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Towards the Hot Sphaleron Rate and Sizable CP Violation in the Standard Model

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Auf dem Weg zu der heißen Sphaleron Rate und beträchtliche CP-Verletzung im Standard-Modell

Zusammenfassung

Wir untersuchen zwei Aspekte des Standard-Modells im Zusammenhang mit Baryogenese an der elektroschwachen Skala. Der erste befasst sich mit CP-Verletzung. Seit einiger Zeit ist es gedacht worden, dass CP-Verletzung im Standard-Modell zu schwach war um die Baryon-Asymmetrie des Universums zu produzieren. Das Argument geht von der kleinen Wert der Jarslkog Determinante, $\sim 10^{-19}$, aber das Letztere ist ein störungstheoretische Berechnung und CP-Verletzung im Experimente kann viel größer sein, z.B. im Kaon-System ist es um 10^{-3}. Mit dem Einsatz der Weltlinie-Methode, leiten wir eine 1-Loop effektive Wirkung durch die Integration die Fermionen ab. Die CP-Verletzung, die zuvor in der Fermion Sektor sich befand, manifestiert sich als Operatoren in der effektive Wirkung, die die CP-Symmetrie verletzen. Wir finden, dass die Operatoren nicht durch die Jarlskog Determinante unterdrückt sind, sondern durch die Jarlskog Invariante, $\sim 10^{-5}$.

Der zweite Teil dieser Arbeit beschäftigt sich mit der Infrarot-Analyse der Bödeker effektive Theorie, die die Dynamik der schwach gekoppelt, nicht-abelschen Eichfelder bei hoher Temperatur mit charakteristischen Impuls-Skala $|\mathbf{k}| \sim g^2 T$ verschlüsselt. Die Motivation dafür ist die mögliche analytische Berechnung der heißen Sphaleron Rate, die direkt zur Rate der Baryonzahl-Verletzung in der symmetrischen Phase proportional ist. Nach der Übertragung von Bödeker effektive Theorie von einer Langevin-Gleichung in einen euklidischen Pfadintegral leiten wir die Dyson-Schwinger-Gleichungen ab. Wir schlagen ein Ansatz zur Lösung der Infrarot-dominierten Gleichungen vor, und finden der erwartende verstärkte Eich-Propagator. Eine analoge Rolle für der Ghost-Propagator in Yang-Mills-Theorie wurde durch den gemischten Propagator gespielt.

Towards the Hot Sphaleron Rate and Sizable CP Violation in the Standard Model

Abstract

In this work we study two aspects of the Standard Model related to baryogenesis at the electroweak scale. The first deals with CP violation. For some time now, it has been thought that CP violation within the Standard Model was too weak to be able to produce the baryon asymmetry of the universe. The argument is based on the small value of the Jarslkog's determinant, $\sim 10^{-19}$, but the latter is a perturbative calculation and CP violation in experiments can be much larger, e.g. in the Kaon system of order 10^{-3}. With the use of the worldline method, we derive a one-loop effective action by integrating out the fermions in the next-to-leading order of a gradient expansion. The CP violation, previously present in the fermion sector, manifests as CP violating operators in the effective action. By treating the fermion masses non-perturbatively, albeit with their derivatives treated perturbatively as befits a gradient expansion, we find the operators not to be suppressed by the Jarlskog determinant, but by the Jarlskog invariant, which is of order 10^{-5}.

The second part of this work deals with the infrared analysis of Bödeker's effective theory, which encodes the dynamics of weakly coupled, non-abelian gauge fields at high temperature with characteristic momentum scale of order $|\mathbf{k}| \sim g^2 T$. The motivation for this is the eventual analytic calculation of the hot sphaleron rate, which is directly proportional to the rate of baryon number violation in the symmetric phase. After transcribing Bödeker's effective theory from a Langevin equation into an Euclidean path integral, we derive Dyson-Schwinger equations. We introduce an ansatz intended to solve the infrared dominated equations, and find the expected enhanced gauge propagator. An analogous role to the ghost propagator in Yang-Mills theory is played by the mixed propagator, which is suppressed.

Contents

Chapter 1

Introduction

The Standard Model of particle physics has been extremely successful in explaining phenomena pertaining to three of the four known fundamental interactions. Despite the current active research to identify and understand phenomena where the model is no longer valid or where it needs extension, any future theory should be able to reproduce the quantitative behaviour presented by the Standard Model (SM) in the region where it is valid.

An important open problem left by the SM is the presence of antimatter. The problem is not in the accurate prediction of its properties or production in the laboratory, but in its almost complete lack of presence in the Universe. In the SM (at low temperatures) matter and antimatter behave almost symmetrically, with only the weak force distinguishing between the two. That our immediate environment is composed only of matter is fortunate, since when a particle and its antiparticle come in contact, they annihilate and produce γ-radiation. If there was a significant presence of antimatter in our vicinity, it would result in its annihilation with an equal amount of matter. Structure formation, e.g. life forms, under these circumstances would be unable to proceed. Nevertheless, one could posit that through a fluke of chance, we find ourselves in a region of the universe relatively devoid of antimatter and structure formation was allowed to proceed in our vicinity. Due to matter and antimatter being almost symmetrical according to the SM, this would imply that there is another region of space where the excess antimatter is found, and where similar structure could have formed. A prediction of this hypothesis would be a specific γ-radiation signature where these two regions come in contact. At the very least there should be a region between these two regions where there is no asymmetry, also with its γ-radiation signature. Neither of these two options are borne out by observations [1]. It is not only our region of space that has an excess of matter, it is the whole visible universe which has such an excess. The experimental conclusion is clear: the observable universe has a matter-antimatter asymmetry. Furthermore, the asymmetry needs to have developed dynamically. Based on observation of the cosmic microwave radiation, there is strong evidence to think that an inflationary period took place in the early history of the universe. During that inflationary period, an initial asymmetry would have been diluted. Therefore, because of inflation, the universe evolves from a configuration with no asymmetry. At some point, conditions changed and an asymmetry was created, and shortly after the electroweak phase transition, this asymmetry was frozen in place.

The lack of quantitative treatment of the problem in the previous paragraph might leave one wondering about the severity of the problem. Two salient points emerge then: what is the magnitude of that asymmetry and what is meant by matter and antimatter being 'almost symmetrical'.

The matter-antimatter asymmetry can be expressed by

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} = 6.21 \pm 0.16 \times 10^{-10}, \tag{1.1}$$

where n_b, $n_{\bar{b}}$ are the baryon and antibaryon densities, respectively, and n_γ is the photon density. All three quantities scale as a^{-3}, where a is the cosmological scale factor, and therefore the asymmetry η remains constant during most of the evolution of the universe after its creation. The numerical value of η presented here was obtained from measurements of the Cosmic Microwave Radiation Background (CMB) performed by the WMAP collaboration [2], but it agrees with analysis of the production of light elements during nucleosynthesis [3]. Even though the number of baryons is much larger than the number of antibaryons, i.e. $n_b \gg n_{\bar{b}}$, the numerical value appears small as a result of the fact that $n_\gamma \gg n_b$ due to pair annihilation. The focus on baryon-antibaryon asymmetry, instead of the more general matter-antimatter is due to the fact that most of the mass of the visible matter in the universe is in the form of baryons. The mechanism by which the universe accrued an excess of baryons is termed baryogenesis.

Expressed like this, the asymmetry does not appear to present an insurmountable problem, and could well in fact be explainable within the SM. After all, it is being claimed that matter and anti-matter are *almost* symmetric. This brings us back to explaining this qualifier. Both the electromagnetic interaction and the strong interaction are invariant under exchange of particles and antiparticles, C symmetry. The weak force, on the other hand, violates it maximally. Both the electromagnetic interaction and strong interaction are also invariant under parity transformation, but the latter is again violated by the weak force. The combination of charge and parity symmetry, CP symmetry, could then be the actual symmetry between particles and antiparticles [4]. CP violation was first observed in Kaon system decays.

So matter and antimatter are symmetric, CP invariant, with respect to the strong and electromagnetic force, and even in most instances where the weak force is involved, but as stated there are weak interactions which violate CP. One can wonder if those interactions are enough to produce the observed asymmetry. To put the question into perspective, the conditions for the creation of the asymmetry need to be presented. Sakharov enumerated the necessary conditions for dynamically generating the asymmetry in 1967[5]:

- Violation of Baryon number conservation.
- Violation of charge conjugation (C) and charge-parity (CP) symmetry.
- Departure from equilibrium.

The first condition is clear. If the baryon number cannot change, then there can be no increase in the number of baryons. If C and CP are both conserved, then the rate for any interaction which produces an excess of baryons will be equal to the conjugate interaction which produces an excess of antibaryons. This would preclude a net excess across the visible universe. The last condition is important if CPT holds, since in that case under equilibrium, both particles and antiparticles have the same thermal distribution.

The SM has a source of Baryon number violation in the form of a weak anomaly [6]. At zero temperature such interactions are mediated by the $SU(2)$ instanton, the sphaleron. They correspond to vacuum to vacuum quantum tunnelling, and the probability is therefore suppressed by a factor of $\exp(-4\pi/\alpha_W)$, which since $\alpha_W \approx 1/30$ is absurdely small. However, with the introduction of temperature, still below the possible phase transition or crossover temperature, it is suppressed by $\exp(-v/gT)$, where v is the Higgs expectation value $\langle H \rangle$. At temperature above the phase transition or crossover temperature, where $v = 0$, the rate is no longer suppressed exponentially by the weak scale, but instead goes like $\sim \alpha_W^5 T^4$ [7]. The baryon number violation rate is directly proportional to the sphaleron rate and it is the latter which will be an object of study in this work. It should be remarked that there are two types of sphaleron transitions. At temperatures below the electroweak phase transition or crossover, where the $v \neq 0$, the sphaleron transition involves thermal fluctuations crossing over the energy barrier, becoming more common as temperature increases. This is reflected in the exponential suppression. At temperatures above the critical value the transition is no longer exponentially suppressed.

The second condition is C and CP violation, the latter being a main subject of this work. Weak interactions violate C maximally and violate CP through the Kobayashi-Maskawa mechanism [8]. It has been often argued in the literature, that CP violation in the SM is too small to be able to generate the asymmetry. The argument rests on the Jarlskog determinant δ_{CP} [9], which is of order 10^{-19}, and hence too small to generate an asymmetry of order 10^{-10}.

This argument, however, is based on the assumption that the observable under consideration is perturbative in the quark masses. The Jarlskog determinant has the following form

$$\delta_{CP} = \text{Im Det}\left[\frac{m_u m_u^\dagger}{v^2}, \frac{m_d m_d^\dagger}{v^2}\right] = J \prod_{i<j} \frac{\tilde{m}_{u,i}^2 - \tilde{m}_{u,j}^2}{v^2} \prod_{i<j} \frac{\tilde{m}_{d,i}^2 - \tilde{m}_{d,j}^2}{v^2} \simeq 10^{-19}, \tag{1.2}$$

where $\tilde{m}_{u/d}^2$ denote the diagonalised mass matrices according to

$$m_d m_d^\dagger = D \tilde{m}_d^2 D^\dagger, \quad m_u m_u^\dagger = U \tilde{m}_u^2 U^\dagger. \tag{1.3}$$

The identity in Eq. (1.2) results then from the relation

$$\text{Im}\left[V_{ab} V_{bc}^\dagger V_{cd} V_{da}^\dagger\right] = J \sum_{e,f} \epsilon_{ace} \epsilon_{bdf}, \quad V = U^\dagger D \tag{1.4}$$

(summation over indices is only performed as explicitly shown) with the Jarlskog invariant J given in terms of the standard parametrisation of the CKM matrix V, with CP violating phase δ, as [9, 10]

$$J = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin(\delta) = (3.0 \pm 0.3) \times 10^{-5}. \tag{1.5}$$

It has been argued that at temperatures of the electroweak scale the CP violation might be only suppressed by the temperature rather than by the Higgs vev v as given in Eq. (1.2), but nevertheless this would be insufficient to be significant in a baryogenesis mechanism unless coherent scattering at a first order phase transition bubble wall and a very distinctive behaviour of the various quarks is assumed [11, 12]. This created a controversial discussion [13].

The Jarlskog invariant J is the first non-trivial phase invariant that can be constructed from the quark sector. A result of this work emphasises that it is J which is a good measure of CP violation in the SM, not δ_{CP}. The small value of δ_{CP} is not due to CP violation being so small in the SM, but rather that the product of mass differences on the scale of v is so small.

For example, CP violation is much larger in the neutral Kaon system than indicated by the Jarlskog determinant. If CP violation in the mixing properties and decay rates of neutral Kaons are considered, the CP-violating effects are suppressed by the Jarlskog invariant J, but not by the Jarlskog determinant δ_{CP}. Experimentally one finds the value [10]

$$\frac{\left\langle \pi^0 \pi^0 | \mathcal{H} | K_L \right\rangle}{\left\langle \pi^0 \pi^0 | \mathcal{H} | K_S \right\rangle} \approx \frac{\left\langle \pi^+ \pi^- | \mathcal{H} | K_L \right\rangle}{\left\langle \pi^+ \pi^- | \mathcal{H} | K_S \right\rangle} \approx 2.2 \times 10^{-3}, \tag{1.6}$$

which is many orders of magnitude larger than the Jarlskog determinant. The reason for this is the well defined quark content of the initial and final states that go into the calculation. The neutral Kaons are composed of a down and an anti-strange quark or a strange and an anti-down quark. If the strange and bottom quarks would be degenerate in mass, the Kaon would be indistinguishable from the B-mesons and CP violation in meson decays would be non-observable. However, the quark masses are not degenerate, and CP violation in the Kaon system is not suppressed by differences in Yukawa couplings as they appear in Eq. (1.2), but rather depends on ratios of Yukawa couplings and not on the small Yukawa couplings themselves. In this sense, CP violation in the Kaon system is a non-perturbative effect in the quark masses and hence does not need to be suppressed by the Jarlskog determinant [12, 14].

In Chapter 2 we construct an effective action by integrating out the fermions using the worldline method as presented in Ref. [15]. Worldline methods in first quantised quantum field theory are ideally adapted for calculating effective actions: One considers the propagation of a particle in some space-time dependent background [16], but in x-space path integral formulation [17]. This method [18], related to the infinite tension limit of String theory [19, 20], was used heavily for the discussion of various effective actions in one-loop [21–24] and two-loop [25–28] order. For example, the high order in the inverse mass calculation of ref. [27] could hardly be done with other methods. In our particular case it presents some computational advantages: it does not involve momentum integrations, avoids the handling of the γ matrix algebra, and can be implemented with a computer algebra program in a straightforward manner.

The action will be expanded in a covariant derivative expansion. In such an expansion, the level of truncation is set by the number of covariant indices. The fermion masses are treated non-perturbatively, but their derivatives are treated perturbatively. In Chapter 3 we present the CP violating operators in the SM. The resulting operators are found not to be suppressed by δ_{CP}, but by J only. Whether they are enough to produce the baryon asymmetry is still under investigation, but two important conclusions can be made. First, as already mentioned, they are only suppressed by J, not δ_{CP}, and second, CP violating terms appear also in the real part of the effective action, contrary to expectations in the literature. This technical point will be discussed in Chapter 3.

The last condition, departure from equilibrium, occurs within the SM at the electroweak phase transition [29], when the electroweak symmetry is broken. A strong first order phase transition, with $v_c/T_c \gtrsim 1$, is needed for electroweak baryogenesis in order to provide a fast freezing out of baryon number after generation in the hot phase. The symmetry is not broken everywhere at the same time, and in the case of a first order phase transition, it produces bubbles nucleating within the hot plasma. The bubbles expand, collide, and merge until they cover the observable universe. Non-equilibrium is achieved by interaction between particles and the moving wall separating regions with broken symmetry from regions with the symmetry restored. However, the strength of the phase transition, and whether it is first order or second order phase transition, or even a crossover is governed by the Higgs mass. The experimental lower bound on the Higgs mass implies that this transition is not strongly first order as required, but is rather a crossover [30]. Therefore, the introduction of new physics is required to produce the out of equilibrium condition.

Since the third condition, and possibly the second condition, for dynamically generating the asymmetry are not met within the SM, explanations of baryogenesis must incorporate physics beyond the SM. There have been several proposals to explain baryogenesis. For a review of some of them see [31]. A crucial matter is that many of them introduce physics at energy ranges that will not be available experimentally for quite some time, e.g. for the case of leptogenesis see [32]. While such limitations do not logically disprove a scenario, it does mean that a scenario which might be falsified sooner, rather than later, deserves great appeal.

One such class of scenarios propose that the electroweak transition occurred as a tachyonic transition at the end of low scale, inverted hybrid inflation [33, 34]. The effective Higgs mass parameter turns negative, not because of a change in temperature, but because of it's coupling to the inflaton field. The transition at the end of electroweak-scale inflation occurs at a temperature close to zero, hence the name: cold electroweak baryogenesis. With the only introduction being an inflaton field driving the zero temperature phase transition, the scenario is very close to the SM.

Given the previous case as motivation, in the first half of this work we consider CP violation in the case of effective theories. As mentioned before, the argument against CP violation being too weak within the SM is not conclusive. In lattice calculations for cold electroweak baryogenesis performed in [34], an assumed CP violating term of the type

$$\frac{3\delta_{CP}}{16\pi^2 M^2}\phi^\dagger\phi\mathrm{tr}\left(A^{\mu\nu}\tilde{A}_{\mu\nu}\right) \tag{1.7}$$

was used, where $A^{\mu\nu}$ is the $SU(2)$ field strength and $\tilde{A}_{\mu\nu}$ its dual. The term is assumed to originate as part of an effective action from higher energy physics, or as a result of integrating out the fermions from within the SM. Part of this work consists of following the second track. Consider the SM at low energies with gauge fields that are weak compared to the energy scale of the quark masses. If the fermionic degrees of freedom are integrated out, a purely bosonic theory describes the physics at low energies. In this case, the CP violation in the quark sector will eventually give rise to higher dimensional operators as first proposed in Ref. [35]. In the present work we will demonstrate that, different from the leading order case [35], in the next-to-leading order of the gradient expansion, the one-loop effective action indeed contains CP violation that exceeds the perturbative bound given in Eq. (1.2), even if they are not of the type shown in Eq. (1.7).

The second part of this thesis deals with baryon number violation. Baryon number is not conserved in electroweak theory because of the chiral anomaly, as mentioned previously. This results in the pure $SU(2)$ vacuum being a periodic structure labeled by an integer Chern-Simons (CS) winding number.

Baryon number violation is not exponentially suppressed in the hot (symmetric) phase, and this is the area of study that concerns us here. The rate of baryon number violation mentioned earlier is directly proportional to the sphaleron rate [36, 37]

$$\Gamma \equiv \lim_{V\to\infty} \lim_{t\to\infty} \frac{\langle (N_{\rm CS}(t) - N_{\rm CS}(0))^2 \rangle}{Vt}. \tag{1.8}$$

To a good approximation, the sphaleron rate in the hot phase can be calculated by working in pure Yang-Mills theory [38]. It is known that the infrared (soft) gauge fields with momenta $\sim g^2T$ are responsible for baryon number violation [7], but momenta of order gT and T play an important role in modifying the dynamics of the soft momenta [39, 40]. Interactions with hard modes lead to Debye screening of the soft non-abelian electric fields, and only the transverse fields determine the sphaleron rate. While static magnetic fields are unscreened, configurations contributing in Eq. 1.8 are not completely static. This leads to Landau damping effects that restrict the *frequency* scale that is relevant to order g^4T. Bödeker has derived an effective theory encoding the relevant physics by integrating out the field modes with momenta of order T and of order gT in a leading logarithmic approximation[41]. The sphaleron rate has been studied using lattice simulations [42]. However, such simulations are kind of a 'black box' that give the answer but hide the way *how* that answer comes about.

While it is possible to study the dynamics of the problem at order gT and T perturbatively, the scale which is most important to the problem at hand, g^2T, is nonperturbative. In this work we wish to provide a complementary, more analytic approach to the non-perturbative physics encoded in Bödeker's effective theory. The emphasis thereby lays not primarily on the accuracy of the results where it is hardly possible to beat the lattice calculations. Our aim is to provide a tool for a deeper understanding of what is really going on in the non-perturbative sector of hot non-abelian gauge theory and during creation of baryon number.

Bödeker's effective theory in its simplest version consists of the following Langevin equation for the $SU(2)$ gauge field with a stochastic driving force, and takes the form

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a = \boldsymbol{\zeta}^a, \tag{1.9}$$

which is written in $A_0 = 0$ gauge and where $\boldsymbol{\zeta}$ is a gaussian white noise stochastic force with the following normalisation

$$\left\langle \boldsymbol{\zeta}^{ai}(t,\mathbf{x}) \boldsymbol{\zeta}^{bj}(t',\mathbf{x}') \right\rangle = 2\sigma T \, \delta^{ij} \delta^{ab} \, \delta(t-t') \, \delta^{D-1}(\mathbf{x}-\mathbf{x}') \tag{1.10}$$

In equilibrium, i.e. the limit $t \to \infty$, the Euclidean probability distribution of the theory is simply the Yang-Mills probability distribution in 3 dimensions. The theory is both local and ultraviolet finite. For our analysis, we do not concern ourselves with initial conditions. If the initial conditions are specified at $t \to -\infty$, then at finite times the dependence of expectation values on them is destroyed by the stochastic force. Or rather, the dependence of the transversal component is damped by the stochastic force. The dependence of the longitudinal component is not damped, but the longitudinal component will drop from gauge invariant calculations. Nevertheless, the introduction of a term intended to raise the effective action from $A_0 = 0$ gauge to a more general flow gauge will also have the result of damping the longitudinal component.

Except for the noise term, which maintains thermal equilibrium, the form of the theory is that of the low frequency limit of the nonabelian Ampere's law in a conducting medium, one in which the constant of proportionality between the current and the nonabelian electric field is σ. The colour conductivity σ can be treated as a constant to next-to-leading order in $\log(1/g)$ because the mean free path for colour changing collisions is shorter than the $1/g^2T$ scale by a factor of $\log(1/g)$ [38].

From Bödeker's theory, an order of magnitude estimate of the sphaleron rate can be made. The spatial characteristic scale of the problem is $R \sim (g^2T)^{-1}$, while the temporal characteristic scale from Eq. (1.9) is $t \sim \sigma R^2 \sim \sigma/g^4T^2$. The colour conductivity is of order $\sigma \sim T/\ln(1/g)$ [43], yielding $t \sim (g^4T\ln(1/g))^{-1}$. Therefore, the sphaleron rate *per unit volume*, Eq. (1.8), has the characteristic size $\Gamma \sim 1/(R^3t) \sim \alpha^5T^4\ln(1/g)$. The α^5 dependence was first noted in [39], and the logarithmic dependence in [41]. It is the more quantitative determination of this relation which the current analysis is intended to lead to.

In Chapter 4 we develop the tools for the study mentioned above. Bödeker's effective theory will be translated into a quantum path integral, and non-perturbative equations will be derived from it. From stochastic quantisation it is known that a LANGEVIN equation can be recast in the form of a path integral [44–46]. This path integral is then reinterpreted as the functional integral formulation of an Euclidean quantum field theory with some given action. The ensuing quantum field theory will still be in $A_0 = 0$ gauge, and we proceed to generalise the result to a larger class of gauges, with a parameter κ determining the choice of gauge.

A direct translation to a path integral results in a complicated action, with interactions of up to order A^6. To reduce the order of the interactions, we introduce an auxiliary field λ, after which the order of the interactions is reduced. The tensor complexity of the vertices is also reduced allowing for tree-level vertices which depend solely on the spatial component of the momentum.

The effective theory thus translated into a path integral we obtain non-perturbative identities available to the theory. While gauge ghosts are not strictly necessary in stochastic quantisation, we introduce them to establish a gauge BRST symmetric formulation, and derive the corresponding WARD–TAKAHASHI identities. We also derive another set of nonperturbative identities, so called Stochastic-Ward identities, which reflect its origin in a stochastic differential equation. Because of the introduction of the auxiliary field, the theory has a much larger variety of different N-point functions compared to regular Yang-Mills theory. Ghost number conservation also helps in reducing that number.

We then proceed to derive the Dyson-Schwinger equations (DSEs) of the theory, offering an approach to the non-perturbative sector that is independent from, and complementary to, the existing lattice studies.

In Chapter 5 we begin a study of the infrared behaviour of Bödeker's theory. In a similar manner to regular Yang-Mills theory, we look for a scaling solution to the DSEs. Because the study of the infrared behaviour of Yang-Mills theory is well advanced (see e.g. [48] and references therein), we will try to draw as much inspiration as possible from those studies. However, there are several differences between the present theory and regular Yang-Mills theory. The first problem we encounter is that, as opposed to the regular Yang-Mills case, Lorentz invariance is broken and we basically have two 'variables': k_0 and $\mathbf{k}$. At the very least, this makes presenting a suitable ansatz more involved.

We start with a very simple ansatz, which involves setting the vertex functions to their tree-level versions and introducing an anomalous dimension to the Feynman propagators found in Appendix C. The self-energies are obtained in the form of an appropriate scaling function multiplied by a scale invariant infinite series. The series introduces an error, which for the simple case, means the ansatz is completely inconsistent.

A generalisation of the previous case is introduced by allowing for the 'coupling' between k_0 and $\mathbf{k}$ to be a parameter to be determined in the analysis, in essence, an effective colour conductivity. With this new ansatz, a suitable value of the parameters is found such that the biggest error introduced by the failure of the chosen full propagators to be fully consistant is tiny.

Chapter 2

Worldline Method

We start by constructing an effective action for a general chiral model. Because of the presence of the chiral anomaly, the action is separated into a real and imaginary part, with the imaginary part containing the chiral anomaly. The imaginary part will furthermore be split into two terms: one which will saturate the anomaly, the so-called Wess-Zumino-Witten (WZW) term, and one which will contain the remaining chiral invariant terms. Once a master formula for each part of the action is presented, there are many possible expansions that could be done, each ordering terms according to a particular rule. In the current work, a type of gradient expansion will be undertaken. The order of the expansion will be set by the number of covariant derivatives present. The expansion therefore maintains gauge invariance at each order, and because the anomalous part has been separated as well, the remaining part maintains chiral invariance as well.

The fields entering the effective action are assumed throughout this chapter to have a general internal structure. In particular, we don't assume the fields to belong to any particular gauge group, and the only restriction will be that the mass-like term be nowhere singular. We do an strict covariant derivative expansion, and since the mass term carries no covariant index, there is no expansion in terms of it. This means that those fields will be considered to all orders. In effect, the calculation will be non-perturbative in the mass term, albeit it will treat its covariant derivatives perturbatively. In order to obtain a close form for the action, we will use the label operator notation. Each mass term will acquire an index indicating where on a particular expression they appear. In effect, they become c-numbers, and this allows for the calculation of the ensuing integrals.

The effective action will constructed using the worldline method. The construction of the real part of the action proceeds in a straightforward manner, and follows [49, 50]. Since the chiral anomaly is contained in the imaginary part of the action, either Lorentz or chiral covariance must be broken. In [50], the imaginary part of the action was constructed by explicitly breaking chiral covariance. Here we will follow the approach laid down in [51]. As mentioned above, the WZW term will saturate the anomaly, and will allow for the construction of an invariant remainder. The invariant remainder will be calculated by matching it to the effective current, which can be expressed as a positive operator, a requirement of the worldline method.

After the construction of the master formulas, the effective current in two and four dimensions is calculated, and the imaginary part of the effective action for the general chiral model is obtained at leading and next-to-leading order.

2.1 Effective Action

We consider a fermion action with the following fields: scalar Φ, pseudoscalar Π, vector A, pseudovector B, and antisymmetric tensor K. We make no assumption on the possible internal matrix structure that these fields can have, except for the restriction that the mass-like term

$\Phi + i\Pi$ cannot vanish. We are concerned then with the following effective action

$$i\,W[\Phi,\Pi,A,B,K] = \log \mathrm{Det}\, i\,[\,i\,\partial\!\!\!/ - \Phi + i\,\gamma^5\Pi + A\!\!\!/ + \gamma^5 B\!\!\!/ + i\,\gamma^\mu\gamma^\nu K_{\mu\nu}], \tag{2.1}$$

and its continuation to Euclidean space. The γ matrices remain unaffected by the continuation, but it is useful to introduce the following notation, $(\gamma_E)_j \equiv i\gamma_j$, $(\gamma_E)_4 \equiv \gamma_0$, and $(\gamma_E)_5 \equiv \gamma_5$. After Wick-rotation, $t \to -it$, one obtains with this new notation

$$\partial\!\!\!/ \to i\partial\!\!\!/_E,\; A\!\!\!/ \to iA\!\!\!/_E,\; B\!\!\!/ \to iB\!\!\!/_E,\; \gamma^\mu\gamma^\nu K_{\mu\nu} \to -(\gamma_E)_\mu(\gamma_E)_\nu K_{E\mu\nu}. \tag{2.2}$$

From now on, the E subscript will be suppressed.

The effective action of Eq. (2.1) now reads

$$-W[\Phi,\Pi,A,B,K] = \log \mathrm{Det}\,[\mathcal{O}], \tag{2.3}$$

with the operator $\mathcal{O}$ in momentum space defined by

$$\mathcal{O} \equiv p\!\!\!/ - i\Phi(x) - \gamma_5\Pi(x) - A\!\!\!/(x) - \gamma_5 B\!\!\!/(x) + \gamma_\mu\gamma_\nu K_{\mu\nu}. \tag{2.4}$$

As in Ref. [49, 51], the real and imaginary parts of the effective action are analysed separately

$$-W^+ - i\,W^- = \log\left(|\mathrm{Det}\,[\mathcal{O}]|\right) + i\,\arg\left(\mathrm{Det}\,[\mathcal{O}]\right). \tag{2.5}$$

A perturbative expansion in weak fields [49] shows that graphs with an even number of γ_5 vertices are real, and graphs with an odd number of γ_5 vertices are imaginary. This will prove useful when the behaviour of the effective action under complex conjugation is explored later on.

2.1.1 Real Part of the Effective Action

Our intention is to obtain a worldline representation for the effective action with manifest chiral and gauge invariance. This is unproblematic for the real part, but it causes certain difficulties for the imaginary part due to the chiral anomaly. In order to familiarise the reader with the worldline method we review the derivation for the real part in four dimensions as it was presented in [49, 50] after previous work in [52].

Construction of a Positive Operator for the Real Part of the Effective Action

In order to use the worldline formalism, one has to rewrite the effective action in terms of a positive operator. This can be done in the following manner

$$W^+ = -\frac{1}{2}\log \mathrm{Det}\,[\mathcal{O}^\dagger\mathcal{O}]. \tag{2.6}$$

The problem with this operator is that it contains terms linear in the γ matrices, what makes the transition to a path integral of Grassman fields problematic. One way to avoid this problem is by doubling the fermion system and exchanging the operator $\mathcal{O}$ for a Hermitian operator Σ yielding

$$W^+ = -\frac{1}{2}\log \mathrm{Det}\,[\mathcal{O}^\dagger\mathcal{O}] = -\frac{1}{4}\log \mathrm{Det}\,[\Sigma^2], \qquad \Sigma \equiv \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O}^\dagger & 0 \end{pmatrix}. \tag{2.7}$$

Since Σ is Hermitian, one can use the Schwinger integral representation of the logarithm without any restrictions,

$$W^+ = \frac{1}{4}\int_0^\infty \frac{dT}{T}\mathrm{Tr}\,\exp(-T\Sigma^2). \tag{2.8}$$

At this point, it is natural to introduce six 8×8 Hermitian Γ_A matrices. These matrices satisfy $\{\Gamma_A,\Gamma_B\} = 2\delta_{AB}$, with $A, B = 1..6$ and are defined as

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & i\,\mathbb{1}_4 \\ -i\,\mathbb{1}_4 & 0 \end{pmatrix}. \tag{2.9}$$

For later use in the imaginary part we also introduce the equivalent of γ_5,

$$\Gamma_7 = -i \prod_{A=1}^{6} \Gamma_A = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \tag{2.10}$$

and Γ_7 anticommutes with all other Γ matrices.

Expressing Σ in terms of these new matrices yields [49],

$$\Sigma = \Gamma_\mu(p_\mu - A_\mu) - \Gamma_6\Phi - \Gamma_5\Pi - i\,\Gamma_\mu\Gamma_5\Gamma_6 B_\mu - i\,\Gamma_\mu\Gamma_\nu\Gamma_6 K_{\mu\nu}. \tag{2.11}$$

The aim is to turn Eq. (2.11) into an expression which is manifestly chiral invariant. This can be achieved by changing to a basis in which $i\,\Gamma_5\Gamma_6$ is diagonal [49] using the following transformation

$$M^{-1} i\Gamma_5\Gamma_6 M = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \qquad M = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_2 \\ 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

In this basis, Σ takes the form

$$\tilde{\Sigma} = M^{-1}\Sigma M = \begin{pmatrix} \gamma_\mu(p_\mu - A^L_\mu) & \gamma_5(-i\,H + \frac{1}{2}\gamma_\mu\gamma_\nu K^s_{\mu\nu}) \\ -\gamma_5(-i\,H^\dagger + \frac{1}{2}\gamma_\mu\gamma_\nu K^{s\dagger}_{\mu\nu}) & \gamma_\mu(p_\mu - A^R_\mu) \end{pmatrix}, \tag{2.13}$$

which is manifestly chiral invariant. Here $A^L = A + B$, $A^R = A - B$, $H = \Phi - i\,\Pi$, $K^s = K - i\,\tilde{K}$ and $\tilde{K}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}K^{\rho\sigma}$ have been defined.

The square of $\tilde{\Sigma}$ constitutes a positive operator which is suitable for the worldline formalism. However, even though this expression contains only even combinations of γ matrices, the coherent state formalism cannot yet be used to transform this expression into a fermionic path integral. In the coherent state formalism, the γ_5 matrices have to be rewritten as a product of the other γ matrices, what would result again in odd combinations. One possible solution of this problem is to enlarge the Clifford space, replacing the γ matrices by Γ matrices

$$\gamma_A \to \Gamma_A = \gamma_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A \in [1 \ldots 5]. \tag{2.14}$$

The matrix Γ_5 is then independent from the other Γ matrices and the coherent state formalism with six (instead of the normal four) operators can be used. The doubling of the Clifford space inside the trace has to be compensated by a factor $\frac{1}{2}$, such that Eq. (2.8) reads

$$W^+ = \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathrm{Tr}\,\exp(-T\hat{\Sigma}^2), \tag{2.15}$$

and the operator $\hat{\Sigma}^2$ is given by

$$\begin{aligned} \hat{\Sigma}^2 = (p - \mathcal{A})^2 + \mathcal{H}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}_{\mu\nu} + \frac{i}{2}\Gamma_\mu\Gamma_\nu(\mathcal{F}_{\mu\nu} + \{\mathcal{H}, \mathcal{K}_{\mu\nu}\} + i\,[\mathcal{K}_{\mu\rho}, \mathcal{K}_{\rho\nu}]) \\ + i\,\Gamma_\mu\Gamma_5(\mathcal{D}_\mu\mathcal{H} + \{p_\nu - \mathcal{A}_\nu, \mathcal{K}_{\mu\nu}\}) - \frac{1}{2}\Gamma_{\mu\rho\sigma}\Gamma_5\mathcal{D}_\mu\mathcal{K}_{\rho\sigma} - \frac{1}{4}\Gamma_{\mu\nu\rho\sigma}\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma}, \end{aligned} \tag{2.16}$$

with enlarged background fields defined by

$$\mathcal{A}_\mu = \begin{pmatrix} A^L_\mu & 0 \\ 0 & A^R_\mu \end{pmatrix}, \; \mathcal{H} = \begin{pmatrix} 0 & i\,H \\ -i\,H^\dagger & 0 \end{pmatrix}, \; \mathcal{K}_{\mu\nu} = \begin{pmatrix} 0 & i\,K^s_{\mu\nu} \\ -i\,K^{s\dagger}_{\mu\nu} & \end{pmatrix}. \tag{2.17}$$

$\Gamma_{A_1 \ldots A_k} \equiv \Gamma_{[A_1} \ldots \Gamma_{A_k]}$ denotes the anti-symmetrised product of k Γ matrices, and the field-strength and the covariant derivative have been defined as

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu], \qquad \mathcal{D}_\mu\chi = \partial_\mu\chi - i\,[\mathcal{A}_\mu, \chi]. \tag{2.18}$$

The $\hat{\Sigma}^2$ operator is seen to be manifestly gauge and chiral invariant. It also contains Γ matrices to even powers only, and is well suited for the worldline path integral representation.

Worldline Path Integral

With the use of the coherent state formalism [49, 53], one can perform the transition from Γ matrices to a path integral over Grassman fields ψ, with the correspondence $\Gamma_A\Gamma_B \to 2\psi_A\psi_B$ and $\Gamma_A\Gamma_B\Gamma_C\Gamma_D \to 4\psi_A\psi_B\psi_C\psi_D$, as long as A, B, C, and D are all different. The final form for the real part of the effective action is

$$W^+ = \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x \int_{AP} \mathcal{D}\psi \,\mathrm{tr}\,\mathcal{P}e^{-\int_0^T d\tau \mathcal{L}(\tau)}. \tag{2.19}$$

Here $\mathcal{N}$ denotes a normalisation constant coming from a momentum integration and AP stands for antiperiodic boundary conditions, which must be fulfilled by the Grassman variables $\psi(T) = -\psi(0)$. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(\tau) =& \frac{\dot{x}^2}{4} + \frac{1}{2}\psi_A\dot{\psi}_A - i\,\dot{x}_\mu\mathcal{A}_\mu + \mathcal{H}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}_{\mu\nu} + 2i\,\psi_\mu\psi_5\left(\mathcal{D}_\mu\mathcal{H} + i\,\dot{x}_\nu\mathcal{K}_{\mu\nu}\right)\\ &+ i\,\psi_\mu\psi_\nu\left(\mathcal{F}_{\mu\nu} + \{\mathcal{H},\mathcal{K}_{\mu\nu}\} + i\,[\mathcal{K}_{\mu\rho},\mathcal{K}_{\rho\nu}]\right) - \psi_\mu\psi_\nu\psi_\rho\left(2\psi_5\mathcal{D}_\mu\mathcal{K}_{\mu\nu} + \psi_\sigma\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma}\right).\end{aligned} \tag{2.20}$$

For the construction of the Green's function for the field $x(\tau)$, the zero modes of the free field operator $\frac{d^2}{d\tau^2}$ must be separated. The fields $x(\tau)$ are split into a constant part and a τ dependent part according to $x(\tau) = x_0 + y(\tau)$, with $\partial_\tau x_0 = 0$ and $\int_0^T d\tau\, y(\tau) = 0$, and the measure in the integral is changed into $\mathcal{D}x = \mathcal{D}y\, d^D x_0$. The Green function is then defined on a subspace orthogonal to the zero modes. The ψ_A fields contain no zero modes since they fulfil antiperiodic boundary conditions. The propagators for the $y(\tau)$ and $\psi_A(\tau)$ fields read

$$\begin{aligned}\langle y(\tau_1)y(\tau_2)\rangle &= \frac{(\tau_1-\tau_2)^2}{T} - |\tau_1-\tau_2|,\\ \langle \psi_A(\tau_1)\psi_B(\tau_2)\rangle &= \frac{1}{2}\delta_{AB}\mathrm{sign}\,(\tau_1-\tau_2)\,.\end{aligned} \tag{2.21}$$

This formalism can then be used to determine the real part of the effective action as discussed in Ref. [50].

2.1.2 Imaginary Part of the Effective Action

As in the case of the real part of the effective action, one requires a positive operator in order to use the Schwinger trick. Even though this is still possible for the imaginary part, gauge and chiral covariance cannot be manifestly conserved due to the chiral anomaly. For example, in Ref. [49] a parameter α is introduced, which breaks the chiral covariance, but leads to a positive operator. However the resulting expression is not appropriate for higher order calculations since the breaking of manifest chiral covariance leads to a large number of contributions in the perturbative expansion of the path integral.

Instead, we present a worldline representation of the effective current for which a manifestly chiral covariant expression exists. This current can then be integrated to obtain the effective action [51, 54]. The integration rather proceeds by matching: First, an ansatz for the effective action is proposed, which has the expected chiral and Lorentz properties. The functional variation of this action is then matched to the covariant current that is obtained using the worldline formalism. This method has the advantage that it is both gauge and chiral covariant at each stage of the calculation. This simplifies higher order calculations tremendously as compared to the formalism presented in [50]. The anomaly only leads to additional complications in the matching procedure of the lowest order contributions as will be discussed in detail in the next section.

Starting point of our analysis is the functional derivative of the imaginary part of the effective action in Eq. (2.5)

$$\delta W^- = \frac{1}{2}\delta\left(\log \mathrm{Det}\,\mathcal{O} - \log \mathrm{Det}\,\mathcal{O}^\dagger\right) = \frac{1}{2}\mathrm{Tr}\left(\delta\mathcal{O}\frac{1}{\mathcal{O}} - \delta\mathcal{O}^\dagger\frac{1}{\mathcal{O}^\dagger}\right). \tag{2.22}$$

This expression can be rewritten in terms of a positive operator that can then employ the worldline representation in combination with the heat kernel formula.

Construction of a Positive Operator for the Imaginary Part of the Effective Action

The expression in Eq. (2.22) can be transformed using the operator Σ defined in Eq. (2.7)

$$\delta W^- = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} 0 & \delta\mathcal{O} \\ -\delta\mathcal{O}^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/\mathcal{O}^\dagger \\ 1/\mathcal{O} & 0 \end{pmatrix}, \tag{2.23}$$

which, with the introduction of a new matrix χ, can be rewritten as

$$\delta W^- = \frac{1}{2}\mathrm{Tr}\,\chi\delta\Sigma\Sigma^{-1}, \tag{2.24}$$

with

$$\Sigma = \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O}^\dagger & 0 \end{pmatrix}, \qquad \chi = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}. \tag{2.25}$$

To produce the positive definite operator Σ^2 in Eq. (2.24), we multiply and divide by Σ, using the cyclic property of the trace and the fact that Σ anticommutes with χ, to obtain

$$\begin{aligned}\delta W^- &= \frac{1}{4}\mathrm{Tr}\,(\chi\delta\Sigma\Sigma + \Sigma\chi\delta\Sigma)\,\Sigma^{-2} \\ &= \frac{1}{4}\mathrm{Tr}\,\chi\,[\delta\Sigma,\Sigma]\,\Sigma^{-2}. \end{aligned} \tag{2.26}$$

Since the last factor is a positive operator, it can be reexpressed as an integral, similar to the expression of the real part of the effective action in Eq. (2.16), namely

$$\delta W^- = \frac{1}{4}\mathrm{Tr}\int_0^\infty dT\chi\,[\delta\Sigma,\Sigma]\,e^{-T\Sigma^2}. \tag{2.27}$$

As in the case of the real part, the chiral covariance can be made manifest by changing to an appropriate basis. With the help of the matrix M in Eq. (2.12), one obtains again

$$\tilde{\Sigma} = \gamma_\mu(p_\mu - \mathcal{A}_\mu) - \gamma_5\mathcal{H} - \frac{i}{2}\gamma_\mu\gamma_\nu\gamma_5\mathcal{K}_{\mu\nu}. \tag{2.28}$$

The additional factors $\chi\,[\delta\Sigma,\Sigma]$ read

$$M^{-1}\chi M = \tilde{\chi} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix} = \chi\,\gamma_5, \tag{2.29}$$

and for the case $\delta\tilde{\Sigma} = -\gamma_\mu\delta\mathcal{A}_\mu$

$$\begin{aligned}\left[\delta\tilde{\Sigma},\tilde{\Sigma}\right] &= -\gamma_{\mu\nu}\{\delta\mathcal{A}_\mu, p_\nu - \mathcal{A}_\nu\} - i\,\mathcal{D}_\mu\delta\mathcal{A}_\mu - \gamma_5\gamma_\mu\{\delta\mathcal{A}_\mu,\mathcal{H}\} \\ &\quad + i\,\gamma_5\gamma_\mu[\delta\mathcal{A}_\nu,\mathcal{K}_{\mu\nu}] - \frac{i}{2}\gamma_5\gamma_{\mu\lambda\sigma}\{\delta\mathcal{A}_\mu,\mathcal{K}_{\lambda\sigma}\}. \end{aligned} \tag{2.30}$$

To use the coherent state formalism, it is again necessary to enlarge the Clifford algebra and to replace the γ matrices by Γ matrices. However, taking into account the factor γ_5 in Eq. (2.29) the imaginary part of the effective action contains only odd combinations of γ matrices. Thus, the replacement

$$\gamma_A \to \Gamma_A = \gamma_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A \in [1\ldots 5] \tag{2.31}$$

has to be compensated by a factor

$$-\frac{i}{2}\Gamma_7\Gamma_6 = \mathbb{1}_4 \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{2.32}$$

The overall factor Γ_7 changes the boundary condition of the fermionic sector from antiperiodic to periodic [49]. This means that the fermionic sector contains zero modes, which have to be separated in the same way as was done for the bosonic sector.

Including the factor in Eq. (2.32) to compensate for the doubling of the Clifford space, one obtains

$$\delta W^- = \frac{i}{8}\mathrm{Tr}\int_0^\infty dT\Gamma_7\Gamma_6\chi w(T)e^{-T\hat{\Sigma}^2}, \tag{2.33}$$

where $\hat{\Sigma}^2$ is given in Eq. (2.16), and the insertion due to the commutator yields

$$\begin{aligned} w(T) &= -\frac{1}{2}\Gamma_5\Gamma_{\mu\nu}\{\delta\mathcal{A}_\mu, p_\nu - \mathcal{A}_\nu\} - i\,\Gamma_5\mathcal{D}_\mu\delta\mathcal{A}_\mu - \Gamma_\mu\{\delta\mathcal{A}_\mu,\mathcal{H}\} \\ &\quad + i\,\Gamma_\mu[\delta\mathcal{A}_\nu,\mathcal{K}_{\mu\nu}] - \frac{i}{2}\Gamma_{\mu\lambda\sigma}\{\delta\mathcal{A}_\mu,\mathcal{K}_{\lambda\sigma}\}. \end{aligned} \tag{2.34}$$

To transform this expression into a worldline path integral, a similar procedure as for the real part of the effective action can be followed. Products of Γ matrices can be replaced by Grassman fields, however in this case the Jacobian of the transformation contains additional contributions from the zero modes

$$\begin{aligned} \mathcal{D}\theta\mathcal{D}\bar{\theta} &\equiv d\theta_3 d\theta_2 d\theta_1 d\bar{\theta}_1 d\bar{\theta}_2 d\bar{\theta}_3 \mathcal{D}\theta'\mathcal{D}\bar{\theta}' \\ &= \frac{1}{J}d\psi_1^0 d\psi_2^0 d\psi_3^0 d\psi_4^0 d\psi_5^0 d\psi_6^0 \mathcal{D}\psi'. \end{aligned} \tag{2.35}$$

The factor J only includes the Jacobian for the zero modes, while the Jacobian for the orthogonal modes is absorbed in the normalisation of the correlation functions of the ψ'_A. J can be calculated from the definition of the Grassman fields ψ in the coherent state formalism [49] and yields in D dimension

$$J = \det\left(\frac{\partial\theta,\bar{\theta}}{\partial\psi}\right) = (-i)^{(D+2)/2}. \tag{2.36}$$

The final result can be expressed as

$$\delta W^- = \frac{1}{8}\,\mathrm{tr}\int_0^\infty dT\mathcal{N}\int\mathcal{D}x\int_P\mathcal{D}\psi\,\chi w(T)\mathcal{P}e^{-\int_0^T d\tau\mathcal{L}(\tau)}. \tag{2.37}$$

The Lagrangian takes the same form as in the real part, Eq. (2.20),

$$\begin{aligned} \mathcal{L}(\tau) =& \frac{\dot{x}^2}{4} + \frac{1}{2}\psi_A\dot{\psi}_A - i\,\dot{x}_\mu\mathcal{A}_\mu + \mathcal{H}^2 + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}_{\mu\nu} + 2i\,\psi_\mu\psi_5(\mathcal{D}_\mu\mathcal{H} + i\,\dot{x}_\nu\mathcal{K}_{\mu\nu}) \\ &+ i\,\psi_\mu\psi_\nu(\mathcal{F}_{\mu\nu} + \{\mathcal{H},\mathcal{K}_{\mu\nu}\} + i\,[\mathcal{K}_{\mu\rho},\mathcal{K}_{\rho\nu}]) - \psi_\mu\psi_\nu\psi_\rho(2\psi_5\mathcal{D}_\mu\mathcal{K}_{\mu\nu} + \psi_\sigma\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma}). \end{aligned} \tag{2.38}$$

and the trivial integration over ψ_6 can be carried out, so that the insertion yields

$$\begin{aligned} w(T) &= -4i\,\psi_5\psi_\mu\psi_\nu\delta\mathcal{A}_\mu\dot{x}_\nu - 2i\,\psi_5\mathcal{D}_\mu\delta\mathcal{A}_\mu - 2\psi_\mu\{\delta\mathcal{A}_\mu,\mathcal{H}\} \\ &\quad + 2i\,\psi_\mu[\delta\mathcal{A}_\nu,\mathcal{K}_{\mu\nu}] - 2i\,\psi_\mu\psi_\lambda\psi_\sigma\{\delta\mathcal{A}_\mu,\mathcal{K}_{\lambda\sigma}\}. \end{aligned} \tag{2.39}$$

The normalisation $\mathcal{N}$ coming from the momentum integration, satisfies

$$\mathcal{N}\int\mathcal{D}x e^{-\int_0^T d\tau\frac{\dot{x}}{4}} = (4\pi T)^{-D/2}\int d^D x. \tag{2.40}$$

The Green function for the bosonic field x is the same as for the real part of the effective action, Eq. (2.21), while the Green function of the Grassman fields ψ_A differs due to the presence of the zero modes. The fermionic fields are split according to $\psi_A(\tau) = \psi_A^0 + \psi_A^{'}(\tau)$, with $\partial_\tau\psi_A^0 = 0$ and $\int_0^T d\tau\psi_A^{'}(\tau) = 0$ and the measure turns into $\mathcal{D}\psi = d\psi_1 d\psi_2 d\psi_3 d\psi_4 d\psi_5 \mathcal{D}\psi'$. The Green function for the $\psi_A^{'}$ fields, defined on a space orthogonal to the zero modes, reads

$$\left\langle\psi_A^{'}(\tau_1)\psi_B^{'}(\tau_2)\right\rangle = \delta_{AB}\left(\frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{(\tau_1-\tau_2)}{T}\right). \tag{2.41}$$

These results can be easily generalised to different dimensions. In two dimension, one obtains an additional overall factor $-i$ from the Jacobian of the zero modes and the fermionic measure reads $\mathcal{D}\psi = d\psi_1 d\psi_2 d\psi_5 \mathcal{D}\psi'$.

The Effective Density

The effective density is obtained by varying with respect to the $\mathcal{H}$ field, so that $\delta\tilde{\Sigma} = -\gamma_5\delta\mathcal{H}$. In comparison to the worldline representation of the covariant current only the insertion changes into

$$\left[\delta\tilde{\Sigma}, \tilde{\Sigma}\right] = -\gamma_5\gamma_\mu\left\{\delta\mathcal{H}, p_\mu - \mathcal{A}_\mu\right\} + \left[\delta\mathcal{H}, \mathcal{H}\right] + \frac{i}{2}\gamma_\mu\gamma_\nu\left[\delta\mathcal{H}, \mathcal{K}_{\mu\nu}\right]. \tag{2.42}$$

The corresponding insertion $w(T)$ in the path integral reads then

$$w(T) = -2i\,\psi_\mu\dot{x}_\mu\delta\mathcal{H} + 2\psi_5\left[\delta\mathcal{H}, \mathcal{H}\right] + 2i\,\psi_\mu\psi_\nu\left[\delta\mathcal{H}, \mathcal{K}_{\mu\nu}\right]. \tag{2.43}$$

Since $\delta\mathcal{A}$ carries an index, the effective current is of one order lower than the effective density and usually results in less terms to calculate. The advantage of the effective density lies in the matching process, since the factors in the effective density consist of the same type as found in the effective action. They both combine the same type of objects, $\mathcal{D}\mathcal{H}$ and $\mathcal{F}$, to the same kind of order, while the effective current combines the terms to a lower order. Besides, there is no distinction between a consistent effective density and a covariant effective density, as there is for the effective current, as will be explained in the next section.

Distinction between the Consistent and the Covariant Current

With Eq. (2.37) an expression for the covariant current which is chiral and gauge covariant was derived. This current cannot be the variation of the effective action, since the effective action contains the chiral anomaly. In fact the covariant current is not a variation of any action. The reason for this is that performing the variation does not commute with the regularisation procedure we used, namely the Schwinger trick. On the other hand, knowing the chiral anomaly, one can reproduce the so-called consistent current that denotes the true variation of the effective action.

To explain the relation between the two currents, we define a general variation

$$\delta_Y = \int dx\, Y_\mu^a(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}, \tag{2.44}$$

so that a gauge variation δ_ξ is given by

$$\delta_\xi = \int dx\,\left(\mathcal{D}_\mu\xi\right)(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}. \tag{2.45}$$

Two subsequent variations have then the commutator $[\delta_Y, \delta_\xi] = \delta_{[Y,\xi]}$ and in order to find the transformation properties of the consistent current, one can apply this commutator to the effective action

$$[\delta_Y, \delta_\xi]W^-[\mathcal{A}_\mu] = \delta_{[Y,\xi]}W^-[\mathcal{A}_\mu]. \tag{2.46}$$

Using the anomalous Ward identity [55]

$$\delta_\xi W^-[\mathcal{A}_\mu] = \int dx\,\xi(x)G[\mathcal{A}_\mu](x), \tag{2.47}$$

with $G[\mathcal{A}_\mu]$ denoting the consistent anomaly, one can evaluate both sides of Eq. (2.46) to obtain

$$\begin{aligned}\int dx\,[Y_\mu, \xi](x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}W^-[\mathcal{A}_\mu] \quad &= \quad \delta_Y\int dx\,\xi(x)G[\mathcal{A}_\mu](x)\\ &\quad -\delta_\xi\int dx\,Y_\mu(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}W^-[\mathcal{A}_\mu].\end{aligned} \tag{2.48}$$

Defining the consistent current as the variation of the effective action

$$\langle j^\mu(x)\rangle = \frac{\delta}{\delta\mathcal{A}_\mu(x)}W^-[\mathcal{A}_\mu], \tag{2.49}$$

one finds

$$\int dx\, Y_\mu(x)\delta_\xi\langle j^\mu(x)\rangle = \int dx\, Y_\mu[\langle j^\mu(x)\rangle,\xi](x) + \int dx\, \xi(x)\delta_Y G[\mathcal{A}_\mu](x). \tag{2.50}$$

Since Y was a general variation this leads to

$$\delta_\xi\langle j^\mu(x)\rangle = [\langle j^\mu(x)\rangle,\xi] + \int dy\, \xi^b(y)\frac{\delta}{\delta\mathcal{A}_\mu(x)}G[\mathcal{A}_\mu](y). \tag{2.51}$$

This shows that only if the anomaly vanishes, the current transforms covariantly. This relation can be used to determine the connection between the consistent current, i.e. the true variation of the action, and the covariant current. The latter is obtained by adding an object $P^\mu[\mathcal{A}_\mu]$, the so-called Bardeen-Zumino (BZ) polynomial [56], to the consistent current so that the sum transforms covariantly

$$\langle \bar{j^\mu}\rangle = \langle j^\mu\rangle + \langle P^\mu\rangle. \tag{2.52}$$

This implies the following gauge transformation property for the BZ polynomial

$$\delta_\xi P^\mu[\mathcal{A}_\mu](x) = [P^\mu[\mathcal{A}_\mu],\xi]\,(x) - \int dy\, \xi(y)\frac{\delta}{\delta\mathcal{A}_\mu(x)}G[\mathcal{A}_\mu](y). \tag{2.53}$$

It can be proven that such an object exists. Using

$$P^\mu[\mathcal{A}_\mu] = \frac{1}{48\pi^2}\epsilon^{\mu\nu\lambda\sigma}\mathrm{tr}\chi\left(\mathcal{A}_\nu\mathcal{F}_{\lambda\sigma} + \mathcal{F}_{\lambda\sigma}\mathcal{A}_\nu + i\,\mathcal{A}_\nu\mathcal{A}_\lambda\mathcal{A}_\sigma\right), \tag{2.54}$$

and the consistent anomaly [55]

$$G[\mathcal{A}_\mu] = \frac{1}{24\pi^2}\epsilon^{\mu\nu\lambda\sigma}\mathrm{tr}\,\chi\,\partial_\mu\left(\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma - \frac{i}{2}\mathcal{A}_\nu\mathcal{A}_\lambda\mathcal{A}_\sigma\right), \tag{2.55}$$

it can be seen that the definition of P^μ in Eq. (2.54) provides a unique polynomial in $\mathcal{A}_\mu$ that satisfies Eq. (2.53). The corresponding functions in two dimensions are given by

$$P^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu}\mathrm{tr}\mathcal{A}_\nu, \quad G[\mathcal{A}_\mu] = \frac{1}{4\pi}\epsilon^{\mu\nu}\mathrm{tr}\,\chi\,\partial_\mu\,\mathcal{A}_\nu. \tag{2.56}$$

As stated above, the path integral in Eq. (2.37) constitutes a worldline representation of the covariant current. To obtain the imaginary part of the effective action from the covariant current one can use the following ansatz

$$W^- = \Gamma_{gWZW} + W_c^-. \tag{2.57}$$

Here, Γ_{gWZW} is an extended gauged Wess-Zumino-Witten action [54, 57, 58], which is chosen to reproduce the correct chiral anomaly, and W_c^- denotes the chiral invariant remainder. The variation of the functional Γ_{gWZW}, consists of a part that saturates the anomaly, namely the BZ polynomial, and a covariant remainder which has to be added to the variation of W_c^- to yield the covariant current.

The Wess-Zumino-Witten action

When the effective action is separated into two parts, it is required of the non-covariant part that it reproduces the anomaly. It is well known that the WZW action has this property.

The ungauged WZW action in four dimension is of the form

$$\Gamma(\mathbf{U}) = \frac{i}{48\pi^2}\int_Q d^5x\,\epsilon^{abcde}\mathrm{tr}\left[\frac{1}{5}\mathbf{U}^{-1}\partial_a\mathbf{U}\mathbf{U}^{-1}\partial_b\mathbf{U}\mathbf{U}^{-1}\partial_c\mathbf{U}\mathbf{U}^{-1}\partial_d\mathbf{U}\mathbf{U}^{-1}\partial_e\mathbf{U}\right], \tag{2.58}$$

where Q is a five-dimensional space with boundary ∂Q equal to the R^4 flat Euclidean space. The matrix $\mathbf{U}$ is a unitary matrix, and is usually related to the case where the mass can be expressed as a constant times that unitary matrix. We are interested in the more general case where the mass matrix is not of this form which is called extended WZW action. In addition, the presence of the background gauge fields makes a gauging of the action mandatory. The gauged extended WZW action can be generally expressed as the integral in five dimensions [54]. Unlike the action itself, the resulting current turns out to be a total derivative in five dimensions, such that it can be represented by an integral over the physical four-dimensional space

$$\begin{aligned}\delta\Gamma_{gWZW} &= \frac{1}{96\pi^2}\int d^4x\,\epsilon^{\mu\nu\lambda\sigma}\mathrm{tr}\,\chi\left[\delta\mathcal{A}_\mu\left(-\mathcal{H}^{-1}\mathcal{D}_\nu\mathcal{H}\mathcal{H}^{-1}\mathcal{D}_\lambda\mathcal{H}\mathcal{H}^{-1}\mathcal{D}_\sigma\mathcal{H}\right.\right.\\ &\quad+\mathcal{D}_\nu\mathcal{H}\mathcal{H}^{-1}\mathcal{D}_\lambda\mathcal{H}\mathcal{H}^{-1}\mathcal{D}_\sigma\mathcal{H}\mathcal{H}^{-1} - i\left\{\mathcal{H}^{-1}\mathcal{D}_\nu\mathcal{H}-\mathcal{D}_\nu\mathcal{H}\mathcal{H}^{-1},\mathcal{F}_{\lambda\sigma}\right\}\\ &\quad+\frac{i}{2}\mathcal{H}\left\{\mathcal{H}^{-1}\mathcal{D}_\nu\mathcal{H},\mathcal{F}_{\lambda\sigma}\right\}\mathcal{H}^{-1}-\frac{i}{2}\mathcal{H}^{-1}\left\{\mathcal{D}_\nu\mathcal{H}\mathcal{H}^{-1},\mathcal{F}_{\lambda\sigma}\right\}\mathcal{H}\\ &\quad\left.\left.-2\left\{\mathcal{A}_\nu,\mathcal{F}_{\lambda\sigma}\right\}-2i\,\mathcal{A}_\nu\mathcal{A}_\lambda\mathcal{A}_\sigma\right)\right], \end{aligned} \tag{2.59}$$

or in two dimensions

$$\delta\Gamma_{gWZW} = \frac{1}{8\pi}\int d^2x\,\epsilon^{\mu\nu}\mathrm{tr}\,\chi\left[\delta\mathcal{A}_\mu\left(-i\,\mathcal{H}^{-1}\mathcal{D}_\nu\mathcal{H}+i\,\mathcal{D}_\nu\mathcal{H}\mathcal{H}^{-1}-2\mathcal{A}_\nu\right)\right]. \tag{2.60}$$

Notice that in both cases the last term in the current denotes the BZ polynomial. The remaining chiral covariant terms have to be subtracted from the covariant current before it is matched to the effective action according to the ansatz made in Eq. (2.57).

2.2 Lowest Order Effective Action

2.2.1 Effective covariant current

Since we are ultimately interested in the Standard Model, and in order to compare our results to [51], we henceforth neglect the antisymmetric field $K_{\mu\nu}$. The fields $\mathcal{A}$ and $\mathcal{H}$ are matrices of some internal group, and we only assume that $\mathcal{H}(x_0)$ is nowhere singular. With this in mind, we restate our result Eq. (2.37) from the last section in D dimensions

$$\delta W^- = -\frac{i^{D/2}}{8}\,\mathrm{tr}\int_0^\infty dT\mathcal{N}\int \mathcal{D}x\int_P \mathcal{D}\psi\,\chi w(T)\mathcal{P}e^{-\int_0^T d\tau\mathcal{L}(\tau)}, \tag{2.61}$$

with

$$\begin{aligned}\mathcal{L}(\tau) &= \frac{\dot{x}^2}{4}+\frac{1}{2}\psi_A\dot{\psi}_A - i\,\dot{x}_\mu\mathcal{A}_\mu+\mathcal{H}^2+2i\,\psi_\mu\psi_5\mathcal{D}_\mu\mathcal{H}+i\,\psi_\mu\psi_\nu\mathcal{F}_{\mu\nu},\\ w(T) &= -4i\,\psi_5\psi_\mu\psi_\nu\delta\mathcal{A}_\mu\dot{x}_\nu-2i\,\psi_5\mathcal{D}_\mu\delta\mathcal{A}_\mu-2\psi_\mu\left\{\delta\mathcal{A}_\mu,\mathcal{H}\right\}. \end{aligned} \tag{2.62}$$

Next, the derivative expansion of the heat kernel is used. In the derivative expansion terms are classified by the number of covariant indices that they carry, so that $\mathcal{D}_\mu\mathcal{H}$ is of first order, while $\mathcal{F}_{\mu\nu}$ is of second order. The worldline formalism is well suited for this expansion, and there are two major advantages compared to the more traditional methods used e.g. in Ref. [51]. First, the tedious manipulations using the γ algebra are avoided. Secondly, the momentum integration is omitted and replaced by the simpler integration in τ space.

The coordinate is split as $x(\tau) = x_0 + y(\tau)$, and we work in the Fock-Schwinger gauge [59], in which $\mathcal{A}(x)\cdot y = 0$. In this gauge, expressions remain gauge covariant and the field $\mathcal{A}$ can be expressed in terms of the field strength tensor $\mathcal{F}_{\mu\nu}$ by

$$\mathcal{A}_\mu(x) = \int_0^1 d\alpha\,\alpha\,\mathcal{F}_{\rho\mu}(x_0+\alpha y)y_\rho. \tag{2.63}$$

All background fields can then be expanded around the point x_0 in terms of covariant derivatives

$$X(x_0 + y(\tau)) = \exp\left(y(\tau)\cdot \mathcal{D}_{x_0}\right) X(x_0), \tag{2.64}$$

where $\mathcal{D}_{x_0}$ refers to the covariant derivative in Eq. (2.18) with respect to x_0. With the expansion of the field strength tensor in terms of covariant derivatives and Eq. (2.63), one can rewrite the field $\mathcal{A}$ as

$$\mathcal{A}_\mu(x) = \frac{1}{2}y_\rho \mathcal{F}_{\rho\mu}(x_0) + \frac{1}{3}y_\alpha y_\rho \mathcal{D}_\alpha \mathcal{F}_{\rho\mu}(x_0) + \frac{1}{4\cdot 2!}y_\alpha y_\beta y_\rho \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{F}_{\rho\mu}(x_0) + \ldots\ . \tag{2.65}$$

Since we will not carry out the integration with respect to x_0 we use the following notation in D dimensions

$$\langle X\rangle_D = -\left(\frac{i}{4\pi}\right)^{D/2} \mathrm{tr}\chi \int d^D x_0 X. \tag{2.66}$$

It is important to remember that χ and $\mathcal{H}$ anticommute; hence, when the cyclic property of the trace is used, a minus sign is generated, for example

$$\begin{aligned}\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{H}\mathcal{F}_{\mu\nu}\mathcal{H}^3\mathcal{F}_{\lambda\sigma}\right\rangle &= -\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{F}_{\mu\nu}\mathcal{H}^3\mathcal{F}_{\lambda\sigma}\mathcal{H}\right\rangle = -\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{H}^3\mathcal{F}_{\lambda\sigma}\mathcal{H}\mathcal{F}_{\mu\nu}\right\rangle \\ &= -\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{H}^3\mathcal{F}_{\mu\nu}\mathcal{H}\mathcal{F}_{\lambda\sigma}\right\rangle .\end{aligned} \tag{2.67}$$

After expanding the mass field $\mathcal{H}(x)^2 = \mathcal{H}^2(x_0) + y_\mu \mathcal{D}_\mu \mathcal{H}^2(x_0) + \ldots$, the field $\mathcal{H}(x_0)$ is treated non-perturbatively. Since all the fields can be matrices of some internal space the resulting expressions normally cannot be expressed in closed form. For this case we use the labeled operator notation [51, 60]. The notation works as follows: In an expression $f(A_1, B_2, \ldots)XY\ldots$, the labels of the operators A, B, ... denote the position of that operator with respect to the remaining operators $XY\ldots$. For instance, for the function $f(A,B) = \alpha(A)\beta(B)$, the expression $f(A_1, B_2)XY$ represents $\alpha(A)X\beta(B)Y$. In the case at hand, the operator appearing in the functions is always $m := \mathcal{H}(x_0)$. If m can be made diagonal, a general function f can be easily interpreted in the basis where m is diagonal. One can also always substitute the recurrance relation $m_{n+1} = m_n - c_n$, where c_n denotes the commutator $[m, .]$ on the $n-$th element, and expand in powers of c_n [51]. Using this notation, Eq. (2.67) can be recast as

$$\begin{aligned}\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{H}\mathcal{F}_{\mu\nu}\mathcal{H}^3\mathcal{F}_{\lambda\sigma}\right\rangle &= \left\langle \epsilon^{\mu\nu\lambda\sigma}m_1 m_2^3\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\sigma}\right\rangle = -\left\langle \epsilon^{\mu\nu\lambda\sigma}m_3 m_2^3\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\sigma}\right\rangle \\ &= -\left\langle \epsilon^{\mu\nu\lambda\sigma}m_2 m_1^3\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\sigma}\right\rangle = -\left\langle \epsilon^{\mu\nu\lambda\sigma}\mathcal{H}^3\mathcal{F}_{\mu\nu}\mathcal{H}\mathcal{F}_{\lambda\sigma}\right\rangle .\end{aligned} \tag{2.68}$$

This notation can also be used to simplify the matrix valued derivative. Using the definition

$$(\nabla f)(m_1, m_2) := \frac{f(m_1) - f(m_2)}{m_1 - m_2}, \tag{2.69}$$

it is possible to prove that

$$\mathcal{D}_\mu f(m) = (\nabla f)(m_1, m_2)\, \mathcal{D}_\mu \mathcal{H}. \tag{2.70}$$

For example, in the polynomial case $f(m) = m^3$ one obtains

$$\begin{aligned}\mathcal{D}_\mu f(m) &= \mathcal{D}_\mu(\mathcal{H}^3) = \mathcal{D}_\mu\mathcal{H}\,\mathcal{H}^2 + \mathcal{H}\mathcal{D}_\mu\mathcal{H}\,\mathcal{H} + \mathcal{H}^2\mathcal{D}_\mu\mathcal{H} \\ &= (m_2^2 + m_1 m_2 + m_1^2)\,\mathcal{D}_\mu\mathcal{H} = \frac{m_1^3 - m_2^3}{m_1 - m_2}\,\mathcal{D}_\mu\mathcal{H} \\ &= (\nabla f)(m_1, m_2)\,\mathcal{D}_\mu\mathcal{H}.\end{aligned} \tag{2.71}$$

As mentioned earlier, non-polynomial expressions can be interpreted as an infinite series. In the case where m is diagonal, so that for $m = \mathrm{diag}(d_1, \ldots, d_n)$ we have

$$\frac{f(m_1) - f(m_2)}{m_1 - m_2}\, X = \frac{f(d_i) - f(d_j)}{d_i - d_j}\, X_{ij}. \tag{2.72}$$

More general, this suggests the following definition for the case with several variables:

$$\nabla_k f(m_1,\ldots,m_n) = \frac{f\left(m_1,\ldots,\hat{m}_{k+1},\ldots,m_n\right) - f\left(m_1,\ldots,\hat{m}_k,\ldots,m_n\right)}{m_k - m_{k+1}}, \tag{2.73}$$

where $\hat{m}_k$ indicates that the corresponding argument is left out.

In the present case, we simplify the notation even further by use of subscripts to refer to the argument of the function, e.g. $f(m_1, m_2) =: f_{12}$. We employ this notation in the following. Additionally, negative arguments will be denoted by underlining the corresponding index, $f(-m_1, m_2) =: f_{\underline{1}2}$. More applications of the labeled operator notation can be found in [51].

The path ordering in Eq. (2.61) is defined by

$$\mathcal{P}\prod_{i=1}^{N}\int_0^T d\tau_i \equiv N!\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\cdots\int_0^{\tau_{N-1}} d\tau_N = N!\int_0^T d\tau_1\cdots\int_0^T d\tau_N\prod_{i-1}^{N-1}\theta(\tau_i-\tau_{i+1}). \tag{2.74}$$

Separating the Lagrangian Eq. (2.62) into $\mathcal{L}(\tau) = \mathcal{L}_0(\tau) + \mathcal{H}^2(x_0) + \mathcal{L}_1(\tau)$, with

$$\begin{aligned}
\mathcal{L}_0(\tau) &= \frac{\dot{x}^2}{4} + \frac{1}{2}\psi_A\dot{\psi}_A, \\
\mathcal{L}_1(\tau) &= -i\,\dot{x}_\mu\mathcal{A}_\mu(x) + 2i\,\psi_\mu\psi_5\mathcal{D}_\mu\mathcal{H}(x) + i\,\psi_\mu\psi_\nu\mathcal{F}_{\mu\nu}(x) + y_\mu\mathcal{D}_\mu\mathcal{H}^2(x_0) + \ldots,
\end{aligned} \tag{2.75}$$

the terms of the expansion of $\mathcal{H}^2(x)$, except the leading term $\mathcal{H}^2(x_0)$, are included in $\mathcal{L}_1(\tau)$, and treated perturbatively. Notice that $\mathcal{L}_0$ commutes with the rest of the Lagrangian, so that the expansion of the path ordered exponential in Eq. (2.61) takes the form

$$\begin{aligned}
\mathcal{P}e^{-\int_0^T d\tau\mathcal{L}(\tau)} = e^{-\int_0^T d\tau\mathcal{L}_0(\tau)}\Bigg(& e^{-T\mathcal{H}^2(x_0)} + \int_0^T d\tau_1 e^{-(T-\tau_1)\mathcal{H}^2(x_0)}\left(-\mathcal{L}_1(\tau_1)\right)e^{-\tau_1\mathcal{H}^2(x_0)} \\
& + \int_0^T d\tau_1\int_0^{\tau_1} d\tau_2 e^{-(T-\tau_1)\mathcal{H}^2(x_0)}\left(-\mathcal{L}_1(\tau_1)\right)e^{-(\tau_1-\tau_2)\mathcal{H}^2(x_0)}\left(-\mathcal{L}_1(\tau_2)\right)e^{-\tau_2\mathcal{H}^2(x_0)} + \ldots\Bigg).
\end{aligned} \tag{2.76}$$

When performing the ψ integrals, the zero modes have to be saturated and at least a factor $\psi_1^0\ldots\psi_D^0\psi_5^0$ is required from the Grassman fields in order to contribute. The first term in Eq. (2.76) lacks the appropriate ψ factor except in two dimensions, where the first term of the insertion Eq. (2.62) already has the appropriate factor. However it contains a factor $\dot{x}_\mu$ which must be contracted with a another x field to form a Green function, hence it does not contribute and can be left out. The rest of Eq. (2.76) can be simplified using the labeled operator notation. Using the expression m_n^2 to denote $\mathcal{H}^2(x_0)$ in the nth position, one obtains

$$\begin{aligned}
\mathcal{P}e^{-\int_0^T d\tau\mathcal{L}(\tau)} &= e^{-\int_0^T d\tau\mathcal{L}_0(\tau)}\Bigg(-\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)}\mathcal{L}_1(\tau_1) \\
&\quad + \int_0^T d\tau_1\int_0^{\tau_1} d\tau_2 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)-\tau_2\left(m_3^2-m_2^2\right)}\mathcal{L}_1(\tau_1)\mathcal{L}_1(\tau_2) + \ldots\Bigg).
\end{aligned} \tag{2.77}$$

The evaluation of the worldline path integral can be summarised as follows: First, all fields in Eq. (2.77) and the insertion are expanded around x_0. Next, the functional integration over the y fields is carried out, generating bosonic Green functions. Then, the ψ integrations are performed saturating the zero modes and generating fermionic Green functions. Finally, the T and τ integrations are performed.

Before presenting the actual calculation, we comment on the behaviour of the effective action under complex conjugation. As noted earlier, in any contribution to the imaginary part of the action the field ψ_5 appears an odd number of times. If one attributes a factor i to the operators $\mathcal{F}$ and $\delta\mathcal{A}$, i.e. they become anti-hermitian, one observes that the remaining expressions in the current in Eq. (2.62) are real. Accordingly, all expressions in W^- are real as long as a factor i is attributed to the operator $\mathcal{F}$. In addition, notice that the effective action has to be an even function in the masses due to chiral invariance.

In order to showcase the method, we present the lowest order calculation in two dimensions. The lowest order contribution coming from the first term in the insertion is given by

$$-4i\,\psi_5\psi_\mu\psi_\nu\dot{y}_\nu(T)\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)}y_\alpha(\tau)\mathcal{D}_\alpha\mathcal{H}^2\,\delta\mathcal{A}_\mu(T). \tag{2.78}$$

Performing the y and ψ integrals one obtains

$$\begin{aligned}&\frac{i}{2}\left\langle\epsilon^{\mu\nu}(m_1+m_2)\int_0^\infty\frac{dT}{T}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)}\dot{g}_B(T,\tau_1)\,\mathcal{D}_\mu\mathcal{H}\,\delta\mathcal{A}_\nu\right\rangle\\ &=\;\frac{i}{2}\left\langle\epsilon^{\mu\nu}\,J_{12}^2(m_1+m_2)\,\mathcal{D}_\mu\mathcal{H}\,\delta\mathcal{A}_\nu\right\rangle.\end{aligned} \tag{2.79}$$

The second term of the insertion does not contribute at lowest order since it is already of second order in derivatives but lacks the appropriate fermionic factor to saturate the zero modes. The third term of the insertion leads only to one contribution of the form

$$-2\psi_\mu\left\{\delta\mathcal{A}_\mu,\mathcal{H}\right\}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)}\left(-2i\,\psi_\nu\psi_5\mathcal{D}_\nu\mathcal{H}\right). \tag{2.80}$$

yielding

$$\begin{aligned}&-\frac{i}{2}\left\langle\epsilon^{\mu\nu}(m_1-m_2)\int_0^\infty\frac{dT}{T}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1\left(m_2^2-m_1^2\right)}\mathcal{D}_\mu\mathcal{H}\,\delta\mathcal{A}_\nu\right\rangle\\ &=\;-\frac{i}{2}\left\langle\epsilon^{\mu\nu}\,J_{12}^1(m_1-m_2)\mathcal{D}_\mu\mathcal{H}\,\delta\mathcal{A}_\nu\right\rangle.\end{aligned} \tag{2.81}$$

The factor (m_1-m_2) results from the anticommutator in Eq. (2.80), and the sign change in the cyclic property of the trace as explained in Eq. (2.67). The integrals J are given in Appendix A.1. The total current is hence given by

$$\delta W^- = -i\left\langle\epsilon^{\mu\nu}A_{12}^1\mathcal{D}_\mu\mathcal{H}\,\delta\mathcal{A}_\nu\right\rangle, \tag{2.82}$$

$$A_{12}^1 = \frac{1}{m_1-m_2}-\frac{m_1m_2\log\left(m_1^2/m_2^2\right)}{(m_1-m_2)(m_1^2-m_2^2)}, \tag{2.83}$$

where the function A_{12}^1 has been defined. This agrees with the results obtained in [51].

2.2.2 Effective Density

The effective density can be obtained in a similar manner to the covariant current, utilising the insertion in Eq. (2.43). Neglecting the antisymmetric tensor $\mathcal{K}$, the insertion is

$$w(T) = -2i\,\psi_\mu\dot{x}_\mu\delta\mathcal{H}+2\psi_5\left[\delta\mathcal{H},\mathcal{H}\right]. \tag{2.84}$$

The contributions to the effective density are

$$\begin{aligned}\delta W^- \;=&\;\left\langle\epsilon^{\mu\nu}\left(\frac{i}{4}\left(J_{12}^1(m_1+m_2)+J_{12}^2(m_1-m_2)\right)\mathcal{F}_{\mu\nu}\right.\right.\\ &\left.\left.-\frac{1}{2}\left(J_{123}^5(m_1+m_3)+J_{123}^6(m_1+m_2)-J_{123}^7(m_2+m_3)\right)\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\right)\delta\mathcal{H}\right\rangle\\ =&\;\left\langle\epsilon^{\mu\nu}\left(-\frac{i}{2}B_{12}^1\mathcal{F}_{\mu\nu}+B_{123}^2\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\right)\delta\mathcal{H}\right\rangle.\end{aligned} \tag{2.85}$$

where the functions B_{12} and B_{123} are given by

$$B^1_{12} = -\frac{1}{m_1+m_2} - \frac{m_1 m_2}{(m_1+m_2)(m_1^2-m_2^2)} \log\left(\frac{m_1^2}{m_2^2}\right), \tag{2.86}$$

$$B^2_{123} = B^R_{123} + B^L_{123}\log(m_1^2) + B^L_{\underline{23}1}\log(m_2^2) + B^L_{\underline{3}12}\log(m_2^2), \tag{2.87}$$

with

$$B^R_{123} = \frac{1}{(m_1-m_2)(m_2-m_3)(m_1+m_3)}, \tag{2.88}$$

$$B^L_{123} = \frac{(m_1^3+m_1m_2m_3)}{(m_1-m_2)(m_1+m_3)(m_1^2-m_2^2)(m_1^2+m_3^2)}, \tag{2.89}$$

in accordance with Ref. [51].

There exists a relation between the effective current and the effective density. Doing a variation of the effective action we have

$$\delta W^-[\mathcal{A}_\mu, \mathcal{H}] = \langle \delta\mathcal{A}_\mu J^A_\mu + \delta\mathcal{H} J^m\rangle, \tag{2.90}$$

with J^A_μ the consistent current and J^m the effective density. A chiral rotation of the fields $\mathcal{A}$ and $\mathcal{H}$ is given by

$$\begin{aligned} \mathcal{A}_\mu &\rightarrow \Omega^{-1}\mathcal{A}_\mu\Omega + i\,\Omega^{-1}\partial_\mu\Omega \\ \mathcal{H} &\rightarrow \Omega^{-1}\mathcal{H}\Omega, \end{aligned} \tag{2.91}$$

where Ω is

$$\Omega(\alpha_L,\alpha_R) = \begin{pmatrix} e^{\alpha_L} & 0 \\ 0 & e^{\alpha_R} \end{pmatrix} \tag{2.92}$$

If we restrict the variation to infinitesimal chiral rotation we have

$$\delta\mathcal{A}_\mu = i\,\mathcal{D}_\mu\alpha, \quad \delta\mathcal{H} = [\mathcal{H},\alpha], \tag{2.93}$$

with

$$\alpha = \begin{pmatrix} \alpha_L & 0 \\ 0 & \alpha_R \end{pmatrix}. \tag{2.94}$$

Plugging the variations into Eq. (2.90) and using Eq. (2.47), we obtain the identity

$$\langle i\,(\mathcal{D}_\mu\alpha)J^A_\mu - \alpha\,\{J^m,\mathcal{H}\}\rangle = \langle \alpha G[\mathcal{A}_\mu]\rangle, \tag{2.95}$$

where G is the consistent anomaly. Plugging in the results from Eqs. (2.52,2.56) and Eqs. (2.82,2.85), it is seen that the identity is obeyed.

2.2.3 Effective Action

We proceed with brief derivation of the imaginary part of the effective action following Ref. [51]. Using the ansatz in Eq. (2.57), the most general functional for W_c^- consistent with chiral and gauge invariance in two dimensions reads

$$W_c^- = \langle \epsilon^{\mu\nu} N_{12}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\rangle\,. \tag{2.96}$$

An additional term proportional to $\mathcal{F}$ could be added but it can be removed by partial integration. Notice that N_{12} is a real function according to the comments made in the last section.

The function N_{12} has some nontrivial restrictions. First of all, the function N_{12} is even in m such that

$$N(-m_1,-m_2) := N_{\underline{12}} = N_{12}. \tag{2.97}$$

Because of the cyclic property of the trace one obtains

$$N_{12} = N_{\underline{3}2} = N_{\underline{2}1} = N_{2\underline{1}}, \tag{2.98}$$

and due to the Hermiticity of W^-

$$N_{12} = -N_{32} = -N_{\underline{1}2} = -N_{21}. \tag{2.99}$$

Varying $W_c^-[\mathcal{A}, \mathcal{H}]$ with respect to $\mathcal{A}$, one obtains

$$\delta W_c^- = -i \left\langle \epsilon^{\mu\nu} \left(-2\,(m_1 + m_2) N_{12}\right) \mathcal{D}_\mu \mathcal{H} \delta \mathcal{A}_\nu \right\rangle. \tag{2.100}$$

Comparing this to Eq. (2.82) and adding the covariant contribution in Eq. (2.60) coming from Γ_{gWZW} one has

$$\frac{1}{m_1 - m_2} - \frac{m_1 m_2 \log\left(m_1^2/m_2^2\right)}{(m_1 - m_2)(m_1^2 - m_2^2)} = \frac{1}{2m_1} - \frac{1}{2m_2} - 2\,(m_1 + m_2) N_{12}, \tag{2.101}$$

which finally leads to

$$N_{12} = \frac{1}{2} \frac{m_1 m_2}{m_1^2 - m_2^2} \left(\frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2} - \frac{1}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right). \tag{2.102}$$

At higher order, the matching of the effective potential to the current potentially becomes more intricate. On the other hand, the anomaly only contributes to the leading order, such that the knowledge of the covariant current (that in higher order coincides with the consistent current) suffices to determine the effective action.

2.2.4 Four Dimensions

For completeness, we also present the results for the effective action and the effective current in four dimensions. The matching procedure proceeds the same way as in Ref. [51], and we do not repeat it here.

The effective current in four dimensions consists of three terms and reads

$$\delta W^-_{d=4} = -i \left\langle \epsilon^{\mu\nu\lambda\sigma} \left(-\frac{i}{2} A^2_{123} \mathcal{F}_{\nu\lambda} \mathcal{D}_\mu \mathcal{H} - \frac{i}{2} A^3_{123} \mathcal{D}_\mu \mathcal{H} \mathcal{F}_{\nu\lambda} - A^4_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \right) \delta \mathcal{A}_\sigma \right\rangle, \tag{2.103}$$

while the effective density can be written as

$$\begin{aligned}\delta W^-_{d=4} = \Bigg\langle \epsilon^{\mu\nu\lambda\sigma} \Bigg(&\frac{1}{4} B^3_{123} \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\sigma} + \frac{i}{2} B^4_{1234} \mathcal{F}_{\lambda\sigma} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \\ &+ \frac{i}{2} B^5_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{F}_{\lambda\sigma} \mathcal{D}_\nu \mathcal{H} + \frac{i}{2} B^6_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{F}_{\lambda\sigma} \\ &- B^7_{12345} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \mathcal{D}_\sigma \mathcal{H} \Bigg) \delta \mathcal{H} \Bigg\rangle. \end{aligned} \tag{2.104}$$

The functions A^2_{123}, A^3_{123}, A^4_{1234}, B^3_{123}, B^4_{1234}, B^5_{1234}, B^6_{1234}, and B^7_{12345} are given in Appendix A.2.

The effective action takes the form

$$W^-_{d=4} = \epsilon^{\mu\nu\lambda\sigma} \left\langle N_{123} \mathcal{F}_{\mu\nu} \mathcal{D}_\lambda \mathcal{H} \mathcal{D}_\sigma \mathcal{H} + N_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \mathcal{D}_\sigma \mathcal{H} \right\rangle \tag{2.105}$$

2.3 Next to Leading Order Effective Action

2.3.1 NLO result in Two Dimensions

In this section we present as a novel result the imaginary part of the effective action in next to leading order and two dimensions. Even though the results are rather lengthy, the evaluation of the worldline path integral involves only very basic integrals such that it can be easily implemented using a computer algebra system.

In two dimensions and in next to leading order, the imaginary part of the effective action takes the form

$$W^c = \epsilon^{\mu\nu}\Big\langle\Big(Q_{12}\mathcal{D}_\mu\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{D}_\nu\mathcal{H} + \frac{i}{2}P_{12}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H} + \widetilde{R}_{123}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H} + \frac{i}{2}\widehat{R}_{123}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H} + M_{1234}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\Big)\Big\rangle. \tag{2.106}$$

At next to leading order the action is chiral invariant and the effective action can hence be immediately obtained by matching with the covariant current that in this order coincides with the consistent current. These functions must have the following properties

$$P_{12} = -P_{\underline{12}} = P_{21}, \quad Q_{12} = Q_{\underline{12}} = -Q_{21}, \tag{2.107}$$

$$\widetilde{R}_{123} = -\widetilde{R}_{\underline{123}} = \widetilde{R}_{\underline{21}3}, \quad \widehat{R}_{123} = \widehat{R}_{\underline{123}} = -\widehat{R}_{\underline{21}3}, \tag{2.108}$$

$$M_{1234} = M_{\underline{1234}} = -M_{\underline{34}12} = M_{4321}. \tag{2.109}$$

We have chosen a rather general imaginary effective action at the required order which preserves gauge and chiral invariance, but we have included a larger number of terms than necessary to perform the matching process with the effective current. In fact, the matching process could be done with solely the functions Q_{12}, $\widetilde{R}_{123}$, $\widehat{R}_{123}$, and M_{1234}. Instead, we have decided to include the additional term P_{12}, in order to have the option of simplifying the action by a judicious choice of this extra function. For example, the extra function can be used to ensure that all functions remain finite at the coincidence limit, as will be explained later on.

The calculation from the worldline formalism leads to the following contributions to the covariant current

$$\delta W^c = -i\,\epsilon^{\mu\nu}\Big\langle I^1_{12}\mathcal{D}_\mu\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + i\,I^2_{12}\mathcal{D}_\alpha\mathcal{F}_{\mu\alpha}\delta\mathcal{A}_\nu + I^3_{123}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu + I^4_{123}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + I^5_{123}\mathcal{D}_\mu\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + I^6_{123}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\mu\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + i\,I^7_{123}\mathcal{F}_{\mu\alpha}\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + i\,I^8_{123}\mathcal{D}_\alpha\mathcal{H}\mathcal{F}_{\mu\alpha}\delta\mathcal{A}_\nu + I^9_{1234}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + I^{10}_{1234}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\delta\mathcal{A}_\nu + I^{11}_{1234}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\mu\mathcal{H}\delta\mathcal{A}_\nu\Big\rangle. \tag{2.110}$$

The coefficient functions are given in Appendix A.3. In order to express the current in this form, partial integration has been used to remove terms of the form $\mathcal{D}\delta\mathcal{A}$. In addition, indices that are contracted with the ϵ tensor have been moved to the left, such that a term of the form $\mathcal{D}_\alpha\mathcal{D}_\mu$ yields a sum of terms of the type $\mathcal{D}_\mu\mathcal{D}_\alpha$ and $\mathcal{F}_{\mu\alpha}$.

The contributions from the variation of Eq. (2.106) can be grouped in three levels, with the first level having only contributions from Q and P; the second level from the previous ones and $\widetilde{R}$ and $\widehat{R}$; the last level with all functions. Adding the contributions from the worldline method and the variation of Eq. (2.106) one obtains for the first level the following two equations

$$\begin{aligned} P_{\underline{2}1} + (m_1+m_2)Q_{12} &= I^1_{12}, \\ (m_1+m_2)P_{12} - (m_1^2-m_2^2)Q_{\underline{2}1} &= I^2_{12}, \end{aligned} \tag{2.111}$$

which have the solution

$$Q_{12} = \frac{I^2_{\underline{2}1}}{m_1^2-m_2^2} - \frac{P_{\underline{2}1}}{m_1+m_2}. \tag{2.112}$$

The solution relies on the following restrictions which must be satisfied by the corresponding terms in the effective current and serve as consistency checks on the worldline calculation

$$(m_1+m_2)I^1_{\underline{2}1} = -I^2_{12}, \quad I^1_{12} = -I^1_{\underline{12}}, \quad I^2_{12} = I^2_{\underline{12}}. \tag{2.113}$$

The matching equations for the next level are

$$
\begin{aligned}
-\nabla_2\left((m_1+m_2)Q_{\underline{2}1}-P_{\underline{2}1}\right)+Q_{12}+Q_{\underline{2}1}+(m_1+m_3)(-\widetilde{R}_{123}+\widetilde{R}_{\underline{3}12}) &= I^3_{123}, &(2.114)\\
\nabla_2\left((m_1+m_2)(Q_{12}+Q_{\underline{2}1})\right)-(m_1+m_3)\widetilde{R}_{\underline{3}12}-2Q_{12}-2Q_{\underline{2}1}+\widehat{R}_{\underline{3}12} &= I^5_{123}, &(2.115)\\
\nabla_2\left((m_1+m_2)P_{12}\right)-(m_1-m_2)\nabla_2\left((m_1+m_2)Q_{\underline{2}1}\right)&\\
-2P_{12}+2(m_1-m_2)Q_{\underline{2}1}+(m_1+m_3)(Q_{13}+2Q_{\underline{3}1})&\\
-\widehat{R}_{123}(m_1+m_3)+(m_1-m_2)(m_1+m_3)\widetilde{R}_{\underline{3}12} &= I^7_{123}, &(2.116)
\end{aligned}
$$

and their complex conjugates.

The first Eq. (2.114) is of the form

$$
\widetilde{R}_{123}-\widetilde{R}_{\underline{3}12}=-\frac{\widetilde{I}^3_{123}}{m_1+m_3}, \tag{2.117}
$$

and a set of solutions to Eqs. (2.114) and (2.115) is hence given by

$$
\begin{aligned}
\widetilde{R}_{123} &= -\frac{1}{2}\left(\frac{\widetilde{I}_{123}}{m_1+m_3}\right)_{123}-\frac{1}{2}\left(\frac{\widetilde{I}_{123}}{m_1+m_3}\right)_{\underline{3}12}-\frac{1}{2}\left(\frac{\widetilde{I}_{123}}{m_1+m_3}\right)_{\underline{2}31} &(2.118)\\
\widehat{R}_{123} &= \widehat{I}_{\underline{2}\underline{3}1}-(m_1-m_2)\widetilde{R}_{123}. &(2.119)
\end{aligned}
$$

The functions $\widetilde{I}$ and $\widehat{I}$ are hereby defined as

$$
\begin{aligned}
\widetilde{I}_{123} &= I^3_{123}+\nabla_2\left((m_1+m_2)Q_{\underline{2}1}-P_{\underline{2}1}\right)-Q_{12}-Q_{\underline{2}1}, &(2.120)\\
\widehat{I}_{123} &= I^5_{123}-\nabla_2\left((m_1+m_2)(Q_{12}+Q_{\underline{2}1})\right)+2Q_{12}+2Q_{\underline{2}1}. &(2.121)
\end{aligned}
$$

The last Eq. (2.116) leads to a constraint on the I functions that is given in Appendix A.3.

The function $\widetilde{R}$ possesses the required symmetries, and it reproduces the effective current correctly, but it is not necessarily finite in the coincidence limit, $m_1 \to -m_3$. One way of solving this problem is to choose the function P appropriately which up to this point remained undetermined. Such a choice is given by

$$
P_{12}=\frac{I^2_{12}}{m_1+m_2}, \quad Q_{12}=0, \tag{2.122}
$$

which leaves $\widetilde{I}$ as

$$
\widetilde{I}_{123} = I^3_{123}-\frac{I^2_{\underline{2}1}}{(m_1-m_2)(m_2-m_3)}+\frac{I^2_{\underline{3}1}}{(m_1-m_3)(m_2-m_3)}. \tag{2.123}
$$

With this choice, $\widetilde{R}$ is finite in the coincidence limit, as can be checked explicitly, and since $\widehat{I}$ is also finite, so is $\widehat{R}$.

For the last level, the following three equations hold

$$
\begin{aligned}
\nabla_1\widehat{R}_{\underline{3}12}-(\nabla_2+\nabla_3)\Big((m_1+m_3)\widetilde{R}_{\underline{3}12}\Big)+2\widetilde{R}_{\underline{3}12}&\\
-2(m_1+m_4)M_{1234} &= I^9_{1234}, &(2.124)\\
(\nabla_3-\nabla_1)\Big(\widetilde{R}_{\underline{3}12}(m_1+m_3)\Big)+\nabla_2\widehat{R}_{\underline{3}12}-2\widetilde{R}_{\underline{3}12}+2\widetilde{R}_{\underline{4}23}&\\
-2(m_1+m_4)M_{\underline{4}123} &= I^{10}_{1234}, &(2.125)\\
(\nabla_1+\nabla_2)\Big((m_1+m_3)\widetilde{R}_{\underline{3}12}\Big)+\nabla_3\widehat{R}_{\underline{3}12}-2\widetilde{R}_{\underline{4}23}&\\
+2(m_1+m_4)M_{1234}) &= I^{11}_{1234}. &(2.126)
\end{aligned}
$$

One of these equations can be used to determine M, while the other two lead again to constraints on the I functions. the sum of the three equations has the especially simple form

$$-2(m_1+m_4)M_{\underline{4}123} \quad = \quad -(\nabla_1+\nabla_2+\nabla_3)\widehat{R}_{\underline{3}12}+I^9_{1234}+I^{10}_{1234}+I^{11}_{1234}. \tag{2.127}$$

Since all previous functions in the effective action have been chosen finite in the coincidence limit, so is M_{1234}. Eqs. (2.124) and (2.126) show that M_{1234} is finite in the limit $m_1 \to m_2$, while Eq. (2.125) shows that M_{1234} is finite in the limit $m_1 \to -m_4$. This concludes the discussion of the next to leading order contributions in two dimensions.

2.3.2 NLO result in Four Dimensions

The imaginary part of the effective action in four dimensions in next-to-leading order in a gradient expansion takes the form

$$\begin{aligned}
W^-_{nlo} \quad = \quad \epsilon^{\mu\nu\lambda\sigma}\Big\langle &+\frac{1}{4}Q^{(1)}_{123}\mathcal{D}_\alpha\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\sigma}\mathcal{D}_\alpha\mathcal{H}+\frac{1}{4}Q^{(2)}_{123}\mathcal{D}_\alpha\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\alpha}\mathcal{D}_\sigma\mathcal{H} \\
&+\frac{1}{4}Q^{(4)}_{123}\mathcal{D}_\alpha\mathcal{F}_{\mu\alpha}\mathcal{F}_{\nu\lambda}\mathcal{D}_\sigma\mathcal{H}+\frac{i}{2}Q^{(5)}_{123}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{D}_\mu\mathcal{H}\mathcal{F}_{\nu\lambda}\mathcal{D}_\sigma\mathcal{H} \\
&+\frac{i}{2}R^{(6)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}+\frac{i}{2}R^{(7)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\sigma\mathcal{H} \\
&+\frac{1}{4}R^{(9)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\alpha}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}+\frac{1}{4}R^{(10)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{F}_{\sigma\alpha}\mathcal{D}_\alpha\mathcal{H} \\
&+\frac{i}{2}R^{(12)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}+\frac{i}{2}R^{(13)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H} \\
&+\frac{1}{4}R^{(14)}_{1234}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{H}\mathcal{F}_{\lambda\sigma}\mathcal{D}_\alpha\mathcal{H}+\frac{1}{4}R^{(15)}_{1234}\mathcal{F}_{\mu\alpha}\mathcal{F}_{\nu\alpha}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H} \\
&+\frac{i}{2}S^{(1)}_{12345}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}+\frac{i}{2}S^{(2)}_{12345}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H} \\
&+\frac{i}{2}S^{(3)}_{12345}\mathcal{F}_{\mu\nu}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\sigma\mathcal{H}+\frac{i}{2}S^{(4)}_{12345}\mathcal{F}_{\mu\nu}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H} \\
&+S^{(7)}_{12345}\mathcal{D}_\alpha\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}+S^{(8)}_{12345}\mathcal{D}_\alpha\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\sigma\mathcal{H} \\
&+T^{(1)}_{123456}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\alpha\mathcal{H}+T^{(2)}_{123456}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H} \\
&+T^{(3)}_{123456}\mathcal{D}_\mu\mathcal{H}\mathcal{D}_\nu\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\mathcal{D}_\lambda\mathcal{H}\mathcal{D}_\sigma\mathcal{H}\mathcal{D}_\alpha\mathcal{H}\Big\rangle+h.c.
\end{aligned} \tag{2.128}$$

The previous expression is not unique. There are a number of operations which can be used to alter the form of W^-_{nlo}. Aside from the trace, partial integration, and hermitian conjugation, it is worth mentioning an identity which is found to be very useful in reducing the number of contributions to the effective action

$$\delta_{\alpha\beta}\epsilon_{\mu_1\mu_2\ldots\mu_d}=\delta_{\alpha\mu_1}\epsilon_{\beta\mu_2\ldots\mu_d}+\delta_{\alpha\mu_2}\epsilon_{\mu_1\beta\ldots\mu_d}+\cdots+\delta_{\alpha\mu_d}\epsilon_{\mu_1\mu_2\ldots\beta} \tag{2.129}$$

As in the two dimensional case, choosing a consistent convention for the order in which the indices contracted with the epsilon tensor and the free index appear reduces the number of terms. As a convention we choose to always commute free indices so that the they appear farthest from $\mathcal{H}$ by means of the identity $\mathcal{D}_\mu\mathcal{D}_\alpha\mathcal{H}=\mathcal{D}_\alpha\mathcal{D}_\mu\mathcal{H}+i[\mathcal{H},\mathcal{F}_{\mu\alpha}]$. Through a judicious set of the transformations just mentioned, W^-_{nlo} is brought into a simpler form than the one obtained originally from the matching procedure, into one which is finite term by term at all coincidence limits. The superscripts of the functions distinguish between different coefficient functions with the same number of arguments. The superscripts are not consecutively numbered what is reminiscent of the fact that we obtained this action by removing some contributions of a more general ansatz.

The system of equations necessary for the matching procedure consists of 97 equations, for the original 28 unknowns in the ansatz. It seems to us unnecessary to present them here, specially considering that the expressions for the effective current at next-to-leading order can be quite unappetising. The explicit functions are also not shown here for space considerations, but they are available as computer files [1]. In order to give the reader an impression of their form, we present the simplest function that is given by

$$
\begin{aligned}
Q^{(2)}_{123} =& \frac{8}{(3(m_1^2-m_2^2)^2(m_1+m_2)(m_2^2-m_3^2)^2(m_1+m_3)(m_2+m_3))} \times \\
&\left(m_1^4\left(m_2^2-m_2m_3+m_3^2\right)\left(m_2^2+4m_2m_3+m_3^2\right)+m_2^4m_3\left(2m_2^3-5m_2^2m_3+m_3^3\right)\right.\\
&+m_1^3m_2m_3(m_2+m_3)\left(3m_2^2-2m_2m_3+3m_3^2\right)\\
&+m_1m_2^3(m_2+m_3)\left(2m_2^3-9m_2^2m_3+3m_3^3\right)\\
&\left.+m_1^2m_2^2\left(-5m_2^4-9m_2^3m_3+11m_2^2m_3^2+m_2m_3^3-2m_3^4\right)\right)\\
&+\frac{8m_1^3\left(m_1^4+m_1^3m_2-3m_1^2m_2^2+3m_1m_2^3+6m_2^3m_3\right)\log\left[\frac{m_1^2}{m_2^2}\right]}{3(m_1^2-m_2^2)^3(m_1+m_2)(m_1^2-m_3^2)(m_1+m_3)}\\
&-\frac{8m_3^3\left(6m_1m_2^3+m_3\left(3m_2^3-3m_2^2m_3+m_2m_3^2+m_3^3\right)\right)\log\left[\frac{m_2^2}{m_3^2}\right]}{3(m_1^2-m_3^2)(m_1+m_3)(m_2^2-m_3^2)^3(m_2+m_3)}.
\end{aligned}
\tag{2.130}
$$

All the other functions, while increasing in complexity as the number of arguments increases, are of this form: rational functions of the masses, eventually multiplied by logarithms of mass ratios. In particular, all functions are homogeneous in their arguments for dimensional reasons

$$Q(a\,m_1, a\,m_2, a\,m_3) = \frac{1}{a^2}Q(m_1, m_2, m_3). \tag{2.131}$$

[1]The complete imaginary part of the effective action can be found at `http://www.thphys.uni-heidelberg.de/~schmidt/Weff_nlo/`

Chapter 3

CP Violation in the Standard Model

In this chapter we will specialise the Effective Action in Eq. (2.1) to the Standard Model. We will proceed in a similar manner to Smit [35] for the leading order case. The functions of the general solution in Eq. (2.128) are decomposed into terms with explicit behaviour under sign flips of the masses $m_i \to -m_i$. Then the trace is taken, and elements which violate CP symmetry are retained, which at this order of the expansion will all have four CKM matrices.

3.1 Preliminaries

3.1.1 The Standard Model

We adopt the same notation and conventions as Smit [35]. As opposed to Salcedo, the gauge fields are taken to be hermitian. The fermion part of the SM action is extended with right-handed neutrino fields, but since the general action assumed in Chapter 2 is of the type $\bar{\psi}\cdots\psi$, it does not allow for mass terms of a Majorana type and we do not include such terms. The SM takes the form [35]

$$S_{\mathrm{F}} = \int d^4x\, \bar{\Psi}\left\{\gamma_\mu\left[\partial_\mu - iA_\mu P_{\mathrm{L}} - iG_\mu - i(Y_{\mathrm{L}}P_{\mathrm{L}} + Y_{\mathrm{R}}P_{\mathrm{R}})B_\mu\right] + \Phi\Lambda P_{\mathrm{R}} + \Lambda^\dagger\Phi^\dagger P_{\mathrm{L}}\right\}\Psi. \qquad (3.1)$$

The gauge field for $U(1)$ is B_μ, for $SU(2)$ it is A_μ, and for $SU(3)$ G_μ. The coupling matrices Y and Λ are, respectively, the $U(1)$ hypercharges and the Yukawa couplings. They, as well as the background fields A_μ and G_μ, are embedded into the group structure in the usual tensor product fashion.

Ψ is a Dirac spinor with the following internal structure

$$\Psi^k, \quad k = (i, c, f), \quad i \in \{u, d\}, \quad c \in \{1, 2, 3\}, \quad f \in \{1, 2, 3\}, \qquad (3.2)$$

with (weak) isospin index i, color index c and family index f, for quarks, and no color index for the leptons. For reasons that will become apparent shortly, we make the $SU(2)$ structure of all the elements in Eq. (3.1) explicit. The $SU(2)$ gauge fields are written in terms of Pauli matrices as $A_\mu = A_\mu^a \tau_a/2$, with

$$(\tau_a)_{kk'} = (\tau_a)_{ii'}\delta_{cc'}\delta_{ff'}, \text{ and } \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3.3)$$

Similarly the $SU(3)$ fields are embedded as $(G_\mu)_{kk'} = (G_\mu)_{cc'}\delta_{ii'}\delta_{ff'}$ [35]. The $U(1)$ hypercharges Y are diagonal matrices, and take the form

$$Y_{\mathrm{L}} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}\pi_q + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\pi_\ell, \quad Y_{\mathrm{R}} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}\pi_q + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\pi_\ell, \qquad (3.4)$$

where π_q and π_ℓ project respectively onto the quark and lepton sectors. In terms of the $SU(2)$ Higgs-doublet $(\varphi^u, \varphi^d)^T$ the Higgs field Φ reads

$$\Phi = \begin{pmatrix} \varphi^{d*} & \varphi^u \\ -\varphi^{u*} & \varphi^d \end{pmatrix}, \quad \Phi \to \Omega\Phi e^{-i\omega\tau_3/2}, \tag{3.5}$$

where its behaviour under gauge transformations has been indicated, $e^{i\omega/6} \in U(1)$, $\Omega \in SU(2)$ [35]. The Yukawa coupling matrix Λ structure is given by

$$\Lambda = \begin{pmatrix} \Lambda_q^u & 0 \\ 0 & \Lambda_q^d \end{pmatrix} \pi_q + \begin{pmatrix} \Lambda_\ell^u & 0 \\ 0 & \Lambda_\ell^d \end{pmatrix} \pi_\ell, \tag{3.6}$$

where the $\Lambda_q^u, \ldots, \Lambda_\ell^d$ are non-trivial matrices in family space.

With these definitions, the fields defined in Eq. (2.13) are then realized as

$$A_\mu^{\rm L} = Y_{\rm L} B_\mu + A_\mu + G_\mu, \quad A_\mu^{\rm R} = Y_{\rm R} B_\mu + G_\mu, \quad H = \Phi\Lambda, \tag{3.7}$$

and

$$\mathcal{D}_\mu \mathcal{H} = \begin{pmatrix} 0 & i\hat{D}_\mu m_{\rm LR} \\ -i\hat{D}_\mu m_{\rm RL} & 0 \end{pmatrix}, \; \mathcal{F}_{\mu\nu} = \begin{pmatrix} F_{\mu\nu}^{\rm L} & 0 \\ 0 & F_{\mu\nu}^{\rm R}, \end{pmatrix} \tag{3.8}$$

where

$$F_{\mu\nu}^{\rm L} = A_{\mu\nu} + Y_{\rm L} B_{\mu\nu} + G_{\mu\nu}, \quad F_{\mu\nu}^{\rm R} = Y_{\rm R} B_{\mu\nu} + G_{\mu\nu}, \tag{3.9}$$

$$\hat{D}_\mu m_{\rm LR} = \left(\partial_\mu \Phi - iA_\mu \Phi + i\Phi \frac{1}{2}\tau_3 B_\mu \right) \Lambda, \quad \hat{D}_\mu m_{\rm RL} = \left(\hat{D}_\mu m_{\rm LR} \right)^\dagger. \tag{3.10}$$

The field strength tensor is defined by $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, and similarly for the other field strengths.

By using unitary gauge, in which $\Phi = h\,\mathbb{1}$ ($\sqrt{2}h$ is the standard-normalised Higgs field), and diagonalising Λ, a basis is obtained in which the coefficient functions appearing in the effective action Eq. (2.128) are normal functions and the trace can be taken directly [35]. Diagonalising Λ then we have

$$\Lambda = V_{\rm L} \lambda V_{\rm R}^\dagger, \quad \lambda = \begin{pmatrix} \lambda_q^u & 0 \\ 0 & \lambda_q^d \end{pmatrix} \pi_q + \begin{pmatrix} \lambda_\ell^u & 0 \\ 0 & \lambda_\ell^d \end{pmatrix} \pi_\ell, \tag{3.11}$$

$$V_{\rm L} = \begin{pmatrix} V_{\rm L}^u & 0 \\ 0 & V_{\rm L}^d \end{pmatrix}_q \pi_q + \begin{pmatrix} V_{\rm L}^u & 0 \\ 0 & V_{\rm L}^d \end{pmatrix}_\ell \pi_\ell, \tag{3.12}$$

and similar for $V_{\rm R}$. Going to unitary gauge

$$\Phi = \Omega h, \quad \Omega \in SU(2), \tag{3.13}$$

and so

$$m_{\rm LR} = U_{\rm L}\, d\, U_{\rm R}^\dagger, \quad U_{\rm L} = \Omega V_{\rm L}, \quad U_{\rm R} = V_{\rm R}, \quad d = h\,\lambda. \tag{3.14}$$

The λ are non-trivial diagonal matrices in family space, $\lambda_q^u = \mathrm{diag}(\lambda_u, \lambda_c, \lambda_t)$, $\lambda_q^d = \mathrm{diag}(\lambda_d, \lambda_s, \lambda_b)$, and similar for the leptons [35]. We concentrate on the quark sector exclusively and omit the subscripts q and ℓ if there is no danger of confusion. The lepton contribution can be done in a similar manner, but since without Majorana masses the lepton sector has no CP violation we will not conduct the analogous calculation.

The matrix elements appearing in (3.8) take the following form in the diagonal basis [35]

$$U_{\rm L}^\dagger \hat{D}_\mu m_{\rm LR} U_{\rm R} = V_{\rm L}^\dagger (h^{-1}\partial_\mu h - iW_\mu + iB_\mu \tau_3/2) V_{\rm L} d \tag{3.15}$$

$$\equiv -iC_\mu d, \tag{3.16}$$

$$U_{\rm R}^\dagger \hat{D}_\mu m_{\rm RL} U_{\rm L} = id\, C_\mu^\dagger, \tag{3.17}$$

$$C_\mu = ih^{-1}\partial_\mu h + W_\mu^a \tilde{\tau}_a/2 - B_\mu \tau_3/2, \tag{3.18}$$

where W_μ is the $SU(2)$ gauge field in unitary gauge,

$$W_\mu = \Omega^\dagger A_\mu \Omega + i\Omega^\dagger \partial_\mu \Omega, \tag{3.19}$$

and

$$\tilde{\tau}_1 = \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}, \quad \tilde{\tau}_2 = \begin{pmatrix} 0 & -iV \\ iV^\dagger & 0 \end{pmatrix}, \quad \tilde{\tau}_3 = \tau_3, \quad V = V_{\rm L}^{u\dagger} V_{\rm L}^d. \tag{3.20}$$

with V being the CKM matrix. Similarly,

$$U_{\rm L}^\dagger F_{\mu\nu}^{\rm L} U_{\rm L} = -i\tilde{W}_{\mu\nu} - iY_{\rm L} B_{\mu\nu} - iG_{\mu\nu}, \tag{3.21}$$

$$U_{\rm R}^\dagger F_{\mu\nu}^{\rm R} U_{\rm R} = -iY_{\rm R} B_{\mu\nu} - iG_{\mu\nu} = F_{\mu\nu}^{\rm R}, \tag{3.22}$$

with

$$\tilde{W}_{\mu\nu} = W_{\mu\nu}^a \tilde{\tau}_a/2. \tag{3.23}$$

In Eq. (3.18) the field Z_μ appears as $Z_\mu = W_\mu^3 - B_\mu$, with coupling constants absorbed,

$$B_\mu = A_\mu - \sin^2\theta_{\rm W} Z_\mu, \quad W_\mu^3 = A_\mu + \cos^2\theta_{\rm W} Z_\mu, \tag{3.24}$$

where A_μ is the photon field with electro-magnetic coupling e absorbed and $\theta_{\rm W}$ is the Weinberg angle.

In a similar manner, the higher order derivatives are obtained. They take the form

$$U_{\rm L}^\dagger \hat{D}_\mu \hat{D}_\nu m_{\rm LR} U_{\rm R} = (-i)^2 C_\mu C_\nu d - i\partial_\mu C_\nu d \tag{3.25}$$

$$\begin{aligned} U_{\rm L}^\dagger \hat{D}_\mu \hat{D}_\nu \hat{D}_\lambda m_{\rm LR} U_{\rm R} = &(-i)^3 C_\mu C_\nu C_\lambda d + (-i)^2 C_\mu \partial_\nu C_\lambda d + (-i)^2 \partial_\mu C_\nu C_\lambda d + (-i)^2 C_\nu \partial_\mu C_\lambda d \\ &-i\partial_\mu \partial_\nu C_\lambda d, \end{aligned} \tag{3.26}$$

with the terms of the type $U_{\rm R}^\dagger \hat{D}_\mu \hat{D}_\nu m_{\rm RL} U_{\rm L}$ obtained by hermitian conjugation.

3.1.2 Decomposition of the general functions

The functions obtained for the general solution, Eq. (2.128), can be taken, at least formally, as defining an infinite series expansion in terms of commutators with the $\mathcal{H}$ field [51]. This interpretation is always available independently of the basis on which one works with. However, in the case at hand, we employ a diagonal basis in which the functions will be normal functions of the elements of the diagonalised mass matrix. In order to diagonalise the general functions, it is necessary to decompose them into parts with well defined behaviour under exchange of the sign of the masses, $m_i \to -m_i$. As an example, we first present the decomposition for the function N_{12} from the LO case in 2 dimensions.

The function N_{12} has the property of being invariant under exchange of all the signs of the masses, i.e. $N_{12} = N_{\underline{12}}$. We can then decompose it in the following way

$$N_{12} = N^{(0)}(m_1^2, m_2^2) + m_1 m_2 N^{(12)}(m_1^2, m_2^2) \tag{3.27}$$

where $N^{(0)}$ and $N^{(12)}$ are functions of m_1^2 and m_2^2 only. In the following we will resume the notation of using subscripts to denote the argument of the functions, just remembering that in the case of the terms from the decomposition the arguments are all to be squared. The value of the decomposition now becomes obvious when working in a diagonal basis. Although $\mathcal{H}$ are not diagonal, the product of two such matrices $\mathcal{H}^2$ will be. This allows to take the trace without having to expand the functions of the general solution in terms of an infinite series. The form of the functions $N^{(0)}$ and $N^{(12)}$ is obtained from the original function N simply by adding combinations of N with different sign of the arguments. Since in the case of this example, N only has two arguments, the solution is simply

$$N_{12}^{(0)} = \frac{N_{12} + N_{1\underline{2}}}{2} \qquad N_{12}^{(12)} = \frac{N_{12} - N_{1\underline{2}}}{2} \tag{3.28}$$

The same procedure can be done for the other functions. Take the case of $R^{(6)}$ from Eq. (2.128) as a further example. It is odd under exchange of all the signs of the masses, i.e. $R^{(6)}_{1234} = -R^{(6)}_{\underline{1234}}$. The decomposition then reads

$$\begin{aligned} R^{(6)}_{1234} =& m_1 R^{(6,1)}_{1234} + m_2 R^{(6,2)}_{1234} + m_3 R^{(6,3)}_{1234} + m_4 R^{(6,4)}_{1234} + m_1 m_2 m_3 R^{(6,123)}_{1234} + m_1 m_2 m_4 R^{(6,124)}_{1234} \\ &+ m_1 m_3 m_4 R^{(6,134)}_{1234} + m_2 m_3 m_4 R^{(6,234)}_{1234} \end{aligned} \tag{3.29}$$

where the new functions $R^{(6,i)}_{1234}$ are functions only of the square of the arguments. They can also be obtained from the original function by adding combinations of the original function, like so

$$R^{(6,1)}_{1234} = \frac{R^{(6)}_{1234} - R^{(6)}_{\underline{1}234} + R^{(6)}_{1\underline{2}34} + R^{(6)}_{12\underline{3}4} - R^{(6)}_{\underline{12}34} - R^{(6)}_{\underline{1}2\underline{3}4} + R^{(6)}_{1\underline{23}4} - R^{(6)}_{\underline{123}4}}{8m_1} \tag{3.30}$$

3.1.3 The Trace Operation

We are now ready to specialise the general solution in Eq. (2.128) to the Standard Model and take the trace of the ensuing objects. Since our general solution cannot handle Majorana masses, we will concentrate on the quark sector. In it, at least 4 CKM matrices are required for CP violation. These are already present at leading order in the term $(\mathcal{DH})^4$ present in Eq. (2.105), but the contraction of space-time indices with the ϵ tensor will project this contribution to zero. Nevertheless, we will start with this term to show how the trace in the more involved next-to-leading order case would proceed.

The term in question is proportional to

$$\epsilon^{\mu\nu\lambda\sigma} \mathrm{tr}\left[\chi N_{1234} \mathcal{D}_\mu \mathcal{H} \mathcal{D}_\nu \mathcal{H} \mathcal{D}_\lambda \mathcal{H} \mathcal{D}_\sigma \mathcal{H}\right] \tag{3.31}$$

First we decompose N_{1234}

$$\begin{aligned} N_{1234} =& N^{(0)}_{1234} + m_1 m_2 N^{(12)}_{1234} + m_1 m_3 N^{(13)}_{1234} + m_1 m_4 N^{(14)}_{1234} + m_2 m_3 N^{(23)}_{1234} + m_2 m_4 N^{(24)}_{1234} \\ &+ m_3 m_4 N^{(34)}_{1234} + m_1 m_2 m_3 m_4 N^{(1234)}_{1234} \end{aligned} \tag{3.32}$$

Now we evaluate the trace for each of the components. For $N^{(0)}$ we have

$$\epsilon^{\mu\nu\lambda\sigma} \sum_{jklm} N^{(0)}_{jklm} \left((C_\mu)_{jk} d_k^2 (C_\nu^\dagger)_{kl} (C_\lambda)_{lm} d_m^2 (C_\sigma^\dagger)_{mj} - d_j^2 (C_\mu^\dagger)_{jk} (C_\nu)_{kl} d_l^2 (C_\lambda^\dagger)_{lm} (C_\sigma)_{mj} \right) \tag{3.33}$$

where the matrix C was defined in Eq. (3.18). The sum over colour indices gives simply a N_c multiplicative factor. Since G is traceless, this also means that the $SU(3)$ field will not appear in CP violating contributions at LO or NLO.

Now consider only the first of the contributions in Eq. (3.33). The function N_{1234} have the properties $N_{1234} = N_{\underline{4}123}$ and $N_{1234} = -N_{4321}$. For $N^{(0)}_{1234}$, these translate directly as $N^{(0)}_{1234} = N^{(0)}_{4123}$ and $N^{(0)}_{1234} = -N^{(0)}_{4321}$, which combined gives $N^{(0)}_{1234} = -N^{(0)}_{1432}$. Using this property, the first contribution gives

$$\begin{aligned} &\epsilon^{\mu\nu\lambda\sigma} \sum_{jklm} N^{(0)}_{jklm} \left((C_\mu)_{jk} d_k^2 (C_\nu^\dagger)_{kl} (C_\lambda)_{lm} d_m^2 (C_\sigma^\dagger)_{mj} \right) = \\ &\epsilon^{\mu\nu\lambda\sigma} \sum_{jklm} \frac{1}{2} \left(N^{(0)}_{jklm} (C_\mu)_{jk} d_k^2 (C_\nu^\dagger)_{kl} (C_\lambda)_{lm} d_m^2 (C_\sigma^\dagger)_{mj} - \left(N^{(0)}_{jklm} (C_\mu)_{jk} d_k^2 (C_\nu^\dagger)_{kl} (C_\lambda)_{lm} d_m^2 (C_\sigma^\dagger)_{mj} \right)^\dagger \right) \end{aligned} \tag{3.34}$$

which means only the imaginary part will contribute. In C, the CKM matrices are accompanied by $W^\pm$. Picking out the term with four CKM matrices gives

$$i N_c \epsilon^{\mu\nu\lambda\sigma} W_\mu^+ W_\nu^- W_\lambda^+ W_\sigma^- \sum_{f\,g\,h\,i} N^{(0)}_{uf,dg,uh,di} d_{dg}^2 d_{di}^2 \mathrm{Im}\left(V_{uf,dg} V^\dagger_{dg,uh} V_{uh,di} V^\dagger_{di,uf} \right) \tag{3.35}$$

From the previous expression it is clear that there is no CP violating contribution coming from W_c^- at leading order: $\epsilon^{\mu\nu\lambda\sigma}W_\mu^+ W_\nu^- W_\lambda^+ W_\sigma^-$ is equal to zero. Regardless, we will take one more step, and that is the substitution of the imaginary part of the product of CKM matrices for the Jarlskog invariant Eq. (1.4)

$$2iN_c J\epsilon^{\mu\nu\lambda\sigma}W_\mu^+ W_\nu^- W_\lambda^+ W_\sigma^- \sum_{f\,g\,h\,i\,j\,k} \epsilon_{uf,uh,uj}\epsilon_{dg,di,dk}N^{(0)}_{uf,dg,uh,di}d_{dg}^2 d_{di}^2 \tag{3.36}$$

One would then only have to do the sum, considering that the function $N^{(0)}$ is now a normal function of the masses, and obtain a numerical value for the factor accompanying the operator $\epsilon^{\mu\nu\lambda\sigma}W_\mu^+ W_\nu^- W_\lambda^+ W_\sigma^-$. As already mentioned, the contribution of this term is exactly zero, and the same will be for its partner in Eq. (3.33), but it showcases already everything what is required to take the trace in the more involved expressions found in the next-to-leading order case.

It turns out that the possible CP violating contribution at LO from W^+ are also zero, as will be shown shortly.

3.2 Magnitude of CP Violation

3.2.1 CP Violation

Before continuing with the calculation, it is important to understand what CP violation will mean for the present effective action, and how the effective action is separated into CP violating and CP non-violating parts, akin to the separation into real and imaginary parts which was done at the beginning of Chapter 2. In the following the WZW term will not be considered since it has already been found to contain no relevant CP violation [35]. First we look to Parity violation. Parity transformations invert the spatial directions, while leaving the temporal direction intact, i.e. $(x^0, \overrightarrow{x}) \to (x^0, -\overrightarrow{x})$. The separation of the effective action into real and imaginary parts in Eq. (2.5) is also a separation into parity even and parity odd parts. The part of the action which remains invariant under a parity transformation is also the real part of the action, and the part that changes sign is the imaginary part of the action [62]. This can be seen by considering Lorentz invariance, and noting that the imaginary part of the action contains the $\epsilon^{\mu\nu\lambda\sigma}$ tensor.

For charge conjugation it is instructive to take the gauge fields in Eq. (3.1) to be anti-hermitian by absorbing the i's that appear therein. With this definition there are no explicit i's produced in the action after taking the Dirac trace. This means that under the exchange of the background fields, including the Yukawa couplings, by their complex conjugates in the effective action, the effective action also becomes complex conjugated. Then, the real part of the action will remain invariant, while the imaginary part will change signs.

From this it is seen that together, complex conjugation and a parity transformation, will leave the effective action invariant. However, the *physical* CP transformation acts only on the dynamical fields, A, B, G, and Φ, not on the Yukawa couplings. The effective action will in general not be invariant under this transformation. This allows the effective action to be separated into two parts depending on whether they are even or odd under CP transformations

$$W = W_+ + W_-. \tag{3.37}$$

Since the action is invariant under full CP transformations, taking a physical CP transformation is equivalent to simply exchanging the Yukawa couplings for their corresponding complex conjugates. This implies that if the Yukawa couplings are real (or if the phase can be transformed away) then there can be no CP violation, a well established result. This is the case for the SM lepton sector, and is the reason why we do not consider it in the calculation. In the quark sector with three or more generations this is not the case, as there exists a phase which cannot be transformed away [8]. Furthermore, using the diagonal basis shown in Eq. (3.11), the part of the Yukawa couplings which can be complex can be regrouped to form the CKM matrices in Eq. (3.20). This shows that a complex CKM matrix is a necessary condition for CP violation.

After all the traces, except for the generation space, are taken, as in Eq. (3.35), the one-loop effective action can be rearranged in the following manner

$$W = \sum_{\chi} \mathrm{tr} f(\Lambda_u, \Lambda_d) \int dx^4 \mathcal{O}_{\chi}(x), \tag{3.38}$$

where the trace refers to generation space, and $\mathcal{O}_{\chi}(x)$ are local operators of the dynamical fields. As seen before, CP violation is equivalent to the substitution $(\Lambda_u, \Lambda_d) \to (\Lambda_u^*, \Lambda_d^*)$, which implies

$$\mathrm{tr} f(\Lambda_u, \Lambda_d) \xrightarrow[CP]{} \mathrm{tr} f(\Lambda_u^*, \Lambda_d^*) = (\mathrm{tr} f(\Lambda_u, \Lambda_d))^* . \tag{3.39}$$

This allows us to write the separation of W into CP odd and CP even parts shown in Eq. (3.37) in terms of the action in Eq. (3.38). The real part of $\mathrm{tr} f(\Lambda_u, \Lambda_d)$ will give the contribution to W_+, while the imaginary part of $\mathrm{tr} f(\Lambda_u, \Lambda_d)$ will give the contribution to W_-

$$W_+ = \sum_{\chi} \mathrm{Re}\,(\mathrm{tr} f(\Lambda_u, \Lambda_d)) \int dx^4 \mathcal{O}_{\chi}(x) \tag{3.40}$$

$$W_- = \sum_{\chi} i\,\mathrm{Im}\,(\mathrm{tr} f(\Lambda_u, \Lambda_d)) \int dx^4 \mathcal{O}_{\chi}(x). \tag{3.41}$$

It is now clear to see what kind of operator $\mathcal{O}$ will have to be in order to violate CP. In the imaginary part of the effective action, W^-, the CP violating part will contain Hermitian operators, while in the real part of the effective action, W^+, the CP violating operators will be anti-Hermitian. This can be taken as a consistency condition for the operators that we will find shortly.

Finally, we return to the Jarlskog invariant J. Eq. (3.41) allows us to see which elements will give CP violation in terms of the CKM matrix. Because of the freedom to rotate the quarks in generation space, the physics are invariant under the transformation $V \to U_1 V U_2$, as long as the appropriate rotation in the quarks is taken as well. Here, U_1 and U_2 are are arbitrary unitary and diagonal matrices. This is what was meant before by transforming the phase away. Since the physics are invariant, only phase invariant combinations of the CKM matrix should appear in Eq. (3.38). It is obvious that terms without CKM matrices would be phase invariant, but would also not have an imaginary part, and hence give no contribution to W_-. Because of the association of the CKM matrix with the gauge fields $W^{\pm}$, not to be confused with the effective action, we need combinations of the gauge fields $W^{\pm}$ to appear in the CP violating sector.

The next possible combination of CKM matrices would be

$$V_{a,b} V^{\dagger}_{b,a}, \tag{3.42}$$

with no summation implied. This invariant once again has no imaginary part. The Jarlskog invariant, Eq. (1.4), is the first non-trivial imaginary contribution coming from a phase invariant combination of CKM matrices

$$\mathrm{Im}\left[V_{ab} V^{\dagger}_{bc} V_{cd} V^{\dagger}_{da}\right] = J \sum_{e,f} \epsilon_{ace} \epsilon_{bdf}. \tag{3.43}$$

Again, summation on the right hand side is not implied. The consequence of this is that we require four CKM matrices to obtain CP violating terms, as previously claimed. It also means that at LO there can be no CP violating contribution. For the imaginary part of the action, we have already seen that the ϵ tensor will project such contributions to zero. In fact, one can make a stronger statement. While the CP violating sector needs charged $SU(2)$ fields, the operators cannot be composed solely from them. For the imaginary part, the same argument as for the LO applies, the ϵ tensor projects such contributions to zero. For the real part, charge conservation means such contributions would have to be of the form

$$\left(\left(W^+_\mu W^+_\mu\right)\left(W^-_\nu W^-_\nu\right)\right)^n \left(W^+_\alpha W^-_\alpha\right)^m, \tag{3.44}$$

which is a Hermitian operator and therefore CP even. At NNLO and higher order there will be phase invariants with more CKM matrices, and in general will not have such a simple relation as in Eq. (3.43), but for NLO we only need to look for contributions containing four CKM matrices.

3.2.2 CP Violation in the Imaginary Part of the Effective Action

We have already seen why there will be no contribution coming from the imaginary part of the effective action at leading order: the ϵ tensor with space-time indices projects such a possible contribution to zero. In the NLO case, this will no longer be the case. With a one-loop effective action it will not be possible to produce the kind of operator that Smit and collaborators were working with in their simulations of cold electroweak baryogenesis [34], but nevertheless there will be CP violating contributions. Interestingly, almost all the contributions cancel amongst themselves, and there is only one contribution to the CP-violating part of the effective action, namely

$$\frac{i}{8(4\pi)^2}\frac{N_c}{16}\frac{J\,\kappa^{CP}}{\tilde{m}_c^2}\epsilon^{\mu\nu\lambda\sigma}\int d^4x\left(\frac{v}{\phi}\right)^2\left(Z_\mu W^+_{\nu\lambda}W^-_\alpha\left(W^+_\sigma W^-_\alpha + W^+_\alpha W^-_\sigma\right) + c.c.\right) \tag{3.45}$$

with J given by Eq. (1.5) and

$$\kappa^{CP} \approx 9.87. \tag{3.46}$$

Finally, notice that the action can always be rewritten in $SU(2)_L$ gauge invariant quantities. For example, the charged gauge fields can be rewritten as

$$W^+_{\mu\nu} = \frac{\phi^\dagger W_{\mu\nu}\tilde{\phi}}{\phi^\dagger\phi},\quad W^-_{\mu\nu} = \frac{\tilde{\phi}^\dagger W_{\mu\nu}\phi}{\phi^\dagger\phi},\quad W^+_\mu = \frac{\phi^\dagger \mathcal{D}_\mu\tilde{\phi}}{\phi^\dagger\phi},\quad W^-_\mu = \frac{\tilde{\phi}^\dagger \mathcal{D}_\mu\phi}{\phi^\dagger\phi}, \tag{3.47}$$

and similarly for the uncharged quantities

$$Z_\mu = W^3_\mu - B_\mu = \frac{\phi^\dagger\mathcal{D}_\mu\phi - \tilde{\phi}^\dagger\mathcal{D}_\mu\tilde{\phi}}{2\phi^\dagger\phi},\quad h^{-1}\partial_\mu h = \frac{\phi^\dagger\mathcal{D}_\mu\phi + \tilde{\phi}^\dagger\mathcal{D}_\mu\tilde{\phi}}{2\phi^\dagger\phi}, \tag{3.48}$$

and

$$W^3_{\mu\nu} = \frac{\phi^\dagger W_{\mu\nu}\phi}{\phi^\dagger\phi}. \tag{3.49}$$

While obtaining the general solution in Eq. (2.128) we performed a series of partial integrations to bring the original result from the matching procedure to a form which would not have spurious divergences at the coincidence limits. One could then think that the simple form of the operator in Eq. (3.45) could be due to this and that it might look different upon partial integration. This is not necessarily the case. Consider the case with only $SU(2)$ fields. Because of the contraction with $\epsilon^{\mu\nu\lambda\sigma}$, $W^+_{\nu\lambda}$ can be replaced by $2\partial_\nu W^+_\lambda$. Partially integrating the ensuing operator will simply result in the same expression. No other form for the operator in Eq. (2.128) could be reached by partial integration in this case.

In calculating the coefficient in Eq. (3.46) we used the full analytic functions, but the final result is too big to present it here. However, in the limit where $\tilde{m}_u \to \tilde{m}_d \to 0$ and $\tilde{m}_b \to \tilde{m}_c$ the result is simpler, and differs by just 1% from the one given in Eq. (3.46). In this limit the contribution takes the following form

$$
\begin{aligned}
\frac{\kappa^{CP}}{\tilde{m}_c^2} \approx\ & \frac{32}{9\tilde{m}_c^2\left(\tilde{m}_c^2-\tilde{m}_s^2\right)^3\left(\tilde{m}_c^2-\tilde{m}_t^2\right)^3\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^2}\times \\
& \Big(\tilde{m}_s^6\tilde{m}_t^6\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^2+3\tilde{m}_c^{14}\left(\tilde{m}_s^2+\tilde{m}_t^2\right) \\
& -5\tilde{m}_c^2\tilde{m}_s^4\tilde{m}_t^4\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^2\left(\tilde{m}_s^2+\tilde{m}_t^2\right)-12\tilde{m}_c^{12}\left(\tilde{m}_s^4+\tilde{m}_t^4\right) \\
& +\tilde{m}_c^4\tilde{m}_s^2\tilde{m}_t^2\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^2\left(13\tilde{m}_s^4+28\tilde{m}_s^2\tilde{m}_t^2+13\tilde{m}_t^4\right)+18\tilde{m}_c^{10}\left(\tilde{m}_s^6+\tilde{m}_t^6\right) \\
& +\tilde{m}_c^8\left(-12\tilde{m}_s^8+37\tilde{m}_s^6\tilde{m}_t^2-74\tilde{m}_s^4\tilde{m}_t^4+37\tilde{m}_s^2\tilde{m}_t^6-12\tilde{m}_t^8\right) \\
& +\tilde{m}_c^6\left(3\tilde{m}_s^{10}-41\tilde{m}_s^8\tilde{m}_t^2+41\tilde{m}_s^6\tilde{m}_t^4+41\tilde{m}_s^4\tilde{m}_t^6-41\tilde{m}_s^2\tilde{m}_t^8+3\tilde{m}_t^{10}\right)\Big) \\
& -\frac{64\,\tilde{m}_c^4\tilde{m}_s^2\tilde{m}_t^2\left(\tilde{m}_c^2-\tilde{m}_t^2\right)\left(\tilde{m}_c^2-3\tilde{m}_s^2+2\tilde{m}_t^2\right)\log\left[\frac{\tilde{m}_s^2}{\tilde{m}_c^2}\right]}{3\left(\tilde{m}_c^2-\tilde{m}_s^2\right)^4\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^3} \\
& +\frac{64\,\tilde{m}_c^4\tilde{m}_s^2\left(\tilde{m}_c^2-\tilde{m}_s^2\right)\tilde{m}_t^2\left(\tilde{m}_c^2+2\tilde{m}_s^2-3\tilde{m}_t^2\right)\log\left[\frac{\tilde{m}_t^2}{\tilde{m}_c^2}\right]}{3\left(\tilde{m}_c^2-\tilde{m}_t^2\right)^4\left(\tilde{m}_s^2-\tilde{m}_t^2\right)^3}.
\end{aligned} \tag{3.50}
$$

Recently, Salcedo has proposed a new method which purports to construct the imaginary, as well as the real part of the effective action by a direct method based on a number of (questionable) steps [63]. Using this method, a calculation of the CP violating contributions was undertaken and found no contribution from the imaginary part [61]. Needless to say, we have rechecked our calculation and found no error. We restate that the large number of consistency checks give us confidence in our result. At this time, we are unable to explain the difference.

3.2.3 CP Violation in the Real Part of the Effective Action

Contrary to expectations in the literature, there can be CP violating terms coming from the real part of the effective action. Calculating the real part is much more straightforward than the imaginary part. On the other hand, since for the imaginary part the worldline expression was for the effective current, the type of τ integrals that appear in the real part will be larger. This is the same disadvantage that calculating the effective density had vis a vis the effective current.

We return to Eq. (2.19)

$$
W^+ = \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\int_{AP}\mathcal{D}\psi\,\mathrm{tr}\,\mathcal{P}e^{-\int_0^T d\tau \mathcal{L}(\tau)}, \tag{3.51}
$$

with the lagrangian given by Eq. (2.20), and is the same used in the imaginary part. Like in the imaginary part, the antisymmetric tensor K is set to zero.

The calculation of W^+ proceeds in a similar manner to the one carried out already for the effective current. The path ordered exponential in Eq. (3.51) is expanded to the desired order in the covariant derivative expansion, in this case to order six in the number of covariant indices. Since there is no matching to be done, the whole procedure can be implemented as a computer algebra program which calculates the result automatically. An advantage of the worldline method.

Calculating the CP violating terms for the real part of the effective action the result is

$$
\frac{i}{8(4\pi)^2}\frac{N_c}{4}\frac{J\,\kappa^{real}}{\tilde{m}_c^2}\int d^4x\left(\frac{\upsilon}{\phi}\right)^2\left(\mathcal{O}_0+\mathcal{O}_1+\mathcal{O}_2\right)+\mathcal{O}\left(D^8\right)+\text{CP invariant terms.} \tag{3.52}
$$

with $\kappa^{real} \approx 0.15322$ and the local operators $\mathcal{O}_i$ defined as

$$\begin{aligned}\mathcal{O}_2 = -4\Big(& 2(\phi_\mu + i\frac{Z_\mu}{2})(\phi_\nu - i\frac{Z_\nu}{2})W^-_\mu W^-_\nu W^+_\alpha W^+_\alpha \\ & +(\phi_\mu + i\frac{Z_\mu}{2})(\phi_\mu + i\frac{Z_\mu}{2})W^+_\nu W^-_\nu W^+_\alpha W^-_\alpha \\ & +(\phi_\mu + i\frac{Z_\mu}{2})(\phi_\nu + i\frac{Z_\nu}{2})W^+_\mu W^+_\nu W^-_\alpha W^-_\alpha \Big) + c.c. \end{aligned} \tag{3.53}$$

$$\begin{aligned}\mathcal{O}_1 = (\phi_\mu + i\frac{Z_\mu}{2})\Big(& +\partial_\mu W^-_\nu \Big(-3W^+_\nu W^-_\alpha W^+_\alpha - 3W^-_\nu W^+_\alpha W^+_\alpha \Big) \\ & +\partial_\mu W^+_\nu \Big(+5W^-_\nu W^-_\alpha W^+_\alpha - 3W^+_\nu W^-_\alpha W^-_\alpha \Big) \\ & +\partial_\nu W^-_\mu \Big(+W^+_\nu W^-_\alpha W^+_\alpha + 6W^-_\nu W^+_\alpha W^+_\alpha \Big) \\ & +\partial_\nu W^+_\mu \Big(+W^-_\nu W^-_\alpha W^+_\alpha - 8W^+_\nu W^-_\alpha W^-_\alpha \Big) \\ & +\partial_\nu W^-_\alpha \Big(-2W^-_\mu W^+_\nu W^+_\alpha + 7W^+_\mu W^-_\nu W^+_\alpha - 7W^+_\mu W^+_\nu W^-_\alpha \Big) \\ & +\partial_\nu W^+_\alpha \Big(+2W^+_\mu W^-_\nu W^-_\alpha + 3W^-_\mu W^+_\nu W^-_\alpha + W^-_\mu W^-_\nu W^+_\alpha \Big) \\ & +\partial_\nu W^+_\nu \Big(-8W^-_\mu W^-_\alpha W^+_\alpha + 7W^+_\mu W^-_\alpha W^-_\alpha \Big) + \partial_\nu W^-_\nu \Big(W^-_\mu W^+_\alpha W^+_\alpha \Big) \\ & +W^-_{\nu\alpha}\Big(+7W^+_\mu W^+_\nu W^-_\alpha \Big) + W^+_{\nu\alpha}\Big(+W^-_\mu W^+_\nu W^-_\alpha \Big) \\ & +W^-_{\mu\nu}\Big(+W^+_\nu W^-_\alpha W^+_\alpha + 6W^-_\nu W^+_\alpha W^+_\alpha \Big) \\ & +W^+_{\mu\nu}\Big(+W^-_\nu W^-_\alpha W^+_\alpha - 2W^+_\nu W^-_\alpha W^-_\alpha \Big)\Big) + c.c. \end{aligned} \tag{3.54}$$

$$\begin{aligned}\mathcal{O}_0 = \partial_\mu W^-_\nu \Big(& +\Big(+\frac{7}{2}\partial_\alpha W^-_\nu - 7\partial_\nu W^-_\alpha \Big) W^+_\mu W^+_\alpha + \frac{11}{2}\partial_\mu W^-_\alpha W^+_\nu W^+_\alpha \\ & +\Big(-\frac{9}{2}\partial_\mu W^-_\nu + \frac{3}{2}\partial_\nu W^-_\mu \Big) W^+_\alpha W^+_\alpha + 2\partial_\alpha W^+_\mu W^+_\nu W^-_\alpha \\ & -2\partial_\nu W^+_\alpha W^-_\mu W^+_\alpha + W^-_{\mu\alpha} W^+_\nu W^+_\alpha + W^-_{\nu\alpha} W^+_\mu W^+_\alpha \\ & +2W^+_{\mu\alpha}\Big(W^-_\nu W^+_\alpha - W^+_\nu W^-_\alpha \Big) + 2W^+_{\nu\alpha}\Big(W^-_\mu W^+_\alpha + W^+_\mu W^-_\alpha \Big)\Big) \\ +\partial_\mu W^-_\mu \Big(& +2\partial_\nu W^+_\alpha \Big(W^+_\nu W^-_\alpha - W^-_\nu W^+_\alpha \Big) - 2\partial_\nu W^-_\alpha W^+_\nu W^+_\alpha \Big) \\ +W^-_{\mu\nu}\Big(& -\frac{1}{2} W^-_{\mu\alpha} W^+_\nu W^+_\alpha - \frac{1}{4} W^-_{\mu\nu} W^+_\alpha W^+_\alpha \Big) + c.c. \end{aligned} \tag{3.55}$$

It is not clear to us why all the individual coefficients for the terms should be integer multiples of the same coefficient. It should be mentioned that this operators are also not in agreement with those found in [61]. It is also not clear that these operators, being P conserving, contribute in the same way as the P-violating operators from the imaginary part, to the parity odd Chern-Simmons number.

3.2.4 Applicability of the expansion

Finding an upper limit of applicability

In this section we discuss some general properties of the effective action in next-to-leading order. First, notice that if written in terms of the gauge field A_μ and the field strength $F_{\mu\nu}$, the coefficient of the effective action has negative mass dimension. Hence, in the limit of vanishing masses, the effective lagrangian diverges. This is not surprising, since the gradient expansion assumes

$$A_\mu \ll m, \quad F_{\mu\nu} \ll m^2. \tag{3.56}$$

This leads to the question what is the range of applicability of our result. In order to discuss this question, we analyse the CP-violating part of a specific term in the effective action. Consider a term of the form

$$R(m_1, m_2, m_3, m_4)\mathcal{D}_\alpha\mathcal{H}\,\mathcal{D}_\alpha\mathcal{H}\,\mathcal{F}_{\mu\nu}\,\mathcal{F}_{\lambda\sigma}. \tag{3.57}$$

This could in principle contain CP violation if all appearing gauge fields are left-handed and charged after symmetry breaking. This yields the contributions

$$\begin{aligned} &\bar{R}(m_1^d, m_2^u, m_3^d, m_4^u)\, A_\alpha^+\, A_\alpha^-\, F_{\mu\nu}^+\, F_{\lambda\sigma}^- \\ &\qquad + \bar{R}(m_1^u, m_2^d, m_3^u, m_4^d)\, A_\alpha^-\, A_\alpha^+\, F_{\mu\nu}^-\, F_{\lambda\sigma}^+, \end{aligned} \tag{3.58}$$

where we used the symmetrisation

$$\bar{R}(m_1, m_2, m_3, m_4) = \frac{1}{16} \sum_{n_i \in \pm m_i} R(n_1, n_2, n_3, n_4)(n_2 - n_1)(n_3 - n_2). \tag{3.59}$$

The symmetrisation ensures that all appearing gauge fields are left-handed. Changing to the mass eigenbasis and using Eq. (1.4) this can be recast as

$$C_1\, A_\alpha^+\, A_\alpha^-\, F_{\mu\nu}^+\, F_{\lambda\sigma}^- + C_2\, A_\alpha^-\, A_\alpha^+\, F_{\mu\nu}^-\, F_{\lambda\sigma}^+, \tag{3.60}$$

where we use the definitions

$$C_1 \;=\; J \sum_{i,k,m\in\text{up}} \; \sum_{j,l,n\in\text{down}} \epsilon_{ikm}\epsilon_{jln}\bar{R}(\tilde{m}_k^d, \tilde{m}_l^u, \tilde{m}_m^d, \tilde{m}_n^u), \tag{3.61}$$

$$C_2 \;=\; -J \sum_{i,k,m\in\text{up}} \; \sum_{j,l,n\in\text{down}} \epsilon_{ikm}\epsilon_{jln}\bar{R}(\tilde{m}_l^u, \tilde{m}_k^d, \tilde{m}_n^u, \tilde{m}_m^d). \tag{3.62}$$

The subscript indicates hereby the quark flavour, up $= \{u, c, t\}$ and down $= \{d, s, b\}$.

Notice that this expression vanishes if two up-type masses or two down-type masses coincide, as required. However, the coefficient can be much larger than the Jarlskog determinant stated in Eq. (1.2) even in units of the light quark masses $\tilde{m}_{u/d}^{-2}$. The largest contribution results typically from the contribution involving only the four lightest quarks.

Let us come back to the question of the range of applicability of the gradient expansion. In principle, one would expect that the largest contributions be proportional to $\tilde{m}_{u/d}^{-2}$ or even larger, e.g. $\tilde{m}_{c/b}^{2}\tilde{m}_{u/d}^{-4}$. In this case, the mass scale that indicates the breakdown of the gradient expansion in Eq. (3.56) would be given by the lightest quarks invalidating the gradient expansion already for very weak external fields. Besides, there might be one more obstacle, namely the physical infrared divergences of the light quarks. The operator under consideration describes a scattering process that is indistinguishable from the same process including a soft quark/anti-quark pair. Hence, the amplitudes can contain contributions that scale as $\log \tilde{m}_u^2$ or $\log \tilde{m}_d^2$ in the massless limit. This would require that the corresponding operators with soft quarks in the initial/final states be taken into account .

Fortunately, it turns out that all appearing CP-violating contributions are finite in the limit of vanishing up/down quark masses and there are only terms that scale as $\mathcal{O}\left(\tilde{m}_c^{-2}, \tilde{m}_b^{-2}, \tilde{m}_t^{-2}\right)$. We hence expect that the range of validity in Eq. (3.56) is at least given by the scale of the charm quark mass.

In fact, the range of applicability can be even larger according to the following argument. For simplification, imagine that there is a common energy scale for the gradient expansion

$$A_\mu \sim \partial_\mu \sim E, \quad F_{\mu\nu} \sim \partial_\mu^2 \sim E^2. \tag{3.63}$$

In the limit of weak fields $E \ll \tilde{m}_c$ we obtain the estimate for CP violation in the effective action

$$W^- \propto J\,\tilde{m}_c^{-2} E^6, \tag{3.64}$$

while in the case of a strong background, $E \gg \tilde{m}_t$, the effective action could be expanded in the quark masses. In this case, following the argument by Jarlskog, one obtains on dimensional grounds an estimate for CP violation similar to the Jarlskog determinant, namely

$$W^- \propto J\,\tilde{m}_t^4 \tilde{m}_b^4 \tilde{m}_c^2 \tilde{m}_s^2 E^{-8}. \tag{3.65}$$

Comparison of these two limits indicates that the transition region is given for energies

$$E \sim (m_t^4 \tilde{m}_b^4 \tilde{m}_c^4 \tilde{m}_s^2)^{1/14} \simeq 5.0 \text{ GeV}. \tag{3.66}$$

and below this value the effective action presented here should indicate the correct order of magnitude of CP violation in the bosonic sector of the SM.

Issues in the Infrared

The conservative limit on the upper bound is not the only problem that is encountered when determining the applicability of our result. The expansion was undertaken with the assumption that the masses are not zero, therefore the infrared behaviour of the solution is also somewhat unclear to assess. The applicability of the gradient expansion in a classical theory, like General Relativity, is more straightforward due to the lack of fluctuations. Imagine a field configuration for the Higgs field which is almost constant, but not zero, almost everywhere, except in a small region in which it dips to zero. In that region, the Higgs might be very small, but its derivative need not be. Is the gradient expansion applicable in this situation? If the expansion will only be used to describe phenomena in the broken phase, then there is no need for further analysis, but if the expansion is to be used in the description of Cold Electroweak Baryogenesis, as we desire, then its applicability is not straightforward. For the baryon number to change after the tachyonic phase transition, the Higgs length necessarily needs to pass through zero [64]. In such isolated points, the Higgs will pass through zero, but its derivative will not be necessarily small. The unitary gauge becomes singular at these points, and needs to be abandoned. To better study this behaviour, we rewrite Eq. (3.45) in a gauge invariant form by use of Eqs. (3.47,3.48,3.49)

$$\frac{i}{8(4\pi)^2}\frac{N_c}{16}\frac{J\,\kappa^{CP}}{\tilde{m}_c^2}\epsilon^{\mu\nu\lambda\sigma}\int d^4x \frac{(\upsilon)^2}{\phi^\dagger\phi}\times \\ \left(\frac{\phi^\dagger \mathcal{D}_\mu \phi - \tilde{\phi}^\dagger \mathcal{D}_\mu \tilde{\phi}}{2\phi^\dagger\phi}\frac{\phi^\dagger W_{\nu\lambda}\tilde{\phi}}{\phi^\dagger\phi}\frac{\tilde{\phi}^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi}\left(\frac{\phi^\dagger \mathcal{D}_\sigma \tilde{\phi}}{\phi^\dagger\phi}\frac{\tilde{\phi}^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi} + \frac{\phi^\dagger \mathcal{D}_\alpha \tilde{\phi}}{\phi^\dagger\phi}\frac{\tilde{\phi}^\dagger \mathcal{D}_\sigma \phi}{\phi^\dagger\phi}\right) + c.c.\right). \tag{3.67}$$

From the previous expression, one can imagine that divergences of up to order ϕ^6 might appear. It turns out that the effective lagrangian can indeed be divergent, but what is not clear is what effect will these divergences have on the effective action. If their effect is noticeable for a particular phenomenon, then the validity of the expansion to describe that phenomenon would be seriously compromised. We make two arguments that suggest that those divergences will not compromise the applicability.

As a first argument, we make a rough approximation to the individual contribution coming from those points to the effective action. We do this by setting up a simple model for the behaviour of such points, similar to that done in [64]. As an approximation, assume that the gauge fields are constant, at least near the points of interest. Now, consider the following parametrisation of the Higgs field by real fields

$$\Phi = \frac{1}{\sqrt{2}}\left(\phi_4 \mathbf{1} + i\phi_a \tau^a\right). \tag{3.68}$$

Without loss of generality, we will consider the point of interest to be located at the origin. As a simple example, we approximate the configuration near the origin by Fourier modes

$$\phi_a(x) = \sin(x \cdot k_\alpha - \epsilon_\alpha), \qquad \alpha = 1, \ldots, 4. \tag{3.69}$$

If all the $\epsilon_\alpha \ll 1$, then the Higgs products $\sqrt{\phi_\alpha \phi_\alpha}$, which appear in the denominators of Eq. (3.67), will be small. To actually get a local minimum, the vectors k_α should span the four dimensional space. Making a linear approximation, the worst of the divergences contribute always as odd polynomials multiplying an even function.

The second argument is that one can reorganise the effective action to introduce a natural infrared cutoff. As the previous argument suggests, the divergences in the effective lagrangian need not translate into divergences for the effective action. Nevertheless, when implementing the operator in Eq. (3.67) on the lattice, having the effective lagrangian blow up will cause problems for the simulation. The introduction of an infrared cutoff then becomes necessary. The easiest cutoff to implement would simply be to replace $\phi^\dagger\phi \to \phi^\dagger\phi + cutoff$. This simple cutoff can be motivated by appealing to thermal masses. However, in the case of cold electroweak baryogenesis, where the phase transition takes place at effectively zero temperature, it is not clear that this cutoff would be appropriate. A more natural cutoff comes from the inclusion of the native scale of the problem.

A gradient expansion is conceptually on solid ground when $\partial\phi/\phi^2 \ll 1$, and this sets a native scale for the problem. Therefore, the following substitution in the denominators of Eq. (3.67)

$$\phi^\dagger\phi \to \phi^\dagger\phi + c\frac{\phi^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi}\frac{\phi^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi}, \tag{3.70}$$

can be a natural infrared cutoff. Whenever $\phi^\dagger\phi$ is much larger than the derivative terms, it will dominate the denominators and Eq. (3.67) will not be much affected, but whenever the original operator would start to blowup, the cutoff would then dominate in the denominator and keep it from doing this. The constant c should be picked in order that the transition from the regime where $\phi^\dagger\phi$ dominates to the regime where $\frac{\phi^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi}\frac{\phi^\dagger \mathcal{D}_\alpha \phi}{\phi^\dagger\phi}$ has a desired behaviour, e.g. does not present a hump.

The introduction of the cutoff can be achieved by reorganising the effective action to the desired order in the expansion. In the present case, where we have only one order of the expansion, the introduction of the cutoff can simply be done by the above mentioned replacement. However, if one were to have even higher orders in the expansion, then the new coefficients at each order could change. As an example, consider a covariant derivative expansion which schematically is of the form

$$c_1\frac{(\mathcal{D}\phi)^2}{\phi^4} + c_2\frac{(\mathcal{D}\phi)^3}{\phi^6} + c_3\frac{(\mathcal{D}\phi)^4}{\phi^8} + \mathcal{O}\left(\mathcal{D}^5\right). \tag{3.71}$$

The expansion could be reorganised to

$$\hat{c}_1\frac{(\mathcal{D}\phi)^2}{\left(\phi^2 + c\frac{(\mathcal{D}\phi)^2}{\phi^2}\right)^2} + \hat{c}_2\frac{(\mathcal{D}\phi)^3}{\left(\phi^2 + c\frac{(\mathcal{D}\phi)^2}{\phi^2}\right)^3} + \hat{c}_3\frac{(\mathcal{D}\phi)^4}{\left(\phi^2 + c\frac{(\mathcal{D}\phi)^2}{\phi^2}\right)^4} + \mathcal{O}\left(\mathcal{D}^5\right), \tag{3.72}$$

where the $\hat{c}$ are defined in terms of the old coefficients, with the old coefficients at every order contributing to the new coefficients at every subsequent order as well.

As mentioned previously, CP violating terms are found first at six order of the covariant derivative expansion, and we have only calculated up to that order. Going beyond this order is not only a difficult technical undertaking, it is also at this time not very encouraging. After all, the operator in Eq. (3.45) is already quite an undertaking to implement on the lattice. Higher order terms would not serve, at this time, for lattice simulations, and so applications for that expansion would be limited.

Regardless of this last technical issue, we believe that these two arguments suggest that the CP violating terms presented in this work should be stable in the infrared. In simulations of cold electroweak baryogenesis, this was the case for the operator $\phi^2 \mathrm{Tr}\, F\tilde{F}$, the behaviour of the operator at these critical points was not very relevant. In the end, the arguments presented here are only tentative, and the real proof will be in the lattice simulation.

Chapter 4

Path Integral Formulation of Bödeker's Theory

We base our analysis on Bödeker's effective theory despite the fact that Bödeker has also derived a generalised Boltzmann–Langevin equation which is valid to all orders in $[\log(1/g)]^{-1}$ [74], of which Bödeker's effective theory is merely the leading logarithmic approximation and the existence of other more general approaches, e.g. [75]. We choose this approximation because of the tractability of the analytic approach within this framework. The more general Boltzmann–Langevin equation not only is far more complicated, but is also not renormalisable by power counting [76]. On the other hand, the effective theory with which we deal here is ultraviolet finite, and is known to still be valid at next-to-leading logarithmic order provided one uses the next-to-leading logarithmic order colour conductivity σ [47].

4.1 Transcription to a Path Integral

4.1.1 Transcription in $A_0 = 0$ Gauge

According to Bödeker's effective theory the dynamics of the soft modes of the gauge field is described to leading logarithmic order by the Langevin equation [41]

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a = \boldsymbol{\zeta}^a \tag{4.1}$$

which is written in $A_0 = 0$ gauge and where $\boldsymbol{\zeta}$ is a gaussian white noise stochastic force. The stochastic force field incorporates the influence of higher momentum modes and has the correlator

$$\left\langle \boldsymbol{\zeta}^{ai}(t, \mathbf{x}) \boldsymbol{\zeta}^{bj}(t', \mathbf{x}') \right\rangle = 2\sigma T \, \delta^{ij} \delta^{ab} \, \delta(t - t') \, \delta^{D-1}(\mathbf{x} - \mathbf{x}') \tag{4.2}$$

reflecting its gaussian white noise character. Here and in the following, the number of spacial dimensions is $D - 1 = 3$, however, we leave D unspecified to allow for dimensional regularisation later. The only physical parameters entering Eqs. (4.1) and (4.2), and therefore the effective theory, are the temperature T, the colour conductivity σ, and the self coupling of the gauge field hidden in the definition of the covariant derivative $\mathbf{D}^{ab} = \delta^{ab}\nabla - g f^{abc} \mathbf{A}^c$.

The procedure of reformulating a Langevin equation like Eq. (4.1) in the form of a field theoretic path integral is well-known [44–46]: According to Eq. (4.1), the gauge field evolves, starting from certain initial conditions, under the influence of the stochastic force. An arbitrary observable of the theory then is defined by some functional of the gauge field $F[\mathbf{A}]$ and given by the expectation value of that functional with respect to the possible realisations of the stochastic force

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\boldsymbol{\zeta} \, F[\mathbf{A}^s[\boldsymbol{\zeta}]] \varrho[\boldsymbol{\zeta}] = \int \mathcal{D}\boldsymbol{\zeta} \, F[\mathbf{A}^s[\boldsymbol{\zeta}]] \exp\left\{ -\frac{1}{4\sigma T} \int dt \, d^{D-1}x \; \boldsymbol{\zeta}^a(t, \mathbf{x}) \cdot \boldsymbol{\zeta}^a(t, \mathbf{x}) \right\} \tag{4.3}$$

Here we have denoted by $\mathbf{A}^s[\boldsymbol{\zeta}]$ the solution of Eq. (4.1) for a specific choice of the stochastic force and the given initial conditions.

To proceed and recast the effective theory of the gauge field into a form resembling the path integral formulation of an 'ordinary' quantum field theory we would rather like to have a path integral running over the gauge field than running over the stochastic force. This can be achieved by inserting unity in an appropriate way. In fact, one has

$$1 = \int \mathcal{D}\mathbf{E}\, \delta(\mathbf{E} - \boldsymbol{\zeta}) = \int_{\text{(i.c.)}} \mathcal{D}\mathbf{A}\, \mathrm{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) \delta(\mathbf{E}[\mathbf{A}] - \boldsymbol{\zeta}) \tag{4.4}$$

where we choose the functional $\mathbf{E}[\mathbf{A}]$ as the left-hand side of Eq. (4.1)

$$\mathbf{E}^a[\mathbf{A}] = \mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a \tag{4.5}$$

The invertibility of $\mathbf{E}[\mathbf{A}]$ is essential to justify the change of variables in Eq. (4.4). It follows from the parabolic nature of the expression and from the restriction to those gauge field configurations in the second path integral satisfying the initial conditions.

Because Eq. (4.4) holds independently of $\boldsymbol{\zeta}$, it can be inserted into the path integral (4.3). The delta function then assures that only those gauge field configurations contribute to the integral that obey $\mathbf{E}[\mathbf{A}] = \boldsymbol{\zeta}$. Due to our choice of $\mathbf{E}[\mathbf{A}]$, however, this is identical to the condition $\mathbf{A} = \mathbf{A}^s[\boldsymbol{\zeta}]$. Thus, after inserting the delta function we may replace $\mathbf{A}^s[\boldsymbol{\zeta}]$ in the path integral simply by the integration variable $\mathbf{A}$, and we are left with

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\boldsymbol{\zeta}\, \varrho[\boldsymbol{\zeta}] \int_{\text{(i.c.)}} \mathcal{D}\mathbf{A}\, \mathrm{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) \delta(\mathbf{E}[\mathbf{A}] - \boldsymbol{\zeta})\, F[\mathbf{A}] \tag{4.6}$$

Moreover, the restriction to field configurations obeying a specific set of initial conditions can be dropped if these initial conditions are specified at $t = -\infty$. This is a consequence of their transversal component always being damped and the fact that any longitudinal contribution drops out whenever a gauge invariant observable is calculated. In case of a gauge variant quantity, however, a damping of the longitudinal component can be achieved by introducing an additional gauge fixing term into the Langevin equation [45]. This will be necessary anyway in the following section in order to generalise from $A_0 = 0$ gauge. Henceforth, we will therefore drop the restriction on the path integration in Eq. (4.6).

At this point, one has two choices. One possibility is to proceed by doing the $\boldsymbol{\zeta}$ integral with the help of the delta function. This results in a theory containing only the gauge field (and perhaps some additional ghost fields to be introduced later), however, at the expense of rather complicated interactions: the functional $\mathbf{E}[\mathbf{A}]$ shows up as argument of the gaussian probability distribution, and since $\mathbf{E}[\mathbf{A}]$ contains terms up to $\mathbf{A}^3$, the action would inherit vertices of up to sixth order.

To avoid this situation, we instead choose to introduce an additional auxiliary field $\boldsymbol{\lambda}$ to represent the delta function

$$\delta(\mathbf{E}[\mathbf{A}] - \boldsymbol{\zeta}) = \int \mathcal{D}\boldsymbol{\lambda}\, \exp\left\{ i \int dt\, d^{D-1}x\, \boldsymbol{\lambda}^a \cdot (\mathbf{E}^a[\mathbf{A}] - \boldsymbol{\zeta}^a) \right\} \tag{4.7}$$

In this way, one can still perform the $\boldsymbol{\zeta}$ integral that becomes gaussian, thereby eliminating the stochastic force field from the theory. One obtains

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda}\, \mathrm{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) F[\mathbf{A}]\, e^{-S[\mathbf{A}, \boldsymbol{\lambda}]} \tag{4.8}$$

with

$$S[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \left[\sigma T\, \boldsymbol{\lambda}^a \cdot \boldsymbol{\lambda}^a - i \boldsymbol{\lambda}^a \cdot \mathbf{E}^a[\mathbf{A}] \right] \tag{4.9}$$

The determinant in Eq. (4.8) need not be taken into account since it can be shown to be a constant in dimensional regularisation (see Appendix B for an explicit calculation). We could, nevertheless, introduce a ghost representation of the determinant referring to the corresponding ghost fields as equation of motion (EOM) ghosts in the following. As a benefit of doing so the action (4.9) would be endowed with a BRST symmetry, allowing to easily obtain a kind of Ward identities (so-called stochastic Ward identities) reflecting the origin of the theory in a stochastic differential equation. Since it is desirable to obtain as many non-perturbative identities as possible in order to find a judicious ansatz for the truncation of the DSEs, introducing EOM ghosts, at first, seems the natural way to proceed.

However there is another type of Ward identities related to gauge invariance. Unfortunately, the gauge ghosts to be introduced to obtain these gauge Ward identities will break the stochastic BRST symmetries. So, instead of introducing EOM ghosts now, we will later introduce gauge ghosts in order to obtain the gauge Ward identities. The stochastic Ward identities will be derived without the help of a BRST symmetry by directly referring to the fundamental structure of the theory that reflects its origin in a stochastic differential equation.

For now, absorbing the constant determinant in the measure, we are left with

$$\langle F[\mathbf{A}]\rangle = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\, F[\mathbf{A}]\, e^{-S[\mathbf{A},\boldsymbol{\lambda}]} \tag{4.10}$$

where the action S is given by Eq. (4.9).

4.1.2 Upgrading to κ Gauge

Bödeker's theory is written in $A_0 = 0$ gauge, and so is our transcription as field theoretic path integral so far. At the end of the day, however, we will be forced to use an approximation to solve the non-perturbative equations obtained, e.g. DSEs, and this approximation might introduce gauge artefacts into the calculation. In order to allow some control over the gauge dependence of the results, we need to base our derivations on a reformulation of Bödeker's equation in a more general gauge.

In [65], Zinn-Justin and Zwanziger have shown that adding a term to Eq. (4.1) that is tangent to the gauge orbit

$$\mathbf{D}^{ab}\times \mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a + \mathbf{D}^{ab}v^b[\mathbf{A}]) = \boldsymbol{\zeta}^a \tag{4.11}$$

has no effect on expectation values of gauge-invariant objects of the form $F[\mathbf{A}]$. This is not the most general modification of Eq. (4.1) which leaves expectation values of gauge invariant objects unchanged [66], but it suffices for our purposes. As long as $v^a[\mathbf{A}]$ contains no time derivatives, the added term has no effect in calculations of gauge invariant objects.

We can reformulate this fact in a different way: Since the non-abelian electric field is given by $\mathbf{E}^a = -\dot{\mathbf{A}}^a - \mathbf{D}^{ab}A^{b0}$, one may rewrite Bödeker's equation in the compact form

$$\mathbf{D}^{ab}\times \mathbf{B}^b - \sigma\mathbf{E}^a = \boldsymbol{\zeta}^a \tag{4.12}$$

which then may be interpreted in any of the so-called flow gauges $A^{a0} = v^a[\mathbf{A}]$ with no time derivatives allowed inside the functional $v^a[\mathbf{A}]$.

The restriction that $v^a[\mathbf{A}]$ does not contain time derivatives plays a more substantial role in our context than in the context of stochastic quantisation which was the object of Zinn-Justin and Zwanziger: In stochastic quantisation the time variable describes a *fictitious* time that is introduced only as a device to reinterpret a given Euclidean quantum field theory as the limit of a stochastic process for large values of the fictitious time [45]. Absence of time derivatives in stochastic quantisation therefore means absence of derivatives with respect to fictitious time and does not pose any restrictions to usual time derivatives. In our context, on the contrary, time is the real, physical time and the restrictions above narrow down the class of possible gauges leading to a well defined Langevin equation.

Moreover, because of the different role of the time variable, we also have a component of the gauge field that is associated with the t variable of the Langevin equation. In stochastic quantisation this is not the case because t is fictitious and the time associated with A_0 is just the zero component of the Euclidean $\mathbf{x}$ vector. To cope with this different structure, to some extent will demand a generalisation of the proof of Zinn-Justin and Zwanziger.

In effect, we not only have to prove that gauge invariant objects of the form $F[\mathbf{A}]$ are left invariant by the introduction of the term $v^a[\mathbf{A}]$, as was shown in [65]. Instead we have to prove the following: Given Bödeker's equation in the form (4.12) and a gauge invariant functional $F[A^0, \mathbf{A}]$, then any choice of a flow gauge leads to the same result. Or put in different words, calculating $\langle F[\,v[\mathbf{A}], \mathbf{A}\,]\rangle$ by means of the equation Eq. (4.11) gives always the same value, independent of $v[\mathbf{A}]$.

We now proceed in a similar manner to [65]. Let us consider the left-hand side of Eq. (4.11) where we add a small variation of the $v^a[\mathbf{A}]$ term. We evaluate this expression for a gauge field that is subject to an arbitrary, infinitesimal gauge transformation $\mathbf{A}'^a = \mathbf{A}^a + \mathbf{D}^{ab}\omega^b$ and find

$$\begin{aligned} &\mathbf{D}'^{ab}\times \mathbf{B}'^b + \sigma(\dot{\mathbf{A}}'^a + \mathbf{D}'^{ab}v^b[\mathbf{A}'] + \mathbf{D}'^{ab}\delta v^b[\mathbf{A}']) \\ &= \; (\delta^{ab} + gf^{abc}\omega^c)\left[\mathbf{D}^{bd}\times \mathbf{B}^d + \sigma(\dot{\mathbf{A}}^b + \mathbf{D}^{bd}v^d[\mathbf{A}])\right] + \sigma\mathbf{D}^{ab}\left[\frac{\partial\omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}]\right] \end{aligned} \tag{4.13}$$

Here we have used

$$\mathbf{D}'^{ab}\times \mathbf{B}'^b \;=\; (\delta^{ab} + gf^{abc}\omega^c)\,\mathbf{D}^{bd}\times \mathbf{B}^d \tag{4.14}$$

$$\dot{\mathbf{A}}'^a \;=\; (\delta^{ab} + gf^{abc}\omega^c)\,\dot{\mathbf{A}}^b \;+\; \mathbf{D}^{ab}\,\frac{\partial\omega^b}{\partial t} \tag{4.15}$$

i.e. the product $\mathbf{D}^{ab}\times \mathbf{B}^b$ transforms covariantly whereas the transformation of $\dot{\mathbf{A}}^a$ has a covariant and non-covariant contribution. In the same way we have split the transformation of $v^a[\mathbf{A}]$ into a covariant and non-covariant part: Starting from

$$v^a[\mathbf{A}'](t,\mathbf{x}) = v^a[\mathbf{A}](t,\mathbf{x}) + \int d^{D-1}y\,\frac{\delta v^a[\mathbf{A}](t,\mathbf{x})}{\delta A^{bi}(t,\mathbf{y})}\,\delta A^{bi}(t,\mathbf{y}) \tag{4.16}$$

we have indeed

$$v^a[\mathbf{A}'](t,\mathbf{x}) = (\delta^{ab} + gf^{abc}\omega^c)\,v^b[\mathbf{A}](t,\mathbf{x}) \;+\; [H[\mathbf{A}]\omega]^a(t,\mathbf{x}) \tag{4.17}$$

where $\delta A^{bi} = D_i^{bc}\omega^c$ has been used and we have introduced the abbreviation

$$[H[\mathbf{A}]\omega]^a(t,\mathbf{x}) = \int d^{D-1}y\,\frac{\delta v^a[\mathbf{A}](t,\mathbf{x})}{\delta A^{bi}(t,\mathbf{y})}\,(D_i^{bc}\omega^c)(t,\mathbf{y}) - gf^{abc}v^b[\mathbf{A}](t,\mathbf{x})\,\omega^c(t,\mathbf{x}) \tag{4.18}$$

Note that the functional derivatives in Eqs. (4.16) and (4.18) are only with respect to a spacial variation because $v^a[\mathbf{A}]$ does not contain any time derivatives (otherwise we would also have to integrate over time). Let us give the explicit form of this somewhat frightening expression for $H[A]\omega$ in the case of the choice $v^a[\mathbf{A}] = -\frac{1}{\kappa}\nabla\cdot\mathbf{A}^a$. One simply obtains

$$[H[\mathbf{A}]\omega]^a(t,\mathbf{x}) = -\frac{1}{\kappa}\,(\mathbf{D}^{ab}\cdot\nabla\omega^b)(t,\mathbf{x}) \tag{4.19}$$

Finally, Eq. (4.17) leads to

$$\mathbf{D}'^{ab}v^b[\mathbf{A}'] = (\delta^{ab} + gf^{abc}\omega^c)\,\mathbf{D}^{bd}v^d[\mathbf{A}] \;+\; \mathbf{D}^{ab}\,[H[\mathbf{A}]\omega]^b \tag{4.20}$$

where it was used that ω is infinitesimal and of course

$$\mathbf{D}'^{ab}\delta v^b[\mathbf{A}'] = \mathbf{D}^{ab}\delta v^b[\mathbf{A}] \tag{4.21}$$

because δv is infinitesimal itself.

Let us now come back to Eq. (4.13) and its meaning. Suppose the gauge field, before the gauge transformation has been performed, was a solution of Bödeker's equation with the $v^a[\mathbf{A}]$ term present, but without the additional $\delta v^a[\mathbf{A}]$ term. In other words, the original gauge field was a solution of Eq. (4.11). We can then replace the first square bracket on the right-hand side of Eq. (4.13) by the stochastic force and find

$$\begin{aligned} &\mathbf{D}'^{ab}\times \mathbf{B}'^{b} + \sigma(\dot{\mathbf{A}}'^{a} + \mathbf{D}'^{ab}v^b[\mathbf{A}'] + \mathbf{D}'^{ab}\delta v^b[\mathbf{A}']) \\ &= \quad \boldsymbol{\zeta}'^a \ + \sigma \mathbf{D}^{ab}\left[\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}]\right] \end{aligned} \tag{4.22}$$

This means, if we subject the original gauge field to an arbitrary, infinitesimal gauge transformation with parameter ω, then the gauge transformed field will be a solution of Eq. (4.22), i.e. of the original equation with v replaced by $v + \delta v$ and the stochastic force transformed in the same way as the gauge field ... but with an ugly additional term on the right-hand side. However, one can play a dirty trick: What was said so far was true for an *arbitrary* gauge transformation. But if we demand ω to be a solution of

$$\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}] = 0 \tag{4.23}$$

then the square bracket on the right of Eq. (4.22) will vanish and we finally arrive at

$$\mathbf{D}'^{ab}\times \mathbf{B}'^{b} + \sigma(\dot{\mathbf{A}}'^{a} + \mathbf{D}'^{ab}v^b[\mathbf{A}'] + \mathbf{D}'^{ab}\delta v^b[\mathbf{A}']) = \boldsymbol{\zeta}'^a \tag{4.24}$$

However, there is a certain subtlety that we want to draw attention to. To clarify this point, let us once again repeat the line of reasoning: Starting with a gauge field being solution of

$$\mathbf{D}^{ab}\times \mathbf{B}^{b} + \sigma(\dot{\mathbf{A}}^{a} + \mathbf{D}^{ab}v^b[\mathbf{A}]) = \boldsymbol{\zeta}^a \tag{4.25}$$

we search for a gauge transformation ω that obeys

$$\frac{\partial \omega^a}{\partial t} + [H[\mathbf{A}]\omega]^a + \delta v^a[\mathbf{A}] = 0 \tag{4.26}$$

(and we can always find such an ω because (4.26) is a linear, inhomogeneous equation with given inhomogeneity $\delta v^a[\mathbf{A}]$). Then the gauge field transformed with *this* ω, $\mathbf{A}'^a = \mathbf{A}^a + \mathbf{D}^{ab}\omega^b$, is a solution of the original equation with v replaced by $v+\delta v$ and the stochastic force also transformed by the same ω

$$\mathbf{D}'^{ab}\times \mathbf{B}'^{b} + \sigma(\dot{\mathbf{A}}'^{a} + \mathbf{D}'^{ab}v^b[\mathbf{A}'] + \mathbf{D}'^{ab}\delta v^b[\mathbf{A}']) = \boldsymbol{\zeta}'^a \tag{4.27}$$

The subtle point is the following: The original gauge field $\mathbf{A}$ is a solution of Eq. (4.25) and thus depends on the stochastic force $\boldsymbol{\zeta}$, of course. But $\mathbf{A}$ is an input of Eq. (4.26) that determines ω. Therefore, ω via $\mathbf{A}$ too depends on $\boldsymbol{\zeta}$. As a consequence of this, $\boldsymbol{\zeta}'$ inherits a non-trivial dependence on $\boldsymbol{\zeta}$: The stochastic force $\boldsymbol{\zeta}'$ not only depends on $\boldsymbol{\zeta}$ because it is the gauge transform of $\boldsymbol{\zeta}$, but also because the gauge transformation itself depends on $\boldsymbol{\zeta}$

$$\boldsymbol{\zeta}'^a = (\delta^{ab} + gf^{abc}\omega^c[\zeta])\,\boldsymbol{\zeta}^b \tag{4.28}$$

We denote by $\mathbf{A}^s[\boldsymbol{\zeta}, v, \mathbf{A}_{\text{ini}}]$ the solution of Eq. (4.11) for the specific realisation $\boldsymbol{\zeta}$ of the stochastic force term and initial conditions $\mathbf{A}_{\text{ini}}$. Correspondingly, let $\mathbf{A}^s[\boldsymbol{\zeta}, v+\delta v, \mathbf{A}_{\text{ini}}]$ denote the solution of this equation with v replaced by $v + \delta v$ and for the same stochastic force and initial conditions. We can then express the contents of Eq. (4.27) in this new notation

$$\mathbf{A}^s[{}^{\omega}\boldsymbol{\zeta}, v+\delta v, {}^{\omega}\mathbf{A}_{\text{ini}}] = {}^{\omega}\mathbf{A}^s[\boldsymbol{\zeta}, v, \mathbf{A}_{\text{ini}}] \tag{4.29}$$

where the superscript ω indicates gauge transformation with the special parameter ω corresponding to the solution on the right-hand side via Eq. (4.26).

After these preparations we can now show that gauge invariant expectation values $\langle F[A^0, \mathbf{A}]\rangle$ are independent of the choice of $v^a[\mathbf{A}]$. To this end, let us write the gauge invariant observable as functional of the non-abelian electric and magnetic field

$$\begin{aligned} \mathbf{E}^a &= -\dot{\mathbf{A}}^a - \mathbf{D}^{ab} A^{b0} \\ \mathbf{B}^a &= \nabla \times \mathbf{A}^a + \tfrac{1}{2} g f^{abc} \mathbf{A}^b \times \mathbf{A}^c \end{aligned} \tag{4.30}$$

We then have

$$\langle F[\mathbf{E},\mathbf{B}]\rangle_{v+\delta v} = \int \mathcal{D}\boldsymbol{\zeta}'\ \varrho[\boldsymbol{\zeta}']\ F\big[\,\mathbf{E}_{v+\delta v}[\mathbf{A}], \mathbf{B}_{v+\delta v}[\mathbf{A}]\,\big]_{\mathbf{A}=\mathbf{A}^s[\boldsymbol{\zeta}', v+\delta v, \mathbf{A}'_{\mathrm{ini}}]} \tag{4.31}$$

with

$$\mathbf{E}^a_{v+\delta v}[\mathbf{A}] = -\dot{\mathbf{A}}^a - \mathbf{D}^{ab} v^b[\mathbf{A}] - \mathbf{D}^{ab}\delta v^b[\mathbf{A}] \tag{4.32}$$

and $\mathbf{B}_{v+\delta v}[\mathbf{A}] = \mathbf{B}_v[\mathbf{A}]$ as in Eq. (4.30). Changing variables according to Eq. (4.28), one obtains

$$\langle F[\mathbf{E},\mathbf{B}]\rangle_{v+\delta v} = \int \mathcal{D}\boldsymbol{\zeta}\ \mathrm{Det}\left(\frac{\delta^\omega\boldsymbol{\zeta}}{\delta\boldsymbol{\zeta}}\right) \varrho[{}^\omega\boldsymbol{\zeta}]\ F\big[\,\mathbf{E}_{v+\delta v}[\mathbf{A}], \mathbf{B}_{v+\delta v}[\mathbf{A}]\,\big]_{\mathbf{A}=\mathbf{A}^s[{}^\omega\boldsymbol{\zeta}, v+\delta v, \mathbf{A}'_{\mathrm{ini}}]} \tag{4.33}$$

We now use independence on the initial conditions, the transformation property (4.29), gauge invariance of $\varrho[\boldsymbol{\zeta}]$ and finally the fact that the determinant is unity (shown in Appendix B). This all together leads to

$$\langle F[\mathbf{E},\mathbf{B}]\rangle_{v+\delta v} = \int \mathcal{D}\boldsymbol{\zeta}\ \varrho[\boldsymbol{\zeta}]\ F\big[\,\mathbf{E}_{v+\delta v}[{}^\omega\mathbf{A}], \mathbf{B}_{v+\delta v}[{}^\omega\mathbf{A}]\,\big]_{\mathbf{A}=\mathbf{A}^s[\boldsymbol{\zeta}, v, \mathbf{A}_{\mathrm{ini}}]} \tag{4.34}$$

Taking into account the transformation properties (4.15), (4.20) and (4.21), we find

$$\begin{aligned} \mathbf{E}^a_{v+\delta v}[{}^\omega\mathbf{A}] &= ({}^\omega\mathbf{E}_v[\mathbf{A}])^a - \mathbf{D}^{ab}\Big[\tfrac{\partial\omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}]\Big] = ({}^\omega\mathbf{E}_v[\mathbf{A}])^a \\ \mathbf{B}^a_{v+\delta v}[{}^\omega\mathbf{A}] &= ({}^\omega\mathbf{B}_v[\mathbf{A}])^a \end{aligned} \tag{4.35}$$

and thus

$$\langle F[\mathbf{E},\mathbf{B}]\rangle_{v+\delta v} = \int \mathcal{D}\boldsymbol{\zeta}\ \varrho[\boldsymbol{\zeta}]\ F\big[\,{}^\omega\mathbf{E}_v[\mathbf{A}], {}^\omega\mathbf{B}_v[\mathbf{A}]\,\big]_{\mathbf{A}=\mathbf{A}^s[\boldsymbol{\zeta}, v, \mathbf{A}_{\mathrm{ini}}]} = \langle F[\mathbf{E},\mathbf{B}]\rangle_v \tag{4.36}$$

because $F[\mathbf{E},\mathbf{B}]$ is a gauge invariant functional.
Consequently, we have shown that Bödeker's equation in $A_0 = 0$ gauge

$$\mathbf{D}^{ab}\times\mathbf{B}^b + \sigma\dot{\mathbf{A}}^a = \boldsymbol{\zeta}^a \tag{4.37}$$

can equivalently be formulated in any flow gauge

$$\mathbf{D}^{ab}\times\mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a + \mathbf{D}^{ab}v^b[\mathbf{A}]) = \boldsymbol{\zeta}^a \tag{4.38}$$

without any time derivatives allowed inside the functional $v^a[\mathbf{A}]$. We will henceforth use the special choice $A^{a0} = v^a[\mathbf{A}] = -\frac{1}{\kappa}\nabla\cdot\mathbf{A}^a$ and refer to it as κ gauge. This is a natural choice for $v^a[\mathbf{A}]$, since it has the lowest order in $\mathbf{A}$, preserves colour invariance, and with $\kappa > 0$ the term $\mathbf{D}^{ab}v^b[\mathbf{A}]$ provides a globally restoring force along gauge orbits [67], while at the same time having the correct dimensions.

4.2 BRST Symmetric Action and Ward-Takahashi Identities

We have argued that in order to derive any reliable statements from our theory, it is essential to gain some control over the gauge dependence possibly introduced by the truncation of the

Dyson–Schwinger equations. This was our main motivation to generalise Bödeker's equation from $A_0 = 0$ gauge to a more general class of flow gauges. In addition to this, the corresponding introduction of a gauge-fixing force has a welcome side-effect: It solves at the same time the problem of undamped longitudinal components of the initial gauge field configuration.

However, the detection of an unphysical gauge dependence is not what we really want; in fact, we would rather like to avoid it. The ultimate goal is to construct a truncation scheme that is physically reasonable and does not (or, realistically speaking, only slightly) violate the gauge symmetry.

To this end, we need identities expressing this symmetry on the level of n-point functions, i.e. we need the Ward-Takahashi identities of the theory. [1]

Any physically reasonable truncation will have to respect these identities. Besides this conceptual importance, we may also hope that some of the Ward identities to be derived in the following will be of some practical use in solving the DSEs: In ordinary QCD, for instance, the full gluon propagator in covariant gauge is restricted to being purely transversal as a consequence of the Ward identities. This leads, of course, to a great simplification in the DSEs of QCD.

In this section, we study three different kinds of non-perturbative identities: gauge Ward identities, i.e. Slavnov-Taylor identities; stochastic Ward identities; and ghost number conservation.

4.2.1 Constructing a BRST Symmetric Action

In Section 4.1.2, we saw that Eq. (4.11) transforms covariantly only under a restricted class of gauge transformations. Obtaining the gauge Ward identities with this restriction turns out to be rather cumbersome. Instead, we will raise the gauge parameter ω to life by introducing into the theory an additional (Grassman valued) field that realizes the constraint on the gauge transformations. The resulting action will be endowed with a BRST symmetry, and we will be able to obtain the gauge Ward identities in a straight-forward manner.

Setting $\delta v^a[\mathbf{A}]$ to zero in Eq. (4.23) we see that Eq. (4.11) transforms covariantly under gauge transformations which obey

$$\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b = 0 \tag{4.39}$$

Note that the introduction of $v^a[\mathbf{A}]$ does not restrict the gauge group further than it already would be. Even without the extra term, the gauge transformations would have to be restricted in order for Eq. (4.11) to be gauge covariant.

The restriction in Eq. (4.39) can be taken into account in the path integral in the following manner. Define a term $\gamma^a[\omega, \mathbf{A}]$ from the left-hand side of Eq. (4.39), which for our choice of $v^a[\mathbf{A}]$ takes the following form

$$\gamma^a[\omega, \mathbf{A}] = \frac{\partial \omega^a}{\partial t} - \frac{1}{\kappa}\,\mathbf{D}^{ab}\cdot \nabla\omega^b \tag{4.40}$$

Perform a change of variables from γ to ω in the following Grassmann integral representation of unity

$$1 = \int \mathcal{D}\gamma\;\delta(\gamma) = \int \mathcal{D}\omega\;\frac{1}{\mathrm{Det}\left(\frac{\delta\gamma[\omega,\mathbf{A}]}{\delta\omega}\right)}\;\delta\Big(\frac{\partial \omega^a}{\partial t} - \frac{1}{\kappa}\,\mathbf{D}^{ab}\cdot \nabla\omega^b\Big) \tag{4.41}$$

Since the determinant is Grassman even, it no longer depends on ω and it can be pulled out of the integral. The determinant is a constant, and can be calculated in a similar manner to the determinant in Eq. (4.8) (see Appendix B for the explicit calculation).

Inserting the integral representation of the Grassmann delta function

$$\delta(\gamma) = \int \mathcal{D}\bar{\omega}\;\exp\left\{\int dx\;\bar{\omega}^a(x)\,\gamma^a(x)\right\} \tag{4.42}$$

[1] In the non-abelian context, these identities are often referred to as Slavnov-Taylor identities. However, following the terminology of Ref. [65], we denote these identities as gauge Ward identities and stochastic Ward identities, corresponding to the two different types that will be encountered in this work.

and absorbing the constant determinant into the measure, we find the identity

$$1 = \int \mathcal{D}\omega\mathcal{D}\bar{\omega}\ \exp\left\{\int dx\ \bar{\omega}^a(x)\Big(\delta^{ab}\frac{\partial}{\partial t} - \frac{1}{\kappa}\,\mathbf{D}^{ab}\cdot\nabla\Big)\omega^b(x)\right\} \tag{4.43}$$

which holds independent of the gauge field $\mathbf{A}$. Therefore, it can be inserted into the path integral representation of the generating functional, Eq. (4.10), leading to

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\ \exp\left\{-S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] + \int dx\ \mathbf{J}^a(x)\mathbf{A}^a(x)\right\} \tag{4.44}$$

with the action now given by

$$S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] = S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]\ +\ S^{(\mathrm{GG})}[\mathbf{A},\omega,\bar{\omega}] \tag{4.45}$$

where $S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]$ is the contribution of the dynamical fields as before

$$S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] = \int dx\ \Big[\sigma T\,\boldsymbol{\lambda}^a\cdot\boldsymbol{\lambda}^a - i\boldsymbol{\lambda}^a\cdot\Big(\mathbf{D}^{ab}\times\mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a - \tfrac{1}{\kappa}\,\mathbf{D}^{ab}\,\nabla\cdot\mathbf{A}^b)\Big)\Big] \tag{4.46}$$

and

$$S^{(\mathrm{GG})}[\mathbf{A},\omega,\bar{\omega}] = \int dx\ \Big[-\bar{\omega}^a\dot{\omega}^a + \frac{1}{\kappa}\,\bar{\omega}^a\mathbf{D}^{ab}\cdot\nabla\omega^b\Big] \tag{4.47}$$

is the new contribution containing the gauge ghosts ω and $\bar{\omega}$.

4.2.2 Gauge Ward Identities

The Slavnov-Taylor identities can be derived by noting that the action (4.45) is invariant under the following BRST transformation

$$\begin{aligned} \delta_\varepsilon\mathbf{A}^a(x) &= \mathbf{D}^{ab}(x)\,\varepsilon\omega^b(x) & \delta_\varepsilon\omega^a(x) &= \tfrac{1}{2}gf^{abc}\varepsilon\omega^c(x)\omega^b(x) \\ \delta_\varepsilon\boldsymbol{\lambda}^a(x) &= gf^{abc}\varepsilon\omega^c(x)\boldsymbol{\lambda}^b(x) & \delta_\varepsilon\bar{\omega}^a(x) &= gf^{abc}\varepsilon\omega^c(x)\bar{\omega}^b(x) + i\varepsilon\sigma\mathbf{D}^{ab}(x)\cdot\boldsymbol{\lambda}^b(x) \end{aligned} \tag{4.48}$$

where ε is a constant Grassmann parameter. Introduce the finite BRST operator s such that the result of acting on a functional of the fields $\mathbf{A}$, $\boldsymbol{\lambda}$, ω and $\bar{\omega}$ is defined as (left) derivative with respect to the parameter ε of the variations in Eq. (4.48). We thus have

$$sF[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}]\ =\ \frac{\partial}{\partial\varepsilon}\,\delta_\varepsilon F[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] \tag{4.49}$$

or conversely

$$\delta_\varepsilon F[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}]\ =\ \varepsilon\,sF[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] \tag{4.50}$$

From Eq. (4.49) one finds the following representation

$$s = \int dx\ \Big[(sA^{ai})\,\frac{\delta}{\delta A^{ai}}\ +\ (s\lambda^{ai})\,\frac{\delta}{\delta\lambda^{ai}}\ +\ (s\omega^a)\,\frac{\delta}{\delta\omega^a}\ +\ (s\bar{\omega}^a)\,\frac{\delta}{\delta\bar{\omega}^a}\Big] \tag{4.51}$$

with the finite BRST transforms of the fundamental fields given by Eq. (4.48)

$$\begin{aligned} s\mathbf{A}^a(x) &= \mathbf{D}^{ab}(x)\,\omega^b(x) & s\omega^a(x) &= \tfrac{1}{2}gf^{abc}\omega^c(x)\omega^b(x) \\ s\boldsymbol{\lambda}^a(x) &= gf^{abc}\omega^c(x)\boldsymbol{\lambda}^b(x) & s\bar{\omega}^a(x) &= gf^{abc}\omega^c(x)\bar{\omega}^b(x) + i\sigma\mathbf{D}^{ab}(x)\cdot\boldsymbol{\lambda}^b(x) \end{aligned} \tag{4.52}$$

The BRST operator s has two essential properties, it annihilates the complete action (4.45)

$$sS[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] = 0 \tag{4.53}$$

expressing the invariance of $S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}]$ under the BRST transformation (4.48), and it's nilpotency

$$s^2 = 0 \tag{4.54}$$

Using the new operator s, we now define the generating functional in the following way

$$\begin{aligned} Z[J,I] \;=\; \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\;\exp\Big\{ -S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] + \int dx\,\Big[\mathbf{A}^a\cdot\mathbf{J}^a_A \;+\; \boldsymbol{\lambda}^a\cdot\mathbf{J}^a_\lambda \;+\; \omega^a J^a_\omega \;+\; \bar{\omega}^a J^a_{\bar{\omega}} \\ + \mathbf{I}^a_{sA}\cdot s\mathbf{A}^a + \mathbf{I}^a_{s\lambda}\cdot s\boldsymbol{\lambda}^a + I^a_{s\omega}s\omega^a + I^a_{s\bar{\omega}}s\bar{\omega}^a\Big]\Big\} \end{aligned} \tag{4.55}$$

Note that ω, $\bar{\omega}$, $s\mathbf{A}$ and $s\boldsymbol{\lambda}$ together with their sources J_ω, $J_{\bar{\omega}}$, $\mathbf{I}_{sA}$, $\mathbf{I}_{s\lambda}$ are Grassmann odd, the remaining quantities Grassmann even.

We proceed to vary the fields in Eq. (4.55) according to Eq. (4.48). The Jacobian of such a transformation is unity due to Eq. (4.52), and the explicit calculation can be seen in Appendix B. We also know that the action is invariant under this change of variables $S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] = S[\mathbf{A}',\boldsymbol{\lambda}',\omega',\bar{\omega}']$. In addition, the source terms of the BRST transformed fields are also invariant due to the nilpotency of s, and the fact that the variations are s-transforms themselves, e.g. $\delta_\varepsilon\mathbf{A}' = \varepsilon s\mathbf{A}'$. Only the source terms of the fundamental fields are not invariant and transform according to

$$\mathbf{A}^a\cdot\mathbf{J}^a_A \;=\; \mathbf{A}'^a\cdot\mathbf{J}^a_A \;+\; \delta_\varepsilon\mathbf{A}'^a\cdot\mathbf{J}^a_A \;=\; \mathbf{A}'^a\cdot\mathbf{J}^a_A \;+\; \varepsilon\, s\mathbf{A}'^a\cdot\mathbf{J}^a_A \tag{4.56}$$

and likewise for the other fields. Thus, under the change of variables (4.48), the integrand in Eq. (4.55) is simply reproduced with all fields replaced by their primed counterparts and an additional factor

$$\exp\Big\{\varepsilon\int dx\,\Big[s\mathbf{A}'^a\cdot\mathbf{J}^a_A \;+\; s\boldsymbol{\lambda}'^a\cdot\mathbf{J}^a_\lambda \;+\; s\omega'^a J^a_\omega \;+\; s\bar{\omega}'^a J^a_{\bar{\omega}}\Big]\Big\} \tag{4.57}$$

generated by the transformation of the fundamental source terms, Eq. (4.56). Because ε is Grassmann odd we have

$$\exp\Big\{\varepsilon\int dx\,\Big[s\mathbf{A}^a\!\cdot\!\mathbf{J}^a_A + s\boldsymbol{\lambda}^a\!\cdot\!\mathbf{J}^a_\lambda + s\omega^a J^a_\omega + s\bar{\omega}^a J^a_{\bar{\omega}}\Big]\Big\} = 1 + \varepsilon\int dx\,\Big[s\mathbf{A}'^a\!\cdot\!\mathbf{J}^a_A + s\boldsymbol{\lambda}'^a\!\cdot\!\mathbf{J}^a_\lambda + s\omega'^a J^a_\omega + s\bar{\omega}'^a J^a_{\bar{\omega}}\Big]$$

Inserted back into the path integral Eq. (4.55), the one just gives $Z[J,I]$, which cancels the left-hand side of the equation. Hence, we obtain

$$0 = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\;\varepsilon\int dx\,\Big[s\mathbf{A}^a\cdot\mathbf{J}^a_A + s\boldsymbol{\lambda}^a\cdot\mathbf{J}^a_\lambda + s\omega^a J^a_\omega + s\bar{\omega}^a J^a_{\bar{\omega}}\Big]\exp\Big\{(\dots)\Big\} \tag{4.58}$$

where the dots represent the exponential in Eq. (4.55). This has to be true for any ε and thus the expression without ε has to vanish itself. Changing the BRST transformed fields for functional derivatives with respect to their sources, we find the following identity

$$\int dx\,\Big[J^{ai}_A(x)\,\frac{\delta}{\delta I^{ai}_{sA}(x)} + J^{ai}_\lambda(x)\,\frac{\delta}{\delta I^{ai}_{s\lambda}(x)} + J^a_\omega(x)\,\frac{\delta}{\delta I^a_{s\omega}(x)} + J^a_{\bar{\omega}}(x)\,\frac{\delta}{\delta I^a_{s\bar{\omega}}(x)}\Big] Z[J,I] = 0 \tag{4.59}$$

Finally, let us transcribe this relation in an identity for the generating functional of one-particle irreducible (1PI) correlation functions. To this end, we first express it by the generating functional of connected correlation functions $W[J,I] = \ln Z[J,I]$. In terms of $W[J,I]$ the relation (4.59) reads

$$\int dx\,\Big[J^{ai}_A(x)\,\frac{\delta W[J,I]}{\delta I^{ai}_{sA}(x)} + J^{ai}_\lambda(x)\,\frac{\delta W[J,I]}{\delta I^{ai}_{s\lambda}(x)} + J^a_\omega(x)\,\frac{\delta W[J,I]}{\delta I^a_{s\omega}(x)} + J^a_{\bar{\omega}}(x)\,\frac{\delta W[J,I]}{\delta I^a_{s\bar{\omega}}(x)}\Big] = 0 \tag{4.60}$$

To define the generating functional of one-particle irreducible correlation functions, we introduce the usual expectation values for the fields in the presence of the external sources

$$\begin{aligned} A^{ai}(x) &= \frac{\delta W[J,I]}{\delta J_A^{ai}(x)} & \omega^a(x) &= -\frac{\delta W[J,I]}{\delta J_\omega^a(x)} \\ \lambda^{ai}(x) &= \frac{\delta W[J,I]}{\delta J_\lambda^{ai}(x)} & \bar{\omega}^a(x) &= -\frac{\delta W[J,I]}{\delta J_{\bar{\omega}}^a(x)} \end{aligned} \tag{4.61}$$

The minus signs in the case of the ghost fields are a consequence of our definition of the generating functional, Eq. (4.55), where we ordered the sources to the right of the fundamental fields.

Assuming that the relations (4.61) can be solved for the sources J, we can define the 1PI generating functional Γ as the Legendre transform of $W[J,I]$ with respect to the sources J. The sources of the BRST transformed fields are not Legendre transformed and play the role of spectators only. With the definition

$$\Gamma[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega};I] = \int dx\, [\mathbf{A}^a\cdot \mathbf{J}_A^a \;+\; \boldsymbol{\lambda}^a\cdot \mathbf{J}_\lambda^a \;+\; \omega^a J_\omega^a \;+\; \bar{\omega}^a J_{\bar{\omega}}^a] - W[J,I] \tag{4.62}$$

one finds

$$\begin{aligned} \frac{\delta\Gamma}{\delta A^{ai}(x)} &= J_A^{ai}(x) & \frac{\delta\Gamma}{\delta\omega^a(x)} &= J_\omega^a(x) \\ \frac{\delta\Gamma}{\delta\lambda^{ai}(x)} &= J_\lambda^{ai}(x) & \frac{\delta\Gamma}{\delta\bar{\omega}^a(x)} &= J_{\bar{\omega}}^a(x) \end{aligned} \tag{4.63}$$

and also

$$\begin{aligned} \frac{\delta\Gamma}{\delta I_{sA}^{ai}(x)} &= -\frac{\delta W}{\delta I_{sA}^{ai}(x)} & \frac{\delta\Gamma}{\delta I_{s\omega}^{a}(x)} &= -\frac{\delta W}{\delta I_{s\omega}^{a}(x)} \\ \frac{\delta\Gamma}{\delta I_{s\lambda}^{ai}(x)} &= -\frac{\delta W}{\delta I_{s\lambda}^{ai}(x)} & \frac{\delta\Gamma}{\delta I_{s\bar{\omega}}^{a}(x)} &= -\frac{\delta W}{\delta I_{s\bar{\omega}}^{a}(x)} \end{aligned} \tag{4.64}$$

which may be used to reexpress the gauge Ward identity (4.60) in terms of Γ

$$\int dx \Big[\frac{\delta\Gamma}{\delta A^{ai}(x)}\,\frac{\delta\Gamma}{\delta I_{sA}^{ai}(x)} + \frac{\delta\Gamma}{\delta\lambda^{ai}(x)}\,\frac{\delta\Gamma}{\delta I_{s\lambda}^{ai}(x)} + \frac{\delta\Gamma}{\delta\omega^{a}(x)}\,\frac{\delta\Gamma}{\delta I_{s\omega}^{a}(x)} + \frac{\delta\Gamma}{\delta\bar{\omega}^{a}(x)}\,\frac{\delta\Gamma}{\delta I_{s\bar{\omega}}^{a}(x)}\Big] = 0 \tag{4.65}$$

4.2.3 Stochastic Ward Identities

We have included in Eq. (4.55) the auxiliary field $\boldsymbol{\lambda}$ and the ghost fields ω and $\bar{\omega}$, all of which were not strictly necessary, but rather were included so as to facilitate our work. They could, in principle, be integrated out and we would be left with Eq. (4.10), except that we have now also introduced sources for the extra fields, as well as for the BRST transformed ones. This would suggest that there could be some sort of relations for the generating functional in Eq. (4.55) resulting from our choice to include the extra fields and sources.

To derive these relations for Bödeker's effective theory, one starts from the generating functional (4.55), including sources of the fundamental as well as the (gauge) BRST transformed fields. Inserting the action and the BRST transforms according to Eqs. (4.45) – (4.47) and Eq. (4.52) with the definitions (4.5) and (4.40) in use, the generating functional $Z[J,I]$ may be written

$$\begin{aligned} Z[J,I] \;=& \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\;\exp\Big\{\int dx\Big[-\sigma T\boldsymbol{\lambda}^a\cdot\boldsymbol{\lambda}^a + i\boldsymbol{\lambda}^a\cdot\big(\mathbf{E}^a[\mathbf{A}] - i\mathbf{J}_\lambda^a + igf^{abc}\omega^b\mathbf{I}_{s\lambda}^c - \sigma\mathbf{D}^{ab}I_{s\bar{\omega}}^b\big) \\ &+ \bar{\omega}^a\big(\gamma^a[\omega,\mathbf{A}] + J_{\bar{\omega}}^a - gf^{abc}\omega^b I_{s\bar{\omega}}^c\big) \;+\; \mathbf{A}^a\cdot\mathbf{J}_A^a + \omega^a J_\omega^a \\ &+ \mathbf{I}_{sA}^a\cdot\mathbf{D}^{ab}\omega^b + I_{s\omega}^a\tfrac{1}{2}gf^{abc}\omega^c\omega^b\Big]\Big\} \end{aligned} \tag{4.66}$$

where terms multiplying $\boldsymbol{\lambda}$ and $\bar{\omega}$ have been collected. Because the exponent is quadratic in the former and linear in the latter, both of these fields can be integrated. One obtains

$$Z[J,I] = \int \mathcal{D}\mathbf{A}\mathcal{D}\omega\ \delta(\gamma')\exp\bigg\{\int dx\Big[-\tfrac{1}{4\sigma T}\,\mathbf{E}'^a\cdot\mathbf{E}'^a + \mathbf{A}^a\cdot\mathbf{J}^a_A + \omega^a J^a_\omega + \mathbf{I}^a_{sA}\cdot\mathbf{D}^{ab}\omega^b + I^a_{s\omega}\tfrac{1}{2}gf^{abc}\omega^c\omega^b\Big]\bigg\} \tag{4.67}$$

with the new functionals $\mathbf{E}'$ and γ' defined as

$$\mathbf{E}'^a[\omega,\mathbf{A};\mathbf{J}_\lambda,\mathbf{I}_{s\lambda},I_{s\bar{\omega}}] = \mathbf{E}^a[\mathbf{A}] - i\mathbf{J}^a_\lambda + igf^{abc}\omega^b\mathbf{I}^c_{s\lambda} - \sigma\mathbf{D}^{ab}I^b_{s\bar{\omega}} \tag{4.68}$$

$$\gamma'^a[\omega,\mathbf{A};J_{\bar{\omega}},I_{s\bar{\omega}}] = \gamma^a[\omega,\mathbf{A}] + J^a_{\bar{\omega}} - gf^{abc}\omega^b I^c_{s\bar{\omega}} \tag{4.69}$$

Hence, when restricting to vanishing sources $\mathbf{J}_A = \mathbf{I}_{sA} = 0$ and $J_\omega = I_{s\omega} = 0$ the exponent becomes purely quadratic in $\mathbf{E}'$. Defining for brevity

$$Z_1[\mathbf{J}_\lambda,J_{\bar{\omega}},\mathbf{I}_{s\lambda},I_{s\bar{\omega}}] = Z[\mathbf{J}_A = 0,\ \mathbf{J}_\lambda,\ J_\omega = 0,\ J_{\bar{\omega}},\ \mathbf{I}_{sA} = 0,\ \mathbf{I}_{s\lambda},\ I_{s\omega} = 0,\ I_{s\bar{\omega}}] \tag{4.70}$$

we have

$$Z_1[\mathbf{J}_\lambda,J_{\bar{\omega}},\mathbf{I}_{s\lambda},I_{s\bar{\omega}}] = \int \mathcal{D}\mathbf{A}\mathcal{D}\omega\ \delta(\gamma')\exp\bigg\{-\frac{1}{4\sigma T}\int dx\ \mathbf{E}'^a\cdot\mathbf{E}'^a\bigg\} \tag{4.71}$$

where $\mathbf{E}'$ and γ' both depend on $\mathbf{A}$ and ω as indicated in Eqs. (4.68) and (4.69). Thus, it is quite natural to attempt a change of variables from $\mathbf{A}$ and ω to $\mathbf{E}'$ and γ'. The Jacobian can be calculated in a similar manner as the Jacobian of Eq. (4.41), and again can be shown to be a constant. The resulting integral is gaussian and evaluates to a constant functional Z_1 leading to

$$Z_1[\mathbf{J}_\lambda,J_{\bar{\omega}},\mathbf{I}_{s\lambda},I_{s\bar{\omega}}] = \text{const.} \tag{4.72}$$

or likewise for $W_1 = \ln Z_1$

$$W_1[\mathbf{J}_\lambda,J_{\bar{\omega}},\mathbf{I}_{s\lambda},I_{s\bar{\omega}}] = \text{const.} \tag{4.73}$$

As a consequence, any combination of functional derivatives with respect to sources chosen from the class $\{\mathbf{J}_\lambda,J_{\bar{\omega}},\mathbf{I}_{s\lambda},I_{s\bar{\omega}}\}$ yields zero when acting on the full generating functionals and evaluated for vanishing sources:

$$\underbrace{\frac{\delta}{\delta\ldots}\frac{\delta}{\delta\ldots}\cdots\frac{\delta}{\delta\ldots}}_{\substack{\text{any combination}\\ \text{of } J_\lambda,\ J_{\bar{\omega}},\ I_{s\lambda},\ I_{s\bar{\omega}}}} W[J,I]\Big|_{J=I=0} = 0 \tag{4.74}$$

with the same relation holding for derivatives of $Z[J,I]$. To obtain a corresponding identity for the 1PI generating functional Γ, note that due to Eq. (4.73) one has on the submanifold defined by the vanishing of the four sources $\mathbf{J}_A, \mathbf{I}_{sA}, J_\omega$ and $I_{s\omega}$

$$\lambda^{ai}(x)\Big|_{\substack{\mathbf{J}_A=\mathbf{I}_{sA}=0\\ J_\omega=I_{s\omega}=0}} = \frac{\delta W_1}{\delta J^{ai}_\lambda(x)} = 0 \qquad \text{and} \qquad \bar{\omega}^a(x)\Big|_{\substack{\mathbf{J}_A=\mathbf{I}_{sA}=0\\ J_\omega=I_{s\omega}=0}} = -\frac{\delta W_1}{\delta J^a_{\bar{\omega}}(x)} = 0 \tag{4.75}$$

So Γ could at most depend on $\mathbf{A}$, ω, $\mathbf{I}_{s\lambda}$ and $I_{s\bar{\omega}}$. However, from Eq. (4.63) we have

$$\frac{\delta\Gamma}{\delta A^{ai}(x)} = J^{ai}_A(x) \qquad\qquad \frac{\delta\Gamma}{\delta\omega^a(x)} = J^a_\omega(x) \tag{4.76}$$

and therefore Γ may not depend on $\mathbf{A}$ or ω anymore. The same conclusion can be reached for the $\mathbf{I}_{s\lambda}$ and $I_{s\bar{\omega}}$ by looking at Eq. (4.64) and Eq. (4.74). Therefore, Γ must be a constant, this leads to

$$\underbrace{\frac{\delta}{\delta\ldots}\frac{\delta}{\delta\ldots}\cdots\frac{\delta}{\delta\ldots}}_{\substack{\text{any combination}\\ \text{of } A,\ \omega,\ I_{s\lambda},\ I_{s\bar{\omega}}}} \Gamma[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega};I]\Big|_{J=I=0} = 0 \tag{4.77}$$

which is the equivalent of the stochastic Ward identity (4.74) in terms of the 1PI generating functional Γ.

4.2.4 Ghost Number Conservation

We will discuss one last symmetry of the action (4.45). The action is invariant under the global transformation

$$\begin{aligned} \omega^a(x) &= e^{i\alpha}\omega'^a(x) \\ \bar{\omega}^a(x) &= e^{-i\alpha}\bar{\omega}'^a(x) \end{aligned} \tag{4.78}$$

of the ghost and anti-ghost fields. In addition to this, subjecting the measure $\mathcal{D}\omega\mathcal{D}\bar{\omega}$ to the transformation (4.78), i.e. to

$$\begin{aligned} &(\omega^a(x_1),\, \bar{\omega}^a(x_1),\, \omega^a(x_2),\, \bar{\omega}^a(x_2),\, \dots) \\ =\;& (e^{i\alpha}\omega'^a(x_1),\, e^{-i\alpha}\bar{\omega}'^a(x_1),\, e^{i\alpha}\omega'^a(x_2),\, e^{-i\alpha}\bar{\omega}'^a(x_2),\, \dots) \end{aligned} \tag{4.79}$$

one finds

$$\mathcal{D}\omega\mathcal{D}\bar{\omega} = \prod_{a,n}[d\omega^a(x_n)\, d\bar{\omega}^a(x_n)] = \prod_{a,n}[d\omega'^a(x_n)\, d\bar{\omega}'^a(x_n)]\; J(\omega',\bar{\omega}') \tag{4.80}$$

with the Jacobian

$$J^{-1}(\omega',\bar{\omega}') = \det\begin{pmatrix} e^{+i\alpha} & 0 & 0 & 0 & \cdots \\ 0 & e^{-i\alpha} & 0 & 0 & \cdots \\ 0 & 0 & e^{+i\alpha} & 0 & \cdots \\ 0 & 0 & 0 & e^{-i\alpha} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 1 \tag{4.81}$$

Hence, the measure is also invariant under the transformation (4.78)

$$\mathcal{D}\omega\mathcal{D}\bar{\omega} = \mathcal{D}\omega'\mathcal{D}\bar{\omega}' \tag{4.82}$$

Together with the invariance of the action, this symmetry leads to ghost number conservation, which poses another restriction on the form of the generating functionals and their derivatives. Indeed, taking the parameter α in Eq. (4.78) to be infinitesimal and performing the corresponding change of variables

$$\begin{aligned} \omega^a(x) &= \omega'^a(x) + i\alpha\, \omega'^a(x) \\ \bar{\omega}^a(x) &= \bar{\omega}'^a(x) - i\alpha\, \bar{\omega}'^a(x) \end{aligned} \tag{4.83}$$

in the defining path integral (4.55) of the generating functional $Z[J,I]$ yields

$$Z[J,I] = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\; \exp\left\{ i\alpha \int dx\, \left[\omega^a J^a_\omega - \bar{\omega}^a J^a_{\bar{\omega}} + \mathbf{I}^a_{sA}\cdot s\mathbf{A}^a + \mathbf{I}^a_{s\lambda}\cdot s\boldsymbol{\lambda}^a + 2I^a_{s\omega} s\omega^a \right] \right\} \exp\left\{ (\dots) \right\}$$

Here we have already renamed the primed symbols again to unprimed ones after the change of variables has been completed. As before, the dots represent the original exponent as it occurs in Eq. (4.55). Using the fact that α is assumed to be infinitesimal, we can expand the first exponential and replace any fields that appear by functional derivatives acting on the exponential (after interchanging the order of the ghost and anti-ghost field and their corresponding sources leading to a minus sign in either case). The derivatives can finally be pulled out of the functional integral and we obtain

$$\int dx\, \left[J^a_\omega(x)\, \frac{\delta}{\delta J^a_\omega(x)} - J^a_{\bar{\omega}}(x)\, \frac{\delta}{\delta J^a_{\bar{\omega}}(x)} + I^{ai}_{sA}(x)\, \frac{\delta}{\delta I^{ai}_{sA}(x)} + I^{ai}_{s\lambda}(x)\, \frac{\delta}{\delta I^{ai}_{s\lambda}(x)} + 2I^a_{s\omega}(x)\, \frac{\delta}{\delta I^a_{s\omega}(x)} \right] Z[J,I] = 0 \tag{4.84}$$

Again, the definition $W[J,I] = \ln Z[J,I]$ implies that the same identity holds for the generating functional $W[J,I]$ of connected correlation functions

$$\int dx\, \left[J^a_\omega \frac{\delta W}{\delta J^a_\omega} - J^a_{\bar{\omega}} \frac{\delta W}{\delta J^a_{\bar{\omega}}} + I^{ai}_{sA} \frac{\delta W}{\delta I^{ai}_{sA}} + I^{ai}_{s\lambda} \frac{\delta W}{\delta I^{ai}_{s\lambda}} + 2I^a_{s\omega} \frac{\delta W}{\delta I^a_{s\omega}} \right] = 0 \tag{4.85}$$

where we have suppressed the space-time argument x and the dependence of W on the sources J and I. This identity in turn can easily be translated to the corresponding restriction on the 1PI generating functional $\Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I]$. By means of Eqs. (4.61), (4.63) and (4.64) one finds

$$\int dx \left[\frac{\delta\Gamma}{\delta\omega^a}\,\omega^a - \frac{\delta\Gamma}{\delta\bar{\omega}^a}\,\bar{\omega}^a + I^{ai}_{sA}\,\frac{\delta\Gamma}{\delta I^{ai}_{sA}} + I^{ai}_{s\lambda}\,\frac{\delta\Gamma}{\delta I^{ai}_{s\lambda}} + 2I^a_{s\omega}\frac{\delta\Gamma}{\delta I^a_{s\omega}} \right] = 0 \tag{4.86}$$

This concludes our derivation of non-perturbative identities for the generating functional (4.55). The explicit form of the identities for lower N-point functions are shown in Appendix (D)

4.3 Dyson–Schwinger Equations

To derive the DSEs, we observe that the path integral of a functional derivative vanishes, i.e.

$$\int \mathcal{D}\phi\, \frac{\delta}{\delta\phi(x)}\, F[\phi] = 0 \tag{4.87}$$

for any functional $F[\phi]$. Hence, in the case of Bödeker's theory, we obtain four different equations by inserting a functional derivative with respect to each of the fields $\mathbf{A}$, $\boldsymbol{\lambda}$, ω or $\bar{\omega}$ into the generating functional

$$\begin{aligned} Z[J,I] \;=\; \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\; \exp\Big\{ -S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] + \int dx \Big[\mathbf{A}^a\cdot\mathbf{J}^a_A \;+\; \boldsymbol{\lambda}^a\cdot\mathbf{J}^a_\lambda \;+\; \omega^a J^a_\omega \;+\; \bar{\omega}^a J^a_{\bar{\omega}} \\ + \mathbf{I}^a_{sA}\cdot s\mathbf{A}^a + \mathbf{I}^a_{s\lambda}\cdot s\boldsymbol{\lambda}^a + I^a_{s\omega} s\omega^a + I^a_{s\bar{\omega}} s\bar{\omega}^a \Big]\Big\} \end{aligned} \tag{4.88}$$

4.3.1 General Dyson–Schwinger Equations

Ghost (ω) and Anti-ghost ($\bar{\omega}$) equations

Starting from the identity

$$0 = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\; \frac{\delta}{\delta\omega^a(x)}\; \exp\big\{(\ldots)\big\} \tag{4.89}$$

where the dots represent the exponent of Eq. (4.88), gives

$$\begin{aligned} 0 \;=\; \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\; \Big[\dot{\bar{\omega}}^a(x) + \tfrac{1}{\kappa}\,\mathbf{D}^{ab}(x)\cdot\nabla\bar{\omega}^b(x) - \tfrac{g}{\kappa}\, f^{abc}\,\bar{\omega}^b(x)\,\nabla\cdot\mathbf{A}^c(x) + J^a_\omega(x) + \nabla\cdot\mathbf{I}^a_{sA}(x) \\ + g f^{abc}\big(-\mathbf{I}^b_{sA}(x)\cdot\mathbf{A}^c(x) - \mathbf{I}^b_{s\lambda}(x)\cdot\boldsymbol{\lambda}^c(x) + I^b_{s\omega}(x)\,\omega^c(x) + I^b_{s\bar{\omega}}(x)\,\bar{\omega}^c(x) \big)\Big] \exp\big\{(\ldots)\big\} \end{aligned} \tag{4.90}$$

Expressing the fields by derivatives acting on the exponential, one obtains

$$\begin{aligned} Z J^a_\omega(x) \;=\; \big(\partial_t + \tfrac{1}{\kappa}\Delta\big)\,\frac{\delta Z}{\delta J^a_{\bar{\omega}}(x)} \;-\; \frac{g}{\kappa} f^{abc}\,\partial_j \frac{\delta^2 Z}{\delta J^b_{\bar{\omega}}(x)\,\delta J^{cj}_A(x)} - Z\,\nabla\cdot\mathbf{I}^a_{sA}(x) \\ + g f^{abc}\Big[I^{bj}_{sA}(x)\,\frac{\delta Z}{\delta J^{cj}_A(x)} + I^{bj}_{s\lambda}(x)\,\frac{\delta Z}{\delta J^{cj}_\lambda(x)} + I^b_{s\omega}(x)\,\frac{\delta Z}{\delta J^c_\omega(x)} + I^b_{s\bar{\omega}}(x)\,\frac{\delta Z}{\delta J^c_{\bar{\omega}}(x)}\Big] \end{aligned} \tag{4.91}$$

or in terms of W

$$\begin{aligned} J^a_\omega(x) \;=\; \big(\partial_t + \tfrac{1}{\kappa}\Delta\big)\,\frac{\delta W}{\delta J^a_{\bar{\omega}}(x)} \;-\; \frac{g}{\kappa} f^{abc}\,\partial_j \Big[\frac{\delta^2 W}{\delta J^b_{\bar{\omega}}(x)\,\delta J^{cj}_A(x)} + \frac{\delta W}{\delta J^b_{\bar{\omega}}(x)}\,\frac{\delta W}{\delta J^{cj}_A(x)}\Big] - \nabla\cdot\mathbf{I}^a_{sA}(x) \\ + g f^{abc}\Big[I^{bj}_{sA}(x)\,\frac{\delta W}{\delta J^{cj}_A(x)} + I^{bj}_{s\lambda}(x)\,\frac{\delta W}{\delta J^{cj}_\lambda(x)} + I^b_{s\omega}(x)\,\frac{\delta W}{\delta J^c_\omega(x)} + I^b_{s\bar{\omega}}(x)\,\frac{\delta W}{\delta J^c_{\bar{\omega}}(x)}\Big] \end{aligned} \tag{4.92}$$

Transcription to the 1PI generating functional Γ yields

$$\begin{aligned}\frac{\delta\Gamma}{\delta\omega^a(x)} &= -\left(\partial_t + \tfrac{1}{\kappa}\Delta\right)\bar{\omega}^a(x) - \frac{g}{\kappa}f^{abc}\,\partial_j\left[\frac{\delta^2 W}{\delta J^b_{\bar{\omega}}(x)\,\delta J^{cj}_A(x)} - \bar{\omega}^b(x)\,A^{cj}(x)\right] - \nabla\cdot\mathbf{I}^a_{sA}(x)\\ &\quad + gf^{abc}\left[I^{bj}_{sA}(x)\,A^{cj}(x) + I^{bj}_{s\lambda}(x)\,\lambda^{cj}(x) - I^b_{s\omega}(x)\,\omega^c(x) - I^b_{s\bar{\omega}}(x)\,\bar{\omega}^c(x)\right]\end{aligned} \tag{4.93}$$

The antighost equation is obtained in a similar manner and reads

$$\frac{\delta\Gamma}{\delta\bar{\omega}^a(x)} = \left(-\partial_t + \tfrac{1}{\kappa}\Delta\right)\omega^a(x) + \frac{g}{\kappa}f^{abc}\,\partial_j\left[\frac{\delta^2 W}{\delta J^b_{\omega}(x)\,\delta J^{cj}_A(x')} - \omega^b(x)A^{cj}(x')\right]_{x'=x} - gf^{abc}I^b_{s\bar{\omega}}(x)\,\omega^c(x) \tag{4.94}$$

where x' is set to x after the space-time derivative is carried out, i.e. the derivative acts on the argument of $J^b_\omega(x)$ only.

Auxiliary field ($\boldsymbol{\lambda}$) equation

To deduce the auxiliary field equation from

$$0 = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\,\frac{\delta}{\delta\lambda^{ai}(x)}\,\exp\{(\dots)\} \tag{4.95}$$

we need, among other things, the functional derivative of the action $S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega}] = S^{(D)}[\mathbf{A},\boldsymbol{\lambda}] + S^{(GG)}[\mathbf{A},\omega,\bar{\omega}]$. However, in the present case the corresponding expression becomes rather cumbersome.

As for the Feynman rules in Appendix C, we want to use a symmetrised $\boldsymbol{\lambda}\mathbf{A}^2$ and $\boldsymbol{\lambda}\mathbf{A}^3$ vertex. The $\boldsymbol{\lambda}$ dependence of the action spreads out over the three contributions to the dynamical action $S^{(D)}[\mathbf{A},\boldsymbol{\lambda}] = S^{(D)}_0[\mathbf{A},\boldsymbol{\lambda}] + S^{(D)}_{\text{int},3}[\mathbf{A},\boldsymbol{\lambda}] + S^{(D)}_{\text{int},4}[\mathbf{A},\boldsymbol{\lambda}]$. The corresponding derivatives can be written in the form

$$\frac{\delta S^{(D)}_0[\mathbf{A},\boldsymbol{\lambda}]}{\delta\lambda^{ai}(x)} = 2\sigma T\lambda^{ai}(x) - i\left[\delta^{ij}(\sigma\partial_t - \Delta) + \left(1-\tfrac{\sigma}{\kappa}\right)\partial_i\partial_j\right]A^{aj}(x) \tag{4.96}$$

$$\begin{aligned}\frac{\delta S^{(D)}_{\text{int},3}[\mathbf{A},\boldsymbol{\lambda}]}{\delta\lambda^{ai}(x)} &= \frac{1}{2!}\,(-ig)f^{abc}\Big[\left(1-\tfrac{\sigma}{\kappa}\right)\left[\delta^{ij}\partial'_k - \delta^{ik}\partial_j\right] + 2\left[\delta^{ij}\partial_k - \delta^{ik}\partial'_j\right]\\ &\qquad + \left[\delta^{jk}\partial'_i - \delta^{kj}\partial_i\right]\Big]A^{bj}(x)A^{ck}(x')\Big|_{x'=x}\end{aligned} \tag{4.97}$$

$$\frac{\delta S^{(D)}_{\text{int},4}[\mathbf{A},\boldsymbol{\lambda}]}{\delta\lambda^{ai}(x)} = \frac{1}{3!}\,(-ig^2)\,V^{abcd}_{ijkl}\,A^{bj}(x)A^{ck}(x)A^{dl}(x) \tag{4.98}$$

and one obtains in terms of the 1PI generating functional

$$\begin{aligned}\frac{\delta\Gamma}{\delta\lambda^{ai}(x)} &= 2\sigma T\lambda^{ai}(x) - i\left[\delta^{ij}(\sigma\partial_t - \Delta) + \left(1-\tfrac{\sigma}{\kappa}\right)\partial_i\partial_j\right]A^{aj}(x)\\ &- \frac{ig}{2!}\,f^{abc}\Big[\left(1-\tfrac{\sigma}{\kappa}\right)\left[\delta^{ij}\partial'_k - \delta^{ik}\partial_j\right] + 2\left[\delta^{ij}\partial_k - \delta^{ik}\partial'_j\right] + \left[\delta^{jk}\partial'_i - \delta^{kj}\partial_i\right]\Big]\\ &\qquad\times\left[\frac{\delta^2 W}{\delta J^{bj}_A(x)\,\delta J^{ck}_A(x')} + A^{bj}(x)\,A^{ck}(x')\right]_{x'=x}\\ &- \frac{ig^2}{3!}V^{abcd}_{ijkl}\left[\frac{\delta^3 W}{\delta J^{bj}_A(x)\,\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)} + 3\,\frac{\delta^2 W}{\delta J^{bj}_A(x)\,\delta J^{ck}_A(x)}\,A^{dl}(x) + A^{bj}(x)\,A^{ck}(x)\,A^{dl}(x)\right]\\ &- gf^{abc}\left[-I^{bi}_{s\lambda}(x)\,\omega^c(x) + i\sigma I^b_{s\bar{\omega}}(x)\,A^{ci}(x)\right] + i\sigma\,\partial_i I^a_{s\bar{\omega}}(x)\end{aligned} \tag{4.99}$$

Gauge field (A) equation

Finally, coming to the gauge field equation

$$0 = \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar{\omega}\,\frac{\delta}{\delta A^{ai}(x)}\,\exp\{(\dots)\} \tag{4.100}$$

and using the derivatives

$$\frac{\delta S_0^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]}{\delta A^{ai}(x)} = -i\left[\delta^{ij}(-\sigma\partial_t - \Delta) + \left(1-\tfrac{\sigma}{\kappa}\right)\partial_i\partial_j\right]\lambda^{aj}(x) \tag{4.101}$$

$$\begin{aligned}\frac{\delta S_{\mathrm{int},3}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]}{\delta A^{ai}(x)} &= -igf^{abc}\Big[-\left(1-\tfrac{\sigma}{\kappa}\right)\left[\delta^{ij}\partial_k' + \delta^{jk}(\partial_i+\partial_i')\right] + 2\left[\delta^{jk}\partial_i' + \delta^{ij}(\partial_k+\partial_k')\right] \\ &\qquad - \left[\delta^{ik}\partial_j' + \delta^{ik}(\partial_j+\partial_j')\right]\Big]\lambda^{bj}(x)A^{ck}(x')\Big|_{x'=x}\end{aligned} \tag{4.102}$$

$$\frac{\delta S_{\mathrm{int},4}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]}{\delta A^{ai}(x)} = \frac{1}{2!}\,(-ig^2)\,V^{dabc}_{lijk}\,\lambda^{dl}(x)A^{bj}(x)A^{ck}(x) \tag{4.103}$$

where the symmetry of V^{abcd}_{ijkl} has been exploited. Together with

$$\frac{\delta S^{(\mathrm{GG})}[\mathbf{A},\omega,\bar{\omega}]}{\delta A^{ai}(x)} = -\frac{g}{\kappa}\,f^{abc}\bar{\omega}^b(x)\,\partial_i\,\omega^c(x) \tag{4.104}$$

one arrives at

$$\begin{aligned}\frac{\delta\Gamma}{\delta A^{ai}(x)} &= -i\left[\delta^{ij}(-\sigma\partial_t - \Delta) + \left(1-\tfrac{\sigma}{\kappa}\right)\partial_i\partial_j\right]\lambda^{aj}(x) \;-\; \frac{ig^2}{2!}\,V^{dabc}_{lijk}\Bigg[\frac{\delta^3 W}{\delta J_\lambda^{dl}(x)\,\delta J_A^{bj}(x)\,\delta J_A^{ck}(x)} \\ &\quad +2\,\frac{\delta^2 W}{\delta J_\lambda^{dl}(x)\,\delta J_A^{bj}(x)}\,A^{ck}(x) \;+\; \lambda^{dl}(x)\,\frac{\delta^2 W}{\delta J_A^{bj}(x)\,\delta J_A^{ck}(x)} \;+\; \lambda^{dl}(x)\,A^{bj}(x)\,A^{ck}(x)\Bigg] \\ &\quad -igf^{abc}\Big[-\left(1-\tfrac{\sigma}{\kappa}\right)\left[\delta^{ij}\partial_k' + \delta^{jk}(\partial_i+\partial_i')\right] + 2\left[\delta^{jk}\partial_i' + \delta^{ij}(\partial_k+\partial_k')\right] \\ &\qquad - \left[\delta^{ik}\partial_j' + \delta^{ik}(\partial_j+\partial_j')\right]\Big]\left[\frac{\delta^2 W}{\delta J_\lambda^{bj}(x)\,\delta J_A^{ck}(x')} \;+\; \lambda^{bj}(x)\,A^{ck}(x')\right]_{x'=x} \\ &\quad -\frac{g}{\kappa}f^{abc}\,\partial_i\left[\frac{\delta^2 W}{\delta J_{\bar{\omega}}^{b}(x')\,\delta J_\omega^{c}(x)} \;+\; \bar{\omega}^b(x')\,\omega^c(x)\right]_{x'=x} + gf^{abc}\left[I_{sA}^{bi}(x)\,\omega^c(x) + i\sigma I_{s\bar{\omega}}^{b}(x)\,\lambda^{ci}(x)\right]\end{aligned} \tag{4.105}$$

4.3.2 Explicit Equations for Lower N-Point Functions

Definitions and General Relations

Concerning the propagators, mixing will occur between the gauge field $\mathbf{A}$ and the auxiliary field $\boldsymbol{\lambda}$, resulting in four possible propagators from the gauge/auxiliary field sector that can be combined into one matrix propagator. These are completed by the propagator of the gauge ghosts. Altogether, we define the full (connected) propagators as

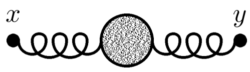
$$G^{(AA)}{}^{ab}_{ij}(x,y) \quad = \quad \left\langle A^{ai}(x)\, A^{bj}(y)\right\rangle_c \;=\; \frac{\delta^2 W[J,I]}{\delta J^{ai}_A(x)\,\delta J^{bj}_A(y)}\bigg|_{J=I=0} \tag{4.106}$$

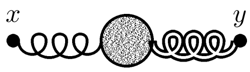
$$G^{(\lambda A)}{}^{ab}_{ij}(x,y) \quad = \quad \left\langle \lambda^{ai}(x)\, A^{bj}(y)\right\rangle_c \;=\; \frac{\delta^2 W[J,I]}{\delta J^{ai}_\lambda(x)\,\delta J^{bj}_A(y)}\bigg|_{J=I=0} \tag{4.107}$$

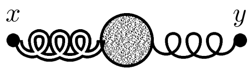
$$G^{(\lambda\lambda)}{}^{ab}_{ij}(x,y) \quad = \quad \left\langle \lambda^{ai}(x)\, \lambda^{bj}(y)\right\rangle_c \;=\; \frac{\delta^2 W[J,I]}{\delta J^{ai}_\lambda(x)\,\delta J^{bj}_\lambda(y)}\bigg|_{J=I=0} \tag{4.108}$$

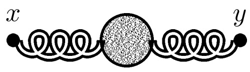
$$G^{(\omega)\,ab}(x,y) \quad = \quad \left\langle \omega^{a}(x)\, \bar\omega^{b}(y)\right\rangle_c \;=\; \frac{\delta^2 W[J,I]}{\delta J^{a}_\omega(x)\,\delta J^{b}_{\bar\omega}(y)}\bigg|_{J=I=0} \tag{4.109}$$

and $G^{(A\lambda)}{}^{ab}_{ij}(x,y) = G^{(\lambda A)}{}^{ba}_{ji}(y,x)$ of course. In graphical representations we denote the gauge field by curly lines, the auxiliary field by double curly lines and the gauge ghosts by dotted lines. Thus, the full propagators are represented by

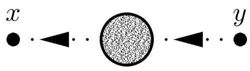
$G^{(AA)}{}^{ab}_{ij}(x,y) \;=\; a,i$  b,j

$G^{(A\lambda)}{}^{ab}_{ij}(x,y) \;=\; a,i$  b,j

$G^{(\lambda A)}{}^{ab}_{ij}(x,y) \;=\; a,i$  b,j

$G^{(\lambda\lambda)}{}^{ab}_{ij}(x,y) \;=\; a,i$  b,j

and finally

$G^{(\omega)\,ab}(x,y) \;=\; a$  b

Besides the propagators, we have to set out our definition for the self-energies. To this end, let us summarise the two left-hand equations of (4.63) in the form

$$J^{ai}_F(x) = \frac{\delta\Gamma[\mathbf{A},\boldsymbol{\lambda},\omega,\bar\omega;I]}{\delta F^{ai}(x)} \tag{4.110}$$

where the index F stands for any of the fields A or λ. Taking the functional derivative of this equation with respect to $J^{bj}_G(y)$, where again $G \in \{A,\lambda\}$, then yields (observing that $\mathbf{A}$, $\boldsymbol{\lambda}$, ω and $\bar\omega$ are functionals of the sources J and I)

$$\begin{aligned}\delta^{ab}\,\delta^{ij}\,\delta_{FG}\,\delta(x-y) \;=\; \int\! dz\Bigg[&\frac{\delta A^{ck}(z)}{\delta J^{bj}_G(y)}\,\frac{\delta^2\Gamma}{\delta A^{ck}(z)\,\delta F^{ai}(x)} + \frac{\delta \lambda^{ck}(z)}{\delta J^{bj}_G(y)}\,\frac{\delta^2\Gamma}{\delta \lambda^{ck}(z)\,\delta F^{ai}(x)} \\ &+ \frac{\delta \omega^{c}(z)}{\delta J^{bj}_G(y)}\,\frac{\delta^2\Gamma}{\delta \omega^{c}(z)\,\delta F^{ai}(x)} + \frac{\delta \bar\omega^{c}(z)}{\delta J^{bj}_G(y)}\,\frac{\delta^2\Gamma}{\delta \bar\omega^{c}(z)\,\delta F^{ai}(x)}\Bigg]\end{aligned} \tag{4.111}$$

Thus, using the Eqs. (4.61) to express the first factor in each term as a second derivative of W and finally setting the sources to zero leads to

$$\delta^{ab}\,\delta^{ij}\,\delta_{FG}\,\delta(x-y) = \tag{4.112}$$

$$\int\! dz\left[G^{(GA)}{}^{bc}_{jk}(y,z)\,\frac{\delta^2\Gamma}{\delta A^{ck}(z)\,\delta F^{ai}(x)}\bigg|_{J=I=0} + \; G^{(G\lambda)}{}^{bc}_{jk}(y,z)\,\frac{\delta^2\Gamma}{\delta \lambda^{ck}(z)\,\delta F^{ai}(x)}\bigg|_{J=I=0}\right]$$

Here, the definitions (4.106) – (4.108) have been used and the terms involving ghost and anti-ghost fields have vanished due to ghost number conservation.

In the following we will often encounter multiple derivatives of the generating functionals W and Γ evaluated for vanishing sources. Let us therefore introduce a shorthand notation where we indicate the fields with respect to which the derivatives are taken as superscripts. Possible Lorentz or colour indices as well as space-time arguments appear in the order of the fields they belong to. For instance, we abbreviate

$$\Gamma^{(\lambda A\bar{\omega})\,abc}_{\;\;ij}(x,y,z) = \left.\frac{\delta^3\Gamma}{\delta\lambda^{ai}(x)\,\delta A^{bj}(y)\,\delta\bar{\omega}^c(z)}\right|_{J=I=0} \tag{4.113}$$

In the case of W, we also use the *fields* as superscripts though the derivatives are taken with respect to the corresponding *sources*, of course.
In this new notation, Eq. (4.112) reads

$$\delta^{ab}\,\delta^{ij}\,\delta_{FG}\,\delta(x-y) = \int\! dz\; G^{(GH)\,bc}_{\;\;jk}(y,z)\,\Gamma^{(HF)\,ca}_{\;\;ki}(z,x) \tag{4.114}$$

where H is a summation index running over the fields A and λ. This equation expresses the fact that the matrix propagator of the gauge/auxiliary field sector

$$\hat{G}^{ab}_{ij}\,(x,y) = \begin{pmatrix} G^{(\lambda\lambda)\,ab}_{\;\;ij}(x,y) & G^{(\lambda A)\,ab}_{\;\;ij}(x,y) \\ G^{(A\lambda)\,ab}_{\;\;ij}(x,y) & G^{(AA)\,ab}_{\;\;ij}(x,y) \end{pmatrix} \tag{4.115}$$

is inverse to the matrix

$$\hat{\Gamma}^{ab}_{ij}\,(x,y) = \begin{pmatrix} \Gamma^{(\lambda\lambda)\,ab}_{\;\;ij}(x,y) & \Gamma^{(\lambda A)\,ab}_{\;\;ij}(x,y) \\ \Gamma^{(A\lambda)\,ab}_{\;\;ij}(x,y) & \Gamma^{(AA)\,ab}_{\;\;ij}(x,y) \end{pmatrix} \tag{4.116}$$

constructed of the second derivatives of Γ. Consequently, the self-energy $\hat{\Pi}^{ab}_{ij}(x,y)$ is determined via the relation

$$\Gamma^{(FG)\,ab}_{\;\;ij}(x,y) \;=\; (\Delta^{-1})^{(FG)\,ab}_{\;\;ij}(x,y) \;+\; \Pi^{(FG)\,ab}_{\;\;ij}(x,y) \tag{4.117}$$

where $(\Delta^{-1})^{(FG)\,ab}_{\;\;ij}(x,y)$ are the components of the inverse free propagator of perturbation theory (see Appendix C, Eqs. (C.6) – (C.9)), and where $F,G\in\{\lambda,A\}$ as before.
Analogously, taking the derivative with respect to $J^b_\omega(y)$ of

$$J^a_\omega(x) = \frac{\delta\Gamma[\mathbf{A},\boldsymbol{\lambda},\omega,\bar{\omega};I]}{\delta\omega^a(x)} \tag{4.118}$$

and performing the same manipulations as described above leads to

$$\delta^{ab}\,\delta(x-y) = -\int\! dz\; G^{(\omega)\,bc}(y,z)\left.\frac{\delta^2\Gamma}{\delta\bar{\omega}^c(z)\,\delta\omega^a(x)}\right|_{J=I=0} \tag{4.119}$$

Hence, we define the self-energy of the gauge ghosts via

$$\Gamma^{(\bar{\omega}\omega)\,ab}(x,y) \;=\; -\Big[(\Delta^{-1})^{(\omega)\,ab}(x,y) \;+\; \Pi^{(\omega)\,ab}(x,y)\Big] \tag{4.120}$$

with the free inverse propagator $(\Delta^{-1})^{(\omega)\,ab}(x,y)$ given in Eq. (C.18). In our graphical representations we denote self-energies and other one-particle irreducible quantities by open circles.

Though generally we are using three-vectors, in the Fourier transformation we use four-vector notation

$$f(x) = \int\frac{d^Dk}{(2\pi)^D}\;e^{-ikx}f(k) \tag{4.121}$$

with $-ikx = -ik_0 t + i\,\mathbf{k}\cdot\mathbf{x}$. The proper vertex functions in momentum space are basically given by the Fourier transforms of the various functional derivatives of the 1PI generating functional Γ. However, due to translational invariance of the theory, all these Fourier transforms contain a delta function expressing momentum conservation at the vertex. It is therefore convenient to pull these delta functions out of the definitions of the vertex functions. In this way, the latter become functions of one momentum variable less than indicated by the number of external legs. For instance, we define

$$(2\pi)^D \delta^D(k_1+k_2+k_3)\,\Gamma^{(\bar\omega\omega G)abc}_{j}(k_1,k_2) = \int dx\,dy\,dz\; e^{-ik_1x-ik_2y-ik_3z}\;\Gamma^{(\bar\omega\omega G)abc}_{j}(x,y,z) \tag{4.122}$$

or equivalently

$$\Gamma^{(\bar\omega\omega G)abc}_{j}(x,y,z) = \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\; e^{-ik_1(z-x)-ik_2(z-y)}\;\Gamma^{(\bar\omega\omega G)abc}_{j}(k_1,k_2) \tag{4.123}$$

Here, the two arguments of the proper vertex function $\Gamma^{(\bar\omega\omega G)abc}_{j}(k_1,k_2)$ refer to the (incoming) momenta along the ghost lines leaving and entering the vertex in this order.

The choice of the $N-1$ momenta that are used as arguments of a vertex with N external legs is, of course, arbitrary and thereby a source of possible confusion. We therefore explicitly list the definitions of the other relevant vertex functions used in this work

$$\Gamma^{(FGH)abc}_{ijk}(x,y,z) = -\int \frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk_3}{(2\pi)^D}\; e^{-ik_2(x-y)-ik_3(x-z)}\;\Gamma^{(FGH)abc}_{ijk}(k_2,k_3) \tag{4.124}$$

with k_2 and k_3 denoting the incoming momenta along the G and H line respectively, and

$$\begin{aligned}\Gamma^{(FGHK)abcd}_{ijkl}(x,y,z,w) \;=\; -\int \frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk_3}{(2\pi)^D}\frac{d^Dk_4}{(2\pi)^D}\; e^{-ik_2(x-y)-ik_3(x-z)} \\ e^{-ik_4(x-w)}\;\Gamma^{(FGHK)abcd}_{ijkl}(k_2,k_3,k_4)\end{aligned} \tag{4.125}$$

with incoming momenta k_2, k_3, k_4 along the G, H and K line. Note the minus signs in the last two equations. The definitions above are chosen in such a way that they reduce at leading order to the corresponding vertices of the Feynman rules, i.e.

$$\Gamma^{(\bar\omega\omega A)abc}_{j}(k_1,k_2) \;=\; \frac{ig}{\kappa}\,f^{abc}k_2^j \;+\; \dots \tag{4.126}$$

$$\Gamma^{(\lambda AA)abc}_{ijk}(k_2,k_3) \;=\; -g\,V^{abc}_{ijk}(\mathbf{k}_2,\mathbf{k}_3) \;+\; \dots \tag{4.127}$$

$$\Gamma^{(\lambda AAA)abcd}_{ijkl}(k_2,k_3,k_4) \;=\; ig^2\,V^{abcd}_{ijkl} \;+\; \dots \tag{4.128}$$

DSE for $\Pi^{(\omega)}(k)$

Let us again start with the ghost equations, being much simpler than the equations for the gauge/auxiliary field sector. By taking the derivative of Eq. (4.94) with respect to $\omega^b(y)$, one finds evaluated for vanishing sources

$$\left.\frac{\delta^2\Gamma}{\delta\omega^b(y)\,\delta\bar\omega^a(x)}\right|_{J=I=0} = \underbrace{\delta^{ab}\big(-\partial_t + \tfrac{1}{\kappa}\Delta\big)\,\delta(x-y)}_{(\Delta^{-1})^{(\omega)\,ab}(x,y)} + \frac{g}{\kappa}f^{ade}\,\partial_j\frac{\delta}{\delta\omega^b(y)}\left.\frac{\delta^2W[J,I]}{\delta J^d_\omega(x)\,\delta J^{ej}_A(x')}\right|_{\substack{x'=x\\ J=I=0}} \tag{4.129}$$

Comparing to the definition of the self-energy of the gauge ghosts in Eq. (4.120) then leads to the relation

$$\Pi^{(\omega)\,ab}(x,y) = \frac{g}{\kappa}f^{ade}\,\partial_j\frac{\delta}{\delta\omega^b(y)}\left.\frac{\delta^2W[J,I]}{\delta J^d_\omega(x)\,\delta J^{ej}_A(x')}\right|_{\substack{x'=x\\ J=I=0}} \tag{4.130}$$

for the gauge ghost self-energy. If we carry out the functional derivative with respect to $\omega^b(y)$, four terms arise because any of the sources $\mathbf{J}_A$, $\mathbf{J}_\lambda$, J_ω and $J_{\bar\omega}$ depends on ω. However, due to ghost number conservation three of these terms vanish when the sources are set to zero and one is left with[2]

$$\Pi^{(\omega)\,ab}(x,y) = \frac{g}{\kappa} f^{ade}\,\partial_j \int dv \left[\frac{\delta^2\Gamma}{\delta\omega^b(y)\,\delta\bar\omega^c(v)}\, \frac{\delta^3 W}{\delta J^c_{\bar\omega}(v)\,\delta J^d_\omega(x)\,\delta J^{ej}_A(x')} \right]_{\substack{x'=x\\ J=I=0}} \tag{4.131}$$

Finally, we express the connected three-point function by its 1PI counterpart

$$W^{(\bar\omega\omega F)\,abc}{}_j(x,y,z) = \int du\,du'du''\; G^{(\omega)\,a'a}(u,x)\,G^{(\omega)\,bb'}(y,u')\,G^{(FG)}{}^{cc'}_{jj'}(z,u'')\,\Gamma^{(\omega\bar\omega G)\,a'b'}{}^{c'}_{j'}(u,u',u'') \tag{4.132}$$

where F represents one of the fields λ or A and G is a summation index taking these two values. The shorthand notation used here was introduced in Eq. (4.113). Note that the order of the ghost and anti-ghost fields in Eq. (4.132) is changed from $W^{(\bar\omega\omega F)}$ to $\Gamma^{(\omega\bar\omega G)}$ and that the (full) gauge ghost propagator is $G^{(\omega)\,ab}(x,y) = W^{(\omega\bar\omega)\,ab}(x,y)$, as defined in Eq. (4.109).

Now, inserting relation (4.132) into Eq. (4.131), using the property (4.119) of the two-point functions and

$$\Gamma^{(\omega\bar\omega G)\,a'b'}{}^{c'}_{j'}(u,u',u'') = -\Gamma^{(\bar\omega\omega G)\,b'a'}{}^{c'}_{j'}(u',u,u'') \tag{4.133}$$

yields the Dyson-Schwinger equation

$$\Pi^{(\omega)\,ab}(x,y) = -\int du'du''\; G^{(AG)}{}^{ee'}_{jj'}(x,u'')\,\frac{g}{\kappa}f^{ade}\,\partial_j G^{(\omega)\,dd'}(x,u')\,\Gamma^{(\bar\omega\omega G)\,d'b}{}^{e'}_{j'}(u',y,u'') \tag{4.134}$$

Using the definition for the momentum space proper vertex Eq.(4.122), we transform to momentum space

$$\Pi^{(\omega)\,ab}(k) = -\int \frac{d^Dk'}{(2\pi)^D}\,\frac{ig}{\kappa} f^{ade} k'^j\, G^{(AG)}{}^{ee'}_{jj'}(k-k')\,G^{(\omega)\,dd'}(k')\,\Gamma^{(\bar\omega\omega G)\,d'b}{}^{e'}_{j'}(-k',k) \tag{4.135}$$

The structure of the Dyson-Schwinger equation (4.135) is illustrated in Fig. 4.1. In Eq. (4.135) the field index G has a summation index taking the values $G=\lambda$ and $G=A$. In the graphical representation of Eq. (4.135) such a summation is symbolised by a solid line. This short-hand notation will become even more important in the other DSEs to follow. Thus, the right-hand side of Fig. 4.1 is a stands for two individual diagrams.

Above we have deduced the Dyson-Schwinger equation of the gauge ghost self-energy from the general *anti-ghost* equation (4.94). A complementary relation can be obtained from the *ghost* equation (4.93). By taking the derivative with respect to $\bar\omega^b(y)$ of Eq. (4.93), one obtains

$$\Pi^{(\omega)\,ab}(k) = -\int \frac{d^Dk'}{(2\pi)^D}\,\frac{ig}{\kappa} f^{dbe} k^j\, G^{(GA)}{}^{e'e}_{j'j}(k-k')\,G^{(\omega)\,d'd}(k')\,\Gamma^{(\bar\omega\omega G)\,ad'}{}^{e'}_{j'}(-k,k') \tag{4.136}$$

[2] It should be clear that x' is set to x only after the space-time derivative is carried out. In order to avoid an extensive use of brackets we decided to assume in this and similar cases some thoughtfulness on the part of the reader.

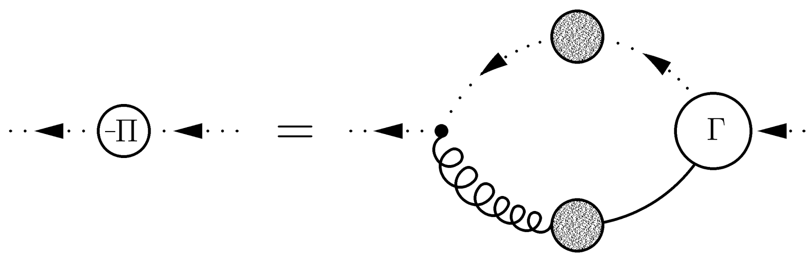


Figure 4.1: DSE of the gauge ghost self-energy, Eqs. (4.135). Filled circles denote full propagators. Empty circles are used for one-particle irreducible quantities, i.e. self-energies and proper vertices. The solid line represents a summation of one graph with the line replaced by a gauge field and a second diagram with an auxiliary field instead.

DSE for $\Pi^{(\lambda\lambda)}(k)$

We come now to the DSEs of the gauge/auxiliary field sector. Taking the derivative with respect to $\lambda^{bj}(y)$ of the auxiliary field equation (4.99) yields after setting the sources to zero

$$\frac{\delta^2\Gamma}{\delta\lambda^{ai}(x)\,\delta\lambda^{bj}(y)}\bigg|_{J=I=0} = \overbrace{2\sigma T\,\delta^{ab}\,\delta^{ij}\,\delta(x-y)}^{(\Delta^{-1})^{(\lambda\lambda)}{}^{ab}_{ij}(x,y)} - \frac{ig^2}{3!}V^{acde}_{iklm}\frac{\delta}{\delta\lambda^{bj}(y)}\,\frac{\delta^3 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)\,\delta J^{em}_A(x)}\bigg|_{J=I=0}$$
$$-\frac{ig}{2!}\,f^{acd}\Big[\big(1-\tfrac{\sigma}{\kappa}\big)\big[\delta^{ik}\partial'_l-\delta^{il}\partial_k\big] + 2\,\big[\delta^{ik}\partial_l-\delta^{il}\partial'_k\big]$$
$$+\big[\delta^{kl}\partial'_i-\delta^{lk}\partial_i\big]\Big]\frac{\delta}{\delta\lambda^{bj}(y)}\,\frac{\delta^2 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x')}\bigg|_{\substack{x'=x\\ J=I=0}} \qquad (4.137)$$

Thus, comparing to Eq. (4.117) one reads off the self-energy component

$$\Pi^{(\lambda\lambda)}{}^{ab}_{ij}(x,y) \;=\; -\frac{ig}{2!}\,f^{acd}\Big[\big(1-\tfrac{\sigma}{\kappa}\big)\big[\delta^{ik}\partial'_l-\delta^{il}\partial_k\big] + 2\,\big[\delta^{ik}\partial_l-\delta^{il}\partial'_k\big]$$
$$+\big[\delta^{kl}\partial'_i-\delta^{lk}\partial_i\big]\Big]\frac{\delta}{\delta\lambda^{bj}(y)}\,\frac{\delta^2 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x')}\bigg|_{\substack{x'=x\\ J=I=0}}$$
$$-\frac{ig^2}{3!}V^{acde}_{iklm}\frac{\delta}{\delta\lambda^{bj}(y)}\,\frac{\delta^3 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)\,\delta J^{em}_A(x)}\bigg|_{J=I=0} \qquad (4.138)$$

To evaluate Eq. (4.138), we have to calculate the remaining functional derivatives and finally transform into momentum space. Let us start with the λ derivative of the connected two-point function. Because we will encounter similar expressions also in the DSEs of the other self-energy components, it is useful to generalise a bit and do the work once and for all. Thus, with F, G and H chosen from the set $\{\lambda, A\}$, we find by means of the chain rule and using ghost number conservation, together with the identities (4.63)

$$\frac{\delta}{\delta F^{bj}(y)}\,\frac{\delta^2 W}{\delta J^{ck}_G(x)\,\delta J^{dl}_H(x')}\bigg|_{J=I=0} = \int\! dv\,\left[\frac{\delta^2\Gamma}{\delta F^{bj}(y)\,\delta K^{em}(v)}\,\frac{\delta^3 W}{\delta J^{em}_K(v)\,\delta J^{ck}_G(x)\,\delta J^{dl}_H(x')}\right]_{J=I=0}$$

The field index K in this equation is summed over the two values λ and A. Expressing the connected three-point function by its one-particle irreducible counterpart

$$W^{(FGH)}{}^{abc}_{ijk}(x,y,z) \;=\; -\int\! du\,du'du''\;G^{(FF')}{}^{aa'}_{i\,i'}(x,u)\,G^{(GG')}{}^{bb'}_{jj'}(y,u')$$
$$G^{(HH')}{}^{cc'}_{kk'}(z,u'')\,\Gamma^{(F'G'H')}{}^{a'b'c'}_{i'j'k'}(u,u',u'') \qquad (4.139)$$

and exploiting the relation (4.114) then leads to the identity

$$\frac{\delta}{\delta F^{bj}(y)} \frac{\delta^2 W}{\delta J_G^{ck}(x)\,\delta J_H^{dl}(x')}\bigg|_{J=I=0} = \\ -\int du'du''\; G^{(GG')}{}^{cc'}_{kk'}(x,u')\, G^{(HH')}{}^{dd'}_{l\,l'}(x',u'')\, \Gamma^{(FG'H')}{}^{bc'd'}_{jk'l'}(y,u',u'') \tag{4.140}$$

Again, doubled field indices are summed over λ and A (which we will assume from now on in all relevant cases). Finally, transforming into momentum space and inserting the definition of the three-point vertex function (4.124) yields

$$\frac{\delta}{\delta F^{bj}(y)} \frac{\delta^2 W}{\delta J_G^{ck}(x)\,\delta J_H^{dl}(x')}\bigg|_{J=I=0} = \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\, e^{-ik(x-y)} e^{ik'(x-x')} G^{(GG')}{}^{cc'}_{kk'}(k-k') \\ G^{(HH')}{}^{dd'}_{l\,l'}(k')\, \Gamma^{(FG'H')}{}^{bc'd'}_{jk'l'}(k'-k,-k') \tag{4.141}$$

Analogously, one can derive a general expression for the fourth functional derivative in Eq. (4.138). Using the chain rule as above, exploiting ghost number conservation and the identity (4.114), translating connected into one-particle irreducible quantities as in Eq. (4.139) and finally introducing the momentum space vertex functions (4.124) and (4.125) leads to

$$\frac{\delta}{\delta E^{bj}(y)} \frac{\delta^3 W}{\delta J_F^{ck}(x)\,\delta J_G^{dl}(x)\,\delta J_H^{em}(x)}\bigg|_{J=I=0} = \int \frac{d^Dk}{(2\pi)^D}\, e^{-ik(x-y)} \int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D} \Big[\\ + G^{(FF')}{}^{cc'}_{kk'}(k-k')\, G^{(GG')}{}^{dd'}_{l\,l'}(k'-k'')\, G^{(HH')}{}^{ee'}_{mm'}(k'')\, \Gamma^{(L'G'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'') \\ G^{(L'K')}{}^{h'g'}_{s'r'}(k')\, \Gamma^{(EF'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k') \\ + G^{(GF')}{}^{dc'}_{lk'}(k-k')\, G^{(HG')}{}^{ed'}_{ml'}(k'-k'')\, G^{(FH')}{}^{ce'}_{km'}(k'')\, \Gamma^{(L'G'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'') \\ G^{(L'K')}{}^{h'g'}_{s'r'}(k')\, \Gamma^{(EF'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k') \\ + G^{(HF')}{}^{ec'}_{mk'}(k-k')\, G^{(FG')}{}^{cd'}_{kl'}(k'-k'')\, G^{(GH')}{}^{de'}_{lm'}(k'')\, \Gamma^{(L'G'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'') \\ G^{(L'K')}{}^{h'g'}_{s'r'}(k')\, \Gamma^{(EF'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k') \\ + G^{(FF')}{}^{cc'}_{kk'}(k-k'-k'')\, G^{(GG')}{}^{dd'}_{l\,l'}(k') \\ G^{(HH')}{}^{ee'}_{mm'}(k'')\, \Gamma^{(EF'G'H')}{}^{bc'd'e'}_{jk'l'm'}(k'+k''-k,-k',-k'') \Big] \tag{4.142}$$

Exploiting the identities (4.141) and (4.142) one can now readily obtain the Dyson-Schwinger equation of the $\Pi^{(\lambda\lambda)}$ self-energy component from Eq. (4.138). One finds

$$\Pi^{(\lambda\lambda)}{}^{ab}_{ij}(k) = -\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\, (-g)V^{acd}_{i\,kl}(\mathbf{k}-\mathbf{k}',\mathbf{k}')\, G^{(AG')}{}^{cc'}_{kk'}(k-k')\, G^{(AH')}{}^{dd'}_{l\,l'}(k') \\ \Gamma^{(\lambda G'H')}{}^{bc'd'}_{jk'l'}(k'-k,-k') \\ -\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\, ig^2 V^{acde}_{iklm}\, G^{(AF')}{}^{cc'}_{kk'}(k-k')\, G^{(AG')}{}^{dd'}_{l\,l'}(k'-k'')\, G^{(AH')}{}^{ee'}_{mm'}(k'') \\ \Gamma^{(L'G'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'')\, G^{(L'K')}{}^{h'g'}_{s'r'}(k')\, \Gamma^{(\lambda F'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k') \\ -\frac{1}{6}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\, ig^2 V^{acde}_{iklm}\, G^{(AF')}{}^{cc'}_{kk'}(k-k'-k'')\, G^{(AG')}{}^{dd'}_{l\,l'}(k')\, G^{(AH')}{}^{ee'}_{mm'}(k'') \\ \Gamma^{(\lambda F'G'H')}{}^{bc'd'e'}_{jk'l'm'}(k'+k''-k,-k',-k'') \tag{4.143}$$

where we have used the symmetry of the vertex V^{acde}_{iklm} in the last three pairs of indices to combine the first three terms arising from Eq. (4.142) into one. We have illustrated Eq. (4.143) in Fig. 4.2.

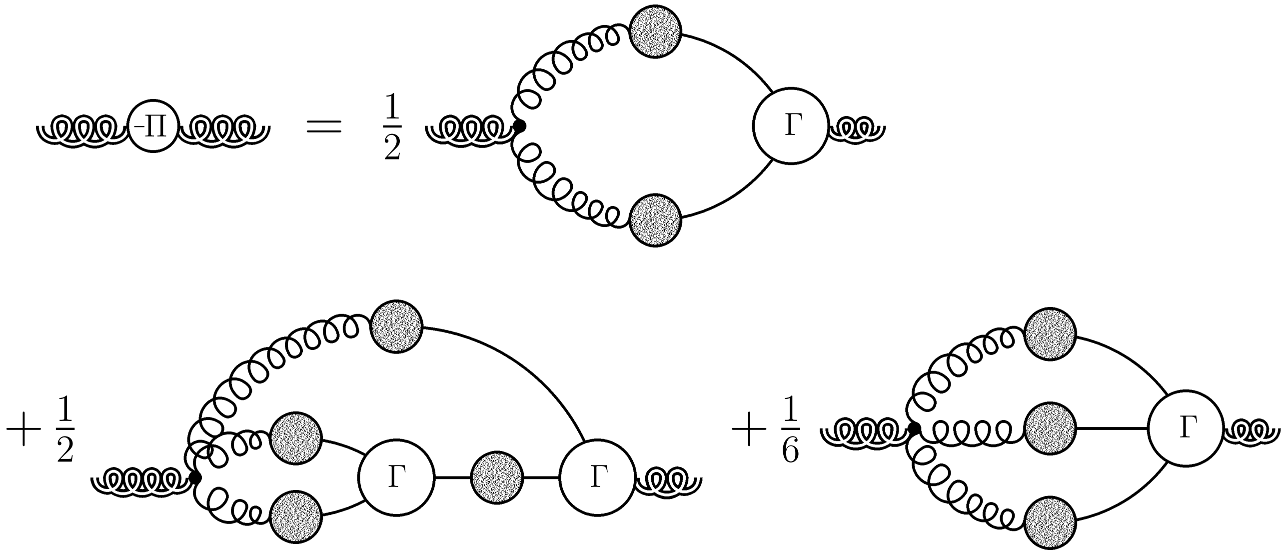


Figure 4.2: Dyson–Schwinger equation of the $\Pi^{(\lambda\lambda)}$ self-energy component, Eq. (4.143).

DSE for $\Pi^{(\lambda A)}(k)$

Taking the derivative of Eq. (4.99) with respect to $A^{bj}(y)$ instead of $\lambda^{bj}(y)$ and afterwards setting the sources to zero leads to the Dyson-Schwinger equation for the $\Pi^{(\lambda A)}$ self-energy component, namely

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta\lambda^{ai}(x)\,\delta A^{bj}(y)}\bigg|_{J=I=0} = {} & \overbrace{-i\,\delta^{ab}\left[(+\sigma\partial_t-\Delta)\,\delta_{ij}+(1-\tfrac{\sigma}{\kappa})\,\partial_i\partial_j\right]\delta(x-y)}^{(\Delta^{-1})^{(\lambda A)ab}_{ij}(x,y)} \\
& -\frac{ig}{2!}\,f^{acd}\Big[\big(1-\tfrac{\sigma}{\kappa}\big)\left[\delta^{ik}\partial'_l-\delta^{il}\partial_k\right]+2\left[\delta^{ik}\partial_l-\delta^{il}\partial'_k\right] \\
& \qquad +\left[\delta^{kl}\partial'_i-\delta^{lk}\partial_i\right]\Big]\,\frac{\delta}{\delta A^{bj}(y)}\,\frac{\delta^2 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x')}\bigg|_{\substack{x'=x\\ J=I=0}} \\
& -\frac{ig^2}{3!}V^{acde}_{iklm}\,\frac{\delta}{\delta A^{bj}(y)}\,\frac{\delta^3 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)\,\delta J^{em}_A(x)}\bigg|_{J=I=0} \\
& -\frac{ig^2}{2!}V^{acdb}_{iklj}\,\delta(x-y)\,\frac{\delta^2 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)}\bigg|_{J=I=0}
\end{aligned} \tag{4.144}$$

Reading off the self-energy component by comparing with Eq. (4.117), and using Eqs. (4.141)–(4.142) one arrives at

$$\begin{aligned}
\Pi^{(\lambda A)ab}_{\;ij}(k) = {} & -\frac{1}{2}\int\frac{d^Dk'}{(2\pi)^D}\,(-g)V^{acd}_{i\,kl}(\mathbf{k}-\mathbf{k}',\mathbf{k}')\,G^{(AG')cc'}_{\;kk'}(k-k')\,G^{(AH')dd'}_{\;l\,l'}(k') \\
& \qquad \Gamma^{(AG'H')bc'd'}_{\;jk'l'}(k'-k,-k') \\
& -\frac{1}{2}\int\frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{acde}_{iklm}\,G^{(AF')cc'}_{\;kk'}(k-k')\,G^{(AG')dd'}_{l\,l'}(k'-k'')\,G^{(AH')ee'}_{\;mm'}(k'') \\
& \qquad \Gamma^{(L'G'H')h'd'e'}_{\;s'l'm'}(k''-k',-k'')\,G^{(L'K')h'g'}_{\;s'r'}(k')\,\Gamma^{(AF'K')bc'g'}_{\;jk'r'}(k'-k,-k') \\
& -\frac{1}{6}\int\frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{acde}_{iklm}\,G^{(AF')cc'}_{\;kk'}(k-k'-k'')\,G^{(AG')dd'}_{\;l\,l'}(k')\,G^{(AH')ee'}_{\;mm'}(k'') \\
& \qquad \Gamma^{(AF'G'H')bc'd'e'}_{\;jk'l'm'}(k'+k''-k,-k',-k'')
\end{aligned}$$

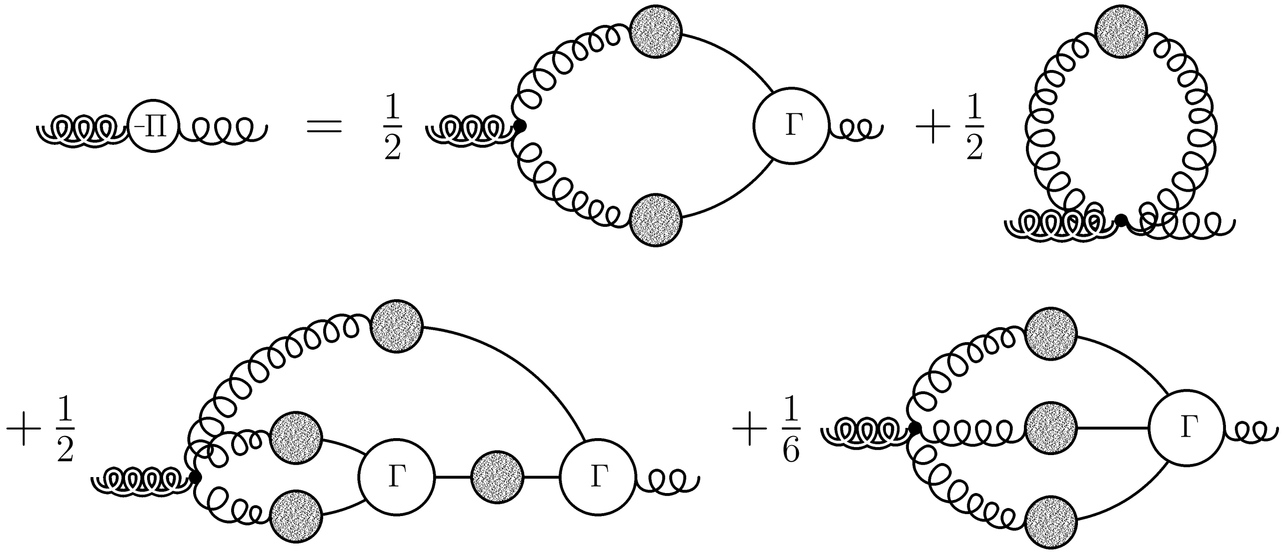


Figure 4.3: Dyson–Schwinger equation of the $\Pi^{(\lambda A)}$ self-energy component, Eq. (4.145).

$$-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\, ig^2 V^{abcd}_{ijkl}\, G^{(AA)}{}^{cd}_{kl}(k') \tag{4.145}$$

which is depicted in Fig. 4.3.

DSE for $\Pi^{(A\lambda)}(k)$

From the gauge field equation (4.105) one obtains by taking the derivative with respect to $\lambda^{bj}(y)$ for vanishing sources

$$\begin{aligned}
\left.\frac{\delta^2\Gamma}{\delta A^{ai}(x)\,\delta\lambda^{bj}(y)}\right|_{J=I=0} = \; & \overbrace{-i\,\delta^{ab}\left[(-\sigma\partial_t - \Delta)\,\delta_{ij} + (1-\tfrac{\sigma}{\kappa})\,\partial_i\partial_j\right]\delta(x-y)}^{(\Delta^{-1})^{(A\lambda)}{}^{ab}_{ij}(x,y)} \\
& -igf^{acd}\Big[-\big(1-\tfrac{\sigma}{\kappa}\big)\left[\delta^{ik}\partial'_l + \delta^{kl}(\partial_i+\partial'_i)\right] + 2\left[\delta^{kl}\partial'_i + \delta^{ik}(\partial_l+\partial'_l)\right] \\
& \qquad - \left[\delta^{il}\partial'_k + \delta^{il}(\partial_k+\partial'_k)\right]\Big]\frac{\delta}{\delta\lambda^{bj}(y)}\left.\frac{\delta^2 W}{\delta J^{ck}_\lambda(x)\,\delta J^{dl}_A(x')}\right|_{\substack{x'=x\\ J=I=0}} \\
& -\frac{ig^2}{2!}V^{eacd}_{mikl}\frac{\delta}{\delta\lambda^{bj}(y)}\left.\frac{\delta^3 W}{\delta J^{em}_\lambda(x)\,\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)}\right|_{J=I=0} \\
& -\frac{ig^2}{2!}V^{bacd}_{jikl}\,\delta(x-y)\left.\frac{\delta^2 W}{\delta J^{ck}_A(x)\,\delta J^{dl}_A(x)}\right|_{J=I=0} \\
& -\frac{g}{\kappa}f^{acd}\,\partial_i\,\frac{\delta}{\delta\lambda^{bj}(y)}\left.\frac{\delta^2 W}{\delta J^{c}_{\bar\omega}(x')\,\delta J^{d}_\omega(x)}\right|_{\substack{x'=x\\ J=I=0}}
\end{aligned} \tag{4.146}$$

As in Eq. (4.145, we have again a self-energy, a tadpole, and terms of the type in Eqs. (4.141)–(4.142), but we also have a new term involving gauge ghosts. It can be calculated in a similar way to the previous cases, and comes out to

$$\begin{aligned}
\frac{\delta}{\delta E^{bj}(y)}\left.\frac{\delta^2 W}{\delta J^{c}_{\bar\omega}(x')\,\delta J^{d}_\omega(x)}\right|_{J=I=0} = & -\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\, e^{-ik(x-y)}e^{ik'(x-x')}\,G^{(\omega)\,c'c}(-k') \\
& G^{(\omega)\,dd'}(k-k')\,\Gamma^{(\bar\omega\omega E)\,d'c'}{}^{b}_{j}(k'-k,-k')
\end{aligned} \tag{4.147}$$

With this, and the previous identities, Eq. (4.146) can be written as

$$\begin{aligned}
\Pi^{(A\lambda)}{}^{ab}_{ij}(k) = &-\int \frac{d^Dk'}{(2\pi)^D}\,(-g)V^{cda}_{kli}(\mathbf{k}',-\mathbf{k})\,G^{(\lambda A)}{}^{cc'}_{kk'}(k-k')\,G^{(AH')}{}^{dd'}_{l\,l'}(k')\\
&\qquad \Gamma^{(\lambda AH')}{}^{bc'd'}_{jk'l'}(k'-k,-k')\\
&-\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(AF')}{}^{cc'}_{kk'}(k-k')\,G^{(AG')}{}^{dd'}_{l\,l'}(k'-k'')\,G^{(\lambda A)}{}^{ee'}_{mm'}(k'')\\
&\qquad \Gamma^{(L'G'A)}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'')\,G^{(L'K')}{}^{h'g'}_{s'r'}(k')\,\Gamma^{(\lambda F'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k')\\
&-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(\lambda A)}{}^{ec'}_{mk'}(k-k')\,G^{(AG')}{}^{cd'}_{kl'}(k'-k'')\,G^{(AH')}{}^{de'}_{lm'}(k'')\\
&\qquad \Gamma^{(L'G'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'')\,G^{(L'K')}{}^{h'g'}_{s'r'}(k')\,\Gamma^{(\lambda AK')}{}^{bc'g'}_{jk'r'}(k'-k,-k')\\
&-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(AF')}{}^{cc'}_{kk'}(k-k'-k'')\,G^{(AG')}{}^{dd'}_{l\,l'}(k')\,G^{(\lambda A)}{}^{ee'}_{mm'}(k'')\\
&\qquad \Gamma^{(\lambda F'G'A)}{}^{bc'd'e'}_{jk'l'm'}(k'+k''-k,-k',-k'')\\
&+\int \frac{d^Dk'}{(2\pi)^D}\,\frac{ig}{\kappa}\,f^{cda}(k-k')^i G^{(\omega)\,c'c}(-k')\,G^{(\omega)\,dd'}(k-k')\,\Gamma^{(\bar\omega\omega\lambda)\,d'c'b}{}_j(k'-k,-k')\\
&-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\,ig^2V^{bacd}_{jikl}\,G^{(AA)}{}^{cd}_{kl}(k')
\end{aligned}\tag{4.148}$$

A graphical representation of this identity can be found in Fig. 4.4.

DSE for $\Pi^{(AA)}(k)$

Finally, we come to the pure gauge field component $\Pi^{(AA)}$. Because $(\Delta^{-1})^{(AA)}{}^{ab}_{ij} = 0$, one has in this case

$$\Pi^{(AA)}{}^{ab}_{ij}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A^{ai}(x)\,\delta A^{bj}(y)}\right|_{J=I=0}\tag{4.149}$$

and thus one obtains from Eq. (4.105) the final identity

$$\begin{aligned}
\Pi^{(AA)}{}^{ab}_{ij}(k) = &-\int \frac{d^Dk'}{(2\pi)^D}\,(-g)V^{cda}_{kli}(\mathbf{k}',-\mathbf{k})\,G^{(\lambda A)}{}^{cc'}_{kk'}(k-k')\,G^{(A\lambda)}{}^{dd'}_{l\,l'}(k')\\
&\qquad \Gamma^{(AA\lambda)}{}^{bc'd'}_{jk'l'}(k'-k,-k')\\
&-\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(AF')}{}^{cc'}_{kk'}(k-k')\,G^{(AG')}{}^{dd'}_{l\,l'}(k'-k'')\,G^{(\lambda A)}{}^{ee'}_{mm'}(k'')\\
&\qquad \Gamma^{(L'G'A)}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'')\,G^{(L'K')}{}^{h'g'}_{s'r'}(k')\,\Gamma^{(AF'K')}{}^{bc'g'}_{jk'r'}(k'-k,-k')\\
&-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(\lambda A)}{}^{ec'}_{mk'}(k-k')\,G^{(AG')}{}^{cd'}_{kl'}(k'-k'')\,G^{(AH')}{}^{de'}_{lm'}(k'')\\
&\qquad \Gamma^{(AG'H')}{}^{h'd'e'}_{s'l'm'}(k''-k',-k'')\,G^{(A\lambda)}{}^{h'g'}_{s'r'}(k')\,\Gamma^{(AA\lambda)}{}^{bc'g'}_{jk'r'}(k'-k,-k')\\
&-\frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}\frac{d^Dk''}{(2\pi)^D}\,ig^2V^{eacd}_{mikl}\,G^{(AF')}{}^{cc'}_{kk'}(k-k'-k'')\,G^{(AG')}{}^{dd'}_{l\,l'}(k')\,G^{(\lambda A)}{}^{ee'}_{mm'}(k'')\\
&\qquad \Gamma^{(AF'G'A)}{}^{bc'd'e'}_{jk'l'm'}(k'+k''-k,-k',-k'')\\
&+\int \frac{d^Dk'}{(2\pi)^D}\,\frac{ig}{\kappa}\,f^{cda}(k-k')^i G^{(\omega)\,c'c}(-k')\,G^{(\omega)\,dd'}(k-k')\,\Gamma^{(\bar\omega\omega A)\,d'c'b}{}_j(k'-k,-k')\\
&-\int \frac{d^Dk'}{(2\pi)^D}\,ig^2V^{eacb}_{mikj}\,G^{(\lambda A)}{}^{ec}_{mk}(k')
\end{aligned}\tag{4.150}$$

which completes our derivation of the DSEs in Bödeker's effective theory.

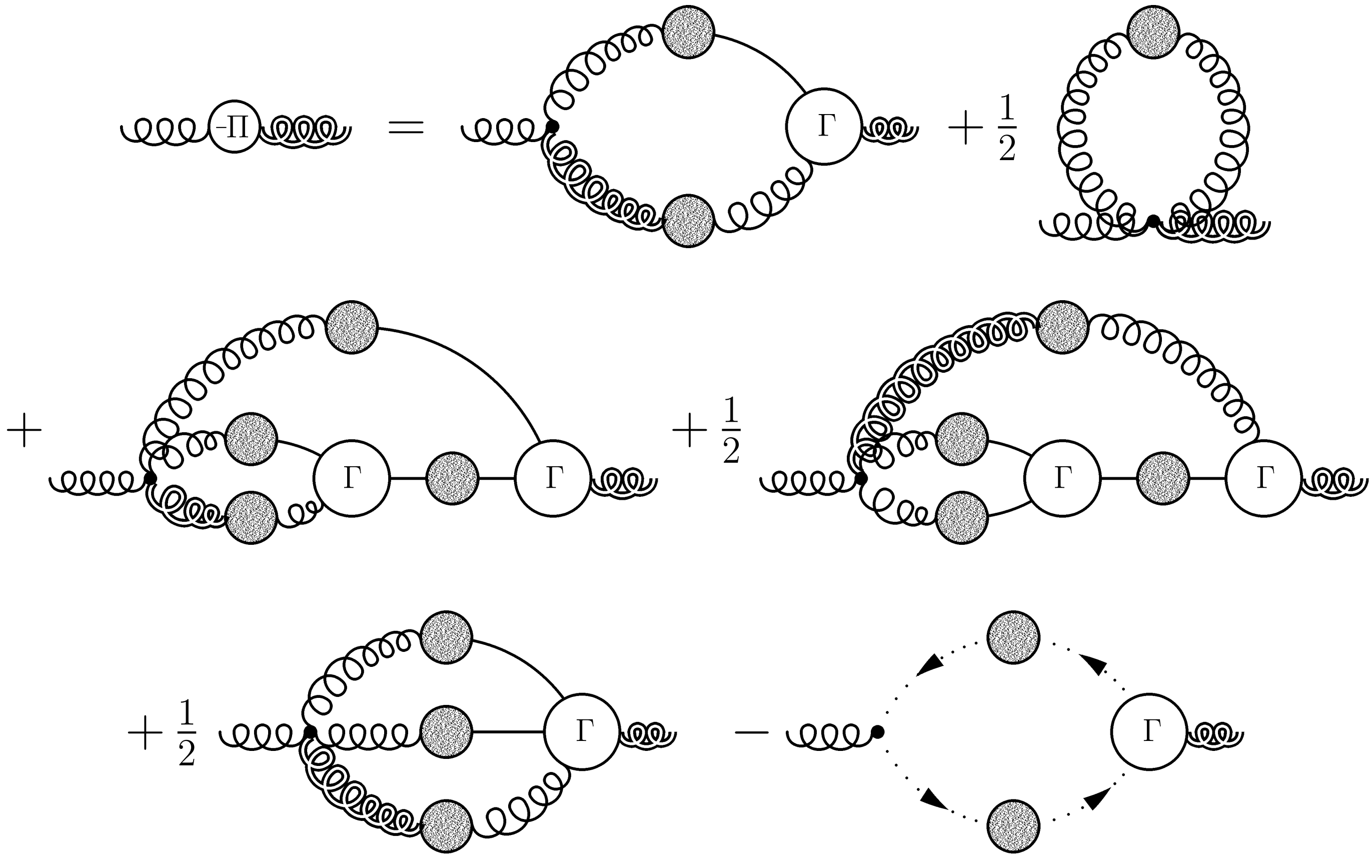


Figure 4.4: DYSON–SCHWINGER equation of the $\Pi^{(A\lambda)}$ self-energy component, Eq. (4.148).

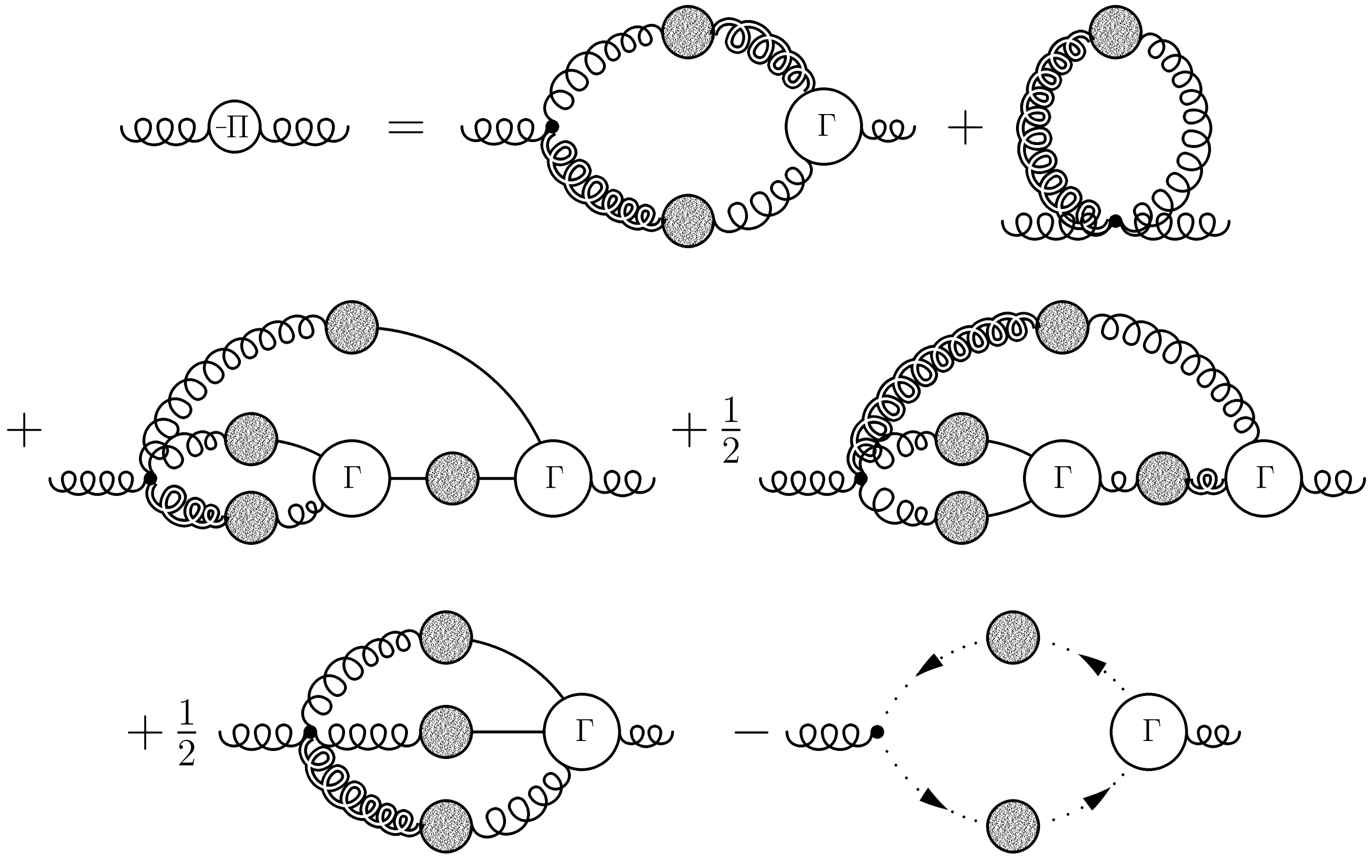


Figure 4.5: Dyson–Schwinger equation of the $\Pi^{(AA)}$ self-energy component, Eq. (4.150).

Chapter 5

Infrared Behaviour of Bodeker's Theory

In this chapter we will be concerned with the infrared behaviour of the previously developed path integral. The purpose of this is to gain a better understanding of the rate of baryon number violations. The $SU(2)$ anomaly

$$\partial_\mu J_B^\mu = N_F \frac{g^2 F^{\mu\nu}\tilde{F}_{\mu\nu}}{32\pi^2} \tag{5.1}$$

is the source of baryon number violation in the SM. The sphaleron rate is defined to be the diffusion constant for this quantity

$$\Gamma \equiv \lim_{V\to\infty} \lim_{t\to\infty} \frac{\left\langle \left(\int dt \int d^3x \frac{g^2 F^{\mu\nu}\tilde{F}_{\mu\nu}}{32\pi^2} \right)^2 \right\rangle}{Vt} = \frac{1}{N_F^2}\Gamma_{N_B}, \tag{5.2}$$

with Γ_{N_B} the diffusion constant for baryon number.

As was already mentioned in the introduction, an order of magnitude estimate gives for the sphaleron rate gives

$$\Gamma \sim \sigma^{-1}(\alpha T)^5 = \kappa' \left(\log \frac{m_D}{g^2 T} + 3.041 + \mathcal{O}(1/\log) \right) \left(\frac{g^2 T^2}{m_D^2} \right) \alpha^5 T^4, \tag{5.3}$$

where the conductivity at next-to-leading log order (NLLO) has been used, which as mentioned before makes Bödeker's theory valid up to that same order [47]. Lattice simulations find a value for κ' of 10.8 ± 0.7 [42].

In an admittedly crude approximation, Moore has attempted to calculate κ' [68]. The quality of the agreement of that calculation with the lattice value is probably fortuitous. The work in this chapter is the first step to a more formal understanding of the infrared behaviour of the effective theory, complementary to the lattice simulations, and eventually to being able to calculate κ' analytically with more rigour.

We start by proposing a simple ansatz by introducing anomalous dimensions to the propagators. As we will see, the ansatz does not close, and an error is generated in the form of a scale invariant infinite sum. With only two anomalous dimensions as parameters, the error cannot be controlled, and the ansatz is clearly seen to fail.

We then generalise the ansatz to allow for effective colour conductivities. While, the four new parameters will still not allow the complete cancellation of error, they are enough to minimise the error as long as the relation between σk_0 and $|\mathbf{k}|^2$ holds in the deep infrared.

The infrared analysis of the DSEs is done under the assumption of infrared dominance. As in infrared Yang-Mills theory, we work under the assumption that the ultraviolet part of the equations, and this includes the inverse tree level propagator, cancels. This might be due to boundary conditions or some other argument, but for this work we'll simply consider it as assumed. The infrared equations therefore, cannot set all the constants, since we can always rescale all others in terms of one of our choice.

Because of the form of the infrared dominated DSEs, the goal of obtaining all the necessary information to be able to calculate the sphaleron rate cannot be accomplished solely within the present analysis. One constant always remains undetermined and needs to be set by comparing to some other data. This we leave for a future work.

5.1 A First Ansatz

There are two important reasons why the search for a scaling solution in the infrared with the present theory presents more difficulties than the analogous work in Yang-Mills theory. First, the broken Lorentz symmetry results in two variables to consider k_0 and $\mathbf{k}$. While a scaling solution in the form of the perturbative propagator with an anomalous power, $1/(k^2)^{1+\alpha}$[1], is a natural ansatz for the standard problem, the presence of the two variables muddles the possible input in the present problem. The second difficulty is the complexity of the tensor structure of the 3-point function, which, at least with the tree-level values, does not allow for a choice of a particular value of the flow gauge in which the transversal component could be solved independently of the longitudinal. In the calculation of the self-energies, both components always mix.

We are looking for scaling solutions to the N-point functions, i.e. a solution with the following behaviour

$$\lim_{\lambda\to 0} V\left(\lambda^2 k_0, \lambda\mathbf{k}\right) = \lambda^\alpha V\left(k_0, \mathbf{k}\right). \tag{5.4}$$

In order to maintain complications to a minimum, we will first set the vertices to their tree-level value. As we will see, the result will have enough complexity to justify this, as least as a pedagogical approach. Later, we will discuss a more suitable ansatz for the vertices. With this choice, there are only three vertices that enter into Eqs. (4.136,4.143, 4.145, 4.150): $\Gamma^{(\bar\omega\omega A)}$, $\Gamma^{(\lambda AA)}$, and $\Gamma^{(\lambda AAA)}$. By the introduction of the auxiliary field, the tree level vertices have no k_0 dependence, and this will simplify the calculation somewhat.

The tree-level propagators are presented in Appendix C. Disregarding multiplicative constants and tensor structure, they are of two forms

$$\frac{1}{-i\kappa' k_0 + |\mathbf{k}|^2}, \qquad \frac{1}{(\kappa')^2 k_0^2 + |\mathbf{k}|^4}, \tag{5.5}$$

where the first one, which for the duration of this chapter will be termed $1/\Delta$, corresponds to the mixed and ghost propagators, and the second one to the pure gauge propagator, $1/\tilde\Delta = 1/|\Delta|^2$. κ' can take the value of σ or κ. Since the vertices have no k_0 dependence, as long as the qualitative distribution of poles in the complex plane is maintained in the infrared behaviour, Eq. (4.150) is automatically fulfilled. It is also clear from Eqs. (4.143, 4.145) that the ghost will not play the role as in Yang-Mills theory, since it only appears in Eq. (4.136). The mixed propagator will play the analogous role to the ghost in Yang-Mills theory.

In contrast to Yang-Mills theory, it is not possible to choose a gauge which will make either of the two remaining propagators be transversal. The tensor structure of the vertex functions always allows for a mixing of the transversal and longitudinal components. For example, in the gauge $\kappa \to 0$, the longitudinal component for the mixed the propagators takes the form

$$\frac{\kappa}{\sigma|\mathbf{k}|^2}, \tag{5.6}$$

but since the vertices contain factors of σ/κ, the longitudinal part will produce finite contributions. Similar behaviour occurs when calculating the infrared critical exponents with time-independent

[1] The usual notation is for the anomalous dimension to be represented by κ, but in the present work, κ has been used for the parameter which governs the flow gauge. To avoid possible confusion, in the present chapter we use α to represent the anomalous dimension

stochastic quantisation [67]. However, if we set $\kappa = \sigma$, then at tree-level the propagators will at least be diagonal, and indeed we will use this choice of gauge. As opposed to Yang-Mills theory, where Landau gauge presents the advantage that the transversal gluon propagator can be calculated independently, we do not specialise the gauge because of a similar advantage. In principle, computations in a general flow gauge will not complicate the calculation of the self-energies by much, it does, however, increase the difficulties encountered during the solving of the algebraic equations that ensue. For the duration of this exercise we will keep to $\kappa = \sigma$ gauge.

We will take as ansatz then, the functions in Eq. (5.5) with anomalous dimensions. Including the tensor structure, the proposed propagators take the following form

$$\begin{aligned} G^{(A\lambda)}{}^{ab}_{ij}(k) = G^{(\lambda A)}{}^{ba}_{ji}(-k) &= P^T_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\Delta Z^T_\lambda(k)} + P^L_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\Delta Z^L_\lambda(k)} \\ &\equiv P^T_{ij}(\mathbf{k})\frac{i\delta^{ab}C^T_\lambda}{(-i\sigma k_0 + |\mathbf{k}|^2)^{1+\alpha_\lambda}} + P^L_{ij}(\mathbf{k})\frac{i\delta^{ab}C^L_\lambda}{(-i\sigma k_0 + |\mathbf{k}|^2)^{1+\alpha_\lambda}}, \\ G^{(AA)}{}^{ab}_{ij}(k) &= P^T_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta}Z^T_A(k)} + P^L_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta}Z^L_A(k)} \\ &\equiv P^T_{ij}(\mathbf{k})\frac{2\sigma T\delta^{ab}C^T_A}{(\sigma^2 k_0^2 + |\mathbf{k}|^4)^{1+\alpha_A}} + P^L_{ij}(\mathbf{k})\frac{2\sigma T\delta^{ab}C^L_A}{(\sigma^2 k_0^2 + |\mathbf{k}|^4)^{1+\alpha_A}}, \end{aligned} \tag{5.7}$$

where $P^T_{ij}(\mathbf{k})$ and $P^L_{ij}(\mathbf{k})$ are respectively the regular transversal and longitudinal projectors in three dimensions.

Because of the mixing, the anomalous dimension will be the same for the transversal and longitudinal components. There is an issue with this ansatz which merits attention at this point. Inputing these forms for the propagators into the appropriate equations, will not result in a closed expression of the desired form, but will contain a scale invariant series which produces an error in the approximation. Of course, if the error is large, then the ansatz is disqualified. The inclusion of a free parameter determining the coupling between k_0 and $\mathbf{k}$ instead of the fixed σ will allow us to minimise the error. We will include such a parameter later on. The only ansatz which we have seen that is able to close the equations exactly, is one of the form $1/(k_0^{\alpha_1}(|\mathbf{k}|^2)^{\alpha_2})$, but we find it unacceptable since its divergence in either direction is wholly independent from the behaviour of the other direction.

Considering the scaling property in Eq. (5.4), the anomalous dimensions must fulfil the relations

$$\begin{aligned} 1 + \alpha_\lambda &= \min\left(\frac{D-3}{2} - 2\alpha_A - \alpha_\lambda, D - 4 - 4\alpha_A - 2\alpha_\lambda, D - 4 - 4\alpha_A - \alpha_\lambda\right), \\ \alpha_\lambda - \alpha_A &= \min\left(\frac{D-5}{4} - 2\alpha_A, \frac{D-5}{2} - 3\alpha_A - \frac{1}{2}\alpha_\lambda, \frac{D-5}{2} - 3\alpha_A\right). \end{aligned} \tag{5.8}$$

As can be easily verified, the two relations are consistent with each other. Because the infrared gauge fields are damped in the plasma, we would expect $\alpha_A < 0$. For the range $-\frac{1}{4} < \alpha_A < 0$, the one-loop graph will be subleading, while for α_A smaller than $-1/4$, the one-loop graph will dominate. Assuming that the latter is true, the relation between α_λ and α_A is then

$$\alpha_\lambda = \frac{D-5}{4} - \alpha_A = -\frac{1}{4} - \alpha_A, \tag{5.9}$$

where the last equality holds for $D = 4$. This result gives an enhanced mixed propagator for the assumed restriction on α_A.

As has been mentioned, for $\alpha_A < -1/4$, the one-loop graphs dominate. The equations that the ansatz in Eq. (5.7) must fulfil are then

$$\begin{aligned} \left(G^{-1}\right)^{(\lambda A)}{}^{ab}_{ij}(k) = \int \frac{d^D k'}{(2\pi)^D}\, (-g)^2 V^{acd}_{i\,k\,l}(\mathbf{k}', \mathbf{k} - \mathbf{k}')\, V^{d'c'b}_{l'k'j}(-\mathbf{k}', \mathbf{k}) \\ \times G^{(AA)}{}^{c\,c'}_{kk'}(k')\, G^{(A\lambda)}{}^{dd'}_{l\,l'}(k - k'), \end{aligned} \tag{5.10}$$

and

$$\left(G^{-1}\right)^{(\lambda A)}{}^{ab}_{ij}(k)G^{(AA)}{}^{ab}_{ij}(k)\left(G^{-1}\right)^{(A\lambda)}{}^{ab}_{ij}(k) = \frac{1}{2}\int \frac{d^Dk'}{(2\pi)^D}(-g)^2 V^{acd}_{i\,k\,l}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\,V^{bd'c'}_{jl'k'}(\mathbf{k}'-\mathbf{k},\mathbf{k}')G^{(AA)}{}^{c\,c'}_{kk'}(k')\,G^{(AA)}{}^{dd'}_{l\,l'}(k-k'). \quad (5.11)$$

The ansatz will not give a closed solution for the equations, but rather will result in the right hand side of Eq. (5.10) behaving schematically like

$$\Pi^{(\lambda A)} = \Delta^{\frac{D-3}{2}-2\alpha_A-\alpha_\lambda}\sum_{n=0}^{\infty} a_n\left(\frac{|\mathbf{k}|^2}{\Delta}\right)^n, \quad (5.12)$$

where $\Delta = -i\sigma k_0 + |\mathbf{k}|^2$, and $a_n = a_n(\alpha_\lambda, \alpha_A; D)$. For the right hand side of Eq. (5.11)

$$\Pi^{(\lambda\lambda)} = \tilde{\Delta}^{\frac{D-5}{4}-2\alpha_A}\sum_{n=0}^{\infty}\tilde{a}_n\left(\frac{|\mathbf{k}|^2}{\tilde{\Delta}^{1/2}}\right)^n, \quad (5.13)$$

where $\tilde{\Delta} = \sigma^2 k_0^2 + |\mathbf{k}|^4$, and similarly $\tilde{a}$ is a function of the parameters. The amplitude of both Δ and $\tilde{\Delta}$ is bounded by 1. Except at the point $k_0 = 0$, $\mathbf{k} \neq 0$, both series converge, but the slower it converges the worse the ansatz in Eq. (5.7) will be. For the moment, we leave this aside, and will come back to it when we introduce a general coupling between k_0 and $\mathbf{k}$.

5.1.1 $\Pi^{(\lambda A)}$ equation

After the previous considerations, we'll now move to solving Eq. (5.10). Applying a transversal projector $P^T_{ij}(\mathbf{k})$ to the $\Pi^{(\lambda A)}$ equation and using the ansatz from Eq. (5.7), results in

$$\Delta Z^T_\lambda(k) = b^{T,TT}_\lambda(k) + b^{T,TL}_\lambda(k) + b^{T,LT}_\lambda(k) + b^{T,LL}_\lambda(k), \quad (5.14)$$

and applying a longitudinal projector

$$\Delta Z^L_\lambda(k) = b^{L,TT}_\lambda(k) + b^{L,TL}_\lambda(k) + b^{L,LT}_\lambda(k) + b^{L,LL}_\lambda(k), \quad (5.15)$$

where

$$b^{F,GH}_\lambda(k) = -\frac{g^2}{P^F_{ii}}\int\frac{d^Dk'}{(2\pi)^D}\frac{\Gamma^{F,GH}_\lambda(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}Z^G_\lambda(k-k')Z^H_A(k'), \quad (5.16)$$

and the $\Gamma^{F,GH}_\lambda$ are given in Eq. (C.29).

The functions $b^{F,GH}_\lambda(k)$ will be calculated in a general form by the following procedure. Using Eq. (5.7), Eq. (5.16) can be written as

$$b^{F,GH}_\lambda(k) = -2i\,\frac{g^2}{P^F_{ii}}\,\sigma T C^G_\lambda C^H_A\int\frac{d^Dk'}{(2\pi)^D}\frac{\Gamma^{F,GH}_\lambda(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \times\frac{1}{(-i\sigma(k_0-k_0')+|\mathbf{k}-\mathbf{k}'|^2)^{1+\alpha_\lambda}}\frac{1}{(\sigma^2 k_0'^2+|\mathbf{k}'|^4)^{1+\alpha_A}}. \quad (5.17)$$

Note that P^F_{ii} equals $D-2$ for $F=T$ and 1 for $F=L$. For convenience, we define $\omega = -2i\,g^2 T N_c C^G_\lambda C^H_A/P^F_{ii}$ and $\hat{k}_0 = -\sigma k_0$, and perform the change of variable $k_0' \to -\frac{k_0'}{\sigma}$,

$$b^{F,GH}_\lambda(k) = \omega\int\frac{d^Dk'}{(2\pi)^D}\frac{\Gamma^{F,GH}_\lambda(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}\frac{1}{\left(i(\hat{k}_0-k_0')+|\mathbf{k}-\mathbf{k}'|^2\right)^{1+\alpha_\lambda}}\frac{1}{(k_0'^2+|\mathbf{k}'|^4)^{1+\alpha_A}}. \quad (5.18)$$

Introduce Gamma function integral representations for the k_0 denominators to obtain

$$b_\lambda^{F,GH}(k) = \frac{\omega}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\alpha_\lambda} x_2^{\alpha_A} x_3^{\alpha_A} \int \frac{d^D k'}{(2\pi)^D} \frac{\Gamma_\lambda^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \exp\left\{-x_1\left(i(\hat{k}_0 - k_0') + |\mathbf{k}-\mathbf{k}'|^2\right) - x_2\left(ik_0' + |\mathbf{k}'|^2\right) - x_3\left(-ik_0' + |\mathbf{k}'|^2\right)\right\}. \quad (5.19)$$

Performing the k_0' integral results in a Dirac delta function $\delta(-x1+x2-x3)$, with which one can do the x_2 integral. Introducing unity in the following form

$$1 = \int_0^\infty dx_4 \delta(x_1 + x_3 - x_4). \quad (5.20)$$

Defining $a = 2 + 2\alpha_A + \alpha_\lambda$, and doing the change of variables $x_1 \to x_4 x_1$ and $x_3 \to x_4 x_3$

$$b_\lambda^{F,GH}(k) = \frac{\omega}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2} \int_0^\infty dx_1 dx_3 dx_4 x_1^{\alpha_\lambda}(x_1+x_3)^{\alpha_A} x_3^{\alpha_A} x_4^{a-1} \delta(x_1+x_3-1)$$
$$\times \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \frac{\Gamma_\lambda^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \exp\left\{-x_4\left(x_1\left(i\hat{k}_0 + |\mathbf{k}-\mathbf{k}'|^2\right) + (x_1+2x_3)|\mathbf{k}'|^2\right)\right\}. \quad (5.21)$$

Perform the x_4 and the x_3 integral to obtain

$$b_\lambda^{F,GH}(k) = \frac{\omega\Gamma(a)}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2} \int_0^1 dx_1 x_1^{\alpha_\lambda}(1-x_1)^{\alpha_A}$$
$$\times \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \frac{\Gamma_\lambda^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \frac{1}{\left(x_1\left(i\hat{k}_0 + |\mathbf{k}-\mathbf{k}'|^2\right) + (2-x_1)|\mathbf{k}'|^2\right)^a}. \quad (5.22)$$

We now introduce the series expansion mentioned in Eq. (5.12). Substitute $i\hat{k_0}$ for $\Delta - |\mathbf{k}|^2$, perform the change of variable $\mathbf{k}' \to \Delta^{1/2}\mathbf{k}'$, and redefine $\mathbf{k}$ as $\mathbf{k} = \hat{\mathbf{k}}\Delta^{1/2}$, where $\hat{\mathbf{k}} = \mathbf{k}/\Delta^{1/2}$ is the variable appearing in the series expansion previously mentioned. By expanding the denominators in terms of $|\hat{\mathbf{k}}|$ we obtain

$$b_\lambda^{F,GH}(k) = \frac{\Delta^{\frac{D-3}{2}-2\alpha_A-\alpha_\lambda}\omega\Gamma(a)}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2} \int_0^1 dx_1 x_1^{\alpha_\lambda}(1-x_1)^{\alpha_A} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}}$$
$$\times \Gamma_\lambda^{F,GH}(\hat{\mathbf{k}},\mathbf{k}')\left(\frac{1}{\left(2|\mathbf{k}'|^2+x_1\right)^a} + 2\frac{\hat{\mathbf{k}}\cdot\mathbf{k}'\left((2+ax_1)|\mathbf{k}'|^2+x_1\right)}{|\mathbf{k}'|^4\left(2|\mathbf{k}'|^2+x_1\right)^{a+1}} + \mathcal{O}\left(|\hat{\mathbf{k}}|^2\right)\right)$$
$$\equiv \frac{\Delta^{\frac{D-3}{2}-2\alpha_A-\alpha_\lambda}\omega\Gamma(a)}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2} \int_0^1 dx_1 x_1^{\alpha_\lambda}(1-x_1)^{\alpha_A} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}}$$
$$\times \sum_i c_i \hat{k}^{n_i} k'^{m_i} \cos(\theta)^{q_i} x_1^{r_i} \frac{1}{\left(2k'^2+x_1\right)^{s_i}}, \quad (5.23)$$

where the second equality defines the coefficients c_i and the appropriate exponents. The final result is

$$b_\lambda^{F,GH}(k) = \frac{\Delta^{\frac{D-3}{2}-2\alpha_A-\alpha_\lambda}\omega\Gamma(2+2\alpha_A+\alpha_\lambda)}{(2\pi)^{D-1}\Gamma(1+\alpha_\lambda)\Gamma(1+\alpha_A)}$$
$$\times \sum_i c_i \hat{k}^{n_i} \frac{\Omega_{q_i}^D \Gamma\left(\frac{D+m_i-1}{2}\right)\Gamma\left(s_i - \frac{D+m_i-1}{2}\right)\Gamma\left(r_i - s_i + \alpha_\lambda + \frac{D+m_i+1}{2}\right)}{2^{\frac{D+m_i+1}{2}}\Gamma(s_i)\Gamma\left(r_i - s_i + \alpha_\lambda + \alpha_A + \frac{D+m_i+3}{2}\right)}. \quad (5.24)$$

Ω corresponds to the angular integral and takes the form

$$\Omega_{q_i}^D = \frac{1+(-1)^{q_i}}{2} \frac{2\pi\Gamma\left(\frac{1+q_i}{2}\right)\Gamma\left(\frac{D-2}{2}\right)}{\Gamma\left(\frac{D+q_i-1}{2}\right)}. \quad (5.25)$$

Finally, we express the result in a slight modification to the notation used in Eq. (5.12)

$$b_\lambda^{F,GH}(k) = -\Delta^{\frac{D-3}{2}-2\alpha_A-\alpha_\lambda} iC_\lambda^G C_A^H \sum_{n=0}^{\infty} a_{\lambda,n}^{F,GH} \left(\frac{|\mathbf{k}|^2}{\Delta}\right)^n. \tag{5.26}$$

At this juncture we will not deal with the error introduced by the failure of the ansatz to be fully consistent, and will deal solely with the leading order a_0 terms, which take the following values

$$\begin{aligned} a_{\lambda,0}^{T,TT} =& \frac{g^2 N_c T\,\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{1}{2}(-1+D-4\alpha_A)\right)\Gamma\left(\frac{3}{2}-\frac{D}{2}+2\alpha_A+\alpha_\lambda\right)}{2^{\frac{D+1}{2}}\pi^{3/2}\Gamma\left(\frac{1}{2}(1+D-2\alpha_A)\right)\Gamma(1+\alpha_\lambda)} \\ =& 2a_{\lambda,0}^{T,LT} = -(2D-4)a_{\lambda,0}^{T,LL} = a_{\lambda,0}^{L,TT} = 2a_{\lambda,0}^{L,LT} = -(2D-4)a_{\lambda,0}^{L,LL} \\ a_{\lambda,0}^{T,TL} =& a_{\lambda,0}^{L,TL} = 0. \end{aligned} \tag{5.27}$$

Returning to Eq. (5.14), we can match orders of Δ to obtain a relation for α_λ in terms of α_A, which has already been shown in Eq. (5.9). After the matching, the result are the two first equations for the C constants of the ansatz, and reads

$$\begin{aligned} \frac{1}{C_\lambda^T} =& C_\lambda^T C_A^T a_{\lambda,0}^{T,TT} + C_\lambda^T C_A^L a_{\lambda,0}^{T,TL} + C_\lambda^L C_A^T a_{\lambda,0}^{T,LT} + C_\lambda^L C_A^L a_{\lambda,0}^{T,LL}, \\ \frac{1}{C_\lambda^L} =& C_\lambda^T C_A^T a_{\lambda,0}^{L,TT} + C_\lambda^T C_A^L a_{\lambda,0}^{L,TL} + C_\lambda^L C_A^T a_{\lambda,0}^{L,LT} + C_\lambda^L C_A^L a_{\lambda,0}^{L,LL}. \end{aligned} \tag{5.28}$$

Rescaling the longitudinal constants, $C_\lambda^L = C_\lambda^T c_\lambda^L$ and $C_A^L = C_A^T c_A^L$, together with Eq. (5.27) reduces the equations to the manegeable form

$$\begin{aligned} 1 =& \left(C_\lambda^T\right)^2 C_A^T a_{\lambda,0}^{T,TT}\left(1+\frac{1}{2}c_\lambda^L - \frac{1}{2D-4}c_\lambda^L c_A^L\right), \\ \frac{1}{c_\lambda^L} =& \left(C_\lambda^T\right)^2 C_A^T a_{\lambda,0}^{T,TT}\left(1+\frac{1}{2}c_\lambda^L - \frac{1}{2D-4}c_\lambda^L c_A^L\right), \end{aligned} \tag{5.29}$$

from which it is obvious that $c_\lambda^L = 1$. The single equation can then be solved by

$$C_A^T = \frac{1}{\left(C_\lambda^T\right)^2 a_{\lambda,0}^{T,TT}\left(\frac{3}{2}-\frac{c_A^L}{2D-4}\right)}. \tag{5.30}$$

5.1.2 $\Pi^{(\lambda\lambda)}$ equation

We will treat Eq. (5.11) in the same manner. The equation after applying a transversal projector $P_{ij}^T(\mathbf{k})$ to the $\Pi^{(\lambda\lambda)}$ equation and using the ansatz from Eq. (5.7), results in

$$\frac{Z_\lambda^T(k)Z_\lambda^T(-k)}{Z_A^T(k)} = b_A^{T,TT}(k) + b_A^{T,TL}(k) + b_A^{T,LT}(k) + b_A^{T,LL}(k). \tag{5.31}$$

The longitudinal equation reads

$$\frac{Z_\lambda^L(k)Z_\lambda^L(-k)}{Z_A^L(k)} = b_A^{L,TT}(k) + b_A^{L,TL}(k) + b_A^{L,LT}(k) + b_A^{L,LL}(k), \tag{5.32}$$

where

$$b_A^{F,GH}(k) = \frac{g^2}{2\,P_{ii}^F}\int \frac{d^Dk'}{(2\pi)^D}\frac{\Gamma_A^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} Z_A^G(k-k')Z_A^H(k'), \tag{5.33}$$

and the $\Gamma_A^{F,GH}$ are given in Eq. (C.31).

The calculation of $b_A^{F,GH}$ proceeds in a similar manner to that of $b_\lambda^{F,GH}$, but here we will not use the integral representation of the Gamma function, and will instead employ a feynman parameter integral directly

$$b_A^{F,GH}(k) = \sigma\omega' \frac{\Gamma(2+2\alpha_A)}{(\Gamma(1+\alpha_A))^2} \int_0^1 dx\, x^{\alpha_A}(1-x)^{\alpha_A} \int \frac{d^D k'}{(2\pi)^D} \frac{\Gamma_A^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \times \frac{1}{\left(x\left(\sigma^2(k_0-k_0')^2+|\mathbf{k}-\mathbf{k}'|^4\right)+(1-x)\left(\sigma^2 k_0'^2+|\mathbf{k}'|^4\right)\right)^{2+2\alpha_A}}, \tag{5.34}$$

where $\omega' = 2\, g^2\sigma T^2 C_A^G C_A^H / P_{ii}^F$. Performing the k_0' integral and making the substitution $\sigma^2 k_0^2 = \tilde{\Delta} - |\mathbf{k}|^4$ gives

$$b_A^{F,GH}(k) = \frac{\omega'}{2\sqrt{\pi}} \frac{\Gamma(\frac{3}{2}+2\alpha_A)}{(\Gamma(1+\alpha_A))^2} \int_0^1 dx\, x^{\alpha_A}(1-x)^{\alpha_A} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \frac{\Gamma_A^{F,GH}(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \times \frac{1}{\left(x(1-x)\tilde{\Delta}+x|\mathbf{k}-\mathbf{k}'|^4+(1-x)|\mathbf{k}'|^4-x(1-x)|\mathbf{k}|^4\right)^{\frac{3}{2}+2\alpha_A}}. \tag{5.35}$$

Defining $a' = \frac{3}{2}+2\alpha_A$, and once again performing a change of variables, but with $\mathbf{k}' \to \tilde{\Delta}^{1/4}\mathbf{k}'$, redefining $\mathbf{k}$ as $\mathbf{k} = \tilde{\mathbf{k}}\tilde{\Delta}^{1/4}$, and expanding in terms of $\tilde{\mathbf{k}}$ results in

$$\begin{aligned} b_A^{F,GH}(k) =& \frac{\tilde{\Delta}^{\frac{D-5}{4}-2\alpha_A}\omega'}{2\sqrt{\pi}} \frac{\Gamma(a')}{(\Gamma(1+\alpha_A))^2} \int_0^1 dx\, x^{\alpha_A}(1-x)^{\alpha_A} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \Gamma_A^{F,GH}(\tilde{\mathbf{k}},\mathbf{k}') \\ &\times \left(\frac{1}{|\mathbf{k}'|^2\left(|\mathbf{k}'|^4+x(1-x)\right)^{a'}} + \frac{2\mathbf{k}\cdot\mathbf{k}'\left(|\mathbf{k}'|^4(1+2ax)+x(1-x)\right)}{|\mathbf{k}'|^4\left(|\mathbf{k}'|^4+x(1-x)\right)^{a'}} + \mathcal{O}\left(|\hat{\mathbf{k}}|^2\right) \right) \\ \equiv& \frac{\tilde{\Delta}^{\frac{D-5}{4}-2\alpha_A}\omega'}{2\sqrt{\pi}} \frac{\Gamma(a')}{(\Gamma(1+\alpha_A))^2} \int_0^1 dx\, x^{\alpha_A}(1-x)^{\alpha_A} \int \frac{d^{D-1}k'}{(2\pi)^{D-1}} \Gamma_A^{F,GH}(\tilde{\mathbf{k}},\mathbf{k}') \\ &\times \sum_i c_i' \hat{k}^{n_i'} k'^{m_i'} \cos(\theta)^{q_i'} x^{r_i'}(1-x)^{r_i''} \frac{1}{\left(|\mathbf{k}'|^4+x(1-x)\right)^{s_i'}}), \end{aligned} \tag{5.36}$$

where the second equality again defines the coefficients c_i' and the appropriate exponents. The result is

$$\begin{aligned} b_A^{F,GH}(k) =& \frac{\tilde{\Delta}^{\frac{D-5}{4}-2\alpha_A}\omega'\Gamma(a')}{8\sqrt{\pi}(2\pi)^{D-1}(\Gamma(1+\alpha_A))^2} \sum_i c_i' \tilde{k}^{n_i'} \Omega_{q_i'}^D \Gamma\left(\frac{D+m_i'-1}{4}\right) \\ &\times \frac{\Gamma\left(\frac{1-D-m_i'}{4}+s_i'\right)\Gamma\left(\frac{3+D+m_i'}{4}+r_i'-s_i'+\alpha_A\right)\Gamma\left(\frac{3+D+m_i'}{4}+r_i''-s_i'+\alpha_A\right)}{\Gamma(s_i')\Gamma\left(\frac{3+D+m_i'}{2}+r_i'+r_i''-2s_i'+2\alpha_A\right)}, \end{aligned} \tag{5.37}$$

where Ω has already been defined in Eq. (5.25). The corresponding expression to Eq. (5.26) is then

$$b_A^{F,GH}(k) = \tilde{\Delta}^{\frac{D-5}{4}-2\alpha_A} C_A^G C_A^H \sum_{n=0}^{\infty} \tilde{a}_{A,n}^{F,GH} \left(\frac{|\mathbf{k}|^2}{\tilde{\Delta}^{1/2}}\right)^n \tag{5.38}$$

The $\tilde{a}_0$ equations are similar to those in Eq. (5.14)

$$\begin{aligned} \tilde{a}_{A,0}^{T,TT} =& \frac{g^2 N_c T^2 \sigma \Gamma(D-1)\Gamma\left(\frac{D-1}{4}-\alpha_A\right)\Gamma\left(\frac{5-D}{4}+2\alpha_A\right)}{2^{3D-7-2\alpha_A}\pi^{\frac{2D-7}{2}}\Gamma\left(\frac{D-1}{4}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D+1}{4}-\alpha_A\right)\Gamma(1+\alpha_A)^2} \\ =& \tilde{a}_{A,0}^{T,TL} = \tilde{a}_{A,0}^{T,LT} = (D-2)\tilde{a}_{A,0}^{T,LL} = \tilde{a}_{A,0}^{L,TT} = \tilde{a}_{A,0}^{L,TL} = \tilde{a}_{A,0}^{L,LT} = (D-2)\tilde{a}_{A,0}^{L,LL}. \end{aligned} \tag{5.39}$$

Upon rescaling the longitudinal constants as before, and using Eq. (5.39), Eq. (5.31) and Eq. (5.32) now read

$$
\begin{aligned}
-\frac{2\sigma T\, C_A^T}{\left(C_\lambda^T\right)^2} &= \left(C_A^T\right)^2 \tilde{a}_{A,0}^{T,TT}\left(1+2c_A^L+(D-2)^2\left(c_A^L\right)^2\right), \\
-\frac{2\sigma T\, C_A^T c_A^L}{\left(C_\lambda^T C_\lambda^L\right)^2} &= \left(C_A^T\right)^2 \tilde{a}_{A,0}^{T,TT}\left(1+2c_A^L+(D-2)^2\left(c_A^L\right)^2\right),
\end{aligned}
\tag{5.40}
$$

which automatically sets $c_A^L = 1$. The remaining equation is a function of α_A

$$
-\sigma T\, a_{\lambda,0}^{T,TT}\left(3-\frac{1}{D-2}\right) = \tilde{a}_{A,0}^{T,TT}\left(3+\frac{1}{D-2}\right), \tag{5.41}
$$

where we have used Eq. (5.30). Solving the previous equation with a general value of D seems not only impossible, but rather unnecessary since we have been keeping it around as a hedge against possible divergences and no longer need it. While the theory is UV finite, this of course does not mean that there can be no need for regularisation. At this point we finally set $D = 4$ to obtain

$$
\frac{80\Gamma\left(\frac{5}{4}-\alpha_A\right)\Gamma\left(-\frac{3}{4}+\alpha_A\right)\Gamma(1+\alpha_A)}{\Gamma\left(\frac{5}{2}-\alpha_A\right)} = \frac{492^{4\alpha_A}\pi\Gamma\left(\frac{3}{4}-\alpha_A\right)\Gamma\left(\frac{1}{4}+2\alpha_A\right)}{\Gamma\left(\frac{11}{4}\right)} \tag{5.42}
$$

which has as solution

$$
\alpha_A = -0.5832. \tag{5.43}
$$

5.2 Improving the Approximation

So far the ansatz is working properly. The gauge propagator is found to be suppressed by the anomalous dimension $\alpha_A = -0.5832$, while the mixed propagator, which plays an analogous role to the ghost propagator in the infrared analysis of QCD, is enhanced with an anomalous dimension of $\alpha_\lambda = 0.3332$, as qualitatively expected from the physics being modelled. Although not demanded, it presents the nice feature of having started with a gauge in which the tree level propagators were diagonal, and by obtaining $C_\lambda^L = C_\lambda^T$ and $C_A^L = C_A^T$ this property would carry over to the infrared behaviour. This property however is misleading, since the $a_{n=2}$ coefficients are not diagonal, i.e. $a_2^{T,GH} \neq a_2^{L,GH}$, and this is true even in perturbation theory.

We now turn to the error which impedes the ansatz being a consistent solution. If we calculate the contributions coming from the $n = 2$ and $n = 4$ terms to the respective self-energies, a catastrophic failure appears

$$
\begin{aligned}
\left|\frac{\Pi_{n=2}^{(\lambda A)T}}{\Pi_{n=0}^{(\lambda A)}}\right| &\approx 2, & \left|\frac{\Pi_{n=2}^{(AA)T}}{\Pi_{n=0}^{(AA)}}\right| &\approx 0.07, \\
\left|\frac{\Pi_{n=4}^{(\lambda A)T}}{\Pi_{n=0}^{(\lambda A)}}\right| &\approx 0.2, & \left|\frac{\Pi_{n=4}^{(AA)T}}{\Pi_{n=0}^{(AA)}}\right| &\approx 0.3
\end{aligned}
\tag{5.44}
$$

The conclusion is that the ansatz is obviously flawed. While we have not yet set a value for the constant C_λ^T, it cannot be done within the current analysis. The increase of value from $\Pi_{n=2}^{(AA)T}$ to $\Pi_{n=4}^{(AA)T}$ might make the reader wonder whether the endeavour is altogether hopeless. The error for $\Pi^{(AA)T}$ presents a hump at $n = 4$, but then decreases monotonically afterwards.

The most direct extension one could do would be to introduce a parameter γ controlling the relation between k_0 and $\mathbf{k}$ in the original ansatz, instead of leaving the fixed σ. The parameters can then be used to try to minimise the error. This we do, and introduce a new ansatz

$$\begin{aligned}G^{(A\lambda)ab}_{ij}(k) = G^{(\lambda A)ba}_{ji}(-k) &= P^T_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\Delta Z^T_\lambda(k)} + P^L_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\Delta Z^L_\lambda(k)} \\ &\equiv P^T_{ij}(\mathbf{k})\frac{i\delta^{ab}C^T_\lambda}{\left(-i\gamma^T_\lambda\sigma k_0 + |\mathbf{k}|^2\right)^{1+\alpha_\lambda}} + P^L_{ij}(\mathbf{k})\frac{i\delta^{ab}C^L_\lambda}{\left(-i\gamma^L_\lambda\sigma k_0 + |\mathbf{k}|^2\right)^{1+\alpha_\lambda}}, \\ G^{(AA)ab}_{ij}(k) &= P^T_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta} Z^T_A(k)} + P^L_{ij}(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta} Z^L_A(k)} \\ &\equiv P^T_{ij}(\mathbf{k})\frac{2\sigma T\delta^{ab}C^T_A}{\left((\gamma^T_A)^2\sigma^2k_0^2 + |\mathbf{k}|^4\right)^{1+\alpha_A}} + P^L_{ij}(\mathbf{k})\frac{2\sigma T\delta^{ab}C^L_A}{\left((\gamma^L_A)^2\sigma^2k_0^2 + |\mathbf{k}|^4\right)^{1+\alpha_A}},\end{aligned} \tag{5.45}$$

With this new ansatz, we redo the calculation. Returning to Eq. (5.22), but now $\omega = -2i\, g^2TN_cC^G_\lambda C^H_A/(\gamma^H_A P^F_{ii})$, and having already done the rescaling $\mathbf{k}' \to \mathbf{k}'\Delta_F^{1/2}$, and $\mathbf{k} = \hat{\mathbf{k}}\Delta_F^{1/2}$, with $\Delta_F = -i\gamma^F_\lambda k_0 + |\mathbf{k}|^2$, the equation takes the form

$$\begin{aligned}b^{F,GH}_\lambda(k) &= \left(\frac{\gamma^H_A}{\gamma^G_\lambda}\right)^{1+\alpha_\lambda} \frac{\omega\Gamma(a)\Delta_F^{\frac{D+1}{2}-a}}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2}\int_0^1 dx_1 x_1^{\alpha_\lambda}(1-x_1)^{\alpha_A} \\ &\times \int \frac{d^{D-1}k'}{(2\pi)^{D-1}}\,\frac{\Gamma^{F,GH}_\lambda(\mathbf{k},\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}\frac{1}{\left(\frac{\left(\gamma^H_A\right)^2}{\gamma^F_\lambda\gamma^G_\lambda}x_1(1-|\hat{\mathbf{k}}|^2) + \frac{\gamma^H_A}{\gamma^G_\lambda}x_1|\mathbf{k}-\mathbf{k}'|^2 + (2-x_1)|\mathbf{k}'|^2\right)^a} \\ &\equiv \left(\frac{\gamma^H_A}{\gamma^G_\lambda}\right)^{1+\alpha_\lambda} \frac{\omega\Gamma(a)\Delta_F^{\frac{D+1}{2}-a}}{\Gamma(1+\alpha_\lambda)\left(\Gamma(1+\alpha_A)\right)^2}\int_0^1 dx_1 x_1^{\alpha_\lambda}(1-x_1)^{\alpha_A}\int\frac{d^{D-1}k'}{(2\pi)^{D-1}} \\ &\times \sum_i c_i\hat{k}^{n_i}k'^{m_i}\cos(\theta)^{q_i}x_1^{r_i}\frac{1}{\left(\left(2-x\left(1-\frac{\gamma^H_A}{\gamma^G_\lambda}\right)\right)k'^2 + \frac{\left(\gamma^H_A\right)^2}{\gamma^F_\lambda\gamma^G_\lambda}x_1\right)^{s_i}},\end{aligned} \tag{5.46}$$

where once again the last equality defines the coefficient and the proper indices. Doing the integrals, the result reads

$$\begin{aligned}b^{F,GH}_\lambda(k) = \frac{\Delta_F^{\frac{D+1}{2}-a}\omega}{(2\pi)^{D-1}\Gamma(1+\alpha_\lambda)\Gamma(1+\alpha_A)}\left(\frac{\gamma^H_A}{\gamma^G_\lambda}\right)^{1+\alpha_\lambda}\sum_i c_i\hat{k}^{n_i}\frac{\Omega^D_{q_i}}{2^{b+1}}\left(\frac{\gamma^F_\lambda\gamma^G_\lambda}{\left(\gamma^H_A\right)^2}\right)^{s_i-b} \\ \times\frac{\Gamma(s_i-b)\Gamma(b)\Gamma(c)\,{}_2F_1\left(b,c;d;\frac{1}{2}\left(1-\frac{\gamma^H_A}{\gamma^G_\lambda}\right)\right)}{\Gamma(d)},\end{aligned} \tag{5.47}$$

with $a = 2+2\alpha_A+\alpha_\lambda$, $b = \frac{D-1+m_i}{2}$, $c = \frac{D+1+m_i}{2}+r_i-s_i+\alpha_\lambda$, and $d = \frac{D+3+m_i}{2}+r_i-s_i+\alpha_A+\alpha_\lambda$. Eq. (5.24) redefines the coefficients $a^{F,GH}_{\lambda,n}$ in Eq. (5.26).

A similar calculation for $\Pi^{(\lambda\lambda)}$ gives

$$\begin{aligned}b^{F,GH}_A(k) = \frac{\tilde{\Delta}_F^{\frac{D+1}{4}-a'}\omega'\Gamma(a')}{8\sqrt{\pi}(2\pi)^{D-1}\left(\Gamma(1+\alpha_A)\right)^2}\sum_i c'_i\tilde{k}^{n'_i}\Omega^D_{q'_i}\left(\frac{\gamma^F_A}{\gamma^G_A}\right)^{r'''+\frac{1-D-m'_i}{4}+s'_i}\Gamma\left(\frac{D+m'_i-1}{4}\right) \\ \times\frac{\Gamma\left(\frac{1-D-m'_i}{4}+s'_i\right)\Gamma(c)\Gamma(d-c)\,{}_2F_1\left(b,c,d;1-\left(\frac{\gamma^G_A}{\gamma^H_A}\right)^2\right)}{\Gamma(s'_i)\Gamma(d)},\end{aligned} \tag{5.48}$$

with $a' = \frac{3}{2}+2\alpha_A$, $b' = \frac{1+D+m'_i}{4} - r''' - s'_i$, $c' = \frac{D+3+m'_i}{4}+r'-s'_i+\alpha_A$, $d' = \frac{D+3+m'_i}{2}+r'+r''-2s'_i+2\alpha_A$, and finally $\omega' = 2\,g^2\sigma T^2C^G_AC^H_A/(\gamma^H_A P^F_{ii})$. All the exponents are defined as in in

Eq. (5.36), but for the new one r''', which corresponds to a term in the summand of the form $(1-x(1-(\gamma_A^G/\gamma_A^H)^2))^{r'''}$

Ideally we would like the new ansatz to be able to minimise the error in Eqs (5.14,5.15) and Eqs (5.31,5.32), without making recourse to the value of the base function in the expansion. The base function $|\mathbf{k}|^2/\tilde{\Delta_F}^{1/2}$ is bounded above by 1, and the larger γ^F is, the larger the volume of the integration space under which it will be small, but we cannot claim that it is small everywhere, and the question once again links to the concept of infrared when one has two momenta. Unfortunately, with the 8 highly nonlinear equations, we were unable to accomplish it this way.

If each individual term in the error series cannot be simultaneously reduced, then the value of the base expansion function is certainly important. For the minimisation procedure we have maintained the same relation between σk_0 and $|\mathbf{k}|^2$ that exists at g^2T, which is certainly within the infrared.

Since the first two terms in the error are the ones that cause more concern by being possibly large, we focus on minimising them. This simplifies the procedure somewhat. The system of equations which fulfil at least a local minimum

$$\begin{array}{cc} \alpha_A = -0.6095, & \alpha_\lambda = 0.3595, \\ \gamma_A^T = 494.0,\ \gamma_A^L = 1.053\times 10^4, & \gamma_\lambda^T = 331.4,\ \gamma_\lambda^L = 1973, \\ C_A^T = \frac{3729}{(C_\lambda^T)^2},\ C_A^L = \frac{1272}{(C_\lambda^T)}, & C_\lambda^L = 11.32 C_\lambda^T, \end{array} \tag{5.49}$$

The error is of course not exactly zero, but is much smaller than in Eq. (5.44). The new values are

$$\begin{array}{cc} \left|\frac{\Pi_{n=2}^{(\lambda A)T}}{\Pi_{n=0}^{(\lambda A)}}\right| \approx 0.003, & \left|\frac{\Pi_{n=2}^{(AA)T}}{\Pi_{n=0}^{(AA)}}\right| \approx 10^{-8}, \\ \left|\frac{\Pi_{n=4}^{(\lambda A)T}}{\Pi_{n=0}^{(\lambda A)}}\right| \approx 0.0004, & \left|\frac{\Pi_{n=4}^{(AA)T}}{\Pi_{n=0}^{(AA)}}\right| \approx 10^{-10} \end{array} \tag{5.50}$$

Similar values hold for the longitudinal components. The high effective conductivity presented for γ_A^L is a consequence of the error in the mixed self-energy equation. They were specially difficult to minimise because the transversal and longitudinal parts are almost inversely related, what minimises the transversal would increase the longitudinal, and vice versa.

As was the case for the first ansatz, the value of all the constants cannot be set within the infrared DSEs, and must look elsewhere to set it. This we leave for future work.

Although we do not include it in our calculations, we mention a further generalisation of the ansatz which could serve to deal with the error.

The contributions to the self energy take the form of a series in $|\mathbf{k}|^2/\tilde{\Delta}^{1/2}$. Since $G^{(AA)}$ appears on both equations as itself, and not as its inverse, it is straightforward to introduce $G^{(AA)}$ as a series in the aforementioned variable

$$G^{(AA)ab}_{ij}(k) = P_{ij}^T(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta}_T^{1+\alpha_A}}\sum_{n=0}^{\infty} C_{A,n}^T\left(\frac{|\mathbf{k}|^2}{\tilde{\Delta}^{1/2}}\right)^n + P_{ij}^L(\mathbf{k})\delta^{ab}\frac{1}{\tilde{\Delta}_L^{1+\alpha_A}}\sum_{n=0}^{\infty} C_{A,n}^L\left(\frac{|\mathbf{k}|^2}{\tilde{\Delta}^{1/2}}\right)^n. \tag{5.51}$$

Its use in the calculation would not necessitate any new formulas to compute, since Eq. (5.48) is general enough to accommodate it. A similar form could also be introduced for the mixed propagators, but in that case the inverse of the propagators do enter the equations, and its form would also need to be computed. At the very least, a new ansatz with the zeroth- and first-order terms seems feasible.

It might very well be that this form of the ansatz is more appropriate, and in fact necessary, to find the correct infrared behaviour of the theory. Notwithstanding, we do not use this further generalisation because until comparisons are made with the lattice data (or experimental values) to determine the remaining constant, this further process would simply seem an exercise on curve fitting.

5.3 An Ansatz for the Vertex Functions

We have not justified our decision to use the tree-level value of the vertex functions in the ansatz. We must admit that we have no physical justification for this. Our reason for doing so was entirely practical. Because of the complicated tensor structure and the possible availability of more varied types of vertices in the present theory compared to Yang-Mills theory, such a pragmatic choice has allowed us to begin calculations, without having to consider the very large task that would entail the analogous ansaetze construction to QCD [69].

We would like to mention, however, a possible form for a vertex ansatz which would not involve proposing new momentum tensor structures. For infrared Yang-Mills theory, there has been shown recently that there is a unique global scaling solution [48]. In that work, a form of the vertex functions was used which can be directly translated to the present case. In the Yang-Mills case, the vertex functions are parametrised in the following manner

$$\Gamma^{(2n,m)}(\vec{p}) = \bar{\Gamma}^{(2n,m)}(\vec{p}) \prod_{i=1}^{2n} \sqrt{Z^{(2,0)}(p_i)} \prod_{i=1}^{m} \sqrt{Z^{(0,2)}(p_{2n+i})}\,, \tag{5.52}$$

where $2n$ is the number of external ghost legs, m the number of gluon legs, $Z^{(2,0)}$ is the ghost dressing function, $Z^{(0,2)}$ corresponds to the gluon one. By means of an analysis of the functional renormalisation group equations and the DSEs, the authors proceed to show that modulo logarithms, the global scaling of the functions $\bar{\Gamma}^{(2n,m)}$ is the canonical one.

The notation that serves so well for the Yang-Mills case, becomes too cumbersome for our purposes, so we just present the gauge-auxiliary 3-point function. The extension to the other propagators is straightforward. For the $\Gamma^{(FGH)}$ of the present case, a possible parametrisation consistant with multiplicative renormalisation would be

$$\begin{aligned}\Gamma^{(FGH)}_{i\,j\,k}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \bar{\Gamma}^{(F'G'H')}_{i'\,j'\,k'}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &\left(P^T_{ii'}\sqrt{Z^{(FF'),T}(\mathbf{k}_1)} + P^L_{ii'}\sqrt{Z^{(FF'),L}(\mathbf{k}_1)}\right)\\ \times&\left(P^T_{jj'}\sqrt{Z^{(GG'),T}(\mathbf{k}_2)} + P^L_{jj'}\sqrt{Z^{(GG'),L}(\mathbf{k}_2)}\right)\\ \times&\left(P^T_{kk'}\sqrt{Z^{(HH'),T}(\mathbf{k}_3)} + P^L_{kk'}\sqrt{Z^{(HH'),L}(\mathbf{k}_3)}\right).\end{aligned} \tag{5.53}$$

While it looks more daunting than the Yang-Mills case in Landau gauge, the main complication is basically bookkeeping, with which by now we are well acquainted in this theory. That $\bar{\Gamma}^{(2n,m)}$ scales canonically does not mean that it therefore takes the tree level form, but such a choice could serve as an ansatz. Even without conducting a similar analysis to that in Ref. [69], the ansatz would then be motivated by multiplicative renormalisation. Of course, the full analysis could certainly lead to important insight into the theory, and would therefore be worth the effort.

There is a potential downside to this proposition however. We did not use Eq. (5.53) as part of the ansatz in this work not because it necessarily complicates the integrals, but because it directly changes the distribution of the poles in the complex plane which we had counted on to ensure that the $\Pi^{(AA)}$ equation be automatically fulfilled. This is not a trivial manner. Adding two new equations to the system, without including any more unknowns, could easily lead to overdetermined system with no solution. If the equation is not automatically fulfilled it might be that the ensuing equations are just redundant, or that the gauge ghost contribution and the mixed propagator loops cancel, thereby saving the situation, but we see we cannot assume that it would. This would have to be checked.

Chapter 6

Conclusions

In this work we have studied two aspects related to electroweak baryogenesis. The first half was concerned with CP violation from within the Standard Model. In the literature it has been argued that CP violation is too small in the SM to explain the baryon asymmetry in the universe. The argument is based on treating the quark masses perturbatively, which results in the Jarlskog determinant, of order 10^{-19} as a measure of CP violation within the SM [70]. We have calculated the CP violating operators appearing in an effective action originating by integrating out the fermions. Interestingly, they were not suppressed by the Jarlskog determinant, but rather by the Jarlskog invariant. This is a crucial point, since had they been suppressed by the Jarlskog determinant they would not serve as CP violation sources for scenarios of electroweak baryogenesis.

In Chapter 2 we derived a master formula for an one-loop fermion effective action for the case of a general chiral model which includes interactions with a scalar, a pseudoscalar, a vector, a pseudovector, and an antisymmetric tensor, although the latter is not used in the SM calculation, and was generally set to zero when calculating the explicit form of the action. The closed form for the effective action was calculated using a covariant derivative expansion, where the number of covariant indices determines the order of the expansion.

The action was first divided into a real (Parity even) and imaginary (Parity odd) part. The imaginary part contains the chiral anomaly and must be dealt with appropriately. In order to obtain a gauge and chiral invariant action, we separated the imaginary part into a Wess-Zumino-Witten (WZW) term, with the property that it saturates the anomaly, and what remains is possible to express in a covariant form. The covariant remainder is calculated by first obtaining the covariant effective current and integrating it. Of course, the covariant current cannot be the functional derivative of a covariant action directly, the anomalous contribution is needed for the integration to be consistent. With the master formula for the covariant current in place, we calculate the leading order and next-to-leading order in both 2 and 4 dimensions for the imaginary part of the action. The results for the 2 dimensional case were presented in [15]. The result was later validated by Salcedo in [63].

The effective action is obtained from the effective current through a matching procedure. A suitable ansatz for the effective action is posited, and its functional variation is taken. The result is matched with the effective current obtained from the worldline method. The worldline method is well suited for the calculation since it avoids momentum integrals, traces over gamma matrices, and can be implemented on a computer algebra program in a straightforward manner. In the leading order case, the WZW term will also give a contribution and it must be included. Our results for the leading order agree with Salcedo [51]. For the next to leading order case, with the use of certain transformations, e.g. Eq. (2.129) or partial integration, the number of terms in the effective current was significantly reduced. Still, in four dimensions, even the reduced expression involves 97 equations, for only 28 unknowns in the original ansatz. Because of partial integration, the trace operation, and other such transformations, not all unknowns are independent and the original ansatz presents some redundancies. The idea behind starting from a larger than necessary ansatz is that a priori it is not clear which set of terms can result in a simpler solution.

If the equations encountered in the matching procedure were to be simple linear equations, then the problem would be relatively simple. Any computer algebra program would likely be able to find the solution, if it exists, but this is not the case. While the label operator notation allows us to compute the integrals in a closed form, it also means that standard algorithms to solve systems of linear equations are not applicable. The system does decompose into different 'levels', with only the unknown functions with the least number of indices contributing to the first level, the functions with the least and next-to-least number of indices contributing to the next level, and so forth. The system can then be solved by solving the first level first, then using those results to reduce the number of unknowns in the second level, and similarly for the other levels.

There are two forms of consistency conditions. That there are many more equations than unknowns means that only a few of them actually need to be solved to find a solution and that the others serve solely as a strong consistency condition that need to be checked. The other type of condition comes from the symmetry that the solutions must observe, e.g. $Q^{(1)}_{123} = -Q^{(1)}_{\underline{123}}$. A correct solution must fulfil these consistency conditions, they are highly nontrivial and give confidence in our results.

The specialisation to the SM was done in Chapter 3. Because at least 4 CKM matrices are required for the operator to be CP violating, no CP violating term is found at leading order. Our calculation shows CP violation in the imaginary part, and also in the real part, at next-to-leading order for the imaginary part of the action, and order six for the real part. Contrary to expectations in the literature, the real part also contains such terms even though it contains no ϵ tensor. The results for the imaginary part were presented in [71]. Our results for the real part are yet unpublished. The explicit form of the real part of the effective action for a general chiral model is not presented because the results for the SM were obtained directly, without the general step. The real part, without a matching procedure, is more straightforward to compute and was implemented as an almost completely automated computer algebra program. It is unclear at this time whether these Parity *conserving*, CP violating terms coming from the real part can influence the Chern-Simmons number.

The validity of the expansion is also studied. For a regime where the Higgs field expectation value is not zero, we can conservatively say that it should be valid at least to an energy range on par with the charm quark mass. As mentioned before though, the effective action was constructed with the motivation of using it for cold electroweak baryogenesis simulations. In such a scenario, the winding number will change when the Higgs length becomes zero at a point [64]. In their simulations, the authors used the CP violating operator in Eq. (1.7). They found that CP violation need not be concentrated around such points for baryogenesis to be viable. On the other hand, the effective lagrangian in Eq. (3.45) diverges around those points. We have argued that the contribution to the effective action at those points coming from Eq. (3.45) might be zero, and that the infrared behaviour of the CP violating terms could then still serve for a simulation. This requires further investigation, and we are collaborating with Dr. Anders Tranberg in the implementation of the operator on the lattice.

In the lattice simulation, a simple infrared cutoff is introduced to keep the effective lagrangian from blowing up. If the results do not depend strongly on the value of the cutoff, then the CP violating operator would serve as it stands. Still, there is the possibility to introduce a more sophisticated infrared cutoff, and we have demonstrated it in Section 3.2.4. It would be interesting to see whether such a cutoff could be obtained directly within the expansion. The idea would be to basically treat not just the Higgs field non-perturbatively, but also its first derivative. It is not clear at this time that such an approach would result in a consistent expansion. Further investigation is needed in this regard.

Another possibility for the infrared stabilisation would be the introduction of temperature. While the phase transition of interest to us occurs at close to zero temperature, even a small temperature might be able to stabilise the integrals in the case of $m_i \to 0$. A first naive attempt, working under the prescription in [72] has resulted in inconsistencies during the matching procedure of the imaginary part of the action. We have not yet attempted such a calculation on the real part of the effective action, where we could directly compare the result with a similar calculation that could be done using Salcedo's formalism.

A last avenue requiring further work is the extension of the formalism to include Majorana masses. Because the assumed form of the fermion action, i.e. $\bar{\Psi} \cdots \Psi$, does not allow for the inclusion of such terms, we have not carried out a similar analysis for the fermion sector. Such an analysis would be of great interest, as it is often used in baryogenesis through leptogenesis scenarios, see [73], and might allow for CP violating terms already at leading order [35].

The second half of this work dealt with another aspect of baryogenesis at the electroweak scale, mainly baryon number violation. As explained before, the rate of baryon number violation is directly related to the sphaleron rate in Eq.(1.8). Since the effects of including the Higgs field results in a decrease of about 20% of the sphaleron rate, a good first approximation is to treat the Yang-Mills theory alone [38]. Bödeker has introduced an infrared effective theory well suited for the study of the sphaleron rate in the hot phase. It has previously been used for lattice calculations [42]. In this work we have provided tools for the study of the effective theory by a more analytic approach. Our goal has not been to compete numerically with the lattice simulations, but to provide a complementary approach.

As such, we have decided to work with Bödeker's original effective action, even though Bödeker has derived a generalised Boltzmann-Langevin equation which is valid to all orders in $[\log(1/g)]^{-1}$ [74], of which Bödeker's effective theory is merely the leading logarithmic approximation and the existence of other more general approaches, e.g. [75]. We choose this approximation because of the tractability of the analytic approach within this framework. The more general theory is not only more complicated, it is not even renormalisable by power counting [76]. The effective theory treated in this work, on the other hand, is is ultraviolet finite, and is known to still be valid at next-to-leading logarithmic order provided one uses the next-to-leading logarithmic order colour conductivity σ [47].

In Chapter 4 we start from Bödeker's effective theory, in the form of a Langevin equation. Bödeker's effective action in $A_0 = 0$ gauge is first expanded to a more general class of flow gauges, and translated into a quantum path integral. In principle, one could derive Dyson-Schwinger equations at this point, but we decided to first simplify the action. A direct translation results in an action with interactions of order A^6, and to simplify this an auxiliary field is introduced. The resulting action has vertices proportional up to A^4.

Since any practical approach will ultimately truncate to an approximation, consideration of gauge artefacts and gauge dependencies is relevant. We further introduce gauge ghosts for the remaining gauge freedom. This enlarged system is endowed with a BRST symmetry reflecting the gauge invariance; and we have derived the corresponding Ward-Takahashi identities. We also derived a second class of restrictions, so-called stochastic Ward identities known from stochastic quantisation [65]. These reflect the characteristic structure of the path integral action induced by its origin in a stochastic differential equation.

Next, Dyson-Schwinger equations for the system are derived. In combination with the gauge and stochastic Ward identities given in Eqs. (D.25) – (D.27), the Dyson-Schwinger equations (4.135), (4.136), (4.143), (4.145), (4.148) and (4.150) provide the necessary tools for an analytic study of the non-perturbative physics encoded in Bödeker's effective theory. These results are presented in [77].

Chapter 5 deals with the analytic study of the infrared behaviour of Bödeker's theory. We do this studying the Dyson-Schwinger equations under the assumption of infrared dominance. Since the infrared behaviour is a much studied problem for Yang-Mills theory, we have tried to draw as much inspiration and guidance from it. Of course, the problem is not Yang-Mills theory in three dimensions, so one can not simply do the same analysis. It is important to mention that studying the infrared behaviour through the Dyson-Schwinger equations, it is not possible to determine completely the amplitude of all the propagators. The amplitudes can be rescaled in terms of one of them, in our case C^T_λ, which remains undetermined.

The first problem to overcome is to determine what is meant by infrared in the theory. With two variables, k_0 and $|\mathbf{k}|$. While the ideal situation would not rely on a relationship between the two, in practice we were forced to consider the characteristic scale of the problem, $\sigma k_0 \sim |\mathbf{k}|^2$ in order to determine leading and subleading behaviour.

The first ansatz that is presented merely assigns an anomalous dimension to the tree level propagators, with the expectation that the gauge propagator will be suppressed. The value of the higher-order vertex functions are taken to be simply the tree-level values. The ansatz does not change the qualitative distribution of the poles in the complex plane, so the structure of the Dyson-Schwinger equations is such that in this case the gauge ghost does not contribute in the $\Pi^{(AA)}$ or $\Pi^{(\lambda A)}$ equations. Instead, the mixed propagator plays an analogous role in the solution to the ghosts in QCD.

We easily see that the ansatz does not satisfy the equations. There will be an error introduced by its failure to close. If the error had been small we could have been satisfied, as the present exercise is merely an approximation and does not intend to compete quantitatively with the lattice calculations. However, this is not the case as can be seen in Eq. (5.44). The ansatz was posited with the same colour conductivity 'coupling' k_0 to $\mathbf{k}$, and this is the obvious change that can be introduced for dealing with the error.

The new ansatz, once again assumes tree-level vertices, and an anomalous dimension for the propagators, but now includes a parameter γ determining the effective colour conductivity that the different propagators feel. All four γ's are found to be large, and specially so for γ_A^L. If the result is not an artefact of having the parameters in a local minima of the highly nonlinear equations which determine them, this could imply a higher resistance in the infrared to magnetic change than would be readily apparent, which in turn would dampen the sphaleron rate. The determination of the parameter set is done by solving a system of nonlinear equations, which expresses very strong dependence to the initial values of the parameters. Establishing whether the parameter set sits on a local minima or not still requires further study.

We have also proposed two possible improvements for the ansatz used in this work. Since the self-energies are obtained as the appropriate scaling factor multiplied by a scale invariant sum, an ansatz for the propagators in terms of this same type of series could solve completely the issue of the error. The formulas presented for the calculation of the self-energies are general enough, that this upgrade would not be difficult to introduce.

The second proposal is for the vertex functions, Eq. (5.53). The form shown was the direct translation of a parametrisation proposed for the infrared Yang-Mills case [69], and with which those authors prove that there is a unique global scaling solution in Landau gauge. As an ansatz, it can be motivated by multiplicative renormalisation, but a full analysis, including the functional renormalisation group equations, could shed light on the theory. The downside of the proposal is that we had counted on the same distribution of the poles in the complex plane as in the tree-level case to guarantee that the Π^{AA} equation was fulfilled. A cancellation of the mixed loop with the gauge ghost loop might have to be arranged, but the gauge ghost is not a simple free parameter to be determined by that equation, since it has its own self-energy equation to fulfil.

However, before improving on the ansatz it is more important to first determine C_λ^T. For this, it is necessary to compare with other data. Of course, comparing with the lattice value for the sphaleron would defeat the purpose of the exercise, so we have to look to other areas of hot non-abelian gauge theory. One possibility lies with magnetic mass and screening [78–81]. Or in applications of the Stochastic Vacuum Model [82–84]. While the need to look elsewhere might be seen as a drawback, we see it as an opportunity to expand the scope of applicability of our work.

Finally, there is still much work to be done in both topics. Non-abelian gauge theories present a plethora of interesting characteristics and phenomena. This work has sought to investigate two related topics within this larger and rich subject. While our main motivation has been the explanation of the baryon asymmetry of the universe with physics as close as possible to the Standard Model, it is easy to see that the applicability of this work is not restricted solely to this area.

Appendix A

Worldline Method

A.1 Integrals used in the calculation

In this section, the function g denotes the bosonic Green function

$$g(T,\tau_1) = \langle y(T)y(\tau_1)\rangle, \tag{A.1}$$

and

$$\dot{g}(T,\tau_1) = \langle \dot{y}(T)y(\tau_1)\rangle = -2\langle \psi_A(T)\psi_A(\tau_1)\rangle, \tag{A.2}$$

where the last expression does not contain a summation over the index A.

A.1.1 Integrals in Two Dimensions

In the calculation in two dimensions the following integrals have been used

$$\begin{aligned}
J^1_{12} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\, e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)} = \frac{\log(m_1^2/m_2^2)}{m_1^2-m_2^2},\\
J^2_{12} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\, e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}\dot{g}(T,\tau_1)\\
&= -\frac{2}{m_1^2-m_2^2} + \frac{(m_1^2+m_2^2)}{(m_1^2-m_2^2)^2}\log\left(\frac{m_1^2}{m_2^2}\right),\\
J^3_{12} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1 e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}\dot{g}(T,\tau_1)g(T,\tau_1)\\
&= -3\frac{m_1^2+m_2^2}{(m_1^2-m_2^2)^3} + \frac{(m_1^4+4m_1^2m_2^2+m_2^4)}{(m_1^2-m_2^2)^4}\log\left(\frac{m_1^2}{m_2^2}\right).
\end{aligned} \tag{A.3}$$

The remaining occurring integrals can be expressed as

$$\begin{aligned}
J^4_{12} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\, e^{-Tm_1^2-\tau_1(m_2^2-m_1^2)}g(T,\tau_1) = \frac{J^2_{12}}{m_1^2-m_2^2},\\
J^5_{123} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)} = -\frac{\nabla_2 J^1_{12}}{m_2+m_3},\\
J^6_{123} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)}\,\dot{g}(T,\tau_1) = -\frac{\nabla_2 J^2_{12}}{m_2+m_3},\\
J^7_{123} &= \int_0^\infty \frac{dT}{T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)}\,\dot{g}(T,\tau_2) = -J^6_{321}.
\end{aligned} \tag{A.4}$$

A.1.2 Integrals in Four Dimensions

In four dimensions the following integrals with three indices are used

$$\begin{aligned}
J^8_{123} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)} \\
&= \frac{m_1^2\log(m_1^2)}{(m_1^2-m_2^2)(m_1^2-m_3^2)} - \frac{m_2^2(m_1^2-m_3^2)\log(m_2^2)}{(m_1^2-m_2^2)(m_1^2-m_3^2)(m_2^2-m_3^2)} \\
&\quad + \frac{m_3^2\log(m_3^2)}{(m_1^2-m_3^2)(m_2^2-m_3^2)}, \\
J^9_{123} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)}\, \dot{g}(T,\tau_1) \\
&= -\frac{m_1^2}{(m_1^2-m_2^2)(m_1^2-m_3^2)} + \frac{m_1^2(m_1^4-m_2^2m_3^2)\log(m_1^2)}{(m_1^2-m_2^2)^2(m_1^2-m_3^2)^2} \\
&\quad - \frac{m_1^2m_2^2\log(m_2^2)}{(m_1^2-m_2^2)^2(m_2^2-m_3^2)} + \frac{m_1^2m_3^2\log(m_3^2)}{(m_1^2-m_3^2)^2(m_2^2-m_3^2)}, \\
J^{10}_{123} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, e^{-Tm_1^2-\sum_{i=1}^2 \tau_i(m_{i+1}^2-m_i^2)}\, \dot{g}(T,\tau_2) = -J^9_{321}.
\end{aligned} \tag{A.5}$$

The integrals with four indices can be expressed as

$$\begin{aligned}
J^{11}_{1234} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3\, e^{-Tm_1^2-\sum_{i=1}^3 \tau_i(m_{i+1}^2-m_i^2)} \\
&= -\frac{\nabla_3 J^8_{123}}{m_3+m_4}, \\
J^{12}_{1234} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2-\sum_{i=1}^3 \tau_i(m_{i+1}^2-m_i^2)}\, \dot{g}(T,\tau_1) \\
&= -\frac{\nabla_3 J^9_{123}}{m_3+m_4}, \\
J^{13}_{1234} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2-\sum_{i=1}^3 \tau_i(m_{i+1}^2-m_i^2)}\, \dot{g}(T,\tau_2) \\
&= -\frac{\nabla_3 J^{10}_{123}}{m_3+m_4}, \\
J^{14}_{1234} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-Tm_1^2-\sum_{i=1}^3 \tau_i(m_{i+1}^2-m_i^2)}\, \dot{g}(T,\tau_3) \\
&= -J^{12}_{4123}.
\end{aligned} \tag{A.6}$$

Finally, the integrals with five indices read

$$\begin{aligned}
J^{15}_{12345} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4\, e^{-Tm_1^2-\sum_{i=1}^4 \tau_i(m_{i+1}^2-m_i^2)} \\
&= -\frac{\nabla_4 J^{11}_{1234}}{m_4+m_5}, \\
J^{16}_{12345} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4\, e^{-Tm_1^2-\sum_{i=1}^4 \tau_i(m_{i+1}^2-m_i^2)}\dot{g}(T,\tau_1) \\
&= -\frac{\nabla_4 J^{12}_{1234}}{m_4+m_5}, \\
J^{17}_{12345} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4\, e^{-Tm_1^2-\sum_{i=1}^4 \tau_i(m_{i+1}^2-m_i^2)}\dot{g}(T,\tau_2) \\
&= -\frac{\nabla_4 J^{13}_{1234}}{m_4+m_5}, \\
J^{18}_{12345} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4\, e^{-Tm_1^2-\sum_{i=1}^4 \tau_i(m_{i+1}^2-m_i^2)}\dot{g}(T,\tau_3) \\
&= -\frac{\nabla_4 J^{14}_{1234}}{m_4+m_5}, \\
J^{19}_{12345} &= \int_0^\infty \frac{dT}{T^2}\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4\, e^{-Tm_1^2-\sum_{i=1}^4 \tau_i(m_{i+1}^2-m_i^2)}\dot{g}(T,\tau_4) \\
&= -\frac{\nabla_3 J^{12}_{1234}}{m_3+m_4}. \qquad \text{(A.7)}
\end{aligned}$$

A.2 Results in four dimensions

In this section, we give the coefficient functions for the effective current and the effective density in four dimensions introduced in section 2.2.4.

The functions of the covariant current are given by

$$\begin{aligned}
A^2_{123} &= \frac{m_1m_2-m_1m_3-m_2m_3}{(m_1+m_2)(m_1-m_3)(m_2-m_3)} \\
&\quad+\frac{m_1^3(m_1(m_2-m_3)-2m_2m_3)\log[m_1^2/m_3^2]}{(m_1+m_2)(m_1-m_3)(m_1^2-m_2^2)(m_1^2-m_3^2)} \\
&\quad+\frac{m_2^3(m_2(m_3-m_1)+2m_1m_3)\log[m_2^2/m_3^2]}{(m_1+m_2)(m_2-m_3)(m_1^2-m_2^2)(m_2^2-m_3^2)}, \qquad \text{(A.8)} \\
A^3_{123} &= -A^1_{321}, \qquad \text{(A.9)}
\end{aligned}$$

and

$$A^4_{1234} = A^R_{1234} + A^L_{1234}\log[m_1^2] + A^L_{2341}\log[m_2^2] + A^L_{3412}\log[m_3^2] + A^L_{4123}\log[m_4^2], \qquad \text{(A.10)}$$

with

$$\begin{aligned}
A^R_{1234} &= \frac{m_1m_2m_3-m_1m_2m_4+m_1m_3m_4-m_2m_3m_4}{(m_1-m_2)(m_1+m_3)(m_1-m_4)(m_2-m_3)(m_2+m_4)(m_3-m_4)}, \\
A^L_{1234} &= \frac{m_1^3(-m_1^3-m_1m_2m_3+m_1m_2m_4-m_1m_3m_4+2m_2m_3m_4)}{(m_1-m_2)(m_1+m_3)(m_1-m_4)(m_1^2-m_2^2)(m_1^2-m_3^2)(m_1^2-m_4^2)}. \qquad \text{(A.11)}
\end{aligned}$$

The explicit functions occurring in the effective density are rather lengthy and hence we display them in terms of the integrals presented in the last section.

$$\begin{aligned}
B^3_{123} &= J^8_{123}(m_1+m_5) - J^9_{123}(m_1-m_3) - J^{10}_{123}(m_2-m_3), \\
B^4_{1234} &= J^{11}_{1234}(m_1+m_4) - J^{12}_{1234}(m_1-m_2) - J^{13}_{1234}(m_2+m_3) + J^{14}_{1234}(m_4+m_3), \\
B^5_{1234} &= J^{11}_{1234}(m_1+m_4) - J^{12}_{1234}(m_1+m_2) - J^{13}_{1234}(m_2-m_3) + J^{14}_{1234}(m_4+m_3), \\
B^6_{1234} &= B^4_{4321}, \\
B^7_{12345} &= J^{15}_{12345}(m_1+m_5) - J^{16}_{12345}(m_1+m_2) + J^{17}_{12345}(m_2+m_3) \\
&\quad -J^{18}_{12345}(m_3+m_4) + J^{19}_{12345}(m_4+m_5). \qquad \text{(A.12)}
\end{aligned}$$

A.3 Covariant Current in NLO 2 dimensions

In this section we summarise the contributions to the covariant current in next to leading order. For the first two levels, they are given by

$$\begin{aligned}
I^1_{12} &= \frac{I^2_{\underline{2}1}}{m_1-m_2}, \quad I^2_{12} = -4(m_1^2-m_2^2)J^3_{12} + 4(m_1+m_2)^2 J^4_{12}, \\
I^3_{123} &= \frac{m_1-m_2+2m_3}{m_2^2-m_3^2}\nabla_2(I^2_{\underline{2}1}) + 8\frac{m_1+m_3}{m_2^2-m_3^2}\nabla_2(m_1m_2J^4_{12}) \\
&\quad +16\frac{m_1m_3J^4_{13}}{m_2^2-m_3^2} - \frac{I^2_{\underline{3}1}}{(m_1-m_3)(m_2-m_3)}, \\
I^5_{123} &= 2\frac{\nabla_2(I^2_{\underline{2}1})}{m_1-m_2} + 3\frac{I^2_{\underline{3}1}}{(m_1-m_3)(m_1-m_2)} - I^3_{123} - \frac{I^7_{\underline{1}32}}{m_1-m_2}, \\
I^7_{123} &= (m_2+m_3)\left(\left(\frac{\nabla_1 I^2_{12}}{m_1+m_2}\right)_{\underline{3}12} - \frac{\nabla_1 I^2_{12}}{m_1+m_2}\right) + \frac{3m_2-m_3}{m_2+m_3}\nabla_2 I^2_{12} \\
&\quad +4(m_2+m_3)(m_1(m_1-m_2-m_3)+m_2m_3)\left(\left(\frac{\nabla_1 J^4_{12}}{m_1+m_2}\right)_{\underline{3}12} - \frac{\nabla_1 J^4_{12}}{m_1+m_2}\right) \\
&\quad -8(m_2+m_3)\left(\left(\frac{\nabla_1(m_1m_2J^4_{12})}{m_1+m_2}\right)_{\underline{3}12} - \frac{\nabla_1(m_1m_2J^4_{12})}{m_1+m_2}\right) \\
&\quad +4(m_2-m_3)\left(\frac{\nabla_2\left((m_1^2-4m_1m_2+m_2^2)J^4_{12}\right)}{m_2+m_3} - \nabla_2\left((m_1-m_2)J^4_{12}\right)\right), \qquad \text{(A.13)}
\end{aligned}$$

and the relations

$$I^4_{123} = -I^3_{321}, \quad I^6_{123} = -I^5_{321}, \quad I^8_{123} = I^7_{321}. \qquad \text{(A.14)}$$

The contributions to the last level read

$$
\begin{aligned}
I^9_{1234} &= -I^{11}_{4321},\\
I^{10}_{1234} &= -\frac{m_1+m_4}{m_3-m_4} I^{11}_{\underline{4}123} + \frac{1}{2}\left(\nabla_2 + \frac{m_1+m_4}{m_3-m_4}\left(\nabla_3\right)_{\underline{4}123}\right) I^5_{123}\\
&\quad + \left(\nabla_2+\nabla_3+\frac{m_1+m_4}{m_3-m_4}\left(\nabla_3\right)_{\underline{4}123}\right)\frac{m_1+m_3}{m_2-m_3}\left(I^3_{\underline{21}3} - (\nabla_2 I^1_{12})_{\underline{21}3}\right)\\
&\quad + \frac{1}{2}\left(\nabla_2+\nabla_3+\frac{m_1+m_4}{m_3-m_4}\left(-\nabla_1+\nabla_2+\nabla_3\right)_{\underline{4}123}\right)\left(I^3_{123} - \nabla_2 I^1_{12}\right)\\
&\quad + \frac{1}{2}\left(-\nabla_1+\nabla_2+\frac{m_1+m_4}{m_3-m_4}\left(\nabla_3\right)_{\underline{4}123}\right)\left(I^3_{321} - (\nabla_2 I^1_{12})_{321}\right)\\
&\quad - \frac{m_1+m_3}{2(m_2-m_3)(m_3-m_4)}\left(I^3_{\underline{21}3} - (\nabla_2 I^1_{12})_{\underline{21}3}\right) + \frac{2}{m_2+m_4}\left(I^3_{432} - (\nabla_2 I^1_{12})_{432}\right)\\
&\quad - \frac{1}{2(m_3-m_4)}\left(I^3_{421} - (\nabla_2 I^1_{12})_{421}\right),\\
I^{11}_{1234} &= \nabla_2 I^4 - \frac{I^4_{124}}{m_3-m_4} - \frac{4(m_1^2-m_3^2)J^3_{13}}{(m_3-m_4)(m_1^2-m_4^2)} - \frac{4(m_3+m_4)J^4_{34}}{m_1^2-m_4^2} - \frac{4J^5_{124}}{m_3-m_4}\\
&\quad - \frac{m_1+m_2}{m_3-m_4}\left(\nabla_2 - \frac{m_2+m_3}{m_3+m_4}\nabla_3\right)\left(\frac{\nabla_1((m_1^2-m_2^2)(I^2_{12}-8m_1m_2J^4_{12}))}{(m_1^2-m_2^2)(m_1+m_2)}\right)\\
&\quad + \frac{(m_3+m_4)\left(\nabla_1(I^2_{12}-8m_1m_2J^4_{12})\right)_{\underline{4}13}}{(m_1^2-m_4^2)(m_1-m_4)} + \frac{4(m_2+m_3)(m_1-m_4)}{m_1+m_2)(m_3-m_4)}\nabla_1 J^4\\
&\quad - \frac{4(m_1+m_2)}{m_3-m_4}\left(-\nabla_2+\frac{m_2+m_3}{m_3+m_4}\nabla_3\right)\frac{(m_1^2+m_3^2)J^4_{13} - (m_2^2-m_3^2)J^4_{\underline{23}}}{m_1^2-m_2^2}\\
&\quad - \frac{4(m_2+m_3)(m_1-m_4)(m_3+m_4)}{m_1+m_2}\nabla_1\frac{\left(\nabla_1 J^4\right)_{\underline{3}12}}{m_1-m_3} - \frac{4(m_1+m_3)}{m_3+m_4}\nabla_3 J^5\\
&\quad - \frac{4(m_1+m_2)(m_2+m_4)}{m_3^2-m_4^2}\nabla_3 J^6.
\end{aligned}
\tag{A.15}
$$

All functions I are finite in the coincidence limit and must fulfil the following constraints due to the behaviour of the terms in the effective action under cyclic permutation and complex conjugation:

$$
\begin{aligned}
\frac{(m_1+m_3)I^3_{\underline{3}12}}{m_2-m_3} + I^3_{321} + \frac{I^2_{13}}{(m_1-m_2)(m_2-m_3)} + \frac{I^2_{\underline{3}1}}{(m_1-m_2)(m_1-m_3)}&\\
-\frac{(m_1+m_3)I^2_{\underline{2}3}}{(m_1-m_2)(m_2^2-m_3^2)} - \frac{I^2_{\underline{3}2}}{(m_1-m_2)(m_2-m_3)} &= 0,\\
I^3_{123} + I^3_{321} - \frac{(m_1+m_3)I^3_{\underline{23}1}}{m_1-m_2} + I^5_{123} + I^5_{321}&\\
+\frac{I^2_{12}}{m_1^2-m_2^2} + \frac{I^2_{\underline{2}1}}{(m_1-m_2)(m_2-m_3)} - \frac{I^2_{\underline{3}1}}{(m_1-m_2)(m_2-m_3)} &= 0,\\
(m_1+m_3)I^3_{\underline{3}12} + (m_1+m_3)I^5_{\underline{3}12} - I^8_{123} + \frac{2I^2_{13}}{m_1-m_2}&\\
+\frac{(3m_1-m_2+2m_3)I^2_{\underline{2}3}}{(m_1-m_2)(m_2+m_3)} &= 0,
\end{aligned}
\tag{A.16}
$$

and

$$
I^3_{123} = I^3_{\underline{123}} = 0, \quad I^5_{123} = I^5_{\underline{123}} = 0, \quad I^7_{123} = -I^7_{\underline{123}} = 0. \tag{A.17}
$$

Appendix B

Calculation of Jacobians

Throughout this work, there appears several times Jacobians as products of change of variables. As is well known from the literature [44], we have claimed that they are constants and have generally absorbed them in the measure. To make this work more self-contained, we provide here a derivation of this claim.

In order to simplify the expressions, we will suppress the colour and space indices until it becomes necessary. The first Jacobian that we encountered was in Eq. (4.4), where the following expression appears

$$\mathrm{Det}\left(\frac{\delta \mathbf{E}}{\delta \mathbf{A}}\right) \tag{B.1}$$

with

$$\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} = \frac{\partial}{\partial t}\left(\mathbb{1} + \frac{1}{2}\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}}\right) \tag{B.2}$$

where $\mathbf{K}$ contains all the terms in the left hand side of Eq.(4.11) without time derivatives. The kernel of the operator $(\partial/\partial t)^{-1}$ is constrained by causality to be $\Theta(t_2 - t_1)$. We then have

$$\mathrm{Det}\left(\frac{\delta \mathbf{E}}{\delta \mathbf{A}}\right) = \text{ const.} \cdot \mathrm{Det}\left(\mathbb{1} + \frac{1}{2}\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}}\right) \tag{B.3}$$

with

$$\left[\frac{1}{2}\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}}\right]_{ij}^{ab}(t,\mathbf{x};t',\mathbf{x}') = \frac{1}{2}\int dt''\ \Theta(t-t'')\,\frac{\delta K_i^a[\mathbf{A}](t'',\mathbf{x})}{\delta A_j^b(t',\mathbf{x}')} \tag{B.4}$$

Since $\mathbf{K}$ contains no time derivatives, the functional derivative produces a delta function in the time variable i.e.

$$\frac{\delta K_i^a[\mathbf{A}](t'',\mathbf{x})}{\delta A_j^b(t',\mathbf{x}')} = \delta(t''-t')\,\frac{\delta_x K_i^a[\mathbf{A}](t',\mathbf{x})}{\delta_x A_j^b(t',\mathbf{x}')} \tag{B.5}$$

where we have introduced the symbol δ_x to denote a variation with respect to the $\mathbf{x}$ dependence only. Hence, we find

$$\left[\frac{1}{2}\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta K[\phi]}{\delta \phi}\right]_{\alpha\beta}(t,\mathbf{x};t',\mathbf{x}') = \frac{1}{2}\,\Theta(t-t')\,\frac{\delta_x K_i^a[\phi](t',\mathbf{x})}{\delta_x A_j^b(t',\mathbf{x}')} \tag{B.6}$$

Coming back to Eq. (B.3) and using $\mathrm{Tr}\ln(\dots) = \ln\mathrm{Det}(\dots)$ in addition to the series expansion of the logarithm, the determinant takes the form

$$\mathrm{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) = \text{ const.} \cdot \exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\frac{1}{2^n}\mathrm{Tr}\left[\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}}\right]^n\right\} \tag{B.7}$$

The trace in this expression can be evaluated with the help of Eq. (B.6). One obtains

$$\mathrm{Tr}\left[\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta\mathbf{K}[\mathbf{A}]}{\delta\mathbf{A}}\right]^n = \int dt_1\cdots dt_n\, d^{D-1}x_1\cdots d^{D-1}x_n\;\Theta(t_1-t_2)\,\frac{\delta_x K^{a_1}_{i_1}[\mathbf{A}](t_2,\mathbf{x}_1)}{\delta_x A^{a_2}_{i_2}(t_2,\mathbf{x}_2)}\;\Theta(t_2-t_3)\,\frac{\delta_x K^{a_2}_{i_2}[\mathbf{A}](t_3,\mathbf{x}_2)}{\delta_x A^{a_3}_{i_3}(t_3,\mathbf{x}_3)}\;\cdots\;\Theta(t_n-t_1)\,\frac{\delta_x K^{a_n}_{i_n}[\mathbf{A}](t_1,\mathbf{x}_n)}{\delta_x A^{a_1}_{i_1}(t_1,\mathbf{x}_1)} \tag{B.8}$$

and thus

$$\mathrm{Tr}\left[\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta K[\phi]}{\delta\phi}\right]^n = \int dt_1\cdots dt_n\;\Theta(t_1-t_2)\,\Theta(t_2-t_3)\,\cdots\,\Theta(t_n-t_1)\,f_n(t_1,t_2,\ldots,t_n) \tag{B.9}$$

if we set

$$f_n(t_1,t_2,\ldots,t_n) = \int d^{D-1}x_1\cdots d^{D-1}x_n\;\frac{\delta_x K^{a_1}_{i_1}[\mathbf{A}](t_2,\mathbf{x}_1)}{\delta_x A^{a_2}_{i_2}(t_2,\mathbf{x}_2)}\,\frac{\delta_x K^{a_2}_{i_2}[\mathbf{A}](t_3,\mathbf{x}_2)}{\delta_x A^{a_3}_{i_3}(t_3,\mathbf{x}_3)}\;\cdots\;\frac{\delta_x K^{a_n}_{i_n}[\mathbf{A}](t_1,\mathbf{x}_n)}{\delta_x A^{a_1}_{i_1}(t_1,\mathbf{x}_1)}$$

for abbreviation. Unless $n = 1$, however, the expression (B.9) vanishes for any function f_n. Therefore, only the first term of the sum in Eq. (B.7) survives and we finally arrive at

$$\mathrm{Det}\left(\frac{\delta\mathbf{E}[\mathbf{A}]}{\delta\mathbf{A}}\right) = \text{const.}\cdot\exp\left\{\frac{1}{2}\,\Theta(0)\int dt\, d^{D-1}x\;\left.\frac{\delta_x K^a_i[\mathbf{A}](t,\mathbf{x})}{\delta_x A^a_b(t,\mathbf{x}')}\right|_{\mathbf{x}'=\mathbf{x}}\right\} \tag{B.10}$$

Our next task is to calculate the functional derivative of $K^a_i[\mathbf{A}]$. To this end, it is easiest to write it down in components which clarifies the structure

$$\begin{aligned}\frac{1}{2}\,K^a_i[\mathbf{A}] \;&=\; gf^{abc}\left[\left(1-\frac{\sigma}{\kappa}\right)A^b_i\partial_j A^c_j + 2A^c_j\partial_j A^b_i + A^b_j\partial_i A^c_j\right]\\ &+\;\left[\left(1-\frac{\sigma}{\kappa}\right)\partial_i\partial_j - \delta_{ij}\Delta\right]A^a_j \;+\; g^2 f^{abc}f^{bde}A^c_j A^d_j A^e_i\end{aligned} \tag{B.11}$$

Obviously, the first term, i.e. the term quadratic in the gauge field, does not contribute to the functional derivative with respect to A^a_i because it always produces a δ^{ab} or δ^{ac} that is contracted with the structure constants f^{abc} in front of the square bracket. The linear term, on the other hand, only contributes a constant that can be absorbed into the constant in Eq. (B.10). Thus, we only have to take care of the third order term which leads to

$$\mathrm{Det}\left(\frac{\delta\mathbf{E}[\mathbf{A}]}{\delta\mathbf{A}}\right) = \text{const.}'\cdot\exp\left\{C_A\,(D-2)\,\Theta(0)\,\delta^{D-1}(0)\,\frac{g^2}{\sigma}\int dt\, d^{D-1}x\;\mathbf{A}^a(t,\mathbf{x})\cdot\mathbf{A}^a(t,\mathbf{x})\right\} \tag{B.12}$$

where $f^{acd}f^{bcd} = C_A\,\delta^{ab}$ as usual. However, in dimensional regularisation $\delta^{D-1}(0)$ gives zero as a consequence of the general rules of D–dimensional integration, and the determinant is simply a constant.

We came acrosss another determinant in Eq. (4.33)

$$\mathrm{Det}\left(\frac{\delta^\omega\boldsymbol{\zeta}}{\delta\boldsymbol{\zeta}}\right) \tag{B.13}$$

we find at first

$$\frac{\delta^\omega\zeta^{ai}(t\,,\mathbf{x}\,)}{\delta\zeta^{bj}(t',\mathbf{x}')} = \delta^{ab}\delta_{ij}\,\delta(t-t')\,\delta^{D-1}(\mathbf{x}-\mathbf{x}') \;+\; gf^{acd}\,\frac{\delta}{\delta\zeta^{bj}(t',\mathbf{x}')}\left[\omega^d[\boldsymbol{\zeta}](t,\mathbf{x})\,\zeta^{ci}(t,\mathbf{x})\right] \tag{B.14}$$

and thus because ω is infinitesimal

$$\mathrm{Det}\left(\frac{\delta^\omega\boldsymbol{\zeta}}{\delta\boldsymbol{\zeta}}\right) = 1+\int dt\, d^{D-1}x\; gf^{acd}\left[\frac{\delta}{\delta\zeta^{ai}(t',\mathbf{x}')}\left[\omega^d[\boldsymbol{\zeta}](t,\mathbf{x})\,\zeta^{ci}(t,\mathbf{x})\right]\right]_{\substack{t'=t\\ \mathbf{x}'=\mathbf{x}}} \tag{B.15}$$

The functional derivative acting on ζ^{ci} produces a δ^{ac} and therefore does not contribute because the Kronecker delta is contracted with the structure constants. To determine the remaining functional derivative of $\omega^d[\boldsymbol{\zeta}]$, let us formally integrate Eq. (4.26)

$$\omega^a(t,\mathbf{x}) = \omega^a(-\infty,\mathbf{x}) - \int_{-\infty}^{t} dt'' \; [H[\mathbf{A}]\omega]^a(t'',\mathbf{x}) - \int_{-\infty}^{t} dt'' \, \delta v^a[\mathbf{A}](t'',\mathbf{x}) \tag{B.16}$$

Since $H[\mathbf{A}]$ and $\delta v^a[\mathbf{A}]$ are local functionals in time, this equation for ω has a causal character, i.e. $\omega(t,\mathbf{x})$ does only depend on the values of the gauge field $\mathbf{A}(t'',\mathbf{x})$ at times $t'' < t$. On the other hand, Eq. (4.25) leads to

$$\sigma\mathbf{A}^a(t,\mathbf{x}) = \sigma\mathbf{A}^a(-\infty,\mathbf{x}) - \int_{-\infty}^{t} dt'' \; \left[\,\mathbf{D}^{ab}\times\mathbf{B}^b + \sigma\mathbf{D}^{ab}v^b[\mathbf{A}]\,\right](t'',\mathbf{x}) + \int_{-\infty}^{t} dt'' \, \boldsymbol{\zeta}^a(t'',\mathbf{x}) \tag{B.17}$$

and $\mathbf{A}(t,\mathbf{x})$ itself only depends on the stochastic force $\boldsymbol{\zeta}(t'',\mathbf{x})$ for $t''< t$. Hence, neither $\mathbf{A}(t,\mathbf{x})$ nor $\omega(t,\mathbf{x})$ have a dependence on $\boldsymbol{\zeta}(t'',\mathbf{x})$ unless $t''< t$ and in taking the functional derivative of Eq. (B.16), we can restrict the integration range accordingly

$$\frac{\delta\omega^a[\boldsymbol{\zeta}](t,\mathbf{x})}{\delta\zeta^{bi}(t',\mathbf{x}')} = -\int_{t'}^{t} dt'' \, \frac{\delta\,[H[\mathbf{A}]\omega]^a(t'',\mathbf{x})}{\delta\zeta^{bi}(t',\mathbf{x}')} - \int_{t'}^{t} dt'' \, \frac{\delta\,\delta v^a[\mathbf{A}](t'',\mathbf{x})}{\delta\zeta^{bi}(t',\mathbf{x}')} \tag{B.18}$$

Evaluating this relation for $t = t'$ as in Eq. (B.15) leads to

$$\left.\frac{\delta\omega^a[\boldsymbol{\zeta}](t,\mathbf{x})}{\delta\zeta^{bi}(t',\mathbf{x}')}\right|_{t=t'} = 0 \tag{B.19}$$

The only way to escape this conclusion would be an integrand that is singular in time. However, if $\delta\omega/\delta\zeta$ appearing under the integral in Eq. (B.18) was singular, the integrated expression would be finite which again is $\delta\omega/\delta\zeta$. Therefore, $\delta\omega/\delta\zeta$ can not be singular. $\delta\mathbf{A}/\delta\zeta$ on the other hand can not be singular neither because of the same argument applied to the functional derivative of Eq. (B.17) with respect to ζ. Thus, we conclude

$$\operatorname{Det}\left(\frac{\delta^{\omega}\boldsymbol{\zeta}}{\delta\boldsymbol{\zeta}}\right) = 1 \tag{B.20}$$

which completes the proof.

During the introduction of gauge ghosts to the path integral, Eq. (4.41), there appears in our work another Jacobian. We can see that it has the same form as the one we have already calculated, but with

$$\frac{1}{2}\,K^a[\omega,\mathbf{A}](t,\mathbf{x}) = -\frac{1}{\kappa}\,(\mathbf{D}^{ab}\cdot\nabla\omega^b)(t,\mathbf{x}) \tag{B.21}$$

Hence, we can rely on our general result for the determinant, Eq. (B.10),

$$\operatorname{Det}\left(\frac{\delta\gamma[\omega,\mathbf{A}]}{\delta\omega}\right) = \text{ const.}\cdot\exp\left\{-\frac{1}{\kappa}\,\Theta(0)\int dt\, d^{D-1}x \, \left.\frac{\delta_x(\mathbf{D}^{ab}\cdot\nabla\omega^b)(t,\mathbf{x})}{\delta_x\omega^a(t,\mathbf{x}')}\right|_{\mathbf{x}'=\mathbf{x}}\right\} \tag{B.22}$$

The functional derivative with respect to spacial variations is given by

$$\frac{\delta_x(\mathbf{D}^{ab}\cdot\nabla\omega^b)(t,\mathbf{x})}{\delta_x\omega^d(t,\mathbf{x}')} = (\delta^{ab}\nabla - gf^{abc}\mathbf{A}^c)\cdot\nabla\delta^D(\mathbf{x}-\mathbf{x}')\,\delta^{bd} \tag{B.23}$$

and thus, evaluated for $d = a$, gives a constant because the $\mathbf{A}$ dependent contribution is set to zero due to the antisymmetry of the structure constants. Note that, this time, we did not have to rely on dimensional regularisation to proof the constancy of the determinant as we had to in the case of $\mathrm{Det}(\delta\mathbf{E}[\mathbf{A}]/\delta\mathbf{A})$.

When we performed the BRST transformation in our derivation of the Ward identities, Eq. (4.48), one more type of determinant appeared. In general, if x_a are Grassmann even and ϑ_i Grassmann odd quantities, a mixed change of variables of the form

$$\begin{aligned} x_a &= x'_a + \varepsilon\, f_a(x', \vartheta') \\ \vartheta_i &= \vartheta'_i + \varepsilon\, \phi_i(x', \vartheta') \end{aligned} \tag{B.24}$$

with ε being a Grassmann odd parameter leads to a Jacobian

$$J = 1 + \varepsilon\, \mathrm{str}(M) \tag{B.25}$$

In this expression, the matrix M under the super trace is given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{\partial f_a}{\partial x'_b} & -\frac{\partial f_a}{\partial \vartheta'_i} \\ \frac{\partial \phi_i}{\partial x'_a} & -\frac{\partial \phi_i}{\partial \vartheta'_j} \end{pmatrix} \tag{B.26}$$

and hence

$$\mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D) = \frac{\partial f_a}{\partial x'_a} + \frac{\partial \phi_i}{\partial \vartheta'_i} \tag{B.27}$$

(See e.g. [44], Section 1.8.2. Note, however, that in our case ε is Grassmann odd which leads to the additional minus signs in the matrix M when ε is commuted with the derivative $\partial/\partial\vartheta$).

In our case, we have two sets of commuting variables, $\mathbf{A}^{ai}(x)$ and $\boldsymbol{\lambda}^{ai}(x)$, and two sets of anti-commuting ones, $\omega^a(x)$ and $\bar{\omega}^a(x)$. Therefore, the Jacobian is given by

$$J = 1 + \varepsilon \int dx \left[\frac{\delta\, sA'^{ai}(x)}{\delta A'^{ai}(x)} + \frac{\delta\, s\lambda'^{ai}(x)}{\delta \lambda'^{ai}(x)} + \frac{\delta\, s\omega'^{a}(x)}{\delta \omega'^{a}(x)} + \frac{\delta\, s\bar{\omega}'^{a}(x)}{\delta \bar{\omega}'^{a}(x)} \right] \tag{B.28}$$

However, any of these functional derivatives vanishes as a short glance at the BRST transformed fields in Eq. (4.52) makes obvious: The derivative always produces a Kronecker delta that is to be contracted with the structure constants. Consequently, the Jacobian of the change of variables (4.48) is unity.

Appendix C

Feynman Rules

The action, as given by Eq. (4.45), is

$$S[\mathbf{A},\boldsymbol{\lambda},\omega,\bar\omega] = S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] + S^{(\mathrm{GG})}[\mathbf{A},\omega,\bar\omega], \tag{C.1}$$

with

$$S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] = \int dx \left[\sigma T\, \boldsymbol{\lambda}^a\cdot\boldsymbol{\lambda}^a - i\boldsymbol{\lambda}^a\cdot\Big(\mathbf{D}^{ab}\times\mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a - \tfrac{1}{\kappa}\,\mathbf{D}^{ab}\,\nabla\cdot\mathbf{A}^b)\Big)\right] \tag{C.2}$$

$$S^{(\mathrm{GG})}[\mathbf{A},\omega,\bar\omega] = \int dx \left[-\bar\omega^a\dot\omega^a + \frac{1}{\kappa}\,\bar\omega^a\mathbf{D}^{ab}\cdot\nabla\omega^b\right] \tag{C.3}$$

C.1 The Propagators

The free, quadratic part of the dynamical action $S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]$ can be cast into the following symmetric form reflecting the mixing that occur between the gauge field $\mathbf{A}$ and the auxiliary field $\boldsymbol{\lambda}$

$$S_0^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] = \int dxdy\,\frac{1}{2}\,(\lambda^{ai}(x), A^{ai}(x))\,(\hat\Delta^{-1})^{\,ab}_{\,ij}\,(x,y)\begin{pmatrix}\lambda^{bj}(y)\\ A^{bj}(y)\end{pmatrix} \tag{C.4}$$

with the matrix

$$(\hat\Delta^{-1})^{\,ab}_{\,ij}\,(x,y) = \begin{pmatrix}(\Delta^{-1})^{(\lambda\lambda)ab}_{\;ij}(x,y) & (\Delta^{-1})^{(\lambda A)ab}_{\;ij}(x,y)\\ (\Delta^{-1})^{(A\lambda)ab}_{\;ij}(x,y) & (\Delta^{-1})^{(AA)ab}_{\;ij}(x,y)\end{pmatrix} \tag{C.5}$$

and

$$(\Delta^{-1})^{(\lambda\lambda)ab}_{\;ij}(x,y) = 2\sigma T\,\delta^{ab}\delta_{ij}\,\delta(x-y) \tag{C.6}$$

$$(\Delta^{-1})^{(\lambda A)ab}_{\;ij}(x,y) = -i\,\delta^{ab}\left[(+\sigma\partial_t-\Delta)\,\delta_{ij} + (1-\tfrac{\sigma}{\kappa})\,\partial_i\partial_j\right]\delta(x-y) \tag{C.7}$$

$$(\Delta^{-1})^{(A\lambda)ab}_{\;ij}(x,y) = -i\,\delta^{ab}\left[(-\sigma\partial_t-\Delta)\,\delta_{ij} + (1-\tfrac{\sigma}{\kappa})\,\partial_i\partial_j\right]\delta(x-y) \tag{C.8}$$

$$(\Delta^{-1})^{(AA)ab}_{\;ij}(x,y) = 0 \tag{C.9}$$

We denote by non-bold symbols combinations of time and space variables, e.g. $\delta(x-y) = \delta(t_x - t_y)\,\delta^{D-1}(\mathbf{x}-\mathbf{y})$. The matrix $\hat\Delta^{-1}$ is symmetric in the following sense

$$(\Delta^{-1})^{(FG)ab}_{\;ij}(x,y) = (\Delta^{-1})^{(GF)ba}_{\;ji}(y,x) \tag{C.10}$$

Hence, the matrix propagator

$$\hat{\Delta}^{\,ab}_{\,ij}\,(x,y) = \begin{pmatrix} \Delta^{(\lambda\lambda)ab}_{\;\;ij}(x,y) & \Delta^{(\lambda A)ab}_{\;\;ij}(x,y) \\ \Delta^{(A\lambda)ab}_{\;\;ij}(x,y) & \Delta^{(AA)abij}(x,y) \end{pmatrix} = \begin{pmatrix} \left\langle \lambda^{ai}(x)\,\lambda^{bj}(y)\right\rangle_0 & \left\langle \lambda^{ai}(x)\,A^{bj}(y)\right\rangle_0 \\ \left\langle A^{ai}(x)\,\lambda^{bj}(y)\right\rangle_0 & \left\langle A^{ai}(x)\,A^{bj}(y)\right\rangle_0 \end{pmatrix}$$

is given by its inverse (C.11)

$$\int d^D y\; \Delta^{(FG)ab}_{\;\;ij}(x,y)\,(\Delta^{-1})^{(GH)bc}_{\;\;jk}(y,z) = \delta^{ac}\,\delta_{ik}\,\delta^{FH}\,\delta^D(x-z) \tag{C.12}$$

or equivalently

$$\Delta^{(FG)ab}_{\;\;ij}(k)\,(\Delta^{-1})^{(GH)bc}_{\;\;jk}(k) = \delta^{ac}\,\delta_{ik}\,\delta^{FH} \tag{C.13}$$

for the momentum space functions

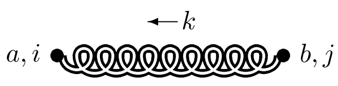
$$\Delta^{(FG)ab}_{\;\;ij}(x,y) = \int \frac{d^Dk}{(2\pi)^D}\; e^{-ik(x-y)}\,\Delta^{(FG)ab}_{\;\;ij}(k) \tag{C.14}$$

$$(\Delta^{-1})^{(FG)ab}_{\;\;ij}(x,y) = \int \frac{d^Dk}{(2\pi)^D}\; e^{-ik(x-y)}\,(\Delta^{-1})^{(FG)ab}_{\;\;ij}(k) \tag{C.15}$$

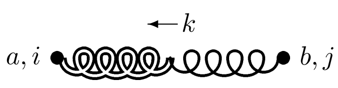
Note again that though we are most of the time dealing with three-vectors, in the Fourier transform we use four-vector notation, i.e. $e^{-ik(x-y)} = e^{-ik_0(x_0-y_0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ leading to

$$(\hat{\Delta}^{-1})^{\,ab}_{\,ij}\,(k) = \begin{pmatrix} 2\sigma T\,\delta^{ab}\delta_{ij} & -i\,\delta^{ab}\left[(-i\sigma k_0 + \mathbf{k}^2)\,\delta_{ij} - (1-\frac{\sigma}{\kappa})\,k_ik_j\right] \\ -i\,\delta^{ab}\left[(+i\sigma k_0 + \mathbf{k}^2)\,\delta_{ij} - (1-\frac{\sigma}{\kappa})\,k_ik_j\right] & 0 \end{pmatrix}. \tag{C.16}$$

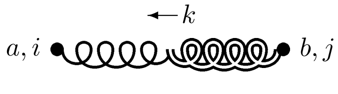
In momentum space, the gauge/auxiliary field propagators are given by:



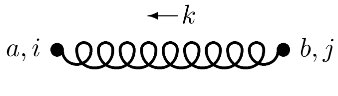
$$\Delta^{(\lambda\lambda)ab}_{\;\;ij}(k) = 0$$



$$\Delta^{(\lambda A)ab}_{\;\;ij}(k) = \frac{i\delta^{ab}}{+i\sigma k_0 + |\mathbf{k}|^2}\left[\delta_{ij} + \left(1-\tfrac{\sigma}{\kappa}\right)\frac{k_ik_j}{+i\sigma k_0 + \frac{\sigma}{\kappa}|\mathbf{k}|^2}\right]$$



$$\Delta^{(A\lambda)ab}_{\;\;ij}(k) = \frac{i\delta^{ab}}{-i\sigma k_0 + |\mathbf{k}|^2}\left[\delta_{ij} + \left(1-\tfrac{\sigma}{\kappa}\right)\frac{k_ik_j}{-i\sigma k_0 + \frac{\sigma}{\kappa}|\mathbf{k}|^2}\right]$$



$$\Delta^{(AA)ab}_{\;\;ij}(k) = \frac{2\sigma T\delta^{ab}}{\sigma^2k_0^2 + |\mathbf{k}|^4}\left[\delta_{ij} + \left(1-\tfrac{\sigma^2}{\kappa^2}\right)\frac{k_ik_j|\mathbf{k}|^2}{\sigma^2k_0^2 + \frac{\sigma^2}{\kappa^2}|\mathbf{k}|^4}\right]$$

For the gauge ghosts, we have the corresponding contribution to the action, Eq. (C.3), comprises the free part

$$S_0^{(\mathrm{GG})}[\omega,\bar{\omega}] = \int dx\, \bar{\omega}^a \big(-\partial_t + \tfrac{1}{\kappa}\,\Delta\big)\, \omega^a \tag{C.17}$$

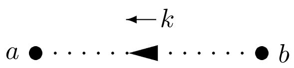
and therefore

$$(\Delta^{-1})^{(\omega)\,ab}(x,y) = \delta^{ab}\big(-\partial_t + \tfrac{1}{\kappa}\,\Delta\big)\,\delta(x-y) \tag{C.18}$$

or in momentum space

$$(\Delta^{-1})^{(\omega)\,ab}(k) = \delta^{ab}\big(ik_0 - \tfrac{1}{\kappa}|\mathbf{k}|^2\big) \tag{C.19}$$

Hence the gauge ghost propagator is given by



$$\Delta^{(\omega)\,ab}(k) = \frac{\kappa\,\delta^{ab}}{i\kappa k_0 - |\mathbf{k}|^2}$$

C.2 The Vertices

For the interacting part of the dynamical action (C.2) we have

$$\begin{aligned} S_{\mathrm{int}}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] \;=\; & \int dx\, \Big\{ -igf^{abc}\lambda^{ai}\Big[\big(1-\tfrac{\sigma}{\kappa}\big)A^{bi}\partial_j A^{cj} + 2A^{cj}\partial_j A^{bi} + A^{bj}\partial_i A^{cj}\Big] \\ & \qquad -ig^2 f^{abc} f^{bde} \lambda^{ai} A^{cj} A^{dj} A^{ei}\Big\} \end{aligned} \tag{C.20}$$

Thus, the theory provides a 3–point vertex containing one auxiliary and two gauge fields and a 4–point vertex of three gauge fields and one auxiliary field. To simplify explicit calculations, it is useful to symmetrise the vertices with respect to the two and three gauge fields in either case. Splitting $S_{\mathrm{int}}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]$ into the contributions corresponding to the 3– and 4–point vertex

$$S_{\mathrm{int}}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] = S_{\mathrm{int},3}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] + S_{\mathrm{int},4}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] \tag{C.21}$$

one obtains

$$\begin{aligned} S_{\mathrm{int},3}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] \;=\; & \int dx\, \frac{1}{2!}\,(-ig) f^{abc}\lambda^{ai}\Big\{\big(1-\tfrac{\sigma}{\kappa}\big)\big[\delta^{ij}A^{bj}\partial_k A^{ck} - \delta^{ik}A^{ck}\partial_j A^{bj}\big] \\ & \qquad + 2\,\big[\delta^{ij}A^{ck}\partial_k A^{bj} - \delta^{ik}A^{bj}\partial_j A^{ck}\big] \\ & \qquad + \;\; \big[\delta^{jk}A^{bj}\partial_i A^{ck} - \delta^{kj}A^{ck}\partial_i A^{bj}\big]\Big\} \end{aligned} \tag{C.22}$$

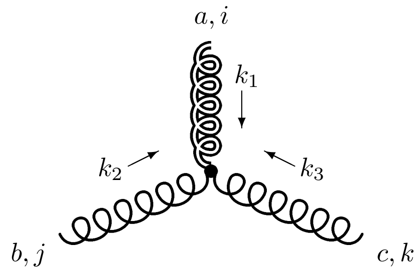
$$S_{\mathrm{int},4}^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}] \;=\; \int dx\, \frac{1}{3!}\,(-ig^2)\, V_{ijkl}^{abcd}\, \lambda^{ai} A^{bj} A^{ck} A^{dl} \tag{C.23}$$

where

$$\begin{aligned} V_{ijkl}^{abcd} \;=\; & f^{ace} f^{bde}(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{kj}) \\ & + f^{abe} f^{cde}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \\ & + f^{ade} f^{bce}(\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl}) \end{aligned} \tag{C.24}$$

Observing that there is an additional minus sign because we have $-S^{(\mathrm{D})}[\mathbf{A},\boldsymbol{\lambda}]$ in the exponent of the generating functional and noting our conventions of the Fourier transform (C.14) of the propagators, we find for the 3–point vertex in momentum space that is symmetrised with respect to the two $\mathbf{A}$ fields

$\boldsymbol{\lambda}\mathbf{A}^2$ vertex



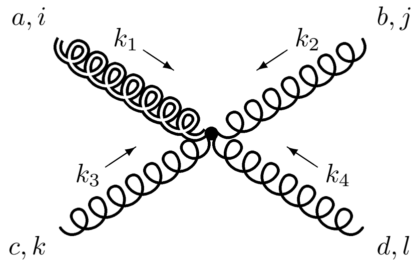
$$-g\,V^{abc}_{ijk}(\mathbf{k}_2,\mathbf{k}_3) \;=\; -g f^{abc}\Big\{\big(1-\tfrac{\sigma}{\kappa}\big)\big(\delta^{ij}k_3^k-\delta^{ik}k_2^j\big) + 2\big(\delta^{ij}k_2^k-\delta^{ik}k_3^j\big) + \delta^{jk}\big(k_3^i-k_2^i\big)\Big\}$$

Momentum conservation is thereby to be understood. By construction, the object $V^{abc}_{ijk}(\mathbf{k}_2,\mathbf{k}_3)$ is symmetric in the last two pairs of indices (and corresponding momenta), i.e.

$$V^{abc}_{ijk}(\mathbf{k}_2,\mathbf{k}_3) = V^{acb}_{ikj}(\mathbf{k}_3,\mathbf{k}_2) \tag{C.25}$$

Analogously, the symmetrised 4–point vertex is found to be

$\boldsymbol{\lambda}\mathbf{A}^3$ vertex



$$ig^2\,V^{abcd}_{ijkl} \;=\; ig^2\Big\{\; f^{ace}f^{bde}(\delta^{ij}\delta^{kl}-\delta^{il}\delta^{kj}) + f^{abe}f^{cde}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk}) + f^{ade}f^{bce}(\delta^{ij}\delta^{kl}-\delta^{ik}\delta^{jl})\Big\}$$

where V^{abcd}_{ijkl} was already introduced in Eq. (C.24) and is symmetric in the last three pairs of indices

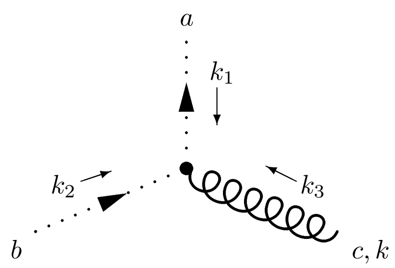
$$V^{abcd}_{ijkl} = V^{abdc}_{ijlk} = V^{acbd}_{ikjl} = V^{acdb}_{iklj} = V^{adbc}_{iljk} = V^{adcb}_{ilkj} \tag{C.26}$$

For the ghost sector, the corresponding interaction term extracted from Eq. (C.3) is given by

$$S^{(\mathrm{GG})}_{\mathrm{int}}[\mathbf{A},\omega,\bar\omega] = \int\! dx\,\tfrac{(-g)}{\kappa}f^{abc}\bar\omega^a(\mathbf{A}^c\cdot\nabla)\,\omega^b = \int\! dx\,\tfrac{(-g)}{\kappa}f^{abc}\bar\omega^a A^{ck}\partial_k\omega^b \tag{C.27}$$

and leads to the momentum space vertex

$\boldsymbol{\omega\bar\omega}\mathbf{A}$ vertex



$$\frac{ig}{\kappa}\,f^{abc}k_2^k$$

C.3 The $\Gamma_\lambda^{F,GH}$ Functions

The $\Gamma_\lambda^{F,GH}(\mathbf{k},\mathbf{k}')$ functions consist of the contraction of the corresponding tensor structures

$$\Gamma_\lambda^{F,GH}(\mathbf{k},\mathbf{k}') = V_{ikl}(\mathbf{k}',\mathbf{k}-\mathbf{k}')V_{l'k'j}(-\mathbf{k}',\mathbf{k})P_{ij}^F(\mathbf{k})P_{ll'}^G(\mathbf{k}-\mathbf{k}')P_{kk'}^H(\mathbf{k}'), \tag{C.28}$$

where V_{ikl} represent solely the spatial momentum tensor structure of the tree level (λAA)-vertex. Their specific values in $\kappa = \sigma$ gauge are

$$\begin{aligned}
\Gamma_\lambda^{T,TT}(\mathbf{k},\mathbf{k}') =& \frac{4\left(-(\mathbf{k}\cdot\mathbf{k}')^2+|\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
&\times\big((-2+D)|\mathbf{k}|^4+(\mathbf{k}\cdot\mathbf{k}')^4-2(-2+D)(\mathbf{k}\cdot\mathbf{k}')^2|\mathbf{k}'|^2 \\
&+(-2+D)|\mathbf{k}'|^4+|\mathbf{k}|^2(-2(-2+D)(\mathbf{k}\cdot\mathbf{k}')^2+(-7+3D)|\mathbf{k}'|^2)\big) \\
\Gamma_\lambda^{T,TL}(\mathbf{k},\mathbf{k}') =& \frac{4(\mathbf{k}\cdot\mathbf{k}')^2((\mathbf{k}\cdot\mathbf{k}')^2-|\mathbf{k}'|^2)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
&\times\big((-2+D)|\mathbf{k}|^4+(\mathbf{k}\cdot\mathbf{k}')^4+|\mathbf{k}|^2(-2(-2+D)(\mathbf{k}\cdot\mathbf{k}')^2+(-3+D)|\mathbf{k}'|^2)\big) \\
\Gamma_\lambda^{T,LT}(\mathbf{k},\mathbf{k}') =& \frac{2(|\mathbf{k}|^2-|\mathbf{k}'|^2)((\mathbf{k}\cdot\mathbf{k}')^2-|\mathbf{k}'|^2)\left((\mathbf{k}\cdot\mathbf{k}')^4+(-3+D)|\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_\lambda^{T,LL}(\mathbf{k},\mathbf{k}') =& \frac{2((\mathbf{k}\cdot\mathbf{k}')^2-|\mathbf{k}'|^2)\left(|\mathbf{k}|^4|\mathbf{k}'|^2-(\mathbf{k}\cdot\mathbf{k}')^4|\mathbf{k}'|^2+|\mathbf{k}|^2\left(-(\mathbf{k}\cdot\mathbf{k}')^4+|\mathbf{k}'|^4\right)\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_\lambda^{L,TT}(\mathbf{k},\mathbf{k}') =& -\frac{2(|\mathbf{k}|^2-2\mathbf{k}\cdot\mathbf{k}')\mathbf{k}\cdot\mathbf{k}'}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
&\times\big((\mathbf{k}\cdot\mathbf{k}')^2-2(-2+D)\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^2+|\mathbf{k}'|^2((-3+D)|\mathbf{k}|^2+(-2+D)|\mathbf{k}'|^2)\big) \\
\Gamma_\lambda^{L,TL}(\mathbf{k},\mathbf{k}') =& \frac{2\mathbf{k}\cdot\mathbf{k}'\left(-|\mathbf{k}|^4|\mathbf{k}'|^2+2(\mathbf{k}\cdot\mathbf{k}')^2(-\mathbf{k}\cdot\mathbf{k}'+|\mathbf{k}'|^2)+|\mathbf{k}|^2\left((\mathbf{k}\cdot\mathbf{k}')^2+2\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^2-2|\mathbf{k}'|^4\right)\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_\lambda^{L,LT}(\mathbf{k},\mathbf{k}') =& \frac{1}{|\mathbf{k}|^2|\mathbf{k}'|^2}\big(2(\mathbf{k}\cdot\mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}'-|\mathbf{k}'|^2)|\mathbf{k}'|^2-|\mathbf{k}|^4\left((\mathbf{k}\cdot\mathbf{k}')^2+2\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^2-3|\mathbf{k}'|^4\right) \\
&+|\mathbf{k}|^6|\mathbf{k}'|^2+|\mathbf{k}|^2\left(2(\mathbf{k}\cdot\mathbf{k}')^3-3(\mathbf{k}\cdot\mathbf{k}')^2|\mathbf{k}'|^2-2\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^4+2|\mathbf{k}'|^6\right)\big) \\
\Gamma_\lambda^{L,LL}(\mathbf{k},\mathbf{k}') =& \frac{(|\mathbf{k}|^2-2\mathbf{k}\cdot\mathbf{k}')\mathbf{k}\cdot\mathbf{k}'(|\mathbf{k}|^2-|\mathbf{k}'|^2)(\mathbf{k}\cdot\mathbf{k}'-|\mathbf{k}'|^2)}{|\mathbf{k}|^2|\mathbf{k}'|^2}
\end{aligned} \tag{C.29}$$

C.4 The $\Gamma_A^{F,GH}$ Functions

The $\Gamma_A^{F,GH}(\mathbf{k},\mathbf{k}')$ functions are defined similar to the mixed case

$$\Gamma_A^{F,GH}(\mathbf{k},\mathbf{k}') = V_{ikl}(\mathbf{k}',\mathbf{k}-\mathbf{k}')V_{jl'k'}(\mathbf{k}'-\mathbf{k},-\mathbf{k}')P_{ij}^F(\mathbf{k})P_{ll'}^G(\mathbf{k}-\mathbf{k}')P_{kk'}^H(\mathbf{k}'). \tag{C.30}$$

Their specific values in $\kappa = \sigma$ gauge are

$$\begin{aligned}
\Gamma_A^{T,TT}(\mathbf{k},\mathbf{k}') =& \frac{4\left(-(\mathbf{k}\cdot\mathbf{k}')^2+|\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
&\times\big((-2+D)|\mathbf{k}|^4+(\mathbf{k}\cdot\mathbf{k}')^2-2(-2+D)\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^2+(-2+D)|\mathbf{k}'|^4 \\
&+|\mathbf{k}|^2(-2(-2+D)\mathbf{k}\cdot\mathbf{k}'+(-7+3D)|\mathbf{k}'|^2)\big) \\
\Gamma_A^{T,TL}(\mathbf{k},\mathbf{k}') =& \frac{4(\mathbf{k}\cdot\mathbf{k}'-|\mathbf{k}'|^2)^2\left((-2+D)|\mathbf{k}|^4+(\mathbf{k}\cdot\mathbf{k}')^2+|\mathbf{k}|^2(-2(-2+D)\mathbf{k}\cdot\mathbf{k}'+(-3+D)|\mathbf{k}'|^2)\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_A^{T,LT}(\mathbf{k},\mathbf{k}') =& \frac{4(\mathbf{k}\cdot\mathbf{k}'-|\mathbf{k}'|^2)^2\left((\mathbf{k}\cdot\mathbf{k}')^2+(-3+D)|\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2}
\end{aligned}$$

$$
\begin{aligned}
\Gamma_A^{T,LL}(\mathbf{k},\mathbf{k}') =& \frac{4(\mathbf{k}\cdot\mathbf{k}' - |\mathbf{k}'|^2)^2\left(-(\mathbf{k}\cdot\mathbf{k}')^2 + |\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_A^{L,TT}(\mathbf{k},\mathbf{k}') =& \frac{(|\mathbf{k}|^2 - 2\mathbf{k}\cdot\mathbf{k}')^2\left((\mathbf{k}\cdot\mathbf{k}')^2 - 2(21+D)\mathbf{k}\cdot\mathbf{k}'|\mathbf{k}'|^2 + |\mathbf{k}'|^2((-3+D)|\mathbf{k}|^2 + (-2+D)|\mathbf{k}'|^2)\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_A^{L,TL}(\mathbf{k},\mathbf{k}') =& \frac{(|\mathbf{k}|^2 - 2\mathbf{k}\cdot\mathbf{k}' + 2|\mathbf{k}'|^2)^2\left(-(\mathbf{k}\cdot\mathbf{k}')^2 + |\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_A^{L,LT}(\mathbf{k},\mathbf{k}') =& \frac{(|\mathbf{k}|^2 - 2\mathbf{k}\cdot\mathbf{k}' + 2|\mathbf{k}'|^2)^2\left(-(\mathbf{k}\cdot\mathbf{k}')^2 + |\mathbf{k}|^2|\mathbf{k}'|^2\right)}{|\mathbf{k}|^2|\mathbf{k}'|^2} \\
\Gamma_A^{L,LL}(\mathbf{k},\mathbf{k}') =& \frac{(|\mathbf{k}|^2 - 2\mathbf{k}\cdot\mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}' - |\mathbf{k}'|^2)^2}{|\mathbf{k}|^2|\mathbf{k}'|^2} \qquad \text{(C.31)}
\end{aligned}
$$

Appendix D

Explicit Consequences of Identities to Lower N-Point Functions

We will now find explicit identities for the lower n-point function from the identities obtained in Section 4.2.

D.1 1-point Functions

Let us start by explicitly writing down the consequences of Ghost number conservation, Eqs. (4.84) – (4.86), to the one-point functions of the theory. Taking the functional derivative of Eq. (4.85) with respect to one of the sources J_ω, $J_{\bar{\omega}}$, I_{sA}, $I_{s\lambda}$ or $I_{s\omega}$ and evaluating for $J = I = 0$ yields

$$\left.\frac{\delta W[J,I]}{\delta J^a_\omega(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta W[J,I]}{\delta J^a_{\bar{\omega}}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta W[J,I]}{\delta I^{ai}_{sA}(x)}\right|_{J=I=0} = 0$$
$$\left.\frac{\delta W[J,I]}{\delta I^{ai}_{s\lambda}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta W[J,I]}{\delta I^a_{s\omega}(x)}\right|_{J=I=0} = 0 \tag{D.1}$$

The same relations follow for the derivatives of $Z[J,I]$ from Eq. (4.84). On the other hand, Eq. (4.63) implies

$$\left.\frac{\delta\Gamma}{\delta A^{ai}(x)}\right|_{J=I=0} = 0 \quad \left.\frac{\delta\Gamma}{\delta\lambda^{ai}(x)}\right|_{J=I=0} = 0 \quad \left.\frac{\delta\Gamma}{\delta\omega^a(x)}\right|_{J=I=0} = 0 \quad \left.\frac{\delta\Gamma}{\delta\bar{\omega}^a(x)}\right|_{J=I=0} = 0 \tag{D.2}$$

and the combination of Eq. (4.64) and (D.1) gives

$$\left.\frac{\delta\Gamma}{\delta I^{ai}_{sA}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta\Gamma}{\delta I^{ai}_{s\lambda}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta\Gamma}{\delta I^a_{s\omega}(x)}\right|_{J=I=0} = 0 \tag{D.3}$$

The last first derivative of Γ can be computed from the stochastic Ward identities, Eq. (4.77)

$$\left.\frac{\delta\Gamma}{\delta I^a_{s\bar{\omega}}(x)}\right|_{J=I=0} = 0 \tag{D.4}$$

Thus, all first derivatives of Γ have to vanish.

D.2 2-point Functions

The consequences of ghost number conservation to the second derivatives of $Z[J, I]$ and $W[J, I]$ are summarised in the following table, indicating for any pair of sources whether the corresponding second derivative (evaluated for $J = I = 0$) is restricted to vanish or not by ghost number conservation

	J_A	J_λ	J_ω	$J_{\bar\omega}$	I_{sA}	$I_{s\lambda}$	$I_{s\omega}$	$I_{s\bar\omega}$
J_A			0	0	0	0	0	
J_λ			0	0	0	0	0	
J_ω	0	0	0		0	0	0	0
$J_{\bar\omega}$	0	0		0			0	0
I_{sA}	0	0	0		0	0	0	0
$I_{s\lambda}$	0	0	0		0	0	0	0
$I_{s\omega}$	0	0	0	0	0	0	0	0
$I_{s\bar\omega}$			0	0	0	0	0	

(D.5)

The analogous result for the second derivatives of the 1PI generating functional $\Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar\omega; I]$ (as well evaluated for vanishing sources $J = I = 0$) is

	A	λ	ω	$\bar\omega$	I_{sA}	$I_{s\lambda}$	$I_{s\omega}$	$I_{s\bar\omega}$
A			0	0	0	0	0	
λ			0	0	0	0	0	
ω	0	0	0				0	0
$\bar\omega$	0	0		0	0	0	0	0
I_{sA}	0	0		0	0	0	0	0
$I_{s\lambda}$	0	0		0	0	0	0	0
$I_{s\omega}$	0	0	0	0	0	0	0	0
$I_{s\bar\omega}$			0	0	0	0	0	

(D.6)

In the following, we will often rely on the information summarised in these tables dropping certain terms that are bound to zero by ghost number conservation from our calculations without further notice. To start with, let us recall the gauge Ward identity in terms of the generating functional of connected correlation functions $W[J, I]$. It was found in Eq. (4.60) to read

$$\int dx \left[J_A^{ai}(x)\, \frac{\delta W[J,I]}{\delta I_{sA}^{ai}(x)} + J_\lambda^{ai}(x)\, \frac{\delta W[J,I]}{\delta I_{s\lambda}^{ai}(x)} + J_\omega^{a}(x)\, \frac{\delta W[J,I]}{\delta I_{s\omega}^{a}(x)} + J_{\bar\omega}^{a}(x)\, \frac{\delta W[J,I]}{\delta I_{s\bar\omega}^{a}(x)} \right] = 0 \tag{D.7}$$

Taking second derivatives, a variety of possibilities arise. For instance, choosing $\delta/\delta J_A^{ai}(x)$ and $\delta/\delta J_A^{bj}(y)$ yields after setting sources to zero

$$\left.\frac{\delta^2 W[J,I]}{\delta J_A^{bj}(y)\, \delta I_{sA}^{ai}(x)}\right|_{J=I=0} + \left.\frac{\delta^2 W[J,I]}{\delta J_A^{ai}(x)\, \delta I_{sA}^{bj}(y)}\right|_{J=I=0} = 0 \tag{D.8}$$

However, due to ghost number conservation both of these terms are zero by themselves. Likewise, the combination of $\delta/\delta J_A^{ai}(x)$ with $\delta/\delta J_\lambda^{bj}(y)$ or $\delta/\delta J_\omega^{b}(y)$ does not lead to any new relation when ghost number conservation is taken into account. The fourth possibility however, combining $\delta/\delta J_A^{ai}(x)$ and a derivative with respect to $J_{\bar\omega}^{b}(y)$, results in the identity

$$\left.\frac{\delta^2 W[J,I]}{\delta J_{\bar\omega}^{b}(y)\, \delta I_{sA}^{ai}(x)}\right|_{J=I=0} + \left.\frac{\delta^2 W[J,I]}{\delta J_A^{ai}(x)\, \delta I_{s\bar\omega}^{b}(y)}\right|_{J=I=0} = 0 \tag{D.9}$$

that will be further exploited in a moment. Considering the combinations of $\delta/\delta J_\lambda^{ai}(x)$ with one of the derivatives $\delta/\delta J_\lambda^{bj}(y)$ or $\delta/\delta J_\omega^{b}(y)$ again only leads to trivial relations in view of ghost number conservation. The pairing of $\delta/\delta J_\lambda^{ai}(x)$ with $\delta/\delta J_{\bar\omega}^{b}(y)$ yields

$$\left.\frac{\delta^2 W[J,I]}{\delta J_{\bar\omega}^{b}(y)\, \delta I_{s\lambda}^{ai}(x)}\right|_{J=I=0} + \left.\frac{\delta^2 W[J,I]}{\delta J_\lambda^{ai}(x)\, \delta I_{s\bar\omega}^{b}(y)}\right|_{J=I=0} = 0 \tag{D.10}$$

However, this relation is a consequence of the two simpler identities

$$\left.\frac{\delta^2 W[J,I]}{\delta J^b_{\bar\omega}(y)\,\delta I^{ai}_{s\lambda}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta^2 W[J,I]}{\delta J^{ai}_{\lambda}(x)\,\delta I^{b}_{s\bar\omega}(y)}\right|_{J=I=0} = 0 \tag{D.11}$$

induced by the stochastic Ward identity (4.74). The remaining possibilities finally, choosing two derivatives with respect to ω, two derivatives with respect to $\bar\omega$, or one with respect to ω, one to $\bar\omega$ again express ghost number conservation only. Hence, up to the level of second derivatives Eq. (D.9) is the only restriction imposed by the gauge BRST symmetry beyond relations that already follow from the stochastic Ward identity or simply are a consequence of ghost number conservation. Implications of the stochastic Ward identities (4.74) and (4.77) are most importantly the vanishing of the auxiliary field propagator to all orders

$$G^{(\lambda\lambda)}{}^{ab}_{ij}(x,y) = \left.\frac{\delta^2 W[J,I]}{\delta J^{ai}_{\lambda}(x)\,\delta J^{bj}_{\lambda}(y)}\right|_{J=I=0} = 0 \tag{D.12}$$

or, equivalently, of the (AA) self-energy component

$$\Pi^{(AA)}{}^{ab}_{ij}(x,y) = \left.\frac{\delta^2 \Gamma}{\delta A^{ai}(x)\,\delta A^{bj}(y)}\right|_{J=I=0} - (\Delta^{-1})^{(AA)}{}^{ab}_{ij}(x,y) = 0 \tag{D.13}$$

where in addition to Eq. (4.77) it was used that the (AA) component of the inverse free propagator is zero too (cf. Eq. (C.9) in the following chapter). Note that Eq. (D.13) is a special case of the general statement that there are no pure gauge field vertices in the theory: All proper vertex functions of the form

$$\Gamma^{(AA\ldots A)}{}^{ab\ldots c}_{ij\ldots k}(x,y,\ldots,z) = \left.\frac{\delta^n \Gamma}{\delta A^{ai}(x)\,\delta A^{bj}(y)\cdots\delta A^{ck}(z)}\right|_{J=I=0} \tag{D.14}$$

vanish as an immediate consequence of the stochastic Ward identity (4.77). Further implications up to second derivatives (neglecting those only expressing ghost number conservation) are

$$\left.\frac{\delta^2 \Gamma}{\delta \omega^b(y)\,\delta I^{ai}_{s\lambda}(x)}\right|_{J=I=0} = 0 \qquad \left.\frac{\delta^2 \Gamma}{\delta A^{ai}(x)\,\delta I^{b}_{s\bar\omega}(y)}\right|_{J=I=0} = 0 \tag{D.15}$$

together with the equivalent identities (D.11),

$$\left.\frac{\delta \Gamma}{\delta I^{a}_{s\bar\omega}(x)}\right|_{J=I=0} = -\left.\frac{\delta W}{\delta I^{a}_{s\bar\omega}(x)}\right|_{J=I=0} = 0 \tag{D.16}$$

and for completeness finally

$$\left.\frac{\delta^2 \Gamma}{\delta I^{a}_{s\bar\omega}(x)\,\delta I^{b}_{s\bar\omega}(y)}\right|_{J=I=0} = -\left.\frac{\delta^2 W}{\delta I^{a}_{s\bar\omega}(x)\,\delta I^{b}_{s\bar\omega}(y)}\right|_{J=I=0} = 0 \tag{D.17}$$

This last identity, however, does not lead to a simple relation among the lower n-point functions because both of the derivatives act on sources of the BRST transformed fields. In general, to make sense of the above identities we will have to translate the derivatives of the I-type to such with respect to sources of the fundamental fields. For instance, one has

$$\begin{aligned}
\frac{\delta Z}{\delta I^{ai}_{sA}(x)} &= \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar\omega\,\left(D^{ab}_i(x)\,\omega^b(x)\right)\exp\Big\{(\ldots)\Big\} \\
&= \int \mathcal{D}\mathbf{A}\mathcal{D}\boldsymbol{\lambda}\mathcal{D}\omega\mathcal{D}\bar\omega\Big(\partial_i\Big(-\frac{\delta}{\delta J^a_\omega(x)}\Big) - g f^{abc}\frac{\delta}{\delta J^{ci}_A(x)}\Big(-\frac{\delta}{\delta J^b_\omega(x)}\Big)\Big)\exp\Big\{(\ldots)\Big\} \\
&= -\partial_i\frac{\delta Z}{\delta J^a_\omega(x)} + g f^{abc}\frac{\delta^2 Z}{\delta J^{ci}_A(x)\,\delta J^b_\omega(x)}
\end{aligned} \tag{D.18}$$

where the dots abbreviate the usual exponent of the generating functional as given in Eq. (4.55). Expressing this identity in terms of $W = \ln Z$ yields

$$\frac{\delta W}{\delta I^{ai}_{sA}(x)} = -\partial_i \frac{\delta W}{\delta J^a_\omega(x)} + g f^{abc}\left(\frac{\delta^2 W}{\delta J^{ci}_A(x)\,\delta J^b_\omega(x)} + \frac{\delta W}{\delta J^{ci}_A(x)}\,\frac{\delta W}{\delta J^b_\omega(x)}\right) \tag{D.19}$$

Analogously, one obtains after some algebra

$$\frac{\delta W}{\delta I^{ai}_{s\lambda}(x)} = g f^{abc}\left(\frac{\delta^2 W}{\delta J^{ci}_\lambda(x)\,\delta J^b_\omega(x)} + \frac{\delta W}{\delta J^{ci}_\lambda(x)}\,\frac{\delta W}{\delta J^b_\omega(x)}\right) \tag{D.20}$$

$$\frac{\delta W}{\delta I^{a}_{s\omega}(x)} = \frac{1}{2} g f^{abc}\left(\frac{\delta^2 W}{\delta J^{c}_\omega(x)\,\delta J^b_\omega(x)} + \frac{\delta W}{\delta J^{c}_\omega(x)}\,\frac{\delta W}{\delta J^b_\omega(x)}\right) \tag{D.21}$$

$$\begin{aligned}\frac{\delta W}{\delta I^{a}_{s\bar\omega}(x)} &= i\sigma\partial_i \frac{\delta W}{\delta J^{ai}_\lambda(x)} - i\sigma g f^{abc}\left(\frac{\delta^2 W}{\delta J^{ci}_A(x)\,\delta J^{bi}_\lambda(x)} + \frac{\delta W}{\delta J^{ci}_A(x)}\,\frac{\delta W}{\delta J^{bi}_\lambda(x)}\right)\\ &\quad + g f^{abc}\left(\frac{\delta^2 W}{\delta J^{c}_\omega(x)\,\delta J^b_{\bar\omega}(x)} + \frac{\delta W}{\delta J^{c}_\omega(x)}\,\frac{\delta W}{\delta J^b_{\bar\omega}(x)}\right)\end{aligned} \tag{D.22}$$

With these substitutions Eq. (D.9) translates to

$$\begin{aligned}&\partial_i G^{(\omega)\,ab}(x,y) - i\sigma\partial_j G^{(A\lambda)}{}^{ab}_{ij}(x,y) = \\ &\quad -g f^{acd} W^{(\bar\omega\omega A)\,bcd}{}_i(y,x,x) - i\sigma g f^{bcd} W^{(AA\lambda)}{}^{acd}_{ijj}(x,y,y) + g f^{bcd} W^{(\bar\omega\omega A)\,cda}{}_i(y,y,x)\end{aligned} \tag{D.23}$$

To further proceed, we express the connected three-point functions by their 1PI counterparts and transform into momentum space. Especially note that we pull out the momentum conserving delta function from the definition of our proper vertices. Hence, only $N-1$ momentum variables appear in the argument of a N-point vertex. For instance, we use $\Gamma^{(\bar\omega\omega G)\,abc}{}_j(k_1,k_2)$ where the superscript G is either the gauge field A or the auxiliary field λ and k_1 and k_2 refer to the (incoming) momenta along the ghost lines leaving and entering the vertex in this order. Accordingly, in $\Gamma^{(FGH)\,abc}{}_{ijk}(k_2,k_3)$ with $F,G,H \in \{A,\lambda\}$ the two arguments k_2 and k_3 refer to the incoming momenta along the G and H line respectively. With these definitions, Eq. (D.23) takes the form

$$\begin{aligned}&ik^i G^{(\omega)\,ab}(k) + \sigma k^j G^{(A\lambda)}{}^{ab}_{ij}(k) = \\ &\quad + G^{(\omega)\,b'b}(k)\int\frac{d^Dk'}{(2\pi)^D}\, g f^{acd}\, G^{(\omega)\,cc'}(k')\, G^{(AF)}{}^{dd'}_{ii'}(k-k')\,\Gamma^{(\bar\omega\omega F)\,c'b'd'}{}_{i'}(-k',k)\\ &\quad - G^{(AF)}{}^{aa'}_{ii'}(k)\int\frac{d^Dk'}{(2\pi)^D}\, g f^{bcd}\Big[G^{(\omega)\,c'c}(k')\, G^{(\omega)\,dd'}(k'-k)\,\Gamma^{(\bar\omega\omega F)\,d'c'a'}{}_{i'}(k-k',k')\\ &\qquad\qquad -i\sigma\, G^{(A\lambda)}{}^{c'c}_{j'j}(k')\, G^{(AG)}{}^{dd'}_{jk'}(k'-k)\,\Gamma^{(FGA)}{}^{a'd'c'}_{i'k'j'}(k-k',k')\Big]\end{aligned} \tag{D.24}$$

The indices F and G in this equation are summation indices taking the two values A and λ. However, as we will show now, the stochastic Ward identity leads to a cancellation among some of the terms involved. To this end, let us express also the identities derived from the stochastic Ward identity in the language of full propagators and proper vertex functions. As mentioned above, identity (D.16) relates the normalisations of the gauge ghost and mixed auxiliary/gauge field propagator

$$g f^{abc}\int\frac{d^Dk}{(2\pi)^D}\left[G^{(\omega)\,cb}(k) - i\sigma G^{(A\lambda)}{}^{cb}_{ii}(k)\right] = 0 \tag{D.25}$$

From the first of the Eqs. (D.15) one obtains after some relabelling

$$\int\frac{d^Dk'}{(2\pi)^D}\, g f^{bcd}\, G^{(A\lambda)}{}^{c'c}_{i'i}(k')\, G^{(\omega)\,dd'}(k'-k)\,\Gamma^{(\bar\omega\omega A)\,d'ac'}{}_{i'}(k-k',-k) = 0 \tag{D.26}$$

from the second equation

$$\int \frac{d^Dk'}{(2\pi)^D}\, gf^{bcd}\, G^{(\omega)\,c'c}(k')\, G^{(\omega)\,dd'}(k'-k)\, \Gamma^{(\bar\omega\omega A)\,d'c'\,a}_{i}(k-k',k')$$
$$-\,i\sigma \int \frac{d^Dk'}{(2\pi)^D}\, gf^{bcd}\, G^{(A\lambda)\,c'c}_{j'j}(k')\, G^{(A\lambda)\,dd'}_{jk'}(k'-k)\, \Gamma^{(A\lambda A)\,ad'c'}_{i\,k'j'}(k-k',k') = 0 \qquad \text{(D.27)}$$

Here we have used $\Gamma^{(FGH)\,abc}_{ijk}(k_2,k_3) = \Gamma^{(GHF)\,bca}_{jki}(k_3,-k_2-k_3)$ in accordance with our definition of the vertex functions.

Let us now come back to Eq. (D.24), that was found to be the expression of the gauge Ward identity on the level of second derivatives. With the summation index F taking the value A, the second integral in Eq. (D.24) consists of three terms: the one with the two ghost propagators and two copies of the second term corresponding to the two possible values $G=\lambda$ and $G=A$. The last of these terms is zero because it contains $\Gamma^{(AAA)}$. Moreover, the remaining two terms cancel each other due to Eq. (D.27) as a consequence of the stochastic Ward identity. Hence, there is only a contribution of the second integral in Eq. (D.24) for $F=\lambda$. The first integral, however, contributes for both choices $F=\lambda$ and $F=A$ (and likewise if F is set to λ in the second integral, G can still take both values $G=\lambda, A$).

The gauge BRST symmetry therefore leads to the following identity to be obeyed by the full propagators and proper vertex functions of the theory

$$ik^i G^{(\omega)\,ab}(k) + \sigma k^j G^{(A\lambda)\,ab}_{i\,j}(k) =$$
$$+\; G^{(\omega)\,b'b}(k) \int \frac{d^Dk'}{(2\pi)^D}\, gf^{acd}\, G^{(\omega)\,cc'}(k')\, G^{(AA)\,dd'}_{i\,i'}(k-k')\, \Gamma^{(\bar\omega\omega A)\,c'b'\,d'}_{i'}(-k',k)$$
$$+\; G^{(\omega)\,b'b}(k) \int \frac{d^Dk'}{(2\pi)^D}\, gf^{acd}\, G^{(\omega)\,cc'}(k')\, G^{(A\lambda)\,dd'}_{i\,i'}(k-k')\, \Gamma^{(\bar\omega\omega\lambda)\,c'b'\,d'}_{i'}(-k',k)$$
$$-\; G^{(A\lambda)\,aa'}_{i\,i'}(k) \int \frac{d^Dk'}{(2\pi)^D}\, gf^{bcd}\Big[G^{(\omega)\,c'c}(k')\, G^{(\omega)\,dd'}(k'-k)\, \Gamma^{(\bar\omega\omega\lambda)\,d'c'\,a'}_{i'}(k-k',k')$$
$$-i\sigma\, G^{(A\lambda)\,c'c}_{j'j}(k')\, G^{(AA)\,dd'}_{jk'}(k'-k)\, \Gamma^{(\lambda AA)\,a'd'c'}_{i'k'j'}(k-k',k')$$
$$-i\sigma\, G^{(A\lambda)\,c'c}_{j'j}(k')\, G^{(A\lambda)\,dd'}_{jk'}(k'-k)\, \Gamma^{(\lambda\lambda A)\,a'd'c'}_{i'k'j'}(k-k',k')\Big] \qquad \text{(D.28)}$$

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