left(s_6^u - s_{10}^u \frac{e_d}{f_d} \right) + v^u w^u z^u \left(s_7^u - s_{11}^u \frac{e_u}{f_u} \right) + s_8^u e^{-\frac{2\pi i m_d}{7}} v^d (w^d)^2 + s_9^u v^u (w^u)^2 \right\} = 0 \tag{C.21}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,u} + \Delta w_{f,u})}{\partial\rho_1^{0u}} &= f_u w^u \delta \eta^u + e_u \left(v^u \delta z_2^u - \frac{v^u z^u}{w^u} \delta w_1^u + z^u \delta v^u \right) + \frac{1}{\Lambda} \left\{ x v^d z^d e^{\frac{2\pi i m_d}{7}} \left(t_1^u - t_2^u \frac{e_d}{f_d} \right) \right. \\
&- t_3^u e^{\frac{\pi i m_d}{7}} (v^d)^3 \left(\frac{e_d z^d}{f_d w^d} \right) - t_4^u (v^u)^3 \left(\frac{e_u z^u}{f_u w^u} \right) + t_5^u e^{\frac{2\pi i m_d}{7}} v^d (w^d)^2 + t_6^u v^u (w^u)^2 + t_7^u e^{\frac{2\pi i m_d}{7}} v^d w^d z^d \\
&\left. + t_8^u v^u w^u z^u - v^d (z^d)^2 e^{\frac{2\pi i m_d}{7}} \left(t_9^u + t_{11}^u \frac{e_d}{f_d} \right) + v^u (z^u)^2 \left(t_{10}^u - t_{12}^u \frac{e_u}{f_u} \right) \right\} = 0 \tag{C.22}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial (w_{f,u} + \Delta w_{f,u})}{\partial \rho_2^{0u}} &= f_u w^u \delta \eta^u + e_u v^u \left(\delta z_1^u - \frac{z^u}{w^u} \delta w_2^u \right) + \frac{1}{\Lambda} \left\{ x v^d z^d e^{-\frac{3\pi i m_d}{7}} \left(t_1^u - t_2^u \frac{e_d}{f_d} \right) \right. \\
&+ t_3^u e^{\frac{3\pi i m_d}{7}} (v^d)^3 \left(\frac{e_d z^d}{f_d w^d} \right) - t_4^u (v^u)^3 \left(\frac{e_u z^u}{f_u w^u} \right) + t_5^u e^{-\frac{3\pi i m_d}{7}} v^d (w^d)^2 + t_6^u v^u (w^u)^2 + t_7^u e^{-\frac{3\pi i m_d}{7}} v^d w^d z^d \\
&\left. + t_8^u v^u w^u z^u - v^d (z^d)^2 e^{-\frac{3\pi i m_d}{7}} \left(t_9^u + t_{11}^u \frac{e_d}{f_d} \right) + v^u (z^u)^2 \left(t_{10}^u - t_{12}^u \frac{e_u}{f_u} \right) \right\} = 0 \quad (\text{C.23})
\end{aligned}$$

Note that we replaced the mass parameter M_ψ^u with the VEV w^u . Analogously, we will replace the dimensionless coupling m_ψ^d with the VEV w^d . We also frequently use the fact that m is an odd integer in order to simplify the phase factors appearing in the formulae.

Similarly, we can deduce another set of equations from the F-terms of the driving fields ψ_i^{0d} , φ_i^{0d} and ρ_i^{0d} which gives rise to the shifts in the VEVs of the flavons ψ_i^d , χ_i^d , ξ_i^d , η^d and σ :

$$\begin{aligned}
\frac{\partial (w_{f,d} + \Delta w_{f,d})}{\partial \psi_1^{0d}} &= b_d \left(v^d \delta w_2^d - w^d \delta v^d \right) + \frac{1}{\Lambda} \left\{ r_1^d v^u x^2 + r_2^d e^{-\frac{\pi i m_d}{7}} v^d w^u x + r_3^d e^{\frac{\pi i m_d}{7}} v^u w^d x \right. \\
&+ v^u (v^d)^2 \left[r_4^d + e^{-\frac{\pi i m_d}{7}} r_5^d - r_{12}^d e^{\frac{3\pi i m_d}{7}} \left(\frac{e_d e_u z^d z^u}{f_d f_u w^d w^u} \right) - r_{13}^d e^{-\frac{\pi i m_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right)^2 \right] \\
&+ v^d w^d w^u \left(e^{\frac{\pi i m_d}{7}} r_6^d + e^{-\frac{\pi i m_d}{7}} r_7^d \right) + r_8^d v^u (w^d)^2 + v^d z^d z^u \left[e^{\frac{2\pi i m_d}{7}} r_9^d + e^{-\frac{2\pi i m_d}{7}} r_{10}^d - e^{\frac{2\pi i m_d}{7}} r_{18}^d \frac{e_d}{f_d} \right. \\
&- \left. e^{-\frac{2\pi i m_d}{7}} r_{20}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] + v^u (z^d)^2 \left[r_{11}^d + e^{-\frac{3\pi i m_d}{7}} r_{21}^d \left(\frac{e_u z^u}{f_u w^u} \right) \right] + v^u w^d z^d \left[r_{14}^d e^{\frac{\pi i m_d}{7}} \right. \\
&- \left. r_{19}^d e^{-\frac{3\pi i m_d}{7}} \left(\frac{e_u z^u}{f_u w^u} \right) \right] + v^d w^u z^d e^{\frac{\pi i m_d}{7}} \left[r_{15}^d - r_{17}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] + r_{16}^d e^{-\frac{2\pi i m_d}{7}} v^d w^d z^u \left. \right\} = 0 \quad (\text{C.24})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial (w_{f,d} + \Delta w_{f,d})}{\partial \psi_2^{0d}} &= e^{-\frac{\pi i m_d}{7}} b_d \left(v^d \delta w_1^d + w^d \delta v^d \right) + \frac{1}{\Lambda} \left\{ r_1^d v^u x^2 + r_2^d v^d w^u x + r_3^d e^{-\frac{\pi i m_d}{7}} v^u w^d x \right. \\
&+ v^u (v^d)^2 \left[r_4^d e^{-\frac{2\pi i m_d}{7}} + e^{-\frac{\pi i m_d}{7}} r_5^d + r_{12}^d e^{\frac{2\pi i m_d}{7}} \left(\frac{e_d e_u z^d z^u}{f_d f_u w^d w^u} \right) - r_{13}^d e^{-\frac{\pi i m_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right)^2 \right] \\
&+ v^d w^d w^u \left(e^{-\frac{2\pi i m_d}{7}} r_6^d + r_7^d \right) + r_8^d v^u (w^d)^2 \\
&+ v^d z^d z^u \left[e^{-\frac{3\pi i m_d}{7}} r_9^d + e^{\frac{\pi i m_d}{7}} r_{10}^d - e^{-\frac{3\pi i m_d}{7}} r_{18}^d \frac{e_d}{f_d} - e^{\frac{\pi i m_d}{7}} r_{20}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] \\
&+ v^u (z^d)^2 \left[r_{11}^d + e^{\frac{3\pi i m_d}{7}} r_{21}^d \left(\frac{e_u z^u}{f_u w^u} \right) \right] + v^u w^d z^d \left[r_{14}^d e^{-\frac{\pi i m_d}{7}} - r_{19}^d e^{\frac{3\pi i m_d}{7}} \left(\frac{e_u z^u}{f_u w^u} \right) \right] \\
&+ v^d w^u z^d e^{-\frac{2\pi i m_d}{7}} \left[r_{15}^d - r_{17}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] + r_{16}^d e^{\frac{\pi i m_d}{7}} v^d w^d z^u \left. \right\} = 0 \quad (\text{C.25})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,d} + \Delta w_{f,d})}{\partial\varphi_1^{0d}} &= e^{\frac{\pi im_d}{7}} \left[a_d \left(v^d \delta w_2^d + w^d \delta v^d \right) + c_d v^d \delta z_2^d - d_d z^d \left(\left[\frac{e_d v^d}{f_d w^d} \right] \delta z_1^d + \delta \eta^d \right) \right] \\
&+ \frac{1}{\Lambda} \left\{ s_1^d e^{\frac{\pi im_d}{7}} x v^u w^d + s_2^d x v^d w^u + x v^u z^d \left[s_3^d e^{\frac{2\pi im_d}{7}} - e^{-\frac{2\pi im_d}{7}} s_6^d \left(\frac{e_u z^u}{f_u w^u} \right) \right] \right. \\
&+ x v^d z^u \left[s_4^d e^{-\frac{\pi im_d}{7}} - s_5^d e^{\frac{3\pi im_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right) \right] + s_7^d (v^d)^2 v^u + s_8^d e^{\frac{\pi im_d}{7}} v^u w^d z^d \\
&+ v^d w^u z^d e^{\frac{2\pi im_d}{7}} \left(s_9^d - s_{13}^d \frac{e_d}{f_d} \right) + s_{10}^d e^{-\frac{\pi im_d}{7}} v^d w^d z^u + s_{11}^d e^{\frac{2\pi im_d}{7}} v^u (w^d)^2 + s_{12}^d v^d w^d w^u \\
&\left. - s_{14}^d e^{-\frac{2\pi im_d}{7}} v^u (w^d)^2 \left(\frac{e_u z^u}{f_u w^u} \right) \right\} = 0 \tag{C.26}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,d} + \Delta w_{f,d})}{\partial\varphi_2^{0d}} &= e^{-\frac{2\pi im_d}{7}} \left[a_d v^d \delta w_1^d + c_d \left(v^d \delta z_1^d + z^d \delta v^d \right) - d_d z^d \left(\left[\frac{e_d v^d}{f_d w^d} \right] \delta z_2^d + \delta \eta^d \right) \right] \\
&+ \frac{1}{\Lambda} \left\{ s_1^d e^{-\frac{\pi im_d}{7}} x v^u w^d + s_2^d e^{-\frac{\pi im_d}{7}} x v^d w^u + x v^u z^d \left[s_3^d e^{-\frac{2\pi im_d}{7}} - e^{\frac{2\pi im_d}{7}} s_6^d \left(\frac{e_u z^u}{f_u w^u} \right) \right] \right. \\
&+ x v^d z^u \left[s_4^d + s_5^d e^{\frac{3\pi im_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right) \right] + s_7^d e^{-\frac{2\pi im_d}{7}} (v^d)^2 v^u + s_8^d e^{-\frac{\pi im_d}{7}} v^u w^d z^d \\
&+ v^d w^u z^d e^{-\frac{3\pi im_d}{7}} \left(s_9^d - s_{13}^d \frac{e_d}{f_d} \right) + s_{10}^d v^d w^d z^u + s_{11}^d e^{-\frac{2\pi im_d}{7}} v^u (w^d)^2 + s_{12}^d e^{-\frac{\pi im_d}{7}} v^d w^d w^u \\
&\left. - s_{14}^d e^{\frac{2\pi im_d}{7}} v^u (w^d)^2 \left(\frac{e_u z^u}{f_u w^u} \right) \right\} = 0 \tag{C.27}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,d} + \Delta w_{f,d})}{\partial\rho_1^{0d}} &= e^{\frac{2\pi im_d}{7}} \left[e_d \left(v^d \delta z_2^d + z^d \delta v^d - \frac{v^d z^d}{w^d} \delta w_1^d \right) - f_d w^d \delta \eta^d \right] + \frac{1}{\Lambda} \left\{ t_1^d e^{\frac{2\pi im_d}{7}} x v^u z^d \right. \\
&+ t_2^d x v^d z^u - t_3^d e^{-\frac{\pi im_d}{7}} x v^u z^u \left(\frac{e_u w^d}{f_u w^u} \right) - t_4^d e^{\frac{3\pi im_d}{7}} x v^d z^d \left(\frac{e_d w^u}{f_d w^d} \right) \\
&- (v^d)^2 v^u \left[t_5^d e^{\frac{2\pi im_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right) + t_6^d e^{-\frac{2\pi im_d}{7}} \left(\frac{e_u z^u}{f_u w^u} \right) \right] + t_7^d e^{\frac{2\pi im_d}{7}} v^u (w^d)^2 + t_8^d e^{\frac{\pi im_d}{7}} v^d w^d w^u \\
&+ w^u z^d v^d e^{\frac{\pi im_d}{7}} \left[t_9^d - t_{14}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] + t_{10}^d v^d w^d z^u + w^d z^d v^u \left[t_{11}^d e^{\frac{3\pi im_d}{7}} - t_{16}^d e^{-\frac{\pi im_d}{7}} \left(\frac{e_u z^u}{f_u w^u} \right) \right] \\
&\left. - t_{12}^d e^{\frac{3\pi im_d}{7}} v^u (z^d)^2 + z^u z^d v^d e^{-\frac{3\pi im_d}{7}} \left(t_{13}^d + t_{15}^d \frac{e_d}{f_d} \right) \right\} = 0 \tag{C.28}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(w_{f,d} + \Delta w_{f,d})}{\partial\rho_2^{0d}} &= e^{-\frac{3\pi im_d}{7}} \left[e_d \left(v^d \delta z_1^d - \frac{v^d z^d}{w^d} \delta w_2^d \right) - f_d w^d \delta \eta^d \right] + \frac{1}{\Lambda} \left\{ t_1^d e^{-\frac{2\pi im_d}{7}} x v^u z^d \right. \\
&+ t_2^d e^{-\frac{\pi im_d}{7}} x v^d z^u - t_3^d e^{\frac{\pi im_d}{7}} x v^u z^u \left(\frac{e_u w^d}{f_u w^u} \right) + t_4^d e^{\frac{3\pi im_d}{7}} x v^d z^d \left(\frac{e_d w^u}{f_d w^d} \right) \\
&+ (v^d)^2 v^u \left[t_5^d e^{\frac{3\pi im_d}{7}} \left(\frac{e_d z^d}{f_d w^d} \right) - t_6^d \left(\frac{e_u z^u}{f_u w^u} \right) \right] + t_7^d e^{-\frac{2\pi im_d}{7}} v^u (w^d)^2 + t_8^d e^{-\frac{2\pi im_d}{7}} v^d w^d w^u \\
&+ w^u z^d v^d e^{-\frac{2\pi im_d}{7}} \left[t_9^d - t_{14}^d \left(\frac{e_d z^d}{f_d w^d} \right) \right] + t_{10}^d e^{-\frac{\pi im_d}{7}} v^d w^d z^u - t_{12}^d e^{-\frac{3\pi im_d}{7}} v^u (z^d)^2 \\
&\left. + w^d z^d v^u \left[t_{11}^d e^{-\frac{3\pi im_d}{7}} - t_{16}^d e^{\frac{\pi im_d}{7}} \left(\frac{e_u z^u}{f_u w^u} \right) \right] + z^u z^d v^d e^{\frac{2\pi im_d}{7}} \left(t_{13}^d + t_{15}^d \frac{e_d}{f_d} \right) \right\} = 0 \tag{C.29}
\end{aligned}$$

One can infer from these equations the generic size of the shifts of the VEVs. In the case of no accidental cancellation among the various terms present here we expect all of them to be of the order VEV^2/Λ which is $\epsilon\text{VEV} \approx \epsilon^2\Lambda$ for all VEVs being of the natural order $\epsilon \cdot \Lambda$ with $\epsilon \approx \lambda^2 \approx 0.04$.

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