Francis S. Macaulay
(1862 – 1937)

MAX NOETHER.

Heidelberger Texte zur Mathematikgeschichte

Max Noether passed away on December 13th, 1921, at Erlangen, where he had been Professor of Mathematics for nearly fifty years. He was born at Mannheim on September 24th, 1844, and took his doctorate at Heidelberg in 1868. He was a corresponding or honorary member of many learned societies, including the Academies of Berlin, Göttingen, Munich, Budapest, Copenhagen, Turin, the Royale Accademia dei Lincei, and the Institut de France. He was elected an honorary member of the London Mathematical Society in 1913 at the same time as Adolf Hurwitz, Paul Painlevé, Corrado Segre, and Woldemar Voigt. He seldom, if ever, visited England, but expressed his regret that he was unable to attend the International Congress of Mathematics at Cambridge in 1912, as be had reached an age when he was averse from travelling.

Although not so well known to fame out of his own country as some other famous mathematicians, he was deservedly held in the highest esteem, and after the death of Clebsch became the recognised leader of the algebraic-geometric school in Germany. He brought himself early to the front by his celebrated and fundamental "Satz aus der Theorie der algebraischen Functionen" (*Math. Annalen*, Vol. 6), which he followed in the next volume with the classical memoir "Ueber die algebraischen Functionen und ihre Anwendung in der Geometric," written in collaboration with A. Brill. This memoir formed the solid foundation, and gave rise to the name, of "Geometry on a Curve," which has been developed and extended, especially in Italy, into a geometry of greater depth and power than anything preceding it. How remarkable these developments have been is clearly shown in H. F. Baker's fascinating review "Some Advances in the Theory of Algebraic Surfaces," published in our *Proceedings*, Ser. 2, Vol. 12, pp. 1–40.

Noether's mind was naturally intuitive, but he mistrusted intuition, and was apt to let anything suggested by it pass out of his thought. It seemed as if this cramped his powers to some extent in his later years. Be found perhaps in his subject that intuition was often liable to mislead. He was of course never content without algebraic or arithmetic proof, but had sometimes to be satisfied with an incomplete proof. Although naturally impatient he would take infinite pains to understand the thought of others, and to give them abundant help out of his own ample resources. There are many, including the writer of this note, who are grateful to him for his help. He had searching and peculiar methods of his own for testing the truth of things.

His name will always be intimately associated with the greatest of mathematical journals, the *Mathematische Annalen*. Scarcely a year passed from the appearance of the 2nd volume in 1870 till the 83rd in 1921 in which he did not make an important contribution to its pages. With the publication of the 42nd volume in 1893 he formally joined the editorial staff, and an obituary notice at the beginning of the 84th volume bears eloquent testimony to the great value of the services he rendered it. He contributed greatly to the advancement of mathematical science in three distinct ways: by the new and fruitful ideas contained in his original researches, by the patient investigation and encouragement he gave

to other writers, and by his acutely critical and detailed historical work. The last consisted of the great work which he and A. Brill compiled in 1894, "Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit," for the third volume of the *Jahresbericht der Deutschen Mathematiker Vereinigung*; and elaborate memorial surveys of the work of many famous mathematicians, beginning with Cayley and Sylvester and ending with Zeuthen, which he contributed to the pages of the *Math. Ann.* The "Entwicklung, etc." may be regarded as the completion and rounding off of his algebraic-geometric work, and in it the Noetherian Theorem comes fully into its own.

Of Noether's original (as distinguished from critical and historical) work the most interesting, because of its far reaching developments, is that which was published in Vols. 6 and 7 of the *Math. Ann.*, to which reference has already been made. This work is still so little known in England, notwithstanding that it is fully reproduced in Clebsch's "Geometrie," that there seems good reason to give an account of what it contains. In his first communication to the *Math. Ann.* (Vol. 2, p. 293) he makes use of a theorem which, up till then, had been taken for granted, viz. that a plane curve f which passed through all the points of intersection of two given curves ϕ, ψ had an equation of the form

$$f \equiv A\phi + B\psi = 0.$$

The real question was whether these purely geometrical conditions were sufficient, and he there gave a formal proof that they were sufficient in the case in which all the points were simple and finitely separated. He was thus led to consider the more general and fundamental problem as to what were the simplest and sufficient conditions which would require a curve f to take the form $A\phi + B\psi$, when ϕ, ψ had common multiple points with contact of any degree of complexity. To this question he gave the answer in Vol. 6, p. 351 (which had been previously published in the *Göttinger Nachrichten*, 1872, p. 490), in a form which soon came to be known as Noether's fundamental theorem. He recognised intuitively that the problem was one of approximation, and that the answer to it could be formulated thus:— The necessary and sufficient conditions that f must satisfy in order that it may be of the form $A\phi + B\psi$, are that f should be capable of being written in the form $A'\phi + B'\psi$ when the origin is moved in succession to each and every point of intersection of ϕ, ψ; where A', B' are ordinary power series which are, for each common point of ϕ, ψ, quite independent of what they may be for all the other points. This approximation of f with $A'\phi + B'\psi$ will have been carried far enough when, on equating coefficients of the terms in f and $A'\phi + B'\psi$ of a sufficiently high degree, the undetermined coefficients of A', B' cannot all be eliminated so as to give a *new* relation among the coefficients of f alone, i.e. a relation which is independent of all the similar relations obtainable from the equating of coefficients of terms of all *lower* degrees. It will be noticed that no conditions have to be satisfied at infinity.

To the proof of this he added a simple rider, of fundamental importance for its applications, that if ϕ, ψ have no common tangent at a common multiple point, of order i for ϕ and j for ψ, then all the relations for the coefficients of f alone corresponding to that intersection of ϕ, ψ come from the equating of coefficients up to degree $i+j-2$, so that it is *sufficient* (but of course not necessary) for f to have a multiple point of order $i+j-1$ at that point. At a later time he extended this, the so-called "simple" case, to the case where ϕ, ψ have higher singularities at a common point, which can be resolved by repeated transformation of type

$$x_1' : x_2' : x_3' = \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}$$

into separate common multiple points coming under the "simple" case. Though much has been written about the theorem these are the only important additions that have been made to it, beyond its extension in various ways to surfaces and hyperspaces. Of these the Lasker-Noether (named by Lasker the Noether-Dedekind) theorem is the most general and fundamental (*Math. Ann.*, Vol. 60, p. 51; and *Cambridge Tracts in Maths., etc.*, No. 19, p. 61).

Brill and Noether's classical memoir "Ueber die algebraischen Functionen, etc.," in Vol. 7 of the *Math. Ann.*, is founded essentially on the investigation of the properties of point-groups $g_R^{(r)}$ on a given curve $f = 0$ cut out by a linear system of curves $\lambda_0\phi_0 + \lambda_1\phi_1 + \ldots + \lambda_r\phi_r = 0$, where $\phi_0, \phi_1, \ldots, \phi_r$ are given linearly independent curves of the same order. Any fixed points common to f, $\phi_0, \phi_1, \ldots, \phi_r$ are left out of account, and R denotes the number of simple points in each group, while r is the *freedom* of the linear series $g_R^{(r)}$, i.e. the number of the R points which can be chosen at will to determine a particular group G_R of the series $g_R^{(r)}$. Thus the curve f of order n can be birationally transformed to a curve F of order R by means of the transformation equations

$$x_1' : x_2' : x_3' = \phi_0 : \phi_1 : \phi_2;$$

and this is possible so long as $r \geq 2$. (The only exceptions are curves for which the genus $p = 0, 1, 2$, which require separate treatment; and hyperelliptic curves, or curves on which exist a series $g_2^{(1)}$, since in this case the transformation is not birational.) Hence one important problem is to find the least value of R for which a series $g_R^{(2)}$ exists, as this value of R gives the lowest value of the order of a curve F into which f can be birationally transformed.

It was shown that the series $g_R^{(r)}$ on f which were of the greatest interest and importance were those which were cut out by a linear system $(\phi_0, \phi_1, \ldots, \phi_r)$ adjoined to f, especially when the curves $\phi_0, \phi_1, \ldots, \phi_r$ were of order $n - 3$. A curve ϕ adjoined to f is a curve having a multiple point of order $i - 1$ at each ordinary multiple point of order i on f, and a corresponding higher singularity at each point of higher singularity on f. If G_R is any given group of R points on f, the series $g_R^{(r)}$ of greatest freedom r, having G_R as one of its groups, can be

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found by drawing any adjoined curve through G_R to cut f again in a group G_Q (apart from the multiple points of f). Then the whole system of adjoined curves through G_Q of the order of ϕ will cut out on f the required series $g_R^{(r)}$ of greatest freedom. *This series $g_R^{(r)}$ is a fixed series, independent of the particular group G_Q by means of which it is derived, i.e. it is a series uniquely dependent on any one of its groups G_R.* This is one way of stating the Brill-Noether theorem of residuation, and, is proved by the help of the rider to Noether's theorem mentioned above. This series $g_R^{(r)}$ is called a *complete coresidual* series (every one of its groups being *residual* to any group G_Q to which any one of its groups G_R is residual) or the linear series of *unrestricted freedom* which any given group G_R determines.

We will suppose now that any series denoted by $g_R^{(r)}$ is a series of unrestricted freedom r unless the contrary is stated. In any group G_R of $g_R^{(r)}$ r points can be chosen freely, and the remaining $R - r$ points are dependent on the r points. Also $R - r \leq p$, where p is the genus of the curve f. *Further, if $R - r < p$, the group G_R lies on an adjoined curve of order $n - 3$.* (This is still more true if the freedom r is restricted, for, on removing the restriction, r would increase and $R - r$ diminish ; but with restricted freedom we may have $R - r \geq p$ and yet G_R on an adjoined curve of order $n - 3$.) Now, taking $R - r < p$, and drawing all the adjoined $(n\text{-}3)$-ics which pass through a fixed group G_R of $g_R^{(r)}$, we determine the series $g_Q^{(q)}$ of freedom q, which is residual to the series $g_R^{(r)}$, where $Q + R = 2p - 2$. The two mutually residual series $g_Q^{(q)}$, $g_R^{(r)}$ may be considered together; or $g_R^{(r)}$ may be considered alone, in which case q must be regarded as a number not less intimately associated with the series $g_R^{(r)}$ than r is; for r is the freedom of the series $g_R^{(r)}$ on f, while q is the freedom of the whole system of adjoined $(n\text{-}3)$-ics through any fixed group G_R of $g_R^{(r)}$. *The numbers R, r, q, p are connected by the relation*

$$R = p - q + r - 1.$$

This theorem is called the Riemann-Roch theorem for plane curves, and is also known under the more definitive title of the Brill-Noether reciprocity theorem. Other ways of stating the theorem are

$$(i) \quad Q - R = 2(q - r), \quad Q + R = 2(p - 1),$$

and

$$(ii) \quad R - 2r = Q - 2q = p - q - r - 1 \geq 0.$$

In seeking the number of relations that must exist between the points of a group G_R on f, in order that the unrestricted series $g_R^{(r)}$ determined bv G_R may be of freedom r, Brill and Noether obtained the formula $(q+1)r$, or $r(p+r-R)$.

This was got by counting equations of condition, and is only valid under stringent limitations, as also the inequality

$$p \geq (q+1)(r+1)$$

derived from it. It is possibly correct for curves f of given genus which are *general of their kind*, and in the memoir it is only applied to such curves, but it requires a strict proof. Assuming the correctness of the formula, then, in fixing the group G_R on f there are $r(p+r-R)$ points dependent on the rest by the equations of condition and $R - r(p+r-R)$ which can be chosen freely. This cannot fall below r, since a group of $g_R^{(r)}$ has r free points even after G_R has been fixed. Hence

$$R - r(p+r-R) \geq r.$$

This inequality determines the least value R can have for a given value of r in a series $g_R^{(r)}$. When $r = 2$, it becomes $R \geq \frac{2}{3}(p+3)$. Hence a *general* curve f of given genus p can be birationally transformed into a curve F of order R equal to the integral part of $\frac{2}{3}(p+4)$. A curve of less order with given genus p, such as a quintic with $p = 5, 6$ or an n-ic with less than $\frac{1}{2}(n-2)(n-4)$ double points, is not a *general* curve of genus p.

The above result is not only interesting, and even surprising, in itself, but gains additional interest from its relation to Riemann's theorem that a curve f of genus p (general of its kind) can be birationally transformed to a curve F whose coefficients depend on $3p - 3$ parameters or moduli. A curve f of order n and genus p has $\frac{1}{2}n(n+3)$ coefficients, which can be reduced to $\frac{1}{2}n(n+3) - 8$ by a collinear transformation; these satisfy a number of independent relations equal to the number of the double points, viz. $\frac{1}{2}(n-1)(n-2) - p$, leaving $3n + p - 9$ effective constants for f. The curve f can be transformed to a curve F of order R as above, and the process of transformation admits of imposing 0, 1, 2 additional geometrical conditions on F (such as requiring it to have 0, 1, 2 points of undulation) according as $p = 0, 1, 2 \bmod 3$. The number of effective constants left for F are therefore $3R + p - 9$, $3R + p - 10$, $3R + p - 11$, i.e. $3p - 3$ in all three cases, since $3R = 2p + 6$, $2p + 7$, $2p + 8$ when $p = 0, 1, 2 \bmod 3$. This really only shows that the number of moduli can be brought as low as $3p - 3$, not that it cannot be brought lower; but other proofs of the result are given in the memoir.

We have dwelt at length on these two memoirs because of their fundamental and historical importance. The greater part of Noether's original work may be said to have been a natural continuation of them. He contributed much besides to the theory of elimination, higher singularities, the geometry of hyperspace, and Abelian and Theta functions.

<div style="text-align: right">F. S. M.</div>